Singular Solutions of a Hénon Equation Involving Nonlinear Gradient Terms

by

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To my parents and my sister, the light of my life.

Abstract

This thesis explores a class of nonlinear elliptic partial differential equations known as Hénon-type equations, employed in various scientific domains including physics, biology, and applied mathematics. These equations are particularly useful in modeling pattern formation, reaction-diffusion processes, and population dynamics. The central focus is on determining solutions to equations of the form:

$$\begin{cases} Lu = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We investigate the existence of positive singular solutions for Hénon-type equations involving nonlinear gradient terms. Specifically, we study equations of the form:

$$\begin{cases} -\Delta u = (1 + g(x))|x|^{\alpha}|\nabla u|^{p}, & \text{in } B_1 \backslash \{0\}, \\ u = 0, & \text{on } \partial B_1, \end{cases}$$

where $p > 1$, $\alpha > 0$, B_1 is the unit ball centered at the origin in $\mathbb{R}^N$, and $g(x)$ is a Hölder continuous function with $g(0) = 0$. We prove the existence of positive singular solutions under various parameters under various conditions.

Moreover, we extend the analysis to exterior domains in $\mathbb{R}^N$, exploring the existence of positive classical solutions to equations of the form:

$$\begin{cases} -\Delta u = |x|^{\alpha}|\nabla u|^{p}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is an exterior domain that does not contain the origin in its closure.

We also consider the case where the solution has two singular points in Ω. In particular, we consider:

$$\begin{cases} -\Delta u = |\nabla u|^p, & \text{in } \Omega\backslash\{\xi_1, \xi_2\}, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $\xi_1, \xi_2 \in \Omega$. This work contributes to understanding singular solutions in Hénon-type equations, offering new insights into their behavior across various domains and parameter ranges.

Contents

Notations and Definitions

Notations

- $\mathbb{R}^N$ is the Euclidean N-dimensional space with points $x = (x_1, ..., x_N)$,
- B_1: unit ball centered at the origin in $\mathbb{R}^N$,
- S^{N-1} denotes the unit $N-1$-dimensional sphere, $\{x \in \mathbb{R}^N : \|x\| = 1\}$,
- ∂U: Boundary of U,
- ΔU: Laplacian of U,
- ∇U: Gradient of U,
- $V \subset\subset U$: V is compactly contained in U if $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact,
- $L^p_{\text{loc}}(U) = \{u : U \to \mathbb{R} : u \in L^p(V) \text{ for each } V \subset\subset U\}$,
- $C(U) = \{u : U \to \mathbb{R} \mid u \text{ is continuous}\}$,
- $C^k(U) = \{u : U \to \mathbb{R} \mid u \text{ is k-times continuously differentiable}\}$,
- $C^{k,\alpha}(U)$: Hölder space $(0 < \alpha \leq 1)$,

$$C^{k,\alpha}(\bar{U}) = \left\{ u \in C^k(\bar{U}) \middle| \, \|u\|_{C^{k,\alpha}(U)} < \infty \right\},$$

where $\|u\|_{C^{k,\alpha}(U)} = \sum_{|j|\leq k} \left\| D^j u \right\|_{C(\bar{U})} + \sum_{|j|=k} \sup_{\substack{x,y \in U, \\ x \neq y}} \dfrac{|D^j u(x) - D^j u(y)|}{|x-y|^\alpha}.$

Definitions and Theorems

Definition 0.1. ***Banach Space***

A Banach space is a normal vector space where every Cauchy sequence converges.

Definition 0.2. ***Hölder Continuous Function***

Let U be open and $0 < \alpha \leq 1$. The function $f : U \to \mathbb{R}$ is Hölder continuous function with exponent α if there are real constants $C \geq 0$, such that

$$|f(x) - f(y)| \leq C \|x - y\|^{\alpha}, \qquad x, y \in U.$$

Note. The Hölder continuous function plays a significant role in ensuring the existence of positive singular solutions in partial differential equations (PDEs) by providing suitable conditions for the existence and behavior of these solutions. Hölder continuous functions have a certain level of smoothness and regularity, which can help smooth the singularity by providing a smoother transition near the singular point. Moreover, Hölder continuous functions can prevent solutions from becoming unbounded near singular points. This controlled growth property is essential for ensuring that positive singular solutions remain bounded and well-behaved about the singularity. Overall, Hölder continuous functions play a crucial role in establishing the existence of these solutions and understanding their properties near singular points.

Definition 0.3. ***Sobolev Space*** *[17, Chapter 5]*

The Sobolev Space $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^{\alpha}u$ exists in the weak sense and belongs to $L^p(U)$.

Definition 0.4. ***Elliptic Operator*** *[17, Chapter 6]*

Let U be an open bounded subset of $\mathbb{R}^n$, and $u : \bar{U} \to \mathbb{R}$. Consider the second-order partial differential operator L defined by

$$Lu = -\sum_{i,j=1}^{n} a_{i,j}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u.$$

We say partial differential operator L is elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \theta|\xi|^2,$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

Definition 0.5. ***Laplace-Beltrami Operator*** *[22, Chapter 2]*

In our work, we will need the Laplace-Beltrami operator $\Delta_{S^{N-1}}$ (which we will on occasion write as Δ_θ) on S^{N-1} but we do not need any profound knowledge of it. For further details on this operator on general manifolds, refer to the above reference. Let Δ_θ denote the Laplace-Beltrami operator on S^{N-1} (0.5). Consider the first few eigenpairs of $-\Delta_\theta\psi_k(\theta) = \lambda_k\psi_k(\theta)$, where $\theta \in S^{N-1}$. The eigenvalues are given by $\lambda_0 = 0, \lambda_1 = N-1, \lambda_2 = 2N$ and where ψ_k is $L^2(S^{N-1})$ normalized. The family $\{\psi_k\}_{k\geq 1}$ forms an orthonormal basis for $L^2(S^{N-1})$, allowing for a suitable expansion of functions on S^{N-1}. Given a function $v(x)$ defined on $B_R = \{x \in \mathbb{R}^N : |x| < R\}$ where $R \in (0, \infty]$, we can write

$$v(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta),$$

where $r = |x|$ and $\theta = \frac{x}{|x|}$. The exact convergence of this series depends on the smoothness properties of $v(x)$. A beneficial fact is that we can express it as follows

$$\Delta_x v(x) = v_{rr} + \frac{N-1}{r}v_r + \frac{\Delta_\theta v}{r^2}.$$

Definition 0.6. ***Cut-off Function***

Let $K_1, K_2 \subset \Omega$ such that $K_1 \subset K_2$, K_1 be compact and K_2 be open. Function $\eta \in C^\infty(\Omega)$ is called a cut-off function if it satisfies the following properties:

- $0 \leq \eta \leq 1$ *in* Ω,
- $\eta = 1$ *in* K_1,
- *supp*$(\eta) \subset K_2$.

Definition 0.7. ***Emden–Fowler Change of Variables***

The Emden–Fowler change of variables is given by

$$v(t, \theta) = r^{\gamma_p} u(r, \sigma) \quad where \quad t = \ln r, \quad \theta = \sigma = x|x|^{-1}.$$

This change of variables can be useful when trying to obtain behaviour of a solution near $r = 0$.

Definition 0.8. ***Positive Singular Solutions***

The partial differential equations (PDEs) we examine are all second-order. By definition, a classical solution is a solution which is at least C^2. If a solution is not a classical solution, it is referred to as a singular solution*. In this work, we will have singular solutions that are unbounded near a point, and we will also have solutions that are bounded near a certain point but are not C^2 within a neighborhood of the points.*

Theorem 0.9. ***Strong Maximum Principle*** *[17, p. 333]*

Let

$$Lu = -\sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u.$$

Assume that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $c \geq 0$ in Ω. Suppose Ω is connected. If $Lu \leq 0$ in Ω and u attains a non-negative maximum over $\bar{\Omega}$ at an interior point, then u is constant within Ω.

Definition 0.10. ***A priori estimate***

Assume the existence of the solution to a PDE, and prove results about the solutions, such as bounds on the size, smoothness, and so on. A priori estimates are the bounds on the solution one gets before one knows the solution exists. Under specific conditions, one can apply these bounds to the solution to show that a solution exists.

Theorem 0.11. ***Method of Continuity*** *[13, Theorem 5.2]*

Let B be a Banach space and V be a normed space. Assume $L_0, L_1 : B \to V$ are bounded linear operators. Define $L_t = (1-t)L_0 + tL_1$ for $t \in [0,1]$. Assume there is a constant $C > 0$ such that for all $t \in [0,1]$, $\|u\|_B \leq C \|L_t(u)\|_V$. Then L_0 is onto if and only if L_1 is onto.

Theorem 0.12. ***Banach fixed point theorem****[17, Theorem 1, Section 9.2]*

Let (X, d) be a complete metric space and $T : X \to X$ is a contraction mapping, i.e., there exists $0 < \gamma < 1$ such that for all $x, y \in X$, we have $d(Tx, Ty) \leq \gamma d(x, y)$. Then T has a unique fixed point x_*, which implies $Tx_* = x_*$.

Theorem 0.13. *[19, Theorem 1]*

In the ball $\Omega = \{x \in \mathbb{R}^N : |x| < R\}$, let $u > 0$ be a positive solution in $C^2(\bar{\Omega})$ of

$$\Delta u + f(u) = 0 \text{ in } \Omega, \quad \text{with} \quad u = 0 \quad \text{on} \quad |x| = R,$$

with f a C^1 function. Then u is radially symmetric and $\dfrac{\partial u}{\partial r} < 0$ for $0 < r < R$.

1

Introduction

The term *Hénon equation* became associated with the partial differential equation $-\Delta u = |x|^\alpha u^{p-1}$, for $\alpha > 0, p > 2$, in honor of French mathematician and astronomer Michel Hénon. Hénon-type equations in PDE refer to a class of partial differential equations that share similar characteristics or structural features with the original Hénon equation. Hénon-type equations have been employed in various scientific fields, including physics, biology, and applied mathematics. They can be used to model phenomena such as reaction-diffusion processes, population dynamics, or complex dynamics in dynamical systems (see [10], [11], [14], [18] and the references therein). For instance, in the introduction of the paper [15], it is established that the existence of stationary stellar dynamics models is equivalent to the solvability of the equation $-\Delta u = f(|x|, u)$ in $\mathbb{R}^3$. Specifically. if $f(|x|, u) = |x|^\alpha |u|^{p-2}u$, then it is a Hénon type equation, where the weight $|x|^\alpha$ represents a black hole located at the center of the cluster (see [23], [15] and the references therein). Additionally, a model describing the dynamics of elliptical galaxies was proposed by the authors in [7], given by the equation $-\Delta u(x) = \phi(r)|u|^{p-2}u$ in $\mathbb{R}^3$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $r = \sqrt{x_1^2 + x_2^2}$.

Extensive research has been conducted on determining a solution to the equation

$Lu = f$ in Ω with $u = 0$ on $\partial\Omega$. Interested readers are encouraged to consult the references provided for further details. [1, 2, 3, 4, 5, 6, 12, 16, 20, 21, 24, 25, 26, 27, 28]. In 1973, Ambrosetti and Rabinowitz [5] studied second-order uniformly elliptic partial differential equations

$$Lu = -\sum_{i,j=1}^{n} (a_{ij} u_{x_i})_{x_j} + c(x)u = g(x, u) \quad x \in \Omega,$$

with $u = 0$ on $\partial\Omega$, and Ω is a smooth bounded domain in $\mathbb{R}^N$. The term $g(x, u)$ is nonlinear, sufficiently regular in u, and its growth must be less than $|u|^p$ for some $1 < p < \frac{N+2}{N-2}$ when $|u|$ is large, while also satisfying additional conditions.

In 1982, Ni [25] investigated an elliptic boundary value problem on the unit ball in $\mathbb{R}^n$, providing valuable insights into nonlinear Dirichlet problems and their practical applications. The problem is

$$\Delta u + b(|x|)f(u) = 0 \quad x \in \Omega,$$

with $u = 0$ on $\partial\Omega$. Among the assumptions, $b(r), f(z)$ are nonnegative Hölder continuous functions. Ni demonstrated that

$$\Delta u + |x|^{\alpha} u^p = 0 \quad x \in B_1,$$

with $u = 0$ on ∂B_1 and $\alpha > 0$ has a positive solution provided $p \in \left(1, \dfrac{N + 2 + 2\alpha}{N - 2}\right)$.

In 1987, Patricio Aviles [6] investigated the following equation, which is relevant to Yang-Mills-Higgs theory as $n = 3$ (see [9],[8]) and astrophysics due to its connection to certain physical phenomena and mathematical models,

$$\Delta u + |x|^{\sigma} u^{\frac{n+\sigma}{n-2}} = 0 \quad \text{in} \quad B_1 \backslash \{0\}.$$

Aviles demonstrated this equation has a nonnegative solution for $\sigma \in (-2, 2)$.

In 2018, Aghajani et al. in [2] established the existence of solutions for the problem

$$\begin{cases} -\Delta u = |\nabla u|^p, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where $0 < p < 1$ and Ω is a bounded smooth domain in $\mathbb{R}^N$, by employing the sub-super solution method.

In 2020, Aghajani and Cowan in [1], considered the existence of positive singular solutions on bounded domains

$$\begin{cases} -\Delta u = (1 + g(x))|\nabla u|^p, & \text{in} \quad B_1, \\ u = 0, & \text{on} \quad \partial B_1, \end{cases}$$

where $g(x)$ is a Hölder continuous function with $\mathrm{g}(0) = 0$. They also explored singular solutions for

$$\begin{cases} -\Delta u = |\nabla u|^p, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

in both cases, Ω is a bounded smooth domain in $\mathbb{R}^N$ and Ω is an exterior domain with smooth boundary in $\mathbb{R}^N$.

In 2023, Aghajani et al. [3] demonstrated the existence of solution for

$$\begin{cases} -\Delta u = (1 + g(x))|\nabla u|^p, & \text{in} \quad \mathbb{R}^N_+, \\ u = 0, & \text{on} \quad \partial\mathbb{R}^N_+, \end{cases}$$

provided $p \in (\frac{4}{3}, 2)$ and g is bounded with uniform radial decay to zero.

2

Singular Solutions of a Hénon Equation Involving Nonlinear Gradient Terms on Bounded Domains

2.1 Introduction

We consider the existence of positive singular solutions of the following Hénon type equations involving nonlinear gradient terms as follows:

$$\begin{cases} -\Delta u = (1+g(x))|x|^{\alpha}|\nabla u|^{p}, & \text{in } B_1\backslash\{0\}, \\ u = 0, & \text{on } \partial B_1, \end{cases} \tag{2.1}$$

where $p > 1$, $\alpha > 0$, B_1 is the unit ball centred at the origin in $\mathbb{R}^N$ and $g \geq 0$ is a Hölder continuous function with $g(0) = 0$.

Here we prove the existence of positive singular solutions of the unperturbed problem (i.e., $g(x)=0$)

$$\begin{cases} -\Delta u = |x|^{\alpha}|\nabla u|^{p}, & \text{in } B_1\backslash\{0\}, \\ u = 0, & \text{on} \quad \partial B_1, \end{cases} \tag{2.2}$$

under various assumptions on the parameters.

2.2 The Unperturbed Problem

In this section we consider the case of $g(x)=0$.

The parameters for Chapter 2. Suppose $N \geq 3$, $\alpha > 0$ and $\frac{\alpha+N}{N-1} < p < 2+\alpha$. The parameters are defined as follows,

$$\xi = (N-1)(p-1), \quad \sigma = \frac{\alpha+2-p}{p-1} = -\tau, \quad \text{and} \quad \beta = \frac{p-1}{p(N-1)-N-\alpha}.$$

Under the above restrictions on p, α and N one can show $\xi > 0$, $\beta > 0$, $p > \frac{\alpha+N}{N-1}$, and $N - \frac{p}{\beta} - \sigma - 2 = -(p-1)\beta^{-1} = -(\xi-\alpha-1)$. Then, for all $t \geq 1$, there is a positive singular radial solution of 2.2 given by $u_t(r) = \int_r^1 \left(ty^{\xi} + \beta y^{\alpha+1}\right)^{\frac{-1}{p-1}} dy$. Moreover, one has the following asymptotic relations:

$$\begin{aligned} &|u_t'(r)| \leq \min\left(\frac{1}{t^{\frac{1}{p-1}} r^{N-1}}, \frac{C_\beta}{r^{\sigma+1}}\right), \\ &r^{\sigma+1}|u_t'(r)| \leq \min\left(\frac{1}{t^{\frac{1}{p-1}} r^{\frac{1}{\beta}}}, C_\beta\right), \\ &\lim_{r\to 0} r^{\sigma+1}u_t'(r) = -C_\beta, \end{aligned}$$

where $C_\beta = \beta^{-\frac{1}{p-1}}$. To see that assume $u'(r) < 0$. Thus, we have

$$-u''(r) - \frac{N-1}{r}u'(r) = r^{\alpha}(-u'(r))^{p}. \tag{2.3}$$

Let $z(r) = -u'(r)$. Therefore, the equation (2.3) changes to a Bernoulli equation,

$$z'(r) + \frac{N-1}{r} z(r) = r^{\alpha} (z(r))^{p}.$$

Because $z(r) \neq 0$, thus

$$(z(r))^{-p} z'(r) + \frac{N-1}{r} (z(r))^{1-p} = r^{\alpha}. \tag{2.4}$$

Let $w(r) = (z(r))^{1-p}$, as a result, the equation (2.4) changes to

$$w'(r) - \frac{(N-1)(p-1)}{r} w(r) = (1-p) r^{\alpha}.$$

We now use the integrating factor method to rewrite this as (we multiplied the above equation by $r^{-\xi}$ and rewrote)

$$\frac{d}{dr}\left(r^{-\xi} w(r)\right) = r^{-\xi} w'(r) - \xi r^{-\xi-1} w(r) = (1-p) r^{\alpha-\xi}. \tag{2.5}$$

Taking integral of equation (2.5), we have

$$r^{-\xi} w(r) = t + \frac{1-p}{\alpha - \xi + 1} r^{\alpha-\xi+1}, \tag{2.6}$$

where t is constant. Therefore, we have $w(r) = t r^{\xi} + \beta r^{\alpha+1}$. Thus,

$$z(r) = w(r)^{\frac{1}{1-p}} = \left(t r^{\xi} + \beta r^{\alpha+1}\right)^{\frac{1}{1-p}} = \frac{1}{\left(t r^{\xi} + \beta r^{\alpha+1}\right)^{\frac{1}{p-1}}}.$$

Therefore,

$$u'(r) = \frac{-1}{\left(t r^{\xi} + \beta r^{\alpha+1}\right)^{\frac{1}{p-1}}}.$$

For all $t \geq 0$, we have $u'(r) < 0$. Taking integral, we have

$$u_t(r) = u(r) - u(1) = \int_1^r u'(y)dy = \int_r^1 \frac{dy}{(ty^\xi + \beta y^{\alpha+1})^{\frac{1}{p-1}}}.$$

This chapter aims to prove the following theorem.

Theorem 2.1. *Suppose $N \geq 3$, $\alpha > 0$ and $\frac{\alpha+N}{N-1} < p < 2+\alpha$. Assume g is Hölder continuous with $g(0) = 0$. Then there is some $t_0 > 0$ (large) such that for all $t \geq t_0$ there is a solution u_t of (2.1).*

2.3 Outline of Approach

We provide the outline of the approach in the case of (2.1). Set $v_t(x) = u_t(r)$ from the unperturbed problem 2.2 and we look for a solution of (2.1) of the form $u(x) = v_t(x)+\phi(x)$ for large t and where ϕ is an unknown. Plug in $u(x) = v_t(x)+\phi(x)$ in equation $-\Delta u = (1+g(x))|x|^\alpha|\nabla u|^p$. Note that $-\Delta v_t = |x|^\alpha|\nabla v_t|^p$. Therefore, we obtain

$$-\Delta\phi = g(x)|x|^\alpha|\nabla v_t + \nabla\phi|^p + |x|^\alpha \left\{|\nabla v_t + \nabla\phi|^p - |\nabla v_t|^p\right\}.$$

By employing the linearization method, we subtract $p|x|^\alpha|\nabla v_t|^{p-2}\nabla v_t \cdot \nabla\phi$ from both side, leading to the following expression:

$$\begin{aligned}-\Delta\phi - p|x|^\alpha|\nabla v_t|^{p-2}\nabla v_t \cdot \nabla\phi =& g(x)|x|^\alpha|\nabla v_t + \nabla\phi|^p \\ &+ |x|^\alpha \Big\{|\nabla v_t + \nabla\phi|^p - |\nabla v_t|^p - p|\nabla v_t|^{p-2}\nabla v_t \cdot \nabla\phi\Big\}.\end{aligned}$$

Let the left-hand side be denoted by $L_t(\phi)$ and the right-hand side by $H_t(\phi)$. Therefore,

$$H_t(\phi) = g(x)|x|^\alpha|\nabla v_t + \nabla\phi|^p + |x|^\alpha \Big\{|\nabla v_t + \nabla\phi|^p - |\nabla v_t|^p - p|\nabla v_t|^{p-2}\nabla v_t \cdot \nabla\phi\Big\}$$

and $L_t(\phi) = -\Delta\phi - p|x|^\alpha|\nabla v_t|^{p-2}\nabla v_t \cdot \nabla\phi$. Writing out the details, we see that ϕ needs to satisfy

$$\begin{cases} L_t(\phi) = H_t(\phi) & \text{in } B_1\backslash\{0\}, \\ \phi = 0, & \text{on} \quad \partial B_1. \end{cases} \tag{2.7}$$

Define the nonlinear operator (which at this point is not well defined) $J_t(\phi) = \psi$ by $L_t(\psi) = H_t(\phi)$ in $B_1\backslash\{0\}$ and $\psi = 0$ on ∂B_1. Our approach to finding a ϕ which satisfies (2.7) is to find a fixed point of J_t. Note we have L_t given explicitly by

$$\begin{aligned} L_t(\phi) &= -\Delta\phi - p|x|^\alpha|\nabla v_t|^{p-2}\nabla v_t \cdot \nabla\phi \\ &= -\Delta\phi + p|x|^\alpha \left(\frac{1}{t|x|^\xi + \beta|x|^{\alpha+1}}\right)^{\frac{p-2}{p-1}} \frac{1}{(t|x|^\xi + \beta|x|^{\alpha+1})^{\frac{1}{p-1}}} \frac{x}{|x|} \cdot \nabla\phi \\ &= -\Delta\phi + p|x|^\alpha \frac{1}{t|x|^\xi + \beta|x|^{\alpha+1}} \frac{x}{|x|} \cdot \nabla\phi \\ &= -\Delta\phi + \frac{p}{\beta + t|x|^{\xi-(\alpha+1)}} \frac{x \cdot \nabla\phi}{|x|^2}. \end{aligned}$$

Using the result from Definition 0.5 we can now write $L_t(\phi)$ as

$$L_t(\phi) = -\phi_{rr} + \left(-(N-1) + \frac{p}{(\beta + tr^{\xi-(\alpha+1)})}\right)\frac{\phi_r}{r} - \frac{1}{r^2}\Delta_\theta\phi.$$

We now define suitable function spaces in which we apply a fixed point argument.

Definition 1. *(Function spaces) Define the norms*

$$\|\phi\|_X = \sup_{0<|x|\le 1}\left\{|x|^\sigma|\phi(x)| + |x|^{\sigma+1}|\nabla\phi(x)|\right\}, \qquad \|f\|_Y = \sup_{0<|x|\le 1}|x|^{\sigma+2}|f(x)|.$$

Let X denote the set of functions with finite norm and which are zero on ∂B_1 and we let Y denote the collection of functions f with a finite norm.

We set X_1 to be the closed subspace of X defined by

$$X_1 = \{\phi \in X : \phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)\},$$

and similar for

$$Y_1 = \{f \in Y : \phi(x) = \sum_{k=1}^{\infty} b_k(r)\psi_k(\theta)\},$$

and note in both cases we have only removed the $k = 0$ modes.

2.4 The Linear Theory

Here we are interested in the linear problem given by

$$\begin{cases} L_t(\phi) = f(x), & \text{in } B_1\backslash\{0\}, \\ \phi = 0, & \text{on } \partial B_1. \end{cases} \tag{2.8}$$

The following result is our main result regarding the linear operator L_t.

Theorem 2.2. *There is some $C_l > 0$ such that for all $t \geq 0$ and all $f \in Y$ there is some $\phi \in X$ which satisfies (2.8) and one has the estimate $\|\phi\|_X \leq C_l \|f\|_Y$.*

Lemma 2.3. *(Relation between σ, $\beta_k^{\pm}$) Assume $\beta_k^{\pm} = -\frac{1}{2}(N-2) \pm \frac{1}{2}\sqrt{(N-2)^2 + 4\lambda_k}$. Then we have $\beta_k^- + \sigma < 0 < \beta_k^+ + \sigma$, for all $k \geq 1$.*

Proof. First, we show that $\beta_k^+ + \sigma > 0$ for all $k \geq 1$. We have

$$-N + 2 + 2\sigma = \frac{1}{p-1}(N - pN + 2 + 2\alpha).$$

Hence,

$$-N + 2 + 2\sigma > 0 \quad \text{if and only if} \quad N + 2 + 2\alpha > pN.$$

Given the assumption $\frac{\alpha+N}{N-1} < p$, we have $pN > N + \alpha + p$. Therefore,

$$-N + 2 + 2\sigma > 0 \quad \text{if and only if} \quad N + 2 + 2\alpha > N + \alpha + p,$$

which is implies $2 + \alpha > p$. Thus, for all $k \geq 1$, $\beta_k^+ + \sigma > 0$.

Next, we demonstrate that $\beta_k^- + \sigma = -\frac{1}{2}(N-2) - \frac{1}{2}\sqrt{(N-2)^2 + 4\lambda_k} + \sigma < 0$, for all $k \geq 1$. Note that

$$\beta_k^- + \sigma < 0 \quad \text{if and only if} \quad -N + 2 + 2\sigma < \sqrt{(N-2)^2 + 4\lambda_k}.$$

We know that $-N + 2 + 2\sigma > 0$. Hence,

$$(-N + 2 + 2\sigma)^2 < (N-2)^2 + 4\lambda_k.$$

Thus, $(N-2)^2 + 4\sigma\,(\sigma - N + 2) < (N-2)^2 + 4\lambda_k$, which implies

$$\sigma\,(\sigma - N + 2) < \lambda_k.$$

It can be seen that $\sigma\,(\sigma - N + 2) = -\frac{\sigma}{\beta} < 0 < N - 1$. Therefore,

$$\sigma\,(\sigma - N + 2) < N - 1 = \lambda_1 \leq \lambda_k,$$

which holds for all $k \geq 1$. $\square$

We will mostly work on the subspace X_1 and in the end, we will combine the $k = 0$ mode back in to obtain the desired result. Because the linear operator L_0 was already examined in [1] we will use a continuation argument to connect L_t to L_0 and hence we need to recall some properties of L_0.

Lemma 2.4. *(Linear theory for L_0) [1]*

1. *If $\phi \in X_1$ with $L_0(\phi) = 0$ in $B_1\backslash\{0\}$ with $\phi = 0$ on ∂B_1, then $\phi = 0$.*

2. *Suppose ϕ has no $k = 0$, $L_0(\phi) = 0$ in $\mathbb{R}^N\backslash\{0\}$ and*

$$\sup_{0<|x|<1} \Big(|x|^{\sigma}|\phi(x)| + |x|^{\sigma+1}|\nabla\phi(x)| \Big) < \infty.$$

Then $\phi = 0$.

3. *There is some $C > 0$ such that for all $f \in Y_1$ there is some $\phi \in X_1$ which solves (2.8) (in the case of $t = 0$). Moreover one has $\|\phi\|_X \leq C\|f\|_Y$.*

Before proceeding, we can find a solution to (2.8) using spherical harmonics to see if

$$\phi(x) = \sum_{k=0}^{\infty} a_k(r)\psi_k(\theta) \quad \text{and} \quad f(x) = \sum_{k=0}^{\infty} b_k(r)\psi_k(\theta),$$

then ϕ solves (2.8) when a_k satisfies

$$a_k''(r) + \left(\frac{N-1}{r} - \frac{p}{\beta r + t r^{\xi-\alpha}}\right) a_k'(r) - \frac{\lambda_k}{r^2} a_k(r) = -b_k(r) \quad 0 < r < 1, \tag{2.9}$$

with $a_k(1) = 0$.

