AMENABILITY PROPERTIES OF BANACH ALGEBRA OF BANACH ALGEBRA-VALUED CONTINUOUS FUNCTIONS

by

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Abstract

In this thesis we discuss amenability properties of the Banach algebra-valued continuous functions on a compact Hausdorff space X. Let A be a Banach algebra. The space of A-valued continuous functions on X, denoted by $C(X, A)$, form a new Banach algebra. We show that $C(X, A)$ has a bounded approximate diagonal (i.e. it is amenable) if and only if A has a bounded approximate diagonal. We also show that if A has a compactly central approximate diagonal then $C(X, A)$ has a compact approximate diagonal. We note that, unlike $C(X)$, in general $C(X, A)$ is not a C^*-algebra, and is no longer commutative if A is not so. Our method is inspired by a work of M. Abtahi and Y. Zhang. In addition to the above investigation, we directly construct a bounded approximate diagonal for $C_0(X)$, the Banach algebra of the closure of compactly supported continuous functions on a locally compact Hausdorff space X.

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TABLE OF CONTENTS

Chapter 1

INTRODUCTION

The notion of amenability for Banach algebras was first introduced by B.E. Johnson in 1972 in [15]. Amenability can be thought of as a fitness condition. As an example, a locally compact group G is finite if and only if every closed cofinite dimensional ideal of $L^1(G)$ has an identity, but is amenable if and only if every such ideal has a bounded approximate identity. All finite dimensional semisimple Banach algebras A have been known to satisfy $H^1(A, X) = \{0\}$ for each Banach A-bimodule X, where $H^1(A, X)$ is the first cohomology group of A with coefficients in X. The Banach algebras A for which $H^1(A, X) = \{0\}$ for each Banach A-bimodule X are called contractible. All contractible Banach algebras constructed so far are finite dimensional and it is still an open problem whether there exists an infinite dimensional contractible Banach algebra. A Banach algebra A is defined to be amenable if $H^1(A, X^*) = \{0\}$ for each Banach A-bimodule X, where X^* is the dual module of X. Although the existence of infinite dimensional contractible Banach algebras is still an open problem, there are many infinite dimensional amenable Banach algebras (see [20], [22]).

In [15] B. E. Johnson defined the notion of homology and cohomology for a Banach algebra A with coefficients in a A-bimodule X and he investigated some of the relations between these two notions. In the same paper, B.E. Johnson considered the case, $A = L^1(G)$ where G is a locally compact topological group and he showed how the cohomology of $L^1(G)$ and the amenability of G are related. Johnson showed that the first cohomology of $L^1(G)$ with coefficients in X^* is zero for each Banach A-bimodule X, if and only if G is an amenable group. We recall that a group G is said to be amenable if there exists an invariant mean on G, i.e. a positive linear functional μ on $L^\infty(G)$ such that $\|\mu\| = \mu(1) = 1$, and $\mu(T_h m) = \mu(m)$, for $m \in L^\infty(G)$, $h \in G$, where T_h is the left translation operator on $L^\infty(G)$ defined by $(T_h m)(g) = m(h^{-1}g)$, for $m \in L^\infty(G)$, $h \in G$.

Due to the relation between amenability of locally compact topological groups and the first cohomology group of $L^1(G)$ with coefficients in X^*, the notion of amenability transferred from groups to algebras. In [14], B. Johnson showed further that a Banach algebra is amenable if and only if it has a bounded approximate diagonal.

Since the notion of amenability was considered, several generalizations of this concept have been introduced in [7], [8], [9]. We thus have notion of approximate amenability, essential amenability, uniform amenability, sequential amenability, weak amenability, weak*-approximate amenability, pseudo-amenability, etc for Banach algebras. Some significant relations between different generalized notions of amenability and equivalent conditions for these notions can be found in the above papers. Examples can be found in [7], [8], [5], [9].

In particular, in [21], the amenability property of $C(X)$ was discussed. It is shown that

$C(X)$ is amenable if X is a non-empty, compact Hausdorff space. Sheinberg proved that the converse is also true, namely every unital, amenable uniform algebra should be of the form of $C(X)$ for some non-empty compact Hausdorff space X; the proof is based on the theorem of B.E. Johnson about amenability of a group and its corresponding group algebra $\ell^1(G)$ and the Weierstrass theorem.

In 2010, M. Abtahi and Y. Zhang constructed a bounded approximate diagonal for $C(X)$ in [1], where X is a non-empty compact space, providing a direct proof of amenability for $C(X)$. The advantage of the direct proof presented by M. Abtahi and Y. Zhang is that it can be generalised to the case of Banach algebra-valued functions. In this thesis we will mainly be concerned with obtaining this generalization.

We show that some amenability properties of $C(X, A)$ are inherited from those of A. In particular, we show that $C(X, A)$ is amenable if and only if A is so. In our proofs, we don't assume A is commutative, unital or a C^*-algebra.

We should also observe that the original proof for $C(X)$ being amenable for a non-empty compact space X does not seem easily extendable to the case $C(X, A)$.

Chapter 2

PRELIMINARIES

One of the important algebraic objects that has the ring and vector space structure at the same time is an algebra. Since an algebra is a vector space, one can define norm on it and if so, it will be called a normed algebra. A normed algebra which is complete is called a Banach algebra. We recall some Banach algebra notions with some examples in the following:

Through out this thesis the symbol $\mathbb{F}$ will be used to denote a field that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

An algebra $(A, +, \cdot)$ over $\mathbb{F}$ is a linear space over $\mathbb{F}$ together with a mapping $(a, b) \to a \cdot b$ of $A \times A$ into A that satisfies the following axioms:

(i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$; (associativity)

(ii) $a \cdot (b + c) = a \cdot b + a \cdot c,\ (a + b) \cdot c = a \cdot c + b \cdot c$; (distribution)

(iii) $(\alpha a) \cdot b = \alpha(a \cdot b) = a \cdot (\alpha b)$,

for all $a, b, c \in A, \alpha \in \mathbb{F}$. The mapping $(a, b) \mapsto a \cdot b$, is called the product in A. The algebra A is called commutative if $a \cdot b = b \cdot a$ for all $a, b \in A$. For convenience $a \cdot b$ is often denoted by ab.

For example $\mathbb{R}$, the set of all real numbers and $\mathbb{C}$, the set of all complex numbers, with the natural addition, multiplication and scalar product, are algebras over $\mathbb{R}$. Let x be an indeterminate and $\mathbb{F}$ be an algebra. The set $\mathbb{F}[x]$ of all polynomials of any order in terms of x with coefficients in $\mathbb{F}$, with the natural addition, multiplication and scalar product, is an algebra over $\mathbb{F}$. Let X be a linear space over $\mathbb{F}$. The space of all linear mapping of X into X with pointwise addition, scalar multiplication and composition product is an algebra over $\mathbb{F}$.

Let E be a set, A an algebra over a field $\mathbb{F}$, α an element of $\mathbb{F}$, and f, g mappings of E into A. The definition of pointwise addition, scalar multiplication and product are given by,

$$(f+g)(s) = f(s)+g(s), \quad (fg)(s) = f(s)g(s), \quad (\alpha f)(s) = \alpha(f(s)) \quad (s \in E), \tag{2.1}$$

The set of all such functions with these operations is an algebra over $\mathbb{F}$.

Let $(A, +, \cdot)$ be an algebra. An algebra norm on A is a norm ρ on A such that $\rho(ab) \leq \rho(a)\rho(b)$, for all $a, b \in A$. A normed algebra is a pair (A, ρ), where A is an algebra and ρ is an algebra-norm on A.

Definition 2.1. *A Banach algebra is a normed algebra (A, ρ) such that the normed linear space A with norm ρ is complete, i.e. every Cauchy sequence in A converges.*

For example the full matrix algebra $M_{n\times n}(\mathbb{C})$ with natural addition, product and scalar multiplication and operator norm is a Banach algebra, for $n \in \mathbb{N}$. We recall that any square matrix corresponds with a linear transformation. Therefore, the norm of a square matrix can be defined to be the operator norm of the corresponding linear transformation. Let X be a topological space and A be a Banach algebra over $\mathbb{F}$. Then $C_b(X, A)$, the set of all bounded continuous functions from X into A with sup-norm, and natural pointwise addition, product and scalar multiplication as defined at (2.1) is a Banach algebra over $\mathbb{F}$. In particular, if X is a locally compact topological space, then with sup-norm $C_{00}(X)$, the set of all complex-valued continuous functions with compact supports on X is a normed algebra. The completion of $C_{00}(X)$ is a Banach algebra, denoted by $C_0(X)$ which is the set of all complex-valued continuous functions vanishing at infinity. Let G be a locally compact Hausdorff topological group and μ be a Haar measure on G. Let $L^1(G)$ be the space of all μ-integrable functions on G. $L^1(G)$ is a Banach algebra with the function addition and scalar multiplication and convolution product (i.e. $(f \star g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y)$ for f, g in $L^1(G)$ and $x, y \in G$). With the norm $\|f\|_1 = \int_G |f|$, $L^1(G)$ is a Banach algebra.

For various topics in Banach algebras, we refer to monographs [6], [2], [17], [12], [19].

2.1 Banach algebra cohomology

Cohomology, formally the dual of homology, is an invariance property for algebras. Cohomology can be used as a tool to classify algebras. The cohomology has more algebraic structure than homology. Banach algebra cohomology for a Banach algebra with coefficients in a Banach bimodule is defined in a similar way to Hochschild cohomology for an abstract algebra, except that one takes the topology into account so that all cochains are continuous.

In this section we recall the definition of nth-order cohomology for a Banach algebra and we will see where the notion of amenability comes from. We recall the notion of module, bimodule and Banach module here.

