ELECTROSTATICS IN A TWO-DIMENSIONAL FINITE GEOMETRY

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ABSTRACT

It has been suggested that our view of physical geometry as being Euclidean be changed and the implications of assuming a finite physical geometry have been discussed by several authors. In this thesis two-dimensional vector analysis is developed for a finite geometry and two-dimensional source-free electrostatics is formulated.

The approach to the vector theory is through complex analysis. The thesis reviews the formalism by which ordinary Euclidean vector analysis is formulated in complex notation and extends this to finite geometry by developing the necessary finite complex theory.

The formulation of source-free electrostatics in the finite geometry follows immediately from the complex theory. It is then seen that the familiar source-free fields of ordinary electrostatics appear as the Pythagorean approximation to certain finite geometry fields.

CHAPTER I

INTRODUCTION TO THE THEORY

(A) INTRODUCTION

One possible approach in the attempt to resolve the difficulties in modern physics is to make a fundamental change in our view of the physical geometry of the world. Any scheme of physics, of course, assumes either explicitly or implicitly a particular physical geometry. We can seek, by changing our assumption concerning the physical geometry, to explain certain phenomena which are puzzling in the old scheme by seeing them as arising naturally from the new world geometry. The example which comes to mind is Einstein's theory of gravitation: the realization that space-time is curved leads to the explanation of gravitation as a direct geometrical consequence. The theory under discussion in this thesis postulates that our physical geometry is a certain mathematically well known structure called a finite geometry.

A finite geometry consists of a finite number of points. Such a structure is obtained by taking space to be co-ordinaatized by a number field of finite order instead of the reals. That is, if we choose the geometrical co-ordinates of each point in our geometry from a number field of finite order instead of from the reals, then the resulting structure is known in mathematics as a finite geometry.[1,2]

The definition of a field and the theory of finite fields can be found in most books on modern algebra.[3] We can characterize a field as the minimum structure on which there can be the usual type of arithmetic involving the four operations of addition, subtraction, multiplication, and division.[4] A finite field, obviously, is a field with a finite number of elements. For a finite field we have the following two fundamental theorems[5]: firstly, that any finite field has order p^n where p is some prime and n some integer and the converse, that for every prime p and integer n there is a field of order p^n; and secondly, that

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[1]H.B. Coish discusses at some length the question of a finite geometry as our physical geometry and gives reference to the mathematics of finite geometries in H.B. Coish, Phys. Rev. 114, No.1, 383 (1959).

[2]A general reference for finite geometry is B. Segre, Lectures on Modern Geometry, (Edizioni Cremonese, Rome, 1961).

[3]For example, R.A. Dean, Elements of Abstract Algebra, (John Wiley & Sons, Inc., New York, 1966).

[4]Shapiro discusses this property and relates it to the use of number fields in physics on p.476 of I.S. Shapiro, Nucl.Phys. 21, 474 (1960).

[5]Ref. 3, p.223.

any two fields of the same (finite) order are isomorphic. Thus we can consider there to be exactly one field of order p^n for every prime p and integer n; this field is called the Galois field of order p^n and is denoted by $GF(p^n)$. Every finite field is some $GF(p^n)$.

A field F is said to be a subfield of another field K if, firstly, K contains F as a subset and, secondly, the operations which make K a field, when restricted to the subset F, make F a field in its own right.[6] For a given prime p, every field $GF(p^n)$ contains GF(p) as a subfield, and GF(p) itself has no proper subfield.[7] Moreover, every field $GF(p^n)$, including GF(p) itself, is of characteristic p, i.e. $px = 0$ for every x in $GF(p^n)$[6]. In this sense GF(p) is primitive and $GF(p^n)$ for $n \geq 2$ is not[8]; $GF(p^n)$ is a complex structure based on the simpler field GF(p) and this causes px to be zero even in the larger field $GF(p^n)$.

In choosing a field to replace the real numbers for our co-ordinates in physics, we can see that the primitive field GF(p) is analogous to the reals and that $GF(p^n)$ with $n \geq 2$ is related to GF(p) in the same way that the complex numbers are related to the reals. We therefore use the field GF(p) to co-ordinatize our geometry. A convenient

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[6]For an explanation of these algebraic concepts see Section 4.3 of Ref.3.

[7]Ref. 3, p.223.

[8]Shapiro discusses this concept in Ref.4.

representation of GF(p) is the field of integers mod p and this is used throughout.

We must also consider that if our postulate of a finite physical geometry is to have any validity it must give the familiar Euclidean geometry as a good approximation in the realm of familiar magnitudes, that is, those magnitudes from the subnuclear to the macroscopic that are within the realm of present experiment. This requirement implies two things. Firstly, the prime p must be so large as to make the points in the geometry numerous enough and dense enough so that the finite geometry is experimentally indistinguishable from a continuous geometry. Jarnefelt[9] has estimated that this would require the order of magnitude of p to be roughly $10^{10^{81}}$. Secondly, there must be some way of approximating an ordering in the field GF(p) chosen to replace the reals. An ordering on a field is, in general terms, a relation satisfying the same axioms as "smaller than or equal to" in the reals. It is a relation which allows us to think of positive and negative elements in the field.[10] The fact that the reals do have an ordering is very basic to Euclidean geometry and our conception of space in the Euclidean sense. Now it is impossible to define an

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[9] G. Jarnefelt, Ann. Acad. Sci. Fennicae, Ser.A.I., No.96 (1951).

[10] For a clear introduction to order, see Sec.3.2, 3.3, and 3.4 of Ref.3.

ordering on GF(p), but to make finite geometry approximate Euclidean geometry in the required way, there must exist a subset of the field on which an ordering can be introduced; it could then be postulated that measurements of familiar magnitudes fall into this ordered subset and it would follow from these measurements alone that the space looks Euclidean.[11]

The ordering can be introduced as follows: We notice that in GF(p), half the non-zero elements are squares and half are non-squares; furthermore, the product of two squares or two non-squares is a square, and the product of a square by a non-square is a non-square.[12] This motivates us to define an element as being "positive" if it is a square and "negative" if it is a non-square. If we choose p so that -1 is a non-square (by taking p of the form[13] $p = 4n-1$), then minus a positive element is a negative element.