Lemma 2.5. *($k = 0$ mode nonhomogenous) There is some $C > 0$ such that for all $b = b(r)$ defined on $(0, 1]$ with $\|b\|_Y = \sup_{0<r\leq 1} r^{\sigma+2}|b(r)| = 1$, (note there is no harm in normalizing b) and all $t \geq 1$, there is some $a = a_t(r)$ which solves (2.9) with $k = 0$, ie.*

$$a_t''(r) + \left(\frac{N-1}{r} - \frac{p}{\beta r + t r^{\xi-\alpha}}\right) a_t'(r) = -b(r) \quad 0 < r < 1, \tag{2.10}$$

with $a_t(1) = 0$. Moreover, one has $\sup_{0<r<1} r^{\sigma}|a_t(r)| \leq C$.

Proof. Consider equation (2.10). Let $a_t'(r) = z_t(r)$. Thus, we need z_t to solve

$$z_t'(r) + \left(\frac{N-1}{r} - \frac{p}{\beta r + t r^{\xi-\alpha}}\right) z_t(r) = -b(r) \quad 0 < r < 1. \tag{2.11}$$

The integrating factor for the equation is $\mu_t(r) = e^{P(r)}$ where

$$P(r) = (N-1)\ln r - \frac{p}{\beta(1+\alpha-\xi)} \left(\ln r^{1+\alpha-\xi} + \ln\left(\frac{t r^{-1-\alpha+\xi} + \beta}{t+\beta}\right)\right),$$

which implies $\mu_t(r) = r^{N-1-\frac{p}{\beta}} \left(\dfrac{tr^{-1-\alpha+\xi}+\beta}{t+\beta}\right)^{\frac{p}{p-1}}$. Thus, we have

$$-\frac{d}{d\tau}\left(\mu_t(\tau)a_t'(\tau)\right) = \mu_t(\tau)b(\tau).$$

Hence, by integrating and using the condition $\mu_t(1) = 1$, we obtain

$$a_t(r) = -\int_r^1 \frac{1}{\mu_t(s)}\left(a_t'(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau\right)ds.$$

We set $a_t'(1) = -\int_{R_t}^1 \mu_t(\tau)b(\tau)d\tau$, where $tR_t^{-1-\alpha+\xi} = 1$. We obtain

$$a_t'(r) = -\frac{1}{\mu_t(r)}\int_{R_t}^r \mu_t(\tau)b(\tau)d\tau.$$

If $r < R_t$, we obtain

$$\begin{aligned}
r^{\sigma+1}|a_t'(r)| &\leq r^{-N+2+\frac{p}{\beta}+\sigma}\left(\frac{tr^{-1-\alpha+\xi}+\beta}{t+\beta}\right)^{-\frac{p}{p-1}}\int_r^{R_t}\frac{\mu_t(\tau)}{\tau^{\sigma+2}}\tau^{\sigma+2}|b(\tau)|d\tau \\
&\leq \left(1+\frac{1}{\beta}\right)^{\frac{p}{p-1}}\left(r^{\frac{p-1}{\beta}}\int_r^{R_t}\tau^{-\frac{p-1}{\beta}-1}d\tau\right) \\
&\leq \frac{\beta}{p-1}\left(1+\frac{1}{\beta}\right)^{\frac{p}{p-1}}\left(1-\left(\frac{r}{R_t}\right)^{\frac{p-1}{\beta}}\right) \\
&\leq \frac{\beta}{p-1}\left(1+\frac{1}{\beta}\right)^{\frac{p}{p-1}}.
\end{aligned}$$

If $R_t < r$, we have $r^{\sigma+1}|a_t'(r)| \leq C\beta\left(1+\dfrac{\beta^{\frac{p}{p-1}}}{p-1}\right)$. To demonstrate it, assume $R_t < r$, and $r=\gamma R_t$, for $1<\gamma<\frac{1}{R_t}$. Note that $tR_t^{\frac{p-1}{\beta}}=1$. Thus, we obtain

$$
\begin{aligned}
r^{\sigma+1}|a_t'(r)| &\le \frac{r^{\frac{p-1}{\beta}}}{(tr^{-1-\alpha+\xi}+\beta)^{\frac{p}{p-1}}}\int_{R_t}^{r}\frac{\tau^{N-1-\frac{p}{\beta}}\left(t\tau^{-1-\alpha+\xi}+\beta\right)^{\frac{p}{p-1}}}{\tau^{\sigma+2}}d\tau\\
&= \frac{\gamma^{\frac{p-1}{\beta}}R_t^{\frac{p-1}{\beta}}}{\left(t\gamma^{\frac{p-1}{\beta}}R_t^{\frac{p-1}{\beta}}+\beta\right)^{\frac{p}{p-1}}}\int_{R_t}^{\gamma R_t}\tau^{N-3-\frac{p}{\beta}-\sigma}\left(t\tau^{-1-\alpha+\xi}+\beta\right)^{\frac{p}{p-1}}d\tau\\
&= \frac{\gamma^{\frac{p-1}{\beta}}}{t\left(\gamma^{\frac{p-1}{\beta}}+\beta\right)^{\frac{p}{p-1}}}\int_{R_t}^{\gamma R_t}\tau^{N-3-\frac{p}{\beta}-\sigma}\left(t\tau^{-1-\alpha+\xi}+\beta\right)^{\frac{p}{p-1}}d\tau.
\end{aligned}
$$

There is a constant $C>0$ such that

$$
\left(t\tau^{-1-\alpha+\xi}+\beta\right)^{\frac{p}{p-1}}\le C\left(t\tau^{-1-\alpha+\xi}\right)^{\frac{p}{p-1}}+C\beta^{\frac{p}{p-1}}.
$$

Therefore,

$$
r^{\sigma+1}|a_t'(r)|\le \frac{C\gamma^{\frac{p-1}{\beta}}}{t\left(\gamma^{\frac{p-1}{\beta}}+\beta\right)^{\frac{p}{p-1}}}\int_{R_t}^{\gamma R_t}\tau^{N-\sigma-3-\frac{p}{\beta}}\left(\left(t\tau^{-1-\alpha+\xi}\right)^{\frac{p}{p-1}}+\beta^{\frac{p}{p-1}}\right)d\tau.
$$

Taking integration, we have

$$
\begin{aligned}
r^{\sigma+1}|a_t'(r)| &\le \frac{C\gamma^{\frac{p-1}{\beta}}}{t\left(\gamma^{\frac{p-1}{\beta}}+\beta\right)^{\frac{p}{p-1}}}\left(\beta t^{\frac{p}{p-1}}\tau^{N-\sigma-2}+\frac{\beta^{\frac{p}{p-1}}}{1-\frac{p}{\beta}}\tau^{N-\sigma-2-\frac{p}{\beta}}\right)\Bigg|_{R_t}^{\gamma R_t}\\
&= \frac{C\gamma^{\frac{p-1}{\beta}}}{\left(\gamma^{\frac{p-1}{\beta}}+\beta\right)^{\frac{p}{p-1}}}\left(\beta(\gamma^{\frac{1}{\beta}}-1)+\frac{\beta^{\frac{p}{p-1}}}{\frac{1-p}{\beta}}(\gamma^{-\frac{p-1}{\beta}}-1)\right)\\
&\le C\beta\left(1+\frac{\beta^{\frac{p}{p-1}}}{p-1}(1-\gamma^{-\frac{p-1}{\beta}})\right)\\
&\le C\beta\left(1+\frac{\beta^{\frac{p}{p-1}}}{p-1}\right).
\end{aligned}
$$

$\square$

Lemma 2.6. *(Kernel for L_t for $t > 0$; X_1) Let $R \in [1, \infty]$ and we suppose $\phi \in X_1$ (with the obvious extension) is such that $L_t(\phi) = 0$ in $B_R \backslash \{0\}$ with $\phi = 0$ on ∂B_R in the case of $R < \infty$. Then $\phi = 0$.*

Proof. Assume ϕ is as in the hypothesis and we write $\phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$ and note we have a_k satisfy

$$a_k''(r) + \left(\frac{N-1}{r} - \frac{p}{\beta r + t r^{\xi - \alpha}} \right) a_k'(r) - \frac{\lambda_k}{r^2} a_k(r) = 0, \quad 0 < r < R, \tag{2.12}$$

and in the case of R finite we have $a_k(R) = 0$. Also note there is some $C_k > 0$ such that

$$\sup_{0<r<R} \left(r^\sigma |a_k(r)| + r^{\sigma+1} |a_k'(r)| \right) \leq C_k < \infty.$$

Fix $k \geq 1$ and set $a(r) = a_k(r)$ and we suppose a is not identically zero. We apply Emden-Fowler change of variables $w(\tau) = r^\sigma a(r)$ where $\tau = \ln(r)$, $\tau \in (-\infty, \ln(R))$ and $\ln(\infty) = \infty$. Then w satisfies

$$w_{\tau\tau} + \left(N - 2 - 2\sigma - \frac{p}{\beta + t e^{\tau(\xi - \alpha - 1)}} \right) w_\tau + f(\tau) w = 0,$$

where $g_k(\tau) = -\lambda_k + \sigma^2 + \sigma \left(2 - N + \frac{p}{\beta + t e^{\tau(\xi - \alpha - 1)}} \right)$ and $-\infty < \tau < \ln(R)$ with $w(\ln(R)) = 0$ in the case of R finite.

Claim. We claim that $r^\sigma |a(r)| \to 0$ as $r \to 0$ and in the case of $R = \infty$ we have $r^\sigma |a(r)| \to 0$ as $r \to \infty$.

If we assume the claim then we have $w(\tau) \to 0$ as $\tau \to -\infty$ and in the case of $R = \infty$ we have $w(\tau) \to 0$ as $\tau \to \infty$. In the case of finite R, we have $w(\ln(R)) = 0$. Since w is not identically zero we can multiply by -1 as assume there is some $-\infty < \tau_0 < \ln(R)$ such that w attains its max (which is positive) at τ_0. At this τ_0

we have $w_{\tau\tau}(\tau_0) \leq 0$ and hence we have

$$g_k(\tau_0)w(\tau_0) = \left(-\lambda_k + \sigma^2 + \sigma\left(2 - N + \frac{p}{\beta + te^{\tau_0(\xi-\alpha-1)}}\right)\right) w(\tau_0) \geq 0,$$

which implies $g_k(\tau_0) \geq 0$. On the other hand, $g_k(\tau)$ is decreasing with respect to τ and k. As a result,

$$g_k(\tau_0) \leq g_k(-\infty) \leq g_1(-\infty) = (N-1)\left(-1 + \sigma\left(p - \frac{\alpha+N}{N-1}\right)\right).$$

As $\sigma\left(p - \dfrac{\alpha+N}{N-1}\right) < 1$, we obtain $g_k(\tau_0) < 0$ but this gives us the desired contradiction.

We now prove the claim. Suppose the result is false and let's assume there is some $r_m \searrow 0$ such that $0 < \varepsilon_0 \leq r_m^\sigma |a(r_m)| \leq C_k$. Now define $a^m(r) := r_m^\sigma a(r_m r)$ for $0 < r < \frac{R}{r_m}$. Note that $|a^m(1)|$ is bounded away from zero and also note that $r^\sigma |a^m(r)| \leq C_k$. Inserting in equation (2.12), we obtain

$$r_m^{\sigma+2} a''(r_m r) + \left(\frac{N-1}{r} - \frac{p}{\beta r + tr^{\xi-\alpha} r_m^{\xi-\alpha-1}}\right) r_m^{\sigma+1} a'(r_m r) - \lambda_k r_m^\sigma \frac{a(r_m r)}{r^2} = 0.$$

As a result,

$$(a^m(r))'' + \left(\frac{N-1}{r} - \frac{p}{\beta r + tr^{\xi-\alpha} r_m^{\xi-\alpha-1}}\right)(a^m(r))' - \lambda_k \frac{a^m(r)}{r^2} = 0. \tag{2.13}$$

Assume $\lim_{m\to\infty} a^m(r) = a^\infty(r)$.

Case 1. Let R be finite and $r_m \searrow 0$ as $m \to \infty$. Passing the limit from equation (2.13), we get

$$(a^\infty(r))'' + \left(\frac{N-1}{r} - \frac{p}{\beta r}\right)(a^\infty(r))' - \frac{\lambda_k}{r^2} a^\infty(r) = 0.$$

Note that $|a^\infty(1)|$ is bounded away from zero. Thus, a^∞ is nonzero with $r^\sigma |a^\infty(r)| +$

$r^{\sigma+1}|(a^\infty)'(r)| \leq C$ which satisfies in $L_0(a^\infty) = 0$. Let $\phi(x) := a^\infty(r)\psi_k(\theta)$. So, $\phi(x)$ is nonzero and in the kernel of L_0 with the desired condition. By Lemma 2.4, this gives us the contradiction.

Case 2. Let R be infinite and there exist $\{r_m\}$ such that $r_m \to \infty$ as $m \to \infty$. Passing the limit from equation (2.13), we get

$$(a^\infty(r))'' + \frac{N-1}{r}(a^\infty(r))' - \frac{\lambda_k}{r^2}a^\infty(r) = 0.$$

The characteristic equation $m^2 + (N-2)\,m - \lambda_k = 0$ has two distinct solutions as $\Delta = (N-2)^2 + 4\lambda_k > 0$ for all $k \geq 1$. Thus, we have

$$\beta_k^\pm = -\frac{1}{2}(N-2) \pm \frac{1}{2}\sqrt{(N-2)^2 + 4\lambda_k}. \tag{2.14}$$

Note that $\beta_k^+ > \beta_k^-$. Consequently, $a_k(r) = C_k r^{\beta_k^+} + D_k r^{\beta_k^-}$. Therefore, $r^\sigma a_k(r) = C_k r^{\beta_k^+ + \sigma} + D_k r^{\beta_k^- + \sigma}$, and thus, there exists some constant $E_k > 0$ such that

$$\left|C_k r^{\beta_k^+ + \sigma} + D_k r^{\beta_k^- + \sigma}\right| \leq E_k \quad \forall\ 0 < r < \infty.$$

Applying Lemma 2.3, we obtain $D_k = 0$ and $C_k = 0$, respectively. As a result, $a_k(r) = 0$ for all $k \geq 1$.

□

Theorem 2.7. *(Apriori estimates on X_1). There is some $C > 0$ such that for all $t_m \geq 0$ and $\{\phi_m\}_m \subset X_1$ if*

$$L_{t_m}(\phi_m) = f^m \qquad B_1\backslash\{0\} \qquad \phi_m = 0 \quad \partial B_1, \tag{2.15}$$

$f^m \in Y_1$, then $\|\phi_m\|_X \leq C\|f^m\|_Y$.

Proof. Assume there are some $t_m \geq 0$, $f^m \in Y_1$ and $\phi^m \in X_1$ with

$$\begin{cases} -\Delta \phi^m + \dfrac{p}{\beta + t_m |x|^{\xi-(\alpha+1)}} \dfrac{x \cdot \nabla \phi^m}{|x|^2} = f^m(x) & \text{in } B_1 \backslash \{0\}, \\ \phi^m = 0 & \text{on } \quad \partial B_1, \end{cases}$$

and

$$\|\phi^m\|_X > m \|f^m\|_Y . \tag{2.16}$$

By normalizing, we can assume $\|\phi^m\|_X = 1$ and $\|f^m\|_Y \to 0$. We claim that $\sup_{x \in B_1} |x|^{\sigma+1} |\nabla \phi^m(x)| \to 0$ and once we have this we can integrate this to show that $\sup_{x \in B_1} |x|^{\sigma} |\phi^m(x)| \to 0$. Towards a contradiction suppose there is some $0 < |x_m| < 1$ and $\varepsilon_0 > 0$ such that

$$\varepsilon_0 \leq |x_m|^{\sigma+1} |\nabla \phi^m(x_m)| \leq 1. \tag{2.17}$$

Let $s_m = |x_m|$.

Case s_m is bounded away from zero. After passing to a subsequence (which we do not rename) $\{\phi^m(x)\}_i$ such that satisfies

$$\begin{cases} -\Delta \phi^m(x) + \dfrac{p}{\beta + t_m |x|^{\xi-(\alpha+1)}} \dfrac{x \cdot \nabla \phi^m(x)}{|x|^2} = f^m(x) & \text{in } B_1 \backslash \{0\}, \\ \phi^m = 0 & \text{on } \quad \partial B_1. \end{cases}$$

For $k \geq 2$ integers define A_k and $\tilde{A}_k$ for $k \geq 2$ as follow:

$$A_k = \left\{ x \in B_1 : \frac{1}{k} < |x| < 1 \right\}, \qquad \tilde{A}_k = \left\{ x \in B_1 : \frac{1}{2k} < |x| < 1 \right\}.$$

It's clear $A_k \subset\subset \tilde{A}_k$ and we can rewrite the equation via

$$-\Delta \phi^m(x) = f^m(x) - \frac{p}{\beta + t_m |x|^{\xi-(\alpha+1)}} \frac{x \cdot \nabla \phi^m(x)}{|x|^2} =: g_m(x) \quad \text{in } \tilde{A}_k,$$

and note that g_m is bounded in $L^\infty(\tilde{A}_k)$ and hence by elliptic regularity we have that

$\{\phi^m\}_m$ is bounded in $C^{1,\delta}(A_k)$. By using a suitable diagonal argument we can pass to another subsequence such that $\phi^m \to \phi$ in $C^{1,\delta}_{loc}(\overline{B_1}\backslash\{0\})$,

$$-\Delta\phi(x) + \frac{p}{\beta + t|x|^{\xi-(\alpha+1)}} \frac{x \cdot \nabla\phi(x)}{|x|^2} = 0 \quad \text{in } B_1\backslash\{0\}, \tag{2.18}$$

with $\phi = 0$ on ∂B_1 in the case of $t < \infty$ and when $t = \infty$ one has ϕ solves $\Delta\phi = 0$ in $B_1\backslash\{0\}$ with $\phi = 0$ on ∂B_1. In either case, we can pass to a subsequence such that $x_m \to x_0$ with $0 < |x_0| \leq 1$ and the above convergence is sufficient to pass to the limit in (2.17) to see that ϕ satisfies $|x_0|^{\sigma+1}|\nabla\phi(x_0)| \geq \varepsilon_0$ and hence ϕ is nonzero.

To demonstrate $\phi \in X_1$, it is sufficient to demonstrate that $\|\phi\|_{\hat{X}} < \infty$ and ϕ has no $k = 0$ mode, i.e, $\phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$. For every fixed x, such that $0 < |x| < 1$, we have $|x|^\sigma|\nabla\phi^{m_i}(x)| \leq 1$. Taking the limit, we obtain $|x|^\sigma|\nabla\phi(x)| \leq 1$. Thus, $\sup_{0<|x|\leq 1} |x|^\sigma|\nabla\phi(x)| \leq 1$. Therefore, $\|\phi\|_{\hat{X}} < \infty$. Moreover, ϕ has no $k = 0$ mode. Using the above convergence we see that for all $0 < r \leq 1$ one has that $\theta \mapsto \phi^m(r\theta)$ converges uniformly on S^{N-1} to $\theta \mapsto \phi(r\theta)$. But then, for all $0 < r \leq 1$, we have

$$0 = \int_{|\theta|=1} \phi^m(r\theta)d\theta \to \int_{|\theta|=1} \phi(r\theta)d\theta,$$

where the first equality follows since $\phi^m \in X_1$. So we see that ϕ has zero average over all spheres for radius $0 < r \leq 1$. Hence, ϕ has no $k = 0$ mode. For $0 \leq t < \infty$, this contradicts our earlier results on the kernel. We consider the case where $t = \infty$. We can write

$$\phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta),$$

where a_k satisfy $a_k(r) = C_k\left(r^{\beta_k^+} - r^{\beta_k^-}\right)$, where $\beta_k^\pm$ is defined in (2.14) and hence we have

$$r^\sigma a_k(r) = C_k\left(r^{\beta_k^+ + \sigma} - r^{\beta_k^- + \sigma}\right),$$

and recall we have show that $\beta_k^+ + \sigma > 0$ and $\beta_k^- + \sigma < 0$ for all $k \geq 1$. It is sufficient to illustrate that $a_0(r) = 0$. We have $\phi^{m_i} \in X_1$. Thus, $a_0^{m_i}(r) = 0$ for each i.

$$a_0(r) = \int_{S^{N-1}} \phi(r\theta)\psi_k(\theta)d\theta = \int_{S^{N-1}} \lim_{m\to\infty} \left(\phi^m(r\theta)\right)\psi_k(\theta)d\theta$$
$$= \lim_{m\to\infty} \int_{S^{N-1}} \phi^m(r\theta)\psi_k(\theta)d\theta = \lim_{m\to\infty} a_0^m(r) = 0.$$

Hence, to ensure that $\phi \in X$, we must have $C_k = 0$ for all $k \geq 1$. This implies that $\phi = 0$, which leads to a contradiction.

Case 2: Assume there is some $x_m \to 0$ such that $|x_m|^{\sigma+1}|\nabla\phi^m(x_m)| \geq \epsilon_0 > 0$. Let $s_m = |x_m|$, so $s_m \to 0$ as $m \to \infty$. Consider the sequence $\{z_m\}_m$, is defined by $z_m := s_m^{-1}x_m$. It is clear that $|z_m| = 1$. Thus,

$$|s_m|^{\sigma+1}|\nabla\phi^m(s_m z_m)| \geq \epsilon_0 > 0.$$

For $0 < |s_m z| < 1$, define $\psi^m(z) := s_m^\sigma(\phi^m(s_m z))$ and note $\psi^m(z_m) = 0$. Note using the bounds on ϕ^m, we obtain

$$|z|^\sigma|\psi^m(z)| \leq 1, \quad \text{and} \quad |z|^{\sigma+1}|\nabla\psi^m(z)| \leq 1, \tag{2.19}$$

with $|\nabla\psi^m(z_m)| \geq \varepsilon_0$. We have

$$-\Delta\phi^m(s_m z) + \frac{p}{\beta + t_m|s_m z|^{\xi-(\alpha+1)}}\frac{s_m z\cdot\nabla\phi^m(s_m z)}{|s_m z|^2} = f^m(s_m z).$$

As $\Delta\psi^m(z) = s_m^{\sigma+2}\Delta\phi^m(s_m z)$ and $\nabla\psi^m(z) = s_m^{\sigma+1}\phi^m(s_m z)$, we obtain

$$-\Delta\psi^m(z) + \frac{p}{\beta + t_m s_m^{\xi-(\alpha+1)}|z|^{\xi-(\alpha+1)}}\frac{z\cdot\nabla\psi^m(z)}{|z|^2} = s_m^{\sigma+2}f^m(s_m z).$$

Thus, we have

$$L_{t_m s_m^{\xi-(\alpha+1)}}(\psi^m(z)) = s_m^{\sigma+2}f^m(s_m z) := g^m(z)$$

in $D_m = \left\{ z : 0 < |z| < \dfrac{1}{s_m} \right\}$. After passing to a subsequence, we can assume

$$t_m s_m^{\xi-(\alpha_1)} \to \tau \in [0, \infty].$$

By using standard compactness results we see that $\psi^m \to \psi$ in $C^{1,\delta}_{\text{loc}}(\mathbb{R}^N \backslash \{0\})$ and note that $g^m \to 0$ in $L^\infty_{\text{loc}}(\mathbb{R}^N \backslash \{0\})$. In particular, we have $L_\tau(\psi) = 0$ in $\mathbb{R}^N \backslash \{0\}$ and we have have there is some $|z_0| = 1$ such that $|\nabla \psi(z_m)| \geq \varepsilon_0$. Moreover, we have $|z|^\sigma |\psi(z)| \leq 1, |z|^{\sigma+1} |\nabla \psi(z)| \leq 1$ in $\mathbb{R}^N \backslash \{0\}$. Also, as before, we can show that ψ has no $k = 0$ mode. Hence, we have $L_\tau(\psi) = 0$ in $\mathbb{R}^N \backslash \{0\}$ with ψ nonzero with the desired properties, which contradicts the earlier kernel results. In the case where $\tau = \infty$, it follows that $\psi(z) = \sum_{k=1}^{\infty} a_k(r) \psi_k(\theta)$ and $a_k(r) = C_k r^{\beta_k^+} + D_k r^{\beta_k^-}$. Consequently, we have $r^\sigma a_k(r) = C_k r^{\beta_k^+ + \sigma} + D_k r^{\beta_k^- + \sigma}$, and note that $\beta_k^+ + \sigma$ and $\beta_k^- + \sigma$ are both nonzero and distinct. Thus, the expression can only be bounded if $C_k = D_k = 0$.

This completes the proof of Claim 1; ie $\sup_{0 < |x| \leq 1} |x|^{\sigma+1} |\phi^m(x)| \to 0$.

$\square$

Corollary 2.8. *There is some $C > 0$ such that for all $t \geq 0$ and all $f \in Y_1$ there is some $\phi \in X_1$ such that $L_t(\phi) = f$ in $B_1 \backslash \{0\}$ with $\phi = 0$ on ∂B_1 and $\|\phi\|_X \leq C \|f\|_Y$.*

Proof. Define the new norm given by

$$\|\phi\|_{\widehat{X}} := \sup_{0 < |x| \leq 1} \left\{ |x|^\sigma |\phi(x)| + |x|^{\sigma+1} |\nabla \phi(x)| + |x|^{\sigma+2} |\Delta \phi(x)| \right\}.$$

Let $\widehat{X_1} := \{ \phi \in X_1 : \|\phi\|_{\widehat{X}} < \infty \}$. Then, $L_0 : \widehat{X_1} \to Y_1$ is an isomorphism. For each $0 < t < \infty$ we can show that $L_t : \widehat{X_1} \to Y_1$ is an isomorphism provided we can show that we have an estimate for all $\tau \in [0, t]$. This relies on a continuation argument that we will now demonstrate.

This follows directly from the previous theorem and the fact that controlling the X norm allows one to control the $\widehat{X}$ norm. Therefore, for all $0 < t < \infty$ there is some C_t such that for all $f \in Y_1$ there is some $\phi \in \widehat{X}_1$ which solves (2.8) and one has $\|\phi\|_{\widehat{X}} \leq C_t \|f\|_Y$. We can then apply the theorem again to see that C_t is bounded above, independent of t.

□

Completion of the proof of Theorem 2.2. Let $f \in Y$ with $\|f\|_Y = 1$ and we write $f(x) = f_0(r) + f_1(x)$ where f_0 is radial and where $f_1 \in Y_1$. Then, there is some $\phi_0 = \phi_0(r)$ and $\phi_1 = \phi_1(x)$ with $\phi_1 \in X_1$ such that $L_t(\phi_i) = f_i$ in $B_1 \backslash \{0\}$ with $\phi_i = 0$ on ∂B_1 and there is some $C > 0$ (independent of t and f, ϕ) such that $\|\phi_1\|_X \leq C\|f_1\|_Y$ and $\|\phi_0\|_X \leq C\|f_0\|_Y$. Set $\phi = \phi_0 + \phi_1$. Then, note by linearity, we have $L_t(\phi) = f$ in $B_1 \backslash \{0\}$ with $\phi = 0$ on ∂B_1 and

$$\|\phi\|_X \leq \|\phi_0\|_X + \|\phi_1\|_X \leq C\left(\|f_0\|_Y + \|f_1\|_Y\right),$$

and hence the proof is complete if we can show there is some $D > 0$ (independent of f) such that

$$\|f_0\|_Y + \|f_1\|_Y \leq D\|f_0 + f_1\|_Y.$$

Given $f \in Y$. Assume $f(x) = \sum_{k=0}^{\infty} b_k(r)\psi_k(\theta) = b_0(r) + \sum_{k=1}^{\infty} b_k(r)\psi_k(\theta) = f_0 + f_1$, where

$$b_k(r) = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} f(r\theta)\psi_k(\theta)d\theta.$$

Thus, we obtain

$$\begin{aligned}
\|f_0\|_Y &= \sup_{0<|x|\leq 1} |x|^{\sigma+2}|b_0(r)| = \sup_{0<|x|\leq 1} |x|^{\sigma+2}\left|\frac{1}{|S^{N-1}|}\int_{|\theta|=1} f(r\theta)d\theta\right| \\
&\leq \sup_{0<|x|\leq 1} |x|^{\sigma+2}\frac{1}{|S^{N-1}|}\int_{|\theta|=1} |f(r\theta)|d\theta \\
&\leq \sup_{0<|x|\leq 1} |x|^{\sigma+2}\frac{\|f\|_Y}{|x|^{\sigma+2}} = \|f\|_Y\,.
\end{aligned}$$

On the other hand, we have $f_1 = f - f_0$. Thus,

$$\|f_1\|_Y = \left\|f - f^0\right\|_Y \leq \|f\|_Y + \|f_0\|_Y \leq 2\|f\|_Y .$$

As a result,

$$\|f_0\|_Y + \|f_1\|_Y \leq D\|f\|_Y ,$$

where $D > 0$ constant independent of f.

