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ASYMPTOTIC ANALYSIS OF THE INFLECTION POINTS OF THE BESSEL FUNCTION $J_\upsilon(x)$

by

THOMAS ROBERT LANG

A thesis
presented to the University of Manitoba
in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE

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ABSTRACT

Recently, L. Lorch and P. Szego have shown that the positive inflection points $j''_{\upsilon,k}$ of the Bessel function $J_\upsilon(x)$ are increasing functions of υ for $k = 1$ when $\upsilon > 0$, and for $k = 2,3,...$ when $0 < \upsilon \leq 3838$. We show that for $k = 2,3,...$, $j''_{\upsilon,k}$ is an increasing function of υ when $\upsilon \geq 10$.

Our method involves the use of an integral representation of $dj''_{\upsilon,k}/d\upsilon$ given by

$$\frac{dj''_{\upsilon,k}}{d\upsilon} = c_{\upsilon,k}\left\{\int_0^\infty \frac{J_\upsilon^2(t)}{t}dt - \int_{j''_{\upsilon,k}}^\infty \frac{J_\upsilon^2(t)}{t}dt - J_\upsilon^2(j''_{\upsilon,k})\right\},$$

where $c_{\upsilon,k}$ is some positive quantity for $k = 2,3,...$ when $\upsilon > 1$. Asymptotic approximations as $\upsilon \to \infty$, complete with error bounds, are found for the integral

$$\int_{j''_{\upsilon,2}}^\infty \frac{J_\upsilon^2(t)}{t}dt$$

and for the quantity

$$J_\upsilon^2(j''_{\upsilon,k}) \quad \text{(fixed k)}.$$

Using these approximations, we show that for $k = 2,3,...$, $dj''_{\upsilon,k}/d\upsilon > 0$ when $\upsilon \geq 10$, and conclude from this result and that of Lorch and Szego that the positive inflection points $j''_{\upsilon,k}$ of $J_\upsilon(x)$ are increasing functions of υ when $\upsilon > 0$.

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I dedicate this small book to my loving parents **Norm & Mary** who have made it all possible for me.

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INTRODUCTION

Let $j_{\upsilon,1}, j_{\upsilon,2}, \ldots$ denote the positive zeros of the Bessel function $J_\upsilon(x)$, and similarly, let $j'_{\upsilon,1}, j'_{\upsilon,2}, \ldots$ denote the positive zeros of $J'_\upsilon(x)$, which are the positive critical points of $J_\upsilon(x)$. It is well-known that when υ is positive, both $j_{\upsilon,k}$ and $j'_{\upsilon,k}$ are increasing functions of υ; see, e.g., [**OLV5**, pp 246 and 248]. Recently, Lorch and Szego [**LOR**] have attempted to show that the same is true for the positive zeros $j''_{\upsilon,1}, j''_{\upsilon,2}, \ldots$ of $J''_\upsilon(x)$, which are the positive inflection points of $J_\upsilon(x)$. They succeeded in proving that this statement holds for k=1, but for k = 2,3,..., they have proved only that it is true when $0 < \upsilon \leq 3838$. The main objective of this thesis is to show that for k = 2,3,..., the statement is also true when $\upsilon > 3838$, and so conclude that $j''_{\upsilon,k}$ is, indeed, an increasing function of υ when $\upsilon > 0$. To settle the case for $\upsilon > 3838$, we begin in the same fashion as Lorch and Szego by examining the steps in the proof that $j_{\upsilon,k}$ and $j'_{\upsilon,k}$ are increasing functions of υ ($\upsilon > 0$). To show, for instance, that $j_{\upsilon,k}$ is an increasing function of υ ($\upsilon > 0$), we merely differentiate the equation $J_\upsilon(j_{\upsilon,k}) = 0$ and utilize the identity

$$\int \frac{J_\upsilon^2(x)}{x}\,dx = \frac{x}{2\upsilon}\left\{ J_\upsilon(x)\,\frac{\partial J'_\upsilon(x)}{\partial \upsilon} - J'_\upsilon(x)\,\frac{\partial J_\upsilon(x)}{\partial \upsilon} \right\}$$

to obtain

$$\frac{dj_{\upsilon,k}}{d\upsilon} = \frac{2\upsilon}{j_{\upsilon,k}\left\{J'_\upsilon(j_{\upsilon,k})\right\}^2} \int_0^{j_{\upsilon,k}} \frac{J_\upsilon^2(x)}{x}\,dx.$$

It is clear, then, that $dj_{\upsilon,k}/d\upsilon > 0$, and so the result for $j_{\upsilon,k}$ follows. In a similar manner, we show that

$$\frac{dj''_{\upsilon,k}}{d\upsilon} = c_{\upsilon,k}\, G(j''_{\upsilon,k}),$$

where

$$G(x) = \int_0^x \frac{J_\upsilon^2(t)}{t}\,dt - J_\upsilon^2(x) = \int_0^\infty \frac{J_\upsilon^2(t)}{t}\,dt - \int_x^\infty \frac{J_\upsilon^2(t)}{t}\,dt - J_\upsilon^2(x),$$

and where

$$c_{\upsilon,k} = \frac{2\upsilon}{(j''_{\upsilon,k})^2 J_\upsilon(j''_{\upsilon,k})\, J'''_\upsilon(j''_{\upsilon,k})}$$

is some positive quantity for $\upsilon > 1$. We then proceed to show that $G(j''_{\upsilon,k}) > 0$ when $\upsilon \geq 10$, and so conclude that $dj''_{\upsilon,k}/d\upsilon > 0$ if $\upsilon \geq 10$, and hence that $j''_{\upsilon,k}$ is an increasing function of υ if $\upsilon \geq 10$.

To show that

$$G(j''_{\upsilon,k}) = \int_0^\infty \frac{J_\upsilon^2(t)}{t}\,dt - \int_{j''_{\upsilon,k}}^\infty \frac{J_\upsilon^2(t)}{t}\,dt - J_\upsilon^2(j''_{\upsilon,k})$$

is positive for $\upsilon \geq 10$, one encounters the problem of obtaining asymptotic approximations for the integral

$$F(\upsilon,k) = \int_{j''_{\upsilon,k}}^\infty \frac{J_\upsilon^2(t)}{t}\,dt \quad \text{as } \upsilon \to \infty \quad \text{(fixed k)},$$

and for the quantity

$$J_\upsilon^2(j''_{\upsilon,k}) \quad \text{as } \upsilon \to \infty \quad \text{(fixed k)}.$$

In this thesis it will be shown that

$$F(\upsilon,2) = \frac{1}{2\upsilon} - \frac{0.812634}{\upsilon^{4/3}} + \frac{\varepsilon_1(\upsilon)}{\upsilon^2},$$

where

$$|\varepsilon_1(\upsilon)| \leq 2.086 \qquad \text{if } \upsilon \geq 10,$$

and that

$$0 \le J_\upsilon^2(j''_{\upsilon,k}) \le \varepsilon_2(\upsilon)/\upsilon^2 \qquad \text{for all } k \ge 2,$$

where

$$0 \le \varepsilon_2(\upsilon) \le 0.215 \qquad \text{if } \upsilon \ge 10.$$

Throughout the thesis, two or more significant figures are given for numbers; the last significant figure is the result of rounding to the nearest digit except for numbers in inequalities, which are rounded to obtain the weakest inequality. Now, it is known that

$$\int_0^\infty \frac{J_\upsilon^2(t)}{t}\,dt = \frac{1}{2\upsilon},$$

and since $F(\upsilon,k) \le F(\upsilon,2)$ for $k = 2,3,\ldots,$ then

$$\begin{aligned} G(j''_{\upsilon,k}) &= \frac{1}{2\upsilon} - F(\upsilon,k) - J_\upsilon^2(j''_{\upsilon,k}) \\ &\ge \frac{1}{2\upsilon} - F(\upsilon,2) - J_\upsilon^2(j''_{\upsilon,k}) \\ &= \frac{0.812634}{\upsilon^{4/3}} + \frac{\varepsilon(\upsilon)}{\upsilon^2}, \qquad k = 2,3,\ldots, \end{aligned}$$

where

$$|\varepsilon(\upsilon)| \le |\varepsilon_1(\upsilon)| + |\varepsilon_2(\upsilon)| \le 2.302 \qquad \text{if } \upsilon \ge 10.$$

From this inequality, one is led to the conclusion that for $k = 2,3,\ldots,$

$$G(j''_{\upsilon,k}) > 0 \text{ when } \upsilon \ge 10.$$

The function $J_\upsilon(x)$ has the asymptotic approximation

$$J_\upsilon(\upsilon x) \sim \frac{1}{\upsilon^{1/3}}\left(\frac{4\zeta}{1-x^2}\right)^{1/4} \mathrm{Ai}(\upsilon^{2/3}\zeta) \qquad \text{as } \upsilon \to \infty,$$

uniformly with respect to x in the interval $(0,\infty)$, where ζ and x are related in a one-to-one manner by the equation

$$\zeta\left(\frac{d\zeta}{dx}\right)^2 = \frac{1-x^2}{x^2},$$

branches being chosen in accordance with the requirement that ζ is real when $x > 0$. To tackle the problem of obtaining an asymptotic approximation for $F(\upsilon,2) = \int_{j''_{\upsilon,2}}^\infty \frac{J_\upsilon^2(t)}{t}\,dt$, we replace the Bessel function $J_\upsilon(t)$ by its asymptotic approximation and formally obtain the integral

$$\int_{\zeta_\upsilon}^{\infty} \varphi^4(-\zeta)\, \mathrm{Ai}^2(-\upsilon^{2/3}\zeta)\, d\zeta\,,$$

where

$$\varphi(\zeta) = \left(\frac{4\zeta}{1 - x^2}\right)^{1/4}.$$

The lower limit of integration $\overline{\zeta}_\upsilon$ depends on υ and approaches 0 as $\upsilon \to \infty$. We are thus led to the consideration of integrals of the form

$$I(\lambda) = \int_0^{\infty} f(t)\, h(\lambda t)\, dt$$

with $f(t) = \varphi^4(-t)$, $h(t) = \mathrm{Ai}^2(-t)$ and $\lambda = \upsilon^{2/3}$ in the present case. A detailed treatment of asymptotic expansions of the Mellin convolution $I(\lambda)$, complete with explicit error bounds, can be found in **[WON1]**. The derivation of the asymptotic expansion of $I(\lambda)$ is given in Chapter 3, §2. The exact relation between $F(\upsilon,2)$ and $I(\lambda)$ is given in §§ 3 and 4 of the same chapter.

Finally, we obtain a bound for $J_\upsilon^2(j''_{\upsilon,k})$ via a one-term asymptotic approximation of $J_\upsilon(\upsilon x)$. In order to determine such a bound, it is necessary to first obtain bounds for the zeros $j''_{\upsilon,k}$ of $J''_\upsilon(t)$. More specifically, it is necessary to obtain bounds for the value ζ corresponding to $x = j''_{\upsilon,k}/\upsilon$. From Bessel's equation

$$x^2W'' + xW' + (x^2 - \upsilon^2)W = 0$$

which is satisfied, for instance, by $W = J_\upsilon(x)$, we find that

$$J''_\upsilon(\upsilon x) = \frac{1 - x^2}{x^2} J_\upsilon(\upsilon x) - \frac{1}{\upsilon x} J'_\upsilon(\upsilon x).$$

Replacing each of $J_\upsilon(\upsilon x)$ and $J'_\upsilon(\upsilon x)$ by their one-term approximations leads to the result

$$J''_\upsilon(\upsilon x) \sim \frac{1}{\upsilon^{1/3}} \frac{4\zeta}{x^2\varphi^3(\zeta)} \mathrm{Ai}(\upsilon^{2/3}\zeta).$$

Tricomi **[TRI]** pointed out that the asymptotic behaviour of the zeros of a function can be deduced from the asymptotic behaviour of the function itself. Hethcote **[HET3]**, moreover, established a theorem for the approximation of real zeros of functions by the zeros of a comparison function. It is this theorem that we apply, with $Ai(\upsilon^{2/3}\zeta)$ as the comparison function, to determine bounds for ζ corresponding to $x = j''_{\upsilon,k}/\upsilon$. The result is

$$\zeta = \upsilon^{-2/3} a_k + \eta_{\upsilon,k},$$

where a_k denotes the $k^{\underline{th}}$ negative zero of Ai(x), and where

$$|\eta_{\upsilon,k}| \leq d_k / \upsilon^{4/3} \qquad \text{if } \upsilon \geq 10,$$

with $d_2 = 0.3743$, and $d_{k+1} < d_k \leq d_2 \ \forall k \geq 2$. This result is dealt with in Chapter 2.

One final remark: although only one-term asymptotic approximations are necessary for our purposes, we derive (see Appendix III) the first few terms of the expansions of $J''_{\upsilon}(\upsilon x)$ and $j''_{\upsilon,k}$. These are useful not only for assessing the "sharpness" of our error bounds (by comparing bounds with coefficients) but also for the sake of having them calculated.

CHAPTER 1

AN INVESTIGATION INTO THE MONOTONICITY PROPERTY OF THE INFLECTION POINTS OF $J_\upsilon(x)$

1. Introduction

Let $j''_{\upsilon,1}$, $j''_{\upsilon,2}$, ... denote, in ascending order of magnitude, the positive zeros of $J''_\upsilon(x)$, which are the positive inflection points of the Bessel function $J_\upsilon(x)$. Since it is true that when υ is positive, both $j_{\upsilon,k}$ and $j'_{\upsilon,k}$ are increasing functions of υ [**OLV5**, pp. 246 and 248], a reasonable question to ask is whether or not the same is true for $j''_{\upsilon,k}$. It is now known [**LOR**] that $j''_{\upsilon,k}$ does increase in υ for k = 1 when $\upsilon > 0$ and for k = 2,3,... when $0 < \upsilon \leq 3838$. Here we shall show that the same is true for k = 2,3,... when $\upsilon > 3838$. Hence, $j''_{\upsilon,k}$ does, indeed, increase in υ when $\upsilon > 0$. Our method is based on asymptotic analysis, and the reason is that we are concerned primarily with large (or moderately large) values of the parameter υ.

We begin this chapter by examining the steps in the proof that $j_{\upsilon,k}$ and $j'_{\upsilon,k}$ are increasing functions of υ ($\upsilon > 0$). This will give us an idea of the method of approach for $j''_{\upsilon,k}$. We apply this method to $j''_{\upsilon,k}$ and, in subsequent chapters, follow up with the eventual result.

2. The Monotonicity of $j_{\upsilon,k}$ and $j'_{\upsilon,k}$

Following in the footsteps of Olver [**OLV5**, pp. 246 - 248], we show that for positive υ, $j_{\upsilon,k}$ is an increasing function of υ. Differentiation of the equation $J_\upsilon(j_{\upsilon,k}) = 0$ with respect to υ yields

$$(2.1) \qquad J'_\upsilon(j_{\upsilon,k})\frac{dj_{\upsilon,k}}{d\upsilon} + \left[\frac{\partial J_\upsilon(x)}{\partial \upsilon}\right]_{x=j_{\upsilon,k}} = 0.$$

To evaluate the second term, we use the identity

$$(2.2)\qquad \int \frac{J_\mu(x)\, J_\upsilon(x)}{x}\, dx = \frac{x\{J_\mu'(x)\, J_\upsilon(x) - J_\mu(x)\, J_\upsilon'(x)\}}{\mu^2 - \upsilon^2} \qquad (\mu^2 \neq \upsilon^2).$$

Since $J_\upsilon(x)$ is a solution of Bessel's differential equation[1]

$$(2.3)\qquad x^2W'' + xW' + (x^2 - \upsilon^2)W = 0$$

we have

$$(2.4)\qquad x^2J_\upsilon''(x) + xJ_\upsilon'(x) + (x^2 - \upsilon^2)J_\upsilon(x) = 0.$$

Using (2.4) one may readily verify that the derivative of the right-hand-side of (2.2) yields the integrand on the left. Noting that the right-hand-side can be written as

$$\frac{x}{\mu + \upsilon}\left\{\frac{J_\upsilon(x)\, [J_\mu'(x) - J_\upsilon'(x)] - J_\upsilon'(x)\, [J_\mu(x) - J_\upsilon(x)]}{\mu - \upsilon}\right\},$$

and letting $\mu \to \upsilon$ in (2.2), we obtain

$$(2.5)\qquad \int \frac{J_\upsilon^2(x)}{x}\, dx = \frac{x}{2\upsilon}\left\{J_\upsilon(x)\, \frac{\partial J_\upsilon'(x)}{\partial \upsilon} - J_\upsilon'(x)\, \frac{\partial J_\upsilon(x)}{\partial \upsilon}\right\}.$$

It is known (see, e.g., [**WAT**, p. 404, Eq. (7)]) that

$$\int_0^\infty \frac{J_\upsilon^2(x)}{x}\, dx = \frac{1}{2\upsilon},$$

and so the integral in (2.5) converges on any interval $(0,x)$, $x > 0$. Provided that $\upsilon > 0$, the integration limits in (2.5) can be set equal to 0 and $j_{\upsilon,k}$. Since $J_\upsilon(x)$ vanishes at $j_{\upsilon,k}$, we have

$$\int_0^{j_{\upsilon,k}} \frac{J_\upsilon^2(x)}{x}\, dx = -\frac{j_{\upsilon,k}}{2\upsilon}\, J_\upsilon'(j_{\upsilon,k}) \left[\frac{\partial J_\upsilon(x)}{\partial \upsilon}\right]_{x=j_{\upsilon,k}}.$$

Then by substitution in (2.1) we obtain

$$(2.6)\qquad \frac{dj_{\upsilon,k}}{d\upsilon} = \frac{2\upsilon}{j_{\upsilon,k}\left\{J_\upsilon'(j_{\upsilon,k})\right\}^2} \int_0^{j_{\upsilon,k}} \frac{J_\upsilon^2(x)}{x}\, dx.$$

[1] There is a considerable amount of literature devoted to Bessel's equation and its solutions; the best-known reference is undoubtedly Watson's [**WAT**].

From (2.6) it is clear that $dj_{\upsilon,k}/d\upsilon > 0$, and so the result for $j_{\upsilon,k}$ follows.

In a similar manner, we show that for υ positive, $j'_{\upsilon,k}$ is an increasing function of υ. Differentiation of the equation $J'_\upsilon(j'_{\upsilon,k}) = 0$ yields

$$(2.7) \qquad J''_\upsilon(j'_{\upsilon,k})\frac{dj'_{\upsilon,k}}{d\upsilon} + \left[\frac{\partial J'_\upsilon(x)}{\partial\upsilon}\right]_{x=j'_{\upsilon,k}} = 0.$$

Since $J'_\upsilon(j'_{\upsilon,k}) = 0$, we have from (2.4) with $x = j'_{\upsilon,k}$

$$(2.8) \qquad J''_\upsilon(j'_{\upsilon,k}) = \left[\left(\upsilon/j'_{\upsilon,k}\right)^2 - 1\right]J_\upsilon(j'_{\upsilon,k}).$$

Furthermore, provided that $\upsilon > 0$, the integration limits in (2.5) can be set equal to 0 and $j'_{\upsilon,k}$, yielding

$$\int_0^{j'_{\upsilon,k}} \frac{J_\upsilon^2(x)}{x}\,dx = \frac{j'_{\upsilon,k}}{2\upsilon}J_\upsilon(j'_{\upsilon,k})\left[\frac{\partial J'_\upsilon(x)}{\partial\upsilon}\right]_{x=j'_{\upsilon,k}}.$$

Substitution of this equation and equation (2.8) in (2.7) yields

$$\frac{dj'_{\upsilon,k}}{d\upsilon} = \frac{j'_{\upsilon,k}}{j'^2_{\upsilon,k} - \upsilon^2}\frac{2\upsilon}{J_\upsilon^2(j'_{\upsilon,k})}\int_0^{j'_{\upsilon,k}} \frac{J_\upsilon^2(x)}{x}\,dx \qquad (\upsilon > 0).$$

Noting that $j'_{\upsilon,1} > \upsilon$ for $\upsilon > 0$ [**OLV5**, p. 246], it follows that $dj'_{\upsilon,k}/d\upsilon > 0$ whenever $\upsilon > 0$, since $j'_{\upsilon,k} \geq j'_{\upsilon,1}$ $\forall k \geq 1$. Hence, $j'_{\upsilon,k}$ is an increasing function of υ $(\upsilon > 0)$.

We are now in a position to consider the monotonicity property of $j''_{\upsilon,k}$.