Let A be an algebra over $\mathbb{F}$, and X a linear space over $\mathbb{F}$. X is called a left A-module if there exists a mapping, called a module multiplication (action), of $A \times X$ into X: $(a, x) \mapsto ax$ which satisfies the following axioms:

(i) $(a_1 + a_2)x = a_1x + a_2x$,

(ii) $a(x_1 + x_2) = ax_1 + ax_2$,

(iii) $a_1(a_2x) = (a_1a_2)x$,

(iv) $(\alpha a)x = \alpha(ax)$,

for $a_1, a_2, a \in A,\ x_1, x_2, x \in X, \alpha \in \mathbb{F}$.

Likewise, X is a right A-module if there exists a mapping of $X \times A$ into X, $(x, a) \mapsto xa$ which satisfies:

(i) $x(a_1 + a_2) = xa_1 + xa_2$,

(ii) $(x_1 + x_2)a = x_1a + x_2a$,

(iii) $(xa_1)a_2 = x(a_1a_2)$,

(iv) $\alpha(xa) = x(\alpha a)$,

for $a_1, a_2, a \in A,\ x_1, x_2, x \in X, \alpha \in \mathbb{F}$.

X is called an A-bimodule if it is both a left A-module and a right A-module and left and right module actions satisfy the associativity law as follows:

$$(a_1 x)a_2 = a_1(xa_2) \quad (a_1, a_2 \in A,\ x \in X).$$

Let A be a Banach algebra over a field, $\mathbb{F}$, and M a normed linear space over $\mathbb{F}$. M is called a normed left A-module if M is a left A-module and also there exists a positive constant K such that

$$\|am\| \leq K\|a\|\|m\|, \quad (a \in A, m \in M).$$

A normed left A-module is called a Banach left A-module if it is complete as a normed linear space. Banach right A-modules are also defined similarly. A Banach A-bimodule is an A-bimodule which is both a left and right Banach A-module.

In this thesis we consider all Banach algebras to be over the field of complex numbers.

Let X, Y, Z be linear spaces over the same field $\mathbb{F}$, a mapping $\phi : X \times Y \to Z$ is called bilinear if it is linear in the first and second coordinate. When $Z = \mathbb{F}$, such a mapping is called a bilinear functional. This definition can be extended to the case of n-linear mapping from $X_1 \times X_2 \times \ldots \times X_n$ into Z. When $X_i,\ i = 1, \ldots, n$, and Z are normed spaces, the set of all bounded n-linear mappings is denoted by $BL(X_1, \ldots, X_n; Z)$.

Let A be a Banach algebra, X a Banach A-bimodule, and $n \in \mathbb{N}$. Denote the Banach space of all bounded n-linear mappings from $\overbrace{A \times \ldots \times A}^{n-times}$ to X by C_n and let $C_0 = X$. For $n \in \mathbb{N}$, define the linear mapping $\delta^n : C_{n-1} \longrightarrow C_n$ as follows,

$$(\delta^1 x)(a) = ax - xa \qquad (x \in X, a \in A)$$

and for $n \geq 2$, $T \in C_{n-1}$, and $a_1, \ldots, a_n \in A$,

$$\begin{aligned}(\delta^n T)(a_1, \ldots, a_n) &= a_1 T(a_2, \ldots, a_n) + \sum_{j=1}^{n-1} (-1)^j T(a_1, \ldots, a_j a_{j+1}, \ldots, a_n) \\ &\quad + (-1)^n T(a_1, \ldots, a_{n-1} a_n).\end{aligned}$$

Let $Z^n(A, X)$ be the kernel of δ^{n+1} and $B^n(A, X)$ be the range of δ^n, for $n \in \mathbb{N}$. By calculation, it can be shown that $\delta^{n+1} \circ \delta^n = 0$ which means that the range of δ^n is included in the kernel of δ^{n+1}. In other words, $B^n(A, X) \subset Z^n(A, X)$. Therefore, the quotient space of $Z^n(A, X)$ modulo $B^n(A, X)$ is meaningful and is called the nth cohomology group of A with coefficients in X and it is denoted by $H^n(A, X)$. Whether $H^n(A, X) = 0$ for certain Banach A-bimodules is a significant property for a Banach algebra to have. B.E. Johnson used it to classify Banach algebras in [15]. It reflects important features of many Banach algebras. Indeed, an amenable Banach algebras is a Banach algebra A that satisfies $H^1(A, X) = 0$ for all dual Banach A-bimodules X. The precise definition will be given in the next section.

2.2 Amenable and contractible Banach algebras

In the previous section we introduced the notion of nth-cohomology of a Banach algebra. In this section we are interested in when $H^1(A, X) = 0$, for certain Banach A-bimodule X. A Banach algebra is called contractible if $H^1(A, X) = 0$ for all Banach A-bimodules X. The only known contractible Banach algebras are the direct sums of finite full matrix algebras. Whether there is an infinite dimensional contractible Banach algebra is still an open problem [23].

If X is a Banach A-bimodule, then X^*, the dual space of X, is automatically a Banach A-bimodule with the module actions defined by

$$< x, af >=< xa, f >, \quad < x, fa >=< ax, f >, \quad f \in X^*, a \in A, x \in X.$$

With these two module actions, X^* is called the dual module of X.

Definition 2.2. *A Banach algebra A is called amenable, if $H^1(A, X^*) = 0$, for all Banach A-bimodules X, where X^* is the dual module of X.*

B.E. Johnson showed in [15] that, a locally compact group G is amenable (i.e. there is a left invariant mean on $L^\infty(G)$) if and only if $H^1\big(L^1(G), X^*\big) = \{0\}$, for each Banach $L^1(G)$-bimodule X.

The condition $H^1(A, X^*) = 0$ means $Z^1(A, X^*) = B^1(A, X^*)$. We note that $Z^1(A, X^*) = ker(\delta^2)$ consists of all bounded linear maps D from A into X^* such that

$$0 = \big(\delta^2(D)\big)(a,b) = aD(b) - D(ab) + D(a)b \qquad (a, b \in A).$$

Let A be a normed algebra and X be a A-bimodule. A linear mapping D from A into X is called a derivation if

$$D(ab) = (Da)b + a(Db) \qquad (a, b \in A).$$

Therefore, $Z^1(A, X^*)$ consists of all continuous derivations from A into X^*.

Given $x \in X$, let δ_x be the mapping from A into X given by

$$\delta_x(a) = ax - xa \qquad (a \in A).$$

It is easy to check that δ_x defines a continuous derivation, called an inner derivation. We note that $\delta_x = \delta^1(x)$. Therefore:

For a Banach algebra A, $Z^1(A, X^*) = B^1(A, X^*)$ for all Banach A-bimodule X (or A is amenable) if and only if every continuous derivation of A into the dual module X^* is inner for any Banach A-bimodule X. Similarly a Banach algebra A is contractible if and only if every continuous derivation from A into any Banach A-bimodule is inner.

Besides Johnson's result about the amenability of the group algebra $L^1(G)$, it is known that a C^*-algebra is amenable if and only if it is nuclear, and a uniform algebra is amenable iff it is isomorphic to $C(X)$ for a compact Hausdorff space X. We will show in chapter 4 that for a Banach algebra A, $C(X, A)$, the algebra of all A-valued continuous function, is amenable if and only if A is amenable. This extends the classic result on $C(X)$.

We have the following definition that is inspired by a result of Gourdeau about amenability of a Banach algebra.

The algebraic tensor product $X \otimes Y$ of the normed spaces X and Y is the linear span of the set $\{x \otimes y : \ x \in X, y \in Y\}$ in $BL(X^*, Y^*; \mathbb{F})$, where

$$x \otimes y(f, g) = f(x)g(y) \quad (f \in X^*, g \in Y^*),$$

where X^* and Y^* are dual space of X and Y, respectively.

Definition 2.3. *[2] Given normed spaces X and Y, the projective tensor norm p on $X \otimes Y$ is defied by*

$$p(u) = inf\Big\{ \sum_i \|x_i\| \|y_i\| : \ u = \sum_i x_i \otimes y_i \Big\}$$

where the infimum is taken over all finite representations of u. We write $\|u\|_p = p(u)$.

It is not difficult to show that p defines an algebra norm on $X \otimes Y$.

Definition 2.4. *[2] The completion of $(X \otimes Y, p)$ is called the projective tensor product of X and Y, and is denoted by $X \hat{\otimes} Y$.*

We have the following significant properties for the algebraic tensor product of normed spaces:

(i) Given a bilinear mapping $\phi : X \times Y \to Z$, there exists a unique linear mapping $\sigma : X \otimes Y \to Z$ such that

$$\sigma(x \otimes y) = \phi(x, y) \qquad (x \in X, y \in Y)$$

Given normed spaces X, Y and Z, a linear mapping $\phi : X \times Y \to Z$ is called bounded if there exists $L > 0$ such that $\|\phi(x, y)\| \leq L\|x\|\|y\|$. In this case the coressponding σ is a bounded linear mapping from $(X \otimes Y, p)$ into Z. A norm α on $X \otimes Y$ is called a cross-norm if

$$\alpha(x \otimes y) = \|x\|\|y\|, \ (x \in X, y \in Y).$$

(ii) A projective tensor norm of two Banach spaces is a cross-norm.