$\square$

2.5 The Nonlinear Theory

We let $0 < \nu < 1$ denote the Hölder exponent of g, so there is some $C_0 > 0$ such that $|g(x)| \leq C_0|x|^\nu$, given that $g(0) = 0$. To prove $J_t(B_R) \subset B_R$, the quantity $H_t(\phi)$ will play a role, where

$$H_t(\phi) = |x|^\alpha \left\{ |\nabla v_t + \nabla\phi|^p - |\nabla v_t|^p - p|\nabla v_t|^{p-2}\nabla v_t \cdot \nabla\phi \right\} + g(x)|x|^\alpha|\nabla v_t + \nabla\phi|^p.$$

Lemma 2.9. *(Into). There is some $C_1^*, C_2^*, C_3^* > 0$ such that for all $0 < R < 1$, $0 < \delta < 1$, $t > 0$ and $\phi \in B_R$, one has*

$$\|H_t(\phi)\|_Y \leq C_1^* R^2 + C_2^* R^p + C_3^* \delta^\nu + A(t, \delta), \tag{2.20}$$

where $\sup_{0<|x|\leq 1} |g(x)| \leq C_g$ and $A(t, \delta) := \dfrac{C_g}{(t\delta^{\xi-\alpha-1} + \beta)^{\frac{p}{p-1}}}$. Note for all $0 < \delta < 1$, we have $A(t, \delta) \to 0$ as $t \to \infty$.

Proof. We write $H_t(\phi) = I_t + K_t$ where

$$I_t = |x|^\alpha \left\{ |\nabla v_t + \nabla\phi|^p - |\nabla v_t|^p - p|\nabla v_t|^{p-2}\nabla v_t \cdot \nabla\phi \right\},$$

and $K_t = g(x)|x|^\alpha |\nabla v_t + \nabla\phi|^p$. First, note we have

$$\|I_t\|_Y = \sup_{0<|x|\le 1} |x|^{\sigma+2+\alpha} \left| |\nabla v_t + \nabla\phi|^p - |\nabla v_t|^p - p|\nabla v_t|^{p-2}\nabla v_t \cdot \nabla\phi \right|.$$

Case 1: Let $p \geq 2$. By applying lemma 6.1, with $x = \nabla v_t$ and $y = \nabla\phi$, we arrive at the following result,

$$\begin{aligned}
\|I_t\|_Y &\le C_{1,1} \sup_{0<|x|\le 1} |x|^{\sigma+2+\alpha} \left\{ |\nabla v_t|^{p-2}|\nabla\phi|^2 + |\nabla\phi|^p \right\} \\
&\le C_{1,1} \sup_{0<|x|\le 1} |x|^{\sigma+2+\alpha}|\nabla v_t|^{p-2}|\nabla\phi|^2 + C_{1,1} \sup_{0<|x|\le 1} |x|^{\sigma+2+\alpha}|\nabla\phi|^p.
\end{aligned}$$

We have $|\nabla v_t| = \left(t|x|^\xi + \beta|x|^{\alpha+1}\right)^{-\frac{1}{p-1}}$. It follows that

$$\begin{aligned}
\|I_t\|_Y &\le C_{1,1} \sup_{0<|x|\le 1} |x|^{\sigma+2+\alpha} \left(t|x|^\xi + \beta|x|^{\alpha+1}\right)^{-\frac{p-2}{p-1}} |\nabla\phi|^2 + C_{1,1} \sup_{0<|x|\le 1} |x|^{\sigma+2+\alpha}|\nabla\phi|^p \\
&\le C_{1,1} \sup_{0<|x|\le 1} |x|^{\sigma+2+\alpha-2(\sigma+1)} \left(t|x|^\xi + \beta|x|^{\alpha+1}\right)^{-\frac{p-2}{p-1}} \left(|x|^{\sigma+1}|\nabla\phi|\right)^2 \\
&+ C_{1,1} \sup_{0<|x|\le 1} |x|^{\sigma+2+\alpha-p(\sigma+1)} \left(|x|^{\sigma+1}|\nabla\phi|\right)^p.
\end{aligned}$$

Recall that $\sigma = \frac{\alpha+2-p}{p-1}$. Consequently, $\sigma + 2 + \alpha - 2(\sigma+1) = 0$. Therefore,

$$\begin{aligned}
\|I_t\|_Y &\le C_{1,1} \sup_{0<|x|\le 1} |x|^{-\sigma+\alpha}|x|^{-(\alpha+1)\frac{p-2}{p-1}} \left(t|x|^{\xi-\alpha-1} + \beta\right)^{-\frac{p-2}{p-1}} \left(|x|^{\sigma+1}|\nabla\phi|\right)^2 \\
&+ C_{1,1} \sup_{0<|x|\le 1} \left(|x|^{\sigma+1}|\nabla\phi|\right)^p \\
&= C_{1,1} \sup_{0<|x|\le 1} \left(t|x|^{\xi-\alpha-1} + \beta\right)^{-\frac{p-2}{p-1}} \left(|x|^{\sigma+1}|\nabla\phi|\right)^2 + C_{1,1} \sup_{0<|x|\le 1} \left(|x|^{\sigma+1}|\nabla\phi|\right)^p \\
&\le C_{1,1}\beta^{-\frac{p-2}{p-1}} R^2 + C_{1,1}R^p.
\end{aligned}$$

Case 2: Let $1 < p < 2$. Similarly, we obtain

$$\|I_t\|_Y \le \hat{C}_{1,1} \sup_{0<|x|\le 1} |x|^{\sigma+2+\alpha}|\nabla\phi|^p = \hat{C}_{1,1} \sup_{0<|x|\le 1} \left(|x|^{\sigma+1}|\nabla\phi|\right)^p \le \hat{C}_{1,1}R^p.$$

On the other hand,

$$
\begin{aligned}
\|K_t\|_Y &= \sup_{0<|x|\le 1} |x|^{\sigma+2}|K_t| = \sup_{0<|x|\le 1} |x|^{\alpha+\sigma+2}|g(x)||\nabla v_t + \nabla\phi|^p \\
&\le \sup_{0<|x|\le 1} |x|^{\alpha+\sigma+2}|g(x)|\left(|\nabla v_t|^p + |\nabla\phi|^p\right) \\
&\le \sup_{0<|x|\le 1} |x|^{\alpha+1+(\sigma+1)(1-p)}|g(x)|\left(|x|^{\sigma+1}|\nabla v_t|\right)^p \\
&+ \sup_{0<|x|\le 1} |x|^{\alpha+1+(\sigma+1)(1-p)}|g(x)|\left(|x|^{\sigma+1}|\nabla \phi|\right)^p \\
&= \sup_{0<|x|\le 1} |g(x)|\left(|x|^{\sigma+1}|\nabla v_t|\right)^p + \sup_{0<|x|\le 1} |g(x)|\left(|x|^{\sigma+1}|\nabla \phi|\right)^p \\
&\le \sup_{0<|x|\le 1} |g(x)|\left(|x|^{\sigma+1}\left(t|x|^{\xi} + \beta|x|^{\alpha+1}\right)^{\frac{-1}{p-1}}\right)^p + C_g R^p \\
&= \sup_{0<|x|\le 1} |g(x)|\left(t|x|^{\xi-\alpha-1} + \beta\right)^{-\frac{p}{p-1}} + C_g R^p.
\end{aligned}
$$

Fix $0 < \delta \ll 1$. By assumption, $g \ge 0$ is an Hölder continuous function with $g(0) = 0$. Thus, there are constants C_0 and $\nu > 0$ such that $|g(x)| \le C_0|x|^\nu$. Thus,

$$
\begin{aligned}
&\sup_{0<|x|\le 1} |g(x)|\left(t|x|^{\xi-\alpha-1} + \beta\right)^{-\frac{p}{p-1}} \\
&\le \sup_{0<|x|\le \delta} |g(x)|\left(t|x|^{\xi-\alpha-1} + \beta\right)^{-\frac{p}{p-1}} + \sup_{\delta<|x|\le 1} |g(x)|\left(t|x|^{\xi-\alpha-1} + \beta\right)^{-\frac{p}{p-1}} \\
&\le \sup_{0<|x|\le \delta} \frac{C_0|x|^\nu}{(t|x|^{\xi-\alpha-1} + \beta)^{\frac{p}{p-1}}} + \sup_{\delta<|x|\le 1} \frac{|g(x)|}{(t|x|^{\xi-\alpha-1} + \beta)^{\frac{p}{p-1}}} \\
&\le \frac{C_0\delta^\nu}{\beta^{\frac{p}{p-1}}} + \frac{C_g}{(t\delta^{\xi-\alpha-1} + \beta)^{\frac{p}{p-1}}} \\
&=: \frac{C_0\delta^\nu}{\beta^{\frac{p}{p-1}}}\delta^\zeta + A(t,\delta),
\end{aligned}
$$

where for all $0<\delta<1$, we have $A(t,\delta)\to 0$ as $t\to\infty$. We obtain

$$\begin{aligned}
\|H_t(\phi)\|_Y &= \sup_{0<|x|<1}|x|^{\sigma+2}|H_t(\phi)| = \sup_{0<|x|<1}|x|^{\sigma+2}|I_t+K_t| \\
&\le \sup_{0<|x|<1}|x|^{\sigma+2}|I_t| + \sup_{0<|x|<1}|x|^{\sigma+2}|K_t| \\
&\le C_{1,1}\beta^{-\frac{p-2}{p-1}}R^2 + C_{1,1}R^p + \frac{C_0\delta^\nu}{\beta^{\frac{p}{p-1}}} + A(t,\delta) + C_gR^p \\
&= \left(C_{1,1}\beta^{-\frac{p-2}{p-1}}R + \left(C_g + C_{1,1}\right)R^{p-1}\right)R + C_0\beta^{-\frac{p}{p-1}}\delta^\nu + A(t,\delta) \\
&= \left(C_1^*R + C_2^*R^{p-1}\right)R + C_3^*\delta^\nu + A(t,\delta).
\end{aligned}$$

□

Lemma 2.10. *(Contraction) There is some $\overline{C}_2>0$ such that for $0<R<1$, $t>1$ and $\phi_1,\phi_2\in B_R$, we have*

$$\|H_t(\phi_2)-H_t(\phi_1)\|_Y \le \left\{\bar{C}_2\left(\delta^\zeta + R^{p-1} + R\right) + B(t,\delta)\right\}\|\phi_2-\phi_1\|_X\,, \tag{2.21}$$

where $B(t,\delta)=\frac{C_g}{t\delta^{\xi-\alpha-1}+\beta}$. Note for all $0<\delta\ll 1$, we have $B(t,\delta)\to 0$ as $t\to\infty$.

Proof. Assume that $H_t(\phi_2)-H_t(\phi_1)=I_1+I_2$ where,

$$\begin{aligned}
&I_1 = g(x)|x|^\alpha\left(|\nabla v_t+\nabla\phi_2|^p - |\nabla v_t+\nabla\phi_1|^p\right), \\
&I_2 = |x|^\alpha\left\{|\nabla v_t+\nabla\phi_2|^p - p|\nabla v_t|^{p-2}\nabla v_t\cdot\nabla\phi_2 - |\nabla v_t+\nabla\phi_1|^p + p|\nabla v|^{p-2}\nabla v_t\cdot\nabla\phi_1\right\}.
\end{aligned}$$

Claim 1. There are constants $C_1, C_g, \hat{C}_1>0$ such that for $0<R<1$, $0<\delta\ll 1$, and $t\ge 0$ and $\phi_1,\phi_2\in B_R$,

$$\sup_{0<|x|\le 1}|x|^{\sigma+2}|I_1| \le \left(C_1\hat{C}_1\delta^\zeta + B(t,\delta) + 2C_1C_gR^{p-1}\right)\|\phi_2-\phi_1\|_X\,.$$

Proof. By Lemma 6.1, we obtain the following bounds

$$
\begin{aligned}
&\sup_{0<|x|\leq 1} |x|^{\sigma+2}|I_1| \leq \sup_{0<|x|\leq 1} \left(|g(x)||x|^{\alpha+\sigma+2} \Big| |\nabla v_t + \nabla\phi_2|^p - |\nabla v + \nabla\phi_1|^p \Big| \right) \\
&\leq C_1 \sup_{0<|x|\leq 1} \Big(|g(x)||x|^{\alpha+\sigma+2} (|\nabla v_t|^{p-1} + |\nabla\phi_2|^{p-1} + |\nabla\phi_1|^{p-1}) |\nabla\phi_2 - \nabla\phi_1| \Big) \\
&\leq C_1 \sup_{0<|x|\leq 1} |g(x)||x|^{\alpha+1} \left(\left(t|x|^{\xi} + \beta|x|^{\alpha+1} \right)^{-1} + |\nabla\phi_2|^{p-1} + |\nabla\phi_1|^{p-1} \right) |x|^{\sigma+1} |\nabla\phi_2 - \nabla\phi_1| \\
&\leq C_1 \sup_{0<|x|\leq 1} |g(x)||x|^{\alpha+1} \left(t|x|^{\xi} + \beta|x|^{\alpha+1} \right)^{-1} |x|^{\sigma+1} |\nabla\phi_2 - \nabla\phi_1| \\
&+ C_1 C_g \sup_{0<|x|\leq 1} \left(|x|^{\alpha+1-(p-1)(\sigma+1)} \left(\left(|x|^{\sigma+1}|\nabla\phi_1| \right)^{p-1} + \left(|x|^{\sigma+1}|\nabla\phi_2| \right)^{p-1} \right) \right) |x|^{\sigma+1} |\nabla\phi_2 - \nabla\phi_1| \\
&\leq C_1 \sup_{0<|x|\leq 1} \frac{|g(x)||x|^{\alpha+1}}{t|x|^{\xi} + \beta|x|^{\alpha+1}} |x|^{\sigma+1} |\nabla\phi_2 - \nabla\phi_1| \\
&+ C_1 C_g \sup_{0<|x|\leq 1} \left(\left(|x|^{\sigma+1}|\nabla\phi_1| \right)^{p-1} + \left(|x|^{\sigma+1}|\nabla\phi_2| \right)^{p-1} \right) |x|^{\sigma+1} |\nabla\phi_2 - \nabla\phi_1| \\
&\leq \left(C_1 \sup_{0<|x|\leq 1} \frac{|g(x)|}{t|x|^{\xi-\alpha-1} + \beta} + 2C_1 C_g R^{p-1} \right) \|\phi_2 - \phi_1\|_X .
\end{aligned}
$$

To determine an upper bound for $\sup_{0<|x|\leq 1} \dfrac{|g(x)|}{t|x|^{\xi-\alpha-1} + \beta}$, fix $0 < \delta \ll 1$. Thus,

$$
\begin{aligned}
\sup_{0<|x|<1} \frac{|g(x)|}{t|x|^{\xi-\alpha-1} + \beta} &\leq \sup_{0<|x|\leq\delta} \frac{|g(x)|}{t|x|^{\xi-\alpha-1} + \beta} + \sup_{\delta<|x|<1} \frac{|g(x)|}{t|x|^{\xi-\alpha-1} + \beta} \\
&\leq \sup_{0<|x|\leq\delta} \frac{|g(x)|}{\beta} + \sup_{\delta<|x|<1} \frac{|g(x)|}{t\delta^{\xi-\alpha-1} + \beta} \\
&\leq C_0 \sup_{0<|x|\leq\delta} \frac{|x|^{\nu}}{\beta} + \frac{C_g}{t\delta^{\xi-\alpha-1} + \beta} \\
&\leq \frac{C_0\delta^{\nu}}{\beta} + \frac{C_g}{t\delta^{\xi-\alpha-1} + \beta} \\
&= \frac{C_0\delta^{\nu}}{\beta} + B(t,\delta),
\end{aligned}
$$

which implies

$$
\sup_{0<|x|\leq 1} |x|^{\sigma+2}|I_1| \leq \left(C_1 \hat{C}_1 \delta^{\nu} + B(t,\delta) + 2C_1 C_g R^{p-1} \right) \|\phi_2 - \phi_1\|_X .
$$

$\square$

Claim 2. There exists a constant $C_3 > 0$ such that for $0 < R < 1$, $t > 1$ and $\phi_1, \phi_2 \in B_R$, we have

$$\sup_{0<|x|<1} |x|^{\sigma+2}|I_2| \leq C_3 \left(R + R^{p-1}\right) \|\phi_2 - \phi_1\|_X .$$

Proof. We derive the subsequent bounds, according to the lemma,

$$\begin{aligned}
&\sup_{0<|x|<1} |x|^{\sigma+2}|I_2| \\
&\leq C_2 \sup_{0<|x|<1} |x|^{\alpha+\sigma+2} \left(|\nabla v_t|^{p-2} \left(|\nabla\phi_2| + |\nabla\phi_1|\right) + |\nabla\phi_2|^{p-1} + |\nabla\phi_1|^{p-1}\right) |\nabla\phi_2 - \nabla\phi_1| \\
&\leq C_2 \sup_{0<|x|<1} |x|^{\alpha+1} \left(t|x|^{\xi} + \beta|x|^{\alpha+1}\right)^{-\frac{p-2}{p-1}} \left(|\nabla\phi_1| + |\nabla\phi_2|\right) \|\phi_2 - \phi_1\|_X \\
&+ C_2 \sup_{0<|x|<1} \left(|x|^{\alpha+1-(p-1)(\sigma+1)} \left(|x|^{\sigma+1}|\nabla\phi_1|\right)^{p-1}\right) \|\phi_2 - \phi_1\|_X \\
&+ C_2 \sup_{0<|x|<1} \left(|x|^{\alpha+1-(p-1)(\sigma+1)} \left(|x|^{\sigma+1}|\nabla\phi_2|\right)^{p-1}\right) \|\phi_2 - \phi_1\|_X .
\end{aligned}$$

Considering that we have $\alpha + 1 - (p-1)(\sigma+1) = 0$. Hence,

$$\begin{aligned}
&\sup_{0<|x|<1} |x|^{\sigma+2}|I_2| \\
&\leq C_2 \|\phi_2 - \phi_1\|_X \left(\beta^{-\frac{p-2}{p-1}} |x|^{\sigma+1} \left(|\nabla\phi_1| + |\nabla\phi_2|\right)\right) \\
&+ C_2 \|\phi_2 - \phi_1\|_X \left(\sup_{0<|x|<1} \left(|x|^{\sigma+1}|\nabla\phi_1|\right)^{p-1} + \sup_{0<|x|<1} \left(|x|^{\sigma+1}|\nabla\phi_2|\right)^{p-1}\right) \\
&\leq C_3 \left(R + R^{p-1}\right) \|\phi_2 - \phi_1\|_X .
\end{aligned}$$

$\square$

In conclusion, based on the assertions made in Claim 1 and Claim 2, we arrive at the following result,

$$
\begin{aligned}
&\|H_t(\phi_2) - H_t(\phi_1)\|_Y = \sup_{0<|x|<1} |x|^{\sigma+2}|I_1 + I_2| \\
&\leq \sup_{0<|x|<1} |x|^{\sigma+2}|I_1| + \sup_{0<|x|<1} |x|^{\sigma+2}|I_2| \\
&\leq \left(C_1\hat{C}_1\delta^{\nu} + B(t,\delta) + 2C_1C_gR^{p-1}\right) \|\phi_2 - \phi_1\|_X + 2C_3\left(R + R^{p-1}\right) \|\phi_2 - \phi_1\|_X \\
&\leq \left(C_1\hat{C}_1\delta^{\nu} + B(t,\delta) + 2C_1C_gR^{p-1} + 2C_3\left(R + R^{p-1}\right)\right) \|\phi_2 - \phi_1\|_X \\
&\leq \left(\bar{C}_2\left(\delta^{\nu} + R^{p-1} + R\right) + B(t,\delta)\right) \|\phi_2 - \phi_1\|_X .
\end{aligned}
$$

□

Completion of the proof of Theorem 2.1. Using (2.21), we obtain

$$
\begin{aligned}
\|J_t(\phi_2) - J_t(\phi_1)\|_X &\leq C_l \|H_t(\phi_2) - H_t(\phi_1)\|_Y \\
&\leq C_l\left(\bar{C}_2\left(\delta^{\nu} + R^{p-1} + R\right) + B(t,\delta)\right) \|\phi_2 - \phi_1\|_X ,
\end{aligned}
$$

and

$$
\begin{aligned}
\|J_t(\phi)\|_X &\leq C_l \|H_t(\phi)\|_Y \\
&\leq C_l\left(C_{1,1}\beta^{-\frac{p-2}{p-1}}R^2 + \left(C_{1,1} + C_g\right)R^p + C_0\delta^{\nu}\beta^{-\frac{p}{p-1}} + A(t,\delta)\right).
\end{aligned}
$$

Thus, J_t is a contraction on B_R if the following conditions satisfy:

$$
\begin{aligned}
&C_l\left(\bar{C}_2\left(\delta^{\nu} + R^{p-1} + R\right) + B(t,\delta)\right) < 1, \\
&C_l\left(C_{1,1}\beta^{-\frac{p-2}{p-1}}R^2 + \left(C_{1,1} + C_g\right)R^p + C_0\delta^{\nu}\beta^{-\frac{p}{p-1}} + A(t,\delta)\right) < R.
\end{aligned}
$$

By taking $0 < R < 1$ small and then taking $0 < \delta < 1$ small and then taking t large, we observe that these two conditions are satisfied.

We can now apply Banach's Contraction mapping principle to find a fixed point, ϕ, of J_t on B_R and hence ϕ satisfies (2.7). Hence, $u_t(x) = u(x) = v_t(r) + \phi(x)$ is a solution of (2.1) provided ϕ satisfies (2.7). We need to verify that we can obtain a strictly positive solution. Note that, we have

$$|x|^{\sigma+1}|\nabla u_t(x)| \geq r^{\sigma+1}|v_t'(r)| - |x|^{\sigma+1}|\nabla\phi(x)| \geq r^{\sigma+1}|v_t'(r)| - R,$$

since $\phi \in B_R$. Now recall for all $t > 0$, we have $\lim_{r\to 0^+} r^{\sigma+1}v_t'(r) = -C_\beta$. Hence, by taking $R > 0$ sufficiently small (relative to C_β), we can assume u_t is nonzero. We now need to show that u_t is positive in $B_1\backslash\{0\}$. For all $t > 0$, there is some $y_t \in (0,1)$ such that $ty^{\xi-\alpha-1} < 1$ for all $y \in (0, y_t)$. Then using the formula for v_t, we see there is some $D = D(\beta, p, \alpha) > 0$ such that for all $0 < r < y_t$, one has

$$r^\sigma v_t(r) \geq D\left(1 - \left(\frac{r}{y_t}\right)^\sigma\right),$$

for all $0 < r < y_t$. We can show that our solution u_t is positive near the origin. We can apply the maximum principle on $\{x : \varepsilon < |x| < 1\}$ and the strong maximum principle to see that u_t is strictly positive. $\square$

2.6 General Bounded Domains

Take Ω as a smooth bounded domain in $\mathbb{R}^N$ which does not contain the origin and suppose $R_0 > 0$ is small such that $B_{10R_0} \subset \Omega$. Without loss of generality, we can assume that $\Omega \subset\subset B_1$; if $\Omega \subset\subset B_{R_1}$ then we can use a radial solution w_{t,R_1} on B_{R_1} as the building block to construct solutions instead of w_t.

We will consider

$$\begin{cases} -\Delta u = |x|^\alpha|\nabla u|^p & \text{in } \ \Omega\backslash\{0\}, \\ u = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{2.22}$$

Theorem 2.11. *Suppose $N \geq 3$, $\alpha > 0$ and $\frac{\alpha+N}{N-1} < p < 2+\alpha$. Then there is some $t_0 > 0$ (large) such that for all $t \geq t_0$ there is a solution u_t of (2.22).*

The parameters for this section are still defined as per Section 2.2.

Definition 2. *(Function spaces) Define the norms*

$$\|\phi\|_X = \sup_{x\in\Omega\backslash\{0\}} \left\{|x|^\sigma|\phi(x)| + |x|^{\sigma+1}|\nabla\phi(x)|\right\}, \quad \textit{and} \quad \|f\|_Y = \sup_{x\in\Omega\backslash\{0\}} |x|^{\sigma+2}|f(x)|,$$

and the respective spaces are all functions with finite norms and, as usual, we impose $\phi = 0$ on $\partial\Omega$ for $\phi \in X$.

The proof of the following theorem is demonstrated in the next section.

Theorem 2.12. *There is some $C_l > 0$ such that for all $t \geq 0$ and all $f \in Y$, there is some $\phi \in X$ which satisfies (2.23) and one has the estimate $\|\phi\|_X \leq C_l \|f\|_Y$.*

WLOG, we assume that $B_{5R_0} \subset \Omega \subset\subset B_1$. The cut-off function $\eta \in C_c^\infty(4B_{R_0})$, $0 \leq \eta \leq 1$ is defined as $\eta = 1$ in B_{2R_0}. Let $\phi \in B_R$ where $0 < R < 1$. Assume $u(x) = u_t(x)\eta(x) + \phi(x)$,

$$-\eta\Delta u_t - u_t\Delta\eta - 2\nabla u_t \cdot \nabla\eta - \Delta\phi = |x|^\alpha|\nabla(u_t\eta) + \nabla\phi|^p.$$

We have $-\Delta u_t = |x|^\alpha|\nabla u_t|^p$. Therefore,

$$-\Delta\phi = -\eta|x|^\alpha|\nabla u_t|^p + u_t\Delta\eta + 2\nabla u_t \cdot \nabla\eta + |x|^\alpha|\nabla(u_t\eta) + \nabla\phi|^p.$$

Subtract $p|x|^\alpha|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla\phi$ from both side, we obtain

$$L_t(\phi) := -\Delta\phi - p|x|^\alpha|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla\phi = H_t(\phi),$$

where

$$H_t(\phi) = -\eta|x|^\alpha|\nabla u_t|^p + u_t\Delta\eta + 2\nabla u_t\cdot\nabla\eta + |x|^\alpha|\nabla(u_t\eta) + \nabla\phi|^p - p|x|^\alpha|\nabla u_t|^{p-2}\nabla u_t\cdot\nabla\phi.$$

2.6.1 The Linear Theory

Consider the linear problem which is given by

$$\begin{cases} L_t(\phi) = f(x), & \text{in } \Omega\backslash\{0\}, \\ \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

where $L_t(\phi) = -\Delta\phi + p|x|^{\alpha}|\nabla u_t|^p \nabla u_t \cdot \nabla\phi$. Define $a_t = \dfrac{px}{|x|^{\xi-\alpha+1}\left(t+\beta|x|^{\alpha+1-\xi}\right)}$. The purpose of this section is the proof of Theorem 2.12. Without loss of generality, we may assume $\|f\|_Y = 1$. As stated in the theorem, we want to demonstrate that there is some $C_l > 0$ such that for all $t \geq 0$ and all $f \in Y$ there is some $\phi \in X$ which satisfies (2.23) and one has the estimate $\|\phi\|_X \leq C_l$. Let ζ be a cut-off function in $C_c^{\infty}(B_{2R_0})$, $0 \leq \zeta \leq 1$ such that $\zeta = 1$ in B_{R_0}. Note that $\eta\zeta = \zeta$. We want to utilize "*inner-outer gluing technique*".

