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SPACES WITH BASES OF FINITE, POINT-FINITE, OR SUBINFINITE RANK

by

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ABSTRACT

In this work we consider the constraints imposed on the topological structure of a topological space X by various order-theoretic properties of the poset ($\underline{B}$,$\subset$), where $\underline{B}$ is an open base for X. Given a space X with an open base $\underline{B}$ such that every fixed antichain of ($\underline{B}$,$\subset$) is finite, we prove that X is hereditarily metacompact. If for every point x of X there is a positive integer n such that every antichain of $((\underline{B})_x,\subset)$ has at most n elements, then we prove X has a Noetherian base with the same property. If in addition X is a β-space in which points are G_δ's, we prove X is developable. We also characterize the regular ccc spaces X which have a base $\underline{B}$ such that the fixed antichains of ($\underline{B}$,$\subset$) have at most n elements, for some integer n.

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CHAPTER I

INTRODUCTION

All topological spaces are assumed to be T_1. We will refer to a base for the open subsets of a topological space as a "base". By a "local base" in a space X we mean a family $\underline{F}$ of open subsets of X such that there exists an $x \in X$ for which $\underline{F}$ is a neighborhood base at x.

We will represent a cardinal by the minimal ordinal of that cardinality. Greek letters are used to denote ordinals. The symbols "ω" and "ω_1" denote the first two infinite cardinals. The cardinality of a set X is denoted by $|X|$.

For a given base $\underline{B}$ of a space X we are interested in the partially ordered set $(\underline{B},\subset)$. We will need the following notation and terminology for partially ordered sets (posets).

Let $(P,\leq)$ be a partially ordered set. If p and q are two elements of P then p is comparable with q if $p \leq q$ or $q \leq p$. Two elements are incomparable if they are not comparable. Two elements p and q of P are compatible in P if there exists an $r \in P$ such that $p \leq r$ and $q \leq r$.

A subset A of P is an antichain if every two members of A are incomparable. A subset C of P is a <u>chain</u> if every two members of C are comparable. A poset is <u>directed</u> if every two elements are compatible.

A partially ordered set $(P,\leq)$ is <u>Noetherian</u> iff every chain of P has a maximal element. An equivalent definition is: A poset $(P,\leq)$ is Noetherian iff every chain of P is well-ordered by ">". The following lemma, in particular, characterizes Noetherian posets with only finite antichains as those posets in which every non-empty subset has at least one, and at most finitely many, maximal elements.

1.1 <u>Lemma</u> [LN] Let $(P,\leq)$ be a poset. The following are equivalent.

(i) P is Noetherian and every antichain has cardinality at most κ.

(ii) Given a subset Q of P, there is a subset Q' of Q such that every member of Q is less than or equal to some member of Q', and $|Q'| < \kappa$. □

The posets that interest us are the posets in which every antichain is finite or in which there is an integer n such that every antichain has at most n elements. The following two theorems are of prime importance for these posets.

1.2 <u>Theorem</u> [D] If $(P,\leq)$ is a poset and $k \in \omega$ is such that every antichain has at most k elements then P can be expressed as the union of at most k chains of P. □

1.3 Theorem [ET],[P] If P is a poset such that every antichain is finite, then P can be expressed as the union of finitely many directed subsets. □

1.4 Lemma If P is a poset whose antichains are finite then there is a chain of P with cardinality $|P|$.

Proof Erdös' theorem is useful here (see [J] for further details on Erdös' theorem). We will partition the two element subsets of P into two sets. Let set I contain the two element subsets with comparable elements. Let set II contain the two element antichains of P. Since P has no infinite antichains, there does not exist an infinite subset H of P, all of whose two element subsets are members of set II. Therefore if $\kappa = |P|$, Erdös' theorem asserts that there is a subset H of P of cardinality κ such that every two element subset of H is a member of set I. Hence H is a chain of P with cardinality κ. □

If X is a set and $\underline{B}$ is a family of subsets of X then we say $\underline{B}$ is an antichain, a chain, Noetherian, or directed if the partially ordered set $(\underline{B},\subset)$ satisfies the respective property.

Let $\underline{B}$ be a family of subsets of a set X. For $x \in X$, denote the family $\{B \in \underline{B}: x \in B\}$ by $(\underline{B})_x$. Define $St(x,\underline{B})$, the star of $\underline{B}$ at x, to be $\cup(\underline{B})_x$. If $K \subset X$, let $\underline{B}|_K$ denote the set $\{B \cap K: B \in \underline{B}\}$. $\underline{B}$ is <u>fixed</u> if $\cap\underline{B} \neq \phi$.

The symbol "$\subset$" will denote set inclusion, but will not necessarily denote proper set inclusion.

1.5 <u>Definition</u> Let κ be a cardinal. The family $\underline{B}$ of subsets of a set X has <u>rank $<\kappa$</u> if every fixed antichain of $\underline{B}$ has less than κ members. If $x \in \cup\underline{B}$, define $r_x(\underline{B})$ to be $\sup\{|\underline{A}|: \underline{A} \subset (\underline{B})_x$ is an antichain$\}$.

Let n be a positive integer. Let $\underline{B}$ be a family of subsets of a set X. We will consider the expression "<u>rank $\leq n$</u>" to be equivalent to the expression "rank $<n+1$". We say $\underline{B}$ has <u>rank n</u>, if $\underline{B}$ has rank $\leq n$ but not rank $<n$. We define $\underline{B}$ to have <u>finite rank</u> if there is a positive integer such that $\underline{B}$ has rank $\leq n$. $\underline{B}$ has <u>point-finite rank</u> if, for every $x \in \cup\underline{B}$, $r_x(\underline{B})$ is finite. $\underline{B}$ has <u>subinfinite rank</u> if $\underline{B}$ has rank $<\omega$.

The concept of rank was defined by Nagata [Na]. Subinfinite rank and point-finite rank were introduced in [GN] as generalizations of finite rank.

We will discuss the properties of spaces which have bases of finite, point-finite, or subinfinite rank. Some common examples of spaces with bases with these properties are given in [GN]. They include the Pixley-Roy hyperspace $F[\mathbb{R}]$ of the real line which has a base of point-finite rank but not a base of finite rank, and the Sorgenfrey line which has a base of subinfinite rank but not a base of point-finite rank [GN].

Now we will define some common topological properties.

Let X be a topological space and $\mathcal{U}$ a family of subsets of X. A family $\mathcal{V}$ of subsets is a <u>partial refinement</u> of $\mathcal{U}$ if every member of $\mathcal{V}$ is a subset of some member of $\mathcal{U}$. A family $\mathcal{V}$ is a <u>refinement</u> of $\mathcal{U}$ if $\mathcal{V}$ is a partial refinement of $\mathcal{U}$ and in addition $\cup\mathcal{V} = \cup\mathcal{U}$. The family $\mathcal{U}$ is <u>point-finite</u> if every point of X is contained in at most finitely many members of $\mathcal{U}$. A space X is <u>metacompact</u> if every open cover of X has a point-finite refinement. A family $\mathcal{U}$ is <u>irreducible</u> if every member of $\mathcal{U}$ contains a point not in any other member of $\mathcal{U}$. The intersection of a decreasing family of subcovers of a point-finite cover is a cover, so by Zorn's lemma, every point-finite cover contains an irreducible subcover.

The main result of Chapter II states that a space with a base of subinfinite rank is metacompact.

This generalizes the result: Every space with a base of point-finite rank is metacompact [GN].

Every regular countably compact metacompact space is compact and every separable metacompact space is Lindelöf. Therefore it becomes clear that a countably compact space with a base of subinfinite rank is compact [G] and a separable space with a base of subinfinite rank is Lindelöf [G].

In Chapter III we study spaces with bases of finite or point-finite rank. It is known [Ny3] that a space with a base of rank 1 has a Noetherian base of rank 1. It has been asked by Nyikos in [GN], [LN], and [Ny4] whether every space with a base of point-finite rank has a Noetherian base of point-finite rank. In Chapter III we give an affirmative answer. With this we can find simpler proofs for many theorems on spaces with bases of point-finite rank. In particular, we can immediately see that every space with a base of point-finite rank is metacompact.

Rank has relevance to metrization and dimension theory, as illustrated by the following theorem.

1.6 <u>Theorem</u> [Na],[Ar1] Let X be metrizable. Then X has a base of rank n iff dim X = n-1. □

There are several important metrization theorems for spaces with bases of finite or point-finite rank.

Arhangel'skii proved that a compact space with a base of rank 1 is metrizable [Ar2], and twelve years later this was extended as follows:

1.7 <u>Theorem</u> [GN] A compact space with a base of point-finite rank is metrizable. □

We now present three classes of spaces, each of which contains the compact spaces and the metrizable spaces.

1.8 <u>Definition</u> [Nm] A sequence $\{\underline{F}_i : i < \omega\}$ of locally finite closed covers of a space X is called a <u>spectral Σ-sequence</u> for X if for every sequence $\{x_i : i < \omega\} \subset X$ and $x \in X$ such that $x_i \in \cap(\underline{F}_i)_x$, then $\{x_i : i < \omega\}$ has a cluster point. A space X which has a spectral Σ-sequence is call a <u>Σ-space</u>.

1.9 <u>Definition</u> [Bo] A space X is a <u>wΔ-space</u> if there is a sequence $\{\underline{G}_i : i < \omega\}$ of open covers of X such that whenever $x \in X$ and $x_i \in St(x,\underline{G}_i)$ for each i, then $\{x_i : i < \omega\}$ has a cluster point.

1.10 <u>Definition</u> [Ho1] A <u>β-space</u> is a space in which there is associated to each point x and each positive integer n an open neighborhood $g(n,x)$ of x such that if $\{x_n: n < \omega\}$ is a sequence in X and $\cap\{g(n,x_n): n < \omega\} \neq \phi$, then $\{x_n: n < \omega\}$ has a cluster point. The function g from $\omega \times X$ to the topology of X is called a <u>β-function</u>.

Every $w\Delta$-space or Σ-space is a β-space [Ho2].

A well-known generalization of a metric space is a developable space.

1.11 <u>Definition</u> A space X is <u>developable</u> if there exists a sequence $\{\mathcal{G}_i: i < \omega\}$ of open covers of X such that for each point $x \in X$, $\{St(x,\mathcal{G}_i): i < \omega\}$ is a local base for x in X.

A regular developable space is also called a <u>Moore space</u>.

Although a first countable regular β-space with a base of point-finite rank may not be metrizable, in Chapter IV we show that it is at least developable.

