MINIMAL WAVE DIGITAL FILTERS BASED ON STRICTLY PSEUDOPASSIVE N-PORT ADAPTORS

by

Ernest D. De Luca

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ABSTRACT

This thesis investigates two important aspects of the design of minimal wave digital filters, namely

1. the synthesis of nominal minimal wave digital filter realizations, and
2. the suppression of zero-input parasitic oscillations that could possibly occur in the filter as a result of the finite-word-length constraint.

Concerning the synthesis problem, an algorithm (termed the direct wave digital (DWD) algorithm) is developed which determines minimal reciprocal wave digital filter realizations directly from transfer function specifications. These realizations are based on pseudolossless n-port adaptors with the general scattering matrix representation introduced by Martens and Meerkötter. The adaptor coefficients are contained in two submatrices, $\underline{N}$ and $\underline{K}$. A method for wave digital cascade synthesis via transfer matrix factorization and the DWD algorithm is described.

The effects of quantizing the coefficients in $\underline{N}$ and $\underline{K}$ are investigated. In particular, it is shown that coefficient quantization generally yields an adaptor with a nondiagonal port reference conductance matrix. It is further demonstrated that the nondiagonal nature of the conductance ma-

trix makes the stability criteria developed for conventional wave digital filters inapplicable. A strictly pseudopassive adaptor structure is then proposed. This structure is derived from the pseudolossless adaptor by simply placing passivity multipliers in the reflected wave branches. Conditions that are sufficient to guarantee nonlinear stability are then given for specific pseudopassive adaptor structures that can realize second-order, fourth-order and symmetric fifth-order reciprocal filter sections.

A general procedure for the design of stable, minimal, reciprocal wave digital filters is then formulated. The hardware implementation of these designs is discussed. Finally, a number of illustrative design examples are presented.

ACKNOWLEDGEMENTS

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Chapter I

INTRODUCTION

In the most general sense, filtering is the process by which an input signal is reshaped or modified to yield an output signal with more desirable characteristics. The requirements that a filter must satisfy may be specified in either the time, or more typically, the frequency domain. In the latter case, filters are often frequency-selective in that a range of frequencies may be attenuated or removed from the input signal, while other frequencies are passed or amplified.

Filtering may be performed on continuous-time or discrete-time signals. A continuous-time signal is one in which the independent variable, time, takes on a continuum of values. A discrete-time signal on the other hand, is one that is defined at discrete instants of time and accordingly, may be viewed as a sequence of numbers. Signals for which both time and amplitude are discrete are called digital signals.

A filter that operates on digital signals is termed a digital filter. More specifically, a digital filter is a computational algorithm or process that transforms an input digital signal or sequence of numbers into another sequence of numbers exhibiting the desired properties. The algorithm may be recursive or nonrecursive in nature.

We may distinguish between two classes of digital filters, namely finite impulse response (FIR) and infinite impulse response (IIR) filters. The IIR class of filters has the greatest flexibility with respect to the locations of attenuation poles and zeroes and hence, is often preferred when attempting to meet stringent attenuation specifications. The design of IIR digital filters comprises four general steps:

1. Solve the approximation problem to determine a system function H(z) that characterizes a linear shift-invariant (LSI) discrete-time system satisfying the given performance specifications.
2. Choose a specific structure for the LSI system that realizes H(z).
3. Impose the finite-word-length constraint on the discrete-time filter.
4. Implement the digital filtering algorithm on a digital computer or with dedicated hardware components.

The approximation problem may be solved directly in the z-domain or alternatively, one may obtain a solution in the continuous-time domain that maps into an appropriate discrete-time system description. This latter approach allows one to utilize the highly advanced analog approximation procedures. There are a number of techniques for effecting a transformation between analog and discrete-time system descriptions. These include the impulse-invariance, the matched z-transform, and the bilinear z-transformation techniques [1,2].

The choice of structure for the LSI discrete-time filter that is to realize the desired system function is complicated by the subsequent finite-word-length restriction which necessitates:

1. quantization or approximation of the nominal filter parameters, and
2. quantization, for example by truncation or roundoff, of the input, output, and intermediate signal quantities within the filter structure.

The errors introduced by these modifications are termed finite-word-length effects. The first modification degrades the frequency response of the digital filter while the second modification introduces nonlinear error into the time response. This latter error may, in some instances, be correlated such that parasitic oscillations (limit cycles) occur at the filter output under zero-input conditions [3]. A distinction is normally made between two types of parasitic oscillations i.e. 1) overflow oscillations which can occur as a result of errors introduced in modifying signals that have exceeded the available range, and 2) granularity or underflow oscillations which may result due to the roundoff or truncation of signal underflow bits. Uncorrelated error in the time response is generally referred to as roundoff noise. It is well known that the relative severity of finite-word-length effects is highly dependent on the filter structure [4,5,6]. Clearly, it is desirable to choose filter structures that minimize these effects, keeping in mind other considerations such as hardware costs and speed limitations.

The most straightforward filter structures that have been established for an arbitrary IIR system function are the direct, forms DI and DII, parallel, and cascade digital structures. The DI and DII forms generally exhibit very poor performance with respect to finite-word-length effects. More practical are the parallel or cascade forms which derive, respectively, from a partial fraction expansion or a factoring of the system function into first and second-order sections. A great deal of interest has been directed towards developing second-order sections that suppress limit cycles [7-11], and have low coefficient sensitivity and low levels of roundoff noise [12-14].

An alternate structure that has received considerable attention is the wave digital filter, introduced by Fettweis and developed by him and his co-workers [15-18]. The wave digital filter is a high-order feedback structure that is derived via the digitization of an analog reference filter. The digitization process utilizes voltage scattering variables and the bilinear z-transformation. The reference filter is typically a resistively terminated LC ladder network, although other classical networks, for example the symmetric lattice or Jaumann structures [19,20], are suitable as prototypes.

The wave digital (WD) structure has the advantages of a low sensitivity of the system function to coefficient quantizations and relatively low levels of roundoff noise [21-24]. Furthermore, Fettweis and Meerkötter [25] have demonstrated, using the concepts of pseudopower and stored pseudopower, that it is possible to suppress all types of zero-input oscillations in WD filters derived from ladder or symmetric lattice prototypes. Disadvantages include that the WD structure requires more adders than conventional designs employing a cascade or parallel connection of direct form low-order sections. Also, WD filters derived from non-minimal ladder reference filters will not have a canonic (i.e. minimum) number of delays. Elimination of the degeneracies within such filters yields structures that are canonic in delays, but for which suppression of limit cycles is no longer easily achieved [26-28].

The standard WD design procedure [17] utilizes basic building blocks, called adaptors, that are interconnected to yield the filter realization. An alternate structure developed by Martens and Meerkötter [29], is one that employs a single n-port adaptor. This method may be applied to realizing WD structures based on prototypes with arbitrary topology. In particular, Lê [30], Martens and Lê [31], Martens and Jarmasz [32], and Jarmasz [33] have applied this technique to deriving limit cycle-free WD filters based on minimal networks including the Brune and Darlington D networks.

This thesis proposes an alternate approach for the realization of minimal WD filters. In particular, a strictly pseudopassive structure based on the n-port adaptor description of Martens and Meerkötter is introduced. Sufficient conditions are given for guaranteeing the complete suppression of zero-input limit cycles within this structure or a cascade of such structures.

Chapter II serves to introduce the basic concepts intrinsic to the WD approach to digital filter design. A general filter derivation is presented which utilizes the theory of scattering variables, the concept of network partitioning via reactance extraction, and the bilinear z-transformation. Various WD filter configurations, including the series-parallel adaptor method introduced by Fettweis, are briefly considered and the reflection-free property is discussed. The n-port adaptor method of Martens and Meerkötter is then described followed by a discussion on the sensitivity, roundoff noise, and nonlinear stability properties of conventional WD filters.

In Chapter III a method of determining nominal WD filter designs is developed. First, we show that a particular scattering matrix synthesis technique may be used to derive n-port adaptor realizations directly from transfer function specifications. Each adaptor realization is described by a constant scattering matrix $\underline{S}_n$. Methods of parameter reduction are then considered, including diagonal transformations on $\underline{S}_n$ and decomposition of $\underline{S}_n$ into a form specified by smaller submatrices. These latter measures result in an n-port adaptor description equivalent to that of Martens and Meerkötter and specified in terms of two submatrices $\underline{N}$ and $\underline{K}$. An example is then presented to illustrate this design algorithm which is termed the "direct wave digital" design technique. Wave digital cascade synthesis via transfer matrix factorization and the "direct wave digital" algorithm is then discussed. Finally, the consequences of quantizing coefficients in $\underline{N}$ and $\underline{K}$ are considered. In particular, we find that the quantized adaptor will in general no longer possess a diagonal reference conductance matrix. As a result, the nonlinear stability of the corresponding digital filter can no longer be guaranteed by the usual methods.

Chapter IV introduces an alternate n-port adaptor structure based on the adaptor description of Martens and Meerkötter. Specifically, simple multipliers are placed in all of the outgoing branches of such an adaptor, resulting in a new strictly pseudopassive adaptor. Although the new structure retains the original nondiagonal reference conductance matrix, we show that the pseudopassive nature of the adaptor may be utilized to ensure complete stability in a corresponding WD filter. Explicit stability criteria, including bounds on the entries of the reference conductance matrix, are then derived for a number of basic adaptors. These stability criteria ensure that a cascade of stable structures will also be stable.

Chapter V presents a WD filter design procedure based on the methods developed in Chapters III and IV. The hardware implementation of WD filtering algorithms is then briefly discussed. The chapter concludes with a number of illustrative examples.

Chapter II

INTRODUCTION TO WAVE DIGITAL FILTERS

The WD filter structure is derived such that it imitates in the digital domain the properties of an analog reference network. It follows that the excellent sensitivity properties that are known to exist for certain classes of analog networks can be retained in a WD realization. In this chapter, some of the basic concepts necessary for the understanding of the WD approach are reviewed. Brief descriptions of the conventional design techniques are presented and some of the special properties of WD filters are described.

2.1 DERIVATION OF WAVE DIGITAL FILTERS

2.1.1 Scattering Variables

Traditionally, an analog system is characterized in terms of the voltages and currents measured at the ports of the network. An alternate representation of a multiport network is one that is defined in terms of wave variables. If v(t) is the voltage, and i(t) is the current associated with a port, referenced as in Fig. 2.1, we define the instantaneous reflected and incident voltage waves, b(t) and a(t) respectively, as

$$b(t) = v(t) - Ri(t) \tag{2.1a}$$

$$a(t) = v(t) + Ri(t) \tag{2.1b}$$

where R is an arbitrarily assigned port reference resistance. Alternatively, in the complex frequency domain

$$B(s) = V(s) - RI(s) \tag{2.2a}$$

$$A(s) = V(s) + RI(s) \tag{2.2b}$$

where s is the complex frequency variable. (In the future, signal quantities may not be expressed explicitly as a function of an independent variable. Meanings should be clear from context.)

The reference filter most commonly utilized in WD filter design is the doubly terminated reactance two-port shown in Fig. 2.2. Such networks exhibit maximum power transfer (zero flat loss) at frequencies within the passband resulting in low passband sensitivity to element variations [34,35]. Consider a scattering variable description of such a two-port,

$$\underline{B}(s) = \underline{S}\ \underline{A}(s) \tag{2.3a}$$

where

$$\underline{B}(s) = \begin{bmatrix} B_1(s) \\ B_2(s) \end{bmatrix} \qquad \underline{A}(s) = \begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix} \qquad \underline{S}(s) = \begin{bmatrix} S_{11}(s) & S_{12}(s) \\ S_{21}(s) & S_{22}(s) \end{bmatrix} \tag{2.3b}$$

If we choose the reference resistances for the source and the load ports to be equal to the value of the source and load resistors R_S and R_L respectively, then

$$B_2(s) = 2V_2(s) \tag{2.4}$$

$$A_1(s) = E_1(s) \tag{2.5}$$

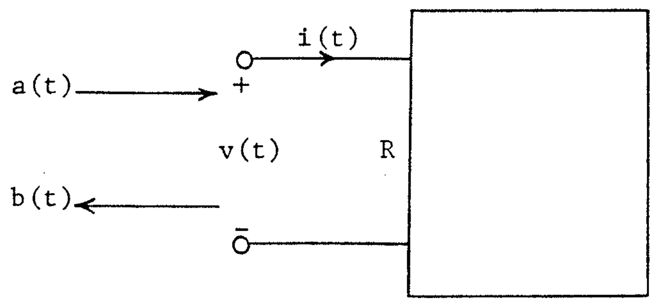


Figure 2.1: Definition of wave variables.

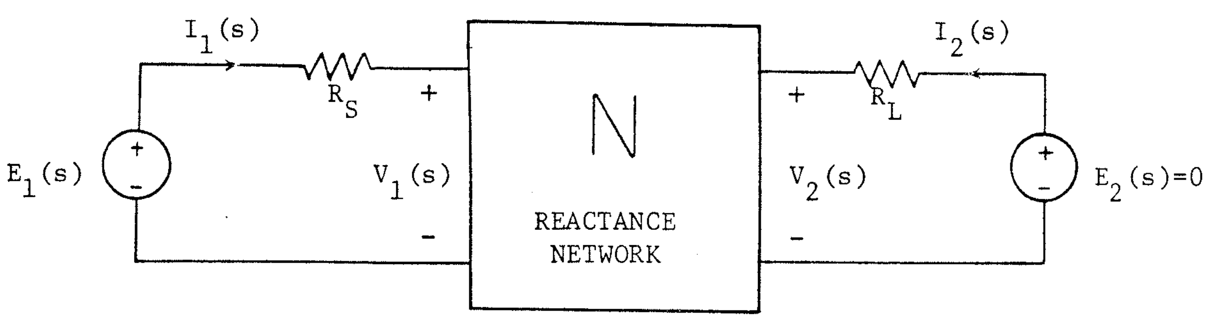


Figure 2.2: Doubly terminated reactance filter.

Under these circumstances, the voltage wave transfer function

$$S_{21}(s) = \left.\frac{B_2(s)}{A_1(s)}\right|_{A_2(s) = 0} = \frac{2V_2(s)}{E_1(s)} \tag{2.6}$$

and the familiar network transfer function

$$H_a(s) = \frac{V_2(s)}{E_1(s)} \tag{2.7}$$

are equivalent except for a frequency-independent gain constant. Clearly the desired frequency response is obtained by realizing either of these functions.

Wave digital filters derive from continuous-time reference filters by applying the bilinear z-transformation

$$s = (z - 1)/(z + 1) \tag{2.8}$$

to the circuit elements of the reference filter. The corresponding discrete-time system is described by

$$\underline{B}(z) = \underline{S}\left(\frac{z - 1}{z + 1}\right) \underline{A}(z) \tag{2.9}$$

where z is the discrete-time frequency variable. The desired z-domain transfer function,

$$H(z) = \frac{B_2(z)}{A_1(z)} = S_{21}\left(\frac{z - 1}{z + 1}\right) \tag{2.10}$$

follows by setting $A_2(z) = 0$. The discrete-time frequency response is given by $H(e^{j\omega T})$ where ω is the digital frequency in radians per second (r/s), and T is the sampling period. The analog frequency ϕ in r/s, and ω are related by

$$\phi = \tan \omega T/2 \tag{2.11}$$

2.1.2 Network Partitioning Via Reactance Extraction

The doubly terminated network illustrated in Fig. 2.2 can be reconfigured as in Fig. 2.3 with N consisting of an interconnection of two reciprocal subnetworks M and $\tilde{M}$. The network $\tilde{M}$ contains all of the reactive elements including k_1 inductors and k_2 capacitors which are uncoupled and have the values $L_1, L_2, \ldots, L_{k_1}, C_1, C_2, \ldots, C_{k_2}$. The (k+2)-port (where $k = k_1 + k_2$) coupling network M contains only direct connections and possibly ideal transformers and is therefore independent of frequency.

Consider a scattering variable description of the network M. We define the port voltage vector $\underline{v}$ and the port current vector $\underline{i}$ such that they can be partitioned with respect to ports connecting resistive, inductive, and capacitive elements, i.e.

$$\underline{v} = \begin{bmatrix} \underline{v}_R \\ \underline{v}_L \\ \underline{v}_C \end{bmatrix} \qquad \underline{i} = \begin{bmatrix} \underline{i}_R \\ \underline{i}_L \\ \underline{i}_C \end{bmatrix} \tag{2.12}$$

Resistive elements are assumed to be in series with a voltage source, possibly of zero value. The wave variables for M are defined by

$$\underline{b} = \underline{v} - \underline{R}\,\underline{i} \tag{2.13a}$$

$$\underline{a} = \underline{v} + \underline{R}\,\underline{i} \tag{2.13b}$$

where $\underline{R}$ is a diagonal matrix of arbitrary port reference resistances. The vectors $\underline{b}$ and $\underline{a}$ and the matrix $\underline{R}$ may be partitioned conformable with the voltage and current vectors,

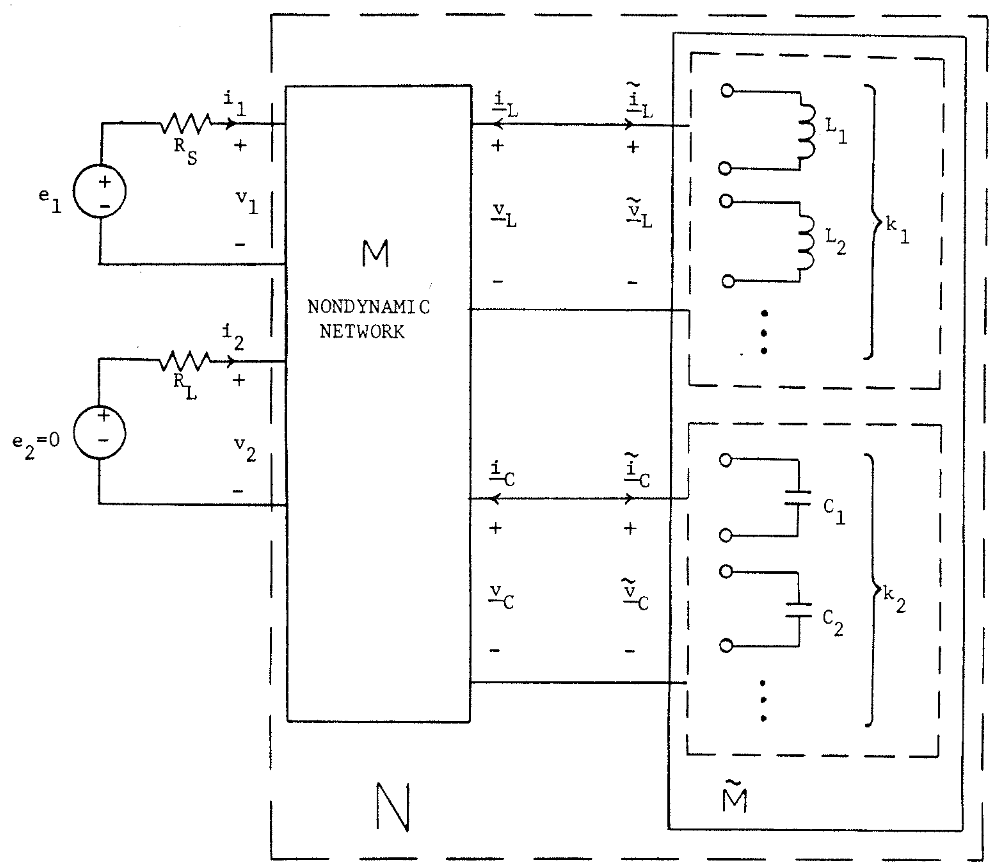


Figure 2.3: The reference filter; partitioned via reactance extraction.

$$\underline{b} = \begin{bmatrix} \underline{b}_R \\ \underline{b}_L \\ \underline{b}_C \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} \underline{a}_R \\ \underline{a}_L \\ \underline{a}_C \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} \underline{R}_R & & \\ & \underline{R}_L & \\ & & \underline{R}_C \end{bmatrix} \tag{2.14}$$

The reflected and incident voltage waves at the ports of M are related by

$$\underline{b} = \underline{S}\,\underline{a} \tag{2.15}$$

where $\underline{S}$ is the scattering matrix describing the network constraints. Since M is a frequency-independent network, $\underline{S}$ will be a real constant matrix.

Similarly, we may define a scattering variable description of the network $\tilde{M}$. The port voltage vector $\tilde{\underline{v}}$ and the port current vector $\tilde{\underline{i}}$ are partitioned such that

$$\tilde{\underline{v}} = \begin{bmatrix} \tilde{\underline{v}}_L \\ \tilde{\underline{v}}_C \end{bmatrix} = \begin{bmatrix} \underline{v}_L \\ \underline{v}_C \end{bmatrix} \qquad \tilde{\underline{i}} = \begin{bmatrix} \tilde{\underline{i}}_L \\ \tilde{\underline{i}}_C \end{bmatrix} = - \begin{bmatrix} \underline{i}_L \\ \underline{i}_C \end{bmatrix} \tag{2.16}$$

The wave variables at the ports of $\tilde{M}$ are defined by

$$\tilde{\underline{b}} = \tilde{\underline{v}} - \tilde{\underline{R}}\,\tilde{\underline{i}} \tag{2.17a}$$

$$\tilde{\underline{a}} = \tilde{\underline{v}} + \tilde{\underline{R}}\,\tilde{\underline{i}} \tag{2.17b}$$

where, partitioned conformable with $\tilde{\underline{v}}$ and $\tilde{\underline{i}}$,

$$\tilde{\underline{b}} = \begin{bmatrix} \tilde{\underline{b}}_L \\ \tilde{\underline{b}}_C \end{bmatrix} \qquad \tilde{\underline{a}} = \begin{bmatrix} \tilde{\underline{a}}_L \\ \tilde{\underline{a}}_C \end{bmatrix} \qquad \tilde{\underline{R}} = \begin{bmatrix} \tilde{\underline{R}}_L & \underline{0} \\ \underline{0} & \tilde{\underline{R}}_C \end{bmatrix} \tag{2.17c}$$

We have the relation

$$\tilde{\underline{B}}(s) = \tilde{\underline{S}}(s)\,\tilde{\underline{A}}(s) \tag{2.18}$$

where $\tilde{\underline{S}}(s)$ represents the scattering matrix of the network $\tilde{M}$. The scattering variable characterization of the overall network N is depicted in Fig. 2.4.

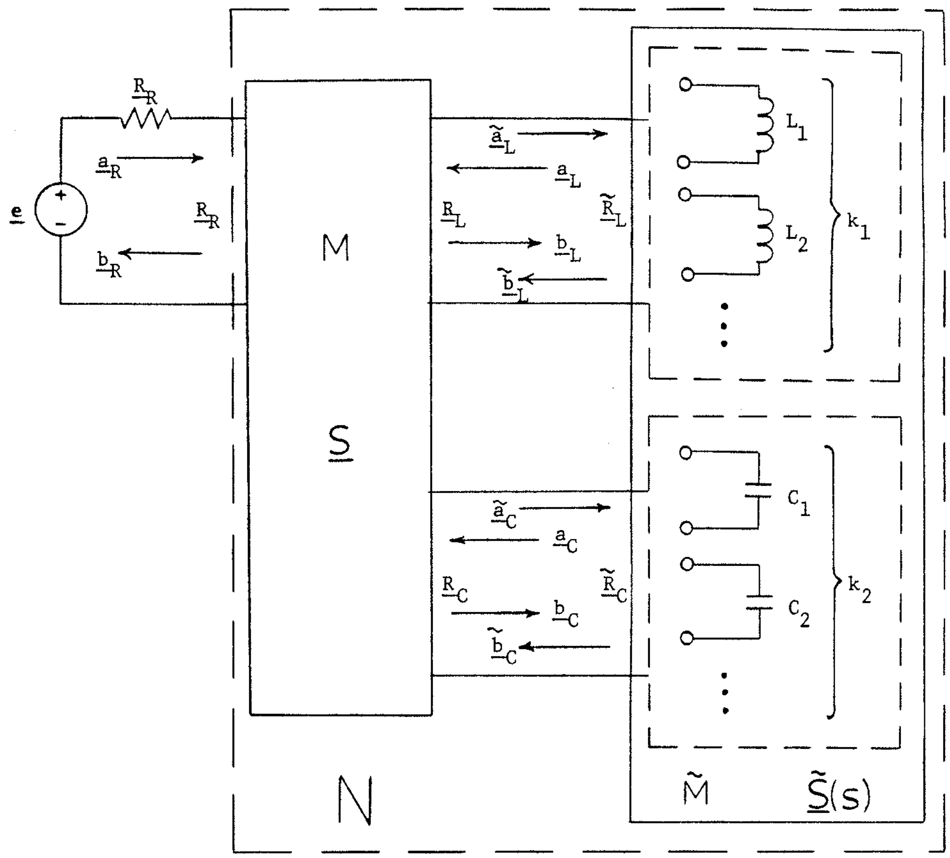


Figure 2.4: The reference filter; characterized in terms of voltage wave variables.

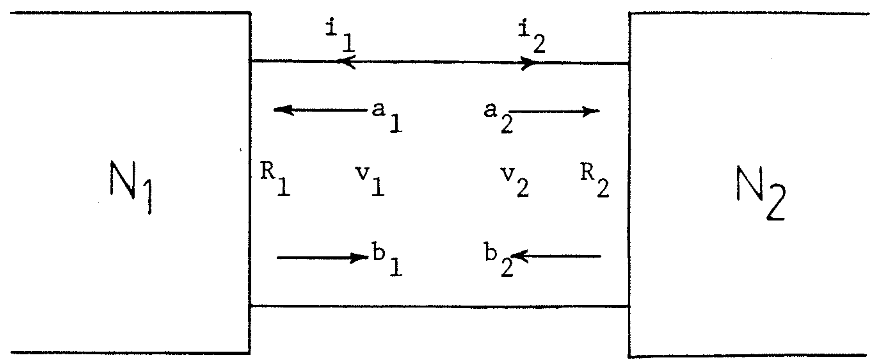


Figure 2.5: Interconnection of ports.

Consider the port-interconnection of two networks as in Fig. 2.5. For our purposes, it is important that the wave variables at the interconnection port be defined such that

$$b_1 = a_2 \tag{2.19a}$$

$$a_1 = b_2 \tag{2.19b}$$

The interconnection physically forces the currents and voltages at the common port to satisfy the constraints

$$v_1 = v_2 \tag{2.20a}$$

$$i_1 = -i_2 \tag{2.20b}$$

The constraints described by (2.19) and (2.20) lead to the requirement that, with respect to Fig. 2.5,

$$R_1 = R_2 \tag{2.21}$$

Equation (2.21) states the necessary compatibility condition for defining wave variables at interconnection ports.

A judicious choice of port reference resistances for the network $\tilde{M}$ is

$$\underline{\tilde{R}}_L = \mathrm{diag}[\ L_1, L_2, \ldots, L_{k_1}\ ] \tag{2.22a}$$

$$\underline{\tilde{R}}_C = \mathrm{diag}[\ 1/C_1, 1/C_2, \ldots, 1/C_{k_2}\ ] \tag{2.22b}$$

With the above choice of reference resistances, $\tilde{M}$ is described by the scattering matrix

$$\underline{\tilde{S}}(s) = \frac{1 - s}{1 + s}\,\underline{\Sigma} \tag{2.23}$$

where $\underline{\Sigma}$ is a polarity matrix defined by

$$\underline{\Sigma} = -\underline{U}_{k_1} \dotplus \underline{U}_{k_2} \tag{2.24}$$

(Here $\dotplus$ denotes direct sum and $\underline{U}_n$ an identity or unit matrix of dimension n×n.)

The port reference resistances for the network M are chosen to satisfy the compatibility condition, i.e.

$$\underline{R} = \begin{bmatrix} \underline{R}_R & & \\ & \underline{\tilde{R}}_L & \\ & & \underline{\tilde{R}}_C \end{bmatrix} \tag{2.25}$$

where $\underline{R}_R = \mathrm{diag}[\ R_S,\ R_L]$. Application of the bilinear z-transformation to the scattering descriptions of M and $\tilde{M}$ yields the discrete-time systems

$$\underline{B}(z) = \underline{S}\ \underline{A}(z) \tag{2.26a}$$

$$\underline{\tilde{B}}(z) = z^{-1}\ \underline{\Sigma}\ \underline{\tilde{A}}(z) \tag{2.26b}$$

or equivalently

$$\underline{b}(n) = \underline{S}\ \underline{a}(n) \tag{2.27a}$$

$$\underline{\tilde{b}}(n) = \underline{\Sigma}\ \underline{\tilde{a}}(n-1) \tag{2.27b}$$

where n is the discrete-time variable. Since the compatibility condition is satisfied at the interconnection ports of M and $\tilde{M}$, we have

$$\begin{bmatrix} \underline{b}_L(n) \\ \underline{b}_C(n) \end{bmatrix} = \begin{bmatrix} \tilde{\underline{a}}_L(n) \\ \tilde{\underline{a}}_C(n) \end{bmatrix} \qquad \begin{bmatrix} \underline{a}_L(n) \\ \underline{a}_C(n) \end{bmatrix} = \begin{bmatrix} \tilde{\underline{b}}_L(n) \\ \tilde{\underline{b}}_C(n) \end{bmatrix} \tag{2.28}$$

The equations (2.27) and (2.28) specify the WD filter realization. From (2.27b) we see that the capacitors and inductors are transformed into delays and delays in series with a sign inverter, respectively. The computational algorithm that determines the signal values at the delays and outputs of the digital filter is described by (2.27a), that is, by $\underline{S}$. A symbolic representation of the digital structure is shown in Fig. 2.6. The derivation outlined above describes in essence the strategy behind WD filter design.

2.1.3 State-Variable Description of Wave Digital Filters

The state-variable description of a digital filter specifies the matrices $\{\underline{A},\underline{B},\underline{C},\underline{D}\}$ where the operation of the filter is given by

$$\underline{y}(n) = \underline{D}\ \underline{u}(n) + \underline{C}\ \underline{x}(n) \tag{2.29a}$$

$$\underline{x}(n+1) = \underline{B}\ \underline{u}(n) + \underline{A}\ \underline{x}(n) \tag{2.29b}$$

The vectors $\underline{x}(n)$, $\underline{u}(n)$, and $\underline{y}(n)$ define the state, the input, and the output of the filter at the n^{th} time instant.

The WD filter realization specified by (2.27) and (2.28) is also described by the system

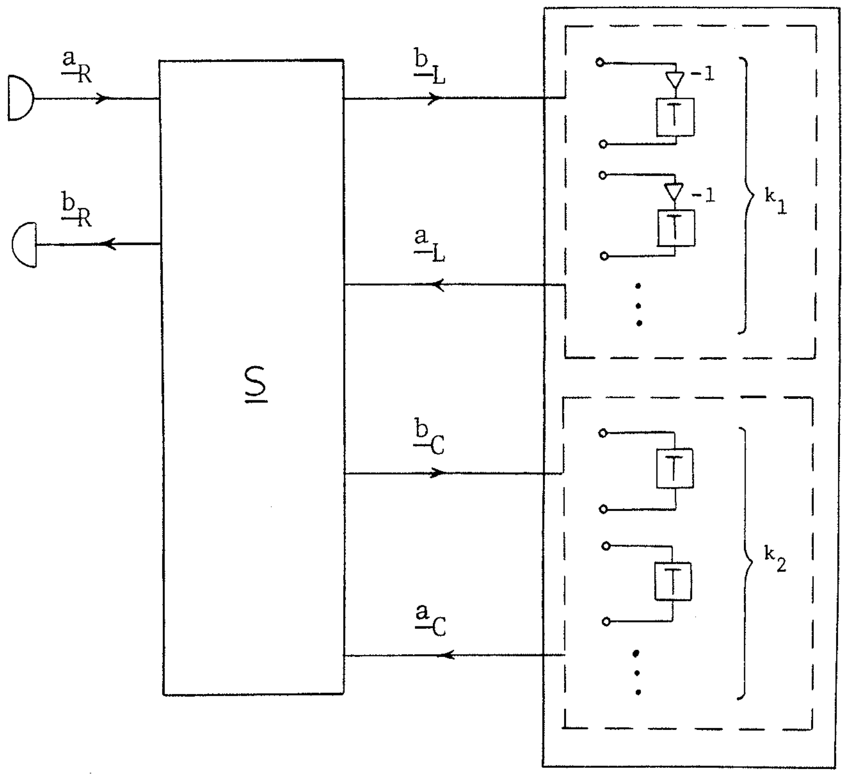


Figure 2.6: The general wave digital filter structure.

$$\begin{bmatrix} \underline{b}_R(n) \\ \underline{b}_D(n) \end{bmatrix} = \begin{bmatrix} \underline{S}_{11} & \underline{S}_{12} \\ \underline{S}_{21} & \underline{S}_{22} \end{bmatrix} \begin{bmatrix} \underline{a}_R(n) \\ \underline{a}_D(n) \end{bmatrix} \tag{2.30a}$$

$$\underline{a}_D(n+1) = \underline{\Sigma}\, \underline{b}_D(n) \tag{2.30b}$$

where

$$\underline{b}_D(n) = [\ \underline{b}_L(n) \quad \underline{b}_C(n)\ ]^T \qquad \underline{a}_D(n) = [\ \underline{a}_L(n) \quad \underline{a}_C(n)\ ]^T \tag{2.30c}$$

and $\underline{S}$ is partitioned conformable with $\underline{b}_R$, $\underline{b}_D$, $\underline{a}_R$, and $\underline{a}_D$. Equivalently,

$$\begin{bmatrix} \underline{b}_R(n) \\ \underline{a}_D(n+1) \end{bmatrix} = \begin{bmatrix} \underline{S}_{11} & \underline{S}_{12} \\ \underline{\Sigma}\, \underline{S}_{21} & \underline{\Sigma}\, \underline{S}_{22} \end{bmatrix} \begin{bmatrix} \underline{a}_R(n) \\ \underline{a}_D(n) \end{bmatrix} \tag{2.31}$$

Clearly, the quadruple $\{\underline{\Sigma}\,\underline{S}_{22}, \underline{\Sigma}\,\underline{S}_{21}, \underline{S}_{12}, \underline{S}_{11}\}$ defines a state-variable description of the wave digital filter. In the z-domain, the system in (2.31) becomes

$$\underline{B}_R(z) = [\, \underline{S}_{11} + \underline{S}_{12}(\, z\underline{U} - \underline{\Sigma}\,\underline{S}_{22})^{-1}\, \underline{\Sigma}\,\underline{S}_{21}\,]\, \underline{A}_R(z) \qquad (2.32)$$

2.2 ALTERNATE WAVE DIGITAL FILTER STRUCTURES

Intrinsic to the WD design procedure is the partitioning of the reference network into subnetworks in the manner illustrated in Fig. 2.3. The WD filter realization follows from the voltage wave scattering representation of the subnetworks. Regarding the scattering representations, we note that the port reference resistances at ports connecting reactive or resistive elements are determined by the element values. Consequently, the scattering variable description of the frequency-independent coupling network will be unique to within a permutation of variables. This seems to imply that there is a corresponding unique digital structure. However, consider the coupling network partitioned into a number of smaller subnetworks. This partitioning, and the subsequent scattering variable descriptions of the resultant subnetworks, are not necessarily unique. Clearly, alternate WD filter structures are possible. This is easily understood with the aid of an example.

