STUDY OF REPEATABLE GAIT FOR PLANAR FIVE-LINK BIPED WALKING ON LEVEL GROUND

BY

BIN MA

A Thesis
Submitted to the Faculty of Graduate Studies
In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF SCIENCE

*****

Department of Mechanical & Industrial Engineering
The University of Manitoba
Winnipeg, Manitoba

Abstract

The main purpose of this thesis is to contribute to the study of the repeatability of bipedal locomotion based on the parametric formulation introduced by Hurmuzlu. Parametric formulation is a systematic method that ties the constraint functions to the resulting joint angle profiles for bipedal robots. Acceptable joint angle profiles for continuous and steady walking must satisfy the additional condition of being repeatable. This extra condition imposes restrictions on the selection of constraint functions, gait parameters and the initial conditions of each step, and can make the synthesis of joint angle profiles extremely challenging. For first order constraint systems, firstly, this thesis presents a correlation between the initial conditions and the gait parameters that will create repeatable gait. Secondly, a general necessary and sufficient condition for producing repeatable gait is proposed. This condition provides a guideline for selecting constraint functions and their associated gait parameters in the context of producing repeatable gait.. For higher order constraint systems, the possibility and challenges to satisfy the repeatability condition are discussed, and some feasible approximate methods are recommended. Time-polynomial technique is used to compare two optimization criteria that can be a part of the constraint systems. This research can not only provide valuable insights into bipedal walking, but also serve as a stepping-stone for employing parametric formulation with various constraint functions.

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES

*****

COPY RIGHT PERMISSION

STUDY OF REPEATABLE GAIT FOR PLANAR FIVE-LINK BIPED WALKING ON LEVEL GROUND

BY

BIN MA

A Thesis / Practicum Submitted to the Faculty of Graduate Studies
of The University of Manitoba
In Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 General introduction

In this section, we mainly review the development of the research of biped robots from the standpoint of the research motivation, mathematical modeling, motion planning and motion control.

1.1.1 Motivation

The biped robots are legged robots that are designed to imitate human-type locomotion. Normal walking is one of the most fundamental motions of a human body. Although each individual walks with personal characteristics, the basic walking pattern is the same. In the process of normal walking, the erect upper body moves forward with one leg supporting the whole body while the other leg swinging forward. As the upper body passes over the supporting leg, the swing leg has to move ahead of the upper body in preparation for touching the ground and becoming the supporting leg.

Interest in biped robots has increased over the last thirty years and the scientific community agrees in considering this displacement platform promising in view of developing robots capable of effectively moving in unstructured and rough environments. Moreover, it is as obvious as interesting that anthropomorphic biped robots will potentially be able to move in all the environments where humans do due to the obstacle avoidance capabilities which do not yet exist. In particular, biped robots, under suitable mechanical design, are potentially capable of producing gaits involving very little input energy (apart from the restitution energy needed to compensate for the losses due to friction and contact) [1]. Nevertheless, we are still far from being able to realize reliable and reasonably affordable biped robots for practical applications. This field of advanced robotics suffers for the lack of knowledge in many involved areas such as mathematical modeling, motion planning and motion control.

In general, dynamic modeling of the biped locomotion system includes three parts: (1) developing the mathematical model, (2) synthesizing the joint angle profiles and (3) designing control algorithms for motion regulation. With the development of computer utilities and computational techniques, dynamic modeling has become a useful mathematical tool for engineers who have been interested in analyzing bipedal locomotion in designing devices of locomotion for the handicapped, understanding the control laws of human walking and deriving the control algorithms of biped locomotion mechanisms. A lot of work needs to be done to improve actuators, sensors, materials, energy accumulator, hardware and software that can be utilized to realize user-friendly biped robots. However, the technology available nowadays is still inadequate.

1.1.2 Mathematical model

The procedure of mathematical modeling of a biped locomotion system includes the development of the structure of the kinematic model and the development of the dynamic equations of motion.

A human locomotion system represents an extremely complex dynamic system both from the aspect of mechanical-structure complexity and control system complexity [2]. For a human, there are more than 300 degrees of freedom involved in the complete skeletal activity. For natural human walking, 20 or more degrees of freedom may be involved [3]. The studies of such a complex system require certain simplifications. It is critical to select mechanical models with few degrees of freedom to keep the equations of motion at a manageable level, yet with enough degrees of freedom to adequately describe the motions of interest.

The first and simplest model used for the study of some of the characteristics of human walking is the inverted pendulum [3] [4]. More complex models with more degrees of freedom were used mainly after the 1980's. Miura *et al* [5] have used a 3-link model (torso and two simple legs) to construct a biped robot the behaviour of which was studied on both the sagittal and frontal planes. Fursho *et al* [6] [7] have used a planar 5-link model as a basis for the construction of their initial biped, which was improved by adding two more links [8].

A 5-link model has been widely employed [9] - [14]. It can be treated as a basis for the construction of the biped, which includes the upper body (one link), thighs (two links) and shanks (two links). The upper body of this biped model, which includes the head, arms and trunk, has been considered as a massive rigid inverted pendulum. The swing motion of the arms and the motion between the thorax and pelvis are ignored. The upper body is connected to two legs with two hinge joints. Each leg consists of two massive rigid links as a thigh and a shank. All links are connected by hinge joints. The feet are considered massless and, therefore, the dynamic structures of the feet are neglected. The five rigid links have five degrees of rotational freedom if the study of the motion of biped robots is constrained in a sagittal plane, and the biped is considered as a planer model.

Once a physical model is set up, the corresponding dynamic equations of a biped robot can be developed. The prevailing methods of the derivation of the dynamic equations are the Euler-Lagrange method and the Lagrangian method [10]. The equations of motion are used as the basis to describe the motion of the biped model and for the development of a control algorithm. An entire process of walking is made up of the single support phase, the impact between the end of the swing leg and the walking surface, the double support phase and the exchange of the supporting leg.

1.1.3 Motion planning

To design the joint angle profiles that describe the human-type locomotion of a biped system is another challenging task. A well-structured approach of designing the joint angle profiles, that ties the resulting gait patterns with the physically coherent parameters, is desired. Hurmuzlu [9] developed a systematic approach that can be followed to formulate objective functions. Such objective functions are cast in terms of the step length, walking speed, maximum clearance of the swing leg, and bias angle of the stance knee that can be used to prescribe the gait of a planar five-link biped system during a single support phase. Hurmuzlu's method is utilized in this thesis with different constraint functions.

A major challenge of using such a systematic approach to obtain the joint angle profiles is solving a set of constraint equations combined with differential and algebraic equations. There is no general way to solve the combination of differential and algebraic equations. Meanwhile, any acceptable joint angle profiles for continuous and steady gait must satisfy the additional condition of being repeatable. This extra condition imposes restrictions on the selection of constraint equations, gait parameters and the initial posture of each step, and makes the synthesis of joint angle profiles extremely challenging.

Walking with the hypertension of the knee joint is not desirable. As such the joint angle profiles obtained must ensure that the knee of the swing leg does not bend backwards during a step.

Recently, many studies have been devoted to optimal trajectory design [11], [14], [23] – [26], [31], [34]. From a practical point of view, it seems reasonable to search for a trajectory that fulfills a certain objective in terms of the gait parameters (such as the walking speed, the step length and the step frequency, etc.) while minimizing the input energy. The problem of determining the existence of these so-called natural gaits is complex and only a few partial analytical results are available [15], which are only for simple systems or a linearized model. The energy-optimal trajectories for highly non-linear equations of a complex robot may be found only numerically.

1.1.4 Motion control

The control of biped walking machines remains a challenge due to the high degree of complexity, organization and efficiency required to maintain balance. To control the walking motion of a biped, control algorithms are applied. The success of the control action in tracking the prescribed motion can be measured by two factors: (1) the periodicity of the resulting gait patterns, and (2) the magnitude of the residual tracking error [13]. It is important to note that the system's response never coincides with the prescribed one throughout the step cycle for any control actions. This is due to system uncertainties and the disturbance caused by the contact event. A successful controller is one that eliminated the uncertainties and disturbance rapidly during the early stage of each step.

Theoretical and applied studies of biped locomotion have provided results on kinematic and dynamic modeling, and control algorithms. Vukobratovic [16] has done one of the most extensive studies of biped locomotion. Raiber [17] built and implemented various control algorithms in running machines having one to four legs. McGeer [18] studied and implemented passive walking machines driven only by gravity. In Japan, biped locomotion has been studied using robotics, control theory, and sensor-based control [19].

The sliding mode control system [10] is known to be highly insensitive to parametric uncertainty and disturbance. The basic idea of the sliding mode control is to transform the original system in one state space into a system in a new state space. A time-varying surface $r(t)$, which is also referred to as a sliding surface, is defined in the new state space by equation $r=0$. This equation represents a set of linear differential equations that has a unique solution so that the tracking error is equal to zero. Thus, the problem of tracing the desired trajectories is reduced to that keeping r at zero. Once the system trajectories lie on the sliding surface, the system trajectories follow the desired one. The control law contains a discontinuous term. Due to the unavoidable delay in switching between the control laws, chattering occurs at the discontinuity surface. In addition, there is a reaching phase problem. In the reaching phase, the system trajectories are sensitive to parameters variations. It is a challenge to design a sliding mode control algorithm that can eliminate all of these problems.

1.2 Literature survey of the repeatability condition for biped robots

Since the emphases of this thesis are on the repeatability of gait and optimization of joint angle profile design, the literature survey is mainly focused on repeatability and optimal trajectory designs.

In joint angle profile designs, there is a lack of systematic methods for synthesizing the joint angle profiles, and most previous methods have been based on trial and error. Tzafestas *et al.* [10] developed a sliding mode control for the motion regulation of a five-link bipedal model walking on level ground and climbing stairs. The joint angle profiles were synthesized based on the effects of gravity on the gait patterns and the requirement that the gait must be repeatable. Furusho and Masubuchi [6] developed numerical and experimental models for five-link bipeds. Their approach is based on linearizing the equations of motion around the vertical equilibrium point and applying proportional plus derivative feedback at the individual joints to track the prescribed joint angle profiles. However, a coherent method for specifying the joint angle profiles was never mentioned. Vukobratovic *et al.* [2] studied locomotion directly. The fundamental aspect of their approach is the adaptation of human walking data to prescribe the motion of the lower limbs. Two immediate problems arise when using human walking data directly. First, a complex dynamic model is required. Secondly, the designer has no freedom to synthesize the joint angle profiles based on tangible gait characteristics, such as the walking speed, step length or step elevation. None of the aforementioned approaches gave a well-defined method for the design of joint angle profiles.

Hurmuzlu [9] developed a parametric formulation that ties the constraint equations and joint angle profiles. The constraint equations are cast in terms of coherent physical characteristics of gait. In light of prevailing locomotion literature, Humuzlu's method fills the gap in the design of walking machines regarding the specification of constraint equations. He further found a correlation between the parameters used in the constraint equations and the outcome of the contact of the limb with the walking surface. Hurmuzlu has applied his method to the design of joint angle profiles for a five-link bipedal model. His five constraint equations lead to four algebraic equations and one first-order ordinary differential equation.

In all of the above design methods, two questions cannot be avoided: 1) what is the repeatability for biped robots, and 2) how can we satisfy the repeatability condition? The repeatability condition must be satisfied by any joint angle profile design. However, the repeatability condition in the previous work was not properly defined. Hurmuzlu's constraint functions [9] can automatically produce a repeatable gait, but Hurmuzlu did not give the definition of the repeatability nor the method of yielding the repeatable gait. Chevallereau *et al.* [11] defined the repeatability condition as: two following steps must be identical and, more precisely, the legs will swap their roles from one half step to the next. Channon *et al.* [20] put forward a principle: at the start of the step, the feet must be at the same height (assuming motion over level ground) and a constant distance apart. Apparently, Chevallereau and Channon's definitions are different. Chevallereau's definition is more general and can be used in any biped robots, because this definition does not have any constraint on the specific biped model. However, there is a lack of precise mathematical interpretation. Channon's definition is more like a specific requirement of the repeatability for a specific biped robot model. It indicates, to a certain extent, how to obtain the repeatability, but does not define what the repeatability is.

To satisfy the repeatability of the locomotion of biped robots, most of the studies in trajectory design concentrate on the techniques that the motion of the biped is approximated, using simple geometric functions. Typically, the motion of the hip and the swing foot is modeled using low-order time polynomials, or Fourier series, and the kinematics of the biped are used to determine the corresponding joint angle profiles. The coefficients of the geometric functions are then optimized numerically so as to minimize a cost function representing the energy consumed during each step. In this way, to satisfy the repeatability condition, the initial and final states of each step are set. Chevallereau *et al.* [11] reduced the number of optimization parameters in their polynomials by setting the initial and final joint angles and angular velocities beforehand. Channon *et al.* [20] included the difference between the distances at the initial and final moments of one step in the energy function and minimized the difference together with the minimization of energy. However, the reasons why such conditions were used were never given.

For satisfying the repeatability condition, Hurmuzlu's parametric formulation [9] is a special case compared with other methods of joint angle profile design. There are one ordinary differential equation and four algebraic equations in Hurmuzlu's constraint equations. To solve the differential equation, only one initial condition is needed.

According to the initial condition given by Hurmuzlu, the solutions to the five constraint equations are repeatable, indicating that the gait is repeatable. Why can such an initial condition lead to a repeatable gait? How do we choose the initial condition when we replace some of the constraint equations in the Hurmuzlu's original constraint equations? These two questions have not been answered in Hurmuzlu's work [9] due to the selecting of the special constraint equations. The restriction of creating repeatable gait, and the lack of rules for selecting proper initial angles, constraint equations and their gait parameters, have severely limited the applications of Hurmuzlu's parametric method [9].

Based on Hurmuzlu's constraint conditions, Wu and Chan [12] [13] designed the joint angle profiles for a 5-link biped using different constraint equations. Due to the improper choosing of the initial conditions, the gait could not be repeated. However, they noticed that the initial angles and certain gait parameters, such as the step length and the step period, are dependent. This observation is a very important stepping stone, because exploring the relationship between the gait parameters and the initial conditions enables us to choose the proper initial conditions.

Repeatability plays an important role in trajectory designs for biped robots. However, its importance has not been recognized and investigated. Our goal is to give a general definition of the repeatability of the locomotion of biped robots, and a general principle of selecting a specific type of constraint systems based on Hurmuzlu's parametric formulation method.

1.3 Literature survey of optimization of biped robot trajectory designs

As is with the development of all other robots, energy efficiency is highly desirable for legged robots. Therefore, it is particularly important to choose an optimization criterion that minimizes energy consumption. For the optimization, many people [21] [22] assumed that the motion is oscillatory, like a pendulum for the mobile leg and like an inverted pendulum for the supporting leg.

In some sense, the optimization of biped robot trajectories can be divided into static optimization and dynamic optimization. To implement a controller for a biped robot, there is a need to generate an admissible and optimal trajectory as a reference input. Previous methods for synthesizing biped walking trajectories include recording human kinematic data using finite state machines merging trajectory control with feedback control laws, and generating gaits from passive interaction of gravity and inertia. This is the static optimization that has been the most common method used to estimate the muscle forces during locomotion [23] [24]. This method is computationally inexpensive, and solutions can be obtained relatively quickly on single-processor computers, even when very detailed models of the body are used. The main disadvantage of static optimization is that the results are heavily influenced by the accuracy of the available experimental data, particularly the measured limb motions [25] [26].

Dynamic optimization is potentially more powerful than static optimization for two reasons. First, because a time-dependent performance criterion can be posed, the goal of the motor task can be included in the formulation of the problem. Secondly, dynamic optimization is inherently a forward dynamics method, and so the problem may be formulated independent of experimental data. These two attributes allow the motor patterns and kinematics of movement to be predicted. The main disadvantage is that dynamic optimization is computationally expensive [26] [27], so much so that previous solutions for walking have been greatly simplified. The earlier solutions confined the motion of the body in the sagittal plane [28]. However, recently, the model of the body [27] was actuated by a significantly greater number of muscles than has been used in any previous dynamic optimization study.

The energy-optimal trajectory for highly nonlinear equations of a complex robot may be found only numerically, and in general, they will be sub-optimal [30]. So far, three methods have been used to approximate the joint angles. They include the time-polynomial [11] [14] method, the Fourier expansions [31] [32] and the alternative method [33].

The time-polynomial functions have been used to determine the Cartesian coordinates of the hip, the swing foot and the trunk angle [11] [14]. The coefficients of the polynomials are optimization parameters. It is then necessary to invert the geometric model to define the joint reference trajectories. To avoid the use of an inverse dynamic model and an inverse geometric model, they chose the joint coordinates as optimized variables. The finite-term polynomial functions were chosen for the joint angle coordinate to limit the number of optimization parameters.

Nagurka [31] and Cabodevila [32] converted the non-linear optimization problem into an algebraic non-linear programming problem by an approximation of each generalized coordinate by an expansion in Fourier series. In nature, the time-polynomial method and the Fourier expansion method consist of using the global dynamic model of the physical model and solving a nonlinear optimization problem with finite time and initial and final state defined.

Roussel [33] proposed an alternative method for energy optimal gait generation. The proposed approach searches for unconstrained trajectories (no particular time or frequency base function are chosen) generated by piecewise constant inputs. A numerical study [34] has shown that, for an equivalent amount of computational burden, the alternative method provides motions with a lower input energy compared with polynomial or Fourier expansions.

In the studies of optimal trajectory design based on the energy minimization, the integrand of the functional is often the summation of square of the control torques. This kind of functional is not directly related to the input energy. Why do people not use the functional that has the physical meaning of energy? Is there any difference between these functionals? A basic understanding is needed.

1.4 Objectives of this thesis

There are two goals in this thesis: (1) to study the repeatability condition of the locomotion of biped robots, and (2) to compare two different optimization criteria from the energy point of view. For the first aspect, a general and precise interpretation of the repeatability condition for biped systems and the methods to satisfy the repeatability condition under various constraint conditions will be needed. For the second aspect, the optimization criterion that will yield a lower input energy will be found. The second aspect is related to the first, because the repeatability condition is a fundamental requirement in the trajectory designs.

In the study of the repeatability condition, a kinematic bipedal model will be applied that consists of five rigid links, which are connected by four pure hinge joints with five degrees of rotational freedom. The motion is confined in the sagittal plane. Such a model includes the single support phase, the phase of the impact with the walking surface and the support end exchange phase. In this thesis, Hurmuzlu's systematic methodology is used, which enables people to have more freedom to control and change the gait pattern of biped robots. Hence, the general repeatability condition will be given based mainly on Hurmuzlu's parametric formulation method and discuss the method to satisfy the repeatability condition. However, proposing the repeatability condition is relatively easier than satisfying them, especially for higher order constraint systems. For higher order constraint systems, the time polynomial technology will be employed to deal with the complexity in satisfying repeatability condition.

In the study of comparison of two different optimization criteria, the trajectories obtained from two different optimization criteria will be substituted into energy functional to calculate the input energy. In this way, the difference between the input energy produced by different optimization criteria will be investigated and the basic understanding of the difference will be gained before they are used in the trajectory designs.

1.5 Thesis organization

The remaining chapters in this thesis are organized as follows. Chapter 2 gives the general repeatability condition for first order constraint systems based on initial-value problems. A first order constraint system is defined by the set of constraint equations that consist of only the first order differential equations and other algebraic equations. The possibility of satisfying the repeatability condition by selecting the proper constraint equations is discussed in Chapter 2. In Chapter 3, the repeatability condition for higher order constraint systems will be discussed. A higher order constraint system is defined by the set of constraint equations that consist of the higher order (higher than one) differential equations and other algebraic equations. Two types of methods will be presented for satisfying the repeatability condition, the method based on initial-value problems and the method based on boundary-value problems. In Chapter 4, two different numerical optimization criteria are compared. The conclusions and recommendations for future work are presented in Chapter 5.

Chapter 2

Repeatability Condition for a Planar Five-Link Biped

2.1 Introduction

Joint angle profiles planning for a bipedal locomotion system consists of generating a set of joint angles that lead to the desired walking pattern. The methodology used in this chapter was introduced by Hurmuzlu in 1993 [9]. The design is aimed at realizing the walking motion of the five-link biped robot on a flat horizontal surface in the sagittal plane with only the single support phase. Hurmuzlu's constraint equations are cast in terms of coherent physical characteristics of gait and used as objective functions by a controller.

Hurmuzlu's five constraint equations lead to four algebraic equations and one first-order ordinary differential equation. Thus, once the initial condition is fixed, the joint angle profiles will be uniquely determined. However, the solution must produce a continuous and repeatable gait on level ground, otherwise the solution is not acceptable. Since the repeatability condition is not included in constraint equations, how do we guarantee that the unique solution to the constraint equations will meet the repeatability condition? This issue has not been addressed previously since the special constraint equations used by Hurmuzlu automatically produced repeatable gait [9]. To the best of our knowledge, the principle of creating a proper set of constraint equations and the corresponding gait parameters that lead to repeatable gait has not been developed.

The objectives of this chapter are (*i*) to investigate the correlation between the initial condition and gait parameters based on a specific class of constraint equations in the context of producing repeatable gait and (*ii*) to develop a general repeatability condition, which can provide a guideline for the selection of constraint equations and their associated gait parameters. This research can provide valuable information about bipedal gait. It is a stepping stone for employing Hurmuzlu's parametric formulation with various constraint equations.