The final result of Chapter IV characterizes the regular ccc space which have a base of finite rank.

CHAPTER II

THE METACOMPACTNESS OF SPACES WITH BASES OF SUBINFINITE RANK

Recall that a family $\underline{U}$ of subsets of a space X has subinfinite rank iff every fixed antichain of $\underline{U}$ is finite.

In this chapter we show that every space with a base of subinfinite rank is hereditarily metacompact. Some of the interesting properties that spaces with a base of subinfinite rank are known to possess can be accounted for by the fact that these spaces are hereditarily metacompact. For example, it has been shown that a countably compact space with a base of subinfinite rank is compact and a separable space with a base of subinfinite rank is hereditarily Lindelöf [G].

2.1 Definition [Au] If $\underline{U}$ is a family of subsets of X then $M \subset \cup\underline{U}$ is maximally distinguished with respect to $\underline{U}$ (max. dist. wrt. $\underline{U}$) if every member of $\underline{U}$ contains at most one element of M and M is maximal in the poset (ordered by inclusion) of subsets of $\cup\underline{U}$ which satisfy this property.

The following lemmata are easily established.

2.2 Lemma If $\underline{U}$ is a family of subsets of a set X then there exists a maximally distinguished set with respect to $\underline{U}$. □

2.3 Lemma Let $\underline{U}$ be a family of subsets of X with subinfinite rank. If M is max. dist. wrt. $\underline{U}$ then $\{St(x,\underline{U}): x \in M\}$ is point-finite. □

2.4 Theorem Let X have a base $\underline{B}$ of subinfinite rank. Then X is hereditarily metacompact.

Proof Let $\underline{W}$ be a family of open sets of X. It is sufficient to show that $\underline{W}$ has an open point-finite refinement.

Choose a family $\{\underline{B}_n: n < \omega\}$ of subsets of $\underline{B}$ and a family $\{M_n: n < \omega\}$ of subsets of X as follows:

(i) $\underline{B}_0 = \{B \in \underline{B}: B \subset W \text{ for some } W \in \underline{W}\}$

and for every $n < \omega$,

(ii) M_n is max. dist. wrt. $\underline{B}_n$

(iii) $\underline{B}_{n+1} = \{B \in \underline{B}_0: B \cap \cup\{M_j: j \leq n\} = \phi\}$

We will first show by induction that $\cup\{M_j: j \leq k\}$ is closed in $\cup\underline{W}$ for every integer k. Assume $\cup\{M_j: j < k\}$ is closed in $\cup\underline{W}$. $\underline{B}_0$ is a base for the points in $\cup\underline{W}$, so by (iii) $\cup\underline{B}_k = \cup\underline{W} \setminus \cup\{M_j: j < k\}$. Every element of $\underline{B}_k$ contains at most one element of M_k, so M_k is closed in $\cup\underline{B}_k$. Therefore $M_k \cup (\cup\underline{W} \setminus \cup\underline{B}_k) = \cup\{M_j: j < k+1\}$ is closed in $\cup\underline{W}$. The induction is complete.

For all $n < \omega$ and $x \in M_n$, it is possible by theorem 1.3 to find an integer k_x and decompose $(\underline{B}_n)_x$ into a family $\{\underline{C}_{x,i}: i < k_x\}$ of finitely many directed subsets. For every $n < \omega$ and $x \in M_n$ we will define a finite family $\{V_{x,i}: i < k_x\}$ of open sets. If $\cup\underline{C}_{x,i} \subset W$ for some $W \in \underline{W}$, then define $V_{x,i}$ to be $\cup\underline{C}_{x,i}$, otherwise choose $V_{x,i} \in \underline{C}_{x,i}$. Consider the set $\{V_{x,i}: x \in \cup\{M_n: n < \omega\},\ i < k_x\}$. We have constructed this to be a partial refinement of $\underline{W}$, in fact, we will be able to show it contains a point-finite refinement of $\underline{W}$.

Define $\{\underline{V}_n: n < \omega\}$ as follows:

(a) $\underline{V}_0 = \{V_{x,i}: x \in M_0,\ i < k_x\}$

(b) $\underline{V}_{n+1} = \{V_{x,i}: x \in M_{n+1},\ i < k_x \text{ and } V_{x,i} \not\subset \cup\{\cup\underline{V}_j: j \leq n\}\ \}$

We claim $\underline{V} = \cup\{\underline{V}_n: n < \omega\}$ is a point-finite refinement of $\underline{W}$. We have already noted that $\underline{V}$ is a partial refinement of $\underline{W}$, and now we will show $\underline{V}$ is a refinement of $\underline{W}$.

Let us prove $\underline{V}$ is at least a cover of M_n for every $n < \omega$. Suppose $x \in M_n$. If $V_{x,0} \in \underline{V}_n$ then $x \in \cup\underline{V}$ and if $V_{x,0} \notin \underline{V}_n$ then by (b) $x \in V_{x,0} \subset \cup\{\underline{V}_j: j < n\}$. Assume $\underline{V}$ is not a cover of $\cup\underline{W}$; we will arrive at a contradiction. Let $z \in \cup\underline{W} \setminus \cup\underline{V}$. We have noted that $z \notin \cup\{M_n: n < \omega\}$. Since $\cup\underline{W} \setminus \cup\underline{B}_n = \cup\{M_j: j < n\}$, we have $z \in \cup\underline{B}_n$ for all $n < \omega$.

By induction we will choose $\{U_n : n < \omega\}$ such that:

(1) $z \in U_n$ and $U_n \in (\underline{\mathcal{B}}_n)_x$ for some $x \in M_n$

(2) $U_n \not\subset U_j$ for all $j < n$.

Suppose $\{U_j : j < n\}$ satisfies (1) and (2). The set $\cup\underline{\mathcal{B}}_n$ contains z, so by the maximality of M_n, there exists $x \in M_n$ such that $z \in \cup(\underline{\mathcal{B}}_n)_x$. There is an $i < k_x$ such that $z \in \cup\underline{\mathcal{C}}_{x,i}$. Since $V_{x,i} \subset \cup\{\cup\underline{\mathcal{V}}_j : j < n\}$ or $V_{x,i} \in \underline{\mathcal{V}}_n$, we have $z \notin V_{x,i}$. Therefore $V_{x,i} \neq \cup\underline{\mathcal{C}}_{x,i}$ and by the definition of $V_{x,i}$, we can conclude that for all $W \in \underline{\mathcal{W}}$, $\cup\underline{\mathcal{C}}_{x,i} \not\subset W$. In particular, for all $j < n$, $\cup\underline{\mathcal{C}}_{x,i} \not\subset U_j$. For each $j < n$, we can choose $U_j' \in \underline{\mathcal{C}}_{x,i}$ so that $U_j' \not\subset U_j$. Since $\underline{\mathcal{C}}_{x,i}$ is directed we can choose $U_n \in \underline{\mathcal{C}}_{x,i}$ which contains $\{z\}$ and each U_j', $j < n$. Then $\{U_j : j \leq n\}$ satisfies (1) and (2).

Let $\{U_n : n < \omega\}$ satisfy (1) and (2). For all $n < \omega$ we have $U_n \cap M_n \neq \phi$. If $n < m$, then by condition (iii) of the definition of $\mathcal{B}_m$, $U_m \cap M_n = \phi$, so $U_n \not\subset U_m$. Condition (2) above states $U_m \not\subset U_n$, therefore the elements of $\{U_n : n < \omega\} \subset (\underline{\mathcal{B}})_z$ are pairwise incomparable, which is a contradiction. Therefore $\underline{\mathcal{V}}$ is a refinement of $\underline{\mathcal{W}}$.

Next we will show the countable family $\{\cup\underline{\mathcal{V}}_n : n < \omega\}$ is point-finite. Let $z \in \cup\underline{\mathcal{V}}$ and define $I = \{n : z \in \cup\underline{\mathcal{V}}_n\}$. For each $n \in I$ we will choose $U_n \in (\underline{\mathcal{B}}_n)_z$ such that $U_n \subset \cup\underline{\mathcal{V}}_n$ but for all $i < n$, $U_n \not\subset \cup\underline{\mathcal{V}}_i$. For each $n \in I$ choose $V_n \in \underline{\mathcal{V}}_n$ such that $z \in V_n$. If $V_n \in \underline{\mathcal{B}}_n$ then define U_n to be V_n. Otherwise

$V_n = \cup\underline{C}$ for some directed set $\underline{C}$ in $(\underline{B}_n)_x$, where $x \in M_n$. By the definition of $\underline{V}_n$, $\cup\underline{C} \not\subset \cup\underline{V}_i$ for $i < n$. We can choose $U \in \underline{C}$ such that $z \in U$ and for each $i \in n \cap I$, we can choose $U_i' \in \underline{C}$ such that $U_i' \not\subset \cup\underline{V}_i$. Since $\underline{C}$ is directed we can choose $U_n \in \underline{C}$ which contains U and U_i', $i \in n \cap I$.

Therefore if n and m are members of I and $n < m$, then $U_n \cap M_n \neq \phi$, while $U_m \cap M_n = \phi$, and $U_m \not\subset \cup\underline{V}_n$ while $U_n \subset \cup\underline{V}_n$. Therefore $\{U_n: n \in I\} \subset (\underline{B})_z$ has pairwise incomparable elements, and since $\underline{B}$ has subinfinite rank, I is finite. Therefore $|\{n: z \in \cup\underline{V}_n\}| < \omega$; in other words, $\{\cup\underline{V}_n: n < \omega\}$ is point-finite.

Finally we can prove $\underline{V} = \cup\{\underline{V}_n: n < \omega\}$ is point-finite. Since $\{\cup\underline{V}_n: n < \omega\}$ is point-finite it will suffice to show that each $\underline{V}_n$ is point-finite. Each member of the family $\underline{V}_n$ contains an element of M_n, so $\underline{V}_n = \cup\{(\underline{V}_n)_x: x \in M_n\}$. The family $\{St(x,\underline{B}_n): x \in M_n\}$ is point-finite by lemma 2.3 and since for $x \in M_n$, $St(x,\underline{V}_n) \subset St(x,\underline{B}_n)$, the family $\{St(x,\underline{V}_n): x \in M_n\}$ is point-finite. For each $x \in M_n$ the set $(\underline{V}_n)_x \subset \{V_{x,i}: i < k_x\}$ is finite, therefore $\cup\{(\underline{V}_n)_x: x \in M_n\}$ is point-finite. $\square$

G. Grabner characterized regular metacompact spaces as those regular spaces in which every open cover has an ω-Noetherian refinement (a family $\underline{U}$ is <u>ω-Noetherian</u> if every well-ordered strictly increasing chain in $\mathcal{U}$ is at most countable) [FG]. It is not true that a space in which every open cover has refinement of subinfinite rank is metacompact. The space ω_1 with the order topology is not metacompact, though every open cover of ω_1 has a refinement of rank 1.