Consider the analog prototype shown in Fig.2.7.

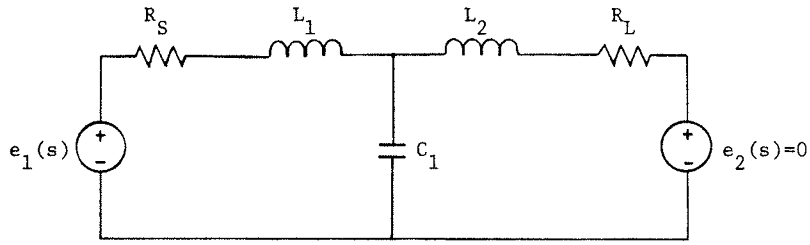


Figure 2.7: An example reference filter.

In Fig. 2.8 we have reconfigured this prototype network in the form introduced in Fig. 2.3. Another possible configuration is illustrated in Fig. 2.9.; the coupling network is viewed as the interconnection of three subsections. The WD design procedure with this latter configuration involves finding a scattering variable description for each of the smaller subsections. As before, the port reference resistances at the ports of the subsections that connect elements are constrained by the associated element values. However, at the interconnection between two of the subsections, the value of the port reference resistance is not similarly restricted. The scattering matrices determined for M_1, M_2, and M_3 describe, in the discrete-time domain, signal-flow networks which interconnected in the manner suggested by Fig. 2.9, determine the computational algorithm of the WD filter realization.

The conventional WD design process [17] utilizes the configuration suggested by Fig. 2.9 to implement filter designs

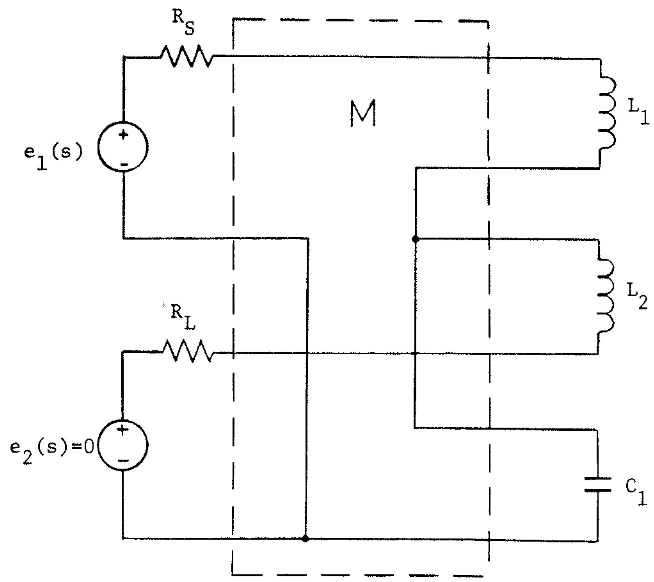


Figure 2.8: The example filter reconfigured as in Fig. 2.3.

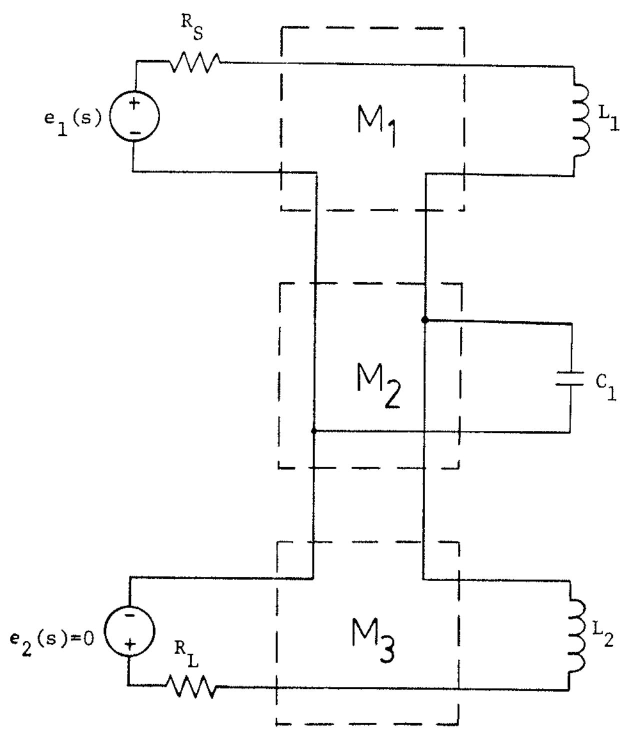


Figure 2.9: An alternate configuration.

based on doubly terminated LC ladder reference filters. The signal-flow subnetworks are termed adaptors. Signal-flow networks derived from analog subsections such as M_1 and M_3 are called series adaptors since they imitate the series connection of analog elements, while similarly, digital networks derived from subsections such as M_2 are called parallel adaptors. These two elementary adaptor types are sufficient for the realization of WD designs based on ladder reference networks. Adaptors have also been derived for the symmetric lattice [19,20], Brune [31,33], and Darlington D [30] sections.

A generalized n-port adaptor has been introduced by Martens and Meerkötter [29]. This method may be applied to a lossless reciprocal frequency-independent network of any topology. The n-port adaptor derives all of the elementary adaptors described above and may be applied to deriving WD realizations based on a single adaptor.

2.3 THE REFLECTION-FREE PROPERTY

Consider the interconnection of two arbitrary frequency-independent lossless networks M and $\tilde{M}$, described by the scattering matrices $\underline{S}$ and $\tilde{\underline{S}}$ (Fig. 2.10). The topological constraints for the networks M and $\tilde{M}$ can be expressed as

$$\begin{bmatrix} \underset{\sim}{b}_o \\ b \end{bmatrix} = \begin{bmatrix} \underset{\sim}{S}_{11} & S_{12} \\ \underset{\sim}{S}_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{a}_o \\ a \end{bmatrix} \qquad \begin{bmatrix} \tilde{b} \\ \underset{\sim}{\tilde{b}}_o \end{bmatrix} = \begin{bmatrix} \tilde{S}_{11} & \underset{\sim}{\tilde{S}}_{12} \\ \tilde{S}_{21} & \underset{\sim}{\tilde{S}}_{22} \end{bmatrix} \begin{bmatrix} \tilde{a} \\ \underset{\sim}{\tilde{a}}_o \end{bmatrix} \qquad (2.33)$$

where $\underline{S}$ and $\underline{\tilde{S}}$ have been partitioned conformable with the wave vectors. The corresponding discrete-time structure is represented in Fig. 2.11 as a signal-flow graph. From Figure 2.11 we see that if the parameters $\tilde{S}_{11}$ and S_{22} are both nonzero, a delay-free path will exist within the signal-flow graph describing the two interconnected adaptors. This condition violates the necessary requirement that for the structure to be computable or realizable, every feedback path must contain at least one delay element [36,37]. Clearly, a desired condition is that one of the reflection coefficients, either $\tilde{S}_{11}$ or S_{22}, be identically zero [17].

Consider the analog network M. We have

$$S_{22} = \left.\frac{b}{a}\right|_{\underline{a}_o = \underline{0}} = \left.\frac{v - Ri}{v + Ri}\right|_{\underline{a}_o = \underline{0}} = \frac{R_d - R}{R_d + R} \qquad (2.34a)$$

where

$$R_d = \left.\frac{v}{i}\right|_{\underline{a}_o = \underline{0}} \qquad (2.34b)$$

The quantity R_d is recognized to be the driving point resistance of the network M at the interconnection port, with all other ports terminated in their port resistances. It follows that if we choose the reference resistance R at the interconnection port to be equal to R_d, we have the condition $S_{22}= 0$. This port is then said to be reflection-free. A similar result will hold for the network $\tilde{M}$ if we wish instead, to ensure $\tilde{S}_{11}= 0$. In summary, a judicious choice of the common port reference resistances assigned to adaptor

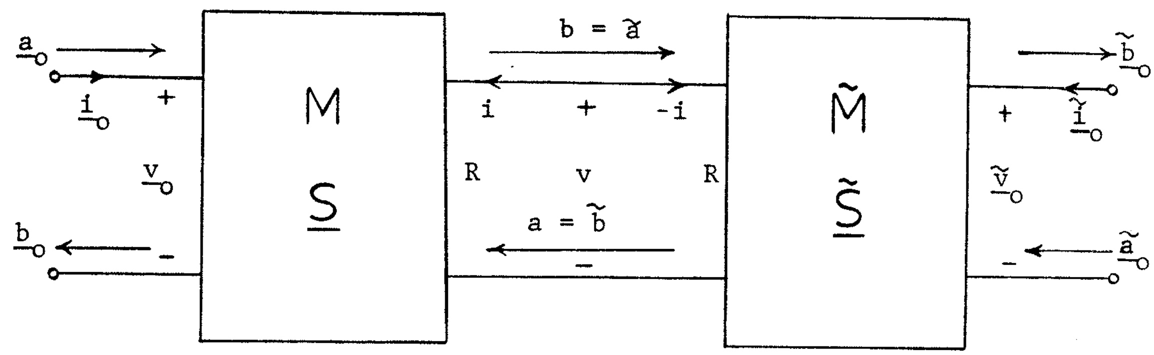


Figure 2.10: Interconnection of two frequency-independent multiport networks.

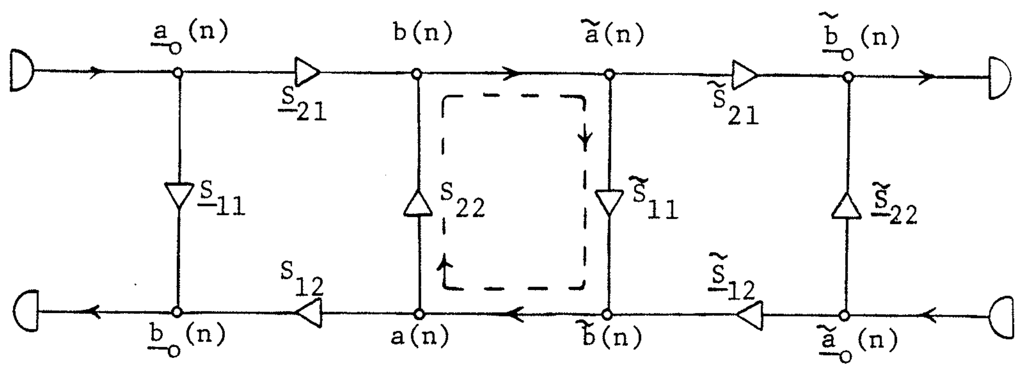


Figure 2.11: A signal-flow graph describing the interconnection of two adaptors.

interconnection ports ensures the realizability of the corresponding digital filter structure.

2.4 THE N-PORT ADAPTOR

The generalized n-port adaptor has been well described in literature [28,29,31]. Consider an n-port reference network of interconnections and possibly ideal transformers. The port voltage vector $\underline{v}$ and the port current vector $\underline{i}$ may be partitioned according to "link" and "tree" ports, i.e.

$$\underline{v} = \begin{bmatrix} \underline{v}_\ell \\ \underline{v}_t \end{bmatrix} \qquad \underline{i} = \begin{bmatrix} \underline{i}_\ell \\ \underline{i}_t \end{bmatrix} \tag{2.35}$$

where the subscripts t and ℓ identify the "tree" and "link" ports, respectively. The corresponding reflected and incident voltage wave vectors, $\underline{b}$ and $\underline{a}$, and the diagonal matrix $\underline{R}$ of reference resistances may be similarly partitioned

$$\underline{b} = \begin{bmatrix} \underline{b}_\ell \\ \underline{b}_t \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} \underline{a}_\ell \\ \underline{a}_t \end{bmatrix} \qquad \underline{R} = \begin{bmatrix} \underline{R}_\ell & \underline{0} \\ \underline{0} & \underline{R}_t \end{bmatrix} \tag{2.36}$$

Martens and Meerkötter [29] have shown that the scattering matrix description of a lossless reciprocal frequency-independent network can be expressed in the form

$$\underline{S} = \begin{bmatrix} 2\underline{Q}_\ell^T \underline{K} - \underline{U} & 2\underline{Q}_\ell^T (\underline{U} - \underline{K} \underline{Q}_\ell^T) \\ 2\underline{K} & \underline{U} - 2\underline{K} \underline{Q}_\ell^T \end{bmatrix} \tag{2.37}$$

where $\underline{S}$ has been partitioned conformable with $\underline{b}$ and $\underline{a}$. The submatrix $\underline{Q}_\ell$ is derived from the "cutset" matrix $\underline{Q}$ of the network by partitioning $\underline{Q}$ according to "link" and "tree" ports, that is $\underline{Q} = [\underline{Q}_\ell \ \underline{U}]$. Clearly $\underline{Q}_\ell$ is dependent only on the topology of the coupling network. The submatrix $\underline{K}$ is defined by

$$\underline{K} = (\underline{G}_t + \underline{Q}_\ell \, \underline{G}_\ell \, \underline{Q}_\ell^T)^{-1} \, \underline{Q}_\ell \, \underline{G}_\ell \tag{2.38}$$

where $\underline{G}_t = \underline{R}_t^{-1}$ and $\underline{G}_\ell = \underline{R}_\ell^{-1}$. Alternate forms for the scattering matrix $\underline{S}$ are given by

$$\underline{S} = \begin{bmatrix} -\underline{U} & \underline{Q}_\ell^T \\ \underline{0} & \underline{U} \end{bmatrix} \begin{bmatrix} -\underline{U} & \underline{0} \\ -2\underline{K} & \underline{U} \end{bmatrix} \begin{bmatrix} -\underline{U} & \underline{Q}_\ell^T \\ \underline{0} & \underline{U} \end{bmatrix} \tag{2.39}$$

$$= \begin{bmatrix} \underline{U} - \underline{Q}_\ell^T \underline{K} & \underline{Q}_\ell^T \\ -\underline{K} & \underline{U} \end{bmatrix} \begin{bmatrix} -\underline{U} & \underline{0} \\ \underline{0} & \underline{U} \end{bmatrix} \begin{bmatrix} \underline{U} & -\underline{Q}_\ell^T \\ \underline{K} & \underline{U} - \underline{K}\,\underline{Q}_\ell^T \end{bmatrix} \tag{2.40}$$

The decomposition of (2.40) serves to display the eigenvalues of the $\underline{S}$ matrix. In (2.39) all of the matrices are self-inverse which ensures that $\underline{S}$ is self-inverse i.e.

$$\underline{S} = \underline{S}^{-1} \tag{2.41}$$

In addition, $\underline{S}$ satisfies [31]

$$\underline{S}^T \, \underline{G} \, \underline{S} = \underline{G} \tag{2.42a}$$

where

$$\underline{G} = \underline{R}^{-1} = \begin{bmatrix} \underline{G}_\ell & \underline{0} \\ \underline{0} & \underline{G}_t \end{bmatrix} \tag{2.42b}$$

is the diagonal positive definite reference conductance matrix. The results (2.41) and (2.42) also imply

$$\underline{S}^T \, \underline{G} = \underline{S} \, \underline{G} \tag{2.43}$$

Properties (2.42) and (2.43) derive from the losslessness and reciprocity of the reference coupling network, respectively. Property (2.42) is very important with respect to the nonlinear stability of WD filter structures.

The matrix $\underline{K}$ has a very useful network interpretation [29]. Consider the n-port reference network with all of the "tree" ports terminated in their port resistances, and all of the "link" ports terminated in their port resistances in series with a voltage source. It follows that

$$\underline{b}_t = 2\underline{v}_t \tag{2.44}$$

$$\underline{a}_t = \underline{0} \tag{2.45}$$

and

$$\underline{a}_\ell = \underline{e}_\ell \tag{2.46}$$

where $\underline{e}_\ell$ denotes the vector of "link" voltage sources. From (2.37) and (2.45)

$$\underline{b}_t = 2\underline{K}\ \underline{a}_\ell \tag{2.47}$$

or, utilizing (2.44) and (2.46)

$$\underline{v}_t = \underline{K}\ \underline{e}_\ell \tag{2.48}$$

Thus the elements of $\underline{K}$ are voltage transfer ratios from the "link" sources to the "tree" branches. This network interpretation of $\underline{K}$ may often be applied in expressing the coefficients of $\underline{K}$ as a useful function of a reduced number of parameters. This is the case for n-ports with a ladder topology.

If a network contains k elements whose values may be chosen independently, then allowing for impedance scaling, the transfer function will have k-1 degrees-of-freedom. An n-port adaptor is said to be canonic in multipliers if the number of multipliers is equal to the number of degrees-of-freedom in the transfer function of the prototype network. Consider a network or a portion therof, with a ladder topology. The $\underline{Q}_\ell$ matrix will have entries that are +1, -1, or 0. All of the multiplier coefficients will be contained in $\underline{K}$. Since $\underline{K}$ is of dimension $t \times \ell$ (t is the number of "tree" ports and ℓ is the number of "link" ports) there may be as many as $t\ell$ coefficients. Equation (2.38) expresses $\underline{K}$ in terms of a canonic number of parameters, that is, the k-1 ratios of independent reference conductance values. But the relation in (2.38) is not useful in the sense that these parameters cannot correspond to multipliers in a hardware realization. However, for the ladder topology it is always possible (through application of the network interpretation) to express $\underline{K}$ in terms of another canonic set of independent parameters with the property that each parameter can correspond to a single multiplier in a hardware implementation [29]. We note that a one-to-one mapping will exist between these, and every two canonic sets of parameters. Clearly, n-port adaptors derived from ladder networks can be made canonic in multipliers.

These results for the class of networks with a ladder topology are the same as those originally derived by Fettweis. In fact, n-port adaptors (for ladder networks) can be determined such that the multipliers are identical to those of the conventional WD structure composed of an interconnection of series and parallel adaptors. Accordingly, the elementary series and parallel adaptors can also be derived by this method. Similar results can be obtained for the class of networks with a lattice or Jaumann structure.

2.5 SPECIAL PROPERTIES OF WAVE DIGITAL FILTERS

The WD class of filters is known to perform well with respect to finite-word-length effects. In particular, WD filters have been found to exhibit attenuation characteristics with low sensitivity to variations in filter coefficients [38-41], and to have relatively low levels of roundoff error [40,42]. Furthermore, Fettweis and Meerkötter [25] have shown that for a large class of WD filters, i.e. those based on LC ladder or lattice prototypes, zero-input oscillations of any type can be completely suppressed. In this section we review the concepts that explain the special properties of the WD structure.

2.5.1 Sensitivity and Roundoff Noise

The transformation that maps an analog reference network into a linear discrete-time WD network preserves many of the desirable properties of the continuous-time network as analogous properties in the discrete-time system. Consider for example, the important class of WD filters that are derived from LC ladder or lattice reference filters. In the previous section we noted that there exists a one-to-one mapping between the set of independent port reference conductance values and a canonic set of multiplier coefficients. However, in a WD realization the independent port conductance values (reflection-free ports have dependent port conductance values) are determined directly from the values of the elements in the reference filter. Clearly a one-to-one mapping exists between these element values and the adaptor multiplier coefficients. Quantization of the multiplier coefficients can thus be interpreted equivalently as a variation in the element values of the reference filter. Element variations degrade the frequency response of the analog prototype and this is mirrored in the discrete-time domain. The coefficient sensitivity of the WD structure will therefore depend on two factors:

1. the sensitivity of the elements in the reference filter to coefficients in the digital structure, and
2. the sensitivity of the reference filter frequency response to variations in the element values.

Accordingly, the sensitivity properties of a WD structure are directly dependent on those of the reference filter. Doubly terminated reactance filters are known to be highly insensitive [34,35] and hence, the relative insensitivities of the associated WD designs.

Fettweis [22] has linked the sensitivity of a structure's attenuation characteristic to coefficient quantization with the roundoff noise generated within the structure. Fettweis contends that signal quantizations can be modelled equivalently as coefficient fluctuations, and argues that a structure that exhibits low coefficient sensitivity will also have reduced levels of roundoff noise. Therefore, WD filters are expected to display superior noise performance and this has been verified experimentally [40,42].

2.5.2 The Concepts of Pseudopower and Stored Pseudopower

Wave digital filters are derived as an interconnection of elementary building blocks. The building blocks, including adaptors, sources and sinks, and delay elements, are derived as wave multiports or wave n-ports. Consider a wave n-port with the scattering variable description

$$\underline{B}(z) = \underline{S}(z)\ \underline{A}(z)$$

and let $\underline{G}$ denote the diagonal reference conductance matrix associated with $\underline{S}(z)$. The instantaneous pseudopower $p(n)$ absorbed through the n-port at the n^{th} time instant is defined as [21]

$$p(n) = \underline{a}^T(n)\ \underline{G}\ \underline{a}(n) - \underline{b}^T(n)\ \underline{G}\ \underline{b}(n) \tag{2.49}$$

An n-port adaptor derived from a lossless reciprocal frequency-independent network is described by a constant scattering matrix $\underline{S}$ that satisfies the property $\underline{S}^T\underline{GS}=\underline{G}$. Hence the pseudopower absorbed by the adaptor has the value

$$p(n) = \underline{a}^T(n)\ (\ \underline{G} - \underline{S}^T\ \underline{G}\ \underline{S}\ )\ \underline{a}(n) \equiv 0 \tag{2.50}$$

In this case we refer to the adaptor as being instantaneously pseudolossless. Similarly, if instead $p(n) \geq 0$ for all n, the adaptor is described as being instantaneously pseudopassive.

Fettweis and Meerkötter [25] have also introduced the concept of stored pseudopower. Consider a single delay (possibly in series with an inverter) connected to a port with a port reference conductance G (Fig. 2.12).

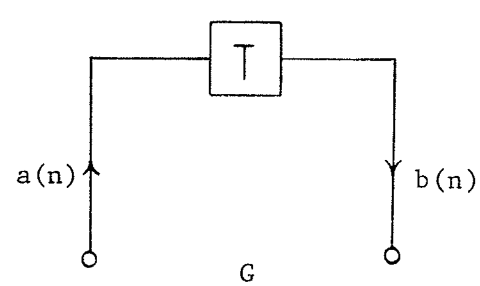


Figure 2.12: A delay connected to a port of port conductance G.

The pseudopower $p_S(n)$ stored in the delay in the time interval $(n,n+1)$ is defined by

$$p_S(n) = G\, a^2(n) \tag{2.51}$$

A general linear WD filter section is represented in Fig. 2.13. Let the n-port adaptor N be described by the scattering system

$$\underline{b} = \underline{S}\, \underline{a} \tag{2.52a}$$

with the ports ordered such that

$$\underline{b} = \begin{bmatrix} \underline{b}_R \\ \underline{b}_D \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} \underline{a}_R \\ \underline{a}_D \end{bmatrix} \qquad \underline{G} = \begin{bmatrix} \underline{G}_R & \underline{0} \\ \underline{0} & \underline{G}_D \end{bmatrix} \tag{2.52b}$$

where the subscripts D and R differentiate between those ports that are terminated in a delay or source and sink, respectively. The matrix $\underline{G}$ is the diagonal matrix of reference conductances and $\underline{\Sigma}$ is the polarity matrix that serves to identify those ports at which an inverter is connected in series with the delay.

The pseudopower stored in the delays of the filter section in the time interval $(n,n+1)$ is given by

$$p_S(n) = (\, \underline{\Sigma}\, \underline{b}_D(n)\,)^T\, \underline{G}_D\, \underline{\Sigma}\, \underline{b}_D(n) \;\; = \underline{b}_D^T(n)\, \underline{G}_D\, \underline{b}_D(n) \tag{2.53}$$

where we note that the wave variables have been defined with respect to the adaptor. (Observe that the vector $\underline{\Sigma}\, \underline{b}_D(n)$ is in fact the "next state" vector of the filter.) The increase in the stored pseudopower at the n^{th} time instant is then

$$\Delta p_S(n) = p_S(n) - p_S(n-1) \tag{2.54}$$

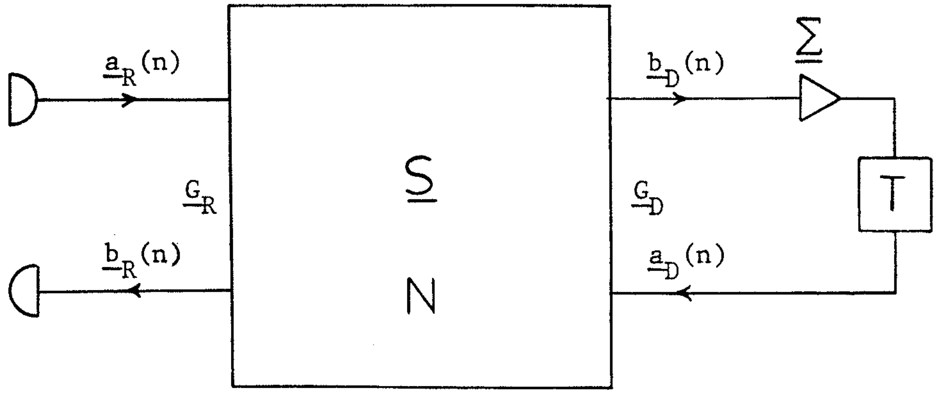


Figure 2.13: A linear wave digital filter section based on an n-port adaptor with a diagonal conductance matrix.

or, since $\underline{a}_D(n) = \underline{\Sigma}\,\underline{b}_D(n-1)$,

$$\Delta p_S(n) = \underline{b}_D^T(n)\ \underline{G}_D(n)\ \underline{b}_D(n) - \underline{a}_D^T(n)\ \underline{G}_D(n)\ \underline{a}_D(n) \tag{2.55}$$

Also consider the pseudopower absorbed by the n-port adaptor N at the n^{th} time instant,

$$p_N(n) = \underline{a}^T(n)\ \underline{G}\ \underline{a}(n) - \underline{b}^T(n)\ \underline{G}\ \underline{b}(n) \tag{2.56}$$

From (2.55) and (2.56)

$$\Delta p_S(n) = -p_N - \underline{b}_R^T(n)\ \underline{G}_R\ \underline{b}_R(n) + \underline{a}_R^T(n)\ \underline{G}_R\ \underline{a}_R(n) \tag{2.57}$$

In the absence of outside signals, i.e. $\underline{a}_R=0$, we have

$$\Delta p_S(n) = -p_N(n) - \underline{b}_R^T(n)\ \underline{G}_R\ \underline{b}_R(n) \tag{2.58}$$

Furthermore, if the adaptor N is pseudopassive we have the condition

$$\Delta p_S(n) \leq -\underline{b}_R^T(n)\ \underline{G}_R\ \underline{b}_R(n) \quad \text{for all } n \tag{2.59}$$

Given that $\underline{G}$ is diagonal and that the port reference conductances are always positive, the quadratic expression on the right hand side of (2.59) is a negative definite function of the filter output variables.

The canonic (in terms of multipliers) n-port adaptors described in Section 2.4 are pseudolossless even after coefficient quantization. Consequently, (2.59) is always satisfied for the WD filters based on these adaptors.

2.5.3 Nonlinear Stability

The analysis presented in the previous section applies for linear operating conditions. In a practical implementation the finite-word-length constraint requires that quantizers be introduced into the linear system. Naturally, we wish the resulting nonlinear filter to be stable, that is, to suppress parasitic oscillations within the structure.

The nonlinear filter is said to be output stable if under zero-input and for arbitrary initial conditions, the output becomes permanently zero in a finite time. Furthermore, a filter is said to be completely stable if under the same conditions, the states become permanently zero in a finite time [25]. Note that complete stability ensures output stability since the output variables are linear functions of the states. Output stability implies freedom from observable oscillations while complete stability implies freedom from both observable and unobservable oscillations.

The nonlinear WD filter structure that results when quantizers are introduced before the delays of the previously linear filter, is depicted in Fig. 2.14.

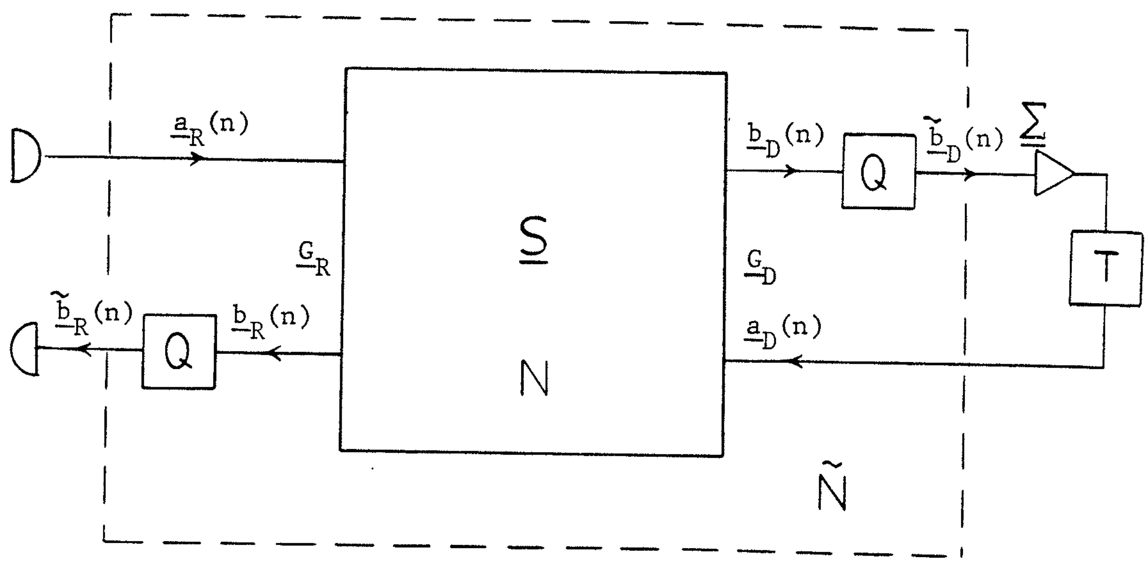


Figure 2.14: A wave digital filter implemented with quantizers.

The reflected wave vector corresponding to the nonlinear adaptor is denoted by $\tilde{\underline{b}}$. The pseudopower $\tilde{p}_S(n)$ stored in the delays of the nonlinear filter in the time interval (n,n+1), and the increase in the stored pseudopower at the

n^{th} time instant, can be determined as in the previous subsection, i.e.

$$\tilde{p}_S = (\underline{\Sigma}\, \underline{\tilde{b}}_D)^T \underline{G}_D \underline{\Sigma}\, \underline{\tilde{b}}_D = \underline{\tilde{b}}_D^T \underline{G}_D \underline{\tilde{b}}_D \tag{2.60}$$

and

$$\Delta\tilde{p}_S = \underline{\tilde{b}}_D^T \underline{G}_D \underline{\tilde{b}}_D - \underline{a}_D^T \underline{G}_D \underline{a}_D \tag{2.61}$$

Note that in order to simplify the notation, we no longer specify quantities explicitly in terms of the discrete-time variable. The pseudopower absorbed by the nonlinear adaptor $\tilde{N}$ is given by

$$\tilde{p}_N = \underline{a}^T \underline{G}\, \underline{a} - \underline{\tilde{b}}^T \underline{G}\, \underline{\tilde{b}} \tag{2.62}$$

From (2.61) and (2.62) and assuming zero-input conditions, we obtain

$$\Delta\tilde{p}_S = -\tilde{p}_N - \underline{\tilde{b}}_R^T \underline{G}_R \underline{\tilde{b}}_R \tag{2.63}$$

The stored pseudopower, as defined in (2.60), is a positive definite function of the "next state" of the WD filter. The quantity $\Delta\tilde{p}_S$ is the first backward difference of the stored pseudopower. If $\Delta\tilde{p}_S$ can be shown to be negative semi-definite under zero-input conditions, the stored pseudopower serves as a Lyapunov function for the nonlinear filter. Fettweis and Meerkötter have utilized this property to derive conditions on numerical operations that guarantee the stability of nonlinear WD filters.

We state the following general theorems based on material in [25].

Theorem 2.1

A digital filter is output stable if there exists a function V(n) such that, for n = 0,1,2,....

i) V(n) is a positive definite function of the signals incident at the delays of the filter at the n^{th} time instant, and

ii) under zero-input conditions, $\Delta V(n)$ (which is defined as $\Delta V(n) = V(n)-V(n-1)$) is nonpositive and moreover, is negative if any signal at the output of the filter at the n^{th} time instant remains nonzero.

Proof:

A digital filter implemented with finite-word-length arithmetic will be a finite state machine. Therefore, under zero-input conditions, the filter signal parameters must either become permanently zero after a finite time, or be sustaining oscillations. However, if one of the output variables is carrying out periodic oscillations with nonzero amplitude, V(n) must decrease within each oscillation cycle by a finite amount. This implies that eventually V(n) must decrease below zero. This contradicts the assumption that V(n) is positive definite. Clearly the filter cannot sustain oscillations at the output and is therefore output stable.

Theorem 2.2

A digital filter is completely stable if there exists a function V(n) such that, for n = 0,1,2,...

i) V(n) is a positive definite function of the signals incident at the delays of the filter at the n^{th} time instant, and

ii) under zero-input conditions, $\Delta V(n)$ is nonpositive and moreover, is negative if any signal incident at the delays of the filter at the n^{th} time instant remains nonzero.

The proof for Theorem 2.2 is similar to that for Theorem 2.1 and is not given here.

