

A NEW APPROACH TO THE REALIZATION OF NONMINIMUM
BIQUADRATIC FUNCTIONS

A Thesis
Presented to
The Faculty of Engineering
The University of Manitoba

In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Electrical Engineering

by
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March 1967



ABSTRACT

In this study, a new approach to the synthesis of driving point impedance functions is considered.

The proposed new technique applies the real part padding method to relocate the even part zeros of a nonminimum biquadratic function so that one of a number of specified favourable conditions will be satisfied.

It has been illustrated that the new approach yields a simpler network as compared with other methods in the classes of techniques that do not use either transformers or gyrators. Various combinations of coefficients of the given biquadratic nonminimum impedance function determine the realizability in one of the specified networks. Proof of the method, as well as illustrative examples are provided in the text.

ACKNOWLEDGMENT

The author wishes to express his gratitude to Dr. H. K. Kim for providing the guidance and assistance necessary for the completion of this study. He is, also, grateful for the financial aid made available by the National Research Council of Canada.

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CHAPTER I

INTRODUCTION

In general, all existing techniques of realizing positive real functions can be classified into two groups. This classification is based on whether or not the manipulation which disturbs the location of the even part zeros is applied. For a positive real function, the zeros of its even part exhibit quadrantal symmetry.

The synthesis techniques belonging to the first group subtract a certain amount of positive real number from the given function so as to cause a set of the quadrantal zeros to be relocated on the imaginary axis in the complex s -plane, prior to the removal of the set. These realizations will, in general, require the use of ideal transformers, as was demonstrated by the Brune (1) realization. However, Bott and Duffin (2) showed that by employing the use of an excessive number of elements, ideal transformers may be avoided in the realization.

The second group does not disturb the even part zeros in the complex s -plane. The methods in this group begin immediately to remove whole sets of zeros either at once, or set by set. This group has an advantage over the first group in that the tedious procedure for finding the maximum amount of permissible real part that may be subtracted from the

given function is eliminated. However, in general, the second group of techniques requires ideal transformers or gyrators for realizations in cascade structures as was demonstrated by Hazony (4). If the constraint on the network structure is eased, a network may be realized with a smaller number of elements than its counterpart in group one which avoids the use of transformers and gyrators. This realization was developed by Kim (5). Synthesis procedures, belonging to this group, advocated by Miyata (6) and Kuh (7) results in simple realizations, requiring no transformers or gyrators and except for the simplest functions, considerably fewer elements are necessary than are required for the techniques discussed previously. Unfortunately, the applicability of the two procedures depends upon the locations of the even part zeros in the s -plane, consequently, only restricted classes of positive real functions may be realized.

The new approach suggested in this investigation applies the concept of real part padding. This manipulation shifts the sets of even part zeros from their original positions, relocating them somewhere between the locations of group one and group two, such that they now occupy favourable regions in the s -plane. When the even part zeros have been placed in these favourable regions, simple realizations are obtained containing reasonable numbers of elements. The conditions corresponding to these favourable regions in which the even part zeros are to be relocated, have been found (8), (9).

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CHAPTER II

DISCUSSION OF THE PROPOSED TECHNIQUE

In this chapter, the function class, upon which the new method will be applied, is defined. With this definition taken into consideration, a discussion of the theoretical basis of the technique will be undertaken.

I. DEFINITION OF THE FUNCTION CLASS

The function class is the general, biquadratic, driving point impedance function $Z(s)$, and may be written as

$$Z(s) = \frac{K(s^2 + a_1 s + a_0)}{s^2 + b_1 s + b_0}. \quad (2-1)$$

The function $Z(s)$ is the driving point impedance of a lumped, linear, passive, bilateral, and time-invariant network. The assumption is made that the quantities a_1 , a_0 , b_1 , b_0 , and K are real, positive, and non-zero numbers. As a result of the above definition and assumptions, $Z(s)$ is a positive real function of the complex variable s . Since the coefficients are positive real numbers, the necessary and sufficient condition that the function $Z(s)$ be a positive real function of the complex variable s is

$$(\sqrt{a_0} - \sqrt{b_0})^2 \leq a_1 b_1. \quad (2-2)$$

Because the coefficients of $Z(s)$ are non-zero, it is automatically minimum reactive and minimum susceptive, since it has no poles or zeros, respectively, on the imaginary axis in

the complex plane. No generality is lost with the assumption of non-zero coefficients, since the case that does occur with zero coefficients may be handled by reactance or susceptance reduction, yielding a simple and economical realization. This case being solved, it is of no further interest.

II. THE PROPOSED TECHNIQUE

The principles underlying the proposed technique are quite simple. It is based on the fact that the zeros of the even part of a positive real function exhibit quadrantal symmetry. For the general biquadratic function $Z(s)$, as defined earlier, a typical set of even part zeros would appear as shown in Figure 1.

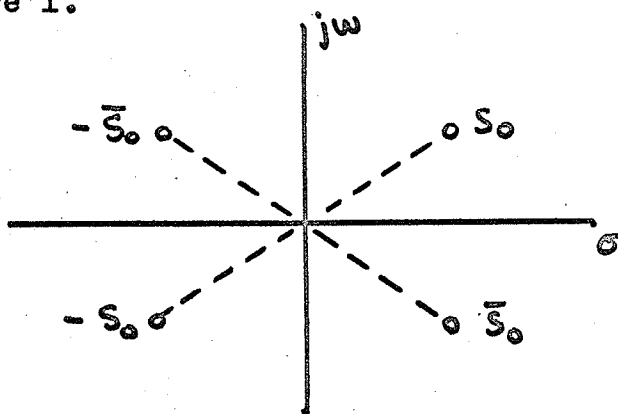


Figure 1. Zeros of the even part of $Z(s)$ exhibiting quadrantal symmetry.

From Figure 1, the zero of the even part of $Z(s)$ located in the first quadrant is defined as

$$s_0 = \sigma_0 + jw_0. \quad (2-3)$$

The real part, σ_0 , and the imaginary part, w_0 , of the even part zero s_0 are positive quantities.

In general, the even part zeros shown will not be located such that the desired realization is possible. However, since the function being dealt with is nonminimum, the even part zeros may be relocated by extracting a portion of the minimum resistance from the function. A remainder function, $Z_1(s)$, may be written as

$$Z_1(s) = Z(s) - R_o,$$

or

$$Z_1(s) = \frac{(K - R_o)s^2 + (a_1K - b_1R_o)s + a_oK - b_oR_o}{s^2 + b_1s + b_o}. \quad (2-4)$$

The remainder function must be positive real, consequently, the value of the extracted resistance, R_o , must not exceed that of the minimum resistance, R_m , of the original function $Z(s)$. The subtraction of R_o from $Z(s)$ tends to shift the even part zeros towards the $j\omega$ -axis. If, as a consequence of shifting even part zeros, one or more of the following relations (8), (9) are satisfied;

$$\sigma_1 = w_1, \quad (2-5)$$

$$\sigma_1 U_1 + w_1 V_1 = 0, \quad V_1 < 0, \quad (2-6)$$

$$\sigma_1 U_1 - w_1 V_1 = 0, \quad V_1 > 0, \quad (2-7)$$

$$V_1 = 0, \quad (2-8)$$

$$\sigma_1 = 0, \quad Z_1(0) = 4Z_1(\infty), \quad (2-9)$$

$$\sigma_1 = 0, \quad Z_1(\infty) = 4Z_1(0), \quad (2-10)$$

then the synthesis of the given function, $Z(s)$, is complete.

The quantities U_1 and V_1 are defined by the equation

$$Z_1(s_1) = U_1 + jV_1, \quad (2-11)$$

where U_1 and V_1 are the real and imaginary parts, respec-

tively, of $Z_1(s)$ evaluated at the even part zero s_1 . From Eq. (2-4), it may be seen that the location of the even part zeros in the complex plane, of the remainder function, is a function of R_0 . A proper selection of R_0 , where possible values of R_0 range from zero to the value of the minimum resistance of $Z(s)$, causes the even part zeros to shift, so that one or more of the conditions in Eqs. (2-5) to (2-10) may be satisfied, resulting in the network shown in Figure 2.

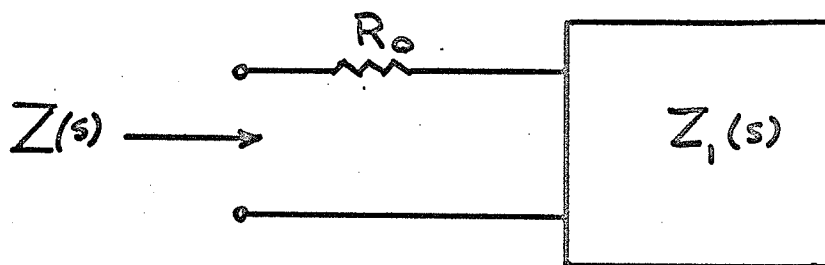


Figure 2. Realization of the given function $Z(s)$.

The networks that realize $Z_1(s)$ are given in Appendix A, and were developed from networks produced by Kim (8). The use of these networks and their associated element values realize $Z_1(s)$ with six elements, consisting of two resistors, two inductors and two capacitors. Consequently, $Z(s)$ is realized with a total of seven elements. If conditions in Eq. (2-9) or Eq. (2-10) are satisfied, then a six element realization, in general, is possible, consisting of three reactive and three resistive elements (9).

The remainder of this study will be concerned with finding expressions for the amount of resistance R_0 , that must be extracted from $Z(s)$ in order to satisfy one of the condi-

tions. Also, an attempt will be made to define the subclass of the general class of functions as defined by $Z(s)$, to which the proposed technique will be amenable. This will be carried out by determining the necessary and sufficient conditions on the coefficients of the given function so that the proposed realization technique may be applied successfully. A flow diagram outlining these conditions will be presented.