We will now present a characterization of metacompact developable spaces.

2.5 Definition [Ar3] A collection of sets will be called perfectly decreasing iff it contains a proper subset of each of its elements.

2.6 Definition [V] A family $\mathcal{B}$ of subsets of X has countable order if whenever $\mathcal{C} \subset \mathcal{B}$ is perfectly decreasing and $x \in \cap\mathcal{C}$ then $\mathcal{C}$ is a local base for x in X.

2.7 Definition [WW] A space X is θ-refinable if for every open cover of X there is a countable family $\{\mathcal{V}_n : n < \omega\}$ of open refinements such that for every point $x \in X$, $(\mathcal{V}_n)_x$ is finite for some $n < \omega$.

2.8 Definition [Na2] A base $\mathcal{B}$ for a space X is uniform, if for every point x of X and every neighborhood U of x, there are at most finitely many $V \in (\mathcal{B})_x$ which are not a subset of U [Na2].

The following proposition examines the properties of a uniform base.

2.9 Proposition A base $\underline{B}$ is uniform if and only if $\underline{B}$ is a Noetherian base of subinfinite rank and countable order.

Proof It is clear that a uniform base is a Noetherian base of subinfinite rank and countable order.

Assume there exists a space X with a Noetherian base $\underline{B}$ of subinfinite rank and countable order which in not uniform. Choose $x \in X$ and a neighborhood U of x such that $\{V \in (\underline{B})_x : V \not\subset U\}$ is infinite. By lemma 1.4, this set contains an infinite chain. Every infinite Noetherian chain contains a perfectly decreasing subchain, and any subset of $\{V \in (\underline{B})_x : V \not\subset U\}$ is clearly not a local base at x. Therefore B does not have countable order; a contradiction.
□

The corollary below strengthens theorem 2.10 of [LN].

2.10 Corollary A space is metacompact and developable iff it has a base of countable order and a base of subinfinite rank.

Proof A space is developable iff it is θ-refinable and has a base of countable order [WW]. Clearly every metacompact space is θ-refinable. By theorem 2.4 every space with a base of subinfinite rank is metacompact. Therefore a space with a base of countable order and a base of subinfinite rank is metacompact and developable.

It can be shown [A1] that a space is metacompact and developable iff it has a uniform base. Proposition 2.9 shows that a uniform base has subinfinite rank and countable order. □

We will apply the following theorem in Chapter IV.

2.11 <u>Theorem</u> Let X have a base of subinfinite rank. If every uncountable subset of X has a countable subset with an accumulation point, then $X \setminus \{p\}$ is Lindelöf for every point p of X.

<u>Proof</u> Assume there is a $p \in X$ such that $X \setminus \{p\}$ is not Lindelöf. Let $\underline{U}$ be a collection of open subset of $X \setminus \{p\}$ which covers $X \setminus \{p\}$ but which does not have a countable subcover.

By theorem 2.4, X is hereditarily metacompact, so we can choose a point-finite refinement $\underline{V}$ of $\underline{U}$. There is an irreducible subfamily of $\underline{V}$ which covers $X \setminus \{p\}$, so we will assume $\underline{V}$ is irreducible. Let $\{V_\alpha : \alpha < \omega_1\}$ be an uncountable subfamily of $\underline{V}$ and for each $\alpha < \omega_1$ choose a point x_α in V_α which is not in any other member of $\underline{V}$. Then $\{x_\alpha : \alpha < \omega_1\}$ is closed and discrete in $X \setminus \{p\}$ and p is an accumulation point in X of any uncountable subset of $\{x_\alpha : \alpha < \omega_1\}$.

Since every neighborhood of p contains all but countably many points of $\{x_\alpha : \alpha < \omega_1\}$, countably many neighborhoods of p have uncountably many points in common. Therefore p does not have a countable local base. Let $\underline{B}$ be a local base of subinfinite rank at p. We can inductively choose $\{B_\alpha : \alpha < \omega_1\} \subset (\underline{B})_p$ to satisfy (i) $x_\beta \notin B_\beta$, and (ii) $B_\beta \not\supset B_\alpha$ if $\alpha < \beta < \omega_1$.

Since $\{B_\alpha : \alpha < \omega_1\}$ has no infinite antichains, by lemma 1.4, there is an uncountable chain $\underline{C} = \{B_\alpha : \alpha \in A\}$. By condition (ii), $B_\beta \not\supset B_\alpha$ for any $\alpha < \beta$, $\underline{C}$ is a well-ordered decreasing chain. There is a countable subset of $\{x_\alpha : \alpha \in A\}$ which has an accumulation point in X. For every countable subset of $\{x_\alpha : \alpha \in A\}$ there is a member of $\underline{C}$ which does not intersect it, therefore $\{x_\alpha : \alpha \in A\}$ has an accumulation point in $X \setminus \{p\}$. But $\{x_\alpha : \alpha \in A\}$ is closed and discrete in $X \setminus \{p\}$. This contradiction proves that $X \setminus \{p\}$ is Lindelöf. $\square$

CHAPTER III

SPACES WITH BASES OF POINT-FINITE RANK

In this chapter we prove that a space with a base of point-finite rank has a Noetherian base of point-finite rank.

Recall that a family of subsets $\underline{B}$ of a space X has rank $\leq n$ if every fixed antichain of $\underline{B}$ has at most n members. If every antichain of $(\underline{B})_x$ has at most n members, we denote this by writing $r_x(\underline{B}) \leq n$. It will be useful to define a property which insists that only some of the fixed antichains of a family $\underline{B}$ have at most n members.

3.1 <u>Definition</u> Let K be a subset of a space X. A family $\underline{B}$ of subsets of X has <u>rank $\leq n$ at K</u> if $r_x(\underline{B}) \leq n$ for every point x of K. If $(\underline{B})_x$ is a local base in X for every point x of K, and $\underline{B}$ has rank $\leq n$ at K, we will say that $\underline{B}$ is a <u>base of rank $\leq n$ at K</u>.

3.2 <u>Definition</u> If $\underline{B}$ is a family of subsets of a set X, then $\underline{B}$ is <u>closed under union of chains</u> if $\cup\underline{C} \in \underline{B}$ for every chain $\underline{C} \subset \underline{B}$. The <u>closure of B under union of chains</u> is the minimal subfamily of the power set of X which contains $\underline{B}$ and is closed under union of chains.

Given an open base of point-finite rank, we will routinely take the closure under union of chains. The next lemma assures us that we will not increase the rank at any point.

3.3 Lemma [Ar2] Let n be a positive integer and let $\underline{B}$ be a family of subsets of a set X. If $x \in X$ and $r_x(\underline{B}) = n$ then the closure of $\underline{B}$ under union of chains also has rank n at x.
□

It follows from corollary 4.5 of [GN] or theorem 2.4 that a space with a base of point-finite rank is metacompact. We require something more for our purposes. Not only do we want a point-finite refinement of any open cover of a space with a base of point-finite rank, but we want the refinement to consist of members from the base. The next lemma says this can be done if the base is closed under union of chains. The lemma differs in only a minor way from theorem 4.4 of [GN].

3.4 Lemma Let $K \subset X$ and let $\underline{B}$ be a base of rank $\leq n$ at K in X. If $\underline{B}$ is closed under union of chains and $\underline{W}$ is an open cover of K in X, then there is a partial refinement $\underline{U} \subset \underline{B}$ of $\underline{W}$ such that $\underline{U}$ covers K and $\underline{U}|_K$ is irreducible.

Proof As every family of finite rank also has subinfinite rank, we know by theorem 2.4 that K is metacompact. Let $\underline{V}$ be a partial refinement of $\underline{W}$ such that $\underline{V}|_K$ is a point-finite cover of K. Define $\underline{B}'$ to be $\{B \in \underline{B}: B \cap K \neq \phi, B \subset V$ for some $V \in \underline{V}\}$. Let $\underline{C}$ be a chain in $\underline{B}'$; we will prove $\cup\underline{C} \in \underline{B}'$. Without loss of generality we can assume all members of $\underline{C}$ have a point x of K in common. Therefore only finitely many members of $\underline{V}$ have as a subset a member of $\underline{C}$, so there exists $V \in \underline{V}$ and a cofinal subchain of $\underline{C}$, all of whose elements are subsets of V. Hence $\cup\underline{C}$ is a subset of some member of $\underline{V}$, and since $\cup\underline{C} \in \underline{B}$, we have by the definition of $\underline{B}'$ that $\cup\underline{C} \in \underline{B}'$.

The union of a maximal chain of $\underline{B}'$ is a maximal member of $\underline{B}'$, so every member of $\underline{B}'$ is a subset of a maximal member of $\underline{B}'$.

Let $\underline{U}$ be the set of maximal members of $\underline{B}'$. Then $\underline{U}$ is a partial refinement of $\underline{W}$ which covers K. Since $\underline{U}$ is an antichain of $\underline{B}'$ and $\underline{B}'$ has rank $\leq n$ at K, every point of K is in at most n members of $\underline{U}$. Every point-finite cover has an irreducible subcover, so a subcover of $\underline{U}$ satisfies the lemma. □

3.5 <u>Definition</u> If $\underline{B}$ is a local base for $x \in X$, and U is a neighborhood of x, then define $r_x(\underline{B},U)$ to be the supremum of the set of cardinalities of the antichains in $\{B \in \underline{B}: x \in B \subset U\}$.

If n is an integer, we will call an antichain with exactly n distinct members, an <u>n-antichain</u>.

3.6 <u>Lemma</u> Let $\underline{B}$ be a base of rank $\le n$ at $x \in X$. If $\underline{U}$ is a finite set of neighborhoods of x and $r_x(\underline{B}, U) = n$ for every $U \in \underline{U}$, then $r_x(\underline{B}, \cap\underline{U}) = n$.