Theorem 2.1 specifies a condition that is sufficient to guarantee the output stability of a nonlinear digital filter. For a WD filter of the type shown in Fig. 2.14, an equivalent condition is that the nonlinear n-port adaptor $\tilde{N}$ be pseudopassive i.e.

$$\tilde{p}_N \geq 0 \tag{2.64}$$

This result follows from (2.63). Since in general, n-port adaptors will be pseudolossless under linear conditions, we can reinterpret (2.64) as the constraint

$$\underline{\tilde{b}}^T \underline{G}\, \underline{\tilde{b}} - \underline{b}^T \underline{G}\, \underline{b} \leq 0 \tag{2.65}$$

Given that $\underline{G}$ is positive definite and diagonal, (2.65) is satisfied if the quantizations are carried out such that

$$|\tilde{b}_i| \leq |b_i| \quad i = 1,2,\ldots,n \tag{2.66}$$

Condition (2.64) or more specifically (2.66), represents a general criterion that will guarantee the output stability of conventional WD filter designs.

We note that the nonlinear adaptor $\tilde{N}$ may be realized as an interconnection of individual adaptors $\tilde{N}_k$, $k = 1,2,\ldots,K$. Let us denote the pseudopower absorbed by $\tilde{N}_k$ as $\tilde{p}_k$. It is shown in [21,25] that

$$\tilde{p}_N = \sum_{k=1}^{K} \tilde{p}_k \tag{2.67}$$

and accordingly, that

$$\tilde{p}_k \geq 0 \qquad k = 1,2,\ldots,K \tag{2.68}$$

is sufficient to guarantee the output stability of the filter. Condition (2.68) is ensured by requiring that the criteria in (2.66) apply at the ports of each adaptor $\tilde{N}_k$.

If a linear WD system based on a pseudopassive n-port adaptor is free from any oscillations (including unobservable ones) and the nonlinear modifications are carried out according to (2.66) (also ensuring $\tilde{b}_i b_i \geq 0$, $i = 1,2,\ldots,n$), the nonlinear filter will be completely stable [25]. However, a pseudolossless reciprocal WD system will be free from all oscillations under linear conditions if and only if it is minimal [28]. Given the criteria in (2.66), it follows that complete stability can only be guaranteed for minimal WD networks. Wave digital filters based on ladder reference networks that have redundant reactive elements, for example those that realize finite attenuation poles, will be nonminimal. Methods for eliminating the redundant states or, equivalently, removing the excess delays have been developed [26-28]. However, these modifications generally change the nature of the WD structure in a manner that invalidates the simple stability criterion in (2.66). Recently, Martens and Jarmasz [32] have demonstrated a method to overcome this problem.

Classen et al. [43] and Meerkötter [44] have also studied the stability of WD filters under forced response conditions. In particular, overflow quantization characteristics (only slightly more restrictive than those suggested by (2.66)) have been specified to guarantee the stability of the forced response.

Chapter III

AN APPROACH TO THE SYNTHESIS OF WAVE DIGITAL FILTERS

The first step in WD filter design is to determine an appropriate analog reference network. The usual procedure for doing so is as follows:

1. Map specifications given in the digital domain to the analog domain.
2. Solve the approximation problem to determine a suitable analog transfer function.
3. Synthesize a doubly terminated reactance network that realizes the desired transfer function.

Normally, the approximation and synthesis problems are both solved with the aid of filter design tables [45]. However when dealing with nonstandard transfer functions, for example those arising in cascade realizations, a direct synthesis remains necessary.

It was noted in Section 2.1 that the voltage wave transfer function $S_{21}(s)$ of a reactance two-port network is related to the voltage transfer function $H_a(s)$ of the corresponding doubly terminated filter by (assuming port reference resistances are chosen appropriately)

$$S_{21}(s) = 2H_a(s)$$

It follows that the transfer function synthesis problem can be reinterpreted as a scattering matrix synthesis problem.

Vongpanitlerd [46] has introduced a simple technique for the synthesis of lossless reciprocal scattering matrices. In Chapter III we show that this same technique may be applied to deriving nominal WD filter designs. In particular, we develop an algorithm based on the method described in [46] that facilitates WD filter synthesis including cascade synthesis.

3.1 THE SCATTERING MATRIX OF A LOSSLESS TWO-PORT

The normalized[1] scattering matrix description $\underline{S}(s)$ of a real lossless two-port may be written in the canonic form [47]

$$\underline{S}(s) = \frac{1}{G(s)} \begin{bmatrix} H(s) & F(s) \\ \sigma F(-s) & -\sigma H(-s) \end{bmatrix} \tag{3.1}$$

where σ is a scalar of value +1 or -1. The polynomials F(s), H(s), and G(s) are termed the canonic polynomials and they have the following properties:

1. They are real polynomials.
2. G(s) is strictly Hurwitz.

---

[1] If the voltage scattering matrix description $\underline{S}(s)$ of an m-port network is derived with the condition that all port reference resistances are each of unit value, then $\underline{S}(s)$ is called the normalized scattering matrix.

3. $F(s)$ is monic i.e. its leading coefficient is equal to unity.

4. $F(s)$, $G(s)$, and $H(s)$ satisfy the equation

$$G(s)\,G(-s) = H(s)\,H(-s) + F(s)\,F(-s) \tag{3.2}$$

which is often called the Feldtkeller equation.

For a reciprocal lossless two-port, $F(s)$ is a pure even or pure odd polynomial that satisfies

$$F(s) = \sigma F(-s) \tag{3.3}$$

Therefore, for the lossless reciprocal case, $\underline{S}(s)$ is a symmetric matrix.

An $m \times m$ real rational scattering matrix is called lossless bounded real (LBR) if [48]

1. $\underline{S}(s)$ has all elements analytic in Re $[s] \geq 0$
2. $\underline{S}^T(-s)\underline{S}(s) = \underline{U}$

It is well known [48] that any $m \times m$ scattering matrix $\underline{S}(s)$ that characterizes a linear, time-invariant, lumped, finite, lossless m-port will be lossless bounded real. One may easily verify that the scattering matrix described by (3.1) satisfies the LBR properties.

3.2 SCATTERING MATRIX SYNTHESIS

The general scattering matrix synthesis problem has been considered in [48,49]. The special case of lossless reciprocal synthesis has been shown to be particularly straightforward [46].

The lossless reciprocal synthesis problem can be briefly stated as follows. Given an $m \times m$ (in our case 2×2) symmetric LBR normalized scattering matrix $\underline{S}(s)$, determine an m-port network N consisting of ideal transformers and positive inductors and capacitors that realizes $\underline{S}(s)$ as its scattering matrix description. Let us assume that the network N which synthesizes the prescribed $\underline{S}(s)$ consists of an interconnection of two lossless subnetworks N_1 and N_2 as depicted in Fig. 3.1. The network N_2 is constrained to contain all of the reactive elements, including k_1 inductors and k_2 capacitors which are uncoupled and whose values are $L_1, L_2, \ldots, L_{k_1}, C_1, C_2, \ldots, C_{k_2}$. The subnetwork N_1 is a nondynamic $(m+k)$-port (where $k = k_1 + k_2$) containing only direct connections and ideal transformers. Note that the structures represented in Figures 2.3 and 3.1 are essentially the same. The network N_1 will have a normalized scattering matrix description, say $\underline{S}_n$. Since N_1 is by assumption a reciprocal lossless frequency-independent network, $\underline{S}_n$ is symmetric and constant, and satisfies the LBR properties.

Assuming knowledge of $\underline{S}_n$, the construction of the network N_1 is relatively straightforward [49]. A synthesis of $\underline{S}(s)$

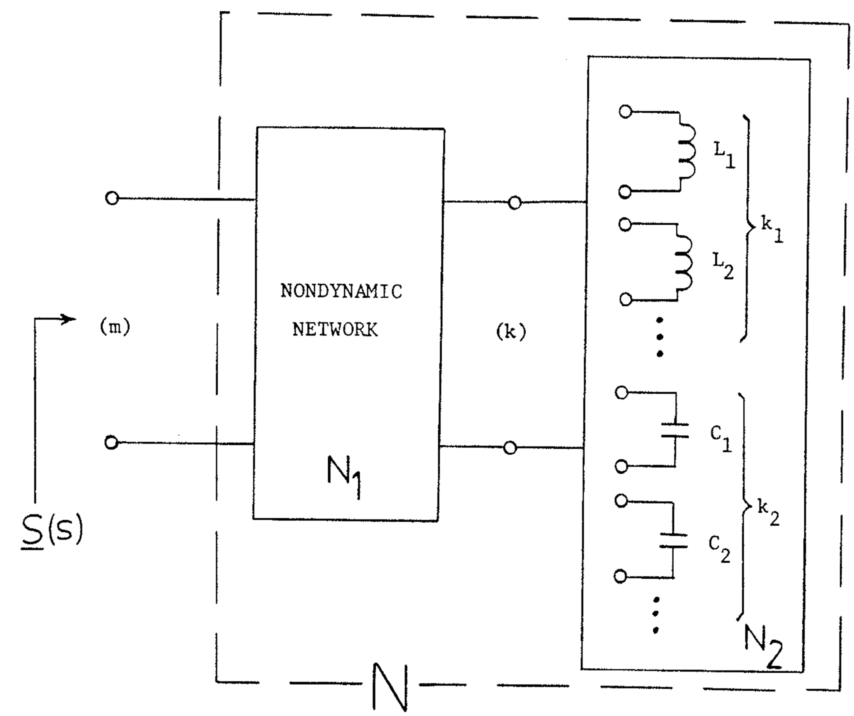


Figure 3.1: Network partitioning via reactance extraction.

follows directly from a synthesis of $\underline{S}_n$ by terminating the ports of N_1 with the appropriate capacitors and inductors as shown in Fig. 3.1. Consequently, the synthesis problem can be restated as follows. Given an $m \times m$ symmetric LBR normalized scattering matrix $\underline{S}(s)$, determine a lossless reciprocal nondynamic (m+k)-port network N_1 (described by the normalized scattering matrix $\underline{S}_n$) which when terminated with the appropriate reactive elements, yields a network N realizing $\underline{S}(s)$.

Vongpanitlerd [46] solved the above synthesis problem with state-variable techniques. In particular, he describes a method for deriving the symmetric constant matrix $\underline{S}_n$ that specifies the coupling network N_1, directly from the prescribed $\underline{S}(s)$ matrix. A characteristic of this synthesis method is that solutions for the network N will utilize the minimum number of reactive elements, the number being the degree of the scattering matrix. In addition, these reactive elements are restricted to be each of unit value. The normalized scattering matrix $\underline{S}_n$ is, by definition, determined with port reference resistances that are also of unit value. Therefore, at the interconnection ports between the networks N_1 and N_2, the port reference resistances R_i, i = 1,2,...,k are related to the values of the reactive elements by

$$R_i = L_i = 1$$

if the i^{th} port is terminated in an inductor, and

$$R_i = 1/C_i = 1$$

if the i^{th} port is terminated in a capacitor. From the discussion of Section 2.1, it is clear that the scattering matrix $\underline{S}_n$ specifies an n-port adaptor for the network N. Since the matrix $\underline{S}_n$ is LBR, it has the property

$$\underline{S}_n^T \underline{S}_n = \underline{U} \tag{3.4}$$

and given $\underline{S}_n$ is symmetric, we also have

$$\underline{S}_n \underline{S}_n = \underline{U} \tag{3.5}$$

Clearly, (3.4) and (3.5) correspond to (2.42) and (2.41) respectively, with $\underline{G} = \underline{U}$.

Having determined $\underline{S}_n$, one may then construct the network N synthesizing $\underline{S}(s)$. Terminating the network with resisitors of unit value yields an analog filter with a voltage transfer function

$$H_a(s) = \frac{1}{2}\frac{F(s)}{G(s)} \tag{3.6}$$

where it is assumed that $\underline{S}(s)$ has the form given in (3.1). Since N incorporates a minimal number of reactive elements, the analog filter may serve as a prototype for minimal WD realizations. However, we have noted that $\underline{S}_n$ already specifies an n-port adaptor, albeit in terms of a large number of parameters (there will be $(2+k)^2$ entries in $\underline{S}_n$). Terminating the ports of this adaptor with the appropriate delays (some in series with inverters), sinks, and sources yields a WD filter realization.

It is apparent that the scattering matrix synthesis method discussed within this section can also be utilized in direct WD filter design. A possible WD filter design algorithm may be as follows:

1. Given a desired analog transfer function $H_a(s)$, ($H_a(s)$ is restricted to be realizable with a doubly terminated reactance filter), let $S_{21}(s) = 2H_a(s)$ where $S_{21}(s) \equiv F(s)/G(s)$.

2. Solve the Feldtkeller equation, i.e. equation (3.2), for the unknown polynomial H(s) and with F(s), H(s), and G(s), construct a scattering matrix $\underline{S}(s)$ in the form of (3.1).

3. Utilize the synthesis algorithm discussed earlier to determine an n-port adaptor (described by $\underline{S}_n$) which when properly terminated, yields a WD filter realization with the desired system function $S_{21}(z-1/z+1)$.

3.3 PARAMETER REDUCTION

In Section 3.2 we discussed the possibility of applying the synthesis algorithm discussed in [46] to the design of WD filter realizations. Specifically, one can directly determine a scattering description $\underline{S}_n$ that describes an n-port adaptor realization. However, it was noted that the constant matrix $\underline{S}_n$ will have $(2+k)^2$ entries (where k is the degree of the filter) and as such, does not represent an attractive adaptor realization. Clearly it is desirable to express $\underline{S}_n$ in terms of a reduced number of parameters.

3.3.1 Decomposition of Scattering Matrices

A self-inverse constant matrix, say $\underline{A}$, can always be decomposed into the form [65]

$$\underline{A} = \underline{P}^T \underline{F} \underline{M} \underline{F} \underline{P} \tag{3.7a}$$

where

$$\underline{F} = \begin{bmatrix} -\underline{U} & \underline{N}^T \\ \underline{0} & \underline{U} \end{bmatrix} \qquad \underline{M} = \begin{bmatrix} -\underline{U} & \underline{0} \\ -2\underline{K} & \underline{U} \end{bmatrix} \tag{3.7b}$$

and $\underline{P}$ is a permutation matrix. The decomposition algorithm is described in Appendix A. Note that (3.7) is essentially the same decomposition as that of (2.39).

The scattering matrix $\underline{S}_n$ will be self-inverse (equation (3.5)) and thus we can apply the decomposition (3.7) to $\underline{S}_n$ to yield

$$\underline{S}_n = \underline{P}^T \underline{F}_n \underline{M}_n \underline{F}_n \underline{P} \tag{3.8}$$

where $\underline{F}_n$ and $\underline{M}_n$ are of the form given in (3.7b) and contain the submatrices $\underline{N}_n$ and $\underline{K}_n$, respectively. Furthermore, since $\underline{S}_n$ is also an orthogonal matrix (equation (3.4)) it can be shown that the following is true:

$$\underline{K}_n = (\underline{U} + \underline{N}_n \underline{N}_n^T)^{-1} \underline{N}_n \tag{3.9}$$

It is apparent that (3.9) corresponds directly to (2.38) with $\underline{G} = \underline{U}$. In fact the matrices $\underline{N}_n$ and $\underline{K}_n$ are directly analogous to the matrices $\underline{Q}_\ell$ and $\underline{K}$ introduced in Section 2.4. It follows that $\underline{N}_n$ characterizes the topology of the network represented by $\underline{S}_n$, and $\underline{K}_n$ will have the same network interpretation discussed earlier. Let the matrix $\underline{N}_n$ be of dimension $t \times \ell$. The matrix $\underline{K}_n$ will be of the same dimension and $\underline{S}_n$ will be of dimension $(t+\ell) \times (t+\ell)$.

3.3.2 Transformations on Scattering Matrices

Let the scattering matrix $\underline{S}_n$ be defined according to the convention that the reflected wave vector $\underline{b}$ and the incident wave vector $\underline{a}$ can be partitioned as

$$\underline{b} = \begin{bmatrix} \underline{b}_R \\ \underline{b}_D \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} \underline{a}_R \\ \underline{a}_D \end{bmatrix} \tag{3.10}$$

where the subscripts R and D identify those ports at which, in a WD filter realization, sources/sinks and delays are connected, respectively. (The polarity matrix $\underline{\Sigma}$ identifies those ports at which an inverter is connected in series with the delay.) We may partition the scattering matrix $\underline{S}_n$ conformable with the $\underline{b}$ and $\underline{a}$ vectors, i.e.

$$\underline{S}_n = \begin{bmatrix} \underline{S}_{11} & \underline{S}_{12} \\ \underline{S}_{21} & \underline{S}_{22} \end{bmatrix} \tag{3.11}$$

A state-variable description of the WD filter associated with $\underline{S}_n$ is given by $\{\underline{\Sigma}\,\underline{S}_{22}, \underline{\Sigma}\,\underline{S}_{21}, \underline{S}_{12}, \underline{S}_{11}\}$. It is well known [51] that if $\{\underline{A},\underline{B},\underline{C},\underline{D}\}$ is a realization of a system function $\underline{H}(z)$, and if $\underline{T}$ is any nonsingular transformation, then $\{\underline{T}^{-1}\underline{AT},\underline{T}^{-1}\underline{B},\underline{CT},\underline{D}\}$ is also a realization. Accordingly, let us consider the range of filter realizations specified by the quadruple

$$\{\, \underline{T}^{-1}\, \underline{\Sigma}\, \underline{S}_{22}\, \underline{T},\ \underline{T}^{-1}\, \underline{\Sigma}\, \underline{S}_{21},\ \underline{S}_{12}\, \underline{T},\ \underline{S}_{11} \,\} \tag{3.12}$$

Define the matrix $\underline{G}$ by

$$\underline{G}^{1/2} = \begin{bmatrix} \underline{U}_2 & \underline{0} \\ \underline{0} & \underline{T} \end{bmatrix} \tag{3.13}$$

If we have the condition

$$\underline{T}^{-1}\,\underline{\Sigma} = \underline{\Sigma}\,\underline{T}^{-1} \tag{3.14}$$

then the transformations suggested in (3.12) can be reinterpreted as the following transformation on $\underline{S}_n$:

$$\underline{G}^{-1/2}\,\underline{S}_n\,\underline{G}^{1/2} \tag{3.15}$$

Since $\underline{S}_n$ is orthogonal, it is obviously true that

$$(\underline{G}^{1/2})^T\,\underline{S}_n^T\,(\underline{G}^{-1/2})^T\,(\underline{G}^{1/2})^T\,\underline{G}^{1/2}\,\underline{G}^{-1/2}\,\underline{S}_n\,\underline{G}^{1/2} = (\underline{G}^{1/2})^T\,\underline{G}^{1/2}$$

or we may write

$$\underline{S}^T\,(\underline{G}^{1/2})^T\,\underline{G}^{1/2}\,\underline{S} = (\underline{G}^{1/2})^T\,\underline{G}^{1/2} \tag{3.16}$$

where

$$\underline{S} \triangleq \underline{G}^{-1/2}\,\underline{S}_n\,\underline{G}^{1/2} \tag{3.17}$$

Furthermore, if we also have the condition that

$$\underline{T}^T\,\underline{T} \text{ is positive definite and diagonal} \tag{3.18}$$

then (3.16) can be written as

$$\underline{S}^T\,\underline{W}\,\underline{S} = \underline{W} \tag{3.19}$$

where $\underline{W} = (\underline{G}^{1/2})^T\underline{G}^{1/2}$ is a positive definite diagonal matrix. From (3.17) it is clear that $\underline{S}$ is self-inverse. It follows that the matrix $\underline{S}$ represented in (3.19) is the scattering description of an n-port adaptor with a port reference conductance matrix $\underline{W}$. We can thus conclude that if a transformation matrix $\underline{T}$ satisfies the conditions in (3.14) and (3.18), the subsequent filter realization specified by (3.12) retains the WD structure.

Consider those cases for which the transformation matrix $\underline{T}$, and thus $\underline{G}$, is diagonal. A diagonal $\underline{T}$ obviously satisfies the conditions (3.14) and (3.18). The matrix $\underline{S}$, defined as in (3.17), is therefore a scattering matrix. Furthermore,

$$\underline{S}^T \underline{G}\, \underline{S} = \underline{G} \tag{3.20}$$

where $\underline{G}$ now represents the diagonal conductance matrix. Utilizing (3.8) in (3.17) allows us to express $\underline{S}$ as

$$\underline{S} = \underline{G}^{-1/2}\, \underline{P}^T\, \underline{F}_n\, \underline{M}_n\, \underline{F}_n\, \underline{P}\, \underline{G}^{1/2}$$

or, since $\underline{P}^T\underline{P} = \underline{U}$,

$$\underline{S} = \underline{P}^T\, \underline{P}\, \underline{G}^{-1/2}\, \underline{P}^T\, \underline{F}_n\, \underline{M}_n\, \underline{F}_n\, \underline{P}\, \underline{G}^{1/2}\, \underline{P}^T\, \underline{P} \tag{3.21}$$

Partition the diagonal matrix $\underline{P}\underline{G}^{1/2}\underline{P}^T$ conformable with $\underline{F}_n$,

$$\underline{P}\, \underline{G}^{1/2}\, \underline{P}^T = \begin{bmatrix} \underline{G}_\ell^{1/2} & \underline{0} \\ \underline{0} & \underline{G}_t^{1/2} \end{bmatrix} \tag{3.22}$$

It is straightforward to show that, given (3.21) and (3.22), $\underline{S}$ may be reexpressed as

$$\underline{S} = \underline{P}^T\, \underline{F}\, \underline{M}\, \underline{F}\, \underline{P} \tag{3.23a}$$

where

$$\underline{F} = \begin{bmatrix} -\underline{U} & \underline{N}^T \\ \underline{0} & \underline{U} \end{bmatrix} \qquad \underline{M} = \begin{bmatrix} -\underline{U} & \underline{0} \\ -2\underline{K} & \underline{U} \end{bmatrix} \tag{3.23b}$$

and

$$\underline{N} = \underline{G}_t^{1/2} \ \underline{N}_n \ \underline{G}_\ell^{-1/2} \tag{3.24a}$$

$$\underline{K} = \underline{G}_t^{-1/2} \ \underline{K}_n \ \underline{G}_\ell^{1/2} \tag{3.24b}$$

From (3.9) and (3.24) we obtain

$$\underline{K} = (\underline{G}_t + \underline{N} \ \underline{G}_\ell \ \underline{N}^T)^{-1} \ \underline{N} \ \underline{G}_\ell \tag{3.25}$$

analogous to the expression in (2.38). (The $\underline{PGP}^T$ matrix is the port conductance matrix of the adaptor when the ports are partitioned according to "link" and "tree" ports instead of source/sink and delay ports.)

The diagonal transformations given in (3.24) may be applied to reducing the number of entries in $\underline{N}_n$ that are not 0,+1, or -1. In this way the scattering matrix $\underline{S}$ can be specified in terms of a reduced number of parameters.

3.4 DESIGN EXAMPLE

Let us clarify some of the concepts introduced in the earlier sections with the aid of an example. Assume that we wish to design a WD filter based on an analog reference filter that exhibits a third-order lowpass Butterworth attenuation characteristic with a 1 r/s cutoff frequency. The desired analog transfer function is given by

$$H_a(s) = \frac{1/2}{s^3 + 2s^2 + 2s + 1} \tag{3.26}$$

A doubly terminated lossless two-port that realizes $H_a(s)$ as a voltage transfer function will have a (normalized) voltage wave tranfer function given by

$$S_{21}(s) \equiv \frac{F(s)}{G(s)} = \frac{1}{s^3 + 2s^2 + 2s + 1} \qquad (3.27)$$

Solving the Feldtkeller equation for the unknown polynomial H(s), we obtain a solution

$$H(s) = s^3 \qquad (3.28)$$

Using (3.27) and (3.28) we construct a scattering matrix $\underline{S}(s)$, i.e.

$$\underline{S}(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}\begin{bmatrix} s^3 & 1 \\ 1 & s^3 \end{bmatrix} \qquad (3.29)$$

The scattering matrix synthesis technique discussed earlier is to be applied to determining a lossless reciprocal network N that realizes $\underline{S}(s)$. With this technique, the network N is regarded as an interconnection of two subnetworks N_1 and N_2, as depicted in Fig. 3.1. The subnetwork N_1 is described by a constant scattering matrix $\underline{S}_n$ and the reactive elements contained in N_2 are all of unit value. One solution for $\underline{S}_n$ is the following:

$$\underline{S}_n = \begin{bmatrix} 1/6 & 1/6 & -\sqrt{2}/3 & -\sqrt{2}/2 & \sqrt{2}/2 \\ 1/6 & 1/6 & -\sqrt{2}/3 & \sqrt{2}/2 & \sqrt{2}/3 \\ -\sqrt{2}/3 & -\sqrt{2}/3 & 1/3 & 0 & 2/3 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 & 0 \\ \sqrt{2}/3 & \sqrt{2}/3 & 2/3 & 0 & 1/3 \end{bmatrix} \qquad (3.30)$$

where the wave vectors $\underline{b}$ and $\underline{a}$ have the form specified by (3.10). The polarity matrix $\underline{\Sigma}$ is given by

$$\underline{\Sigma} = \begin{bmatrix} -\underline{U}_2 & \underline{0} \\ \underline{0} & \underline{U}_1 \end{bmatrix} \tag{3.31}$$

A WD filter realization can be derived directly from $\underline{S}_n$ and $\underline{\Sigma}$, that is $\{\underline{\Sigma}\,\underline{S}_{22}, \underline{\Sigma}\underline{S}_{21}, \underline{S}_{12}, \underline{S}_{11}\}$ where $\underline{S}_n$ has been partitioned conformable with $\underline{b}$ and $\underline{a}$ as in (3.11).

Let us decompose $\underline{S}_n$ into the form suggested by (3.8). One such decomposition yields

$$\underline{N}_n = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 1 \end{bmatrix}, \quad \underline{K}_n = \begin{bmatrix} -\sqrt{2}/4 & \sqrt{2}/6 \\ \sqrt{2}/4 & \sqrt{2}/6 \\ 0 & 1/3 \end{bmatrix}, \quad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_2 \\ \underline{U}_3 & \underline{0} \end{bmatrix} \tag{3.32}$$

Consider the diagonal transformations described by (3.24). Choose

$$\underline{G}_\ell^{1/2} = \mathrm{diag}[\, -\sqrt{2}/2,\ \sqrt{2}/2 \,] \tag{3.33a}$$

$$\underline{G}_t^{1/2} = \mathrm{diag}[\, 1,\ 1,\ \sqrt{2}/2 \,] \tag{3.33b}$$

such that

$$\underline{N} = \underline{G}_t^{1/2}\ \underline{N}_n\ \underline{G}_\ell^{-1/2} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \tag{3.34a}$$

$$\underline{K} = \underline{G}_t^{-1/2}\ \underline{K}_n\ \underline{G}_\ell^{1/2} = \begin{bmatrix} 1/4 & 1/6 \\ -1/4 & 1/6 \\ 0 & 1/3 \end{bmatrix} \tag{3.34b}$$

The matrices $\underline{N}$ and $\underline{K}$ specify an alternate n-port adaptor realization. The adaptor scattering matrix $\underline{S}$ is defined as in (3.23). The matrix $\underline{S}$ satisfies the equation $\underline{S}^T\underline{GS} = \underline{G}$ with

$$\underline{G} = \underline{P}^T \left(\underline{G}_\ell \dotplus \underline{G}_t \right) \underline{P} = \mathrm{diag}[\ 1,\ 1,\ 1/2,\ 1/2,\ 1/2\ ] \tag{3.35}$$

Note that the matrix $\underline{G}$ in (3.35) is of the form specified in (3.13) with

$$\underline{T} = \mathrm{diag}[\ \sqrt{2}/2,\ -\sqrt{2}/2,\ \sqrt{2}/2\ ] \tag{3.36}$$

The WD filter realization based on this alternate adaptor is described by the quadruple $\{\underline{\Sigma}\,\underline{T}^{-1}\underline{S}_{22}\underline{T}, \underline{\Sigma}\,\underline{T}^{-1}\underline{S}_{21}, \underline{S}_{12}\underline{T}, \underline{S}_{11}\}$ which in turn can be characterized by only three unique parameters, i.e. the three unique entries in $\underline{K}$. In a practical filter realization, these three parameters will correspond to multiplier coefficients and accordingly will be quantized to a finite word-length. A hardware implementation of the filter will only require three multipliers.

This section has served to illustrate the WD design process proposed in Section 3.2 and developed in Section 3.3. Henceforth, for the sake of clarity, we shall refer to this design process as the "direct wave digital" (DWD) design algorithm.

3.5 CASCADE SYNTHESIS

Let $\underline{S}(s)$ represent the normalized scattering matrix description of a real lossless two-port network N. We may write

$$\underline{B}(s) = \underline{S}(s)\ \underline{A}(s) \tag{3.37}$$

where $\underline{B}(s)$ is the reflected wave vector and $\underline{A}(s)$ is the incident wave vector. The system in (3.37) may be reexpressed in the form

$$\underline{Y}(s) = \underline{T}(s)\ \underline{X}(s) \tag{3.38a}$$

where

$$\underline{Y}(s) = [\ B_1(s) \quad A_1(s)\ ]^T \qquad \underline{X}(s) = [\ A_2(s) \quad B_2(s)\ ]^T \tag{3.38b}$$

The matrix $\underline{T}(s)$ is called the transfer matrix of the network N. For real lossless two-ports, the canonic form of the transfer matrix is given by [47]

$$\underline{T}(s) = \frac{1}{F(s)} \begin{bmatrix} \sigma G(-s) & H(s) \\ \sigma H(-s) & G(s) \end{bmatrix} \tag{3.39}$$

where the parameter σ and polynomials $F(s)$, $H(s)$, and $G(s)$ are the same as those represented in (3.1). The canonic forms of $\underline{S}(s)$ and $\underline{T}(s)$ are very simply related.

Consider the cascade connection of two real lossless two-port networks N_1 and N_2 described by the transfer matrices $\underline{T}_1(s)$ and $\underline{T}_2(s)$, respectively. The wave variables at the ports of N_1 and N_2 are constrained as follows:

$$\underline{Y}_1(s) = \underline{T}_1(s)\ \underline{X}_1(s) \tag{3.40a}$$

$$\underline{Y}_2(s) = \underline{T}_2(s)\ \underline{X}_2(s) \tag{3.40b}$$

At the interconnection port we have the additional constraint

$$\underline{X}_1(s) = \underline{Y}_2(s) \tag{3.41}$$

From (3.40) and (3.41) we obtain

$$\underline{Y}_1(s) = \underline{T}_1(s)\ \underline{T}_2(s)\ \underline{X}_2(s) \tag{3.42}$$

We recognize that the matrix $\underline{T}(s)$ defined by

$$\underline{T}(s) = \underline{T}_1(s)\ \underline{T}_2(s) \tag{3.43}$$

is the transfer matrix for the two-port cascade network. From $\underline{T}(s)$ a scattering description of the cascade network, say $\underline{S}(s)$, is easily determined.

Conversely, consider a lossless network N described by the scattering matrix $\underline{S}(s)$ or the transfer matrix $\underline{T}(s)$. Factorization of $\underline{T}(s)$ into the form suggested by (3.43) allows one to represent N as a cascade connection of two subnetworks described by the transfer matrices $\underline{T}_1(s)$ and $\underline{T}_2(s)$.

It is well known that the cascade synthesis of lossless two-ports may be accomplished via factorization of the transfer matrix [50,52]. Fettweis [50] has published a thorough study of the subject which includes a relatively straightforward approach to solving the factorization problem. Let us examine briefly some aspects of the factorization process.

The transfer matrix $\underline{T}(s)$ of a lossless two-port network N, is to be represented as a product of the transfer matrices $\underline{T}_1(s)$ and $\underline{T}_2(s)$. These latter matrices are restricted to have the form

$$\underline{T}_i(s) = \frac{1}{F_i(s)} \begin{bmatrix} \sigma G_i(-s) & H_i(s) \\ \sigma H_i(-s) & G_i(s) \end{bmatrix} \qquad i = 1,2 \qquad (3.44)$$

where the polynomials $F_i(s)$, $H_i(s)$, and $G_i(s)$ have the same properties as the polynomials $F(s)$, $H(s)$, and $G(s)$. The first step in the matrix factorization is the following polynomial factorization:

$$F(s) = F_1(s)\, F_2(s) \qquad (3.45)$$

Let the degree of the network N be designated by n, that is, $n = \deg G(s)$. Given (3.2), we can state the following constraints:

$$n \geq \deg H(s) \quad \text{and} \quad n \geq \deg F(s) \qquad (3.46)$$

Consequently, given that $n_i = \deg G_i(s)$,

$$n_i \geq \deg H_i(s) \quad \text{and} \quad n_i \geq \deg F_i(s) \qquad i = 1,2 \qquad (3.47)$$

In addition,

$$n = n_1 + n_2 \qquad (3.48)$$

The transmission zeros (attenuation poles) of the network N are defined by the polynomial $F(s)$ and the degree n. The distribution of these transmission zeros among the networks represented by $\underline{T}_1(s)$ and $\underline{T}_2(s)$ is determined by (3.48) and the factorization (3.45).

Through repeated factorizations, $\underline{T}(s)$ may be represented as a product of R matrices, i.e.

$$\underline{T}(s) = \underline{T}_1(s)\ \underline{T}_2(s)\ \ldots\ \underline{T}_R(s) \tag{3.49}$$

Each transfer matrix $\underline{T}_i(s)$ defines a lossless network N_i in the cascade realization of N. Corresponding to (3.49) we have the factorization

$$F(s) = F_1(s)\ F_2(s)\ \ldots\ F_R(s) \tag{3.50}$$

The factorization in (3.50) determines the minimum degree that each section in the cascade network must have. For reciprocal lossless (reactance) networks, F(s) is a pure even or pure odd polynomial. The zeros of F(s) necessarily form a quad in the s-plane, arranged symmetrically about the origin [35]. Exceptions to this are when zeros occur as conjugate pairs on the imaginary axis, or as positive-negative pairs on the real axis. If F(s) is factored into as many polynomials as possible, the maximum degree of any polynomial will be four. It follows that any arbitrary lossless network N can be realized as a cascade of basic networks of zeroth, first, second, and fourth order.

In summary, given a normalized scattering matrix $\underline{S}(s)$ describing a lossless network N, transfer matrix factorization can be utilized to specify N as a cascade of lossless subnetworks N_i with the respective scattering descriptions $\underline{S}_i(s)$. In the context of WD cascade synthesis, each subnetwork will represent the reference filter for a WD filter section. The DWD design approach appears to be especially convenient in the reciprocal case since the section designs can be determined directly from the scattering matrices $\underline{S}_i(s)$.