3. The Monotonicity of $j''_{\upsilon,k}$: Initial Procedure

Proceeding in a manner similar to that outlined in §2 above, we begin by considering the equation $J''_\upsilon(j''_{\upsilon,k}) = 0$. Differentiation of this equation with respect to υ yields

$$(3.1) \qquad J'''_\upsilon(j''_{\upsilon,k})\frac{dj''_{\upsilon,k}}{d\upsilon} + \left[\frac{\partial J''_\upsilon(x)}{\partial\upsilon}\right]_{x=j''_{\upsilon,k}} = 0.$$

From Bessel's equation (2.4), we have

$$J''_\upsilon(x) = -\frac{1}{x}J'_\upsilon(x) - \left(1 - \frac{\upsilon^2}{x^2}\right)J_\upsilon(x).$$

Differentiation with respect to υ yields

$$(3.2) \qquad \frac{\partial J''_\upsilon(x)}{\partial\upsilon} = -\frac{1}{x}\frac{\partial J'_\upsilon(x)}{\partial\upsilon} + \frac{2\upsilon}{x^2}J_\upsilon(x) - \left(1 - \frac{\upsilon^2}{x^2}\right)\frac{\partial J_\upsilon(x)}{\partial\upsilon}.$$

Again from (2.4) we have

$$1 - \frac{\upsilon^2}{x^2} = -\frac{J''_\upsilon(x) + \frac{1}{x}J'_\upsilon(x)}{J_\upsilon(x)},$$

and so (3.2) becomes

$$(3.3) \qquad \frac{\partial J''_\upsilon(x)}{\partial\upsilon} = -\frac{1}{xJ_\upsilon(x)}\left\{J_\upsilon(x)\frac{\partial J'_\upsilon(x)}{\partial\upsilon} - J'_\upsilon(x)\frac{\partial J_\upsilon(x)}{\partial\upsilon} - \frac{2\upsilon}{x}J_\upsilon^2(x) - xJ''_\upsilon(x)\frac{\partial J_\upsilon(x)}{\partial\upsilon}\right\}.$$

If we let

$$G(x) = \frac{x}{2\upsilon}\left[J_\upsilon(x)\frac{\partial J'_\upsilon(x)}{\partial\upsilon} - J'_\upsilon(x)\frac{\partial J_\upsilon(x)}{\partial\upsilon}\right] - J_\upsilon^2(x),$$

then (cf. (2.5))

$$(3.4) \qquad G(x) = \int_0^x \frac{J_\upsilon^2(x)}{x}dx - J_\upsilon^2(x),$$

and

$$(3.5) \qquad \frac{\partial J''_\upsilon(x)}{\partial\upsilon} = -\frac{1}{xJ_\upsilon(x)}\left\{\frac{2\upsilon}{x}G(x) - x\,J''_\upsilon(x)\frac{\partial J_\upsilon(x)}{\partial\upsilon}\right\}.$$

Thus, provided that $\upsilon > 0$, we obtain from (3.5) with $x = j''_{\upsilon,k}$

$$(3.6) \qquad \left[\frac{\partial J''_\upsilon(x)}{\partial\upsilon}\right]_{x=j''_{\upsilon,k}} = -\frac{2\upsilon}{(j''_{\upsilon,k})^2 J_\upsilon(j''_{\upsilon,k})}G(j''_{\upsilon,k}),$$

since $J''_\upsilon(j''_{\upsilon,k}) = 0$. Substitution of (3.6) in (3.1) yields for $\upsilon > 0$,

$$(3.7) \qquad \frac{dj''_{\nu,k}}{d\upsilon} = c_{\upsilon,k}\, G(j''_{\upsilon,k}) ,$$

where

$$(3.8) \qquad c_{\upsilon,k} = \frac{2\upsilon}{(j''_{\upsilon,k})^2\, J_\upsilon(j''_{\upsilon,k})\, J'''_\upsilon(j''_{\upsilon,k})} .$$

It is known (see **[LOR]**) that for k = 2,3,...,

$$(3.9) \qquad c_{\upsilon,k} > 0 \qquad\qquad \text{if } \upsilon > 1.$$

Thus, to show that for k = 2,3,..., $dj''_{\upsilon,k}/d\upsilon > 0$ if $\upsilon \geq 10$, we must show (in light of (3.7)) that for k = 2,3,...,

$$(3.10) \qquad G(j''_{\upsilon,k}) > 0 \qquad\qquad \text{whenever } \upsilon \geq 10.$$

CHAPTER 2
THE INFLECTION POINTS OF $J_\upsilon(x)$

1. Introduction

In §2 of this chapter, it is shown that $J''_\upsilon(\upsilon x)$ has the asymptotic approximation

$$(1.1) \qquad J''_\upsilon(\upsilon x) \sim \frac{1}{\upsilon^{1/3}} \frac{4\zeta}{x^2\varphi^3(\zeta)} \mathrm{Ai}(\upsilon^{2/3}\zeta), \qquad \text{as } \upsilon \to \infty,$$

uniformly with respect to x in the interval $(0,\infty)$, where ζ and x are related in a one-to-one manner by the equation

$$(1.2) \qquad \zeta\left(\frac{d\zeta}{dx}\right)^2 = \frac{1-x^2}{x^2},$$

branches being chosen in accordance with the requirement that ζ is real when x is positive. Explicit expressions for ζ in terms of x are given in §2 (see (2.4) and (2.5)). The derivation of (1.1) relies on Olver's expansions for $J_\upsilon(\upsilon x)$ and $J'_\upsilon(\upsilon x)$, both valid for $\upsilon > 0$ and $x > 0$. Very precise bounds for the remainder terms associated with these expansions have been constructed by him. Since our work involves the use of these delicate error bounds, we have included them in §2.

Tricomi **[TRI]** pointed out that the asymptotic behaviour of the zeros of a function can be deduced from the asymptotic behaviour of the function itself. The goal of this chapter is to establish bounds for the value ζ corresponding to $x = j''_{\upsilon,k}/\upsilon$. The asymptotic behaviour of the zeros $j''_{\upsilon,k}$ of $J''_\upsilon(t)$ can then be deduced from these bounds. To determine such bounds, we apply a theorem[1] of Hethcote (see **[HET3]**; see also **[HET1]**) which was derived from a method of Gatteschi **[GAT]**. This theorem approximates real zeros of functions by the zeros of a comparison function. The comparison function in our case, as evidenced by (1.1), is the Airy function $\mathrm{Ai}(\upsilon^{2/3}\zeta)$. The final result is

$$(1.3) \qquad \zeta = \upsilon^{-2/3}a_k + \eta_{\upsilon,k},$$

---

[1] This theorem is duplicated in Appendix IV.

where a_k denotes the $k^{\underline{th}}$ negative zero of the Airy function Ai(x), and where

(1.4) $\quad |\eta_{\upsilon,k}| \leq d_k / \upsilon^{4/3} \qquad$ if $\upsilon \geq 10$,

with $d_2 = 0.3743$, and $d_{k+1} < d_k \leq d_2 \ \forall k \geq 2$.

The bound on $\eta_{\upsilon,k}$ is established in §§3 and 5; the cases k = 2,3,...,9 are dealt with in §3, and $k \geq 10$ in §5. The reason why we split these cases in such a fashion is simple: for k = 2,3,...,9, accurate numerical tables of values associated with the Airy function exist[1], and so these cases can be dispensed with numerically. On the other hand, the cases $k \geq 10$ need to be treated in a different manner. The manner in which we treat these cases is via asymptotic analysis, since we are interested in values of $k \to \infty$.

The bounds for ζ corresponding to $x = j''_{\upsilon,k}/\upsilon$ for the cases $k \geq 10$ can be determined by again applying Hethcote's theorem. In order to do so, it is necessary to first determine bounds for the zeros a_k and a'_k of Ai(x) and Ai'(x), respectively. In §4 of this chapter, we establish a bound for a'_k (a bound for a_k is already known.) Furthermore, in our investigations we find it necessary to compute bounds for the value $\rho_k = \frac{1}{4}(a_k - a'_{k+1})$ and for the values $\alpha_k = -(a_k - \rho_k)$ and $\beta_k = -(a_k + \rho_k)$. These bounds are established in §4.

We begin now with our discussion on $J''_\upsilon(\upsilon x)$.

2. Derivation of the Asymptotic Approximation of $J''_\upsilon(\upsilon x)$: Olver's Error Bounds

Bessel's equation (1.2.4) with x replaced by υx yields

$$J''_\upsilon(\upsilon x) = \frac{1 - x^2}{x^2} J_\upsilon(\upsilon x) - \frac{1}{\upsilon x} J'_\upsilon(\upsilon x). \tag{2.1}$$

From (2.1) the asymptotic approximation (1.1) is obtained by replacing each of $J_\upsilon(\upsilon x)$ and $J'_\upsilon(\upsilon x)$ by the leading terms of their uniform asymptotic expansions. The uniform asymptotic expansions of $J_\upsilon(\upsilon x)$ and $J'_\upsilon(\upsilon x)$, valid for $\upsilon > 0$ and $x > 0$, are given by

[1] See [**ABR**, pp. 476 - 478]. These tables are in part the work of F.W.J. Olver.

$$J_\upsilon(\upsilon x) \sim \frac{\varphi(\zeta)}{\upsilon^{1/3}}\left\{ \mathrm{Ai}(\upsilon^{2/3}\zeta)\left[1 + \frac{A_1(\zeta)}{\upsilon^2} + \frac{A_2(\zeta)}{\upsilon^4} + \cdots\right] + \frac{\mathrm{Ai}'(\upsilon^{2/3}\zeta)}{\upsilon^{4/3}}\left[B_0(\zeta) + \frac{B_1(\zeta)}{\upsilon^2} + \cdots\right]\right\} \tag{2.2}$$

and

$$J_\upsilon'(\upsilon x) \sim -\frac{\psi(\zeta)}{\upsilon^{2/3}}\left\{ \frac{\mathrm{Ai}(\upsilon^{2/3}\zeta)}{\upsilon^{2/3}}\left[C_0(\zeta) + \frac{C_1(\zeta)}{\upsilon^2} + \cdots\right] + \mathrm{Ai}'(\upsilon^{2/3}\zeta)\left[1 + \frac{D_1(\zeta)}{\upsilon^2} + \frac{D_2(\zeta)}{\upsilon^4} + \cdots\right]\right\} \tag{2.3}$$

respectively, where

$$\zeta = \left\{\frac{3}{2}\int_x^1 \frac{(1-x^2)^{1/2}}{x}\,dx\right\}^{2/3} = \left\{\frac{3}{2}\ln\frac{1+(1-x^2)^{1/2}}{x} - \frac{3}{2}(1-x^2)^{1/2}\right\}^{2/3}, \qquad 0 < x \le 1, \tag{2.4}$$

$$\zeta = -\left\{\frac{3}{2}\int_1^x \frac{(x^2-1)^{1/2}}{x}\,dx\right\}^{2/3} = -\left\{\frac{3}{2}(x^2-1)^{1/2} - \frac{3}{2}\sec^{-1}x\right\}^{2/3}, \qquad x \ge 1, \tag{2.5}$$

$$\varphi(\zeta) = \left(\frac{4\zeta}{1-x^2}\right)^{1/4}, \tag{2.6}$$

$$\psi(\zeta) = 2/\{x\varphi(\zeta)\}. \tag{2.7}$$

The coefficients $A_s(\zeta)$ and $B_s(\zeta)$ in (2.2), and $C_s(\zeta)$ and $D_s(\zeta)$ in (2.3), satisfy a set of recurrence relations, and are holomorphic functions in a region containing the real axis.

These results are due to Olver, and can be found in **[OLV2]** and **[OLV5**, Ch. 11]. Precise bounds for the remainder terms in each of the series (2.2) and (2.3) have also been constructed by him; see **[OLV4]** and **[OLV5]**. To state these results, we first recall from **[OLV3**, p. 750] the modulus functions M(x) and N(x), and the weight function E(x) associated with the Airy functions Ai(x) and Bi(x); see also **[OLV5**, p. 395].

$$E(x) = \exp(\tfrac{2}{3}x^{3/2}), \qquad x > 0,$$
(2.8)
$$E(x) = 1, \qquad x \le 0; \qquad E^{-1}(x) = 1/E(x),$$

$$M(x) = \{E^2(x)\,Ai^2(x) + E^{-2}(x)\,Bi^2(x)\}^{1/2},$$
(2.9)
$$N(x) = \{E^2(x)\,Ai'^2(x) + E^{-2}(x)\,Bi'^2(x)\}^{1/2},$$

(2.10) $$\lambda = \max_{(-\infty,\infty)}\left\{\pi\,|x|^{1/2}\,M^2(x)\right\} = 1.430,$$

(2.11) $$\mu = \max_{(-\infty,0)}\left\{\pi\,|x|^{1/2}\,M^2(x)\right\} = 1 \qquad ([\mathbf{OLV3}, \text{p. } 751]).$$

Olver's results then state that

(2.12) $$J_\upsilon(\upsilon x) = \frac{1}{1+\delta_{2n+1}}\frac{\varphi(\zeta)}{\upsilon^{1/3}}\left\{Ai(\upsilon^{2/3}\zeta)\sum_{s=0}^{n}\frac{A_s(\zeta)}{\upsilon^{2s}} + \frac{Ai'(\upsilon^{2/3}\zeta)}{\upsilon^{4/3}}\sum_{s=0}^{n-1}\frac{B_s(\zeta)}{\upsilon^{2s}} + \varepsilon_{2n+1,1}(\upsilon,\zeta)\right\},$$

where

(2.13) $$|\,\delta_{2n+1}\,| \le 2\exp\left\{\frac{2\lambda}{\upsilon}\,\mathcal{V}_{-\infty,\infty}\,(|\zeta|^{1/2}B_0)\right\}\upsilon^{-2n-1}\mathcal{V}_{-\infty,\infty}\,(|\zeta|^{1/2}B_n),$$

(2.14) $$|\,\varepsilon_{2n+1,1}(\upsilon,\zeta)\,| \le \frac{2M(\upsilon^{2/3}\zeta)}{E(\upsilon^{2/3}\zeta)} \exp\left\{\frac{2\lambda}{\upsilon}\,\mathcal{V}_{\zeta,\infty}\,(|\zeta|^{1/2}B_0)\right\} \frac{\mathcal{V}_{\zeta,\infty}\,(|\zeta|^{1/2}B_n)}{\upsilon^{2n+1}},$$

and

(2.15) $$J_\upsilon'(\upsilon x) = -\frac{1}{1+\delta_{2n+1}}\frac{\psi(\xi)}{\upsilon^{2/3}}\left\{\frac{Ai(\upsilon^{2/3}\zeta)}{\upsilon^{2/3}}\left[\sum_{s=0}^{n-1}\frac{C_s(\zeta)}{\upsilon^{2s}} + \frac{C_n(\zeta)-\zeta B_n(\zeta)}{\upsilon^{2n}}\right]\right.$$

$$\left. + Ai'(\upsilon^{2/3}\zeta)\sum_{s=0}^{n}\frac{D_s(\zeta)}{\upsilon^{2s}} + \eta_{2n+1,1} + \frac{\varphi'(\zeta)}{\varphi(\zeta)}\frac{\varepsilon_{2n+1,1}}{\upsilon^{2/3}}\right\},$$

where

(2.16) $$|\,\eta_{2n+1,1}(\upsilon,\zeta)\,| \le \frac{2N(\upsilon^{2/3}\zeta)}{E(\upsilon^{2/3}\zeta)} \exp\left\{\frac{2\lambda}{\upsilon}\,\mathcal{V}_{\zeta,\infty}\,(|\zeta|^{1/2}B_0)\right\} \frac{\mathcal{V}_{\zeta,\infty}\,(|\zeta|^{1/2}B_n)}{\upsilon^{2n+1}}.$$

In (2.13), (2.14), and (2.16), we have used $\mathcal{V}_{a,b}(f)$ to denote the total variation of a function $f(\zeta)$ on an interval (a,b). Numerical computations of the following values are found in [**OLV2**, p. 9] and [**OLV4**, p. 207]:

(2.17) $$\mathcal{V}_{-\infty,\infty}\{|\zeta|^{1/2}B_0(\zeta)\} = 0.1051,$$

(2.18) $$\upsilon_0 = 2\lambda\mathcal{V}_{-\infty,\infty}(|\zeta|^{1/2}B_0) = 0.30.$$

For our purpose, it suffices to take n = 0 in (2.12) - (2.16). Thus, (2.12) gives

(2.19) $$J_\upsilon(\upsilon x) = \frac{1}{1+\delta_1}\frac{\varphi(\zeta)}{\upsilon^{1/3}}\left[Ai(\upsilon^{2/3}\zeta) + \varepsilon_{1,1}(\upsilon,\zeta)\right],$$

and since [**OLV1**, p. 342]

(2.20) $$C_0(\zeta) = \chi(\zeta) + \zeta B_0(\zeta),$$

where

(2.21) $$\chi(\zeta) = \varphi'(\zeta)/\varphi(\zeta),$$

(2.15) gives

$$(2.22) \qquad J_\upsilon'(\upsilon x) = -\frac{1}{1+\delta_1}\frac{2}{\upsilon^{2/3}\, x\, \varphi(\zeta)}\left\{ Ai'(\upsilon^{2/3}\zeta) + \eta_{1,1} + \frac{\chi(\zeta)}{\upsilon^{2/3}}\left[Ai(\upsilon^{2/3}\zeta) + \varepsilon_{1,1}\right]\right\}.$$

Substitution of (2.19) and (2.22) in (2.1) leads to the result

$$(2.23) \qquad J_\upsilon''(\upsilon x) = \frac{1}{1+\delta_1}\frac{1}{\upsilon^{1/3}}\frac{4}{x^2\varphi^3(\zeta)}\left[\left\{\zeta + \frac{G(\zeta)}{\upsilon^2}\right\}\left\{Ai(\upsilon^{2/3}\zeta) + \varepsilon_{1,1}\right\}\right.$$

$$\left. + \frac{H(\zeta)}{\upsilon^{4/3}}\left\{Ai'(\upsilon^{2/3}\zeta) + \eta_{1,1}\right\}\right],$$

where

$$(2.24) \qquad H(\zeta) = \tfrac{1}{2}\varphi^2(\zeta), \qquad\qquad G(\zeta) = H(\zeta)\chi(\zeta) = \tfrac{1}{2}\varphi(\zeta)\varphi'(\zeta).$$

3. Bounds for $\zeta(j''_{\upsilon,k}/\upsilon)$, $k = 2,3,\ldots,9$

Provided that ζ does not vanish, (2.23) can be written as

$$(3.1) \qquad (1+\delta_1)\upsilon^{1/3}\frac{x^2\varphi^3(\zeta)}{4\zeta}J_\upsilon''(\upsilon x) = \left\{1 + \frac{G(\zeta)}{\upsilon^2\zeta}\right\}\left[Ai(\upsilon^{2/3}\zeta) + \varepsilon_{1,1}\right]$$

$$+ \frac{H(\zeta)}{\upsilon^{4/3}\zeta}\left[Ai'(\upsilon^{2/3}\zeta) + \eta_{1,1}\right]$$

$$= Ai(\upsilon^{2/3}\zeta) + \varepsilon_3(\upsilon,\zeta),$$

where

$$(3.2) \qquad \varepsilon_3(\upsilon,\zeta) = \varepsilon_{1,1}(\upsilon,\zeta) + \frac{G(\zeta)}{\upsilon^2\zeta}\left[Ai(\upsilon^{2/3}\zeta) + \varepsilon_{1,1}(\upsilon,\zeta)\right]$$

$$+ \frac{H(\zeta)}{\upsilon^{4/3}\zeta}\left[Ai'(\upsilon^{2/3}\zeta) + \eta_{1,1}(\upsilon,\zeta)\right].$$

In view of (2.17) and (2.18), (2.13) simplifies to

$$(3.3) \qquad |\delta_1| \le 2\, e^{0.30/\upsilon}\,\frac{0.1051}{\upsilon} \le \frac{0.22}{\upsilon} \qquad\qquad \text{if } \upsilon \ge 10.$$

For $\zeta < 0$, we also have from (2.14)

$$(3.4) \qquad |\varepsilon_{1,1}(\upsilon,\zeta)| \le \frac{2\,e^{0.30/\upsilon}(0.1051)\upsilon^{-1}}{\sqrt{\pi}\ (-\upsilon^{2/3}\zeta)^{1/4}},$$

on account of (2.11), and from (2.16),

$$(3.5) \qquad |\eta_{1,1}(\upsilon,\zeta)| \le 2\,e^{0.30/\upsilon}(0.1051)\upsilon^{-1}\, N(\upsilon^{2/3}\zeta), \qquad \zeta < 0.$$

Since $\varphi(\zeta)$ is a nonnegative increasing function in $(-\infty,0]$[1], and $\varphi(0) = 2^{1/3}$, we have from (2.24)

$$(3.6) \qquad H(\zeta) \le (1/2)^{1/3} = 0.794, \qquad \zeta \le 0.$$

Furthermore, since $|\varphi'(\zeta)/\varphi(\zeta)| \le 0.160$, [**OLV2**, p. 10] $\qquad -\infty \le \zeta \le \infty$,

$$(3.7) \qquad |G(\zeta)| \le \frac{0.160}{2}\varphi^2(\zeta) \le 0.08(2^{2/3}) = 0.127, \qquad \zeta \le 0.$$

By (2.9) and (2.11),

$$(3.8) \qquad |Ai(x)| \le M(x) \le 1/\{\sqrt{\pi}\,(-x)^{1/4}\}, \qquad x < 0.$$

Moreover, we find (see [**OLV3**, p. 750, Eq. (2.07)]) that

$$(3.9) \qquad |Ai'(x)| \le N(x), \qquad x < 0.$$

Taking absolute values of both sides of (3.2), we find by a direct substitution of the bounds (3.4) - (3.9) that, for $\zeta < 0$,

$$(3.10) \qquad |\varepsilon_3(\upsilon,\zeta)| \le \frac{2e^{0.30/\upsilon}(0.1051)\upsilon^{-1}}{\sqrt{\pi}\,(-\upsilon^{2/3}\zeta)^{1/4}} + \frac{0.127}{\upsilon^{4/3}(-\upsilon^{2/3}\zeta)}\,\frac{1+2e^{0.30/\upsilon}(0.1051)\upsilon^{-1}}{\sqrt{\pi}\,(-\upsilon^{2/3}\zeta)^{1/4}}$$
$$+ \frac{0.794}{\upsilon^{2/3}(-\upsilon^{2/3}\zeta)}\left[1+2e^{0.30/\upsilon}(0.1051)\upsilon^{-1}\right]N(\upsilon^{2/3}\zeta)$$
$$= \frac{1}{\upsilon^{2/3}\tau^{1/4}}\left\{\frac{0.2102e^{0.30/\upsilon}}{\sqrt{\pi}\,\upsilon^{1/3}} + \frac{1+0.2102e^{0.30/\upsilon}\upsilon^{-1}}{\tau^{1/2}}\left[\frac{0.127}{\sqrt{\pi}\,\upsilon^{2/3}\tau^{1/2}} + \frac{0.794\,N(-\tau)}{\tau^{1/4}}\right]\right\},$$

where, for brevity, we have set

$$(3.11) \qquad \tau = -\upsilon^{2/3}\zeta.$$

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[1] See Appendix I.