This chapter is organised as follows. A five-link bipedal system is briefly described and five constraint equations, used by Hurmuzlu [9], are then outlined in Section 2.2. Explicit relationships between the initial angles and the gait parameters, based on Hurmuzlu's constraint equations, will be developed in Section 2.3. These relationships will be generalized to a certain class of constraint equations. In Section 2.4, a general sufficient and necessary condition for repeatable gait will be established and proven. The application of this repeatability condition for selecting constraint equations will be discussed in the same section. A further discussion on the sufficient and necessary condition is made in Section 2.5.

2.2 Description of the five-link biped robot

The kinematic model of the five-link biped robot and related background information are briefly presented in this section. The biped robot studied here is modeled for walking on a flat horizontal plane only. The five-link biped model is shown in Figure 2.1.

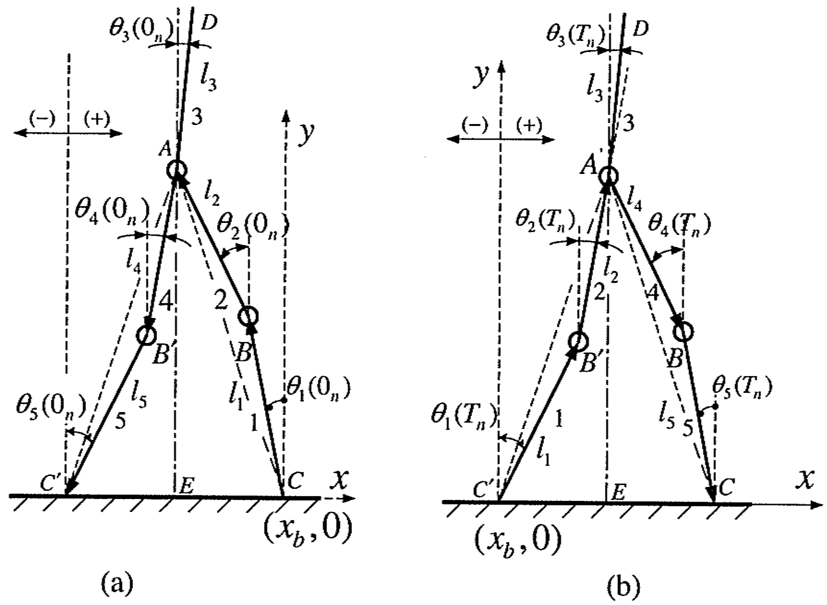

Figure 2.1 Posture of a 5-link biped (a) at start-of-step and (b) at end-of-step

This bipedal model consists of five rigid links, one link for the upper body (link 3) and two links for the thighs (link2 and 4), and the two links for the shanks (link 1 and 5). The parameters shown in Figure 2.1 are as follows:

m_i: the mass of link i

l_i: the length of link i, where $l_4 = l_2$ and $l_5 = l_1$

d_i: the distance between the centre of mass and the lower joint of link i

I_i: the moment of inertia of link i with respect to the axis which passes through the centre of mass of link i and perpendicular to the sagittal plane

θ_i: the absolute angle of link i with respect to the vertical line. Such a reference line is located on the lower joint of the link. We define the positive angle such that the link is at the right side of the vertical axis, and define the negative angle such that the link is at the left-hand-side of the vertical axis. For example, $\theta_1(0_n)$ shown in Figure 2.1 is negative and $\theta_5(0_n)$ is positive.

These links are connected to one another by four purely hinge joints. There are two joints at the hip and two joints at the knees. To simplify our analysis, the following assumptions of this model are made in advance:

(1) The feet of the biped are massless.

(2) There is a point contact between the tip of the supporting leg and the walking surface.

(3) The left side and the right side of the biped are symmetric.

(4) The biped is constrained in the sagittal plane.

(5) There is sufficient friction between the foot and the walking surface to prevent slippage.

Although the dynamics of the feet is neglected by assuming massless feet and point contact between the tip of the lower limb and the walking surface, a torque to be applied on the supporting ankle during walking is still needed.

The biped moves steadily in the sagittal plane, that is, the joint angle profile is continuous and repeatable. Only the single support phase is considered in Hurmuzlu's work [9]. Figures 2.1(a) and 2.1(b) show the postures at the start-of-step and the end-of-step, respectively. $\theta_i(0_n)$ and $\theta_i(T_n)$ $(i = 1,2,...,5)$ are absolute angles at the beginning and end of the n^{th} step. The upward vectors represent the supporting leg, and the downward vectors represent the swing leg. The single support phase consists of a swing phase and an impact phase. The swing phase is a continuous forward motion during which the biped is pivoted on one limb (stance limb) and the other limb (swing limb) is moving in the forward direction. The impact phase arises when the swing limb comes into contact with the walking surface.

2.2.1 Hurmuzlu's constraint equations:

The five constraint equations used by Hurmuzlu to determine the joint angle profiles for a five-link biped robot, walking on the horizontal surface, are [9]:

(1) Erect Body Posture: The biped maintains an erect posture during locomotion. Referring to Figure 2.1, this relation can be written as

$$\theta_3(t)=0\,. \tag{2.1}$$

(2) Walking speed: The walking speed has been defined as the speed of the centre of mass of the upper body in the horizontal direction. According to the kinematical relation, the corresponding constraint equation is

$$S_2=l_1\cos\theta_1\dot{\theta}_1+l_2\cos\theta_2\dot{\theta}_2+d_3\cos\theta_3\dot{\theta}_3-V_p=0\,, \tag{2.2}$$

where V_P is the desired walking speed.

(3) Trajectory of the Swing Limb: A parabola has been chosen as the trajectory of the tip of the swing leg in advance. By choosing the step length S_L and the maximum tip clearance H_m, the trajectory of the swing limb can be determined below:

$$S_3=y_T-(\frac{4H_m}{S_L^2}x_T^2+H_m)=0\,, \tag{2.3}$$

where x_T and y_T are the co-ordinates of the tip of the swinging leg. The step length, S_L is defined by the horizontal displacement of the tip of the swinging leg in the single support phase.

(4) Bias of the Stance Knee: The stance knee has been fixed at a certain angle during the single support phase. The corresponding relation is

$$S_4 = (\theta_2 - \theta_1) - \sigma = 0, \tag{2.4}$$

where σ is a given bias angle.

(5) Coordination of the Motion of the Limbs: To define the direction of the swing limb's motion, the constraint equation has been given as

$$S_5 = x_T - 2x_{c3} = 0, \tag{2.5}$$

where x_{c3} and x_T are the x-coordinates of the centre of mass of the upper body and the tip of the swing leg, respectively. Equation (2.5) also implies that the tip moves twice as fast as the centre of mass of the upper body.

These five equations were only used to define the joint trajectory or the gait pattern of the five-link biped robot. Control torques were not considered for this stage.

2.2.2 Requirement of repeatable gait

The general definition of repeatability is given in this section. Challenges for satisfying the repeatability condition corresponding to a certain type of constraint systems are further discussed.

For a continuous and steady gait, a basic requirement is that the gait is repeatable. In this thesis, a general definition of repeatability is written as $P^{(n)}(kT+t) = P^{(n)}((k+1)T+t)$, where $k, n = 0,1,2,...$ $P(t)$ is the posture of the biped and $P^{(n)}(t)$ is its nth order derivative with respect to time at instant t, and $0 \le t \le T$, where T is the step period. "Posture" here refers to all the joint angles of the biped, and the derivatives with respect to time of the posture refer to the derivatives of all the joint angles with respect to time. According to this definition, the locomotion of each point on each link must be periodic with a period of T.

Due to the continuity of the gait, at least one of the requirements of repeatable gait on flat ground indicates that the posture at the end of each step must be the same as the one at the beginning of each step. For the case of the 5-link bipedal model, shown in Figure 2.1, we have

$$\begin{aligned} &\theta_1(0_n)=\theta_5(T_n),\ \theta_2(0_n)=\theta_4(T_n), \\ &\theta_3(0_n)=\theta_3(T_n),\ \theta_4(0_n)=\theta_2(T_n),\ \theta_5(0_n)=\theta_1(T_n) \end{aligned} \tag{2.6}$$

To have the solution to further satisfy equations (2.6), the initial angles, constraint equations and their associated gait parameters must be carefully selected. In other words, if there does not exist a principle to examine the chosen constraint equations or guide us to select the constraint equations, we will never know whether the solutions to the constraint equations produce a repeatable gait before the solutions are solved. Satisfaction of the requirement of repeatable gait can make the design of joint angle profiles extremely challenging. For example, in Hurmuzlu's constraint equations, equations (2.1) and (2.3)-(2.5) are algebraic equations and equation (2.2) is a first order ordinary differential equation. Thus, only one initial angle of the n^{th} step, for example, $\theta_1(0_n)$, can be chosen and the rest of the initial angles will be fixed by equations (2.1) and (2.3)-(2.5). Once the initial angle is chosen, the solution to the constraint equations (2.1)-(2.5) is uniquely determined. However, an arbitrarily chosen initial angle probably cannot make the final solution satisfy the requirement of repeatable gait. Therefore, it is necessary to have an in-depth discussion on how to establish a set of constraint equations and make their solutions satisfy the requirement of repeatable gait. To the best knowledge, no such research has been documented.

2.3 Correlation between the initial angles and gait parameters for repeatable gait

In this section, the relationship between the initial angles and the gait parameters associated with the constraint equations used by Hurmuzlu [9] will be investigated. This relationship will be generalized to a certain class of constraint equations. According to equation (2.1) and (2.4), equation (2.2) can be written as

$$\frac{d\theta_1}{dt} = \frac{V_p}{l_1 \cos\theta_1 + l_2 \cos(\theta_1 - \sigma)}. \tag{2.7}$$

Note that the right-hand side of equation (2.7) satisfies the Lipschitz condition. There exists a unique solution to equation (2.7). Solving equation (2.7), we have

$$\sin\theta_1(t) = \frac{(V_p t + C_0)(l_1 + l_2 \cos\sigma)}{A'C'^2} - \frac{l_2 \sin\sigma\sqrt{A'C'^2 - (V_p t + C_0)^2}}{A'C'^2}, \tag{2.8}$$

where C_0 is a constant shown below:

$$C_0 = l_1 \sin\theta_1(0_n) + l_2 \sin[\theta_1(0_n) + \sigma], \tag{2.9}$$

and $A'C' = \sqrt{l_1^2 + l_2^2 + 2l_1 l_2 \cos\sigma}$ (see Figure 2.1). Since $\theta_1(t_n)$ must be real on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, it requires that

$$A'C'^2 - (V_p t + C_0)^2 \geq 0.$$

Hence, the step period

$$T \le \frac{A'C' - C_0}{V_p} \quad . \tag{2.10}$$

Based on the geometric relation (see Figure 1(a)), it is easy to prove that $A'C' > C_0$ always holds. Equation (2.10) implies that there is an upper boundary for a step period with V_P not approaching zero. According to equation (2.5), the upper boundary for the step length also exists:

$$S_L = 2V_P T \le 2(A'C' - C_0). \tag{2.11}$$

2.3.1 The relationship among the initial angles, step period and stance knee bias angle

In this section, we develop the explicit relationship between the initial angles, the step period, T, and stance knee bias angle, σ, using the constraint equations (2.1) to (2.5). In general, the step period involves two parts: the period of single support phase T_1, the period of the double support phase T_2, where $T=T_1+T_2$. Since only the single support phase is considered here, the step period in the following discussion includes only the period of single support phase. For convenience, we replace T_1 by T.

At the final moment of the single support phase, the following angular relation defines that the swinging foot touchs the ground (refer to Figure 2.1):

$$y_e(T_n;\theta_i) = l_1 \cos\theta_1(T_n) + l_2 \cos\theta_2(T_n) + l_4 \cos\theta_4(T_n) + l_5 \cos\theta_5(T_n) = 0, \tag{2.12}$$

where $y_e(T_n;\theta_i)$ is the y co-ordinate of the tip of the swing leg. Since the step length is given in Hurmuzlu's constraint equations [9]. From Figure 2.1, $CC' = \frac{1}{2} S_L$, and we have

$$l_1 \sin\theta_1(T_n) + l_2 \sin\theta_2(T_n) + l_4 \sin\theta_4(T_n) + l_5 \sin\theta_5(T_n) - \frac{S_L}{2} = 0 . \tag{2.13}$$

From equations (2.1), (2.2) and (2.4), we may find the final values of every angle. Equation (2.8) can also be written as

$$t_n = \frac{1}{V_p}\{ l_1 \sin\theta_1(t_n) + l_2 \sin[\theta_1(t_n) + \sigma] - C_0 \} . \tag{2.14}$$

Replacing $\theta_1(t_n)$ by $\theta_1(T_n)$, we have the step period of single support phase. Since Hurmuzlu's five constraint equations satisfy the requirement of repeatable gait (the related proof will be shown in section 2.4.2), using equations (2.6), the relationship between the step period and initial posture is obtained:

$$T = \frac{1}{V_p}\{ l_1 \sin\theta_5(0_n) + l_2 \sin[\theta_5(0_n) + \sigma] - C_0 \} . \tag{2.15}$$

Note that C_0 is a function of $\theta_1(0_n)$, as shown in equation (2.9), and $\theta_5(0_n)$ is a function of $\theta_1(0_n)$. Thus the relationship between the step period and the initial condition is established in (2.15).

2.3.2 The relationship among the initial angles, step length and stance knee bias angle

In this section, we shall discuss the relationship among the initial angles, step length, S_L, and stance knee bias angle, σ, using Hurmuzlu's constraint equations [9]. This relationship can be determined based on the geometry shown in Figure 2.1.

Since, in Hurmuzlu's work, the upper body is assumed to be in the upright position and the mass centre of the upper body to be the midpoint of the tips of the two legs, it follows that the hip joint remains the midpoint of the tips of the two legs. From the geometric relationship shown in Figure 2.1, we have $|\theta_1(0)| = \angle EAC - \angle ACB$ and

$$\angle EAC = \arcsin(\frac{S_L}{4AC}) \tag{2.16}$$

$$\angle ACB = \arcsin(\frac{l_2}{AC}\sin(\sigma)) \tag{2.17}$$

From equations (2.16) and (2.17), we have

$$\theta_1(0) = -\arcsin(\frac{S_L}{4AC}) + \arcsin(\frac{l_2}{AC}\sin(\sigma)) \tag{2.18}$$

where $AC = \sqrt{l_1^2 + l_2^2 + 2l_1 l_2 \cos\sigma}$. Equation (2.18) shows the explicit relationship among the initial angle, $\theta_1(0)$, step length, S_L, and supporting knee bias angle, σ. It indicates that if the link lengths and gait parameters, S_L and σ, are given, the initial angle is fixed, as is the solution to the constraint equations (2.1)-(2.5) for the following step. On the other hand, for a given physical system with steady gait, once the initial angle and the bias angle of the supporting knee are chosen, the step length cannot be selected arbitrarily.

Since equation (2.18) has been derived based on the geometrical relationship at the beginning of the step among the gait parameters and link lengths and only equations (2.3)-(2.5) were involved, the relationship between the initial angle, $\theta_1(0)$, and gait parameters, shown in equation (2.18), is valid for more general constraint equations than those used by Hurmuzlu. This class of constraint equations should include stance knee bias angle and step length. Furthermore, equation (2.18) is also valid when the double support phase is considered.

2.4 The general repeatability condition for first order constraint system

In this section, a sufficient and necessary condition for repeatable gait for a 5-link bipedal system (repeatability condition) is developed. This condition will provide a guideline for selecting constraint equations and their associated gait parameters, which lead to a proper initial condition for repeatable gait.

For convenience, we define all the constraint equations as a constraint system, and the highest order of the differential equations as the order of the constraint system. According to this definition, Hurmuzlu's five constraint equations form a first order constraint system. However, our discussions in this section are not restricted in Hurmuzlu's constraint equations, and they are valid for any first order constraint system.

2.4.1 Repeatability condition

Theorem 2.1: For the 5-link bipedal system shown in Figure 2.1, given a set of constraint equations, which consists of at least one first-order ordinary differential equation, let T and $P(t_n)$ be the step period and the instant posture during the n^{th} step, respectively, where $0 \leq t_n \leq T$. Triangle $\Delta ACC'$ can be determined by connecting the hip joint A and the tips of two legs, C and C', as shown in Figure 2.1. Assuming that hyperextension of the knee of the swing leg does not occur, the sufficient and necessary conditions that leads to a repeatable gait are: (*i*) $\Delta ACC'(0_n) \cong \Delta ACC'(T_n)$ and (*ii*) $\theta_3(t)$ is a periodic function with period of T, where 0_n and T_n denote the beginning and end of the n^{th} step.

Note:

1. Since the joint angle profiles are always continuous, it follows that the posture at the end of each step must be identical to the one at the beginning of the next step, guaranteeing that triangle $\Delta ACC'(T_n) \cong \Delta ACC'(0_{n+1})$.

2. Here, we require that there exists at least one first order ordinary differential equation in the constraint equations. The solution to the differential equation is uniquely dependent on the initial value. This characteristic enables us not to examine the solutions at each time instant, but to examine only the initial or final value of the solution during a step. That means, as long as the initial or final posture is repeatable, the solutions to the constraint system is repeatable.

3. Although two relatively independent repeatability conditions are proposed, (one is for the lower limbs and the other is for the upper body), sometimes only the first condition is considered if and only if $\theta_3(t) = \theta_3(\theta_k(t))$, where $k \neq 3$. In other words, as long as there exists a one-to-one relationship between the locomotion of the upper body and the lower limbs, the locomotion of the upper body must be periodic when the locomotion of the lower limbs is periodic and the periods are equal. To simplify the following proof of the necessary and sufficient condition, we assume that there exists a one-to-one functional relationship between the upper body and the lower limbs.

Proof: Sufficiency

Referred to Figure 2.1, since $\Delta ACC'(0_n) \cong \Delta ACC'(T_n) \cong \Delta ACC'(0_{n+1})$, we have

$$AC(0_n) = AC(T_n) = AC(0_{n+1}) \tag{2.19a}$$

$$\angle ACC'(0_n) = \angle ACC'(T_n) = \angle ACC'(0_{n+1}) \tag{2.19b}$$

Since all links are rigid, $l_2 = l_4$ and $l_1 = l_5$, we have $\Delta ABC(0_n) \cong \Delta ABC(T_n) \cong \Delta ABC(0_{n+1})$. Thus,

$$\angle ACB(0_n) = \angle ACB(T_n) = \angle ACB(0_{n+1}) \tag{2.20a}$$

$$\angle ABC(0_n) = \angle ABC(T_n) = \angle ABC(0_{n+1}) \tag{2.20b}$$

Since neither of the knees can be bent backward, from the geometrical relationship shown in Figure 2.1, we have

$$\theta_1(0_n) = \frac{\pi}{2} - \angle ACC'(0_n) - \angle ACB(0_n) \tag{2.21a}$$

$$\theta_1(0_{n+1}) = \frac{\pi}{2} - \angle ACC'(0_{n+1}) - \angle ACB(0_{n+1}) \tag{2.21b}$$

From the relationships shown in (2.19b) and (2.20a), we have

$$\theta_1(0_n) = \theta_1(0_{n+1}) \tag{2.22}$$

Similarly, we can prove that $\theta_5(0_n) = \theta_5(0_{n+1})$. From Figure 2.1, we see that

$$\theta_2(0_n) = \theta_1(0_n) + \pi - \angle ABC(0_n) \tag{2.23a}$$

$$\theta_2(0_{n+1}) = \theta_1(0_{n+1}) + \pi - \angle ABC(0_{n+1}) \tag{2.23b}$$

From equations (2.20b) and (2.22), we have

$$\theta_2(0_n) = \theta_2(0_{n+1}) \tag{2.24}$$

Following the same procedure, we can prove that $\theta_4(0_n) = \theta_4(0_{n+1})$.

Since we have $\theta_3(t) = \theta_3(\theta_k(t))$, $k \neq 3$, and $\theta_k(0_n) = \theta_k(0_{n+1})$, $k \neq 3$, we finally have $\theta_3(0_n) = \theta_3(T_n)$. Based on the above discussion, all initial angles of the n^{th} and $(n+1)^{th}$ step are the same, guaranteeing that the solutions of the n^{th} and $(n+1)^{th}$ step are identical and that the gait is repeatable.

Necessity:

If the gait is repeatable, we have $P(t_n) = P(t_{n+1})$ where $0 \le t_n \le T$. This indicates that motion of the upper body must be repeatable during each step, *i.e.*, $\theta_3(t)$ is a periodic function with period *T*. For repeatable gait, the step length remains the same, leading to

$$CC'(0_n) = CC'(0_{n+1}) = CC'(T_n) \tag{2.25}$$

Since the link lengths are fixed and $\angle ABC(0_n) = \angle ABC(0_{n+1}) = \angle ABC(T_n)$, it follows that $\Delta ABC(0_n) \cong \Delta ABC(0_{n+1}) \cong \Delta ABC(T_n)$. Thus,

$$AC(0_n) = AC(0_{n+1}) = AC(T_n) \tag{2.26a}$$

$$AC'(0_n) = AC'(0_{n+1}) = AC'(0_n) \tag{2.26b}$$

According to equations (2.25) and (2.26),

$$\Delta ACC'(0_n) \cong \Delta ACC'(T_n) \tag{2.27}$$

Thus, the repeatability conditions have been proven.

Remarks:

1. The above repeatability condition is based on the assumption that at least one constraint equation is a differential equation, the highest order of the differential equations is one, and the solution is uniquely determined by the initial angles.

