# Normal Uniform Deformations of Monotropic Hyperelastic Rods

By

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A Thesis
Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree of

**MASTER OF SCIENCE**

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**THE UNIVERSITY OF MANITOBA**

**FACULTY OF GRADUATE STUDIES**
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# Normal Uniform Deformations of Monotropic Hyperelastic Rods

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# Abstract

A directed rod theory describes the deformation of a long, thin body called a rod. The rod is modeled as curve (the rod axis) with additional structure provided by a triad of vectors at each point along the rod axis. A hyperelastic rod is one which is associated with a scalar function called the strain energy density. Field equations and constitutive restrictions form the theory describing the deformation of hyperelastic rods. The field equations are derived from a variational principle relating the virtual work (which depends on the strain energy density) to an arbitrary virtual displacement of the rod. Two constitutive restrictions are assumed to apply to the rod undergoing deformation. Material frame indifference states that strain energy density is invariant under a rotation of the rod system following a deformation. Monotropic symmetry states that strain energy density is invariant under special rotations and reflections prior to deformation. Uniform rods are those which have a constant twist about the rod axis described by a constant skew-symmetric tensor. Normal rods are those in which the cross-section of the rod is perpendicular to the rod axis at every point. Normal uniform rods have a limited number of possible shapes: straight, circular and helical. Normal uniform deformations are those deformations in which the initial and final configurations of the rod are normal and uniform. Four normal uniform deformations are solved where the initial state is straight and untwisted.

# Acknowledgements

Professor Harley Cohen - My thesis advisor who taught me tensor analysis and the rudiments of continuum mechanics and whose guidance, corrections and encouragement pushed me to succeed in writing this thesis.

Faculty and staff of the Department of Mathematics, University of Manitoba

Fellow graduate students

Family - My parents for their love, encouragement and prayers and instilling the values in me necessary to accomplishment difficult goals.

- My sisters and their families for their love and support.

Friends - Particularly John and Janelle Sadowski who have looked after me, encouraged me and been true friends throughout the duration of this work.


# Dedication

This thesis is dedicated to my Lord and Savior, Jesus Christ:

> "For by Him all things were created, both in the heavens and on earth, visible and invisible, whether thrones or dominions or rulers or authorities - all things have been created by Him and for Him. And He is before all things, and in Him all things hold together."
>
> Colossians 1:16-17

# Contents

# Chapter 1

# Introduction

Continuum mechanics is concerned with the behavior of materials, especially how they deform under the effects of forces and loads. Although it is possible to develop theories for bodies of arbitrary shape, it is useful to take advantage of specific shapes, at least in a general kind of a way. One possibility is rod theory. In rod theory, the bodies which are studied are long in one dimension and small in the remaining dimensions.

Rod theory, in turn, may be studied in more than one way. One obvious way is to consider the rod as a material body in space and model the deformation of the body as a transformation in the mathematical sense. Thus each point in the body is mapped from its location before deformation to its location after deformation. A theory based on this approach is called a derived theory.

An alternative approach is directed rod theory. In this case, we take advantage of the general shape of the rod by considering it as a curve in space with additional structure at each point along the curve. The curve is a one dimensional mathematical object. The additional structure provides the substance of the rod turning it into a three dimensional object. A deformation of a rod is a transformation from one curve to another and a transformation of the additional structure point by point along the curves.

## 1.1 Objective

The goals of this thesis are threefold:

1. develop the theory of directed rods based on a variational principle applied to a scalar function,
2. define and describe normal uniform rods and normal uniform deformations and
3. solve four specific normal uniform deformations.

These three goals are intended to reproduce the results of a paper written by H. Cohen [1]. Cohen's paper omits the details of items 1 and 3 listed above. This thesis provides those details as well as a thorough explanation of item 2.

## 1.2 Development of the Theory

Theories describing the deformation of material bodies can be divided into three parts: geometry of the body, field equations and constitutive restrictions.

There are two distinct types of theories of material deformation. In a dynamical theory, the various mathematical quantities of the model are considered to be functions of time. A static theory is not concerned with the behavior of the body over time, only its initial and final states. This thesis develops a static theory.