The bound (3.10) can be further simplified by noting that for $\tau \geq 1$, $\tau^{-1/4}N(-\tau) \leq 0.60$ [**OLV3**, p.752]. Then, for $\upsilon \geq 1$ and $\tau \geq 1$ [1],

$$\frac{0.127}{\sqrt{\pi}\,\upsilon^{2/3}\tau^{1/2}} + \frac{0.794\,N(-\tau)}{\tau^{1/4}} \leq 0.072 + 0.477 \leq 0.55,$$

and so

$$|\,\varepsilon_3(\upsilon,\zeta)\,| \leq \frac{1}{\upsilon^{2/3}\tau^{1/4}}\left\{\frac{0.2102e^{0.30/\upsilon}}{\upsilon^{1/3}\sqrt{\pi}} + \frac{0.55}{\tau^{1/2}} + \frac{0.2102e^{0.30/\upsilon}(0.55)}{\upsilon^{1/3}(\upsilon^{2/3}\tau^{1/2})}\right\} \tag{3.12}$$

$$\leq \frac{1}{\upsilon^{2/3}\tau^{1/4}}\left\{\frac{0.55}{\tau^{1/2}} + \frac{0.24e^{0.30/\upsilon}}{\upsilon^{1/3}}\right\}.$$

Let us now set $\upsilon x = j''_{\upsilon,k}$ in (3.1). Since $J''_\upsilon(j''_{\upsilon,k}) = 0$,

$$0 = \mathrm{Ai}(\upsilon^{2/3}\zeta) + \varepsilon_3(\upsilon,\zeta). \tag{3.13}$$

If, in the usual notation, the $k^{\underline{th}}$ negative zero of the Airy function Ai(x) is denoted by a_k, we deduce that the corresponding value of ζ is given by

$$\zeta = \upsilon^{-2/3}a_k + \eta_{\upsilon,k}, \tag{3.14}$$

where

$$\eta_{\upsilon,k} = O(\upsilon^{-4/3}) \qquad \text{(fixed k)}. \tag{3.15}$$

This result will be made more rigorous in the following discussion. Also, since this value of ζ depends on υ and k, we shall henceforth denote ζ by $\zeta_{\upsilon,k}$. Furthermore, for brevity, we shall suppress the dependence of $\eta_{\upsilon,k}$ on υ and write $\eta_{\upsilon,k}$ simply as η_k.

Therefore, let

$$\zeta_{\upsilon,k} \equiv \zeta(j''_{\upsilon,k}/\upsilon) = \upsilon^{-2/3}a_k + \eta_k. \tag{3.16}$$

We now establish a bound for $\zeta_{\upsilon,k}$. In order to determine such a bound, we consider equation (3.1) with $\upsilon^{2/3}\zeta$ as the independent variable, and write

$$f(\upsilon^{2/3}\zeta) = \mathrm{Ai}(\upsilon^{2/3}\zeta) + \varepsilon_3(\upsilon,\zeta), \tag{3.17}$$

[1] That $\tau \geq 1$ is not really a restriction for our purposes, as we shall see later; cf. the remarks following (3.19).

where

(3.18) $$f(\upsilon^{2/3}\zeta) = (1+\delta_1)\upsilon^{1/3}\frac{x^2\varphi^3(\zeta)}{4\zeta}J''_\upsilon(\upsilon x)$$

is a continuous function in $-\infty < \zeta < 0$.

Then, on the interval $I_k = [a_k - \rho_k, a_k + \rho_k]$, where $\rho_k > 0$ is chosen small enough so that $m_k \equiv \min_{I_k} | Ai'(x) | > 0$, we have $-\upsilon^{2/3}\zeta \geq -(a_k + \rho_k) \equiv \beta_k$, say, and so from (3.12),

(3.19) $$E_k \equiv \max_{I_k} | \varepsilon_3(\upsilon,\zeta) | \leq \frac{1}{\upsilon^{2/3}\beta_k^{1/4}}\left\{\frac{0.55}{\beta_k^{1/2}} + \frac{0.24e^{0.30/\upsilon}}{\upsilon^{1/3}}\right\},$$

provided that $\beta_k \geq 1$, $\upsilon \geq 1$.

Let us now restrict our attention to the case k = 2, and choose $\rho_2 = \frac{1}{4}(a_2 - a'_3)$. Then, since $a_2 = -4.08795$, $a'_3 = -4.82010$ [**ABR**, p.478], $\rho_2 = 0.18304$, and so $\beta_2 = 3.90491 \geq 1$. Hence the bound (3.19) on $\varepsilon_3(\upsilon,\zeta)$ applies, provided that $\upsilon \geq 1$. If $\upsilon \geq 10$ and $\rho_2 = 0.18304$, (3.19) gives

(3.20) $$E_2 = \max_{I_2} | \varepsilon_3(\upsilon,\zeta) | \leq 0.27965/\upsilon^{2/3} \leq 0.06025.$$

Furthermore, from tables of Airy functions [*op. cit.*, p. 477],

(3.21) $$m_2 = \min_{I_2} | Ai'(x) | = 0.74713 > 0,$$

(3.22) $$E_2 < M_2 \equiv \min \{|Ai(a_2 - \rho_2)|, |Ai(a_2 + \rho_2)|\} = 0.14359.$$

Since a_2 is a zero of Ai(x), and $\min_{I_2} | Ai'(x) | > 0$, then $Ai(a_2 - \rho_2)$ and $Ai(a_2+ \rho_2)$ are of opposite signs. From (3.17) and inequality (3.22), we further deduce that $f(a_2 - \rho_2)$ and $f(a_2+ \rho_2)$ must also be of opposite signs. Hence, if $\upsilon \geq 10$, there exists at least one zero $\upsilon^{2/3}\zeta_{\upsilon,2}$ of $f(\upsilon^{2/3}\zeta)$ in the interval $I_2 = [-4.27099, -3.90491]$. If we now set $\upsilon^{2/3}\zeta = \upsilon^{2/3}\zeta_{\upsilon,2}$ in (3.17), we obtain

(3.23) $$0 = Ai(\upsilon^{2/3}\zeta_{\upsilon,2}) + \varepsilon_3(\upsilon,\zeta_{\upsilon,2}).$$

Setting

(3.24) $$\upsilon^{2/3}\zeta_{\upsilon,2} = a_2 + \upsilon^{2/3}\eta_2,$$

we find from (3.23) and Taylor's formula [*op. cit.*, p. 14], (3.25) $\quad 0 = Ai'(\xi_2)\upsilon^{2/3}\eta_2 + \varepsilon_3(\upsilon,\zeta_{\upsilon,2}),$

where $\quad \xi_2 \in (a_2 - \rho_2, a_2 + \rho_2) \subseteq I_2.$

Since $Ai'(x) \neq 0$ on I_2, we have from (3.25),

(3.26) $$|\upsilon^{2/3}\eta_2| \leq E_2 / m_2 \leq \frac{0.27965}{\upsilon^{2/3}} \frac{1}{0.74713} \leq \frac{0.3743}{\upsilon^{2/3}} \quad \text{if } \upsilon \geq 10,$$

by virtue of (3.20) - (3.22). Hence, from (3.24) and (3.26),

(3.27) $$\zeta_{\upsilon,2} = \upsilon^{-2/3}a_2 + \eta_2,$$

where

(3.28) $$|\eta_2| \leq d_2 / \upsilon^{4/3} \qquad \text{if } \upsilon \geq 10, \qquad \text{with } d_2 = 0.3743.$$

By a similar argument, numerical computations show that, with the choice $\rho_k = \frac{1}{4}(a_k - a'_{k+1})$, then for $k = 3,4,\ldots,9$,

(3.29) $$\zeta_{\upsilon,k} = \upsilon^{-2/3}a_k + \eta_k,$$

where

(3.30) $$|\eta_k| \leq d_k / \upsilon^{4/3}, \qquad \text{if } \upsilon \geq 10,$$

with
$$d_2 = 0.37430, \quad d_3 = 0.28836, \quad d_4 = 0.24212, \quad d_5 = 0.21239,$$
$$d_6 = 0.19131, \quad d_7 = 0.17540, \quad d_8 = 0.16287, \quad d_9 = 0.15268.$$

The values which are necessary in these numerical computations, namely, a_k, a'_{k+1}, ρ_k, E^*_k, M_k and m_k, are compiled in a table which we exhibit later (see TABLE 1 below); here, $E^*_k = \upsilon^{2/3}E_k$, where E_k is given by (3.19), $M_k = \max\{|Ai(a_k - \rho_k)|, |Ai(a_k + \rho_k)|\}$ and $m_k = \min_{I_k} |Ai'(x)|$. The cases $k = 2,3,\ldots 9$ are taken care of in this manner due to the fact that accurate numerical tables of values associated with the Airy functions exist for the range of arguments $k = 1,2,\ldots,10$ [*op. cit.*, pp. 476 - 478]. For the remaining cases ($k \geq 10$), we must proceed in a slightly different manner. The approach we take is again via asymptotic analysis, since we are interested in values of $k \to \infty$.

Our conclusion will again be given by (3.29), with η_k bounded by $d_k / \upsilon^{4/3}$, valid for $\upsilon \geq 10$, where $\{d_k\}$ forms a monotonically decreasing sequence. The proof of this fact will be exhibited in the next two sections.

k	$-a_k$	$-a'_{k+1}$	ρ_k	E^*_k	M_k	m_k
2	4.08795	4.82010	0.18304	0.27965	0.14359	0.74713
3	5.52056	6.16331	0.16069	0.23158	0.13570	0.80309
4	6.78671	7.37218	0.14637	0.20447	0.13007	0.84450
5	7.94413	8.48849	0.13609	0.18642	0.12575	0.87771
6	9.02265	9.53545	0.12820	0.17326	0.12227	0.90563
7	10.04017	10.52766	0.12187	0.16309	0.11936	0.92981
8	11.00852	11.47506	0.11663	0.15492	0.11688	0.95120
9	11.93602	12.38479	0.11219	0.14816	0.11472	0.97043

TABLE 1.

4. Bounds for Zeros of the Airy Functions Ai(x), Ai'(x)

In his paper **[HET2]**, Hethcote has established the bound

$$(4.1) \qquad a_k = -\left[\frac{3\pi}{8}(4k-1)\right]^{2/3}(1+\sigma_k),$$

where

$$(4.2) \qquad |\sigma_k| \leq 0.130\left[\frac{3\pi}{8}(4k-1.051)\right]^{-2} \qquad \text{if } k \geq 1,$$

for the zeros a_k of the Airy function Ai(x). He further points out that the bound on σ_k is quite good since 0.130 is only slightly greater than the coefficient 5/48 of the next term in the asymptotic expansion of a_k. In a similar manner, we shall establish the bound

$$(4.3) \qquad a'_k = -\left[\frac{3\pi}{8}(4k-3)\right]^{2/3}(1+\tau_k),$$

where

$$(4.4) \qquad |\tau_k| \leq 0.165\left[\frac{3\pi}{8}(4k-3.0382)\right]^{-2} \qquad \text{if } k \geq 2,$$

for the critical points a'_k of Ai(x). The bound on τ_k is also quite reasonable since 0.165 is only slightly greater than the magnitude of the coefficient -7/48 of the next term in the asymptotic expansion of a'_k (cf. [**ABR**, p. 450, Eq. 10.4.95]). The derivation of this bound now follows.

The asymptotic expansion of Ai'(x) is given by

$$\text{(4.5)} \qquad \mathrm{Ai}'(-x) = \frac{x^{1/4}}{\sqrt{\pi}}\left\{\sin\left(\xi - \tfrac{\pi}{4}\right)P(\xi) + \cos\left(\xi - \tfrac{\pi}{4}\right)Q(\xi)\right\},$$

where

$$\text{(4.6)} \qquad \xi = \tfrac{2}{3}x^{3/2},$$

$$\text{(4.7)} \qquad P(\xi) \sim \sum_{s=0}^{\infty} (-)^s A_{2s}(\upsilon)/\xi^{2s}, \qquad Q(\xi) \sim \sum_{s=0}^{\infty} (-)^s A_{2s+1}(\upsilon)/\xi^{2s+1}, \qquad \upsilon = 2/3,$$

$$\text{(4.8)} \qquad A_s(\upsilon) = \frac{(4\upsilon^2 - 1^2)(4\upsilon^2 - 3^2)\cdots\left\{4\upsilon^2 - (2s-1)^2\right\}}{s!\,8^s};$$

see [**OLV5**, p. 394]. The first few coefficients of $A_s(2/3)$ are given by $A_0(2/3) = 1$, and

$$\text{(4.9)} \qquad \begin{aligned} &A_1(2/3) = 7/72, && A_2(2/3) = -455/10368 \\ &A_3(2/3) = 95095/2239488, && A_4(2/3) = -40415375/644972544. \end{aligned}$$

It is known [*Ibid.*] that the error in truncating the expansions for $P(\xi)$ and $Q(\xi)$ is less than the first omitted term, provided that the following term is of opposite sign. Hence,

$$\text{(4.10)} \qquad P(\xi) = 1 + \frac{455}{10368}\frac{1}{\xi^2} + \hat{\varepsilon}_4(\xi),$$

where

$$\text{(4.11)} \qquad |\hat{\varepsilon}_4(\xi)| \le \frac{40415375}{644972544}\frac{1}{\xi^4},$$

$$\text{(4.12)} \qquad Q(\xi) = \bar{\varepsilon}_1(\xi),$$

where

$$\text{(4.13)} \qquad |\bar{\varepsilon}_1(\xi)| \le \frac{7}{72}\frac{1}{\xi}.$$

We now apply Hethcote's corollary[1] [**HET3**, Cor. 1, p. 147] in order to determine a bound for a'_k. To this end, let

$$(4.14) \qquad f(\xi) = \frac{\sqrt{\pi}\,Ai'(-x)}{x^{1/4}}, \qquad\qquad \psi = -\frac{\pi}{4},$$

as in the corollary. Then

$$(4.15) \qquad \begin{aligned} f(\xi) &= \sin(\xi+\psi)P(\xi) + \cos(\xi+\psi)Q(\xi) \\ &= \sin(\xi+\psi)\left[1 + \frac{455}{10368}\xi^{-2} + \hat{\varepsilon}_4(\xi)\right] + \cos(\xi+\psi)\,\bar{\varepsilon}_1(\xi) \end{aligned}$$

in view of (4.10) and (4.12). Furthermore, (4.15) can be written as

$$(4.16) \qquad f(\xi) = \sin(\xi+\psi) + \hat{\varepsilon}(\xi)$$

where

$$(4.17) \qquad |\,\hat{\varepsilon}(\xi)\,| \le \frac{7}{72}\frac{1}{\xi} + \frac{455}{10368}\frac{1}{\xi^2} + \frac{40415375}{644972544}\frac{1}{\xi^4}$$

in view of (4.11), (4.13) and the fact that sin x and cos x are bounded in absolute value by 1.

If $\xi \ge 3.88$, then from (4.17), $|\,\hat{\varepsilon}(\xi)\,| \le 0.02825$. If $\rho = 0.03$, then

$$\hat{E} = \max |\,\hat{\varepsilon}(\xi)\,| \le 0.02825 < \sin\rho = 0.03.$$

Hence, $\exists$ a zero d_n of $f(\xi)$ in the interval $[n\pi - \psi - \rho,\, n\pi - \psi + \rho] = [n\pi + \pi/4 - 0.03,\, n\pi + \pi/4 + 0.03]$ if $n \ge 1$. The assumption that $\xi \ge 3.88$ is justified, since $d_1 \in [3.89, 3.96]$. Note also that since $a'_1 = -1.01879$, $a'_2 = -3.24820$, then $d_1 = \frac{2}{3}(-a'_2)^{3/2}$. Furthermore, on the interval, $\xi \ge n\pi + \pi/4 - 0.03$, and so

$$|\,d_n - (n\pi + \pi/4)\,| \le \hat{E}/\cos\rho \le \frac{3.88(0.02825)}{n\pi + \pi/4 - 0.03}\,\frac{1}{\cos(0.03)},$$

or equivalently,

$$(4.18) \qquad |\,d_n - (n\pi + \pi/4)\,| \le 0.1097/(n\pi + \pi/4 - 0.03) \qquad \text{if } n \ge 1.$$

[1] For a statement of this corollary, see Appendix IV.

If we now compare the zeros of Ai'(x) to those of f(ξ), we find that

(4.19) $$d_n = \tfrac{2}{3}(-a'_{n+1})^{3/2}.$$

Thus,

(4.20) $$\left| (-a'_{n+1})^{3/2} - \tfrac{3\pi}{8}(4n+1) \right| \le \frac{3}{2\pi}(0.1097)/(n + 0.24045).$$

Hence,

(4.21) $$(-a'_{n+1})^{3/2} = \tfrac{3\pi}{8}(4n+1) + \tau^*_{n+1},$$

where

(4.22) $$|\tau^*_{n+1}| \le \frac{3}{2\pi}(0.1097)/(n + 0.24045) \qquad \text{if } n \ge 1,$$

and so

(4.23) $$-a'_{n+1} = \left[\frac{3\pi}{8}(4n+1)\right]^{2/3}(1+\tilde{\tau}_{n+1})^{2/3},$$

where

(4.24) $$|\tilde{\tau}_{n+1}| = |\tau^*_{n+1}| / \{ \tfrac{3\pi}{8}(4n+1) \}$$
$$\le 0.2469\left[\frac{3\pi}{8}(4n+0.9618)\right]^{-2}, \qquad n \ge 1.$$

If $n \ge 1$, (4.24) gives $|\tilde{\tau}_{n+1}| \le 0.00723$ and so, by the Mean Value Theorem,

(4.25) $$-a'_{n+1} = \left[\frac{3\pi}{8}(4n+1)\right]^{2/3}(1+\tau_{n+1}),$$

where

(4.26) $$|\tau_{n+1}| \le \frac{2}{3}\left[\frac{1}{1-0.00723}\right]^{1/3}(0.2469)\left[\frac{3\pi}{8}(4n+0.9618)\right]^{-2}$$
$$\le 0.165\left[\frac{3\pi}{8}(4n+0.9618)\right]^{-2} \qquad \text{if } n \ge 1.$$

The result (4.3) - (4.4) now follows by replacing n by k - 1 in (4.25) and (4.26).