This definition does not lead to an ordering of GF(p) because the set of positives is not closed under addition (i.e. the ordering is not transitive). However, if the prime p is of the form[11] $p = 8x\prod_{i=1}^{k} q_i - 1$ where x is an odd integer and $\prod_{i=1}^{k} q_i$ is the product of the first k odd primes,

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[11]See p.384 of Ref.1 for a more detailed discussion of these matters with references to the literature.

[12]L.E. Dickson, Linear Groups (Dover Publications, Inc., New York, 1958), Sec. 61, p.44.

[13]P. Kustaanheimo, Ann. Acad. Sci. Fennicae Ser. A.I., No.129 (1952).

then the first N elements of GF(p) are squares, with $N \sim q_k \sim \ln p$. Thus for an enormous p, $p \sim 10^{10^{100}}$, N is also enormous, $N \sim 10^{100}$.

This set consisting of the first N elements of GF(p) is the ordered subset mentioned earlier; it is called the Pythagorean region. Only elements from the Pythagorean region can be recognized as observable measurements from the point of view of a Euclidean geometry.

When we formulate any physical theory in our finite geometry, we must also discuss the special case of the theory, called the Pythagorean approximation, under the assumption that all measurements fall into the Pythagorean region, and we must relate this to our view of the same phenomenon when a Euclidean space is assumed.

This thesis is an attempt to formulate electrostatics in a two-dimensional finite geometry. After a review of some basic mathematical properties of finite fields in the latter half of this introductory chapter, the mathematics required for electrostatics is developed in Chapter 2. The electrostatics is discussed in Chapter 3.

(B) FINITE FIELD ANALYSIS

We give here several basic properties and identities of finite field elements all of which will be used later.

1. We have Fermat's Theorem:[14]

For $x \in GF(p^n)$, $\quad x^{p^n} = x$

$$\text{or} \qquad x^{p^n-1} = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad (1.1)$$

Thus any polynomial over $GF(p^n)$ can be reduced to an equivalent form with degree smaller than p^n.

2. The multiplicative group of nonzero elements in $GF(p^n)$ is cyclic.[15] This means that there exists in $GF(p^n)$ a generator or primitive element ε such that every nonzero element of $GF(p^n)$ is some power of ε smaller than p^n,

$$\text{i.e. } GF(p^n) - \{0\} = \{\varepsilon^k \mid k=1,2\ldots, p^n-1\} \qquad (1.2)$$

3. Using 2, we can evaluate a sum of the form

$$\sum_{\substack{x \in GF(p^n) \\ x \neq 0}} x^r$$

$$\sum_{\substack{x \in GF(p^n) \\ x \neq 0}} x^r = \sum_{k=1}^{p^n-1} (\varepsilon^k)^r = \sum_{k=1}^{p^n-1} (\varepsilon^r)^k$$

If $\varepsilon^r \neq 1$, i.e. if $r \neq m\,(p^n-1)$ for any integer m, then we have a geometric progression which can be evaluated by the standard formula valid in any field.

$$\sum_{k=1}^{p^n-1} (\varepsilon^r)^k = \varepsilon^r \frac{(\varepsilon^r)^{p^n-1} - 1}{\varepsilon^r - 1} = 0$$

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[14] Ref. 3, p.86.

[15] Ref. 3, p.224.

If $\varepsilon^r = 1$, i.e. if $r = m(p^n-1)$ for some integer m, then we have,

$$\sum_{k=1}^{p^n-1} 1 = p^n-1 = -1$$

Thus
$$\sum_{\substack{x \in GF(p^n) \\ x \neq 0}} x^r = \begin{cases} -1 & \text{if } r = 0,\ p^n-1,\ 2(p^n-1),\ldots \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

4. We have in GF(p) Wilson's theorem[16]

$$(p-1)! = -1 \tag{1.4}$$

and its generalization[17]

$$(p-1-k)! = \frac{(-1)^{k+1}}{k!} \tag{1.5}$$

5. The binomial expansion is valid in any field. That is, we can expand the expression $(x+y)^r$ as

$$(x+y)^r = \sum_{k=0}^{r} C_k^r\, x^{r-k}\, y^k \quad \text{where } C_k^r \text{ is the binomial} \tag{1.6}$$

coefficient (an integer)

$$C_k^r = \frac{r!}{(r-k)!k!}$$

6. In the field $GF(p^n)$,

$$(x + y)^p = \sum_{k=0}^{p} C_k^p \quad x^{p-k}y^k$$

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[16]Ref.3, p.87.

[17]Obtained by writing $(p-1)! = (p-1)(p-2)\ldots(p-k)(p-1-k)!$ which becomes $-1 = (-1)^k\, 1\cdot 2\cdot \ldots\cdot k\ (p-1-k)!$

Now $C_k^p = \dfrac{p!}{(p-k)!\ k!}$

$GF(p^n)$ is of characteristic p (i.e. $px = 0$ for all $x \in GF(p^n)$) and so the only non-zero terms in the binomial expansion are those for $k = 0$ and $k = p$.

Hence,

$$(x + y)^p = x^p + y^p \text{ in } GF(p^n) \qquad (1.7)$$

7. Any single-valued function from $GF(p^n)$ into $GF(p^n)$ can be represented uniquely as a polynomial over $GF(p^n)$ of degree smaller than p^n. This is done as follows. Suppose the function f is given by its value $f(x')$ at every x' in $GF(p^n)$.

$$\text{Now } (x-x')^{p^n-1} = \begin{cases} 1 \text{ for all } x \neq x' \\ 0 \text{ for } x = x' \end{cases}$$

$$\therefore\ 1-(x-x')^{p^n-1} = \begin{cases} 0 \text{ for } x \neq x' \\ 1 \text{ for } x = x' \end{cases}$$

Therefore we can write,

$$f(x) = \sum_{x' \in GF(p^n)} \left[1-(x-x')^{p^n-1}\right] f(x') \qquad (1.8)$$

which can be explicitly expanded to a polynomial of degree p^n-1 or less using paragraph 5 above.