2. The above condition for repeatable joint angle profiles is not restricted to the bipedal system with only the single support phase. This condition can be used for gait, which involves the double support phase as long as the set of constraint equations for double support phase consists of at least one differential equation where the highest order is one.

3. The restriction that the hyperextension of the knee of the swing leg does not occur is important to keep the above proposed condition sufficient. It is not a strong restriction since, in most bipedal walking, neither knee is bent backwards. The same restriction has been used in other works.

4. Note that during the impact phase, the angles of each link will not change, so we do not consider the effect of impact as long as the constraint system is first order. However, for any higher order constraint system, we must consider the impact. This point will be discussed in Chapter 3.

5. The above proof is not related to Hurmuzlu's five constraint equations, so the necessary and sufficient condition is valid for not only Hurmuzlu's constraint system, but also for any other first order constraint system.

For the constraint equations used by Hurmuzlu [9], due to the constraint of the co-ordination of the motion of the limbs, as shown in equation (2.5), $\Delta ACC'(0_n)$ and $\Delta ACC'(T_n)$ must be an isosceles triangle. The parabolic trajectory of the tip of the swinging limb during the single support phase, as shown in equation (2.3), guarantees that the lengths of $CC'(0_n)$ and $CC'(T_n)$ remain the same. The constraint of fixed stance knee bias angle, described by equation (2.4), guarantees that the length of $AC'(0_n)$ and $AC'(T_n)$ are fixed. Therefore, triangles $\Delta ACC'(0_n)$ and $\Delta ACC'(T_n)$ remain identical for every step and the repeatability condition is automatically satisfied.

2.4.2 The compatibility of Theorem 2.1 with the solution to a first order initial-value problem

In this section, we shall discuss the compatibility of Theorem 2.1 with the solution to a first order initial-value problem. The key point of this discussion is to answer how to make a first order constraint system satisfy the repeatability condition. When the solution to a first order constraint system produces a repeatable gait, we say that the solution is compatible with Theorem 2.1.

In fact, Theorem 2.1 explicitly shows the restriction on the solutions to the constraint equations at the end of each step. According to Theorem 2.1, if the repeatability condition is satisfied, the final posture is fixed when the initial posture of the biped is given. Here we do not consider the solutions to the differential equations. In other words, the final conditions obtained from Theorem 2.1 are not solved from the differential equations. Two different concepts should be distinguished: one is the final posture given by Theorem 2.1, and the other one is the final values of the solutions to the constraint system. The former one is dictated by the requirement of repeatability condition; the latter one is the result of solving the constraint system that includes the differential equation. However, for a given constraint system, there do not exist two different sets of solutions. Even though the repeatability condition is independent of any constraint systems, the two sets of solutions respectively obtained from Theorem 2.1 and the differential equations must be compatible. This is the compatibility of Theorem 2.1 and the initial-value problem. If and only if the compatibility is satisfied, we can conclude that the repeatability condition is satisfied by the constraint system.

Hurmuzlu's five constraint equations can be taken as an example. Note that when all the algebraic equations are given, the solution to an initial-value problem is determined only by the differential equation and the corresponding initial condition. Once the differential equation and the initial condition are given, every point in the solution space at any instant is fixed. Hurmuzlu gave the parameters and the relative angles immediately before the contact of the *(n-1)th* step [9],

Table 2.1. The parameters of planar five-link biped

Number of the link	1	2	3	4	5
Length L_i (m)	0.5	0.5	1.0	0.5	0.5

$$\phi_1^-(T_{n-1}) = -\arcsin[\frac{S_L}{4\cos\frac{\sigma}{2}}] - \frac{\sigma}{2} \tag{2.28a}$$

$$\phi_2^-(T_{n-1}) = \phi_5^-(T_{n-1}) = \sigma \tag{2.28b}$$

$$\phi_3^-(T_{n-1}) = -\phi_1^-(T_{n-1}) - \phi_2^-(T_{n-1}) \tag{2.28c}$$

$$\phi_4^-(T_{n-1}) = -\arcsin[\frac{S_L}{4\cos\frac{\sigma}{2}}] - \frac{\sigma}{2} \tag{2.28d}$$

where the relative angles $\phi_i(T_{n-1}) = \theta_{i+1}(T_{n-1}) - \theta_i(T_{n-1})$, $(i = 2,3,4)$; $\phi_1(T_{n-1}) = \theta_1(T_{n-1})$. The angles are obtained from geometric relations at the instant when the swing leg touches the ground. If the posture does not change during the following exchange phase, according to the continuity condition, we may derive the initial condition of the supporting leg for the next step, say the n^{th} step,

$$\phi_1(0_n) = -\arcsin[\frac{S_L}{4\cos\frac{\sigma}{2}}] + \frac{\sigma}{2}. \tag{2.29}$$

The initial condition (2.29) determines the final solution to the Hurmuzlu's constraint system during the n^{th} step. It is easy to show that at the final moment of the single support phase in the the n^{th} step, we have

$$\phi_1^-(T_n) = -\arcsin[\frac{S_L}{4\cos\frac{\sigma}{2}}] - \frac{\sigma}{2}, \tag{2.30a}$$

$$\phi_2^-(T_n) = \phi_5^-(T_n) = \sigma, \tag{2.30b}$$

$$\phi_3^-(T_n) = -\phi_1^-(T_n) - \phi_2^-(T_n), \tag{2.30c}$$

$$\phi_4^-(T_n) = -\arcsin[\frac{S_L}{4\cos\frac{\sigma}{2}}] - \frac{\sigma}{2}. \tag{2.30d}$$

The above solutions solved from the deferential and algebraic equations, (2.30a)-- (2.30d) are the same as the given angles (2.28a)-- (2.28d). The corresponding joint angles are repeated after one step period. The repeatability condition is automatically satisfied by Hurmuzlu's constraint system.

The initial angle θ_1 is calculated when the hip is the mid-point of the tips of link 1 and 5. According to equation (2.5), at the end of the step, the hip will get to the middle of the feet again. Furthermore, according to equation (2.4), we can look at the supporting leg, link 1 and 2, as one rigid body rotating around the contact point. Apparently, the final point of A and its initial point are symmetric to the vertical line through the contact point. It is easy to prove that the final posture is the same as the initial posture if the step length is fixed. Thus, equations (2.4) and (2.5) guarantee that the solution to Hurmuzlu's constraint system leads to a repeatable gait.

From the above example, we see that in Hurmuzlu's constraint system, two equations are important for satisfaction of the repeatability condition. One is about the stance knee bias angle (equation (2.4)) and the other one is about the relationship between the horizontal displacement of the hip and of the tip of the swing leg (equation (2.5)). If we rewrite these two equations into a general form, we have

$$S_4 = h(\sigma(T), \theta_1(T)) - h(\sigma(0), \theta_1(0)) = 0\,, \tag{2.31}$$

where h is the height of the hip. When the link lengths and the initial angle are given, the hip height is a function of stance knee bias angle,

$$S_5 = \int_0^T \dot{x}_T dt - 2\int_0^T \dot{x}_{c3} dt = 0\,, \tag{2.32}$$

where $\dot{x}_T$ and $\dot{x}_{c3}$ are respectively the velocity of the tip of swing leg and the velocity of the hip along the horizontal direction. Equation (2.31) indicates that we do not have to restrict the stance knee bias angle to be a constant. It can be a function of time as long as the hip is at the same height at the initial and final moments. Equation (2.32) shows that after a step period, the displacement of the hip must be half of that of the tip of the swing leg. Thus, Hurmuzlu's equations (2.4) and (2.5) represent only a special case, making the initial and final posture symmetric. Equation (2.32) dose not require that the hip must be in the middle of the feet along x-coordinate all the time during a step, but it requires that the displacement of the tip of the swing leg must be twice as the displacement of the hip along x-coordinate.

Lemma 2.1: Given a first order constraint system for a five-link biped robot shown in Figure 2.1, if the solution to a constraint system yields a repeatable gait, two conditions are satisfied: 1) at the initial and final moments during each step, the hip height must be identical, and 2) the horizontal displacement of the hip is half of that of the tip of the swing leg.

Proof:

At the initial and final states, the hip point and the tips of two legs are connected shown in Figure 2.1. Since the links are rigid and the stance knee bias angles at the initial and final moments are specified, a simplified model is shown in Figure 2.2.

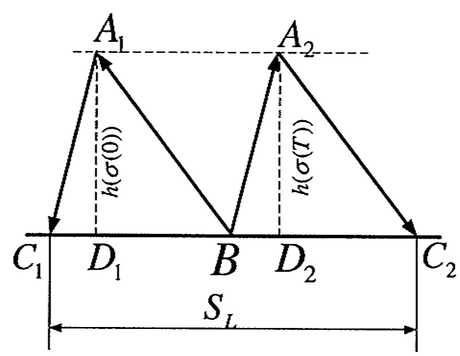

Figure 2.2 The displacement of the hip during a step

In Figure 2.2, the upward vectors, $\mathbf{BA_1}$ and $\mathbf{BA_2}$ stand for the supporting leg and the downward vectors, $\mathbf{A_1C_1}$ and $\mathbf{A_2C_2}$ stand for the swing leg. A_1 and A_2 are the hip at the initial and final moments.

Since the solution to a constraint system leads to a repeatable gait, the final posture must be the same as the initial posture, which indicates $\Delta A_1C_1B \cong \Delta A_2BC_2$. So we have the following three relationships:

1) $A_1D_1 = A_2D_2$;

2) $BC_1 = BC_2 = S_L / 2$;

3) $A_1A_2 = BC_1$.

The relation 1) means the hip height is identical at the initial and final moments and the relation 2) and 3) means that the horizontal displacement of the hip is half of that of the tip of the swinging leg. Therefore the Lemma 2.1 is proven.

We note that the inverse proposition of Lemma 2.1 does not hold. The reason is that the two conditions shown in equations (2.31) and (2.32) do not have any restrictions on the distance between the feet at the initial and final moments. It is easy to show that if the distance between the feet at the final moment is not equal to that at the initial moment, the repeatability condition cannot be satisfied.

However, if we prescribe that the distance between the feet at the final moment is equal to that at the initial moment, we shall have another theorem repeatability of gait.

Theorem 2.2: Given a first order constraint system for a five-link biped robot shown in Figure 2.1, the sufficient and necessary conditions that lead to a repeatable gait are: 1) the distance between the feet must be identical at the initial and final moments of a step; 2) at the initial and final moments during one step, the hip height remains identical; and 3) the horizontal displacement of the hip is half of the step length.

Proof:

At the initial and final states, we connect the hip point to the tips of two legs shown in Figure 2.1. Since the links are rigid and the stance knee bias angles at each time instant are specified, we have a simplified model shown in Figure 2.2.

Necessity: According to Lemma 2.1, when repeatability condition is satisfied by the constraint system, conditions 2) and 3) shown in equation (2.31) and (2.32) must be satisfied; Meanwhile, in light of Theorem 2.1, $\Delta A_1C_1B \cong \Delta A_2BC_2$ in Figure 2.2, the distance between the feet must be constant at the initial and final moments. Thus, if a repeatable gait is produced, the three conditions are satisfied.

Sufficiency: Given: 1) the distance between the feet remains identical at the initial and final moments of a step: $BC_1 = BC_2$ shown in Figure 2.2; 2) the height of hip is always a constant at the initial and final moments: $h(\sigma(T)) - h(\sigma(0)) = 0$; 3) the horizontal displacement of hip is half of the step length: $\int_0^T \dot{x}_T dt - 2\int_9^T \dot{x}_{c3} dt = 0$.

a)$\because$ $h(\sigma(T)) - h(\sigma(0)) = 0$, $\therefore$ $A_1A_2 // C_1C_2$;

b) $\because$ $\int_0^T \dot{x}_T dt - 2\int_9^T \dot{x}_{c3} dt = 0$, $\therefore$ $A_1A_2 = \frac{1}{2}C_1C_2$;

c) $\because$ $BC_1 = BC_2$,$\therefore$ $A_1A_2 = BC_1 = BC_2$.

From conditions a) and c), we know that $A_1C_1BA_2$ is parallelogram, so $A_1C_1 // A_2B, A_1C_1 = A_2B$. Hence $\angle A_1C_1B = \angle A_2BC_2$, and then $\Delta A_1C_1B \cong \Delta A_2BC_2$. According to Theorem 2.1, the repeatability condition is satisfied.

Remarks:

1. Theorem 2.2 is a further extension of Theorem 2.1 in that Theorem 2.1 gives the condition of repeatability, but does not show how to satisfy it. Theorem 2.2 indicates that the restrictions on the identical hip height at the initial and final moments and the horizontal displacement of the hip being half of the step length shown in equations (2.31) and (2.32) are absolutely necessary, because they guarantee that we can obtain the proper final posture that leads to the repeatable gait.

2. The restrictions on the identical hip height and the horizontal displacement of the hip being half of the step length shown in equations (2.31) and (2.32) are general expressions. They may have other variation forms, but not all the variation forms can guarantee that the distance between the feet remains identical at the initial and final moments. If the invariance of the distance between the feet at the initial and final moments is not satisfied by the constraint equations, it must be satisfied by the initial condition, otherwise the repeatability condition will be violated.

3. Equations (2.31) and (2.32) are the most general expressions. To satisfy the repeatability condition, we do not have to use the special forms derived from them to have the unnecessary restrictions. This provides us more freedom to establish our constraint system.

2.4.3 Selection of constraint equations satisfying the repeatability condition

Theorem 2.1 and Theorem 2.2 can be used to examine whether the constraint equations will produce repeatable gait once they are chosen based on tangible gait characteristics. Furthermore, Theorem 2.2 also provides a guideline for selecting constraint equations for five-link bipeds, which will be demonstrated through the following example.

The walking speed will first be selected as one of the constraint equations; it provides a first-order ordinary differential equation. The upper body movement can be used as another constraint function, which is not necessarily restrained at the upright position. To satisfy the repeatability condition, the equations that indicate the restrictions on the hip height and displacement must be included. If the condition of the invariance of the distance between the feet cannot be satisfied by all the constraint equations, it must and can be met by choosing the initial angle and setting the initial distance between feet as half step length.

Based on the above discussion, one set of possible constraint equations that produce repeatable gait can be selected as: walking speed, upper body movement with $\theta_3(0_n) = \theta_3(T_n)$ and equations (2.31) and (2.32) or their other variation forms. The fifth constraint function can be selected without the restriction of the repeatability condition.

2.5 A further discussion about Theorem 2.1 and Theorem 2.2

In this section, we are discussing the restriction of applying Theorem 2.1 and 2.2.

All of our discussion in this chapter is based on initial-value problems, and we are trying to obtain a periodic solution with a period of step period to the constraint system when the initial condition is given. The subsequent gait is precisely described by the differential equation(s) and affected only by the initial condition as long as the constraint system is given.

The restriction of applying Theorem 2.1 and Theorem 2.2 lies in the order of initial-value problems. Given an n^{th} order initial-value problem, say

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{(n-1)} y}{dt^{(n-1)}} + \ \dots a_1 \frac{dy}{dt} + a_0 = 0, \tag{2.33}$$

to uniquely solve this differential equation, we must give the initial values,

$$y(0) = b_0, \ y'(0) = b, \ \dots, \ y^{(n-1)}(0) = b_{n-1}. \tag{2.34}$$

In joint angle profile designs, we define joint angle as a zero order physical quantity, and define its nth order derivative with respect to time as nth order physical quantity. For Hurmuzlu's constraint equations and any other constraint equations that consist of at least one first order differential equation, the only initial value(s) is (are) the initial angle(s). In this situation, we consider only the initial posture, instead of the initial angular velocities or angular accelerations, of the biped robot to satisfy the repeatability condition. Theorem 2.1 and Theorem 2.2 can be applied into any first order initial-value problem. For higher order constraint systems, Theorems 2.1 and 2.2 are not enough to produce a repeatable gait, but we still require that the three conditions in Theorem 2.2 must be satisfied.

In Chapter 3, we shall discuss another type of problem, the two-point boundary-value problem. In nature, that type of problem is still an initial-value problem, but it is treated as a boundary value problem. To satisfy the repeatability condition, the initial and final states of each step are fixed, and the approximate solutions to the constraint system are obtained. In this sense, it can be called a fitting problem. Chapter 3 will present a detail discussion.

2.6 Summary

The parametric formulation, developed by Hurmuzlu [9], is a systematic method of designing joint angle profiles for bipedal models. However, steady and continuous gait requires that the joint angle profiles are repeatable, imposing an additional requirement on the solution. Consequently, specific constraint equations, gait parameters and initial angles have to be chosen. Thus, it is highly desirable to develop a technique for specifying the constraint equations, gait parameters and initial angles such that repeatable gait can be produced.

In this chapter, two explicit relationships between the initial angles and gait parameters, used in a specific class of constraint equations, have been developed as shown in equations (2.15) and (2.18), which include (*i*) the relationship among the initial angles and step period, stance knee bias angle and walking speed, and (*ii*) the relationship among the initial angle and step length and stance knee bias angle. From these results, we found that if the step length and stance knee bias angle are selected, the initial angles are fixed, which determine the solution uniquely. Furthermore, with a fixed stance knee bias angle, increasing the step length requires an increase in the value of the initial angle, $\theta_1(0_n)$. In addition, to show the relationship between the initial angles and gait parameters, equations (2.15) and (2.18) further show the dependence among the gait parameters. For example, the walking speed, step length and step period are related. Once two of these are selected, the third one will be determined.

The necessary and sufficient condition for producing repeatable gait was developed. The condition is valid for the types of constraint equations, which consist of an algebraic and at least one first-order ordinary differential equation, whose solutions are uniquely determined by the initial angles. This condition can be used in two ways. One is to examine whether the constraint equations will produce repeatable gait once they are chosen based on tangible gait characteristics and other design criteria; the other one is to provide a guideline for selecting constraint equations and their associated gait parameters in the context of producing repeatable gait. This point has been demonstrated through an example.

Note that in some previous work on designing optimal gait [20], the repeatability condition was considered by including the error in the joint angles at the beginning and the end of each step in some type of functional and minimizing them. This method works well when optimization is the only constraint function. If additional and improper constraint equations are selected, inherently violating the repeatability condition, the approach of minimising the error in the joint angles at the beginning and end of each step will not yield repeatable gait. For this method, we shall have further discussion in the next chapter.

It is believed that this research can provide valuable information about bipedal gait. It is a stepping stone for employing Hurmuzlu's parametric formulation with various constraint equations. However, some challenges will be introduced when this work is extended to sets of constraint equations involving higher order ordinary differential equations. The application of the repeatability condition in the higher order constraint equations will be discussed in the next chapter.

Chapter 3

The repeatability condition for higher order constraint systems

3.1 Introduction

In Chapter 2, the question of how to satisfy the repeatability condition for the first order constraint systems has been successfully answered. However, in many cases, we have to deal with higher order constraint systems. For example, if a optimization criterion is introduced into the constraint system, the constraint system will be a higher order one. Thus, we have to extend the repeatability condition to higher order constraint systems. In this chapter, we shall discuss the possibility and challenge in applying the repeatability condition to higher order constraint systems.

The study of the repeatability condition considers a series of continuous steps. For the constraint equations involving initial value problems, the repeatable gait requires the same initial state vector of each step. If the double support phase is not considered, two continuous steps are connected by the impact phase. Figure 3.1 shows the effect of impact in one step. In this figure, P, V, A, ... indicate posture, angular velocities, angular accelerations, etc. The indices $m-1$, m, $m+1$, ... indicate the $(m-1)^{th}$, m^{th}, $(m+1)^{th}$, ... step. The indices i and f stand for the initial and final instant, respectively. All the physical quantities, except the joint angles (posture), will change because of the impact. Due to the relationships between the physical quantities before and after the impact, the final state vector of one step is related to the initial state vector of the next step. Thus, the study of the repeatability condition is confined in one step, i.e., the proper initial state vectors is found such that the final state vectors will lead to the identical initial state vectors for the next step. Therefore, developing the relationships between the physical quantities before and after the impact, and finding the desired final state vector of one step are the key points in the study of the repeatability condition for higher order constraint systems.

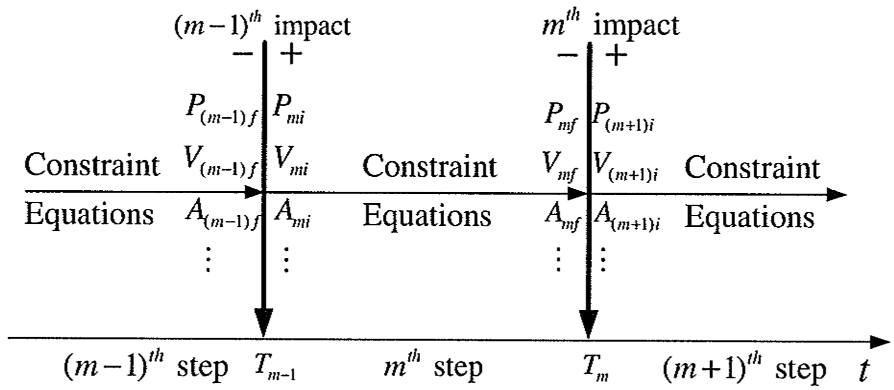

Figure 3.1 The effect of impact on the steps

So far the methods of satisfying the repeatability condition for higher order constraint systems can be divided into two types: 1) the methods based on initial-value problem, and 2) the methods based on two-point boundary value problem. The former type is an extension of the method we have discussed in Chapter 2. The solutions to the constraint equations are uniquely determined by the initial conditions. However, most prevailing methods of studying bipedal walking with some considerations of repeatable gait belong to the latter type. In these methods, the step period, and the initial and final state vectors of the step are defined. The joint angles are approximated by a certain form of series, such as time polynomials, Fourier series, etc. The initial and final state vectors are given like boundary conditions to determine some of the unknown coefficients of the series, and the rest of the coefficients can be determined by solving the constraint equations. In this way, the differential equations can be converted into algebraic equations. The major advantages of the second type of methods are the low computation expense, and the fact that these methods can easily produce repeatable gait. However, on the other hand, the major disadvantage is that some undesirable characteristics inherent to the chosen series are imposed to the joint angles.