(iii) Let A, B be normed algebras over $\mathbb{F}$. Then the projective tensor norm on $A \otimes B$ is an algebra norm, when $A \otimes B$ is equipped with the product defined by

$$(a \otimes b).(c \otimes d) = ac \otimes bd.$$

(iv) If A is Banach algebra, then $A \hat{\otimes} A$ is a Banach A-bimodule, where the module actions are given naturally by

$$a \cdot (c \otimes d) = ac \otimes d, \qquad (c \otimes d) \cdot a = c \otimes da.$$

There are other ways to characterize amenability for Banach algebras. Among them we highlight the following characterization:

Definition 2.5. *Let A be a Banach algebra. An approximate diagonal for A is a bounded net $\{m_\alpha\}$ in $A \hat{\otimes} A$ that satisfies $am_\alpha - m_\alpha a \longrightarrow 0$ and $\pi(m_\alpha)a \longrightarrow a$ for all $a \in A$ in the respective norm topologies.*

Theorem 2.6 (B. Johnson in [14])**.** *A Banach algebra is amenable if and only if it has a bounded approximate diagonal.*

Note 2.7. *A Banach algebra A is contractible if and only if there is an element $u \in A\hat{\otimes}A$ for which $au = ua$ and $\pi(u)a = a$ for all $a \in A$. The element u is called a diagonal for A.*

2.3 Generalized notions of amenability

As we know one way of characterizing an amenable Banach algebra is the existence of a bounded approximate diagonal. To generalize the notion of amenability one can consider the Banach algebra with an approximate diagonal (not necessarily bounded). Such a Banach algebra is called a pseudo-amenable Banach algebra. Due to Gourdeau in [10], a Banach algebra A is amenable if and only if any bounded derivation D from A into any Banach A-bimodule X is approximately inner (or, equivalently, weakly approximately inner). Namely, there exists a bounded net $(x_i) \subset X$ such that $D(a) = lim_i ax_i - x_i a$ for all $a \in A$. Dropping the boundedness requirement for the net (x_i), one would get another generalization of amenability of Banach algebras, called approximately amenable. Later we will see some relations between these generalized notions of amenability. To be precise, we give the following formal definition.

Definition 2.8. *A Banach algebra A is called* pseudo-amenable *if there is a net (not necessarily bounded) $\{u_\alpha\} \subset A\hat{\otimes}A$, called approximate diagonal, such that $au_\alpha - u_\alpha a \to 0$ and $\pi(u_\alpha)a \to a$ for each $a \in A$.*

Since $A\otimes A$ is dense in $A\hat{\otimes}A$, it is not hard to see that if there is an approximate diagonal (u_α) for A, then the net (u_α) may be chosen from $A\otimes A$.

Let A be a Banach algebra, and let X be a Banach A-bimodule. A continuous derivation $D : A \longrightarrow X$ is *approximately inner* if there is a net (x_α) in X such that

$$D(a) = lim_\alpha(a.x_\alpha - x_\alpha.a) \qquad (a \in A),$$

i.e. if $D = lim_\alpha \delta_{x_\alpha}$ in strong-operator topology.

Definition 2.9. *[9] A Banach algebra A is called* approximately amenable *if, for every Banach A-bimodule X, every continuous derivation from A into the dual bimodule X^* is approximately inner.*

Definition 2.10. *[9] A Banach algebra A is called* approximately contractible *if every continuous derivation from A into each Banach A-bimodule X is approximately inner.*

Definition 2.11. *[2] The* unitization *of a normed algebra A over a field $\mathbb{F}$, denoted by $A+\mathbb{F}$, is the normed algebra consisting of the set $A\times\mathbb{F}$ with addition, scalar multiplication and product defined by*

$$\begin{aligned}(x,\alpha)+(y,\beta) &= (x+y,\alpha+\beta),\\ \beta(x,\alpha) &= (\beta x,\beta\alpha),\\ (x,\alpha)(y,\beta) &= (xy+\alpha y+\beta x,\alpha\beta)\end{aligned}$$

and with the norm defined by

$$\|(x,\alpha)\| = \|x\| + |\alpha|.$$

It is easy to see that $A+\mathbb{F}$ is a normed algebra with unit element $(0,1)$, and $\|(0,1)\| = 1$.

- [9, Theorem 3.1.] For a Banach algebra A the following statements are equivalent:

  (i) A is approximately amenable;

  (ii) A is weak* approximately amenable.

  (iii) A is approximately contractible;

  (iv) the unitization of A is pseudo-amenable.

- [4] Let A be a Banach algebra which has a bounded central approximate identity and an approximate diagonal. Then A is approximately contractible, and so $A^\sharp$ has an approximate diagonal.

- [7, Proposition 2.4.] A Banach algebra A is approximately amenable if and only if $A^\sharp$ is approximately amenable.

- [7, Proposition 2.6.] A Banach algebra A is approximately contractible if and only if any one of the following equivalent conditions holds.

  (1) there is a net $(M_\alpha) \subset A^\sharp \hat{\otimes} A^\sharp$ such that for each $a \in A^\sharp$, $a.M_\alpha - M_\alpha.a \to 0$ and $\pi(M_\alpha) \to e$.

  (2) there are nets $(M''_\alpha) \subset A\hat{\otimes}A$, $(F_\alpha), (G_\alpha) \subset A$, such that for all $a \in A$,

(a) $a.M''_\alpha - M''_\alpha.a + F_\alpha \otimes a - a \otimes G_\alpha \to 0$;

(b) $a.F_\alpha \to a, G_\alpha.a \to a$

(c) $\pi(M''_\alpha).a - F_\alpha.a - G_\alpha.a \to 0$

- [7, Theorem 2.1.] A is approximately amenable if and only if either of the following equivalent conditions hold:

  (i) there is a net $(M_\alpha) \subset (A^\sharp \hat{\otimes} A^\sharp)^{**}$ such that for each $a \in A^\sharp$, $a.M_\alpha - M_\alpha.a \to 0$ and $\pi^{**}(M_\alpha) \to e$;

  (ii) there is a net $(M_\alpha) \subset (A^\sharp \hat{\otimes} A^\sharp)^{**}$ such that for each $a \in A^\sharp$, $a.M_\alpha - M_\alpha.a \to 0$ and $\pi^{**}(M_\alpha) = e$ for every α.

2.4 Relations between different notions of amenability

For the relations among these generalized notions of amenability we have the following known results:

- Any approximately amenable commutative Banach algebra is pseudo-amenable.

- [7, Proposition 2.3.] Suppose that A is approximately amenable and has a bounded approximate identity, and that B is amenable. Then $A\hat{\otimes}B$ is approximately amenable.

- [7, Proposition 2.7.] Suppose that A and B are approximately amenable and each has a bounded approximate identity. Then $A \oplus B$ is approximately amenable.

- [7, Proposition 2.2.] Suppose that a Banach algebra A is approximately amenable and $\Phi : A \to B$ is continuous epimorphism. Then B is approximately amenable.

- [9, Proposition 3.3.] Suppose that A is a Banach algebra having a central approximately identity. If A is approximately amenable, then it is pseudo-amenable.

- [7, Theorem 2.3.] Suppose that A^{**} is approximately amenable. Then A is approximately amenable.

- [9] Suppose A^{**} is pseudo-amenable. Then so is A.

- [9, Proposition 4.1.] For a locally compact group G, $L^1(G)$ is pseudo-amenable if and only if G is amenable.

- [9, Proposition 4.2.] $M(G)$ is pseudo-amenable if and only if G is discrete and amenable. $L^1(G)^{**}$ is pseudo-amenable if and only if G is finite.

- If a finite-dimensional Banach algebra is approximately amenable, then it is already amenable.

- [7, Lemma 2.2.] Suppose that A is approximately amenable. Then A has left and right approximate identities. In particular A^2 is dense in A.

- [7, Theorem 3.1.] $M(G)$ is approximately amenable if and only if G is discrete and amenable.

- [7, Theorem 3.2.] $L^1(G)$ is approximately amenable if and only if G is amenable.

- [7, Theorem 3.3.] $L^1(G)^{**}$ is approximately amenable if and only if G is finite.

- [4, Proposition 2.4.] Let $\mathcal{S}$ be one of the following classes of Banach algebras: approximately amenable, approximately contractible, sequentially approximately amenable, sequentially approximately contractible, boundedly approximately amenable, boundedly approximately contractible. Let A be a Banach algebra. Then $A \in \mathcal{S}$ if and only if $A^\sharp \in \mathcal{S}$.

- [4, Corrollary 7.3.] The full group C^*-algebra of a locally compact group Γ is approximately amenable if and only if Γ is amenable.

  Let Γ be a discrete group, with convolution algebra $\ell^1(\Gamma)$. Given $p \in (1, \infty)$ we may consider the left regular representation of Γ on $\ell^p(\Gamma)$, and this gives an injective continuous algebra homomorphism $\theta_p : \ell^1(\Gamma) \to \mathcal{B}(\ell^p(\Gamma))$. The norm closure in $\mathcal{B}\big(\ell^\infty(\Gamma)\big)$ of the range of θ_p is denoted by $PF_p(\Gamma)$.

- [4, Theorem 7.1.] Let Γ be a discrete group. Then the following are equivalent:

  (i) Γ is amenable;

  (ii) $PF_p(\Gamma)$ is amenable for all $p \in (1, \infty)$;

  (iii) $PF_p(\Gamma)$ is approximately amenable for some $p \in (1, \infty)$;

  (iv) $PF_p(\Gamma)$ is pseudo-amenable for some $p \in (1, \infty)$.

- [4, Theorem 5.3.] The following are equivalent for a locally compact group G.

  (i) The group G is compact.

  (ii) There is a Segal algebra on G which is pseduo-contractible.

  (iii) All Segal algebras on G are pseudo-contractible.

- [4, Proposition 3.1.] Let A be a Banach algebra which has a bounded central approximate identity and an approximate diagonal. Then A is approximately contractible, and so $A^\sharp$ has an approximate diagonal.

- [4, Theorem 5.5.] If S(G) is approximately amenable or pseudo-amenable then G is an amenable group.