CHAPTER III

THE REALIZATION OF $Z(s)$

This chapter will be devoted to determining the amount of resistance that must be removed from $Z(s)$ in order to force the remainder function, $Z_1(s)$, into the class of functions that will satisfy at least one of the desired relations contained in Eqs. (2-5) to (2-10). Consideration of the positive real property of $Z_1(s)$ yields the necessary and sufficient conditions on the coefficients of the given function, $Z(s)$, required for the specific relation to be satisfied.

I. DETERMINATION OF THE RESISTANCE R_o

Before developing specific expressions for R_o , it will be convenient to determine the quantities o_1 , w_1 , U_1 , and V_1 , in terms of the coefficients and the gain constant of $Z(s)$ and as functions of R_o .

Equation (2-4) may be rewritten in an alternate form as the following,

$$Z_1(s) = \frac{(K - R_o) \left[s^2 + \frac{(a_1 K - b_1 R_o)s}{(K - R_o)} + \frac{(a_o K - b_o R_o)}{(K - R_o)} \right]}{s^2 + b_1 s + b_o}, \quad (3-1)$$

or

$$Z_1(s) = \frac{K'(s^2 + a_1' s + a_o')}{s + b_1 s + b_o}. \quad (3-2)$$

The coefficients in Eq. (3-2), a' , a' , and K' are defined as

$$a_1' = \frac{a_1 K - b_1 R_o}{K - R_o}, \quad (3-3)$$

$$a_o' = \frac{a_o K - b_o R_o}{K - R_o}, \quad (3-4)$$

$$\text{and } K' = K - R_o. \quad (3-5)$$

The function $Z_1(s)$ must be positive real, consequently, the quantities a_1' and a_o' , and the gain constant K' , must be positive numbers. Furthermore, the coefficients of $Z_1(s)$ must satisfy the relation

$$(\sqrt{a_o'} - \sqrt{b_o'})^2 \leq a_1' b_1, \quad (3-6)$$

to ensure the positive realness of the function.

The even part of $Z_1(s)$ is given by

$$\begin{aligned} \text{Ev } Z_1(s) &= \frac{N_1(-s^2)}{D_1(-s^2)}, \\ \text{or } K' \left[s^4 + s^2 \left\{ \frac{K(a_o + b_o - a_1 b_1) - R_o(2b_o - b_1^2)}{K'} \right\} \right. \\ &\quad \left. + \frac{b_o(a_o K - b_o R_o)}{K'} \right] \\ \text{Ev } Z_1(s) &= \frac{\quad}{s^4 + (2b_o - b_1^2)s^2 + b_o^2}. \quad (3-7) \end{aligned}$$

From Figure 1, it may be seen that the even part numerator of $Z_1(s)$ can be written as

$$N(-s^2) = (s - s_1)(s + s_1)(s - \bar{s}_1)(s + \bar{s}_1),$$

or as

$$N(-s^2) = s^4 - 2(\sigma_1^2 - w_1^2)s^2 + (\sigma_1^2 - w_1^2)^2 + 4\sigma_1^2 w_1^2, \quad (3-8)$$

when the substitution

$$s_1 = \sigma_1 + jw_1$$

is made. Comparison of the coefficients of s in Eq. (3-7) with the like coefficients of Eq. (3-8) yields the equations

$$-2(\sigma_1^2 - w_1^2)(K - R_0) = K(a_0 + b_0 - a_1 b_1) - R_0(2b_0 - b_1^2),$$

and

$$(\sigma_1^2 - w_1^2)^2(K - R_0) = b_0(a_0 K - b_0 R_0). \quad (3-10)$$

The simultaneous solution of Eqs. (3-9) and (3-10) for σ_1 and w_1 leads to the following expressions,

$$\sigma_1 = \sqrt{\left[\frac{b_0(a_0 K - b_0 R_0)}{4(K - R_0)} \right]^{1/2} - \frac{K(a_0 + b_0 - a_1 b_1) - R_0(2b_0 - b_1^2)}{4(K - R_0)}}, \quad (3-11)$$

$$w_1 = \sqrt{\left[\frac{b_0(a_0 K - b_0 R_0)}{4(K - R_0)} \right]^{1/2} + \frac{K(a_0 + b_0 - a_1 b_1) - R_0(2b_0 - b_1^2)}{4(K - R_0)}}. \quad (3-12)$$

Equations (3-11) and (3-12) are the desired form for the real and imaginary parts, respectively, of the even part zero of $Z_1(s)$ located in the first quadrant of the complex plane.

Expressions for U_1 and V_1 , the real and imaginary parts of $Z_1(s)$ evaluated at s_1 , respectively, may be found by substituting $s_1 = \sigma_1 + jw_1$ into Eq. (3-1) and separating out the real and imaginary parts. The expressions obtained are

$$U_1 = \frac{K(\sigma_1^2 - w_1^2 + a_1 \sigma_1 + a_0)(\sigma_1^2 - w_1^2 + b_1 \sigma_1 + b_0) + Kw_1^2(2\sigma_1 + a_1)(2\sigma_1 + b_1)}{(\sigma_1^2 - w_1^2 + b_1 \sigma_1 + b_0)^2 + w_1^2(2\sigma_1 + b_1)^2} - R_0, \quad (3-13)$$

$$V_1 = \frac{w_1 K(2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1 \sigma_1 + b_0) - w_1 K(2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1 \sigma_1 + a_0)}{(\sigma_1^2 - w_1^2 + b_1 \sigma_1 + b_0)^2 + w_1^2(2\sigma_1 + b_1)^2}. \quad (3-14)$$

If the expressions contained in Eqs. (3-11) and (3-12) are

substituted into Eqs. (3-13) and (3-14) for σ_1 and w_1 , respectively, then the equations for U_1 and V_1 become explicit functions of the resistance R_0 .

The relations for the parameters contained in the conditions (2-5) to (2-10) have now been determined. The next step is to find the resistance, R_0 , which, when extracted from the given function, $Z(s)$, will yield a positive real remainder function, $Z_1(s)$, that belong to the subclass of functions that satisfies one of the conditions given in Eqs. (2-5) and (2-10).

Case I: $\sigma_1 = w_1$.

When Eq. (3-11) is equated to Eq. (3-12) and the resulting equation solved for R_0 , the result obtained, in terms of the coefficients and the gain constant of $Z(s)$, is

$$R_1 = \frac{K(a_0 + b_0 - a_1 b_1)}{(2b_0 - b_1^2)}. \quad (3-15)$$

The value of R_0 that solves this particular condition is denoted as R_1 .

Case II: $\sigma_1 U_1 + w_1 V_1 = 0$, $V_1 < 0$.

Substitution of Eqs. (3-13) and (3-14) into Eq. (2-6) yields

$$\begin{aligned} & \sigma_1 K(\sigma_1^2 - w_1^2 + a_1 \sigma_1 + a_0)(\sigma_1^2 - w_1^2 + b_1 \sigma_1 + b_0) \\ & \quad + \sigma_1^2 K w_1 (2\sigma_1 + a_1)(2\sigma_1 + b_1) \\ & \quad - R_0 \sigma_1 (\sigma_1^2 - w_1^2 + b_1 \sigma_1 + b_0)^2 - R_0 \sigma_1 w_1^2 (2\sigma_1 + b_1)^2 \\ & \quad + w_1^2 K (2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1 \sigma_1 + b_0) \end{aligned}$$

$$-w_1^2 K(2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_0) = 0. \quad (3-16)$$

The substitution of Eqs. (3-11) and (3-12), for σ_1 and w_1 , respectively, into Eq. (3-16) yields the result

$$R_2 = \frac{K(a_1 b_1 - a_0 + b_0)}{b_1^2}, \quad (3-17)$$

when solved for the resistance R_0 . The value of the resistance to be subtracted in order to satisfy the specified condition is designated as R_2 .

The mathematical manipulations between Eqs. (3-16) and (3-17) are contained in Appendix B.

Case III: $\sigma_1 U_1 - w_1 V_1 = 0$, $V_1 > 0$.

The expressions for U_1 and V_1 contained in Eqs. (3-13) and (3-14), respectively, may be substituted into Eq. (2-7) yielding

$$\begin{aligned} & \sigma_1 K(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_0)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0) \\ & + \sigma_1 w_1^2 K(2\sigma_1 + a_1)(2\sigma_1 + b_1) \\ & - \sigma_1 R_0(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0)^2 - \sigma_1 w_1^2 R_0(2\sigma_1 + b_1)^2 \\ & - w_1^2 K(2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0) \\ & + w_1^2 K(2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_0) = 0. \end{aligned} \quad (3-18)$$

Equation (3-18) may be solved for R_0 by substituting the appropriate relations for σ_1 and w_1 . The resulting formula for the resistance is

$$R_3 = \frac{K(a_1 b_1 + a_0 - b_0)}{b_1^2}. \quad (3-19)$$

Where R_3 is defined as that value of R_0 which causes $Z_1(s)$ to be in the subclass of functions that satisfies the relation, $\sigma_1 U_1 - w_1 V_1 = 0$, at its even part zero.

The details of the derivation of Eq. (3-19) are contained in Appendix C.

Case IV: $V_1 = 0$.

From Eq. (3-14) it may be seen that the imaginary part of $Z_1(s_1)$ can be made equal to zero in two cases. These cases are,

$$1. (2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0) - (2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_0) = 0, \quad (3-20)$$

and

$$2. w_1 = 0. \quad (3-21)$$

After substitution into Eq. (3-20) for σ_1 and w_1 , it becomes obvious that there is no value of R_0 which will satisfy this equation. Equation (3-20) may be satisfied, regardless of the value of R_0 , only if the coefficients of $Z(s)$ obey the equality

$$a_0 = b_0. \quad (3-22)$$

This is demonstrated in Appendix D. Consequently, a sufficient condition that the given function be of the subclass of functions which has V_1 set to zero is given by Eq. (3-22). In this case, resistance need not be subtracted since the function automatically satisfies the condition V_1 equal to zero.