With the bounds for a_k and a'_k established, we are now in a position to determine a bound for the value

(4.27) $$\rho_k = \frac{1}{4}(a_k - a'_{k+1}).$$

It is known **[HET2**, p. 73**]** that

(4.28) $$a_k \le -\left[\frac{3\pi}{8}(4k-1)\right]^{2/3}.$$

Hence, from (4.27), we obtain

(4.29) $$4\rho_k \le \left[\frac{3\pi}{8}(4k)\right]^{2/3}\left\{\left[1+\frac{1}{4k}\right]^{2/3}(1+\tau_{k+1}) - \left[1-\frac{1}{4k}\right]^{2/3}\right\}, \qquad k \ge 1,$$

using (4.25) and (4.28). By the Mean Value Theorem, simple estimations yield

$$\left(1+\frac{1}{4k}\right)^{2/3} \le 1+\frac{2}{3}\frac{1}{4k}, \qquad k \ge 1,$$

and

$$\left(1-\frac{1}{4k}\right)^{2/3} \ge 1-\frac{2}{3}\left[1-\frac{1}{4k}\right]^{-1/3}\frac{1}{4k} \ge 1-\frac{2}{3}\frac{1}{4k-1}, \qquad k \ge 1.$$

Thus,

(4.30) $$4\rho_k \le \left(\frac{3\pi k}{2}\right)^{2/3}\left\{\frac{4}{3}\frac{1}{4k-1} + \left(1+\frac{1}{6k}\right)\tau_{k+1}\right\}.$$

Hence, if $k \ge 10$, (4.26) and (4.30) give

$$4\rho_k \le \left(\frac{3\pi k}{2}\right)^{2/3}\left\{\frac{4}{3}\frac{1}{4k-1} + \frac{61}{60}(0.165)\left[\frac{3\pi}{8}(4k+0.9618)\right]^{-2}\right\}$$

$$\le \left(\frac{3\pi k}{2}\right)^{2/3}\frac{1.3365}{4k-1}$$

$$\le 0.964k^{-1/3},$$

and so

(4.31) $$\rho_k \le 0.241k^{-1/3} \qquad \text{if } k \ge 10.$$

In what is to follow, we shall further require bounds for the quantities α_k and β_k defined by

$$(4.32) \qquad \alpha_k \equiv -(a_k - \rho_k), \qquad \beta_k \equiv -(a_k + \rho_k),$$

and shall now establish these bounds. From (4.27), we have

$$(4.33) \qquad \beta_k = -\tfrac{5}{4}a_k + \tfrac{1}{4}a'_{k+1}.$$

From (4.1) and (4.3), we find that

$$(4.34) \qquad \beta_k = \frac{5}{4}\left[\frac{3\pi}{8}(4k-1)\right]^{2/3}(1+\sigma_k) - \frac{1}{4}\left[\frac{3\pi}{8}(4k+1)\right]^{2/3}(1+\tau_{k+1}), \qquad k \geq 1,$$

with bounds for σ_k, τ_k given by (4.2) and (4.4) respectively. Writing the second term in square brackets as $\frac{3\pi}{8}(4k-1) + \frac{3\pi}{4}$, (4.34) becomes

$$(4.35) \qquad \beta_k = \frac{1}{4}\left[\frac{3\pi}{8}(4k-1)\right]^{2/3}\left\{5(1+\sigma_k) - \left[1 + \frac{2}{4k-1}\right]^{2/3}(1+\tau_{k+1})\right\}.$$

From Taylor's formula, we have

$$(4.36) \qquad (1+x)^{2/3} = 1 + \tfrac{2}{3}x + \tfrac{1}{9}(1+\xi)^{-4/3}x^2, \qquad 0 < \xi < x,$$

and so

$$(4.37) \qquad \left[1 + \frac{2}{4k-1}\right]^{2/3} = 1 + \frac{4}{3}\frac{1}{4k-1} + e_k,$$

where

$$(4.38) \qquad |e_k| \leq \frac{4}{9}\frac{1}{(4k-1)^2}, \qquad k \geq 1.$$

Substitution of (4.37) in (4.35) yields

$$(4.39) \qquad \beta_k = \tfrac{1}{4}B_k\{5(1+\sigma_k) - (1 + \tfrac{4}{3}\tfrac{1}{4k-1} + e_k)(1+\tau_{k+1})\}$$

$$= B_k(1 - \tfrac{1}{3}\tfrac{1}{4k-1} + \mu_k^*),$$

where

$$(4.40) \qquad B_k \equiv \left[\frac{3\pi}{8}(4k-1)\right]^{2/3},$$

and

$$(4.41) \qquad \mu_k^* = \tfrac{5}{4}\sigma_k - \tfrac{1}{4}e_k - \tfrac{1}{4}\left(1 + \tfrac{4}{3}\tfrac{1}{4k-1} + e_k\right)\tau_{k+1}.$$

Hence,

$$(4.42) \qquad \frac{2}{3}\beta_k^{3/2} = \frac{2}{3}B_k^{3/2}\left(1 - \frac{1}{3}\frac{1}{4k-1} + \mu_k^*\right)^{3/2}.$$

Considering each of the bounds (4.2), (4.4) and (4.38), we find that

$$(4.43) \qquad \begin{cases} |\sigma_k| \le \dfrac{0.130\,(8/3\pi)^2}{(4k-1.051)^2} \le \dfrac{0.094}{(4k-1.051)^2}, & k \ge 1, \\[2ex] |\tau_{k+1}| \le \dfrac{0.165\,(8/3\pi)^2}{(4k+0.9618)^2} \le \dfrac{0.119}{(4k-1.051)^2}, & k \ge 1, \\[2ex] |e_k| \le \dfrac{4}{9}\dfrac{1}{(4k-1)^2} \le \dfrac{4}{9}\dfrac{1}{(4k-1.051)^2}, & k \ge 1, \end{cases}$$

and so, from (4.41),

$$(4.44) \qquad \begin{aligned} |\mu_k^*| &\le \tfrac{5}{4}|\sigma_k| + \tfrac{1}{4}|e_k| + \tfrac{1}{4}\left(1 + \tfrac{4}{3}\tfrac{1}{4k-1} + |e_k|\right)|\tau_{k+1}| \\ &\le \frac{0.2584}{(4k-1.051)^2} + \frac{0.040}{(4k-1.051)^3} + \frac{0.014}{(4k-1.051)^4} \qquad \text{if } k \ge 1. \end{aligned}$$

Furthermore,

$$(4.45) \qquad |\mu_k^*| \le 0.260/(4k-1.051)^2 \qquad \text{if } k \ge 10.$$

If we now let

$$(4.46) \qquad \tilde{\mu}_k = -\tfrac{1}{3}\tfrac{1}{4k-1} + \mu_k^*,$$

then

$$(4.47) \qquad |\tilde{\mu}_k| \le 0.341/(4k-1.051) \le 0.00880 \qquad \text{if } k \ge 10.$$

Hence, Taylor's formula applied to the last term on the right-hand-side of (4.42) leads to

$$(4.48) \qquad \frac{2}{3}\beta_k^{3/2} = \frac{2}{3}B_k^{3/2}\left(1 - \frac{1}{2}\frac{1}{4k-1} + \mu_k\right),$$

where

$$(4.49)\qquad |\mu_k| \le \frac{3}{2}|\mu_k^*| + \frac{3}{8}(1 - 0.00880)^{-1/2}\tilde{\mu}_k^2$$

$$\le \frac{0.4338}{(4k - 1.051)^2} \qquad \text{if } k \ge 10,$$

in view of (4.45) and (4.47). Recalling now the definition of B_k given in (4.40), we have

$$(4.50)\qquad \frac{2}{3}B_k^{3/2}|\mu_k| \le 0.3412/(4k - 1.051) \qquad \text{if } k \ge 10,$$

and so from (4.48)

$$(4.51)\qquad \frac{2}{3}\beta_k^{3/2} = k\pi - \frac{3\pi}{4} + \frac{3\pi}{8} + \theta_2(k),$$

where

$$(4.52)\qquad |\theta_2(k)| \le 0.342/(4k - 1.051) \qquad \text{if } k \ge 10.$$

In a similar manner, we establish a bound for α_k, where, again from (4.27),

$$(4.53)\qquad \alpha_k = -\frac{3}{4}a_k - \frac{1}{4}a'_{k+1}.$$

Following exactly similar steps as above, we find that

$$(4.54)\qquad \frac{2}{3}\alpha_k^{3/2} = \frac{2}{3}B_k^{3/2}\left(1 + \frac{1}{3}\frac{1}{4k - 1} + \eta_k^*\right)^{3/2},$$

where B_k is defined by (4.40), and

$$(4.55)\qquad \eta_k^* = \frac{3}{4}\sigma_k + \frac{1}{4}e_k + \frac{1}{4}\left(1 + \frac{4}{3}\frac{1}{4k-1} + e_k\right)\tau_{k+1}.$$

Using the bounds given in (4.43), we have

$$(4.56)\qquad |\eta_k^*| \le \frac{0.2114}{(4k - 1.051)^2} + \frac{0.040}{(4k - 1.051)^3} + \frac{0.014}{(4k - 1.051)^4} \qquad \text{if } k \ge 1,$$

so

$$(4.57)\qquad |\eta_k^*| \le 0.213/(4k - 1.051)^2 \qquad \text{if } k \ge 10.$$

If we let

$$(4.58)\qquad \tilde{\eta}_k = -\frac{1}{3}\frac{1}{4k-1} + \eta_k^*,$$

then

$$(4.59)\qquad |\tilde{\eta}_k| \le 0.339/(4k - 1.051) \le 0.00871 \qquad \text{if } k \ge 10.$$

Hence,

$$(4.60)\qquad \frac{2}{3}\alpha_k^{3/2} = \frac{2}{3}B_k^{3/2}\left(1+\frac{1}{2}\frac{1}{4k-1}+\eta_k\right),$$

where

$$(4.61)\qquad |\eta_k| \le \frac{3}{2}|\eta_k^*| + \frac{3}{8}(1-0.00871)^{-1/2}\tilde{\eta}_k^2$$

$$\le \frac{0.3628}{(4k-1.051)^2} \qquad \text{if } k \ge 10.$$

Finally, we have

$$(4.62)\qquad \frac{2}{3}\alpha_k^{3/2} = k\pi + \frac{\pi}{4} - \frac{3\pi}{8} + \theta_1(k),$$

where

$$(4.63)\qquad |\theta_1(k)| \le 0.286/(4k-1.051) \qquad \text{if } k \ge 10,$$

in light of (4.60), (4.61) and the definition of B_k in (4.40).

5. Bounds for $\zeta(j''_{\upsilon,k}/\upsilon)$, $k \ge 10$

Our goal now is to show that for $k \ge 10$,

$$(5.1)\qquad \zeta_{\upsilon,k} = \upsilon^{-2/3}a_k + \eta_k,$$

where

$$(5.2)\qquad |\eta_k| \le d_k/\upsilon^{4/3} \qquad \text{if } \upsilon \ge 10,$$

with $\{d_k\}$ forming a monotonically decreasing sequence.

The cases $k = 2,3,\ldots,9$ have been dealt with in §3.

As in §3, let us consider the interval $I_k = [a_k - \rho_k, a_k + \rho_k]$. On this interval $-\upsilon^{2/3}\zeta \ge -(a_k + \rho_k) = \beta_k$. From tables [**ABR**, p. 478] we find $a_{10} = -12.82878$. Hence, from (4.31), $\beta_k \ge -a_{10} - 0.241/10^{1/3} \ge 12.716$ if $k \ge 10$, and so the bound (3.19) on $\varepsilon_3(\upsilon,\zeta)$ applies, provided that $\upsilon \ge 1$. If $\upsilon \ge 10$, then for $k \ge 10$, $\beta_k^{1/2} \ge 1$, and so (3.19) gives

(5.3) $$E_k \equiv \max_{I_k} |\, \varepsilon_3(\upsilon,\zeta) \,| \;\le\; 0.1433/\beta_k^{1/4}.$$

We shall now show that

(5.4) $$M_k \equiv \min\left\{|Ai(-\alpha_k)|, |Ai(-\beta_k)|\right\} \;\ge\; 0.2045/\beta_k^{1/4} \qquad \text{if } k \ge 10,$$

and so conclude from this inequality and inequality (5.3) that

(5.5) $$E_k \;<\; M_k \qquad \text{for all } k \ge 10.$$

To establish inequality (5.4) we consider the asymptotic expansion of Ai(-x) given by

(5.6) $$Ai(-x) \;=\; \frac{1}{\sqrt{\pi}\, x^{1/4}}\left\{\cos(\xi - \tfrac{\pi}{4})\,[1+\tilde{\tilde{\varepsilon}}_2(\xi)] - \sin(\xi - \tfrac{\pi}{4})\,\tilde{\varepsilon}_1(\xi)\right\}$$

$$= \frac{\cos(\tfrac{2}{3}x^{3/2} - \tfrac{\pi}{4}) \;+\; \tilde{\varepsilon}(\xi)}{\sqrt{\pi}\;x^{1/4}},$$

where

(5.7) $$\tilde{\varepsilon}(\xi) \;=\; \cos(\xi - \tfrac{\pi}{4})\,\tilde{\tilde{\varepsilon}}_2(\xi) \;-\; \sin(\xi - \tfrac{\pi}{4})\,\tilde{\varepsilon}_1(\xi), \qquad \xi = \tfrac{2}{3}x^{3/2},$$

with

(5.8) $$|\,\tilde{\varepsilon}_1(\xi)\,| \le |\,A_1(1/3)\,|\,\xi^{-1} \;=\; \tfrac{5}{72}\xi^{-1},$$

(5.9) $$|\,\tilde{\tilde{\varepsilon}}_2(\xi)\,| \le |\,A_2(1/3)\,|\,\xi^{-2} \;=\; \tfrac{385}{10368}\xi^{-2}.$$

These results are given in Olver [**OLV5**, p. 394]. The bounds (5.8) and (5.9) are the magnitudes of the first omitted terms in the expansions of $P(\xi)$ and $Q(\xi)$ given by (4.7) with $\upsilon = 1/3$ replacing $\upsilon = 2/3$; cf. [***Ibid.***]; also compare with the asymptotic expansion of Ai'(x) given in §4. From (5.7), we find that

(5.10) $$|\,\tilde{\varepsilon}(\xi)\,| \le |\,\tilde{\varepsilon}_1(\xi)\,| + |\,\tilde{\tilde{\varepsilon}}_2(\xi)\,| \;\le\; 0.0024 \qquad \text{if } \xi \ge 30.$$

If we now set $x = \beta_k$ in (5.6), where β_k is defined by (4.32), then $\xi = \xi_k = \tfrac{2}{3}\beta_k^{3/2}$. From (4.52) we have

(5.11) $$|\,\theta_2(k)\,| \;\le\; 0.0088 \qquad \text{if } k \ge 10.$$

Using (5.11) we find from (4.51) that

(5.12) $$\tfrac{2}{3}\beta_k^{3/2} \;\ge\; 10\pi \;-\; \frac{3\pi}{8} \;-\; 0.0088 \;\ge\; 30, \qquad k \ge 10.$$

Since $\cos(k\pi + \ell) = \cos k\pi \cos \ell - \sin k\pi \sin \ell = (-1)^k \cos \ell \quad \forall k \in \mathbb{Z}$,

then from (4.51)

(5.13) $\qquad \cos(\xi_k - \pi/4) = \cos[(k-1)\pi + 3\pi/8 + \theta_2(k)] = \pm \cos[3\pi/8 + \theta_2(k)].$

But now, from (5.11),

(5.14) $\qquad 1.1692 \le 3\pi/8 + \theta_2(k) \le 1.1869 \qquad$ if $k \ge 10$,

and so

(5.15) $\qquad 1 \ge |\cos(\frac{2}{3}\beta_k^{3/2} - \frac{\pi}{4})| \ge \cos(1.1869) \ge 0.3745 \qquad$ if $k \ge 10$,

In light of (5.12), inequality (5.10) applies, so (5.6) with $x = \beta_k$ yields

(5.16) $$|\mathrm{Ai}(-\beta_k)| \ge \frac{|\cos(\xi_k - \frac{\pi}{4})| - |\tilde{\varepsilon}(\xi)|}{\sqrt{\pi}\;\beta_k^{1/4}} \ge \frac{0.2099}{\beta_k^{1/4}} \qquad \text{if } k \ge 10,$$

by virtue of inequalities (5.10) and (5.15).

If we now set $x = \alpha_k$ in (5.6), where α_k is defined by (4.32), then $\xi = \xi_k = \frac{2}{3}\alpha_k^{3/2}$. Proceeding in a similar manner, we find that if $k \ge 10$, (4.63) gives

(5.17) $\qquad |\theta_1(k)| \le 0.0074,$

and so

(5.18) $\qquad 1 \ge |\cos(\frac{2}{3}\alpha_k^{3/2} - \frac{\pi}{4})| = |\cos[-3\pi/8 + \theta_1(k)]| \ge 0.3758 \qquad$ if $k \ge 10$,

in view of (4.62) and inequality (5.17). Furthermore, since $\alpha_k \ge \beta_k$, then $\xi_k = \frac{2}{3}\alpha_k^{3/2} \ge 30$ if $k \ge 10$ (see (5.12)), and so (5.6) with $x = \alpha_k$ yields

(5.19) $\qquad |\mathrm{Ai}(-\alpha_k)| \ge 0.2106/\alpha_k^{1/4} \qquad$ if $k \ge 10$,

by virtue of inequalities (5.10) and (5.18). Moreover, from (4.32), we deduce that $\alpha_k = \beta_k + 2\rho_k$, and so

(5.20) $$|\mathrm{Ai}(-\alpha_k)| \ge \frac{0.2106}{\beta_k^{1/4}}\left\{1 + \frac{2\rho_k}{\beta_k}\right\}^{-1}.$$

Recalling that $a_{10} = -12.82878$, we have from (4.31),

$\beta_k \ge \beta_{10} = -a_{10} - \rho_{10} \ge 12 \quad \forall k \ge 10$. Thus, (5.21) $$|\,Ai(-\alpha_k)\,| \geq \frac{0.2106(1.0187)^{-1}}{\beta_k^{1/4}} \geq \frac{0.2067}{\beta_k^{1/4}} \qquad \text{whenever } k \geq 10.$$

Hence, (5.16) and (5.21) together imply the result (5.4), and consequently, (5.5) holds.

If we now choose ρ_k as in (4.27), then

(5.22) $$m_k = \min_{I_k} |\,Ai'(x)\,| \geq 0.5174\,\beta_k^{1/4},$$

provided that $k \geq 10$. To see this, let us again consider the asymptotic approximation of $Ai'(-x)$ given in (4.5), with the coefficients $P(\xi)$ and $Q(\xi)$ given by (4.10) and (4.12), respectively. Specifically, we have

(5.23) $$Ai'(-x) = \frac{x^{1/4}}{\sqrt{\pi}} f(\xi),$$

where ξ is given by (4.6), and $f(\xi)$ by (4.16).

If we set $x = \beta_k$ in (5.23), then $\xi = \xi_k = \frac{2}{3}\beta_k^{3/2}$. Since $\sin(k\pi + \ell) = \pm \sin \ell\ \forall k \in \mathbb{Z}$, then $\sin(\frac{2}{3}\beta_k^{3/2} - \frac{\pi}{4}) = \pm \sin[3\pi/8 + \theta_2(k)]$, in light of (4.51), and so

(5.24) $$1 \geq |\sin(\tfrac{2}{3}\beta_k^{3/2} - \tfrac{\pi}{4})| \geq \sin(1.1692) \geq 0.9204 \qquad \text{if } k \geq 10,$$

by virtue of (5.14). Furthermore, since $\xi_k = \frac{2}{3}\beta_k^{3/2} \geq 30$ for $k \geq 10$ (see (5.12)), then (4.17) gives

(5.25) $$|\,\hat{\varepsilon}(\xi)\,| \leq 0.0033,$$

and so we obtain from (5.23) with $x = \beta_k$

(5.26) $$|\,Ai'(-\beta_k)\,| \geq \frac{\beta_k^{1/4}}{\sqrt{\pi}}(0.9204 - 0.0033) \geq 0.5174\beta_k^{1/4} \qquad \text{whenever } k \geq 10.$$

In a similar manner, if we set $x = \alpha_k$ in (5.23), we find that

(5.27) $$|\,Ai'(-\alpha_k)\,| \geq \frac{\alpha_k^{1/4}}{\sqrt{\pi}}\left\{|\sin(-3\pi/8 + \theta_1(k))| - |\hat{\varepsilon}(\xi)|\right\} \geq 0.5177\alpha_k^{1/4}$$

in view of (4.62), (5.17) and (5.25). Furthermore, since $\alpha_k \geq \beta_k\ \forall k$, we have

(5.28) $$|\,Ai'(-\alpha_k)\,| \geq 0.5177\beta_k^{1/4} \qquad \text{whenever } k \geq 10.$$

Since it is known[1] that for $k \geq 10$, $a'_{k+1} < -\alpha_k < a_k < -\beta_k < a'_k$, inequality (5.22) now follows from (5.26) and (5.28).