### 1.2.1 Geometry

The first part of the theory consists of defining the geometry used to describe the body. Directed rod theory is based on the idea of considering the rod as a curve in space with additional structure associated at each point of the curve. The theory of curves is a well established part of differential geometry and we utilize the knowledge of curves (e.g. Frenet frames) throughout the thesis. The additional structure is provided by the introduction of three linearly independent members of the tangent space at each point of the curve. These vectors are called directors and together with the curve describing the rod, form the basic geometry of the directed rod. All other geometric quantities are defined as tensor products among the directors, their derivatives and basis vectors of the physical space in which the rod is embedded. Thus the other geometric quantities in the model are second order tensors.

We can distinguish between two types of second order tensors used in the geometric description of directed rods. One type describes the behavior of the directors in a directed rod and the other kind describes the deformation of a directed rod from one state to another.

### 1.2.2 Field Equations

The second part of the theory is the field equations. This is the physics of materials: it provides fundamental equations describing how a body behaves under the application of forces or loads. The field equations are often called Euler's Laws.

To derive field equations, we introduce several physical quantities such as the vectors describing forces and loads. Although the theory may be developed in terms of these vectors, we choose to work with second order tensors defined as tensor products of the physical vectors with the directors and their derivatives. We then state several postulates to describe how physical quantities are related to geometric ones. From these postulates, the fundamental equations of the theory (the field equations) are obtained.

The field equations may be derived in more than one way. They are usually founded on a general concept of conservation or balance of quantities, as described by Leigh [7]. For example, the theory includes the postulate that the total mass of the rod must be conserved under a deformation and would include an integral equation corresponding to this postulate. In fact, Leigh provides a general balance integral, which may be thought of as a template to generate the various balance laws forming the field equations. Each one of these conservation or balance laws is a postulate of the theory.

In this thesis, the field equations are derived in a different fashion, although reference to at least one balance law is necessary. We employ a variational method to derive the field equations, as done by Cohen [2] and Ericksen [4]. The variational approach is based on the postulate there exists a strain energy density function $\varepsilon$ associated with the rod. Materials in which the strain energy density is assumed to exist are called hyperelastic and it is this class of rods we are solely concerned with in this thesis. The quantity $\varepsilon$ is a scalar which is assumed to be a function of the deformation tensor and its first derivative with respect to the parameter of the curve describing the rod axis. This defines $\varepsilon$ as a constitutive relation or equation (for non-hyperelastic theories, the constitutive equations are usually expressed as the stress tensor as a function of the deformation tensor and, possibly, its derivatives). The exact functional form of $\varepsilon$ is not specified and depends upon the material of which the rod is composed. Applying a variation principle to $\varepsilon$, we obtain Euler's equations in a form relating the derivatives of $\varepsilon$ to physical quantities such as the stress tensor and tensor-moments.

### 1.2.3 Constitutive Restrictions

The third part of the theory is the constitutive restrictions. When deriving the field equations, the constitutive relation $\varepsilon$ is introduced. The constitutive restrictions are constraints imposed on $\varepsilon$ indicating some type of symmetry, either in space or in the material of the rod. The constitutive restrictions may be defined by providing a specific form for the function (e.g. linear). Alternatively, the constitutive restrictions may be assumptions about properties of the function $\varepsilon$ without stating the function explicitly.

In the model discussed in this thesis, there are two properties described by the constitutive restrictions: material frame-indifference and material symmetry. Material frame-indifference describes the effect of rotating the rod after it has been deformed while material symmetry describes the effect of rotations of the rod prior to deformation. Although the two constitutive properties seem similar mathematically, they have quite different implications physically. Material frame-indifference is really concerned with the symmetry of the space in which the rod is embedded. Material symmetry, as the name suggests, is concerned with symmetry inherent in the body itself.

## 1.3 Normal Uniform Rods

An arbitrary deformation of a directed rod is described by a system of nonlinear differential equations (the field equations) and these may be very hard or impossible to solve. However, certain special cases may occur in which the differential equations can be solved in closed form.

In fact, as Ericksen [4] first showed, there exist special geometries of the rod which turn the differential equations into algebraic relationships which can be solved for exact answers. Building on Erickson's work, we define normal uniform rods. Erickson showed the axis of such rods can only assume a small class of familiar shapes - straight lines, circular arcs and helices.