Solving the inner and outer problems. Our goal is to find a solution ϕ of (2.23). If we take $\phi = \eta\psi_1 + \psi_2$ then a computation shows that ϕ satisfies (2.23) if ψ_1, ψ_2 satisfy the following system of equations:

$$\begin{cases} L^t(\psi_1) = \zeta f(x) + \zeta a_t \cdot \nabla\psi_2, & \text{in } B_1\backslash\{0\}, \\ \psi_1 = 0, & \text{on } \partial B_1, \end{cases} \tag{2.24}$$

and

$$-\Delta\psi_2 + (1-\zeta)a_t \cdot \nabla\psi_2 = (1-\zeta)f(x) + \psi_1\Delta\eta + 2\nabla\psi_1 \cdot \nabla\eta - a_t \cdot (\psi_1\nabla\eta)\,, \tag{2.25}$$

in $\Omega\backslash\{0\}$ with $\psi_2 = 0$ on $\partial\Omega$. Note that, we are dealing with two different domains in $\mathbb{R}^N$ we will denote the space by $X(B_1), Y(B_1)$ and $X(\Omega), Y(\Omega)$ where $X(\Omega)$ denotes the set of functions that have finite norm

$$\sup_{x\in\Omega}\left(|x|^{\sigma}|\phi(x)|+|x|^{\sigma+1}|\nabla\phi(x)|\right)<\infty,$$

and which are zero on $\partial\Omega$ and we let $Y(\Omega)$ denotes the collection of functions f with finite norm

$$\sup_{x\in\Omega}|x|^{\sigma+2}|f(x)|<\infty.$$

We will first solve the outer problem (2.25). Let C_0 denote the constant from Theorem 2.2 and let B_{2C_0} denote the closed ball centered at the origin in X with radius $2C_0$ (here the norm is on B_1). Let $\psi_1\in B_{2C_0}$ and suppose $\psi_2:=\psi_2(\psi_1,f)$ solves equation (2.25). Now taking this ψ_2 and inserting it into the right hand side of (2.24), we consider $\hat{\psi}_1$ solves problem (2.24) with ψ_1 replaced with $\hat{\psi}_1$. If we can show that $\hat{\psi}_1=\psi_1$ then we are done and hence this suggests we define the nonlinear mapping $T_t(\psi_1)=\hat{\psi}_1$. Thus, $T_t:X\to X$. Our goal is to find a fixed point of T_t.

Lemma 2.13. *There is finite $C_\dagger>0$ such that for ψ_2 satisfies in equation (2.25) we have $\sup_{\Omega}|\nabla\psi_2|\le C_\dagger$ and the constant is uniform for $\psi_1\in B_{2C_0}$.*

Proof. Note that there exists a constant $C_1>0$ such that for all $t>0$ and all bounded h, we have

$$\begin{cases} -\Delta\psi_2+\left(\dfrac{p(1-\zeta)x}{t|x|^{\xi-\alpha+1}+\beta|x|^2}\right)\cdot\nabla\psi_2=h(x), & \text{in }\ \Omega\backslash\{0\},\\ \psi_2=0, & \text{on }\partial\Omega.\end{cases}$$

The solution ψ_2 satisfies

$$\sup_{\Omega}|\nabla\psi_2|\le C_1\|h\|_{L^\infty}. \tag{2.26}$$

Thus, it suffices to demonstrate that there exists $C>0$ independent of t such that

$$\sup_{\Omega\backslash\{0\}}\left|\frac{p(1-\zeta)x}{t|x|^{\xi-\alpha+1}+\beta|x|^2}\right|\le C.$$

We have

$$\sup_{\Omega\setminus\{0\}}\left|\frac{p(1-\zeta)x}{t|x|^{\xi-\alpha+1}+\beta|x|^2}\right| \le \sup_{\Omega\setminus B_{R_0}}\frac{p(1-\zeta)|x|^{-1}}{t|x|^{\xi-\alpha-1}+\beta} \le \frac{p}{R_0} = C.$$

Applying inequality (2.26) for equation (2.25), we have

$$\begin{aligned}
&\sup_{\Omega}|\nabla\psi_2| \le C_1\left\|(1-\zeta)f(x)+\psi_1\Delta\eta+2\nabla\psi_1\cdot\nabla\eta-\frac{px\cdot\psi_1\nabla\eta}{t|x|^{\xi-\alpha+1}+\beta|x|^2}\right\|_{L^\infty}\\
&\le C_1\left\|(1-\zeta)f(x)\right\|_{L^\infty}+C_1\left\|\psi_1\Delta\eta\right\|_{L^\infty}\\
&+2C_1\left\|\nabla\psi_1\cdot\nabla\eta\right\|_{L^\infty}+C_1\left\|\frac{px\cdot\psi_1\nabla\eta}{t|x|^{\xi-\alpha+1}+\beta|x|^2}\right\|_{L^\infty}.
\end{aligned}$$

Recall that $\eta\in C_c^\infty(4B_{R_0})$, $0\le\eta\le 1$ is defined as $\eta=1$ in B_{2R_0} and $\zeta\in C_c^\infty(B_{2R_0})$, $0\le\zeta\le 1$ such that $\zeta=1$ in B_{R_0}. Therefore,

$$\begin{aligned}
\sup_{\Omega}|\nabla\psi_2| \le C_1 &\sup_{R_0<|x|<2R_0}|f(x)|+\tilde{C}_1\sup_{R_0<|x|<2R_0}|\psi_1|+\tilde{C}_1\sup_{R_0<|x|<2R_0}|\nabla\psi_1|\\
&+pC_1\sup_{R_0<|x|<2R_0}\frac{|\psi_1|}{t|x|^{\xi-\alpha}+\beta|x|}\\
&=C_1\sup_{R_0<|x|<2R_0}\frac{|x|^{\sigma+2}|f(x)|}{|x|^{\sigma+2}}+C_1\sup_{R_0<|x|<2R_0}\frac{|x|^{\sigma}|\psi_1|}{|x|^{\sigma}}\\
&+2C_1\sup_{R_0<|x|<2R_0}\frac{|x|^{\sigma+1}|\nabla\psi_1|}{|x|^{\sigma+1}}+pC_1\sup_{R_0<|x|<2R_0}\frac{|x|^{\sigma}|\psi_1|}{t|x|^{\xi-\alpha+\sigma}+\beta|x|^{\sigma+1}}\\
&\le\bar{C}_1\sup_{\Omega}|x|^{\sigma+2}|f(x)|+\bar{C}_2\sup_{\Omega}|x|^{\sigma}|\psi_1|+\bar{C}_3\sup_{\Omega}|x|^{\sigma+1}|\nabla\psi_1|+\bar{C}_4\sup_{\Omega}|x|^{\sigma}|\psi_1|\\
&\le\bar{C}_1+\bar{C}_5\left\|\psi_1\right\|_X:=C_\dagger.
\end{aligned}$$

□

This lemma is essential for establishing the proof of the subsequent theorem.

Lemma 2.14. *There is some $\tilde{C}>0$ such that for all $\psi_1,\tilde{\psi}_1\in B_{2C_0}$, if we define $T_t(\psi_1)=\psi_2$ and $T_t(\tilde{\psi}_1)=\tilde{\psi}_2$, then there exists $\tilde{C}>0$ such that*

$$\sup_{\Omega}|\nabla(\tilde{\psi}_2-\psi_2)|\le\tilde{C}\left\|\tilde{\psi}_1-\psi_1\right\|_X. \tag{2.27}$$

Proof. Note that $\tilde{\psi}_2 - \psi_2$ satisfies the following equation

$$
\begin{aligned}
&-\Delta(\tilde{\psi}_2 - \psi_2) + \frac{p(1-\zeta)x \cdot \nabla(\tilde{\psi}_2 - \psi_2)}{t|x|^{\xi-\alpha+1} + \beta|x|^2} \\
&= (\tilde{\psi}_2 - \psi_1)\Delta\eta + 2\nabla(\tilde{\psi}_2 - \psi_1) \cdot \nabla\eta - \frac{px \cdot (\tilde{\psi}_2 - \psi_1)\nabla\eta}{t|x|^{\xi-\alpha+1} + \beta|x|^2},
\end{aligned}
$$

in $\Omega\backslash\{0\}$ where $\tilde{\psi}_2 - \psi_2 = 0$ on $\partial\Omega$. By using inequality 2.26,

$$
\begin{aligned}
&\sup_{\Omega} |\nabla(\tilde{\psi}_2 - \psi_2)| \leq C_1 \left\| (\tilde{\psi}_1 - \psi_1)\Delta\eta + 2\nabla(\tilde{\psi}_1 - \psi_1) \cdot \nabla\eta - \frac{px \cdot (\tilde{\psi}_1 - \psi_1)\nabla\eta}{t|x|^{\xi-\alpha+1} + \beta|x|^2} \right\|_{L^\infty} \\
&\leq C_1 \left\| (\tilde{\psi}_1 - \psi_1)\Delta\eta \right\|_{L^\infty} + 2C_1 \left\| \nabla(\tilde{\psi}_1 - \psi_1) \cdot \nabla\eta \right\|_{L^\infty} + C_1 \left\| \frac{px \cdot (\tilde{\psi}_1 - \psi_1)\nabla\eta}{t|x|^{\xi-\alpha+1} + \beta|x|^2} \right\|_{L^\infty} \\
&\leq C_1 \sup_{2R_0<|x|<4R_0} |\tilde{\psi}_1 - \psi_1| + 2C_1 \sup_{2R_0<|x|<4R_0} |\nabla(\tilde{\psi}_1 - \psi_1)| \\
&+ C_1 \sup_{2R_0<|x|<4R_0} \frac{p|\tilde{\psi}_1 - \psi_1|}{t|x|^{\xi-\alpha} + \beta|x|} \\
&= C_1 \sup_{2R_0<|x|<4R_0} \frac{|x|^{\sigma}|\tilde{\psi}_1 - \psi_1|}{|x|^{\sigma}} + 2C_1 \sup_{2R_0<|x|<4R_0} \frac{|x|^{\sigma+1}|\nabla(\tilde{\psi}_1 - \psi_1)|}{|x|^{\sigma+1}} \\
&+ C_1 p \sup_{2R_0<|x|<4R_0} \frac{|x|^{\sigma}|\tilde{\psi}_1 - \psi_1|}{t|x|^{\xi-\alpha+\sigma} + \beta|x|^{\sigma+1}} \\
&\leq \tilde{C}_1 \sup_{\Omega} |x|^{\sigma}|\tilde{\psi}_1 - \psi_1| + \tilde{C}_2 \sup_{\Omega} |x|^{\sigma+1}|\nabla(\tilde{\psi}_1 - \psi_1)| + \tilde{C}_3 \sup_{\Omega} |x|^{\sigma}|\tilde{\psi}_1 - \psi_1| \\
&\leq \tilde{C} \left\| \tilde{\psi}_1 - \psi_1 \right\|_X .
\end{aligned}
$$

□

Theorem 2.15. *$T_t : X \to X$ is a contraction mapping for all t sufficiently large.*

Proof. **Into.** We desire to demonstrate that $T_t(B_{2C_0}) \subset B_{2C_0}$. Let ψ_2 is satisfied in

(2.25). There exits $C_0 > 0$ such that

$$\begin{aligned}
\|T_t(\psi_1)\|_X &\leq C_0 \left\|\zeta\left(f(x) + a_t \cdot \nabla\psi_2\right)\right\|_Y \\
&\leq C_0 \|f(x)\|_Y + C_0 \|a_t \cdot \nabla\psi_2\|_Y \\
&\leq C_0 + C_0 \sup_{\Omega} |x|^{\sigma+2}|a_t||\nabla\psi_2| \\
&\leq C_0 + C_0 \left(\sup_{\Omega} |x|^{\sigma+2}|a_t|\right) \sup_{\Omega} |\nabla\psi_2| \\
&\leq C_0 + C_0 \left(\sup_{x\in\Omega} \frac{p|x|^{\sigma+2}}{t|x|^{\xi-\alpha-1}+\beta}\right) \sup_{\Omega} |\nabla\psi_2|.
\end{aligned}$$

Hence, $\|T_t(\psi_1)\|_X \leq C_0 + C_0 C_\dagger \delta_t$, where $\delta_t = \sup_{x\in\Omega} \frac{p|x|^{\sigma+2}}{t|x|^{\xi-\alpha-1}+\beta}$. As a result, for t large enough, we have $\hat{\psi}_1 \in B_{2C_0}$.

Contraction. Let $\psi_1, \tilde{\psi_1} \in B_{2C_0}$ and we let $\psi_2, \tilde{\psi_2}$ solve (2.25) and we set $T_t(\psi_1) = \hat{\psi}_1$ and $T_t(\tilde{\psi_1}) = \hat{\tilde{\psi}}_1$. We demonstrate that there is a $0 < \gamma < 1$ such that $\left\|T_t(\tilde{\psi_1}) - T_t(\psi_1)\right\|_X \leq \gamma \left\|\tilde{\psi_1} - \psi_1\right\|_X$. Applying Lemma 2.14, we obtain we are dealing with two different domains

$$\begin{aligned}
\left\|T_t(\tilde{\psi_1}) - T_t(\psi_1)\right\|_X &\leq C_0 \left\|\zeta a_t \nabla(\tilde{\psi_2} - \psi_2)\right\|_Y \\
&\leq C_0 \sup_{x\in\Omega} |x|^{\sigma+2}|a_t||\nabla(\tilde{\psi_1} - \psi_1)| \\
&\leq C_0 \delta_t \sup_{\Omega} |\nabla(\tilde{\psi_1} - \psi_1)| \\
&\leq C_0 \delta_t \left\|\tilde{\psi_1} - \psi_1\right\|_X.
\end{aligned}$$

For t large enough ($t \geq pC_0 \sup_{x\in\Omega} |x|^{\sigma+2-\xi+\alpha+1}$), we have $C_0\delta_t < 1$. This implies T_t is a Contraction mapping on B_{2C_0} therefore, there is some $\psi_1 \in B_{2C_0}$ such that $T_t(\psi_1) = \psi_1$. Let $x_0 \in \partial\Omega$. Point out that $\Omega \subset B_1$. Thus, $\psi_2(x_0) = 0$. Define

$g(t) = \psi_2(x_0 + t(x - x_0)) = u(x_t)$. Thus, $g'(t) = \nabla\psi_2(x_t) \cdot (x - x_0)$ which implies

$$\begin{aligned} |\psi_2(x)| &= |\psi_2(x) - \psi_2(x_0)| = |g(1) - g(0)| \le \int_0^1 |g'(t)|dt \\ &= \int_0^1 |\nabla\psi_2(x_t)||x - x_0|dt \le \sup_{z\in\Omega} |\nabla u(z)||x - x_0| \le \sup_{z\in\Omega} |\nabla u(z)| = C_\dagger. \end{aligned}$$

This finding confirms that

$$\begin{aligned} \|\phi\|_X &= \|\eta\psi_1 + \psi_2\|_X \le \|\eta\psi_1\|_X + \|\psi_2\|_X \\ &= \sup_{0<|x|\le 1} \Big\{|x|^\sigma|\eta\psi_1| + |x|^{\sigma+1}|\nabla(\eta\psi_1)|\Big\} + \|\psi_2\|_X \\ &\le \sup_{0<|x|\le 1} \Big\{|x|^\sigma|\psi_1| + |x|^{\sigma+1}|\eta\nabla\psi_1 + \psi_1\nabla\eta|\Big\} + \|\psi_2\|_X \\ &\le C \sup_{0<|x|\le 1} \Big\{|x|^\sigma|\psi_1| + |x|^{\sigma+1}|\nabla\psi_1|\Big\} + C \sup_{0<|x|\le 1} |x|^{\sigma+1}|\psi_1\nabla\eta| + \|\psi_2\|_X \\ &\le C\,\|\psi_1\|_X + C \sup_{0<|x|\le 1} |x|^{\sigma+1}|\psi_1\nabla\eta| + \|\psi_2\|_X \\ &\le 2CC_0 + C\tilde{C} + C_\dagger := C_l. \end{aligned}$$

□

2.7 The Nonlinear Theory

Recall that

$$H_t(\phi) = -\eta|x|^\alpha|\nabla u_t|^p + u_t\Delta\eta + 2\nabla u_t\cdot\nabla\eta + |x|^\alpha|\nabla(u_t\eta) + \nabla\phi|^p - p|x|^\alpha|\nabla u_t|^{p-2}\nabla u_t\cdot\nabla\phi.$$

Define $J_t(\phi) = \psi$, where $L_t(\psi) = H_t(\phi)$, in $\Omega\backslash\{0\}$ with $\psi = 0$ on $\partial\Omega$.

Theorem 2.16. *Assume $0 < R < 1$ small sufficiently. Let B_R be a ball with center at origin and radius R in space X. Then $J_t : B_R \to B_R$ is a contraction mapping for t large enough.*

Proof. Assume $H_t(\phi) = I_1 + I_2 + I_3$, where

$$I_1 = |x|^\alpha \left(|\nabla(u_t\eta) + \nabla\phi|^p - |\nabla(u_t\eta)|^p - p|\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta)\cdot\nabla\phi \right),$$
$$I_2 = p|x|^\alpha \left(|\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta) - |\nabla u_t|^{p-2}\nabla u_t \right)\cdot\nabla\phi,$$
$$I_3 = |x|^\alpha \left(|\nabla(u_t\eta)|^p - \eta|\nabla u_t|^p \right) + u_t\Delta\eta + 2\nabla u_t\cdot\nabla\eta.$$

Into. Let $0 < R < 1$, $p \geq 2$ and $\phi \in B_R$. By Lemma 6.1, there is some $C_{1,1}$ such that

$$\begin{aligned}
\|I_1\|_Y &= \sup_\Omega |x|^{\alpha+\sigma+2}\left| |\nabla(u_t\eta) + \nabla\phi|^p - |\nabla(u_t\eta)|^p - p|\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta)\cdot\nabla\phi \right| \\
&\leq C_{1,1}\sup_\Omega |x|^{\alpha+\sigma+2}\left(|\nabla(u_t\eta)|^{p-2}|\nabla\phi|^2 + |\nabla\phi|^p \right) \\
&\leq C_{1,1}\sup_\Omega |x|^{\alpha+\sigma+2}|\nabla(u_t\eta)|^{p-2}|\nabla\phi|^2 + C_{1,1}\sup_\Omega \left(|x|^{\sigma+1}|\nabla\phi| \right)^p \\
&\leq C_{1,1}\sup_\Omega \left(|x|^{\frac{\alpha+1}{p-1}}|\nabla(u_t\eta)| \right)^{p-2}\left(|x|^{\sigma+1}|\nabla\phi| \right)^2 + C_{1,1}\sup_\Omega \left(|x|^{\sigma+1}|\nabla\phi| \right)^p \\
&\leq C_{1,1}\sup_\Omega \left(|x|^{\sigma+1}|\nabla(u_t\eta)| \right)^{p-2} R^2 + C_{1,1}R^p.
\end{aligned}$$

Define $\mathcal{E}_t(x) := (|x|^{\sigma+1}|\nabla(u_t\eta)|)^{p-2}$. By considering that $\eta = 0$ on $\Omega\backslash B_{4R_0}$, we obtain

$$\begin{aligned}
\sup_\Omega \mathcal{E}_t &\leq \sup_{|x|\leq 2R_0}\left(|x|^{\sigma+1}|\nabla u_t| \right)^{p-2} + \sup_{2R_0\leq|x|\leq 4R_0}\left(|x|^{\sigma+1}|\nabla(u_t\eta)| \right)^{p-2} \\
&= \sup_{|x|\leq 2R_0}\left(\frac{1}{t|x|^{\xi-\alpha-1} + \beta} \right)^{\frac{p-2}{p-1}} + \sup_{2R_0\leq|x|\leq 4R_0}\left(|x|^{\sigma+1}|\nabla(u_t\eta)| \right)^{p-2} \\
&\leq C_\beta^{p-2} + \sup_{2R_0\leq|x|\leq 4R_0}\left(|x|^{\sigma+1}|\eta\nabla u_t + u_t\nabla\eta| \right)^{p-2} \\
&\leq C_\beta^{p-2} + \sup_{2R_0\leq|x|\leq 4R_0}\left(|x|^{\sigma+1}|\eta||\nabla u_t| \right)^{p-2} + \sup_{2R_0\leq|x|\leq 4R_0}\left(|x|^{\sigma+1}|u_t||\nabla\eta| \right)^{p-2} \\
&\leq \mathcal{C}_t,
\end{aligned}$$

and note that $\mathcal{C}_t$ is bounded above independently of t (for large t). Thus, $\|I_1\|_Y \leq C_{1,1}\mathcal{C}_tR^2 + C_{1,1}R^p$.

We now show there is some $\zeta_t \to 0$ as $t \to \infty$ such that $\|I_2\|_Y \leq \zeta_t R$. Define

$$E_t(x) := p|x|^{\alpha+1}\left(|\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta) - |\nabla u_t|^{p-2}\nabla u_t\right).$$

Note that $E_t(x) = 0$ for $|x| < 2R_0$ and since u_t and $|\nabla u_t|$ go to zero uniformly away from the origin as $t \to \infty$, it follows that $\sup_{x\in\Omega} |E_t(x)| \to 0$ as $t \to \infty$. Therefore,

$$\begin{aligned}
\|I_2\|_Y &= \sup_{\Omega} |x|^{\sigma+2}|I_2| = p\sup_{\Omega} |x|^{\alpha+\sigma+2}\left|\left(|\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta) - |\nabla u_t|^{p-2}\nabla u_t\right)\cdot\nabla\phi\right| \\
&\leq p\sup_{\Omega} |x|^{\alpha+\sigma+2}\left||\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta) - |\nabla u_t|^{p-2}\nabla u_t\right||\nabla\phi| \\
&= \sup_{2R_0<|x|, x\in\Omega} \left(p|x|^{\alpha+1}\left||\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta) - |\nabla u_t|^{p-2}\nabla u_t\right|\right)\left(|x|^{\sigma+1}|\nabla\phi|\right) \\
&= \sup_{2R_0<|x|, x\in\Omega} |E_t(x)|\left(|x|^{\sigma+1}|\nabla\phi|\right) \\
&\leq R\sup_{2R_0<|x|, x\in\Omega} |E_t(x)|,
\end{aligned}$$

which proves the desired result.

By definition of η, I_3 is zero whenever $|x| \geq 4R_0$ or $|x| \leq 2R_0$. Thus,

$$\|I_3\|_Y = \sup_{2R_0<|x|<4R_0} |x|^{\sigma+2}\left||x|^{\alpha}|\nabla(u_t\eta)|^p - \eta|x|^{\alpha}|\nabla u_t|^p + u_t\Delta\eta + 2\nabla u_t\cdot\nabla\eta\right|,$$

and again using facts about u_t and ∇u_t for large t we see that $\|I_3\|_Y = \zeta_t' \to 0$ as $t \to \infty$.

Therefore,

$$\begin{aligned}
\|H_t(\phi)\|_Y &= \sup_{\Omega} |x|^{\sigma+2}|H_t(\phi)| \\
&= \sup_{\Omega} |x|^{\sigma+2}|I_1 + I_2 + I_3| \\
&\leq C_{1,1}\mathcal{C}_t R^2 + C_{1,1}R^p + CR\zeta_t + C\zeta_t'.
\end{aligned}$$

Hence,

$$\|\psi_2\|_X \leq C_l \|H_t(\phi)\|_Y \leq C_l(\mathcal{C}_{1,1}\mathcal{C}_t R^2 + C_{1,1}R^p + CR\zeta_t + C\zeta_t').$$

Since $R \leq 1$, we can write this as less than or equal $C_{1,1}\mathcal{C}_t R^2 + C_{1,1}R^p + C\zeta_t$. Thus, for $J_t(B_R) \subset B_R$ it is sufficient that we have roughly

$$C_l C(R^2 + R^p + \zeta_t) \leq R, \tag{2.28}$$

and note we are using the common practice of renaming constants.

Contraction. Let $0 < R < 1$, $p \geq 2$ and $\phi_1, \phi_2 \in B_R$, such that $\psi_i = J_t(\phi_i)$ for $i = 1, 2$.

$$\begin{aligned}
&\|I_1(\phi_2) - I_1(\phi_1)\|_Y \\
&\leq C_{1,2} \sup_\Omega |x|^{\alpha+\sigma+2} \left(|\nabla(u_t\eta)|^{p-2}(|\nabla\phi_1| + |\nabla\phi_2|) + |\nabla\phi_1|^{p-1} + |\nabla\phi_2|^{p-1}\right) |\nabla\phi_2 - \nabla\phi_1| \\
&\leq C_{1,2} \sup_\Omega \left(|x|^{\sigma+1}|\nabla(u_t\eta)|\right)^{p-2} \left(|x|^{\sigma+1}|\nabla\phi_1| + |x|^{\sigma+1}|\nabla\phi_2|\right) \sup_\Omega |x|^{\sigma+1}|\nabla\phi_2 - \nabla\phi_1| \\
&+ C_{1,2} \sup_\Omega \left(\left(|x|^{\sigma+1}|\nabla\phi_1|\right)^{p-1} + \left(|x|^{\sigma+1}|\nabla\phi_2|\right)^{p-1}\right) \sup_\Omega |x|^{\sigma+1}|\nabla\phi_2 - \nabla\phi_1|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|I_1(\phi_2) - I_1(\phi_1)\|_Y &\leq 2C_{1,2}R^{p-1} \|\phi_2 - \phi_1\|_X + (2C_{1,2}R\mathcal{C}_t) \|\phi_2 - \phi_1\|_X \\
&\leq \left(2C_{1,2}R^{p-1} + 2C_{1,2}R\mathcal{C}_t\right) \|\phi_2 - \phi_1\|_X .
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\|I_2(\phi_2) - I_2(\phi_1)\|_Y \\
&\leq \sup_{2R_0<|x|<4R_0} \left(p|x|^{\alpha+1} \left| |\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta) - |\nabla u_t|^{p-2}\nabla u_t \right| \right) \left(|x|^{\sigma+1} \left| \nabla\phi_2 - \nabla\phi_1 \right| \right) \\
&\leq \sup_{2R_0<|x|<4R_0} |E_t(x)| \sup_{2R_0<|x|<4R_0} |x|^{\sigma+1} \left| \nabla\phi_2 - \nabla\phi_1 \right| \\
&\leq \left(\sup_{2R_0<|x|<4R_0} |E_t(x)| \right) \|\phi_2 - \phi_1\|_X .
\end{aligned}
$$

Hence,

$$
\|\psi_2 - \psi_1\|_X \leq C_l \|H_t(\phi_2) - H_t(\phi_1)\|_Y \leq \gamma_1 \|\phi_2 - \phi_1\|_X ,
$$

where $\gamma_1 = 2C_lC_{1,2}\left(R^{p-1} + R\mathcal{C}_t\right) + C_l \left(\sup_{2R_0<|x|<4R_0} |E_t(x)| \right)$. As a result, J_t is contraction if

$$
\gamma_1 = 2C_lC_{1,2}\left(R^{p-1} + R\mathcal{C}_t\right) + C_l \left(\sup_{2R_0<|x|<4R_0} |E_t(x)| \right) \leq 1. \tag{2.29}
$$

For equations (2.28) and (2.29) to hold, it suffices to choose $0 < R < 1$ to be sufficiently small and then t to be sufficiently large.

□

Note 1. *If we may assume $1 < p \leq 2$, then, in the proof of "into" part, we have*

$$
\|I_1\|_Y \leq \hat{C}_{1,1} \sup_{\Omega} \left(|x|^{\sigma+1}|\nabla\phi| \right)^p \leq \hat{C}_{1,1}R^p.
$$

Also, in the proof of "contraction" part, we obtain

$$
\begin{aligned}
&\|I_1(\phi_2) - I_1(\phi_1)\|_Y \leq \hat{C}_{1,2} \sup_{\Omega} |x|^{\alpha+\sigma+2} \left(|\nabla\phi_1|^{p-1} + |\nabla\phi_2|^{p-1} \right) |\nabla\phi_2 - \nabla\phi_1| \\
&\leq \hat{C}_{1,2} \sup_{\Omega} \left(\left(|x|^{\sigma+1}|\nabla\phi_1| \right)^{p-1} + \left(|x|^{\sigma+1}|\nabla\phi_2| \right)^{p-1} \right) \sup_{\Omega} |x|^{\sigma+1}|\nabla\phi_2 - \nabla\phi_1| \\
&\leq 2\hat{C}_{1,2}R^{p-1} \|\phi_2 - \phi_1\|_X .
\end{aligned}
$$

By applying Banach's Contraction mapping principle, we obtain a fixed point, ϕ, of J_t on B_R such that ϕ satisfies $L_t(\phi) = H_t(\phi)$, in $\Omega\backslash\{0\}$ with $\phi = 0$ on $\partial\Omega$. Thus, $u(x) = u_t(x)\eta(x) + \phi(x)$ is a solution of (2.22) provided ϕ satisfies (2.7). The solution is strictly positive, and the proof follows the same reasoning as we used at the end of Section 2.5.