In this chapter, the applications and difference between the two types of methods will be discussed. For convenience, we define the joint angle as zero order physical quantity, and consequently, define the joint angle's n^{th} order derivative with respect to time as n^{th} order physical quantity. According to this definition, the angular velocity and angular acceleration are the first and second order physical quantities, respectively.

3.2 The repeatability condition for higher order constraint systems

In this section, we shall give the repeatability condition for higher order constraint systems. Because the constraint systems are an initial-value problem, the solutions to the constraint systems are affected only by the choice of initial conditions.

Given an n^{th} order initial-value problem,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{(n-1)} y}{dt^{(n-1)}} + \ \dots \ a_1 \frac{dy}{dt} + a_0 = 0,$$

$$y(0) = b_0,\ y'(0) = b,\ \dots,\ y^{(n-1)}(0) = b_{n-1},$$

if $n > 1$, to determine the solution, we must consider all the initial conditions, from zero order to $(n-1)^{th}$ order.

According to the theory of initial-value problems, when the differential equation is given, its solution is influenced only by the initial conditions. For a repeatable gait solved from a constraint system, the solution to the differential equation must be identical or periodic with a period of step period. Therefore, the initial conditions must be repeatable or periodic if the solution is repeatable. From this point, we give a general repeatability condition as follows.

Lemma 3.1: For a five link biped robot, if the given constraint system is an n^{th} order initial value problem, the sufficient and necessary condition that leads to a repeatable gait is: During two successive steps, say the m^{th} and $(m+1)^{th}$ step, $I_m^{(k)} = I_{m+1}^{(k)}$, $k = 0,\ 1,\ ...,\ n\text{-}1$, where $I_m^{(k)}$ stands for the k^{th} order initial physical quantity of the m^{th} step.

If $n = 1$, we shall have Theorem 2.1. We mean that Theorem 2.1 is a special case of Lemma 3.1. However, compared with Theorem 2.1, Lemma 3.1 is only a pure mathematical interpretation of the repeatability condition. From the following discussion, we shall see that the satisfaction of such a repeatability condition is challenging.

3.3 The change of the values of physical quantities after the impact phase

Normally, the impact phase plays an important role in relating the initial and final state vectors. Any two successive steps are connected by the impact phase and the initial state vector of one step can be dictated by the final state vector of its previous step when we do not consider the double support phase. Obtaining the relationship of the state vectors before and after the impact phase is a key factor in satisfying the repeatability condition, since the relationship enables us to satisfy the repeatability condition by studying only the information of one step, including the final state vector of the step. Therefore, the main task of this section is to find the final state vector from the given initial state vector of one step.

We make three assumptions here:

1) the planar biped robot consists of five links;

2) only single support phase is considered;

3) the step ends with a single impact.

The impact of a robot with its environment is a common phenomenon in robotic systems. Industrial robots have to perform some desired tasks that require the robot to come in contact with the object under processing and exert upon it a suitable force. The impact phenomenon in legged robots takes place when the swing leg comes into contact with the ground. In the following discussion, we do not consider the double support phase. Therefore, after the impact phase, a new step starts, and the state vector after the impact is the state vector of the initial state vector of the next step.

Due to the assumption that impact takes place in an infinitely small time interval and is perfectly plastic, the tip of the swing leg does not leave the walking surface after the impact and its velocity immediately becomes zero. In this situation, the joint angles are unchanged before and after the impact. According to the repeatability condition, we know exactly the angles at the final moment of the single support phase.

Meanwhile, the impact with the environment causes a sudden change in the joint angular velocities. It is therefore required to compute the joint angular velocities just before each collision. Actually, at the moment when the robot comes into contact with the environment, a geometric constraint is enforced to the system.

Let $\mathbf{X}_e$ be the instantaneous position of the free end of the swing leg which comes into contact with the ground, with respect to the fixed coordinate frame shown in Figure 2.1 (b). We can express it as

$\mathbf{X}_e = \mathbf{X}_e(\theta_1, \theta_2, ..., \theta_5) = \begin{pmatrix} x_e \\ y_e \end{pmatrix}$, where $\boldsymbol{\theta} = \boldsymbol{\theta}(\theta_1, \theta_2, ..., \theta_5)^T$ is the vector of generalized coordinates of the system. A generalized constraint force $\mathbf{F}_\delta$ is introduced into the system [14] when the collision occurs at a given contact point, i.e., $\mathbf{X}_e(\theta) = \mathbf{X}_s$, where $\mathbf{X}_s$ is the contact point, and

$$\mathbf{F}_\delta = \left(\frac{\partial \mathbf{X}_e}{\partial \boldsymbol{\theta}} \right)^T \boldsymbol{\lambda}, \tag{3.1}$$

where $\boldsymbol{\lambda}$ is a vector of Lagrangian multipliers.

If the dynamic model before the collision is

$$\mathbf{D}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \mathbf{G}(\boldsymbol{\theta}) = \mathbf{T}_\theta. \tag{3.2}$$

where

$$\begin{aligned}
&\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_5)^T, \\
&\mathbf{T}_\theta = (T_{\theta 1}, T_{\theta 2}, ..., T_{\theta 5})^T, \\
&\mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = col\Big(\sum_{j=1(j \neq i)}^{5} (h_{ijj}(\dot{\theta}_j)^2) \Big), \\
&\mathbf{G}(\boldsymbol{\theta}) = col(G_i(\boldsymbol{\theta})), \\
&\mathbf{D}(\boldsymbol{\theta}) = (D_{ij}(\theta)),\ i, j = 1, 2, ..., 5.
\end{aligned}$$

The equation of motion just during the impact is

$$\mathbf{D}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \mathbf{G}(\boldsymbol{\theta}) = \mathbf{F}_\delta + \mathbf{T}_\theta. \tag{3.3}$$

Here we assume that the collision occurs during an infinitesimal duration. Since the joint angular velocities are finite and their integral over an infinitesimal time interval is zero, the joint positions remain unchanged.

Integrating equation (3.3) over the infinitesimal time interval during the impact $[t_0, t_0 + \Delta t]$ (t_0 is the instant of impact), we have

$$\lim_{\delta t \to 0} \int_{t_0}^{t_0+\delta t} \mathbf{D}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} dt + \lim_{\delta t \to 0} \int_{t_0}^{t_0+\delta t} [\mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \mathbf{G}(\boldsymbol{\theta}) - \mathbf{T}_\theta] dt = \lim_{\delta t \to 0} \int_{t_0}^{t_0+\delta t} \mathbf{F}_\delta dt .$$

Therefore,

$$\mathbf{D}(\boldsymbol{\theta})\Delta\dot{\boldsymbol{\theta}} = \lim_{\delta t \to 0} \int_{t_0}^{t_0+\delta t} \mathbf{F}_\delta dt = \mathbf{J}^T \lim_{\delta t \to 0} \int_{t_0}^{t_0+\delta t} \boldsymbol{\lambda} dt ,$$

where $\mathbf{J} = \left(\frac{\partial \mathbf{X}_e}{\partial \boldsymbol{\theta}} \right)$. In addition, we have $\mathbf{V}_e = \mathbf{J}\dot{\boldsymbol{\theta}}$. Hence,

$$\mathbf{V}_e - \mathbf{V}_{contact} = \mathbf{J}_a \dot{\boldsymbol{\theta}}_a - \mathbf{V}_{contact} .$$

We note that $\mathbf{V}_{contact} = 0$, so that

$$\mathbf{J}_a [\dot{\boldsymbol{\theta}}_a(t_0 + \delta t) - \dot{\boldsymbol{\theta}}_a(t_0)] = \mathbf{V}_e(t_0 + \delta t) - \mathbf{V}_e(t_0) ,$$

and

$$\mathbf{J}_a \Delta\dot{\boldsymbol{\theta}}_a = \Delta \mathbf{V}_e .$$

It is easy to get

$$\lim_{\delta t \to 0} \int_{t_0}^{t_0+\delta t} \boldsymbol{\lambda} dt = [\mathbf{J}\mathbf{D}^{-1}\mathbf{J}^T]^{-1} \Delta \mathbf{V}_e .$$

Finally, we can show that

$$\Delta\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}_{after} - \dot{\boldsymbol{\theta}}_{before} = \mathbf{D}^{-1}\mathbf{J}^T[\mathbf{J}\mathbf{D}^{-1}\mathbf{J}^T]^{-1} \Delta \mathbf{V}_e = -\mathbf{D}^{-1}\mathbf{J}^T[\mathbf{J}\mathbf{D}^{-1}\mathbf{J}^T]^{-1} \mathbf{V}_{before} . \quad (3.4)$$

Equation (3.4) also can be written as

$$\dot{\boldsymbol{\theta}}_{before} = \dot{\boldsymbol{\theta}}_{after} + \mathbf{D}^{-1}\mathbf{J}^{T}[\mathbf{J}\mathbf{D}^{-1}\mathbf{J}^{T}]^{-1}\mathbf{V}_{before}. \tag{3.5}$$

Since $\mathbf{V}_{before} = \mathbf{J}\dot{\boldsymbol{\theta}}_{before}$, we have

$$\dot{\boldsymbol{\theta}}_{before} = \{\mathbf{E} - \mathbf{D}^{-1}\mathbf{J}^{T}[\mathbf{J}\mathbf{D}^{-1}\mathbf{J}^{T}]^{-1}\mathbf{J}\}^{-1}\dot{\boldsymbol{\theta}}_{after}, \tag{3.6}$$

where $\mathbf{E}$ is the identity matrix.

Equation (3.6) indicates that there exists a one to one relationship between $\dot{\boldsymbol{\theta}}_{before}$ and $\dot{\boldsymbol{\theta}}_{after}$. Here, $\dot{\boldsymbol{\theta}}_{before}$ is the final joint angular velocity vector of one step and $\dot{\boldsymbol{\theta}}_{after}$ is the initial joint angular velocity vector of the next step. For a repeatable gait, if the initial joint angular velocities of one step are given, the initial joint angular velocities of the next step ($\dot{\boldsymbol{\theta}}_{after}$) are fixed too. According to equation (3.6), $\dot{\boldsymbol{\theta}}_{before}$ or the angular velocities at the final moment of the previous step can be found.

For the change of the higher order (> 1) physical quantities, we notice a fact that the values of second order physical quantities (angular accelerations) can be determined by dynamic equations (3.2) and (3.3) if the angles and angular velocities before and after the impact are given. We give a general form of the dynamic equations before and after the impact,

$$\mathbf{D}(q)\ddot{\mathbf{q}} + \mathbf{h}(q,\dot{q})\dot{\mathbf{q}} + \mathbf{G}(q) = \mathbf{F}, \tag{3.7}$$

where $\mathbf{F}$ is the generalized force which includes control torque and any force introduced during the impact. Assuming that $\mathbf{F}$ is not time variant, and does not depend on $\ddot{\mathbf{q}}$ and the higher order physical quantities. Equation (3.7) can be written as

$$\ddot{\mathbf{q}} = \mathbf{D}^{-1}(q)[\mathbf{F} - \mathbf{h}(q,\dot{q})\dot{\mathbf{q}} - \mathbf{G}(q)]. \tag{3.8}$$

Because the generalized force **F** is assumed not to be the function of $\ddot{q}$ and the higher order physical quantities, the initial and final values of $\ddot{\mathbf{q}}$ are determined by the initial and final values of $\mathbf{q}$ and $\dot{\mathbf{q}}$. Consequently, the values of the initial and final higher order physical quantities can be found by differentiating equation (3.8). Therefore, the relationship of the joint angle velocities before and after the impact is the most fundamental one.

All the discussion in this section is to establish the relationships between the initial and final physical quantities. These relationships are crucial conditions in satisfying the repeatability condition. Among these relationships, the relationship of the joint angle velocities before and after the impact is the most fundament relationship.

3.4 The method of satisfying the repeatability condition for higher order constraint systems based on initial-value problems

The method discussed in this section is an extension of that used in the first order constraint systems, i.e., we investigate the possibility of satisfying the repeatability condition by setting only the initial state vector of one step. In this section, we shall discuss this method's application and challenges in higher order constraint systems.

According to the discussion in Section 3.3, the initial and final joint angular accelerations and other higher order physical quantities can be determined by the dynamic equation if the initial and final joint angles and angular velocities are given. Therefore, in the following discussion, the initial and final state vectors mean the initial and final joint angles and angular velocities.

According to Lemma 3.1, for a repeatable gait, the initial state vectors in each step must be identical. Meanwhile, there exists a relationship between the final state vector of the current step and the initial state vector of the next step, so the final state vector of one step is fixed when the initial state vector of any one step is given.

As an initial value problem, the problem is that the solutions to the constraint equations are determined by the given initial state vector of the step. Therefore, the final state vector of one step is also determined by the initial state vector when we solve the constraint equations. How can we guarantee that the final state vector obtained by solving the constraint equations will lead to the identical initial state vector of the next step that is identical with the given initial state vector?

In some sense, this problem can be solved by the trial-and-error method. In this way, we try to give the initial state vector and solve the constraint equations, and examine whether the final state vector will lead to an identical initial state vector of the next step. Otherwise, we try another initial state vector until the final state vector can make initial state vector of the next step identical to the given initial state vector. However, this method is not feasible for highly complex nonlinear bipedal systems. At this point, we intend to correlate the initial state vectors with constraint equations and their assorted gait parameters in the context of generating repeatable gait, so that by selecting proper constraint equations and the gait parameters the repeatable gait can be achieved. However, we are not able to establish a principle like Lemma 2.2 to satisfy the repeatability condition by choosing certain equations, step parameters etc. for higher order constraint systems.

3.5 The methods of satisfying the repeatability condition for higher order constraint systems based on boundary-value problems

We have discussed the challenge when we employ the method based on initial-value problems. Due to this difficulty, this type of method is not feasible to deal with the constraint system whose order is higher than one. In this situation, another type of method must be employed.

The basic idea can be explained as follows. We expand the joint angles into a certain form of finite series, such as

$$q_i(t) = \sum_{k=0}^{m} a_k f_k(t), \tag{3.9}$$

where a_k is the k^{th} coefficient that is independent of time and $f_k(t)$ is the k^{th} function of time, and m is a finite number. Since the initial state vectors of each step are identical according to the repeatability condition, the final state vectors of each step are fixed when the impact phase is considered. For one step, the initial and final state vectors are fixed at the same time. Thus, we can convert the initial-value problem to a two-point boundary-value problem and treat the initial and final state vectors as the boundary conditions. Set $t = 0$, we arbitrarily choose the initial angle as $q_i(0) = \sum_{k=0}^{m} a_k f_k(0)$. Consequently, we can have its derivatives with respect to time at $t = 0$. Similarly, given $t = T$, where T is the step period, the final joint angle is determined by the initial angles and the impact effect as $q_i(T) = \sum_{k=0}^{m} a_k f_k(T)$. Differentiating the joint angles at the final moment, we obtain the derivatives with respect to time. In this way, some of the unknown coefficients of the series are determined. Substituting the joint angle in the constraint equations and the dynamic equation, the constraint equations can be solved by finding desired remaining coefficients.

Based on this idea, several methods to meet the repeatability condition have been proposed. Channon *et al.* [20] considered the repeatability condition by introducing the error of the step length of each step into the optimization problem and minimizing it. This error is restricted to a small enough value in each step. The basic idea is that if the error is so small that it can be accepted, the gait was believed repeatable. However, it is not true that a repeatable step length will lead to a repeatable gait pattern. It might violate the definition of repeatability given in Chapter 2. Chevallereau [11] expanded the joint angle into a time polynomial and considered the joint angles and angular velocities at the same time. Repeatability is satisfied by fixing the same initial and final state vectors (joint angles and angular velocities) in each step. Fourier expansions [31] [32] and alternative methods [33] were also employed.

Once the initial state vector is given, the final state vector is determined by the repeatability condition and the impact effect. We repeat the procedure of setting the initial and final state vectors as the boundary conditions in each step, so we can have repeatable solutions to the constraint equations.

The advantages of this type of method are: 1) the partial repeatability up to the certain order, but not all the order of the physical quantities is easily satisfied, and 2) the computation expense is low. While the main disadvantages of this type of method are: 1) some undesirable characteristics inherent to the chosen series are imposed on the joint angles, and 2) the repeatability of higher order physical quantities cannot be satisfied when the specific form of series is given.

Unlike this type of method, the method based on initial-value problem considers a sequence of continuous steps. The results obtained in one step must be brought to the next step. The effect of the initial conditions will be forwarded to all the following steps. Any deviation from the exact solutions will be enlarged in the following steps, leading to a change of the original gait. If the calculation is accurate, we do not have to be concerned about the deviation of periodicity of the solutions. A series of steps is precisely prescribed by the differential equations and the corresponding initial state vector. For some simpler systems, like a first order constraint system, this kind of method is straightforward and shows the beauty of prediction. However, for most complicated system, this method faces serious challenges. Chan [12] [13] introduced a 2^{nd} order differential equation into the constraint system, and the method based on initial-value problems was employed to satisfy the repeatability condition. Because the initial and final state vectors are not on the same solution curves, the gait could not be repeated well and finally the repeatability condition is violated. The main advantage of the method based on initial-value problems is that if this method is feasible, the repeatability of all the orders of physical quantities can be satisfied. While the disadvantage is that, it is difficult to make the final state vector determined by the initial state vector compatible with that determined by the repeatability condition and the impact effect.

3.6 Summary

In this chapter, the repeatability condition for higher order constraint systems has been given and two types of methods of satisfaction of the repeatability condition for higher order constraint systems have been discussed. In principle, the method based on initial-value problem can be applied for any order constraint systems, and provide exact solutions, but this type of method faces great difficulty. So far, the repeatability condition has not been successfully satisfied using the method based on initial-value problems for higher order constraint systems. Nevertheless, the methods based on boundary-value problems can be applied in higher order constraint systems. The partial repeatability up to the certain order, but not all the orders of the physical quantities is easily satisfied. The time-polynomial method used in Chapter 4 can handle the problem of partially satisfying the repeatability condition for higher order systems.

Chapter 4

Comparison of two minimization criteria based on input energy

4.1 Introduction

The minimization criterion

$$\min \ J_1 = \int_{t_0}^{t_0+T} \sum_{i=1}^{n} T_i^2 dt, \tag{4.1}$$

has often been employed for optimal trajectory designs to minimize joint torques. Functional does not have any physical meaning, however, it is believed that if J_1 is minimized, the joint accelerations will be small, implying that the resulting walking motions will be smooth and energy consumption low [24].

From an energy point of view, the minimization criterion

$$\min J_0 = \int_{t_0}^{t_0+T} \sum_{i=1}^{n} | T_i \dot{q}_i | dt, \tag{4.2}$$

should be the most properly proposed criterion to minimize the input energy, because the physical meaning of functional J_0 is energy or input work. Nevertheless, due to the non-linearity and non-differentiability at certain points, to directly minimize J_0 is not feasible.

To avoid the non-differentiability problem, the minimization criterion

$$\min J_2 = \int_{t_0}^{t_0+T} \sum_{i=1}^{n} (T_i \dot{q}_i)^2 dt, \tag{4.3}$$

can be treated as an approximation of criterion (4.2), but strictly speaking, these two criteria are different in mathematics. Like functional J_1, functional J_2 does not have any physical meaning either. From an energy point of view, for the same physical system and on the same conditions, that criterion (4.3) might produce lower input energy than criterion (4.1) is expected. Meanwhile, from a control torques point of view, that criterion (4.1) might produce lower control torques than criterion (4.3) is anticipated.

In this chapter, functionals J_1 and J_2 defined in (4.1) and (4.3) will be discussed from the standpoint of minimizing energy. A workable way is to obtain the trajectories from minimizing functional J_1 and J_2, and then input them into functional J_0 defined in (4.2) to calculate the total energy input. Thus, we can compare the difference of input energy.

The physical model used in this chapter is two-link pendulums, which is shown in Figure 4.1. The locomotion of the two pendulums is confined in the saggital plane and the initial and final points are given. The tip of the link is driven from initial point B_0 to the final point B_T.

Given the link length L_i, mass M_i, inertia I_i, the location of the centre of mass d_i, and the distance between B_0 and B_T, we shall calculate the minimal input energy corresponding to the optimization criteria (4.1) and (4.3). The detail derivation of the dynamic equations is shown in Appendix A. The optimization will lead to 4th order ordinary differential equations shown in Appendix B.

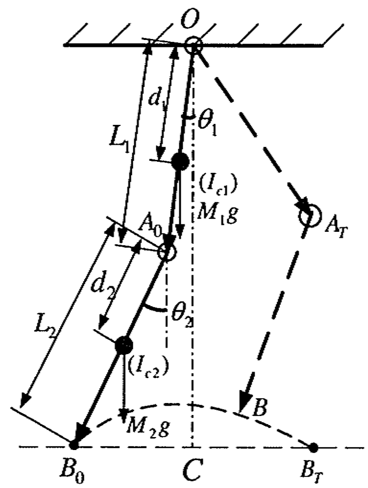

Figure 4.1 The two link pendulums model

4.2 The polynomial technique

Because of the mathematical and dynamic difficulties we discussed in Chapter 3, it is not applicable to use the method based on initial-value problem to solve the constraint systems for higher order cases.