Since a_k is a zero of Ai(x), and $\min_{I_k} | Ai'(x) | > 0$, then $Ai(a_k - \rho_k)$ and $Ai(a_k + \rho_k)$ are of opposite signs. From (3.17) and inequality (5.5), it follows that $f(a_k - \rho_k)$ and $f(a_k + \rho_k)$ are also of opposite signs. Hence, if $\upsilon \geq 10$, there exists at least one zero $\upsilon^{2/3}\zeta_{\upsilon,k}$ of $f(\upsilon^{2/3}\zeta)$ in the interval I_k. If we now set $\upsilon^{2/3}\zeta = \upsilon^{2/3}\zeta_{\upsilon,k}$ in (3.17), and let $\upsilon^{2/3}\zeta_{\upsilon,k} = a_k + \upsilon^{2/3}\eta_k$, then from Taylor's formula,

$$(5.29) \qquad 0 = Ai'(\xi_k)\, \upsilon^{2/3}\eta_k + \varepsilon_3(\upsilon,\zeta_{\upsilon,k}),$$

where $\xi_k \in (a_k - \rho_k, a_k + \rho_k) \subseteq I_k$. Since $Ai'(x) \neq 0$ on I_k, we thus have, for $k \geq 10$,

$$(5.30) \qquad | \upsilon^{2/3}\eta_k | \leq E_k / m_k$$

$$\leq \frac{1}{\upsilon^{2/3}\beta_k^{1/4}} \left\{ \frac{0.55}{\beta_k^{1/2}} + \frac{0.24e^{0.30/10}}{10^{1/3}} \right\} \frac{1}{0.5174\, \beta_k^{1/4}}$$

if $\upsilon \geq 10$, by virtue of (3.19) and (5.22). Furthermore, from (4.31) and (4.32), we have $\beta_k \geq \beta_{10} \geq 12.716 \ \forall k \geq 10$, and so

$$(5.31) \qquad | \eta_k | \leq d_k / \upsilon^{4/3} \qquad \text{if } \upsilon \geq 10,$$

where $\quad d_k = 0.5200 / \beta_k^{1/2} \leq 0.146 \qquad$ for $k \geq 10$.

It is now clear that $\{d_k\}_{k=10}^{\infty}$ forms a monotonically decreasing sequence, and so our goal (5.1) - (5.2) has been attained. Furthermore, in view of (3.30) and (5.31), the result (1.3) - (1.4) now follows.

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[1] For a proof that $a'_{k+1} < -\alpha_k < a_k < -\beta_k < a'_k$ for $k \geq 10$, see Appendix II.

CHAPTER 3
THE MONOTONICITY PROPERTY OF $j''_{\upsilon,k}$

1. Introduction

The goal of the present chapter is to prove that for $k \geq 2$, $j''_{\upsilon,k}$ is an increasing function of υ whenever $\upsilon \geq 10$. As noted in Chapter 1, to prove this result, we must show that for $k = 2,3,\ldots$, $G(j''_{\upsilon,k}) > 0$ whenever $\upsilon \geq 10$, where $G(j''_{\upsilon,k})$ is determined by (1.3.4). Specifically,

$$(1.1) \qquad G(j''_{\upsilon,k}) = \int_0^{j''_{\upsilon,k}} \frac{J_\upsilon^2(x)}{x}\,dx - J_\upsilon^2(j''_{\upsilon,k})$$

$$= \int_0^{\infty} \frac{J_\upsilon^2(x)}{x}\,dx - \int_{j''_{\upsilon,k}}^{\infty} \frac{J_\upsilon^2(x)}{x}\,dx - J_\upsilon^2(j''_{\upsilon,k})\,.$$

Recalling that

$$\int_0^{\infty} \frac{J_\upsilon^2(x)}{x}\,dx = \frac{1}{2\upsilon}\,,$$

and setting

$$(1.2) \qquad F(\upsilon,k) = \int_{j''_{\upsilon,k}}^{\infty} \frac{J_\upsilon^2(x)}{x}\,dx\,,$$

we have from (1.1)

$$(1.3) \qquad G(j''_{\upsilon,k}) = \frac{1}{2\upsilon} - F(\upsilon,k) - J_\upsilon^2(j''_{\upsilon,k}).$$

To show that (1.3) is positive for $\upsilon \geq 10$, we encounter the problem of obtaining asymptotic approximations for the integral $F(\upsilon,k)$, defined by (1.2), for large υ and fixed k, and for the quantity $J_\upsilon^2(j''_{\upsilon,k})$, also for large υ and fixed k.

Sections 2, 3 and 4 of this chapter are dedicated to determining the asymptotic behaviour of $F(\upsilon,2)$ as $\upsilon \to \infty$. From (1.2), it is clear that $F(\upsilon,k) \leq F(\upsilon,2)$ $\forall k \geq 2$, since $j''_{\upsilon,k+1} \geq j''_{\upsilon,k}$ $\forall k$. Hence, it suffices for our purposes to consider only the case $k = 2$. Section 5 is dedicated to determining the asymptotic behaviour of $J_\upsilon^2(j''_{\upsilon,k})$ as $\upsilon \to \infty$.

To determine the asymptotic behaviour of

$$F(\upsilon,2) = \int_{j''_{\upsilon,2}}^{\infty} \frac{J_\upsilon^2(t)}{t}\,dt\,,$$

we replace the Bessel function $J_\upsilon(t)$ in the integrand by its asymptotic approximation (2.2.19) and formally obtain the integral

$$\int_{\bar{\zeta}_\upsilon}^{\infty} \varphi^4(-\zeta)\,\mathrm{Ai}^2(-\upsilon^{2/3}\zeta)\,d\zeta\,,$$

where $\varphi(\zeta)$ is defined by (2.2.6). The lower limit of integration $\bar{\zeta}_\upsilon$ depends on υ and approaches 0 as $\upsilon \to \infty$ (see §4; specifically, compare with eq. (4.7)), and so we are led to the consideration of integrals of the form

$$(1.4) \qquad I(\lambda) = \int_0^{\infty} f(t)\,\mathrm{Ai}^2(-\lambda t)\,dt\,,$$

where λ is a large positive parametre, and f(t) is a C^∞- function in $0 < t < \infty$. Section 2 is dedicated to the study of the asymptotic behaviour of the integral in (1.4). To determine this behaviour, we apply a recent result in asymptotics dealing with the treatment of asymptotic expansions of Mellin convolutions (of which (1.4) is a particular example), complete with explicit error bounds, which can be found in a paper by Wong (see [**WON1**]). Section 3 then applies the result of section 2 to a specific integral, namely, the integral

$$F_1(\upsilon) = \int_0^{\infty} \varphi^4(-\zeta)\,\mathrm{Ai}^2(-\upsilon^{2/3}\zeta)\,d\zeta\,.$$

In this case, $\lambda = \upsilon^{2/3}$ is the large positive parametre, and $f(t) = \varphi^4(-t)$ is the C^∞- function in $(0,\infty)$.

Section 4 establishes the relationship between $F(\upsilon,2)$ and the function $F_1(\upsilon)$ studied in section 3. It concludes with a bound for $F(\upsilon,2)$, and the result is

(1.5) $$F(\upsilon,2) = \frac{1}{2\upsilon} - \frac{0.812634}{\upsilon^{4/3}} + \frac{\varepsilon_1(\upsilon)}{\upsilon^2},$$

where $|\varepsilon_1(\upsilon)| \le 2.086$ if $\upsilon \ge 10$.

In section 5 we determine bounds for $J^2_\upsilon(j''_{\upsilon,k})$ via a one-term asymptotic approximation of $J_\upsilon(\upsilon x)$ given by (2.2.19). The result is

(1.6) $$0 \le J^2_\upsilon(j''_{\upsilon,k}) \le \varepsilon_2(\upsilon)/\upsilon^2, \qquad k = 2,3,\ldots,$$

where $0 \le \varepsilon_2(\upsilon) \le 0.215$ if $\upsilon \ge 10$.

Section 6 concludes with the fact that $G(j''_{\upsilon,k}) > 0$ whenever $\upsilon \ge 10$, and realizes the goal of this thesis and that of Lorch and Szego, namely, to show that for $k = 1,2,3,\ldots$, $j''_{\upsilon,k}$ <u>is an increasing function of</u> υ <u>whenever</u> $\upsilon > 0$.

2. The Asymptotic Expansion of $I(\lambda)$

It is well-known that

$$Ai(-z) = \frac{1}{2}\sqrt{\frac{z}{3}}\left[e^{i\pi/6} H^{(1)}_{1/3}(\zeta) + e^{-i\pi/6} H^{(2)}_{1/3}(\zeta)\right],$$

where $\zeta = \frac{2}{3} z^{3/2}$; see [**ABR**, p. 447]. Hence, we may write

(2.1) $$Ai^2(-z) = h_1(z) + h_2(z) + h_3(z)$$

with

$$h_1(z) = \frac{z}{12} e^{i\pi/3}\left[H^{(1)}_{1/3}(\zeta)\right]^2, \qquad h_2(z) = \frac{z}{12} e^{-i\pi/3}\left[H^{(2)}_{1/3}(\zeta)\right]^2,$$

and

$$h_3(z) = \frac{z}{6} H^{(1)}_{1/3}(\zeta) H^{(2)}_{1/3}(\zeta) = \frac{z}{6}\left[J^2_{1/3}(\zeta) + Y^2_{1/3}(\zeta)\right].$$

Furthermore, since

$$H^{(1)}_\upsilon(z) = J_\upsilon(z) + i\,Y_\upsilon(z), \qquad H^{(2)}_\upsilon(z) = \overline{H^{(1)}_\upsilon(z)},$$

we have

$$\left[H_\upsilon^{(1)}(z)\right]^2 + \left[H_\upsilon^{(2)}(z)\right]^2 = 2\left[J_\upsilon^2(z) - Y_\upsilon^2(z)\right],$$

and

$$\left[H_\upsilon^{(1)}(z)\right]^2 - \left[H_\upsilon^{(2)}(z)\right]^2 = 4i\, J_\upsilon(z)\, Y_\upsilon(z).$$

Hence,

$$\left[H_\upsilon^{(1)}(z)\right]^2 = \left[J_\upsilon^2(z) - Y_\upsilon^2(z)\right] + 2i\, J_\upsilon(z)\, Y_\upsilon(z),$$

and

$$\left[H_\upsilon^{(2)}(z)\right]^2 = \left[J_\upsilon^2(z) - Y_\upsilon^2(z)\right] - 2i\, J_\upsilon(z)\, Y_\upsilon(z).$$

The asymptotic expansion of $h_3(z)$ can be obtained from that of $J_\upsilon^2 + Y_\upsilon^2$. More precisely, we have

$$h_3(z) \sim \frac{1}{2\pi} \sum_{s=0}^{\infty} 1 \cdot 3 \cdot 5 \cdots (2s-1) \left(\frac{3}{2}\right)^{2s} A_s(1/3)\, z^{-3s-1/2}, \tag{2.2}$$

where $A_s(\upsilon)$ is given by (2.4.8); cf. [**OLV5**, p. 342]. Furthermore, it is known that the remainder after n terms is of the same sign as, and is numerically less than, the n+1^st term. From the asymptotic expansions of the Hankel functions $H_\upsilon^{(1)}$ and $H_\upsilon^{(2)}$, we also have

$$h_1(z) \sim \frac{1}{4\pi} \exp\left\{ i\left(\tfrac{4}{3} z^{3/2} - \tfrac{\pi}{2}\right)\right\} \sum_{s=0}^{\infty} i^s \left(\tfrac{3}{2}\right)^s C_s(1/3)/z^{(3s+1)/2} \tag{2.3}$$

and

$$h_2(z) \sim \frac{1}{4\pi} \exp\left\{ -i\left(\tfrac{4}{3} z^{3/2} - \tfrac{\pi}{2}\right)\right\} \sum_{s=0}^{\infty} (-i)^s \left(\tfrac{3}{2}\right)^s C_s(1/3)/z^{(3s+1)/2} \tag{2.4}$$

where $$C_s(\upsilon) = \sum_{\ell=0}^{s} A_\ell(\upsilon) A_{s-\ell}(\upsilon).$$

The first few coefficients of $A_s(1/3)$ are given by $A_0(1/3) = 1$, and

(2.5) $A_1(1/3) = -5/72$ $\qquad$ $A_2(1/3) = 385/10368$

$A_3(1/3) = -85085/2239488$ $\qquad$ $A_4(1/3) = 27227200/644972544$.

Bounds for the remainders associated with the expansions (2.3) and (2.4) can be constructed from those of the Hankel functions; see [*op. cit.*, pp. 266 - 269].

Insertion of (2.1) in (1.4) gives

(2.6) $$I(\lambda) = I_1(\lambda) + I_2(\lambda) + I_3(\lambda),$$

where

(2.7) $$I_i(\lambda) = \int_0^\infty f(t)\, h_i(\lambda t)\, dt, \qquad i = 1,2,3.$$

Throughout this section we shall assume that f(t) is an infinitely differentiable function in $(0,\infty)$ with an asymptotic expansion of the form

(2.8) $$f(t) \sim \sum_{s=0}^{\infty} a_s\, t^{s+\alpha-1}, \qquad \text{as } t \to 0^+,$$

where $0 < \alpha \leq 1$. We further assume that the asymptotic expansion of the derivatives of f(t) can be obtained by termwise differentiation of (2.8), and that for each j = 0,1,2...,

(2.9) $$f^{(j)}(t) = O(t^{-1-\varepsilon}), \qquad \text{as } t \to \infty,$$

where ε is some fixed nonnegative number.

From (2.8) it follows that the Mellin transform of f(t) defined by

(2.10) $$M[f; z] = \int_0^\infty t^{z-1} f(t)\, dt, \qquad 1 - \alpha < \mathrm{Re}\, z < 1 + \varepsilon,$$

can be analytically continued to a meromorphic function in the half-plane Re $z < 1 + \varepsilon$, with simple poles at $z = 1 - s - \alpha$ of residue a_s, s = 0,1,2,...; see [**EVG**, p. 211]. Throughout this discussion, the notation M[f; z] is used to denote not only the integral in (2.10) but also its analytic continuation.

The Mellin transforms of $h_i(t)$ can be obtained from integral tables [**MAR**, p. 199, Eq. 23(1), p. 203, Eq. 32(1), and p. 209, Eq. 45(1)][1], and we have

(2.11) $$M[h_1; z] = -\frac{3^{s-2}}{4\pi^2} e^{i\pi s/2} \frac{1}{\Gamma(s)} \Gamma(\tfrac{s}{2} + \tfrac{1}{3})\, \Gamma(\tfrac{s}{2} - \tfrac{1}{3})\, \Gamma^2(\tfrac{s}{2})$$

(2.12) $$M[h_2; z] = -\frac{3^{s-2}}{4\pi^2} e^{-i\pi s/2} \frac{1}{\Gamma(s)} \Gamma(\tfrac{s}{2} + \tfrac{1}{3})\, \Gamma(\tfrac{s}{2} - \tfrac{1}{3})\, \Gamma^2(\tfrac{s}{2})$$

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[1] We also make reference here to [**ERD**, p. 333, Eq. (40)] and [**OBE**, p. 102, Eq. (10.41)], but note that the transforms given in these references are <u>incorrect</u>; however, Marichev's transforms stand as correct.

$$(2.13) \qquad M[h_3; z] = \frac{3^{s-2}}{\pi^2} \cos(\tfrac{\pi}{3}) \frac{\Gamma(\frac{s}{2})\,\Gamma(\frac{s}{2}+\frac{1}{3})\,\Gamma(\frac{s}{2}-\frac{1}{3})}{\Gamma(1-\frac{s}{2})\,\Gamma(s)} \frac{\pi}{\sin \pi s},$$

where $s = \frac{2}{3}(z+1)$.

We are now ready to apply the results in [**WON1**, §2]. For each $n \geq 1$, we set

$$(2.14) \qquad f(t) = \sum_{s=0}^{n-1} a_s\, t^{s+\alpha-1} + f_n(t).$$

By our assumption (2.9),

$$f_n^{(j)} = O(t^{n+\alpha-j-1}), \qquad \text{as } t \to 0^+,$$

for $j = 0,1,2,\ldots$. Similarly, we write (2.2) in the form

$$(2.15) \qquad h_3(t) = \sum_{s=0}^{n-1} b_s\, t^{-s-1/2} + h_{3,n}(t)$$

with $b_{3s+1} = b_{3s+2} = 0$, and

$$(2.16) \qquad b_{3s} = \frac{1 \cdot 3 \cdot 5 \cdots (2s-1)}{2\pi} \left(\frac{3}{2}\right)^{2s} A_s(1/3), \qquad s = 0,1,2,\ldots.$$

By an earlier remark, we also have

$$(2.17) \qquad |\, h_{3,n}(t)\,| \leq |\, b_n \,|\, t^{-n-1/2} \qquad \text{for } t > 0, \text{ if } n = 0,3,6,\ldots.$$

If $\alpha \neq \frac{1}{2}$ then it follows from Theorem 1 in [*Ibid.*] that

$$(2.18) \qquad I_3(\lambda) = \sum_{s=0}^{n-1} a_s\, M[h_3; s+\alpha]\, \lambda^{-s-\alpha} + \sum_{s=0}^{n-1} b_s\, M[f; 1-s-\tfrac{1}{2}]\, \lambda^{-s-1/2} + \delta_{3,n}(\lambda),$$

whereas if $\alpha = \frac{1}{2}$ then we obtain from Theorem 2 in [*Ibid.*]

$$(2.19) \qquad I_3(\lambda) = \sum_{s=0}^{n-1} c_s\, \lambda^{-s-1/2} + (\ell n\, \lambda) \sum_{s=0}^{n-1} a_s b_s\, \lambda^{-s-1/2} + \delta_{3,n}(\lambda),$$

where

$$c_s(\alpha) = a_s b_s^* + a_s^* b_s,$$

$$a_s^* = \lim_{z \to s+1/2} \left\{ M[f; 1-z] + \frac{a_s}{z-s-1/2} \right\}, \qquad b_s^* = \lim_{z \to s+1/2} \left\{ M[f; 1-z] + \frac{b_s}{z-s-1/2} \right\};$$

cf. [WON3, §3]. In both cases, the remainder is given by

$$(2.20)\qquad \delta_{3,n}(\lambda) = \int_0^\infty f_n(t)\, h_n(\lambda t)\, dt.$$

Bounds for $\delta_{3,n}(\lambda)$ can also be found in [*Ibid.*]. In particular, if $\alpha > \frac{1}{2}$ then from (2.17),

$$(2.21)\qquad |\,\delta_{3,n}(\lambda)\,| \le \frac{|\,b_n\,|}{\lambda^{n+1/2}} \int_0^\infty |\,f_n(t)\,|\, t^{-n-1/2}\, dt, \qquad \text{if } n = 0,3,6,\ldots.$$

To the oscillatory integrals $I_1(\lambda)$ and $I_2(\lambda)$, we apply the result in [WON1, §4] which gives

$$(2.22)\qquad I_i(\lambda) = \sum_{s=0}^{n-1} a_s\, M[h_i\, ;\, s+\alpha]\, \lambda^{-s-\alpha} + \delta_{i,n}(\lambda)$$

for i = 1,2, where

$$(2.23)\qquad \delta_{i,n}(\lambda) = \frac{(-1)^n}{\lambda^n} \int_0^\infty f_n^{(n)}(t)\, h_i^{(-n)}(\lambda t)\, dt$$

and $h_i^{(-n)}(t)$ denotes an n^{th} iterated integral of $h_i(t)$. In the case of $h_1(t)$, we can write

$$h_1^{(-n)}(t) = \frac{(-1)^n}{(n-1)!} \int_t^{t+\infty e^{i\pi/3}} (w-t)^{n-1}\, h_1(w)\, dw\,;$$

on the path of integration, $w = t + \rho e^{i\pi/3}$ and ρ varies from 0 to ∞.