8. It is possible to define in GF(p) the analogues of both differentiation and integration.

First we consider the derivative. Let f be a function from GF(p) into GF(p). We define

$$\frac{df(x)}{dx} \quad \text{as} \quad - \sum_{\substack{h \neq 0 \\ h \in GF(p)}} \frac{f(x+h) - f(x)}{h} \tag{1.9}$$

To show that this expression indeed has the properties of a derivative we apply it to the general polynomial

$$\sum_{k=0}^{p-1} a_k x^k \;:$$

$$\frac{d}{dx} \sum_{k=0}^{p-1} a_k x^k = -\sum_{h\neq 0} h^{-1} \sum_{k=0}^{p-1} a_k \, (x+h)^k + \sum_{h\neq 0} h^{-1} \sum_{k=0}^{p-1} a_k x^k$$

$$= - \sum_{h\neq 0} \sum_{k=0}^{p-1} \sum_{m=0}^{k} h^{-1} a_k C_m^k \, x^{k-m} \, h^m$$

$$+ \sum_{h\neq 0} h^{-1} \sum_{k=0}^{p-1} a_k x^k$$

$$= -\sum_{k=0}^{p-1} \sum_{m=0}^{k} a_k C_m^k \, x^{k-m} \sum_{h\neq 0} h^{m-1}$$

$$+ \sum_{k=0}^{p-1} a_k x^k \sum_{h\neq 0} h^{-1}$$

Now $\displaystyle\sum_{h\neq 0} h^{m-1} = \begin{cases} -1 & m = 1 \\ 0 & \text{otherwise} \end{cases}$

and $\displaystyle\sum_{h\neq 0} h^{-1} = 0$

Thus
$$\frac{d}{dx} \sum_{k=0}^{p-1} a_k x^k = \sum_{k=0}^{p-1} k a_k x^{k-1}, \text{ since } C_1^k = k, \quad (1.10)$$

exactly as expected of a derivative.

Finally, we define a definite integral for the function f that satisfies the restriction that it be of degree p-2 or less:

$$\int_0^X f\, dx = - \sum_{h \neq 0} \sum_{t=1}^{Xh^{-1}} hf\,(th) \qquad (1.11)$$

Again, we examine the properties of the expression by applying it to the polynomial $\sum_{k=0}^{p-2} a_k x^k$.

$$\int_0^X \sum_{k=0}^{p-2} a_k x^k dx = - \sum_{h \neq 0} \sum_{t=1}^{Xh^{-1}} h \sum_{k=0}^{p-2} a_k t^k h^k$$

$$= - \sum_{k=0}^{p-2} a_k \sum_{h \neq 0} h^{k+1} \sum_{t=1}^{Xh^{-1}} t^k$$

Now $\displaystyle\sum_{t=1}^{Xh^{-1}} t^k = \frac{(Xh^{-1})^{k+1}}{k+1}$ + lower powers of (Xh^{-1}).[18]

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[18] H.T. Davis, The Summation of Series (The Principia Press of Trinity University, San Antonio, Texas, 1962) p.62.

It can be seen that for the lower powers of h^{-1}, once they are multiplied by h^{k+1}, the summation over h will give zero. Only the first term, $\frac{(Xh^{-1})^{k+1}}{k+1}$ will lead to a non-vanishing sum; it is given by

$$- \sum_{k=0}^{p-2} \sum_{h\neq 0} \frac{a_k X^{k+1}}{k+1} \text{, and because}$$

$$\sum_{h\neq 0} 1 = -1 \text{, this becomes } \sum_{k=0}^{p-2} a_k \frac{X^{k+1}}{k+1} \tag{1.12}$$

This is exactly the result expected of a definite integral.

Note that our derivative and definite integral are the inverse operations of each other in the expected way.

$$\text{i.e. } \frac{d}{dx} \int_0^x f = f$$

Note also that in our definitions of both the derivative and integral, the operation $-\sum_{h\neq 0}$ plays the part in GF(p) of $\lim_{h\to 0}$ in ordinary analysis. The analogue of the limit process at a point in the finite field is thus a global operation, involving a sum over all values in the field and depending on the value of the function everywhere. This means, of course, that we can take a limit and therefore a derivative and definite integral only for functions defined everywhere in the field.

CHAPTER 2

FINITE COMPLEX ANALYSIS AND TWO-DIMENSIONAL VECTOR ANALYSIS

It is apparent that for the formulation of any two-dimensional vector theory, complex number notation and complex function theory can be very useful. This is true in ordinary analysis where one represents the two-dimensional Euclidean space by the complex number system; vector operations then correspond in a convenient way to operations with complex quantities.[19] An analogous situation exists when one is dealing with a finite field GF(p) instead of the reals. An analogous extension of GF(p) can be considered as the "finite complex field" and finite complex analysis can be developed. Then, under the hypothesis of finite geometry, one represents a plane in space by this complex field.

In this chapter, complex number theory and complex analysis are developed for finite geometry and in Section (E) vector analysis for finite geometry is formulated in these terms.

(A) THE COMPLEX FINITE FIELD

By the complex finite field we mean the field GF(p^2), which can be considered as the Cartesian product of the field

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[19]See Section (E) of this Chapter for a summary of this theory.

GF(p) with itself, or, very loosely, as the "plane" whose abscissa and ordinate are the "lines" GF(p). If the prime p is of the form 4n-1 (n an integer)[20], then the element -1 is not a square in GF(p). In this case, and we take this to be the case hereafter, we can consider the complex field $GF(p^2)$ to be the "real" field GF(p) with $i = \sqrt{-1}$ adjoined, just as in ordinary complex analysis.[21] The usual complex notation can be used here, i.e. the element $z \in GF(p^2)$ can be written as $z = x + iy$ with x and y in GF(p).

The algebra of finite complex numbers is in many ways similar to that of ordinary complex numbers. There are, however, several new features introduced as a result of the finiteness of the field, and some of these are given below.

1. $$i^p = i^{4n-1} = (i^{4n})(i^{-1}) = (+1)\left(\frac{1}{i}\right) = -(-1)\left(\frac{1}{i}\right) = -\frac{i^2}{i} = -i \qquad (2.1)$$

2. If $z = x + iy$, then $$z^p = (x + iy)^p$$
$$= x^p + (iy)^p \text{ by Par. 6. of Chap. 2(B)}$$
$$= x^p + i^p y^p = x - iy = z^* \qquad (2.2)$$

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[20] See Ref. 13.

[21] Note, however, the following important difference between finite and ordinary complex analysis: The finite complex field $GF(p^2)$ is not algebraically closed; there is no Fundamental Theorem of Algebra which would say that every polynomial over $GF(p^2)$ has a zero in $GF(p^2)$. Thus there exist algebraic extensions of $GF(p^2)$, obtained by adjoining to $GF(p^2)$ the roots of irreducible polynomials over $GF(p^2)$. In fact, $GF(p^n)$ exists for every n.

