# LOCALIZATION AND TOEPLITZ OPERATORS WITH COMPLEX BOREL MEASURE SYMBOLS ON WEIGHTED BERGMAN SPACES

by

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# Abstract

We extend the definition of weak localization to the weighted Bergman spaces of the unit ball, $L^2_a(\mathbb{B}_n, dV_\alpha)$, for $\alpha > -1$. We prove that a Toeplitz operator with a complex Borel measure symbol whose total variation is Carleson is weakly localized on $L^2_a(\mathbb{B}_n, dV_\alpha)$. We extend the definitions of strongly localized and sufficiently localized operators defined in [37], which is the paper of the Ph.D. candidate and Prof. N. Zorboska, to the weighted Bergman spaces $L^2_a(\mathbb{B}_n, dV_\alpha)$ and show that they are also weakly localized. We also show that bounded Toeplitz operators with $BMO^1$ symbols are strongly (and therefore also weakly) localized. Finally, we prove that the Toeplitz operators induced by the complex Borel measures with Carleson total variation are weakly localized on the spaces $L^2_a(\omega)$ with the weight $\omega$ in the class $\mathcal{E}$.

**To my beloved mother**

**and**

**the memory of my father**

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# Introduction

The characterization of bounded and compact Toeplitz operators on the Bergman spaces via their Berezin transforms has been widely studied. See for example [7], [52], [41], [24], and [15]. These types of problems are usually referred to as the Reproducing Kernel Thesis (RKT). Namely, since the normalized kernel functions form a dense subset of the space, the RKT asks when we can describe the behaviour of the operator by checking its behaviour on the kernel functions. In [7], Axler and Zheng proved that if $T$ is a finite sum of finite products of Toeplitz operators with bounded symbols on the Bergman space of the open unit disk, then $T$ is compact if and only if the Berezin transform of $T$ vanishes on the boundary of the disk. Then there were several generalizations to Toeplitz operators with unbounded symbols (see for example [52]). In [41], Suárez proved that an operator on $L_a^2(\mathbb{B}_n, dV_\alpha)$ is compact if and only if it belongs to the Toeplitz algebra and its Berezin transform vanishes on the boundary of the ball.

In [15], Isralowitz, Mitkovski, and Wick introduced the class of "weakly localized operators" including the Toeplitz operators with bounded symbols. They also showed that the property of weak localization allows describing the compactness of an operator via its Berezin transform. The main idea for their work came from the so-called "sufficiently localized operators" on the Fock space defined by Xia and Zheng in [43]. Afterwards, a long list of papers using similar ideas have been published on both Bergman and Fock spaces. See for example [42], [16], and [37]. Our goal is to show that some large classes of Toeplitz operators with possibly unbounded symbols are contained in the class of weakly localized operators.

Toeplitz operators with complex Borel measure symbols were first studied by Luecking in [23]. This class of Toeplitz operators includes the class of Toeplitz operators with integrable symbols. We are specifically interested in the Toeplitz operators with complex Borel measure symbols whose total variation is a Carleson measure. Carleson measures were first introduced for the Hardy spaces $H^p$ by Carleson in his solution of the corona problem [12]. Corresponding versions of Carleson measures were later introduced for other functions spaces such as the Bergman and the Dirichlet spaces. Being a Carleson measure is a geometric property which is nicely matched with the geometric decomposition of the open unit ball. This interesting connection is our motivation for considering Toeplitz operators induced by Carleson measures.

The theory of $BMO$ spaces in the hyperbolic metric was introduced by Zhu in his Ph.D. thesis [51] and then extended by Békollé, Berger, Coburn, and Zhu in [9]. Toeplitz operators with $BMO$ symbols constitute another interesting class of Toeplitz operators containing those with bounded and with non-negative symbols. The $BMO$ spaces provide a class of possibly unbounded symbols that can, under certain assumptions, induce bounded Toeplitz operators on the Bergman spaces as shown by Zorboska in [52].

The theory of Bergman spaces began with S. Bergman in [11], and it was extended to the large Bergman spaces by Lin and Rochberg in [20]. This class of Bergman spaces is called large since it contains the classical weighted Bergman spaces on the open unit disk with the standard weights $\omega_\alpha(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$, and many others. However, in chapter 3, we will focus on a subclass of the large Bergman spaces with the weights in the class $\mathcal{E}$, which does not contain the standard weights. Weak localization of operators on a class of weighted Bergman spaces with radial decreasing weights was defined in [4]. We use the same definition of weak localization for operators on the Bergman space $L^2_a(\omega)$ with the weight $\omega$ in the class $\mathcal{E}$, and show that our result from chapter 2 on weak localization of Toeplitz operators induced by complex Borel measures with Carleson total variation on the standard weighted Bergman spaces also holds true in the setting of this subclass of large Bergman spaces.

The organization of the thesis is as follows. In chapter 1, we review some definitions and theorems we need for presenting our work. Chapter 2 is devoted to our results on localization of some classes of Toeplitz operators acting on the standard weighted Bergman spaces $L_a^2(\mathbb{B}_n, dV_\alpha)$, $\alpha > -1$, that are extensions of the same results derived by Prof. N. Zorboska and myself in [37], proven for the non-weighted Bergman spaces. In chapter 3, we prove that our main result from chapter 2 holds true also for the Bergman spaces $L_a^2(\omega)$, with the weight $\omega$ in the class of exponential-type weights $\mathcal{E}$. Finally, in chapter 4, we state some related open problems and mention a few topics for further investigation.

# Contents

# Chapter 1

# Preliminaries

In this chapter, we recall some definitions, notations and fundamental theorems which will be used in the next chapters. Some references are provided for basic complex analysis, basic functional analysis, and measure theory. One can consult those for more details. See [2], [14], [34], and [36].

Throughout this manuscript, the notation $A(x) \lesssim B(x)$ means that there is a positive constant $C$, independent of $x$, such that $A(x) \leq CB(x)$, and the notation $A(x) \approx B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$. We first review reproducing kernel Hilbert spaces and some basic properties of kernel functions which will be used in chapters 2 and 3.

## 1.1 Reproducing Kernel Hilbert Spaces

**Definition 1.1.1.** Let $X$ be any set and let $F(X, \mathbb{C})$ be the vector space of all complex valued functions defined on $X$ endowed with the usual operations of addition and scalar multiplication. By [3], a Hilbert space $\mathcal{H}$ *is called a* ***reproducing kernel Hilbert space (RKHS) on*** $X$ over $\mathbb{C}$ if

(i) $\mathcal{H}$ is a vector subspace of $F(X, \mathbb{C})$,

(ii) for any $x \in X$ the linear evaluation functional, $E_x : \mathcal{H} \longrightarrow \mathbb{F}$ defined by

$E_x(f) = f(x)$ is bounded.

If $\mathcal{H}$ is a RKHS on $X$, then by the Riesz representation theorem for Hilbert spaces, any evaluation functional $E_x$ comes from an inner product with a unique vector, $K_x \in \mathcal{H}$. Hence, for $x \in X$, we have $f(x) = \langle f, K_x \rangle_{\mathcal{H}}$, for all $f \in \mathcal{H}$.

The function $K_x$ is called the **reproducing kernel** for the point $x$ and the function defined by $K(x, y) = K_y(x)$ is called the **reproducing kernel** for $\mathcal{H}$. Kernel functions have the following properties:

(1) $K(x, y) = K_y(x) = \langle K_y, K_x \rangle_{\mathcal{H}}$,

(2) $K(y, x) = \overline{K(x, y)}$,

(3) $\|E_x\| = \|K_x\| = \sqrt{K(x, x)}$.

See [1], [3], and [31] for further studies in reproducing kernel Hilbert spaces. In this thesis, we deal with two classes of weighted Bergman spaces which are reproducing kernel

Hilbert spaces. Other significant reproducing kernel Hilbert spaces which have been widely studied, but will not be covered here, are Fock spaces, Hardy spaces and Dirichlet spaces.

**Definition 1.1.2.** Let $\mathcal{H}$ be a RKHS on $X$ over $\mathbb{C}$ which contains constant functions and let $T$ be a linear operator on $\mathcal{H}$. The Berezin transform of $T$ is denoted by $\widetilde{T}$ and is defined on $X$ by

$$\widetilde{T}(x) = \frac{\langle TK_x, K_x \rangle_{\mathcal{H}}}{K(x,x)}, \quad x \in X.$$

It is clear that if the operator is bounded, then so is its Berezin transform. The Berezin transform also uniquely characterizes the operator [48, Proposition 6.2], and can serve as a tool for describing and determining some of the properties of the operator.

## 1.2 Complex Measures and Polar Representation

Next, we will review some basic facts about complex measures and a consequence of the Radon-Nikodym theorem that will be addressed later.

**Definition 1.2.1.** Let $\mathfrak{B}$ be a *σ-algebra in a set* $X$*. A* ***complex measure*** $\nu$ on $\mathfrak{B}$ is a complex function on $\mathfrak{B}$ such that

$$\nu(E) = \sum_{j=1}^{\infty} \nu(E_j), \qquad E \in \mathfrak{B}$$

for any partition $\{E_j\}_{j=1}^{\infty}$ *of* $E$*.*

**Definition 1.2.2.** Let $\nu$ be a complex measure on a $\sigma$*-algebra* $\mathfrak{B}$ on a set $X$. The **total variation** of $\nu$ is denoted by $|\nu|$, and is defined by

$$|\nu|(E) = \sup\left\{\sum_{j=1}^{\infty} |\nu(E_j)| \;:\; \{E_j\}_{j=1}^{\infty} \textit{ is a partition of } E\right\}.$$

The total variation $|\nu|$ of a complex measure $\nu$ is a positive measure on $\mathfrak{B}$ and it has the following properties:

- $|\nu|(X) < \infty$, and
- $|\nu(E)| \leq |\nu|(E), \quad \forall\, E \in \mathfrak{B}$.

See [36, Theorem 6.2] and [36, Theorem 6.4]. Hence, any complex Borel measure on every $\sigma$-algebra is bounded, since

$$|\nu(E)| \leq |\nu|(E) \leq |\nu|(X) < \infty, \quad \forall\, E \in \mathfrak{B}.$$

Note that, while an infinite positive measure is not a complex measure (since infinity is not a complex number), a finite positive measure $\mu$ may be regarded as a complex measure. It is clear that if $\mu$ is a finite positive measure, then $|\mu| = \mu$.

The following theorem is a consequence of the Radon-Nikodym theorem. It states that the unique measurable function that represents the complex measure $\nu$ in terms of its total variation is unimodular. As the complex measure $\nu$ is represented as the product of its modulus and a unimodular function, this is usually called the polar representation or polar decomposition of $\nu$ [36, Theorem 6.12].

**Theorem 1.2.3.** **[36]** Let $\nu$ be a complex measure on a $\sigma$-algebra $\mathfrak{B}$ in a set $X$. Then there exists a measurable function $h$ with $|h(x)| = 1$, for all $x$ in $X$, such that $d\nu = h\, d|\nu|$.

## 1.3 Weighted Bergman Spaces on the Unit Ball

Let $\mathbb{C}^n$ denote the Euclidean space of complex dimension $n$. Componentwise operations of addition and scalar multiplication turn $\mathbb{C}^n$ into a vector space. For

$$z = (z_1, z_2, \cdots, z_n) \qquad \text{and} \qquad w = (w_1, w_2, \cdots, w_n)$$

in $\mathbb{C}^n$, we define the inner product of $z$ and $w$ by

$$\langle z, w \rangle = z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n,$$

where $\overline{w}_i$ is the complex conjugate of $w_i$. Thus, the norm of $z$ in $\mathbb{C}^n$ is

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}.$$

The space $\mathbb{C}^n$ equipped with the inner product above is a Hilbert space over $\mathbb{C}$.

Let $\mathbb{B}_n$ denote the open unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ in $\mathbb{C}^n$ and $\mathbb{S}_n$ denote its boundary $\{z \in \mathbb{C}^n : |z| = 1\}$, called the unit sphere in $\mathbb{C}^n$. We let $dV$ denote normalized volume Lebesgue measure on $\mathbb{C}^n$ such that $V(\mathbb{B}_n) = 1$. For $\alpha > -1$, we define

$$dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \, dV(z) \quad \text{with} \quad c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}, \tag{1.1}$$

where $\Gamma$ is the Euler gamma function, so that $V_\alpha(\mathbb{B}_n)$=1. For the following definitions, statements, and more details we will refer to [35] and [50].

For $1 \le p < \infty$, the **weighted Bergman space** $L^p_a(\mathbb{B}_n, dV_\alpha)$ is the set of all holomorphic functions $f$ on $\mathbb{B}_n$ such that

$$\|f\|_{p,\alpha} = \left( \int_{\mathbb{B}_n} |f(z)|^p \, dV_\alpha(z) \right)^{\frac{1}{p}} < \infty.$$

When $p = 2$, we simply write $\|f\|_\alpha$ instead of $\|f\|_{2,\alpha}$. We also let $L^p(\mathbb{B}_n, dV_\alpha)$ stand for the standard Lebesgue space on $\mathbb{B}_n$ with respect to the measure $V_\alpha$, and $L^\infty(\mathbb{B}_n, dV_\alpha)$ for the space of bounded functions on $\mathbb{B}_n$ with respect to $V_\alpha$.

By [50, Theorem 2.7], the weighted Bergman space $L^2_a(\mathbb{B}_n, dV_\alpha)$ is a reproducing kernel Hilbert space with the kernel $K^\alpha$ given by

$$K^\alpha(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.$$

We will denote the corresponding reproducing kernel function at $z$ by $K_z^\alpha$, and the normalized reproducing kernel function at $z$ by $k_z^\alpha$. That is,

$$K_z^\alpha(w) = \frac{1}{(1-\langle z, w\rangle)^{n+1+\alpha}} \qquad \text{and} \qquad k_z^\alpha(w) = \frac{(1-|z|^2)^{(n+1+\alpha)/2}}{(1-\langle z, w\rangle)^{n+1+\alpha}}. \tag{1.2}$$

For $0 \neq z \in \mathbb{B}_n$, $P_z$ will be the orthogonal projection onto the one dimensional subspace $\mathbb{C}z$ generated by $z$ and $Q_z = I - P_z$ its complementary projection onto $\mathbb{C}^n \ominus \mathbb{C}z$. We have that

$$P_z(w) = \frac{\langle w, z\rangle}{|z|^2}\, z, \qquad w \in \mathbb{C}^n,$$

and

$$Q_z(w) = w - \frac{\langle w, z\rangle}{|z|^2}\, z, \qquad w \in \mathbb{B}_n.$$

For $0 \neq z \in \mathbb{B}_n$, the mapping

$$\varphi_z(w) = \frac{z - P_z(w) - \sqrt{1-|z|^2}\, Q_z(w)}{1-\langle z, w\rangle},$$

represents the automorphism of $\mathbb{B}_n$, that interchanges the points 0 and $z$, [50, Lemma 1.2]. For $z = 0$, We define $\varphi_z(w) = -w$.

The following lemma from [50] gives two important properties of the automorphisms $\varphi_z$.

**Lemma 1.3.1.** **[50]** For any $z \in \mathbb{B}_n$, the mapping $\varphi_z$ satisfies

$$1 - \left|\varphi_z(w)\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z, w\rangle|^2}\,, \tag{1.3}$$

and

$$\varphi_z \circ \varphi_z = id. \tag{1.4}$$

Consequently, we have the interesting identity

$$\left|\langle k_z^\alpha, k_w^\alpha\rangle_\alpha\right| = \frac{1}{\left\|K^\alpha_{\varphi_z(w)}\right\|_\alpha}, \qquad \forall\, z, w \in \mathbb{B}_n, \tag{1.5}$$

and the change of variable formula

$$\int_{\mathbb{B}_n} f\Big(\varphi_z(w)\Big)\left|k_z^\alpha(w)\right|^2 dV_\alpha(w) = \int_{\mathbb{B}_n} f(\zeta)\, dV_\alpha(\zeta), \tag{1.6}$$

for all $f \in L^1(\mathbb{B}_n, dV_\alpha)$, which will be subsequently referred to in chapter 2.

Throughout this thesis, the measure $d\lambda$ will denote the **Möbius invariant measure** on $\mathbb{B}_n$,

$$d\lambda(z) = \frac{dV(z)}{(1-|z|^2)^{n+1}},$$

and $d\sigma$ will represent the normalized surface measure on $\mathbb{S}_n$, that is,

$$\sigma(E) = \frac{1}{v(\mathbb{B}_n)}\, v\Big(\Big\{tx : x \in E,\ t \in [0,1]\Big\}\Big),$$

for any Borel measurable subset $E$ of $\mathbb{S}_n$, where $v$ is the volume Lebesgue measure on $\mathbb{C}^n$ without normalization.

The **pseudo-hyperbolic and hyperbolic metrics** on $\mathbb{B}_n$ are defined by

$$\rho(z,w) = \Big|\varphi_z(w)\Big| \qquad \text{and} \qquad \beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|},$$

respectively. For $z \in \mathbb{B}_n$, we denote the **pseudo-hyperbolic ball** with radius $0 < r < 1$ centered at $z$ by

$$D_\rho(z,r) = \{w \in \mathbb{B}_n : \rho(z,w) < r\},$$

and the **hyperbolic ball** with radius $r > 0$ centred at $z$ by

$$D(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}.$$

Thus,

$$D(z,r) = \{w \in \mathbb{B}_n : \rho(z,w) < \tanh r\} = D_\rho(z, \tanh r),$$

and therefore we can easily represent any of these balls in the other form. In fact, the pseudo-hyperbolic and the hyperbolic balls are Euclidean balls such that

$$V_\alpha\Big(D(w,r)\Big) \approx V_\alpha\Big(D(z,r)\Big) \approx \Big(1-|z|^2\Big)^{n+1+\alpha} \quad \text{and} \quad ||K_w^\alpha||_\alpha \approx ||K_z^\alpha||_\alpha, \tag{1.7}$$

for any $w \in D(z,r)$, with the equivalence constants depending only on $r$. For details see [50, Lemma 1.24]. An interesting property of the pseudo-hyperbolic and the hyperbolic metrics is that they are Möbius invariant [50, Proposition 1.21 and Corollary 1.22].

**Proposition 1.3.2.** **[50]** *The pseudo-hyperbolic and the hyperbolic metrics are invariant under automorphism of $\mathbb{B}_n$, that is,*

$$\rho(\varphi(z), \varphi(w)) = \rho(z,w) \qquad \text{and} \qquad \beta(\varphi(z), \varphi(w)) = \beta(z,w),$$

*for all $z$, $w \in \mathbb{B}_n$, and $\varphi \in Aut(\mathbb{B}_n)$.*

The following covering lemma gives a decomposition of the open unit ball into hyperbolic balls, which is very helpful when we deal with integrals with respect to Carleson measures [50, Theorem 2.23].

**Theorem 1.3.3.** **[50]** *There is a positive integer M such that for any $0 < r \leq 1$, there exists a sequence $\{z_i\}_{i=1}^{\infty} \subseteq \mathbb{B}_n$ such that*

(1) $\mathbb{B}_n = \displaystyle\bigcup_{i=1}^{\infty} D(z_i, r)$;

(2) $D(z_i, r/4) \bigcap D(z_j, r/4) = \emptyset, \quad \forall\, i \neq j$;

(3) *Any point in $\mathbb{B}_n$ belongs to at most M of the hyperbolic balls $D(z_i, 2r)$.*

**Definition 1.3.4.** **[50]** A sequence $\{z_i\}_{i=1}^{\infty} \subseteq \mathbb{B}_n$ satisfying the conditions of Theorem 1.3.3 is called an *$r$-lattice on $\mathbb{B}_n$.*

The **Forelli-Rudin estimates** on the open unit ball and the sphere are our essential tools for estimating integrals in chapter 2. See [35, Proposition 1.4.10] or [50, Theorem 1.12].

**Theorem 1.3.5.** **[35]** Suppose $z \in \mathbb{B}_n$, c is real, and $t > -1$. Consider the integrals

$$\mathcal{I}_c(z) = \int_{\mathbb{S}_n} \frac{1}{|1 - \langle z, \zeta \rangle|^{n+c}} \, d\sigma(\zeta),$$

and

$$\mathcal{J}_{c,t}(z) = \int_{\mathbb{B}_n} \frac{(1 - |w|^2)^t}{|1 - \langle z, w \rangle|^{n+1+t+c}} \, dV(w).$$

(1) If $c < 0$, then $\mathcal{I}_c$ and $\mathcal{J}_{c,t}$ are both bounded in $\mathbb{B}_n$.

(2) If $c = 0$, then

$$\mathcal{I}_c(z) \approx \mathcal{J}_{c,t}(z) \approx \log \frac{1}{1 - |z|^2}$$

as $|z| \longrightarrow 1^-$.

(3) If $c > 0$, then

$$\mathcal{I}_c(z) \approx \mathcal{J}_{c,t}(z) \approx \frac{1}{(1 - |z|^2)^c}$$

as $|z| \longrightarrow 1^-$.

The notation $A(z) \approx B(z)$ as $|z| \longrightarrow 1^-$, means that the ratio $\dfrac{A(z)}{B(z)}$ has a positive finite limit as $|z| \longrightarrow 1^-$.

## 1.4 Large Bergman Spaces and Exponential-Type Weights

In this section, we state some definitions and theorems regarding large Bergman spaces and its subclass corresponding to the class of exponential-type weights $\mathcal{E}$. Throughout this section, we consider the open unit disk as the domain. This class of spaces is called large as it contains all Bergman spaces with standard weights. However, later in chapter 3, we will only work with a subclass of the large Bergman spaces, namely the class of spaces that corresponds to the class $\mathcal{E}$, which doesn't contain the Bergman spaces with standard weights.