After carefully defining normal uniform rods and deriving several mathematical properties of such rods, we solve four of the transformations in which the rod is normal and uniform both prior to and after deformation. These are called normal uniform deformations. Even though these will be algebraic equations (as opposed to differential equations), it still requires considerable effort to find solutions. In all four problems, we must use both field equations and constitutive restrictions to obtain the desired solution.

## 1.4 Comparison With the Literature

As already suggested, there are several ways to build a theory of directed rods. It seems appropriate to highlight some of the differences between theories as they appear in the literature and the approach used in this thesis.

1. As already indicated, the deformation of directed rods may be modelled dynamically or statically. A dynamical theory includes time as a variable whereas the static theory ignores time and considers the deformation in terms of a state before deformation (the reference configuration) and after (the deformed configuration). This theory considers the static model. An example of a dynamical model may be found in reference [3]. Examples of static models may be found in references [1] , [2] and [4].

2. The stress tensors appearing in the theory can be formed from tensor products of vectors both in the deformed configuration (Cauchy stress tensor) or one vector from the reference configuration and the other from the deformed configuration (Piola-Kirchoff stress tensor). This thesis uses the Piola-Kirchoff stress tensor. References [1], [2], [3] and [4] use the Cauchy stress tensor.

3. There are at least two ways of deriving the field equations: from balance laws or from a variational principle. This thesis derives the field equations from a variational principle. Reference [1] does not derive the field equations, but simply states them based on the balance laws. Reference [3] derives the field equations using balance laws. References [2] and [4] use a variational principle to obtain the field equations.

4. Reference [4] introduced the basic idea of uniform rods. Reference [1] and this thesis apply rod theory to normal uniform rods.

These are the major differences among directed rod theories. There are other differences as well, some of which are mentioned in this thesis as they occur.

## 1.5 Notation

Letters in calligraphic type will denote sets. Second order tensors will be appear as upper case bold letters. Lower case bold letters denote vectors. Regular Latin letters represent indices, scalars, matrices or functions. Greek letters may represent indices, scalar quantities or operators.

The Einstein summation notation is assumed - there is a sum over any index which appears as both a subscript and a superscript in the same expression. Occasionally, the sum must be done on the same index appearing twice as a subscript or twice as a superscript. In this case, the summation symbol will be employed or one of the repeated symbols with be enclosed in parenthesis. Latin and Greek letters used as indices indicate summation from 1 to 3 and 1 to 2 respectively. Other notation conventions will be mentioned as they appear.

Equations are numbered only if they are referred to subsequently in the text.

# Chapter 2

# Rod Geometry

The goal of this chapter is to describe the geometry of a directed rod. This is done in two sections. The first section reviews tensors and proves some results required in deriving the field equations. The second section describes the geometry of a directed rod before and after the rod has been deformed. Directors and three tensors are introduced in this section. Some useful lemmas regarding these tensors are stated and proved.

## 2.1 Mathematical Preliminaries

In this section, we prove some general results to be used later in the thesis. We use real vector spaces although many of the following comments would apply to vector spaces over any field. Much of the information for this section including the notation was taken from Bowen and Wang [10].

Recall that every finite-dimensional vector space $\mathcal{V}$ has a dual space $\mathcal{V}^*$ consisting of the set of linear transformations to real numbers on $\mathcal{V}$. Thus if $\mathbf{u}^* \in \mathcal{V}^*$, $\mathbf{a}, \mathbf{b} \in \mathcal{V}$ and $\alpha, \beta \in \Re$, then $\mathbf{u}^*$ is a linear map on the members of $\mathcal{V}$ and thus $\mathbf{u}^*(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{u}^*(\mathbf{a}) + \beta\mathbf{u}^*(\mathbf{b})$. Members of the dual space are often called covectors. The scalar product $\langle , \rangle : \mathcal{V}^* \times \mathcal{V} \rightarrow \Re$ is defined by $\langle \mathbf{u}^*, \mathbf{v} \rangle = \mathbf{u}^*(\mathbf{v})$ for every $\mathbf{u}^* \in \mathcal{V}^*$ and $\mathbf{v} \in \mathcal{V}$. It is straightforward to show the scalar product is bilinear and definite.