However, a biquadratic function with the coefficients

a_0 and b_0 equal may be easily realized by a four element network consisting of one inductor, one capacitor, and two resistors, regardless of the numerical value of the other coefficients, a_1 and b_1 . Consequently, a simpler realization is available than that produced by the proposed technique.

The resistance required to meet the second case may be found by solving Eq. (3-22) for R_0 . The expression that is obtained is identical to the expression for the minimum resistance of $Z(s)$ except that the formula holds for an opposite set of inequalities. This appears in a discussion of the minimum resistance of a function and is given in Appendix E. The resistance required for condition (3-22) to hold is;

$$R_4 = \frac{K}{b_1(4b_0 - b_1^2)} \left[\frac{b_1(a_0 + b_0) + a_1(2b_0 - b_1^2)}{+ 2\sqrt{b_0(a_0 - b_0)^2 + b_0(a_1 - b_1)(a_0b_1 - a_1b_0)}} \right], \quad (3-23)$$

or, if $b_1^2 - 4b_0$ equals zero, the solution for R_4 then reduces to the following,

$$R_4 = \frac{K}{b_1^2} \left[\frac{(a_0 - b_0)^2 + a_1b_1(a_1b_1 - 2a_0 - 2b_0)}{a_1b_1 - 2(a_0 - b_0)} \right]. \quad (3-24)$$

The formulas shown in Eqs. (3-23) and (3-24) are valid only if the following sets of conditions hold,

$$a_0 > b_0, \quad (3-25)$$

and

$$a_1b_1 \geq (a_0 - b_0) + b_1^2, \quad (3-26)$$

or if,

$$a_0 < b_0, \quad (3-27)$$

and

$$a_1 b_1 \geq (b_0 - a_0) + b_1^2 a_0 / b_0. \quad (3-27)$$

However, it is shown in Appendix E that if the coefficients of the function satisfy either one of the above two sets of conditions, then a five element realization is possible (four if the equality is satisfied).

The proposed method will realize the two cases, in general, with six elements and seven elements, respectively. Consequently, the subclass of functions which satisfy the condition V_1 equal to zero may be realized by techniques which yield more economical realizations than the new method. An attempt to obtain this subclass by this method would not be justified.

Case V: $\sigma_1 = 0$, $Z_1(0) = 4Z_1(\infty)$.

The condition that the real part σ_1 be zero is a requirement that $Z_1(s)$ be a minimum function. In terms of the coefficients of $Z_1(s)$, the requirement is

$$(\sqrt{a'_0} - \sqrt{b_0})^2 = a'_1 b_1. \quad (3-28)$$

Also, in terms of the coefficients of $Z_1(s)$, the requirement that $Z_1(0)$ be equal to $4Z_1(\infty)$ is

$$a'_0 = 4b_0. \quad (3-29)$$

Substitution of Eq. (3-29) into Eq. (3-28) yields

$$a'_0 = 4b_0 = 4a'_1 b_1. \quad (3-30)$$

Equation (3-30) may be solved for R_0 equal to R_5 , and the result is

$$R_0 = R_5 = \frac{1}{3b_0} \frac{K}{K}$$

$$R_5 = \frac{K}{3b_0}(4b_0 - a_0), \quad (3-31)$$

provided the coefficients of $Z(s)$ satisfy the equation

$$b_0(a_0 - b_0) = 3b_0a_1b_1 - b_1^2(4b_0 - a_0). \quad (3-32)$$

Consequently, if Eq. (3-32) is satisfied by the given function, then a resistance, R_5 , may be subtracted and the resulting function will be in the subclass of functions that satisfies the conditions in Eq. (2-9).

Case VI: $\sigma_1 = 0$, $4Z_1(0) = Z_1(\infty)$.

The requirement that $4Z_1(0)$ be equal to $Z_1(\infty)$ in terms of the coefficients of $Z_1(s)$ is

$$4a_0' = b_0. \quad (3-33)$$

Equation (3-33) substituted into Eq. (3-28) gives the result,

$$4a_0' = b_0 = 4a_1'b_1. \quad (3-34)$$

The solution of Eq. (3-34) in terms of R_0 equal to R_6 yields the equation,

$$R_6 = \frac{K}{3b_0}(4a_0 - b_0), \quad (3-35)$$

which is valid provided the coefficients of $Z(s)$ satisfy the relation

$$b_0(b_0 - a_0) = 3b_0a_1b_1 - b_1^2(4a_0 - b_0). \quad (3-36)$$

If the relation specified in Eq. (3-36) is satisfied, then the resistance R_6 may be subtracted from $Z(s)$ leaving a remainder which is in the subclass of functions that satisfies the conditions in Eq. (2-10).

In both Cases V and VI, in general, a six element

realization is possible, consisting of three reactive and three resistive elements, and are shown, together with corresponding element values, in Appendix A.

II. NECESSARY AND SUFFICIENT CONDITIONS ON THE COEFFICIENTS OF $Z(s)$, AND EXAMPLES OF THE REALIZATION TECHNIQUE

In the previous section, specific formulas for the amount of resistance required to be extracted from $Z(s)$ in order to satisfy a particular condition given by Eqs. (2-5) to (2-10) have been developed. It now remains to define the subclass of functions that will satisfy at least one of the conditions. The necessary and sufficient conditions on the coefficients of the given function, $Z(s)$, such that application of the proposed technique will yield a remainder function, $Z_1(s)$, that is in a favourable subclass, will be determined through positive real considerations of $Z_1(s)$, and by the restriction that the resistance to be extracted must be a positive real quantity.

As was stated earlier, to ensure the positive realness of $Z_1(s)$, the quantities a_1' , a_0' , and K' as given by Eqs. (3-3), (3-4), and (3-5), respectively, must be positive. Also, Eq. (3-6) must be satisfied by the coefficients of $Z_1(s)$.

Case I: $\sigma_1 = w_1$.

Since the use of negative resistance is to be avoided, R_1 , given by Eq. (3-15), must be positive. This consideration

tion gives rise to the following two sets of conditions,

$$\begin{aligned} \text{A. } a_0 + b_0 - a_1 b_1 &\geq 0, \\ 2b_0 - b_1^2 &> 0, \end{aligned} \quad (3-37)$$

and

$$\begin{aligned} \text{B. } a_0 + b_0 - a_1 b_1 &\leq 0, \\ 2b_0 - b_1^2 &\leq 0. \end{aligned} \quad (3-38)$$

Equation (3-15) substituted into Eq. (3-3) leads to the expression,

$$\frac{K(2b_0 - b_1^2) - K(a_0 + b_0 - a_1 b_1)}{2b_0 - b_1^2} > 0. \quad (3-39)$$

If the conditions in set A hold, the conditions in set B

$$a_1 b_1 - a_0 + b_0 > b_1^2, \quad (3-40)$$

is obtained from Eq. (3-39), but if the conditions in set B hold, then the condition

$$b_1^2 > a_1 b_1 - a_0 + b_0, \quad (3-41)$$

is obtained.

The substitution of Eq. (3-15) into Eq. (3-4) leads to the result,

$$a_1(2b_0 - b_1^2) > b_1(a_0 + b_0 - a_1 b_1), \quad (3-42)$$

providing the conditions in set A hold. If set B holds then the result is

$$a_1(b_1^2 - 2b_0) > b_1(a_1 b_1 - a_0 - b_0). \quad (3-43)$$

Similarly, Eq. (3-15) substituted into Eq. (3-5) yields the following inequalities,

$$b_0(a_1 b_1 + a_0 - b_0) > a_0 b_1^2, \quad (3-44)$$

if case A holds, or,

$$a_0(b_1^2 - a_0 - b_0) > b_0(a_1 b_1 + a_0 - b_0)$$

$$a_0 b_1^2 > (a_1 b_1 - a_0 - b_0) b_0 \quad (3-45)$$

if the conditions in case B hold.

It may be seen by substituting Eq. (3-15) into Eq. (3-6) that this inequality is automatically satisfied, consequently, no new restrictions on the coefficients of $Z(s)$ is introduced.

In summation, the necessary and sufficient conditions on the coefficients of $Z(s)$ such that application of the proposed method will yield a function $Z_1(s)$ which satisfies the condition σ_1 equal to w_1 are

$$\begin{aligned} \text{Case A: } & a_0 + b_0 - a_1 b_1 \geq 0, \\ & 2b_0 - b_1^2 > 0, \\ & a_1 b_1 - a_0 + b_0 > b_1^2, \\ & a_1(2b_0 - b_1^2) > b_1(a_0 + b_0 - a_1 b_1), \\ & b_0(a_1 b_1 + a_0 - b_0) > a_0 b_1^2, \end{aligned} \quad (3-46)$$

and

$$\begin{aligned} \text{Case B: } & a_1 b_1 - a_0 - b_0 \geq 0, \\ & b_1^2 - 2b_0 > 0, \\ & b_1^2 > a_1 b_1 - a_0 + b_0, \\ & a_1(b_1^2 - 2b_0) > b_1(a_1 b_1 - a_0 - b_0), \\ & a_0 b_1^2 > b_0(a_1 b_1 + a_0 - b_0). \end{aligned} \quad (3-47)$$

If the coefficients in either Set A or Set B are satisfied by the coefficients of the given function $Z(s)$, then a realization in the form of the network shown in Figure A-3(a) is possible.