2.8 Conclusion of Chapter 2.

In Chapter 2, we began by obtaining positive solutions (2.1) in the case of g is a Hölder continuous function on B_1 with $g(0) = 0$. The idea was the approximate solution $u_t(x)$ becomes closer to a true solution the larger t gets and for sufficiently large t we can utilize a fixed point argument to prove the existence of a solution. In the case of a general domain, this same idea is at play but here we need to utilize a gluing procedure.

3

A Hénon Problem Involving the Gradient on Exterior Domains

Under various assumptions on the parameters $p > 1, \alpha, N$ we prove the existence of positive classical solutions of

$$\begin{cases} -\Delta u = |x|^\alpha |\nabla u|^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is an exterior domain in $\mathbb{R}^N$ which does not contain the origin.

The parameters for Chapter 3. In this section, there are two ranges of parameters p, α, N and they will be classified by whether κ is positive or negative, see Section 3.1. The choice of σ (that appears in the definition of the function spaces) will depend on the sign of κ; see the end of Section 3.1.

3.1 Introduction

In this chapter, we examine the existence of positive classical solutions of (3.1) where $p > 1$ and Ω is an exterior domain in $\mathbb{R}^N$ which does not contain 0 in its closure. We begin by considering an explicit example in the complement of the unit ball.
For $N \geq 2$ and $\alpha > 0$, define

$$u_t(r) = \int_1^r \tau^{1-N} \left(\kappa \tau^{\frac{p-1}{\kappa}} + t\right)^{\frac{-1}{p-1}} d\tau,$$

where $\kappa = \dfrac{p-1}{N+\alpha-p(N-1)}$. We consider two distinct cases of parameters

1. If $1 < p < \frac{N+\alpha}{N-1}$, i.e., $\kappa > 0$. Then for all $t > 0$, u_t solves (3.1) in the case of the exterior unit ball.

2. If $\frac{N+\alpha}{N-1} < p < \alpha + 2$, i.e., $\kappa < 0$. Then u_t solves (3.1) in the case of the exterior unit ball.

In both cases, we have $u_t, |\nabla u_t| \to 0$ uniformly on $\Omega \cap B_R$ as $t \to \infty$, where B_R is the open ball radius R centered at the origin in $\mathbb{R}^N$. We assume $u'(r) > 0$ to demonstrate this. Thus we have

$$-u''(r) - \frac{N-1}{r} u'(r) = r^\alpha (u'(r))^p. \tag{3.2}$$

Let $v(r) = u'(r)$, thus the equation (3.2) change to a Bernoulli equation,

$$-v'(r) - \frac{N-1}{r} v(r) = r^\alpha v(r)^p.$$

Assume $v(r) \neq 0$, thus

$$v(r)^{-p} v'(r) + \frac{N-1}{r} v(r)^{1-p} = -r^\alpha. \tag{3.3}$$

Let $w(r) = v(r)^{1-p}$. As a result, equation (3.3) changes to

$$w'(r) - \frac{(N-1)(p-1)}{r} w(r) = (p-1)r^{\alpha}. \tag{3.4}$$

Multiply equation (3.4) by $r^{-(N-1)(p-1)}$. Therefore, we have

$$\frac{d}{dr}\left(r^{-(N-1)(p-1)}w(r)\right) = (p-1)r^{-(N-1)(p-1)+\alpha} = (p-1)r^{\frac{p-1}{\kappa}-1}. \tag{3.5}$$

Note that $p \neq \dfrac{\alpha+N}{N-1}$. Taking integral from equation (3.5), we have

$$r^{-(N-1)(p-1)}w(r) = t + \kappa r^{\frac{p-1}{\kappa}},$$

where t is constant. Therefore, we have $w(r) = tr^{(N-1)(p-1)} + \kappa r^{\alpha+1}$. Thus,

$$v(r) = w(r)^{\frac{1}{1-p}} = \left(tr^{(N-1)(p-1)} + \kappa r^{\alpha+1}\right)^{\frac{1}{1-p}}.$$

Therefore, $u_t'(r) = \dfrac{1}{r^{N-1}\left(t+\kappa r^{\frac{p-1}{\kappa}}\right)^{\frac{1}{p-1}}}$.

Case 1. Let $1 < p < \frac{N+\alpha}{N-1}$. Then $\kappa > 0$. For all $t \geq 0$, we have $u_t'(r) > 0$. Taking integral, we have

$$u_t(r) = u_t(r) - u_t(1) = \int_1^r u_t'(y)dy = \int_1^r \frac{d\tau}{\tau^{N-1}\left(t+\kappa\tau^{N+\alpha-p(N-1)}\right)^{\frac{1}{p-1}}}.$$

Case 2. Let $\frac{N+\alpha}{N-1} < p$. Then $\kappa < 0$. For all $t \geq 0$ large enough, we have $u'(r) > 0$. Taking integral, we have

$$u_t(r) = \int_1^r \frac{1}{\tau^{N-1}\left(t+\kappa\tau^{N+\alpha-p(N-1)}\right)^{\frac{1}{p-1}}} d\tau.$$

Theorem 3.1. *Suppose $N \geq 2$, $\alpha > 0$. Assume Ω is an exterior domain in $\mathbb{R}^N$ which does not contain 0 in its closure.*

1. *Let $p \in (1, \frac{N+\alpha}{N-1})$, such that $p > \max\{\frac{\alpha+1}{N-1}, \frac{\alpha+N+1}{N}\}$. Then, there exist solution u_t of equation (3.1).*

2. *Let $p \in (\frac{N+\alpha}{N-1}, \alpha+2)$. Then, there is some $t_0 > 0$ (large) such that for all $t \geq t_0$ there is a solution u_t of (3.1).*

Our approach to finding a solution of (3.1) will be to linearize around u_t. Let B_R denote a closed ball centered at the origin in X with radius R. Let R_0 be very large and assume $0 \leq \eta \leq 1$ be a cut-off function such $\eta = 0$ in B_{R_0} and $\eta = 1$ for $|x| > R_0 + 1$ (note we are allowing more general cut-off functions than before). We look for a solution of (3.1) of the form $u(x) = u_t(x)\eta(x) + \phi(x)$. Then note u satisfies (3.1) provided ϕ satisfies

$$\begin{cases} L_t(\phi) = H_t(\phi) & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $L_t(\phi) = -\Delta\phi - p|x|^\alpha |\nabla u_t|^{p-2}\nabla u_t \cdot \nabla\phi$ and

$$H_t(\phi) = -\eta|x|^\alpha|\nabla u_t|^p + u_t\Delta\eta + 2\nabla u_t\cdot\nabla\eta + |x|^\alpha|\nabla(u_t\eta) + \nabla\phi|^p - p|x|^\alpha|\nabla u_t|^{p-2}\nabla u_t\cdot\nabla\phi.$$

To do this, one needs to carefully choose suitable spaces to work on and then analyze the linearized operator L_t, which can explicitly be computed as

$$L_t(\phi) = -\Delta\phi - \frac{p}{\kappa + t|x|^{-\frac{p-1}{\kappa}}} \frac{x\cdot\nabla\phi}{|x|^2}.$$

The function spaces. Define the norms

$$\|\phi\|_X := \sup_{x\in\Omega} |x|^{\sigma+1}|\nabla\phi(x)|, \qquad \|f\|_Y := \sup_{x\in\Omega} |x|^{\sigma+2}|f(x)|,$$

and to define the space X, we impose $\phi = 0$ on $\partial\Omega$. The only thing left to do is to pick the parameter σ. Note in case 1 ($\kappa > 0$), we have for all $t > 0$ that

$$u_t'(r) r^{N-1+\frac{1}{\kappa}} \to \kappa^{\frac{-1}{p-1}},$$

as $r \to \infty$. In case 2 ($\kappa < 0$) we have

$$u_t'(r) r^{N-1} \to t^{\frac{-1}{p-1}},$$

as $r \to \infty$. We will pick $\sigma > 0$ sufficiently large that if $\phi \in B_R$ for some R and $u(x) = u_t(r) + \phi(x)$ that we see u is nonzero for large enough $|x|$. Thus, towards this note that if $\phi \in B_R$ (closed ball in X of radius R centered at the origin) we have

$$|\nabla u(x)| \geq |u_t'(r)| - |\nabla \phi(x)| \geq u_t'(r) - \frac{R}{|x|^{\sigma+1}},$$

and we want this positive for large $|x|$. In case 1 we see its sufficient to take $\sigma > \frac{\alpha-p+2}{p-1}$ and in case 2 we need $\sigma > N-2$. With the above in mind, we take $\sigma = \frac{\alpha-p+2}{p-1} + \varepsilon$ in case 1 and in case 2 we take $\sigma = N-2+\varepsilon$, where in both cases we are taking $\varepsilon > 0$ small.

3.2 The Linear Theory

In this section, we want to solve

$$\begin{cases} L_t(\phi) = f(x) & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

3.2.1 The Case $\kappa < 0$

In this section, we take $\sigma = N - 2 + \varepsilon$ where $\varepsilon > 0$ is small. Define $K_t(\phi) = \dfrac{-px \cdot \nabla \phi(x)}{|x|^{-\frac{p-1}{\kappa}+2}\left(t + \kappa |x|^{\frac{p-1}{\kappa}}\right)}$. Thus, $L_t(\phi) = -\Delta\phi + K_t(\phi)$.

Lemma 3.2. *There is some ε_t for all $\phi \in X$ such that $\varepsilon_t \to 0$ as $t \to \infty$ and $\|K_t(\phi)\|_Y \le \varepsilon_t \|\phi\|_X$.*

Proof. Let $\phi \in X$. Therefore,

$$\|K(\phi)\|_Y \le p \sup_{x\in\Omega} \left| \frac{|x|^{\frac{p-1}{\kappa}}}{\left(t + \kappa |x|^{\frac{p-1}{\kappa}}\right)} \right| \|\phi\|_X =: p\varepsilon_t \|\phi\|_X.$$

Since $\kappa < 0$, we see that for large t, this supremum is attained at some $x \in \partial\Omega$ and from this we see that $\varepsilon_t \to 0$ as $t \to \infty$. □

Define $\hat{X} := \{\phi : \|\phi\|_{\hat{X}} < \infty, \text{ and } \phi = 0 \text{ on } \partial\Omega\}$ where

$$\|\phi\|_{\hat{X}} = \sup_{x\in\Omega} \left\{ |x|^{\sigma+1} |\nabla\phi| + |x|^{\sigma+2} |\Delta\phi| \right\}.$$

Lemma 3.3 ([1, Theorem 3])**.** *The mapping $\Delta : \hat{X} \to Y$ is one-to-one and onto and continuous with the continuous inverse.*

Theorem 3.4. *There is some $C_l > 0$ and large t_0 such that for all $t > t_0$, for all $f \in Y$, there is some $\phi \in X$ which satisfies*

$$\begin{cases} L_t(\phi)(x) = f(x), & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Moreover, we have $\|\phi\|_X \le C_l \|f\|_Y$.

Proof. We prove this directly. Recall $L_t(\phi) = -\Delta\phi + K_t(\phi)$. Fix $f \in Y$ and consider the nonlinear mapping on X by $T_t(\phi) = \psi$ where ψ satisfies

$$\begin{cases} -\Delta\psi + K_t(\phi) = f(x), & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

We demonstrate that T_t is a contraction on X for large enough t. Let $\phi_i \in X$ and $\psi_i = T_t(\phi_i)$ and note we have $-\Delta(\psi_2 - \psi_1) = -K_t(\phi_2 - \phi_1)$ in Ω with $\psi_2 - \psi_1 = 0$ on $\partial\Omega$. Then by applying the mapping results for $-\Delta$, we have

$$\|\psi_2 - \psi_1\|_X \leq C\|K_t(\phi_2 - \phi_1)\|_Y \leq C\varepsilon_t\|\phi_2 - \phi_1\|_X,$$

and this shows the desired result. We also easily obtain the desired bound from the bound on Δ and the smallness of ε_t. $\square$

3.2.2 The Case $\kappa > 0$

Lemma 3.5. *(Onto mapping of Δ) Suppose $N - 2 < \sigma < N - 1$. Then there is $C > 0$ such that for all $f \in Y$ there is some $\phi \in X$ such that*

$$\begin{cases} \Delta\phi = f, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Moreover, one has $\|\phi\|_X \leq C\|f\|_Y$.

Proof. The proof is similar to the proof of Theorem 3 in [1]. $\square$

Assumptions on parameters for the remainder of this section. We assume $\alpha > 0$ and

$$\frac{\alpha + N + 1}{N} < p < \frac{N + \alpha}{N - 1}, \tag{3.11}$$

and then we take $\sigma = \frac{\alpha - p + 2}{p - 1} + \varepsilon$ where $\varepsilon > 0$ is small and hence $\sigma \in (N - 2, N - 1)$.

Lemma 3.6. *Let* $a_t = \dfrac{-px}{|x|^{-\frac{p-1}{\kappa}+2}\left(t+\kappa|x|^{\frac{p-1}{\kappa}}\right)}$. *There is some* $C > 0$ *and* $t_0 > 0$ *(large) such that for all* $g \in Y$ *there is some* ψ *which satisfies*

$$\begin{cases} \Delta\psi + (1-\eta)a_t \cdot \nabla\psi = g & \textit{in} \;\; \Omega, \\ \psi = 0, & \textit{on} \;\; \partial\Omega. \end{cases} \tag{3.12}$$

Moreover, we have $\|\psi\|_X \le C\|g\|_Y$. *For* $R_0 > 0$ *(large), set* $Y_{R_0} = \{g \in Y : g(x) = 0 \; \forall |x| \ge R_0\}$. *Let* ψ_g *denote the solution. Then, for all* $g \in Y_{R_0}$ *with* $\|g\|_Y = 1$ *and* $t > t_0$, *we have* $\sup\limits_{|x|\to\infty} |x|^{\sigma+1}|\nabla\psi_g(x)| = 0$ *uniformly in g; i.e. for all* $\varepsilon > 0$ *there is some* $R_\varepsilon > 0$ *such that for all* $t > t_0$, *for all* $g \in Y_{R_0}$ *with* $\|g\|_Y = 1$, *we have* $|x|^{\sigma+1}|\nabla\psi_g(x)| < \varepsilon$ *for all* $|x| > R_\varepsilon$.

Proof. We can either perturb Δ or use a fixed point argument, we will use the second method. Define the nonlinear mapping $T_t(\psi) = \hat{\psi}$ via

$$\begin{cases} \Delta\hat{\psi} + (1-\eta)a_t \cdot \nabla\psi = g & \text{in} \;\; \Omega, \\ \hat{\psi} = 0, & \text{on} \;\; \partial\Omega. \end{cases} \tag{3.13}$$

We claim that T_t is a contraction on X for sufficiently large t. Note that T_t is continuous from X to X. We have $\Delta(\hat{\psi}_2 - \hat{\psi}_1) = (1-\eta)a_t \cdot \nabla(\psi_1 - \psi_2)$ in Ω, with zero Dirichlet boundary condition. Using the mapping properties of Δ, we have

$$\begin{aligned} \|\hat{\psi}_2 - \hat{\psi}_1\|_X &\le C\|(1-\eta)a_t \cdot \nabla(\psi_1 - \psi_2)\|_Y \\ &= \sup_{\Omega \cap supp(1-\eta)} \left\{ |x||a_t(x)||x|^{\sigma+1}|\nabla(\psi_1 - \psi_2)| \right\} \\ &= \sup_{\Omega \cap B_{R_0+1}} \left\{ |x||a_t(x)||x|^{\sigma+1}|\nabla(\psi_1 - \psi_2)| \right\} \\ &\le \hat{\varepsilon}_t \|\psi_1 - \psi_2\|_X, \end{aligned}$$

where $\hat{\varepsilon}_t = \sup\limits_{\Omega \cap B_{R_0+1}} \dfrac{p}{\left(t|x|^{-\frac{p-1}{\kappa}} + \kappa\right)}$. Note that, $\hat{\varepsilon}_t \to 0$ as $t \to \infty$. Hence, we have T_t as a contraction on X for large enough t. This gives the desired result and we also get the desired bound after applying Banach's fixed point theorem. We now suppose the final part is false. Then, there is some $g_m \in Y_{R_0}$ with $\|g_m\|_Y = 1$ and $t_m > t_0$ and $|x_m|^{\sigma+1}|\nabla\psi_{g_m}(x_m)|$ bounded away from zero. Set $\psi_m(x) = \psi_{g_m}$ and set $\zeta_m(z) = s_m^\sigma (\psi_m(s_m z) - \psi_m(s_m z_m))$ where $s_m = |x_m|$ and $z_m s_m = x_m$ and ζ_m is defined for $z \in D_m = \{z \in \mathbb{R}^N : s_m z \in \Omega\}$. Also note that $\zeta_m(z_m) = 0$ and $|\nabla\zeta_m(z_m)|$ is bounded away from zero and there is some $C > 0$ such that $|z|^{\sigma+1}|\nabla\zeta_m(z)| \le C$ on D_m. A computation shows that

$$\Delta\zeta_m(z) + (1-\eta(s_m z))s_m a_{t_m}(s_m z)\cdot\nabla\zeta_m(z) = s_m^{\sigma+2} g_m(s_m z) \qquad \text{in } D_m.$$

By standard arguments, we see that $\zeta_m \to \zeta$ in $C^{1,\gamma}_{loc}(\mathbb{R}^N\backslash\{0\})$ where $\Delta\zeta = 0$ in $\mathbb{R}^N\backslash\{0\}$ with $\zeta(z_\infty) = 0$ and $|\nabla\zeta(z_\infty)| \neq 0$ (here $z_m \to z_\infty$) and $|z|^{\sigma+1}|\nabla\zeta(z)| \le C$. But for our choice of σ, this can't happen. $\square$

To proceed, we will introduce the following quantities,

$$\beta_k^\pm = -\frac{1}{2}(N-2+\frac{p}{\kappa}) \pm \frac{1}{2}\sqrt{\left(N-2+\frac{p}{\kappa}\right)^2 + 4\lambda_k}, \tag{3.14}$$

$$\gamma_k^\pm = -\frac{1}{2}(N-2) \pm \frac{1}{2}\sqrt{(N-2)^2 + 4\lambda_k}. \tag{3.15}$$

Lemma 3.7. *(Relation between σ, $\beta_k^\pm$ and $\gamma_k^\pm$) Assume $\sigma = \dfrac{\alpha-p+2}{p-1} + \varepsilon$ and $\sigma \in (N-2, N-1)$.*

1. *For all $k \ge 0$ one has $\sigma + \beta_k^+ > 0$ and $\sigma + \beta_k^- < 0$.*

2. *For all $k \ge 1$ one has $\gamma_k^- + \sigma < 0 < \gamma_k^+ + \sigma$.*

Proof. We have $\beta_k^- \le \beta_0^-$ for each $k \ge 0$. Therefore, $\sigma + \beta_k^- < \sigma + \beta_0^- = \varepsilon - \frac{p-1}{\kappa}$. If we choose ε small enough such that $\varepsilon < \frac{p-1}{\kappa} = \alpha + N - pN + p$, then for each $k \ge 1$, we have $\sigma + \beta_k^- \neq 0$. Let show $\beta_k^+ + \sigma > 0$ for all $k \ge 1$. Note that $\kappa > 0$ and $N > 2$.

We have

$$2\beta_k^+ + 2\sigma \geq 2\beta_0^+ + 2\sigma = \left|N - 2 + \frac{p}{\kappa}\right| + 2\sigma - (N - 2 + \frac{p}{\kappa}) = 2\sigma > 0.$$

For part 2, it's obvious that $\gamma_k^+ + \sigma > 0$. We demonstrate that $\gamma_k^- + \sigma < 0$ for all $k \geq 1$. Note that

$$\gamma_k^- + \sigma = -\frac{1}{2}(N-2) - \frac{1}{2}\sqrt{(N-2)^2 + 4\lambda_k} + \sigma < 0,$$

if $-N + 2 + 2\sigma < \sqrt{(N-2)^2 + 4\lambda_k}$. We know that $-N + 2 + 2\sigma > 0$, we have the left-hand side is positive. Thus,

$$(-N + 2 + 2\sigma)^2 < (N-2)^2 + 4\lambda_k.$$

As $\sigma < N - 1$, we have $\sigma(\sigma - N + 2) < \lambda_1$. Thus, $\sigma(\sigma - N + 2) < \lambda_k$ for each $k \geq 1$. $\square$

Lemma 3.8. *(Kernel of Δ)*

1. *Suppose that $\Delta\phi = 0$ in B_1^c with $\phi = 0$ on ∂B_1 such that $\sup_{|x|>1} |x|^{\sigma+1}|\nabla\phi| < \infty$. Then, $\phi = 0$.*

2. *Assume $\Delta\phi = 0$ in $\mathbb{R}^N \backslash \{0\}$ such that $\sup_{|x|>0} |x|^{\sigma+1}|\nabla\phi| < \infty$. Then, ϕ is constant.*

Proof. Suppose $\phi(x) = \sum_{k=0}^{\infty} a_k(r)\psi_k(\theta)$ solves $\Delta\phi = 0$ in B_1^c with $\phi = 0$ on ∂B_1 or $\Delta\phi = 0$ in $\mathbb{R}^N \backslash \{0\}$ and in both cases, we assume we have the desired gradient bound. Then, we have

$$a_k''(r) + \frac{N-1}{r}a_k'(r) - \frac{\lambda_k}{r^2}a_k(r) = 0 \quad \text{for } r \in I_i, i = 0, 1,$$

where $I_1 = (1, \infty)$ in the first case (with $a_k(1) = 0$) and $I_0 = (0, \infty)$ in the second case. It is straightforward to verify the result for $k=0$. The general solution in this case is $a_0(r)=C_0r^{-N+2}+D_0$, where C_0, D_0 are constant. Recall that $\sigma+2-N\in(0,1)$. In the first case, we use $a_0(1)=0$ to see that $a_0(r)=C(1-r^{2-N})$. We now use the fact that $r^{\sigma+1}a_0'(r)$ is bounded for large r to see that we must have $C=0$ and hence $a_0(r)=0$ in the first case. In the second case, this same argument shows that a_0 is a constant. We now suppose $k\geq 1$ and a_k satisfy the Euler equation with characteristic equation $\gamma^2+(N-2)\gamma-\lambda_k=0$. The general solution of the equation is given by $a_k(r)=C_kr^{\gamma_k^+}+D_kr^{\gamma_k^-}$, where $\gamma_k^{\pm}$ is given in (3.15). We have

$$r^{\sigma+1}a_k'(r)=C_k\gamma_k^+r^{\sigma+\gamma_k^+}+D_k\gamma_k^-r^{\sigma+\gamma_k^-}.$$

Note that $\gamma_k^{\pm}\neq 0$ and by Lemma 3.7, we have $\gamma_k^-+\sigma<0<\gamma_k^++\sigma$. If a_k satisfies the desired gradient bound, we should have $C_k=D_k=0$ in both cases. $\square$

Lemma 3.9. *(Kernel of L_0 on B_1^c and $\mathbb{R}^N$)*

1. *Suppose* $\sup\limits_{|x|>1}|x|^{\sigma+1}|\nabla\phi(x)|<\infty$, $L_0(\phi)=0$ *in* B_1^c *with* $\phi=0$ *on* ∂B_1 *and no* $k=0$ *mode. Then,* $\phi=0$.

2. *Suppose* $\sup\limits_{|x|>0}|x|^{\sigma+1}|\nabla\phi(x)|<\infty$, *has no* $k=0$ *mode and is such that* $L_0(\phi)=0$ *in* $\mathbb{R}^N\backslash\{0\}$. *Then,* $\phi=0$.

Proof. Suppose $\phi(x)=\sum\limits_{k=1}^{\infty}a_k(r)\psi_k(\theta)$ solves $L_0(\phi)=0$ in B_1^c with $\phi=0$ on ∂B_1 or $L_0(\phi)=0$ in $\mathbb{R}^N\backslash\{0\}$ and in both cases we assume we have the desired gradient bound. Then we have

$$a_k''(r)+\left(N-1+\frac{p}{\kappa}\right)\frac{a_k'(r)}{r}-\lambda_k\frac{a_k(r)}{r^2}=0 \quad \text{for } r\in I,$$

where $I=(1,\infty)$ in the first case (with $a_k(1)=0$) and $I=(0,\infty)$ in the second case. This equation is of Euler form and hence the characteristic equation is β^2+

$\left(N-2+\frac{p}{\kappa}\right)\beta-\lambda_k=0$ has the roots given by $\beta_k^{\pm}$. Therefore, there are A_k, B_k such that $a_k(r)=A_k r^{\beta_k^-}+B_k r^{\beta_k^+}$. We now separate the two cases.

1. By assumption $\phi=0$ on ∂B_1. Thus, we have $a_k(r)=A_k\left(r^{\beta_k^-}-r^{\beta_k^+}\right)$. By Lemma 3.7, we have $\beta_k^+ + \sigma > 0$ for each $k \geq 1$. Therefore,

$$\sup_{1<r<\infty} r^{\sigma+1}|a_k'(r)| = |A_k| \sup_{1<r<\infty} r^{\beta_k^+ + \sigma}\left|\beta_k^- r^{-(\beta_k^+ - \beta_k^-)} - \beta_k^+\right| \leq C_k, \tag{3.16}$$

where to see the existence of a finite C_k note that we have a_k via

$$a_k(r)=\int_{|\theta|=1}\phi(r\theta)\psi_k(\theta)d\theta,$$

and a_k obtains a bound from ϕ. The only way (3.16) can be bounded (after considering sending $r\to\infty$ and $\beta_k^+\neq 0$) is for $A_k=0$. Thus, $a_k=0$.

2. For all $k\geq 1$, we have $\sigma+\beta_k^- < 0 < \sigma+\beta_k^+$. We have $a_k(r)=A_k r^{\beta_k^-}+B_k r^{\beta_k^+}$. Note that we have

$$r^{\sigma+1}a_k'(r)=\beta_k^- A_k r^{\beta_k^-+\sigma}+\beta_k^+ B_k r^{\beta_k^++\sigma}.$$

We want to demonstrate that $A_k=B_k=0$. Recall that We have $\sup\limits_{0<r<\infty} r^{\sigma+1}|a_k'(r)|<\infty$. Let one of A_k, B_k is zero and the other one is nonzero. Without loss of generality, we may assume $A_k=0$ and $B_k\neq 0$. (The proof is similar for $B_k=0$ and $A_k\neq 0$). Thus, $a_k'(r)=B_k\beta_k^+ r^{\beta_k^+-1}$. Note that $\beta_k^+\neq 0$. If $\beta_k^++\sigma>0$ (respectively $\beta_k^++\sigma<0$) Then, $\sup\limits_{0<r<\infty} r^{\sigma+\beta_k^+}=\infty$ as $r\to\infty$ (respectively, $\sup\limits_{0<r<\infty} r^{\sigma+\beta_k^+}=\infty$ as $r\to 0^+$) which contradicts the conditions $\sup\limits_{0<r<\infty} r^{\sigma+1}|a_k'(r)|<\infty$. Let $A_k, B_k\neq 0$. By Lemma 3.7, we have $\beta_k^-+\sigma<0<\beta_k^++\sigma$. Therefore,

$$\sup_{0<r<\infty}|A_k\beta_k^- r^{\sigma+\beta_k^-}+B_k\beta_k^+ r^{\sigma+\beta_k^+}| = \sup_{0<r<\infty} r^{\sigma+\beta_k^+}|A_k\beta_k^- r^{\beta_k^- - \beta_k^+}+B_k\beta_k^+|\to\infty,$$

as $r \to \infty$. Also,

$$\sup_{0<r<\infty} |A_k \beta_k^- r^{\sigma+\beta_k^-} + B_k \beta_k^+ r^{\sigma+\beta_k^+}| = \sup_{0<r<\infty} r^{\sigma+\beta_k^-} |A_k \beta_k^- + B_k \beta_k^+ r^{\beta_k^+ - \beta_k^-}| \to \infty,$$

as $r \to 0^+$. That contradiction shows that $A_k = B_k = 0$, which implies $a_k(r) = 0$. $\square$

We require some preliminary results before we can establish this result.

Lemma 3.10. *(L_0 nonhomogenous ode's) For $k \geq 1$, there is some C_k and for all b_k with $\sup_{1<r<\infty} r^{\sigma+2}|b_k(r)| = 1$, there is some a_k which satisfies*

$$a_k''(r) + \left(N - 1 + \frac{p}{\kappa}\right) \frac{a_k'(r)}{r} - \lambda_k \frac{a_k(r)}{r^2} = -b_k(r) \quad 1 < r < \infty, \tag{3.17}$$

with $a_k(1) = 0$. Moreover, we have $\sup_{1<r<\infty} r^\sigma |a_k(r)| \leq C_k$.