Chevallereau *et al* [11] introduced an approximate technique. At the point of the satisfaction of the repeatability condition, this technique belongs to the method based on two-point boundary-value problem that we discussed in Chapter 3. Due to the convenience in satisfaction of the repeatability condition for higher order constraint systems, we employ this method in this chapter. However, the main task of this chapter is to compare two optimization criteria (4.1) and (4.3). We have discussed the repeatability problem in detail in Chapter 3, so in this chapter, we only arbitrarily set the initial and final state of the locomotion of the swing links. In practical research of the biped gait, the final state is determined by the relationship of the physical quantities before and after the impact.

Each angle is approximated by a polynomial with respect to time. The polynomials take the form of

$$q_i = a_{i0} + a_{i1}t + a_{i2}t^2 + \ldots + a_{ik}t^k\,, \tag{4.4}$$

where, i denotes the i^{th} angle, and k is a finite number which indicates the highest order of the polynomials, and, a_{ij}, $j = 0,\ 1,\ \ldots\ , k$ are the parameters to be determined. The repeatability condition can be employed to reduce the number of optimization parameters. Some of these parameters can be determined in advance according to certain initial and final conditions, and the left uncertain parameters are for the minimization.

The determination of k is in accordance with some physical and mathematical constraint conditions. Compared with Channon's work [20], besides the joint angles, the angular velocities at the initial and final moment in one step were also taken into account by Chevallereau *et al* [11]. Assume that we have the initial angles and angular velocities,

$$q_i(0) = a_{i0}\,, \tag{4.5a}$$

$$\dot{q}_i(0) = a_{i1}\,. \tag{4.5b}$$

At the same time, according to the relationships based on the repeatability condition, we fix the final angles and angular velocities as

$$q_i(T) = a_{i0} + a_{i1}T + a_{i2}T^2 + \ldots + a_{ik}T^k\,, \tag{4.5c}$$

$$\dot{q}_i(T) = a_{i1} + a_{i2}T + \ldots + a_{ik}T^{k-1}\,. \tag{4.5d}$$

Now we find that if we consider the angular velocities, we can determine four of the parameters in polynomial (4.4). Therefore, the smallest number that k can take is three. However, that $k = 3$ is not desirable for the minimization, because there is no freedom for the minimization. Hence, the lowest order of the polynomial (4.2) is four.

4.3 The numerical strategy to minimize functional J_1 and J_2

Minimizing or maximizing a single function of several variables is a very common and important problem in all branches of science and engineering. This problem occurs naturally as part of many optimization codes that are used to find the "best" combination of variables to maximize the profit of a business, minimize the energy in an atomic configuration, find the most favourable or shortest path along a surface, or optimize the fit of a multi-parameter function to data.

Basically, two types of algorithms are used to find the minimum of a multivariable function. The first is to attempt to bracket the point of the minimum with the hope of successively reducing the volume of the bracketing region until sufficient accuracy is obtained. This method is known as the downhill simplex method. Simplex methods are not highly efficient and are appropriate only if the number of variables is not too large, say less than six [40]. The second, more complicated, and usually more efficient, class of methods attempts to use information contained in the derivative of the function to follow the function downhill, perhaps along a curving path, until the function reaches a local minimum. These types of procedures are called directional set methods or quasi-Newton methods [40], and they have numerous variations. In this thesis, we apply the quasi-Newton method for comparing the minimization criteria (4.1) and (4.3).

4.3.1 Quasi-Newton method for minimization of multivariable function

Give a multidimensional Taylor series expansion for a function $f(\vec{x})$ expanded about the point $\vec{x}_0$,

$$f(\vec{x}) = f(\vec{x}_0) + (\vec{x} - \vec{x}_0) \cdot \nabla f_0 + \frac{1}{2}(\vec{x} - \vec{x}_0) \cdot \mathbf{H}_0 \cdot (\vec{x} - \vec{x}_0) + \cdots, \tag{4.6}$$

where the function and its derivatives on the right-hand side are to be evaluated at $\vec{x}_0$. The matrix $\mathbf{H}_0$ is the set of second order partials,

$$(\mathbf{H}_0)_{ij} = \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\vec{x}=\vec{x}_0}, \tag{4.7}$$

and is known as the Hessian of the function f at the point $\vec{x}_0$.

If f is a multivariable function, a minimum is characterized by $\nabla f = 0$, and its Hessian is positive, definite, and symmetric. The iteration relationship is presented by

$$\vec{x}_{k+1} = \vec{x}_k - \vec{\mathbf{H}}_k^{-1} \cdot \nabla f_k. \tag{4.8}$$

This step will have to be iterated until changes in the projected minimum differ by less than a convergence tolerance. In quasi-Newton methods [40], each iteration begins with an attempt to build up successively better and better approximations to $\vec{\mathbf{H}}_k^{-1}$ using only computed values of f and ∇f.

4.3.2 Determination of some of the parameters in the polynomials

As the discussion in Section 4.2, some of the parameters in the polynomial can be determined by initial and final conditions. If the highest order of the polynomial (4.4) is four, we can rewrite the polynomial (4.4) as

$$q_i = a_{i0} + a_{i1}t + a_{i2}t^2 + a_{i3}t^3 + x_i t^4,\ i = 1, 2, \tag{4.9}$$

where x_i is the parameter to be optimized and time $t \in [t_0, t_0 + T]$. Furthermore, we can have the corresponding derivatives with respect to time,

$$\dot{q}_i = a_{i1} + 2a_{i2}t + 3a_{i3}t^2 + 4x_i t^3,\ i = 1,2; \tag{4.10}$$

$$\ddot{q}_i = 2a_{i2} + 6a_{i3}t + 12x_i t^2,\ i = 1,2. \tag{4.11}$$

We set the initial angles and angular velocities as

$$q_i(0) = q_{i0},\ i = 1,2; \tag{4.12}$$

$$\dot{q}_i(0) = \dot{q}_{i0},\ i = 1,2. \tag{4.13}$$

Assume that the final angles and angular velocities are given by the continuity condition and repeatability condition. Now we let

$$q_i(T) = q_{iT},\ i = 1,2; \tag{4.14}$$

$$\dot{q}_i(T) = \dot{q}_{iT},\ i = 1,2. \tag{4.15}$$

Considering equations (4.9)—(4.11), we have relations as

$$a_{i0} = q_{i0},\ i = 1,2, \tag{4.16}$$

$$a_{i1} = \dot{q}_{i0},\ i = 1,2, \tag{4.17}$$

$$q_i(T) = a_{i0} + a_{i1}T + a_{i2}T^2 + a_{i3}T^3 + x_i T^4,\ i = 1,2, \tag{4.18}$$

$$\dot{q}_i(T) = a_{i1} + 2a_{i2}T + 3a_{i3}T^2 + 4x_i T^3,\ i = 1,2, \tag{4.19}$$

Rewriting equation (4.18) and (4.19), we have

$$(q_{iT} - q_{i0} - \dot{q}_{i0}T)/T^2 = a_{i2} + a_{i3}T + x_i T^2,\ i = 1,2, \tag{4.20}$$

$$(\dot{q}_{iT} - \dot{q}_{i0})/T = 2a_{i2} + 3a_{i3}T + 4x_i T^2,\ i = 1,2, \tag{4.21}$$

Let $\alpha_i = (q_{iT} - q_{i0} - \dot{q}_{i0}T)/T^2$, and $\beta_i = (\dot{q}_{iT} - \dot{q}_{i0})/T$, $i = 1,2$. Solving equations (4.20) and (4.21), we have

$$a_{i2} = 3\alpha_i - \beta_i + x_i T^2,\ i = 1,2, \tag{4.22}$$

$$a_{i3} = (\beta_i - 2\alpha_i - 2x_i T^2)/T,\ i = 1,2. \tag{4.23}$$

Finally, four of the five parameters for each joint angle from a_{i0} to a_{i3} have been determined. Here we note that for the 4th order polynomials, to determine 4 of the 5 parameters, the zero and first order initial and final conditions are needed and any higher order initial and final conditions are not necessary any more. That is why we take only the zero and first order initial and final conditions into account. On the other hand, the lowest order of the polynomial is determined by the order of the initial and final conditions that we are considering. Generally, the order of the polynomial N can be expressed by

$$N = (2 \times N_2 + F), \tag{4.24}$$

where N_2 is the highest order of the initial and final conditions considered by us, and F is the number of the parameters in the polynomial to be optimized. The total degree of freedom of optimization can be written as

$$N = N_1 \times F, \tag{4.25}$$

where N_1 is the number of the links.

The two functionals J_1 and J_2 also can be written as the form

$$J_1 = \int_{t_0}^{t_0+T} g_1(x_1, x_2; t)dt, \tag{4.26}$$

$$J_2 = \int_{t_0}^{t_0+T} g_2(x_2, x_2; t)dt, \tag{4.27}$$

where $g_1(x_1, x_2; t) = \sum_{i=1}^{2} T_i^2$, and $g_2(x_1, x_2; t) = \sum_{i=1}^{2} (T_i \dot{q}_i)^2$, $i = 1,2$. In this way, we treat the functionals as multivariable functions.

4.3.3 The gradient and Hessian of the integrands of J_1 and J_2

To use Quasi-Newton method, the gradient and Hessian of the function to be minimized need to be derived. Take gradient at the both sides of function (4.26) and (4.27), and we have

$$\nabla J_i = \nabla \int_{t_0}^{t_0+T} g_i(x_1, x_2; t)dt = \int_{t_0}^{t_0+T} \nabla g_i(x_1, x_2; t)dt . \tag{4.28a}$$

Relation (4.28a) shows that the gradient of the functional is the integration of the gradient of the functional's integrand on the original interval. Take gradient at both sides of equation (4.28a) once more, we have

$$\nabla\nabla J_i = \mathbf{H}_i = \nabla\nabla \int_{t_0}^{t_0+T} g_i(x_1, x_2; t)dt = \int_{t_0}^{t_0+T} \nabla\nabla g_i(x_1, x_2; t)dt . \tag{4.28b}$$

Equation (4.28b) is the expression of the Hessian of the functional.

The gradient of the integrand is written as

$$\nabla g_i = \frac{\partial g_i}{\partial x_j} = \frac{\partial g_i}{\partial q_k}\frac{\partial q_k}{\partial x_j} + \frac{\partial g_i}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial x_j} + \frac{\partial g_i}{\partial \ddot{q}_k}\frac{\partial \ddot{q}_k}{\partial x_j}, \tag{4.29}$$

where $i, j, k = 1,2$. Recall equations (4.9), (4.10) and (4.11), and we note that

$$\frac{\partial q_k}{\partial x_j} = 0, \ \frac{\partial \dot{q}_k}{\partial x_j} = 0, \ \frac{\partial \ddot{q}_k}{\partial x_j} = 0, \ k \neq j . \tag{4.30}$$

Hence, equation (4.29) can be written as

$$\nabla g_i = \frac{\partial g_i}{\partial x_j} = \frac{\partial g_i}{\partial q_j}\frac{\partial q_j}{\partial x_j} + \frac{\partial g_i}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial x_j} + \frac{\partial g_i}{\partial \ddot{q}_j}\frac{\partial \ddot{q}_j}{\partial x_j}, \tag{4.31}$$

where

$$\frac{\partial q_j}{\partial x_j} = t^4, \frac{\partial \dot{q}_j}{\partial x_j} = 4t^3, \frac{\partial \ddot{q}_j}{\partial x_j} = 12t^2 . \tag{4.32}$$

Now let $i=1$, and we have $g_1(x_1,x_2;t)=\sum_{i=1}^{2}T_i^2(x_1,x_2;t)=T_1^2(x_1,x_2;t)+T_2^2(x_1,x_2;t)$.

Substituting g_1 into equation (4.31), we have

$$\nabla g_1 = 2(A_1\frac{\partial q_1}{\partial x_1}+A_2\frac{\partial \dot{q}_1}{\partial x_1}+A_3\frac{\partial \ddot{q}_1}{\partial x_1}), \tag{4.33}$$

where

$$A_1 = T_1[\gamma\cos q_1+\sigma\cos(q_1+q_2)]g+T_2\sigma\cos(q_1+q_2)g\,, \tag{4.34a}$$

$$A_1 = \beta\sin q_2(T_2\dot{q}_1-T_1\dot{q}_2)\,, \tag{4.34b}$$

$$A_3 = T_1(\alpha+\beta\, cos\, q_2)+T_2(\delta+\frac{\beta}{2}cos\, q_2). \tag{4.34c}$$

Let $i=2$, then $g_2(x_1,x_2;t)=\sum_{i=1}^{2}(T_i\dot{q}_i)^2=(T_1\dot{q}_1)^2+(T_2\dot{q}_2)^2$. Substituting g_2 into equation (4.31), we have

$$\nabla g_2 = 2(B_1\frac{\partial q_1}{\partial x_1}+B_2\frac{\partial \dot{q}_1}{\partial x_1}+B_3\frac{\partial \ddot{q}_1}{\partial x_1}), \tag{4.35}$$

where

$$\begin{aligned} B_1 = {} & T_1[-\beta\sin q_2\ddot{q}_1-\frac{\beta}{2}\sin q_2\ddot{q}_2-\beta\cos q_2\dot{q}_1\dot{q}_2-\frac{\beta}{2}\cos q_2\dot{q}_2^2+\sigma\cos(q_1+q_2)g] \\ & +T_2[-\frac{\beta}{2}\sin q_2\ddot{q}_1+\frac{\beta}{2}\cos q_2\dot{q}_1^2+\sigma\cos(q_1+q_2)g], \end{aligned} \tag{4.36a}$$

$$B_1 = \beta\sin q_2[T_2\dot{q}_1-T_1(\dot{q}_1+\dot{q}_2)]\,, \tag{4.36b}$$

$$B_3 = T_1(\delta+\frac{\beta}{2}cos\, q_2)+T_2\delta\,. \tag{4.36c}$$

Given any instant t, from equation (4.33) and (4.35) we can obtain the gradients of functions g_1 and g_2.

In equation (4.31), we let $C_{i1} = \frac{\partial g_i}{\partial q_j}\frac{\partial q_j}{\partial x_j}$, $C_{i2} = \frac{\partial g_i}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial x_j}$, $C_{i3} = \frac{\partial g_i}{\partial \ddot{q}_j}\frac{\partial \ddot{q}_j}{\partial x_j}$. Now we have the Hessian as

$$
\begin{aligned}
\nabla\nabla g_i &= \sum_{k=1}^{3}[\frac{\partial C_{ik}}{\partial q_j}\frac{\partial q_j}{\partial x_j} + \frac{\partial C_{ik}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial x_j} + \frac{\partial C_{ik}}{\partial \ddot{q}_j}\frac{\partial \ddot{q}_j}{\partial x_j}] \\
&= \frac{\partial^2 g_i}{\partial q_j^2}(\frac{\partial q_j}{\partial x_j})^2 + \frac{\partial^2 g_i}{\partial \dot{q}_j^2}(\frac{\partial \dot{q}_j}{\partial x_j})^2 + \frac{\partial^2 g_i}{\partial \ddot{q}_j^2}(\frac{\partial \ddot{q}_j}{\partial x_j})^2 \\
&\quad + 2\frac{\partial^2 g_i}{\partial \dot{q}_j \partial q_j}\frac{\partial q_j}{\partial x_j}\frac{\partial \dot{q}_j}{\partial x_j} + 2\frac{\partial^2 g_i}{\partial \ddot{q}_j \partial q_j}\frac{\partial q_j}{\partial x_j}\frac{\partial \ddot{q}_j}{\partial x_j} + 2\frac{\partial^2 g_i}{\partial \ddot{q}_j \partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial x_j}\frac{\partial \ddot{q}_j}{\partial x_j}
\end{aligned}
\quad . \quad (4.37)
$$

The Hessian, the second order derivatives of function g_i, shows the complexity, which prevents us from applying it into a relatively more complicated function g_i. In our cases, the dynamic equations are extraordinaryly complicated, so we are going to discuss an approximate method to calculate the Hessian.

In this thesis, we approximate the Hessian by replacing the differential expressions by the difference expressions. Given any instant t, we can always find the function $g_i = g_i(x_1, x_2; t)$. We have the derivatives of g_i with respect to x_1 and x_2 at the given point $\vec{x}_c$ in the finite difference form (given the difference step h),

$$\left.\frac{\partial g_i}{\partial x_1}\right|_{\vec{x}=\vec{x}_c} = \frac{g_i(x_{1c}+h, x_{2c}) - g_i(x_{1c}-h, x_{2c})}{2h}, \quad (4.38a)$$

$$\left.\frac{\partial g_i}{\partial x_2}\right|_{\vec{x}=\vec{x}_c} = \frac{g_i(x_{1c}, x_{2c}+h) - g_i(x_{1c}, x_{2c}-h)}{2h}, \quad (4.38b)$$

$$\left.\frac{\partial^2 g_i}{\partial x_1^2}\right|_{\vec{x}=\vec{x}_c} = \frac{g_i(x_{1c}+h, x_{2c}) + g_i(x_{1c}-h, x_{2c}) - 2g_i(x_{1c}, x_{2c})}{h^2}, \quad (4.39a)$$

$$\left.\frac{\partial^2 g_i}{\partial x_1 \partial x_2}\right|_{\bar{x}=\bar{x}_c} = \frac{g_i(x_{1c}+h, x_{2c}+h) - g_i(x_{1c}, x_{2c}+h) - g_i(x_{1c}+h, x_{2c}) + g_i(x_{1c}, x_{2c})}{h^2}, \quad (4.39b)$$

$$\left.\frac{\partial^2 g_i}{\partial x_2^2}\right|_{\bar{x}=\bar{x}_c} = \frac{g_i(x_{1c}, x_{2c}+h) + g_i(x_{1c}, x_{2c}-h) - 2g_i(x_{1c}, x_{2c})}{h^2}. \quad (4.39c)$$

In our cases, equation $\left.\frac{\partial^2 g_i}{\partial x_1 \partial x_2}\right|_{\bar{x}=\bar{x}_c} = \left.\frac{\partial^2 g_i}{\partial x_2 \partial x_1}\right|_{\bar{x}=\bar{x}_c}$ always holds. In this way, at any given instant t, we can easily calculate the integrant's gradient

$$\nabla g_i = \begin{pmatrix} \frac{\partial g_i}{\partial x_1} \\ \frac{\partial g_i}{\partial x_2} \end{pmatrix}, \text{ and the Hessian } \nabla\nabla g_i = \begin{pmatrix} \frac{\partial^2 g_i}{\partial x_1^2} & \frac{\partial^2 g_i}{\partial x_1 \partial x_2} \\ \frac{\partial^2 g_i}{\partial x_2 \partial x_1} & \frac{\partial^2 g_i}{\partial x_2^2} \end{pmatrix}.$$

4.3.4 The numerical strategy to minimize functional J_1 and J_2

The goal of the simulation is to minimize functionals J_1 and J_2, and obtain the corresponding trajectories based on the same conditions. After that, we input the two trajectories into functional J_0 to compute the input energy, and compare the two results. We shall see which one of the two minimization criteria will lead to lower input energy. The strategy of the whole procedure is presented by the numerical flowchart in Figure 4.2.

First, we set all the physical parameters, such as the length, mass, inertia, etc., of the links. At the same time, we must give the initial conditions, such as the initial angle and angular velocity of each link. According to the continuity and repeatability condition, we also need to fix the final conditions, such as the final angle and angular velocity of each link.