Thus taking the p^{th} power of a finite complex number is equivalent to taking its complex conjugate.

3. For any $z = x + iy$,

$$z^{p+1} = z^* z = x^2 + y^2 \qquad (2.3)$$

Thus z^{p+1} is the analogue of the modulus-squared of the complex number z. Note, however, that using the term "modulus" can be misleading in view of the next paragraph.

4. In GF(p), there are p+1 solutions to $x^2 + y^2 = 1$
and p+1 solutions to $x^2 + y^2 = -1$ [22]

This is a striking feature of finite field analysis: Besides what we may consider usual directions[23] $\omega = \lambda + i\mu$ with $\lambda^2 + \mu^2 = 1$, there are "strange" types of directions $\xi = \alpha + i\beta$ with $\alpha^2 + \beta^2 = -1$.

5. The ω's and ξ's defined above can be used as the basis of a polar form for complex numbers. Any element z of $GF(p^2)$ can be represented as one of

$$z = r\omega$$
$$\text{or} \quad z = r\xi \qquad (2.4)$$

with $r \in GF(p)$ and r positive or zero and $\omega^{p+1} = 1$, $\xi^{p+1} = -1$. The set of all ω's and ξ's give all possible directions of

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[22] Ref. 12, Sec. 64, p. 46.

[23] When vector analysis is formulated in complex variable notation, a complex number of unit modulus represents a unit vector or direction. See Section (E) of this Chapter.

complex numbers and can be considered as forming the "unit circle". In this sense it is analagous to the set of $e^{i\phi}$'s for all angles ϕ in the ordinary complex plane. Note, however, that the $\wp$- directions have no strict analogues in that $\wp^*\wp = -1$; they do, though, represent half the possible directions in $GF(p^2)$ and must be included in the unit circle.

6. The ω's form a group under multiplication[24]. Moreover the group is cyclic[25]: there is a generator ω_0, such that each ω is some power, smaller than $p+1$, of ω_0

i.e. $$\{\omega\} = \{\omega_0^n \mid n = 0,1,\ldots, p\} \tag{2.5}$$

The product of two $\wp$'s is an ω and the product of a $\wp$ by an ω is a $\wp$. Thus if ω_0 is a generator of the ω-group and $\wp_0$ is any $\wp$, then the set of all $\wp$'s can be written as

$$\{\wp\} = \{\wp_0\,\omega_0^n \mid n = 0,1,\ldots,p\} \tag{2.6}$$

In what follows, if we wish to sum over the whole unit circle, including both the ω's and the $\wp$'s, we shall use the symbol $\mathfrak{F}$.

7. We can use the above to evaluate sums of the form

$$\sum_{\omega} \omega^n \quad \text{and} \quad \sum_{\wp} \wp^n$$

The argument is similar to that in paragraph (3) of Chapter 1(B).

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[24]The axioms of a group can all be verified: the product of two ω's is also an ω; the complex number $z=1$ is an ω and is the unit of the group; every element ω_1 has an inverse, because $\omega_1^{p+1}=1$ and so $\omega_1^{-1} = \omega_1^p$

[25]The set of ω's is a subgroup of $GF(p^2) - \{0\}$ which is a cyclic group. By Theorem 14, P.52 of Ref.3, the set of ω's is also cyclic.

$$\sum_{\omega} \omega^n = \sum_{k=0}^{p} (\omega_0^{\,k})^n$$

$$= \sum_{k=0}^{p} (\omega_0^{\,n})^k$$ which is a geometric progression if and only if $n \neq 0, p+1, 2(p+1), \ldots$

$$= \begin{cases} p+1 = 1 & \text{if } n = 0,\ p+1, 2(p+1), \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (2.7)$$

and $$\sum_{\wp} \wp^n = \sum_{k=0}^{p} (\wp_0\, \omega_0^{\,k})^n$$

$$= \wp_0^{\,n} \sum_{k=0}^{p} (\omega_0^{\,n})^k$$

$$= \begin{cases} +1 & \text{if } n = 0, 2(p+1), 4(p+1), \ldots \\ -1 & \text{if } n = (p+1), 3(p+1), \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (2.8)$$

We can use these to evaluate sums of the form

$$\sum_{\xi} \xi^n$$

$$\sum_{\xi} \xi^n = \sum_{\omega} \omega^n + \sum_{\wp} \wp^n$$

$$= \begin{cases} 2 & \text{if } n = 0,\ 2(p+1), \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (2.9)$$

8. Of course we have for the field $GF(p^2)$ any result mentioned in Chapter 1 for the general finite field $GF(p^n)$.

These include:

(a) $z^{p^2} = z$

(b) multiplicative group of $GF(p^2)$ is cyclic

(c) single-valued functions from $GF(p^2)$ into $GF(p^2)$ can be written as polynomials with degree less than p^2.

(B) ANALYTIC FUNCTIONS IN $GF(p^2)$

We consider now a function f defined on the complex plane and with values in the complex plane.

In ordinary complex analysis we can consider f to be a function of the two real variables x and y, with $z = x + iy$. We can also consider the domain of f to be extended so that x and y take on all complex as well as real values (by writing f explicitly as a function of $z = x + iy$ and formally taking x and y as complex). Under this condition we can consider the equations

$$z = x + iy$$
$$z^* = x - iy$$

to be a change from one set of independent complex variables x and y to another, z and z^*. The chain rule gives us

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial z^*} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \qquad (2.10)$$

Now it can be seen[26] that the Cauchy-Riemann equations on f,

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[26] L.V. Ahfors, Complex Analysis, (McGraw-Hill, Inc., 1966) p.27.

the condition that f be analytic, are exactly equivalent to the condition

$$\frac{\partial f}{\partial z^*} = 0$$

or that f not be an explicit function of z^* but only of z. If f is analytic the total derivative $\frac{df}{dz}$ exists and is equal to $\frac{\partial f}{\partial z}$.