Let $H(\mathbb{D})$ denote the space of holomorphic complex valued functions on the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. A **weight** is a non-negative function $\omega$ in $L^1(\mathbb{D}, dA)$, where $dA = \dfrac{dx\,dy}{\pi}$ is the normalized Lebesgue measure on the unit disk. We say that a weight $\omega$ is **radial** if $\omega(z) = \omega(|z|)$, for all $z \in \mathbb{D}$. We consider some specific classes of radial weights which will be used later. The following definitions are from [4], [20] and [6].

**Definition 1.4.1.** We say that a positive function $\tau$ defined on $\mathbb{D}$ belongs to the **class $\mathcal{L}$** if it satisfies the following two conditions:

(A) there exists $c_1 > 0$ such that $\tau(z) \leq c_1(1 - |z|)$ for all $z \in \mathbb{D}$.

(B) there exists $c_2 > 0$ such that $|\tau(z) - \tau(\zeta)| \leq c_2|z - \zeta|$ for all $z, \zeta \in \mathbb{D}$.

We let $D(\delta\tau(z))$ denote the Euclidean disk centered at $z \in \mathbb{D}$ with radius $\delta\tau(z)$ for $\delta \in (0, \mathfrak{M}_\tau)$, where

$$\mathfrak{M}_\tau = \frac{\min(1, c_1^{-1}, c_2^{-1})}{16}. \tag{1.8}$$

Thus, as shown in [29, Lemma 2.1], if $\tau \in \mathcal{L}$ and $\zeta \in D(\delta\tau(z))$ with $\delta \in (0, \mathfrak{M}_\tau)$, then by using the Lipschitz condition B,

$$\frac{1}{2}\tau(z) \leq \tau(\zeta) \leq 2\tau(z). \tag{1.9}$$

The above estimate will be used frequently in chapter 3.

**Definition 1.4.2.** Let $\Delta$ denote the classical Laplace operator. We say that a weight $\omega$ belongs to the **class** $\mathcal{L}^*$ if it is of the form $\omega = e^{-2\varphi}$, where $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$, and $(\Delta\varphi(z))^{-\frac{1}{2}} \asymp \tau(z)$, with $\tau$ being a function in the class $\mathcal{L}$.

The following examples can be found in [20], [29], and [4].

**Example 1.4.3.** The standard weights $\omega_\alpha(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$, are in the class $\mathcal{L}^*$with the associated strictly subharmonic functions

$$\varphi_\alpha(z) = -\frac{\alpha}{2}\log(1 - |z|^2).$$

We also have that $(\Delta\varphi_\alpha(z))^{-\frac{1}{2}} = \dfrac{1 - |z|^2}{\sqrt{2\alpha}}$. So, we let $\tau(z) = 1 - |z|^2$, which satisfies the conditions (A) and (B) in the definition of the class $\mathcal{L}$.

**Example 1.4.4.** For $\alpha \geq 0$, $\sigma > 0$, and $c > 0$, the weights

$$\omega_{\alpha,\sigma}(z) = (1-|z|^2)^{\alpha} \exp\left(\frac{-c}{(1-|z|^2)^{\sigma}}\right),$$

belong to the class $\mathcal{L}^*$with the associated strictly subharmonic functions

$$\varphi_{\alpha,\sigma}(z) = -\frac{\alpha}{2}\log(1-|z|^2) + \frac{c}{2}(1-|z|^2)^{-\sigma}.$$

Since $\Delta\varphi(z) \approx (1-|z|^2)^{-2-\sigma}$, we may let $\tau_{\sigma}(z) = (1-|z|^2)^{1+\frac{\sigma}{2}}$, which is in the class $\mathcal{L}$.

**Example 1.4.5.** For $\beta > 0$ and $\gamma > 1$, the weights

$$\omega_{\beta,\gamma}(z) = \exp\left(-\beta\left(\log\frac{e}{1-|z|}\right)^{\gamma}\right),$$

lie in the class $\mathcal{L}^*$with the corresponding strictly subharmonic functions

$$\varphi_{\beta,\gamma}(z) = \frac{\beta}{2}\left(\log\frac{e}{1-|z|}\right)^{\gamma}.$$

We have that

$$\Delta\varphi_{\beta,\gamma}(z) \approx (1-|z|)^{-2}\left(\log\frac{e}{1-|z|}\right)^{\gamma-1},$$

and hence, we may consider that

$$\tau_\gamma(z) = (1 - |z|)\Big(\log \frac{e}{1-|z|}\Big)^{\frac{1-\gamma}{2}}.$$

**Definition 1.4.6.** [6] The **class** $\mathcal{E}$ consists of the weights $\omega \in \mathcal{L}^*$ for which an associated function $\tau$ satisfies the following condition:

(C) For any $n \geq 1$, there exist constants $b_n > 0$ and $0 < t_n < \frac{1}{n}$ such that

$$\tau(z) \leq \tau(\zeta) + t_n|z-\zeta|, \quad \textit{for } |z-\zeta| > b_n\tau(\zeta).$$

**Example 1.4.7.** **[6]** $(i)$ The class $\mathcal{E}$ contains the family of exponential-type weights

$$\omega_\sigma(z) = \exp\left(\frac{-c}{(1-|z|^2)^\sigma}\right), \quad \sigma > 0,\, c > 0,$$

with the associated strictly subharmonic functions

$$\varphi_\sigma(z) = \frac{c}{2}(1-|z|^2)^{-\sigma},$$

and with $\tau_\sigma(z) = (1-|z|^2)^{1+\frac{\sigma}{2}}$, which is in the class $\mathcal{L}$. Note that this is a subclass of Example 1.4.4 with $\alpha = 0$. The inclusion of this family of rapidly decreasing weights in the class is the reason of choosing the name $\mathcal{E}$ (from exponential) for this class of weights.

$(ii)$ For $p > 0$, any radial weight $\omega$ in the class $\mathcal{E}$, and any non-vanishing holomorphic function $f$ in $L^p_a(\omega)$, the non-radial weight $\omega_{p,f}(z) = |f(z)|^p\,\omega(z)$, is in the class $\mathcal{E}$.

**Remark 1.4.8.** The standard weights $\omega_\alpha(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, are not in the class $\mathcal{E}$.

**Proof.** Suppose $\omega_\alpha \in \mathcal{E}$. So, an associated function $\tau(z) = k(1-|z|^2)$, satisfies the condition $C$, for some $k > 0$. Thus, for any $n \geq 1$, there exist constants $b_n > 0$ and $0 < t_n < \frac{1}{n}$ such that

$$\tau(z) \leq \tau(\zeta) + t_n|z-\zeta|, \quad \text{for } |z-\zeta| > b_n\tau(\zeta).$$

Fix $n \geq \dfrac{1}{k}$. Let $\zeta > 0$ be in $\mathbb{D}$, close enough to 1 such that $z = \zeta - (1+\dfrac{1}{n})b_n\tau(\zeta)$ is positive. Then,

$$|z-\zeta| = (1+\frac{1}{n})b_n\tau(\zeta) > b_n\tau(\zeta),$$

and hence, by the condition (C),

$$\tau(z) \leq \tau(\zeta) + t_n|z-\zeta|,$$

which implies that

$$\zeta^2 - z^2 \leq \frac{t_n}{k}|z-\zeta| < \frac{1}{nk}|z-\zeta| \leq |z-\zeta|.$$

Since $\zeta$ and $z$ are positive numbers, we have that $\zeta - z = |z-\zeta|$ and hence, we get that

$$\zeta + z < 1.$$

If $\zeta \longrightarrow 1^-$, then $\tau(\zeta) \longrightarrow 0$, and $z \longrightarrow 1^-$. Thus, we reach a contradiction, and therefore $\omega_\alpha$ is not in the class $\mathcal{E}$. $\square$

The following covering lemma due to Oleinik [27] gives a geometric decomposition of $\mathbb{D}$ in tems of $\tau \in \mathcal{L}$ and $\delta \in (0, \mathfrak{M}_\tau)$.

**Lemma 1.4.9.** **[27]** Let $\tau \in \mathcal{L}$ and $\delta \in (0, \mathfrak{M}_\tau)$. Then there exist a sequence $\{z_i\}_i \subseteq \mathbb{D}$, and a positive integer $N$ such that the following conditions are satisfied:

(1) $\mathbb{D} = \bigcup\limits_i D(\delta\tau(z_i))$;

(2) $z_i \notin D(\delta\tau(z_j)), \quad \forall\, i \neq j$;

(3) $\tilde{D}(\delta\tau(z_i)) \subseteq D(3\delta\tau(z_i))$, where $\tilde{D}(\delta\tau(z_i)) = \bigcup\limits_{z \in D(\delta\tau(z_i))} D(\delta\tau(z))$, for $i = 1, 2, \ldots$;

(4) Any point of $\mathbb{D}$ belongs to at most $N$ disks of the covering $\Big\{D(3\delta\tau(z_i))\Big\}_i$.

**Definition 1.4.10.** A sequence $\{z_i\}_i \subseteq \mathbb{D}$ satisfying the conditions of Lemma 1.4.9 is called a $(\delta, \tau)$-lattice on $\mathbb{D}$.

**Definition 1.4.11.** For $1 \leq p < \infty$, and any weight function $\omega$, the Lebesgue space $L^p(\omega) = L^p(\mathbb{D}, \omega dA)$ is the set of all complex valued functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{L^p(\omega)} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

and the corresponding weighted Bergman space is the set of all holomorphic functions in $L^p(\omega)$, that is, $L^p_a(\omega) = L^p(\omega) \cap H(\mathbb{D})$. When $p = 2$, we simply write $\|f\|_\omega$ instead of

$\|f\|_{L^2(\omega)}$.

The following lemma from [29] gives a generalized sub-mean value property of holomorphic functions. It yields that the Bergman space $L^p_a(\omega)$ is complete and the point evaluation functionals are bounded on $L^2_a(\omega)$.

**Lemma 1.4.12.** **[29]** *Let $\omega \in \mathcal{L}^*$, $1 \leq p < \infty$, and $\beta \in \mathbb{R}$. Then there exists a constant $C \geq 1$ such that*

$$|f(z)|^p\,\omega(z)^\beta \leq \frac{C}{\delta^2\tau(z)^2}\int_{D(\delta\tau(z))} |f(\zeta)|^p\,\omega(\zeta)^\beta\,dA(\zeta),$$

*for any $f \in H(\mathbb{D})$, any $z \in \mathbb{D}$, and any $\delta \in (0, \mathfrak{M}_\tau)$.*

Thus, the weighted Bergman space $L^2_a(\omega)$ equipped with the inner product

$$\langle f, g\rangle_\omega = \int_{\mathbb{D}} f(z)\,\overline{g(z)}\,\omega(z)\,dA(z),$$

is a reproducing kernel Hilbert space and $L^2_a(\omega)$ is a closed subspace of $L^2(\omega)$. We denote the corresponding kernel function at $z \in \mathbb{D}$ by $K^\omega_z$ and the normalized kernel at $z \in \mathbb{D}$ by $k^\omega_z$.

For $1 \leq p < \infty$, the projection $P_\omega : L^p(\omega) \longrightarrow L^p_a(\omega)$ is defined by

$$P_\omega f(z) = \int_{\mathbb{D}} f(\zeta)\,\overline{K^\omega_z(\zeta)}\,\omega(\zeta)\,dA(\zeta), \quad \forall z \in \mathbb{D}.$$

The operator $P_\omega$ is not necessarily bounded on $L^p(\omega)$. When $p = 2$, the operator $P_\omega$ is the orthogonal Bergman projection from $L^2(\omega)$ onto the closed subspace $L^2_a(\omega)$, and therefore, it is bounded.

**Definition 1.4.13.** [**6**] For a given weight $\omega$, we denote the set of all holomorphic functions $f$ on the unit disk with

$$\|f\|_{L^\infty_a(\omega^{1/2})} = \sup_{z\in\mathbb{D}} |f(z)|\,\omega(z)^{1/2} < \infty,$$

by $L^\infty_a(\omega^{1/2})$, and the set of all functions in $L^\infty_a(\omega^{1/2})$ for which

$$\lim_{|z|\to 1^-} |f(z)|\,\omega(z)^{1/2} = 0,$$

by $L^0_a(\omega^{1/2})$.

The following lemma from [6] gives a nice and important property of the reproducing kernels of $L^2_a(\omega)$, when the weight is in the class $\mathcal{E}$.

**Lemma 1.4.14.** [**6**] Let $\omega \in \mathcal{E}$ and $z \in \mathbb{D}$. Then $K^\omega_z \in L^0_a(\omega^{1/2})$.

The main motivation for choosing the class of weights $\mathcal{E}$ for the localization property on large Bergman spaces comes from the following two lemmas from [20] and [6], and Lemma 1.5.7 which will be presented later. Considering the fact that there are no representations for the reproducing kernel functions of $L^2_a(\omega)$, the following estimates for the reproducing kernel functions are our essential tools for estimating integrals in chapter 3. See [20, Lemma 3.5] and [6, Theorem 3.1].

**Lemma 1.4.15.** **[20]** *Let* $\omega \in \mathcal{E}$. *Then*

$$\|K_z^\omega\|_\omega \approx \tau(z)^{-1}\,\omega(z)^{-\frac{1}{2}}, \quad z \in \mathbb{D}.$$

**Lemma 1.4.16.** **[6]** *Let* $\omega \in \mathcal{E}$. *Then for each* $M \geq 1$, *there exists a constant* $C_M > 0$ *such that*

$$|K_z^\omega(\zeta)| \leq C_M\,\frac{1}{\tau(z)}\,\frac{1}{\tau(\zeta)}\,\omega(z)^{-\frac{1}{2}}\,\omega(\zeta)^{-\frac{1}{2}}\,\left(\frac{\min(\tau(z),\tau(\zeta))}{|z-\zeta|}\right)^M,$$

*for any* $z, \zeta \in \mathbb{D}$.

Combining Lemma 1.4.15 and Lemma 1.4.16, we obtain the following estimate.

**Lemma 1.4.17.** **[20]** **[6]** *Let* $\omega \in \mathcal{E}$. *Then for each* $M \geq 1$, *there exists a constant* $C'_M > 0$ *such that*

$$\left|\langle k_z^\omega, k_\zeta^\omega\rangle\right| \leq C'_M\,\left(\frac{\min(\tau(z),\tau(\zeta))}{|z-\zeta|}\right)^M,$$

*for any* $z, \zeta \in \mathbb{D}$.

The following estimate from [6] will be used in the proof of Lemma 1.5.7.

**Lemma 1.4.18.** **[6]** *Let* $\omega \in \mathcal{E}$, *and let* $K_z^\omega$ *be the reproducing kernel for* $L_a^2(\omega)$ *at* $z \in \mathbb{D}$.

Then there exists some constant $C > 0$ such that

$$\int_{\mathbb{D}} |K_z^{\omega}(\zeta)|\, \omega(\zeta)^{\frac{1}{2}}\, dA(\zeta) \leq C\, \omega(z)^{-\frac{1}{2}}.$$

Carleson measures for weighted Bergman spaces provide us with an important class of symbols of Toeplitz operators which will be defined later.

**Definition 1.4.19.** A finite positive Borel measure $\mu$ on $\mathbb{D}$ is said to be a Carleson measure for the Bergman space $L_a^2(\omega)$ if there exist a constant $C > 0$ such that

$$\int_{\mathbb{D}} \left|f(z)\right|^2 \omega(z)\, d\mu(z) \leq C \int_{\mathbb{D}} \left|f(z)\right|^2 \omega(z)\, dA(z),$$

for all $f \in L_a^2(\omega)$.

The following theorem gives equivalent characterizations of Carleson measures for $L_a^2(\omega)$ in terms of the disks $D(\delta\tau(z))$. See [20, Theorem 2.4] and [20, Theorem 2.8].

**Theorem 1.4.20.** **[20]** Let $\omega \in \mathcal{E}$ and let $\mu$ be a finite positive Borel measure on $\mathbb{D}$. Then the following conditions are equivalent:

(1) $\mu$ is a Carleson measure for the Bergman space $L_a^2(\omega)$.

(2) There exists some constant $\gamma \in (0, \mathfrak{M}_\tau)$ such that

$$\sup_{z\in\mathbb{D}} \frac{\mu\Big(D(\gamma\tau(z))\Big)}{\tau(z)^2} < \infty.$$

(3) There exists some constant $\delta \in (0, \mathfrak{M}_\tau)$ such that for every $(\delta, \tau)$-lattice $\Big\{D(\delta\tau(z_i))\Big\}_i$ on $\mathbb{D}$,

$$\sup_i \frac{\mu\Big(D(\delta\tau(z_i))\Big)}{\tau(z_i)^2} < \infty.$$

## 1.5 Toeplitz Operators

### 1.5.1 Toeplitz Operators on $L_a^2(\mathbb{B}_n, dV_\alpha)$

A significant class of operators on the classical Bergman spaces which has been broadly studied is the class of Toeplitz operators. In this section, we recall the definitions of Toeplitz operators with integrable symbols and Toeplitz opertors induced by complex Borel measures on the weighted Bergman spaces $L_a^2(\mathbb{B}_n, dV_\alpha)$, $\alpha > -1$. Most of what follows, and more, can be found in [50].

**Definition 1.5.1.** Let $\alpha > -1$ and f be a measurable function in $L^1(\mathbb{B}_n, dV_\alpha)$. The **Toeplitz operator** $T_f$ on $L_a^2(\mathbb{B}_n, dV_\alpha)$ is defined by

$$T_f\, h(z) = P_\alpha(fh)(z) = \int_{\mathbb{B}_n} \frac{f(w)\, h(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}}\, dV_\alpha(w), \quad h \in L_a^2(\mathbb{B}_n, dV_\alpha),$$

where $P_\alpha$ is the projection of $L^1(\mathbb{B}_n, dV_\alpha)$ onto $L_a^1(\mathbb{B}_n, dV_\alpha)$. Since $H^\infty$ is dense in $L_a^2(\mathbb{B}_n, dV_\alpha)$, the Toeplitz operator $T_f$ is densely defined on $L_a^2(\mathbb{B}_n, dV_\alpha)$.

It is not clear in general when the Toeplitz operator $T_f$ is bounded on $L_a^2(\mathbb{B}_n, dV_\alpha)$.

However, $T_f$ is bounded on $L_a^2(\mathbb{B}_n, dV_\alpha)$ whenever $f \in L^\infty(\mathbb{B}_n)$. There is also a characterization of bounded Toeplitz operators with $BMO$ symbols in terms of their Berezin transform, that is a generalization of the result in [52] for the classical Bergman space on the disk which will be presented later.

**Definition 1.5.2.** Let $\alpha > -1$ and f be a measurable function in $L^1(\mathbb{B}_n, dV_\alpha)$. The **Berezin transform** of $f$ is defined to be the Berezin transform of $T_f$, that is,

$$\widetilde{f}(z) = \widetilde{f}^\alpha(z) = \langle T_f k_z^\alpha, k_z^\alpha \rangle_\alpha = \int_{\mathbb{B}_n} f(w) \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2(n+1+\alpha)}} \, dV_\alpha(w).$$

Next, we will define Toeplitz operators induced by complex Borel measures.

**Definition 1.5.3.** For a complex Borel measure $\nu$ on $\mathbb{B}_n$, the **Toeplitz operator** $T_\nu$ on $L_a^2(\mathbb{B}_n, dV_\alpha)$ is defined by

$$T_\nu g(z) = \int_{\mathbb{B}_n} \frac{g(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}} \, d\nu(w), \quad g \in L_a^2(\mathbb{B}_n, dV_\alpha).$$

As before, the Toeplitz operator $T_\nu$ is densely defined on $L_a^2((\mathbb{B}_n, dV_\alpha)$, and it can be unbounded in general. It is clear that if $f$ is a measurable function in $L^1(\mathbb{B}_n, dV_\alpha)$, and if $d\nu = f\, dV_\alpha$, then $T_\nu = T_f$.

For positive measures, the boundedness of the corresponding Toeplitz operators is completely characterized in terms of Carleson measures. See, for example, [25, Lemma 2.2].

**Definition 1.5.4.** For $p \geq 1$, a finite positive Borel measure $\mu$ on $\mathbb{B}_n$ is said to be a **Carleson measure** for the Bergman space $L^p_a(\mathbb{B}_n, dV_\alpha)$ if $L^p_a(\mathbb{B}_n, dV_\alpha)$ is continuously embedded into $L^p(\mathbb{B}_n, d\mu)$; that is, if there exist a constant $C > 0$, independent of $f$, such that

$$\int_{\mathbb{B}_n} \left|f(z)\right|^p d\mu(z) \leq C \int_{\mathbb{B}_n} \left|f(z)\right|^p dV_\alpha(z),$$

for all $f \in L^p_a(\mathbb{B}_n, dV_\alpha)$.

We notice that the continuity condition of the embedding is redundant. In fact, by the closed graph theorem, if $L^p_a(\mathbb{B}_n, dV_\alpha) \subseteq L^p(\mathbb{B}_n, d\mu)$, then the embedding is automatically continuous. Thus, $\mu$ is a Carleson measure for $L^p_a(\mathbb{B}_n, dV_\alpha)$ if

$$\int_{\mathbb{B}_n} \left|f(z)\right|^p d\mu(z) < \infty, \quad \forall\, f \in L^p_a(\mathbb{B}_n, dV_\alpha).$$

Actually, the property of being a Carleson measure for $L^p_a(\mathbb{B}_n, dV_\alpha)$ is independent of $p$, and only depends on $\alpha$. The following theorem gives equivalent characterizations of Bergman Carleson measures on $\mathbb{B}_n$ in terms of hyperbolic balls. See [25, Lemma 2.2] and [50, Theorem 2.25].

**Theorem 1.5.5.** **[25]** **[50]** Suppose that $\mu$ is a finite positive Borel measure on $\mathbb{B}_n$ and $r > 0$. Then the following conditions are equivalent:

(1) $\mu$ is a Carleson measure for the Bergman space $L^2_a(\mathbb{B}_n, dV_\alpha)$.