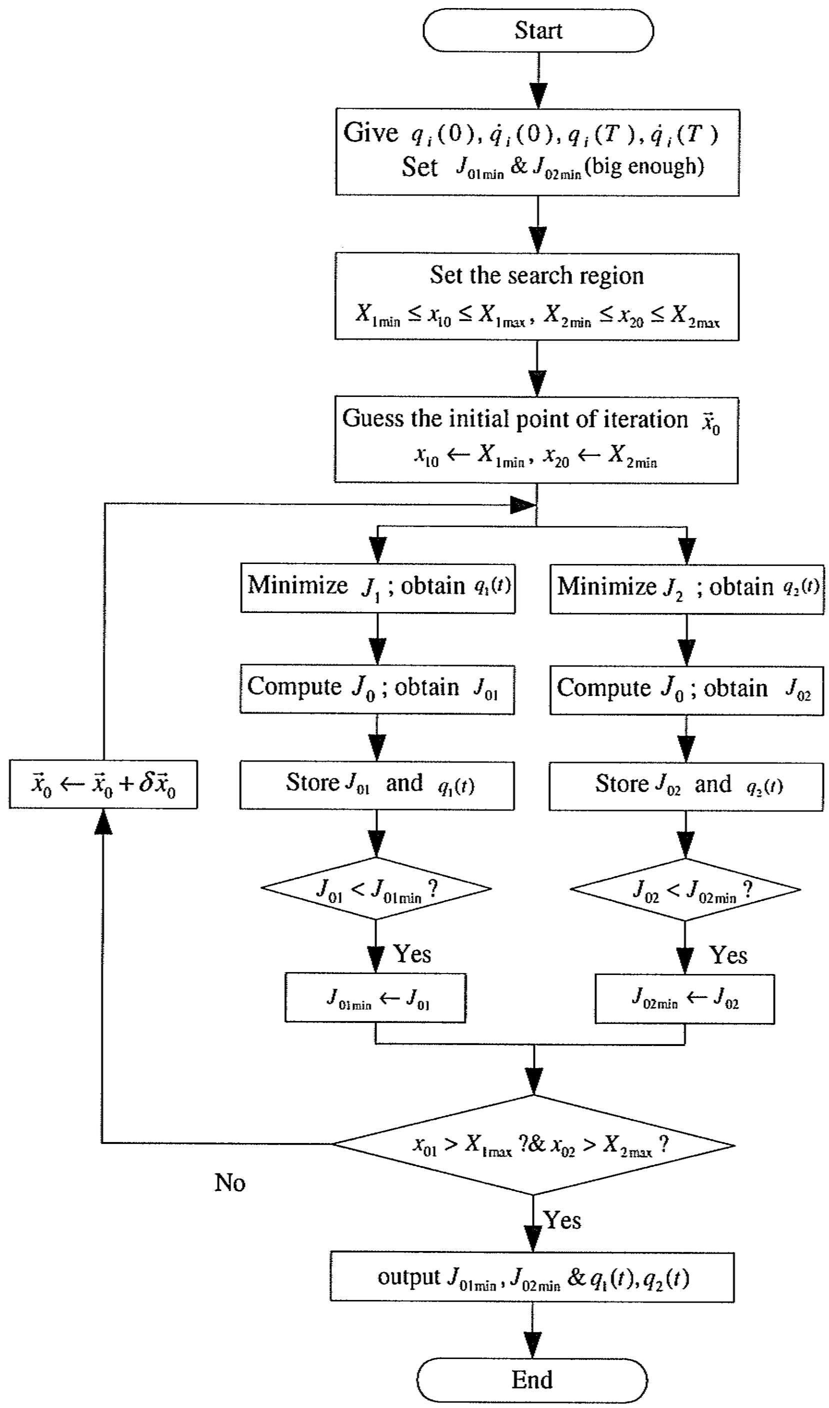

Figure 4.2 Numerical strategy of the comparison of the two minimization criteria

The minimization procedure is fulfilled by a series of iterations. Therefore, for the iteration, we arbitrarily give the initial position of the minimizer to start the iteration. Unlike analytical methods, one problem of the numerical minimization method is that we cannot find the minima on the infinite interval, which indicates that it is only a local minimization procedure. We set a range of the positions of the guessing minimizers, and say that the minimum is the lowest value when the guessing minimizer is located within the given range.

In our case, because it is a 2-dimensional problem (we have only parameters x_1 and x_2 to be minimized), we set a rectangular region for the initial guessing minimizers. The iteration will start from a certain guessing point within this region and converge to a minimizer. The different initial points may make the iteration converge to different minimizers. In this way, we are trying to find all the minima corresponding to all the initial guessing minimizers within the region we set, and obtain the smallest one from all the minima. The specific procedure is described by Figure 4.3.

We note that the gradient and Hessian of the functionals cannot be obtained directly. We calculate the gradient and Hessian of the corresponding integrant at each instant t. The gradient and Hessian of the functionals are finally obtained through the integration of the integrants' gradient and Hessian on interval $[0,T]$. The principle is presented in section 4.3.3. The numerical integration method used in this thesis is the Simpson integration.

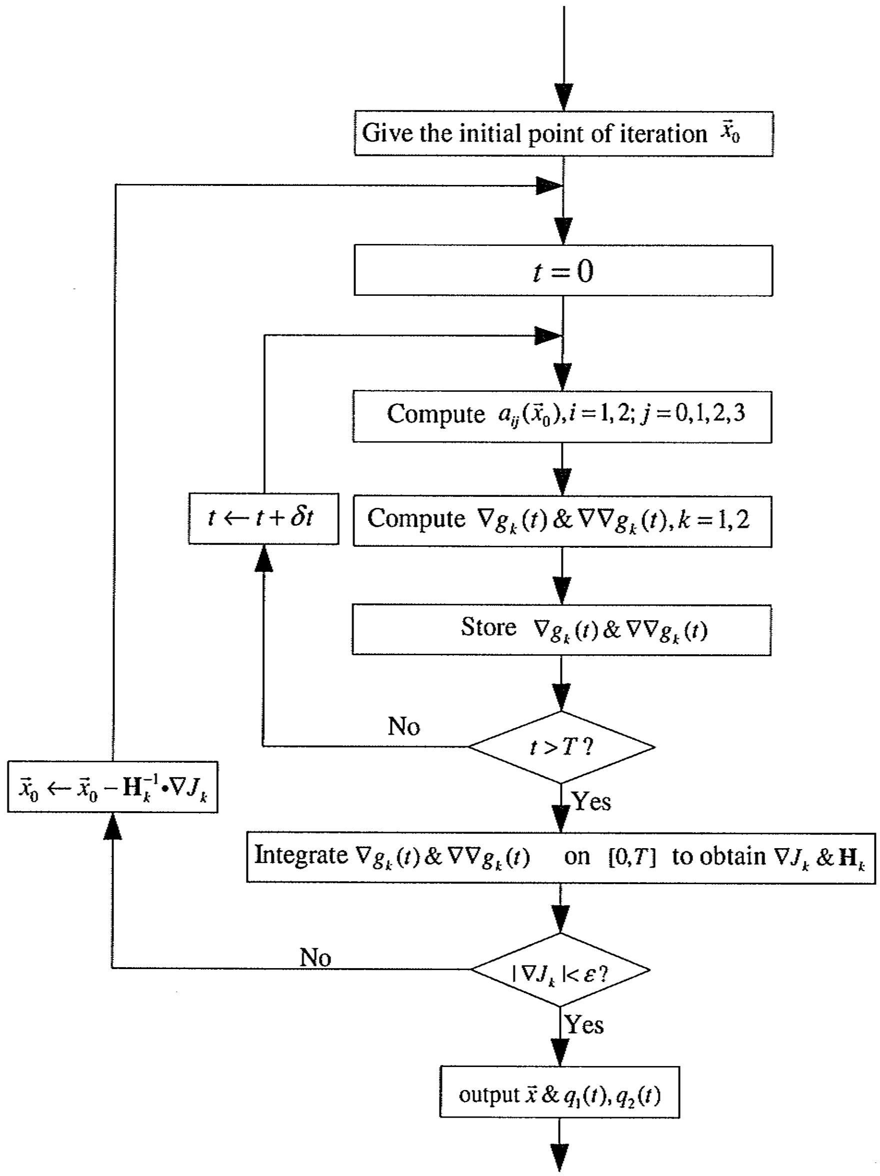

Figure 4.3 Numerical strategy of minimizing the functional J_k, $k = 1,2$

4.4 Results and discussions

In this section, the results of a simulation study are presented for the two-link pendulums. The major task of this simulation is to investigate under certain conditions, which performance function (functional) will yield lower input energy by using the polynomial technique. Two sets of joint profiles are obtained by the minimization of functional J_1 and J_2. These joint profiles will be used to calculate the total input energy by putting them into functional J_0.

In this chapter, we are interested only in the difference of the two functionals, J_1 and J_2. In other words, for the time being, we are discussing mainly the mathematical problems instead of the practical physical walking model. Therefore, for the sake of simplicity, we shall not introduce any constraint conditions that are related to the bipedal walking pattern.

4.4.1 The simulation study of the joint angle profiles

In our simulations, we replace J_1 and J_2 by $J_1^{'}$ and $J_2^{'}$ respectively, and $J_1^{'}$ and $J_2^{'}$ are defined as:

$$J_1^{'} = J_1 / S_L,$$

$$J_2^{'} = J_2 / S_L,$$

where $S_L = l_1 \sin(q_{1T}) + l_2 \sin(q_{1T} + q_{2T}) - l_1 \sin(q_{10}) - l_2 \sin(q_{10} + q_{20})$ [11].

By using the polynomial technique, each joint angle can be presented by a fourth order polynomial with respect to time t. Five terms are contained in each polynomial.

Intensive numerical simulations are carried out to generate different sets of joint angle profiles with different time period T.

We know that the repeatability and continuity condition is related to the practical biped system, including the physical system and the constraint system. Here we give only certain values of the initial and final conditions to finish the simulations, because the real values of the initial and final conditions can be determined by certain mathematical relationships. Furthermore, these mathematical relationships can be fixed as long as the practical biped system is given, at least for the constraint system that is no higher than second order. In this chapter, we do not consider the actual mathematical relationships representing the repeatability and continuity condition, but we do consider the repeatability and continuity condition by fixing the initial and final conditions of the physical system and reducing the number of the optimization parameters in the polynomials.

The values of the parameters m_i, I_i, l_i, and d_i of the two-link pendulum are listed in Table 4.1 and used for generating the joint angle profiles.

Table 4.1 Parameters of the two-link pendulum

Link	Mass, m_i (kg)	Moment of inertia, I_i (kgm^2)	Length, l_i (m)	Location of centre of mass, d_i (m)
1	5.28	3.3×10^{-2}	0.302	0.066
2	2.23	3.3×10^{-2}	0.332	0.143

To compare the two minimization criteria at different speeds of movement, we choose the cases with their periods as follows:

(1). T=0.3s, (2). T=0.4s, (high speed cases)

(3). T=0.5s, (4). T=0.6s, (moderate speed cases)

(5). T=0.7s, (6). T=1.0s. (low speed cases)

The initial angles and angular velocities, the final angles and angular velocities are used for all cases:

$q_1(0) = q_{10} = -0.1745$ (rad), $q_2(0) = q_{20} = -0.1745$ (rad),

$\dot{q}_1(0) = \dot{q}_{10} = 0.1$ (rad/s), $\dot{q}_2(0) = \dot{q}_{20} = -0.1$ (rad/s);

$q_1(T) = q_{1T} = 0.1745$ (rad), $q_2(T) = q_{2T} = 0.1745$ (rad),

$\dot{q}_1(T) = \dot{q}_{1T} = 0.1$ (rad/s), $\dot{q}_2(T) = \dot{q}_{2T} = 0$ (rad/s).

The parameters in the dynamic equations (A.4) shown in Appendix A are shown below:

$\alpha = 0.931986,$
$\beta = 0.192610,$
$\gamma = 1.021940,$
$\delta = 0.437971,$
$\sigma = 0.318890.$

In the minimization part, the region of the initial guessing point is set as $x_{01} \in [-30,30]$ $(1/s^4)$, $x_{02} \in [-30,30]$ $(1/s^4)$. The searching is carried out in a way that we change the value of x_{01} with the difference of 0.2 on the condition that x_{02} is fixed first; after finishing the searching along the axis of x_{01}, change the value of x_{02} with the difference of 0.2. The larger the region is the more expensive the computation will be. Due to the high efficiency of the Quasi-Newton method, the region we give above is affordable. From the searches, we find that in the given region, we have only one minimizer for each functional.

4.4.2 The joint angle profiles

According to equation (4.13) and (4.14), we directly give the two parameters of the polynomials. In our case, we arbitrarily give

$a_{10} = -0.1745$,

$a_{11} = 0.1$,

$a_{20} = -0.1745$

$a_{21} = -0.1$.

Through the minimization of functional $J_1^{'}$, with the various period we have the different sets of parameters in the polynomials (4.9). These parameters are shown in Table 4.2. Note that $a_{14}(x_1)$ and $a_{24}(x_2)$ are the coordinates of the minimizers. Similarly, through the minimization of functional $J_2^{'}$, we can also obtain Table 4.3 as follows.

Table 4.2 The parameters of the polynomials corresponding to minimizing $J_1^{'}$

	T = 0.3(s)	T = 0.4(s)	T = 0.5(s)	T = 0.6(s)	T = 0.7(s)	T = 1.0(s)
$a_{12}(1/s^2)$	10.62935	5.758775	3.527035	2.326962	1.615863	0.69144
$a_{13}(1/s^3)$	-23.593311	-9.477256	-4.538032	-2.403469	-1.362339	-0.386617
$a_{14}(x_1)(1/s^4)$	-0.068666	-0.226316	-0.247023	-0.227556	-0.189191	-0.055758
$a_{22}(1/s^2)$	12.297182	7.086894	4.669174	3.351238	2.544273	1.294572
$a_{23}(1/s^3)$	-26.934414	-11.742854	-6.306589	-3.873278	-2.586367	-0.89288
$a_{24}(x_2)(1/s^4)$	-0.055716	0.261931	0.321534	0.30284	0.247797	0.047374

Table 4.3 The parameters of the polynomials corresponding to minimizing $J_2^{'}$

	T = 0.3(s)	T = 0.4(s)	T = 0.5(s)	T = 0.6(s)	T = 0.7(s)	T = 1.0(s)
$a_{12}(1/s^2)$	10.452179	5.654101	3.473352	2.310486	1.622563	0.576247
$a_{13}(1/s^3)$	-22.412173	-8.953888	-4.323298	-2.348549	-1.381481	-0.156231
$a_{14}(x_1)(1/s^4)$	-2.037229	-0.880526	-0.461756	-0.273323	-0.175518	-0.17095
$a_{22}(1/s^2)$	12.88987	7.481516	4.932251	3.520715	2.651149	1.470607
$a_{23}(1/s^3)$	-30.885671	-13.715963	-7.358895	-4.438202	-2.891728	-1.244951
$a_{24}(x_2)(1/s^4)$	6.529712	2.728318	1.37384	0.77361	0.465912	0.22341

Substitute these parameters into the polynomial (4.9), we have the joint angle profiles under different conditions. Figures 4.4a and 4.4b show the joint angles during one period at $T = 0.5s$. Figures 4.5a and 4.5b show the joint angular velocities at T = 0.5s. Here, we note that these two sets of corresponding curves are very close.

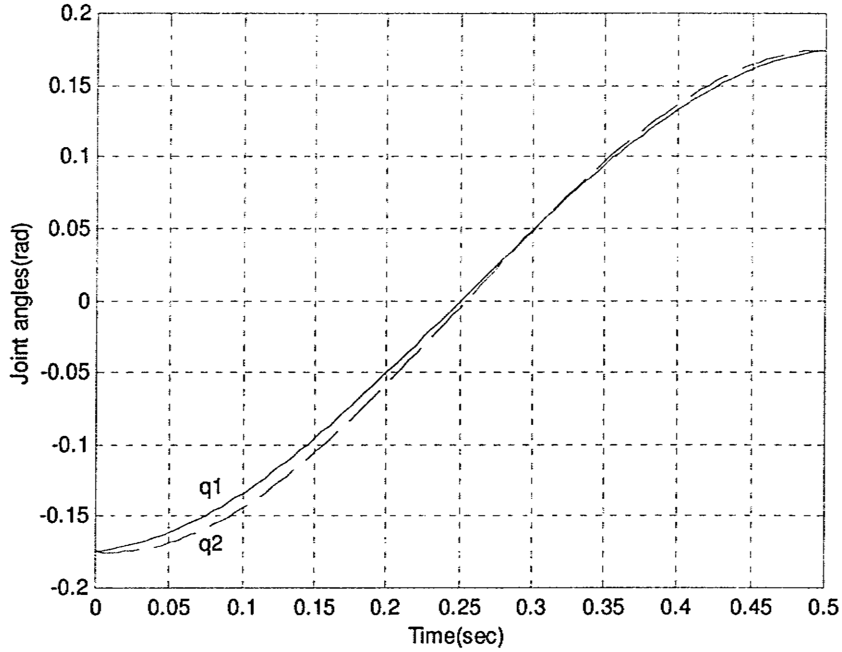

Figure 4.4a The joint angle displacements obtained by min $J_1^{'}$ at T=0.5s

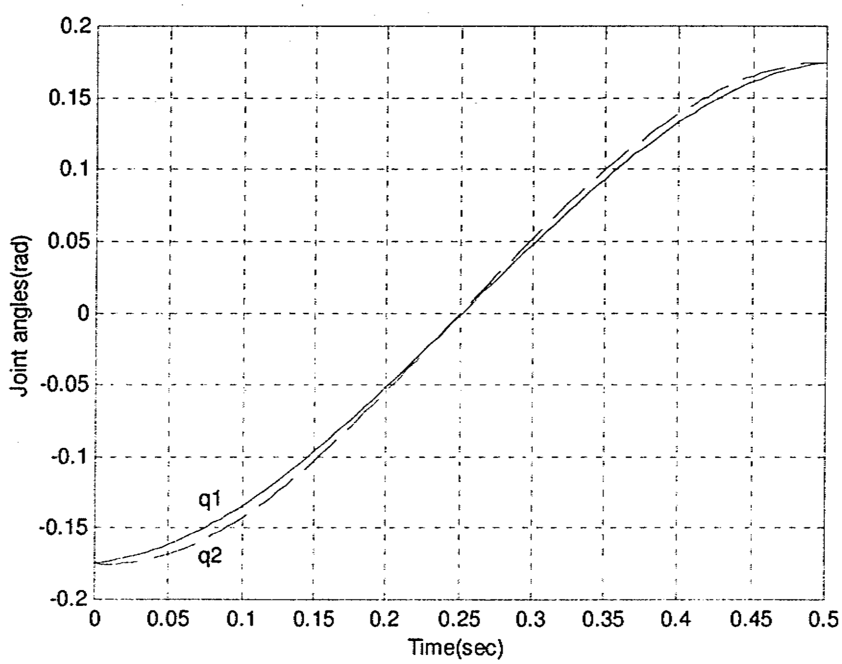

Figure 4.4b The joint angle displacements obtained by min $J_2^{'}$ at T=0.5s

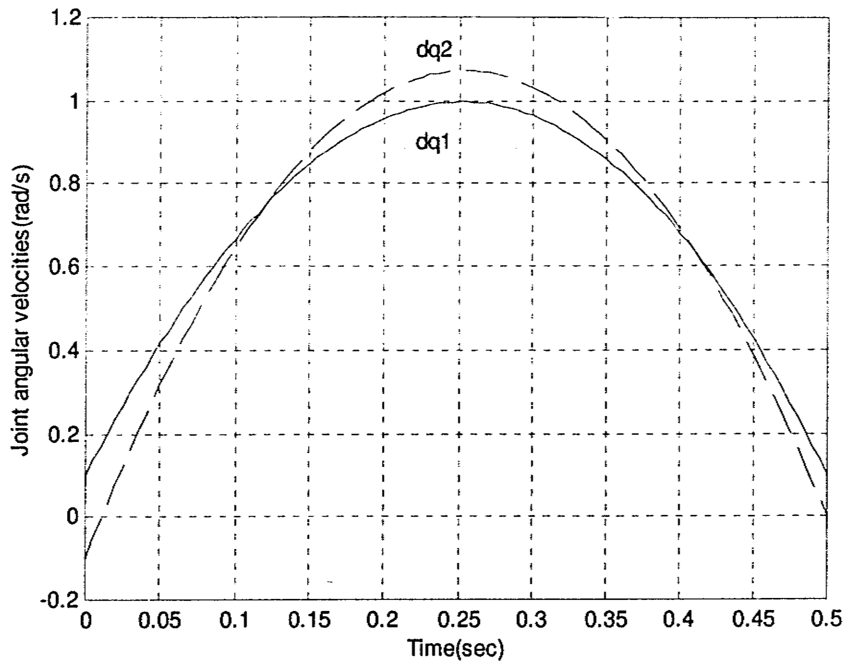

Figure 4.5a The joint angular velocities obtained by min $J_1^{'}$ at T=0.5s

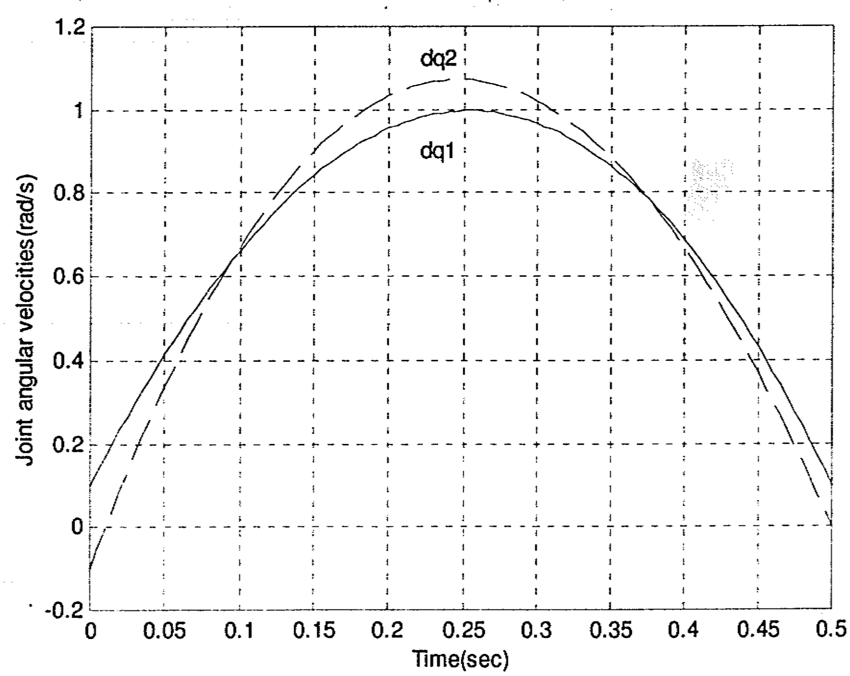

Figure 4.5b The joint angular velocities obtained by min $J_2^{'}$ at T=0.5s

4.4.3 The minima of $J_1^{'}$ and $J_2^{'}$ and the corresponding energy input J_0

After the searching for the minimum is finished starting from all the knots within the given region, all the minima are obtained. the smallest values among the minima is chosen as the minima of the functional $J_1^{'}$ and $J_2^{'}$ shown in Table 4.4.