Proof. The general solution to the homogeneous equation is given by $a_k^h(r) = A_k r^{\beta_k^-} + B_k r^{\beta_k^+}$. We want to apply the method of variation of parameters to find particular solutions to non-homogeneous differential equations. The particular solution is given by $a_k^p(r) = h_k(r) r^{\beta_k^-} + g_k(r) r^{\beta_k^+}$ such that h_k and g_k are functions of r. The solution to equation (3.17) is given by

$$a_k(r) = a_k^h(r) + a_k^p(r) = A_k r^{\beta_k^-} + B_k r^{\beta_k^+} + h_k(r) r^{\beta_k^-} + g_k(r) r^{\beta_k^+}.$$

The Wronskian of $r^{\beta_k^-}$ and $r^{\beta_k^+}$ is

$$\mathcal{W}(r) = \begin{vmatrix} r^{\beta_k^-} & r^{\beta_k^+} \\ \beta_k^- r^{\beta_k^- - 1} & \beta_k^+ r^{\beta_k^+ - 1} \end{vmatrix} = \left(\beta_k^+ - \beta_k^-\right) r^{\beta_k^- + \beta_k^+ - 1} \neq 0.$$

Therefore, h_k and g_k are satisfy in the following equations

$$
\begin{aligned}
h_k'(r) &= -\mathcal{W}(r)^{-1} r^{\beta_k^+}(-b_k(r)) = \frac{r^{1-\beta_k^- - \beta_k^+}}{\beta_k^+ - \beta_k^-} r^{\beta_k^+} b_k(r) = \frac{r^{1-\beta_k^-}}{\beta_k^+ - \beta_k^-} b_k(r), \\
g_k'(r) &= \mathcal{W}(r)^{-1} r^{\beta_k^-}(-b_k(r)) = -\frac{r^{1-\beta_k^- - \beta_k^+}}{\beta_k^+ - \beta_k^-} r^{\beta_k^-} b_k(r) = -\frac{r^{1-\beta_k^+}}{\beta_k^+ - \beta_k^-} b_k(r).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
a_k(r) &= A_k r^{\beta_k^-} + B_k r^{\beta_k^+} + r^{\beta_k^-} \int_1^r \frac{\tau^{1-\beta_k^-}}{\beta_k^+ - \beta_k^-} b_k(\tau) d\tau - r^{\beta_k^+} \int_\infty^r \frac{\tau^{1-\beta_k^+}}{\beta_k^+ - \beta_k^-} b_k(\tau) d\tau \\
&= r^{\beta_k^-} \left(A_k + \frac{1}{\beta_k^+ - \beta_k^-} \int_1^r \tau^{1-\beta_k^-} b_k(\tau) d\tau \right) + r^{\beta_k^+} \left(B_k - \frac{1}{\beta_k^+ - \beta_k^-} \int_\infty^r \tau^{1-\beta_k^+} b_k(\tau) d\tau \right).
\end{aligned}
$$

As $a_k(1) = 0$, we obtain $A_k + B_k + \int_1^\infty \frac{\tau^{1-\beta_k^+}}{\beta_k^+ - \beta_k^-} b_k(\tau) d\tau = 0$. Assume $|b_k(r)| \leq \hat{b}_k r^{-(\sigma+2)}$ and $B_k = 0$, which implies

$$
\begin{aligned}
\sup_{1 \leq r < \infty} r^{\sigma + \beta_k^-} |A_k| &= \sup_{1 \leq r < \infty} r^{\sigma + \beta_k^-} \left| \frac{1}{\beta_k^+ - \beta_k^-} \int_1^\infty \tau^{1-\beta_k^+} b_k(\tau) d\tau \right| \\
&\leq \frac{\hat{b}_k}{\beta_k^+ - \beta_k^-} \sup_{1 \leq r < \infty} r^{\sigma + \beta_k^-} \int_1^\infty \tau^{-1-\sigma-\beta_k^+} d\tau \\
&< \hat{A}_k.
\end{aligned}
$$

Thus,

$$
a_k(r) = r^{\beta_k^-} \left(A_k + \frac{1}{\beta_k^+ - \beta_k^-} \int_1^r \tau^{1-\beta_k^-} b_k(\tau) d\tau \right) - r^{\beta_k^+} \left(\frac{1}{\beta_k^+ - \beta_k^-} \int_\infty^r \tau^{1-\beta_k^+} b_k(\tau) d\tau \right).
$$

Therefore,

$$
\begin{aligned}
&\sup_{1 \leq r < \infty} r^\sigma |a_k(r)| \\
&\leq \hat{A}_k + \frac{\hat{b}_k}{\beta_k^+ - \beta_k^-} \sup_{1 \leq r < \infty} \left(r^{\sigma + \beta_k^-} \int_1^r \tau^{-1-\sigma-\beta_k^-} d\tau + r^{\sigma + \beta_k^+} \int_\infty^r \tau^{-1-\sigma-\beta_k^+} d\tau \right) \\
&\leq \hat{A}_k + \frac{\hat{b}_k}{\beta_k^+ - \beta_k^-} \sup_{1 \leq r < \infty} \left(\frac{r^{-\sigma-\beta_k^-}}{\sigma + \beta_k^-} \right) = C_k.
\end{aligned}
$$

$\square$

Definition 3. *Let $A \in \{B_1^c, \mathbb{R}^N \backslash \{0\}\}$ and we define the norm on $X = X(A)$ via $\|\phi\|_X := \sup_{x \in A} |x|^{\sigma+1} |\nabla \phi(x)|$ and in the case of $A = B_1^c$ we impose $\phi = 0$ on $|x| = 1$. In both cases, we define the subspace X_1 to be the set of functions in X with no $k = 0$ mode, ie.*

$$\phi(x) = \sum_{k=1}^{\infty} a_k(r) \psi_k(\theta),$$

and we similarly define Y_1, i.e,

$$f(x) = f = \sum_{k=1}^{\infty} b_k(r) \psi_k(\theta).$$

Lemma 3.11. *Suppose $\phi \in X_1$, and $L_t(\phi) = 0$ in B_1^c with $\phi = 0$ on ∂B_1 or $L_t(\phi) = 0$ in $\mathbb{R}^N \backslash \{0\}$. Then, $\phi = 0$.*

Proof. We suppose ϕ as in the hypothesis and we write $\phi(x) = \sum_{k=1}^{\infty} a_k(r) \psi_k(\theta)$. Then note that, for all $k \geq 1$, we have

$$a_k''(r) + \left(N - 1 + \frac{p}{\kappa + t r^{-\frac{p-1}{\kappa}}} \right) \frac{a_k'(r)}{r} - \lambda_k \frac{a_k(r)}{r^2} = 0 \quad \text{ for } r \in I_i, \tag{3.18}$$

where $I_1 = (1, \infty)$ and $I_0 = (0, \infty)$. In the first case, we add the boundary condition with $a_k(1) = 0$. Fix $k \geq 1$ and set $a(r) = a_k(r)$. By the bound ϕ, there is some $C_k > 0$ such that $\sup_{r \in I_i} r^{\sigma+1} |a'(r)| \leq C_k$. We show that $\sup_{r \in I_i} r^{\sigma} |a(r)| < \infty$. Since $\phi \in X_1$, we can easily show that $\phi \in \widehat{X_1}$; where we add the $\sup_x |x|^{\sigma+2} |\Delta \phi|$ term to the norm. There is also some C_k such that

$$\sup_{r \in I_i} r^{\sigma+2} \left| a''(r) + \frac{(N-1)}{r} a'(r) - \frac{\lambda_k}{r^2} a(r) \right| \leq C_k.$$

By the gradient bound, we easily see that a has a limit as $r \to \infty$, call it T. Define

$h(r)$ such that

$$a''(r) + \frac{(N-1)}{r}a'(r) - \frac{\lambda_k}{r^2}a(r) = \frac{h(r)}{r^{\sigma+2}},$$

and note h is bounded on $(0,\infty)$. Multiply both side by r^{N-1}, we obtain

$$r^{N-1}a''(r) + (N-1)r^{N-2}a'(r) - \lambda_k r^{N-3}a(r) = r^{N-3-\sigma}h(r),$$

which implies

$$\lambda_k r^{N-3}a(r) = \frac{d}{dr}(r^{N-1}a'(r)) - r^{N-3-\sigma}h(r).$$

Now for large R integrate over $(R, 2R)$,

$$\left|\int_R^{2R} \lambda_k r^{N-3}a(r)dr\right| \leq \left|\int_R^{2R} \frac{d}{dr}(r^{N-1}a'(r))dr\right| + \left|\int_R^{2R} r^{N-3-\sigma}h(r)dr\right|,$$

which implies

$$\lambda_k\left|\int_R^{2R} r^{N-3}T dr\right| \leq \tilde{C}R^{N-2-\sigma}.$$

Therefore, $|T| \leq \hat{C}R^{-\sigma}$. When sending $R \to \infty$; this shows $T = 0$. We can integrate the a' bound from ∞ to get the desired bound on a.

$$r^\sigma|a(r)| \leq r^\sigma \int_r^\infty |a'(\tau)|d\tau \leq C_k r^\sigma \int_r^\infty \tau^{-\sigma-1}d\tau = \frac{C_k}{\sigma}.$$

We demonstrate that $a = 0$. We will show later that $\lim_{r\to\infty} r^\sigma|a(r)| = 0$ and in the case of I_0, we also have $\lim_{r\to 0} r^\sigma|a(r)| = 0$. Define $w(\tau) = r^\sigma a(r)$, where $\tau = \ln(r)$ where $\tau \in (0,\infty)$ (in the first case) and $\tau \in \mathbb{R}$ in the second case. Note in the first case, we have $w(0) = w(\infty) = 0$ and in the second case, we have $w(-\infty) = w(\infty) = 0$. A computation shows that $w(\tau)$ satisfies

$$w_{\tau\tau} + \left(-2\sigma + N - 2 + \frac{p}{\kappa + te^{\tau(-\frac{p-1}{\kappa})}}\right)w_\tau + C_k(\tau)w = 0,$$

where $C_k(\tau) = \sigma^2 - \sigma \left(N - 2 + \frac{p}{\kappa + te^{\tau(-\frac{p-1}{\kappa})}} \right) - \lambda_k$. We now assume that w is not identically zero. After observing the limiting behavior of w, we can suppose there is some $\tau_0 \in (0, \infty)$ in the first case, and some $\tau_0 \in \mathbb{R}$ in the second case such that w has a maximum at τ_0 with $w(\tau_0) > 0$ (multiply by -1 if necessary). Note at τ_0, we have $w_\tau(\tau_0) = 0$ and $w_{\tau\tau}(\tau_0) \leq 0$. Then, we have

$$C_k(\tau_0)w(\tau_0) = \left(\sigma^2 - \sigma \left(N - 2 + \frac{p}{\kappa + te^{\tau_0(-\frac{p-1}{\kappa})}} \right) - \lambda_k \right) w(\tau_0) = -w_{\tau\tau}(\tau_0) \geq 0. \tag{3.19}$$

We have $\sigma < N - 1$. Thus, we obtain

$$C_k(0) = -\frac{\sigma p}{\kappa + t} + \sigma^2 - \sigma(N-2) - \lambda_k \leq (\sigma + 1)(\sigma + 1 - N) < 0.$$

Also,

$$\begin{aligned} \lim_{\tau \to \infty} C_k(\tau) &= \sigma^2 - \sigma (N-2) - \lambda_k - \lim_{\tau \to \infty} \left(\frac{\sigma p}{\kappa + te^{\tau(-\frac{p-1}{\kappa})}} \right) \\ &\leq \sigma^2 - \sigma (N-2) - \lambda_k < 0. \end{aligned}$$

Note we have $C_k'(\tau) = \dfrac{-t\sigma p(p-1)}{\kappa(\kappa + te^{-\tau\frac{p-1}{\kappa}})^2} e^{-\tau\frac{p-1}{\kappa} - 1} e^{\tau} < 0$, which implies that $C_k(\tau_0) < 0$ and this leads to a contradiction. We prove the earlier assumed results for $r^\sigma |a(r)|$. We aim to show

$$\lim_{r \to \infty} r^\sigma |a(r)| = 0, \quad \text{in} \;\; I_0, I_1,$$

and

$$\lim_{r \to 0} r^\sigma |a(r)| = 0, \quad \text{in} \;\; I_0.$$

Assume not. Suppose there is some $\{r_m\}_m$ such that $r_m^\sigma |a(r_m)|$ is bounded away from zero. Consider $a^m(r) := r_m^\sigma a(r_m r)$ where $D_m = \{r : r_m r \in I_i\}$ and note in both cases I_0, I_1 we have $D_m \to (0, \infty)$. Also note that $|a^m(1)|$ is bounded away from zero and $r^\sigma|a^m(r)| \leq E_k$ on D_m. Thus, inserting in equation (3.18), we have

$$r_m^{\sigma+2}a''(r_m r) + \left(\frac{N-1}{r} + \frac{p}{r\left(\kappa + tr_m^{-\frac{p-1}{\kappa}-1}r^{-\frac{p-1}{\kappa}}\right)}\right) r_m^{\sigma+1}a'(r_m r) - \frac{\lambda_k}{r^2}r_m^\sigma a(r_m r) = 0,$$

for $r \in D_m$, which implies

$$(a^m(r))'' + \left(\frac{N-1}{r} + \frac{p}{r\left(\kappa + tr_m^{-\frac{p-1}{\kappa}-1}r^{-\frac{p-1}{\kappa}}\right)}\right)(a^m(r))' - \frac{\lambda_k}{r^2}a^m(r) = 0, \qquad \text{for } r \in D_m.$$

Note that $\{a^m\}_m \subset C^{1,\gamma}(A_k)$ where $A_k = \{1 \leq r \leq k\}$. By diagonal argument, there is a subsequence (which we don't rename) such that $a^m \to a^\infty$ in $C^{1,\frac{\gamma}{2}}(A_k)$ as $m \to \infty$. Using the convergence, we can pass to the limit in the equation.
Assume $\lim_{m\to\infty} a^m(r) = a^\infty(r)$.
Case 1. Let $r_m \to \infty$. Thus,

$$(a^\infty(r))'' + \left(\frac{N-1}{r} + \frac{p}{r\kappa}\right)(a^\infty(r))' - \frac{\lambda_k}{r^2}a^\infty(r) = 0 \qquad r \in (0,\infty).$$

The characteristic equation is given by $\chi^2 + \left(\frac{\alpha+2-p}{p-1}\right)\chi - \lambda_k = 0$. The characteristic equation has two distinct solutions as $\Delta = \left(\frac{\alpha+2-p}{p-1}\right)^2 + 4\lambda_k > 0$, for all $k \geq 1$. Thus, we have

$$\beta_k^\pm = -\frac{1}{2}\left(\frac{\alpha+2-p}{p-1}\right) \pm \frac{1}{2}\sqrt{\left(\frac{\alpha+2-p}{p-1}\right)^2 + 4\lambda_k}. \tag{3.20}$$

Note that $a^\infty(r) = C_k r^{\beta_k^+} + D_k r^{\beta_k^-}$. Thus, we have $r^\sigma a^\infty(r) = C_k r^{\beta_k^+ + \sigma} + D_k r^{\beta_k^- + \sigma}$. As $r^\sigma|a^m(r)| \leq E_k$ on D_m, we obtain

$$\left|C_k r^{\beta_k^+ + \sigma} + D_k r^{\beta_k^- + \sigma}\right| \leq E_k \quad \forall\ 0 < r < \infty.$$

By Lemma 3.7, $\beta_k^- + \sigma < 0$, and $\beta_k^+ + \sigma > 0$ imply $D_k = 0$ and $C_k = 0$, respectively. As a result, $a^\infty(r) = 0$ for all $k \geq 1$. Contradiction.

Case 2. Let $r_m \to 0$. Thus,

$$(a^\infty(r))'' + \frac{N-1}{r}\,(a^\infty(r))' - \frac{\lambda_k}{r^2} a^\infty(r) = 0 \qquad r \in (0, \infty).$$

The characteristic equation $m^2 + (N-2)\,m - \lambda_k = 0$ has two distinct solutions as $\Delta = (N-2)^2 + 4\lambda_k > 0$ for all $k \geq 1$. Thus, we have

$$\gamma_k^{\pm} = -\frac{1}{2}(N-2) \pm \frac{1}{2}\sqrt{(N-2)^2 + 4\lambda_k}. \tag{3.21}$$

We have, $a^\infty(r) = C_k r^{\gamma_k^+} + D_k r^{\gamma_k^-}$. We have, $r^\sigma a^\infty(r) = C_k r^{\gamma_k^+ + \sigma} + D_k r^{\gamma_k^- + \sigma},$. Thus, there is some constant $E_k > 0$ such that

$$\left| C_k r^{\gamma_k^+ + \sigma} + D_k r^{\gamma_k^- + \sigma} \right| \leq E_k \quad \forall\ 0 < r < \infty.$$

By Lemma 3.7, $\gamma_k^- + \sigma < 0$, and $\gamma_k^+ + \sigma > 0$ imply $D_k = 0$ and $C_k = 0$, respectively. As a result, $a^\infty(r) = 0$ for all $k \geq 1$. Contradiction. $\square$

Note 2. *Let $\phi \in X$ and consider this for $1 < r < \infty$. Note that $\int_{|\theta|=1} \psi_k(\theta) \cdot 1 d\theta = 0$ for $k \geq 1$. We have*

$$\int_{|\theta|=1} \phi(r\theta) d\theta = a_0(r) \int_{|\theta|=1} \psi_0(\theta) d\theta + \sum_{k=1}^{\infty} a_k(r) \int_{|\theta|=1} \psi_k(\theta) d\theta = C a_0(r).$$

Therefore, $\phi \in X$ then $\phi \in X_1$ if and only if $\int_{|\theta|=1} \phi(r\theta) d\theta = 0$.

Proposition 3.12. *There is some $C_0 > 0$ such that for all $t > 0$ and $f \in Y_1$, there* *is some $\phi \in X_1$ such that*

$$\begin{cases} L_t(\phi) = f & \textit{in} \;\; B_1^c, \\ \phi = 0, & \textit{on} \;\; \partial B_1. \end{cases} \tag{3.22}$$

Moreover, $\|\phi\|_X \leq C_0 \|f\|_Y$.

Proof. The proof is based on the method of continuity (see Theorem 0.11), since we have the result for $t = 0$. Thus, we assume we do not have the desired estimate. Therefore, there is $t_m > 0$ such that $t_m \to t \in [0, \infty]$ as $m \to \infty$ and such that $L_{t_m}(\phi_m) = f_m$ in B_1^c with $\phi_m = 0$ on ∂B_1 and $\|\phi_m\|_X = 1$ and $\|f_m\|_Y \to 0$. There exists some x_m such $|x_m|^{\sigma+1} |\nabla \phi_m(x_m)| \geq \frac{3}{4}$. Let $s_m = |x_m|$.

Case s_m bounded. Using compactness, we have $\phi_m \to \phi$ in $C^{1,\delta}_{loc}(B_1^c)$ and $L_t(\phi) = 0$ (we are allowing for the case of $t = \infty$ here) in B_1^c with $\phi = 0$ on ∂B_1. Moreover, we have $|\nabla \phi(x)| \neq 0$ and $|x|^{\sigma+1} |\nabla \phi(x)| \leq 1$. In the case of $t \in [0, \infty)$, we obtain

$$-\Delta \phi - p \frac{x \cdot \nabla \phi}{\kappa |x|^2 + t|x|^{2 - \frac{p-1}{\kappa}}} = 0 \text{ in } B_1^c \text{ with } \phi = 0 \text{ on } \partial B_1.$$

Using Lemma 3.11, we have $\phi = 0$. In the case of $t = \infty$, we have $-\Delta \phi = 0$ in B_1^c with $\phi = 0$ on ∂B_1. By Lemma 3.5, $\phi = 0$ which implies $|\nabla \phi(x)| = 0$, which contradicts $|\nabla \phi(x)| \neq 0$ in both cases.

Case $s_m \to \infty$. Define $|z_m| = 1$ by $s_m z_m = x_m$ and define $\psi_m(z) = s_m^\sigma \phi_m(s_m z)$ for $z \in D_m = \{z : s_m z \in B_1^c\}$. Note that $|\psi_m(z_m)| = 0$ and $|z|^{\sigma+1} |\nabla \psi_m(z)| \leq 1$ and also $|\nabla \psi_m(z_m)|$ is bounded away from zero. We have

$$-\Delta \phi_m(s_m z) - \frac{p}{\kappa + t_m |s_m z|^{-\frac{p-1}{\kappa}}} \frac{(s_m z) \cdot \nabla \phi_m(s_m z)}{|s_m z|^2} = f_m(s_m z).$$

As $\Delta\psi_m(z) = s_m^{\sigma+2}\Delta\phi_m(s_m z)$ and $\nabla\psi_m(z) = s_m^{\sigma+1}\phi_m(s_m z)$, we obtain

$$-\Delta\psi_m(s_m z) - \frac{p}{\kappa + t_m|s_m z|^{-\frac{p-1}{\kappa}}}\frac{z\cdot\nabla\psi_m(z)}{|z|^2} = s_m^{\sigma+2}f_m(s_m z).$$

Then, note we have

$$L_{t_m s_m^{\frac{-(p-1)}{\kappa}}}(\psi_m)(z) = g_m(z) = s_m^{\sigma+2}f_m(s_m z) \quad z\in D_m.$$

Note that $g_m \to 0$ in $L^\infty_{loc}(\mathbb{R}^N\backslash\{0\})$. We now suppose $t_m s_m^{\frac{-(p-1)}{\kappa}} \to \tau \in [0,\infty]$. When $\tau = 0$, we obtain a contradiction with the kernel result for L_0 on the full space, as stated in Lemma 3.9. When $0 < \tau < \infty$, we encounter a contradiction with the kernel result for L_τ. Lastly, when $\tau = \infty$, we obtain a contradiction with the result for the Laplacian, as outlined in Lemma 3.8.

□

Lemma 3.13. *($k = 0$ mode nonhomogenous) There is some $C > 0$ such that for $p > \frac{\alpha+1}{N-1}$, one have $r^{\sigma+1}|a_t'(r)| \le C$.*

Proof. We have

$$a_t''(r) + \left(N - 1 + \frac{p}{\kappa + tr^{-\frac{p-1}{\kappa}}}\right)\frac{a_t'(r)}{r} = -b(r) \quad 1 < r < \infty, \tag{3.23}$$

with $a_t(1) = 0$. Without loss of generality, assume that $\sup\limits_{1<r<\infty} r^{\sigma+2}|b(r)| \le 1$. Let $x_t(r) = a_t'(r)$. Thus, the equation (3.23) changes to a first order differential equation,

$$x_t'(r) + p(r)x_t(r) = -b(r) \quad 1 < r < \infty, \tag{3.24}$$

where $p(r) = r^{-1}\left(N - 1 + p(\kappa + tr^{-\frac{p-1}{\kappa}})^{-1}\right)$. The integrating factor for the equation is $\mu_t(r) = e^{P(r)}$, where

$$P(r) = \int_1^r p(\tau)d\tau \quad = (N-1)\ln r + \frac{p}{p-1}\ln\left(\frac{t+\kappa r^{\frac{p-1}{\kappa}}}{t+\kappa}\right),$$

which implies $\mu_t(r) = r^{N-1}\left(\frac{t+\kappa r^{\frac{p-1}{\kappa}}}{t+\kappa}\right)^{\frac{p}{p-1}}$.

Thus, we have $-\dfrac{d}{d\tau}\left(\mu_t(\tau)a_t'(\tau)\right) = \mu_t(\tau)b(\tau)$. Hence, by taking integral and considering that $\mu_t(1) = 1$, we obtain

$$\mu_t(1)a_t'(1) - \mu_t(s)a_t'(s) = -\int_s^1 \mu_t(\tau)b(\tau)d\tau,$$

which implies

$$a_t(r) = -\int_r^1 \frac{1}{\mu_t(s)}\left(a_t'(1) + \int_s^1 \mu_t(\tau)b(\tau)d\tau\right)ds.$$

We set $a_t'(1) = -\int_{R_t}^1 \mu_t(\tau)b(\tau)d\tau$, where $tR_t^{-\frac{p-1}{\kappa}} = 1$. We obtain

$$a_t'(r) = -\frac{1}{\mu_t(r)}\int_{R_t}^r \mu_t(\tau)b(\tau)d\tau.$$

Case 1: Assume $r < R_t$. We have

$$r^{\sigma+1}|a_t'(r)| \le \hat{C}\left(\frac{1+\kappa}{\kappa}\right)^{\frac{p}{p-1}},$$

provided $\sigma+1-\frac{p}{\kappa}>0$, i.e., $p>\frac{\alpha+1}{N-1}$. To demonstrate it,

$$\begin{aligned}
r^{\sigma+1}|a_t'(r)| &\le r^{\sigma+2-N}\left(\frac{t+\kappa r^{\frac{p-1}{\kappa}}}{t+\kappa}\right)^{-\frac{p}{p-1}}\int_r^{R_t}\mu_t(\tau)|b(\tau)|d\tau\\
&\le Cr^{\sigma+2-N}\left(t+\kappa r^{\frac{p-1}{\kappa}}\right)^{-\frac{p}{p-1}}\int_r^{R_t}\tau^{-\sigma-2}\left(t+\kappa\tau^{\frac{p-1}{\kappa}}\right)^{\frac{p}{p-1}}d\tau\\
&\le C\left(\frac{1+\kappa}{\kappa}\right)^{\frac{p}{p-1}}r^{-(N-1)}\frac{1}{\sigma+1-\frac{p}{\kappa}}\left(1-\left(\frac{r}{R_t}\right)^{\sigma+1-\frac{p}{\kappa}}\right)\\
&\le \hat{C}\left(\frac{1+\kappa}{\kappa}\right)^{\frac{p}{p-1}}.
\end{aligned}$$

Case 2: Assume $R_t<r$, and $r=\gamma R_t$, for $\gamma>1$. Note that $tR_t^{-\frac{p-1}{\kappa}}=1$. Therefore,

$$r^{\sigma+1}|a_t'(r)|\le\frac{C_1}{t\left(1+\kappa\gamma^{\frac{p-1}{\kappa}}\right)^{\frac{p}{p-1}}}\int_{R_t}^{\gamma R_t}\tau^{-\sigma-2}\left(t+\kappa\tau^{\frac{p-1}{\kappa}}\right)^{\frac{p}{p-1}}d\tau,$$

which implies

$$r^{\sigma+1}|a_t'(r)|\le\frac{C_3\gamma^{-(\sigma+1)}(\gamma^{\frac{p}{\kappa}}-1)}{\left(1+\kappa\gamma^{\frac{p-1}{\kappa}}\right)^{\frac{p}{p-1}}}\le C_4\gamma^{-(\sigma+1)}\le C.$$

□

Corollary 3.14. *There is some $C_0>0$ such that for all $t>0$ and $f\in Y$, there is some $\phi\in X$ that satisfies (3.22) provided $p>\frac{\alpha+1}{N-1}$. Moreover, one has $\|\phi\|_X\le C_0\|f\|_Y$.*

Proof. Let $f\in Y$ with $\|f\|_Y=1$. Assume

$$f(x)=\sum_{k=0}^{\infty}b_k(r)\psi_k(\theta)=b_0(r)+\sum_{k=1}^{\infty}b_k(r)\psi_k(\theta)=f_0+f_1,$$

where

$$b_k(r)=\frac{1}{|S^{N-1}|}\int_{S^{N-1}}f(r\theta)\psi_k(\theta)d\theta.$$

Then, we can write $f(x)=f_0(r)+f_1(x)$ where f_0 is radial $f_1\in Y_1$. By Proposition

3.12 and Lemma 3.13, there is some $C_0 > 0$ such that for all $t > 0$ there is some $\phi_1 \in X_1$ such that $L_t(\phi_1) = f_1$ in B_1^c, and $\phi_1 = 0$, on ∂B_1. Moreover, $\|\phi_1\|_X \leq C_0\|f_1\|_Y$. Set $\phi = \phi_0 + \phi_1$. Then note by linearity, we have $L_t(\phi) = f$ in B_1^c with $\phi = 0$ on ∂B_1 and

$$\|\phi\|_X \leq \|\phi_0\|_X + \|\phi_1\|_X \leq C_0 \left(\|f_0\|_Y + \|f_1\|_Y\right),$$

and hence the proof is complete if we can show there is some $D > 0$ (independent of f) such that

$$\|f_0\|_Y + \|f_1\|_Y \leq D\|f_0 + f_1\|_Y = D\|f\|_Y.$$

We can prove this using the same technique as we did in the case of B_1; we omit the details.