Example of Case I. Consider the application of the

technique to the following biquadratic, driving-point impedance,

$$Z(s) = \frac{2(s^2 + 2.5s + 3)}{s^2 + 2s + 4}.$$

Inspection of the coefficients of $Z(s)$ shows that the conditions in Set A are valid, consequently, the network in Figure A-2(a) is applicable, if a resistance of the value R_1 is subtracted from $Z(s)$. The value of R_1 as calculated using Eq. (3-15) is

$$R_1 = 1 \text{ ohm.}$$

The remainder function, $Z_1(s)$, is

$$Z_1(s) = \frac{s^2 + 3s + 2}{s^2 + 2s + 4}.$$

Use of Eqs. (3-11) and (3-12) enable the calculation of σ_1 and w_1 to be made. The results are

$$\sigma_1 = 1.19, \quad w_1 = 1.19.$$

It may be seen that this satisfies the required condition. The value of s_1 ($s_1 = 1.19 + j1.19$), substituted into $Z_1(s)$ yields

$$Z_1(s_1) = 1.014 + j0.174.$$

To determine the element values in the realized network,

(1) Calculate the values of the multiplicative constants given in Figure A-2(a),

$$K_2 = 2, \quad K_3 = 1.$$

(2) Calculate the value of the parameters given in Eqs. (A-1) to (A-5),

Leaf 22 omitted.

$$- 2\sqrt{b_0(K - R_2)(a_0K - b_0R_2)} \leq a_1b_1K - b_1^2R_2.$$

Equation (3-17) substituted into the above expression produces the inequality,

$$a_0 \geq b_0. \quad (3-52)$$

By inspection of the five conditions derived above, it may be seen that the inequalities in Eqs. (3-51) and (3-52) are superfluous. Consequently, the necessary and sufficient conditions that the subtraction of R_2 , specified in Eq. (3-17), from $Z(s)$ will yield a new positive real function which satisfies the case, $\sigma_1 U_1 + w_1 V_1 = 0$, are

$$\begin{aligned} a_0 &> b_0, \\ a_1b_1 &\geq a_0 - b_0, \end{aligned}$$

and

$$b_1^2 > a_1b_1 - a_0 + b_0.$$

If this set of conditions is satisfied, then a realization as shown in Figure A-2(b) is possible.

Example of Case II. The impedance to be realized is

$$Z(s) = \frac{4(s^2 + s + 1.75)}{s^2 + s + 1.0}.$$

An inspection of the coefficients of $Z(s)$ indicates that the relations specified in Eqs. (3-48), (3-49), and (3-50) are satisfied. Consequently, a realization in the form of the network in Figure A-2(b) is positive.

The value of R_2 is found using Eq. (3-17), and the result is

$$R_2 = 1 \text{ ohm.}$$

The remainder function, $Z_1(s)$, is

$$\begin{aligned} Z_1(s) &= Z(s) - 1, \\ &= \frac{3(s^2 + s + 2)}{s^2 + s + 1}. \end{aligned}$$

The values of σ_1 and w_1 are found from Eqs. (3-11) and (3-12) and are as follows,

$$\sigma_1 = 0.455, \text{ and } w_1 = 1.098.$$

This value of s_1 ($s_1 = 0.455 + j 1.098$), substituted into $Z_1(s)$ gives

$$Z_1(s_1) = 3.29 - j1.37.$$

The values calculated for σ_1 , w_1 , U_1 , and V_1 , substituted into Eq. (2-6), show that this condition is satisfied. In order to determine the element values in the realized network, the following calculations are made.

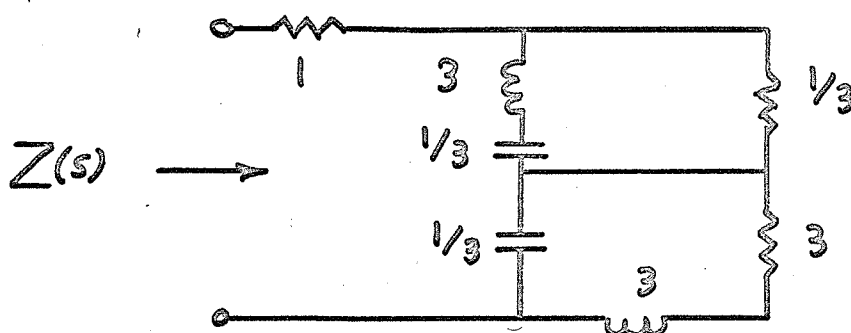


Figure 4. The realization of $Z(s)$ for Case II.

(1) Calculate the values of the multiplicative constants given in Figure A-2(b),

$$K_1 = 0.50, \quad K_3 = 0.50.$$

(2) Calculate the value of the parameters given in Eqs. (A-1) to (A-5),

$A = 0.50$, $B = 3.0$, $C = 1.0$, $D = 0.167$, $F = 6.0$, respectively. The network shown in Figure 4, with the corresponding element values, realizes $Z(s)$.

Case III: $\sigma_1 U_1 - w_1 V_1 = 0$, $V_1 > 0$.

The fact that R_3 , given by Eq. (3-19), must be positive leads to the condition,

$$a_1 b_1 \geq b_0 - a_0. \quad (3-53)$$

The substitution of the expression for R_3 into Eqs. (3-3), (3-4), (3-5), and (3-6) yields the inequalities shown below,

$$b_1^2 > a_1 b_1 + a_0 - b_0, \quad (3-54)$$

$$b_0 > a_0, \quad (3-55)$$

$$\text{and} \quad a_0 b_1^2 > b_0(a_1 b_1 + a_0 - b_0), \quad (3-56)$$

and

$$b_0 \geq a_0, \quad (3-57)$$

respectively. It may be seen that the conditions given in Eqs. (3-54) and (3-57) are redundant, consequently, the relations developed above may be reduced to the following set of conditions,

$$a_0 < b_0,$$

$$a_1 b_1 \geq b_0 - a_0,$$

and

$$a_0 b_1^2 > b_0(a_1 b_1 + a_0 - b_0).$$

This set of relations are the necessary and sufficient conditions that the subtraction of R_3 , specified in Eq. (3-19), from $Z(s)$ will yield a new positive real function which sat-

satisfies the case, $\sigma_1 U_1 - w_1 V_1 = 0$. The network shown in Figure A-2(c) will then realize the new function $Z_1(s)$.

Example of Case III. The given impedance to be realized is

$$Z(s) = \frac{2(s^2 + 1.5s + 2)}{s^2 + 2s + 3}.$$

Inspection of the coefficients indicate that the inequalities (3-53), (3-55), and (3-56) are satisfied, thus the network in Figure A-2(c) is applicable.

The value of R_3 is determined using Eq. (3-19), and the result is

$$R_3 = 1 \text{ ohm.}$$

The remainder function is

$$\begin{aligned} Z_1(s) &= Z(s) - 1, \\ &= \frac{s^2 + s + 1}{s^2 + 2s + 3}. \end{aligned}$$

The values of σ_1 and w_1 , calculated from Eqs. (3-11) and (3-12) are

$$\sigma_1 = 0.605, \text{ and } w_1 = 1.169.$$

The substitution of s_1 into $Z_1(s)$ yields

$$Z_1(s_1) = 0.478 + j0.248.$$

It may be noted that the values of σ_1 , w_1 , U_1 , and V_1 substituted into Eq. (2-7) show that this condition is satisfied. The following calculations are made in order to determine the element values of the network in Figure A-2(c).

(1) The values of the multiplicative constants in Fig-

ure A-2(c) are

$$K_1 = 1.0, K_2 = 2.0.$$

(2) The values of the parameters given in Eqs. (A-1) to (A-5) are

$A = 1.0$, $B = 0.33$, $C = 0.33$, $D = 2.0$, $F(s) = 0.33$, respectively. The network shown in Figure 5 will realize the given function $Z(s)$.

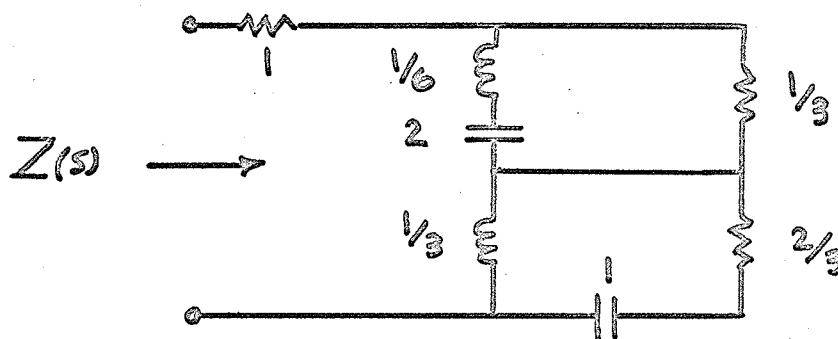


Figure 5. The realization of $Z(s)$ for Case III.

Case IV: $V_1 = 0$.

The necessary and sufficient conditions for this case were developed earlier in this study. These are contained in Eq. (3-22), Eqs. (3-25) and (3-26), and Eqs. (3-27a) and (3-27b). However, for these conditions, realizations using a smaller number of elements are possible. Consequently, although this case can be obtained, no advantage is offered by doing so.

Case V: $\sigma_1 = 0$, $Z_1(0) = 4Z_1(\infty)$.

Since the resistance R_5 , given by Eq. (3-31), must be positive the following condition applies,

$$4b_0 \geq a_0. \quad (3-58)$$

Equation (3-31) substituted into Eqs. (3-3) to (3-6) yields the relations

$$a_0 > b_0, \quad (3-59)$$

and

$$b_0(a_0 - b_0) = 3b_0a_1b_1 - b_1^2(4b_0 - a_0). \quad (3-60)$$

The relations contained in Eqs. (3-58) to (3-60) represent the necessary and sufficient conditions that R_5 , subtracted from $Z(s)$, will yield a new function, $Z_1(s)$, which satisfies Case V. The network that realizes $Z_1(s)$ is given in Figure A-2(e). An example problem is worked out below.