It is readily verified that

$$\mathrm{Im}(w^{3/2}) \ge \left(\frac{\sqrt{3}}{2}\right)^{3/2} \rho^{3/2}.$$

In view of the well-known result [OLV5, p. 266 ff]

$$|\,H_{1/3}^{(1)}(\zeta)\,| \le \left|\sqrt{\frac{2}{\pi\zeta}}\, e^{i\zeta}\right|, \qquad 0 \le \arg\zeta \le \pi,$$

it follows that

$$|\,h_1(w)\,| \le \frac{1}{4\pi}\, t^{-1/2} \exp\left\{-\frac{2^{1/2}}{3^{1/4}}\, \rho^{3/2}\right\}.$$

Consequently,

$$| h_1^{(-n)}(t) | \le \frac{1}{(n-1)!} \frac{t^{-1/2}}{6\pi} \Gamma(2n/3) \left(\frac{\sqrt{3}}{2} \right)^{n/3} .$$

Similarly, we can write

$$h_2^{(-n)}(t) = \frac{(-1)^n}{(n-1)!} \int_t^{t+\infty e^{-i\pi/3}} (w - t)^{n-1} h_2(w)\, dw.$$

Using the estimate [*op. cit.*, p. 267]

$$| H_{1/3}^{(2)}(\zeta) | \le \left| \sqrt{\frac{2}{\pi\zeta}} e^{-i\zeta} \right|, \qquad -\pi \le \arg \zeta \le 0,$$

we have

$$| h_2^{(-n)}(t) | \le \frac{1}{(n-1)!} \frac{t^{-1/2}}{6\pi} \Gamma(2n/3) \left(\frac{\sqrt{3}}{2} \right)^{n/3} .$$

Thus, if $\frac{1}{2} < \alpha \le 1$, then (2.23) gives

(2.24) $$| \delta_{i,n}(\lambda) | \le \frac{c_n}{\lambda^{n+1/2}} \int_0^\infty t^{-1/2} | f_n^{(n)}(t) |\, dt, \qquad i = 1,2,$$

where

(2.25) $$c_n = \frac{1}{(n-1)!\, 6\pi} \Gamma(2n/3) \left(\frac{\sqrt{3}}{2} \right)^{n/3} .$$

3. A Special Case

We now apply the results of the previous section to the integral

(3.1) $$F_1(\upsilon) = \int_0^\infty \varphi^4(-\zeta)\, Ai^2(-\upsilon^{2/3}\zeta)\, d\zeta ,$$

where $\varphi(\zeta)$ is given by (2.2.6). In the notations of §2, we have $f(t) = \varphi^4(-t)$ and $\lambda = \upsilon^{2/3}$. The Taylor expansion of f(t) is given by

(3.2) $$f(t) = \varphi^4(-t) = 2^{4/3} - \frac{8}{5} t + \frac{12}{35} (2^{2/3})\, t^2 - \cdots.$$

Hence, condition (2.8) is satisfied by this function, and $\alpha = 1$ in the present case. Furthermore, it is readily verified that $f^{(j)}(t) = O(t^{-2-j})$ as $t \to \infty$ for j = 0,1,2,..., and so

(2.9) is also satisfied by this function. We shall take n = 2 in (2.14) and (2.15). The coefficients a_s in (2.14) are determined by (3.2), and b_s in (2.15) by (2.16). The first few values are given by

$$(3.3) \qquad a_0 = 2^{4/3}, \qquad a_1 = -\tfrac{8}{5}, \qquad b_0 = \tfrac{1}{2\pi}, \qquad b_1 = 0.$$

From the definition of the Mellin transform given in (2.10), we have

$$M[f; 1/2] = \int_0^\infty t^{-1/2} f(t)\, dt = \int_0^\infty t^{-1/2} \varphi^4(-t)\, dt.$$

The last integral can be evaluated explicitly. To see this, we note that from (2.2.5) and (2.2.6),

$$(3.4) \qquad \int_0^\infty \varphi^4(-\zeta)\, \zeta^{-1/2}\, d\zeta = 4\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\, dx = 2\pi,$$

the relationship between $d\zeta$ and dx being deducible from (2.1.2), and given by

$$(3.5) \qquad d\zeta/dx = 2/\{x\varphi^2(\zeta)\}.$$

Thus,

$$(3.6) \qquad M[f; 1/2] = 2\pi.$$

Furthermore, the remaining Mellin transforms may be calculated directly from (2.11) - (2.13) from which we find (see also [**ABR**, p. 446, Eq. 10.4.4])

$$(3.7) \qquad \begin{aligned} M[h_1; 1] + M[h_2; 1] + M[h_3; 1] &= -Ai'^2(0), \\ M[h_1; 2] + M[h_2; 2] + M[h_3; 2] &= -\tfrac{1}{3} Ai(0)Ai'(0). \end{aligned}$$

From (2.6) we find that

$$(3.8) \qquad F_1(\upsilon) = \upsilon^{-1/3} - 2^{4/3}Ai'^2(0)\upsilon^{-2/3} + \tfrac{8}{15} Ai(0)Ai'(0)\upsilon^{-4/3} + \delta(\upsilon)$$

in light of (2.7), (2.18) and (2.22). The remainder $\delta(\upsilon)$ is given by

$$(3.9) \qquad \delta(\upsilon) = \delta_{1,2}(\lambda) + \delta_{2,2}(\lambda) + \delta_{3,2}(\lambda),$$

where $\lambda = \upsilon^{2/3}$, and $\delta_{3,2}$ and $\delta_{i,2}$, i = 1,2 are as defined by (2.20) and (2.23), respectively.

We now simplify the remainder $\delta_{3,2}(\lambda)$ where, from (2.20),

$$\delta_{3,2}(\lambda) = \int_0^\infty f_2(t)\, h_{3,2}(\lambda t)\, dt.$$

By Taylor's theorem (cf. (3.2)),

$$f(t) = \varphi^4(-t) = 2^{4/3} - \tfrac{8}{5} t + f_2(t) \tag{3.10}$$

where

$$f_2(t) = \tfrac{1}{2} f''(\xi)\, t^2 = \tfrac{1}{2} (\varphi^4)''(-\xi)\, t^2, \qquad 0 < \xi < t. \tag{3.11}$$

Since $[\varphi^4(-\zeta)]''$ is a positive decreasing function in $0 \le \zeta < \infty$ [1], we have $0 < (\varphi^4)''(-\xi) \le (\varphi^4)''(0) = \frac{24}{35} 2^{2/3} = 1.09$. Also, since $b_2 = 0$ in (2.15), we have $h_{3,2}(t) = h_{3,3}(t)$, and so (2.17) holds for $h_{3,2}(t)$, $t \in (0,\infty)$. Thus,

$$|\, h_{3,2}(t)\, | \le \frac{5}{64\pi} t^{-3-1/2} \le \frac{5}{64\pi} t^{-2-1/2} \qquad \text{if } t \ge 1,$$

in view of (2.5) and (2.16). Since $1/\lambda \to 0$ as $\lambda \to \infty$, it is convenient to split the interval of integration into $(0,1/\lambda]$ and $[1/\lambda,\infty)$ so that

$$\delta_{3,2}(\lambda) = \int_0^{1/\lambda} f_2(t)\, h_{3,2}(\lambda t)\, dt + \int_{1/\lambda}^\infty f_2(t)\, h_{3,2}(\lambda t)\, dt.$$

Since $h_3(t)$ is positive, we have

$$\left| \int_0^{1/\lambda} f_2(t)\, h_{3,2}(\lambda t)\, dt \right| \le \frac{1.09}{2} \int_0^{1/\lambda} t^2 \left[\tfrac{1}{2\pi} - h_3(\lambda t)\right] dt \tag{3.12}$$

$$\le \frac{0.545}{2\pi} \int_0^{1/\lambda} t^2\, dt \le \frac{0.029}{\lambda^3}$$

in light of (2.15), and

$$\left| \int_{1/\lambda}^\infty f_2(t)\, h_{3,2}(\lambda t)\, dt \right| \le \frac{5}{64\pi} \lambda^{-5/2} \int_{1/\lambda}^\infty f_2(t)\, t^{-5/2}\, dt \tag{3.13}$$

$$\le \frac{5}{64\pi} \lambda^{-5/2} \int_0^\infty f_2(t)\, t^{-5/2}\, dt.$$

[1] See Appendix I.

To bound the last integral, we note that from (3.11),

$$\int_0^1 f_2(t)\, t^{-5/2}\, dt \le \frac{1.09}{2} \int_0^1 t^{-1/2}\, dt = 1.09,$$

and from (3.4),

$$\int_1^\infty f(t)\, t^{-5/2}\, dt \le \int_1^\infty f(t)\, t^{-1/2}\, dt < 2\pi . \tag{3.14}$$

Also, a straightforward calculation gives

$$\int_1^\infty [2^{4/3} - \tfrac{8}{5} t]\, t^{-5/2}\, dt = \tfrac{2}{3}(2^{4/3}) - \tfrac{16}{5} = -1.520 .$$

In light of (3.10),

$$\int_1^\infty f_2(t)\, t^{-5/2}\, dt < 2\pi + 1.520 = 7.803$$

and so

$$0 < \int_0^\infty f_2(t)\, t^{-5/2}\, dt < 1.09 + 7.803 \le 8.894.$$

Hence,

$$\left| \int_{1/\lambda}^\infty f_2(t)\, h_{3,2}(\lambda t)\, dt \right| \le 0.222\lambda^{-5/2}. \tag{3.15}$$

Combining (3.12) and (3.15), we obtain

$$| \delta_{3,2}(\lambda) | \le \frac{0.029}{\lambda^3} + \frac{0.222}{\lambda^{5/2}} . \tag{3.16}$$

The bounds for $\delta_{i,2}$, $i = 1,2$, are given by (2.24) from which we find

$$| \delta_{i,2}(\lambda) | \le \frac{0.044}{\lambda^{5/2}} \int_0^\infty t^{-1/2}\, | f_2^{(2)}(t) |\, dt. \tag{3.17}$$

To simplify these bounds, we note that since
$f_2^{(2)}(t) = f^{(2)}(t)$ and $0 \le f^{(2)}(t) \le (\varphi^4)''(-t) \le \frac{24}{35}(2^{2/3})$,

$$\int_0^1 t^{-1/2}\, | f_2^{(2)}(t) |\, dt \le 2.178 . \tag{3.18}$$

On the other hand, integration by parts twice gives

$$(3.19)\qquad \int_1^{\infty} t^{-1/2}\,|\,f_2^{(2)}(t)\,|\,dt = \int_1^{\infty} t^{-1/2}\,f^{(2)}(t)\,dt = \frac{3}{4}\int_1^{\infty} t^{-5/2}\,f(t)\,dt - f'(1) - \frac{1}{2}f(1)$$

by virtue of the order relation as $t \to +\infty$ given in (2.9). From (3.5) we find by straightforward differentiation that

$$\frac{d}{d\zeta}\varphi^4(\zeta) = -\frac{4(x^2-1) + 4\zeta x^2\varphi^2(\zeta)}{(x^2-1)^2}.$$

From Olver's tables [**OLV2**, pp. 38 and 41] we find

$\varphi(-1) = 1.0821991971$ and $x(-1) = 1.9789626178$. Hence,

$f(1) = \varphi^4(-1) = 1.371604273$ and $f'(1) = -(\varphi^4)'(-1) = -0.785580091$.

Consequently, it follows from (3.19) and (3.14) that

$$(3.20)\qquad \int_1^{\infty} t^{-1/2}\,|\,f_2^{(2)}(t)\,|\,dt \le \frac{3}{4}(2\pi) + 0.0998 \le 4.813.$$

Coupling (3.18) and (3.20), we obtain

$$(3.21)\qquad |\,\delta_{i,2}(\lambda)\,| \le 0.310\lambda^{-5/2}, \qquad\qquad i = 1,2.$$

Finally, a combination of (3.9), (3.16) and (3.21) gives

$$(3.22)\qquad |\,\delta(\upsilon)\,| \le \frac{0.842}{\upsilon^{5/3}}\left\{1 + \frac{0.0345}{\upsilon^{1/3}}\right\}$$

$$\le \frac{0.856}{\upsilon^{5/3}} \qquad\qquad \text{for } \upsilon \ge 10.$$

4. The Relationship Between $F(\upsilon,2)$ and $F_1(\upsilon)$: A Bound for $F(\upsilon,2)$

We now turn our attention to the integral defined in (1.2). Since $j''_{\upsilon,k} \ge j''_{\upsilon,2}$ $\forall k \ge 2$, it is clear that $F(\upsilon,k) \le F(\upsilon,2)$ $\forall k \ge 2$. Hence, for our purposes, it suffices to take $k = 2$ in (1.2), and consider the integral

$$(4.1)\qquad F(\upsilon,2) = \int_{j''_{\upsilon,2}}^{\infty} \frac{J_\upsilon^2(t)}{t}\,dt.$$

In (4.1) we first make the change of variable $t = \upsilon x$ and replace $J_\upsilon^2(\upsilon x)$ by its asymptotic approximation which we now establish: squaring both sides of (2.2.19) gives

$$(4.2)\qquad J_\upsilon^2(\upsilon x) = \frac{1}{(1+\delta_1)^2}\frac{\varphi^2(\zeta)}{\upsilon^{2/3}}\left[\mathrm{Ai}^2(\upsilon^{2/3}\zeta) + \varepsilon^*(\upsilon,\zeta)\right],$$

where $\qquad \varepsilon^*(\upsilon,\zeta) = 2\mathrm{Ai}(\upsilon^{2/3}\zeta)\,\varepsilon_{1,1}(\upsilon,\zeta) + \varepsilon_{1,1}^2(\upsilon,\zeta).$

Since ζ is negative in our case, it follows from (2.3.4) and (2.3.8) that

$$(4.3)\qquad |\,\varepsilon^*(\upsilon,\zeta)\,| \le \frac{0.134e^{0.30/\upsilon}\upsilon^{-1}}{(-\upsilon^{2/3}\zeta)^{1/2}}\left[1+\frac{0.1051e^{0.30/\upsilon}}{\upsilon}\right]$$

$$\le \frac{0.140}{\upsilon^{4/3}(-\zeta)^{1/2}} \qquad\qquad \text{if } \upsilon \ge 10.$$

Equation (4.2) can be further simplified to

$$(4.4)\qquad J_\upsilon^2(\upsilon x) = \varphi^2(\zeta)\frac{\mathrm{Ai}^2(\upsilon^{2/3}\zeta)}{\upsilon^{2/3}} + \tilde{\varepsilon}(\upsilon,\zeta)$$

with

$$(4.5)\qquad \tilde{\varepsilon}(\upsilon,\zeta) = \frac{\varphi^2(\zeta)}{\upsilon^{2/3}}\left\{\frac{\varepsilon^*(\upsilon,\zeta) - \delta_1(2+\delta_1)\,\mathrm{Ai}^2(\upsilon^{2/3}\zeta)}{(1+\delta_1)^2}\right\}.$$

By (2.3.3) we find $|1+\delta_1| \ge 1 - 0.022$ if $\upsilon \ge 10$, and so a combination of (2.3.8) and (4.3) gives

$$(4.6)\qquad |\,\tilde{\varepsilon}(\upsilon,\zeta)\,| \le \varphi^2(\zeta)\frac{0.292}{\upsilon^2(-\zeta)^{1/2}} \qquad\qquad \text{if } \upsilon \ge 10.$$

After replacing $J_\upsilon^2(\upsilon x)$ in (4.1) by its approximation (4.4), we then make ζ the variable of integration. Since $j''_{\upsilon,2} > j'_{\upsilon,1} > \upsilon$ (see [**LOR**, (2.4)] and [**OLV5**, p. 246]), the point $x = x_{\upsilon,2} \equiv j''_{\upsilon,2}/\upsilon$ is greater than 1 and its image $\zeta = \zeta_{\upsilon,2}$ under the transformation (2.2.5) is negative. The result is

$$(4.7) \qquad F(\upsilon,2) = \frac{1}{2\upsilon^{2/3}} \int_{\bar{\zeta}_\upsilon}^{\infty} \varphi^4(-\zeta)\, Ai^2(-\upsilon^{2/3}\zeta)\, d\zeta + \rho_1(\upsilon),$$

where $\bar{\zeta}_\upsilon = -\zeta_{\upsilon,2}$, $\varphi(\zeta)$ is the function defined by (2.2.6), and

$$(4.8) \qquad \rho_1(\upsilon) = \frac{1}{2} \int_{\bar{\zeta}_\upsilon}^{\infty} \tilde{\varepsilon}(\upsilon,-\zeta)\, \varphi^2(-\zeta)\, d\zeta.$$

Since $\bar{\zeta}_\upsilon > 0$, it follows from (4.6) that

$$(4.9) \qquad |\rho_1(\upsilon)| \le \frac{0.292}{2\upsilon^2} \int_0^{\infty} \varphi^4(-\zeta)\, \zeta^{-1/2}\, d\zeta \le \frac{0.918}{\upsilon^2}, \qquad \upsilon \ge 10,$$

in light of (3.4). For convenience, let us now set

$$(4.10) \qquad F^*(\upsilon) = -\int_0^{\bar{\zeta}_\upsilon} \varphi^4(-\zeta)\, Ai^2(-\upsilon^{2/3}\zeta)\, d\zeta$$

so that we may write (4.7) as

$$(4.11) \qquad F(\upsilon,2) = \frac{1}{2\upsilon^{2/3}} [F_1(\upsilon) + F^*(\upsilon)] + \rho_1(\upsilon),$$

where $F_1(\upsilon)$ is defined by (3.1).