□

We now move on to our main linear result on an exterior domain.

Theorem 3.15. *There is some $C > 0$ such that for all $t > 0$ (Large) and $f \in Y$, there is some ϕ such that*

$$\begin{cases} L_t(\phi) = f & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.25}$$

Moreover, $\|\phi\|_X \leq C\|f\|_Y$.

Let $L_t(\phi) = -\Delta\phi - a_t(x) \cdot \nabla\phi$. Let R_0 be very large and consider $\eta = 0$ in B_{R_0} and $\eta = 1$ for $|x| > R_0 + 1$. Now take δ such that $\delta = 0$ in B_{R_0-1} and $\delta = 1$ for $|x| \geq R_0$ and hence we have $\eta\delta = \eta$. We look for a solution of $L_t(\phi) = f$ in Ω with $\phi = 0$ on $\partial\Omega$. We want to use "*inner-outer gluing technique*". Look for a solution in the form $\phi = \eta\varphi + \psi$, where φ and ψ solve the inner and the outer problems, respectively. A computation shows that ϕ satisfies the equation if φ, ψ satisfy the system

$$\begin{cases} \Delta\varphi + a_t \cdot \nabla\varphi = -\delta f - \delta a_t \cdot \nabla\psi & \text{in } B_1^c, \\ \varphi = 0, & \text{on } \partial B_1, \end{cases} \tag{3.26}$$

and

$$\begin{cases} \Delta\psi + (1-\eta)a_t \cdot \nabla\psi = -(1-\eta)f - 2\nabla\varphi\cdot\nabla\eta - (\Delta\eta)\varphi - (a_t\cdot\nabla\eta)\varphi & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

We will solve this system via a fixed point argument. We now outline the procedure. Let C_0 be from Lemma 3.12 and suppose $\varphi \in B_{2C_0} \subset X(B_1^c)$ and let ψ satisfy (3.27). Assume $\hat{\varphi}$ satisfy

$$\begin{cases} \Delta\hat{\varphi} + a_t \cdot \nabla\hat{\varphi} = -\delta f - \delta a_t \cdot \nabla\psi & \text{in } B_1^c, \\ \varphi = 0, & \text{on } \partial B_1. \end{cases} \tag{3.28}$$

This defines the nonlinear mapping $T_t(\varphi) = \hat{\varphi}$ and if we can show T_t has a fixed point then we are done. Hence, our goal is to show that T_t is a contraction mapping on $B_{2C_0} := \{\phi \in X(B_1^c) : \|\phi\|_{X(B_1^c)} \leq 2C_0\}$.

The proof of Theorem 3.15. Let $\varepsilon > 0$ be small and without loss of generality, lets take $\|f\|_Y = 1$ and $\varphi \in B_{2C_0} \subset X(B_1^c)$ and let ψ satisfy (3.27). By Lemma 3.6, there is some $R_\varepsilon > 0$ (independent of t,f, φ) such that $|x|^{\sigma+1}|\nabla\psi(x)| \leq \varepsilon$ for all $|x| \geq R_\varepsilon$. Also, note there is some $C > 0$ such that $\sup_\Omega |x|^{\sigma+1}|\nabla\psi(x)| \leq C$ (where C is uniform over the allowed φ and t). We aim to solve the outer problem (3.27). Assume φ is given such that $\|\varphi\|_X < \infty$. Suppose $\psi := \psi(\varphi, f)$ solves equation (3.27) and $\|\psi\|_X < \infty$. Inserting in (3.26), we consider $\hat{\varphi}$ solves problem (3.26). This defines the nonlinear mapping $T_t(\varphi) = \hat{\varphi}$. Therefore, $T_t : X \to X$. Our goal is to find a fixed point of T_t. Let C_0 denote the constant from Proposition 3.12. We demonstrate that T_t is a contraction on B_{2C_0} (the closed ball centered at the origin in X with radius $2C_0$). Then, we get the desired result by *Banach Fixed Point Theorem*.

Into. We will demonstrate that $T_t(B_{2C_0}) \subset B_{2C_0}$, i.e, $\hat{\varphi} \in B_{2C_0}$ for sufficiently large t. Let $\varphi \in B_{2C_0}$ and suppose ψ satisfies (3.27) and $\hat{\varphi}$ satisfies (3.26) with φ replaced with $\hat{\varphi}$. Then, we have

$$\begin{aligned}\|\hat{\phi}\|_{X(B_1^c)} &\leq C_0\|\delta f\|_{Y(B_1^c)} + C_0\|\delta a_t \cdot \nabla\psi\|_{Y(B_1^c)} \\ &\leq C_0 + C_0\|\delta a_t \cdot \nabla\psi\|_{Y(B_1^c)}.\end{aligned}$$

Hence, if we can show this last term is small for large t we would have the desired result. Define $A_R(t) = \sup_{\Omega\cap B_R} |x||a_t(x)|$ and note that for each fixed large R we have $A_R(t) \to 0$ as $t \to \infty$. Then note we have

$$\begin{aligned}\|\delta a_t \cdot \nabla\psi\|_{Y(B_1^c)} &\leq \sup_{\Omega\cap B_{R_\varepsilon}} |x||a_t(x)||x|^{\sigma+1}|\nabla\psi(x)| + \sup_{|x|>R_\varepsilon} |x||a_t(x)||x|^{\sigma+1}|\nabla\psi(x)| \\ &\leq CA_{R_\varepsilon}(t) + \left\{\sup_{|x|>R_\varepsilon} |x||a_t(x)|\right\}\varepsilon \\ &\leq CA_{R_\varepsilon}(t) + C_{10}\varepsilon,\end{aligned}$$

which can be made small by taking $\varepsilon > 0$ small (fixed) and then taking t large.

Contraction. We aim to show that there exists $0 < \gamma < 1$ such that

$$\|T_t(\varphi_2) - T_t(\varphi_1)\|_{X(B_1^c)} \leq \gamma \|\varphi_2 - \varphi_1\|_{X(B_1^c)}.$$

First, we will prove the following claim.

Claim. We claim for all $0 < \tilde{\gamma} \ll 1$, there is some $R_{\tilde{\gamma}} > R_0 + 1$ such that for all $t > t_0$, for all $\varphi \in B_{2C_0}$ with ψ satisfying

$$\begin{cases} \Delta\psi + (1-\eta)a_t \cdot \nabla\psi = -2\nabla\varphi \cdot \nabla\eta - (\Delta\eta)\varphi - (a_t \cdot \nabla\eta)\varphi =: g_t(\varphi) & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega. \end{cases}$$

We have $\sup_{|x|>R_{\tilde{\gamma}}} |x|^{\sigma+1}|\nabla\psi| \leq \tilde{\gamma}\|\varphi\|_X$. By using a normalization argument we can assume $\|\varphi\|_X = 1$ and then note that $\|g_t\|_{Y(\Omega)} \leq C$, where C is independent of t and ϕ. Then we can apply Lemma 3.6 to complete the proof of the claim. We now show

T_t is a contraction for sufficiently large t. Let $\varphi_i \in B_{2C_0}$ and let ψ_i satisfy (3.27) with $\varphi = \varphi_i$. Let $\hat{\varphi}_i = T_t(\varphi_i)$ for $i = 1, 2$. We have

$$\Delta(\hat{\varphi}_2 - \hat{\varphi}_1) + a_t \cdot \nabla(\hat{\varphi}_2 - \hat{\varphi}_1) = -\delta a_t \cdot \nabla(\psi_2 - \psi_1) \quad \text{in } B_1^c,$$

with $\hat{\varphi}_2 - \hat{\varphi}_1 = 0$ on ∂B_1. Applying Proposition 3.12, we obtain

$$\begin{aligned}
\|\hat{\varphi}_2 - \hat{\varphi}_1\|_X &\leq C_0 \|\delta a_t \cdot \nabla(\psi_2 - \psi_1)\|_Y \\
&\leq C_0 \sup_{\mathrm{supp}(\delta) \cap B_{R_\varepsilon}} |x|^{\sigma+2} |a_t| |\nabla(\psi_2 - \psi_1)| + C_0 \sup_{|x| > R_\varepsilon} |x|^{\sigma+2} |a_t| |\nabla(\psi_2 - \psi_1)| \\
&\leq C_1 \|\psi_2 - \psi_1\|_X A_{R_\varepsilon}(t) + C_1 \sup_{|x| > R_\varepsilon} |x|^{\sigma+1} |\nabla(\psi_2 - \psi_1)| \\
&\leq C_2 \|\varphi_2 - \varphi_1\|_X A_{R_\varepsilon}(t) + C_2 \varepsilon \|\varphi_2 - \varphi_1\|_X \\
&\leq C_3 \left(A_{R_\varepsilon}(t) + \varepsilon\right) \|\varphi_2 - \varphi_1\|_X \\
&\leq \gamma \|\varphi_2 - \varphi_1\|_X,
\end{aligned}$$

where $\gamma = C_3 \left(A_{R_\varepsilon}(t) + \varepsilon\right)$. Choosing $\varepsilon > 0$ small and t big enough, we obtain $\gamma < 1$ and this completes the proof of T_t being a contraction for large enough t. With the completion of this step, we are now prepared to finalize the proof. Recall that $\phi = \eta\varphi + \psi$. We know $\|\varphi\|_{X(B_1^c)} \leq 2C_0$ and by Lemma 3.6, we have $\|\psi\|_{X(\Omega)} < C_1$. As a result, we have

$$\begin{aligned}
\|\phi\|_{X(\Omega)} = \|\eta\varphi + \psi\|_{X(\Omega)} &= \sup_{x \in \Omega} |x|^{\sigma+1} |\nabla(\eta\varphi + \psi)| \\
&\leq \sup_{x \in \Omega} |x|^{\sigma+1} |\nabla(\eta\varphi)| + \sup_{x \in \Omega} |x|^{\sigma+1} |\nabla(\psi)| \\
&\leq \sup_{x \in \Omega} |x|^{\sigma+1} |\eta \nabla\varphi + \varphi \nabla\eta| + \|\psi\|_{X(\Omega)} \\
&\leq \sup_{|x| > R_0} |x|^{\sigma+1} |\nabla\varphi| + \sup_{R_0 < |x| < R_0 + 1} |x|^{\sigma+1} |\varphi \nabla\eta| + \|\psi\|_{X(\Omega)} \\
&\leq \|\varphi\|_{X(B_1^c)} + \sup_{R_0 < |x| < R_0 + 1} |x|^{\sigma+1} |\varphi \nabla\eta| + \|\psi\|_{X(\Omega)} \\
&\leq C.
\end{aligned}$$

3.3 The fixed point argument

We are looking for a solution u of (3.1) of the form $u(x) = u_t(x)\eta(x) + \phi(x)$ and this implies that ϕ to satisfy in equation (3.6). Recall that

$$H_t(\phi) = -\eta|x|^\alpha|\nabla u_t|^p + u_t\Delta\eta + 2\nabla u_t\cdot\nabla\eta + |x|^\alpha|\nabla(u_t\eta) + \nabla\phi|^p - p|x|^\alpha|\nabla u_t|^{p-2}\nabla u_t\cdot\nabla\phi.$$

Note that for u to be positive, we need ϕ to satisfy $u_t(x)\eta(x) + \phi(x) > 0$ in Ω. Thus, to find a solution ϕ, we seek a fixed point of the nonlinear mapping J_t where $J_t(\phi) = \psi$ where

$$\begin{cases} L_t(\psi) = H_t(\phi) & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.29}$$

Let $B_R = \{\phi \in X : \|\phi\|_X \le R\}$. We demonstrate that $J_t : B_R \to B_R$ is a contraction mapping if R is satisfied under specific conditions. We can write $H_t(\phi) = I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= |x|^\alpha \left(|\nabla(u_t\eta) + \nabla\phi|^p - |\nabla(u_t\eta)|^p - p|\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta)\cdot\nabla\phi\right), \\ I_2 &= p|x|^\alpha \left(|\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta) - |\nabla u_t|^{p-2}\nabla u_t\right)\cdot\nabla\phi, \\ I_3 &= |x|^\alpha \left(|\nabla(u_t\eta)|^p - \eta|\nabla u_t|^p\right) + u_t\Delta\eta + 2\nabla u_t\cdot\nabla\eta. \end{aligned}$$

Into. Assume $\phi \in B_R$. First note that

$$\|H_t(\phi)\|_Y = \sup_{x\in\Omega}|x|^{\sigma+2}|H_t(\phi)| \le \sup_{x\in\Omega}|x|^{\sigma+2}|I_1| + \sup_{x\in\Omega}|x|^{\sigma+2}|I_2| + \sup_{x\in\Omega}|x|^{\sigma+2}|I_3|.$$

Recall that $0 \le \eta \le 1$ be a cut-off function such $\eta = 0$ in B_{R_0} and $\eta = 1$ for $|x| > R_0 + 1$ for R_0 very large. Define $\zeta_t := \|I_2\|_Y$ and $\xi_t := \|I_3\|_Y$. For $\eta = 0, 1$, we have $|I_3| = 0$. For $0 < \eta < 1$, we have $u_t, \nabla u_t \to 0$ as $t \to \infty$. Therefore, $\xi_t \to 0$ as $t \to \infty$. Define $D_t(x) := p|x|^{\alpha+1}\left(|\nabla(u_t\eta)|^{p-2}\nabla(u_t\eta) - |\nabla u_t|^{p-2}\nabla u_t\right)$, and note that

$D_t(x) = 0$ on $|x| > R_0 + 1$. We have $\delta_t := \sup_{|x|<R_0+1} |D_t| \to 0$ as $t \to \infty$ on bounded region $|x| > R_0 + 1$. As we have $\|I_2\|_Y \leq R\delta_t$, which implies $\zeta_t \to 0$ as $t \to \infty$. Assume that $p \geq 2$. Therefore, by Lemma 6.1, there is some $C_{1,1} > 0$ such that

$$
\begin{aligned}
\|I_1\|_Y &\leq C_{1,1} \sup_{\Omega} |x|^{\alpha+\sigma+2} \left(|\nabla(u_t\eta)|^{p-2} |\nabla\phi|^2 + |\nabla\phi|^p \right) \\
&\leq C_{1,1} \sup_{\Omega} \left(|x|^{\frac{\alpha+1}{p-1}} |\nabla(u_t\eta)| \right)^{p-2} \left(|x|^{\sigma+1} |\nabla\phi| \right)^2 + C_{1,1} \sup_{\Omega} \left(|x|^{\sigma+1} |\nabla\phi| \right)^p \\
&\leq C_{1,1} \sup_{\Omega} \left(|x|^{\sigma+1} |\nabla(u_t\eta)| \right)^{p-2} R^2 + C_{1,1} R^p.
\end{aligned}
$$

Define $\mathcal{E}_t(x) := (|x|^{\sigma+1}|\nabla(u_t\eta)|)^{p-2}$. We obtain

$$
\begin{aligned}
\sup_{\Omega} \mathcal{E}_t &\leq \sup_{|x|>R_0+1} \left(|x|^{\sigma+1} |\nabla u_t| \right)^{p-2} + \sup_{R_0<|x|<R_0+1} \left(|x|^{\sigma+1} |\nabla(u_t\eta)| \right)^{p-2} \\
&\leq \kappa^{-\frac{p-2}{p-1}} + \sup_{R_0<|x|<R_0+1} \left(|x|^{\sigma+1} |\eta\nabla u_t + u_t \nabla\eta| \right)^{p-2} \\
&\leq \kappa^{-\frac{p-2}{p-1}} + \sup_{R_0<|x|<R_0+1} \left(|x|^{\sigma+1} \eta |\nabla u_t| \right)^{p-2} + \sup_{R_0<|x|<R_0+1} \left(|x|^{\sigma+1} |u_t| |\nabla\eta| \right)^{p-2} \\
&\leq \mathcal{C}_t.
\end{aligned}
$$

Thus, $\|I_1\|_Y \leq C_{1,1}\mathcal{C}_t R^2 + C_{1,1}R^p$. Therefore, $\|H_t(\phi)\|_Y \leq R\left(C_{1,1}\mathcal{C}_t R + C_{1,1}R^{p-1}\right) + \zeta_t + \xi_t$. Assume $1 \leq p < 2$. Using lemma 6.1, there is some $\hat{C}_{1,1}$ such that

$$
\|I_1\|_Y \leq \hat{C}_{1,1} \sup_{x\in\Omega} |x|^{\sigma+2+\alpha} |\nabla\phi|^p \leq \hat{C}_{1,1} R^p \sup_{x\in\Omega} |x|^{-(p-1)(\sigma - \frac{\alpha+2-p}{p-1})}.
$$

Therefore, there exists $\hat{C}_t > 0$ such that $\|I_1\|_Y \leq \hat{C}_t \hat{C}_{1,1} R^p$. Thus, $\|H_t(\phi)\|_Y \leq R\left(\hat{C}_t \hat{C}_{1,1} R^{p-1}\right) + \zeta_t + \xi_t$.

Contraction. Assume $\phi_1, \phi_2 \in B_R$ and $p \geq 2$. Therefore, there exists $C_{1,2} > 0$ such that

$$
\begin{aligned}
&|H_t(\phi_2) - H_t(\phi_1)| \\
&\le C_{1,2}|x|^\alpha \left(|\nabla(u_t\eta)|^{p-2} \left(|\nabla\phi_2| + |\nabla\phi_1|\right) + |\nabla\phi_2|^{p-1} + |\nabla\phi_1|^{p-1} \right) |\nabla\phi_2 - \nabla\phi_1| \\
&+ p|x|^\alpha ||\nabla(\eta u_t)|^{p-2}\nabla(\eta u_t) - |\nabla u_t|^{p-2}\nabla u_t||\nabla\phi_2 - \nabla\phi_1|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|H_t(\phi_2) - H_t(\phi_1)\|_Y \le 2C_{1,2}R \sup_{x\in\Omega} |x|^{\alpha-\sigma}|\nabla(u_t\eta)|^{p-2} \|\phi_2 - \phi_1\|_X \\
&+ 2C_{1,2}R^{p-1} \sup_{x\in\Omega} |x|^{\alpha+1-(\sigma+1)(p-1)} \|\phi_2 - \phi_1\|_X + \delta_t \|\phi_2 - \phi_1\|_X .
\end{aligned}
$$

Define

$$
C_* := \delta_t + 2C_{1,2}R^{p-1} \sup_{x\in\Omega} |x|^{\alpha+1-(\sigma+1)(p-1)} + 2C_{1,2}R \sup_{x\in\Omega} |x|^{\alpha-\sigma}|\nabla(u_t\eta)|^{p-2}.
$$

Note that $\alpha + 1 - (\sigma + 1)(p - 1) < 0$ since $\sigma > \frac{\alpha+2-p}{p-1}$. On the other hand,

$$
\sup_{x\in\Omega} |x|^{\alpha-\sigma}|\nabla(u_t\eta)|^{p-2} = \sup_{x\in\Omega} |x|^{\frac{p-2}{p-1}}\mathcal{E}_t(x) \le (R_0 + 1)^{\frac{p-2}{p-1}}\mathcal{C}_t.
$$

As a result, $C_* < \infty$. Therefore, $\|H_t(\phi_2) - H_t(\phi_1)\|_Y \le C_* \|\phi_2 - \phi_1\|_X$.
Assume $\phi_1, \phi_2 \in B_R$ and $1 \le p < 2$. Therefore, there exists $\hat{C}_{1,2}$ such that

$$
\begin{aligned}
|H_t(\phi_2) - H_t(\phi_1)| &\le \hat{C}_{1,2}|x|^\alpha \left(|\nabla\phi_2|^{p-1} + |\nabla\phi_1|^{p-1} \right) |\nabla\phi_2 - \nabla\phi_1| \\
&\quad + p|x|^\alpha \Big||\nabla(\eta u_t)|^{p-2}\nabla(\eta u_t) - |\nabla u_t|^{p-2}\nabla u_t\Big||\nabla\phi_2 - \nabla\phi_1|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|H_t(\phi_2) - H_t(\phi_1)\|_Y \\
&\le \hat{C}_{1,2} \sup_{x\in\Omega} |x|^{\alpha+1-(\sigma+1)(p-1)} \left((|x|^{\sigma+1}|\nabla\phi_2|)^{p-1} + (|x|^{\sigma+1}\nabla\phi_1|)^{p-1} \right) \|\phi_2 - \phi_1\|_X \\
&+ \delta_t \|\phi_2 - \phi_1\|_X .
\end{aligned}
$$

Therefore,

$$
\|H_t(\phi_2) - H_t(\phi_1)\|_Y \le \left(\delta_t + 2\hat{C}_{1,2} R^{p-1} \sup_{x\in\Omega} |x|^{\alpha+1-(\sigma+1)(p-1)} \right) \|\phi_2 - \phi_1\|_X .
$$

Define $\hat{C}_* := \delta_t + 2\hat{C}_{1,2} R^{p-1} \sup_{x\in\Omega} |x|^{\alpha+1-(\sigma+1)(p-1)}$. Therefore, $\|H_t(\phi_2) - H_t(\phi_1)\|_Y \le \hat{C}_* \|\phi_2 - \phi_1\|_X$.

Case 1. Let $p > 2$. we have

$$
\|J_t(\phi_2) - J_t(\phi_1)\|_X \le C_l C_* \|\phi_2 - \phi_1\|_X ,
$$

and

$$
\|J_t(\phi)\|_X \le C_l \left(C_{1,1}\mathcal{C}_t R^2 + C_{1,1} R^p \right) + \zeta_t + \xi_t.
$$

Thus, J_t is a contraction on B_R if $C_l C_* < 1$ and $C_l \left(C_{1,1}\mathcal{C}_t R^2 + C_{1,1} R^p \right) + \zeta_t + \xi_t < R$.

Case 2. Let $1 \le p < 2$, we have

$$
\|J_t(\phi_2) - J_t(\phi_1)\|_X \le C_l \hat{C}_* \|\phi_2 - \phi_1\|_X , \qquad \|J_t(\phi)\|_X \le C_l \hat{C}_t \hat{C}_{1,1} R^p + \zeta_t + \xi_t.
$$

Thus, J_t is a contraction on B_R if $C_l \hat{C}_* < 1$ and $C_l \hat{C}_t \hat{C}_{1,1} R^p + \zeta_t + \xi_t < R$. To obtain the desired result, choose R small enough and then take t large. We can now apply Banach's contraction mapping principle to find a fixed point, ϕ, of J_t on B_R and hence ϕ satisfies (3.29). Consequently $u_t(x) = u(x) = v_t(r) + \phi(x)$ is a solution of (3.1). In the case of κ positive and negative, we carefully selected σ, to ensure that when ϕ is sufficiently small (in the space X), the solution $u(x)$ remains positive for large $|x|$. This established that u was nonzero, one could now apply a maximum principle argument on $\Omega \cap B_R$ (for large R) to see that u is positive on the remainder of Ω and this shows that u is strictly positive.

3.4 Conclusion of Chapter 3.

In Chapter 3, we examined the case of (3.1) for an exterior domain that does not include the origin. Here, the key building block of solutions is the explicit solution of the problem on B_1^c. A parameter κ comes up that involves the other parameters and the sign of κ plays a major role in our analysis. In the case of $\kappa < 0$, we can show for large t that our linearized operator is a small perturbation of the Laplacian and this allows fairly easy analysis of the linear problem (once the correct spaces are chosen). The case of $\kappa > 0$ is much harder and here one needs to perform a detailed analysis of the linearized operator on B_1^c and then use a gluing procedure to prove the result on a general exterior domain.

4

Nonlinear Gradient Problem with Two Blow-up Points

In this chapter, we consider the following elliptic problem

$$\begin{cases} -\Delta u = |\nabla u|^p, & \text{in } \Omega \backslash \{\xi_1, \xi_2\}, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω a bounded domain in $\mathbb{R}^N$, $N \geq 3$ with $\xi_1, \xi_2 \in \Omega$ two distinct points and $\frac{N}{N-1} < p < 2$. We prove the existence of positive classical solutions with $\lim_{|x| \to \xi_i} u(x) = \infty$. In particular, we show the following.

Theorem 4.1. *Suppose $N \geq 3$, and $\frac{N}{N-1} < p < 2$. Then there is some $t_0 > 0$ (large) such that for all $t \geq t_0$, there is a solution u_t of (4.1) with $\lim_{|x| \to \xi_i} u(x) = \infty$.*

Our solutions will be built using the following problem. Let B_1 denote the unit ball centered at the origin in $\mathbb{R}^N$ for $N \geq 3$. Then for $\frac{N}{N-1} < p < 2$, we define $\alpha := (p-1)(N-1)$, $\beta := \frac{p-1}{\alpha - 1}$, $\sigma := \frac{2-p}{p-1}$ and note $\alpha > 1$. Then

$$w_t(r) := \int_r^1 \frac{dy}{(\beta y + t y^\alpha)^{1/(p-1)}}, \qquad t > -\beta,$$

is a positive singular solution of $-\Delta w = |\nabla w|^p$ in $B_1 \backslash \{0\}$ with $w = 0$ on ∂B_1. We require some asymptotic properties of w_t. First note that

$$w_t'(r) = \frac{-1}{(\beta r + t r^\alpha)^{\frac{1}{p-1}}}, \qquad \text{and if we set } C_\beta := \frac{1}{\beta^{\frac{1}{p-1}}}.$$

Hence, we obtain

$$|w_t'(r)| \leq \min\left\{ \frac{C_\beta}{r^{\frac{1}{p-1}}}, \frac{1}{t^{\frac{1}{p-1}} r^{N-1}} \right\}. \tag{4.2}$$

Therefore,

$$r^{\sigma+1}|w_t'(r)| \leq \min\left\{ C_\beta, \frac{1}{t^{\frac{1}{p-1}} r^{N-2-\sigma}} \right\}. \tag{4.3}$$

Thus, we see for any $t > 0$ we have $\lim_{r\to 0} r^{\sigma+1} u_t'(r) = -C_\beta$ and u_t, u_t' converge uniformly to zero away from the origin.

Without loss of generality, we assume there exists some $\rho > 0$ such that $B_\rho(\xi_1) \cap B_\rho(\xi_2)$ is empty and we also assume $B_\rho(\xi_i) \subset \Omega$. We assume $0 \leq \eta_i \in C_c^\infty(B_{\frac{\rho}{4}}(\xi_i))$ radial about ξ_i, with $\eta_i = 1$ in $B_{\frac{\rho}{8}}(\xi_i)$. Let $0 \leq \delta_i \in C_c^\infty(B_{\frac{\rho}{2}}(\xi_i))$ radial about ξ_i with $\delta_i = 1$ in $B_{\frac{\rho}{4}}(\xi_i)$. Note that we have $\eta_i \delta_i = \eta_i$. Consider $w_{t,i}(x) = w_t(x - \xi_i)$ and we look for a solution of (4.1) of the form

$$u(x) = \eta_1(x) w_{t,1}(x) + \eta_2(x) w_{t,2}(x) + \phi(x),$$

with $\phi = 0$ on $\partial\Omega$. To simplify the notation, let $w_i = w_{t,i}$ for the following computations. Define

$$L_t(\phi) = -\Delta\phi - p|\nabla(\eta_1 w_1) + \nabla(\eta_2 w_2)|^{p-2} \left\{ \nabla(\eta_1 w_1) + \nabla(\eta_2 w_2) \right\} \cdot \nabla\phi.$$

Then, we need ϕ to satisfy

$$L_t(\phi) = I_{1,t}(\phi) + K_{2,t} + K_{3,t} \quad \text{ in } \Omega,$$

with $\phi = 0$ on $\partial\Omega$, where

$$
\begin{aligned}
I_{1,t}(\phi) = & |\nabla(\eta_1 w_1 + \eta_2 w_2) + \nabla\phi|^p \\
& - |\nabla(\eta_1 w_1 + \eta_2 w_2)|^p - p|\nabla(\eta_1 w_1 + \eta_2 w_2)|^{p-2}(\nabla(\eta_1 w_1 + \eta_2 w_2)) \cdot \nabla\phi,
\end{aligned}
$$

and

$$K_{2,t} = |\nabla(\eta_1 w_1 + \eta_2 w_2)|^p - |\nabla w_1|^p \eta_1 - |\nabla w_2|^p \eta_2.$$

Note that, if the supports of η_1, η_2 do not overlap, we can express this as

$$K_{2,t} = |\nabla(\eta_1 w_1)|^p + |\nabla(\eta_2 w_2)|^p - |\nabla w_1|^p \eta_1 - |\nabla w_2|^p \eta_2,$$

and

$$K_{3,t} = 2\nabla\eta_1 \cdot \nabla w_1 + 2\nabla\eta_2 \cdot \nabla w_2 + w_1 \Delta\eta_1 + w_2 \Delta\eta_2.$$

Note that $K_{i,t} \to 0$ uniformly in Ω as $t \to \infty$. To find a ϕ, we will apply a fixed point argument. Given ϕ define the operator $J_t(\phi) = \psi$ by

$$L_t(\psi) = I_{1,t}(\phi) + K_{2,t} + K_{3,t} \quad \text{in } \Omega \backslash \{\xi_1, \xi_2\}, \tag{4.4}$$

with $\psi = 0$ on $\partial\Omega$. We will show J_t has a suitable fixed point and then argue that u is positive and blows up at the indicated points. To do this, we need to work within suitable function spaces. Define the norms

$$\|f\|_Y = \sup_\Omega \left\{ |x - \xi_1|^{2+\sigma} + |x - \xi_2|^{2+\sigma} \right\} |f(x)|, \tag{4.5}$$

$$\|\phi\|_X = \sup_\Omega \left\{ |x - \xi_1|^{\sigma+1} + |x - \xi_2|^{\sigma+1} \right\} |\nabla\phi(x)|, \tag{4.6}$$

and we impose on the function space X that $\phi = 0$ on $\partial\Omega$. Here $\sigma = \frac{2-p}{p-1}$.