To complete this survey section we discuss examples of Banach algebras which are either with or without a certain amenability property:

(i) [9, Proposition 2.1.] Let $\{A_i :\ i \in I\}$ to be a collection of pseudo-amenable/pseudo contractible Banach algebras. Then so is the $\bigoplus_{i\in I}^{p} A_i$ for $p \geq 1$ or $p = 0$

(ii) [4, Proposition 5.1] The Feichtinger algebra on an infinite compact abelian group is not approximately amenable.

(iii) [4] Let $\mathbb{F}_2$ denote the free group on two generators. Then the Fourier algebra on $\mathbb{F}_2$, $A(\mathbb{F}_2)$, is not approximately amenable.

(iv) [3, Theorem 2.1.] Let $1 \leq p < \infty$ and let $\mathcal{S}_p(H)$ denote the ideal of Schatten class operators on an infinite-dimensional Hilbert space H. Then $\mathcal{S}_p(H)$ is not approximately amenable.

(v) [3, Theorem 2.5.] If a Banach algebra $\mathcal{A}$ contains a sort of net, called SUM configuration, then $\mathcal{A}$ is not approximately amenable.

(vi) [3, Example 2.7.] Let $\{B_n\}$ be a sequence of Banach algebras, such that for each n the algebra B_n has an identity element $\mathbf{1}_n$, and suppose $\|\mathbf{1}_n\| \to \infty$ as $n \to \infty$. Let $\mathcal{A}$ be the completion of the algebraic direct sum $\bigoplus_n B_n$ in either c_0 or ℓ^p-norms for $1 \leq p < \infty$. Then $\mathcal{A}$ is not approximately amenable.

(vii) [3, Example 2.8.] Let S be a Brandt semigroup over a group G with infinite index and consider the convolution algebra $\ell^1(S)$. Then, $\ell^1(\mathcal{S})$ is not approximately amenable.

(viii) [3, Example 2.10.] Let $\mathbb{N}_{min}$ denote the semilattice whose underlying set is $\mathbb{N}$ and where the product of two elements is defined to be their minimum. Any function $\omega : \mathbb{N} \to [1, \infty)$ satisfying $\omega(1) = 1$ defines a weight function on $\mathbb{N}_{min}$, and via the Gelfand transform the weighted convolution algebra $\ell^1(\mathbb{N}_{min}, \omega)$ is isomorphic to the Feinstein algebra A_ω. In the case where $\omega(n) \to \infty$ as $n \to \infty$, $\ell^1(\mathbb{N}_{min}, \omega)$ is not approximately amenable.

(ix) [3, Theorem 3.1.] Let $d \in \mathcal{N}$ and let $\mathcal{A}$ be a proper Segal subalgebra of $L^1(\mathbb{R}^d)$. Then $\mathcal{A}$ is not approximately amenable.

Chapter 3

AMENABILITY PROPERTIES OF $C(X)$

In this chapter we consider the algebras $C(X), C_0(X)$, and $C^b(X)$ for a locally compact space X and we will discuss an original and direct constructive proof of the amenability of $C(X)$, where X is a compact Hausdorff space.

Let S be a non-empty set. The set of all bounded complex-valued functions on S is denoted by $\ell^\infty(S)$. It is a unital subalgebra of $\mathbb{C}^S$ with pointwise operations. The norm on $\ell^\infty(S)$ is defined by

$$|f|_S = sup\{|f(s)| : \ s \in S\} \qquad (f \in \ell^\infty(S)).$$

Then, $|.|_S$ is an algebra norm on $\ell^\infty(S)$, called the *uniform norm* on S. Moreover, $(\ell^\infty(S), |.|_S)$ is a commutative, unital Banach algebra.

Let X be a non-empty topological space. Then, $C(X)$, the set of all continuous complex valued functions, $C^b(X)$, the set of all bounded continuous complex valued functions, and $C_0(X)$, the set of continuous complex valued functions which vanishes at infinity are subalgebras of $\mathbb{C}^X$. Here a function $f \in C(X)$ is said to vanish at infinity if and only if $\{x \in X| \ |f(x)| \geq \varepsilon\}$ is compact for each $\varepsilon > 0$. It can be observed that $C^b(X) = \ell^\infty(X) \cap C(X)$ and $C_0(X)$ are commutative Banach algebras as subalgebras of $(\ell^\infty(X), |.|_X)$. The subspace $C(X, \mathbb{R})$ of $C(X)$ is a closed, real subalgebra.

3.1 Amenability of $C(X)$ for a compact Hausdorff space X

Let X be a locally compact Hausdorff space, Then $C_0(X)$, the space of complex-valued continuous functions on X which vanish at infinity is a commutative Banach algebra with the pointwise algebra operations and supremum norm. The unitization of $C_0(X)$ is $C(\bar{X})$, for some compact Hausdorff space $\bar{X}$. In fact, $\bar{X}$ is homeomorphic to the one-point compactification of X. Since the amenability of the unitization of a Banach algebra is equivalent to the amenability of the original Banach algebra, one can assume that X is compact and study $C(X)$ to get the amenability of the general $C_0(X)$.

Let X be a non-empty topological space, and let A be an algebra of functions on X. Then A is a function algebra on X if A separates strongly the points of X and if the A-topology [6, page 470] on X is the given topology of X. The algebra A is a Banach function algebra on X if A is also a Banach algebra with respect to some norm. A uniform algebra on X is a function algebra on X which is a closed subalgebra of $(C^b(X), |.|_X)$. By [6, proposition 4.1.2] for a non-empty, locally compact space Ω, a subalgebra A of $C_0(\Omega)$ is a function algebra on Ω if it separates strongly the points of Ω.

We recall that for a non-empty set E. the set $\mathbb{C}^S$ is a commutative and unital algebra with respect to the pointwise operations. An algebra of functions on S is a subalgebra of $\mathbb{C}^S$. A subset E of $\mathbb{C}^S$ separates the points of S if, for each $s, t \in S$ with $s \neq t$, there exists $f \in E$ with $f(s) \neq f(t)$, and E separates strongly the points of S if, further, for each $s \in S$, there exists $f \in E$ with $f(s) \neq 0$.

In [6, Theorem 5.6.2], one can find a proof to the following. If Ω is a non-empty compact space, then the uniform algebra $C(\Omega)$ is amenable, and if A is a unital, amenable uniform algebra, then $A \simeq C(\Omega)$, for some compact space Ω. The proof uses B. Johnson's deep research about the equivalency of the amenability of a group G and the amenability of its group algebra $\ell^1(G)$.

The proof starts with defining $G = (C(\Omega, \mathbb{R}), +)$. Since G is an abelian group, it is amenable and thus $\ell^1(G)$ is amenable. The mapping

$$\theta : \sum_{h \in G} \alpha_h \delta_h \mapsto \sum_{h \in G} \alpha_h e^{ih}$$

defines a continuous homomorphism from $\ell^1(G)$ into $C(\Omega)$. By the Stone-Weierstrass theorem, $B = \theta(\ell^1(G))$ is dense in $C(\Omega)$. Since $C(\Omega)$ contains a dense subalgebra which is the image of an amenable Banach algebra, $C(\Omega)$ is amenable.

3.2 Constructive proof of the amenability of $C(X)$

In the previous section we discussed an abstract proof for the amenability of $C(X)$ with X being a compact space. Due to B.E. Johnson, the amenability of a Banach algebra is equivalent to either the existence of a bounded approximate diagonal or the existence of a virtual diagonal.

In 2009 M. Abtahi and Y. Zhang gave a new proof of the amenability of $C(X)$ for a compact Hausdroff space X by constructing a bounded approximate diagonal for $C(X)$.

The proof is based on the partition of unity for a compact Hausdorff space and the following well-known Grothendieck inequality.

Lemma 3.1. *[11] Let X be a compact space. There is a constant $c > 0$, independent of X, such that for every $u = \sum_{k=1}^{n} v_k \otimes w_k \in C(X) \otimes C(X)$, we have*

$$\|u\|_p \leq c\Big(\Big\| \sum_{i=1}^{n} |v_i|^2 \Big\| + \Big\| \sum_{j=1}^{n} |w_j|^2 \Big\|\Big),$$

where $\|.\|_p$ denotes the projective tensor norm and $\|.\|$ denotes the uniform norm on $C(X)$.

Here we briefly go over the construction of a bounded approximate diagonal for $C(X)$ given in [1]. For a given $\varepsilon > 0$ and a finite subset F of $C(X)$, by compactness of X, one can choose $x_1, \ldots, x_n \in X$ and neighborhoods V_i of x_i such that $X \subset \cup_{i=1}^{n} V_i$ and $|a(s) - a(x_i)| < \varepsilon$, for $s \in V_i$ and $a \in F$, $i = 1, \ldots, n$. By the partition of unity for the compact space X, there exist non-negative real valued continuous functions $h_1, \ldots, h_n$ on X for which

$$h_1 + \ldots + h_n = 1 \text{ and } supp(h_i) \subset V_i, \qquad i = 1, \ldots, n.$$

Then, new functions u_i's and $u_{(F,\varepsilon)}$ are defined by

$$u_i = \sqrt{h_i}\ i = 1, \ldots, n \text{ and } u_{(F,\varepsilon)} = \textstyle\sum_{i=1}^{n} u_i \otimes u_i \in C(X)\hat{\otimes}C(X).$$

For simplicity we denote $u_{(F,\varepsilon)}$ by u. The function u has the following properties.

(1) $\pi(u)a = a, \quad \big(a \in C(X)\big)$

(2) $\|au - ua\|_p < \varepsilon, \quad (a \in F)$

(3) $\|u\|_p \leq 2c,$ where c is the constant ensured in Lemma 3.1.