We can use this idea to extend to finite geometry the concept of an analytic function. We have in $GF(p^2)$ that $z^* = z^p$. Now f can be written as a polynomial in z of degree p^2-1 or smaller. It is apparent that the analogue of the condition $\frac{\partial f}{\partial z^*} = 0$ in ordinary analysis is the condition in $GF(p^2)$ that f be a polynomial of degree smaller than p; in this case f would contain powers of z only and not of $z^* = z^p$. Accordingly we define the analytic functions in $GF(p^2)$ to be the polynomials in z of degree p-1 or smaller.[27]

It will be seen in what follows that our definition of analyticity fits into the complex finite theory as it should.

(C) THE DERIVATIVE IN $GF(p^2)$

It is possible to introduce a derivative in $GF(p^2)$ in much the same way as was done for GF(p) in Chapter 1. It

[27] The reader will note when he reads Ref. 28 that our definition of analyticity is based on a completely different argument from that of Kustaanheimo, who first introduced the concept in $GF(p^2)$ in Ref. 28. Kustaanheimo's argument is based on what he considers the analogue in $GF(p^2)$ of the directional derivative $D_u f(z)$. His directional derivative, however, does not have the property that $D_{\hat{x}} f(z)$ and $D_{\hat{y}} f(z)$ be the components of $\frac{\partial f(z)}{\partial z}$ for non-analytic f, and for this important reason we do not introduce the idea at all.

was first introduced by Kustaanheimo[28]. He used an argument from ordinary complex analysis to motivative his definition of Df(z) as the average of the quotients $\frac{f(z+h) - f(z)}{h}$ for all non-zero values of h in GF(p^2):

$$Df(z) = \frac{1}{p^2-1} \sum_{h \neq 0} \frac{f(z+h) - f(z)}{h}$$

$$= - \sum_{h \neq 0} \frac{f(z+h) - f(z)}{h} \qquad (2.11)$$

As in Chapter 1, we can see that this quantity is indeed the analogue of a derivative by applying it to the general polynomial

$$\sum_{k=0}^{p^2-1} a_k z^k$$

We get

$$D \sum_{k=0}^{p^2-1} a_k z^k = - \sum_{h\neq 0} \sum_{k=0}^{p^2-1} a_k \left[\frac{(z+h)^k - z^k}{h}\right]$$

$$= - \sum_{h\neq 0} \sum_{k=0}^{p^2-1} \sum_{m=0}^{k} a_k C^k_m z^{k-m} h^{m-1}$$

$$+ \sum_{h\neq 0} \sum_{k=0}^{p^2-1} a_k z^k h^{-1}$$

[28]P. Kustaanhemo and B. Quist, Ann. Acad. Scient. Fennicae Ser. A.I., No.137.

Now we can see from Paragraph 3 of Chapter 1(B) that

$$\sum_{h \neq 0} h^{m-1} = \begin{cases} -1 & m = 1 \\ 0 & m \neq 1 \end{cases}$$

and $\displaystyle\sum_{h \neq 0} h^{-1} = 0$

Thus,

$$D \sum_{k=0}^{p^2-1} a_k z^k = \sum_{k=0}^{p^2-1} a_k C_1^k z^{k-1}$$

$$= \sum_{k=0}^{p^2-1} k a_k z^{k-1} \qquad (2.12)$$

For an analytic function this certainly corresponds to the total derivative $\frac{d}{dz}$ in ordinary analysis. Moreover, we can show that for a non-analytic function (which has no total derivative in ordinary analysis) this derivative Df(z) corresponds to the partial $\frac{\partial}{\partial z}$ with constant z* discussed in the last section. Consider any non-analytic term $a_k z^k$ in the polynomial. It can be written as $a_k z^k = a_k z^{mp+\ell}$ for some $m > 0$ and some $\ell < p$. Now $D(a_k z^k) = k a_k z^{k-1} = \ell a_k z^{k-1}$ since $mp + \ell = \ell$ in $GF(p^2)$.

But the analogue of the partial derivative $\frac{\partial}{\partial z}$ would be

$$\frac{\partial}{\partial z}(a_k z^k) = \frac{\partial}{\partial z}(a_k z^{\ell} z^{mp}) = \frac{\partial}{\partial z}(a_k z^{\ell} [z^*]^m)$$

$$= a_k \ell z^{\ell-1} [z^*]^m = \ell a_k z^{\ell-1+mp}$$

$$= \ell a_k z^{k-1}$$

and this is equal to $D\,(a_k z^k)$.

Thus for a non-analytic function f Kustaanheimo's derivative $Df(z)$ corresponds to $\frac{\partial f}{\partial z}$. It is to be kept in mind that in order to properly think of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial z^*}$ as partial derivatives, the function $f(x+iy)$ must be thought of as being extended to depend on two complex variables x and y. In what follows we shall often write $\frac{\partial}{\partial z}$ for Kustaanhemo's derivative and the preceding remark applies in each case.

Note that in the definition of the derivative, the sum

$$-\sum_{\substack{h \in GF(p^2) \\ h \neq 0}}$$

plays the part of $\lim\limits_{h \to 0}$ in the ordinary complex plane. This is exactly the same analogy as for the real derivative, mentioned in Chapter 1.

Note also that we have assumed that the function f is of degree p^2-1 or less. In what follows we always take arguments of derivatives to be reduced to this form.

As Kustaanhemo points out, $\frac{\partial}{\partial z}$ is linear, as in ordinary analysis. However, the product rule and chain rule do not hold in general. We have

$$\frac{\partial}{\partial z}(fg) = f\,\frac{\partial g}{\partial z} + g\,\frac{\partial f}{\partial z}$$

and

$$\frac{\partial}{\partial z}(f \circ g) = \frac{\partial f(g(z))}{\partial g}\,\frac{\partial g}{\partial z}$$

only when the degree of the combined expressions fg and f0g are respectively less than p^2

It will be useful to write the derivative in polar form. By definition

$$\frac{\partial f}{\partial z} = -\sum_{h\neq 0} \frac{f(z+h)}{h} + f(z) \sum_{h\neq 0} h^{-1}$$

We have that the second term is zero, so

$$\frac{\partial f}{\partial z} = -\sum_{h\neq 0} h^{-1} f(z+h)$$

This can be written as

$$\frac{\partial f}{\partial z} = -\frac{1}{2} \sum_{r\neq 0} \sum_{\xi} (r\xi)^{-1} f(z+r\xi) \qquad (2.13)$$

where $\sum_{\xi}$ is a sum over the whole unit circle and the factor $\frac{1}{2}$ takes account of the fact that every point is summed twice in $\sum_{r\neq 0} \sum_{\xi}$

We will now show that if f is analytic, the derivative $\frac{\partial f}{\partial z}$ can be written as a sum over just the elements in a single circle about z instead of a sum over the whole plane.