3.6 WAVE DIGITAL CASCADE SYNTHESIS: THE REFLECTION-FREE PROPERTY

Transfer matrix factorization may be utilized in representing a lossless reciprocal two-port network as a cascade of two-port subnetworks. Since each two-port subnetwork is described by a normalized scattering matrix, we may apply the DWD algorithm to determine WD realizations for each subnetwork in the cascade. These WD sections are then interconnected to yield a WD filter design.

Wave digital sections are interconnected at the adaptor level. It has been demonstrated earlier, that at the interconnection of two adaptors, the following conditions are necessary:

1. port reference resistances satisfy the compatibility criterion, and
2. one of the adaptor ports is reflection-free.

Regarding the DWD design technique, the adaptor scattering matrices are determined with the condition that reference conductances (resistances) are of unit value at interconnection (source and sink) ports. This implies that the compatibility criterion is always satisfied at interconnections. However, these adaptors will not in general have reflection-free ports. In this section we discuss how this situation may be rectified.

Consider the n-port adaptor corresponding to an arbitrary WD section in a cascade realization. Let the adaptor scattering matrix, say $\underline{S}$, be specified in terms of the submatrices $\underline{N}$, $\underline{K}$, and $\underline{P}$ as suggested in (3.23). For the sake of simplicity assume the wave variables have been ordered such that $\underline{P} = \underline{U}$. Accordingly, $\underline{S}$ has the decomposition

$$\underline{S} = \underline{F}\,\underline{M}\,\underline{F} \tag{3.51}$$

The submatrix $\underline{K}$ is given by (3.25) where

$$\underline{G} = \underline{G}_{\ell} \dotplus \underline{G}_{t} \tag{3.52}$$

is the port reference conductance matrix. Assume that the i^{th} port of the adaptor is an interconnection port and let the reference resistance at that port be designated by R_i. Port i will be reflection-free if the scattering matrix element S_{ii} is identically zero. As demonstrated in Section 2.3,

$$S_{ii} = \frac{R_{di} - R_i}{R_{di} + R_i} \tag{3.53}$$

where R_{di} is the driving point resistance at port i with all other ports terminated in their reference resistances. Clearly the condition

$$R_i = R_{di}$$

guarantees the reflection-free property. Equation (3.53) can be rewritten as

$$R_{di} = R_i \frac{1 + S_{ii}}{1 - S_{ii}} \tag{3.54}$$

The reference resistance at an interconnection port of an adaptor designed via the DWD algorithm will be of unit value. In other words, the adaptor scattering matrix $\underline{S}$ will have been determined with the reference resistance $R_i = 1$. Let us determine another scattering matrix $\underline{S}'$ based on a new reference resistance at port i, R_i', given by

$$R_i' = R_{di} \tag{3.55}$$

where R_{di} is determined by (3.54). All other reference resistances remain the same as those for $\underline{S}$. Therefore, the port reference conductance matrix $\underline{G}'$ corresponding to $\underline{S}'$ has the property

$$G_{jj}' = G_{jj} \qquad j = 1,2,\ldots,n \qquad j \neq i \tag{3.56a}$$

$$G_{ii}' = 1/R_{di} \tag{3.56b}$$

The scattering matrices $\underline{S}'$ and $\underline{S}$ describe the same coupling network, hence $\underline{S}'$ has the decomposition

$$\underline{S}' = \underline{F}\,\underline{M}'\,\underline{F} \tag{3.57}$$

where $\underline{F}$, or more specifically $\underline{N}$, remains unchanged from (3.51). The matrix $\underline{M}'$ is specified by the submatrix $\underline{K}'$ which can be determined as follows:

$$\underline{K}' = (\underline{G}_t' + \underline{N}\, \underline{G}_\ell'\, \underline{N}^T)^{-1}\, \underline{N}\, \underline{G}_\ell' \tag{3.58a}$$

where

$$\underline{G}' = \underline{G}_\ell' \dotplus \underline{G}_t' \tag{3.58b}$$

Port i of the new adaptor determined by $\underline{S}'$ will be reflection-free. In order to maintain the compatibility condition, a similar process must be carried out with the interconnected network, that is, the new scattering description based on the changed reference resistance (at the interconnection port) must be determined.

The procedure described above can be utilized to ensure that, at each port-interconnection of adaptors, one adaptor has that port reflection-free. It must be emphasized that the modifications entailed in this procedure do not affect the linear behaviour of the cascade realization as viewed from the output.

The reflection-free condition places constraints on the entries of the $\underline{K}$ matrix. Suppose the matrices $\underline{N}$ and $\underline{K}$ are of dimension $t \times \ell$. With regards to the form specified in (3.51), it is easily shown that if port i is one of the first ℓ ports, $S_{ii} = 0$ requires that

$$\sum_{r=1}^{t} n_{ri}\, k_{ri} = 1/2 \tag{3.59}$$

where n_{ri} and k_{ri} are entries of $\underline{N}$ and $\underline{K}$, respectively. Alternatively, if port i is one of the last t ports, $S_{ii} = 0$ requires that

$$\sum_{r=1}^{\ell} k_{ir} n_{ir} = 1/2 \tag{3.60}$$

As will be seen later, these equations become useful when considering coefficient quantization.

3.7 BASIC ADAPTORS FOR MINIMAL RECIPROCAL FILTER SECTIONS

3.7.1 The Quantized Adaptor

The DWD synthesis technique yields digital filter designs based on n-port adaptors. The design algorithm essentially assumes infinite precision representation of the adaptor coefficients. However, for practical realizations it is necessary that these coefficients be quantized to a finite accuracy.

Nominal adaptor designs are described by a constant scattering matrix, say $\underline{S}_o$. We shall assume that wave variables are ordered such that we may partition as follows

$$\underline{b} = \begin{bmatrix} \underline{b}_R \\ \underline{b}_L \\ \underline{b}_C \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} \underline{a}_R \\ \underline{a}_L \\ \underline{a}_C \end{bmatrix} \tag{3.61}$$

where the subscripts R, L, and C identify those ports to which sources and sinks, delays in series with inverters, and delays are connected, respectively. As discussed earlier, $\underline{S}_o$ and the corresponding diagonal conductance matrix $\underline{G}_o$ satisfy the equation

$$\underset{\sim}{S}_o^T \, \underset{\sim}{G}_o \, \underset{\sim}{S}_o = \underset{\sim}{G}_o \tag{3.62}$$

Furthermore, we have shown that $\underset{\sim}{S}_o$ can be specified in terms of submatrices, that is

$$\underset{\sim}{S}_o = \underline{P}^T \, \underset{\sim}{F}_o \, \underset{\sim}{M}_o \, \underset{\sim}{F}_o \, \underline{P} \tag{3.63a}$$

where

$$\underset{\sim}{F}_o = \begin{bmatrix} -\underline{U} & \underset{\sim}{N}_o^T \\ \underline{0} & \underline{U} \end{bmatrix} \qquad \underset{\sim}{M}_o = \begin{bmatrix} -\underline{U} & \underline{0} \\ -2\underset{\sim}{K}_o & \underline{U} \end{bmatrix} \tag{3.63b}$$

Let us consider the entries of $\underset{\sim}{N}_o$ and $\underset{\sim}{K}_o$ to be the coefficients of the adaptor. This is the same approach taken by Lê [30], and Martens and Lê [31]. Quantization of the entries in $\underset{\sim}{N}_o$ and $\underset{\sim}{K}_o$ yields the matrices $\underline{N}$ and $\underline{K}$, respectively. The submatrices $\underline{N}$ and $\underline{K}$ determine a quantized adaptor described by a scattering matrix $\underline{S}$ where

$$\underline{S} = \underline{P}^T \, \underline{F} \, \underline{M} \, \underline{F} \, \underline{P} \tag{3.64}$$

and the matrices $\underline{F}$ and $\underline{M}$ have the same form specified by (3.63b) with $\underset{\sim}{N}_o$ and $\underset{\sim}{K}_o$ replaced by $\underline{N}$ and $\underline{K}$, respectively. The independent entries in $\underset{\sim}{N}_o$ and $\underset{\sim}{K}_o$ do not, in general, specify a canonic set of parameters. Consequently, $\underline{S}$ no longer satisfies an equation such as (3.62) for a diagonal port conductance matrix. However, it is always possible to determine a symmetric positive definite matrix, say $\underline{G}$, such that

$$\underline{S}^T \, \underline{G} \, \underline{S} = \underline{G} \tag{3.65}$$

Since $\underline{S}$ is very "close" to $\underline{S}_o$, we can expect $\underline{G}$ to be "close" to $\underline{G}_o$. In fact $\underline{G}$ is generally near-diagonal, that is, off-diagonal terms are much smaller in magnitude than diagonal terms.

We may express $\underline{S}$ in the form that displays the eigenvalues (see (2.40)), i.e.

$$\underline{S} = \underline{P}^T \underline{T} \underline{\Lambda} \underline{T}^{-1} \underline{P} \tag{3.66a}$$

where, given that $\underline{N}$ is of dimension $t \times \ell$,

$$\underline{T} = \begin{bmatrix} \underline{U} - \underline{N}^T \underline{K} & \underline{N}^T \\ -\underline{K} & \underline{U} \end{bmatrix} \qquad \underline{\Lambda} = \begin{bmatrix} -\underline{U}_\ell & \underline{0} \\ \underline{0} & \underline{U}_t \end{bmatrix} \tag{3.66b}$$

Substituting (3.66a) into (3.65), we obtain the equation

$$(\underline{P}^T \underline{T} \underline{\Lambda} \underline{T}^{-1} \underline{P})^T \underline{G} (\underline{P}^T \underline{T} \underline{\Lambda} \underline{T}^{-1} \underline{P}) = \underline{G}$$

or

$$\underline{\Lambda} (\underline{T}^T \underline{P} \underline{G} \underline{P}^T \underline{T}) \underline{\Lambda} = \underline{T}^T \underline{P} \underline{G} \underline{P}^T \underline{T} \tag{3.67}$$

Assume a block diagonal $\underline{PGP}^T$ matrix that is partitioned conformable with the $\underline{T}$ matrix, i.e.

$$\underline{P} \underline{G} \underline{P}^T = \begin{bmatrix} \underline{G}_\ell & \underline{0} \\ \underline{0} & \underline{G}_t \end{bmatrix} \tag{3.68}$$

Then

$$\underline{T}^T \underline{P} \underline{G} \underline{P}^T \underline{T} = \begin{bmatrix} (\underline{U} - \underline{K} \underline{N}^T) \underline{G}_\ell (\underline{U} - \underline{N}^T \underline{K}) + \underline{K}^T \underline{G}_t \underline{K} & [\underline{N} \underline{G}_\ell (\underline{U} - \underline{N}^T \underline{K}) - \underline{G}_t \underline{K}]^T \\ \underline{N} \underline{G}_\ell (\underline{U} - \underline{N}^T \underline{K}) - \underline{G}_t \underline{K} & \underline{N} \underline{G}_\ell \underline{N}^T + \underline{G}_t \end{bmatrix} \tag{3.69}$$

To satisfy (3.67) the off-diagonal terms in (3.69) must be identically zero. Therefore we require

$$\underline{N}\,\underline{G}_\ell\,(\,\underline{U} - \underline{N}^T\,\underline{K}\,) = \underline{G}_t\,\underline{K}$$

or

$$\underline{G}_t\,\underline{Q} = \underline{N}\,\underline{G}_\ell \tag{3.70a}$$

where

$$\underline{Q} \triangleq \underline{K}\,(\,\underline{U} - \underline{N}^T\,\underline{K}\,)^{-1} \tag{3.70b}$$

It follows that any $\underline{G}_\ell$ and $\underline{G}_t$ satisfying (3.70) also define a $\underline{G}$ matrix that satisfies (3.67) and thus (3.65). In other words, solutions to (3.70) are also solutions to (3.65).

In (3.70), the unknown quantities are the elements of $\underline{G}_\ell$ and $\underline{G}_t$. Given that the square matrices $\underline{G}_\ell$ and $\underline{G}_t$ are of dimension $\ell \times \ell$ and $t \times t$, respectively, there are $\ell^2 + t^2$ unknown quantities that must be determined. If we constrain ourselves to only symmetric solutions, the number of unknowns is reduced to $\ell(\ell+1)/2 + t(t+1)/2$.

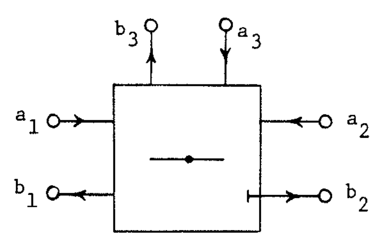
The matrix equation in (3.70) can be expressed equivalently as a system of ℓt homogeneous linear equations. For symmetric $\underline{G}_\ell$ and $\underline{G}_t$ there are more unknowns then linear equations with the consequence that the system is undetermined. This generally allows us to find solutions in which some off-diagonal terms in $\underline{G}_\ell$ and $\underline{G}_t$ are exactly zero.

3.7.2 First-Order Sections

The nominal n-port adaptor corresponding to a first-order section designed via the DWD technique is described by a 3×3 scattering matrix. The decomposition in (3.63) invariably yields adaptor realizations that are equivalent to the three-port series or parallel adaptors introduced by Fettweis [15]. The series adaptor realizations employed in the examples presented within this thesis have the following structure:

$$\underline{N}^T = [\,1 \quad 1\,]\,, \qquad \underline{K} = (k_{ij})_{2\times 1}\,, \qquad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_1 \\ \underline{U}_2 & \underline{0} \end{bmatrix} \tag{3.71}$$

It can be shown that if port 1 is reflection-free $k_{11} = 1/2$, and if port 2 is reflection-free, $k_{21} = 1/2$. The symbol for this adaptor is shown below.



The stroked output indicates that in this case, port 2 is reflection-free. The adaptor coefficients are all contained in $\underline{K}$ and they are canonic in number. Therefore, the quantized adaptor always has a diagonal reference conductance matrix associated with the adaptor scattering matrix. That this is true, is easily demonstrated through (3.70). Assume that the conductance matrix has a diagonal form with

$$\underline{G}_t = \mathrm{diag}[\, g_1,\, g_2 \,] \;, \qquad \underline{G}_\ell = \mathrm{diag}[\, g_3 \,] \tag{3.72}$$

The matrix $\underline{Q}$, defined in (3.70b), will be of dimension 2×1 and can be represented by

$$\underline{Q} = \begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} \tag{3.73}$$

Equation (3.70a) can be rewritten as

$$\underline{Q} = \underline{G}_t^{-1}\, \underline{N}\, \underline{G}_\ell \tag{3.74}$$

Substituting (3.72) and (3.73) into (3.74), we obtain

$$\begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} = \begin{bmatrix} g_3/g_1 \\ g_3/g_2 \end{bmatrix} \tag{3.75}$$

Clearly, (3.75) can always be satisfied for arbitrary entries in $\underline{Q}$ and therefore diagonal solutions for $\underline{G}_\ell$ and $\underline{G}_t$ always exist. Accordingly, one can always determine a diagonal $\underline{G}$ to satisfy (3.65).

Similar results hold for the parallel adaptor. We do not discuss them here.

3.7.3 Second-Order Sections

The reciprocal second-order sections designed with the DWD algorithm are based on a four-port adaptor. The decomposition in (3.63) invariably yields adaptor realizations that are at most, a variation of the standard WD adaptor for a Brune or Darlington C section [31]. One general adaptor realization that may be obtained via the DWD approach has the structure specified below.

$$\underline{N}^T = \begin{bmatrix} 1 & 1 \\ 1 & n \end{bmatrix}, \quad \underline{K} = (k_{ij})_{2\times2}, \quad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_2 \\ \underline{U}_2 & \underline{0} \end{bmatrix} \tag{3.76}$$

Nominal WD filter designs based on this adaptor can also be derived by conventional methods, from an analog reference filter with the structure depicted in Fig. 3.2.

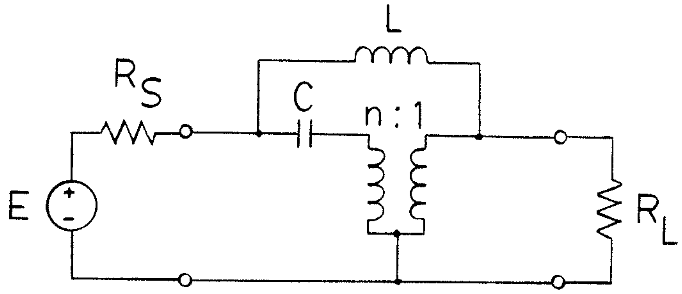


Figure 3.2: Brune reference network.

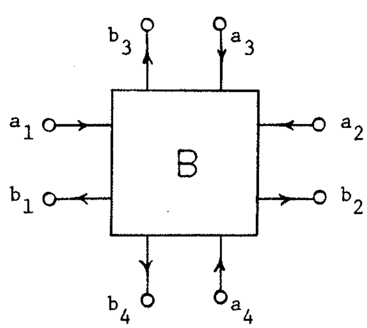
If port 1 of the adaptor specified by (3.76) is to be reflection-free, the entries in $\underline{K}$ must satisfy the following:

$$k_{11} + k_{12} = 1/2 \tag{3.77}$$

Similarly, if port 2 is to be reflection-free,

$$k_{21} + nk_{22} = 1/2 \tag{3.78}$$

The symbol for this adaptor is shown below.



In general, the quantized adaptor scattering matrix $\underline{S}$ will not satisfy (3.65) with a diagonal $\underline{G}$ matrix. To see this, assume a diagonal solution exists, that is

$$\underline{G}_t = \text{diag}[\ g_1,\ g_2\ ] \qquad \underline{G}_\ell = \text{diag}[\ g_3,\ g_4\ ] \tag{3.79}$$

The matrix $\underline{Q}$, defined in (3.70b), will be of dimension 2×2 and may be represented as

$$\underline{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{3.80}$$

Equation (3.74) becomes

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} g_3/g_1 & g_4/g_1 \\ g_3/g_2 & ng_4/g_2 \end{bmatrix} \tag{3.81}$$

There are only three independent conductance ratios. However, in (3.81) four equalities must be satisfied. From (3.81) we can derive the following constraint on the entries of $\underline{Q}$:

$$n\frac{q_{21}}{q_{22}} = \frac{q_{11}}{q_{12}} \tag{3.82}$$

Utilizing the definition for $\underline{Q}$, (3.82) can be rewritten as

$$n = \frac{k_{11}\ (\ k_{22} - \det \underline{K}\ )}{k_{12}\ k_{21} + \det \underline{K}\ (\ k_{12} + k_{21} + k_{22}\ )} \tag{3.83}$$

where $\det\underline{K}$ designates the determinant of $\underline{K}$. In general, (3.83) is not satisfied by the quantized parameters and thus a diagonal solution for $\underline{G}$ does not usually exist.

Solutions can be found with the following form:

$$\underline{G}_t = \text{diag}[\ g_1,\ g_2\ ]\ ; \qquad \underline{G}_\ell = \begin{bmatrix} g_3 & g_o \\ g_o & g_4 \end{bmatrix} \tag{3.84}$$

We can show this by again considering (3.74), i.e.

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} (g_3 + g_o)/g_1 & (g_4 + g_o)/g_1 \\ (g_3 + ng_o)/g_2 & (ng_4 + g_o)/g_2 \end{bmatrix} \tag{3.85}$$

There are now four independent ratios of conductances, ensuring that (3.85) can always be satisfied for arbitrary entries in $\underline{Q}$. Solving (3.85) yields

$$\frac{g_1}{g_2} = \frac{q_{21} + q_{22}}{nq_{12} + q_{11}} \tag{3.86a}$$

$$\frac{g_3}{g_2} = \frac{(n-1)\; q_{11}\, q_{21} + n \det \underline{Q}}{(nq_{12} + q_{11})\;(n-1)} \tag{3.86b}$$

$$\frac{g_4}{g_2} = \frac{(n-1)\; q_{12}\, q_{22} + \det \underline{Q}}{(nq_{12} + q_{11})\;(n-1)} \tag{3.86c}$$

$$\frac{g_o}{g_2} = \frac{nq_{21}\, q_{12} - q_{11}\, q_{22}}{(nq_{12} + q_{11})\;(n-1)} \tag{3.86d}$$

The conductance matrix for this adaptor is given by

$$\underline{G} = \underline{P}^T\, (\, \underline{G}_\ell \dotplus \underline{G}_t\,)\, \underline{P} = \underline{G}_t \dotplus \underline{G}_\ell \tag{3.87}$$

Similarly, one may show that solutions for $\underline{G}$ can be obtained with the alternate form

$$\underline{G}_t = \begin{bmatrix} g_1 & g_o \\ g_o & g_2 \end{bmatrix}, \qquad \underline{G}_\ell = \mathrm{diag}[\, g_3,\, g_4\,] \tag{3.88}$$

The conductance ratios, given in terms of the entries of $\underline{Q}$, are then

$$\frac{g_1}{g_2} = \frac{(n-1)\; q_{21}\, q_{22} + \det \underline{Q}}{n \det \underline{Q} + (n-1)\; q_{11}\, q_{12}} \tag{3.89a}$$

$$\frac{g_3}{g_2} = \frac{(nq_{21} + q_{11}) \det \underline{Q}}{n \det \underline{Q} + (n-1)\, q_{11}\, q_{12}} \tag{3.89b}$$

$$\frac{g_4}{g_2} = \frac{(q_{22} + q_{12}) \det \underline{Q}}{n \det \underline{Q} + (n-1)\, q_{11}\, q_{12}} \tag{3.89c}$$

$$\frac{g_o}{g_2} = \frac{q_{11}\, q_{22} - n\, q_{12}\, q_{21}}{n \det \underline{Q} + (n-1)\, q_{11}\, q_{12}} \tag{3.89d}$$

3.7.4 Fourth-Order Sections

Minimal reciprocal fourth-order WD sections will employ a six-port adaptor. The DWD technique can be used to obtain adaptor designs, or alternatively, a wave adaptor derived for the standard Darlington D section [30] may be employed. Note that for this higher-order section, a large variety of WD adaptor realizations (i.e. analog prototypes) exists. This follows from the discussion of Subsection 2.1.2; in the fourth-order case, it is possible to apply a variety of non-diagonal transformations to an initial WD adaptor design and thus obtain many significantly different adaptor realizations.

One adaptor realization obtained with the DWD algorithm, has the following general form:

$$\underline{N}^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & n_1 & n_2 \end{bmatrix}, \qquad \underline{K} = (k_{ij})_{3\times 3}\ ; \qquad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_3 \\ \underline{U}_3 & \underline{0} \end{bmatrix} \tag{3.90}$$

Here again, the quantized adaptor scattering description $\underline{S}$ does not usually satisfy (3.65) with a diagonal $\underline{G}$ matrix. However, solutions for $\underline{G}$ can be found where

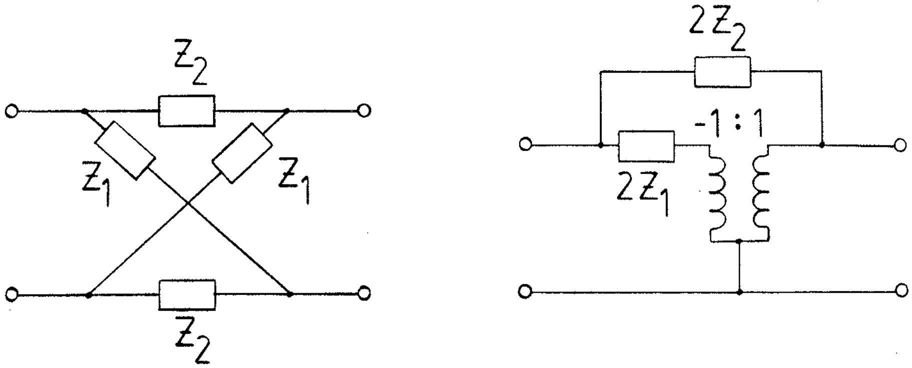
$$\underline{G}_t = \begin{bmatrix} g_1 & g_{o12} & 0 \\ g_{o12} & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix}, \qquad \underline{G}_\ell = \begin{bmatrix} g_4 & g_{o45} & g_{o46} \\ g_{o45} & g_5 & g_{o56} \\ g_{o46} & g_{o56} & g_6 \end{bmatrix} \tag{3.91}$$

This may be demonstrated by same methods employed in the previous sections. The matrix $\underline{G}$ is then given by

$$\underline{G} = \underline{G}_t \dotplus \underline{G}_\ell \tag{3.92}$$

3.7.5 Wave Digital Filters Based on the Jaumann Structure; The Fifth-Order Case

Fettweis, Levin, and Sedlmeyer [19], and Nouta [20] have derived WD adaptors for the symmetric lattice, and Jaumann structures, respectively. The Jaumann structure [47], one form of which is depicted in Fig 3.3b, is a well known lossless two-port equivalent of the symmetric lattice (Fig 3.3a) and can be used to realize symmetric networks (i.e. networks with the property $H(s) = -\sigma H(-s)$). It is known [53] that filter networks exhibiting a Butterworth, Chebyshev, inverse Chebyshev, or elliptic frequency response of odd order will be symmetric in the matched case. The Jaumann structure is therefore suitable for realizing filters with this large class of transfer functions.



(a) (b)

Figure 3.3: (a) The symmetric lattice. (b) The Jaumann structure.

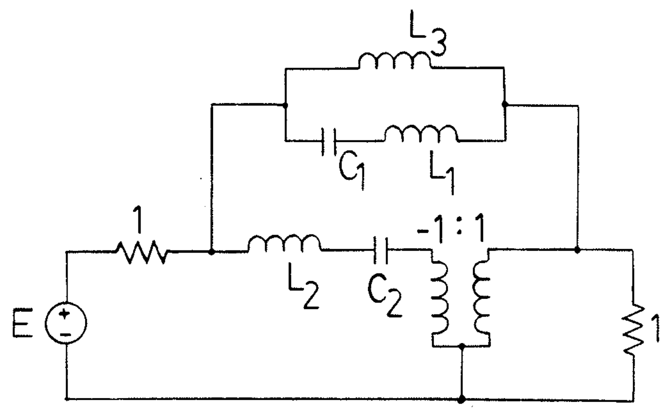


Figure 3.4: A fifth-order reference filter with the Jaumann structure.

A nominal n-port adaptor description for the fifth-order (Jaumann) reference network represented in Fig. 3.4, is specified by

$$\underline{N}_{\underset{\sim}{o}}^{T} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \qquad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_3 \\ \underline{U}_4 & \underline{0} \end{bmatrix} \tag{3.93a}$$

and

$$\underline{K}_o = (\underline{G}_{to} + \underline{N}_o \, \underline{G}_{\ell o} \, \underline{N}_o^T)^{-1} \, \underline{N}_o \, \underline{G}_{\ell o} \tag{3.93b}$$

where

$$\underline{G}_{to} = \mathrm{diag}[\ 1,\ 1,\ 1/L_1,\ 1/L_2\ ] \tag{3.93c}$$

$$\underline{G}_{\ell o} = \mathrm{diag}[\ 1/L_3,\ C_1,\ C_2\ ] \tag{3.93d}$$

Given knowledge of $\underline{G}_o$, a nominal adaptor design follows directly. The same results can also be obtained using the DWD algorithm. The form of $\underline{N}_o$ imposes constraints on the entries of $\underline{K}_o$, that is, given

$$\underline{K}_o = (k_{ij})_{4\times 3} \tag{3.94a}$$

we find

$$\begin{array}{lll} k_{11} = -k_{21} & k_{12} = -k_{22} & k_{13} = k_{23} \\ k_{33} = 0 & k_{41} = 0 & k_{42} = 0 \end{array} \tag{3.94b}$$

The coefficients in $\underline{K}_o$ must be quantized to a finite precision yielding the matrix $\underline{K}$, which with $\underline{N} = \underline{N}_o$ and $\underline{P}$, determines the quantized adaptor design. If the entries in $\underline{K}$ retain the form specified in (3.94b), one can show (Appendix B) that the adaptor scattering matrix $\underline{S}$ satisfies equation (3.65) with a matrix $\underline{G}$ of the form

$$\underline{G} = \underline{G}_t \dotplus \underline{G}_\ell \tag{3.95a}$$

$$\underline{G}_t = \mathrm{diag}[\ g_1,\ g_2,\ g_3,\ g_4\ ]\ , \qquad \underline{G}_\ell = \begin{bmatrix} g_5 & g_o & 0 \\ g_o & g_6 & 0 \\ 0 & 0 & g_7 \end{bmatrix} \tag{3.95b}$$

Chapter IV

NONLINEAR STABILITY OF WAVE DIGITAL REALIZATIONS WITH NEAR-DIAGONAL REFERENCE CONDUCTANCE MATRICES

In Chapter III we developed the DWD synthesis method for determining minimal reciprocal WD filter designs based on pseudolossless n-port adaptors. It was demonstrated that when the entries of $\underline{N}$ and $\underline{K}$ are chosen as the adaptor coefficients, these nominal WD designs will generally be noncanonic in multipliers with the consequence that reference conductance matrices associated with quantized adaptor scattering descriptions are usually nondiagonal. Clearly these filter sections are not conventional WD structures and thus the stability criterion described in Section 2.5 does not apply.

Moon [54] has shown that for a special class of filter structures, a near-diagonal quadratic form can suffice as a Lyapunov function. In this chapter we propose a general strategy, motivated in part by the concepts presented in [54], that may be employed to ensure the stability of filters based on the adaptors introduced in Chapter III.

4.1 THE CONCEPTS OF PSEUDOPOWER AND STORED PSEUDOPOWER EXTENDED TO WAVE DIGITAL STRUCTURES WITH NONDIAGONAL CONDUCTANCE MATRICES

Consider a quantized filter section of the type introduced in Chapter III. The n-port adaptor corresponding to this section is described by a scattering matrix, say $\underline{S}$, which has a reference conductance matrix $\underline{G}$ associated with it. The matrices $\underline{S}$ and $\underline{G}$ satisfy the relation $\underline{S}^T\underline{G}\underline{S}=\underline{G}$. Assume $\underline{S}$ is defined such that the wave vectors can be partitioned in the familiar manner

$$\underline{b} = \begin{bmatrix} \underline{b}_R \\ \underline{b}_D \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} \underline{a}_R \\ \underline{a}_D \end{bmatrix} \tag{4.1}$$

where the subscripts R and D identify those ports connecting sources and sinks, and delays, respectively. We shall consider only those adaptor realizations for which there exists a port reference matrix $\underline{G}$ that is block diagonal with respect to the source and sink and delay ports, i.e. when $\underline{G}$ is partitioned conformably with the wave vectors

$$\underline{G} = \begin{bmatrix} \underline{G}_R & \underline{0} \\ \underline{0} & \underline{G}_D \end{bmatrix} \tag{4.2}$$

The matrix $\underline{G}$ is also assumed to be positive-definite and symmetric.

Further assume that the ports corresponding to the delays are ordered such that we can partition as follows:

$$\underline{b}_D = \begin{bmatrix} \underline{b}_L \\ \underline{b}_C \end{bmatrix} \qquad \underline{a}_D = \begin{bmatrix} \underline{a}_L \\ \underline{a}_C \end{bmatrix} \qquad \underline{G}_D = \begin{bmatrix} \underline{G}_L & \underline{G}_o \\ \underline{G}_o^T & \underline{G}_C \end{bmatrix} \tag{4.3}$$

The subscripts L and C differentiate between those ports that have a delay with and without an inverter connected in series, respectively. The polarity matrix, when partitioned conformable with $\underline{G}_D$, is then

$$\underline{\Sigma} = \begin{bmatrix} -\underline{U} & \underline{0} \\ \underline{0} & \underline{U} \end{bmatrix} \tag{4.4}$$

Similarly, we can distinguish between the two ports connecting sources and sinks, i.e.

$$\underline{b}_R = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \underline{a}_R = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \underline{G}_R = \begin{bmatrix} g_1 & g_{o12} \\ g_{o12} & g_2 \end{bmatrix} \tag{4.5}$$

A general linear filter section is represented in Fig. 4.1. Note that the inverters are now placed after the delays.

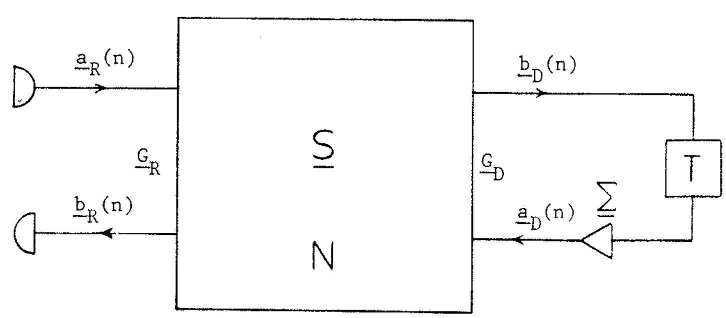


Figure 4.1: A linear wave digital filter section based on an n-port adaptor with a nondiagonal conductance matrix.