It follows that, the net $\{u_{(F,\varepsilon)}\}$ is indeed a bounded approximate diagonal for $C(X)$.

Chapter 4

AMENABILITY PROPERTIES OF $C(X, A)$

Let X be a compact Hausdorff space and A be a Banach algebra. The space of all continuous functions from X into A with pointwise operations and sup norm is a Banach algebra, denoted by $C(X, A)$.

The original proof of the amenability of $C(X)$ discussed in the previous chapter due to Sheinberg [21] does not seem to work for Banach algebra-valued functions. The Stone-Weierstrass Theorem and the theorem of B. Johnson about equivalency of amenability of a group and that of its group algebra will not apply here. But the technique provided in [1] as discussed in Section 3.2 helps us to investigate amenability of $C(X, A)$ directly. In this chapter, we prove that if X is compact Hausdorff and A is an amenable Banach algebra (not necessarily commutative or unital), then the Banach algebra $C(X, A)$ has a bounded approximate diagonal and therefore is amenable.

Note 4.1. *Let A be an algebra, X be a set and f be a mapping from X into the field of complex numbers. For $a \in A$ the mapping af is defined by, $(af)(x) = f(x)a$ for $x \in X$.*

Note 4.2. *Let A be a Banach algebra and the net $(\alpha_\lambda)_{\lambda\in\Lambda}$ be bounded approximate diagonal. Then $sup\{\|\alpha_\lambda\|;\ \lambda \in \Lambda\} \geq 1$. If $(e_\lambda)_{\lambda\in\Lambda}$ is a bounded approximate identity for A, then for a nonzero element of $a \in A$, $\big(sup_\lambda\|e_\lambda\|\big)\|a\| = \big(sup_\lambda\|e_\lambda\|\|a\|\big) \geq lim_\lambda\|e_\lambda a\| = \|a\|$.*

Therefore $sup_\lambda \|e_\lambda\| \geq 1$. *In particular, when* $(\alpha_\lambda)_{\lambda\in\Lambda} \in A\hat{\otimes}A$ *is a bounded approximate diagonal for* A*, then* $\|\pi(\alpha_\lambda)\| \leq \|\alpha_\lambda\|$ *and by taking supremum over* $\lambda \in \Lambda$ *one gets* $sup_\lambda \|\pi(\alpha_\lambda)\| \leq sup_\lambda \|\alpha_\lambda\|$. *Since* $\big(\pi(\alpha_\lambda)\big)_{\lambda\in\Lambda}$ *is a bounded approximate identity for* A $1 \leq sup_\lambda \|\alpha_\lambda\|$.

4.1 The bounded approximate diagonal for $C(X, A)$

Let A be an amenable Banach algebra and let X be a locally compact space. Then we will prove that $C(X, A)$, the Banach algebra of A-valued continuous functions on X is amenable. This will be done by constructing a bounded approximate diagonal for $C(X, A)$.

Theorem 4.3. *Let* X *be a compact Hausdorff space and let* A *be an amenable Banach algebra. Then,* $C(X, A)$ *has a bounded approximate diagonal.*

Proof. Since A is amenable, it has a bounded approximate diagonal, say, $\alpha = (\alpha_\lambda)_{\lambda\in\Lambda} \subset A\hat{\otimes}A$. Denote $M = sup\{\|\alpha_\lambda\|_p,\ \lambda \in \Lambda\}$. Let F be a finite subset of $C(X, A)$, and $\varepsilon > 0$ be given. Since X is compact, there exist $y_1, \dots, y_{l'} \in X$, and open subsets $V'_1, \dots, V'_{l'}$ of X such that $y_i \in V'_i$, $X = \bigcup_{i=1}^{l'} V'_i$ and $\|a(s) - a(y_i)\| < \varepsilon/12Mc$ for $a \in F$ and $s \in V'_i$, $i = 1, 2, \dots, l'$. Here c is the constant used in Lemma 3.1. From [18, Theorem 2.13] there exist $h'_k \in C(X),\ k = 1, 2, \dots, l'$ such that $0 \leq h'_k \leq 1$, $\sum_1^{l'} h'_k = 1$ and $supp(h'_k) \subset V'_k,\ k = 1, \dots, l'$. Let $M' = max\{\|a\|;\ a \in F\}$.

By continuity of elements of F, continuity of $h'_1, \cdots .h'_n$ and compactness of X, there exists neighborhoods $V_1, \dots V_l$ of $x_1, \dots, x_l \in X$, respectively, such that $X = \bigcup_{i=1}^{l} V_i$,

$\|a(s) - a(x_i)\| < \varepsilon/4M$ and $|h'(x_i) - h'(s)| < \frac{\varepsilon}{8MM'l'}$, for $a \in F$, $h' \in \{h'_1, \cdots, h'_{l'}\}$ and $s \in V_i$, $i = 1, 2, \ldots, l$, and a partition of unity for X such that $h_i \in C(X)$, $0 \leq h_i \leq 1$, $\sum_i^l h_i = 1$, $supp(h_i) \subset V_i,\ i = 1, \ldots, l$, and for all $k = 1, \ldots, l'$,

$$\Big\| h'_k\big(\sum_i u_i \otimes u_i\big) - \big(\sum_i u_i \otimes u_i\big) h'_k \Big\|_p < \frac{\varepsilon}{4MM'l'}. \tag{4.1}$$

where $u_i := \sqrt{h_i}$, $i = 1, \ldots, l$.

The sets $\{a(x_i),\ a \in F,\ i = 1, 2, \ldots l\}$ and $\{a(y_i),\ a \in F,\ i = 1, 2, \ldots l'\}$ are finite subsets of A, since F is finite. Since α is an approximate diagonal for A, there exists $\lambda_0 = \lambda_{0(F,\varepsilon)} \in \Lambda$ such that, for $\lambda_0 \leq \lambda, a \in F,\ i = 1, \ldots, l, j = 1, \ldots, l'$, we have

$$\big\| a(y_j)\alpha_\lambda - \alpha_\lambda a(y_j) \big\|_p < \frac{\varepsilon}{8l'c}, and\ \big\| \pi(\alpha_\lambda) a(x_i) - a(x_i) \big\| < \frac{\varepsilon}{4}. \tag{4.2}$$

Suppose $\alpha_{\lambda_0} = \sum_{j=1}^{m_{\lambda_0}} \alpha_j^{\lambda_0} \otimes \beta_j^{\lambda_0} \in A \otimes A$. Then we show that $U_{(F,\varepsilon)} := \sum_{i,j} u_i \alpha_j^{\lambda_0} \otimes u_i \beta_j^{\lambda_0} \in C(X, A) \otimes C(X, A)$ satisfies the following, where for simplicity we denote $U_{(F,\varepsilon)}$ by $U = \sum_{i,j} u_i \alpha_j \otimes u_i \beta_j$:

(1) $\|U\|_p \leq 2Mc$,

(2) $\|\pi(U)a - a\| < \varepsilon,\ a \in F$,

(3) $\|aU - Ua\|_p < \varepsilon,\ a \in F$.

Let $T : \Big(C(X) \hat{\otimes} C(X)\Big) \otimes (A \hat{\otimes} A) \to C(X, A) \hat{\otimes} C(X, A)$ be the linear operator defined by

$$(v \otimes w) \otimes (\alpha \otimes \beta) \mapsto v\alpha \otimes w\beta.$$

We can easily observe that $\|T\| \leq 1$. Then we have:

$$\begin{aligned}
\|U\|_p &= \Big\| \sum_{i,j} u_i\alpha_j \otimes u_i\beta_j \Big\|_p \\
&= \Big\| \sum_{i,j} T\Big((u_i \otimes u_i) \otimes (\alpha_j \otimes \beta_j) \Big) \Big\|_p \\
&= \Big\| T\Big(\sum_i (u_i \otimes u_i) \otimes \sum_j (\alpha_j \otimes \beta_j) \Big) \Big\|_p \\
&\leq \Big\| \sum_i (u_i \otimes u_i) \Big\|_p \Big\| \sum_j (\alpha_j \otimes \beta_j) \Big\|_p \\
&\leq M \Big\| \sum_i (u_i \otimes u_i) \Big\|_p \\
&\leq 2Mc.
\end{aligned} \tag{4.3}$$

For the last inequality we used Lemma 3.1. Therefore (1) is proved. For $a \in F$,

$$\begin{aligned}
\|\pi(U)a - a\| &= \Big\| \sum_{i,j} u_i\alpha_j u_i\beta_j a - a \Big\| \\
&= \Big\| \sum_{i,j} \big[u_i^2\alpha_j\beta_j\big(a - a(x_i)\big) + u_i^2\alpha_j\beta_j a(x_i) \big] - a \Big\| \\
&= \Big\| \sum_{i,j} h_i\alpha_j\beta_j\big(a - a(x_i)\big) + \sum_{i,j} \big[h_i\alpha_j\beta_j a(x_i) \big] - a \Big\| \\
&= \Big\| \sum_{i,j} \alpha_j\beta_j h_i\big(a - a(x_i)\big) + \sum_i h_i \big[\sum_j \alpha_j\beta_j a(x_i) - a \big] \Big\| \\
&\leq \Big\| \sum_{i,j} \alpha_j\beta_j h_i\big(a - a(x_i)\big) \Big\| + \Big\| \sum_i h_i \big[\sum_j \alpha_j\beta_j a(x_i) - a \big] \Big\|.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\Big\| \sum_{i,j} \alpha_j \beta_j h_i\big(a - a(x_i)\big) \Big\| &\leq \Big\| \sum_j \alpha_j \beta_j \Big\| \Big\| \sum_i h_i\big(a - a(x_i)\big) \Big\| \\
&\leq M \Big\| \sum_i h_i\big(a - a(x_i)\big) \Big\| \\
&\leq M sup_{s \in X} \Big\| \sum_i h_i(s)\big(a(s) - a(x_i)\big) \Big\| \\
&\leq M sup_{s \in X} \sum_i h_i(s) \big\| \big(a(s) - a(x_i)\big) \big\| \\
&\leq M sup_{s \in X} \sum_i h_i(s) \frac{\varepsilon}{4M} \\
&= \frac{\varepsilon}{4}
\end{aligned}
$$

and,

$$
\begin{aligned}
\Big\| \sum_i h_i \big[\sum_j \alpha_j \beta_j a(x_i) - a \big] \Big\| &\leq \Big\| \sum_i h_i \big[\sum_j \alpha_j \beta_j a(x_i) - a(x_i) \big] \Big\| + \Big\| \sum_i h_i \big(a(x_i) - a \big) \Big\| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4M} \\
&< \frac{\varepsilon}{2}
\end{aligned}
$$

using (4.2) and the last conclusion. Therefore, for $a \in F$ we have shown

$$
\big\| \pi(U) a - a \big\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

Therefore (2) holds.