Table 4.4 The minima of $J_1^{'}$ and $J_2^{'}$

	T = 0.3(s)	T = 0.4(s)	T = 0.5(s)	T = 0.6(s)	T = 0.7(s)	T = 1.0(s)
$J_1^{'}$ ($N^2 \cdot m \cdot s$)	466.997209	165.308624	66.722865	28.016363	11.372175	0.529302
$J_2^{'}$ ($N^2 \cdot m \cdot s$)	326.6044	64.45067	16.3597	4.619277	1.296667	0.036489

The relationship of minima of $J_1^{'}$ and the period can be approximately formed by a first order exponential decay fit as

$$J_1^{'}(T) = y_{10} + \lambda_{11} e^{-T/C_{11}}, \tag{4.40}$$

where $y_{10} = 0$, $\lambda_{11} = 9434.75991$, $C_{11} = 0.09972$. Similarly, the functional relation of $J_2^{'}$ and period can be approximately formed as

$$J_2^{'}(T) = y_{20} + \lambda_{21} e^{-T/C_{21}}, \tag{4.41}$$

where $y_{20} = 0$, $\lambda_{21} = 39609.03449$, $C_{21} = 0.06252$. We find that the minima of $J_1^{'}$ and $J_2^{'}$ decay very quickly with the growth of the period. If the step period is long enough, the minima are not sensitive to the step period. The change of the minima of $J_1^{'}$ and $J_2^{'}$ with respect to period can be shown by Figure 4.6.

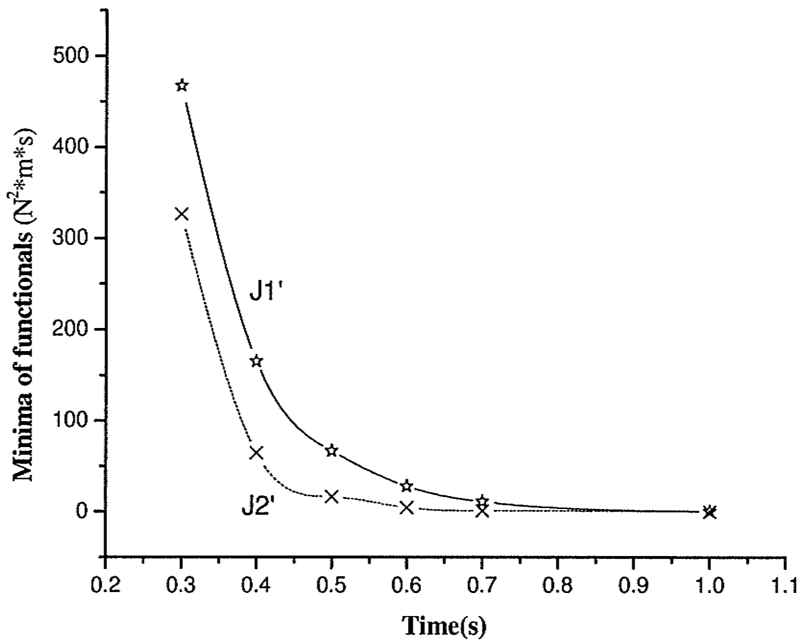

Figure 4.6 The change of minima with respect to time period

Substitute the trajectories obtained by minimizing $J_1^{'}(T)$ and $J_2^{'}(T)$ shown in Table 4.3 into J_0. Then we have the input energy J_{01} and J_{02} corresponding to the minimization of $J_1^{'}(T)$ and $J_2^{'}(T)$ shown in Table 4.5.

Table 4.5 The input energy corresponding to the minimization of $J_1^{'}$ and $J_2^{'}$

	T = 0.3s	T = 0.4s	T = 0.5s	T = 0.6s	T = 0.7s	T = 1.0s
J_{01} (N·m)	7.130984	3.675099	2.082147	1.220277	0.702591	0.133177
J_{02} (N·m)	7.130218	3.674315	2.081391	1.219557	0.701874	0.132988

These results indicate that minimizing $J_2^{'}$ yields lower input energy than minimizing $J_1^{'}$. In other words, the solution to criterion (4.3) is closer to that for criterion (4.2). However, on the conditions we give in this thesis, the difference of the input energy corresponding to criterion (4.1) and (4.3) is very small.

Given different sets of initial and final conditions, we do the numerical simulations. The initial and final conditions are given by Table 4.6, and the energy input corresponding to each set of initial and final conditions is shown in Table 4.7.

Table 4.6 The initial and final conditions

	Set 1	Set 2	Set 3	Set 4	Set 5
q_{10} (rad)	-0.1745	-0.1745	-0.1745	-0.349	-0.5235
q_{1T} (rad)	0.1745	0.1745	0.1745	0.349	0.5235
$\dot{q}_{10}$ (rad/s)	0.2	0.3	0.3	0.1	0.1
$\dot{q}_{1T}$ (rad/s)	0.1	0.1	0.1	0.1	0.1
q_{20} (rad)	-0.1745	-0.1745	-0.1745	-0.349	-0.5235
q_{2T} (rad)	0.1745	0.1745	0.1745	0.349	0.5235
$\dot{q}_{20}$ (rad/s)	-0.2	-0.3	-0.8	-0.1	-0.1
$\dot{q}_{2T}$ (rad/s)	0	0	0	0	0

Table 4.7 The energy input corresponding to the different initial and final conditions

	Set 1	Set 2	Set 3	Set 4	Set 5
J_{01} $(N \cdot m)$	7.089321	7.057729	7.839321	28.980290	65.710478
J_{02} $(N \cdot m)$	7.086355	7.048216	7.754962	28.979526	65.707727

From Table 4.7, we find that even though the initial and final conditions are changed, J_{02} is smaller than J_{01} and they are still close.

4.4.4 The control torques

As long as we get the joint angle profiles, it is easy to calculate the control torques. Bringing the trajectories into the dynamic equations (A.4) shown in Appendix A, we have the control torques as shown in Figure 4.7a -- 4.7f. From these figures, we note that the control torques yielded by the different minimization criteria are always very close to each other. Nevertheless, we also note that criterion (4.3) always yields little bit larger control torques at the initial and final moment than criterion (4.1).

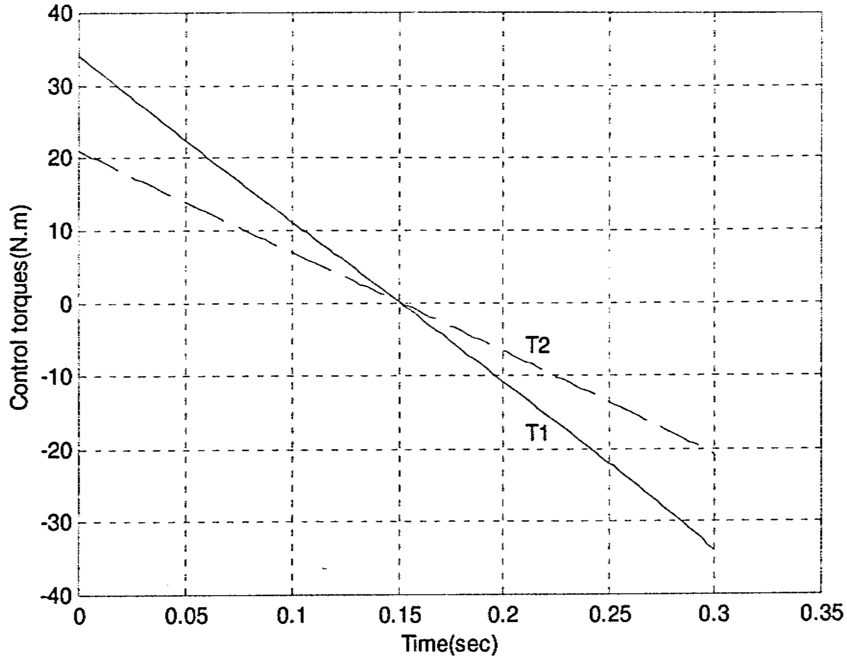

Figure 4.7a The control torques obtained by min $J_1^{'}$ at T = 0.3s

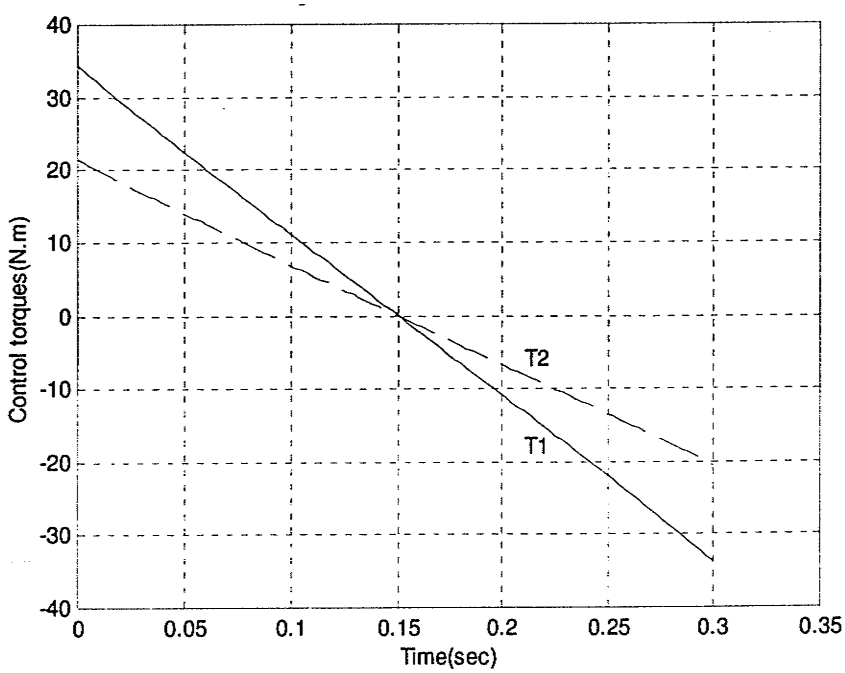

Figure 4.7b The control torques obtained by min $J_2^{'}$ at T = 0.3s

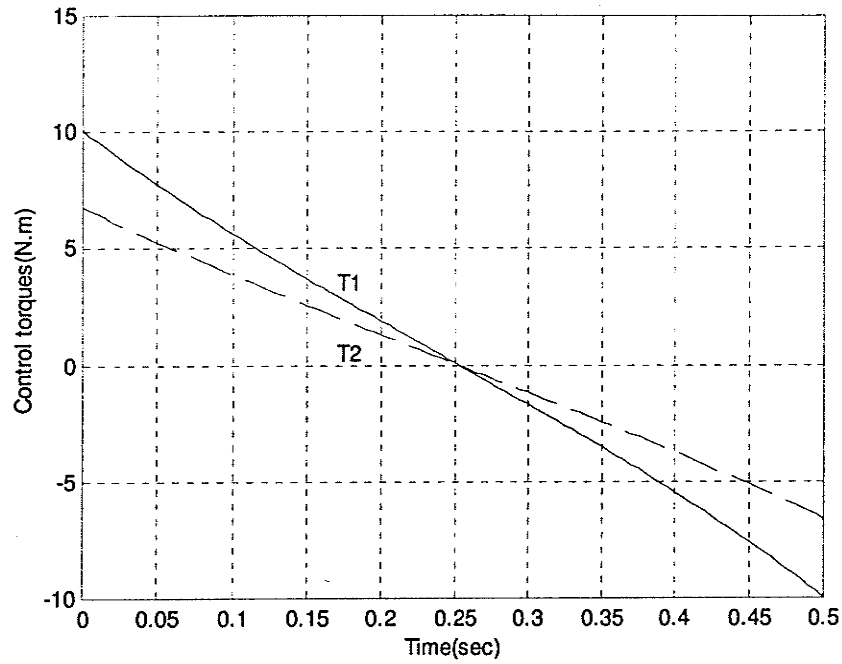

Figure 4.7c The control torques obtained by min $J_1^{'}$ at T = 0.5s

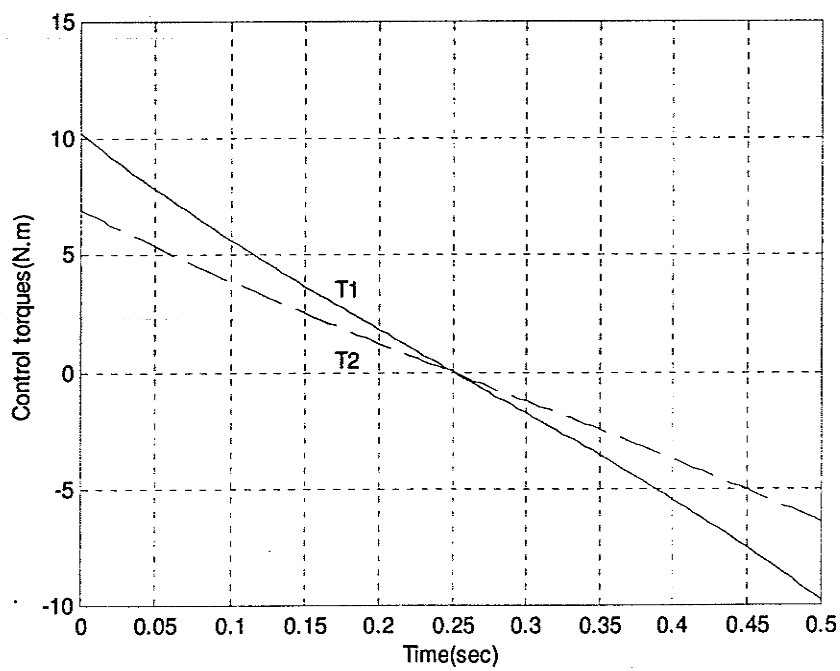

Figure 4.7d The control torques obtained by min $J_2^{'}$ at T = 0.5s

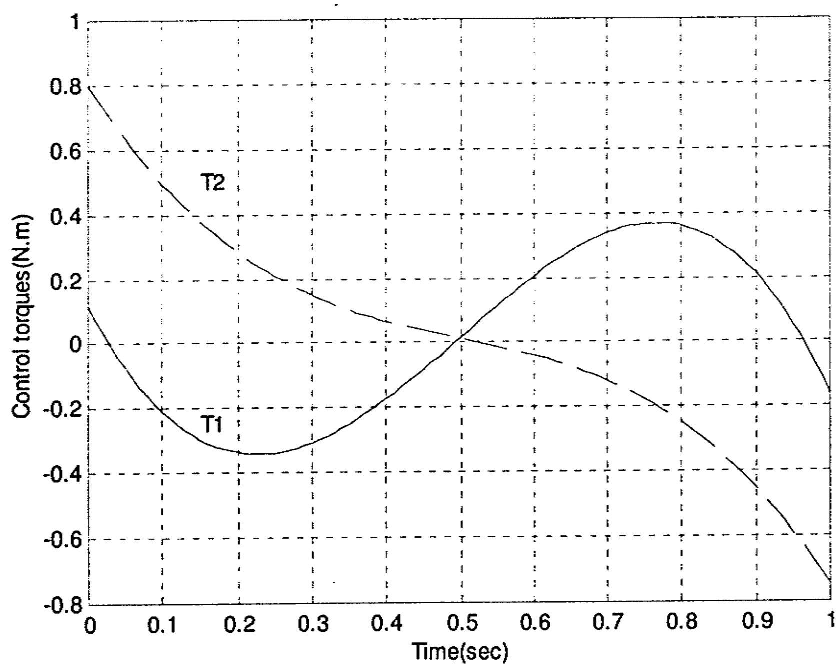

Figure 4.7e The control torques obtained by min $J_1^{'}$ at T = 1.0s

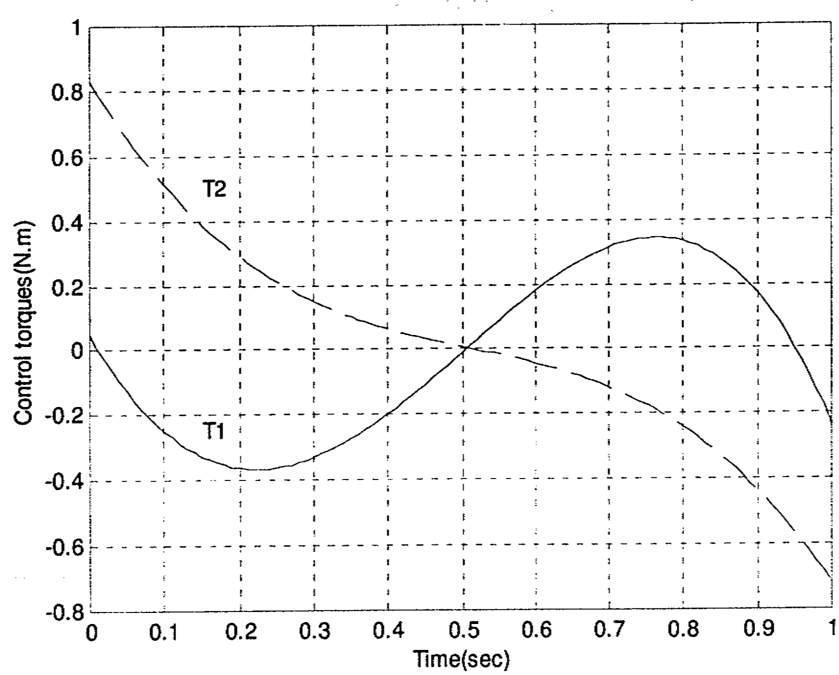

Figure 4.7f The control torques obtained by min $J_2^{'}$ at T = 1.0s

4.5 Discussions

In this chapter, we have applied the polynomial technique to two higher order optimization problems. We used fourth order polynomial, and the degree of freedom of the optimizations is two. In this method, the angles and angular velocities at the initial and final moments are set. On this condition, if the impact effect is considered, the repeatability condition will be partially satisfied, as the discussions in Chapter 3.

The goal of this chapter is to investigate which one of the two optimization criteria (4.1) and (4.3) will yield lower input energy. The simulation results from the specific example used in this thesis showed that optimization criterion (4.3) always yields lower input energy under the given conditions. That means the solution to the criterion (4.3) is closer to that of the criterion (4.2). However, we also noted that there is no significant difference between the two energy curves yielded by optimization criteria (4.1) and (4.3). More research needs to be carried out to draw any definite conclusions.

The reasons that there is no significant difference between criteria (4.1) and (4.3) probably lie in two facts. First, the physical model we chose in this thesis is two-link pendulums. For such a model, gravity plays a dominant role in the entire locomotion compared with the control torques. Different models should be introduced to do the simulations. Secondly, there is a lack of additional restrictions on the locomotion of the links. For example, the tip of the swing leg must be above the ground; the knee cannot bend backward. In this chapter, we discuss only the mathematical problems, so we do not consider those additional constraint conditions for the time being.

When period T changes, the costs that are expressed by functional J_1, J_2, and J_0 vary greatly. We find that all the changes of the costs can be approximately expressed by exponential decay functions. The shorter the period is, the higher the costs will be.

In this chapter, we applied finite difference expressions to approximate all the derivatives. The computation expense depends on the total degree of freedom of optimization, i.e., the value of $N_1 \times F$ shown in equation (4.23). To calculate the 2nd order difference, we need nine nodes when $N_1 \times F = 2$; we need 81 nodes when $N_1 \times F = 4$. In general, the number of nodes that we need to calculate the 2nd difference is $3^{(N_1 \times F)}$. The Quasi-Newton method requries us to calculate the 2nd derivatives of the functions, so the higher the total degree of freedom of optimization is, the more expensive the computation is.

The polynomial method is only a kind of approximate method. In principle, the factors that could affect the computation precision lie in two aspects: 1) the highest order of the initial and final conditions we take into account; 2) the number of the parameters to be optimized in each polynomial. So far, we still do not know how these two aspects affect the precision of our approximate solutions. Further investigation is needed.

Chapter 5

Conclusions

5.1 Conclusions

In this thesis, our major research work consists of the following two parts: (1) study of the repeatability condition for the planar five link biped robot based on Hurmuzlu's parametric formulation method, and (2) comparison of the two optimization criteria. Our final goal is to investigate how to make an initial value problem based constraint system satisfy the repeatability condition. For the latter aspect, we hope to gain some understanding of these two optimization criteria before we take them as one of the constraint equations. Contributions that have been made to each part are detailed below.

(1) The repeatability condition based on initial value problems:

A strict mathematical definition of repeatability of the gait pattern of a biped robot is given in this thesis. This definition is universal and can be used in any constraint system.

For first order and higher order constraint systems, we give the general repeatability condition for creating repeatable gait as shown by Theorem 2.1 and Lemma 3.1. The major advantage of these propositions of the repeatability condition is that we can use them to examine if a given constraint system will produce the repeatable gait. The major disadvantage is that they do not provide us constructive information about the selection of constraint systems to satisfy the repeatability condition.

For first order constraint systems, Theorem 2.2 successfully tells us what kind of constraint systems can provide the corresponding repeatable gait. According to Theorem 2.2, two constraint equations (one is about the hip height and the other one is about the horizontal displacement of the hip) are crucial to satisfy the repeatability condition. Thus, any properly designed constraint system must consist of those two equations or their variations. In addition, at the same time, as long as the constraint system consists of those two equations or their variations, the constraint systems will yield a repeatable gait. However, Theorem 2.2 is not applicable to higher order constraint systems.