Example of Case V. The impedance to be realized is

$$Z(s) = \frac{3(s^2 + 2s + 6)}{s + 2s + 4}.$$

A study of the relations contained in Eqs. (3-58) to (3-60) indicates that a realization in the form of Figure A-2(e) is possible. From Eq. (3-31), the resistance that must be subtracted is

$$R_5 = 2.5 \text{ ohms.}$$

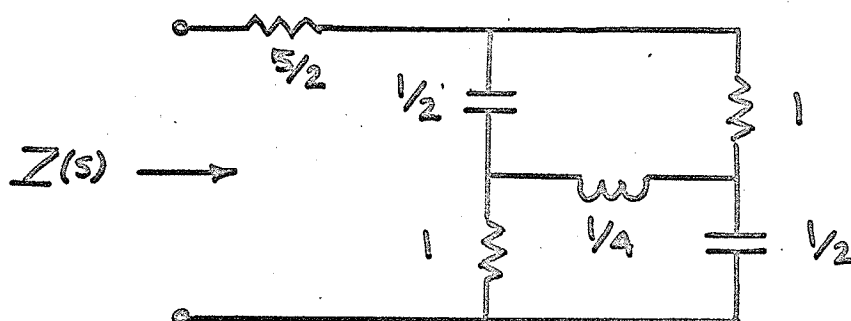


Figure 6. The realization of $Z(s)$ for Case V.

The remainder function, $Z_1(s)$, is

$$Z_1(s) = \frac{0.5(s^2 + 2s + 16)}{(s^2 + 2s + 4)}.$$

The resulting realization is the resistor R_5 in series with a five element minimum resistance bridge and is given, along with the appropriate element values, in Figure 6.

Case VI: $\sigma_1 = 0$, $Z_1(\infty) = 4 Z_1(0)$.

Consideration of the fact that R_6 , given by Eq. (3-35), must be positive leads to the relation,

$$4a_0 \geq b_0. \quad (3-61)$$

The substitution of Eq. (3-35) into Eqs. (3-3) to (3-6) yields the relations,

$$b_0 > a_0, \quad (3-62)$$

and

$$b_0(b_0 - a_0) = 3b_0a_1b_1 - b_1^2(4a_0 - b_0). \quad (3-63)$$

Equations (3-61) to (3-63) are the necessary and sufficient conditions that the subtraction of R_6 from $Z(s)$ will yield a new function which satisfies Case VI. The network shown in Figure A-2(f) realizes $Z(s)$.

Example of Case VI. The impedance to be realized is

$$Z(s) = \frac{2(s^2 + s5/6 + 1)}{s^2 + 2s + 2}.$$

Since the relations in Eqs. (3-61) to (3-63) are satisfied, a realization of the type in Figure A-2(f) is possible.

From Eq. (3-35), the resistance that must be extracted is

$$R_6 = 2/3 \text{ ohms.}$$

The remainder function is

$$Z_1(s) = \frac{4}{3} \frac{(s^2 + 0.25s + 0.50)}{s^2 + 2s + 2}.$$

The realization that results is the resistor R_6 in series with a minimum resistance five element bridge and is shown in Figure 7.

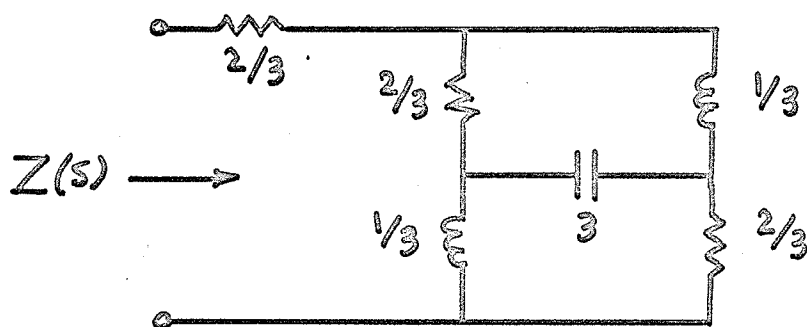


Figure 7. The realization of $Z(s)$ for Case VI.

CHAPTER VI

DISCUSSIONS AND CONCLUSIONS

The development of the proposed technique consisted of an attempt to the conditions specified in six cases, Eqs. (2-5) to (2-10), by subtracting, from a given biquadratic driving-point impedance, $Z(s)$, a resistance of a suitable value such that a new positive real function, $Z_1(s)$, is produced whose even part zero locations satisfies the conditions specified in one of the cases. Associated with each of the six cases is a network which will realize $Z_1(s)$ without transformers or gyrators and with an economical number of elements. The realization of $Z(s)$ then consists of the extracted resistance R_0 in series with the network representing $Z_1(s)$. In general, if Case I to VI are used, seven elements are required, and if Cases V or VI are used, a six element realization is needed.

In determining the necessary and sufficient conditions on the coefficients of the given nonminimum function such that subtraction of the appropriate resistance yields a $Z_1(s)$ which satisfies one of the Cases I to VI, it was found that the entire subclass of functions which could be realized by Case IV, $V_1 = 0$, could also be realized more economically by other networks. Case IV realizes this subclass of functions with seven elements (six if the equalities in Eqs. (3-22), (3-26), and (3-27) are satisfied), while the other networks,

as described in Chapter III and Appendix D, realize the same subclass with five elements (four if the equalities in Eqs. (3-22), (3-26), and (3-27) are satisfied). This results in a saving of two elements over the entire subclass. Consequently, application of the proposed method on this group of functions offers no advantage.

Furthermore, it may be noted, upon investigation of the necessary and sufficient conditions required for Case I, $o_1 = w_1$, that this case synthesizes a group of functions which is covered completely by Cases II, III, and IV. Therefore, Case I is superfluous, and may be discarded.

A comparison of the number of elements required by several existing techniques and the proposed method is given in Table I. None of the techniques listed require ideal transformers or gyrators.

TABLE I

COMPARISON OF THE PROPOSED TECHNIQUE WITH EXISTING METHODS
ON THE BASIS OF THE NUMBER OF ELEMENTS REQUIRED

Method	Number of elements required in general
Bott-Duffin	9
Modified Bott-Duffin	8
Miyata $n + 1$ ladder breakdown	9
Miyata tree breakdown	7
Proposed method	7

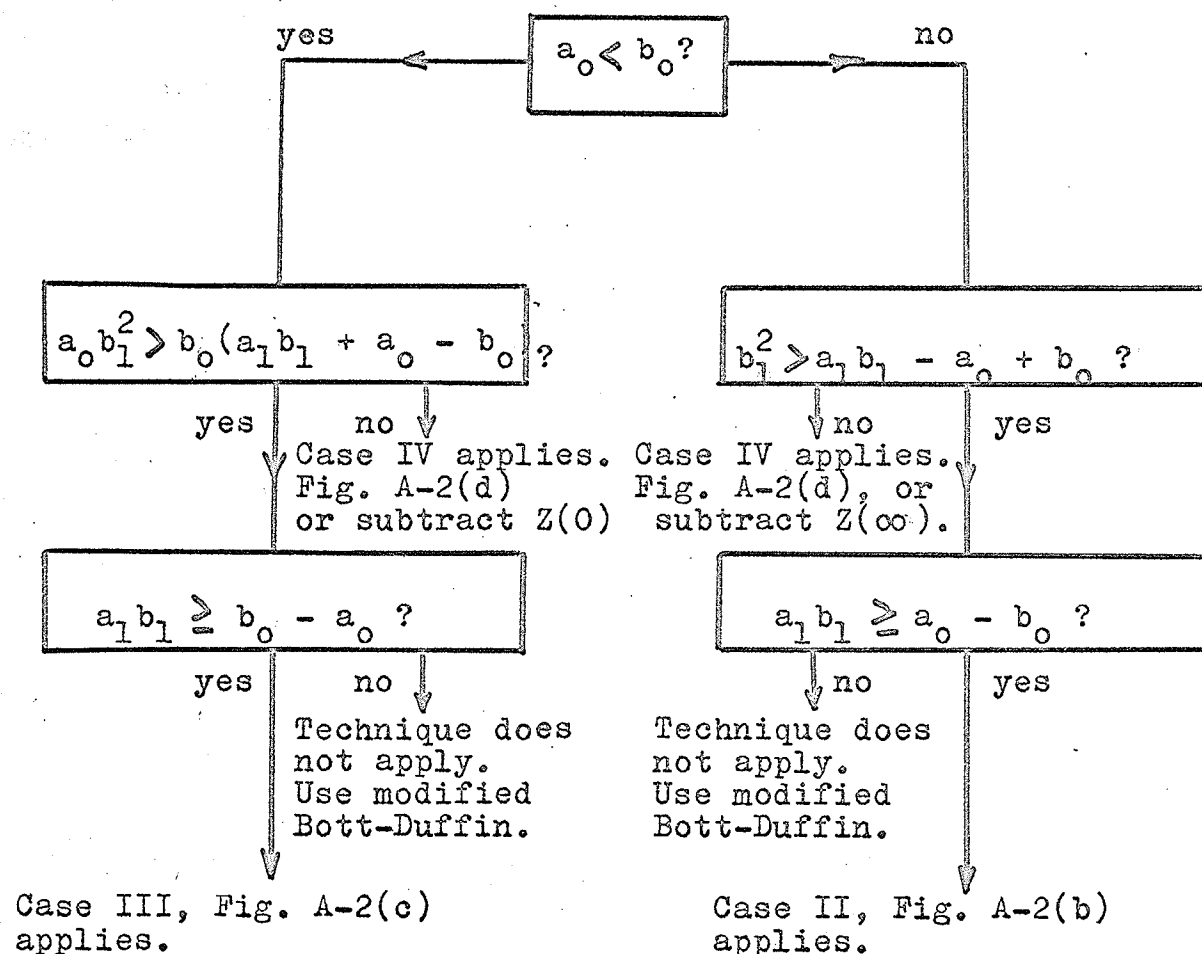
On the basis of Table I, it is evident that the new technique is economical with respect to the number of elements required. However, as is evident in the flow chart given in Figure 8,

the proposed technique will not cover all the biquadratic functions. This method, along with the Miyata technique, lacks the generality of application which the Bott and Duffin procedure possesses. In spite of this drawback, it may be concluded that when the new procedure applies, a simple realization results with an economical number of elements.