We will show that if f is analytic, we can write

$$\frac{\partial f}{\partial z} = \frac{1}{2} \sum_{\xi} f(z+R\xi)(R\xi)^{-1} \text{ for any } R \neq 0 \qquad (2.14)$$

Write f as

$$f = \sum_{k=0}^{p-1} a_k z^k.$$

Then

$$\frac{1}{2}\sum_{\xi} f(z+R\xi)(R\xi)^{-1} = \frac{1}{2}\sum_{\xi} R^{-1}\xi^{-1} \sum_{k=0}^{p-1} a_k (z+R\xi)^k$$

$$= \frac{1}{2}\sum_{\xi} \sum_{k=0}^{p-1} \sum_{m=0}^{k} R^{-1} \xi^{-1} a_k C_m^k z^{k-m} R^m \xi^m$$

$$= \frac{1}{2} \sum_{k=0}^{p-1} \sum_{m=0}^{k} a_k C_m^k z^{k-m} R^{m-1} \sum_{\xi} \xi^{m-1}$$

and $$\sum_{\xi} \xi^{m-1} = \begin{cases} 2 & m=1 \\ 0 & m\neq 1 \end{cases}$$

Hence we get

$$\frac{1}{2} \sum_{k=0}^{p-1} a_k C_1^k z^{k-1} \cdot 2 = \sum_{k=0}^{p-1} k a_k z^{k-1}$$

$$= \frac{\partial f}{\partial z}$$

This is an important property of an analytic function and will be related in the next section to the Cauchy representation formula for derivatives.

(D) INTEGRATION IN GF(p^2)

No attempt is made here to develop a general line integral in GF(p^2), because we are not so much interested in developing a general complex analysis as we are in having just enough theory to make possible a preliminary formulation of electrostatics. For this purpose it is necessary only to find a line integral around a very special closed contour– a circle – and it would remain another problem to put this in the framework of a more general complex analysis.

We can find the analogue of a closed circle integration quite easily. In ordinary analysis we have in polar form $z = r e^{i\phi}$ and for a circular contour, $dz = i r e^{i\phi} d\phi$

So $$\oint_{\odot} f\, dz = \int_{0}^{2\pi} f\, i r e^{i\phi}\, d\phi$$

We have that in $GF(p^2)$, the ξ's (the ω's plus the δ's) form the unit circle and hence are the analogue of $e^{i\phi}$. Now the integration with respect to ϕ is really a sum over all directions, the number of directions being $\frac{2\pi}{d\phi}$

Thus $\frac{1}{2\pi} \oint_{\odot} f\,dz = \int_0^{2\pi} f ir\, e^{i\phi} \frac{d\phi}{2\pi}$

becomes in finite geometry, when we sum instead of integrating over all directions,

$$\sum_{\xi} f\, ir\xi \;\frac{1}{\text{number of directions}}$$

The number of directions is $2(p+1) = 2$ in $GF(p^2)$. Thus the analogue of

$$\frac{1}{\pi} \oint_{\odot} f\,dz \quad \text{is} \quad i\sum_{\xi} r\xi\, f(z+r\xi) \text{ in finite geometry} \tag{2.15}$$

where the circle has centre z and radius r. Note that we have been able to find the analogue of $\frac{1}{\pi}$ times the integral and not the integral itself.

The above argument, of course, can be considered at best a motivation for the definition of our contour integral. The validity of the definition depends not on arguments of formal analogy, but on whether our expression acts like a contour integral and fits into our complex analysis as a contour integral. This indeed seems to be the case.

Firstly, let us apply the closed - circle integration to an analytic function

$$\sum_{k=0}^{p-1} a_k z^k$$

We get

$$i\sum_{\xi} r\xi \sum_{k=0}^{p-1} a_k\,(z+r\xi)^k = i\sum_{\xi}\sum_{k=0}^{p-1}\sum_{m=0}^{k} a_k C_m^k z^{k-m} r^{m+1} \xi^{m+1}$$

and $\sum \xi^{m+1} = 0$ for all m in the summation, by Paragraph 7 of Chapter 2(A). Thus the line integral of an analytic function around any closed circle is zero, as required if our definitions are to have any meaning. The analytic functions actually satisfy a stronger condition. If the summation $\sum_{\xi}$ is broken up into the two summations $\sum_{\omega} + \sum_{\varsigma}$, it can be seen that for an analytic function both summations $\sum_{\omega}$ and $\sum_{\varsigma}$ vanish by themselves.

Secondly, let us integrate the function $\frac{f(z+t)}{t}$, with f analytic, about a closed circle with centre at z. We get

$$i\sum_{\xi} r\xi \sum_{k=0}^{p-1} a_k \frac{(z+r\xi)^k}{r\xi}$$

$$= i\sum_{\xi}\sum_{k=0}^{p-1}\sum_{m=0}^{k} a_k C_m^k\, z^{k-m}\, r^m\, \xi^m$$

Here $\sum_{\xi} \xi^m$ vanishes except for m = 0 when its value is 2.

Thus we have

$$2i \sum_{k=0}^{p-1} a_k C_0^k\, z^k = 2i \sum_{k=0}^{p-1} a_k z^k$$

$$= 2i\ f(z) \qquad (2.16)$$

This gives the exact analogue of the Cauchy integral formula in ordinary complex analysis.[29]

$$f(z) = \frac{1}{2\pi i} \oint_{\circleddash} \frac{f(z+t)}{t} \quad dt$$

Finally, if we integrate the function $\frac{f(z+t)}{t^2}$ about a closed circle with centre at z we get

$$i \sum_{\xi} r\xi \; \frac{f(z+r\xi)}{r^2 \xi^2} \quad = i \sum_{\xi} \frac{f(z+r\xi)}{r\xi}$$

$$= i \; ? \; \frac{df}{dz} \quad \text{if } f \text{ is analytic} \qquad (2.17)$$

This has the form of Cauchy's representation form for the derivative.[30]

(E) TWO-DIMENSIONAL VECTOR ANALYSIS

We shall first review some notation from ordinary vector analysis and afterwards extend these concepts to a finite geometry using the complex analysis discussed earlier.

(1) As mentioned earlier, two-dimensional vector analysis can be represented very conveniently in complex notation.