If a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_N\}$ is chosen for $\mathcal{V}$, the dual basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \ldots, \mathbf{e}^N\}$ for $\mathcal{V}^*$ is defined by $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$. In terms of the scalar product, $\langle \mathbf{e}^i, \mathbf{e}_j \rangle = \delta^i_j$. If $\mathbf{a}, \mathbf{v} \in \mathcal{V}$, $\mathbf{u}^*, \mathbf{b}^* \in \mathcal{V}^*$, the tensor product of $\mathbf{a}$ and $\mathbf{b}^*$ is denoted $\mathbf{a}\otimes\mathbf{b}^*$ and is a mapping $\mathbf{a}\otimes\mathbf{b}^* : \mathcal{V}^* \times \mathcal{V} \to \Re$ defined by $\mathbf{a}\otimes\mathbf{b}^*(\mathbf{u}^*, \mathbf{v}) \equiv \langle\mathbf{u}^*, \mathbf{a}\rangle\langle\mathbf{b}^*, \mathbf{v}\rangle$. The tensor product is also bilinear and definite. Note that the tensor product may be considered as a mapping of a covector and vector to a real number or as a mapping of a vector to another vector. That is, the tensor product is equivalently an endomorphism on the vector space defined by $\mathbf{a}\otimes\mathbf{b}^*\mathbf{v} = \langle\mathbf{b}^*, \mathbf{v}\rangle\mathbf{a}$.

The identity endomorphism is that endomorphism which maps any vector to itself. The relationship between the identity endomorphism and the tensor product is given in the following lemma.

**Lemma 1** *If* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_N\}$ *is a basis for* $\mathcal{V}$ *with dual basis* $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \ldots, \mathbf{e}^N\}$*, then* $\mathbf{e}_i\otimes\mathbf{e}^i$ *is the identity endomorphism on* $\mathcal{V}$.

**Proof.** Let $\mathbf{v}\in\mathcal{V}$ so $\mathbf{v} = v^j\mathbf{e}_j$. Then

$$\mathbf{e}_i\otimes\mathbf{e}^i\mathbf{v} = \langle\mathbf{e}^i, \mathbf{v}\rangle\mathbf{e}_i = \langle\mathbf{e}^i, v^j\mathbf{e}_j\rangle\mathbf{e}_i = v^j\langle\mathbf{e}^i, \mathbf{e}_j\rangle\mathbf{e}_i = v^j\delta^i_j\mathbf{e}_i = v^j\mathbf{e}_j = \mathbf{v}.$$

Since $\mathbf{e}_i\otimes\mathbf{e}^i$ maps an arbitrary vector $\mathbf{v}$ to itself, it follows that $\mathbf{e}_i\otimes\mathbf{e}^i$ is the identity endomorphism. ■

A dual space is itself a vector space with the same dimension as the original vector space. Thus $\mathcal{V}$ and $\mathcal{V}^*$ are isomorphic. In general there is no natural (i.e. canonical) isomorphism between them. If an inner product

$$\cdot : \mathcal{V}\times\mathcal{V} \to \Re$$

is defined for the vector space, we say $\mathcal{V}$ is an inner product space. This establishes a canonical isomorphism given by $\mathbf{a}\cdot\mathbf{b} = \langle\mathbf{a}, \mathbf{b}\rangle$ . In this case, any member of $\mathcal{V}$ or $\mathcal{V}^*$ may be expressed in terms of a basis from either vector space. This thesis uses inner product spaces and hence the notation $\mathbf{a}\cdot\mathbf{b}$ will be used exclusively. Notice this means the tensor product acting as an endomorphism may now be written as $\mathbf{a}\otimes\mathbf{b}\mathbf{v} = (\mathbf{b}\cdot\mathbf{v})\mathbf{a}$ where $\mathbf{a}, \mathbf{b}, \mathbf{v}\in\mathcal{V}$.

In an inner product space, the dual basis becomes the reciprocal basis. If the bases defined above are dual bases, in the inner product space they become reciprocal bases. Hence $\mathbf{e}^i\cdot\mathbf{e}_j = \delta^i_j$.

If a vector space has an inner product defined on it, then it is always possible to find an orthonormal basis for that space. The dual basis of an orthonormal basis is also orthonormal.