<u>Proof</u> For every $U \in \underline{U}$, we can choose an n-antichain $\underline{A}_U \subset (\underline{B})_x$ whose members are subsets of U. There are no n+1-antichains in $\cup\{\underline{A}_U: U \in \underline{U}\}$ since $r_x(\underline{B}) \le n$, so by theorem 1.2 we write $\cup\{\underline{A}_U: U \in \underline{U}\}$ as the union of at most n chains. Without loss of generality we can assume these chains are pairwise disjoint. Since $\cup\{\underline{A}_U: U \in \underline{U}\}$ contains a n-antichain, our partition consists of exactly n chains. Define $\underline{A}$ to be the set of minimal members of each of the n chains we have chosen. Therefore $\underline{A}$ has n elements and we will show it is an antichain whose members are subsets of $\cap\underline{U}$.

For every $U \in \underline{U}$, $\underline{A}_U$ has n incomparable sets so each of the chains we have chosen contains exactly one member of $\underline{A}_U$. Therefore every member of $\underline{A}$ is a subset of some member of $\underline{A}_U$. Assume A and B are two distinct members of $\underline{A}$. If $A \in \underline{A}_U$, we can choose $B' \in \underline{A}_U$ such that $B \subset B'$. If $B' = A$ then B is a proper subset of A, hence $A \not\subset B$. If A is incomparable to B', then $A \not\subset B$. Therefore $A \not\subset B$ and similarly we can show $B \not\subset A$. Thus $\underline{A}$ is an n-antichain.

If $A \in \underline{A}$ then A is a subset of some member of $\underline{A}_U$ for each U $\in \underline{U}$, and every member of $\underline{A}_U$ is a subset of U. Therefore every member of $\underline{A}$ is a subset of $\cap\underline{U}$. □

3.7 Definition Let $\underline{B}$ be a family of subsets of X. We will say a subfamily $\underline{U} \subset \underline{B}$ is n-wide with respect to $\underline{B}$ iff for every two members V and U of $\underline{U}$, if V is a proper subset of U then there is an n-antichain $\underline{A} \subset \underline{B}$ between V and U, that is, $V \subset A \subset U$ for every $A \in \underline{A}$. If it is clear from context to which family of sets we refer, we will simply say $\underline{U}$ is n-wide.

3.8 Lemma Let $\underline{B}$ be a base for K in X, and suppose $\underline{U} \subset \underline{B}$ is n-wide. If $x \in K$ and $r_x(\underline{B}) \leq n$ then either $\cap(\underline{U})_x$ is a neighborhood of x or $(\underline{U})_x$ is a local base in X for x.

Proof Suppose $x \in K$ and $r_x(\underline{B}) \leq n$. Assume $\cap(\underline{U})_x$ is not a neighborhood of x. Let $B \in \underline{B}$ be a neighborhood of x. We will find a member of $(\underline{U})_x$ which is a subset of B.

Let $\underline{A}$ be a maximal antichain of $\{W \in (\underline{U})_x : B \not\subset W\}$. This set is not empty since $\cap(\underline{U})_x$ does not have the neighborhood B as a subset. Also $\underline{A}$ has at most n members, since $r_x(\underline{B}) \leq n$. Therefore $\cap\underline{A}$ is a neighborhood of x and since $\cap(\underline{U})_x$ is not a neighborhood of x, we can choose $U_0 \in (\underline{U})_x$ such that

$B \cap \cap \underline{A} \not\subset U_0$.

Since $B \not\subset U_0 \in (\underline{U})_x$ and $\underline{A}$ is a maximal antichain of the members of $(\underline{U})_x$ for which B is not a subset, we can choose a member U_1 of $\underline{A}$ which is comparable to U_0. We can rule out the possiblity that $U_1 \subset U_0$, since $\cap\underline{A} \not\subset U_0$; therefore $U_0 \subset U_1$. Since $\underline{U}$ is an n-wide family, we can choose an n-antichain $\underline{A}' \subset \underline{B}$ between U_0 and U_1. The point x is in the set $B \cap \cap\underline{A}'$ and $r_x(\underline{B}) \leq n$, so we can choose $A \in \underline{A}'$ such that A and B are comparable. B cannot be a subset of A since B is not a subset of U_1. Hence A, and therefore $U_0 \in (\underline{U})_x$, is a subset of B. This proves $(\underline{U})_x$ is a local base at x. □

Suppose $\underline{B}$ is a base of rank $\leq$n for X. If $\underline{U} \subset \underline{B}$ is n-wide with respect to $\underline{B}$, it is clear from the definition of wideness that any subfamily of $\underline{U}$ is n-wide with respect to $\underline{B}$. Therefore, by the previous lemma, if $\underline{U}' \subset \underline{U}$ then either (1) $\cap\underline{U}'$ is open or (2) $\cap(\underline{U}') = \{x\}$ and $\underline{U}'$ is a local base for some $x \in X$. Since every subfamily of a base of rank 1 is 1-wide, we have a generalization of lemma 1.4 in [Ny3], which is stated below:

Let $\underline{B}$ be a rank 1 base for a space X, and let $B' \subset B$. Then either

(1) $\cap\underline{B}'$ is open

or (2) $\cap\underline{B}'$ is a singleton $\{x\}$ and $\underline{B}'$ is a local base for x in X.

The next result is the key description of the structure of a base of finite rank. For a point x and a neighborhood U of x, recall that $r_x(\underline{B},U)$ is defined to be the rank of $\{B \in (\underline{B})_x : B \subset U\}$ at x.

3.9 Lemma Let n be a positive integer. Let $K \subset X$ and let $\underline{B}$ be a base of rank $\leq n$ at K in X. If $\underline{B}$ is closed under union of chains then there exists a Noetherian n-wide family $\underline{U} \subset \underline{B}$ such that for every point $x \in K$ either

(i) $(\underline{U})_x$ is a local base

or (ii) $r_x(\underline{B},U) < n$ for some $U \in (\underline{U})_x$.

Proof Let $\underline{B}$ be a base for $K \subset X$ such that $\underline{B}$ is closed under union of chains and for all $x \in K$, $r_x(\underline{B}) \leq n$. We will assume every member of $\underline{B}$ intersects K. For convenience we will assume K contains no isolated points of X. This will not result in a loss of generality since by simply adding open singletons to the family $\underline{U}$ which we will construct for a set K without isolated points, we obtain the desired family for any subset of X.

We wish to inductively choose antichains $\underline{U}_\alpha \subset \underline{B}$, for every ordinal α, that satisfy conditions (a)-(e) below.

(a) $\underline{U}_0$ is a cover of K,

(b) $\underline{U}_\alpha$ is an antichain and a subfamily of $\underline{B}$,

(c) $\underline{U}_{\alpha+1}$ is a partial refinement of $\underline{U}_\alpha$ and a cover of $\{x \in K \cap (\cup\underline{U}_\alpha) : r_x(\underline{B}, \cap(\underline{U}_\alpha)_x) = n\}$,

(d) for every $V \in \underline{U}_{\alpha+1}$, $U \in \underline{U}_\alpha$, if $V \cap U \cap K \neq \phi$ then V is a proper subset of U and there is an n-antichain of $\underline{B}$ between V and U,

(e) if α is a limit ordinal, then $\underline{U}_\alpha$ is a partial refinement of $\underline{W}_\alpha = \{\mathrm{int}_X \cap (\cup\{\underline{U}_\delta : \delta < \alpha\})_x : x \in \cup\underline{U}_\delta$ for all $\delta < \alpha\}$, and $\underline{U}_\alpha$ is a cover of $(\cup\underline{W}_\alpha) \cap K$.

By lemma 3.4, we can choose a family $\underline{U}_0 \subset \underline{B}$ such that $\underline{U}_0|_K$ is an irreducible cover of K.

Assume α is a limit ordinal and we have defined $\{\underline{U}_\delta : \delta < \alpha\}$. By lemma 3.4 we can find a subfamily $\underline{U}_\alpha$ of $\underline{B}$ which is a partial refinement of the collection $\underline{W}_\alpha$ defined in (e) above which, when restricted to $\cup\underline{W}_\alpha \cap K$, is an irreducible cover. $\underline{U}_\alpha$ satisfies (b) and (e).

Assume we have defined $\underline{U}_\alpha$. We will choose $\underline{U}_{\alpha+1}$ so that (b)-(d) are satisfied.

If $x \in K \cap \cup\underline{U}_\alpha$ and $r_x(\underline{B}, \cap(\underline{U}_\alpha)_x) = n$, then choose an n-antichain $\underline{A}_x \subset \{B \in \underline{B}: x \in B \subset \cap(\underline{U}_\alpha)_x\}$. If n = 1 then we explicitly demand that the member of $\underline{A}_x$ is a proper subset of $\cap(\underline{U}_\alpha)_x$. We can do this as x is assumed not to be isolated in X. Define $\underline{T}_\alpha$ to be the collection of all sets of the form $\cap\underline{A}_x$. Clearly $\underline{T}_\alpha$ is a cover of $\{x \in K \cap (\cup\underline{U}_\alpha) : r_x(\underline{B},\cap(\underline{U}_\alpha)_x) = n\}$ and a partial refinement of $\underline{U}_\alpha$.

By lemma 3.4 we can choose $\underline{U}_{\alpha+1} \subset \underline{B}$ which is a partial refinement of $\underline{T}_\alpha$ and an irreducible cover of $(\cup\underline{T}_\alpha) \cap K$. Therefore $\underline{U}_{\alpha+1}$ satisfies (b) and (c)

Let us show (d) is satisfied. Let $V \in \underline{U}_{\alpha+1}$. V is a subset of some member of $\underline{T}_\alpha$, which is the intersection of an n-antichain $\underline{A} \subset \underline{B}$ whose members are subsets of $\cap(\underline{U}_\alpha)_x$ for some $x \in K \cap \cup\underline{U}_\alpha$. For every member U of $\underline{U}_\alpha$ such that $U \cap \cap\underline{A} \cap K \neq \phi$ we claim $\cup\underline{A} \subset U$. This is clearly true if $U \in (\underline{U}_\alpha)_x$, and in fact we will prove $U \in (\underline{U}_\alpha)_x$.

Since $\cap\underline{A} \cap U \cap K \neq \phi$ and $r_z(\underline{B}) \leq n$ for all $z \in K$, we can choose $A \in \underline{A}$ such that A and U are comparable. If $A \subset U$ then a fortiori, $x \in U$ and we are done. But if we assume $x \notin U$ and $U \subset A$ then U is a proper subset of A and since A is a subset of every member of $(\underline{U}_\alpha)_x$, U is then a proper subset of some member of $(\underline{U}_\alpha)_x$. This is not possible since $\underline{U}_\alpha$ is an antichain.