(2) There exists a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2(n+1+\alpha)}}\, d\mu(w) \leq C,$$

for all $z \in \mathbb{B}_n$.

(3) There exists a constant $C_r > 0$ such that

$$\mu\Big(D(z,r)\Big) \leq C_r V_\alpha\Big(D(z,r)\Big),$$

for all $z \in \mathbb{B}_n$.

(4) There exists a constant $C_r > 0$ such that

$$\mu\Big(D(z_i,r)\Big) \leq C_r V_\alpha\Big(D(z_i,r)\Big),$$

for all $i \geq 1$, where $\{z_i\}_{i=1}^{\infty}$ is an $r$-lattice on $\mathbb{B}_n$.

(5) The Toeplitz operator $T_\mu$ is bounded on $L_a^2(\mathbb{B}_n, dV_\alpha)$.

Suppose that $\nu$ is a complex Borel measure such that its total variation $|\nu|$ is a Carleson measure for $L_a^2(\mathbb{B}_n, dV_\alpha)$. The real part and the imaginary part of $\nu$ are signed measures which may be written as the difference of two positive measures by the Jordan Decomposition.

Hence, $\nu$ may be decomposed as

$$\nu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4), \tag{1.10}$$

where each $\mu_j$ is a positive Borel measure and $|\nu| \approx \sum_{j=1}^{4} \mu_j$. So, by Theorem 1.5.5, each $\mu_j$ is a Carleson measure and $T_{\mu_j}$ is bounded on $L_a^2(\mathbb{B}_n, dV_\alpha)$. Therefore, the Toeplitz operator

$$T_\nu = (T_{\mu_1} - T_{\mu_2}) + i(T_{\mu_3} - T_{\mu_4})$$

is bounded on $L_a^2(\mathbb{B}_n, dV_\alpha)$.

### 1.5.2 Toeplitz Operators on $L_a^2(\omega)$

**Definition 1.5.6.** Let $\nu$ be a complex Borel measure on $\mathbb{D}$ and let $\omega \in \mathcal{E}$. The Toeplitz operator $T_\nu^\omega$ on $L_a^2(\omega)$ is defined by

$$T_\nu^\omega f(z) = \int_{\mathbb{D}} f(\zeta)\, \overline{K_z^\omega(\zeta)}\, \omega(\zeta)\, d\nu(\zeta), \quad \forall z \in \mathbb{D}.$$

Generally, there is no guarantee that the above integral converges. But for a positive Borel measure $\nu$ on $\mathbb{D}$, imposing the condition

$$\int_{\mathbb{D}} \left|K_z^\omega(\zeta)\right|^2 \omega(\zeta)\, d\nu(\zeta) < \infty,$$

on $\nu$ ensures that $T_\nu^\omega$ is well defined. See [4] and [6] for more details.

If $\nu$ is a complex Borel measure such that its total variation $|\nu|$ is a Carleson measure for $L_a^2(\omega)$, $\omega \in \mathcal{E}$, then

$$\int_{\mathbb{D}} \left|K_z^\omega(\zeta)\right|^2 \omega(\zeta)\, d|\nu|(\zeta) \lesssim \int_{\mathbb{D}} \left|K_z^\omega(\zeta)\right|^2 \omega(\zeta)\, dA(\zeta) < \infty,$$

since $K_z^\omega \in L_a^2(\omega)$, and hence, $T_{|\nu|}^\omega$ is well defined and bounded. This immediately implies that $T_\nu^\omega$ is also well defined. By the Jordan decomposition $\nu$ may be decomposed as

$$\nu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4),$$

where each $\mu_i$ is a positive Carleson measure. Since $T_\nu^\omega$ is a linear combination of four Toeplitz operators with positive Carleson measure symbols, it is a bounded operator.

Combining a few of the results from [6] stated in section 1.4, and some ideas from the proof of a similar statement in [4] for another class of weights, we have the following.

**Lemma 1.5.7.** Let $\omega \in \mathcal{E}$ and $\nu$ be a complex Borel measure on $\mathbb{D}$ such that $|\nu|$ is a Carleson measure for $L_a^2(\omega)$. Then

$$\langle T_\nu^\omega f, g\rangle_\omega = \int_{\mathbb{D}} f(\xi)\, \overline{g(\xi)}\, \omega(\xi)\, d\nu(\xi), \tag{1.11}$$

for all $f, g \in L_a^2(\omega)$.

**Proof.** By the Jordan decomposition $\nu$ may be decomposed as

$$\nu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4),$$

where each $\mu_i$ is a positive Carleson measure. So, it suffices to show that (1.11) holds for positive Carleson measures $\mu$.

Let $\mu$ be a positive Carleson measure for $L^2_a(\omega)$. Suppose $f$ is in $L^2_a(\omega)$ and $g$ is in $L^\infty_a(\omega^{1/2})$. Then by Lemma 1.4.18, the Cauchy–Schwarz inequality, and using the assumption that $\mu$ is a Carleson measure for $L^2_a(\omega)$, we have that

$$\begin{aligned}
I &= \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |g(z)|\, |K_z^\omega(\xi)|\, \omega(z)\, dA(z) \right) |f(\xi)|\, \omega(\xi)\, d\mu(\xi) \\
&\leq \|g\|_{L^\infty_a(\omega^{1/2})} \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |K_z^\omega(\xi)|\, \omega(z)^{\frac{1}{2}}\, dA(z) \right) |f(\xi)|\, \omega(\xi)\, d\mu(\xi) \\
&\lesssim \|g\|_{L^\infty_a(\omega^{1/2})} \int_{\mathbb{D}} |f(\xi)|\, \omega(\xi)^{\frac{1}{2}}\, d\mu(\xi) \\
&\lesssim \|g\|_{L^\infty_a(\omega^{1/2})} \left( \int_{\mathbb{D}} |f(\xi)|^2\, \omega(\xi)\, d\mu(\xi) \right)^{\frac{1}{2}} \\
&\lesssim \|g\|_{L^\infty_a(\omega^{1/2})} \|f\|_\omega < \infty.
\end{aligned}$$

Hence, by Fubini's theorem, we get that

$$\begin{aligned}\langle T_\mu^\omega f, g\rangle_\omega &= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} f(\xi)\,\overline{K_z^\omega(\xi)}\,\omega(\xi)\,d\mu(\xi)\right)\overline{g(z)}\,\omega(z)\,dA(z)\\ &= \int_{\mathbb{D}} f(\xi)\left(\int_{\mathbb{D}} \overline{g(z)}\,K_\xi^\omega(z)\,\omega(z)\,dA(z)\right)\,\omega(\xi)\,d\mu(\xi)\\ &= \int_{\mathbb{D}} f(\xi)\,\overline{g(\xi)}\,\omega(\xi)\,d\mu(\xi).\end{aligned}$$

By Lemma 1.4.14, $K_z^\omega \in L_a^0(\omega^{1/2}) \subseteq L_a^\infty(\omega^{1/2})$, for all $z \in \mathbb{D}$. Since the set of finite linear combinations of the reproducing kernels is dense in $L_a^2(\omega)$, the set $L_a^\infty(\omega^{1/2})$ is dense in $L_a^2(\omega)$, and therefore, we conclude that (1.11) holds for $\mu$. $\square$

# Chapter 2

# Localization on the Weighted Bergman Spaces of the Unit Ball

## 2.1 Weak localization on $L_a^2(\mathbb{B}_n, dV_\alpha)$

In [15], the notions of weak localization on Bergman and Fock spaces were introduced by Isralowitz, Mitkovski, and Wick. They proved that the Toeplitz operators with bounded symbols are weakly localized and therefore the Toeplitz algebra is contained in the $C^*$-algebra generated by weakly localized operators. In [42], Xia established that these two algebras are actually the same, that is, the $C^*$-algebra generated by weakly localized operators is equal to the Toeplitz algebra.

In this chapter, we show that also some large classes of Toeplitz operators containing Toeplitz operators with bounded symbols are included in the class of weakly localized operators. To present our work in a more general context, we first extend the notion of weak localization, introduced in [15], to the weighted Bergman spaces of the open unit ball $L^2_a(\mathbb{B}_n, dV_\alpha)$. It is mentioned in [15] that this extension is possible and is left to the reader, but without the details given. This chapter is based on the same results that hold in the case of the Bergman space on the unit ball, published as a joint paper with my supervisor in [37]. The results here are extensions of those results, and follow the methods and the ideas used in that paper. We prove that a Toeplitz operator with complex Borel measure symbol, whose total variation is Carleson, is weakly localized on $L^2_a(\mathbb{B}_n, dV_\alpha)$. We define strongly localized and sufficiently localized operators on $L^2_a(\mathbb{B}_n, dV_\alpha)$, and show that they are also weakly localized. Then we show that bounded Toeplitz operators with $BMO^1$ symbols are strongly (and therefore also weakly) localized.

**Definition 2.1.1.** Let $\alpha > -1$ and $\frac{n-1-\alpha}{n+1+\alpha} < a < 1$. We say that a bounded linear operator $T$ on $L^2_a(\mathbb{B}_n, dV_\alpha)$ is $a, \alpha$-**weakly localized** if the following conditions are satisfied:

$$(i) \sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\|K_z^\alpha\|_\alpha^a}{\|K_w^\alpha\|_\alpha^a}\, d\lambda(w) < \infty, \ \ and \tag{2.1}$$

$$\sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} \left|\langle T^*k_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\|K_z^\alpha\|_\alpha^a}{\|K_w^\alpha\|_\alpha^a}\, d\lambda(w) < \infty. \tag{2.2}$$

$$(ii) \lim_{r\to\infty} \sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\|K_z^\alpha\|_\alpha^a}{\|K_w^\alpha\|_\alpha^a}\, d\lambda(w) = 0, \ \ and \tag{2.3}$$

$$\lim_{r\to\infty} \sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} \left|\langle T^*k_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\|K_z^\alpha\|_\alpha^a}{\|K_w^\alpha\|_\alpha^a}\, d\lambda(w) = 0, \tag{2.4}$$

where $d\lambda$ is the Möbius invariant measure on $\mathbb{B}_n$,

$$d\lambda(z) = \frac{dV(z)}{(1-|z|^2)^{n+1}}.$$

As mentioned before, we have that

$$D(z,r) = \{w \in \mathbb{B}_n : \rho(z,w) < \tanh r\} = D_\rho(z, \tanh r),$$

and therefore, the statements of part $(ii)$ can also be given in terms of complements of pseudo-hyperbolic balls $D_\rho(z,r)$. In that case we take the limits as $r \to 1^-$.

In this section, we will show that the Toeplitz operators induced by a complex measure with Carleson total variation, are $a, \alpha$-weakly localized on $L_a^2(\mathbb{B}_n, dV_\alpha)$ for any $\frac{n-1-\alpha}{n+1+\alpha} < a < 1$. This is an extension of the non-weighted Bergman space case, published in [37].

We first provide a weighted version of [15, Lemma 2.1].

**Lemma 2.1.2.** Let $\alpha > -1$ and $\frac{n-1-\alpha}{n+1+\alpha} < a < 1$. Then

$$\lim_{r\to 1^-} \sup_{z\in\mathbb{B}_n} \int_{D_\rho(z,r)^c} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, dV_\alpha(w) = 0.$$

**Proof.** By the change of variable $w = \varphi_z(\xi)$ and identity (1.5), we have that

$$\begin{aligned}
I(z,r,a) &= \int_{D_\rho(z,r)^c} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, dV_\alpha(w) \\
&= \int_{D_\rho(0,r)^c} \left|K_z^\alpha(\varphi_z(\xi))\right| \frac{\left\|K_{\varphi_z(\xi)}^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \left|k_z^\alpha(\xi)\right|^2 dV_\alpha(\xi) \\
&= \int_{D_\rho(0,r)^c} \left|\langle k_z^\alpha, k_{\varphi_z(\xi)}^\alpha\rangle_\alpha\right| \frac{\left\|K_{\varphi_z(\xi)}^\alpha\right\|_\alpha^{2-a}}{\left\|K_z^\alpha\right\|_\alpha^{-a}} \left|k_z^\alpha(\xi)\right|^2 dV_\alpha(\xi) \\
&= \int_{D_\rho(0,r)^c} \left\|K_\xi^\alpha\right\|_\alpha^{-1} \left|\langle k_z^\alpha, k_\xi^\alpha\rangle_\alpha\right|^{a-2} \left\|K_z^\alpha\right\|_\alpha^{a} \left|k_z^\alpha(\xi)\right|^2 dV_\alpha(\xi) \\
&= \int_{D_\rho(0,r)^c} \left\|K_\xi^\alpha\right\|_\alpha \left|\langle k_z^\alpha, k_\xi^\alpha\rangle_\alpha\right|^{a} \left\|K_z^\alpha\right\|_\alpha^{a} dV_\alpha(\xi) \\
&= \int_{D_\rho(0,r)^c} \frac{\left(1-|\xi|^2\right)^{\frac{n+1+\alpha}{2}(a-1)+\alpha}}{\left|1-\langle z,\xi\rangle\right|^{(n+1+\alpha)a}}\, c_\alpha\, dV(\xi),
\end{aligned}$$

and since $D_\rho(0,r)$ is equal to the Euclidean ball of radius $r$ centered at the origin, by integration in polar coordinates, we see that

$$\begin{aligned} I(z,r,a) &= c_\alpha \int_r^1 \int_{\mathbb{S}_n} \frac{\left(1-R^2\right)^{\frac{n+1+\alpha}{2}(a-1)+\alpha}}{\left|1-\langle Rz,\zeta\rangle\right|^{(n+1+\alpha)a}} R^{2n-1} d\sigma(\zeta)\, dR \\ &= c_\alpha \int_r^1 \mathcal{I}_{(n+1+\alpha)a-n}(Rz) \frac{R^{2n-1}}{\left(1-R^2\right)^{\frac{n+1+\alpha}{2}(1-a)-\alpha}}\, dR, \end{aligned}$$

where

$$\mathcal{I}_c(z) = \int_{\mathbb{S}_n} \frac{1}{|1-\langle z,\zeta\rangle|^{n+c}}\, d\sigma(\zeta).$$

By Forelli-Rudin estimates, we get that

$$\mathcal{I}_{(n+1+\alpha)a-n}(Rz)) \lesssim \begin{cases} 1 & if \quad (n+1+\alpha)a-n<0 \\ \log \frac{1}{1-|Rz|^2} & if \quad (n+1+\alpha)a-n=0 \\ \frac{1}{(1-|Rz|^2)^{(n+1+\alpha)a-n}} & if \quad (n+1+\alpha)a-n>0, \end{cases}$$

and therefore, we have that

$$I(z,r,a) \lesssim \begin{cases} \int_r^1 \frac{R^{2n-1}}{\left(1-R^2\right)^{\frac{n+1+\alpha}{2}(1-a)-\alpha}}\, dR & if \quad \frac{n-1-\alpha}{n+1+\alpha} < a < \frac{n}{n+1+\alpha} \\ \\ \int_r^1 \log\left(\frac{1}{1-R^2}\right) \frac{R^{2n-1}}{\left(1-R^2\right)^{\frac{n+1+\alpha}{2}(1-a)-\alpha}}\, dR & if \quad a = \frac{n}{n+1+\alpha} \\ \\ \int_r^1 \frac{R^{2n-1}}{(1-R^2)^{(\frac{n+1+\alpha}{2})a-\frac{n-1+\alpha}{2}}}\, dR & if \quad \frac{n}{n+1+\alpha} < a < 1. \end{cases}$$

Using change of variables and integration by parts, we see that these three functions are integrable over $(0,1)$ as $\frac{n-1-\alpha}{n+1+\alpha} < a < 1$. By taking supremum over $z \in \mathbb{B}_n$ and then letting $r \to 1^-$, we get that

$$\lim_{r\to 1^-} \sup_{z\in\mathbb{B}_n} \int_{D_\rho(z,r)^c} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, dV_\alpha(w) = 0. \qquad \square$$

The proof of the main theorem of this section relies on the following lemma. The general idea for the proof of the lemma is based on the geometric decomposition of $\mathbb{B}_n$ into hyperbolic balls, Theorem 1.3.3 and the characterization of Carleson measures in terms of hyperbolic balls, Theorem 1.5.5.

**Lemma 2.1.3.** Suppose that $\alpha > -1$ and $\frac{n-1-\alpha}{n+1+\alpha} < a < 1$. If $\mu$ is a Carleson measure for $L_a^2(\mathbb{B}_n, dV_\alpha)$, then there exists a constant $C > 0$ such that

$$(a) \int_{D(z,r)^c} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \, d\mu(w) \le C \int_{D(z,\frac{r}{4})^c} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \, dV_\alpha(w),$$

for all $z \in \mathbb{B}_n$, and for all $r > 1$; and

$$(b) \int_{\mathbb{B}_n} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \, d\mu(w) \le C \int_{\mathbb{B}_n} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \, dV_\alpha(w),$$

for all $z \in \mathbb{B}_n$.

**Proof.** Fix $s \le \frac{1}{4}$, and suppose $\{z_i\}_{i=1}^\infty$ is an $s$-lattice on $\mathbb{B}_n$ coming from Theorem 1.3.3. Hence, we have that

$$\mathbb{B}_n = \bigcup_{i=1}^\infty D(z_i, s); \qquad D(z_i, s/4) \bigcap D(z_j, s/4) = \emptyset, \quad \forall\, i \ne j, \tag{2.5}$$

and that each point of $\mathbb{B}_n$ belongs to at most $M$ hyperbolic balls $D(z_i, 2s)$. For $w \in D(z_i, s)$, we have that $D(w, s) \subseteq D(z_i, 2s)$, and that

$$\left(1 - |w|^2\right)^{n+1+\alpha} \approx V_\alpha\Big(D(w,s)\Big) \approx V_\alpha\Big(D(z_i,s)\Big) \approx \left(1 - |z_i|^2\right)^{n+1+\alpha},$$

by (1.7). Hence, since $|K_z^\alpha|$ is subharmonic, for any $i \in \mathbb{N}$ and any $w \in D(z_i, s)$, we get

$$\left|K_z^\alpha(w)\right| \le \frac{1}{V\Big(D(w,s)\Big)} \int_{D(w,s)} \left|K_z^\alpha(\zeta)\right| dV(\zeta)$$

$$
\begin{aligned}
&\lesssim \frac{1}{V\Big(D(z_i,s)\Big)} \int_{D(w,s)} \Big|K_z^\alpha(\zeta)\Big|\, dV(\zeta)\\
&\lesssim \frac{1}{(1-|z_i|^2)^{n+1}} \int_{D(w,s)} \Big|K_z^\alpha(\zeta)\Big|\, dV(\zeta)\\
&\lesssim \frac{1}{(1-|z_i|^2)^{n+1+\alpha}} \int_{D(w,s)} \Big|K_z^\alpha(\zeta)\Big|(1-|\zeta|^2)^\alpha\, dV(\zeta)\\
&\lesssim \frac{1}{V_\alpha\Big(D(z_i,s)\Big)} \int_{D(z_i,2s)} \Big|K_z^\alpha(\zeta)\Big|\, dV_\alpha(\zeta),
\end{aligned}
$$

where the fourth inequality comes from

$$
(1-|z_i|^2)^\alpha \approx (1-|w|^2)^\alpha \approx (1-|\zeta|^2)^\alpha, \quad \forall\, w \in D(z_i,s), \forall\, \zeta \in D(w,s),
$$

which holds by (1.7). Since we also have that

$$
\Big\|K_w^\alpha\Big\|_\alpha \approx \Big\|K_{z_i}^\alpha\Big\|_\alpha \quad \textit{and} \quad \Big\|K_{z_i}^\alpha\Big\|_\alpha \approx \Big\|K_\zeta^\alpha\Big\|_\alpha, \quad \forall\, w \in D(z_i,s), \forall\, \zeta \in D(z_i,2s),
$$

we get that

$$
\begin{aligned}
\Big|K_z^\alpha(w)\Big| \frac{\Big\|K_w^\alpha\Big\|_\alpha^{1-a}}{\Big\|K_z^\alpha\Big\|_\alpha^{1-a}} &\lesssim \frac{1}{V_\alpha\Big(D(z_i,s)\Big)} \int_{D(z_i,2s)} \Big|K_z^\alpha(\zeta)\Big| \frac{\Big\|K_w^\alpha\Big\|_\alpha^{1-a}}{\Big\|K_z^\alpha\Big\|_\alpha^{1-a}}\, dV_\alpha(\zeta)\\
&\lesssim \frac{1}{V_\alpha\Big(D(z_i,s)\Big)} \int_{D(z_i,2s)} \Big|K_z^\alpha(\zeta)\Big| \frac{\Big\|K_\zeta^\alpha\Big\|_\alpha^{1-a}}{\Big\|K_z^\alpha\Big\|_\alpha^{1-a}}\, dV_\alpha(\zeta),
\end{aligned}
$$

for any $w \in D(z_i, s)$. If for $z \in \mathbb{B}_n$ and $i \in \mathbb{N}$, we define

$$S_i(z,s) = \sup_{w \in D(z_i,s)} \left| K_z^\alpha(w) \right| \frac{\left\| K_w^\alpha \right\|_\alpha^{1-a}}{\left\| K_z^\alpha \right\|_\alpha^{1-a}},$$

then we can rewrite the inequality obtained above as

$$S_i(z,s)\, V_\alpha\Big(D(z_i,s)\Big) \lesssim \int_{D(z_i,2s)} \left| K_z^\alpha(\zeta) \right| \frac{\left\| K_\zeta^\alpha \right\|_\alpha^{1-a}}{\| K_z^\alpha \|_\alpha^{1-a}}\, dV_\alpha(\zeta), \quad \forall\, i \in \mathbb{N}. \tag{2.6}$$

$(a)$ Let $z \in \mathbb{B}_n$, and $r > 1$ be arbitrary. Thus, we have that $s \le \frac{1}{4} < \frac{r}{4}$. We define the family

$$J_{z,r} = \Big\{ j \in \mathbb{N} \,:\, D(z_j,s) \bigcap D(z,r)^c \ne \emptyset \Big\}.$$

Since $\{z_i\}_{i=1}^\infty$ is an $s$-lattice on $\mathbb{B}_n$, and by the definition of $J_{z,r}$, we have that

$$D(z,r)^c \subseteq \bigcup_{j \in J_{z,r}} D(z_j,s); \quad D(z_j,s/4) \bigcap D(z_k,s/4) = \emptyset, \quad \forall j,k \in J_{z,r}\ (j \ne k), \tag{2.7}$$

and that each point of $\mathbb{B}_n$ belongs to at most $M$ hyperbolic balls $D(z_j,2s)$, $j \in J_{z,r}$, for some positive integer $M$, independent of $z$ and $r$, coming from Theorem 1.3.3.