A vector (or vector function) $\underline{V} = V_x \hat{\underline{x}} + V_y \hat{\underline{y}}$ is represented by the complex number (or complex function)

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[29]Ref.26, p.119.

[30]Ref.26, p.120.

$V = V_x + iV_y$. In what follows we shall always use this notation; underlined letters stand for vectors and the same letters without underlining stand for the corresponding complex quantities.

(2) The complex path integral $\int_c V^* dz$ can be written

$$\int_c V^* \, dz = \int_c (V_x - iV_y)(dx + idy)$$

$$= \int_c (V_x dx + V_y dy) + i \int_c (V_x dy - V_y dx)$$

$$= \int_c \underline{V} \cdot \underline{dr} + i \int_c (\underline{V} \times \underline{dr}) \cdot \underline{k}$$

$$= \int_c \underline{V} \cdot \underline{dr} + i \int_c (\underline{k} \times \underline{V}) \cdot \underline{dr} \qquad (2.18)$$

(3) It is well known that the divergence and curl can be defined by

$$\operatorname{div} \underline{E} = \lim_{V \to 0} \frac{\oint_S \underline{E} \cdot \hat{n} dS}{V}$$

$$\text{and } \operatorname{curl} \underline{E} = \lim_{V \to 0} \frac{\oint_S (\hat{n} \times \underline{E}) dS}{V}$$

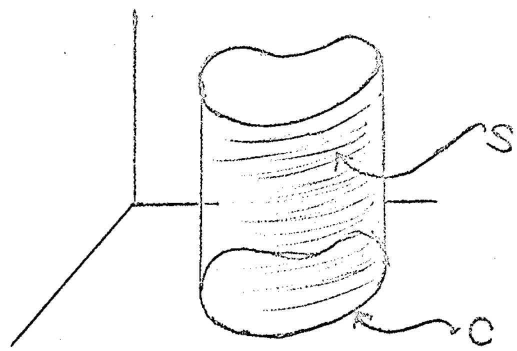
where V is the volume enclosed by the surface S.

To reduce these expressions for the case of a two-dimensional vector $\underline{E}$, consider S to be the cylinder (the right-cylinder) of unit height generated by the contour C, which lies in two-dimensions and encloses area A. Then it can easily be shown that

$$\oint_S \underline{E}\cdot \hat{n}dS = \oint_c (\underline{k} \times \underline{E})\cdot \underline{dr}$$

and

$$\oint_S \underline{k}\cdot(\hat{n} \times \underline{E})dS = \oint_c \underline{E}\cdot\underline{dr}$$



The first gives for the divergence

$$\operatorname{div} \underline{E} = \lim_{A\to 0} \frac{\oint_c (\underline{k} \times \underline{E})\cdot\underline{dr}}{A} \tag{2.19}$$

and the second gives, for the only non-vanishing component of the curl,

$$(\operatorname{curl} \underline{E})\cdot\underline{k} = \lim_{A\to 0} \frac{\oint_c \underline{E}\cdot\underline{dr}}{A} \tag{2.20}$$

In what follows we shall write curl $\underline{E}$ to mean (curl $\underline{E}$)$\cdot\underline{k}$.

(4) Using (2) and (3) we get

$$\lim_{A\to 0} \frac{\oint_c E^* \, dz}{A} = (\operatorname{curl} \underline{E}) + i\,(\operatorname{div} \underline{E}) \tag{2.21}$$

(5) We have for the partial derivative $\frac{\partial}{\partial z}$

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

When this is applied to a function $E = E_x + iE_y$ it can easily be seen that it results in

$$\frac{\partial E}{\partial z} = \frac{1}{2}\,\text{div}\,\underline{E} + \frac{i}{2}\,\text{curl}\,\underline{E} \tag{2.22}$$

The preceeding paragraphs have been a review of certain concepts from ordinary complex analysis. The next step is to extend the formalism in a consistent manner to the finite complex plane. We wish to obtain, firstly, expressions for curl $\underline{E}$ and div $\underline{E}$ in finite geometry and, secondly, to show that these expressions fit into a formalism in analogy with the ordinary case above.

Now, the extension to finite geometry of curl $\underline{E}$ and div $\underline{E}$ seems obvious: we can use the identity (2.21) from ordinary analysis, with a circle for contour C and putting in the finite geometry analogue of limit and closed-circle integration. This gives

$$\lim_{A \to 0} \frac{\oint_{\odot} E^* dz}{A} = \lim_{r \to 0} \pi \frac{\frac{1}{\pi}\oint_{\odot} E^* dz}{\pi r^2}$$

$$= \lim_{r \to 0} \frac{1}{r^2}\left(\frac{1}{\pi}\oint_{\odot} E^* dz\right)$$

which becomes in finite geometry

$$-i \sum_{r \neq 0} \frac{1}{r^2} \sum_{\xi} r\xi E^*(z + r\xi)$$

However, a simple consideration shows that this is not exactly the natural analogue desired here. As we saw in paragraph (2) above we can write in ordinary analysis

$$E^* dz = \underline{E} \cdot d\underline{r} + i(\underline{k} \times \underline{E}) \cdot d\underline{r}$$

Both $\underline{E} \cdot \underline{dr}$ and $(\underline{k} \times \underline{E}) \cdot \underline{dr}$ are invariants under linear transformations, and therefore $E^* dz$ too is a scalar. However, in the extension to finite geometry considered above, the analogous quantity $r\xi E^*(z + r\xi)$ is not an invariant because of the existence of "strange" directions; it reverses sign under a transformation $z' = \varsigma_0 z$ where ς_0 is a "strange" direction. A simple invariant extension is $r\xi^{\frac{p^2-1}{2}} \xi E^*(z+r\xi)$;

$$\xi^{\frac{p^2-1}{2}} = \begin{array}{ll} 1 & \text{for } \xi = \omega \\ -1 & \text{for } \xi = \varsigma \end{array}.$$

It can then be easily seen that $r\xi^{\frac{p^2-1}{2}} \xi E^*(z+r\xi)$ is indeed a scalar.