Therefore every member of $\underline{A}$ is a subset of U for any $U \in \underline{U}_\alpha$ such that $\cap\underline{A} \cap U \cap K \neq \phi$. Since $V \subset \cap\underline{A}$, $\underline{A}$ is an antichain between V and any member of U of $\underline{U}_\alpha$ which intersects $V \cap K$. Therefore $\underline{U}_{\alpha+1}$ satisfies (d).

We have shown $\underline{U}_{\alpha+1}$ satisfies conditions (b)-(d) and the induction is complete.

Suppose $\underline{U}_\alpha$ has been defined for every ordinal α and conditions (a)-(e) are satisfied. We will show there exists an ordinal η such that $\underline{U}_\alpha = \phi$ for all ordinals $\alpha \geq \eta$. Suppose $V \in \underline{U}_\beta$, $U \in \underline{U}_\alpha$, $\alpha < \beta$, and $V \cap U \cap K \neq \phi$. It follows from (c) and (e) that $\underline{U}_\beta$ is a partial refinement of $\underline{U}_{\alpha+1}$, so we can choose $V' \in \underline{U}_{\alpha+1}$ such that $V \subset V'$. Therefore $V' \cap U \cap K \neq \phi$ and by (d) we can infer that V is a proper subset of U and there is an n-antichain between V and U. Therefore $\underline{U}_\alpha \cap \underline{U}_\beta = \phi$, and for any ordinal η, $\{\underline{U}_\alpha\colon \alpha \leq \eta\}$ is n-wide in $\underline{B}$. Since the families $\underline{U}_\alpha$ are pairwise disjoint subsets of $\underline{B}$, we can choose an ordinal η such that $\underline{U}_\eta = \phi$. We will show the n-wide family $\underline{U} = \cup\{\underline{U}_\alpha\colon \alpha \leq \eta\}$ is Noetherian and satisfies (i) and (ii) of the lemma.

Assume $\underline{C} \subset \underline{U}$ is a chain. Define δ to be $\min\{\alpha < \eta\colon \underline{C} \cap \underline{U}_\alpha \neq \phi\}$. If $\alpha < \beta \leq \eta$ then no member of U_α is a subset of any member of U_β. Therefore $\underline{C} \cap \underline{U}_\delta$ is a singleton set whose member is the maximal member of $\underline{C}$. Therefore $\underline{U}$ is Noetherian.

Let $x \in K$. We will prove either $(\underline{U})_x$ is a local base for x in X, or there exists $U \in (\underline{U})_x$ such that $r_x(\underline{B},U) < n$. Let α be the minimal ordinal such that $x \notin \cup\underline{U}_\alpha$. Assume α is a limit ordinal. We will show $(\underline{U})_x$ is a local base for x. Since $x \notin \cup\underline{U}_\alpha$, by (e) we have $x \notin \mathrm{int}_X \cap(\cup\{\underline{U}_\delta : \delta < \alpha\})_x$. Therefore, since $\underline{U}$ is n-wide, lemma 3.8 states that $(\cup\{\underline{U}_\delta : \delta < \alpha\})_x$ is a local base at x.

Now assume α is equal to $\delta+1$ for some ordinal δ. By (b) $(\underline{U}_\delta)_x$ is finite and by (c) $r_x(\underline{B}, \cap(\underline{U}_\delta)_x) < n$, since $x \notin \cup\underline{U}_{\delta+1}$. By lemma 3.6 we can conclude that there exists $U \in (\underline{U}_\delta)_x$ such that $r_x(\underline{B}, U) < n$. □

3.10 Lemma Let n be a positive integer. Let $K \subset X$ and let $\underline{B}$ be a base of rank $\leq n$ at K. If $\underline{B}$ is closed under union of chains then there exists a subset of $\underline{B}$ which is a Noetherian base for K in X.

Proof Let n be a positive integer. Assume for every subset L of X and for every base of rank $<n$ at L that is closed under union of chains, we can find a Noetherian subset which is a base for L in X.

Let $K \subset X$ have a base $\underline{B}$ of rank $\leq n$ that is closed under union of chains. Using the induction assumption, we will exhibit a subcollection of $\underline{B}$ that is a Noetherian base for K in X.

We will insist that every member of $\underline{B}$ intersects K. By lemma 3.9 we can choose a Noetherian n-wide family $\underline{U} \subset \underline{B}$ such that for all $x \in K$ either (i) $(\underline{U})_x$ is a local base at x or (ii) $r_x(\underline{B},U) < n$ for some $U \in \underline{U}$ which contains x.

For each $U \in \underline{U}$, define F_U to be $\{x \in U \cap K: r_x(\underline{B},U) < n\}$, and define B'_U to be $\{B \in \underline{B}: B \cap F_U \neq \phi, B \subset U\}$. The family $\underline{B}'_U$ is a base of rank $<n$ at F_U and since $\underline{B}$ is closed under union of chains, so is $\underline{B}'_U$. By our induction assumption, we can choose a Noetherian subfamily $\underline{B}_U$ of $\underline{B}'_U$ which is a base for F_U in X.

Let $\underline{C}$ be a chain in $\cup\{\underline{B}_U: U \in \underline{U}\}$. We will show that $\underline{C}$ has a maximum member.

Define $\underline{D}$ to be $\{U \in \underline{U}: \underline{C} \cap \underline{B}_U \neq \phi\}$. Any set of n+1 members of $\underline{D}$ have in common a subset which is a member of $\underline{C}$, and therefore have a point of K in the intersection. Since $\underline{B}$ has rank $\leq n$ at K, there are at most n maximal members in the set $\underline{D}$. For each maximal member U of $\underline{D}$, let C_U be the maximum member of $\underline{C} \cap \underline{B}_U$. Suppose C is a member of $\underline{C}$. We will prove $\underline{C}$ has a maximum member by proving C is a subset of one of the members from the finite set $\{C_U: U$ is maximal in $\underline{D}\}$.

Choose $V \in \underline{D}$ such that $C \in \underline{B}_V$. If V is maximal in $\underline{D}$, then $C \subset C_V$, so suppose V is not maximal. There is a maximal member U of $\underline{D}$ which has V as a subset, since $\underline{U}$, and therefore $\underline{D}$, is Noetherian. Since $\underline{U}$ is n-wide there is an n-antichain $\underline{A} \subset \underline{B}$ between V and U. Therefore $r_z(\underline{B},U) = n$ for every $z \in V \cap K$, and this implies that $V \cap F_U = \phi$. Since $C \subset V$ and $C_U \cap F_U \neq \phi$, we cannot have $C_U \subset C$. But C_U and C are members of a chain, so $C \subset C_U$. Hence $\cup\{\underline{B}_U: U \in \underline{U}\}$ is Noetherian.

The union of two Noetherian families is Noetherian, so $\underline{U} \cup \cup\{\underline{B}_U: U \in \underline{U}\}$ is Noetherian. If $x \in K$ and $\underline{U}$ is not a local base for x then by (ii) of lemma 3.9, there exists some $U \in \underline{U}$, such that $r_x(\underline{B},U) < n$. Then $\underline{B}_U$ is a local base for x. Hence $\underline{U} \cup \cup\{\underline{B}_U: U \in \underline{U}\}$ is a Noetherian base for K in X. □

We are now able to answer a question of P. Nyikos appearing in [GN], [LN] and [Ny4].

3.11 <u>Theorem</u> A T_1 space with a base of (point-)finite rank has a Noetherian base of (point-)finite rank.

<u>Proof</u> Let X have a base $\underline{B}$ of finite or point-finite rank. By lemma 3.3 we can assume, without loss of generality, that $\underline{B}$ is closed under union of chains.

If $\underline{B}$ has finite rank, then lemma 3.10 states that there is a Noetherian subset of $\underline{B}$ which is a base for X. This Noetherian base has finite rank since $\underline{B}$ does, so in this case we are done.

Suppose $\underline{B}$ has point-finite rank. For every integer n, define X_n to be $\{x \in X: r_x(\underline{B}) = n\}$. Clearly $\underline{B}$ is a base of rank $\leq n$ at X_n. By lemma 3.10 we can choose a Noetherian subset $\underline{B}'_n$ of $\underline{B}$ which is a base for X_n in X. We will suppose every member of $\underline{B}'_n$ intersects X_n.

For each integer n, define $\underline{B}_n$ to be $\{B \in \underline{B}'_n: B \cap \cup\{X_i: i < n\} = \phi\}$. If $x \notin \cup\{X_i: i < n\}$ then there is an n-antichain $\underline{A}$ in $(\underline{B})_x$. The set $\cap\underline{A}$ is a neighborhood of x which does not intersect $\cup\{X_i: i < n\}$. Therefore $\cup\{X_i: i < n\}$ is closed in X which implies $\underline{B}_n$ is a base in X for X_n. Therefore $\cup\{\underline{B}_n: n < \omega\}$ is a base for $\cup\{X_n: n < \omega\} = X$.

Since $\underline{B}_n$ is a subset of $\underline{B}'_n$, $\underline{B}_n$ is Noetherian. We claim $\cup\{\underline{B}_n: n < \omega\}$ is Noetherian. Let $\underline{C}$ be a chain in this base. Define m to be the minimal integer n such that $\underline{C} \cap \underline{B}_n \neq \phi$. Let C be the maximal member of $\underline{C} \cap \underline{B}_m$. We can show C is maximal in $\underline{C}$. If $D \in \underline{C} \setminus \underline{B}_m$, then $D \in \underline{C} \cap \underline{B}_{m'}$ for some $m' > m$. Therefore $D \cap X_m = \phi$ and since $C \cap X_m \neq \phi$, C is not a subset of D. Therefore $D \subset C$ and C is the maximal element of $\underline{C}$. □

Since the maximal members of any subcollection of a Noetherian base of point-finite rank is point-finite, it is easy to see from theorem 3.11 that a space with a base of point-finite rank is hereditarily metacompact, a result proven in [GN].

3.12 Corollary A regular space X with a base of point-finite rank has a base $\underline{B}$ such that for every point $x \in X$, the cardinality of $(\underline{B})_x$ is not greater than the cellularity of X.

Proof By theorem 3.11 we can choose a Noetherian base $\underline{B}'$ of point-finite rank. For each $U \in \underline{B}'$ consider the set $[U] = \{V \in \underline{B}' : cl_X V = cl_X U\}$. Since $\underline{B}'$ is Noetherian every member of $[U]$ is a subset of a maximal member of $[U]$. Define $\underline{B}$ to be $\{U \in \underline{B}' : U \text{ is maximal in } [U]\}$.