Since $3s < \frac{3r}{4}$, and since the hyperbolic distance from each $z_j$ to $D(z,r)^c$ is less than $s$, we have that the hyperbolic distance from $\bigcup_{j \in J_{z,r}} D(z_j,2s)$ to $z$ is greater than $r - 3s > \frac{r}{4}$. Thus

$$\bigcup_{j \in J_{z,r}} D(z_j,2s) \subseteq D(z,r/4)^c. \tag{2.8}$$

Thus, since $\mu$ is a Carleson measure, and by (2.6), (2.7), (2.8) and Theorem 1.5.5,

$$\begin{aligned}
\int_{D(z,r)^c} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, d\mu(w) &\leq \int_{\bigcup_{j\in J_{z,r}} D(z_j,s)} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, d\mu(w) \\
&\leq \sum_{j\in J_{z,r}} \int_{D(z_j,s)} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, d\mu(w) \\
&\leq \sum_{j\in J_{z,r}} S_j(z,s)\, \mu\Big(D(z_j,s)\Big) \\
&= \sum_{j\in J_{z,r}} S_j(z,s)\, \frac{\mu\Big(D(z_j,s)\Big)}{V_\alpha\Big(D(z_j,s)\Big)}\, V_\alpha\Big(D(z_j,s)\Big) \\
&\lesssim \sum_{j\in J_{z,r}} S_j(z,s)\, V_\alpha\Big(D(z_j,s)\Big) \\
&\lesssim \sum_{j\in J_{z,r}} \int_{D(z_j,2s)} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, dV_\alpha(\zeta) \\
&\lesssim \int_{D(z,\frac{r}{4})^c} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, dV_\alpha(\zeta).
\end{aligned}$$

The last inequality comes from the fact that each $\zeta \in \mathbb{B}_n$ belongs to at most $M$ of the hyperbolic balls $D(z_j, 2s)$.

($b$) Since $\mu$ is a Carleson measure, and by (2.5), (2.6) and Theorem 1.5.5,

$$\int_{\mathbb{B}_n} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, d\mu(w) = \int_{\bigcup_{i=1}^\infty D(z_i,s)} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, d\mu(w)$$

$$
\begin{aligned}
&\le \sum_{i=1}^{\infty} \int_{D(z_i,s)} \left|K_z^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \, d\mu(w) \\
&\le \sum_{i=1}^{\infty} S_i(z,s)\, \mu\Big(D(z_i,s)\Big) \\
&= \sum_{i=1}^{\infty} S_i(z,s)\, \frac{\mu\Big(D(z_i,s)\Big)}{V_\alpha\Big(D(z_i,s)\Big)}\, V_\alpha\Big(D(z_i,s)\Big) \\
&\lesssim \sum_{i=1}^{\infty} S_i(z,s)\, V_\alpha\Big(D(z_i,s)\Big) \\
&\lesssim \sum_{i=1}^{\infty} \int_{D(z_i,2s)} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \, dV_\alpha(\zeta) \\
&\lesssim \int_{\mathbb{B}_n} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \, dV_\alpha(\zeta).
\end{aligned}
$$

The last inequality holds since each $\zeta \in \mathbb{B}_n$ belongs to at most $M$ of the hyperbolic balls $D(z_i, 2s)$. $\square$

Before presenting the main theorem of this section, we state a version of the Forelli-Rudin estimate on $\mathbb{B}_n$ in terms of kernel functions of the weighted Bergman space $L_a^2(\mathbb{B}_n, dV_\alpha)$.

For $\alpha > -1$ and $\frac{n-1-\alpha}{n+1+\alpha} < a < 1$, it is easy to check that

$$
t = \frac{-n-1+\alpha}{2} + \Big(\frac{n+1+\alpha}{2}\Big)a > -1, \qquad c = \Big(\frac{n+1+\alpha}{2}\Big)(1-a) > 0,
$$

and $n+1+t+c = n+1+\alpha$. Therefore by part (3) of Theorem 1.3.5, we get that

$$\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{\left|K_z^\alpha(w)\right|}{\left\|K_z^\alpha\right\|_\alpha^{1-a}\left\|K_w^\alpha\right\|_\alpha^{1+a}}\,d\lambda(w) = c_\alpha^{-1}\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n}\left|K_z^\alpha(w)\right|\frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\,dV_\alpha(w) < \infty. \tag{2.9}$$

This implies that the integrals appearing in Lemma 2.1.3 are finite. We will apply this form of the Forelli-Rudin estimate on $\mathbb{B}_n$ in our main theorem, presented in the following subsection.

### 2.1.1 Weak Localization and Toeplitz operators with Complex Measure Symbols on $L_a^2(\mathbb{B}_n, dV_\alpha)$

In this subsection we present the main theorem of this chapter. As mentioned before, this is the extension of one of the results of a joint paper with my supervisor [37].

**Theorem 2.1.4.** Let $\alpha > -1$ and $\nu$ be a complex Borel measure on $\mathbb{B}_n$ such that $|\nu|$ is a Carleson measure for $L_a^2(\mathbb{B}_n, dV_\alpha)$. Then the Toeplitz operator $T_\nu$ is $a,\alpha$-weakly localized on the Bergman space $L_a^2(\mathbb{B}_n, dV_\alpha)$, for any $\frac{n-1-\alpha}{n+1+\alpha} < a < 1$.

**Proof.** Since $|\nu|$ is a Carleson measure, the Toeplitz operator $T_\nu$ is bounded on the Bergman space $L_a^2(\mathbb{B}_n, dV_\alpha)$. Let $\frac{n-1-\alpha}{n+1+\alpha} < a < 1$ be arbitrary and fixed. By Fubini's theorem, and Theorem 1.2.3, we have that

$$\left|\langle T_\nu k_z^\alpha, k_w^\alpha\rangle_\alpha\right| = \left|\int_{\mathbb{B}_n} k_z^\alpha(\zeta)\,\overline{k_w^\alpha(x)}\,d\nu(\zeta)\right| \le \int_{\mathbb{B}_n}\left|k_z^\alpha(\zeta)\,k_w^\alpha(\zeta)\right|d|\nu|(\zeta).$$

Hence, by Fubini's theorem, the Forelli-Rudin estimate (2.9), and part ($b$) of Lemma 2.1.3, we get that

$$\begin{aligned}
\int_{\mathbb{B}_n} \left|\langle T_\nu k_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d\lambda(w) &\le \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \left|k_z^\alpha(\zeta)\, k_w^\alpha(\zeta)\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d|\nu|(\zeta)\, d\lambda(w) \\
&= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \left|k_z^\alpha(\zeta)\, k_w^\alpha(\zeta)\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d\lambda(w)\, d|\nu|(\zeta) \\
&= \int_{\mathbb{B}_n} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \int_{\mathbb{B}_n} \frac{\left|K_\zeta^\alpha(w)\right|}{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}\left\|K_w^\alpha\right\|_\alpha^{1+a}}\, d\lambda(w)\, d|\nu|(\zeta) \\
&\le \left(\sup_{\zeta\in\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{\left|K_\zeta^\alpha(w)\right|}{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}\left\|K_w^\alpha\right\|_\alpha^{1+a}}\, d\lambda(w)\right)\int_{\mathbb{B}_n} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, d|\nu|(\zeta) \\
&\lesssim \int_{\mathbb{B}_n} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\|K_z^\alpha\|_\alpha^{1-a}}\, dV_\alpha(\zeta).
\end{aligned}$$

Invoking the Forelli-Rudin estimate (2.9) one more time, we get that

$$\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n} \left|\langle T_\nu k_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d\lambda(w) < \infty.$$

Since $T_\nu^* = T_{\overline{\nu}}$, and $|\nu| = |\overline{\nu}|$, replacing $T_\nu$ by $T_{\overline{\nu}}$ in the proof above, we get that

$$\sup_{z\in\mathbb{B}_n}\int_{\mathbb{B}_n} \left|\langle T_\nu^* k_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d\lambda(w) < \infty.$$

Thus, we have shown that conditions (2.1) and (2.2) from the definition of weak localization hold for the operator $T_\nu$. Next, we prove that that conditions (2.3) and (2.4) also hold for the operator $T_\nu$. By Fubini's theorem, we have that

$$\begin{aligned}
\int_{D(z,r)^c} \left|\langle T_\nu k_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d\lambda(w) &\le \int_{D(z,r)^c}\int_{\mathbb{B}_n} \left|k_z^\alpha(\zeta)\, k_w^\alpha(\zeta)\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d|\nu|(\zeta)\, d\lambda(w) \\
&= \int_{\mathbb{B}_n}\int_{D(z,r)^c} \left|k_z^\alpha(\zeta)\, k_w^\alpha(\zeta)\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d\lambda(w)\, d|\nu|(\zeta) \\
&= I_1(z,r) + I_2(z,r),
\end{aligned}$$

where

$$I_1(z,r) = \int_{D(z,\frac{r}{2})^c}\int_{D(z,r)^c} \left|k_z^\alpha(\zeta)\, k_w^\alpha(\zeta)\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d\lambda(w)\, d|\nu|(\zeta),$$

and

$$I_2(z,r) = \int_{D(z,\frac{r}{2})}\int_{D(z,r)^c} \left|k_z^\alpha(\zeta)\, k_w^\alpha(\zeta)\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d\lambda(w)\, d|\nu|(\zeta).$$

By the Forelli-Rudin estimate (2.9) and part $(a)$ of Lemma 2.1.3, we see that

$$\begin{aligned}
I_1(z,r) &\le \int_{D(z,\frac{r}{2})^c}\int_{\mathbb{B}_n} \left|k_z^\alpha(\zeta)\, k_w^\alpha(\zeta)\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\, d\lambda(w)\, d|\nu|(\zeta) \\
&= \int_{D(z,\frac{r}{2})^c} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \int_{\mathbb{B}_n} \frac{\left|K_\zeta^\alpha(w)\right|}{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}\left\|K_w^\alpha\right\|_\alpha^{1+a}}\, d\lambda(w)\, d|\nu|(\zeta) \\
&\lesssim \int_{D(z,\frac{r}{2})^c} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}}\, d|\nu|(\zeta)
\end{aligned}$$

$$\lesssim \int_{D(z,\frac{r}{8})^c} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\|K_z^\alpha\|_\alpha^{1-a}} \, dV_\alpha(\zeta).$$

So, by Lemma 2.1.2, we get that

$$\lim_{r\to\infty} \sup_{z\in\mathbb{B}_n} I_1(z,r) = 0.$$

To find an upper bound for $I_2(z,r)$, we first notice that

$$D(\zeta, \frac{r}{2}) \subseteq D(z,r), \quad \forall\, \zeta \in D(z, \frac{r}{2}).$$

Thus,

$$\begin{aligned}
I_2(z,r) &\leq \int_{D(z,\frac{r}{2})} \int_{D(\zeta,\frac{r}{2})^c} \left|k_z^\alpha(\zeta)\, k_w^\alpha(\zeta)\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a} \, d\lambda(w)\, d|\nu|(\zeta) \\
&= \int_{D(z,\frac{r}{2})} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \int_{D(\zeta,\frac{r}{2})^c} \frac{\left|K_w^\alpha(\zeta)\right|}{\left\|K_w^\alpha\right\|_\alpha^{1+a}\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}} \, d\lambda(w)\, d|\nu|(\zeta) \\
&= \int_{D(z,\frac{r}{2})} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \int_{D(\zeta,\frac{r}{2})^c} \left|K_\zeta^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}} \, c_\alpha^{-1} dV_\alpha(w)\, d|\nu|(\zeta) \\
&\lesssim \left( \sup_{\zeta\in\mathbb{B}_n} \int_{D(\zeta,\frac{r}{2})^c} \left|K_\zeta^\alpha(w)\right| \frac{\left\|K_w^\alpha\right\|_\alpha^{1-a}}{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}} \, dV_\alpha(w) \right) \left( \int_{\mathbb{B}_n} \left|K_z^\alpha(\zeta)\right| \frac{\left\|K_\zeta^\alpha\right\|_\alpha^{1-a}}{\left\|K_z^\alpha\right\|_\alpha^{1-a}} \, d|\nu|(\zeta) \right)
\end{aligned}$$

$$\lesssim \Big( \sup_{\zeta \in \mathbb{B}_n} \int_{D(\zeta, \frac{r}{2})^c} \left| K_\zeta^\alpha(w) \right| \frac{\left\| K_w^\alpha \right\|_\alpha^{1-a}}{\left\| K_\zeta^\alpha \right\|_\alpha^{1-a}} \, dV_\alpha(w) \Big) \Big( \int_{\mathbb{B}_n} \left| K_z^\alpha(\zeta) \right| \frac{\left\| K_\zeta^\alpha \right\|_\alpha^{1-a}}{\left\| K_z^\alpha \right\|_\alpha^{1-a}} \, dV_\alpha(\zeta) \Big),$$

where the last inequality comes from part $(b)$ of Lemma 2.1.3. We use one more time the Forelli-Rudin estimate (2.9) to get that

$$I_2(z,r) \lesssim \sup_{\zeta \in \mathbb{B}_n} \int_{D(\zeta, \frac{r}{2})^c} \left| K_\zeta^\alpha(w) \right| \frac{\left\| K_w \right\|_\alpha^{1-a}}{\left\| K_\zeta^\alpha \right\|_\alpha^{1-a}} \, dV_\alpha(w),$$

and therefore, using Lemma 2.1.2 one more time, we obtain

$$\lim_{r \to \infty} \sup_{z \in \mathbb{B}_n} I_2(z,r) = 0.$$

Thus,

$$\lim_{r \to \infty} \sup_{z \in \mathbb{B}_n} \int_{D(z,r)^c} \left| \langle T_\nu k_z^\alpha, k_w^\alpha \rangle_\alpha \right| \frac{\left\| K_z^\alpha \right\|_\alpha^a}{\left\| K_w^\alpha \right\|_\alpha^a} \, d\lambda(w) = 0.$$

Since $T_\nu^* = T_{\overline{\nu}}$, and $|\nu| = |\overline{\nu}|$, replacing $T_\nu$ by $T_{\overline{\nu}}$ in the proof above gives

$$\lim_{r \to \infty} \sup_{z \in \mathbb{B}_n} \int_{D(z,r)^c} \left| \langle T_\nu^* k_z^\alpha, k_w^\alpha \rangle_\alpha \right| \frac{\left\| K_z^\alpha \right\|_\alpha^a}{\left\| K_w^\alpha \right\|_\alpha^a} \, d\lambda(w) = 0.$$

Hence, we have proven that conditions (2.3) and (2.4) from the definition of weak localization hold for the operator $T_\nu$ and the proof is complete. $\square$

## 2.2 Strong and Sufficient Localization

Recall that for any $z \in \mathbb{B}_n$, the weighted composition operator $U_z^\alpha$ defined by

$$U_z^\alpha f = (f \circ \varphi_z) k_z^\alpha$$

is a selfadjoint unitary operator on $L_a^2(\mathbb{B}_n, dV_\alpha)$, $\alpha > -1$, and

$$\left| U_z^\alpha k_w^\alpha(\zeta) \right| = \left| k_{\varphi_z(w)}^\alpha(\zeta) \right|, \quad \forall\, z, w, \zeta \in \mathbb{B}_n. \tag{2.10}$$

See [39, Page 116] and [24, Page 2036].

**Definition 2.2.1.** We say that a bounded operator $T$ on the weighted Bergman space $L_a^2(\mathbb{B}_n, dV_\alpha)$, $\alpha > -1$, is $\alpha$**-sufficiently localized** if there exists $2 + \dfrac{n}{1+\alpha} < p < \infty$ such that

$$\sup_{z \in \mathbb{B}_n} \left\| U_z^\alpha T U_z^\alpha \mathbb{1} \right\|_{p,\alpha} < \infty \quad \textit{and} \quad \sup_{z \in \mathbb{B}_n} \left\| U_z^\alpha T^* U_z^\alpha \mathbb{1} \right\|_{p,\alpha} < \infty,$$

and we say that $T$ *is* $\alpha$**-strongly localized** if the above conditions hold for any $1 \le p < \infty$.

These types of conditions were first used by Axler and Zheng for characterization of compact Toeplitz operators with bounded symbols in terms of their Berezin transform [7]. The idea was used later by other authors for larger classes of Toeplitz operators (see [52], [24]). In [52], Zorboska used the same conditions to show that the characterization holds true for compact Toeplitz operators with $BMO$ symbols. In [41], Suárez gave a complete characterization of compact operators on the Bergman spaces $L_a^p(\mathbb{B}_n, dV)$, for $1 < p < \infty$, in terms of the Berezin transform. Namely, he showed that an operator on $L_a^p(\mathbb{B}_n, dV_\alpha)$ is compact if and only if it belongs to the Toeplitz algebra and its Berezin transform vanishes on the boundary of the ball.

We recall that the **Fock space** $\mathcal{F}^2(\mathbb{C}^n)$ is the set of all entire functions in $L^2(\mathbb{C}^n, d\Lambda)$, where $d\Lambda(z) = e^{-|z|^2} dV(z)$ is Gaussian measure on $\mathbb{C}^n$. It is well known that $\mathcal{F}^2(\mathbb{C}^n)$ is a reproducing kernel Hilbert space with $K(z,w) = e^{\langle z,w\rangle}$. For this and for more details, see [45]. Xia and Zheng introduced the notion of "sufficiently localized operator" in [43] for operators on the Fock space. A bounded operator $T$ on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ is said to be sufficiently localized if there exist some constants $2n < \beta < \infty$ and $C > 0$ such that

$$\left|\langle Tk_z, k_w\rangle\right| \leq \frac{C}{(1 + |z-w|)^\beta},$$

for all $z, w \in \mathbb{C}^n$. They showed that Toeplitz operators with bounded symbols are sufficiently localized on $\mathcal{F}^2(\mathbb{C}^n)$, and therefore, the $C^*$-algebra generated by sufficiently localized operators contains the Fock Toeplitz algebra. This observation motivated further investigation of localization properties of operators both on the Bergman, and on the Fock-type spaces (see [43], [24], [15], [42], [8], [16]). In the joint paper [37], my supervisor and I, determined the results of this section for the non-weighted Bergman spaces on the unit ball. It is clear that every $\alpha$-strongly localized operator is $\alpha$-sufficiently localized, and we will show that every sufficiently $\alpha$-localized operator is also $\alpha, a$-weakly localized for $a = \frac{n}{2(1+\alpha)+n}$. Thus, the labels of sufficient and strong localization make sense. We show that the sufficient localization implies the weak localization in a few steps, first by proving the following three lemmas.