Finally, we show $\|aU - Ua\|_p < \varepsilon,\ a \in F$. For $a \in F$, let $\hat{a}(y) = \sum_{k=1}^{l'} a(y_k){h'}_k(y)$. Then, $\hat{a} \in C(X, A)$ and

$$\begin{aligned}
\|a(y) - \hat{a}(y)\|_A &= \Big\| \sum_{k=1}^{l'} \big(a(y) - a(y_k)\big){h'}_k(y) \Big\|_A \\
&\leq \sum_{k=1}^{l'} \big\| \big(a(y) - a(y_k)\big) \big\|_A {h'}_k(y) \\
&\leq \frac{\varepsilon}{12Mc} \sum_{k=1}^{l'} {h'}_k(y) \\
&= \frac{\varepsilon}{12Mc}.
\end{aligned}$$

Taking supremum over X, we have $\|a - \hat{a}\| < \frac{\varepsilon}{12Mc}$, $a \in F$. Hence,

$$\begin{aligned}
\big\|aU - Ua\big\|_p &\leq \big\|aU - \hat{a}U\big\|_p + \big\|\hat{a}U - U\hat{a}\big\|_p + \big\|U\hat{a} - Ua\big\|_p \\
&\leq \big\|a - \hat{a}\big\| \big\|U\big\|_p + \big\|\hat{a}U - U\hat{a}\big\| + \big\|U\big\|_p \big\|\hat{a} - a\big\| \\
&< 2\frac{\varepsilon}{12Mc} 2Mc + \big\|\hat{a}U - U\hat{a}\big\|_p \qquad (4.4) \\
&= \frac{\varepsilon}{3} + \big\|\hat{a}U - U\hat{a}\big\|_p,
\end{aligned}$$

where in the third inequality we used (1).

But for the second term in (4.4) we have,

$$\begin{aligned}
\big\|\hat{a}U - U\hat{a}\big\|_p &= \Big\| \sum_{k=1}^{l'} a(y_k)h'_k U - U \sum_{k=1}^{l'} a(y_k)h'_k \Big\|_p \\
&= \Big\| \sum_{k=1}^{l'} a(y_k)h'_k \sum_{i,j}^{l,m} u_i\alpha_j \otimes u_i\beta_j - \sum_{i,j}^{l,m} u_i\alpha_j \otimes u_i\beta_j \sum_{k=1}^{l'} a(y_k)h'_k \Big\|_p
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \sum_{i,j,k} \big(a(y_k) h'_k u_i \alpha_j \otimes u_i \beta_j - u_i \alpha_j \otimes u_i \beta_j a(y_k) h'_k \big) \Big\|_p \\
&= \Big\| \sum_k T\Big(\sum_i \big(h'_k u_i \otimes u_i \big) \otimes \sum_j \big(a(y_k) \alpha_j \otimes \beta_j \big) \Big) \\
&\qquad\qquad - \sum_k T\Big(\sum_i \big(u_i \otimes u_i h'_k \big) \otimes \sum_j \big(\alpha_j \otimes \beta_j a(y_k) \big) \Big) \Big\|_p \\
&= \Big\| \sum_k T\Big(\sum_i \big(h'_k u_i \otimes u_i - u_i \otimes u_i h'_k \big) \otimes \sum_j \big(a(y_k) \alpha_j \otimes \beta_j \big) \Big) \\
&\qquad\qquad - \sum_k T\Big(\sum_i \big(u_i \otimes u_i h'_k \big) \otimes \sum_j \big(\alpha_j \otimes \beta_j a(y_k) - a(y_k) \alpha_j \otimes \beta_j \big) \Big) \Big\|_p \\
&\leq \sum_k \Big(\Big\| \sum_i \big(h'_k u_i \otimes u_i - u_i \otimes u_i h'_k \big) \Big\|_p \big\| a(y_k) \big\|_A \Big\| \sum_j \big(\alpha_j \otimes \beta_j \big) \Big\|_p \\
&\qquad\qquad + \Big\| (\sum_i \big(u_i \otimes u_i h'_k \big) \Big\|_p \Big\| \sum_j \big(\alpha_j \otimes \beta_j a(y_k) - a(y_k) \alpha_j \otimes \beta_j \big) \Big\|_p \Big) \\
&\leq \sum_k \big(\frac{\varepsilon}{4MM'l'} \, M'M + 2c \frac{\varepsilon}{8l'c} \big) \\
&= \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\
&= \frac{\varepsilon}{2}.
\end{aligned}
$$

The last inequality in calculation above is by using (4.1), (4.2) and Lemma 3.1. Therefore,

$$
\begin{aligned}
\big\| aU - Ua \big\| &\leq \big\| aU - \hat{a}U \big\| + \big\| \hat{a}U - U\hat{a} \big\| + \big\| U\hat{a} - Ua \big\| \\
&\leq \big\| a - \hat{a} \big\| \big\| U \big\|_p + \big\| \hat{a}U - U\hat{a} \big\| + \big\| U \big\|_p \big\| \hat{a} - a \big\| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}
$$

Therefore, $\|aU - Ua\| < \varepsilon$ for all $a \in F$. Thus $\{U_{(F,\varepsilon)}\}$, in which F is a finite subset of $C(X, A)$ and $\varepsilon > 0$, forms a bounded approximate diagonal for $C(X, A)$. □

4.2 The converse of Theorem 4.3

In this section we show that the converse of Theorem 4.2 is also true. Therefore not only amenability of A implies amenability of $C(X, A)$ but also amenability of $C(X, A)$ implies that of A.

Lemma 4.4. *Let A and B be Banach algebras and $\Gamma : A \to B$ be a continuous surjective Banach algebra homomorphism. Assume A has a bounded approximate diagonal. Then so does B.*

Proof. Let $\{U_\alpha\}$ be a bounded approximate diagonal for A. The mapping

$$\Gamma \times \Gamma : A \times A \to B \otimes B, \quad (a_1, a_2) \mapsto \Gamma(a_1) \otimes \Gamma(a_2)$$

is a bounded bilinear mapping which defines a bounded linear mapping

$$\Gamma \otimes \Gamma : A \otimes A \to B \otimes B, \quad a_1 \otimes a_2 \mapsto \Gamma(a_1) \otimes \Gamma(a_2).$$

Let $V_\alpha = \Gamma \otimes \Gamma(U_\alpha)$. Then,

$$\begin{aligned} \pi(V_\alpha)b &= \pi\big(\Gamma \otimes \Gamma(U_\alpha)\big)\Gamma(a) \\ &= \Gamma\big(\pi(U_\alpha)\big)\Gamma(a) \\ &= \Gamma\big(\pi(U_\alpha)a\big), \end{aligned}$$

where $b = \Gamma(a)$. Since $\pi(U_\alpha)$ is an approximate identity for A, $\pi(V_\alpha)$ is an approximate identity for B.

Moreover,

$$\begin{aligned} bV_\alpha - V_\alpha b &= \Gamma(a)\big(\Gamma \otimes \Gamma\big)(U_\alpha) - \big(\Gamma \otimes \Gamma\big)(U_\alpha)\Gamma(a) \\ &= \big(\Gamma \otimes \Gamma\big)(aU_\alpha - U_\alpha a), \end{aligned}$$

where $b = \Gamma(a)$. But $\|aU_\alpha - U_\alpha a\| \to 0$, hence

$$\|bV_\alpha - V_\alpha b\| \to 0$$

Therefore, $\{V_\alpha\}$ is a bounded approximate diagonal for B. □

In fact, Lemma **??** is true and well-known if Γ is merely Banach algebra homomorphism with dense range. But we only need our Lemma **??** to show the main result of this section below.

Theorem 4.5. *Let X be a locally compact Hausdorff space and A be a Banach algebra. If $C(X, A)$ is amenable, then so is A.*

Proof. Since $C(X, A)$ is amenable, it has a bounded approximate diagonal. To show that A is amenable, we can equivalently prove that A has a bounded approximate diagonal. Fix, $x_0 \in X$, and define $\Gamma : C(X, A) \to A, \quad f \mapsto f(x_0)$. Then, Γ is a continuous surjective Banach algebra homomorphism. By Lemma 4.4, since $C(X, A)$ has a bounded approximate diagonal, so does A. □

Chapter 5

MISCELLANEOUS RESULTS AND QUESTIONS

A Banach algebra is amenable if and only if it has a bounded approximate diagonal. By dropping the boundedness restriction in the definition of an approximate diagonal for a Banach algebra, one can obtain new notions such as unbounded approximate diagonals, compactly-invariant approximate diagonals, or central compactly-invariant approximate diagonals. In this chapter we define the notion of a compactly-invariant approximate diagonal and we give some example of Banach algebras that satisfy to this new notion. In particular, we show that if A possesses this kind of approximate diagonal, then so does $C(X, A)$. For the sake of completeness, we also give a direct proof of the amenability of $C_0(X)$.