We shall now consider the integral in (4.10). In view of (2.3.27) and (2.3.28) we can write $F^*(\upsilon)$ in the form

$$(4.12) \qquad F^*(\upsilon) = F_2(\upsilon) + \rho_2(\upsilon),$$

where

$$(4.13) \qquad F_2(\upsilon) = -\frac{1}{\upsilon^{2/3}} \int_{a_2}^{0} \varphi^4(\upsilon^{-2/3}\tau)\, Ai^2(\tau)\, d\tau,$$

and

$$(4.14) \qquad \rho_2(\upsilon) = -\frac{1}{\upsilon^{2/3}} \int_{a_2 + \upsilon^{2/3}\eta_2}^{a_2} \varphi^4(\upsilon^{-2/3}\tau)\, Ai^2(\tau)\, d\tau.$$

Recalling that $0 \le \varphi(\zeta) \le \varphi(0) = 2^{1/3}$ for $\zeta \in (-\infty,0]$, and noting that $|Ai(\zeta)| \le 0.53566$ for $\zeta \in (-\infty,0]$ (see [**ABR**, pp. 446 and 478]), we have from (2.3.27) and (2.3.28)

$$(4.15) \qquad |\rho_2(\upsilon)| \le 0.278/\upsilon^{4/3} \qquad \text{for } \upsilon \ge 10.$$

To evaluate the integral in (4.13), we use the Taylor expansion (cf. (3.10) and (3.11))

$$\varphi^4(\zeta) = 2^{4/3} + \tfrac{8}{5}\zeta + R_2(\zeta),$$

where

$$R_2(\zeta) = \frac{\zeta^2}{2!}(\varphi^4)''(\xi), \qquad \zeta < \xi < 0.$$

By the remark following (3.11), we have $0 \le [\varphi^4(\zeta)]'' \le 1.09$ for $-\infty < \zeta \le 0$, and so

(4.16) $$| R_2(\zeta) | \le 0.55\zeta^2, \qquad -\infty < \zeta < 0.$$

Using the fact that Ai(z) satisfies the differential equation $w'' - zw = 0$, we have by integration by parts

$$M_0(z) \equiv \int Ai^2(z)\,dz = z\,Ai^2(z) - Ai'^2(z)$$

$$M_1(z) \equiv \int zAi^2(z)\,dz = \tfrac{1}{3}[z\,M_0(z) + Ai(z)\,Ai'(z)]$$

$$M_2(z) \equiv \int z^2Ai^2(z)\,dz = \tfrac{1}{5}[3z\,M_1(z) + z\,Ai(z)\,Ai'(z) - Ai^2(z)],$$

from which it follows that

$$M_0(0) = -Ai'^2(0), \qquad M_1(0) = \tfrac{1}{3}Ai(0)\,Ai'(0), \qquad M_2(0) = -\tfrac{1}{5}Ai^2(0),$$
$$M_0(a_2) = -Ai'^2(a_2), \qquad M_1(a_2) = -\tfrac{1}{3}a_2\,Ai'^2(a_2), \qquad M_2(a_2) = -\tfrac{1}{5}a_2^2\,Ai'^2(a_2).$$

Consequently, we obtain

(4.17) $$F_2(\upsilon) = -\frac{1}{\upsilon^{2/3}}\left\{\int_{a_2}^{0}\left[2^{4/3} + \tfrac{8}{5}(\upsilon^{-2/3}\tau)\right]Ai^2(\tau)\,d\tau + \int_{a_2}^{0} R_2(\upsilon^{-2/3}\tau)\,Ai^2(\tau)\,d\tau\right\}$$

$$= \frac{2^{4/3}}{\upsilon^{2/3}}\left[Ai'^2(0) - Ai'^2(a_2)\right] - \frac{8}{15\upsilon^{4/3}}\left[Ai(0)\,Ai'(0) + a_2\,Ai'^2(a_2)\right] + \rho_3(\upsilon)$$

where

(4.18) $$| \rho_3(\upsilon) | \le \frac{0.55}{5\upsilon^2}[\,a_2^2\,Ai'^2(a_2) - Ai^2(0)].$$

From numerical tables [*op. cit.*, pp. 476 and 478], we find

$a_2^2\,Ai'^2(a_2) - Ai^2(0) = 10.65253$, and so

(4.19) $\quad | \rho_3(\upsilon) | \le 1.172 / \upsilon^2.$

Coupling (4.11) and (4.12), we have

$$(4.20) \qquad F(\upsilon,2) = \frac{1}{2\upsilon^{2/3}}[F_1(\upsilon) + F_2(\upsilon) + \rho_2(\upsilon)] + \rho_1(\upsilon).$$

Substitution of (3.8) and (4.17) in (4.20) yields

$$(4.21) \qquad F(\upsilon,2) = \frac{1}{2\upsilon} - \frac{2^{1/3}Ai'^2(a_2)}{\upsilon^{4/3}} - \frac{4}{15}\frac{a_2Ai'^2(a_2)}{\upsilon^2} + \rho(\upsilon),$$

where $\qquad \rho(\upsilon) = \rho_1(\upsilon) + [\delta(\upsilon) + \rho_2(\upsilon) + \rho_3(\upsilon)] / 2\upsilon^{2/3}.$

From (3.22), (4.9), (4.15) and (4.19), it follows that

$$| \rho(\upsilon) | \le \frac{1.057}{\upsilon^2}\left[1 + \frac{0.4050}{\upsilon^{1/3}} + \frac{0.5540}{\upsilon^{2/3}}\right] \le \frac{1.382}{\upsilon^2} \qquad \text{if } \upsilon \ge 10.$$

The approximation formula (1.5) is obtained from (4.21) with

$$\frac{\varepsilon_1(\upsilon)}{\upsilon^2} = -\frac{4}{15}\frac{a_2Ai'^2(a_2)}{\upsilon^2} + \rho(\upsilon).$$

5. Bounds for $J_\upsilon^2(j''_{\upsilon,k})$, $k \ge 2$

If we set $\upsilon x = j''_{\upsilon,k}$ in (2.2.19) we obtain

$$(5.1) \qquad J_\upsilon(j''_{\upsilon,k}) = \frac{1}{1+\delta_1}\frac{\varphi(\zeta_{\upsilon,k})}{\upsilon^{1/3}}\left[Ai(\upsilon^{2/3}\zeta_{\upsilon,k}) + \varepsilon_{1,1}(\upsilon,\zeta_{\upsilon,k})\right]$$

where $\upsilon^{2/3}\zeta_{\upsilon,k} \in I_k = [-\alpha_k, -\beta_k]$ is given by (2.3.16) with α_k and β_k defined by (2.4.32). On the interval I_k, $\upsilon^{2/3}\zeta_{\upsilon,k} \le -\beta_k < 0$ and so from (2.3.4),

$$(5.2) \qquad \max_{I_k} | \varepsilon_{1,1}(\upsilon,\zeta) | \le \frac{0.1186e^{0.30/\upsilon}}{\upsilon\,\beta_k^{1/4}} \le \frac{0.0568}{\upsilon^{2/3}\,\beta_k^{1/4}} \qquad \text{if } \upsilon \ge 10.$$

Also as remarked earlier (see §2.5; in particular, note the remark preceding (2.5.29)), we have

$$Ai(\upsilon^{2/3}\zeta_{\upsilon,k}) = Ai'(\xi_k)\upsilon^{2/3}\eta_k, \qquad \xi_k \in (a_k - \rho_k, a_k + \rho_k) \subseteq I_k.$$

A bound for η_k is given by (2.3.38) for k = 2,3,...,9 and by (2.5.2) for k ≥ 10.

Since $\xi_k \in [a'_{k+1}, a'_k]$[1], we have $|Ai'(\xi_k)| \le |Ai'(a_k)|$ by virtue of the fact that a_k is a local extremum point of Ai'(x) [2], and lies between the zeros a'_{k+1} and a'_k of Ai'(x). Hence, $|Ai(\upsilon^{2/3}\zeta_{\upsilon,k})| \le |Ai'(a_k)|\,|\upsilon^{2/3}\eta_k|$. Restricting our attention again to the case k = 2, we have from (2.3.26) for $\upsilon \ge 10$,

$$|\upsilon^{2/3}\eta_2| \le \frac{0.3743}{\upsilon^{2/3}} = \frac{0.53223}{(-a_2)^{1/4}\upsilon^{2/3}},$$

and so

$$|Ai(\upsilon^{2/3}\zeta_{\upsilon,2})| \le \frac{0.53223}{\upsilon^{2/3}}\frac{|Ai'(a_2)|}{(-a_2)^{1/4}}.$$

Recalling again that $|Ai'(x)|/|x|^{1/4} \le 0.60$ for x < 0 (see the remark following (2.3.11)), we have

(5.3) $$|Ai(\upsilon^{2/3}\zeta_{\upsilon,2})| \le 0.3194/\upsilon^{2/3}.$$

Furthermore, we recall that $0 \le \varphi(\zeta) \le 2^{1/3}$ for $-\infty < \zeta \le 0$, and note that, from (2.3.3), $|1+\delta_1| \ge 1 - 0.022$ if $\upsilon \ge 10$. Thus, since $\beta_2 = -(a_2 + \rho_2) = 3.90491$, (5.1) gives

(5.4) $$|J_\upsilon(j''_{\upsilon,2})| \le \frac{2^{1/3}}{1-0.022}\frac{1}{\upsilon}\left\{0.3194 + \frac{0.0568}{(3.90491)^{1/4}}\right\}$$

$$\le \frac{0.4636}{\upsilon} \qquad \text{if } \upsilon \ge 10$$

in light of the bounds (5.2) and (5.3). Hence,

(5.5) $$0 \le J^2_\upsilon(j''_{\upsilon,2}) \le e_2/\upsilon^2, \qquad \text{with } e_2 = 0.215.$$

In an exactly similar manner, numerical computations show that for k = 3,4,...,9,

(5.6) $$0 \le J^2_\upsilon(j''_{\upsilon,k}) \le e_k/\upsilon^2,$$

with $e_2 = 0.21458$, $e_3 = 0.15175$, $e_4 = 0.12074$, $e_5 = 0.10191$,
$e_6 = 0.08911$, $e_7 = 0.07977$, $e_8 = 0.07261$, $e_9 = 0.06692$.

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[1] See Appendix II.
[2] Ai(x) satisfies the differential equation w'' - xw = 0. Hence, $Ai''(a_k) = 0$.

For the values $k \geq 10$, we follow a similar argument as above, except that we replace the bound on $| \upsilon^{2/3}\eta_2 |$ by the bound

$$| \upsilon^{2/3}\eta_k | \leq \frac{0.5200}{\upsilon^{2/3}\beta_k^{1/2}}, \qquad k \geq 10,$$

given by (2.5.31). The result on combining this bound with the bound (5.2) is, for $k \geq 10$,

$$| J_\upsilon(j_{\upsilon,k}) | \leq \frac{2^{1/3}}{1 - 0.022} \frac{1}{\upsilon\beta_k^{1/4}} \left\{ 0.5200 \, | Ai'(a_k) | / \beta_k^{1/4} + 0.0568 \right\} \text{ if } \upsilon \geq 10.$$

Furthermore, since $\beta_k = -a_k(1 + \rho_k/a_k)$, we have from (2.4.31) and [**Abr**, p. 478] for $k \geq 10$,

$$\beta_k \geq - a_k\left(1 - \frac{0.1119}{12.82878}\right) \geq -0.9912 \, a_k.$$

Thus, $| Ai'(a_k) | / \beta_k^{1/4} \leq 0.60(0.9912)^{1/4} \leq 0.60$, and so for $k \geq 10$,

(5.7) $$| J_\upsilon(j''_{\upsilon,k}) | \leq \frac{0.476}{\upsilon\beta_k^{1/4}} \qquad \text{if } \upsilon \geq 10.$$

Hence,

(5.8) $$0 \leq J^2_\upsilon(j''_{\upsilon,k}) \leq \frac{0.227}{\upsilon^2\beta_k^{1/2}} \leq \frac{0.064}{\upsilon^2} \qquad \text{if } \upsilon \geq 10, k \geq 10.$$

A combination of (5.6) and (5.8) shows that

(5.9) $$0 \leq J^2_\upsilon(j''_{\upsilon,k}) \leq e_k / \upsilon^2 \qquad \text{whenever } \upsilon \geq 10,$$

with $e_2 = 0.215$, and $e_{k+1} < e_k \leq e_2 \; \forall k \geq 2$. Consequently, (1.6) follows.

6. Proof of the Monotonicity of $j''_{\upsilon,k}$: Conclusion

We are now in a position to verify the property that for $k \geq 2$, $j''_{\upsilon,k}$ is an increasing function of υ whenever $\upsilon \geq 10$. Recalling (1.3), we have

$$G(j''_{\upsilon,k}) = \frac{1}{2\upsilon} - F(\upsilon,k) - J^2_\upsilon(j''_{\upsilon,k})$$

$$\geq \frac{1}{2\upsilon} - F(\upsilon,2) - J^2_\upsilon(j''_{\upsilon,k})$$

$$= \frac{0.812634}{\upsilon^{4/3}} + \frac{\varepsilon(\upsilon)}{\upsilon^2},$$

where $\quad |\varepsilon(\upsilon)| \leq |\varepsilon_1(\upsilon)| + |\varepsilon_2(\upsilon)| \leq 2.302 \quad$ if $\upsilon \geq 10$,

in light of (1.5) and (1.6). Consequently, if $\upsilon \geq 10$,

$$G(j''_{\upsilon,k}) \geq \frac{0.812634}{\upsilon^{4/3}}\left[1 - \frac{2.83276}{10^{2/3}}\right] = \frac{0.31668}{\upsilon^{4/3}} > 0.$$

Furthermore, for $k = 2,3,\ldots$, $c_{\upsilon,k} > 0$ if $\upsilon \geq 1$ (see (1.3.9)), and so, recalling (1.3.7), we have for $k = 2,3,\ldots$,

$$\frac{dj''_{\upsilon,k}}{d\upsilon} = c_{\upsilon,k}\, G(j''_{\upsilon,k}) > 0 \qquad \text{if } \upsilon \geq 10.$$

Thus, $j''_{\upsilon,k}$ is an increasing function of υ if $\upsilon \geq 10$. Combining this result with that of Lorch and Szego [**LOR**], we have for $k = 1,2,3,\ldots$,

$j''_{\upsilon,k}$ *is an increasing function of* υ *if* $\upsilon > 0$, and so our task is complete.

The bounds that we have established throughout this thesis are not as sharp as they could be, and they can, indeed, be improved; we believe that our asymptotic argument can be used to show that $G(j''_{\upsilon,k})$ is positive for all $\upsilon \geq 7$ and for $k = 2,3,\ldots$. For slightly sharper bounds in certain cases, see [**WON4**].

APPENDIX I

(i) Monotonicity of $\varphi(\zeta)$:

We shall demonstrate that the analytic function

$$(1) \qquad \varphi(\zeta) = \left(\frac{4\zeta}{1 - x^2}\right)^{1/4}$$

is a nonnegative, increasing function in $-\infty < \zeta \leq 0$. First, from the relation (2.2.5), it is readily verified that $\varphi(\zeta) = O((-\zeta)^{-1/2})$ as $\zeta \to -\infty$ (also compare with [**OLV2**, p.40]), and so $\varphi(-\infty) = 0$. Furthermore, $\varphi(0) = 2^{1/3}$ (cf. [*op. cit.*, p. 38]). Next, we show that $\varphi(\zeta)$ is increasing in $(-\infty,0)$ by showing that the derivative of the function $f(\zeta) = \varphi^4(\zeta)$ is positive on this interval. It then follows that $\varphi(\zeta)$ is a nonnegative, increasing function in $-\infty < \zeta \leq 0$.

Using (3.3.5), we have by straightforward differentiation (cf. the remark following (3.3.19)),

$$(2) \qquad f'(\zeta)' = \frac{-4(x^2 - 1) - 4(-\zeta)x^2\varphi^2(\zeta)}{(x^2 - 1)^2}.$$

If $f'(\zeta) > 0$ for $-\infty < \zeta < 0$, then $f(\zeta)$ is increasing in $(-\infty,0)$. Hence, we show that $f'(\zeta) > 0$ by considering the numerator of the right-hand-side of (2). Replacing $\varphi(\zeta)$ in (2) by the expression in (1), we find that the numerator can be written as

$$4(x^2 - 1)^{-1/2}[-(x^2 - 1)^{3/2} + 2(-\zeta)^{3/2}\, x^2].$$

This expression is positive if the function $g(\zeta)$ defined by

$$g(\zeta) = -(x^2 - 1)^{3/2} + 2(-\zeta)^{3/2}\, x^2$$

is positive. Noting that $\sec^{-1}x = \tan^{-1}\sqrt{x^2 - 1}$ if $x > 1$, we have from (2.2.5)

$$(3) \qquad x^{-2}g(\zeta) = -\frac{(x^2 - 1)^{3/2}}{x^2} + 3[(x^2 - 1)^{1/2} - \tan^{-1}(x^2 - 1)^{1/2}].$$

Since (3) is a function of $u = (x^2 - 1)^{1/2}$, for convenience, we write

$$(4) \qquad x^{-2}g(\zeta) = h(u) = -\frac{u^3}{1 + u^2} + 3(u - \tan^{-1}u),$$

and note that

$$\zeta = 0 \leftrightarrow x = 1 \leftrightarrow u = 0,$$

$$\zeta = -\infty \leftrightarrow x = \infty \leftrightarrow u = \infty.$$

Hence, we show that $h(u) > 0$ for $u > 0$. Now $h(0) = 0$, and

$$h'(u) = 3\left[1 - \frac{1}{1+u^2}\right] - \frac{3u^2(1+u^2) - 2u^4}{(1+u^2)^2} = \frac{2u^4}{(1+u^2)^2} > 0 \qquad \text{if } u > 0.$$

Thus, $h(u) > 0$ if $u > 0$, and consequently, $g(\zeta) > 0$ if $\zeta < 0$. Retracing our steps, we find the demonstration complete.

(ii) Monotonicity of $[\varphi^4(\zeta)]''$:

We now show that the function $[\varphi^4(-\zeta)]''$ is a positive, decreasing function in $0 \le \zeta < \infty$ or, equivalently, that $[\varphi^4(\zeta)]''$ is a positive, increasing function in $-\infty < \zeta \le 0$. Continuing to differentiate* the function $f(\zeta) = \varphi^4(\zeta)$ defined in (i) above, we find that

(5) $$f''(\zeta) = -20(-\zeta)^{1/2}(u^{-5} + u^{-3}) + 8(-\zeta)^2(5u^{-8} + 8u^{-6} + 3u^{-4}),$$

(6) $$f'''(\zeta) = 10(-\zeta)^{-1/2}(u^{-5} + u^{-3}) - 36(-\zeta)[5u^{-8} + 8u^{-6} + 3u^{-4}] + 32(-\zeta)^{5/2}[10u^{-11} + 22u^{-9} + 15u^{-7} + 3u^{-5}],$$

where, again for convenience, we have set $u = (x^2 - 1)^{1/2}$. From (5) and the relation (2.2.5), we find that $f''(-\infty) = 0$, and $f''(0) = \frac{24}{35}(2^{2/3})^\dagger$. To show that $f''(\zeta)$ is increasing in $(-\infty,0)$, we show that $f'''(\zeta) > 0$ for $-\infty < \zeta < 0$. To this end, let us recall (2.2.5), and write

(7) $$\begin{aligned} g(u) &= \tfrac{1}{2}(-\zeta)^{1/2}u^{11}f'''(\zeta) \\ &= 5(u^6 + u^8) - 27(5u^3 + 8u^5 + 3u^7)[u - \tan^{-1}u] \\ &\qquad + 36(10 + 22u^2 + 15u^4 + 3u^6)[u - \tan^{-1}u]^2. \end{aligned}$$

Then, to show that $f'''(\zeta) > 0$ for $\zeta < 0$, we show that $g(u) > 0$ for $u > 0$.

One can show that $g(u) > 0$ for $u > 0$ by symbolic computation in the following way. It is clear that the function $g(u)$ has the form

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* It should be noted that the computations involved in attaining these derivatives are extremely tedious, and a careful check has been made of them.

† The value of $f''(0)$ may also be determined by differentiation of the Taylor expansion (3.3.2).

$$g(u) \equiv g_0(u) = G_{00}(u) + G_{01}(u)W + G_{02}(u)W^2,$$

where $G_{0,i}(u)$, $i = 0,1,2$, are polynomials in u, and $W = W(u) = u - \tan^{-1}u$. Specifically,

$$G_{00}(u) = 5u^6 + 5u^8$$

$$G_{01}(u) = -135u^3 - 216u^5 - 81u^7$$

$$G_{02}(u) = 360 + 792u^2 + 540u^4 + 108u^6 .$$

Differentiation of $g(u) = g_0(u)$ yields

$$g_1(u) \equiv (1 + u^2)\, g_0'(u) = G_{10}(u) + G_{11}(u)W + G_{12}(u)W^2,$$

$$g_2(u) \equiv (1 + u^2)\, g_1'(u) = G_{20}(u) + G_{21}(u)W + G_{22}(u)W^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

where $\qquad G_{i+1,j}(u) = (1 + u^2)\, G_{i,j}'(u) + (j + 1)u^2\, G_{i,j+1}(u), \qquad i = 0,1,2,\ldots,\ j = 0,1,2,$

with $G_{ij}(u) = 0$ if $j > 2$. Consequently, we define the sequence $\{g_i(u)\}$ as

$$g_0(u) = g(u), \text{ and}$$

$$g_{i+1}(u) = (1 + u^2)\, g_i'(u)$$

$$= G_{i+1,0}(u) + G_{i+1,1}(u)W + G_{i+1,2}(u)W^2, \qquad i = 0,1,2,\ldots .$$

Then each $g_i(u)$ has the same form as $g_0(u)$.

If $g(0) = 0$ and $g'(u) > 0$ for $u > 0$, then $g(u) > 0$ for $u > 0$; the same applies to each $g_i(u)$. Hence, to show that $g(u) > 0$ when $u > 0$, we determine that value of i for which $g_j(0) = 0\ \forall\ 0 \le j \le i$, and for which it is clear that $g_i(u) > 0$ for $u > 0$. Then $g_{i-1}'(u) > 0$ $(u > 0)$, since $1 + u^2 > 0$, and consequently, $g_{i-1}(u) > 0$ for $u > 0 \Rightarrow \cdots \Rightarrow g_0(u) = g(u) > 0$ for $u > 0$. Applying this procedure to the function $g(u)$ given in (7), we find that $g_0(0) = g_1(0) = \cdots = g_{17}(0) = 0$, and that the polynomials $G_{17,0}(u)$, $G_{17,1}(u)$ and $G_{17,2}(u)$ each have large <u>nonnegative</u> coefficients[1]. Furthermore, it is clear from the Mean Value Theorem that $W(u) = u - \tan^{-1}u > 0$ for $u > 0$, and so we conclude from the above argument that $g_0(u) > 0$ for $u > 0$. Hence, we are done.

[1] Due to the fact that these coefficients are large, we shall not include them here. The necessary computations were performed symbolically on an IBM 360 mainframe.