**Lemma 2.2.2.** Let $\alpha > -1$ and $2 \leq p < \infty$. Suppose $T$ is a bounded operator on $L^p_a(\mathbb{B}_n, dV_\alpha)$. Then

$$\left\|U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{p,\alpha}^p = c_\alpha \int_{\mathbb{B}_n} \left|\langle T k_z^\alpha, k_w^\alpha \rangle_\alpha\right|^p \left\|K^\alpha_{\varphi_z(w)}\right\|_\alpha^{p-2} d\lambda(w).$$

**Proof.** Using the identities (2.10) and (1.5), and by the change of variable $w = \varphi_z(\zeta)$, we have that

$$\begin{aligned}
\left\|U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{p,\alpha}^p &= \int_{\mathbb{B}_n} \left|\langle U_z^\alpha T U_z^\alpha \mathbb{1}, K_\zeta^\alpha \rangle_\alpha\right|^p dV_\alpha(\zeta) \\
&= \int_{\mathbb{B}_n} \left|\langle U_z^\alpha T k_z^\alpha, k_\zeta^\alpha \rangle_\alpha\right|^p \left\|K_\zeta^\alpha\right\|_\alpha^p dV_\alpha(\zeta) \\
&= \int_{\mathbb{B}_n} \left|\langle T k_z^\alpha, U_z^\alpha k_\zeta^\alpha \rangle_\alpha\right|^p \left\|K_\zeta^\alpha\right\|_\alpha^p dV_\alpha(\zeta) \\
&= \int_{\mathbb{B}_n} \left|\langle T k_z^\alpha, k_{\varphi_z(\zeta)}^\alpha \rangle_\alpha\right|^p \left\|K_\zeta^\alpha\right\|_\alpha^p dV_\alpha(\zeta) \\
&= \int_{\mathbb{B}_n} \left|\langle T k_z^\alpha, k_w^\alpha \rangle\right|^p \left\|K_{\varphi_z(w)}^\alpha\right\|_\alpha^p \left|k_z^\alpha(w)\right|^2 dV_\alpha(w)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle\right|^p \left\|K^\alpha_{\varphi_z(w)}\right\|_\alpha^{p-2} \left\|K^\alpha_{\varphi_z(w)}\right\|_\alpha^2 \left|k_z^\alpha(w)\right|^2 dV_\alpha(w) \\
&= \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^p \left\|K^\alpha_{\varphi_z(w)}\right\|_\alpha^{p-2} \left|\langle k_z^\alpha, k_w^\alpha\rangle_\alpha\right|^{-2} \left|\langle k_z^\alpha, k_w^\alpha\rangle_\alpha\right|^2 \left\|K_w^\alpha\right\|_\alpha^2 dV_\alpha(w) \\
&= c_\alpha \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^p \left\|K^\alpha_{\varphi_z(w)}\right\|_\alpha^{p-2} d\lambda(w).
\end{aligned}
$$

**Lemma 2.2.3.** Let $\alpha > -1$ and $T$ be a linear operator on the weighted Bergman space $L_a^2(\mathbb{B}_n, dV_\alpha)$ such that

$$\sup_{z\in\mathbb{B}_n} \left\|U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{p,\alpha} < \infty,$$

for some $2 + \dfrac{n}{1+\alpha} < p < \infty$, then for $a = \dfrac{n}{2(1+\alpha)+n} \in \left(\dfrac{n-1-\alpha}{n+1+\alpha}, 1\right)$, we have that

$$\sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a} d\lambda(w) < \infty.$$

**Proof.** Let $2 + \dfrac{n}{1+\alpha} < p < \infty$, and $q = \dfrac{p}{p-1}$. Then for $a = \dfrac{n}{2(1+\alpha)+n}$, we see that

$$
\begin{aligned}
I(z,p) &= \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a} d\lambda(w) \\
&= \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right| \frac{\left(\left\|K_z^\alpha\right\|_\alpha \left\|K_w^\alpha\right\|_\alpha\right)^{\frac{p-2}{p}}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)\frac{2-p}{p}}} \frac{\left\|K_z^\alpha\right\|_\alpha^{a-\frac{p-2}{p}}}{\left\|K_w^\alpha\right\|_\alpha^{a+\frac{p-2}{p}}} \frac{d\lambda(w)}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)\frac{p-2}{p}}},
\end{aligned}
$$

and by Hölder's inequality, we have $I(z,p) \leq I_1(z,p)^{\frac{1}{p}}\, I_2(z,p)^{\frac{1}{q}}$, where

$$I_1(z,p) = \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^p \frac{\left(\left\|K_z^\alpha\right\|_\alpha \left\|K_w^\alpha\right\|_\alpha\right)^{p-2}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(2-p)}}\, d\lambda(w),$$

and

$$I_2(z,p) = \int_{\mathbb{B}_n} \frac{\left\|K_z^\alpha\right\|_\alpha^{\left(\frac{(a-1)p+2}{p}\right)q} \left\|K_w^\alpha\right\|_\alpha^{\left(\frac{(a+1)p-2}{-p}\right)q}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{p-2}{p})q}}\, d\lambda(w).$$

Using the identity (1.5) and Lemma 2.2.2, we get that

$$\begin{aligned}
I_1(z,p) &= \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^p \frac{\left(\left\|K_z^\alpha\right\|_\alpha \left\|K_w^\alpha\right\|_\alpha\right)^{p-2}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(2-p)}}\, d\lambda(w) \\
&= \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^p \left(\frac{\left\|K_z^\alpha\right\|_\alpha \left\|K_w^\alpha\right\|_\alpha}{\left|1-\langle z,w\rangle\right|^{-(n+1+\alpha)}}\right)^{p-2} d\lambda(w) \\
&= \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^p \left(\left|\langle k_z^\alpha, k_w^\alpha\rangle_\alpha\right|^{-1}\right)^{p-2} d\lambda(w) \\
&= \int_{\mathbb{B}_n} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^p \left\|K_{\varphi_z(w)}^\alpha\right\|_\alpha^{p-2} d\lambda(w) \\
&= c_\alpha^{-1} \left\|U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{p,\alpha}^p.
\end{aligned}$$

Thus, we have

$$I_1(z,p)^{\frac{1}{p}} \leq c_\alpha^{-\frac{1}{p}} \left\|U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{p,\alpha}.$$

Next, we use the Forelli-Rudin estimates to show that $I_2(z,p)$ is bounded above by a constant independent of $z$. We have that

$$\begin{aligned}
I_2(z,p) &= \int_{\mathbb{B}_n} \frac{\left\|K_z^\alpha\right\|_\alpha^{\left(\frac{(a-1)p+2}{p}\right)q} \left\|K_w^\alpha\right\|_\alpha^{\left(\frac{(a+1)p-2}{-p}\right)q}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{p-2}{p})q}}\, d\lambda(w) \\
&= \int_{\mathbb{B}_n} \frac{\left\|K_z^\alpha\right\|_\alpha^{\frac{(a-1)p+2}{p-1}} \left\|K_w^\alpha\right\|_\alpha^{\frac{(a+1)p-2}{1-p}}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{p-2}{p-1})}}\, d\lambda(w) \\
&= \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{(\frac{n+1+\alpha}{2})\frac{(1-a)p-2}{p-1}}\,(1-|w|^2)^{(\frac{n+1+\alpha}{2})\frac{(a+1)p-2}{p-1}}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{p-2}{p-1})}(1-|w|^2)^{n+1}}\, dV(w) \\
&= \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{(\frac{n+1+\alpha}{2})\frac{(1-a)p-2}{p-1}}\,(1-|w|^2)^{(\frac{n+1+\alpha}{2})\frac{(a+1)p-2}{p-1}-(n+1)}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{p-2}{p-1})}}\, dV(w).
\end{aligned}$$

Setting

$$c = \Big(\frac{n+1+\alpha}{2}\Big)\frac{(1-a)p-2}{p-1} \quad \text{and} \quad t = \Big(\frac{n+1+\alpha}{2}\Big)\frac{(a+1)p-2}{p-1} - (n+1),$$

and using the assumption that $2+\dfrac{n}{1+\alpha} < p < \infty$, and $a = \dfrac{n}{2(1+\alpha)+n}$, it is easy to verify that $c>0$ and $t>-1$. We also have that

$$n+1+t+c = \Big(n+1+\alpha\Big)\Big(\frac{p-2}{p-1}\Big),$$

and hence, by the Forelli-Rudin estimates, there exists $C > 0$ such that

$$I_2(z,p) \leq C, \quad \forall z \in \mathbb{B}_n.$$

Therefore, we obtain that

$$I(z,p) \leq c_\alpha^{-\frac{1}{p}} C^{1-\frac{1}{p}} \left\| U_z^\alpha T U_z^\alpha \mathbb{1} \right\|_{p,\alpha},$$

and by taking the supremum over all $z \in \mathbb{B}_n$, and using the assumption, we get that

$$\sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} \left| \langle T k_z^\alpha, k_w^\alpha \rangle_\alpha \right| \frac{\left\| K_z^\alpha \right\|_\alpha^a}{\left\| K_w^\alpha \right\|_\alpha^a} \, d\lambda(w) < \infty. \qquad \square$$

**Lemma 2.2.4.** Let $\alpha > -1$ and $T$ be a linear operator on the weighted Bergman space $L_a^2(\mathbb{B}_n, dV_\alpha)$ such that

$$\sup_{z\in\mathbb{B}_n} \left\| U_z^\alpha T U_z^\alpha \mathbb{1} \right\|_{p,\alpha} < \infty,$$

for some $2 + \dfrac{n}{1+\alpha} < p < \infty$. Then

$$\lim_{r\to 1^-} \sup_{z\in\mathbb{B}_n} \left\| \chi_{D_\rho(0,r)^c} U_z^\alpha T U_z^\alpha \mathbb{1} \right\|_{q,\alpha} = 0,$$

for all $q$ with $2 + \dfrac{n}{1+\alpha} < q < p$.

**Proof.** Let $2 + \dfrac{n}{1+\alpha} < q < p$ be fixed, $s = \dfrac{p}{q}$ and $s' > 1$ be such that $\dfrac{1}{s} + \dfrac{1}{s'} = 1$. By applying Hölder's inequality with $s$ and $s'$, we find that

$$\begin{aligned}
\left\|\chi_{D_\rho(0,r)^c} U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{q,\alpha}^q &= \int_{D_\rho(0,r)^c} \left|U_z^\alpha T U_z^\alpha \mathbb{1}(w)\right|^q dV_\alpha(w) \\
&\leq \left(\int_{D_\rho(0,r)^c} \left|U_z^\alpha T U_z^\alpha \mathbb{1}(w)\right|^p dV_\alpha(w)\right)^{\frac{q}{p}} \left(\int_{D_\rho(0,r)^c} dV_\alpha(w)\right)^{1-\frac{q}{p}} \\
&\leq \left\|U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{p,\alpha}^q \left(\int_{\mathbb{B}_n \setminus r\overline{\mathbb{B}_n}} dV_\alpha(w)\right)^{1-\frac{q}{p}} \\
&\leq (nc_\alpha)^{1-\frac{q}{p}} \left\|U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{p,\alpha}^q \quad (1-r^2)^{(1+\alpha)(1-\frac{q}{p})}.
\end{aligned}$$

Hence, since $(1+\alpha)(\frac{1}{q} - \frac{1}{p}) > 0$, we obtain that

$$\lim_{r\to 1^-} \sup_{z\in\mathbb{B}_n} \left\|\chi_{D_\rho(0,r)^c} U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{q,\alpha} \leq (nc_\alpha)^{\frac{1}{q}-\frac{1}{p}} \sup_{z\in\mathbb{B}_n} \left\|U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{p,\alpha} \lim_{r\to 1^-} (1-r^2)^{(1+\alpha)(\frac{1}{q}-\frac{1}{p})} = 0,$$

which is the desired conclusion. □

We are ready now to prove that the sufficient localization implies the weak localization.

**Theorem 2.2.5.** Suppose that $T$ is a linear operator acting on the weighted Bergman space $L_a^2(\mathbb{B}_n, dV_\alpha)$, $\alpha > -1$. If $T$ is $\alpha$-sufficiently localized, then it is $a,\alpha$-weakly localized for $a = \dfrac{n}{2(1+\alpha)+n}$.

**Proof.** Let $T$ be $\alpha$-sufficiently localized on the weighted Bergman space $L_a^2(\mathbb{B}_n, dV_\alpha)$. Thus, there exists $2 + \dfrac{n}{1+\alpha} < p < \infty$ such that

$$\sup_{z\in\mathbb{B}_n} \left\| U_z^\alpha T U_z^\alpha \mathbb{1} \right\|_{p,\alpha} < \infty \quad \text{and} \quad \sup_{z\in\mathbb{B}_n} \left\| U_z^\alpha T^* U_z^\alpha \mathbb{1} \right\|_{p,\alpha} < \infty,$$

By Lemma 2.2.3, we see that $T$ and $T^*$ satisfy condition $(i)$ from the definition of weak localization for $a = \dfrac{n}{2(1+\alpha)+n}$. We will show that condition $(ii)$ from the definition of weak localization in terms of pseudo-hyperbolic balls also holds for $a = \dfrac{n}{2(1+\alpha)+n}$. That is, if $T$ is $\alpha$-sufficiently localized, then

$$\lim_{r\to 1^-} \sup_{z\in\mathbb{B}_n} \int_{D_\rho(z,r)^c} \left| \langle T k_z^\alpha, k_w^\alpha \rangle_\alpha \right| \frac{\left\| K_z^\alpha \right\|_\alpha^a}{\left\| K_w^\alpha \right\|_\alpha^a} \, d\lambda(w) = 0,$$

and

$$\lim_{r\to 1^-} \sup_{z\in\mathbb{B}_n} \int_{D_\rho(z,r)^c} \left| \langle T^* k_z^\alpha, k_w^\alpha \rangle_\alpha \right| \frac{\left\| K_z^\alpha \right\|_\alpha^a}{\left\| K_w^\alpha \right\|_\alpha^a} \, d\lambda(w) = 0.$$

Since $T$ is $\alpha$-sufficiently localized if and only if $T^*$ is $\alpha$-sufficiently localized, it is enough to show that the first limit vanishes.

Let $q$ be any number between $2 + \frac{n}{1+\alpha}$ and $p$, and $q'$ be the conjugate exponent of $q$. We have that

$$
\begin{aligned}
I &= \sup_{z\in\mathbb{B}_n}\int_{D_\rho(z,r)^c}\left|\langle Tk_z^\alpha,k_w^\alpha\rangle_\alpha\right|\frac{\left\|K_z^\alpha\right\|_\alpha^a}{\left\|K_w^\alpha\right\|_\alpha^a}\,d\lambda(w)\\
&= \int_{D_\rho(z,r)^c}\left|\langle Tk_z^\alpha,k_w^\alpha\rangle_\alpha\right|\frac{\left(\left\|K_z^\alpha\right\|_\alpha\left\|K_w^\alpha\right\|_\alpha\right)^{\frac{q-2}{q}}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)\frac{2-q}{q}}}\frac{\left\|K_z^\alpha\right\|_\alpha^{a-\frac{q-2}{q}}}{\left\|K_w^\alpha\right\|_\alpha^{a+\frac{q-2}{q}}}\frac{d\lambda(w)}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)\frac{q-2}{q}}},
\end{aligned}
$$

and by Hölder's inequality, we have $I\le I_1(z,q)^{\frac{1}{q}}\,I_2(z,q)^{\frac{1}{q'}}$, where

$$
I_1(z,q)=\int_{D_\rho(z,r)^c}\left|\langle Tk_z^\alpha,k_w^\alpha\rangle_\alpha\right|^q\frac{\left(\left\|K_z^\alpha\right\|_\alpha\left\|K_w^\alpha\right\|_\alpha\right)^{q-2}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(2-q)}}\,d\lambda(w),
$$

and

$$
I_2(z,q)=\int_{D_\rho(z,r)^c}\frac{\left\|K_z^\alpha\right\|_\alpha^{\left(\frac{(a-1)q+2}{q}\right)q'}\left\|K_w^\alpha\right\|_\alpha^{\left(\frac{(a+1)q-2}{-q}\right)q'}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{q-2}{q})q'}}\,d\lambda(w).
$$

Using the identity (1.5), we get that

$$
\begin{aligned}
I_1(z,q) &= \int_{D_\rho(z,r)^c}\left|\langle Tk_z^\alpha,k_w^\alpha\rangle_\alpha\right|^q\frac{\left(\left\|K_z^\alpha\right\|_\alpha\left\|K_w^\alpha\right\|_\alpha\right)^{q-2}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(2-q)}}\,d\lambda(w)\\
&= \int_{D_\rho(z,r)^c}\left|\langle Tk_z^\alpha,k_w^\alpha\rangle_\alpha\right|^q\left(\frac{\left\|K_z^\alpha\right\|_\alpha\left\|K_w^\alpha\right\|_\alpha}{\left|1-\langle z,w\rangle\right|^{-(n+1+\alpha)}}\right)^{q-2}\,d\lambda(w)
\end{aligned}
$$

$$= \int_{D_\rho(z,r)^c} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^q \left(\left|\langle k_z^\alpha, k_w^\alpha\rangle_\alpha\right|^{-1}\right)^{q-2} d\lambda(w)$$

$$= \int_{D_\rho(z,r)^c} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^q \left\|K_{\varphi_z(w)}^\alpha\right\|_\alpha^{q-2} d\lambda(w).$$

Next we show that

$$\left\|\chi_{D_\rho(0,r)^c} U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{q,\alpha}^q = c_\alpha \int_{D_\rho(z,r)^c} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^q \left\|K_{\varphi_z(w)}^\alpha\right\|_\alpha^{q-2} d\lambda(w).$$

Using the identities (2.10) and (1.5), and by the change of variable $w = \varphi_z(\zeta)$ together with the fact that $\varphi_z\Big(D_\rho(0,r)\Big) = D_\rho(\varphi_z(0), r) = D_\rho(z,r)$, we have that

$$\begin{aligned}
\left\|\chi_{D_\rho(0,r)^c} U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{q,\alpha}^q &= \int_{D_\rho(0,r)^c} \left|\langle U_z^\alpha T U_z^\alpha \mathbb{1}, K_\zeta^\alpha\rangle_\alpha\right|^q dV_\alpha(\zeta) \\
&= \int_{D_\rho(0,r)^c} \left|\langle U_z^\alpha T k_z^\alpha, k_\zeta^\alpha\rangle_\alpha\right|^q \left\|K_\zeta^\alpha\right\|_\alpha^q dV_\alpha(\zeta) \\
&= \int_{D_\rho(0,r)^c} \left|\langle T k_z^\alpha, U_z^\alpha k_\zeta^\alpha\rangle_\alpha\right|^q \left\|K_\zeta^\alpha\right\|_\alpha^q dV_\alpha(\zeta) \\
&= \int_{D_\rho(0,r)^c} \left|\langle T k_z^\alpha, k_{\varphi_z(\zeta)}^\alpha\rangle_\alpha\right|^q \left\|K_\zeta^\alpha\right\|_\alpha^q dV_\alpha(\zeta) \\
&= \int_{D_\rho(z,r)^c} \left|\langle T k_z^\alpha, k_w^\alpha\rangle\right|^q \left\|K_{\varphi_z(w)}^\alpha\right\|_\alpha^q \left|k_z^\alpha(w)\right|^2 dV_\alpha(w) \\
&= \int_{D_\rho(z,r)^c} \left|\langle T k_z^\alpha, k_w^\alpha\rangle\right|^q \left\|K_{\varphi_z(w)}^\alpha\right\|_\alpha^{q-2} \left\|K_{\varphi_z(w)}^\alpha\right\|_\alpha^2 \left|k_z^\alpha(w)\right|^2 dV_\alpha(w)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{D_\rho(z,r)^c} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^q \left\|K^\alpha_{\varphi_z(w)}\right\|_\alpha^{q-2} \left|\langle k_z^\alpha, k_w^\alpha\rangle_\alpha\right|^{-2} \left|\langle k_z^\alpha, k_w^\alpha\rangle_\alpha\right|^{2} \left\|K_w^\alpha\right\|_\alpha^2 dV_\alpha(w) \\
&= c_\alpha \int_{D_\rho(z,r)^c} \left|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\right|^q \left\|K^\alpha_{\varphi_z(w)}\right\|_\alpha^{q-2} d\lambda(w).
\end{aligned}
$$

Hence,

$$
I_1(z,q) = c_\alpha^{-1} \left\|\chi_{D_\rho(0,r)^c} U_z^\alpha T U_z^\alpha \mathbb{1}\right\|_{q,\alpha}^q.
$$

Next, we use the Forelli-Rudin estimates to find an upper bound for $I_2(z,q)$.

$$
\begin{aligned}
I_2(z,q) &= \int_{D_\rho(z,r)^c} \frac{\left\|K_z^\alpha\right\|_\alpha^{\left(\frac{(a-1)q+2}{q}\right)q'} \left\|K_w^\alpha\right\|_\alpha^{\left(\frac{(a+1)q-2}{-q}\right)q'}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{q-2}{q})q'}} d\lambda(w) \\
&= \int_{D_\rho(z,r)^c} \frac{\left\|K_z^\alpha\right\|_\alpha^{\frac{(a-1)q+2}{q-1}} \left\|K_w^\alpha\right\|_\alpha^{\frac{(a+1)q-2}{1-q}}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{q-2}{q-1})}} d\lambda(w) \\
&= \int_{D_\rho(z,r)^c} \frac{(1-|z|^2)^{(\frac{n+1+\alpha}{2})\frac{(1-a)q-2}{q-1}} (1-|w|^2)^{(\frac{n+1+\alpha}{2})\frac{(a+1)q-2}{q-1}}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{q-2}{q-1})}(1-|w|^2)^{n+1}} dV(w) \\
&= \int_{D_\rho(z,r)^c} \frac{(1-|z|^2)^{(\frac{n+1+\alpha}{2})\frac{(1-a)q-2}{q-1}} (1-|w|^2)^{(\frac{n+1+\alpha}{2})\frac{(a+1)q-2}{q-1}-(n+1)}}{\left|1-\langle z,w\rangle\right|^{(n+1+\alpha)(\frac{q-2}{q-1})}} dV(w).
\end{aligned}
$$

Setting

$$
c = \left(\frac{n+1+\alpha}{2}\right)\frac{(1-a)q-2}{q-1} \quad \text{and} \quad t = \left(\frac{n+1+\alpha}{2}\right)\frac{(a+1)q-2}{q-1} - (n+1),
$$

and using the assumption that $2+\dfrac{n}{1+\alpha} < q < \infty$, and $a = \dfrac{n}{2(1+\alpha)+n}$, it is easy to verify that $c > 0$ and $t > -1$. We also have that

$$n+1+t+c = \Big(n+1+\alpha\Big)\Big(\frac{q-2}{q-1}\Big),$$

and hence, by the Forelli-Rudin estimates, there exists $C > 0$ such that

$$I_2(z,q) \leq C, \quad \forall z \in \mathbb{B}_n.$$

Therefore, we obtain that

$$\begin{aligned} I &= \sup_{z\in\mathbb{B}_n} \int_{D_\rho(z,r)^c} \Big|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\Big| \frac{\Big\|K_z^\alpha\Big\|_\alpha^a}{\Big\|K_w^\alpha\Big\|_\alpha^a}\, d\lambda(w) \\ &\leq I_1(z,q)^{\frac{1}{q}}\, I_2(z,q)^{\frac{1}{q'}} \\ &= C^{1-\frac{1}{q}}\ c_\alpha^{-\frac{1}{q}} \Big\|\chi_{_{D_\rho(0,r)^c}} U_z^\alpha T U_z^\alpha \mathbb{1}\Big\|_{q,\alpha}, \end{aligned}$$

Taking supremum of both sides over $\mathbb{B}_n$, letting $r \to 1^-$, and applying Lemma 2.2.4, we conclude that

$$\lim_{r\to 1^-} \sup_{z\in\mathbb{B}_n} \int_{D_\rho(z,r)^c} \Big|\langle Tk_z^\alpha, k_w^\alpha\rangle_\alpha\Big| \frac{\Big\|K_z^\alpha\Big\|_\alpha^a}{\Big\|K_w^\alpha\Big\|_\alpha^a}\, d\lambda(w) = 0,$$

for $a = \dfrac{n}{2(1+\alpha)+n}$. Therefore, $T$ is $a, \alpha$-weakly localized. $\square$

## 2.3 Toeplitz Operators with BMO Symbols

We start this section with the definition of $BMO$ spaces. For what follows and for more details about $BMO$ spaces see [30], [47], [48], and [50].