Let us extend the concepts of pseudopower and stored pseudopower to WD structures with nondiagonal port reference conductance matrices. The pseudopower $p_S(n)$ stored in the delays of a general filter section (Fig. 4.1) in the time interval (n,n+1) is defined by

$$p_S(n) = \underline{b}_D^T(n)\ \underline{G}_D\ \underline{b}_D(n) \qquad (4.6)$$

analogous to (2.53). Clearly, $p_S(n)$ is a positive definite quadratic function. The increase in pseudopower at the n^{th} time instant is given by

$$\Delta p_S(n) = p_S(n) - p_S(n-1) \qquad (4.7)$$

Since $\underline{\Sigma a}_D(n) = \underline{b}_D(n-1)$, we may also write

$$\Delta p_S(n) = \underline{b}_D^T(n)\ \underline{G}_D\ \underline{b}_D(n) - (\ \underline{\Sigma}\ \underline{a}_D(n)\ )^T\ \underline{G}_D\ \underline{\Sigma}\ \underline{a}_D(n)$$

or equivalently

$$\Delta p_S(n) = \underline{b}_D^T(n)\ \underline{G}_D\ \underline{b}_D(n) - \underline{a}_D^T(n)\ \underline{G}_D\ \underline{a}_D(n)$$
$$- \underline{a}_D^T(n)\ (\ \underline{\Sigma}\ \underline{G}_D\ \underline{\Sigma} - \underline{G}_D\ )\ \underline{a}_D(n) \qquad (4.8)$$

The pseudopower absorbed by the adaptor N at the n^{th} time instant is defined as

$$p_N(n) = \underline{a}^T(n)\ \underline{G}\ \underline{a}(n) - \underline{b}^T(n)\ \underline{G}\ \underline{b}(n) \qquad (4.9)$$

in analogy to (2.56). Introduce the matrix $\underline{\varepsilon}_D$ where

$$\underline{\varepsilon}_D = \underline{G}_D - \underline{\Sigma}\ \underline{G}_D\ \underline{\Sigma} = \begin{bmatrix} \underline{0} & 2\underline{G}_o \\ 2\underline{G}_o^T & \underline{0} \end{bmatrix} \qquad (4.10)$$

Utilizing (4.9) and (4.10), (4.8) can be rewritten as

$$\Delta p_S = -p_N + \underline{a}_D^T\ \underline{\varepsilon}_D\ \underline{a}_D - \underline{b}_R^T\ \underline{G}_R\ \underline{b}_R + \underline{a}_R^T\ \underline{G}_R\ \underline{a}_R \qquad (4.11)$$

where in order to simplify the notation, quantities are no longer expressed explicitly in terms of the discrete-time variable n. By assumption, the adaptor scattering matrix satisfies the relation $\underline{S}^T\underline{G}\underline{S} = \underline{G}$. It is easy to verify that this implies p_N is identically zero or equivalently, that N is pseudolossless. Therefore, in the absence of outside signals ($\underline{a}_R = \underline{0}$) (4.11) reduces to

$$\Delta p_S = \underline{a}_D^T \, \underline{\varepsilon}_D \, \underline{a}_D - \underline{b}_R^T \, \underline{G}_R \, \underline{b}_R \qquad (4.12)$$

From (4.12) it is evident that Δp_S is an indefinite quadratic function. Consequently, the stored pseudopower p_S cannot be utilized as in Section 2.5, to prove the stability of the filter section.

4.2 PSEUDOPASSIVE WAVE DIGITAL FILTERS

The stored pseudopower, defined as in (4.6), is not a suitable function for proving the stability of the general linear system depicted in Fig. 4.1. A similar conclusion may be drawn with respect to the subsequent nonlinear system that results from inserting quantizers before the delays of the linear filter. In this section we introduce an alternate WD realization for which the stored pseudopower is a suitable function for proving stability.

4.2.1 The Pseudopassive n-Port Adaptor

Consider a modified linear filter section in which a linear multiplier of value α is introduced into all the reflected wave branches of the pseudolossless adaptor N (Fig. 4.2). The modified n-port adaptor N_P will have a scattering description $\underline{S}_P$. Clearly $\underline{S}_P$ is related to $\underline{S}$, the scattering description of N, by the relation

$$\underline{S}_P = \alpha\, \underline{S} \tag{4.13}$$

The matrix $\underline{S}_P$ satisfies the equations

$$\underline{S}_P^T\, \underline{G}\, \underline{S}_P = \alpha^2\, \underline{G} \tag{4.14}$$

$$\underline{S}_P^T\, \underline{G} = \underline{G}\, \underline{S}_P \tag{4.15}$$

where $\underline{G}$ is the port reference matrix corresponding to N.

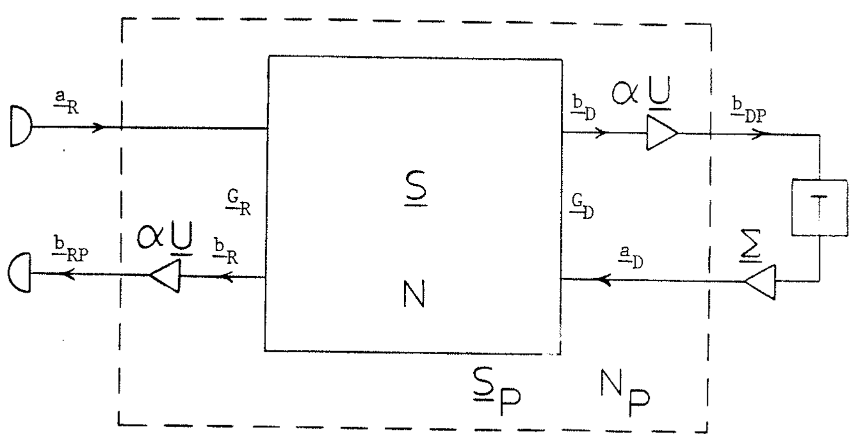


Figure 4.2: A linear wave digital filter based on a strictly pseudopassive n-port adaptor.

Constrain the value of the multiplier as follows:

$$0 < \alpha < 1 \tag{4.16}$$

(We will see later that α is normally chosen very close to unity.) It follows from (4.14) and (4.16) that

$$\underline{S}_P^T \, \underline{G} \, \underline{S}_P < \underline{G} \tag{4.17}$$

Let $\underline{G}$ be the port reference matrix associated with $\underline{S}_P$. Then from (4.9) and (4.17) it is evident that the n-port adaptor N_P is strictly pseudopassive. The reflected wave vector corresponding to the passified adaptor N_P is denoted by $\underline{b}_P$ and is related to the reflected wave vector $\underline{b}$ of the adaptor N by

$$\underline{b}_P = \alpha \, \underline{b} \tag{4.18}$$

(The incident wave vector $\underline{a}$ is common to N and N_P.) The vector $\underline{b}_P$ may be partitioned conformable with $\underline{b}$, i.e.

$$\underline{b}_P = \begin{bmatrix} \underline{b}_{RP} \\ \underline{b}_{DP} \end{bmatrix} \qquad \underline{b}_{DP} = \begin{bmatrix} \underline{b}_{LP} \\ \underline{b}_{CP} \end{bmatrix} \qquad \underline{b}_{RP} = \begin{bmatrix} b_{1P} \\ b_{2P} \end{bmatrix} \tag{4.19}$$

The pseudopower $p_{SP}(n)$ stored in the delays of the passified filter section in the time interval (n,n+1) is given by

$$p_{SP}(n) = \underline{b}_{DP}^T(n) \, \underline{G}_D \, \underline{b}_{DP}(n) \tag{4.20}$$

The increase in the pseudopower at the n^{th} time instant is then

$$\Delta p_{SP}(n) = \underline{b}_{DP}^T(n) \, \underline{G}_D \, \underline{b}_{DP}(n) - \left(\underline{\Sigma} \, \underline{a}_D(n) \right)^T \underline{G}_D \, \underline{\Sigma} \, \underline{a}_D(n) \tag{4.21}$$

The pseudopower $p_{NP}(n)$ absorbed by the adaptor N_P at the n^{th} time instant is given by

$$p_{NP}(n) = \underline{a}^T(n)\ \underline{G}\ \underline{a}(n) - \underline{b}_P^T(n)\ \underline{G}\ \underline{b}_P(n) \tag{4.22}$$

From (4.21) and (4.22) we obtain a result similar to (4.11)

$$\Delta p_{SP} = -p_{NP} + \underline{a}_D^T\ \underline{\varepsilon}_D\ \underline{a}_D - \underline{b}_{RP}^T\ \underline{G}_R\ \underline{b}_{RP} + \underline{a}_R^T\ \underline{G}_R\ \underline{a}_R \tag{4.23}$$

where $\underline{\varepsilon}_D$ has been defined earlier. Substituting (4.18) into (4.22) yields a new expression for the absorbed pseudopower, i.e.

$$p_{NP} = \underline{a}^T\ \underline{G}\ \underline{a} - \alpha^2\ \underline{b}^T\ \underline{G}\ \underline{b}$$

or

$$p_{NP} = p_N + (1 - \alpha^2)\ \underline{b}^T\ \underline{G}\ \underline{b} \tag{4.24}$$

where p_N is the pseudopower absorbed by the adaptor N. Considering (4.24) in (4.23) yields

$$\Delta p_{SP} = -p_N - (1 - \alpha^2)\ \underline{b}^T\ \underline{G}\ \underline{b} + \underline{a}_D^T\ \underline{\varepsilon}_D\ \underline{a}_D - \underline{b}_{RP}^T\ \underline{G}_R\ \underline{b}_{RP} + \underline{a}_R^T\ \underline{G}_R\ \underline{a}_R \tag{4.25}$$

Given that the n-port adaptor N is pseudolossless, (4.25) becomes under zero-input conditions

$$\Delta p_{SP} = -(1 - \alpha^2)\ \underline{b}^T\ \underline{G}\ \underline{b} + \underline{a}_D^T\ \underline{\varepsilon}_D\ \underline{a}_D - \underline{b}_{RP}^T\ \underline{G}_R\ \underline{b}_{RP} \tag{4.26}$$

Comparing (4.26) and (4.12), it is evident that the expression corresponding to the passified linear filter section has an additional negative definite term. If the contribution of this term ensures the condition

$$\Delta p_{SP} \leq -\underline{b}_{RP}^T \, \underline{G}_R \, \underline{b}_{RP} \tag{4.27}$$

then the stored pseudopower p_{SP} can be used to prove the stability of the linear system and possibly the stability of the related nonlinear filter.

For reasons that will be apparent later, we introduce the matrix $\underline{\varepsilon}_R$ defined by

$$\underline{\varepsilon}_R = \begin{bmatrix} 0 & g_{o12} \\ g_{o12} & 0 \end{bmatrix} \tag{4.28}$$

where g_{o12} is the off-diagonal term of $\underline{G}_R$. Also, we introduce the function $F(n)$ where

$$F \triangleq -p_{NP} + \underline{a}_D^T \, \underline{\varepsilon}_D \, \underline{a}_D - \underline{b}_{RP}^T \, \underline{\varepsilon}_R \, \underline{b}_{RP} + \underline{a}_R^T \, \underline{\varepsilon}_R \, \underline{a}_R \tag{4.29}$$

Utilizing the definition (4.29), (4.25) becomes

$$\Delta p_{SP} = F - \sum_{i=1}^{2} \left(b_{iP}^2 - a_i^2 \right) g_i \tag{4.30}$$

where g_i is the i^{th} diagonal term in $\underline{G}_R$ (see (4.5)). Clearly the condition

$$F(n) \leq 0 \qquad \text{for all } n \tag{4.31}$$

is sufficient to ensure that under zero-input, Δp_{SP} is negative as long as the output remains nonzero.

4.2.2 Cascade Realizations

Consider the cascade connection of K filter sections of the type depicted in Fig. 4.2. The wave vectors and the conductance matrix corresponding to the ports of the k^{th} pseudo-passive adaptor $N_P{}^k$ that connect sources and sinks are denoted by $\underline{b}_{RP}{}^k$, $\underline{a}_R{}^k$, and $\underline{G}_R{}^k$, respectively, where

$$\underline{b}_{RP}^{k} = \begin{bmatrix} b_{1P}^{k} \\ b_{2P}^{k} \end{bmatrix} \qquad \underline{a}_{R}^{k} = \begin{bmatrix} a_{1}^{k} \\ a_{2}^{k} \end{bmatrix} \qquad \underline{G}_{R}^{k} = \begin{bmatrix} g_{1}^{k} & g_{o12}^{k} \\ g_{o12}^{k} & g_{2}^{k} \end{bmatrix} \tag{4.32}$$

The adaptors corresponding to the k^{th} and $k+1^{st}$ filter sections are interconnected according to the requirement (Fig. 4.3)

$$b_{2P}^{k} = a_{1}^{k+1} \tag{4.33a}$$

$$a_{2}^{k} = b_{1P}^{k+1} \tag{4.33b}$$

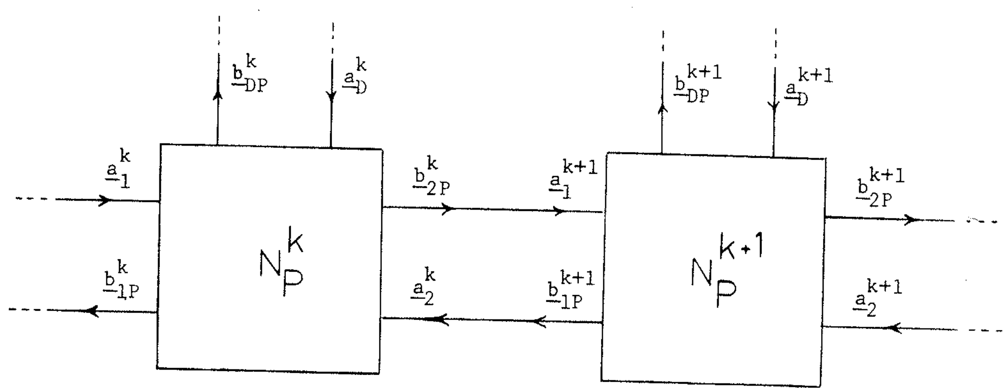


Figure 4.3: Interconnection of adaptors.

Since the port reference matrix $\underline{G}_R{}^k$ is not generally diagonal, an individual reference conductance (resistance) cannot be assigned to each interconnection port of $N_P{}^k$. The compatibility condition that was required with conventional WD sections, i.e. (2.21), does not apply here. Let us define an alternate compatibility condition for the port reference matrices $\underline{G}_R{}^k$, $k = 1,2,\ldots,K$; at an interconnection port we require (Fig. 4.3)

$$g_2^k = g_1^{k+1} \qquad k = 1,2,\ldots,K-1 \tag{4.34}$$

If the off-diagonal terms in $\underline{G}_R{}^k$ are of zero value, condition (4.34) becomes the same as that for the conventional WD sections.

Let the pseudopower stored in the delays of the k_{th} cascaded filter section in the time interval $(n,n+1)$ be represented by $p_{SP}{}^k$. The pseudopower stored in the delays of the complete filter in the time interval $(n,n+1)$ is then

$$p_{ST} = \sum_{k=1}^{K} p_{SP}^k \tag{4.35}$$

The increase in the stored pseudopower at the n^{th} time instant is given by

$$\Delta p_{ST} = \sum_{k=1}^{K} \Delta p_{SP}^k \tag{4.36}$$

or, utilizing (4.30), by

$$\Delta p_{ST} = \sum_{k=1}^{K} F^k - \sum_{k=1}^{K} \sum_{i=1}^{2} \{ (b_{iP}^k)^2 - (a_i^k)^2 \} g_i^k \tag{4.37}$$

where F^k is the function F defined for the k^{th} adaptor. With the constraints (4.33) and (4.34), the expression in (4.37) reduces to

$$\Delta p_{ST} = \sum_{k=1}^{K} F^k - \left[(b_{1P}^1)^2 g_1^1 + (b_{2P}^K)^2 g_2^K - (a_1^1)^2 g_1^1 - (a_2^K)^2 g_2^K \right] \tag{4.38}$$

It follows from (4.38) that if we guarantee

$$F^k \le 0 \qquad k = 1,2,\ldots,K \tag{4.39}$$

then under zero-input conditions, Δp_{ST} will be negative for a nonzero output.

4.2.3 Linear Stability

Consider the function F corresponding to an arbitrary filter section of the type depicted in Fig. 4.2. In general, we wish F to satisfy the condition

$$F \le 0 \tag{4.40}$$

Utilizing the definition in (4.29), the constraint in (4.40) can be reexpressed as

$$-(1-\alpha^2)\,\underline{b}^T\,\underline{G}\,\underline{b} + \underline{a}_D^T\,\underline{\varepsilon}_D\,\underline{a}_D - \underline{b}_{RP}^T\,\underline{\varepsilon}_R\,\underline{b}_{RP} + \underline{a}_R^T\,\underline{\varepsilon}_R\,\underline{a}_R \le 0 \tag{4.41}$$

or as

$$-\eta_1(1-\alpha^2)\,\underline{b}^T\,\underline{G}\,\underline{b} - \eta_2(1-\alpha^2)\,\underline{b}^T\,\underline{G}\,\underline{b} + \underline{a}_D^T\,\underline{\varepsilon}_D\,\underline{a}_D$$
$$- \underline{b}_{RP}^T\,\underline{\varepsilon}_R\,\underline{b}_{RP} + \underline{a}_R^T\,\underline{\varepsilon}_R\,\underline{a}_R \le 0 \tag{4.42}$$

where

$$\eta_1 + \eta_2 = 1 \qquad \eta_1 > 0, \quad \eta_2 > 0 \tag{4.43}$$

The first two terms in (4.42) are negative definite while the other terms are indefinite. Since N is pseudolossless, we have the property

$$\underline{b}^T \, \underline{G} \, \underline{b} = \underline{a}^T \, \underline{G} \, \underline{a} \tag{4.44}$$

Employing this relationship in (4.42) yields

$$-\eta_1(1 - \alpha^2) \, \underline{b}^T \, \underline{G} \, \underline{b} - \underline{b}_{RP}^T \, \underline{\varepsilon}_R \, \underline{b}_{RP} + \underline{a}^T \, [\, (\underline{\varepsilon}_R \dotplus \underline{\varepsilon}_D) - \eta_2(1 - \alpha^2) \, \underline{G} \,] \, \underline{a} \le 0 \tag{4.45}$$

or, applying (4.18),

$$-\eta_1 \frac{(1 - \alpha^2)}{\alpha^2} \, \underline{b}_{DP}^T \, \underline{G}_D \, \underline{b}_{DP} - \underline{b}_{RP}^T \, [\, \underline{\varepsilon}_R + \eta_1 \frac{(1 - \alpha^2)}{\alpha^2} \, \underline{G}_R \,] \, \underline{b}_{RP} + \underline{a}^T \, [\, (\, \underline{\varepsilon}_R \dotplus \underline{\varepsilon}_D) - \eta_2(1 - \alpha^2) \, \underline{G} \,] \, \underline{a} \le 0 \tag{4.46}$$

Equation (4.46) is satisfied if the following are true:

$$\eta_1 \frac{(1 - \alpha^2)}{\alpha^2} \, \underline{G}_R + \underline{\varepsilon}_R \ge 0 \tag{4.47a}$$

$$\eta_2 \, (1 - \alpha^2) \, \underline{G} - (\underline{\varepsilon}_R \dotplus \underline{\varepsilon}_D) \ge 0 \tag{4.47b}$$

that is, the matrices defined in (4.47) are positive semi-definite. Assuming (4.47) holds, we then have

$$F \le -\eta_1 \frac{(1 - \alpha^2)}{\alpha^2} \, \underline{b}_{DP}^T \, \underline{G}_D \, \underline{b}_{DP} \tag{4.48}$$

which satisfies (4.40)

If (4.48) holds for every filter section in a cascade realization, Δp_{ST} will always be nonpositive. Moreover, Δp_{ST} will be negative as long as any of the signals incident at the delays of the filter remain nonzero. By Theorem 2.2, the linear filter will be completely stable.

4.2.4 Nonlinear Stability

Consider the filter section of Fig 4.2, implemented with finite-word-length arithmetic. Quantizers are introduced before the delays in the linear system, resulting in a nonlinear filter realization (Fig. 4.4). The reflected wave vector corresponding to the nonlinear adaptor $\tilde{N}_P$ is denoted by $\underline{\tilde{b}}_P$.

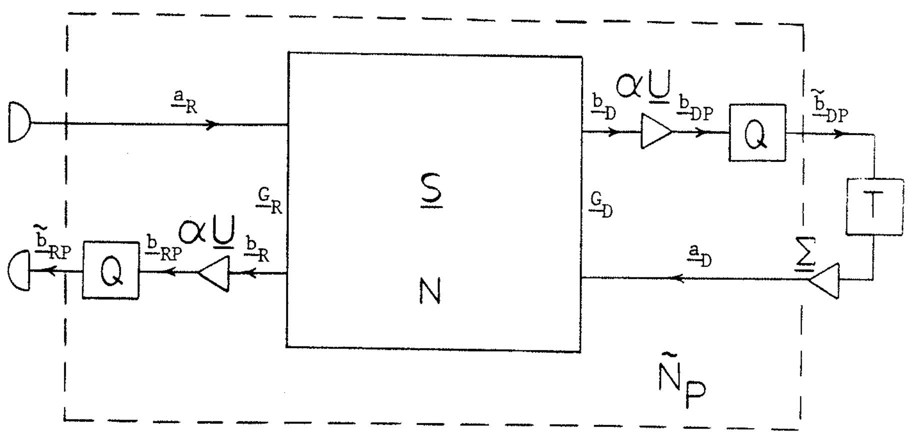


Figure 4.4: A filter section based on a pseudopassive adaptor and implemented with quantizers.

The stored pseudopower $\tilde{p}_{SP}$ and the increase in the stored pseudopower $\Delta\tilde{p}_{SP}$ for the nonlinear filter are given by, respectively,

$$\tilde{p}_{SP} = \underline{\tilde{b}}_{DP}^{T} \, \underline{G}_D \, \underline{\tilde{b}}_{DP} \tag{4.49}$$

and

$$\Delta\tilde{p}_{SP} = \underline{\tilde{b}}_{DP}^{T} \, \underline{G}_D \, \underline{\tilde{b}}_{DP} - (\underline{\Sigma} \, \underline{a}_D)^{T} \, \underline{G}_D \, \underline{\Sigma} \, \underline{a}_D \tag{4.50}$$

Similarly, the pseudopower absorbed by the nonlinear adaptor is given by

$$\tilde{p}_{NP} = \underline{a}^T \underline{G}\, \underline{a} - \underline{\tilde{b}}_P^T \underline{G}\, \underline{\tilde{b}}_P \tag{4.51}$$

From (4.50) and (4.51), we obtain a result analogous to (4.30), i.e.

$$\Delta\tilde{p}_{SP} = \tilde{F} - \sum_{i=1}^{2} (\tilde{b}_{iP}^2 - a_i^2)\, g_i \tag{4.52}$$

where

$$\tilde{F} \triangleq -\tilde{p}_{NP} + \underline{a}^T \underline{\varepsilon}_D\, \underline{a}_D - \underline{\tilde{b}}_{RP}^T \underline{\varepsilon}_R\, \underline{\tilde{b}}_{RP} + \underline{a}_R^T \underline{\varepsilon}_R\, \underline{a}_R \tag{4.53}$$

A cascade of K of these nonlinear filter sections will have a total stored pseudopower $\tilde{p}_{ST}$ and the increase in the stored pseudopower is given by

$$\Delta\tilde{p}_{ST} = \sum_{k=1}^{K} \Delta\tilde{p}_{SP}^{k} \tag{4.54}$$

where the superscript k is used to identify quantities corresponding to the k^{th} section. The expression in (4.54) will reduce to a form similar to (4.38) (the constraints in (4.33) also hold for the quantized variables), i.e.

$$\Delta\tilde{p}_{ST} = \sum_{k=1}^{K} \tilde{F}^k - \left[(\tilde{b}_{1P}^1)^2\, g_1^1 + (\tilde{b}_{2P}^K)^2\, g_2^K - (a_1^1)^2\, g_1^1 - (a_2^K)^2\, g_2^K \right] \tag{4.55}$$

Clearly, we desire the condition

$$\tilde{F}^k \leq 0 \qquad k = 1,2,\ldots,K \tag{4.56}$$

The function $\tilde{F}$ corresponding to a nonlinear adaptor $\tilde{N}_P$ obviously satisfies the relation

$$\tilde{F} = F + (\tilde{F} - F) \tag{4.57}$$

where F has been defined earlier for the linear subnetwork N_P (Fig. 4.4). Given that (4.48) holds, F is negative as long as the linear response $\underline{b}_{DP}$ remains nonzero. Accordingly, if we guarantee that for all nonzero $\underline{b}_{DP}$

$$\tilde{F} - F < -F \tag{4.58}$$

then $\tilde{F}$ will also be negative for nonzero $\underline{b}_{DP}$. Since signal quantizations within $\tilde{N}_P$ will always be made such that the following is true:

$$\underline{b}_{DP} = \underline{0} \Rightarrow \underline{\tilde{b}}_{DP} = \underline{0} \tag{4.59}$$

$\tilde{F}$ will also remain negative for the condition $\underline{\tilde{b}}_{DP} \neq \underline{0}$. If (4.58) holds for every section in a cascade realization, then

$$\underline{\tilde{b}}^k_{DP} \neq \underline{0} \quad \text{for any } k,\ k = 1,2,\ldots,K \quad \Rightarrow \quad \Delta\tilde{p}_{ST} < 0 \tag{4.60}$$

and by Theorem 2.2, the cascade realization will be completely stable.

Assuming (4.48) holds, (4.58) is satisfied if, for all nonzero $\underline{b}_{DP}$,

$$\tilde{F} - F < \eta_1 \frac{(1 - \alpha^2)}{\alpha^2} \underline{b}^T_{DP} \underline{G}_D \underline{b}_{DP} \tag{4.61}$$

Utilizing the definitions for F and $\tilde{F}$, we may reexpress the condition in (4.61) as

$$-\tilde{p}_{NP} + p_{NP} - \underline{\tilde{b}}_{RP}^{T}\, \underline{\varepsilon}_{R}\, \underline{\tilde{b}}_{RP} + \underline{b}_{RP}^{T}\, \underline{\varepsilon}_{R}\, \underline{b}_{RP} < \eta_1 \frac{(1 - \alpha^2)}{\alpha^2}\, \underline{b}_{DP}^{T}\, \underline{G}_{D}\, \underline{b}_{DP}$$

or

$$\underline{\tilde{b}}_{DP}^{T}\, \underline{G}_{D}\, \underline{\tilde{b}}_{DP} - \underline{b}_{DP}^{T}\, \underline{G}_{D}\, \underline{b}_{DP} + \underline{\tilde{b}}_{RP}^{T}\, (\underline{G}_{R} - \underline{\varepsilon}_{R})\, \underline{\tilde{b}}_{RP} - \underline{b}_{RP}^{T}\, (\underline{G}_{R} - \underline{\varepsilon}_{R})\, \underline{b}_{RP} < \eta_1 \frac{(1 - \alpha^2)}{\alpha^2}\, \underline{b}_{DP}^{T}\, \underline{G}_{D}\, \underline{b}_{DP} \qquad (4.62)$$

Note that $\underline{G}_R - \underline{\varepsilon}_R$ is a positive definite diagonal matrix. If quantizations are performed such that

$$|\tilde{b}_{RPi}| \le |b_{RPi}| \qquad i = 1,2 \qquad (4.63)$$

then

$$\underline{\tilde{b}}_{RP}^{T}\, (\underline{G}_{R} - \underline{\varepsilon}_{R})\, \underline{\tilde{b}}_{RP} - \underline{b}_{RP}^{T}\, (\underline{G}_{R} - \underline{\varepsilon}_{R})\, \underline{b}_{RP} \le 0 \qquad (4.64)$$

The constraint in (4.62) becomes

$$\left[\, 1 + \eta_1 \frac{(1 - \alpha^2)}{\alpha^2} \,\right] \underline{b}_{DP}^{T}\, \underline{G}_{D}\, \underline{b}_{DP} - \underline{\tilde{b}}_{DP}^{T}\, \underline{G}_{D}\, \underline{\tilde{b}}_{DP} > 0 \qquad (4.65)$$

Define the scalar quantities γ_1 and γ_2 by

$$\gamma_1 = \eta_1 \frac{(1 - \alpha^2)}{\alpha^2} \qquad (4.66a)$$

$$\gamma_2 = \eta_2\, (1 - \alpha^2) \qquad (4.66b)$$

Equations (4.47) and (4.65) can then be expressed as

$$\gamma_1\, \underline{G}_{R} + \underline{\varepsilon}_{R} \ge 0 \qquad (4.67a)$$

$$\gamma_2\, \underline{G} - (\underline{\varepsilon}_{R} \dotplus \underline{\varepsilon}_{D}) \ge 0 \qquad (4.67b)$$

$$(1 + \gamma_1)\, \underline{b}_{DP}^{T}\, \underline{G}_{D}\, \underline{b}_{DP} - \underline{\tilde{b}}_{DP}^{T}\, \underline{G}_{D}\, \underline{\tilde{b}}_{DP}^{T} > 0 \qquad (4.67c)$$

The analysis presented within this section has shown that if the constraints (4.63) and (4.67) are satisfied for each nonlinear filter section in a cascade realization, then the overall filter will be completely stable.

4.3 NONLINEAR STABILITY: BOUNDS ON THE ENTRIES OF THE CONDUCTANCE MATRIX

In Section 4.2 we introduced a general filter section based on a passified n-port adaptor. It was shown that the nonlinear stability of the filter section or a cascade of such sections is guaranteed if the constraints (4.63) and (4.67) are satisfied. In this section, the requirements specified by (4.67) are expressed in terms of constraints on the entries of the port conductance matrix $\underline{G}$.

4.3.1 General Bounds

Consider first the requirement of (4.67a), that is, the matrix $\gamma_1\underline{G}_R + \underline{\varepsilon}_R$ be positive semi-definite. Utilizing the general definitions for the matrices $\underline{\varepsilon}_R$ and $\underline{G}_R$, we obtain

$$\gamma_1\underline{G}_R + \underline{\varepsilon}_R = \begin{bmatrix} \gamma_1 g_1 & (\gamma_1+1)g_{o12} \\ (\gamma_1+1)g_{o12} & \gamma_1 g_2 \end{bmatrix} \qquad (4.68)$$

A symmetric matrix is positive semi-definite if all of the principal minors of the matrix are nonnegative [51]. Therefore, the above matrix is positive semi-definite if the following are true:

$$\gamma_1 g_1 \geq 0 , \qquad \gamma_1 g_2 \geq 0 \qquad (4.69a)$$

$$\gamma_1^2 g_1 g_2 - (\gamma_1+1)^2 g_{o12}^2 \geq 0 \tag{4.69b}$$

The constraints in (4.69a) are always satisfied, since by assumption, $\underline{G}_R$ is positive definite. Therefore (4.67a) is satisfied if (4.69b) holds, or equivalently

$$\frac{(g_1 \; g_2)^{1/2}}{|g_{o12}|} \geq \frac{1 + \gamma_1}{\gamma_1} \tag{4.70}$$

The requirement (4.67b) can be expressed as the two seperate constraints

$$\gamma_2 \, \underline{G}_R - \underline{\varepsilon}_R \geq 0 \tag{4.71a}$$

$$\gamma_2 \, \underline{G}_D - \underline{\varepsilon}_D \geq 0 \tag{4.71b}$$

The condition in (4.71a) is similar in form to that of (4.67a). Accordingly, (4.71a) is satisfied if the following holds:

$$\frac{(g_1 \; g_2)^{1/2}}{|g_{o12}|} \geq \frac{1 - \gamma_2}{\gamma_2} \tag{4.72}$$

The conditions specified by (4.71b) and (4.67c) involve the submatrix $\underline{G}_D$. The form of $\underline{G}_D$ is dependent on the order of the realization. Results will be derived for the general second and fourth-order filter sections. We also consider the special fifth-order section introduced in Chapter III that is based on the Jaumann structure.

4.3.2 The Second-Order Case

A second-order section of the type introduced in Chapter III, will have a 2×2 $\underline{G}_D$ matrix associated with the n-port adaptor. Let $\underline{G}_D$ have a form consistent with (4.3), i.e.

$$\underline{G}_D = \begin{bmatrix} g_3 & g_{o34} \\ g_{o34} & g_4 \end{bmatrix} \tag{4.73}$$

By assumption, $\underline{\Sigma}$ = diag[-1, 1]. Therefore, by (4.10)

$$\underline{\varepsilon}_D = \begin{bmatrix} 0 & 2g_{o34} \\ 2g_{o34} & 0 \end{bmatrix} \tag{4.74}$$

Utilizing (4.73) and (4.74), we obtain

$$\gamma_2 \underline{G}_D - \underline{\varepsilon}_D = \begin{bmatrix} \gamma_2 g_3 & (\gamma_2 - 2) g_{o34} \\ (\gamma_2 - 2) g_{o34} & \gamma_2 g_4 \end{bmatrix} \tag{4.75}$$

Condition (4.71b) requires that this matrix be positive semi-definite or, applying the arguments used earlier, that equivalently

$$\frac{(g_3 \, g_4)^{1/2}}{|g_{o34}|} \geq \frac{2 - \gamma_2}{\gamma_2} \tag{4.76}$$

Next consider condition (4.67c) for the second-order case. Up to this point the output of a quantizer with input x has been denoted by $\tilde{x}$. Alternatively, the output of a quantizer can be represented as x+e where e represents the error introduced by the quantization operation (Fig. 4.5). Assume henceforth that quantizers implement a signal modification scheme that has the following properties (with respect to Fig. 4.5):

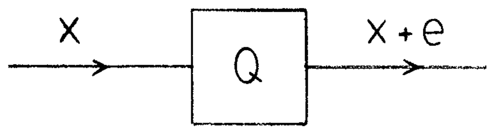


Figure 4.5: Quantizer model.

$$\mathrm{sgn}(\ e\ ) = -\mathrm{sgn}(\ x\ ) \tag{4.77a}$$

$$\mathrm{sgn}(\ x+e\ ) = \mathrm{sgn}(\ x\ ) \tag{4.77b}$$

Note that this quantization characteristic satisfies the conditions specified by (4.63). The condition in (4.67c) is satisfied for the second-order section if, for all nonzero $\underline{x}$,

$$(1 + \gamma_1)\ \underline{x}^T\ \underline{G}_D\ \underline{x} - \underline{\tilde{x}}^T\ \underline{G}_D\ \underline{\tilde{x}} > 0 \tag{4.78}$$

Let the vector $\underline{\tilde{x}}$ be represented as $\underline{x}+\underline{e}$ where $\underline{e}$ is the error vector. The expression in (4.78) becomes

$$(1 + \gamma_1)\ \underline{x}^T\ \underline{G}_D\ \underline{x} - (\underline{x} + \underline{e})^T\ \underline{G}_D\ (\underline{x} + \underline{e}) > 0 \tag{4.79}$$

Given (4.77), (4.79) is true if the entries of $\underline{G}_D$ satisfy the following criteria

$$\frac{(\ g_3\ g_4\ )^{1/2}}{|g_{o34}|} > \frac{1 + \gamma_1}{\gamma_1}\ , \qquad \frac{g_3}{|g_{o34}|} \geq 2\ , \qquad \frac{g_4}{|g_{o34}|} \geq 2 \tag{4.80}$$

This result is derived in Appendix C. It follows that the constraint in (4.67c) can be replaced with those in (4.77) and (4.80).