If $x \in X$ and W is a neighborhood of x, then we can choose $U \in \underline{B}'$ such that $x \in U \subset cl_X U \subset W$. There exists a maximal member V of $[U]$ which contains U. Therefore $V \in \underline{B}$ and since $x \in U \subset V \subset cl_X U \subset W$, we have shown $\underline{B}$ is a base for X.

For any $x \in X$, by lemma 1.4 we can choose a chain $\underline{C}$ of $(\underline{B})_x$ with the same cardinality as the cardinality of $(\underline{B})_x$. Since $\underline{C}$ is Noetherian, let $\{C_\alpha : \alpha < \beta\}$ be a well-ordering of $\underline{C}$ which agrees with the order given by reverse set inclusion. Since the C_α's are comparable, they have distinct closures in X. Therefore $\{C_\alpha \setminus cl_X C_{\alpha+1} : \alpha < \beta\}$ is a collection of pairwise disjoint non-empty open subsets of X, and $|\beta| =$

$|(\underline{B})_x|$. □

The following results are immediate consequences of 3.12. A regular hereditarily Lindelöf space with a base of point-finite rank has a point-countable base [G]. A regular separable space with a base of point-finite rank is metrizable [G]. A regular space with calibre ω_1 and a base of finite rank is metrizable and separable [GN].

Metacompactness is not preserved under finite products. For example, the Sorgenfrey line is metacompact, in fact Lindelöf, but its square is separable and has an uncountable closed discrete subset. The Sorgenfrey line also has a base of subinfinite rank, so in corollary 3.13 below, "point-finite" cannot be replaced by "subinfinite".

The next corollary describes a sufficient condition for the product of finitely many metacompact spaces to be metacompact.

3.13 Corollary The product of a finite family of spaces with bases of point-finite rank is metacompact.

Proof By theorem 3.11, a space with a base of point-finite rank has a Noetherian base of point-finite rank. Theorem 2.5 of [GN] states that a product of finitely many spaces with Noetherian bases of subinfinite rank (and in particular point-finite rank) has a Noetherian base of subinfinite rank. Therefore the product, by theorem 2.4, is metacompact. $\square$

CHAPTER IV

SPACES WITH BASES OF POINT-FINITE RANK THAT ARE METRIZABLE

In this chapter we will prove that a regular first countable β-space with a base of point-finite rank is developable and we characterize those regular ccc spaces which have a base of finite rank.

Let us recall the definition of a β-space [Ho1]. A $\underline{\beta\text{-space}}$ is a space in which there is associated to each point x and each positive integer n an open neighborhood g(n,x) of x such that if $\{x_n : n < \omega\}$ is a sequence in X and $\cap\{g(n,x_n) : n < \omega\} \neq \phi$, then $\{x_n : n < \omega\}$ has a cluster point. The function g from $\omega \times X$ to the topology of X is called a $\underline{\beta\text{-function}}$.

Listed in [Ho1] are characterizations of developable spaces, $w\Delta$-spaces, semi-stratifiable spaces, and weak-Nagata spaces given in terms of a function from $\omega \times X$ to the topology of X. It is clear from the characterizations given that these classes are all contained in the class of β-spaces.

Suppose X is a β-space and g is a β-function for X. If for every $x \in X$ and $n < \omega$, f(n,x) is a subset of g(n,x) and an open neighborhood of x, then f is also a β-function for X. Therefore, without loss of generality we will assume that, for each $x \in X$, $\{g(n,x): n < \omega\}$ is a decreasing sequence of neighborhoods for x.

4.1 Lemma Let g is a β-function for a space X such that $\{g(n,x): n < \omega\}$ is a decreasing sequence, for all $x \in X$. If $\cap\{g(n,x_n): n < \omega\} \neq \phi$, then every infinite subset of $\{x_n: n < \omega\}$ has an accumulation point in X.

Proof Suppose $\cap\{g(n,x_n): n < \omega\} \neq \phi$ and A is an infinite subset of ω. Define y_n to be x_n if $n \in A$, otherwise define y_n to be $x_{n'}$, for some $n' \in A$ which is larger than n. Since $g(n,y_n) \supset g(n',x_{n'})$, we have $\cap\{g(n,y_n): n < \omega\} \supset \cap\{g(n,x_n): n \in A\} \neq \phi$. Therefore $\{y_n: n < \omega\} = \{x_n: n \in A\}$ has a cluster point. $\square$

4.2 Definition A constructible subset of a space X is the intersection of two subsets of X, one open in X and the other closed in X.

4.3 Lemma Let X be a regular β-space in which every point is a G_δ. If a constructible subset K of X has a Noetherian base $\underline{B}$ of rank $\leq n$ at K, then there is a subfamily $\underline{U}$ of $\underline{B}$ of countable order (see Definition 2.6) such that for all $x \in K$ either (i) $(\underline{U})_x$ is a local base for x or (ii) $r_x(\underline{B},U) < n$ for some $U \in \underline{U}$.

Proof Let K be a constructible subset of X which does not contain any isolated points of X. Choose an open subset V_K of X and a closed subset F_K of X such that $K = V_K \cap F_K$. Assume $\underline{B}'$ is a Noetherian base of rank $\leq n$ at K. Define $\underline{B}$ to be $\{B \in \underline{B}': cl_X B \subset V_K,\ B \cap K \neq \phi\}$. $\underline{B}$ is a base of rank $\leq n$ at K; it is Noetherian since $\underline{B}'$ is. We will construct a subfamily of $\underline{B}$ that is a base for K in X of countable order.

Let g be a β-function for X such that $\{g(i,x): i < \omega\}$ is a decreasing sequence for every $x \in X$. Since every point x of X is a G_δ, we will assume $\cap\{g(i,x): i < \omega\} = \{x\}$.

We will inductively construct a n-wide subfamily of $\underline{B}$ (recall Definition 3.7). We will define $\underline{U}_i \subset \underline{B}$ and $D_i \subset K$ for $i < \omega$ to satisfy the following:

(1) $\underline{U}_0$ is a cover of K

(2) $\underline{U}_i \subset \underline{B}$ is an antichain

(3) $\underline{U}_{i+1}$ is a cover of $\{x \in K: r_x(\underline{B},U) = n$ for all $U \in (\cup\{\underline{U}_j: j \leq i\})_x\}$

For each $x \in D_i$, we choose $f(i,x) \in \underline{B}$ such that $\underline{U}_i = \{f(i,x): x \in D_i\}$ and such that:

(4) $x \in f(i,x) \subset g(i,x)$

(5) $f(i,x)$ is a proper subset $\cap(\cup\{\underline{U}_j: j < i\})_x$ and there exists an n-antichain of $(\underline{B})_x$ between $f(i,x)$ and $\cap(\cup\{\underline{U}_j: j < i\})_x$

(6) if $y \in D_j$, $j < i$, $x \in f(j,y)$, and $y \neq x$ then $cl_X f(i,x) \subset f(j,y) \setminus \{y\}$

Let us define $\underline{U}_0$ and D_0. For each $x \in K$ choose $f(0,x) \in \underline{B}$ such that $x \in f(0,x) \subset g(0,x)$. Define $\underline{U}_0$ to be the set of all maximal members of $\{f(0,x): x \in K\}$. For each member U of $\underline{U}_0$, choose $x_U \in K$ such that $f(0,x_U) = U$ and define D_0 to be $\{x_U: U \in \underline{U}_0\}$.

Suppose a subset D_j of K, and an antichain $\underline{U}_j = \{f(j,x): x \in D_j\}$ have been chosen satisfying (1)-(6) for all $j \leq i$. We will choose $\underline{U}_{i+1}$.

Define $\underline{V}_i$ to be $\cup\{\underline{U}_j: j \leq i\}$ and define E_i to be $\cup\{D_j: j \leq i\}$. Suppose $x \in K$. Each $\underline{U}_j$, $j \leq i$, is an antichain and has finite rank at K, so the set $(\underline{V}_i)_x$ is finite. Therefore $\{y \in E_i: x \in f(j,y) \in \underline{U}_j\}$ is finite and $\cap(\underline{V}_i)_x$ is a neighborhood of x in X.

Suppose further that $r_x(\underline{B}, \cap(\underline{V}_i)_x) = n$. By definition we can choose an n-antichain $\underline{A}_x \subset (\underline{B})_x$ such that $\cup\underline{A}_x \subset \cap(\underline{V}_i)_x$. No isolated point of X is a point of K, so now we can choose a neighborhood $f(i+1,x)$ of x whose closure in X is a proper subset of the neighborhood $\cap\underline{A}_x \cap g(i+1,x)$ and which does not intersect $\{y \in E_i: x \in f(j,y) \in \underline{U}_j,\ j \leq i,\ y \neq x\}$. Then $f(i+1,x)$ satisfies (4)-(6).

Define $\underline{U}_{i+1}$ to be the set of all maximal members of $\{f(i+1,x)\colon x \in K,\ r_x(\underline{B}, \cap(\underline{V}_i)_x) = n\}$. For each member U of $\underline{U}_{i+1}$, choose $x_U \in K$ such that $f(i+1,x_U) = U$ and define D_{i+1} to be $\{x_U\colon U \in \underline{U}_{i+1}\}$. Then $\underline{U}_{i+1}$ and D_{i+1} satisfy all required conditions. This completes the induction.

If $\underline{U}_i$ has been defined as above for all $i < \omega$, we claim $\underline{U} = \cup\{\underline{U}_i\colon i < \omega\}$ is n-wide in $\underline{B}$ and and is the required base.

If $j < i$ and $x \in D_i$ then, by (5) and (3), we have $x \in \cup\underline{U}_j$. Then by (5) again, $\underline{U}_i$ is a partial refinement of $\underline{U}_j$, since $(\underline{U}_j)_x \neq \phi$ for all $x \in D_i$.

Suppose $x \in D_i$, $y \in D_j$, $f(i,x) \in \underline{U}_i$, $f(j,y) \in \underline{U}_j$ and $f(i,x)$ is a proper subset of $f(j,y)$. Assume $i \leq j$. Then $\underline{U}_j$ is a partial refinement of $\underline{U}_i$ and $f(i,x)$, as a proper subset of $f(j,y) \in \underline{U}_j$, must be a proper subset of some member of $\underline{U}_i$. But this is a contradiction since $\underline{U}_i$ is an antichain.