**Definition 2.3.1.** **[30]** For $\alpha > -1$, $r > 0$, and $f \in L^1(\mathbb{B}_n, dV_\alpha)$, we define the averaging function $\widehat{f}_r$ *on* $\mathbb{B}_n$ by

$$\widehat{f}_r(z) = \frac{1}{V_\alpha(D(z,r))} \int_{D(z,r)} f(w)\, dV_\alpha(w).$$

For $1 \leq p < \infty$, we say that the function $f \in L^1(\mathbb{B}_n, dV_\alpha)$ belongs to $BMO_\alpha^{p,r}(\mathbb{B}_n)$ if

$$\sup_{z\in\mathbb{B}_n} \left[\frac{1}{V_\alpha(D(z,r))} \int_{D(z,r)} \left|f(w) - \widehat{f}_r(z)\right|^p dV_\alpha(w)\right]^{1/p} < \infty.$$

By [50, Page 195] and [47, Page 386], $BMO_\alpha^{p,r}(\mathbb{B}_n)$ is independent of $r$. Thus, we may write $BMO_\alpha^p(\mathbb{B}_n)$ instead of $BMO_\alpha^{p,r}(\mathbb{B}_n)$. Equivalently, $f \in BMO_\alpha^p(\mathbb{B}_n)$ if and only if

$$\|f\|_{BMO_\alpha^p}^p = \sup_{z\in\mathbb{B}_n} \left\|f \circ \varphi_z - \widetilde{f}(z)\right\|_{p,\alpha} < \infty,$$

where $\varphi_z$ is the automorphism of $\mathbb{B}_n$, that interchanges the points 0 and $z$, and $\widetilde{f}$ is the Berezin transform of $f$ [47, Theorem 5].

We notice that $\|.\|_{BMO_\alpha^p}$ is a semi-norm on $BMO_\alpha^p$. The $BMO_\alpha^p(\mathbb{B}_n)$ spaces equipped with the norm

$$|||f|||_{p,\alpha} = |\tilde{f}(0)| + \|f\|_{BMO^p_\alpha},$$

are Banach spaces and we have that

$$L^\infty(\mathbb{B}_n, dV_\alpha) \subseteq BMO^p_\alpha(\mathbb{B}_n) \subseteq L^p(\mathbb{B}_n, dV_\alpha), \qquad 1 \leq p < \infty,$$

and

$$BMO^p_\alpha(\mathbb{B}_n) \subseteq BMO^q_\alpha(\mathbb{B}_n) \subseteq BMO^1_\alpha(\mathbb{B}_n), \qquad 1 \leq q < p.$$

The space $BMO^1_\alpha(\mathbb{B}_n)$ also contains the non-negative functions in $L^1(\mathbb{B}_n, dV_\alpha)$. For more details on the $BMO$ spaces see [30], [47], and [48].

In [52], Zorboska showed that bounded and compact Toeplitz operators with $BMO^1$ symbols on the Bergman space of the open unit disk can be characterized by their Berezin transform. This was generalized later by Zhang, Liu, and Lu to the weighted Bergman spaces of the open unit ball in [44].

**Theorem 2.3.2.** **[44]** Let $f \in BMO^1_\alpha(\mathbb{B}_n)$. Then

(1) $T_f$ is bounded on $L^2_a(\mathbb{B}_n, dV_\alpha)$ if and only if $\widetilde{f}$ is bounded on $\mathbb{B}_n$.

(2) $T_f$ is compact if and only if $\widetilde{f}(z) \longrightarrow 0$, as $|z| \longrightarrow 1^-$.

Hence, the $BMO^p_\alpha(\mathbb{B}_n)$ spaces provide a class of possibly unbounded symbols that might induce bounded Toeplitz operators on the classical Bergman spaces.

**Definition 2.3.3.** [**50**] The Bloch space of $\mathbb{B}_n$ is denoted by $\mathcal{B}$ and is defined to be the set of all holomorphic functions in $\mathbb{B}_n$ such that

$$\|f\| = \sup_{z\in\mathbb{B}_n}(1-|z|^2)|\nabla f(z)| < \infty,$$

where

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), ..., \frac{\partial f}{\partial z_n}(z)\Big),$$

and $|\nabla f(z)|$ is called the holomorphic gradient of $f$ *at* $z$.

It is clear that $\|.\|$ does not distinguish constants. In fact, $\|f\| = 0$ if and only if $f$ is a constant. The Bloch space $\mathcal{B}$ endowed with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{B}_n}(1-|z|^2)|\nabla f(z)|,$$

is a Banach space.

The semi-norm of the Bloch space is invariant under the action of the automorphism group of $\mathbb{B}_n$, and the Bloch space is the largest space of holomorphic functions with this property. Furthermore, by [30, Corollary 3.7], the set of holomorphic functions in $BMO^p_\alpha(\mathbb{B}_n)$ is precisely the Bloch space $\mathcal{B}$, that is,

$$BMO^p_\alpha(\mathbb{B}_n)\cap H(\mathbb{B}_n) = \mathcal{B}, \quad 1\le p<\infty.$$

The Bloch space is also nicely connected to the Bergman metric. The set of all holomorphic functions that are Lipschitz from $\mathbb{B}_n$, with the hyperbolic metric, to $\mathbb{C}$, with the Euclidean metric, is exactly the Bloch space. We have the following theorem based on the proof of [50, Theorem 2.16].

**Theorem 2.3.4.** **[50]** For $1 \leq p < \infty$ *and* $\alpha > -1$, the Bloch space is continuously embedded in $L^p(\mathbb{B}_n, dV_\alpha)$.

In [19], Li and Luecking showed that the Bergman projection $P$ maps $BMO^p(\mathbb{B}_n)$ continuously onto the Bloch space $\mathcal{B}$ and this was generalized to the weighted case in [44].

**Lemma 2.3.5.** **[44]** For $1 \leq p < \infty$ and $\alpha > -1$, the Bergman Projection $P_\alpha$ is a bounded operator from $BMO^p_\alpha(\mathbb{B}_n)$ onto the Bloch space $\mathcal{B}$.

More details on the Bloch spaces can be found in [50], [9], and [46].

### 2.3.1 Localization of Bounded Toeplitz Operators with BMO Symbols

The content of this section was done in [37] by Prof. N. Zorboska and myself for the non-weighted Bergman space on the unit ball. The ideas and methods of the proofs here follow closely the ideas from [37].

**Theorem 2.3.6.** *Let $f \in BMO^1_\alpha(\mathbb{B}_n)$ with $\widetilde{f} \in L^\infty(\mathbb{B}_n)$. Then the Toeplitz operator $T_f$ is $\alpha$-strongly localized and so also $a, \alpha$-weakly localized on the Bergman space $L^2_a(\mathbb{B}_n, dV_\alpha)$ for $a = \dfrac{n}{2(1+\alpha)+n}$.*

**Proof.** For any $f \in BMO^1_\alpha(\mathbb{B}_n)$ with $\tilde{f} \in L^\infty(\mathbb{B}_n)$ we have that $f \circ \varphi_z \in BMO^1_\alpha(\mathbb{B}_n)$, for all $z \in \mathbb{B}_n$, and that $U^\alpha_z T_f U^\alpha_z \mathbb{1} = T_{f\circ\varphi_z}\mathbb{1}$ [44, Page 2138]. By Lemma 2.3.5, the Bergman projection $P_\alpha$ is a bounded operator from $BMO^1(\mathbb{B}_n)$ onto the Bloch space $\mathcal{B}$ and by Theorem 2.3.4, the Bloch space is continuously included in $L^p(\mathbb{B}_n, dV_\alpha)$, for any $1 \le p < \infty$. Thus, we have that

$$\begin{aligned}
\left\|U^\alpha_z T_f U^\alpha_z \mathbb{1}\right\|_{p,\alpha} &= \left\|T_{f\circ\varphi_z}\mathbb{1}\right\|_{p,\alpha} \\
&= \left\|P_\alpha(f\circ\varphi_z)\right\|_{p,\alpha} \\
&\lesssim \left\|P_\alpha(f\circ\varphi_z)\right\|_{\mathcal{B}} \\
&\lesssim \left\|f\circ\varphi_z\right\|_{BMO^1_\alpha} \\
&\lesssim ||f||_{BMO^1_\alpha} + \|\tilde{f}\|_\infty,
\end{aligned}$$

for any $1 \le p < \infty$, by [44, Page 2138]. Thus, we get that

$$\sup_{z\in\mathbb{B}_n} \left\|U^\alpha_z T_f U^\alpha_z \mathbb{1}\right\|_p \lesssim ||f||_{BMO^1_\alpha} + ||\tilde{f}||_\infty < \infty, \quad \forall\, 1 \le p < \infty.$$

To show that the second condition also holds, we notice that $U_z^\alpha T_f^* U_z^\alpha \mathbb{1} = T_{\bar{f}\circ\varphi_z}\mathbb{1}$, and that $\bar{f} \in BMO_\alpha^1(\mathbb{B}_n)$, with $\tilde{\bar{f}} = \bar{\tilde{f}} \in L^\infty(\mathbb{B}_n)$. So, we get that

$$\sup_{z\in\mathbb{B}_n} \left\| U_z^* T_f^* U_z \mathbb{1} \right\|_p < \infty, \quad \forall\, 1 \leq p < \infty.$$

Hence, $T_f$ is $\alpha$-strongly localized. But since then $T_f$ is also $\alpha$-sufficiently localized, by Theorem 2.2.5, it follows that $T_f$ is $a, \alpha$-weakly localized for $a = \dfrac{n}{2(1+\alpha)+n}$. $\square$

Recall that the Toeplitz algebra $\mathcal{T}_\alpha$ is the (operator norm) closed subalgebra of $B\Big(L_a^2(\mathbb{B}_n, dV_\alpha)\Big)$ generated by $\Big\{T_f : f \in L^\infty, (\mathbb{B}_n, dV_\alpha)\Big\}$. In [42], Xia has proved that the $C^*$-algebra generated by the set of weakly localized operators on $L_a^2(\mathbb{B}_n, dV)$ is the Bergman Toeplitz algebra $\mathcal{T}_0$.

**Theorem 2.3.7.** **[42]** For any $\frac{n-1}{n+1} < a < 1$, the $C^*$-algebra generated by $a, 0$-weakly localized operators on $L_a^2(\mathbb{B}_n, dV)$ is the Toeplitz algebra $\mathcal{T}_0$.

**Remark 2.3.8.** In the remark on [42, Page 810], it is mentioned that with some trivial modifications the theorem also holds for the weighted Bergman spaces of the unit ball. Thus, for any $\frac{n-1-\alpha}{n+1+\alpha} < a < 1$, the $C^*$-algebra generated by the set of $a, \alpha$-weakly localized operators on $L_a^2(\mathbb{B}_n, dV_\alpha)$ is the Toeplitz algebra $\mathcal{T}_\alpha$. Hence, by [25, Theorem 1.1], if a linear operator $T$ is $a, \alpha$-weakly localized on $L_a^2(\mathbb{B}_n, dV_\alpha)$, $\alpha > -1$, such that $\widetilde{T}(z) \longrightarrow 0$ as $|z| \longrightarrow 1^-$, then $T$ is compact on $L_a^2(\mathbb{B}_n, dV_\alpha)$.

# Chapter 3

# Localization on Bergman Spaces with Exponential-Type Weights

The notion of weak localization of operators on a class of weighted Bergman spaces with special radial decreasing weights was defined in [4], where it was also shown that any weakly localized operator on such weighted Bergman spaces is bounded. The thesis also contains a comment stating that the Toeplitz operators with bounded symbols on such weighted Bergman spaces are weakly localized. However, that work has not been published and was not available to us. Moreover, this does not impact our results since we deal with a different class of weights, and with Toeplitz operators in a larger class with possibly unbounded symbols.

In this chapter, we consider the definition of weak localization from [4], for the Bergman spaces $L^2_a(\omega)$ with $\omega$ in $\mathcal{E}$, and prove that any Toeplitz operator induced by a complex Borel measure with Carleson total variation is weakly localized on $L^2_a(\omega)$. This gives a large class of weakly localized operators on $L^2_a(\omega)$ with $\omega$ in $\mathcal{E}$. We prove the theorem in a few steps. Note that, as mentioned in chapter 1, the class $\mathcal{E}$ does not contain the standard weights $\omega_\alpha(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, and so our main results from chapter 2 and chapter 3 are independent.

## 3.1 Geometry of the Sets $D_m(z)$

In this section we will define a special kind of sets, related to the class of weights $\mathcal{E}$, depending on the function $\tau$, that was defined and discussed in section 1.4.

Let $z, \zeta \in \mathbb{D}$, and let $\tau$ be a positive function such that the corresponding weight $\omega$ is in the class $\mathcal{E}$. We denote

$$d_\tau(z,\zeta) := \frac{|z-\zeta|}{\min(\tau(z),\tau(\zeta))}\,.$$

We always let $\delta \in (0, \mathfrak{M}_\tau)$ be fixed, where

$$\mathfrak{M}_\tau = \frac{\min(1, c_1^{-1}, c_2^{-1})}{16}\,,$$

and $c_1$ and $c_2$ are given in Definition 1.4.1. For $m \geq 1$, and for $z \in \mathbb{D}$, we define

$$D_m(z) = \Big\{\zeta \in \mathbb{D} \,:\, d_\tau(z,\zeta) < 2^m\delta\Big\}.$$

We notice that $d_\tau$ is not a metric on the open unit disk and therefore it is less convenient to work with the sets $D_m(z)$ rather than the Euclidean disks. Nevertheless, they are needed in order to properly describe the weak localization for the specific subclass of large weighted Bergman spaces. We present two lemmas regarding the geometry of the sets $D_m(z)$ which will be used for the proof of the main result of this chapter.

**Lemma 3.1.1.** Let $z \in \mathbb{D}$ and $m \geq 1$. For $\tau \in \mathcal{L}$ and $\delta \in (0, \mathfrak{M}_\tau)$, let $\{z_j\}_{j=1}^{\infty} \subseteq \mathbb{D}$ be a $(\delta, \tau)$-lattice on $\mathbb{D}$. Then

$$\bigcup_{j \in J_{m,z}} D(3\delta\tau(z_j)) \subseteq D_{m-4}(z)^c, \, \forall \, m \geq 7,$$

where

$$J_{m,z} = \Big\{ j \, : \, D(\delta\tau(z_j)) \cap D_m(z)^c \neq \emptyset \Big\}.$$

**Proof.** By the definition of the family $J_{m,z}$, and by property (1) of a $(\delta, \tau)$-lattice, we have that

$$D_m(z)^c \subseteq \bigcup_{j \in J_{m,z}} D(\delta\tau(z_j)). \tag{3.1}$$

Pick $\zeta \in \bigcup_{j \in J_{m,z}} D(3\delta\tau(z_j))$. Then $\zeta \in D(3\delta\tau(z_j))$, for some $j \in J_{m,z}$. By the Lipschitz condition (B), and since $\delta c_2 < 1/16$, we get that

$$|\tau(\zeta) - \tau(z_j)| \leq c_2 |\zeta - z_j| \leq c_2 (3\delta\tau(z_j)) \leq \frac{3}{16}\tau(z_j).$$

Hence, $\tau(\zeta) \leq 2\tau(z_j)$ and similarly $\tau(z_j) \leq 2\tau(\zeta)$. We need to prove that $\zeta \in D_{m-4}(z)^c$. We will show that $\zeta \in D_{m-4}(z)$ leads to a contradiction. If $\zeta \in D_{m-4}(z)$, then we have that

$$\frac{|z-\zeta|}{\min(\tau(z),\tau(\zeta))} < 2^{m-4}\delta.$$

Let $y_j \in D(\delta\tau(z_j)) \cap D_m(z)^c \neq \emptyset$, then

$$\begin{aligned}|z-y_j| &\leq |z-\zeta| + |\zeta - z_j| + |z_j - y_j| \\ &< 2^{m-4}\delta \ \min(\tau(z),\tau(\zeta)) + 3\delta\tau(z_j) + \delta\tau(z_j) \\ &= 2^{m-4}\delta \ \min(\tau(z),\tau(\zeta)) + 4\delta\tau(z_j).\end{aligned}$$

There are two cases: either $\min(\tau(z),\tau(\zeta)) = \tau(z)$ or $\min(\tau(z),\tau(\zeta)) = \tau(\zeta)$. We show that in each case we reach a contradiction.

Case 1. If $\min(\tau(z),\tau(\zeta)) = \tau(z)$, then

$$\begin{aligned}|z-y_j| &< 2^{m-4}\delta \ \min(\tau(z),\tau(\zeta)) + 4\delta\tau(z_j) \\ &= 2^{m-4}\delta\tau(z) + 4\delta\tau(z_j) \\ &\leq 2^{m-4}\delta\tau(z) + 8\delta\tau(y_j) \\ &\leq 2^{m-4}\delta\tau(z) + 8\delta\left[c_2\,|z-y_j| + \tau(z)\right] \\ &< 2^{m-4}\delta\tau(z) + \frac{1}{2}|z-y_j| + 8\delta\tau(z),\end{aligned}$$

where the second inequality comes from the fact that $y_j \in D(\delta\tau(z_j))$ plus an inequality proved earlier, the third inequality from the Lipschitz condition (B), and the fourth one from $\delta c_2 < \frac{1}{16}$.

Thus, we obtain that

$$\frac{1}{2}|z - y_j| < 2^{m-4}\delta\tau(z) + 8\delta\tau(z) \leq 2^{m-4}\delta\tau(z) + 2^{m-4}\delta\tau(z) = 2^{m-3}\delta\tau(z),$$

as $m \geq 7$, and so

$$|z - y_j| < 2^{m-2}\delta\tau(z).$$

On the other hand, since $\tau(z) \leq \tau(\zeta) \leq 2\tau(z_j)$, and $y_j \in D(\delta\tau(z_j))$, we get that

$$|z - y_j| < 2^{m-2}\delta\tau(z) \leq 2^{m-1}\delta\tau(z_j) < 2^m\delta\tau(y_j).$$

Hence,

$$|z - y_j| < 2^m\delta \min(\tau(z), \tau(y_j)),$$

and this means that $y_j \in D_m(z)$ which violates $y_j \in D_m(z)^c$.

Case 2. If $\min(\tau(z), \tau(\zeta)) = \tau(\zeta)$, then

$$\begin{aligned}
|z-y_j| &< 2^{m-4}\delta\,\min(\tau(z),\tau(\zeta)) + 4\delta\tau(z_j) \\
&= 2^{m-4}\delta\,\tau(\zeta) + 4\delta\tau(z_j) \\
&< 2^{m}\delta\tau(y_j).
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
|z-y_j| &< 2^{m-4}\delta\,\min(\tau(z),\tau(\zeta)) + 4\delta\tau(z_j) \\
&= 2^{m-4}\delta\,\tau(\zeta) + 4\delta\tau(z_j) \\
&\le 2^{m-4}\delta\tau(\zeta) + 8\delta\tau(\zeta) \\
&< 2^{m}\delta\tau(z).
\end{aligned}$$

So, we again get that

$$|z-y_j| < 2^m\delta\,\min(\tau(z),\tau(y_j)),$$

and this gives that $y_j \in D_m(z)$ which negates $y_j \in D_m(z)^c$ again.

We conclude that $\zeta \in D_{m-4}(z)^c$, and therefore, we have shown that for $m \geq 7$,

$$\bigcup_{j\in J_{m,z}} D(3\delta\tau(z_j)) \subseteq D_{m-4}(z)^c. \qquad \square$$

We notice that by the condition (A), the positive function $\tau$ vanishes on the boundary of the unit disk, and so by the definition of the set $D_m(z)$, the points which are close enough to the boundary are not in $D_m(z)$. Since the area of $D(\delta\tau(z_j))$ vanishes as $z_j$ approaches the boundary, there are infinitely many elements of the lattice $\{z_j\}_{j=1}^{\infty}$ near the boundary, and therefore we conclude that the set $J_{m,z}$ is infinite.

**Lemma 3.1.2.** *Let $\delta \in (0, \mathfrak{M}_\tau)$, $z \in \mathbb{D}$ and $m \geq 4$. Then, $D_{\frac{m}{4}}(\xi) \subseteq D_m(z)$, for any $\xi \in D_{\frac{m}{4}}(z)$.*

**Proof.** Fix $\xi \in D_{\frac{m}{4}}(z)$ and let $\zeta \in D_{\frac{m}{4}}(\xi)$ be arbitrary. Thus, we have that

$$\frac{|z-\xi|}{\min(\tau(z), \tau(\xi))} < 2^{\frac{m}{4}}\delta \qquad \text{and} \qquad \frac{|\xi-\zeta|}{\min(\tau(\xi), \tau(\zeta))} < 2^{\frac{m}{4}}\delta. \tag{3.2}$$

Combining the first one with the Lipschitz condition (B), we get that

$$\tau(\xi) < (1 + 2^{\frac{m}{4}}\delta c_2)\,\tau(z) \qquad \text{and} \qquad \tau(z) < (1 + 2^{\frac{m}{4}}\delta c_2)\,\tau(\xi). \tag{3.3}$$

Similarly, the second inequality in (3.2) and the Lipschitz condition (B) gives that

$$\tau(\zeta) < (1 + 2^{\frac{m}{4}}\delta c_2)\,\tau(\xi) \qquad \text{and} \qquad \tau(\xi) < (1 + 2^{\frac{m}{4}}\delta c_2)\,\tau(\zeta). \tag{3.4}$$

We need to show that $\zeta \in D_m(z)$. By (3.2), we have that

$$|z-\zeta| \leq |z-\xi| + |\xi-\zeta| < 2^{\frac{m}{4}}\delta \min(\tau(z),\tau(\xi)) + 2^{\frac{m}{4}}\delta \min(\tau(\xi),\tau(\zeta))$$

There are four cases and we show that $\zeta \in D_m(z)$ in any case.