4.3.3 Block Diagonal Conductance Matrices

In some instances the matrix $\underline{G}_D$ may be block diagonal. Consider for example,

$$\underline{G}_D = \mathrm{diag}[\, \underline{G}_1, \underline{G}_2, \ldots, \underline{G}_N \,] \tag{4.81}$$

Examining the constraint specified in (4.71b), it should be obvious that if (4.81) is true, then (4.71b) can be replaced by the equivalent requirement

$$\gamma_2 \, \underline{G}_i - \underline{\varepsilon}_i \geq 0 \qquad i = 1,2,\ldots,N \tag{4.82}$$

where

$$\underline{\varepsilon}_i = \underline{G}_i - \underline{\Sigma}_i \, \underline{G}_i \, \underline{\Sigma}_i \tag{4.83}$$

and $\underline{\Sigma}$ has been partitioned conformably with $\underline{G}_D$, i.e.

$$\underline{\Sigma} = \mathrm{diag}[\, \underline{\Sigma}_1, \underline{\Sigma}_2, \ldots, \underline{\Sigma}_N \,] \tag{4.84}$$

Also consider the requirement of (4.67c). If (4.81) is true, then (4.67c) is satisfied if, for all nonzero $\underline{x}$,

$$(1 + \gamma_1) \, \underline{x}^T \, \underline{G}_i \, \underline{x} - \underline{\tilde{x}}^T \, \underline{G}_i \, \underline{\tilde{x}} > 0 \qquad i = 1,2,\ldots,N \tag{4.85}$$

If the submatrix $\underline{G}_j$ is diagonal, then by (4.83) $\underline{\varepsilon}_j = \underline{0}$ and (4.82) is immediately satisfied for the case i=j. Furthermore, (4.85) will also be satisfied for the case i=j since, by assumption, the signal quantization characteristic satisfies (4.77). Therefore, the diagonal submatrices of $\underline{G}_D$ automatically satisfy the stability constraints and need not be considered.

4.3.4 The Fourth-Order Case

A fourth-order section of the type introduced in Section 3.5, will have a 4×4 $\underline{G}_D$ matrix associated with the n-port adaptor. The solution for $\underline{G}_D$ derived in Chapter III has the form

$$\underline{G}_D = \mathrm{diag}[\ \underline{G}_1,\ \underline{G}_2\ ] \tag{4.86a}$$

where

$$\underline{G}_1 = g_3 \qquad \underline{G}_2 = \begin{bmatrix} g_4 & g_{o45} & g_{o46} \\ g_{o45} & g_5 & g_{o56} \\ g_{o46} & g_{o56} & g_6 \end{bmatrix} \tag{4.86b}$$

The polarity matrix for this filter section is given by

$$\underline{\Sigma} = \mathrm{diag}[\ -1,\ -1,\ 1,\ 1\ ]$$

Since $\underline{G}_1$ is diagonal, we need only consider the matrix $\underline{G}_2$. Utilizing the definition (4.83), we obtain

$$\gamma_2\ \underline{G}_2 - \underline{\varepsilon}_2 = \begin{bmatrix} \gamma_2 g_4 & (\gamma_2-2)g_{o45} & (\gamma_2-2)g_{o46} \\ (\gamma_2-2)g_{o45} & \gamma_2 g_5 & \gamma_2 g_{o56} \\ (\gamma_2-2)g_{o46} & \gamma_2 g_{o56} & \gamma_2 g_6 \end{bmatrix} \tag{4.87}$$

To satisfy (4.71b) the above matrix must be positive semi-definite. Let us express the matrix as the sum of three matrices, that is

$$\gamma_2 \underline{G}_2 - \underline{\varepsilon}_2 = \underline{M}_1 + \underline{M}_2 + \underline{M}_3 \tag{4.88a}$$

where

$$\underline{M}_1 = \begin{bmatrix} (\gamma_2 g_4)/2 & (\gamma_2-2)g_{o45} & 0 \\ (\gamma_2-2)g_{o45} & (\gamma_2 g_5)/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{M}_2 = \begin{bmatrix} (\gamma_2 g_4)/2 & 0 & (\gamma_2-2)g_{o46} \\ 0 & 0 & 0 \\ (\gamma_2-2)g_{o46} & 0 & (\gamma_2 g_6)/2 \end{bmatrix} \tag{4.88b}$$

$$\underline{M}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (\gamma_2 g_5)/2 & \gamma_2 g_{o56} \\ 0 & \gamma_2 g_{o56} & (\gamma_2 g_6)/2 \end{bmatrix}$$

If $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$ are each positive semi-definite, then the matrix in (4.87) will also be. Employing the same arguments applied in Subsection 4.3.1, we find that (4.71b) is satisfied if

$$\frac{(g_4 \; g_5)^{1/2}}{|g_{o45}|} \geq \frac{2(2 - \gamma_2)}{\gamma_2} \tag{4.89a}$$

$$\frac{(g_4 \; g_6)^{1/2}}{|g_{o46}|} \geq \frac{2(2 - \gamma_2)}{\gamma_2} \tag{4.89b}$$

$$\frac{(g_5 \; g_6)^{1/2}}{|g_{o56}|} \geq 2 \tag{4.89c}$$

Now consider condition (4.67c). We require that, for all nonzero $\underline{x}$,

$$(1 + \gamma_1)\ \underline{x}^T\ \underline{G}_2\ \underline{x} - \tilde{\underline{x}}^T\ \underline{G}_2\ \tilde{\underline{x}} > 0 \tag{4.90}$$

Given that $\underline{x}=[x_1\ \ x_2\ \ x_3]^T$, introduce the following vectors:

$$\underline{x}_{12} = [\ x_1\ \ x_2\ ]^T \qquad \underline{x}_{13} = [\ x_1\ \ x_3\ ]^T \qquad \underline{x}_{23} = [\ x_2\ \ x_3\ ]^T \tag{4.91}$$

Utilizing the definitions in (4.91), (4.90) can be reexpressed as

$$\begin{aligned} &(1 + \gamma_1)\ \underline{x}_{12}^T\ \underline{W}_1\ \underline{x}_{12} - \tilde{\underline{x}}_{12}^T\ \underline{W}_1\ \tilde{\underline{x}}_{12} \\ &+ (1 + \gamma_1)\ \underline{x}_{13}^T\ \underline{W}_2\ \underline{x}_{13} - \tilde{\underline{x}}_{13}^T\ \underline{W}_2\ \tilde{\underline{x}}_{13} \\ &+ (1 + \gamma_1)\ \underline{x}_{23}^T\ \underline{W}_3\ \underline{x}_{23} - \tilde{\underline{x}}_{23}^T\ \underline{W}_3\ \tilde{\underline{x}}_{23} > 0 \end{aligned} \tag{4.92a}$$

where

$$\underline{W}_1 = \begin{bmatrix} g_4/2 & g_{o45} \\ g_{o45} & g_5/2 \end{bmatrix} \qquad \underline{W}_2 = \begin{bmatrix} g_4/2 & g_{o46} \\ g_{o46} & g_6/2 \end{bmatrix} \qquad \underline{W}_3 = \begin{bmatrix} g_5/2 & g_{o56} \\ g_{o56} & g_6/2 \end{bmatrix} \tag{4.92b}$$

Clearly, if for nonzero $\underline{x}_{12}$, $\underline{x}_{13}$, and $\underline{x}_{23}$ we have

$$(1 + \gamma_1)\ \underline{x}_{12}^T\ \underline{W}_1\ \underline{x}_{12} - \tilde{\underline{x}}_{12}^T\ \underline{W}_1\ \tilde{\underline{x}}_{12} > 0 \tag{4.93a}$$

$$(1 + \gamma_1)\ \underline{x}_{13}^T\ \underline{W}_2\ \underline{x}_{13} - \tilde{\underline{x}}_{13}^T\ \underline{W}_2\ \tilde{\underline{x}}_{13} > 0 \tag{4.93b}$$

$$(1 + \gamma_1)\ \underline{x}_{23}^T\ \underline{W}_3\ \underline{x}_{23} - \tilde{\underline{x}}_{23}\ \underline{W}_3\ \tilde{\underline{x}}_{23} > 0 \tag{4.93c}$$

then (4.92) will also be satisfied. Furthermore, notice that each equation in (4.93) has essentially the same form as that of (4.78). Therefore we can apply the results derived for the second-order case. Accordingly, the requirement described by (4.67c) will be satisfied for this fourth-order section if

$$\frac{(g_4 g_5)^{1/2}}{|g_{o45}|} > \frac{2(1+\gamma_1)}{\gamma_1}, \quad \frac{g_4}{|g_{o45}|} \geq 4, \quad \frac{g_5}{|g_{o45}|} \geq 4 \tag{4.94a}$$

$$\frac{(g_4 g_6)^{1/2}}{|g_{o46}|} > \frac{2(1+\gamma_1)}{\gamma_1}, \quad \frac{g_4}{|g_{o46}|} \geq 4, \quad \frac{g_6}{|g_{o46}|} \geq 4 \tag{4.94b}$$

$$\frac{(g_5 g_6)^{1/2}}{|g_{o56}|} > \frac{2(1+\gamma_1)}{\gamma_1}, \quad \frac{g_5}{|g_{o56}|} \geq 4, \quad \frac{g_6}{|g_{o56}|} \geq 4 \tag{4.94c}$$

and the signal quantizations satisfy (4.77).

4.3.5 Fifth-Order Filters Based on the Jaumann Structure

The special fifth-order section introduced in Section 3.7 has a 5×5 $\underline{G}_D$ matrix associated with the n-port adaptor. As discussed in Subsection 3.7.5, a general solution for $\underline{G}_D$ has the form

$$\underline{G}_D = \text{diag}[\, \underline{G}_1, \underline{G}_2, \underline{G}_3 \,] \tag{4.95a}$$

where

$$\underline{G}_1 = \text{diag}[\, g_3, g_4 \,], \quad \underline{G}_2 = \begin{bmatrix} g_5 & g_{o56} \\ g_{o56} & g_6 \end{bmatrix}, \quad \underline{G}_3 = g_7 \tag{4.95b}$$

Since $\underline{G}_1$ and $\underline{G}_3$ are diagonal, it is only necessary to consider the matrix $\underline{G}_2$. Utilizing the definition (4.83), we obtain

$$\underline{\varepsilon}_2 = \begin{bmatrix} 0 & 2g_{o56} \\ 2g_{o56} & 0 \end{bmatrix} \tag{4.96}$$

Observe that $\underline{G}_2$ and $\underline{\varepsilon}_2$ have the same form as the matrices $\underline{G}_D$ and $\underline{\varepsilon}_D$ ((4.73) and (4.74)) corresponding to the second-order section. Consequently the results derived for that case can also be applied here. Accordingly, (4.71b) and (4.67c) are satisfied for this special fifth-order section if

$$\frac{(g_5 \, g_6)^{1/2}}{|g_{o56}|} \geq \frac{2 - \gamma_2}{\gamma_2} \tag{4.97a}$$

$$\frac{(g_5 \, g_6)^{1/2}}{|g_{o56}|} > \frac{1 + \gamma_1}{\gamma_1} \, , \quad \frac{g_5}{|g_{o56}|} \geq 2 \, , \quad \frac{g_6}{|g_{o56}|} \geq 2 \tag{4.97b}$$

and the signal quantizations satisfy (4.77).

4.4 SIMPLIFIED STABILITY CRITERIA

Let us summarize the results developed in the previous sections. Nominal n-port adaptor designs, possibly derived via the DWD algorithm, are described by a constant scattering matrix that will have a diagonal port reference conductance matrix associated with it. If the entries of the submatrices $\underline{N}$ and $\underline{K}$ are chosen to be the coefficients of the adaptor, coefficient quantization generally yields a digital structure that is no longer the image of an analog prototype. Specifically, the scattering matrix description of the quantized adaptor will be based on a nondiagonal reference conductance matrix.

In Section 4.1 the concepts of pseudopower and stored pseudopower were extended to filter structures based on these quantized adaptors. However, it was shown that unlike conventional WD realizations, the stored pseudopower cannot be used to demonstrate linear or nonlinear stability. We then proposed a modified adaptor that is derived by placing passivity multipliers in the reflected wave branches of the original quantized adaptor. The modified adaptor was demonstrated to be strictly pseudopassive. Furthermore it was shown that for this class of adaptor, the stored pseudopower can be utilized to prove the stability of the associated nonlinear filter if certain conditions are satisfied. In particular, for each filter section in a cascade realization, constraints are placed on

1. the entries of the reference conductance matrix and the value of the passivity multiplier α (see (4.67)), and
2. the signal quantizations performed within the nonlinear filter (see (4.63) and (4.67c)).

In Section 4.3, these constraints were interpreted as bounds on the entries of the conductance matrix and as explicit conditions (i.e. (4.77)) on the signal quantization characteristic. In this section, the bounds on the entries of the conductance matrix are expressed in a more simplified and practical form.

4.4.1 Second-Order Sections

Let us review the results presented earlier for the second-order section. The reference conductance matrix associated with the passified adaptor has the form

$$\underline{G} = \left[\begin{array}{cc|cc} g_1 & g_{o12} & & \\ g_{o12} & g_2 & \multicolumn{2}{c}{\underline{0}} \\ \hline & & g_3 & g_{o34} \\ \multicolumn{2}{c|}{\underline{0}} & g_{o34} & g_4 \end{array}\right] \tag{4.98}$$

In order to guarantee the stability of a cascade realization employing such a filter section, the constraints specified by (4.70), (4.72), (4.76), and (4.80) must all be satisfied. Utilizing the definitions for γ_1 and γ_2, these constraints are listed again below.

$$\frac{(g_1 g_2)^{1/2}}{|g_{o12}|} \geq 1 + \frac{\alpha^2}{\eta_1(1 - \alpha^2)} \tag{4.99a}$$

$$\frac{(g_1 g_2)^{1/2}}{|g_{o12}|} \geq \frac{1}{\eta_2(1 - \alpha^2)} - 1 \tag{4.99b}$$

$$\frac{(g_3 g_4)^{1/2}}{|g_{o34}|} \geq \frac{2}{\eta_2(1 - \alpha^2)} - 1 \tag{4.100a}$$

$$\frac{(g_3 g_4)^{1/2}}{|g_{o34}|} > 1 + \frac{\alpha^2}{\eta_1(1 - \alpha^2)} \tag{4.100b}$$

$$\frac{g_3}{|g_{o34}|} \geq 2 \;, \qquad \frac{g_4}{|g_{o34}|} \geq 2 \tag{4.100c}$$

As has been discussed in Chapter III, the G matrix associated with the adaptor of the second-order section can be determined with either $g_{o12}= 0$ or $g_{o34}= 0$. If $g_{o12}= 0$, the constraints in (4.99) may be ignored and similarly if $g_{o34}= 0$, (4.100) no longer applies.

Assume $g_{o12}= 0$ and therefore consider (4.100). The parameters η_1 and η_2 are as yet unspecified and may be assigned values arbitrarily, subject to (4.43). It is clearly desirable to choose these values such that the maximum value of the bounds in (4.100a,b) is minimized. Since in general $\alpha \simeq 1$, the constant terms in these bounds will be of negligible significance. A close to optimal choice for the values of η_1 and η_2 is therefore 1/3 and 2/3, respectively. The lower bound in (4.100a) becomes

$$\frac{3}{1 - \alpha^2} - 1 < \frac{3}{1 - \alpha^2} \tag{4.101}$$

and the lower bound in (4.100b) becomes

$$1 + \frac{3\alpha^2}{1 - \alpha^2} < \frac{3}{1 - \alpha^2} \tag{4.102}$$

Therefore, we may replace (4.100) with the sufficient conditions

$$\frac{(g_3 \, g_4 \,)^{1/2}}{|g_{o34}|} \geq \frac{3}{1 - \alpha^2} \, , \quad \frac{g_3}{|g_{o34}|} \geq 2, \quad \frac{g_4}{|g_{o34}|} \geq 2, \quad g_{o12} = 0 \tag{4.103}$$

Alternatively, assume $g_{o34}= 0$ and thus consider (4.99). Applying an argument similar to the one discussed above yields the result that (4.99) can be replaced by the sufficient conditions

$$\frac{(g_1\ g_2\ )^{1/2}}{|g_{o12}|} \geq \frac{2}{1-\alpha^2}\ , \qquad g_{o34} = 0 \tag{4.104}$$

Utilizing the results in (3.86) and (3.89), (4.103) and (4.104) can be reexpressed as, respectively,

$$\frac{f_3\ f_4}{f_o} \geq \frac{3}{1-\alpha^2}\ , \quad \frac{(f_3)^2}{f_o} \geq 2\ , \quad \frac{(f_4)^2}{f_o} \geq 2 \tag{4.105}$$

and

$$\frac{f_1\ f_2}{f_o} \geq \frac{2}{1-\alpha^2} \tag{4.106}$$

where

$$f_1 \triangleq \left|(n-1)q_{21}q_{22} + \det \underline{Q}\right|^{1/2} \tag{4.107a}$$

$$f_2 \triangleq \left|(n-1)q_{11}q_{12} + n \det \underline{Q}\right|^{1/2} \tag{4.107b}$$

$$f_3 \triangleq \left|(n-1)q_{11}q_{21} + n \det \underline{Q}\right|^{1/2} \tag{4.107c}$$

$$f_4 \triangleq \left|(n-1)q_{12}q_{22} + \det \underline{Q}\right|^{1/2} \tag{4.107d}$$

$$f_o \triangleq \left|nq_{21}q_{12} - q_{11}q_{22}\right| \tag{4.107e}$$

(The q_{ij} are elements of the matrix $\underline{Q} = \underline{K}(\underline{U} - \underline{N}^T\underline{K})^{-1}$, and n is the coefficient in $\underline{N}$.) It is sufficient for stability that the parameters of each second-order section in a cascade realization satisfy either (4.105) or (4.106) and that signal quantizations satisfy (4.77).

4.4.2 Fourth-Order Sections

For the purposes of reference, the form of the conductance matrix corresponding to the passive adaptor of the fourth-order section is reshown below.

$$\underline{G} = \left[\begin{array}{cc|cccc} g_1 & g_{o12} & & & & \\ g_{o12} & g_2 & & & \underline{0} & \\ \hline & & g_3 & 0 & 0 & 0 \\ & & 0 & g_4 & g_{o45} & g_{o46} \\ & \underline{0} & 0 & g_{o45} & g_5 & g_{o56} \\ & & 0 & g_{o46} & g_{o56} & g_6 \end{array}\right] \tag{4.108}$$

In order to guarantee the stability of a cascade realization employing such a filter section, the constraints specified by (4.70), (4.72), (4.89), and (4.94) must all be satisfied. Utilizing the same approach introduced in the previous sub-section, these constraints can be replaced (choosing $\eta_1 = 1/3$ and $\eta_2 = 2/3$) with the sufficient conditions

$$\frac{(g_1\ g_2)^{1/2}}{|g_{o12}|} \geq \frac{3}{1-\alpha^2} \tag{4.109a}$$

$$\frac{(g_4\ g_5)^{1/2}}{|g_{o45}|} \geq \frac{6}{1-\alpha^2}, \quad \frac{g_4}{|g_{o45}|} \geq 4, \quad \frac{g_5}{|g_{o45}|} \geq 4 \tag{4.109b}$$

$$\frac{(g_4\ g_6)^{1/2}}{|g_{o46}|} \geq \frac{6}{1-\alpha^2}, \quad \frac{g_4}{|g_{o46}|} \geq 4, \quad \frac{g_6}{|g_{o46}|} \geq 4 \tag{4.109c}$$

$$\frac{(g_5\ g_6)^{1/2}}{|g_{o56}|} \geq \frac{6}{1-\alpha^2}, \quad \frac{g_5}{|g_{o56}|} \geq 4, \quad \frac{g_6}{|g_{o56}|} \geq 4 \tag{4.109d}$$

4.4.3 Fifth-Order Filters Based on the Jaumann Structure

The form of the port reference matrix associated with the passive adaptor of this special fifth-order filter is as shown below.

$$\underline{G} = \left[\begin{array}{c|ccc} \mathrm{diag}[\, g_1,\ g_2,\ g_3,\ g_4 \,] & & \underline{0} & \\ \hline & g_5 & g_{o56} & 0 \\ \underline{0} & g_{o56} & g_6 & 0 \\ & 0 & 0 & g_7 \end{array}\right] \tag{4.110}$$

In order to guarantee the stability of the digital filter, the constraints specified by (4.97) must be satisfied. The constraints (4.70) and (4.72) can be ignored due to the diagonal nature of $\underline{G}_R$. Using the same approach taken with respect to the second-order section yields the result that (4.97) can be replaced with the sufficient conditions

$$\frac{(\, g_5\, g_6 \,)^{1/2}}{|g_{o56}|} \geq \frac{3}{1 - \alpha^2}\ , \qquad \frac{g_5}{|g_{o56}|} \geq 2\ , \qquad \frac{g_6}{|g_{o56}|} \geq 2 \tag{4.111}$$

or, utilizing (B.4), with

$$\frac{f_5\, f_6}{f_o} \geq \frac{3}{1 - \alpha^2}\ , \qquad \frac{(f_5)^2}{f_o} \geq 2\ , \qquad \frac{(f_6)^2}{f_o} \geq 2 \tag{4.112}$$

where

$$f_5 \triangleq \left| q_1(q_4 + q_5) - q_2 q_4 \right|^{1/2} \tag{4.113a}$$

$$f_6 \triangleq \left| q_2 q_5 \right|^{1/2} \tag{4.113b}$$

$$f_o \triangleq \left| q_2 q_4 \right|^{1/2} \tag{4.113c}$$

and the q_i are defined, with respect to the matrix $\underline{Q}$, in Appendix B.

Chapter V

DESIGN OF MINIMAL WAVE DIGITAL FILTERS

In Chapter V we present a procedure, based on the concepts developed in Chapters III and IV, for the design of minimal WD filters. The alternatives regarding the susequent hardware implementation of these filters are then discussed. Lastly, three examples that illustrate the design method are presented.

5.1 DESIGN PROCEDURE

The following steps describe a procedure for the design of minimal reciprocal WD filters.

1. Map specifications given in the z-domain to specifications in the s-domain via equation (2.11).
2. Solve the approximation problem (usually with the aid of filter tables) to determine an analog transfer function $H_a(s)= F(s)/G(s)$ that meets or exceeds specifications. Of course, $H_a(s)$ must be realizable as a reciprocal network.
3. Solve the Feldtkeller equation (i.e. (3.2)) for the polynomial $H(s)$.
4. Construct a normalized two-port scattering matrix $\underline{S}(s)$ having the canonic form shown in (3.1). Derive

the transfer matrix $\underline{T}(s)$ from $\underline{S}(s)$, and factor $\underline{T}(s)$ as $\underline{T}(s)=\underline{T}_1(s)\underline{T}_2(s)\ldots\underline{T}_K(s)$ in order to obtain a WD cascade realization utilizing the basic adaptors introduced in Chapter III.

5. Obtain the two-port scattering matrices $\underline{S}_i(s)$, $i=1,2,\ldots,K$ corresponding to the transfer matrices $\underline{T}_i(s)$, $i=1,2,\ldots,K$ and utilize the DWD algorithm to derive a nominal WD section realization for each two-port description $\underline{S}_i(s)$. Each section will be based on an n-port adaptor described by a constant scattering matrix $\underline{S}_o$, or equivalently by $\underline{N}_o$, $\underline{K}_o$, and $\underline{P}$. The matrix $\underline{S}_o$ satisfies the relation $\underline{S}_o^T\underline{G}_o\underline{S}_o=\underline{G}_o$ where $\underline{G}_o$ is diagonal and positive definite.

6. Redefine the reference conductances such that at each port-interconnection of two adaptors, one adaptor has that port reflection-free. Each redetermined adaptor is described by a new scattering matrix $\underline{S}_o'$ or alternatively, by the submatrices $\underline{N}_o$, $\underline{K}_o'$, and $\underline{P}$. The matrix $\underline{S}_o'$ satisfies the relation $(\underline{S}_o')^T\underline{G}_o'\underline{S}_o'=\underline{G}_o'$ where $\underline{G}_o'$ is diagonal and positive definite.

7. Determine quantized, stable designs according to the following steps:

   a) Quantize the entries in $\underline{N}_o$ and $\underline{K}_o'$ of each adaptor to a given precision to obtain a quantized adaptor described by $\underline{S}$. (Note that the reflection-free property must be maintained in the quantized adaptor.) The matrix $\underline{S}$ satisfies the relation $\underline{S}^T\underline{G}\underline{S}=\underline{G}$ where $\underline{G}$ is positive definite and diagonal or symmetric and near-diagonal.

b) For those adaptors with nondiagonal $\underline{G}$, determine the value of the passivity multiplier α that satisfies the appropriate stability criteria (Section 4.4).

c) Passify, as necessary, the pseudolossless adaptors to obtain pseudopassive adaptors (described by $\underline{S}_P=\alpha\underline{S}$).

d) Check the frequency response of the resultant WD cascade realization to ensure specifications are met. If specifications are not satisfied, return to step (a) and increase the precision of quantizations.

8. A possible alternative to step (7) is as follows:

a) Choose for each adaptor that is noncanonic in multipliers, a reasonable value of the passivity multiplier α. (The parameter α is normally chosen to be very close to unity in value in order to minimize any adverse effects on the frequency response.)

b) Quantize the entries in $\underline{N}_o$ and $\underline{K}_o'$ of each adaptor. For those adaptors with a subsequently nondiagonal $\underline{G}$, quantize to a precision that ensures the appropriate stability criteria are satisfied.

c) Passify, as necessary, the pseudolossless adaptors to obtain pseudopassive adaptors.

d) Check the frequency response of the resultant WD cascade realization to ensure specifications are met. If the specifications are not satisfied, either return to step (b) and increase the precision of the quantizations or return to step (a) and choose a value of α that is closer to unity in value.

9. Scale the quantized filter structure in order to maximize the signal-to-noise ratio at the quantization points while keeping the probability of overflow within acceptable bounds [5,6]. Scaling in WD filters is achieved by inserting the digitized equivalents of two-port transformers (with turns ratio n/l where n is a power of 2) into the signal-flow graph [25].

10. Implement the filtering algorithm on a digital computer or as dedicated hardware. Signal quantizations must conform to the constraints specified by (4.77). For example, a quantization characteristic that implements magnitude truncation of underflow bits and saturation arithmetic for overflows is suitable.

5.2 IMPLEMENTATION

The basic arithmetic operation intrinsic to digital filtering algorithms is an inner product of the form

$$y = \sum_{i=1}^{n} a_i x_i \tag{5.1}$$

where x_i is an input or intermediate signal variable, a_i represents a coefficient of the filter, and y is an output or another intermediate signal variable. Implementation of the inner product may be achieved by two basic approaches. The most straightforward approach is to use a digital multiplier or adder to directly implement every multiplication or addition operation performed within (5.1). Alternatively, distributed arithmetic techniques can be applied. One such technique, developed independently by Crosier et al. [55] and Peled and Liu [56], utilizes memory devices to record precalculated partial products, eliminating the need for actual hardware multipliers. Another approach employing distributed arithmetic is that developed by Moon and Martens [57]. This technique implements an inner product using only adders, inverters and shifters.

Consider an arbitrary pseudopassive n-port adaptor described by $\underline{S}_p$ and designed via the procedures described earlier. The adaptor must be realized as a device that performs the arithmetic operation

$$\underline{b} = \underline{S}_p \, \underline{a} \tag{5.2}$$

The above operation can be expressed as the following system of inner products:

$$b_j = \sum_{i=1}^{n} S_{Pij}\, a_i \qquad j = 1,2,\ldots,n \tag{5.3}$$

where b_j is the reflected wave variable at the j^{th} port and a_i is the incident wave variable at the i^{th} port. To implement the system in (5.3) requires as many as n^2 digital multipliers. If $\underline{S}_P$ is represented in the familiar form

$$\underline{S}_P = \alpha\underline{S} = \alpha\ \underline{P}^T\ \underline{F}\ \underline{M}\ \underline{F}\ \underline{P} \tag{5.4}$$

then the required number of digital multipliers can be reduced to a value that is equal to the number of entries in $\underline{K}$, plus twice the number of entries in $\underline{N}$ that are not +1,-1, or 0, plus the number of passivity multipliers (usually n), subtract the number of reflection-free ports (usually 1). In general, this number of multipliers is more than twice as great as the canonic number. Accordingly, WD filters based on pseudopassive adaptors of the type discussed will not be cost-competitive with other filter structures if implementations utilize discrete multipliers.

Alternatively, distributed arithmetic techniques can be used to implement the n inner products described in (5.3). That WD filters can be implemented with distributed arithmetic is well known [57-60]. Moreover, it is recognized that this form of implementation can compare favorably with canonic multiplier based designs [57,60].

Utilizing the relation $\underline{S}_P=\alpha\underline{S}$, rewrite (5.3) as

$$b_j = \sum_{i=1}^{n} \alpha \, S_{ij} \, a_i \qquad j = 1,2,\ldots,n \tag{5.5}$$

In order that the system in (5.5) be implementable, it is necessary that the coefficients αS_{ij} be finite-word-length binary numbers. However, it is not necessary that α and the S_{ij} be individually finite-word-length binary. We may exploit this property to keep the wordlength of the products αS_{ij} at a minimum. This is best illustrated with an example.

The value of α is generally close to unity. Consider the case where a value for α of 127/128 is appropriate. The matrix $\underline{S}$ has the familiar decomposition $\underline{P}^T\underline{F}\ \underline{M}\ \underline{F}\ \underline{P}$ where the only entries in $\underline{F}$ and $\underline{M}$ that are not 0, +1, or -1 are restricted to the submatrices $\underline{N}$ and $\underline{K}$. Assume the entries of $\underline{N}$ have been quantized to be finite-word-length binary and the entries of $\underline{K}$ have been approximated to be rational numbers with a denominator of $2^m \cdot 127$, where m is an integer. It follows that $\underline{S}$ has the form

$$\underline{S} = \frac{1}{127}\underline{S}_B \tag{5.6}$$

where the entries of $\underline{S}_B$ are binary numbers. The matrix $\underline{S}_P$ is given by

$$\underline{S}_P = \alpha\,\underline{S} = \frac{1}{128}\underline{S}_B \tag{5.7}$$

and clearly, $\underline{S}_P$ will have binary entries. Furthermore, it should be clear that the wordlength of these entries will in general be shorter, utilizing this quantization scheme, than if α and the S_{ij} are individually finite-word-length binary.

5.3 ILLUSTRATIVE EXAMPLES

We shall demonstrate the design procedure developed in Section 5.1 with the aid of several examples.

5.3.1 First Example

Consider the following attenuation scheme:

$$\begin{aligned} 0.5\text{dB} > A(\omega T) > -0.5\text{dB} \quad &\text{for} \quad \omega T \,\varepsilon\, [\, 0,\pi/10 \text{ r} \,] \\ A(\omega T) > 80.0\text{dB} \quad &\text{for} \quad \omega T \,\varepsilon\, [\, 3\pi/10,\pi \text{ r} \,] \end{aligned} \tag{5.8}$$

where $A(\omega T)$ is the attenuation in dB, ω is the digital frequency in r/s, and T is the sampling period. A fifth-order digital filter meeting these specifications is implemented in the Hewlett-Packard Model 3582A spectrum analyzer as a decimation filter [62]. We wish to design a WD filter by the procedure of Section 5.1 to satisfy these same specifications.

The tolerance scheme described by (5.8) maps via (2.11) into the following scheme in the analog domain:

$$\begin{aligned} 0.5\text{dB} > A(\phi) > -0.5\text{dB} \quad &\text{for} \quad \phi \,\varepsilon\, [\, 0,0.15838 \text{ r/s} \,] \\ A(\phi) > 80.0\text{dB} \quad &\text{for} \quad \phi \,\varepsilon\, [\, 0.50953,\infty \text{ r/s} \,] \end{aligned} \tag{5.9}$$

where ϕ is the analog frequency in r/s. A standard fifth-order C052519 elliptic characteristic [45], frequency scaled by the factor 0.16589, will satisfy the specifications in

(5.9). The frequency-normalized analog transfer function is given by

$$2H_a(s) = F(s)/G(s) \tag{5.10}$$

where

$$F(s) = (s^2 + 26.33561) (s^2 + 10.37590)$$

$$G(s) = 1053.536\ s^5 + 1443.012\ s^4 + 2323.643\ s^3 + 1752.470\ s^2 + 1027.410\ s + 273.2556$$

The desired transfer function is in fact $H_a(s/0.16589)$, however, it is more convenient to frequency scale realizations at a later step. The Feldtkeller equation is solved to yield

$$H(s) = 1053.536\ s^5 + 32.00857\ s^4 + 1335.894\ s^3 + 23.97851\ s^2 + 343.3522\ s \tag{5.11}$$

The normalized two-port scattering matrix $\underline{S}(s)$ and thus the transfer matrix $\underline{T}(s)$ follow directly from the polynomials $F(s)$, $H(s)$, and $G(s)$. That the matrix $\underline{S}(s)$ may be realized as a cascade of three subnetworks is evident from the factorization

$$F(s) = F_1(s)\ F_2(s)\ F_3(s) \tag{5.12}$$

where

$$F_1(s) = s^2 + 26.33561\ ,\quad F_2(s) = 1\ ,\quad F_3(s) = s^2 + 10.37590$$

and

$$n_1 = 2\ ,\quad n_2 = 1\ ,\quad n_3 = 2 \tag{5.13}$$

The polynomial $F_i(s)$ and the section degree n_i specify the transmission zeros of the i^{th} section. The transfer matrix $\underline{T}(s)$ is factored, consistent with (5.12) and (5.13), as

$$\underline{T}(s) = \underline{T}_1(s)\ \underline{T}_2(s)\ \underline{T}_3(s) \tag{5.14}$$

where $\underline{T}_i(s)$ describes the i^{th} section. One solution to (5.14) yields

$$\begin{aligned}
H_1(s) &= 25.06908\ s^2 + 2.102403\ s \\
G_1(s) &= 25.08902\ s^2 + 35.68219\ s + 26.33561 \\
H_2(s) &= 1.151566\ s \\
G_2(s) &= 1.151566\ s + 1 \\
H_3(s) &= -9.092471\ s^2 + 0.2607053\ s \\
G_3(s) &= 9.1472961\ s^2 + 13.00535\ s + 10.37590
\end{aligned} \tag{5.15}$$

The section scattering matrices $\underline{S}_i(s)$, $i=1,2,3$ are determined directly from the polynomials $F_i(s)$, $H_i(s)$, and $G_i(s)$. The DWD algorithm can be used at this point to derive nominal WD realizations based on the analog networks represented by the $\underline{S}_i(s)$. Frequency-scaled adaptor designs[2] (that conform with the basic structures described in Section 3.7) are presented below.

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[2] If a reference conductance G corresponds to a port in the reference filter that connects an inductor, frequency scaling by the factor k_f is achieved by scaling G as Gk_f. Similarly, if G corresponds to a port connecting a capacitor, G is scaled as G/k_f.