This motivates us to define in finite geometry

$$\text{curl } \underline{E} + i \text{ div } \underline{E} = -i \sum_{r \neq 0} \frac{1}{r^2} \sum_{\xi} r \xi^{\frac{p^2-1}{2}} \xi E^*(z+r\xi)$$

$$= -i \sum_{r \neq 0} \sum_{\xi} r^{-1} \xi^{\frac{p^2-1}{2}} \xi E^*(z+r\xi)$$

We have an independent expression for the derivative $\frac{\partial E}{\partial z}$ in $GF(p^2)$ and we can check the consistency of the above procedure using paragraph (5) above.

We have (curl $\underline{E}$) + i(div $\underline{E}$) $= -i \sum_{r \neq 0} \sum_{\xi} r^{-1} \xi^{\frac{p^2-1}{2}} \xi E^*(z+r\xi)$

We take the complex conjugate of both sides and multiply through by i to get:

$$(\text{div } \underline{E}) + i(\text{curl}\underline{E}) = i\left[i \sum_{r \neq 0} \sum_{\xi} r^{-1} \xi^{\frac{p^2-1}{2}} \xi^* E(z+r\xi)\right]$$

since $\xi^{\frac{p^2-1}{2}}$ is real. Now $\xi^* = \xi^p = \xi^{p+1}\xi^{-1}$ and $\xi^{\frac{p^2-1}{2}} \xi^{p+1} = 1$

Hence $\xi^{\frac{p^2-1}{2}} \xi^* = \xi^{-1}$ and (div$\underline{E}$) + i(curl$\underline{E}$) $= -\sum_{r \neq 0} \sum_{\xi} r^{-1} \xi^{-1} E(z+r\xi)$

Now this is exactly twice the polar form for Kustaanheimo's derivative, and our vector analysis in GF(p^2) is consistent with paragraph (5).

CHAPTER 3

ELECTROSTATICS

In this chapter we discuss source-free electrostatics in the two-dimensional finite geometry employing the vector analysis developed in the last chapter. The first problem, of course, is to formulate electrostatics and the second is to examine the fields, indicating their nature in the Pythagorean approximation.

R. A. Beth[31] has shown that, in ordinary space, Maxwell's equations on the fields E and H are exactly equivalent to the condition of analyticity (the Cauchy-Riemann equations) for two certain functions constructed from the components of E and H. For a charge free region these are[32]

$$E' = E_y + iE_x \quad \text{and} \quad H' = H_y + iH_x \tag{3.1}$$

Now $E' = iE^*$ and $H' = iE^*$, so Maxwell's equations in a source free region are equivalent to the condition that E^* and H^* be analytic.

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[31] R. A. Beth, J. Appl. Phys. 37, 2568 (1966).

[32] Note that our notation differs slightly from that used by Beth. Beth uses the symbol E for the function $E_y + iE_x$, but we have already established the convention that $E = E_x + iE_y$.

The same situation holds in finite geometry. The conditions that

$$\operatorname{div} \underline{E} = 0 \qquad \text{and} \qquad \operatorname{curl} \underline{E} = 0$$ [33]

are equivalent to the condition that

$$\frac{\partial E}{\partial z} = 0 \quad ,$$

which, when we take the complex conjugate of both sides, becomes

$$\frac{\partial E^*}{\partial z^*} = 0 .$$

This is exactly the condition that E* be an analytic function in z. Thus we can say that the fields for source-free space in finite geometry are the complex conjugates of the analytic functions, or the functions of the form

$$E = \sum_{k=0}^{p-1} b_k z^{kp} \tag{3.2}$$

Of course, the above is also true for source-free magnetic fields.

Finally, we consider these fields in the Pythagorean approximation. We give not a rigorous and complete treatment but an outline of the investigation. We will attempt to see the fields of ordinary electrostatics as certain finite geometry fields in the Pythagorean approximation.

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[33] The definition of div $\underline{E}$ and curl $\underline{E}$ in $GF(p^2)$ is given in Chapter 2(E).

This can be accomplished very simply as follows: any source-free field in ordinary geometry can be approximated by a polynomial function of form

$$E^* = \sum_{k=1}^{K} b_k z^k$$

By taking K large enough (immensely large if necessary but finite) we can represent any source-free field in this way to within any experimental error. We give K an appropriately large value and consider it constant. The experimentally observed values of z and E range from the very small to the very large but are finite in both directions. We choose units for z and E so that all experimental values are very large and experimentally indistinguishable from integers (complex integers).

We now postulate a finite geometry with basic number field GF(p). In the integer representation of GF(p) the first N integers (N~lnp) form a Pythagorean subset, and we take N large enough so that all the (integral) experimental values for z and E are smaller than N. We can then consider the integral values of z and E as representing elements of the Pythagorean subset of $GF(p^2)$. Further, the coefficients b_k will be, experimentally, rational numbers and can also be considered to represent finite field numbers, not necessarily Pythagorean.

We now consider the equation

$$E^* = \sum_{k=1}^{K} b_k z^k$$

as an equation in the finite number field. According to our formulation of electrostatics, E is a source-free field in the finite geometry. Moreover, to an observer who assumed a Euclidean geometry and therefore could perceive only Pythagorean elements, this field would appear as a usual source-free electrostatics field of ordinary geometry, according to our discussion above. We can thus say that every source-free field of ordinary Euclidean electrostatics would arise as the Pythagorean approximation to some source - free finite geometry field.

There will, of course, be many other source-free fields in the finite geometry which would not be recognized as ordinary source-free fields in the Pythagorean approximation. The existence of these fields would predict new effects within the framework of electrostatics; their study is a further problem and not part of this thesis.

CONCLUSION

We have given a formulation of source-free electrostatics for a finite geometry, the Pythagorean approximation to this theory, and the complex analysis required for the electrostatics.

The work could be continued on several fronts. One could develop further the complex analysis in $GF(p^2)$; this would involve defining the line integral for a general type of curve, not necessarily closed, as well as for closed contours more general than the circle. The first problem here, of course, is to establish the meaning of the geometrical terms in a finite geometry. The possibility of functions analytic in only a region or containing singularities might be considered. This seems necessary for a theory of electrostatics that is to include sources.

Electrostatics, of course, could be extended in other ways than the inclusion of sources. The uniqueness of the solutions under various conditions must be considered; this would involve the analogue of the Dirichlet problem and its Pythagorean approximation.

Finally, there remains the investigation of the source-free finite geometry fields which in the Pythagorean approximation would not be recognized as source-free fields at all. These would be seen in the observable region as new effects hitherto not considered part of source-free electrostatics.