Case 1. If $\min(\tau(z),\tau(\xi)) = \tau(\xi)$ and $\min(\tau(\xi),\tau(\zeta)) = \tau(\xi)$, then

$$|z-\zeta| < 2^{\frac{m}{4}}\delta\,\tau(\xi) + 2^{\frac{m}{4}}\delta\,\tau(\xi) < 2^{m}\delta\,\tau(\xi) \leq 2^{m}\delta \min(\tau(z),\tau(\zeta)),$$

and hence, $\zeta \in D_m(z)$.

Case 2. If $\min(\tau(z),\tau(\xi)) = \tau(z)$ and $\min(\tau(\xi),\tau(\zeta)) = \tau(\xi)$, then by using (3.3), we get that

$$\begin{aligned}
|z-\zeta| &< 2^{\frac{m}{4}}\delta\,\tau(z) + 2^{\frac{m}{4}}\delta\,\tau(\xi) \\
&< 2^{\frac{m}{4}}\delta\,\tau(z) + 2^{\frac{m}{4}}\delta\,(1+2^{\frac{m}{4}}\delta c_2)\,\tau(z) \\
&< 2^{m}\delta\,\tau(z) \\
&= 2^{m}\,\delta \min(\tau(z),\tau(\zeta)),
\end{aligned}$$

since $\delta c_2 < \frac{1}{16}$, which gives us $\zeta \in D_m(z)$.

Case 3. If $\min(\tau(z),\tau(\xi))=\tau(\xi)$ and $\min(\tau(\xi),\tau(\zeta))=\tau(\zeta)$, then by (3.4),

$$\begin{aligned}
|z-\zeta| &< 2^{\frac{m}{4}}\delta\,\tau(\xi)+2^{\frac{m}{4}}\delta\,\tau(\zeta)\\
&< 2^{\frac{m}{4}}\delta\,(1+2^{\frac{m}{4}}\delta c_2)\,\tau(\zeta)+2^{\frac{m}{4}}\delta\,\tau(\zeta)\\
&< 2^{m}\delta\,\tau(\zeta)\\
&= 2^{m}\,\delta\,\min(\tau(z),\tau(\zeta)),
\end{aligned}$$

since $\delta c_2 < \frac{1}{16}$, which implies that $\zeta\in D_m(z)$.

Case 4. If $\min(\tau(z),\tau(\xi))=\tau(z)$ and $\min(\tau(\xi),\tau(\zeta))=\tau(\zeta)$, then by (3.3),

$$\begin{aligned}
|z-\zeta| &< 2^{\frac{m}{4}}\delta\,\tau(z)+2^{\frac{m}{4}}\delta\,\tau(\zeta)\\
&\le 2^{\frac{m}{4}}\delta\,\tau(z)+2^{\frac{m}{4}}\delta\,\tau(\xi)\\
&< 2^{\frac{m}{4}}\delta\,\tau(z)+2^{\frac{m}{4}}\delta\,(1+2^{\frac{m}{4}}\delta c_2)\,\tau(z)\\
&< 2^{m}\delta\,\tau(z),
\end{aligned}$$

and similarly, by (3.4), $|z-\zeta|<2^m\delta\,\tau(\zeta)$. Putting these two together, we obtain that

$$|z-\zeta| < 2^m\,\delta\,\min(\tau(z),\tau(\zeta)),$$

or $\zeta\in D_m(z)$ which is the desired conclusion. So, we have shown that

$$D_{\frac{m}{4}}(\xi) \subseteq D_m(z), \quad \forall\, \xi \in D_{\frac{m}{4}}(z). \qquad \square$$

## 3.2 Weak Localization on $L^2_a(\omega)$ with $\omega \in \mathcal{E}$

The definition of weak localization of operators on a class of weighted Bergman spaces with specific radial decreasing weights was given in [4]. We will use the same definition also for the Bergman spaces $L^2_a(\omega)$ with the weight in the class $\mathcal{E}$.

**Definition 3.2.1.** For $\omega \in \mathcal{E}$, a linear operator $T$ on $L^2_a(\omega)$ is said to be weakly localized if the following conditions hold:

$$(i)\ \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left|\langle Tk_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} < \infty,\ \ \mathit{and}$$

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left|\langle T^* k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} < \infty.$$

$$(ii)\ \lim_{m\to\infty} \sup_{z\in\mathbb{D}} \int_{D_m(z)^c} \left|\langle Tk_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} = 0,\ \ \mathit{and}$$

$$\lim_{m\to\infty} \sup_{z\in\mathbb{D}} \int_{D_m(z)^c} \left|\langle T^* k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} = 0,$$

where

$$D_m(z) = \left\{\zeta \in \mathbb{D} \,:\, d_\tau(z,\zeta) = \frac{|z-\zeta|}{\min(\tau(z),\tau(\zeta))} < 2^m\delta\right\}.$$

We create the tools we need for the proof of our main result in this chapter, namely of Theorem 3.3.1, in three lemmas. First, we recall that the Forelli-Rudin estimates were essential tools for estimating integrals in chapter 2. The following lemma will play the role of Forelli-Rudin estimates for the space $L^2_a(\omega)$.

**Lemma 3.2.2.** Let $\omega \in \mathcal{E}$. Then

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left|\langle k^\omega_z, k^\omega_\zeta\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} < \infty.$$

**Proof.** We split the integral into two integrals, one over the disk $D(\delta\tau(z))$, and the other over its complement. We show that each of these integrals is bounded above by a constant independent of $z$.

$$I(z) = \int_{\mathbb{D}} \left|\langle k^\omega_z, k^\omega_\zeta\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} = I_1(z) + I_2(z),$$

where

$$I_1(z) = \int_{D(\delta\tau(z))} \left|\langle k^\omega_z, k^\omega_\zeta\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2},$$

and

$$I_2(z) = \int_{D(\delta\tau(z))^c} \left|\langle k^\omega_z, k^\omega_\zeta\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2}.$$

By Lemma 1.4.17 with $M = 1$, we have that

$$
\begin{aligned}
I_1(z) &= \int_{D(\delta\tau(z))} \left|\langle k_z^\omega, k_\zeta^\omega \rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \\
&\lesssim \int_{D(\delta\tau(z))} \frac{\min(\tau(z), \tau(\zeta))}{|z-\zeta|} \frac{dA(\zeta)}{\tau(\zeta)^2} \\
&\leq \int_{D(\delta\tau(z))} \frac{1}{|z-\zeta|} \frac{dA(\zeta)}{\tau(\zeta)} \\
&\approx \int_{D(\delta\tau(z))} \frac{1}{|z-\zeta|} \frac{dA(\zeta)}{\tau(z)} \\
&= \tau(z)^{-1} \int_{|\zeta|<\delta\tau(z)} \frac{1}{|\zeta|} \, dA(\zeta).
\end{aligned}
$$

Integrating in polar coordinates, we get that

$$
\begin{aligned}
I_1(z) &\lesssim \tau(z)^{-1} \int_{|\zeta|<\delta\tau(z)} \frac{1}{|\zeta|} \, dA(\zeta) \\
&= \tau(z)^{-1} \int_0^{2\pi} \int_0^{\delta\tau(z)} r^{-1} \, \frac{r \, dr \, d\theta}{\pi} \\
&= \tau(z)^{-1} \left(2\delta\tau(z)\right) \\
&= 2\delta < \infty.
\end{aligned}
$$

To estimate the second integral, we use Lemma 1.4.17 with $M = 3$.

$$
\begin{aligned}
I_2(z) &= \int_{D(\delta\tau(z))^c} \left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \\
&\lesssim \int_{D(\delta\tau(z))^c} \left(\frac{\min(\tau(z),\tau(\zeta))}{|z-\zeta|}\right)^3 \frac{dA(\zeta)}{\tau(\zeta)^2} \\
&\leq \tau(z) \int_{D(\delta\tau(z))^c} \frac{1}{|z-\zeta|^3}\, dA(\zeta) \\
&\leq \tau(z) \sum_i \int_{G_i} \frac{1}{|z-\zeta|^3}\, dA(\zeta),
\end{aligned}
$$

where

$$
G_i = \Big\{\zeta \in \mathbb{D} \,:\, 2^{i-1}\,\delta\,\tau(z) \leq |z-\zeta| < 2^i\,\delta\,\tau(z)\Big\},
$$

for $i = 1, 2, \ldots$ Hence, we obtain that

$$
\begin{aligned}
I_2(z) &\lesssim \tau(z) \sum_i \int_{G_i} \frac{1}{|z-\zeta|^3}\, dA(\zeta) \\
&\leq \tau(z) \sum_i \int_{G_i} \frac{1}{(2^{i-1}\delta\tau(z))^3}\, dA(\zeta) \\
&= \frac{\tau(z)^{-2}}{\delta^3} \sum_i \frac{1}{2^{3i-3}}\, A(G_i) \\
&= \frac{\tau(z)^{-2}}{\delta^3} \sum_i \frac{1}{2^{3i-3}} \left[\Big(2^i\delta\tau(z)\Big)^2 - \Big(2^{i-1}\delta\tau(z)\Big)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\tau(z)^{-2}}{\delta^3} \sum_i \frac{1}{2^{3i-3}}\, \delta^2\, \tau(z)^2 \left[2^{2i} - 2^{2i-2}\right] \\
&= \frac{3}{\delta} \sum_i \frac{1}{2^{i-1}} \; < \infty.
\end{aligned}
$$

Therefore, we conclude that

$$
\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} < \infty. \qquad \square
$$

Next, we create the corresponding versions of Lemma 2.1.2 and Lemma 2.1.3 from chapter 2, which will be used in the proof of the main theorem of this chapter. We recall that $\delta \in (0, \mathfrak{M}_\tau)$ is always fixed and

$$
D_m(z) = \left\{\zeta \in \mathbb{D} \,:\, d_\tau(z,\zeta) = \frac{|z-\zeta|}{\min(\tau(z), \tau(\zeta))} < 2^m\delta\right\}
$$

for $m \geq 1$, and for $z \in \mathbb{D}$, where

$$
\mathfrak{M}_\tau = \frac{\min(1, c_1^{-1}, c_2^{-1})}{16}\,.
$$

**Lemma 3.2.3.** Let $\omega \in \mathcal{E}$. Then

$$
\lim_{m\to\infty} \sup_{z\in\mathbb{D}} \int_{D_m(z)^c} \left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} = 0.
$$

**Proof.** By the definition of the set $D_m(z)^c$, we see that

$$I(z,m) = \int_{D_m(z)^c} \left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2}$$
$$\leq I_1(z,m) + I_2(z,m),$$

where

$$I_1(z,m) = \int_{|z-\zeta|\geq 2^m\delta\tau(z)} \left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2}\ ,$$

and

$$I_2(z,m) = \int_{|z-\zeta|\geq 2^m\delta\tau(\zeta)} \left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2}.$$

We show that each of these integrals is bounded above by a constant independent of $z$, that tends to zero as $m\to\infty$.

Applying Lemma 1.4.17 with $M = 3$ , we get that

$$\begin{aligned}
I_1(z,m) &= \int_{|z-\zeta|\geq 2^m\delta\tau(z)} \left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2}\\
&\lesssim \int_{|z-\zeta|\geq 2^m\delta\tau(z)} \left(\frac{\min(\tau(z),\tau(\zeta))}{|z-\zeta|}\right)^3 \ \frac{dA(\zeta)}{\tau(\zeta)^2}\\
&\leq \tau(z)\int_{|z-\zeta|\geq 2^m\delta\tau(z)} \frac{1}{|z-\zeta|^3}\, dA(\zeta)\\
&\leq \tau(z)\sum_j \int_{R_{m,j}} \frac{1}{|z-\zeta|^3}\, dA(\zeta),
\end{aligned}$$

where $R_{m,j} = \Big\{\zeta \in \mathbb{D} \,:\, 2^{m+j}\delta\,\tau(z) \le |z-\zeta| \le 2^{m+j+1}\delta\,\tau(z)\Big\}$. So, we have that

$$\begin{aligned}
I_1(z,m) &\lesssim \tau(z) \sum_j \int_{R_{m,j}} \frac{1}{|z-\zeta|^3}\, dA(\zeta) \\
&\le \tau(z) \sum_j \int_{R_{m,j}} \frac{1}{(2^{m+j}\delta\tau(z))^3}\, dA(\zeta) \\
&= \frac{\tau(z)^{-2}}{\delta^3} \sum_j \frac{1}{2^{3(m+j)}}\, A(R_{m,j}) \\
&= \frac{\tau(z)^{-2}}{\delta^3} \sum_j \frac{1}{2^{3(m+j)}} \left[ \Big(2^{m+j+1}\delta\tau(z)\Big)^2 - \Big(2^{m+j}\delta\tau(z)\Big)^2 \right] \\
&= \frac{\tau(z)^{-2}}{\delta^3} \sum_j \frac{1}{2^{3(m+j)}}\, \delta^2\, \tau(z)^2 \left[ 2^{2(m+j+1)} - 2^{2(m+j)} \right] \\
&= \frac{3}{2^m\delta} \sum_j \frac{1}{2^j} \\
&\lesssim \frac{1}{2^m} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty\,.
\end{aligned}$$

To estimate the second integral, we apply Lemma 1.4.17 with $M = 4$.

$$\begin{aligned}
I_2(z,m) &= \int_{|z-\zeta|\ge 2^m\delta\tau(\zeta)} \Big|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\Big| \frac{dA(\zeta)}{\tau(\zeta)^2} \\
&\lesssim \int_{|z-\zeta|\ge 2^m\delta\tau(\zeta)} \left( \frac{\min(\tau(z),\tau(\zeta))}{|z-\zeta|} \right)^4 \frac{dA(\zeta)}{\tau(\zeta)^2} \\
&\le \frac{\tau(z)}{2^m\delta} \int_{|z-\zeta|\ge 2^m\delta\tau(\zeta)} \frac{1}{|z-\zeta|^3}\, dA(\zeta)\,.
\end{aligned}$$

By the Lipschitz condition (B), if $|z-\zeta| \geq 2^m\delta\,\tau(\zeta)$, then

$$|z-\zeta| \geq 2^m\delta\,\tau(\zeta) \geq 2^m\delta\Big[\tau(z) - c_2|z-\zeta|\Big] = 2^m\delta\,\tau(z) - 2^m\delta c_2|z-\zeta|,$$

which implies that,

$$(1+2^m\delta\,c_2)|z-\zeta| \geq 2^m\delta\,\tau(z),$$

and using the fact that $\delta c_2 < \frac{1}{16}$, we obtain that

$$|z-\zeta| > \frac{2^m\delta\,\tau(z)}{1+2^{m-4}} \geq \frac{2^m\delta\,\tau(z)}{2^{m-3}} = 8\delta\,\tau(z),$$

for all $m \geq 4$. On the other hand, as shown in the proof of Lemma 3.2.2, we have that

$$\tau(z)\int_{|z-\zeta|\geq\delta\tau(\zeta)} \frac{1}{|z-\zeta|^3}\,dA(\zeta) \lesssim 1,$$

and so, we get that

$$\begin{aligned} I_2(z,m) &\lesssim \frac{\tau(z)}{2^m\delta}\int_{|z-\zeta|\geq 2^m\delta\tau(\zeta)} \frac{1}{|z-\zeta|^3}\,dA(\zeta) \\ &\leq \frac{\tau(z)}{2^m\delta}\int_{|z-\zeta|>8\delta\tau(z)} \frac{1}{|z-\zeta|^3}\,dA(z) \\ &\lesssim \frac{1}{2^m} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty\,. \end{aligned}$$

Therefore, we have proven that

$$\lim_{m\to\infty}\sup_{z\in\mathbb{D}}\int_{D_m(z)^c}\left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right|\frac{dA(\zeta)}{\tau(\zeta)^2}=0,$$

which is the desired conclusion. □

Note that by Lemma 3.2.2 and Lemma 3.2.3, it follows that the identity operator is weakly localized on the Bergman space $L_a^2(\omega)$ with $\omega\in\mathcal{E}$.

**Lemma 3.2.4.** Let $\mu$ be a Carleson measure for the weighted Bergman space $L_a^2(\omega)$ with $\omega\in\mathcal{E}$. Then there exists a constant $C$ such that

(a) $\displaystyle\int_{\mathbb{D}}\left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right|\frac{d\mu(\zeta)}{\tau(\zeta)^2}\le C\int_{\mathbb{D}}\left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right|\frac{dA(\zeta)}{\tau(\zeta)^2},\quad \forall\, z\in\mathbb{D}.$

(b) $\displaystyle\int_{D_m(z)^c}\left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right|\frac{d\mu(\zeta)}{\tau(\zeta)^2}\le C\int_{D_{m-4}(z)^c}\left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right|\frac{dA(\zeta)}{\tau(\zeta)^2},\quad \forall\, m\ge 7,\ \forall\, z\in\mathbb{D}.$

**Proof.** Fix $\delta\in(0,\mathfrak{M}_\tau)$. By Theorem 1.4.9, there exists a sequence $\{z_i\}$ such that

$$\mathbb{D}=\bigcup_i D(\delta\tau(z_i));\tag{3.5}$$

and any point of $\mathbb{D}$ belongs to at most $N$ disks of the covering $\{D(3\delta\tau(z_i))\}_i$. By Lemma 1.4.12, there exists a constant $C\ge 1$ such that

$$|k_z^\omega(\zeta)|\,\omega(\zeta)^{\frac{1}{2}}\le\frac{C}{\delta^2\tau(\zeta)^2}\int_{D(\delta\tau(\zeta))}|k_z^\omega(\xi)|\,\omega(\xi)^{\frac{1}{2}}\,dA(\xi),$$

for any $z \in \mathbb{D}$, and for any $\zeta \in D(\delta\tau(z_i))$. We also have that

$$D(\delta\tau(\zeta)) \subseteq D(3\delta\tau(z_i)), \quad \forall\, \zeta \in D(\delta\tau(z_i)). \tag{3.6}$$

Hence, by (1.9) and (3.6), we get that

$$\begin{aligned}
|k_z^\omega(\zeta)| \,\frac{\omega(\zeta)^{\frac{1}{2}}}{\tau(\zeta)} &\lesssim \frac{1}{\tau(\zeta)^2} \int_{D(\delta\tau(\zeta))} |k_z^\omega(\xi)|\, \omega(\xi)^{\frac{1}{2}}\, \frac{dA(\xi)}{\tau(\zeta)} \\
&\lesssim \frac{1}{\tau(\zeta)^2} \int_{D(\delta\tau(\zeta))} |k_z^\omega(\xi)|\, \omega(\xi)^{\frac{1}{2}}\, \frac{dA(\xi)}{\tau(\xi)} \\
&\leq \frac{1}{\tau(\zeta)^2} \int_{D(3\delta\tau(z_i))} |k_z^\omega(\xi)|\, \omega(\xi)^{\frac{1}{2}}\, \frac{dA(\xi)}{\tau(\xi)} \\
&\lesssim \frac{1}{\tau(z_i)^2} \int_{D(3\delta\tau(z_i))} |k_z^\omega(\xi)|\, \omega(\xi)^{\frac{1}{2}}\, \frac{dA(\xi)}{\tau(\xi)}\,,
\end{aligned}$$

for any $\zeta \in D(\delta\tau(z_i))$. Thus,

$$|k_z^\omega(\zeta)| \,\frac{\omega(\zeta)^{\frac{1}{2}}}{\tau(\zeta)} \lesssim \frac{1}{\tau(z_i)^2} \int_{D(3\delta\tau(z_i))} |k_z^\omega(\xi)|\, \frac{\omega(\xi)^{\frac{1}{2}}}{\tau(\xi)}\, dA(\xi)\,, \tag{3.7}$$

for any $\zeta \in D(\delta\tau(z_i))$. For $i \in \mathbb{N}$, and $z \in \mathbb{D}$, we define

$$S_i(z) = \sup_{\zeta \in D(\delta\tau(z_i))} |k_z^\omega(\zeta)|\, \frac{\omega(\zeta)^{\frac{1}{2}}}{\tau(\zeta)}\,,$$

which is finite by (3.7) and Lemma 3.2.2. By (3.7), we get that

$$S_i(z)\,\tau(z_i)^2 \lesssim \int_{D(3\delta\tau(z_i))} |k_z^\omega(\xi)|\,\frac{\omega(\xi)^{\frac{1}{2}}}{\tau(\xi)}\,dA(\xi)\,, \tag{3.8}$$

for any $i \in \mathbb{N}$, and any $z \in \mathbb{D}$.