4.1 The Linear Analysis

Recall that η_i is identically one near ξ with support near ξ. Using this, we see that we can write L_t as

$$L_t(\phi) = -\Delta\phi - p\eta_1^{p-1}|\nabla w_1|^{p-2}\nabla w_1 \cdot \nabla\phi - p\eta_2^{p-1}|\nabla w_2|^{p-2}\nabla w_2 \cdot \nabla\phi + \delta_t(x) \cdot \nabla\phi,$$

where $|\delta_t| \to 0$ uniformly in Ω. Note by redefining δ_t (it still converges uniformly to zero), we can write

$$L_t(\phi) = -\Delta\phi - p\eta_1|\nabla w_1|^{p-2}\nabla w_1 \cdot \nabla\phi - p\eta_2|\nabla w_2|^{p-2}\nabla w_2 \cdot \nabla\phi + \delta_t(x) \cdot \nabla\phi,$$

(note we just changed the powers on η_i). Now set

$$\tilde{L}_t(\phi) = \Delta\phi + p\eta_1|\nabla w_1|^{p-2}\nabla w_1 \cdot \nabla\phi + p\eta_2|\nabla w_2|^{p-2}\nabla w_2 \cdot \nabla\phi = \Delta\phi + a_t(x) \cdot \nabla\phi.$$

To solve $\tilde{L}_t(\phi) = f$ in Ω with $\phi = 0$ on $\partial\Omega$, we look for a solution of the form $\phi = \eta_1\varphi_1 + \eta_2\varphi_2 + \psi$ where φ_i are defined on $B_1(\xi_i)$ and ψ is defined on Ω. Then, it is sufficient that

$$\Delta\varphi_1 + a_t \cdot \nabla\varphi_1 = \delta_1 f + \delta_1 a_t \cdot \nabla\psi \quad \text{in } B_1(\xi_1), \tag{4.7}$$

and

$$\Delta\varphi_2 + a_t \cdot \nabla\varphi_2 = \delta_2 f + \delta_2 a_t \cdot \nabla\psi \quad \text{in } B_1(\xi_2), \tag{4.8}$$

with $\varphi_i = 0$ on $\partial B_1(\xi_i)$ and we need ψ to satisfy

$$\begin{aligned}\Delta\psi + (1-\eta_1-\eta_2)a_t \cdot \nabla\psi &= (1-\eta_1-\eta_2)f - \varphi_1\Delta\eta_1 - 2\nabla\eta_1 \cdot \nabla\varphi_1 \\ &- \varphi_2\Delta\eta_2 - 2\nabla\eta_2 \cdot \nabla\varphi_2 - \varphi_1 a_t \cdot \nabla\eta_1 - \varphi_2 a_t \cdot \nabla\eta_2 \qquad \text{in } \Omega,\end{aligned} \tag{4.9}$$

with $\psi = 0$ on $\partial\Omega$. We now outline the fixed point argument to find a solution to the system. Note there are three spaces here. We have an X, Y over $B_1(\xi_i)$ and also over Ω. To indicate the space we will write $Y(B_1(\xi_i))$, $Y(\Omega)$ and similarly for X. We now outline the idea.

Claim 1. There is some $C_0 > 0$ and $t_0 > 0$ such that for all $t > t_0$ and $f \in Y(B_1(\xi_i))$ there is some $\varphi_i \in X(B_1(\xi_i))$ which solves $\Delta\varphi_i + a_t \cdot \nabla\varphi_i = f$ in $B_1(\xi_i)$ with $\varphi_i = 0$ on $\partial B_1(\xi_i)$. Moreover, we have $\|\varphi\|_{X(B_1(\xi_i))} \leq C_0\|f\|_{Y(B_1(\xi_i))}$. We will work on the space $X(B_1(\xi_1)) \times X(B_1(\xi_2))$ with norm given by

$$\|(\varphi_1, \varphi_2)\| = \|\varphi_1\|_{X(B_1(\xi_1))} + \|\varphi_2\|_{X(B_1(\xi_2))}.$$

Consider the space

$$\mathcal{F} := \left\{(\varphi_1, \varphi_2) \in X(B_1(\xi_1)) \times X(B_1(\xi_2)) : \|(\varphi_1, \varphi_2)\| \leq 10C_0\right\}.$$

Let $f \in Y(\Omega)$ with $\|f\|_Y = 1$ and let $(\varphi_1, \varphi_2) \in \mathcal{F}$ and let ψ satisfy (4.9).

Claim 2. There is some $C_1 > 0$, independent of f and $t > t_0$ such that $\sup\limits_{\Omega} |\nabla\psi| \leq C_1$.

We now define $T_t(\varphi_1, \varphi_2) = (\widehat{\varphi}_1, \widehat{\varphi}_2)$, where

$$\Delta\widehat{\varphi}_1 + a_t \cdot \nabla\widehat{\varphi}_1 = \delta_1 f + \delta_1 a_t \cdot \nabla\psi \quad B_1(\xi_1), \tag{4.10}$$

and

$$\Delta\widehat{\varphi}_2 + a_t \cdot \nabla\widehat{\varphi}_2 = \delta_2 f + \delta_2 a_t \cdot \nabla\psi \quad B_1(\xi_2). \tag{4.11}$$

Then, note we have

$$\|\widehat{\varphi}_i\|_{X(B_1(\xi_i))} \leq C_0\|\delta_i f\|_{Y(B_1(\xi_i))} + C_0\|\delta_i a_t \cdot \nabla\psi\|_{Y(B_1(\xi_i))}.$$

Claim 3. We claim that for large t, $\|\delta_i a_t \cdot \nabla\psi\|_{Y(B_1(\xi_i))}$ is small uniformly in f and also in $(\varphi_1, \varphi_2) \in \mathcal{F}$.

Using the above claims we easily see that $T_t(\varphi_1, \varphi_2) \in \mathcal{F}$ for large t. This shows the mapping is into.

We now show the mapping is a contraction. Let $(\varphi_1, \varphi_2) \in \mathcal{F}$ and we let ψ_{12} solve (4.9) and then let $(\widehat{\varphi}_1, \widehat{\varphi}_2)$ solve (4.10) and (4.11). We now let $(\varphi_3, \varphi_4) \in \mathcal{F}$ where φ_3 plays the role of φ_1 and φ_4 plays the role of φ_2 and we let ψ_{34} solve

$$\Delta\psi_{34} + (1 - \eta_1 - \eta_2)a_t \cdot \nabla\psi_{34} = F, \tag{4.12}$$

in Ω with $\psi_{34} = 0$ on $\partial\Omega$, where

$$F = (1-\eta_1-\eta_2)f - \varphi_3\Delta\eta_1 - 2\nabla\eta_1\cdot\nabla\varphi_3 - \varphi_4\Delta\eta_2 - 2\nabla\eta_2\cdot\nabla\varphi_4 - \varphi_3 a_t\cdot\nabla\eta_1 - \varphi_4 a_t\cdot\nabla\eta_2.$$

Then, note we have

$$\begin{aligned}\|T(\varphi_1, \varphi_2) - T(\varphi_3, \varphi_4)\| &= \|(\widehat{\varphi}_1, \widehat{\varphi}_2) - (\widehat{\varphi}_3, \widehat{\varphi}_4)\| \\ &= \|\widehat{\varphi}_1 - \widehat{\varphi}_3\|_{X(B_1(\xi_1))} + \|\widehat{\varphi}_2 - \widehat{\varphi}_4\|_{X(B_1(\xi_2))}.\end{aligned}$$

Using the equations, we see that we have

$$\|\widehat{\varphi}_1 - \widehat{\varphi}_3\|_{X(B_1(\xi_1))} \le C_0\|\delta_1 a_t \cdot \nabla(\psi_{12} - \psi_{34})\|_{Y(B_1(\xi_1))}, \tag{4.13}$$

and

$$\|\widehat{\varphi}_2 - \widehat{\varphi}_4\|_{X(B_1(\xi_2))} \le C_0\|\delta_2 a_t \cdot \nabla(\psi_{12} - \psi_{34})\|_{Y(B_1(\xi_2))}. \tag{4.14}$$

Also, note we have

$$\begin{aligned}
&\Delta(\psi_{12}-\psi_{34})+(1-\eta_1\eta_2)a_t\cdot\nabla(\psi_{12}-\psi_{34})\\
&=(\Delta\eta_1)(\varphi_3-\varphi_1)+2\nabla\eta_1\cdot\nabla(\varphi_3-\varphi_1)\\
&+(\Delta\eta_2)(\varphi_4-\varphi_2)+2\nabla\eta_2\cdot\nabla(\varphi_4-\varphi_2)+(a_t\cdot\nabla\eta_1)(\varphi_3-\varphi_1).
\end{aligned}$$

Arguing as in the proof of Claim 2, there is some $C>0$ (independent of parameters) such that

$$\sup_{\Omega}|\nabla(\psi_{12}-\psi_{34})|\leq C\|\varphi_3-\varphi_1\|_{X(B_1(\xi_1))}+\|\varphi_4-\varphi_3\|_{X(B_1(\xi_2))}.$$

We now estimate the right-hand side of (4.13). Let $\gamma>0$ be very small and note we have

$$\begin{aligned}
\|\delta_1 a_t\cdot\nabla(\psi_{12}-\psi_{34})\|_{Y(B_1(\xi_1))} &= \sup_{|x-\xi_1|\leq 1}|x-\xi_1|^{2+\sigma}|\delta_1||a_t||\nabla(\psi_{12}-\psi_{34})|\\
&\leq \sup_{|x-\xi_1|\leq\gamma}|x-\xi_1|^{2+\sigma}|\delta_1||a_t||\nabla(\psi_{12}-\psi_{34})|\\
&\quad+\sup_{\gamma<|x-\xi_1|\leq 1}|x-\xi_1|^{2+\sigma}|\delta_1||a_t||\nabla(\psi_{12}-\psi_{34})|.
\end{aligned}$$

Now, note the first term on the right we can write it as

$$\begin{aligned}
&\sup_{|x-\xi_1|\leq\gamma}|x-\xi_1|^{2+\sigma}|\delta_1||a_t||\nabla(\psi_{12}-\psi_{34})|\\
&=\sup_{|x-\xi_1|\leq\gamma}|x-\xi_1|^{1+\sigma}|\delta_1|\left\{|x-\xi_1||a_t|\right\}|\nabla(\psi_{12}-\psi_{34})|\\
&\leq\gamma^{\sigma+1}C\sup_{\Omega}|\nabla(\psi_{12}-\psi_{34})|\\
&\leq C\gamma^{\sigma+1}C\left\{\|\varphi_3-\varphi_1\|_{X(B_1(\xi_1))}+\|\varphi_4-\varphi_3\|_{X(B_1(\xi_2))}\right\}.
\end{aligned}$$

For the second term, we have

$$\sup_{\gamma<|x-\xi_1|\leq 1}|x-\xi_1|^{2+\sigma}|\delta_1||a_t||\nabla(\psi_{12}-\psi_{34})|\leq C\left\{\sup_{\gamma\leq|x-\xi_1|\leq 1}|a_t||\delta_1|\right\}\sup_{\Omega}|\nabla(\psi_{12}-\psi_{34})|,$$

and hence we arrive at

$$\begin{aligned}&\|\delta_1 a_t\cdot\nabla(\psi_{12}-\psi_{34})\|_{Y(B_1(\xi_1))}\\&\leq C\left\{\gamma^{\sigma+1}+\sup_{\gamma\leq|x-\xi_1|\leq 1}|a_t|\delta_1\right\}\left\{\|\varphi_3-\varphi_1\|_{X(B_1(\xi_1))}+\|\varphi_4-\varphi_3\|_{X(B_1(\xi_2))}\right\}.\end{aligned}$$

From this, we see that

$$\|\widehat{\varphi}_1-\widehat{\varphi}_3\|_{X(B_1(\xi_1))}\leq C_0C\left\{\gamma^{\sigma+1}+\sup_{\gamma\leq|x-\xi_1|\leq 1}|a_t|\delta_1\right\}\|(\varphi_1,\varphi_2)-(\varphi_3,\varphi_4)\|.$$

One can estimate $\|\widehat{\varphi}_2-\widehat{\varphi}_4\|_{X(B_1(\xi_2))}$ similarly to see arrive at

$$\|T_t(\varphi_1,\varphi_2)-T_t(\varphi_3,\varphi_4)\|\leq C_0CA(\gamma,t)\|(\varphi_1,\varphi_2)-(\varphi_3,\varphi_4)\|,$$

where

$$A(\gamma,t)=2\gamma^{\sigma+1}+\sup_{\gamma\leq|x-\xi_1|\leq 1}|a_t|\delta_1+\sup_{\gamma\leq|x-\xi_2|\leq 1}|a_t|\delta_2,$$

and note by taking $\gamma>0$ small and then taking t large we can make $A(\gamma,t)$ small and hence T_t is a contraction on $\mathcal{F}$ and hence there is a fixed point of T_t. Therefore, we have a solution φ_1,φ_2,ψ of our system. Using the bounds on these functions we obtain the needed bound on ϕ and hence we have proved the linear theory up to proving the various claims.

Proof of Claim 1. Note that in $B_1(\xi_1)$ that $a_t=p\eta_1|\nabla w_1|^{p-2}\nabla w_1$ and hence, except for the η_1 term, this is exactly the linear operator we have examined in the scalar problem with one blow up point that we examine on the unit ball centered at the origin. Translating the problem to the $B_1(0)$ the linear problem we need to examine is

$$\Delta\phi(x) - \frac{\eta_1(x)px \cdot \nabla\phi(x)}{\beta|x|^2 + t|x|^{\alpha+1}} = f, \quad B_1(0),$$

with $\phi = 0$ on ∂B_1. Taking $\eta_1(x)$ to be a radial cut-off, we can perform a similar argument as in our previous works, we omit the details. This completes the proof of Claim 1.

Proof of Claim 2. After noting the supports of the various functions, to prove the result we need to show that for $\varepsilon_0 > 0$ (fixed and small) there is some $C > 0$ and $t_0 > 0$ such that for all $g \in L^\infty(\Omega)$ with $\sup_\Omega |g| = 1$ and with $g = 0$ on $B_\varepsilon(\xi_1) \cup B_\varepsilon(\xi_2)$ and $t > t_0$ we have a solution of

$$\Delta\psi + (1 - \eta_1 - \eta_2)a_t \cdot \nabla\psi = g \qquad \text{in } \Omega,$$

with $\psi = 0$ on $\partial\Omega$ and $\sup_\Omega |\nabla\psi| \leq C$. Note the advection term converges to zero uniformly on Ω. Hence this result follows directly from theory for Δ. As a result, the proof of Claim 2 is completed.

Proof of Claim 3. This is essentially proved above.

We can now obtain the desired result for L_t since it's a small perturbation of the above operator.

4.2 The Fixed Point Argument

Let $B_R = \{\phi \in X : \|\phi\|_X \leq R\}$. We show that for suitable $0 < R < 1$ and t large that J_t (see (4.4)) is a contraction on B_R and hence has a fixed point. We have

$$\hat{H}_t(\phi) = I_{1,t}(\phi) + K_{2,t} + K_{3,t} \quad \text{ in } \Omega,$$

$$\begin{aligned} I_{1,t}(\phi) &= |\nabla(\eta_1 w_1 + \eta_2 w_2) + \nabla\phi|^p - |\nabla(\eta_1 w_1 + \eta_2 w_2)|^p \\ &\quad - p|\nabla(\eta_1 w_1 + \eta_2 w_2)|^{p-2}(\nabla(\eta_1 w_1 + \eta_2 w_2)) \cdot \nabla\phi, \end{aligned}$$

$$K_{2,t} = |\nabla(\eta_1 w_1)|^p + |\nabla(\eta_2 w_2)|^p - |\nabla w_1|^p \eta_1 - |\nabla w_2|^p \eta_2,$$

and

$$K_{3,t} = 2\nabla\eta_1 \cdot \nabla w_1 + 2\nabla\eta_2 \cdot \nabla w_2 + w_1 \Delta\eta_1 + w_2 \Delta\eta_2.$$

Lemma 4.2. *Let $\phi \in B_R$, there is $C_0 > 0$ such that $\|I_{1,t}(\phi)\|_Y \leq C_0 R^p$.*

Proof. Assume $\phi \in B_R$. There is some $C_{1,1}$ such that for

$$\begin{aligned}
\|I_{1,t}\|_Y &= \sup_\Omega \left\{|x-\xi_1|^{2+\sigma} + |x-\xi_2|^{2+\sigma}\right\} |I_{1,t}| \\
&\leq C_{1,1} \sup_\Omega \left\{|x-\xi_1|^{2+\sigma} + |x-\xi_2|^{2+\sigma}\right\} |\nabla\phi|^p \\
&\leq C_{1,1} \sup_\Omega \frac{|x-\xi_1|^{2+\sigma} + |x-\xi_2|^{2+\sigma}}{(|x-\xi_1|^{1+\sigma} + |x-\xi_2|^{1+\sigma})^p} \sup_\Omega \left(\left\{|x-\xi_1|^{1+\sigma} + |x-\xi_2|^{1+\sigma}\right\} |\nabla\phi|\right)^p \\
&\leq C_{1,1} \sup_\Omega \frac{|x-\xi_1|^{2+\sigma} + |x-\xi_2|^{2+\sigma}}{(|x-\xi_1|^{2+\sigma} + |x-\xi_2|^{2+\sigma})^p} \sup_\Omega \left(\left\{|x-\xi_1|^{1+\sigma} + |x-\xi_2|^{1+\sigma}\right\} |\nabla\phi|\right)^p \\
&\leq C_{1,1} \sup_\Omega \left(|x-\xi_1|^{2+\sigma} + |x-\xi_2|^{2+\sigma}\right)^{1-p} \sup_\Omega \left(\left\{|x-\xi_1|^{1+\sigma} + |x-\xi_2|^{1+\sigma}\right\} |\nabla\phi|\right)^p \\
&\leq C_0 R^p.
\end{aligned}$$

□

Theorem 1. *(Fixed point argument) $J_t : X \to X$ is a contraction mapping for sufficiently large t.*

Proof. **Into.** Let $\phi \in B_R$ and $\psi = J_t(\phi)$. Then note that we have

$$L_t(\psi) = I_{1,t}(\phi) + \varepsilon_t, \qquad \Omega \backslash \{\xi_1, \xi_2\},$$

where $\varepsilon_t(x) = K_{2,t} + K_{3,t} \to 0$ unifomrly in Ω. Using Lemma (4.2), there is some $C_0 > 0$ such that $\|I_{1,t}(\phi)\|_Y \leq C_0 R^p$ and hence there is some C_2 such that $\|\psi\|_X \leq C_2 R^p + C_2 \varepsilon_t$, where now $\varepsilon_t = \|\varepsilon_t\|_{L^\infty}$ and hence J_t is into B_R provided

$$C_2 R^p + C_2 \varepsilon_t \leq R. \tag{4.15}$$

Contraction. Let $\phi_i \in B_R$ and $\psi_i = J_t(\phi_i)$. Then, we have

$$L_t(\psi_2 - \psi_1) = I_{1,t}(\phi_2) - I_{1,t}(\phi_1) \text{ in } B_R.$$

By Lemma (6.1), we have

$$|I_{1,t}(\phi_2) - I_{1,t}(\phi_1)| \leq C_0(|\nabla\phi_1|^{p-1} + |\nabla\phi_2|^{p-1})|\nabla\phi_2 - \nabla\phi_1|,$$

and from this we have

$$\|I_{1,t}(\phi_2) - I_{1,t}(\phi_1)\|_Y \leq C(\Omega)R^{p-1}\|\phi_2 - \phi_1\|_X,$$

and hence we see that J_t is a contraction on B_R is the previous condition holds and

$$C_1(\Omega)R^{p-1} \leq \frac{3}{4}.$$

We obtain

$$\|J_t(\phi_2) - J_t(\phi_1)\|_X \leq C_l \|H_t(\phi_2) - H_t(\phi_1)\|_Y \leq \frac{3}{4}C_l \|\phi_2 - \phi_1\|_X,$$

and

$$\|J_t(\phi)\|_X \leq C_l \|H_t(\phi)\|_Y \leq C_l (C_2R^p + C_2\varepsilon_t).$$

Thus, J_t is a contraction on B_R if the following conditions satisfy:

$$C_l \leq \frac{4}{3},$$
$$C_l (C_2R^p + C_2\varepsilon_t) \leq R.$$

By taking $0 < R < 1$ small and then t large one can see that J_t is a contraction on B_R and hence has a fixed point. In conclusion, there exists some $t_0 > 0$ (sufficiently large) such that for all $t \geq t_0$, there is a solution u_t of (4.1). The solution is strictly positive, and the proof follows the same reasoning as we used at the end of Section 2.5.

$\square$

4.3 Conclusion of Chapter 4.

In Chapter 4, we examined the same equation as we did in Chapter 2 (ie. (2.22)) but here want the solution to blow up at two points instead of one. Here we utilize a different approximate solution and the gluing process required for the linear analysis is much more complicated.

5

Future Work

In this chapter, we outline several potential problems for future research. The proposed problems involve extending current results to more complex settings, including domains with multiple singularities and different boundary conditions such as *Neumann boundary condition* and *Robin boundary condition*.

We begin by considering smooth bounded domains that exclude the origin and examine the behavior of solutions in such settings. Let Ω be a smooth bounded domain in $\mathbb{R}^N$ which does not contain the origin. We will consider

$$\begin{cases} -\Delta u = (1+g(x))|x|^{\alpha}|\nabla u|^{p}, & \text{in } \ \Omega, \\ u = 0, & \text{on } \ \partial\Omega, \end{cases}$$

where $p > 1$ and $g(x)$ is a Hölder continuous function with $g(0) = 0$.
We then extend our focus to problem with the exterior domain. Assume Ω is an exterior domain in $\mathbb{R}^N$ which does not contain the origin.

$$\begin{cases} -\Delta u = (1+g(x))|x|^{\alpha}|\nabla u|^{p}, & \text{in } \ \Omega, \\ u = 0, & \text{on } \ \partial\Omega, \end{cases}$$

where $p > 1$, $\alpha > 0$, Ω is bounded and $g \geq 0$ is a Hölder continuous function with

$g(0) = 0$.

Next, we propose the study of an equation in a domain with multiple singularities, exploring how the presence of multiple distinct points affects the nature of the solution. Consider the existence of positive singular solutions of the following Hénon type equations involving nonlinear gradient terms as follows:

$$\begin{cases} -\Delta u = |\nabla u|^p, & \text{in } \Omega\backslash\{\xi_1, \xi_2, \dots, \xi_m\}, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u = (1 + g(x))|x|^\alpha |\nabla u|^p, & \text{in } \Omega\backslash\{\xi_1, \xi_2, \dots, \xi_m\}, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω a bounded domain in $\mathbb{R}^N$, $\alpha > 0$ with $\xi_1, \xi_2, \dots, \xi_m \in \Omega$ are finite distinct points.

We consider Hénon-type equations with nonlinear gradient terms with different boundary conditions, investigating the conditions under which positive singular solutions can exist. We consider the following problems

$$\begin{cases} -\Delta u = (1 + g(x))|x|^\alpha |\nabla u|^p, & \text{in } \Omega\backslash\{0\}, \\ u(x) + \beta \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u = (1 + g(x))|x|^\alpha |\nabla u|^p, & \text{in } \Omega\backslash\{0\}, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p > 1$, $\alpha > 0$, Ω is bounded and $g \geq 0$ is a Hölder continuous function with $g(0) = 0$.

Finally, we can also apply the p-Laplacian to the problems mentioned above, where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$.

6

Appendix

Lemma 6.1. *1. Suppose $p > 2$. Then there is some $C_{1,1}$ such that for all $x \neq 0$ and all y*

$$\left| |x+y|^p - |x|^p - p|x|^{p-2}x \cdot y \right| \leq C_{1,1}|x|^{p-2}|y|^2 + C_{1,1}|y|^p.$$

2. Suppose $1 < p \leq 2$. Then there is some $\hat{C}_{1,1}$ such that for all $x \neq 0$ and all y

$$\left| |x+y|^p - |x|^p - p|x|^{p-2}x \cdot y \right| \leq \hat{C}_{1,1}|y|^p.$$

3. For and $x, y, z \in \mathbb{R}^N$, and $p > 2$, there exists $C_{1,2}$ such that

$$\begin{aligned} &\left| |x+y|^p - p|x|^{p-2}x \cdot y - |x+z|^p + p|x|^{p-2}x \cdot z \right| \\ &\leq C_{1,2}\left(|x|^{p-2}\left(|y| + |z|\right) + |y|^{p-1} + |z|^{p-1} \right)|y-z|. \end{aligned}$$

4. For and $x, y, z \in \mathbb{R}^N$, and $1 < p \leq 2$, there exists $\hat{C}_{1,2}$ such that

$$\left| |x+y|^p - p|x|^{p-2}x \cdot y - |x+z|^p + p|x|^{p-2}x \cdot z \right| \leq \hat{C}_{1,2}\left(|y|^{p-1} + |z|^{p-1} \right)|y-z|.$$

Proof. These results are essentially known so we will give some sketches of possible proofs.

1. By dividing by $|x|^p$, it follows that to prove the desired result, it suffices to show that

$$\Big||x+y|^p - 1 - px\cdot y\Big| \leq C|y|^2 + C|y|^p \qquad \forall x,y\in\mathbb{R}^N, \text{ with } |x|=1. \tag{6.1}$$

It's clear the result will hold for large $|y|$; hence, we only need to consider the case of bounded $|y|$, so let's restrict $|y|\leq R$. We can now look at the Taylor series for the function $f(x)=|x|^p$ expanded at x. If one prefers they can instead write $|x+y|^p$ as $(|x+y|^2)^{\frac{p}{2}}$ and use scalar results since that is more familiar to most people; we will do the second one. So we are left with trying to show that

$$\Big|(1+2x\cdot y+|y|^2)^{\frac{p}{2}} - 1 - px\cdot y\Big| \leq C|y|^2 + C|y|^p, \tag{6.2}$$

for all $|y|\leq R$ and $|x|=1$. You can now apply the results from scalar series, such as the Binomial series or Taylor series, to obtain the desired result.

2. This follows from the same reasoning as 1.

3. Consider

$$g(t) = |x+y+t(z-y)|^p - tp|x|^{p-2}x\cdot(z-y),$$

for $0\leq t\leq 1$ and note that

$$g'(t) = \Big(p|x+y+t(z-y)|^{p-2}(x+y+t(z-y)) - p|x|^{p-2}x\Big)\cdot(z-y).$$

We now write out $|g(1)-g(0)| \leq \int_0^1 |g'(t)|dt$ to arrive at the desired result.

□

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