5.1 Compactly approximate diagonals

The notion of compactly-invariant approximate diagonal was first introduced by Y. Zhang in 2003. We recall this notion in this section and we states some examples of Banach algebras which have (central) compactly-invariant approximate diagonal.

Definition 5.1. *Let A be a Banach algebra. A* compactly-invariant approximate diagonal *for A is a net $\{u_\alpha\}_{\alpha\in\Lambda} \subset A\hat{\otimes}A$ for which $au_\alpha - u_\alpha a \to 0$ and $\pi(u_\alpha)a \to a$, uniformly on compact sets of A.*

If, in addition, $au_\alpha = u_\alpha a$ for $\alpha \in \Lambda$ and $a \in A$, then $\{u_\alpha\}$ is called a central *compactly-invariant approximate diagonal.*

Before we go over some examples, we recall the notions of c_0 and ℓ^p-direct sums of Banach algebras.

Let $\{A_i;\ i \in I\}$ be a collection of Banach algebras. Then the ℓ_p-direct sum is

$$\bigoplus^p{}_{i\in I} A_i = \left\{a \in \prod_{i\in I} A_i;\ \|a\|_p = \left(\sum_i \|a(i)\|^p\right)^{\frac{1}{p}} < \infty\right\}$$

where $1 \le p < \infty$, and the c_0-direct sum is

$$\bigoplus^0{}_{i\in I} A_i = \left\{a \in \prod_{i\in I} A_i;\ lim_i\ a(i) = 0,\ \|a\|_\infty = sup_i \|a(i)\| < \infty\right\}.$$

With coordinatewise algebraic operations, these are Banach algebras.

The Banach algebra ℓ^p or c_0-direct sum of amenable Banach algebras has a compactly-invariant approximate diagonal. Any Segal algebra on an amenable locally compact SIN group has a compactly-invariant approximate diagonal. Moreover, $\mathcal{K}(H)$, the Banach algebras of compact operators on a Hilbert space H, Segal algebras on a compact group, and c_0 or ℓ^p-direct sums of contractible Banach algebras have a central compactly-invariant approximate diagonal. For a locally compact group G, $L^1(G)$ has a compactly-invariant approximate diagonal if and only if G is amenable.

5.2 Compactly-invariant approximate diagonals for $C(X,A)$

Let X be a compact Hausdorff space and A be a Banach algebra (not necessarily commutative or unital). In this section we construct a compactly-invariant approximate diagonal for $C(X,A)$, assuming A has a central compactly-invariant approximate diagonal.

Using a standard topological argument one can get the following Lemma.

Lemma 5.2. *Let A be a Banach algebra and X be a compact Hausdorff space. Then the mapping*

$$\begin{aligned} \Gamma :& C(X,A) \times X \longrightarrow A \\ &(f,x) \mapsto f(x), \end{aligned}$$

is continuous, where $C(X,A) \times X$ is equipped with the product topology.

Let $\mathcal{K}$ be a compact subset of $C(X,A)$. As a consequence of Lemma 5.2, the image of Γ restricted to $\mathcal{K} \times X$ is compact in A, i.e. the set $\{f(x) : \ x \in X, f \in \mathcal{K}\}$ is compact in A. Moreover, with the assumptions in the above lemma, the mapping

$$\begin{aligned} \Gamma :& \mathcal{K} \times X \longrightarrow \mathbb{R} \\ &(f,x) \mapsto \|f(x)\| \end{aligned}$$

is bounded.

Theorem 5.3. *Let A be a Banach Algebra, and X be a compact Hausdorff space. If A has a central compactly-invariant approximate diagonal, then $C(X,A)$ has a compactly-invariant approximate diagonal.*

Proof. Let $\varepsilon > 0$ to be given. Let $\mathcal{K}$ be a compact subset of $C(X,A)$ and $K = \{f(x) | \ f \in \mathcal{K}, \ x \in X\}$, then by Lemma 5.2 K is compact in A and $M' = sup\{\|f(x)\|; \ f \in \mathcal{K}, x \in X\} < \infty$ exists. Let $(a_\lambda)_{\lambda\in\Lambda} \in A\hat{\otimes}A$ be a compactly approximate diagonal for A. There exists $\lambda_0 \in \Lambda$ such that

$$ka_{\lambda_0} = a_{\lambda_0}k, \quad and \quad \|\pi(a_{\lambda_0})k - k\| < \frac{\varepsilon}{2}, \ k \in K. \tag{5.1}$$

Let $a_{\lambda_0} = \sum_{j=1}^{m} \alpha_j \otimes \beta_j$ be a representation of a_{λ_0} and $M = max\{\|a_{\lambda_0}\|_p, 1\}$. Let $c > 0$ to be the constant given in Lemma 3.1. Since $\mathcal{K}$ and K are both compact, there exist $y_i \in X$ and open neighborhoods $V_i' \subset X$ of y_i, $i = 1, \dots l'$ such that $X = \bigcup_{i=1}^{l'} V_i'$ and $\|f(s) - f(y_i)\| < \frac{\varepsilon}{8Mc}$, for $f \in \mathcal{K}$ and $s \in V_i', \ i = 1, \dots l'$. Let $h_1', \dots, h_{l'}' \in C(X)$ be a partition of unity for X in which $supp \ h_i' \subset V_i'$. As with the proof of Theorem 4.3, by construction of an approximate diagonal for $C(X)$ given in [1], there exists $x_1, \dots, x_l \in X$ and open neighborhoods V_i of x_l, $i = 1, \dots, l$ such that $X = \bigcup_{i=1}^{l} V_i$ and $\|f(s) - f(x_i)\|_A < \frac{\varepsilon}{2MM'l'}$ for $s \in V_i$, $f \in \mathcal{K}$, $i = 1, \dots, l$, and a partition of unity, $h_1, \dots, h_l$ for $X = \bigcup_{i=1}^{l} V_i$ such that $supp(h_i) \subset V_i$, and for $k = 1, \dots, l'$

$$\Big\| h_k' \Big(\sum_i^l u_i \otimes u_i \Big) - \Big(\sum_i^l u_i \otimes u_i \Big) h_k' \Big\|_p < \frac{\varepsilon}{2MM'l'}, \tag{5.2}$$

where $u_i = \sqrt{h_i}, \ i = 1, \dots, l$. Define $u = \sum_{i=1}^{l} \sum_{j=1}^{m} u_i\alpha_j \otimes u_i\beta_j$.

For any $f \in \mathcal{K}$, we show the following:

(1) $\|\pi(u)f - f\| < \varepsilon$,

(2) $\|fu - uf\|_p < \varepsilon$.

$$\begin{aligned}
\|\pi(u)f - f\| &= \Big\|\Big(\sum_{i,j} u_i\alpha_j u_i\beta_j\Big)f - f\Big\| \\
&= \Big\|\Big(\sum_{i,j} h_i\alpha_j\beta_j\Big)f - f\Big\| \\
&= \Big\|\Big(\sum_{i} h_i \sum_j \alpha_j\beta_j\Big)f - f\Big\| \\
&= sup_{t\in X}\Big\|\sum_{i} h_i(t) \sum_j \alpha_j\beta_j f(t) - f(t)\Big\| \\
&= sup_{t\in X}\|\pi(a_{\lambda_0})f(t) - f(t)\| \\
&\leq sup_{k\in K}\|\pi(a_{\lambda_0})k - k\| \\
&< \varepsilon
\end{aligned}$$

by (5.1).

For $f \in \mathcal{K}$, define $\hat{f} = \sum_i f(y_i)h_i'$.

$$\|fu - uf\|_p \leq \|fu - \hat{f}u\|_p + \|\hat{f}u - u\hat{f}\|_p + \|u\hat{f} - uf\|_p$$

$$\|f(y) - \hat{f}(y)\|_A = \|f(y) - \sum_i f(y_i)h_i'(y)\|_A$$

$$= \Big\| \sum_i \big(f(y) - f(y_i)\big) h_i'(y) \Big\|_A$$

$$\leq \sum_i \big\| f(y) - f(y_i) \big\|_A h_i'(y)$$

$$< \frac{\varepsilon}{8Mc}.$$

Taking supremum over y we have

$$\|f - \hat{f}\| \leq \frac{\varepsilon}{8Mc}$$

As in the proof of Theorem 4.3, $\|u\|_p \leq 2Mc$.