Therefore, if $f(i,x)$ and $f(j,y)$ are two distinct members of $\underline{U}$ and $f(i,x)$ is a proper subset of $f(j,y)$, then $i > j$. Furthermore by (5), there is an n-antichain of $(\underline{B})_x$ between $f(i,x)$ and $f(j,y)$. This proves $\underline{U}$ is n-wide.

Let $\underline{C} \subset \underline{U} = \cup\{\underline{U}_i\colon i \in \omega\}$ be an infinite chain. To prove $\underline{U}$ has countable order, we will show that either $\cap\underline{C} = \phi$ or $\underline{C}$ is a local neighborhood base in X for some point of K.

Each $\underline{U}_i$ is an antichain, so the chain $\underline{C}$ will contain at most one member of $\underline{U}_i$. Therefore there is an infinite set $A \subset \omega$ such that $\underline{C} = \{f(i,x_i): i \in A,\ x_i \in D_i\}$.

Suppose there exists an x such that $\{i \in A: x_i = x\}$ is infinite. We have chosen $\{g(i,x): i < \omega\}$ to be a strictly decreasing sequence whose intersection is the set $\{x\}$, therefore by (4) $\cap\{f(i,x): i \in A\} = \{x\}$. But $\underline{U}$ is n-wide and $\{x\}$ is not open in X (K contains no isolated points of X), so by lemma 3.8, $\{f(i,x_i) \in \underline{C}: x_i = x\}$ is a local base for x in X.

If no element is repeated infinitely often in $\{x_i: i \in A\}$ then there is an infinite subset $A' \subset A$ such that, if $\{i,j\} \subset A'$ and $i \neq j$, then $x_i \neq x_j$. Suppose $\cap\underline{C} \neq \phi$. Then $\{x_i: i \in A'\}$ has a cluster point $c \in X$, by our choice of the β-function and lemma 4.1. We have proven earlier that, if $i < j$ then $f(i,x_i) \not\subset f(j,x_j)$. Therefore $f(j,x_j) \subset f(i,x_i)$ for any $\{i,j\} \subset A'$, $i < j$, and furthermore by (6) $cl_X f(j,x_j) \subset f(i,x_i) \setminus \{x_i\}$. Therefore $\cap\underline{C}$ does not intersect $\{x_k: k \in A'\}$, but since $f(i,x_i)$ contains all but finitely many points of $\{x_k: k \in A'\}$, we have $c \in \cap\underline{C}$. Since $\{x_k: k \in \omega\}$ is a subset of the closed set F_K, the cluster point c is contained in F_K. Since $c \in \cap\underline{C}$ and every member of $\underline{C}$ is a subset of V_K, we have $c \in V_K$. Therefore $c \in V_K \cap F_K = K$. But $\cup\{\underline{U}_i: i < \omega\}$ is n-wide and $r_c(\underline{B}) \leq n$, so by

3.8, $\cap\underline{C}$ is a neighborhood of c or $\underline{C}$ is a local base at c. Since $\cap\underline{C}$ does not meet $\{x_i: i \in A'\}$, $\cap\underline{C}$ cannot be a neighborhood of the cluster point c, so $\underline{C}$ is a local base for c in X and $\cap\underline{C} = \{x\}$.

Therefore every infinite chain of $\underline{U} = \cup\{\underline{U}_i: i < \omega\}$ has empty intersection or is a local base. Therefore $\underline{U}$ has countable order.

Finally we will show (i) and (ii) in the statement of the lemma are satisfied. Suppose $x \in K$ and $r_x(\underline{B}, U) = n$ for all $U \in (\underline{U})_x$. We will show $(\underline{U})_x$ is a local base for x in X. By (3), $x \in \cup\underline{U}_i$ for all $i < \omega$ so $(\underline{U})_x$ is infinite. $\underline{B}$ has finite rank at x, so by lemma 1.4 there is an infinite chain in $(\underline{U})_x$. We have proven every infinite chain of $\underline{U}$ is either a local base or has empty intersection, therefore $(\underline{U})_x$ is a local base for x.

We have proven the lemma assuming that K is a constructible subset of X without any isolated points of X. If K′ is any constructible subset of X and $\underline{B}$ is a Noetherian base of rank $\leq n$ at K′, then define K to be $K' \setminus \{x: \{x\} \text{ open in } X\}$. Then K is a constructible subset of X and $\underline{B}$ is a base of rank $\leq n$ at K, so we can find a subset of $\underline{B}$ to satisfy the lemma for K. By simply adding all the open singletons to this subset, we obtain the desired family for K′. □

4.4 Lemma Let X be a regular β-space in which every point is a G_δ. If a constructible subset K of X has a Noetherian base $\underline{B}$ of rank $\le n$ at K, then there is a subfamily of $\underline{B}$ with countable order which is a base for K in X.

Proof We will prove the lemma by induction on n. Let $n \ge 0$ be an integer. Assume we have shown the lemma is true for every integer less than n.

Let K be a constructible subset of X and suppose $\underline{B}$ is a Noetherian base of rank $\le n$ at K. We will assume every member of $\underline{B}$ intersects K. By lemma 4.3, we can choose a subfamily $\underline{U}$ of $\underline{B}$ with countable order such that for all $x \in K$, either $(\underline{U})_x$ is a local base at x, or $r_x(\underline{B},U) < n$ for some $U \in \underline{U}$.

For each $U \in \underline{U}$, define K_U to be $\{x \in U \cap K: r_x(\underline{B}_K,U) < n\}$. Every point of U that is not in K_U is in the intersection of an n-antichain which does not intersect K_U, therefore K_U is closed in $U \cap K$, and is a constructible subset of X. Furthermore $\{B \in \underline{B}: B \subset U\}$ is a base of rank $<n$ at K_U. By our induction assumption, we can choose a subset $\underline{B}_U$ of $\{B \in \underline{B}: B \subset U\}$ which is a base of countable order for K_U in X.

We will show $\underline{U} \cup \cup\{\underline{B}_U: U \in \underline{U}\}$ is a base of countable order for K in X.

Let $\underline{C} \subset \cup\{\underline{B}_U\colon U \in \underline{U}\}$ be an infinite chain. Assume that the set $\underline{C}' = \{U \in \underline{U}\colon C \subset U$ for some $C \in \underline{C}\}$ is infinite. Since the intersection of every subcollection of $\underline{C}'$ with n+1 members contains as a subset a member of $\underline{C}$, and therefore contains an point of K, there are no n+1-antichains in $\underline{C}'$. Therefore by lemma 1.4 there is an infinite chain in $\underline{C}'$. Since it is also Noetherian, there is a perfectly decreasing subchain. $\underline{U}$ has countable order so $\cap\underline{C}' = \phi$ or $\underline{C}'$ is a local base for some point of X. Therefore $\cap\underline{C} = \phi$ or $\underline{C}$ is a local base for some point.

Assume $\{U \in \underline{U}\colon C \subset U$ for some $C \in \underline{C}\}$ is finite. Then the members of $\underline{C}$ are distributed among only finitely many families $\underline{B}_U$, $U \in \underline{U}$. Therefore there exists $U \in \underline{U}$ such that $\underline{C} \cap \underline{B}_U$ is infinite. $\underline{B}_U$ is Noetherian and has countable order, so the chain $\underline{C} \cap \underline{B}_U$ has empty intersection or is a local base. Therefore $\underline{C}$ has empty intersection or is a local base.

We have shown $\cup\{\underline{B}_U\colon U \in \underline{U}\}$ has countable order. The union of two families of countable order also has countable order, so $\underline{U} \cup \cup\{\underline{B}_U\colon U \in \underline{U}\}$ has countable order. If $x \in K$ and $\underline{U}$ is not a base at x, then there exists $U \in \underline{U}$ such that $r_x(\underline{B},U) <$ n, and therefore $\underline{B}_U$ is a base for x. Hence $\underline{U} \cup \cup\{\underline{B}_U\colon U \in \underline{U}\}$ is a base of countable order for K in X. $\square$

Recall that a Moore space is a regular space with a developement (see Definition 1.11).

4.5 <u>Theorem</u> Let X be a regular β-space in which every point is a G_δ. If X has a base of point-finite rank, then X is a Moore space.

<u>Proof</u> By theorem 3.11 we can assume X has a Noetherian base $\underline{B}$ of point-finite rank.

For each $n < \omega$ define K_n to be $\{x \in X: r_x(\underline{B}) \leq n\}$. Then $\underline{B}$ is a base of rank $\leq n$ at K_n. If $r_x(\underline{B}) > n$ then x is contained in the intersection of an n+1-antichain of $\underline{B}$, and this intersection does not intersect K_n. Therefore K_n is closed in X, so we can apply lemma 4.4 to find a base $\underline{B}_n \subset \underline{B}$ of countable order for K_n in X.

Define $\underline{B}'_0$ to be the empty set and for $n > 0$, define $\underline{B}'_n$ to be $\{B \in \underline{B}_n: B \cap K_{n-1} = \phi\}$. Each $\underline{B}'_n$ is a base for $K_n \setminus K_{n-1}$ in X and has countable order since $\underline{B}_n$ does. We claim $\cup\{\underline{B}'_n: n < \omega\}$ is a base of countable order for X.

Let $\underline{C} \subset \cup\{\underline{B}'_n: n < \omega\}$ be a perfectly decreasing chain. If, for every $n < \omega$, there exist a member of $\underline{C}$ which does not intersect K_n then $\cap\underline{C} \subset \cap\{X \setminus K_n: n < \omega\} = \phi$. If there exists an $n < \omega$ such that every member of $\underline{C}$ intersects K_n

then $\underline{C} \subset \cup\{\underline{B}_i': i \le n\}$. Therefore there is an i such that $\underline{C} \cap \underline{B}_i'$ is infinite. Since $\underline{B}_i'$ has countable order, either $\underline{C} \cap \underline{B}_i'$ is a local base or its intersection is empty. Hence either $\underline{C}$ is a local base or $\cap\underline{C} = \phi$.

Therefore X has a base of countable order, and since X has a base of point-finite rank, by corollary 2.10 X is developable. □

The requirement in the theorem above that X be a β-space can not be dropped. The Michael line, the space obtained by declaring each irrational of the real line to be isolated, is a space with a base of rank 1 (see example 5.1 of [LN]). It is first countable, but it is not developable. The "two arrows" space of Alexandroff, which is I x 2 with the lexigraphic order (I denotes the unit interval), is a compact first countable space with a base of subinfinite rank [GN]. Since every collection-wise normal Moore space is metrizable [Bi], the two arrows space is not a Moore space. Therefore the property "point-finite rank" cannot be replaced by the weaker property "subinfinite rank" in the theorem above.