Adaptor 1.

$$\underline{N}_o = \begin{bmatrix} 1 & 1 \\ 1 & 50.1580 \end{bmatrix} , \qquad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_2 \\ \underline{U}_2 & \underline{0} \end{bmatrix}$$

$$\underline{G}_o = \underline{G}_t \dotplus \underline{G}_\ell = \text{diag}[\ 1,\ 1,\ 0.1156209,\ 0.0031808\ ]$$

$$\underline{K}_o = (\ \underline{G}_t + \underline{N}_o\ \underline{G}_\ell\ \underline{N}_o^T\ )^{-1}\ \underline{N}_o\ \underline{G}_\ell$$

The polarity matrix for the WD section realization is $\underline{\Sigma}$=diag[-1,1], implying that a delay in series with an inverter is connected at port 3 and a delay is connected at port 4 of the adaptor.

Adaptor 2.

$$\underline{N}_o = [\ 1 \quad 1\ ]^T , \qquad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_1 \\ \underline{U}_2 & \underline{0} \end{bmatrix}$$

$$\underline{G}_o = \underline{G}_t \dotplus \underline{G}_\ell = \text{diag}[\ 1,\ 1,\ 0.0720258\ ]$$

$$\underline{K}_o = (\ \underline{G}_t + \underline{N}_o\ \underline{G}_\ell\ \underline{N}_o^T\ )^{-1}\ \underline{N}_o\ \underline{G}_\ell$$

The polarity matrix for the WD section realization is $\underline{\Sigma}$=diag[-1].

Adaptor 3.

$$\underline{N}_o = \begin{bmatrix} 1 & 1 \\ 1 & 0.054825 \end{bmatrix} , \qquad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_2 \\ \underline{U}_2 & \underline{0} \end{bmatrix}$$

$$\underline{G}_o = \underline{G}_t \dotplus \underline{G}_\ell = \text{diag}[\ 1,\ 1,\ 0.1297453,\ 8.288391\ ]$$

$$\underline{K}_o = (\ \underline{G}_t + \underline{N}_o\ \underline{G}_\ell\ \underline{N}_o^T\ )^{-1}\ \underline{N}_o\ \underline{G}_\ell$$

The polarity matrix for the section realization is $\underline{\Sigma}=\text{diag}[-1,1]$.

Reference conductances must be redefined in order to ensure that at each port-interconnection of adaptors, one adaptor has that port reflection-free. Choosing the reference conductances such that port 1 of adaptor 2 and port 1 of adaptor 3 are reflection-free, we obtain new adaptor realizations described by

Adaptor 1.

$$\underline{K}_o' = \begin{bmatrix} 0.100703 & -0.001965 \\ 0.010735 & 0.019548 \end{bmatrix}$$

$$\underline{G}_o' = \text{diag}[\ 1,\ 0.0713928,\ 0.1156209,\ 0.0031808\ ]$$

Adaptor 2.

$$\underline{K}_o' = \begin{bmatrix} 0.5 \\ 0.004394 \end{bmatrix}$$

$$\underline{G}_o' = \text{diag}[\ 0.0713928,\ 8.122602,\ 0.0720258\ ]$$

Adaptor 3.

$$\underline{K}_o' = \begin{bmatrix} 0.003946 & 0.496054 \\ 0.110370 & 0.142587 \end{bmatrix}$$

$$\underline{G}_o' = \text{diag}[\ 8.122602,\ 1,\ 0.1297453,\ 8.288391\ ]$$

where the submatrices $\underline{N}_o$ and $\underline{P}$ remain for each adaptor, unchanged from before.

The nominal WD realization follows directly from the above adaptor descriptions and has the structure represented in Fig. 5.1. The final realization based on pseudopassive adaptors will have the structure shown in Fig. 5.2. The passivity multipliers present in the output (sink) branches of the pseudopassive filter simply contribute a gain factor and may be deleted. This leaves 6 passivity multipliers that must be realized. With the 11 multipliers required to realize the pseudolossless adaptors, a total of 17 multipliers are necessary to implement the structure in Fig 5.2. This is more than twice as many as the canonic number. A multiplier/adder implementation is clearly not economical. Accordingly, a practical hardware implementation will employ distributed arithmetic techniques.

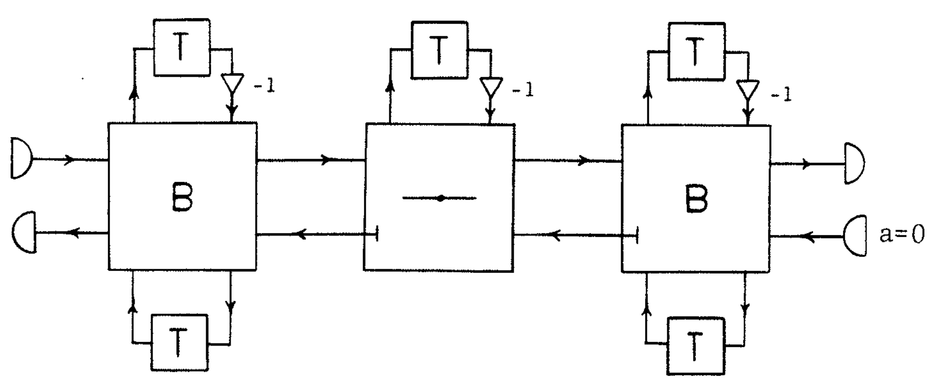


Figure 5.1: Pseudolossless WD filter realization.

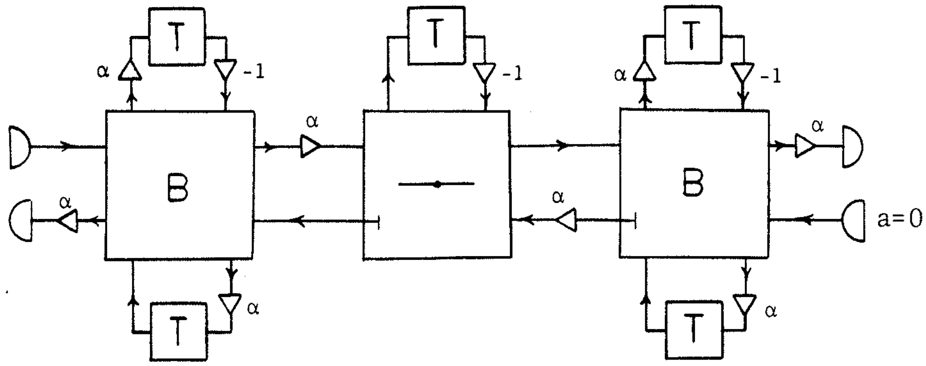


Figure 5.2: Pseudopassive WD filter realization.

For this example, quantization of the coefficients was achieved with the aid of a simple optimization algorithm. The algorithm follows the procedure suggested in step 8 of Section 5.1 and employs an exhaustive search about each nominal coefficient value in order to obtain filter coefficients with short wordlength. The following results were obtained.

<u>Adaptor 1.</u>

$$\underline{N} = \begin{bmatrix} 1 & 1 \\ 1 & 50 \end{bmatrix}, \quad \underline{K} = \begin{bmatrix} 51/510 & -1/510 \\ 5/510 & 5/255 \end{bmatrix}, \quad \alpha = 255/256$$

Let us verify that this value of α satisfies one of the stability criteria given in Section 4.4, that is, either (4.105) or (4.106). The matrix $\underline{Q}$ for this adaptor is given by

$$\underline{Q} = \begin{bmatrix} 0.113785 & 0.002188 \\ 1.341348 & 2.006552 \end{bmatrix} \tag{5.16}$$

It follows that

$$\frac{f_3 f_4}{f_o} = 35.23 < \frac{3}{1 - (\frac{255}{256})^2} = 384.75$$

$$\frac{f_1 f_2}{f_o} = 473.26 > \frac{2}{1 - (\frac{255}{256})^2} = 256.50$$

We see that the condition in (4.106) is satisfied although that in (4.105) is not. The adaptor scattering matrix follows from $\underline{N}$, $\underline{K}$, $\underline{P}$, and α, i.e.

$$\underline{S}_p = 2^{-8} \begin{bmatrix} 205 & -1 & 51 & 1 \\ -15 & -250 & 5 & 10 \\ 445 & 4 & -199 & 9 \\ -290 & 249 & 301 & 244 \end{bmatrix} \qquad (5.17)$$

The reflected wave variable b_1 corresponding to this adaptor also represents an output signal of the complete filter. Since this particular output is not generally utilized, it is not necessary that we implement the inner product associated with row 1 of $\underline{S}_p$.

<u>Adaptor</u> <u>2</u>.

$$\underline{N} = [\,1 \quad 1\,]^T , \qquad \underline{K} = \begin{bmatrix} 1/2 \\ 1/256 \end{bmatrix}$$

Note that k_{11}=1/2, as required by the reflection-free constraint. The adaptor scattering matrix follows from $\underline{N}$, $\underline{K}$, and $\underline{P}$, i.e.

$$\underline{S} = 2^{-7}\begin{bmatrix} 0 & -128 & 128 \\ -1 & 127 & 1 \\ 127 & 127 & 1 \end{bmatrix} \tag{5.18}$$

Adaptor 3.

$$\underline{N} = \begin{bmatrix} 1 & 1 \\ 1 & 1/16 \end{bmatrix}, \quad \underline{K} = \begin{bmatrix} 1/255 & 253/510 \\ 32/255 & 40/255 \end{bmatrix}, \quad \alpha = 255/256$$

Note that $k_{11} + k_{12} = 1/2$, as required by the reflection-free constraint. Let us verify that either (4.105) or (4.106) is satisfied for this adaptor. The matrix $\underline{Q}$ is given by

$$\underline{Q} = \begin{bmatrix} 0.018400 & 1.028283 \\ 0.151133 & 0.517172 \end{bmatrix} \tag{5.19}$$

Evaluating the terms in (4.105) and (4.106) yields

$$\frac{f_3\ f_4}{f_o} = 441.25 > \frac{3}{1-(\frac{255}{256})^2}, \quad \frac{(f_3)^2}{f_o} = 59.52 > 2, \quad \frac{(f_4)^2}{f_o} = 3271.29 > 2$$

$$\frac{f_1\ f_2}{f_o} = 389.44 > \frac{2}{1-(\frac{255}{256})^2}$$

Clearly, the stability conditions are satisfied.

For this adaptor, the incident wave variable a_2 also represents the input signal (to the complete filter) that is normally set to be identically zero. Given $a_2=0$, the values of the elements in the second column of $\underline{S}_P$ are irrelevant and can be set to zero. The remaining entries in $\underline{S}_P$ are as follows:

$$\underline{S}_p = 2^{-8} \begin{bmatrix} 0 & 0 & 2 & 253 \\ -144 & 0 & 64 & 80 \\ 111 & 0 & -189 & 333 \\ 246 & 0 & 6 & 3 \end{bmatrix} \tag{5.20}$$

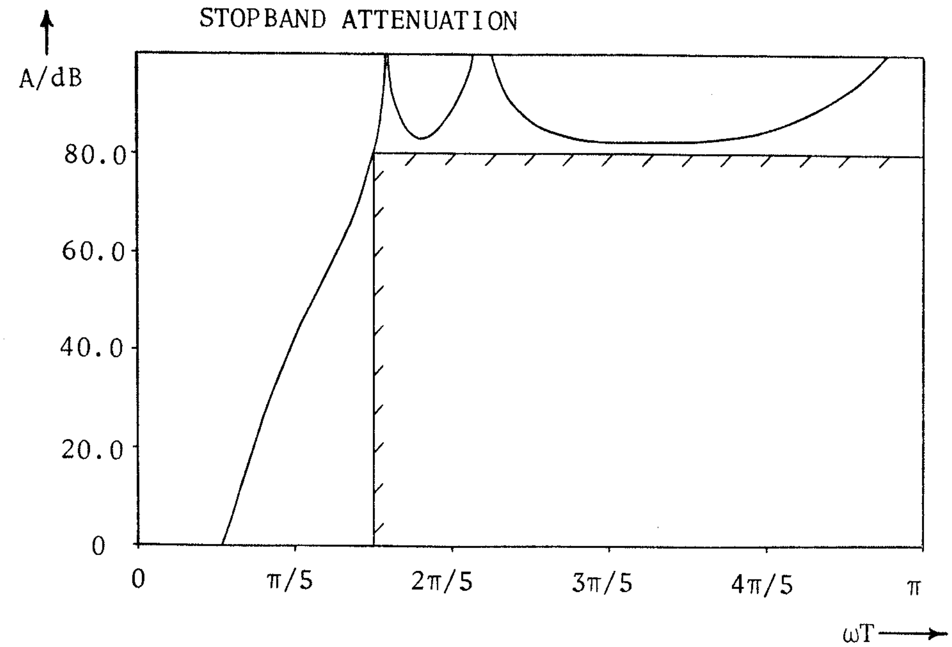
The pseudopassive WD filter based on the adaptors described by (5.17), (5.18), and (5.20) exhibits an attenuation characteristic with a minimum attenuation of 1.791 dB. Multiplying the filter output by an additional factor of 5/4 yields the characteristic depicted in Fig. 5.3. Clearly this characteristic satisfies the specifications. Simulations of this filter under finite-word-length and zero-input conditions showed no parasitic oscillations as expected.

5.3.2 Second Example

Interpolation is the process by which the sampling rate of a discrete-time signal is increased without any conversion to a continuous-time waveform. Integral to an interpolater is a lowpass filter, necessary for filtering out the baseband. An example of a tolerance scheme such a filter might need to satisfy in the passband is given in Fig. 5.4 [63]. In the stopband the requirement is that

$$A(\omega T) > 40.0\text{dB} \quad \text{for} \quad \omega T \; \varepsilon \; [\, 0.14375\pi, \; \pi \text{ r} \,]$$

We wish to design a WD filter with an attenuation characteristic that satisfies these specifications when normalized to have 0 dB of attenuation at $\omega T = \pi/40$ r.



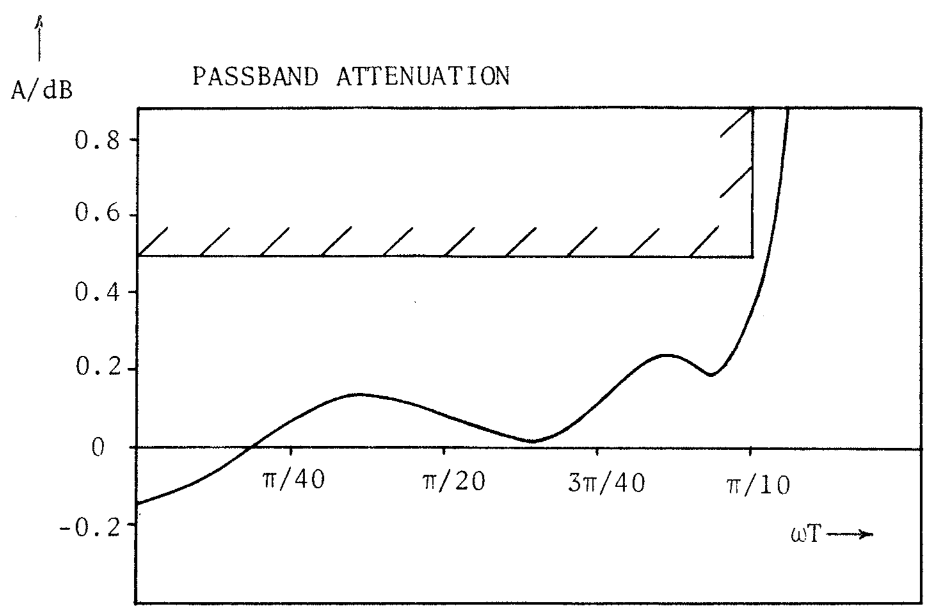


Figure 5.3: Stop- and passband attenuation for the first example.

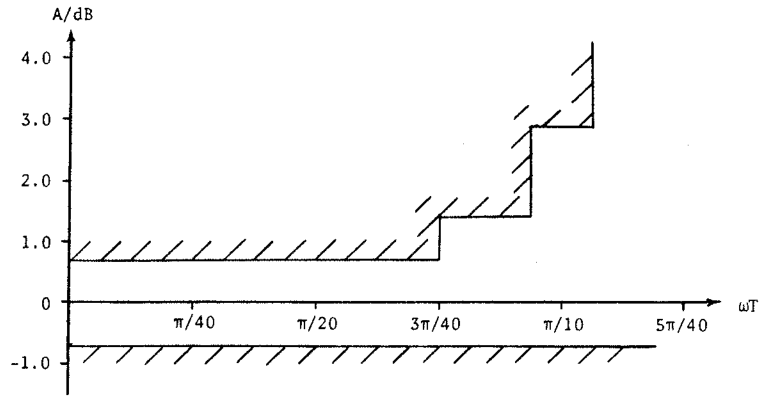


Figure 5.4: Passband tolerance scheme.

A standard fifth-order C052548 elliptic characteristic [45], frequency scaled by the factor 0.1685, will map (under the bilinear z-transformation) to a suitable discrete-time system function. The frequency-normalized analog transfer function is F(s)/G(s), where

$$F(s) = (s^2 + 4.120337) (s^2 + 1.936578)$$

$$G(s) = 20.71831\ s^5 + 27.97496\ s^4 + 47.34040\ s^3 + 37.12756\ s^2 + 23.84069\ s + 7.979355 \qquad (5.21)$$

Solving the Feldtkeller equation yields the polynomial

$$H(s) = 20.71831\ s^5 + 0.4977016\ s^4 + 28.48388\ s^3 + 0.4076646\ s^2 + 8.516512\ s \qquad (5.22)$$

We wish to realize the WD design as a cascade of three sub-networks. Accordingly, we factorize F(s) as

$$F(s) = F_1(s)\ F_2(s)\ F_3(s) \qquad (5.23)$$

where

$$F_1(s) = s^2 + 4.120337 \ , \quad F_2(s) = 1 \ , \quad F_3(s) = s^2 + 1.936578$$

Furthermore, let

$$n_1 = 2, \qquad n_2 = 1, \qquad n_3 = 2 \tag{5.24}$$

Factorization of the transfer matrix yields

$$\begin{aligned}
H_1(s) &= 3.436532 \ s^2 + 0.01879064 \ s \\
G_1(s) &= 3.579072 \ s^2 + 4.610167 \ s + 4.120337 \\
H_2(s) &= -1.172634 \ s \\
G_2(s) &= -1.172634 \ s - 1 \\
H_3(s) &= 1.060669 \ s^2 + 0.2127835 \ s \\
G_3(s) &= -1.457744 \ s^2 - 1.348403 \ s - 1.936578
\end{aligned} \tag{5.25}$$

The DWD algorithm is then applied to determine WD section realizations for each $\underline{S}_i(s)$, i=1,2,3. After redefinition of reference conductances to ensure port 2 of adaptor 1 and port 1 of adaptor 3 are reflection-free, we obtain the following nominal frequency-scaled adaptor designs:

<u>Adaptor</u> <u>1</u>.

$$\underset{\sim}{N}_o = \begin{bmatrix} 1 & 1 \\ 1 & 7.01560 \end{bmatrix} , \qquad \underline{P} = \begin{bmatrix} \underline{0} & \underset{\sim}{U}_2 \\ \underset{\sim}{U}_2 & \underline{0} \end{bmatrix}$$

$$\underset{\sim}{K}_o' = \begin{bmatrix} 0.113330 & 0.060428 \\ -0.000735 & 0.071375 \end{bmatrix}$$

$$\underline{G}_o' = \text{diag}[\ 1,\ 7.605779,\ 0.1499855,\ 0.1827478\ ]$$

$$\underline{\Sigma} = \text{diag}[\ -1,\ 1\ ]$$

Adaptor 2.

$$\underline{N}_o = [\ 1 \quad 1\ ]^T \quad , \quad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_1 \\ \underline{U}_2 & \underline{0} \end{bmatrix} \quad , \quad \underline{K}_o' = \begin{bmatrix} 0.009209 \\ 0.015888 \end{bmatrix}$$

$$\underline{G}_o' = \text{diag}[\ 7.605779,\ 4.408466,\ 0.0718468\ ]$$

$$\underline{\Sigma} = \text{diag}[\ -1\ ]$$

Adaptor 3.

$$\underline{N}_o = \begin{bmatrix} 1 & 1 \\ 1 & 0.397076 \end{bmatrix} \quad , \quad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_2 \\ \underline{U}_2 & \underline{0} \end{bmatrix}$$

$$\underline{K}_o' = \begin{bmatrix} -0.020847 & 0.520847 \\ 0.119639 & 0.700197 \end{bmatrix}$$

$$\underline{G}_o' = \text{diag}[\ 4.408466,\ 1,\ 0.2873439,\ 13.16116\ ]$$

$$\underline{\Sigma} = \text{diag}[\ -1,\ 1\ ]$$

The nominal WD realization follows from the above adaptor designs and has a structure similar to that shown in Fig. 5.1, although in this case, port 2 of adaptor 1 (instead of port 1 of adaptor 2) is reflection-free. Quantization of the nominal coefficients was again achieved with the aid of the optimization algorithm mentioned earlier. The quantized adaptor designs are described below.

Adaptor 1.

$$\underline{N} = \begin{bmatrix} 1 & 1 \\ 1 & 7 \end{bmatrix}, \qquad \underline{K} = \begin{bmatrix} 7/63 & 7/126 \\ 0 & 9/126 \end{bmatrix}, \qquad \alpha = 63/64$$

For this adaptor, the quantized coefficients have by chance satisfied the dependency relation (3.83). Consequently, the reference conductance matrix $\underline{G}$ is diagonal and conditions (4.105) and (4.106) are automatically satisfied. From $\underline{N}$, $\underline{K}$, $\underline{P}$, and α we obtain

$$\underline{S}_P = 2^{-6} \begin{bmatrix} 42 & -63 & 14 & 7 \\ -9 & 0 & 0 & 9 \\ 96 & 0 & -49 & 16 \\ 42 & 378 & 14 & 7 \end{bmatrix} \tag{5.26}$$

As discussed earlier, the output signal b_1 of this first filter section is not generally utilized. Therefore, the inner product associated with row 1 of $\underline{S}_P$ can be disregarded.

Adaptor 2.

$$\underline{N} = [\,1 \quad 1\,]^T , \qquad \underline{K} = \begin{bmatrix} 1/128 \\ 5/256 \end{bmatrix}$$

The adaptor scattering matrix $\underline{S}$ is given by

$$\underline{S} = 2^{-7} \begin{bmatrix} 126 & -2 & 2 \\ -5 & 123 & 5 \\ 249 & 249 & -121 \end{bmatrix} \tag{5.27}$$

Adaptor 3.

$$\underline{N} = \begin{bmatrix} 1 & 1 \\ 1 & 1/5 \end{bmatrix}, \qquad \underline{K} = \begin{bmatrix} -1/102 & 26/51 \\ 13/102 & 91/102 \end{bmatrix}, \qquad \alpha = 255/256$$

The matrix $\underline{Q}$ associated with this adaptor is calculated to be

$$\underline{Q} = \begin{bmatrix} 0.019520 & 1.722998 \\ 0.212289 & 3.816268 \end{bmatrix} \tag{5.28}$$

Evaluating the terms in (4.105) and (4.106) yields

$$\frac{f_3\ f_4}{f_o} = 436.68 > \frac{3}{1 - (\frac{255}{256})^2}, \quad \frac{(f_3)^2}{f_o} = 45.99 > 2, \quad \frac{(f_4)^2}{f_o} = 4146.57 > 2$$

$$\frac{f_1\ f_2}{f_o} = 211.26 < \frac{2}{1 - (\frac{255}{256})^2}$$

As required, one of the stability conditions is satisfied. The adaptor scattering matrix is given by

$$\underline{S}_P = 2^{-8} \begin{bmatrix} 0 & 0 & -5 & 260 \\ 520 & 0 & 65 & 455 \\ -265 & 0 & -195 & 715 \\ 151 & 0 & 8 & 96 \end{bmatrix} \tag{5.29}$$

where as discussed previously, the entries of the second column in $\underline{S}_P$ are irrelevant under the condition $a_2=0$ and can be set to zero. The WD filter based on the adaptors specified by (5.26), (5.27) and (5.29) exhibits the magnitude-normalized attenuation characteristic depicted in Fig. 5.5.

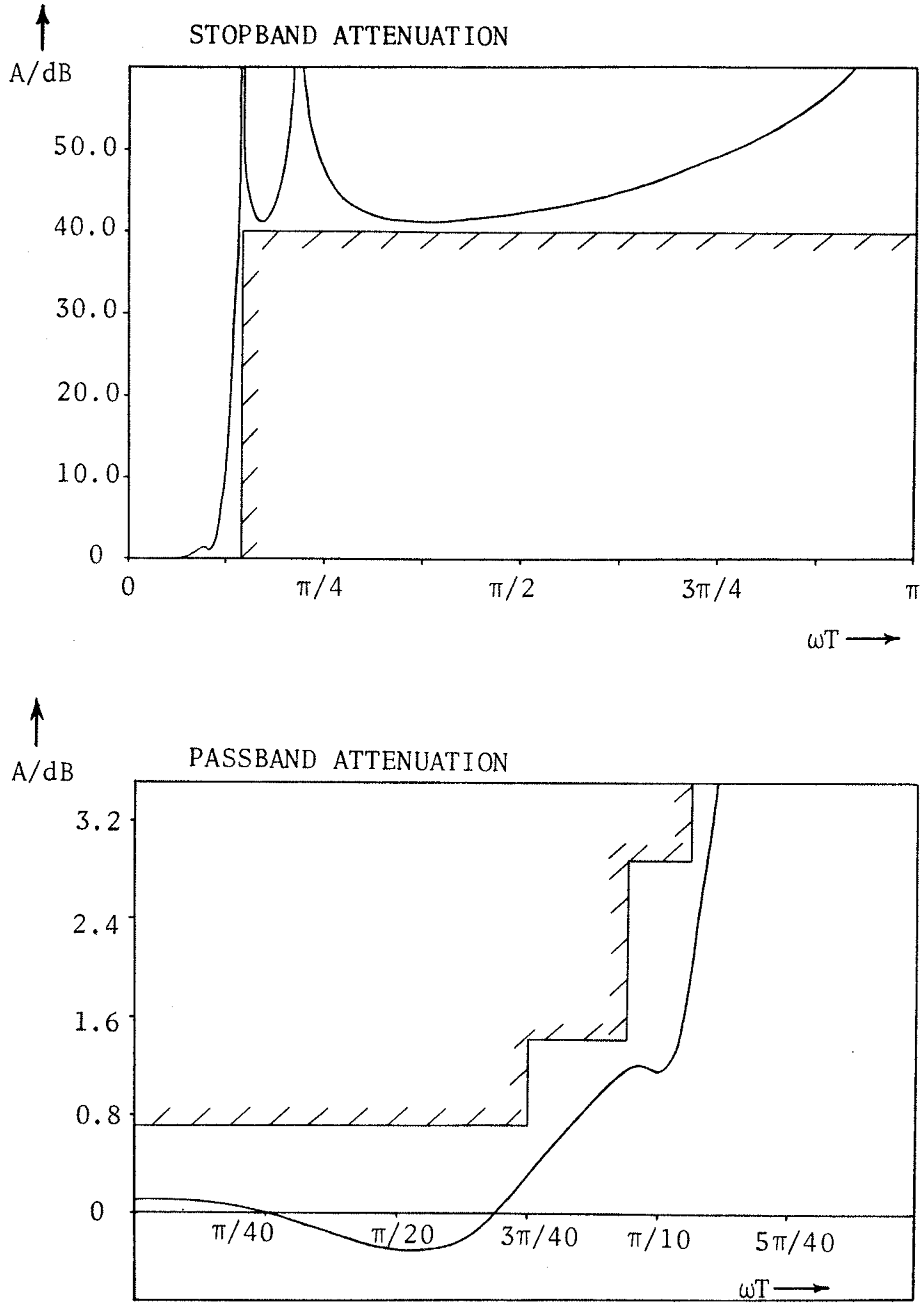


Figure 5.5: Stop- and passband attenuation for the second example.

Clearly the specifications have been satisfied. Simulations of the filter showed no zero-input parasitic oscillations as expected.

5.3.3 Third Example

Let us design another lowpass filter to meet the following specifications [64]:

$$\begin{aligned} 0.3 \text{ dB} > A(\omega T) > 0 \text{ dB} \quad &\text{for} \quad \omega T \in [\, 0,\ 0.2125\pi \text{ r} \,] \\ A(\omega T) > 32.15 \text{ dB} \quad &\text{for} \quad \omega T \in [\, 0.2875\pi,\ \pi \text{ r} \,] \end{aligned} \tag{5.30}$$

An interpolater that increases the sampling rate by a factor of 4 will utilize a filter of this type. An appropriate analog response is a fifth-order C051549 elliptic characteristic [45], frequency scaled by the factor 0.36677. The frequency-normalized analog transfer function is F(s)/G(s), where

$$\begin{aligned} F(s) &= s^4 + 5.827753\, s^2 + 7.411629 \\ G(s) &= 11.07190\, s^5 + 18.88720\, s^4 + 31.34882\, s^3 \\ &\quad + 28.52800\, s^2 + 18.91279\, s + 7.411629 \end{aligned} \tag{5.31}$$

Solving the Feldtkeller equation yields[3]

$$\begin{aligned} H(s) = 11.07190\, s^5 &+ 0.0005884\, s^4 + 15.28445\, s^3 \\ &+ 0.0004510\, s^2 + 4.604570\, s \end{aligned} \tag{5.32}$$

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[3] A network with an odd order elliptic characteristic has the property $H(s)=-\sigma H(-s)$. Clearly, this condition is not strictly satisfied in (5.11), (5.22), or (5.32). This is due to initial small errors in the coefficients of F(s) and G(s).

We wish the WD realization to be based on a single Jaumann adaptor. A nominal frequency-scaled adaptor design, obtained with the DWD algorithm, is given below.

$$\underline{N}_o^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \qquad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_3 \\ \underline{U}_4 & \underline{0} \end{bmatrix}$$

$$\underline{K}_o = \begin{bmatrix} 0.087896 & 0.031897 & 0.105143 \\ -0.087896 & -0.031897 & 0.105143 \\ -0.150157 & 0.799685 & 0 \\ 0 & 0 & 0.709809 \end{bmatrix}$$

$$\underline{G}_o = \text{diag}[\ 1,\ 1,\ 0.0491078,\ 0.1482181,\ 0.1155896,\ 1.315835,\ 0.2876522\ ]$$

$$\underline{\Sigma} = \text{diag}[\ -1,\ -1,\ -1,\ 1,\ 1\ ]$$

Quantization of the coefficients was achieved in the straightforward manner suggested in step 8 of the design procedure in Section 5.1. The quantized adaptor design is described by $\underline{N}=\underline{N}_o$, $\underline{K}$, $\underline{P}$, and α, where

$$\underline{K} = \frac{1}{511}\begin{bmatrix} 45 & 16 & 54 \\ -45 & -16 & 54 \\ -77 & 409 & 0 \\ 0 & 0 & 363 \end{bmatrix}, \qquad \alpha = 511/512$$

Let us verify that the given value of α satisfies the stability criteria in (4.112). For this adaptor design,

$$\underline{Q} = \begin{bmatrix} 0.115578 & 0.281407 & 1.349997 \\ -0.115578 & -0.281407 & 1.349997 \\ -0.002513 & 5.841708 & 0 \\ 0 & 0 & 9.074986 \end{bmatrix} \tag{5.33}$$

Evaluating the terms in (4.112) yields

$$\frac{f_5 \; f_6}{f_o} = 790.53 > \frac{3}{1 - (\frac{511}{512})^2} = 768.75$$

$$\frac{(f_5)^2}{f_o} = 268.84 > 2, \quad \frac{(f_6)^2}{f_o} = 2324.60 > 2$$

as required. The adaptor scattering matrix is described by

$$\underline{S}_P = 2^{-9} \begin{bmatrix} 281 & 14 & -32 & -108 & 90 & 32 & 108 \\ 14 & 281 & 32 & -108 & -90 & -32 & 108 \\ -664 & 664 & -307 & 0 & -154 & 818 & 0 \\ -726 & -726 & 0 & -215 & 0 & 0 & 726 \\ 778 & -778 & -64 & 0 & -331 & 64 & 0 \\ 114 & -114 & 140 & 0 & 26 & 371 & 0 \\ 80 & 80 & 0 & 80 & 0 & 0 & 431 \end{bmatrix} \tag{5.34}$$

The reflected wave variable b_1 represents a normally unused output of the WD filter and consequently the inner product associated with row 1 of $\underline{S}_P$ can be disregarded. Furthermore, given $a_2=0$, the entries in the second column of $\underline{S}_P$ can be set to zero. The filter attenuation characteristic (normalized to be 0dB at $\omega T=0$) satisfies the specifications as shown in Fig.5.6. Simulations revealed no parasitic oscillations under zero-input conditions.