$(a)$ By Lemma 1.4.15, and (3.5) we obtain that

$$\begin{aligned}
\int_{\mathbb{D}} \left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{d\mu(\zeta)}{\tau(\zeta)^2} &\lesssim \int_{\mathbb{D}} |k_z^\omega(\zeta)|\,\frac{\omega(\zeta)^{\frac{1}{2}}}{\tau(\zeta)}\,d\mu(\zeta)\\
&\lesssim \int_{\bigcup_{i=1}^n D(\delta\tau(z_i))} |k_z^\omega(\zeta)|\,\frac{\omega(\zeta)^{\frac{1}{2}}}{\tau(\zeta)}\,dA(\zeta)\\
&\leq \sum_{i=1}^\infty \int_{D(\delta\tau(z_i))} |k_z^\omega(\zeta)|\,\frac{\omega(\zeta)^{\frac{1}{2}}}{\tau(\zeta)}\,d\mu(\zeta)\\
&\leq \sum_{i=1}^\infty S_i(z)\,\mu\Big(D(\delta\tau(z_i))\Big)\\
&= \sum_{i=1}^\infty S_i(z)\,\frac{\mu\Big(D(\delta\tau(z_i))\Big)}{\tau(z_i)^2}\,\tau(z_i)^2,
\end{aligned}$$

but since $\mu$ is a Carleson measure for $L_a^2(\omega)$, by Theorem 1.4.20 and (3.8), and since $\{D(3\delta\tau(z_i))\}_i$ is a covering of the disk with bounded multiplicity,

$$\lesssim \sum_{i=1}^\infty S_i(z)\,\tau(z_i)^2$$

$$
\begin{aligned}
&\lesssim \sum_{i=1}^{\infty} \int_{D(3\delta\tau(z_i))} |k_z^{\omega}(\xi)| \, \frac{\omega(\xi)^{\frac{1}{2}}}{\tau(\xi)} \, dA(\xi) \\
&\lesssim \int_{\bigcup_{i=1}^{\infty} D(3\delta\tau(z_i))} |k_z^{\omega}(\xi)| \, \frac{\omega(\xi)^{\frac{1}{2}}}{\tau(\xi)} \, dA(\xi) \\
&= \int_{\mathbb{D}} |k_z^{\omega}(\xi)| \, \frac{\omega(\xi)^{\frac{1}{2}}}{\tau(\xi)} \, dA(\xi) \\
&\lesssim \int_{\mathbb{D}} \left| \langle k_z^{\omega}, k_\xi^{\omega} \rangle_{\omega} \right| \frac{dA(\xi)}{\tau(\xi)^2} \, ,
\end{aligned}
$$

where the last inequality comes from Lemma 1.4.15. Therefore, we have shown that

$$
\int_{\mathbb{D}} \left| \langle k_z^{\omega}, k_\zeta^{\omega} \rangle_{\omega} \right| \frac{d\mu(\zeta)}{\tau(\zeta)^2} \leq C \int_{\mathbb{D}} \left| \langle k_z^{\omega}, k_\zeta^{\omega} \rangle_{\omega} \right| \frac{dA(\zeta)}{\tau(\zeta)^2}, \qquad \forall\, z \in \mathbb{D}.
$$

$(b)$ For $m \geq 7$ and $z \in \mathbb{D}$, we define the family

$$
J_{m,z} = \Big\{ j \ : \ D(\delta\tau(z_j)) \cap D_m(z)^c \neq \emptyset \Big\}.
$$

So, by the definition and property (1) of a $(\delta, \tau)$-lattice, we have that

$$
D_m(z)^c \subseteq \bigcup_{j \in J_{m,z}} D(\delta\tau(z_j)),
$$

and therefore, by Lemma 3.1.1, we get that

$$\bigcup_{j\in J_{m,z}} D(3\delta\tau(z_j)) \subseteq D_{m-4}(z)^c,\ \forall\, m \geq 7. \tag{3.9}$$

Now, we are ready to prove that $(b)$ holds. By Lemma 1.4.15, (3.1), and (3.9) we obtain that

$$\begin{aligned}
\int_{D_m(z)^c} \left|\langle k_z^\omega, k_\zeta^\omega\rangle_\omega\right| \frac{d\mu(\zeta)}{\tau(\zeta)^2} &\lesssim \int_{D_m(z)^c} |k_z^\omega(\zeta)|\, \frac{\omega(\zeta)^{\frac{1}{2}}}{\tau(\zeta)}\, d\mu(\zeta) \\
&\lesssim \int_{\bigcup_{j\in J_{m,z}} D(\delta\tau(z_j))} |k_z^\omega(\zeta)|\, \frac{\omega(\zeta)^{\frac{1}{2}}}{\tau(\zeta)}\, dA(\zeta) \\
&\leq \sum_{j\in J_{m,z}} \int_{D(\delta\tau(z_j))} |k_z^\omega(\zeta)|\, \frac{\omega(\zeta)^{\frac{1}{2}}}{\tau(\zeta)}\, d\mu(\zeta) \\
&\leq \sum_{j\in J_{m,z}} S_j(z)\, \mu\Big(D(\delta\tau(z_j))\Big) \\
&= \sum_{j\in J_{m,z}} S_j(z)\, \frac{\mu\Big(D(\delta\tau(z_j))\Big)}{\tau(z_j)^2}\, \tau(z_j)^2,
\end{aligned}$$

but since $\mu$ is a Carleson measure for $L_a^2(\omega)$, by Theorem 1.4.20 and (3.8), and since $\{D(3\delta\tau(z_j))\}_j$ is a covering of the disk with bounded multiplicity,

$$\lesssim \sum_{j\in J_{m,z}} S_j(z)\, \tau(z_j)^2$$

$$
\begin{aligned}
&\lesssim \sum_{j\in J_{m,z}} \int_{D(3\delta\tau(z_j))} |k_z^\omega(\xi)| \, \frac{\omega(\xi)^{\frac{1}{2}}}{\tau(\xi)} \, dA(\xi) \\
&\lesssim \int_{\bigcup_{j\in J_{m,z}} D(3\delta\tau(z_j))} |k_z^\omega(\xi)| \, \frac{\omega(\xi)^{\frac{1}{2}}}{\tau(\xi)} \, dA(\xi) \\
&\leq \int_{D_{m-4}(z)^c} |k_z^\omega(\xi)| \, \frac{\omega(\xi)^{\frac{1}{2}}}{\tau(\xi)} \, dA(\xi) \\
&\lesssim \int_{D_{m-4}(z)^c} \left| \langle k_z^\omega, k_\xi^\omega \rangle_\omega \right| \frac{dA(\xi)}{\tau(\xi)^2}.
\end{aligned}
$$

We notice that the last two inequalities come from (3.9) and Lemma 1.4.15, respectively. Therefore, we have shown that

$$
\int_{D_m(z)^c} \left| \langle k_z^\omega, k_\zeta^\omega \rangle_\omega \right| \frac{d\mu(\zeta)}{\tau(\zeta)^2} \leq C \int_{D_{m-4}(z)^c} \left| \langle k_z^\omega, k_\zeta^\omega \rangle_\omega \right| \frac{dA(\zeta)}{\tau(\zeta)^2}, \quad \forall\, m \geq 7, \, \forall\, z \in \mathbb{D},
$$

and the proof is complete. $\square$

## 3.3 Weak Localization and Toeplitz operators induced by Complex Measures on $L_a^2(\omega)$ with $\omega \in \mathcal{E}$

The following theorem is the corresponding version of Theorem 2.1.4 for Bergman spaces $L_a^2(\omega)$ with $\omega \in \mathcal{E}$. It is the main result of this chapter. The general idea is similar, however, some technical difficulties will come up when we deal with the set $D_m(z)^c$, as $d_\tau$ is not a metric and so $D_m(z)^c$ is not a disk.

**Theorem 3.3.1.** Let $\omega \in \mathcal{E}$ and $\nu$ be a complex Borel measure on $\mathbb{D}$ such that $|\nu|$ is a Carleson measure for $L^2_a(\omega)$. Then the Toeplitz operator $T^\omega_\nu$ is weakly localized on $L^2_a(\omega)$.

**Proof.** By Lemma 1.5.7, we have that

$$\left|\langle T^\omega_\nu k^\omega_z, k^\omega_\zeta\rangle_\omega\right| = \left|\int_{\mathbb{D}} k^\omega_z(\xi)\,\overline{k^\omega_\zeta(\xi)}\,\omega(\xi)\,d\nu(\xi)\right| \leq \int_{\mathbb{D}} \left|k^\omega_z(\xi)\right|\left|k^\omega_\zeta(\xi)\right|\omega(\xi)\,d|\nu|(\xi).$$

Therefore, by Fubini's theorem and Lemma 1.4.15, we get that

$$\begin{aligned}
\int_{\mathbb{D}} \left|\langle T^\omega_\nu k^\omega_z, k^\omega_\zeta\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} &\leq \int_{\mathbb{D}}\int_{\mathbb{D}} \left|k^\omega_z(\xi)\right|\left|k^\omega_\zeta(\xi)\right|\omega(\xi)\,d|\nu|(\xi)\,\frac{dA(\zeta)}{\tau(\zeta)^2} \\
&= \int_{\mathbb{D}}\int_{\mathbb{D}} \left|k^\omega_z(\xi)\right|\left|k^\omega_\zeta(\xi)\right|\omega(\xi)\,\frac{dA(\zeta)}{\tau(\zeta)^2}\,d|\nu|(\xi) \\
&= \int_{\mathbb{D}}\int_{\mathbb{D}} \left|\langle k^\omega_z, k^\omega_\xi\rangle_\omega\right|\left|\langle k^\omega_\zeta, k^\omega_\xi\rangle_\omega\right| \left\|K^\omega_\xi\right\|^2_\omega \omega(\xi)\,\frac{dA(\zeta)}{\tau(\zeta)^2}\,d|\nu|(\xi) \\
&\approx \int_{\mathbb{D}}\int_{\mathbb{D}} \left|\langle k^\omega_z, k^\omega_\xi\rangle_\omega\right|\left|\langle k^\omega_\zeta, k^\omega_\xi\rangle_\omega\right| \;\frac{dA(\zeta)}{\tau(\zeta)^2}\,\frac{d|\nu|(\xi)}{\tau(\xi)^2} \\
&= \int_{\mathbb{D}} \left|\langle k^\omega_z, k^\omega_\xi\rangle_\omega\right| \int_{\mathbb{D}} \left|\langle k^\omega_\zeta, k^\omega_\xi\rangle_\omega\right| \;\frac{dA(\zeta)}{\tau(\zeta)^2}\,\frac{d|\nu|(\xi)}{\tau(\xi)^2} \\
&\leq \left(\sup_{\xi\in\mathbb{D}} \int_{\mathbb{D}} \left|\langle k^\omega_\zeta, k^\omega_\xi\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2}\right)\left(\int_{\mathbb{D}} \left|\langle k^\omega_z, k^\omega_\xi\rangle_\omega\right| \frac{d|\nu|(\xi)}{\tau(\xi)^2}\right).
\end{aligned}$$

By Lemma 3.2.2 and part $(a)$ of Lemma 3.2.4, we conclude that

$$\begin{aligned}\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\left|\langle T_\nu^\omega k_z^\omega, k_\zeta^\omega\rangle_\omega\right|\frac{dA(\zeta)}{\tau(\zeta)^2} &\leq \left(\sup_{\xi\in\mathbb{D}}\int_{\mathbb{D}}\left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right|\frac{dA(\zeta)}{\tau(\zeta)^2}\right)\left(\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right|\frac{d|\nu|(\xi)}{\tau(\xi)^2}\right)\\ &\lesssim \left(\sup_{\xi\in\mathbb{D}}\int_{\mathbb{D}}\left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right|\frac{dA(\zeta)}{\tau(\zeta)^2}\right)\left(\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right|\frac{dA(\xi)}{\tau(\xi)^2}\right)\\ &< \infty.\end{aligned}$$

Therefore, as the adjoint of $T_\nu^\omega$ is $T_{\overline{\nu}}^\omega$, and $|\overline{\nu}| = |\nu|$, we have established that the condition $(i)$ from the definition of weak localization is satisfied.

Next, we use Lemma 3.1.2 to prove that the condition $(ii)$ in the Definition 3.2.1 also holds. By Fubini's theorem and Lemma 1.4.15, we obtain that

$$\begin{aligned}\int_{D_m(z)^c}\left|\langle T_\nu^\omega k_z^\omega, k_\zeta^\omega\rangle_\omega\right|\frac{dA(\zeta)}{\tau(\zeta)^2} &\leq \int_{D_m(z)^c}\int_{\mathbb{D}}\left|k_z^\omega(\xi)\right|\left|k_\zeta^\omega(\xi)\right|\omega(\xi)\,d|\nu|(\xi)\,\frac{dA(\zeta)}{\tau(\zeta)^2}\\ &= \int_{\mathbb{D}}\int_{D_m(z)^c}\left|k_z^\omega(\xi)\right|\left|k_\zeta^\omega(\xi)\right|\omega(\xi)\,\frac{dA(\zeta)}{\tau(\zeta)^2}\,d|\nu|(\xi)\\ &= \int_{\mathbb{D}}\int_{D_m(z)^c}\left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right|\left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right|\left\|K_\xi^\omega\right\|_\omega^2\omega(\xi)\,\frac{dA(\zeta)}{\tau(\zeta)^2}\,d|\nu|(\xi)\\ &\approx \int_{\mathbb{D}}\int_{D_m(z)^c}\left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right|\left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right|\,\frac{dA(\zeta)}{\tau(\zeta)^2}\,\frac{d|\nu|(\xi)}{\tau(\xi)^2}\\ &= I_1(z,m) + I_2(z,m),\end{aligned}$$

where

$$I_1(z,m) = \int_{D_{m/4}(z)}\int_{D_m(z)^c}\left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right|\left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right|\,\frac{dA(\zeta)}{\tau(\zeta)^2}\,\frac{d|\nu|(\xi)}{\tau(\xi)^2},$$

and

$$I_2(z,m) = \int_{D_{m/4}(z)^c} \int_{D_m(z)^c} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \frac{d|\nu|(\xi)}{\tau(\xi)^2}.$$

By Lemma 3.1.2, and part $(a)$ of Lemma 3.2.4,

$$\begin{aligned}
I_1(z,m) &= \int_{D_{m/4}(z)} \int_{D_m(z)^c} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \frac{d|\nu|(\xi)}{\tau(\xi)^2} \\
&\le \int_{D_{m/4}(z)} \int_{D_{m/4}(\xi)^c} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \frac{d|\nu|(\xi)}{\tau(\xi)^2} \\
&\le \int_{\mathbb{D}} \int_{D_{m/4}(\xi)^c} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \frac{d|\nu|(\xi)}{\tau(\xi)^2} \\
&\le \left( \sup_{\xi\in\mathbb{D}} \int_{D_{m/4}(\xi)^c} \left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \right) \left( \int_{\mathbb{D}} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \frac{d|\nu|(\xi)}{\tau(\xi)^2} \right) \\
&\le \left( \sup_{\xi\in\mathbb{D}} \int_{D_{m/4}(\xi)^c} \left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \right) \left( \int_{\mathbb{D}} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\xi)}{\tau(\xi)^2} \right).
\end{aligned}$$

So, by Lemma 3.2.2 and Lemma 3.2.3, we get that

$$\sup_{z\in\mathbb{D}} I_1(z,m) \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.$$

Applying Lemma 3.2.2 and part $(b)$ of Lemma 3.2.4, we get that

$$\begin{aligned}
I_2(z,m) &= \int_{D_{m/4}(z)^c} \int_{D_m(z)^c} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \frac{d|\nu|(\xi)}{\tau(\xi)^2} \\
&\le \int_{D_{m/4}(z)^c} \int_{\mathbb{D}} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \frac{d|\nu|(\xi)}{\tau(\xi)^2}
\end{aligned}$$

$$
\begin{aligned}
&\le \left( \sup_{\xi\in\mathbb{D}} \int_{\mathbb{D}} \left|\langle k_\zeta^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\zeta)}{\tau(\zeta)^2} \right) \left( \int_{D_{m/4}(z)^c} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \frac{d|\nu|(\xi)}{\tau(\xi)^2} \right) \\
&\lesssim \int_{D_{m/4}(z)^c} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \frac{d|\nu|(\xi)}{\tau(\xi)^2} \\
&\lesssim \int_{D_{m'}(z)^c} \left|\langle k_z^\omega, k_\xi^\omega\rangle_\omega\right| \frac{dA(\xi)}{\tau(\xi)^2}\,,
\end{aligned}
$$

for large enough $m$, where $m' = \dfrac{m}{4} - 4$. Hence, by Lemma 3.2.3,

$$
\sup_{z\in\mathbb{D}} I_2(z,m) \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.
$$

Therefore, we have proven that the condition $(ii)$ from the definition of weak localization also holds and the proof is complete. $\square$

# Chapter 4

# Some Open Problems and Further Research

In this chapter, we list some problems regarding localization of operators acting on the classical Bergman space, or on the Bergman spaces with weights in $\mathcal{E}$, that we have not been able to solve.

## 4.1 Weak Localization and Compactness on $L^2_a(\omega)$

In [15], it was proven that any weakly localized operator with a vanishing Berezin transform is compact on the Bergman space of the unit ball. One may ask whether the same statement holds true for the weakly localized operators on the Bergman spaces $L^2_a(\omega)$ with $\omega \in \mathcal{E}$. This question was also posed in [4] for another class of weighted Bergman spaces. We have not had time to attempt this question yet, and this is probably a hard question that would require a lot of time and effort.

## 4.2 Strong and Sufficient Localization on $L_a^2(\omega)$

Following the work done in [37], in chapter 2, strong and sufficient localization were defined on the classical weighted Bergman spaces of the open unit ball. A natural question is how we should define strong and sufficient localization on weighted Bergman spaces $L_a^2(\omega)$, with $\omega \in \mathcal{E}$. Due to the connection between compactness and localization, we believe that we first need to have a deeper study of compactness of Toeplitz operators on such spaces in order to explore the appropriate definitions of strong and sufficient localization. At this point, we are not sure how hard it would be to build up such constructions.

## 4.3 Localization of BMO($\tau$)-Toeplitz Operators

Another interesting problem is whether the Toeplitz operators with BMO($\tau$) symbols, as defined in [4], are weakly localized on $L_a^2(\omega)$ with $\omega \in \mathcal{E}$.

For $1 \leq p < \infty$, and $\delta \in (0, \mathfrak{M}_\tau)$, the space $BMO_\delta^p(\tau)$ consists of integrable functions $f$ on $\mathbb{D}$ such that

$$\|f\|_{BMO_\delta^p(\tau)} = \sup_{z\in\mathbb{D}} \left( \frac{1}{\delta^2\, \tau(z)^2} \int_{D(\delta\tau(z))} \left| f(\zeta) - \widehat{f}_\delta(z) \right|^p dA(\zeta) \right)^{1/p} < \infty,$$

where

$$\widehat{f}_\delta(z) = \frac{1}{\delta^2\,\tau(z)^2}\int_{D(\delta\tau(z))} f(\zeta)\,dA(\zeta).$$

One may attack this problem either directly through the definition of weak localization on weighted Bergman spaces $L_a^2(\omega)$, or by a similar approach as used in chapter 2, for the classical weighted Bergman spaces.

**Problem 1.** Determine whether the Toeplitz operators with BMO($\tau$) symbols are weakly localized on $L_a^2(\omega)$ with $\omega \in \mathcal{E}$.

## 4.4 Localization of Composition and Weighted Composition Operators

In chapter 2, we have shown that some classes of Toeplitz operators are weakly localized. As a result, multiplication operators are also weakly localized on the classical weighted Bergman spaces. A natural question arising is what we can say about other important classes of operators, such as, for example, the class of weighted composition operators. We give an example and a remark regarding localization of composition operators on the classical Bergman space $L_a^2(\mathbb{D}, dA)$.

We recall that for a non-constant holomorphic function $\varphi$ mapping $\mathbb{D}$ into itself, the composition operator $C_\varphi$ is defined on $L^2_a(\mathbb{D}, dA)$ by

$$C_\varphi f = f \circ \varphi,$$

and for a bounded holomorphic function $g : \mathbb{D} \longrightarrow \mathbb{C}$ and $\varphi$ as above, the weighted composition operator $W_{g,\varphi}$ is defined by

$$W_{g,\varphi} h = g(h \circ \varphi),$$

for all $h \in L^2_a(\mathbb{D}, dA)$.

**Example 4.4.1.** **[26]** A unitary composition operator defined on $L^2_a(\mathbb{D}, dA)$ is weakly localized if and only if it is the identity operator.

Suppose $C_\varphi$ is a unitary operator on $L^2_a(\mathbb{D}, dA)$. If $C_\varphi$ is the identity operator, then $C_\varphi = T_{\mathbb{1}}$, and hence by Theorem 2.1.4 it is weakly localized.

Conversely, suppose that $C_\varphi \neq Id$ is weakly localized. Since $C_\varphi$ is unitary, $\varphi$ is of the form $\varphi(z) = \lambda z$ with $|\lambda| = 1$ (see [38]), and since it is not the identity operator, we have that $\lambda \neq 1$. The Berezin transform of $C_\varphi$ is

$$\widetilde{C_\varphi}(z) = \frac{(1 - |z|^2)^2}{(1 - \lambda |z|^2)^2},$$

and since $\lambda \neq 1$, we get that

$$\lim_{|z|\to 1^-} \widetilde{C_\varphi}(z) = \lim_{|z|\to 1^-} \frac{(1-|z|^2)^2}{(1-\lambda|z|^2)^2} = 0\,.$$

If $C_\varphi$ is weakly localized, it is in the Toeplitz algebra by Xia'theorem, and since, its Berezin transform vanishes on the boundary of the unit disk, it is compact on $L_a^2(\mathbb{D}, dA)$, by [41, Theorem 9.5]. This is a contradiction, since $C_\varphi$ is unitary.

**Problem 2.** Which weighted composition operators are weakly (strongly) localized on $L_a^2(\mathbb{D}, dA)$?

We notice that if the composition operator $C_\varphi$ is weakly localized on $L_a^2(\mathbb{D}, dA)$, then by the fact that multiplication operators are weakly localized, and by [15, Proposition 2.3 ], we conclude that the weighted composition operator $W_{g,\varphi}$ is also weakly localized on $L_a^2(\mathbb{D}, dA)$, for any $g \in H^\infty$. The converse is also true if $g$ does not vanish on $\mathbb{D}$.

**Problem 3.** Which composition operators are weakly (strongly) localized on $L_a^2(\mathbb{D}, dA)$?

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