4.6 Corollary A collectionwise normal Σ-space (or wΔ-space) with a base of point-finite rank is metrizable.

Proof Every wΔ-space or Σ-space is a β-space [Ho2]. We will show every point of a Σ-space or $w\Delta$-space with a base of point-finite rank is a G_δ. Then we can apply theorem 4.5 to prove X is developable. Since a collectionwise normal developable space is metrizable [Bi], we will then be done.

Suppose X is a $w\Delta$-space, and $x \in X$. It follows immediately from the definition of a $w\Delta$-space that every point of X has a collection $\{U_n: n < \omega\}$ of neighborhoods such that, for any sequence $\{x_n: n < \omega\}$ of X, if $x_n \in U_n$ for all n, then $\{x_n: n < \omega\}$ has an accumulation point. Since X is regular we can choose a closed G_δ V_n contained in U_n and containing x. Therefore the closed set $\cap\{V_n: n < \omega\}$ is a countably compact G_δ containing x. Every base of point-finite rank is a base of subinfinite rank, so by theorem 2.11, x is a G_δ of $\cap\{V_n: n < \omega\}$, and hence x is a G_δ in X.

Now suppose X is a Σ-space, and let $\{\underline{F}_n: n < \omega\}$ be a spectral Σ-sequence for X. For each point of X and for every integer n we can choose a neighborhood V_n which intersect only finitely many members of $\underline{F}_n$. Let U_n be a closed G_δ such that $x \in U_n \subset V_n$. Using the method used in the proof of theorem 2 of [GZ], we will show every uncountable subset of the closed G_δ, $\cap\{U_n: n < \omega\}$, has a countable subset with an accumulation point.

Let A be an uncountable subset of $G = \cap\{U_n: n < \omega\}$. For every n, only finitely many members of $\underline{F}_n$ meet G, so the set

$\{\cap(\mathcal{F}_n)_x\colon x \in G,\ n < \omega\}$ is countable. If for every $x \in G$, there is an integer $n(x)$ such that $\cap(\mathcal{F}_{n(x)})_x \cap A$ is finite, then $\{\cap(\mathcal{F}_{n(x)})_x\colon x \in G\}$ would be a countable cover of A with each member containing only finitely many elements of A. This contradiction shows there exists a point y such that $A \cap \cap(\mathcal{F}_n)_y$ is infinite for every $n < \omega$. We can inductively choose $\{a_n\colon n < \omega\} \subset A$ as follows: $a_0 \in \cap(\mathcal{F}_0)_y \cap A$, and $a_{n+1} \in \cap(\mathcal{F}_{n+1})_y \cap A \setminus \{a_i\colon i \le n\}$. By the definition of a spectral sequence, $\{a_n\colon n < \omega\}$ has an accumulation point. Therefore every uncountable subset of G has a countable subset with an accumulation point, so by theorem 2.11, every point of G is a G_δ in G, and hence in X. $\square$

We cannot weaken "collectionwise normal" to "normal" in corollary 4.6. For example, consider the following subspace of Heath's "tangent V space" [He]. Let X be the set of all points in the Euclidean half-plane above the x-axis plus ω_1 points of the x-axis. A basic open neighborhood of a point x on the x-axis is a pair of line segments of equal length extending up from x at an angle of 45 degrees. All points not on the x-axis are isolated. This describes a base of rank 2 for X. X is a Σ-space and a $w\Delta$-space, though it is not metrizable since X is not collection-wise normal. If we add Martin's axiom and the negation of the continuum hypothesis (see [Je]) to the usual axioms of set theory, then the space X is normal. The proof is identical to that given in [R] to show the square of a κ-Sorgenfrey line is normal if $\omega < \kappa < 2^{\omega}$. Therefore in some models of set theory, X is a normal non-metrizable Σ-space and $w\Delta$-space with a base of rank 2. On the other hand, if the set-theoretic axiom PMEA is consistent (though it may not be, see [Ny5]), then every normal first countable space is collectionwise normal.

Corollary 5.14 of [GN] states that a ccc space is a finite dimensional metric space iff it is a collectionwise normal Σ-space with a base of finite rank. Corollary 4.6 together with theorem 1.6 show that this result holds for all spaces, not just the ccc spaces.

In the remainder of this chapter we characterize the regular ccc spaces with a base of finite rank.

4.7 <u>Definition</u> A <u>Souslin line</u> is a ccc non-separable space that can be embedded in a linearly ordered space. A <u>Souslin tree</u> is any uncountable poset $(P,<)$ with no uncountable antichains or chains, such that for every $x \in P$, $\{y \in P: y < x\}$ is a chain well-ordered by "$<$".

A Souslin line exists if and only if a Souslin tree exists [M]. The existence of a Souslin tree is consistent with and independent of the usual axioms of set theory [Je2].

It is shown in [Ny2] that a ccc space with a base of rank 1 is non-metrizable if and only if there exists a Souslin tree. We extend this to spaces with bases of finite rank.

4.8 Theorem If a regular, ccc space X has a base of finite rank, and no open subset of X is separable, then a dense subset of X is a Souslin line.

Proof Let G be an open subset of X. Define n to be the minimal integer m such that there exists an open subset W′ of G with a base $\underline{B}_W{}'$ of rank $\leq m$. Choose an open subset W of G which has a base $\underline{B}$ of rank $\leq n$.

We will inductively choose a dense subset of W.

Suppose we have chosen $\{x_\alpha : \alpha < \beta\} \subset W$ such that $r_{x\alpha}(\underline{B}\ , W \setminus cl_X\{x_\gamma : \gamma < \alpha\}) = n$ for every $\alpha < \beta$. Assume the set $V_\beta = W \setminus cl_X\{x_\alpha : \alpha < \beta\}$ is not empty. There exists a point $x \in V_\beta$ such that $r_x(\underline{B}, V_\beta) = n$, otherwise the family $\{B \in \underline{B} : B \subset V_\beta\}$ is a base of rank $<n$ for V_β, contradicting the minimality of n. Therefore we can choose $x_\beta \in V_\beta$ such that $r_{x\beta}(\underline{B}, V_\beta) = n$.

In this way we will choose a dense subset $\{x_\alpha : \alpha < \eta\}$ of W. For every $\alpha < \eta$, we can choose an n-antichain $\underline{A}_\alpha \subset \{B \in \underline{B} : x_\alpha \in B \subset W \setminus cl_X\{x_\gamma : \gamma < \alpha\}\}$ since the rank of $\{B \in \underline{B} : B \subset W \setminus \{x_\gamma : \gamma < \alpha\}\}$ at x_α is exactly n. Also, by corollary 3.12, X is first-countable, so for each x_α, we can choose a decreasing local base $\{U_{\alpha,i}: i < \omega\} \subset \underline{B}$ such that $U_{\alpha,i} \subset \cap\underline{A}_\alpha$ for each i.

We claim $\{U_{\alpha,i}: \alpha < \eta,\ i < \omega\}$ has rank 1. Suppose $\alpha < \beta$ and $U_{\alpha,i}$ intersects $U_{\beta,j}$. Then $U_{\alpha,i}$ intersects $\cap\underline{A}_\beta$ which has $U_{\beta,j}$ as a subset. $\underline{B}$ has rank $\leq n$ at W, so $U_{\alpha,i}$ is comparable to some member of the n-antichain $\underline{A}_\beta$. The point x_α is not contained in any member of $\underline{A}_\beta$. Therefore $U_{\alpha,i}$ must have some member of $\underline{A}_\beta$ as a subset, so $U_{\beta,j}$ is a subset of $U_{\alpha,i}$.

Therefore $\{U_{\alpha,i}: \alpha < \eta,\ i < \omega\}$ is a base of rank 1 for $\{x_\alpha: \alpha < \eta\}$. Since $\{x_\alpha: \alpha < \eta\}$ is a dense subset of W, it is non-separable and ccc. Every space with a base of rank 1 can be embedded as a subspace of a totally ordered set with the usual order topology [Ny3], therefore $\{x_\alpha: \alpha < \eta\}$ is a Souslin line dense in an open subset of G.

We can choose a maximal cellular family of open subsets of X which contain a dense Souslin line, and since every open set of X contains an open subset with a dense Souslin line, the union of this family is dense in X. In this way we can choose a countable family of pairwise disjoint Souslin lines whose union is dense in X and is a Souslin line. $\square$

Let us show that every ccc non-separable space X has an open subset with no open separable subsets. Choose a maximal cellular (possibly empty) family $\underline{V}$ of separable open subsets of X. Since X is ccc, $\underline{V}$ is countable, so $\cup\underline{V}$ is separable. Thus $X \setminus cl_X \cup \underline{V}$ is a non-empty open subset of X, and by the maximality of $\underline{V}$, every open subset of $X \setminus cl_X \cup \underline{V}$ is non-separable.

Therefore any ccc non-separable space X with a base of finite rank contains an open subset with a dense Souslin line.

A space X has <u>precalibre</u> ω_1 if every uncountable family of open subsets of X contains an uncountable filter-base. In [IN] it is proven that the set-theoretic axiom K (every ccc space has precalibre ω_1) implies that every regular space with a base of finite rank is separable. It is well known that a Souslin line does not have precalibre ω_1, so this result is a consequence of theorem 4.8.

4.9 <u>Corollary</u> The following statements are equivalent.

(1) X is a regular ccc space with a base of rank $\leq n$ and does not contain a Souslin line.

(2) X is a regular space with a base of rank $\leq n$ and X^2 is ccc.

(3) X is a separable metric space of dimension n-1.

<u>Proof</u> Recall that theorem 1.6 states that a metrizable

space has a base of rank n iff it has dimension n-1. Therefore (3) implies (1).

Suppose X is a regular ccc space with a base of rank $\leq$n. If X^2 is not ccc then X is not separable, so by the remark following theorem 4.8, X contains a Souslin line. Therefore (1) implies (2).

If X^2 is ccc, then no open subset of X contains densely a Souslin line, since it is well known that the square of a Souslin line is not ccc. By the remark following theorem 4.8, a non-separable ccc space with a base of finite rank contains an open subset with a dense Souslin line. Therefore, if X is a regular space with a base of finite rank and X^2 is ccc, then X is separable, and a separable space with a base of point-finite rank is metrizable [G]. Therefore (2) implies (3). □

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