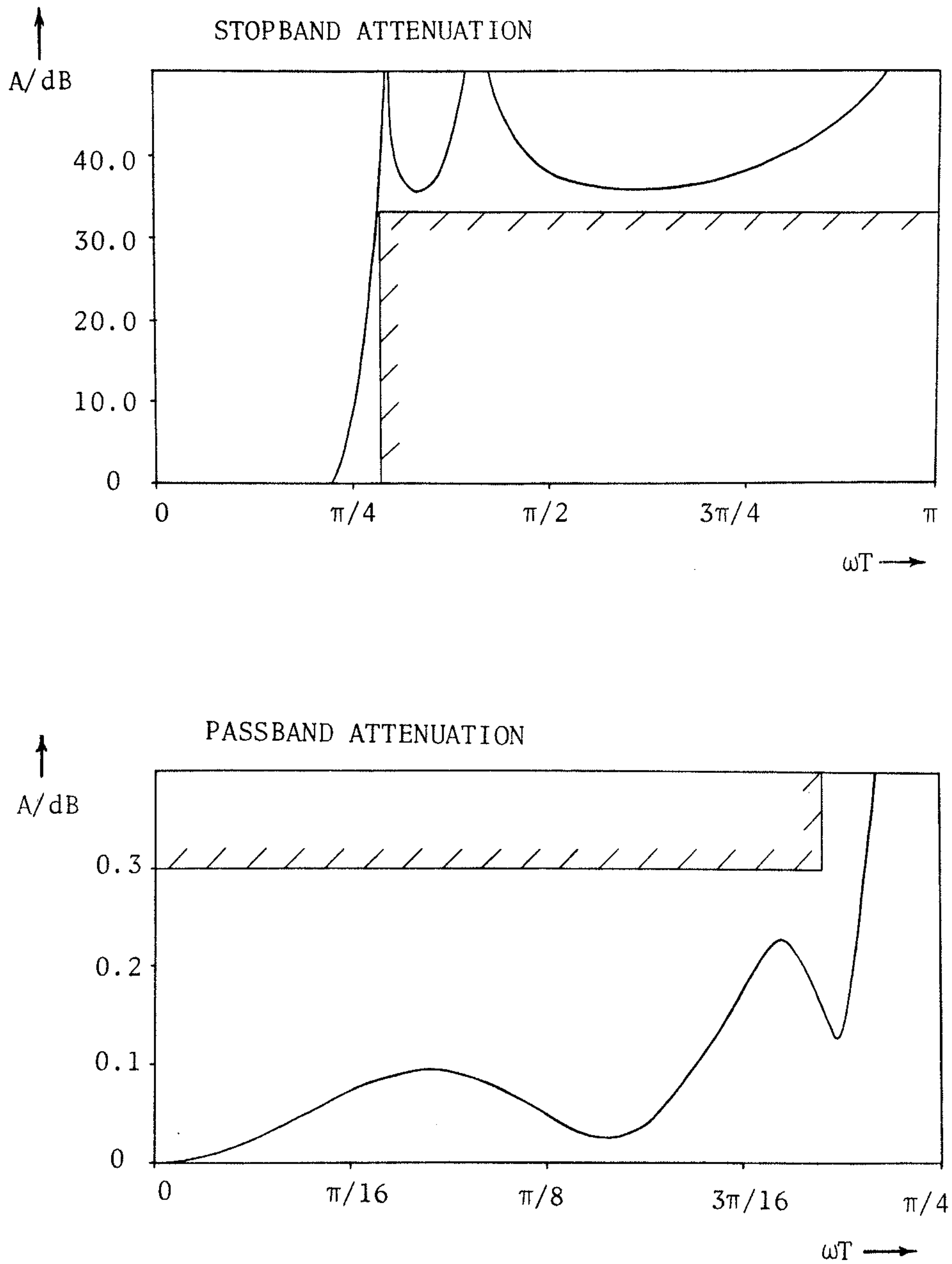

Figure 5.6: Stop- and passband attenuation for the third example.

Chapter VI

CONCLUSIONS AND RECOMMENDATIONS

This thesis has proposed methods for the design of minimal reciprocal WD filters that suppress all types of zero-input parasitic oscillations. First, the DWD algorithm was developed to facilitate the synthesis of minimal WD realizations based on reciprocal pseudolossless n-port adaptors. This algorithm determines nominal designs directly from transfer function specifications and thus provides an alternative to conventional synthesis procedures which utilize an analog reference filter as a starting point. An advantage of the DWD algorithm is that it is amenable to implementation on a digital computer and therefore can be a useful design tool.

The problem of guaranteeing the stability of WD filters under finite-word-length conditions was then considered. It was demonstrated that after quantization of coefficients in $\underline{N}$ and $\underline{K}$, the pseudolossless WD realization does not usually retain a diagonal reference conductance matrix. As a result, parasitic oscillations are no longer necessarily eliminated by applying magnitude truncation (or any other comparably simple quantization scheme) at the delays and adaptor interconnections. A strictly pseudopassive n-port adaptor structure was then proposed; and conditions for guaranteeing the stability of filters utilizing these adaptors were derived. These conditions include restrictions on the entries of the port conductance matrix and constraints on the signal quantization characteristic (magnitude truncation is a suitable scheme). Explicit stability formulae were developed for pseudopassive adaptor structures that can realize second-order and symmetric fifth-order reciprocal filter sections. Stability formulae for an adaptor to realize reciprocal fourth-order sections were partially developed.

The solution for the second-order section allows us to determine stable minimal WD cascade networks that can closely realize the important class of transfer functions that includes the Butterworth, Chebyshev, inverse Chebyshev, and elliptic characteristics. In contrast, the procedures developed by Fettweis [26] and Meerkötter [27] for obtaining minimal WD networks that realize finite attenuation poles, yield designs for which stability is not guaranteed by the usual methods of signal quantization. Other methods of obtaining stable minimal WD filters have been proposed by Ashley [28], and Martens and Lê [31]. However, these require more complicated signal quantization schemes which can also lead to degradation of the filter performance. Thus, the design procedure developed within this thesis does have advantages. The approach developed recently by Martens and Jarmasz [32] also yields minimal WD realizations that are guaranteed stable when quantizers employ magnitude truncation.

A disadvantage of our proposed design method is the necessity of distributed arithmetic techniques for implementations to be economical. Moreover, distributed arithmetic techniques utilizing stored products [55,56] are not readily applied to WD cascade realizations, which cannot generally be pipelined [60]. However, the hardware structure developed by Moon and Martens [57] is generally applicable and preliminary indications are that implementations can be reasonably economical. The determination of coefficient quantization schemes that are tailored towards reducing the complexity of subsequent distributed arithmetic implementations is an area that requires more investigation.

Another disadvantage of this design method is that the introduction of the passivity multipliers invariably causes some (deterministic) degradation in the filter frequency response. One possible approach to this problem is to predistort the nominal realization to compensate for the effects caused by the subsequent introduction of the passivity multipliers. Some success has already been obtained with this method.

The basic strategy utilized within this thesis to obtain stable digital filters is not restricted to WD realizations. Preliminary studies have indicated that the same approach can be applied to realizing stable single-input-output state-variable digital filters of arbitrary order. Whether stable state-variable cascade (in the network sense) realizations can be determined, warrants further research.

Appendix A

DECOMPOSITION OF SELF-INVERSE MATRICES

In this appendix we demonstrate that a self-inverse, real, constant matrix $\underline{S}$ can always be decomposed as

$$\underline{S} = \underline{P}^T \underline{F} \underline{M} \underline{F} \underline{P} \tag{A.1a}$$

where

$$\underline{F} = \begin{bmatrix} -\underline{U} & \underline{N}^T \\ \underline{0} & \underline{U} \end{bmatrix}, \qquad \underline{M} = \begin{bmatrix} -\underline{U} & \underline{0} \\ -2\underline{K} & \underline{U} \end{bmatrix} \tag{A.1b}$$

and where $\underline{N}$ and $\underline{K}$ are real matrices of dimension $t \times \ell$ ($\underline{S}$ is of dimension $t+\ell$), $\underline{P}$ is a permutation matrix, and $\underline{U}$ represents a unit matrix of appropriate dimension. Furthermore, a method for achieving the decomposition is described.

Consider the nontrivial case where $\underline{S} \neq \pm\underline{U}$. Let

$$\underline{S}^+ \triangleq \frac{1}{2} (\underline{S} + \underline{U}) \tag{A.2}$$

Then

$$\begin{aligned} (\underline{S}^+)^2 - \underline{S}^+ = \underline{S}^+ (\underline{S}^+ - \underline{U}) &= \frac{1}{2} (\underline{S} + \underline{U}) \frac{1}{2} (\underline{S} - \underline{U}) \\ &= \frac{1}{4} (\underline{S}^2 - \underline{U}) = \underline{0} \end{aligned} \tag{A.3}$$

This result demonstrates that $\underline{S}^+$ will be singular. Accordingly, there exists a permutation matrix $\underline{P}$ and a nonsingular matrix $\underline{V}$ such that

$$\underline{P}\ \underline{S}^+\ \underline{V} = \begin{bmatrix} \underline{0} & \underline{N}^T \\ \underline{0} & \underline{U} \end{bmatrix} \tag{A.4}$$

Since $\underline{P}^T\underline{P} = \underline{U}$,

$$\underline{P}\ \underline{S}^+\ \underline{V} = \underline{P}\ \underline{S}^+\ \underline{P}^T\ \underline{P}\ \underline{V} \tag{A.5}$$

Define

$$\underline{W} = (\ \underline{P}\ \underline{V}\ )^{-1} \tag{A.6}$$

Then from (A.4) and (A.5)

$$\underline{P}\ \underline{S}^+\ \underline{P}^T = \begin{bmatrix} \underline{0} & \underline{N}^T \\ \underline{0} & \underline{U} \end{bmatrix} \underline{W} = \begin{bmatrix} \underline{0} & \underline{N}^T \\ \underline{0} & \underline{U} \end{bmatrix} \begin{bmatrix} \underline{W}_{11} & \underline{W}_{12} \\ \underline{W}_{21} & \underline{W}_{22} \end{bmatrix} = \begin{bmatrix} \underline{N}^T \\ \underline{U} \end{bmatrix} \begin{bmatrix} \underline{W}_{21} & \underline{W}_{22} \end{bmatrix} \tag{A.7}$$

and

$$(\ \underline{P}\ \underline{S}^+\ \underline{P}^T\ )^2 = \begin{bmatrix} \underline{N}^T \\ \underline{U} \end{bmatrix} \begin{bmatrix} \underline{W}_{21} & \underline{W}_{22} \end{bmatrix} \begin{bmatrix} \underline{N}^T \\ \underline{U} \end{bmatrix} \begin{bmatrix} \underline{W}_{21} & \underline{W}_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{N}^T \\ \underline{U} \end{bmatrix} \begin{bmatrix} \underline{W}_{21}\ \underline{N}^T + \underline{W}_{22} \end{bmatrix} \begin{bmatrix} \underline{W}_{21} & \underline{W}_{22} \end{bmatrix} \tag{A.8}$$

Given (A.3), it follows that

$$(\ \underline{P}\ \underline{S}^+\ \underline{P}^T\ )^2 - \underline{P}\ \underline{S}^+\ \underline{P}^T = \underline{P}\ \{\ (\underline{S}^+)^2 - \underline{S}^+\ \}\ \underline{P}^T = \underline{0} \tag{A.9}$$

and utilizing (A.7) and (A.8) in (A.9), we obtain

$$\begin{bmatrix} \underline{N}^T \\ \underline{U} \end{bmatrix} \begin{bmatrix} \underline{W}_{21}\ \underline{N}^T + \underline{W}_{22} - \underline{U} \end{bmatrix} \begin{bmatrix} \underline{W}_{21} & \underline{W}_{22} \end{bmatrix} = \underline{0} \tag{A.10}$$

Since $\underline{W}$ is nonsingular, $[\ \underline{W}_{21}\ \ \underline{W}_{22}]$ has maximum row rank and this implies

$$\underline{W}_{21}\ \underline{N}^T + \underline{W}_{22} - \underline{U} = \underline{0}$$

or,

$$\underline{W}_{22} = \underline{U} - \underline{W}_{21}\, \underline{N}^T \tag{A.11}$$

Utilizing (A.11) in (A.7) yields

$$\underline{P}\; \underline{S}^{+}\; \underline{P}^T = \begin{bmatrix} \underline{N}^T \\ \underline{U} \end{bmatrix} \begin{bmatrix} \underline{W}_{21} & \underline{U} - \underline{W}_{21}\, \underline{N}^T \end{bmatrix} \tag{A.12}$$

From (A.2) and (A.12),

$$\begin{aligned} \underline{S} &= \underline{P}^T \left(2\, \underline{P}\; \underline{S}^{+}\; \underline{P}^T - \underline{U} \right) \underline{P} \\ &= \underline{P}^T \begin{bmatrix} 2\underline{N}^T\underline{W}_{21} - \underline{U} & 2\underline{N}^T(\underline{U} - \underline{W}_{21}\underline{N}^T) \\ 2\underline{W}_{21} & \underline{U} - 2\underline{W}_{21}\underline{N}^T \end{bmatrix} \underline{P} \\ &= \underline{P}^T \begin{bmatrix} -\underline{U} & \underline{N}^T \\ \underline{0} & \underline{U} \end{bmatrix} \begin{bmatrix} -\underline{U} & \underline{0} \\ -2\underline{W}_{21} & \underline{U} \end{bmatrix} \begin{bmatrix} -\underline{U} & \underline{N}^T \\ \underline{0} & \underline{U} \end{bmatrix} \underline{P} \end{aligned} \tag{A.13}$$

Clearly (A.13) is the result we require with $\underline{K} \equiv \underline{W}_{21}$. The submatrices $\underline{P}$ and $\underline{N}$ are determined through (A.4) and $2\underline{K}$ is then the $t \times \ell$ submatrix in the lower left hand corner of $\underline{PSP}^T$.

Appendix B

A SOLUTION FOR THE CONDUCTANCE MATRIX OF THE JAUMANN ADAPTOR

Consider the general fifth-order Jaumann prototype network introduced in Subsection 3.7.5. The adaptor scattering representation is specified by

$$\underline{N} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \underline{K} = (\,k_{ij})_{4\times 3}\,, \quad \underline{P} = \begin{bmatrix} \underline{0} & \underline{U}_3 \\ \underline{U}_4 & \underline{0} \end{bmatrix}$$

where the entries of $\underline{K}$ have the property

$$k_{11} = -k_{21} \qquad k_{12} = -k_{22} \qquad k_{13} = k_{23}$$

$$k_{33} = 0 \qquad k_{41} = 0 \qquad k_{42} = 0$$

The matrix $\underline{Q}$, where $\underline{Q} = \underline{K}\,(\underline{U} - \underline{N}^T\underline{K})^{-1}$, will have the general form

$$\underline{Q} = \begin{bmatrix} q_1 & q_2 & q_3 \\ -q_1 & -q_2 & q_3 \\ q_4 & q_5 & 0 \\ 0 & 0 & q_6 \end{bmatrix}$$

Consider equation (3.80), that is

$$\underline{Q} = \underline{G}_t^{-1}\,\underline{N}\,\underline{G}_\ell \tag{B.1}$$

for the situation

$$\underline{G}_t = \text{diag}[\ g_1,\ g_2,\ g_3,\ g_4\ ]\ , \qquad \underline{G}_\ell = \begin{bmatrix} g_5 & g_o & 0 \\ g_o & g_6 & 0 \\ 0 & 0 & g_7 \end{bmatrix} \tag{B.2}$$

Equation (B.1) becomes

$$\begin{bmatrix} q_1 & q_2 & q_3 \\ -q_1 & -q_2 & q_3 \\ q_4 & q_5 & 0 \\ 0 & 0 & q_6 \end{bmatrix} = \begin{bmatrix} (g_5 + g_o)/g_1 & (g_6 + g_o)/g_1 & g_7/g_1 \\ -(g_5 + g_o)/g_2 & -(g_6 + g_o)/g_2 & g_7/g_2 \\ g_o/g_3 & g_6/g_3 & 0 \\ 0 & 0 & g_7/g_4 \end{bmatrix} \tag{B.3}$$

A solution to (B.3) is given by

$$\frac{g_2}{g_1} = 1 \tag{B.4a}$$

$$\frac{g_3}{g_1} = \frac{q_2}{q_4 + q_5} \tag{B.4b}$$

$$\frac{g_4}{g_1} = \frac{q_3}{q_6} \tag{B.4c}$$

$$\frac{g_5}{g_1} = q_1 - \frac{q_2 q_4}{q_4 + q_5} \tag{B.4d}$$

$$\frac{g_6}{g_1} = \frac{q_2 q_5}{q_4 + q_5} \tag{B.4e}$$

$$\frac{g_7}{g_1} = q_3 \tag{B.4f}$$

$$\frac{g_o}{g_1} = \frac{q_2 q_4}{q_4 + q_5} \tag{B.4g}$$

Clearly, (B.2) and (B.4) specify a solution for $\underline{G} = \underline{G}_t \dotplus \underline{G}_\ell$.

Appendix C

DERIVATION OF BOUNDS ON THE CONDUCTANCE MATRIX

In this appendix we consider the condition

$$(1+\gamma)\underline{x}^T\underline{W}\,\underline{x} - (\underline{x}+\underline{e})^T\underline{W}(\underline{x}+\underline{e}) > 0 \qquad \text{(C.1)}$$

where γ is a positive scalar, $\underline{W}$ is a 2×2 positive definite symmetric matrix, and where the elements of $\underline{x} = [\ x_1 \quad x_2]^T$ and $\underline{e} = [\ e_1 \quad e_2]^T$ satisfy

$$\mathrm{sgn}(e_i) = -\,\mathrm{sgn}(x_i) \qquad \text{(C.2a)}$$

$$\mathrm{sgn}(x_i + e_i) = \mathrm{sgn}(x_i) \qquad \text{(C.2b)}$$

Bounds on γ and the entries of $\underline{W}$ are derived which are sufficient to guarantee that (C.1) holds for all nonzero $\underline{x}$.

Equation (C.1) reduces to the following:

$$\gamma\underline{x}^T\underline{W}\,\underline{x} - 2\underline{x}^T\underline{W}\,\underline{e} - \underline{e}^T\underline{W}\,\underline{e} > 0 \qquad \text{(C.3)}$$

Let us decompose $\underline{W}$ as

$$\underline{W} = \underline{W}_D + \underline{W}_O \qquad \text{(C.4a)}$$

where

$$\underline{W}_D = \begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix} \qquad \underline{W}_O = \begin{bmatrix} 0 & W_{12} \\ W_{12} & 0 \end{bmatrix} \qquad \text{(C.4b)}$$

Utilizing (C.4), (C.3) can be reexpressed as

$$\gamma\, \underline{x}^T \underline{W}_D\, \underline{x} + \gamma\, \underline{x}^T \underline{W}_0\, \underline{x} - 2\underline{x}^T \underline{W}_D\, \underline{e}$$
$$> 2\, \underline{x}^T \underline{W}_0\, \underline{e} + \underline{e}^T \underline{W}_D\, \underline{e} + \underline{e}^T \underline{W}_0\, \underline{e} \qquad (C.5)$$

Given (C.2), it follows that

$$-\underline{x}^T \underline{W}_D\, \underline{e} \geq \underline{e}^T \underline{W}_D\, \underline{e} \qquad (C.6)$$

Therefore, (C.5) is satisfied if

$$\gamma\, \underline{x}^T \underline{W}_D\, \underline{x} + \gamma\, \underline{x}^T \underline{W}_0\, \underline{x} - \underline{x}^T \underline{W}_D\, \underline{e} > (2\underline{x} + \underline{e})^T \underline{W}_0\, \underline{e} \qquad (C.7)$$

The term on the right hand side of (C.7) is maximum for arbitrary $\underline{x}$ when we have

$$(2x_i + e_i)\, e_j\, W_{12} \geq 0 \quad i = 1,2, \quad j = 1,2 \quad i \neq j \qquad (C.8)$$

If (C.8) holds and (C.2) is true, then

$$x_i\, x_j\, W_{12} \leq 0 \quad i = 1,2 \quad j = 1,2 \quad i \neq j \qquad (C.9)$$

for which the left hand side of (C.7) is minimum. Clearly, (C.8) and (C.9) define the worst case for satisfying (C.7). Assuming (C.8) and (C.9) hold, we expand (C.7) as

$$\begin{aligned} &\gamma\, W_{11}\, x_1^2 + \gamma\, W_{22}\, x_2^2 - 2\gamma |x_1\, x_2\, W_{12}| \\ &\quad + |x_1\, e_1|\, W_{11} + |x_2\, e_2|\, W_{22} \\ &\quad > -2|e_1\, e_2\, W_{12}| + 2|x_1\, e_2\, W_{12}| + 2|x_2\, e_1\, W_{12}| \end{aligned} \qquad (C.10)$$

Utilizing the property

$$W_{11}\, x_1^2 + W_{22}\, x_2^2 \geq 2\, W_{11}^{1/2}\, W_{22}^{1/2}\, |x_1\, x_2| \qquad (C.11)$$

we can replace (C.10) with the condition

$$2\gamma\{W_{11}^{1/2}\,W_{22}^{1/2} - |W_{12}|\}\,|x_1\,x_2| + |x_1\,e_1|W_{11} + |x_2\,e_2|W_{22}$$
$$+\,2|e_1\,e_2\,W_{12}| > 2|x_1\,e_2\,W_{12}| + 2|x_2\,e_1\,W_{12}| \qquad \text{(C.12)}$$

Consider the special case $|x_1| \geq |x_2|$. The condition in (C.12) is satisfied if

$$2\gamma\{W_{11}^{1/2}\,W_{22}^{1/2} - |W_{12}|\}\,|x_1\,x_2| + |x_1\,e_1|\{W_{11} - 2|W_{12}|\}$$
$$+\,|x_2\,e_2|W_{22} + 2|e_1\,e_2\,W_{12}| > 2|x_1\,e_2\,W_{12}| \qquad \text{(C.13)}$$

If we further constrain

$$W_{11} \geq 2|W_{12}| \quad , \qquad W_{22} \geq 2|W_{12}| \qquad \text{(C.14)}$$

then all the terms on the left hand side of (C.13) are positive definite. Clearly (C.13) is satisfied if

$$2\gamma\{W_{11}^{1/2}\,W_{22}^{1/2} - |W_{12}|\}\,|x_1\,x_2| > 2|x_1\,e_2\,W_{12}|$$

or, since (C.2) ensures $|x_2| \geq |e_2|$, if

$$2\gamma\{W_{11}^{1/2}\,W_{22}^{1/2} - |W_{12}|\}\,|x_1\,e_2| > 2|x_1\,e_2\,W_{12}| \qquad \text{(C.17)}$$

For $|x_1\,e_2\,W_{12}| \neq 0$, (C.17) can be expressed as

$$\frac{\gamma\{W_{11}^{1/2}\,W_{22}^{1/2} - |W_{12}|\}}{|W_{12}|} > 1 \qquad \text{(C.18)}$$

or equivalently,

$$\frac{W_{11}^{1/2}\,W_{22}^{1/2}}{|W_{12}|} > \frac{1+\gamma}{\gamma} \qquad \text{(C.19)}$$

For the alternate case $|x_2| \geq |x_1|$, the identical result is obtained. Therefore we can conclude that (C.1) is satisfied for all nonzero $\underline{x}$ if (C.14) and (C.19) hold and signal quantizations satisfy (C.2).

REFERENCES

1. A.V. Oppenheim and R.W. Schafer, Digital Signal Processing. New Jersey: Prentice-Hall, 1975.

2. L.R. Rabiner and B. Gold, Theory and Application of Digital Signal Processing. New Jersey: Prentice-Hall, 1975.

3. T.M.C.M. Classen, W.F.G. Mecklenbräuker, and J.B.H. Peek, "Effects of Quantization and Overflow in Recursive Digital Filters," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-24, pp. 517-529, Dec. 1976.

4. C.M. Rader and B. Gold, "Effects of Parameter Quantization on the Poles of a Digital Filter," Proc. IEEE, vol. 55, pp. 688-689, May 1967.

5. L.B. Jackson, "On the Interaction of Roundoff Noise and Dynamic Range in Digital Filters," Bell Syst. Tech. J., vol. 49, pp. 159-184, Feb. 1970.

6. C.T. Mullis and R.A. Roberts, "Synthesis of Minimum Roundoff Noise Fixed-Point Digital Filters," IEEE Trans. Circuits Syst., vol. CAS-23, pp. 551-562, Sept. 1976.

7. K. Meerkötter and W. Wegener, "A New Second-Order Digital Filter Without Parasitic Oscillations," Arch. Elek. Übertragung, vol. 29, pp. 312-314, 1975.

8. G. Veerkroost and H.J. Butterweck, "Suppression of Parasitic Oscillations in Wave Digital Filters and Related Structures by Means of Controlled Rounding," Arch. Elek. Übertragung, vol. 30, pp. 181-186, May 1976.

9. C.W. Barnes and A.T. Fam, "Minimum Norm Recursive Digital Filters that are Free of Overflow Limit Cycles," IEEE Trans. Circuits Syst., vol. CAS-24, pp. 569-574, Oct. 1977.

10. A. Antoniou, Digital Filters: Analysis and Design. pp. 333-341, New York: Mcgraw-Hill, 1979.

11. C. Eswaran and V. Ganapathy, "On the Stability of Digital Filters Designed Using the Concept of Generalized-Immitance Converter," IEEE Trans. Circuits Syst., vol. CAS-28, pp. 745-747, July 1981.

12. L.B. Jackson, A.G. Lindgren and Y. Kim, "Optimal Synthesis of Second-Order State-Space Structures for Digital Filters," IEEE Trans. Circuits Syst., vol. CAS-26, pp.149-153, Mar. 1979.

13. C.W. Barnes, "Roundoff Noise and Overflow in Normal Digital Filters," IEEE Trans. Circuits Syst., vol. CAS-26, pp. 154-159, Mar. 1979.

14. W.L. Mills, C.T. Mullis, and R.A. Roberts, "Low Roundoff Noise and Normal Realizations of Fixed-Point IIR Digital Filters," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-29, pp. 893-903, Aug. 1981.

15. A. Fettweis, "Digital Filter Structures Related to Classical Filter Networks," Arch. Elek. Übertragung, vol. 25, pp. 79-89, Feb. 1971.

16. A. Fettweis, "Some Principles of Designing Digital filters Imitating Classical Filter Structures," IEEE Trans. Circuit Theory, vol. CT-18, pp. 314-316, Mar. 1971.

17. A. Sedlmeyer and A. Fettweis, "Digital Filters with True Ladder Configuration," Int. J. Circuit Theory and Applications, vol. 1, pp. 5-10, Mar. 1973.

18. A. Fettweis and K. Meerkötter, "On Adaptors for Wave Digital Filters," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-23, pp. 516-525, Dec. 1975.

19. A. Fettweis, H. Levin, and A. Sedlmeyer, "Wave Digital Lattice Filters," Int. J. Circuit Theory and Applications, vol. 2, pp. 203-211, June 1974.

20. R. Nouta, "The Jaumann Structure in Wave Digital Filters," Int. J. Circuit Theory and Applications, vol. 2, pp. 163-174, June 1974.

21. A. Fettweis, "Pseudopassivity, Sensitivity, and Stability of Wave Digital Filters," IEEE Trans. Circuit Theory, vol. CT-19, pp. 668-673, Nov. 1972.

22. A. Fettweis, "On the Connection Between Multiplier Wordlength Limitation and Roundoff Noise in Digital Filters," IEEE Trans. Circuit Theory, vol. CT-19, pp. 486-491, Sept. 1972.

23. A. Fettweis, "Roundoff Noise and Attenuation Sensitivity in Digital Filters with Fixed-Point Arithmetic," IEEE Trans. Circuit Theory, vol. CT-20, pp. 174-175, Mar. 1973.

24. A. Fettweis, "On Sensitivity and Roundoff Noise in Wave Digital Filters," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-22, pp.383-384, Oct. 1974.

25. A. Fettweis and K. Meerkötter, "Suppression of Parasitic Oscillations in Wave Digital Filters," IEEE Trans. Circuits Syst., vol. CAS-22, pp. 239-246, Mar. 1975; also "Correction to 'Suppression of Parasitic Oscillations in Wave Digital Filters,'" IEEE Trans. Circuits Syst., vol. CAS-22, p. 575, June 1975.

26. A. Fettweis, "Canonic Realizations of Ladder Wave Digital Filters," Int. J. Circuit Theory and Applications, vol. 3, pp. 321-332, 1975.

27. K. Meerkötter, "Canonic Realization of Wave Digital Filters Involving Unit Elements," Int. J. Circuit Theory and Applications, vol. 5, pp.395-402, Oct. 1977.

28. A.T. Ashley, "Minimal Wave Digital Filter Realizations: The n-Port Approach," Ph.D. Thesis, University of Manitoba, Canada, May 1978.

29. G.O. Martens and K. Meerkötter, "On n-Port Adaptors for Wave Digital Filters with Applications to a Bridged-Tee Filter," Proc. 1976 IEEE Symp. Circuits and Systems, Munich, pp.514-517.

30. H.H. Lê, "Wave Digital Adaptors for Brune, Darlington C and D, and Twin T Sections," Ph.D. Thesis, University of Manitoba, Canada, Aug. 1977.

31. G.O. Martens and H.H. Lê, "Wave Digital Adaptors for Reciprocal Second-Order Sections," IEEE Trans. Circuits Syst., vol. CAS-25, pp. 1077-1083, Dec. 1978.

32. G.O. Martens and M.R. Jarmasz, "Design of Digital Filters with a Diagonal Lyapunov Function Based on Elliptic Reference Filters," Proc. 1982 IEEE Symp. Circuits and Systems, Rome, pp. 527-530.

33. M.R. Jarmasz, "Design of Canonic Wave Digital Filters which Supress All Zero-Input Parasitic Oscillations," M.Sc. Thesis, University of Manitoba, Canada, 1983.

34. H.J. Orchard, "Loss Sensitivity in Singly and Doubly Terminated Filters," IEEE Trans. Circuits Syst., vol. CAS-26, pp. 293-297, May 1979.

35. G.C. Temes and S.K. Mitra, Modern Filter Theory and Design. New York: John Wiley and Sons, 1973.

36. R.E. Crochieré and A.V. Oppenheim, "Analysis of Linear Digital Networks," Proc. IEEE, vol. 63, pp. 581-595, Apr. 1975.

37. A. Fettweis, "Realizability of Digital Filter Networks," Arch. Elek. Übertragung, vol. 30, pp. 90-96, Feb. 1976.

38. R.E. Crochieré, "Digital Ladder Structures and Coefficient Sensitivity," IEEE Trans. Audio Electroacoust., vol. AU-20, pp. 240-246, Oct. 1972.

39. K. Renner and S.C. Gupta, "On the Design Of Wave Digital Filters with Low Sensitivity Properties," IEEE Trans. Circuit Theory, vol. CT-20, pp. 555-567, Sept. 1973.

40. W.H. Ku and S. Ng, "Floating-Point Coefficient Sensitivity and Roundoff Noise of Recursive Digital Filters Realized in Ladder Structures," IEEE Trans. Circuits Syst., vol. CAS-22, pp. 927-936, Dec. 1975.

41. W. Wegener, "Design of Wave Digital Filters with Very Short Coefficient Word Lengths," Proc. 1976 IEEE Symp. Circuits and Systems, Munich, pp. 473-476.

42. K. Renner and S.C. Gupta, "Reduction of Roundoff Noise in Wave Digital Filters," IEEE Trans. Circuits Syst., vol. CAS-21, pp. 305-310, Mar. 1974.

43. T.M.C.M. Classen, W.F.G. Mecklenbräuker, and J.B.H. Peek, "On the Stability of the Forced Response of Digital Filters with Overflow Nonlinearities," IEEE Trans. Circuits Syst., vol. CAS-22, pp. 692-696, Aug. 1975.

44. K. Meerkötter, "Incremental Pseudopassivity of Wave Digital Filters," Proc. First European Signal Processing Conf., Lausanne, pp. 27-31, Sept. 1980.

45. R. Saal, Handbook of Filter Design. AEG Telefunken, Backnang, F.R. Germany, 1979.

46. S. Vongpanitlerd, "Reciprocal Lossless Synthesis Via State-Variable Techniques," IEEE Trans. Circuit Theory, vol. CT-17, pp. 630-632, Nov. 1970.

47. V. Belevitch, Classical Network Theory. San Francisco: Holden Day, 1968.

48. B.D.O. Anderson and S. Vongpanitlerd, Network Analysis and Synthesis: A Modern Systems Theory Approach. New Jersey: Prentice-Hall, 1973.

49. S. Vongpanitlerd and B.D.O. Anderson, "Scattering Matrix Synthesis Via Reactance Extraction," IEEE Trans. Circuit Theory, vol. CT-17, pp. 511-517, Nov. 1970.

50. A. Fettweis, "Factorization of Transfer Matrices of Lossless Two-Ports," IEEE Trans. Circuit Theory, vol. CT-17, pp. 86-94, Feb. 1970.

51. C.T. Chen, Introduction to Linear System Theory. New York: Holt, Rinehart and Winston, 1970.

52. R. Nouta, "Wave Digital Cascade Synthesis," Int. J. Circuit Theory and Applications, vol. 3, pp. 231-247, Sept. 1975.

53. L. Weinberg, Network Analysis and Synthesis. New York: McGraw-Hill, 1962.

54. P.R. Moon, "On Near Diagonal Lyapunov Functions in Digital Filters: Limit-Cycle Suppression and Roundoff-Noise Reduction," Internal Report TR81-1, Dept. of Elec. Eng., University of Manitoba, Jan. 1981.

55. A. Croisier, D.J. Esteban, M.E. Levilion, and V. Rizo, "Digital Filters for PCM Encoded Signals," U.S. Patent 3 777 130, Dec. 1973.

56. A. Peled and B. Liu, "A New Hardware Realization of Digital Filters," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-22, pp. 456-462, Dec. 1974.

57. P.R. Moon and G.O. Martens, "A Digital Filter Structure Requiring Only m-Bit Delays, Shifters, Inverters, and m-Bit Adders Plus Simple Logic Circuitry," IEEE Trans. Circuits Syst., vol. CAS-27, pp. 901-908, Oct. 1980.

58. T.M.C.M. Classen, W.F.G. Mecklenbräuker, and J.B.H. Peek, "Some Considerations on the Implementation of Digital Systems for Signal Processing," Philips Res. Repts., vol. 30, pp. 73-84, 1975.

59. E. Simonyi and A. Heszberger, "A Modular Approach to Multiplexed Wave Digital Matrix Filters," Proc. 1980 IEEE Symp. Circuits and Systems, Houston, pp. 1125-1128.

60. L. Wanhammer, "An Approach to LSI Implementation of Wave Digital Filters," Ph.D. Dissertation, Linkoping University, Sweden, April 1981.

61. M. Arjmand and R.A. Roberts, "On Comparing Hardware Implementations of Fixed-Point Digital Filters," IEEE Circuits Syst. Magazine, vol. 3, pp 2-8, June 1981.

62. L.A. Schmidt, "Designing Programmable Digital Filters for LSI Implementation," Hewlett-Packard Journal, Sept. 1981.

63. W. Wegener, "Entwurf von Wellendigitalfiltern mit minimalen Realisierungsaufwand," Ph.D. Dissertation, Ruhr-Universität, Bochum, F.R. Germany, 1979.

64. K. Meerkötter, "Beiträge zur Theorie der Wellendigitalfilter," Ph.D. Dissertation, Ruhr-Universität, Bochum, F.R. Germany, 1979.

65. G.O. Martens, Private Communication, 1981.

66. L.R. Rabiner et. al., "Terminology in Digital Signal Processing," IEEE Trans. Audio Electroacoust., vol. AU-20, pp. 322-337, Dec. 1972.