SUGGESTIONS FOR FURTHER STUDIES

The obvious suggestion is to attempt to generalize the method to include all positive real biquadratic functions and possibly extend the method to cover functions of higher degrees. The extension to functions of higher degree may possibly be carried out by some combination of this method with Kim's transformerless zero removal synthesis (5).

Figure 8. Flow chart of the synthesis procedure as applied to the general biquadratic driving-point impedance, given by Eq. (2-1)



Special Cases:

(1) $a_0 = b_0$, simple four element realizations possible.

(2) If, $a_0 < b_0 \leq 4a_0$,

Or if, $b_0 < a_0 \leq 4b_0$,

$b(b_0 - a_0) = 3b_0 a_1 b_1 - b_1^2(4a_0 - b_0)$, $b_0(a_0 - b_0) = 3b_0 a_1 b_1 - b_1^2(4b_0 - a_0)$
 then Case VI, Fig. A-2(f), applies.
 then Case V, Fig. A-2(e), applies.

APPENDICES

APPENDIX A

NETWORK REALIZATIONS ASSOCIATED WITH EACH CASE

The following results are taken from a paper written by Kim (8), and are reproduced here for reference purposes. The general network is shown in Figure A-1.

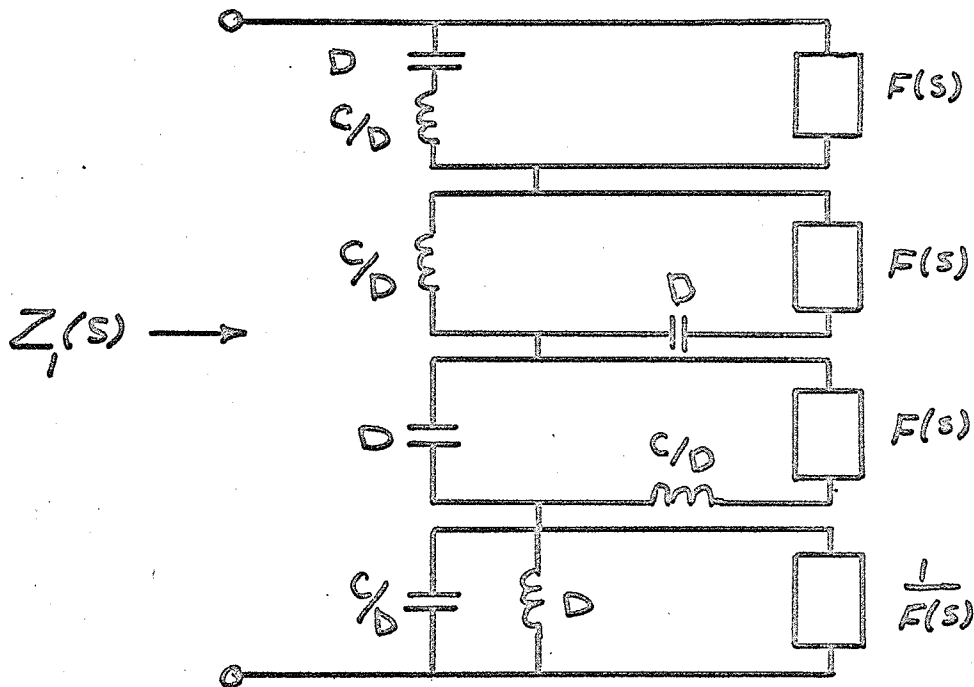


Figure A-1. The general network as derived by Kim.

The parameters shown in the network of Figure A-1 are defined as follows,

$$A = \frac{w_1 U_1 + \sigma_1 V_1}{(\sigma_1^2 + w_1^2)(w_1 U_1 - \sigma_1 V_1)}, \quad (A-1)$$

$$B = \frac{2\sigma_1 w_1 (U_1^2 + V_1^2)}{(\sigma_1^2 + w_1^2)(w_1 U_1 + \sigma_1 V_1)}, \quad (A-2)$$

$$C = \frac{w_1 U_1 - \sigma_1 V_1}{(\sigma_1^2 + w_1^2)(w_1 U_1 + \sigma_1 V_1)}, \quad (A-3)$$

$$D = \frac{2\sigma_1 w_1}{(\sigma_1^2 + w_1^2)(w_1 U_1 - \sigma_1 V_1)} \quad (\text{A-4})$$

The term $F(s)$ is given by

$$F(s) = \frac{(Cs^2 + 1)Z_1(s) - Bs}{(As^2 + 1) - DsZ_1(s)} \quad (\text{A-5})$$

Hazony (4) states that $F(s)$ is a positive real function with a rank that is four less than that of $Z_1(s)$. Since the $Z_1(s)$ under consideration is a biquadratic, the $F(s)$ in this case will be a constant and may be realized as a resistor.

The introduction of a positive multiplicative constant, K_1 , into the network of Figure A-1 can be carried out in such a way that one of the series connections may be removed. This yields a realization consisting of three series connections with the impedance level of each of the series networks raised by its corresponding K . These constants are such that if the even part zeros of $Z_1(s)$ are suitable located, then one of the constants will be zero. This leads to the cases given in Eqs. (2-5) to (2-8), with a corresponding network consisting of two of the series connections. In general, this results in a six element realization for $Z_1(s)$. The resulting networks are shown in Figure A-2(a), (b), (c), and (d).

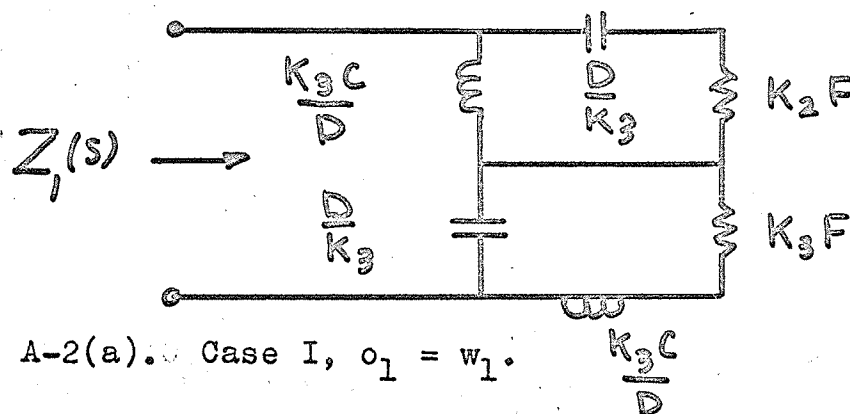


Figure A-2(a). Case I, $\sigma_1 = w_1$.

The multiplicative constants in Figure A-2(a) are

$$K_2 = \frac{(U_1 + V_1)^2}{(U_1 - V_1)^2}, \text{ and } K_3 = 1. \quad (\text{A-6})$$

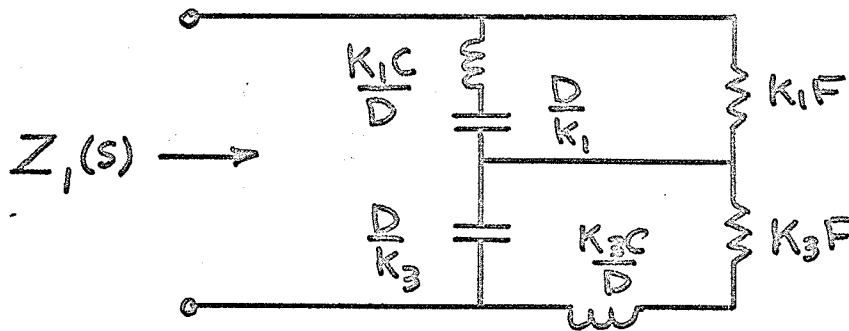


Figure A-2(b). Case II, the condition $\sigma_1 U_1 + w_1 V_1 = 0$, $V_1 < 0$ is satisfied.

The multiplicative constants in Figure A-2(b) are

$$K_3 = \frac{2U_1 D(-V_1)}{(w_1 U_1 + \sigma_1 V_1)C}, \text{ and } K_1 = A/C. \quad (\text{A-7})$$

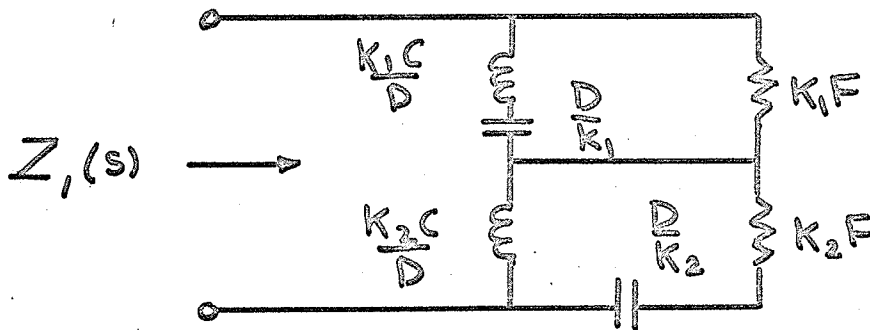


Figure A-2(c). Case III, the condition $\sigma_1 U_1 - w_1 V_1 = 0$, $V_1 > 0$, is satisfied by $Z_1(s)$.