Therefore we have,

$$\|fu - uf\|_p \leq \|fu - \hat{f}u\|_p + \|\hat{f}u - u\hat{f}\|_p + \|u\hat{f} - uf\|_p$$

$$\leq 2\|f - \hat{f}\|\|u\|_p + \|\hat{f}u - u\hat{f}\|_p$$

$$< 2(2Mc)\frac{\varepsilon}{8Mc} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

In fact, with T defined as it was in the proof of Theorem 4.3,

$$\|\hat{f}u - u\hat{f}\|_p \leq \Big\| \sum_k f(y_k) h_k' \sum_{i,j} \alpha_j u_i \otimes \beta_j u_i - \sum_{i,j} \alpha_j u_i \otimes \beta_j u_i \sum_k f(y_k) h_k' \Big\|$$

$$\leq \Big\| \sum_{i,j,k} f(y_k) h_k' \alpha_j u_i \otimes \beta_j u_i - \alpha_j u_i \otimes \beta_j u_i f(y_k) h_k' \Big\|$$

$$
\begin{aligned}
&\leq \Big\| \sum_{k}^{l'} T(\sum_i (h'_k u_i \otimes u_i) \otimes \sum_j (f(y_k)\alpha_j \otimes \beta_j)) \\
&\qquad - T(\sum_i (u_i \otimes u_i h'_k) \otimes \sum_j (\alpha_j \otimes \beta_j) f(y_k)) \Big\| \\
&= \Big\| \sum_k T\Big(\sum_i (h'_k u_i \otimes u_i - u_i \otimes u_i h'_k) \otimes \sum_j f(y_k)(\alpha_j \otimes \beta_j) \Big) \\
&\qquad + T\Big(\sum_i (u_i \otimes u_i h'_k) \otimes \sum_j \big(f(y_k)(\alpha_j \otimes \beta_j) - (\alpha_j \otimes \beta_j) f(y_k) \big) \Big) \Big\| \\
&< \sum_k \Big(\Big\| \sum_i (h'_k u_i \otimes u_i - u_i \otimes u_i h'_k) \Big\| \|f(y_k)\| \Big\| \sum_j (\alpha_j \otimes \beta_j) \Big\| \\
&\qquad + \Big\| \sum_i (u_i \otimes u_i h'_k) \Big\| \Big\| \sum_j \big(f(y_k)(\alpha_j \otimes \beta_j) - (\alpha_j \otimes \beta_j) f(y_k) \big) \Big\| \Big) \\
&< l' \big(\frac{\varepsilon}{2M'Ml'} M'M + 0 \big) \\
&= \frac{\varepsilon}{2},
\end{aligned}
$$

using (5.1) and (5.2).

Therefore the net $\{u_{(\mathcal{K},\varepsilon)}\}$ is a compactly-invariant approximate diagonal for $C(X, A)$, where $\varepsilon > 0$, and $\mathcal{K}$ is a compact subset of $C(X, A)$. $\square$

5.3 Approximate diagonals for $C_0(X)$

Let X be a locally compact Hausdorff space. It is a known result that $C_b(X)$ is an amenable Banach algebra. It was first proved by Sheinburg [21] and later a direct proof was provided by M. Abtahi and Y. Zhang [1]. In [1] for X compact, the amenability of $C(X)$ was proved by constructing a bounded approximate diagonal for $C(X)$. For a locally compact space

X, the Banach algebra $C_0(X)$ of all continuous functions on X vanishing at infinity can be unitized and the unitization will be homomorphic to C(Y) for some compact Hausdorff space Y, which we already know is amenable. So as a Banach algebra $C_0(X)$ is amenable since its unitization is amenable. In this section we directly prove the amenability of $C_0(X)$ by constructing a bounded approximate diagonal for $C_0(X)$.

Theorem 5.4. *Let X be a locally compact Hausdorff space. Then, $C_0(X)$ has a bounded approximate diagonal.*

Let X be a locally compact Hausdorff space. Let F be a finite subset of $C_0(X)$ and let $\varepsilon > 0$ be given. Since F is finite, there exists a compact set $K \subset X$ such that $|a(s)| < \varepsilon/3$ for $a \in F$ and $s \in X\backslash K$. Let O be an open subset of X containing K and with compact closure K_1. The set K_1 can be covered by finitely many V_i, neighborhood of $x_i,\ i = 1, \ldots, n$ such that $|a(x_i) - a(s)| < \varepsilon/5c$ for $a \in F, s \in V_i, i = 1, \ldots, n$, where c is the constant described in Lemma 3.1 which is independent of K and of F. By [18] there exist $h'_i \in C_0(X),\ i = 1, 2, \ldots, n$ such that $0 \leq h'_i \leq 1$, $\sum_1^n h'_i = 1$ on K_1 and $supp(h'_i) \subset V_i,\ i = 1, \ldots, n$.

As X is Hausdorff, it is completely regular. Since K and $X\backslash O$ are closed and K is compact, there exists a continuous function $f : X \longrightarrow [0, 1]$ such that $f \equiv 1$ on K and $f \equiv 0$ on $X\backslash O$. Define $h_i : X \longrightarrow \mathbb{R}$ by $h_i = fh'_i,\ i = 1, \ldots, n$. Then $supp(h_i) \subset V_i \cap K_1,\ i = 1, 2, \ldots n$ and $\sum_i^n h_i = 1$ on K. Let $u_i := \sqrt{h_i}$ and let $u = u_{(\varepsilon,F)} := \sum_{i=1}^n u_i \otimes u_i$. We show $\{u_{(\varepsilon,F)}\}$ is a bounded approximate diagonal for $C_0(X)$.

Precisely we show:

(1) $\|u\|_p \leq 2c$;

(2) $\|\pi(u)a - a\| < \varepsilon, \ a \in F$;

(3) $\|au - ua\|_p < \varepsilon, \ a \in F$,

where c is the constant ensured in Lemma 3.1.

For (1) by Lemma 3.1 we have:

$$\begin{aligned}\|u\|_p &\leq c \left(\Big\| \sum_{i=1}^{n} |u_i|^2 \Big\| + \Big\| \sum_{i=1}^{n} |u_i|^2 \Big\| \right) \\ &= 2c \Big\| \sum_{i=1}^{n} |u_i|^2 \Big\| \\ &= 2c \Big\| \sum_{i=1}^{n} h_i \Big\| \\ &= 2c \ sup_{x \in K_1} \Big| \sum_{i=1}^{n} h_i(x) \Big| \\ &\leq 2c \ sup_{x \in K_1} |f(x)| \Big| \sum_{i=1}^{n} h'_i(x) \Big| \\ &\leq 2c.\end{aligned}$$

Moreover,

$$a \cdot u - u \cdot a = \sum_{i=1}^{n} \Big[\Big(a - a(x_k) \Big) u_i \otimes u_i - u_i \otimes u_i \Big(a - a(x_k) \Big) \Big].$$

Since $supp(u_i) \subset V_i$ and $|a(x) - a(x_i)| < \varepsilon/5c$ for $x \in V_i$, we have

$$\big|\big(a(x)-a(x_i)\big)u_i\big| < \frac{\varepsilon}{5c}|u_i|.$$

Similarly $\big|u_i\big(a(x)-a(x_i)\big)\big| < \frac{\varepsilon}{5c}|u_i|$. Denote $\eta = \varepsilon/5c$ and apply Lemma 3.1 again, to obtain:

$$\begin{aligned}
\|a.u-u.a\|_p &\leq \Big\|\sum_{i=1}^{n}\frac{\big(a-a(x_i)\big)}{\sqrt{\eta}}u_i\otimes\sqrt{\eta}u_i\Big\| + \Big\|\sum_{i=1}^{n}\sqrt{\eta}u_i\otimes\frac{\big(a-a(x_i)\big)}{\sqrt{\eta}}u_i\Big\| \\
&\leq c\,\Big[\Big\|\sum_{i=1}^{n}\frac{\big|\big(a-a(x_i)\big)u_i\big|^2}{\eta}\Big\| + \Big\|\sum_{i=1}^{n}\eta|u_i|^2\Big\|\Big] + c\,\Big[\Big\|\sum_{1}^{n}\eta|u_i|^2\Big\| + \\
&\qquad \Big\|\sum_{i=1}^{n}\frac{\big|\big(a-a(x_i)\big)u_i\big|^2}{\eta}\Big\|\Big] \\
&\leq 4c\eta \\
&= 4c\frac{\varepsilon}{5c} \\
&< \varepsilon.
\end{aligned}$$

We finally show $\|\pi(u)a-a\| \leq \varepsilon$, for $a \in F$.

We have,

$$\pi(u) = \sum_{i=1}^{n}u_i^2 = \sum_{i=1}^{n}h_i = f\sum_{i=1}^{n}h_i'.$$

Also,

$$\pi(u)a-a = \Big[(\sum_{i=1}^{n}h_i')fa-fa\Big] + \Big[fa-a\Big] \tag{5.3}$$

Since $supp(f) \subset K_1$ and $\sum_1^n h_i' = 1$ on K_1, we see that the first term on the right hand side of the equation 5.3 is zero.

Since $f \equiv 1$ on K and $|f| \leq 1$, we have

$$\begin{aligned} \|\pi(u)a - a\| &= \|fa - a\| \\ &= sup_{x \in X \setminus K} \Big| \Big(fa\Big)(x) - a(x) \Big| \\ &\leq 2 sup_{x \in X \setminus K} |a(x)| \\ &< \varepsilon. \end{aligned}$$

Therefore we can conclude $\{u_{(\varepsilon,F)}\},\ \varepsilon > 0$ and F a finite subset of $C_0(X)$, is a bounded approximate diagonal satisfying $\|u_{(\varepsilon,F)}\| \leq 2c$.

5.4 Open Questions

We complete this thesis with couple of questions for future study.

Question 5.5. *Let X be a compact Hausdorff space. If the Banach algebra A has merely an approximate diagonal, is it true that $C(X, A)$ has an approximate diagonal?*

Question 5.6. *For a compact Hausdorff space X and a Banach algebra A, if A has a central approximate diagonal (not necessarily bounded), is it true that $C(X, A)$ has an approximate diagonal?*

Question 5.7. *For a locally compact Hausdorff space X and a Banach algebra A, if A*

has a compactly-invariant approximate diagonal, is it true that $C(X, A)$ has a compactly-invariant approximate diagonal?

Question 5.8. *For a locally compact Hausdorff space X and a Banach algebra A, is it true that $C_0(X, A)$ has a bounded approximate diagonal?*

Question 5.9. *For a locally compact Hausdorff space X and a Banach algebra A, if A is approximately amenable, what can we say about $C(X, A)$?*

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