APPENDIX II

We show that for $k \geq 10$,

$$a'_{k+1} < -\alpha_k < a_k < -\beta_k < a'_k,$$

where

$$-\alpha_k = a_k - \rho_k, \qquad -\beta_k = a_k + \rho_k$$

with

$$\rho_k = \tfrac{1}{4}(a_k - a'_{k+1}).$$

By a simple application of Rolle's Theorem, together with the fact that $Ai''(x) = xAi(x)$, we have $a'_{k+1} < a_k < a'_k$, where a'_{k+1} and a'_k are consecutive zeros of $Ai'(x)$. (This fact can also be deduced from the graph of $Ai(-x)$ given in [**ABR**, p.446].) Hence, $\rho_k > 0$, and so it is clear that $a'_{k+1} < -\alpha_k < a_k < -\beta_k$. All we need show now is the fact that $-\beta_k < a'_k$ for $k \geq 10$, and to see this, we recall that $d_{k-1} = \frac{2}{3}(-a'_k)^{3/2}$ (see (2.4.19)). Consequently, it follows from (2.4.18) that if $k \geq 2$,

$$\text{(1)} \qquad \tfrac{2}{3}(-a'_k)^{3/2} = k\pi - \frac{3\pi}{4} + \psi(k),$$

where

$$\text{(2)} \qquad |\psi(k)| \leq \frac{0.1097}{k\pi - 3\pi/4 - 0.03} \leq \frac{0.140}{4k - 3.01}.$$

A combination of (1) and (2.4.51) gives

$$\Delta_k = \tfrac{2}{3}\beta_k^{3/2} - \tfrac{2}{3}(-a'_k)^{3/2} = \frac{3\pi}{8} + \theta_2(k) - \psi(k).$$

From (2.5.11) and (2), we have

$$\theta_2(k) \geq -0.0088 \quad \text{and} \quad \psi(k) \leq 0.0038 \quad \text{for } k \geq 10.$$

Hence, $\Delta_k \geq 3\pi/8 - 0.0088 - 0.0039 > 0$, and we are done.

APPENDIX III

(i) Asymptotic Expansion of $J''_\upsilon(\upsilon x)$:

The asymptotic expansion of $J''_\upsilon(\upsilon x)$ is given by

$$(1)\qquad J''_\upsilon(\upsilon x) \sim \frac{4}{x^2}\left(\frac{1-x^2}{4\zeta}\right)^{3/4}\left\{\frac{Ai(\upsilon^{2/3}\zeta)}{\upsilon^{1/3}}\sum_{k=0}^{\infty} E_k(\zeta)/\upsilon^{2k} + \frac{Ai'(\upsilon^{2/3}\zeta)}{\upsilon^{5/3}}\sum_{k=0}^{\infty} F_k(\zeta)/\upsilon^{2k}\right\},$$

uniform with respect to x in the interval $(0,\infty)$, valid for $\upsilon > 0$ and $x > 0$, where the coefficients $E_k(\zeta)$ and $F_k(\zeta)$ are holomorphic functions in a region containing the real axis. Explicit expressions for $E_k(\zeta)$ and $F_k(\zeta)$ are given by

$$(2)\qquad E_k(\zeta) = \zeta\sum_{s=0}^{2k} \mu_s\, \zeta^{-3s/2}\, w_{2k-s}(t),$$

$$(3)\qquad F_k(\zeta) = -\zeta^{1/2}\sum_{s=0}^{2k+1} \lambda_s\, \zeta^{-3s/2}\, w_{2k-s+1}(t),$$

where

$$(4)\qquad t = (1-x^2)^{-1/2},$$

$$(5)\qquad \lambda_s = \frac{(2s+1)(2s+3)\cdots(6s-1)}{s!\,(144)^s}, \qquad \mu_s = -\frac{6s+1}{6s-1}\lambda_s,$$

$$(6)\qquad w_k(t) = v_k(t) - \tfrac{t}{2}\{(t^2+1) - 2t(t^2-1)v'_{k-1}(t)\},$$

with $v_k(t)$ defined by [**ABR**, p. 366, Eqs. 9.3.13 and 9.3.14], and λ_s and μ_s by [**ABR**, p. 368, Eq. 9.3.41]; see also [**OLV1**]. Specifically, we have

$$E_0(\zeta) = \zeta,$$

$$F_0(\zeta) = \frac{21t+5t^3}{24}\zeta^{1/2} - \frac{5}{48\zeta},$$

$$E_1(\zeta) = \frac{513t^2 - 798t^4 + 385t^6}{1152}\zeta + \frac{147t+35t^3}{1152}\zeta^{-1/2} - \frac{455}{4608}\zeta^{-2},$$

each valid for $0 \le x \le 1$. When $1 \le x \le \infty$, (2) and (3) again hold, provided that the two-valued branchs are interpreted using the relations (7.20) and (7.21) given in [**OLV2**, p. 16].

To derive the above expansion (1), we insert the uniform expansions $J_\upsilon(\upsilon x)$ and $J'_\upsilon(\upsilon x)$, derived by Olver (see (2.2.2) and (2.2.3)), in Bessel's differential equation

$$J''_\upsilon(\upsilon x) = \frac{1-x^2}{x^2} J_\upsilon(\upsilon x) - \frac{1}{\upsilon x} J'_\upsilon(\upsilon x),$$

and get

$$J''_\upsilon(\upsilon x) \sim \frac{1-x^2}{x^2}\left(\frac{4\zeta}{1-x^2}\right)^{1/4}\left\{\frac{\mathrm{Ai}(\upsilon^{2/3}\zeta)}{\upsilon^{1/3}}\sum_{k=0}^{\infty} A_k(\zeta)/\upsilon^{2k} + \frac{\mathrm{Ai}'(\upsilon^{2/3}\zeta)}{\upsilon^{5/3}}\sum_{k=0}^{\infty} B_k(\zeta)/\upsilon^{2k}\right\}$$

$$- \frac{1}{\upsilon x}\left(-\frac{2}{x}\right)\left(\frac{1-x^2}{4\zeta}\right)^{1/4}\left\{\frac{\mathrm{Ai}(\upsilon^{2/3}\zeta)}{\upsilon^{4/3}}\sum_{k=0}^{\infty} C_k(\zeta)/\upsilon^{2k} + \frac{\mathrm{Ai}'(\upsilon^{2/3}\zeta)}{\upsilon^{2/3}}\sum_{k=0}^{\infty} D_k(\zeta)/\upsilon^{2k}\right\}.$$

The term to the left of the first set of curly brackets can be written as

$$\frac{4}{x^2}\left(\frac{1-x^2}{4\zeta}\right)^{3/4}\zeta,$$

and to the left of the second set as

$$\frac{4}{x^2}\left(\frac{1-x^2}{4\zeta}\right)^{3/4}\left(\frac{\zeta}{1-x^2}\right)^{1/2}\frac{1}{\upsilon}.$$

If we now set

(7) $$E_k(\zeta) = \zeta A_k(\zeta) + \zeta^{1/2}\, t\, C_{k-1}(\zeta),$$

(8) $$F_k(\zeta) = \zeta B_k(\zeta) + \zeta^{1/2}\, t\, D_k(\zeta),$$

then the asymptotic expansion (1) follows. The explicit expressions (2) and (3) of $E_k(\zeta)$ and $F_k(\zeta)$, respectively, can be deduced from the explicit expressions for $A_k(\zeta)$, $B_k(\zeta)$, $C_k(\zeta)$ and $D_k(\zeta)$ given by Olver (see [**ABR**, pp. 368 and 369, Eqs. 9.3.40 and 9.3.46]; see also [**OLV1**]) using (7) and (8).

(ii) Asymptotic Expansion of $j''_{\upsilon,k}$:

The following is a brief, formal derivation of the asymptotic expansion of $j''_{\upsilon,k}$; the method used is similar in spirit to Olver's method of deriving the asymptotic expansions of j_υ and $j'_{\upsilon,k}$ (see and compare with [**OLV1**, pp. 343 - 345]).

From (1) it follows that for large υ the values of ζ corresponding to the zeors of $J''_\upsilon(\upsilon x)$ satisfy the asymptotic equation

$$(9) \qquad 0 \sim W(\zeta) \equiv Ai(\upsilon^{2/3}\zeta)\sum_{k=0}^{\infty} E_k(\zeta)/\upsilon^{2k} + \frac{Ai'(\upsilon^{2/3}\zeta)}{\upsilon^{4/3}}\sum_{k=0}^{\infty} F_k(\zeta)/\upsilon^{2k}.$$

Let us denote the corresponding values of ζ by $\zeta_{\upsilon,k}$, and set $\zeta_{\upsilon,k} = \alpha + \eta$, where $\alpha = \upsilon^{-2/3}a_k$, and η is some quantity whose asymptotic expansion we now determine.

Since $W(\zeta_{\upsilon,k}) = W(\alpha + \eta)$, from (9) we derive by expansion

$$(10) \qquad W(\alpha) + \eta W'(\alpha) + \frac{\eta^2}{2} W''(\alpha) + \cdots \sim 0,$$

Formal differentiation of the asymptotic series (9) yields by induction

$$W^{(2m)}(\zeta) = \upsilon^{2m}\, Ai(\upsilon^{2/3}\zeta)\sum_{r=0}^{\infty} E_r^{2m}(\zeta)/\upsilon^{2r} \;+\; \upsilon^{2m-4/3}\, Ai'(\upsilon^{2/3}\zeta)\sum_{r=0}^{\infty} F_r^{2m}(\zeta)/\upsilon^{2r}$$

$$W^{(2m+1)}(\zeta) = \upsilon^{2m}\, Ai(\upsilon^{2/3}\zeta)\sum_{r=0}^{\infty} E_r^{2m+1}(\zeta)/\upsilon^{2r} \;+\; \upsilon^{2m+2/3}\, Ai'(\upsilon^{2/3}\zeta)\sum_{r=0}^{\infty} F_r^{2m+1}(\zeta)/\upsilon^{2r},$$

where

$$E_k^0 = E_k, \qquad F_k^0 = F_k,$$

and

$$E_k^{2m} = \frac{d}{d\zeta} E_{k-1}^{2m-1} + \zeta F_k^{2m-1}, \qquad E_k^{2m+1} = \frac{d}{d\zeta} E_k^{2m} + \zeta F_k^{2m},$$

$$F_k^{2m} = E_{k-1}^{2m-1} + \zeta F_k^{2m-1}, \qquad F_k^{2m+1} = E_k^{2m} + \frac{d}{d\zeta} F_{k-1}^{2m}.$$

Hence,

$$W^{(2m)}(\alpha) = \upsilon^{2m-4/3}\,\mathrm{Ai}'(a_k)\sum_{r=0}^{\infty} F_r^{2m}(\alpha)/\upsilon^{2r},$$
$$W^{(2m+1)}(\alpha) = \upsilon^{2m+2/3}\,\mathrm{Ai}'(a_k)\sum_{r=0}^{\infty} F_r^{2m+1}(\alpha)/\upsilon^{2r},$$

and substitution of these values in (10) leads to the asymptotic equality

$$\sum_{k=0}^{\infty} F_k^0(\alpha)/\upsilon^{2k} + \eta\upsilon^2\sum_{k=0}^{\infty} F_k^1(\alpha)/\upsilon^{2k} + \frac{\eta^2\upsilon^2}{2!}\sum_{k=0}^{\infty} F_k^2(\alpha)/\upsilon^{2k} + \frac{\eta^3\upsilon^4}{3!}\sum_{k=0}^{\infty} F_k^3(\alpha)/\upsilon^{2k} + \cdots \sim 0.$$

Since $F_0^1(\zeta) = E_0(\zeta) = \zeta$, then $F_0^1(\alpha) = \alpha$. Hence, by reversion of series, we have

$$\eta \sim -\frac{f_0}{\upsilon^2 f_1} - \frac{f_2 f_0^2}{2\upsilon^4 f_1^3} - \frac{\frac{1}{2}f_2^2 - \frac{1}{6}\upsilon^2 f_1 f_3}{\upsilon^6 f_1^5} f_0^3 - \cdots,$$

where, for brevity, we let $f_\ell = \sum_{k=0}^{\infty} F_k^\ell(\alpha)/\upsilon^{2k}$. Thus,

$$\eta \sim \frac{\eta_1}{\upsilon^2} + \frac{\eta_2}{\upsilon^4} + \frac{\eta_3}{\upsilon^6} + \cdots,$$

where, since the leading coefficient of f_1 is α,

$$\eta_1 = -\alpha^{-1}F_0, \qquad \eta_2 = -\alpha^{-1}(F_1 + \eta_1 F_1^1 + \tfrac{1}{2}\eta_1^2 F_0^2 + \tfrac{1}{6}\eta_1^3 F_0^3), \; \ldots.$$

Expanding $f_\ell(\alpha)$ about the origin, and collecting like terms, we find that

$$\eta = \eta_{\upsilon,k} \sim -\frac{1}{a_k}\left(\frac{1}{2}\right)^{1/3}\upsilon^{-4/3} - \left\{\frac{3}{7} + \frac{1}{a_k^3}\right\}\left(\frac{1}{2}\right)^{2/3}\upsilon^{-2} - \left\{\frac{86a_k}{1575} + \frac{59}{210a_k^2} + \frac{1}{a_k^5}\right\}\upsilon^{-8/3}.$$

The $k^{\underline{th}}$ positive zero $j''_{\upsilon,k}$ of $J''_\upsilon(x)$, is given by

(11) $$j''_{\upsilon,k} = \upsilon x(\alpha + \eta) = \upsilon x(\alpha) + \upsilon\eta x'(\alpha) + \upsilon\frac{\eta^2 x''(\alpha)}{2!} + \cdots.$$

From (2.1.2), the expansion of $x(\zeta)$ is found to be (cf. **[OLV1**, p.336])

(12) $$x(\zeta) = 1 - 2^{-1/3}\zeta + \frac{3}{20}2^{1/3}\zeta^2 + \frac{1}{700}\zeta^3 - \frac{479}{252000}2^{2/3}\zeta^4 + \cdots.$$

Higher order differentiation of (12), and insertion in (11), leads to the result

$$(13) \qquad j''_{\upsilon,k} = \upsilon - 2^{-1/3} a_k \upsilon^{1/3} + \left(\frac{1}{2}\right)^{2/3} \left\{ \frac{3\,a_k^3 + 10}{10 a_k} \right\} \upsilon^{-1/3} + \frac{a_k^6 - 60 a_k^3 + 350}{700 a_k^3} \upsilon^{-1} + O(\upsilon^{-5/3}).$$

Specifically, we have

$$j''_{\upsilon,1} = \upsilon + 1.8557571\upsilon^{1/3} + 0.7637185\upsilon^{-1/3} - 0.1430921\upsilon^{-1} + \cdots,$$

$$j''_{\upsilon,2} = \upsilon + 3.2446076\upsilon^{1/3} + 3.0041417\upsilon^{-1/3} - 0.1906263\upsilon^{-1} + \cdots.$$

To verify that (13) is actually the asymptotic representation of $j''_{\upsilon,k}$, we have taken $\upsilon x = j''_{\upsilon,k}$ in (1) and used the corresponding value $\zeta = \zeta_{\upsilon,k} = \upsilon^{-2/3} a_k + \eta$ to show by expansion that every coefficient vanishes.

Another quantity of interest is the value $J_\upsilon(j''_{\upsilon,k})$, whose asymptotic representation is obtained by taking $\upsilon x = j''_{\upsilon,k}$ in (2.2.2) and, using the corresponding value $\zeta = \zeta_{\upsilon,k} = \upsilon^{-2/3} a_k + \eta$, showing by expansion that

$$J_\upsilon(j''_{\upsilon,k}) \sim -\frac{Ai'(a_k)}{\upsilon\, a_k} \left\{ 1 + \left(\frac{1}{2}\right)^{1/3} \left(\frac{3a_k}{5} + \frac{1}{a_k^2} \right) \upsilon^{-2/3} + 2^{1/3} \left(\frac{69 a_k^2}{700} + \frac{107}{140 a_k} + \frac{1}{a_k^4} \right) \upsilon^{-4/3} \right\}.$$

In particular, we have

$$J_\upsilon(j''_{\upsilon,1}) \sim 0.29991\upsilon^{-1}\{1 - 0.96827\upsilon^{-2/3} + 0.30924\upsilon^{-4/3} + \ldots\},$$

$$J_\upsilon(j''_{\upsilon,2}) \sim -0.19646\upsilon^{-1}\{1 - 1.89927\upsilon^{-2/3} + 1.84437\upsilon^{-4/3} + \ldots\}.$$

APPENDIX IV

The following is excerpted from **[HET3]**:

THEOREM 1. *In the interval* $[b - \rho, b + \rho]$, *suppose* $f(x) = g(x) + \varepsilon(x)$, *where* $f(x)$ *is continuous*, $g(x)$ *is differentiable*, $g(b) = 0$, $m = \min | g'(x) | > 0$, *and*

$$E = \max | \varepsilon(x) | < \min \{| g(b - \rho) |, | g(b + \rho) |\}.$$

Then there exists a zero c *of* $f(x)$ *in the interval such that* $| c - b | \leq E / m$.

COROLLARY 1. *In the interval* $[n\pi - \psi - \rho, n\pi - \psi + \rho]$ *where* $\rho < \pi/2$, *suppose* $f(x) = \sin(x + \psi) + \varepsilon(x)$, $f(x)$ *is continuous and* $E = \max | \varepsilon(x) | < \sin \rho$. *Then there exists a zero* c *of* $f(x)$ *in the interval such that* $| c - (n\pi - \psi) | \leq E / \cos \rho$.

REFERENCES

[ABR] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions*. Dover Publications, Inc., New York, 1972

[ERD] A. Erdélyi, ed., *et. al.*. *Tables of integral transforms*, vol. 1. McGraw-Hill Book Company, Inc., New York, 1954

[EVG] M. A. Evgrafov. *Analytic functions*. Nauka Pub., Moscow, 1965; English transl., Saunders, Philadelphia, 1966

[GAT] L. Gatteschi. On the zeros of certain functions with applications to Bessel functions. *Nederl. Akad. Wetensch. Proc. Ser. A* **55**(1952), pp. 224 - 229

[HET1] Herbert W. Hethcote. *Asymptotic Approximations with error bounds for zeros of Airy and cylendrical functions*. Ph. D. Thesis, University of Michigan, 1968

[HET2] _____. Bounds for zeros of some special functions. *Proc. Amer. Math. Soc.* **25**(1970), pp. 72 - 74

[HET3] _____. Error bounds for asymptotic approximations of zeros of transcendental functions. *SIAM J. Math. Anal.* **1**(1970), pp. 147 -152

[LOR] Lee Lorch and Peter Szego. On the points of inflection of Bessel functions I. To appear.

[MAR] O. I. Marichev. *Handbook of integral transforms of higher trancendental functions, theory and algorithmic tables*. Ellis Horwood Ltd., West Sussex, England, 1983

[OBE] F. Oberhettinger. *Tables of Mellin transforms*. Springer-Verlag, New York and Berlin, 1974

[OLV1] F. W. J. Olver. The asymptotic expansion of Bessel functions of large order. *Philos. Trans. Roy. Soc. London Ser A* **247**(1954), pp. 307 - 327

[OLV2] _____. *Tables for Bessel functions of moderate or large orders*. Math. Tables Nat. Phys. Lab., vol. 6, Her Majesty's Stationery Office, London, 1962

[OLV3] _____. Error bounds for first approximations in turning-point problems. *J. Soc. Indust. Appl. Math.* **11**(1963), pp. 748 - 772

[OLV4] _____. Error bounds for asymptotic expansions in turning-point problems. *Ibid.* **12**(1964), pp. 200 - 214

[OLV5] _____. *Asymptotics and special functions.* Academic press, New York, 1974

[TRI] F. G. Tricomi. Sugli zeri delle funzioni di cui si conosce una rappresentazione asintotica. *Ann. Mat. Pura Appl.* **26**(1947), pp. 163 - 184

[WAT] G. N. Watson. *A treatise on the theory of Bessel functions*, 2nd ed. Cambridge Univ. Press, London and New York, 1944

[WON1] R. Wong. Explicit error terms for asymptotic expansions of Mellin convolutions. *J. Math. Anal. Appl.* **72**(1979), pp. 740 - 756

[WON2] _____. Error bounds for asymptotic expansions of integrals. *SIAM Rev.* **22**(1980), pp. 401 - 435

[WON3] _____. Applications of some recent results in asymptotic expansions. *Proc. 12th Winnipeg Conf. on Numerical Methods of Computing, Congr. Numer.* **37**(1983), pp. 145 - 182

[WON4] _____ and T. Lang. On the points of inflection of Bessel functions II. To be submitted to *SIAM J. Math. Anal.*