The multiplicative constants in Figure A-2(c) are

$$K_2 = \frac{2U_1 D}{C(w_1 V_1 + \sigma_1 V_1)}, \text{ and } K_1 = 1. \quad (\text{A-8})$$

The multiplicative constants in Figure A-2(d) are given by

$$K_1 = 1, \text{ and } K_4 = U_1^2.$$

(A-9)

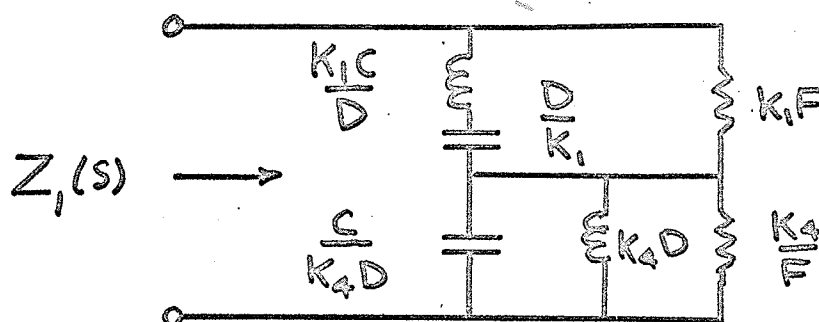


Figure A-2(d). Case IV, the condition $V_1 = 0$ is satisfied by $Z_1(s)$.

The networks that realize the cases given in Eqs. (2-9) and (2-10) are taken from a paper by Seshu (9), and are shown in Figures A-2(e) and (f).

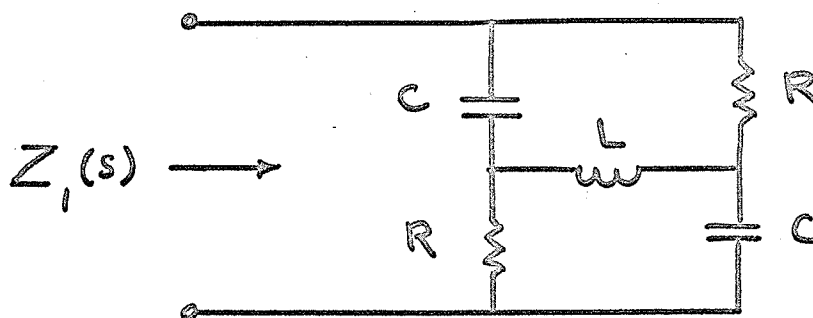


Figure A-2(e). Case V, the condition $\sigma_1 = 0$, $Z_1(0) = 4Z_1(\infty)$ is satisfied.

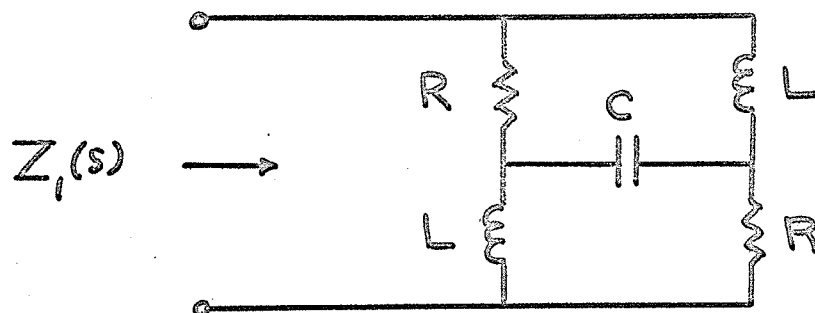
The element values in Figure A-2(e) are

$$R = 2K, C = \frac{1}{2a_1}K, \text{ and } L = \frac{4a_1 K}{a_0}. \quad (\text{A-10})$$

The element values in Figure A-2(f) are

$$R = K/2, C = \frac{1}{a_1}K, \text{ and } L = \frac{a_1 K}{a_0}. \quad (\text{A-11})$$

Figure A-2(f). Case VI, the condition $\sigma_1 = 0$, $4Z_1(0) = Z_1(\infty)$ is satisfied.



APPENDIX B

THE DEVELOPMENT OF THE EXPRESSION FOR R_2

In this Appendix, the development of Eq. (3-17), the expression for R_2 , is discussed.

Equation (3-16) is rewritten in the following form,

$$R_2 = \frac{Kw_1^2(2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0) - Kw_1^2(2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_0) + K\sigma_1(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_0)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0) + K\sigma_1w_1^2(2\sigma_1 + a_1)(2\sigma_1 + b_1)}{\sigma_1(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0)^2 + \sigma_1w_1^2(2\sigma_1 + b_1)^2} \quad (B-1)$$

Equation (B-2) is derived from Eq. (B-1) by multiplying out the terms in Eq. (B-1) and cancelling where possible,

$$\frac{R_2 D_0}{K} = a_0 b_0 \sigma_1 + (\sigma_1^2 + w_1^2)^2 \sigma_1 + a_1 b_1 \sigma_1 (\sigma_1^2 + w_1^2) + \sigma_1^3 (a_0 + b_0) + \sigma_1 w_1^2 (b_0 - 3a_0) + b_1 (\sigma_1^2 + w_1^2)^2 + a_1 (\sigma_1^4 - w_1^4) + a_1 b_0 (\sigma_1^2 + w_1^2) + a_0 b_1 (\sigma_1^2 - w_1^2). \quad (B-2)$$

The quantity $\frac{R_2 D_0}{K}$ is set equal to the denominator of Eq. (B-1).

The expressions for σ_1 and w_1 , Eqs. (3-11) and (3-12), respectively, may be substituted into Eq. (B-2). The resulting equation is

$$\sigma_1 \left[2b_0(a_0 K - b_0 R_2) + b_1(a_1 K - b_1 R_2) \sqrt{\frac{b_0(a_0 K - b_0 R_2)}{K - R_2}} + K(b_0 - a_0) \sqrt{\frac{b_0(a_0 K - b_0 R_2)}{K - R_2}} - (a_0 K - b_0 R_2) \left\{ \frac{K(a_0 + b_0 - a_1 b_1) - R_2(2b_0 - b_1^2)}{K - R_2} \right\} \right]$$

$$\begin{aligned}
& + b_1 b_0 (a_0 K - b_0 R_2) + b_0 (a_1 K - b_1 R_2) \sqrt{\frac{b_0 (a_0 K - b_0 R_2)}{K - R_2}} \\
& - b_1 (a_0 K - b_0 R_2) \left[\frac{K(a_0 + b_0 - a_1 b_1) - R_2(2b_0 - b_1^2)}{2(K - R_2)} \right] \\
& - (a_1 K - b_1 R_2) \left[\frac{K(a_0 + b_0 - a_1 b_1) - R_2(2b_0 - b_1^2)}{K - R_2} \right] \sqrt{\frac{b_0 (a_0 K - b_0 R_2)}{4(K - R_2)}} \\
& = 0. \tag{B-3}
\end{aligned}$$

After performing some algebraic manipulation on Eq. (B-3), it becomes apparent that the term

$$K(a_1 b_1 - a_0 + b_0) - b_1^2 R_2$$

may be factored out of Eq. (B-3), leading to the equation,

$$\begin{aligned}
& \left[K(a_1 b_1 - a_0 + b_0) - b_1^2 R_2 \right] \left[2\sigma_1 \sqrt{\frac{b_0 (a_0 K - b_0 R_2)}{4(K - R_2)}} \right. \\
& \left. + \frac{\sigma_1 (a_0 K - b_0 R_2)}{K - R_2} + \frac{(a_1 K - b_1 R_2)}{K - R_2} + \frac{(a_0 K - b_0 R_2)}{K - R_2} \right] = 0. \tag{B-4}
\end{aligned}$$

From Eqs. (3-3), (3-4), and (3-5), it may be seen that the quantities $K - R_2$, $a_0 K - b_0 R_2$, and $a_1 K - b_1 R_2$ are positive, also σ_1 is positive, by definition. Consequently, Eq. (B-4) can be satisfied only if,

$$K(a_1 b_1 - a_0 + b_0) - b_1^2 R_2 = 0. \tag{B-5}$$

The solution of Eq. (B-5) for R_2 yields the desired result,

$$R_2 = \frac{K(a_1 b_1 - a_0 + b_0)}{b_1^2}. \tag{B-6}$$

APPENDIX C

THE DEVELOPMENT OF THE EXPRESSION FOR R_3

In this Appendix, a development of Eq. (3-19) from the expression in Eq. (3-18) is presented.