Due to the mathematical and dynamic difficulties, associated with higher order constraint systems, it is not feasible to apply the method based on initial-value problem to study the repeatability of the biped locomotion. We apply the polynomial technique to approximate each joint angle. This technique enables us to introduce the initial and final conditions into the same solution curve to the higher order differential equations, and then to fit the solution curve between the initial and final moment. To some extent, it is much easier to satisfy the repeatability condition in higher order constraint systems, but the repeatability condition here is fundamentally different from that described in Lemma 3.1. The polynomial technique isolates a step from a series of continuous steps, and it gets the deviation of the exact gait under control in each step. This is a very effective and direct method in higher order constraint systems. However, the disadvantage is that the computation precision can be affected by two aspects: 1) the highest order of the initial and final conditions, and 2) the number of parameters to be optimized in each polynomial. So far, we do not know how these two aspects affect the computation precision.

(2) The comparison of the two optimization criteria:

Optimization can be one of the constraint equations, but before we use that, we must understand its physical meaning and know what kind of physical quantity is optimized. In this thesis, we have compared two optimization criteria (the minimization of functionals).

The functional in which the integrand is the square of control torques was minimized to minimize the input energy. The simulation results show that if the control torque is replaced by input power in the integrand, the corresponding minimal input energy will be lower. However, under the conditions given in this thesis, we also find that the difference between the corresponding minima is not significant. All the conclusions are based on the conditions and physical model we chose in this thesis. We are not drawing a general conclusion here.

5.2 Future work

The results in this thesis are only the first step in the development of the study of repeatability condition. Since minimization of energy is a basic characteristic of human natural walking (Winter *et al.* 1976), the methodology of joint angle planning can be extended to design the prescribed motion with optimization of energy. However, the simplest optimization could yield the joint angle profiles that are meaningless. Therefore, we expect in the future to combine Hurmuzlu's constraint equations with different optimization criteria. In this study, the methods based on boundary-value problems are promising. They enable us to easily satisfy the repeatability condition, but we still need to investigate the factors that affect the computation precision.

References

[1] Roussel, L., Canudas-de-Wit, C., Goswami, A., "Generation of energy optimal compete gait cycles for biped robots," Proceedings of the IEEE, International Conference on Robotics & Automation, Leuven, Belgium, pp. 2036- 2041, 1998.

[2] Vukobratovic, M., Borovac, Surla, B. D., and Stokic, D., *Scientific Fundamentals of Robotics 7. Biped Locomotion: Dynamics Stability, Control and Application* (Springer-Verlag, New York, 1990).

[3] Hemami, H., Wil, C. and Goliday, G. L., "The inverted pendulum and biped stability", *Math. Biosci.* **34** (1977), 95-110.

[4] Hemami, H. and Katbab, A., "Constrained inverted pendulum model for evaluating upright postural stability", *ASME J. Dyn. Syst. Meas. Contr.* **104** (1982), 343-349.

[5] Miura, H. and Shinoyama, I., "Dynamic walk of a biped," Intl. J. Robotics Res. **3**(2), 1984.

[6] Furusho, J. and Masubuchi, M., "Control of a dynamic biped locomotion system for steady walking," ASME J. Dyn. Syst. Meas. Contr. **108**(1986), pp. 111-118.

[7] Furusho, J. and Masubuchi, M., "A theoretically motivated reduced-order mode for the control of dynamic biped locomotion," ASME J. Dyn. Syst. Meas. Contr. 109(1987), pp. 155-163.

[8] Yamada, M., Furusho, J. and Sano, A., "Dynamic control of walking robot with kick-action," Proc. 1985 Intl. Conf. On Advanced Robotics (ICAR'85), pp. 405-412, Tokyo, 1985.

[9] Hurmuzlu, Y., "Dynamics of Bipedal Gait: Part I - Objective Functions and the Contact Event of a Planar Five-Link Biped," *Journal of Applied Mechanics* **60**, 331-336 (1993).

[10] Tzafestas, S., Raibert, M. and Tzafestas, C., "Robust Sliding-mode Control Applied to a 5-Link Biped Robot," *Journal of Intelligent and Robotic Systems* 15, 67-133 (1996).

[11] Chevallereau, C., Aoustin, Y., "Optimal reference trajectories for walking and running of a biped robot," Robotica, **19**, pp. 557-569, 2001.

[12] Wu, Q. and Chan, C. Y. A., "Design of energy efficient joint profiles for a planar five-link biped robot", *Proceedings of 2001 IEEE International Symposium on Computational Intelligence in Robotics and Automation* (*IEEE CIRA 2001*), 35-40 (2001).

[13] Chan, C. Y. A., 2000, "Dynamic modeling, control and simulation of a planar five-link bipedal walking system", University of Manitoba Master of Science Thesis.

[14] Channon, P. H. S., Hopkins, H. and Pham, D. T., "Derivation of optimal walking motions for a bipedal walking robot," Robotica, **10** (part 2), pp. 165-172 (1992).

[15] Francois, C. and Samson, C., "Running with constant energy," Proceedings of IEEE International Conference on Robotics and Automation, pp. 131-136, San Diego, USA, 1994.

[16] Vukobratovich, M., "Legged locomotion robots and anthropomorphic mechanisms," Beograd, Yugoslavia: Mihailo Pupin Institute, 1976.

[17] Raibert, M. H., "Legged robots that balance," MIT Press, Cambridge, Mass., 1986.

[18] McGeer, T., “Passive dynamic walking,” Inter. J. of Robotics Research, pp. 62-82, April, 1990.

[19] Furusho, J. and Sano A., “Sensor-based control of a nine-link biped,” Inter. J. Robotics Research, pp. 83-98, April, 1990.

[20] Channon, P. H., Hopkins, S. H. and Pham, D.T., "A variational approach to the optimization of gait for a bipedal robot", *Journal of Mechanical Engineering Science* **210**, 177-186 (1996).

[21] Yang, J. S. and Shahabuddin, A., “Trajectory planning and controld for a five-degree-of-freedom biped locomotion system,” Proceedings of the A. M. C., pp 3105-3109, June 1994.

[22] Kajita, S., Yamaura, T. and Kobayashi, A., “Dynamic walking control of a biped robot along a potential energy conserving orbit,” IEEE Transactions on Robotics and Automation, **8(4)**: 431-438, August 1992.

[23] Growninshield, R. D., and Brand, R. A., 1981, “A physiologically based criterion of muscle force prediction in locomotion,” J. Biomech., **14**, pp. 793-801.

[24] Patriarco, A. B., Mann, R. W., Simon, S. R., and Mansour, J. M., 1981, “An evaluation of the approached of optimization methods in the prediction of muscle forces during human gait,” J. Biomech., **30**, pp. 513-525.

[25] Davy, D. T., and Audu, M. L., 1987, “A dynamic optimization technique for predicting muscles in the swing phase of gait,” J. Biomech., **20**, pp. 187-201.

[26] Anderson, F. C., Zieggler, J. M., Pandy, M. G., and Whalen, R. T., 1995, “Application of High-Performance Computing to Numerical Simulation of Human Movement,” ASME J. Biomech. Eng., **117**, pp 1555-157.

[27] Yamaguchi, G. T., and Zajac, F. E., 1990, "Restoring Unassisted Natural Gait to Paraplegics Via Functional Neuromuscular Stimulations: A Computer Simulation Study", IEEE Trans. Biomed. Eng., **37**, pp 886-902.

[28] Chow, C.K., and Jacobson, D. H., 1971, "Studies of human locomotion via optimal programming," Math. Biosci., **10**, pp. 239-306.

[29] Anderson, F. C., and Pandy, M. G., 2001, "Static and dynamic optimization solutions for gait are practically equivalent," J. Biomech., **34**, pp. 153-161.

[30] Yen, V. and Nagurka, M. L., "Suboptimal trajecotory planning of a five-link human locomotion model," Biomechanics of Normal and Prosthetic Gait, pp. 17-22, Boston, MA, 1987. ASME Winter Annual Meeting.

[31] Nagurka, M. L. and Yen, V., "Fourier-based optimal control of nonlinear dynamic systems," Journal of Dynamic Systems, Measurement, and Control, **112**, pp 17-26, March 1990.

[32] Cabodevila, G. and Abba, G., "Quasi optimal gait for a biped robot using genetic algorith," 1997, *IEEE International Conference on Systems, Man and Cybernetics*, Orlando, Floride, U.S.A., October 1997.

[33] Roussel, L., Canudas-de-Wit, C. and Goswami, A., "Generation of energy optimal compete gait cycles for biped robots," Proceedings of the IEEE, International Conference on Robotics & Automation, Leuven, Belgium, pp. 2036- 2041, 1998.

[34] Roussel, L., Canudas-de-Wit, C. and Goswami, A., "Comparative study of method for energy-optimal gait generation for biped robots," International Conference on Informatics and Control, pp. 1205-1212, St. Petersburg, June 1997.

[35] Anderson, F. and Pandy, M. G., "Dynamic Optimization of Human Walking", Journal of Biomechanical Engineering **123**, pp 381-390 (2001).

[36] Hurmuzlu, Y., "Dynamics of Bipedal Gait: Part II - Stability Analysis of a Planar Five-Link Biped, "*Journal of Applied Mechanics* **60**, 337-343 (1993).

[37] Furusho, J. and Masubuchi, M., "A Theoretically Motivated Reduced Order Model for the Control of Dynamic Biped Locomotion," *Journal of Dynamic Systems, Measurement, and Control* **109**, 155-163 (1987).

[38] Furusho, J. and Masubuchi, M., "Control of a Dynamical Biped Locomotion System for Steady Walking," *Journal of Dynamic Systems, Measurement, and Control* **108**, 111-118 (1986).

[39] Shih, C., Zhu, Y., and Gruver, W. A., "Optimization of the biped robot trajectory," Proc. of 1991 IEEE Int. Conf. on Systems, Man and Cybernetics, Charlottesville, VA, Oct 1991, pp. 899-904.

[40] Borse, G. J., "Numerical methods with MATLAB: a resource for scientists and engineers", Boston : PWS Publishing, c1997.

Appendix A

The equations of motion of two-link pendulums

The two-link pendulums model is shown in the Figure A.1.

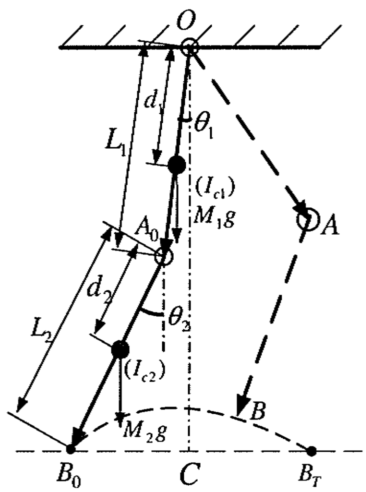

Figure A.1 The two link pendulums model

The two links are represented by two downward vectors. That means these two links indicate the swing leg of a five-link biped robot during the single support phase. The suspending point O stands for the hip of a five-link biped robot.

The derivation of the equations of motion for the two pendulums follows the standard procedure of the Lagrangian equation. The Lagrangian is:

$$L = U - \sum_{i=1}^{2} K_i ,$$

where the potential energy can be written as

$$U = -m_1 d_1 g \cos\theta_1 - m_2 (l_1 + d_2) g \cos\theta_2 ,$$

and the kinetic energy of each link is

$$K_i = \frac{1}{2} m_i v_{ci}^2 + \frac{1}{2} I_{ci} \dot{\theta}_i^2$$

The total kinetic is

$$K = \frac{1}{2}(I_{c1} + m_1 d_1^2 + m_2 l_2^2)\dot{\theta}_1^2 + \frac{1}{2}(I_{c2} + m_2 d_2^2)\dot{\theta}_2^2 + m_2 l_1 d_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2.$$

According to the Lagrangian equation, we have

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_i}) - \frac{\partial L}{\partial \theta_i} = T_i,$$

where $i = 1,\ 2$. Accordingly,

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = &-(I_{c1} + m_1 d_1^2 + m_2 l_2^2)\dot{\theta}_1 - m_2 l_1 d_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 \\ &+ m_2 l_1 d_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_2\end{aligned}$$

$$\frac{\partial L}{\partial \theta_1} = (m_1 d_1 + m_2 l_1)\sin\theta_1 + m_2 l_1 d_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = -(I_{c1} + m_1 d_1^2 + m_2 l_2^2)\dot{\theta}_1 - m_2 l_1 d_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2 l_1 d_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_2} = m_1 g d_1 \sin\theta_1 - m_2 l_1 d_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$

Finally, we have the control torques as

$$\begin{aligned}T_1 = &(I_{c1} + m_1 d_1^2 + m_2 L_1^2)\ddot{\theta}_1 + m_2 L_1 d_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 \\ &+ m_2 L_1 d_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + (m_1 d_1 + m_2 L_1) g \sin\theta_1,\end{aligned} \tag{A.1a}$$

$$\begin{aligned}T_2 = &m_2 L_1 d_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + (I_{c2} + m_2 d_2^2)\ddot{\theta}_2 \\ &- m_2 L_1 d_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 + m_2 g d_2 \sin\theta_2\end{aligned}. \tag{A.1b}$$

If we use the relative angles, they can be expressed as follows,

$$q_1 = \theta_1,\ q_2 = \theta_2 - \theta_1;\ \theta_1 = q_1,\ \theta_2 = q_1 + q_2. \tag{A.2}$$

Meanwhile, the transformation of control torques from an absolute angle system to a relative angle system is

$$T_{qi} = \sum_{j=1}^{2} T_{\theta j} \frac{\partial \theta_j}{\partial q_i} .$$

Thus, we have $T_1 = T_{q1} = T_{\theta 1} + T_{\theta 2}$, and $T_2 = T_{q2} = T_{\theta 2}$. It is easy to have

$$\begin{aligned} T_1 = & (\alpha + \beta \cos q_2)\ddot{q}_1 + (\delta + \frac{\beta}{2}\cos q_2)\ddot{q}_2 - \beta \sin q_2 \dot{q}_1 \dot{q}_2 \\ & - \frac{\beta}{2}\sin q_2 \dot{q}_2^2 + [\gamma \sin q_1 + \sigma \sin(q_1 + q_2)]g \end{aligned} , \tag{A.3a}$$

$$T_2 = (\delta + \frac{\beta}{2}\cos q_2)\ddot{q}_1 + \delta \ddot{q}_2 + \frac{\beta}{2}\sin q_2 \dot{q}_1^2 + \sigma \sin(q_1 + q_2) g . \tag{A.3b}$$

The dynamical equations also can be written in matrix form as

$$\begin{pmatrix} \alpha + \beta \cos q_2 & \delta + \frac{\beta}{2}\cos q_2 \\ \delta + \frac{\beta}{2}\cos q_2 & \delta \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \frac{\beta}{2}\sin q_2 \begin{pmatrix} -2\dot{q}_2 & -\dot{q}_2 \\ \dot{q}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} \gamma \sin q_1 + \sigma \sin(q_1 + q_2) \\ \sigma \sin(q_1 + q_2) \end{pmatrix} g$$

$$= \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}, \tag{A.4}$$

where

$\alpha = I_{c1} + m_1 d_1^2 + m_2 L_1^2 + m_2 d_2^2 + I_{c2}$,

$\beta = 2 m_2 L_1 d_2$, $\delta = I_{c2} + m_1 d_2^2$,

and $\gamma = m_1 d_1 + m_2 L_1$, $\sigma = m_2 d_2$.

Appendix B

The discussion of the order of the optimal design problems

The following discussions are based on the two-link pendulum model. According to this model as given in Appendix A, we design a control algorithm for control torques T_1 and T_2.

For functional $J_1 = \int_{t_0}^{t_0+T} \sum_{i=1}^{n} T_i^2 dt$ and $J_2 = \int_{t_0}^{t_0+T} \sum_{i=1}^{n} (T_i \dot{q}_i)^2 dt$, where $T_i, \dot{q}_i$ are control torque and joint angular velocities respectively, we let the integrands

$$f_1 = T_1^2 + T_2^2, \tag{B.1}$$

and

$$f_2 = (T_1\dot{q}_1)^2 + (T_2\dot{q}_2)^2. \tag{B.2}$$

Applying the classical method of calculus of variation, we will obtain Euler equations in the end. In our case, if the highest order of the derivative of other constraint equations is lower than that of the final Euler equations, we can omit other constraint equations for the time being and have the Euler equations for a unconstraint system. We have Euler equations,

$$\frac{\partial f_j}{\partial q_k} - \frac{d}{dt}\frac{\partial f_j}{\partial \dot{q}_k} + \frac{d^2}{dx^2}\frac{\partial f_j}{\partial \ddot{q}_k} = 0, \tag{B.3}$$

where $j, k = 1, 2$.

When we take $j = 1$, we have

$$\frac{\partial f_1}{\partial q_k}-\frac{d}{dt}\frac{\partial f_1}{\partial \dot{q}_k}+\frac{d^2}{dx^2}\frac{\partial f_1}{\partial \ddot{q}_k}=0\,.$$

Expanding the above equation, and considering equation (B.1), we finally have

$$2T_1\left(\frac{\partial T_1}{\partial q_k}-\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}_k}+\frac{d^2}{dx^2}\frac{\partial T_1}{\partial \ddot{q}_k}\right)-2\frac{dT_1}{dt}\frac{\partial T_1}{\partial \dot{q}_k}+2\frac{d^2T_1}{dt^2}\frac{\partial T_1}{\partial \ddot{q}_k}$$

$$+2T_2\left(\frac{\partial T_2}{\partial q_k}-\frac{d}{dt}\frac{\partial T_2}{\partial \dot{q}_k}+\frac{d^2}{dx^2}\frac{\partial T_2}{\partial \ddot{q}_k}\right)-2\frac{dT_2}{dt}\frac{\partial T_2}{\partial \dot{q}_k}+2\frac{d^2T_2}{dt^2}\frac{\partial T_2}{\partial \ddot{q}_k}=0\,. \tag{B.4}$$

From equations (A.4), the highest order of the first and the fourth term is 2, the highest order of the second and the fifth term is 3, and the highest order of the third and the sixth term is 4. Therefore, the highest order of equations (B.4) is 4.

When we take $j=2$ in equations (B.3), we have

$$2P_1\left(\frac{\partial P_1}{\partial q_k}-\frac{d}{dt}\frac{\partial P_1}{\partial \dot{q}_k}+\frac{d^2}{dx^2}\frac{\partial P_1}{\partial \ddot{q}_k}\right)-2\frac{dP_1}{dt}\frac{\partial P_1}{\partial \dot{q}_k}+2\frac{d^2P_1}{dt^2}\frac{\partial P_1}{\partial \ddot{q}_k}$$

$$+2P_2\left(\frac{\partial P_2}{\partial q_k}-\frac{d}{dt}\frac{\partial P_2}{\partial \dot{q}_k}+\frac{d^2}{dx^2}\frac{\partial P_2}{\partial \ddot{q}_k}\right)-2\frac{dT_2}{dt}\frac{\partial P_2}{\partial \dot{q}_k}+2\frac{d^2P_2}{dt^2}\frac{\partial P_2}{\partial \ddot{q}_k}=0\,,$$

where $P_1=T_1\dot{q}_1$ and $P_2=T_2\dot{q}_2$. Expand above equations, and then we have

$$g_1+g_2=0\,, \tag{B.5}$$

where

$$\begin{aligned} g_1 = {} & 2P_1\{\dot{q}_1\frac{\partial T_1}{\partial q_k}-\frac{d}{dt}(T_1\frac{\partial \dot{q}_1}{\partial \dot{q}_k}+\dot{q}_1\frac{\partial T_1}{\partial \dot{q}_k})+\frac{d^2}{dt^2}(\dot{q}_1\frac{\partial T_1}{\partial \ddot{q}_k})\} \\ & -2(\dot{q}_1\frac{dT_1}{dt}+T_1\ddot{q}_1)(T_1\frac{\partial \dot{q}_1}{\partial \dot{q}_k}+\dot{q}_1\frac{\partial T_1}{\partial \dot{q}_k})+2\dot{q}_1\frac{\partial T_1}{\partial \ddot{q}_k}(\dot{q}_1\frac{d^2T_1}{dt^2}+T_1\dddot{q}_1) \end{aligned}\,, \tag{B.6a}$$

and

$$
\begin{aligned}
g_1 = 2P_2\{\dot{q}_1 \frac{\partial T_2}{\partial q_k} - \frac{d}{dt}(T_2 \frac{\partial \dot{q}_2}{\partial \dot{q}_k} + \dot{q}_1 \frac{\partial T_2}{\partial \dot{q}_k}) + \frac{d^2}{dt^2}(\dot{q}_2 \frac{\partial T_2}{\partial \ddot{q}_k})\} \\
-2(\dot{q}_2 \frac{dT_2}{dt} + T_2\ddot{q}_2)(T_2 \frac{\partial \dot{q}_2}{\partial \dot{q}_k} + \dot{q}_1 \frac{\partial T_2}{\partial \dot{q}_k}) + 2\dot{q}_2 \frac{\partial T_2}{\partial \ddot{q}_k}(\dot{q}_2 \frac{d^2 T_2}{dt^2} + T_2\dddot{q}_2)
\end{aligned}
\quad . \qquad \text{(B.6b)}
$$

It is easy to show that the highest order will be obtained from $\frac{d^2T_1}{dt^2}$ and $\frac{d^2T_2}{dt^2}$ in equation (B.6a) and (B.6b), and the highest order is 4.

If our constraint system includes the problem (3.7) or (3.8) as one of the constraint equations and the order of derivative of other constraint equations is lower than 4, we say the highest order of the whole constraint system is 4.