Equation (3-18) is rewritten in the following form,

$$R_3 = \frac{K\sigma_1(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_0)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0) + K\sigma_1 w_1^2(2\sigma_1 - a_1)(2\sigma_1 - b_1) - Kw_1^2(2\sigma_1 + a_1)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0) + Kw_1^2(2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a_1\sigma_1 + a_0)}{\sigma_1(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0)^2 + w_1^2\sigma_1(2\sigma_1 + b_1)^2} \quad (C-1)$$

Equation (C-2) may be derived from Eq. (C-1) by multiplying out and cancelling terms where possible in Eq. (C-1). The result is

$$R_3\sigma_1(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0)^2 + R_3\sigma_1 w_1^2(2\sigma_1 + b_1)^2 = + K[a_1 b_1 \sigma_1(\sigma_1^2 + w_1^2) + a_0 b_0 \sigma_1 + a_0 b_1(\sigma_1^2 + w_1^2) + a_1 b_0(\sigma_1^2 - w_1^2) + \sigma_1(\sigma_1^2 - w_1^2)(a_0 + b_0) + a_1(\sigma_1^2 + w_1^2)^2 + \sigma_1(\sigma_1^2 + w_1^2)^2 + b_1(\sigma_1^2 - w_1^2)(\sigma_1^2 + w_1^2) + 2\sigma_1 w_1^2(a_0 - b_0)] \quad (C-2)$$

The substitution of the expression for σ_1 and w_1 , given by Eqs. (3-11) and (3-12), respectively, into Eq. (C-2) yields the following result after some manipulation,

$$2\sigma_1 b_1(b_1 R_3 - a_1 K) \sqrt{\frac{b_0(a_0 K - b_0 R_3)}{4(K - R_3)}} - 2b_0(a_0 K - b_0 R_3)\sigma_1 - \sigma_1 [b_0 K(a_0 + b_0 - a_1 b_1) - R_3 b_0(2b_0 - b_1^2)] - 2K(a_0 - b_0) \sqrt{\frac{b_0(a_0 K - b_0 R_3)}{4(K - R_3)}}$$

$$\begin{aligned}
&= (a_1 K - b_1 R_3) b_0 \frac{(a_0 K - b_0 R_3)}{K - R_3} \\
&+ b_1 (a_0 K - b_0 R_3) \sqrt{\frac{b_0 (a_0 K - b_0 R_3)}{K - R_3}} \\
&- b_1 \left[K(a_0 + b_0 - a_1 b_1) - R_3(2b_0 - b_1^2) \right] \sqrt{\frac{b_0 (a_0 K - b_0 R_3)}{4(K - R_3)}} \\
&- b_0 (a_1 K - b_1 R_3) \left[\frac{K(a_0 - b_0 - a_1 b_1) - R_3(2b_0 - b_1^2)}{2(K - R_3)} \right] \quad (C-3)
\end{aligned}$$

It becomes evident, after some algebraic manipulations, that the expression

$$\begin{aligned}
&K(a_1 b_1 + a_0 - b_0) - R_3 b_1^2, \\
&\text{may be factored out of Eq. (C-3). The resulting equation is} \\
&\left[K(a_1 b_1 + a_0 - b_0) - R_3 b_1^2 \right] \left[\sigma_1 \sqrt{\frac{b_0 (a_0 K - b_0 R_3)}{K - R_3}} + \sigma_1 b_0 \right. \\
&\left. + b_1 \sqrt{\frac{b_0 (a_0 K - b_0 R_3)}{K - R_3}} + b_0 \frac{(a_1 K - b_1 R_3)}{2(K - R_3)} \right] = 0. \quad (C-4)
\end{aligned}$$

Since σ_1 is positive, and because of the positive real requirement on the new function $Z_1(s)$ which causes the quantities $K - R_3$, $a_0 K - b_0 R_3$, and $a_1 K - b_1 R_3$ to be positive, Eq. (C-4) can only be satisfied if the following equation holds,

$$K(a_1 b_1 + a_0 - b_0) - R_3 b_1^2 = 0. \quad (C-5)$$

The solution of Eq. (C-5) yields the desired expression for the resistance R_3 ,

$$R_3 = \frac{K(a_1 b_1 + a_0 - b_0)}{b_1^2} \quad (C-6)$$

APPENDIX D

THE MINIMUM RESISTANCE OF A BIQUADRATIC IMPEDANCE FUNCTION

In the discussion of the proposed method, a simple technique for determining an expression for the minimum resistance of a biquadratic impedance was found.

The minimum resistance of an impedance function is defined as the smallest value that the even part of the function along the imaginary axis in the complex s-plane. The removal of this resistance shifts a set of quadrantal zeros of the even part to the imaginary axis. The even part zeros of a biquadratic minimum function would appear as in Figure D-1.

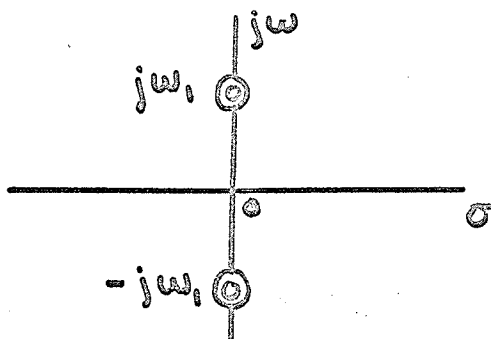


Figure D-1. The even part zeros of a minimum biquadratic function.

From Figure D-1, it may be seen that σ_1 , the real part of the even part zero in the first quadrant, is zero. Equation (3-11), the expression for σ_1 , set to zero yields the result,

$$R_m^2 b_1^2 (b_1^2 - 4b_0) - 2KR_m [b_1^2 (a_0 - b_0) - a_1 b_1 (b_1^2 - 2b_0)] + K^2 (a_0 - b_0)^2 + a_1 b_1 K^2 (a_1 b_1 - 2a_0 - 2b_0) = 0. \quad (D-1)$$

Equation (D-1) solved for R_m , the minimum resistance, yields

$$R_m = \frac{K}{b_1(4b_0 - b_1^2)} \left[b_1(a_0 + b_0) + a_1(2b_0 - b_1^2) - 2\sqrt{b_0(a_0 - b_0)^2 + b_0(a_1 - b_1)(a_0b_1 - a_1b_0)} \right]. \quad (D-2)$$

If the term $b_1^2 - 4b_0$ equals zero, then the expression for R_m reduces to

$$R_m = K \frac{[(a_0 - b_0)^2 + a_1b_1(a_1b_1 - 2a_0 - 2b_0)]}{b_1^2(a_1b_1 - 2a_0 - 2b_0)} \quad (D-3)$$

The above expressions for R_m are valid only when the minimum real part of $Z(jw)$ occurs at a finite, non-zero value of w along the jw -axis. Kuh (7) found that the minimum real part of $Z(jw)$ is found at infinite w if the following conditions are satisfied,

$$a_0 > b_0, \quad (D-4)$$

$$a_1b_1 \geq (a_0 - b_0) + b_1^2.$$

In this case the minimum resistance would be

$$R_m = Z(\infty) = K. \quad (D-5)$$

If the following relations are satisfied,

$$a_0 < b_0, \quad (D-6)$$

$$a_1b_1 \geq (b_0 - a_0) + a_0b_1^2/b_0$$

Then the minimum real part occurs at zero frequency. The minimum resistance would be in this case,

$$R_m = Z(0) = Ka_0/b_0. \quad (D-7)$$

It may be noted, that if a biquadratic impedance satisfies either the conditions in Eq. (D-4) or those in Eq.

(D-5), then, in general, a five element (four if the equality is satisfied) realization is possible. The realization procedure, in this case, starts with the removal of either $Z(0)$ or $Z(\infty)$ as a series resistance and may be completed by identification of components.

APPENDIX E

DISCUSSION OF CASE IV: $V_1 = 0$

The development of Eq. (3-22) from Eq. (3-20) is discussed.

The imaginary part of $Z_1(s_1)$ may be written as shown below,

$$V_1 = w_1 K' \frac{[(2\sigma_1 + a'_1)(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0) - (2\sigma_1 + b_1)(\sigma_1^2 - w_1^2 + a'_1\sigma_1 + a'_0)]}{(\sigma_1^2 - w_1^2 + b_1\sigma_1 + b_0)^2 + w_1^2(2\sigma_1 + b_1)^2} \quad (E-1)$$

If the expressions for K' , a'_1 , and a'_0 , as given by Eqs. (3-5), (3-3), and (3-4), respectively, are substituted into Eq. (E-1) then it becomes identical to Eq. (3-14). Since the condition V_1 equal to zero is desired, the numerator of the expression in Eq. (E-1) is set to zero. This leads to the relation,

$$2\sigma_1(b_0 - a'_0) + (\sigma_1^2 + w_1^2)(b_1 - a'_1) + a'_1 b_0 - a'_0 b_1 = 0. \quad (E-2)$$

Equation (E-2) may be rewritten as follows,

$$2\sigma_1(b_0 - a'_0) + a'_1(b_0 - \sqrt{a'_0 b_0}) + b_1(\sqrt{a'_0 b_0} - a'_0) = 0. \quad (E-3)$$

In order to see if a solution to Eq. (E-3) exists, it is examined for the following three cases,

$$(1) \quad b_0 > a'_0,$$

$$(2) \quad b_0 < a'_0,$$

and

$$(3) \quad b_0 = a'_0.$$

In the first case, it may be seen that for the specified condition, all three terms of the left hand side of Eq.

(E-3) are positive. As a result, no solution is possible.

In the second case, all three terms are negative for the specified condition. Therefore, no value of R_0 can be found which will satisfy Eq. (E-3).

However, if the condition $b_0 = a'_0$ is satisfied, then the left hand side of Eq. (E-3) becomes zero, with the result that it is satisfied identically.

The substitution of a'_0 , given by Eq. (3-4), into cases (1), (2), and (3) yields the results,

$$(a) \quad b_0 > a_0,$$

$$(b) \quad b_0 < a_0,$$

and

$$(c) \quad b_0 = a_0,$$

respectively. It follows from this, that unless the given function has the coefficients b_0 and a_0 equal, Eq. (E-3) is not satisfied. There is no value of R_0 which can be extracted with the intent of making the two coefficients equal each other. Consequently, if a function has a_0 and b_0 equal, it automatically has V_1 equal to zero. A sufficient condition that V_1 be equal to zero is

$$a_0 = b_0. \tag{E-4}$$

This proves Eq. (3-22).

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