The set of endomorphisms on a vector space $\mathcal{V}$ is denoted $\mathcal{L}(\mathcal{V},\mathcal{V})$. $\mathcal{L}(\mathcal{V},\mathcal{V})$ is also a vector space over the same field as $\mathcal{V}$. If $\mathcal{V}$ has dimension $N$, then $\mathcal{L}(\mathcal{V},\mathcal{V})$ has dimension $N^2$.

If $\mathbf{A} \in \mathcal{L}(\mathcal{V},\mathcal{V})$, then the transpose of $\mathbf{A}$ is denoted $\mathbf{A}^{\mathrm{T}}$ and defined to be the endomorphism such that $\mathbf{u}\cdot(\mathbf{A}\mathbf{v}) = (\mathbf{A}^{\mathrm{T}}\mathbf{u})\cdot\mathbf{v}$ for all $\mathbf{u},\mathbf{v} \in \mathcal{V}$. Let $\mathbf{A},\mathbf{B},\mathbf{0},\mathbf{I} \in \mathcal{L}(\mathcal{V},\mathcal{V})$, $\mathbf{0}$ and $\mathbf{I}$ are the zero and identity endomorphism respectively and $\mathbf{a},\mathbf{b} \in \mathcal{V}$. The transpose operation can be shown to have the following properties:

1. If $\mathbf{A},\mathbf{B} \in \mathcal{L}(\mathcal{V},\mathcal{V})$, then $(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$,
2. $(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$,
3. $\mathbf{0}^{\mathrm{T}} = \mathbf{0}$,
4. $\mathbf{I}^{\mathrm{T}} = \mathbf{I}$,
5. $(\mathbf{A}\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$,
6. $(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1}$ if $\mathbf{A}$ is invertible.

**Lemma 2** *If* $\mathbf{a},\mathbf{b} \in \mathcal{V}$, $(\mathbf{a}\otimes\mathbf{b})^{\mathrm{T}} = \mathbf{b}\otimes\mathbf{a}$.

**Proof.** Let $\mathbf{A} = \mathbf{a}\otimes\mathbf{b}$ and $\mathbf{u},\mathbf{v} \in \mathcal{V}$. Then

$$\mathbf{u}\cdot(\mathbf{A}\mathbf{v}) = \mathbf{u}\cdot(\mathbf{a}\otimes\mathbf{b}\mathbf{v}) = \mathbf{u}\cdot(\mathbf{b}\cdot\mathbf{v})\,\mathbf{a} = (\mathbf{u}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{v}) = ((\mathbf{u}\cdot\mathbf{a})\,\mathbf{b})\cdot\mathbf{v}.$$

But

$$(\mathbf{A}^{\mathrm{T}}\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{A}\mathbf{v}) = ((\mathbf{u}\cdot\mathbf{a})\,\mathbf{b})\cdot\mathbf{v}$$

so

$$\begin{aligned}(\mathbf{A}^{\mathrm{T}}\mathbf{u})\cdot\mathbf{v} - ((\mathbf{u}\cdot\mathbf{a})\,\mathbf{b})\cdot\mathbf{v} &= 0\\ (\mathbf{A}^{\mathrm{T}}\mathbf{u} - (\mathbf{u}\cdot\mathbf{a})\,\mathbf{b})\cdot\mathbf{v} &= 0.\end{aligned}$$

Since $\mathbf{u}$ and $\mathbf{v}$ are arbitrary, this implies

$$\mathbf{A}^{\mathrm{T}}\mathbf{u} - (\mathbf{u}\cdot\mathbf{a})\,\mathbf{b} = \mathbf{0} \text{ and } \mathbf{A}^{\mathrm{T}}\mathbf{u} = (\mathbf{u}\cdot\mathbf{a})\,\mathbf{b}.$$

But $\mathbf{b} \otimes \mathbf{a}\mathbf{u} = (\mathbf{u} \cdot \mathbf{a})\,\mathbf{b}$ so $\mathbf{A}^{\mathrm{T}}\mathbf{u} = \mathbf{b} \otimes \mathbf{a}\mathbf{u}$. Since $\mathbf{u}$ is arbitrary, it follows that $\mathbf{A}^{\mathrm{T}} = \mathbf{b} \otimes \mathbf{a}$. Thus $(\mathbf{a} \otimes \mathbf{b})^{\mathrm{T}} = \mathbf{b} \otimes \mathbf{a}$. ■

**Lemma 3** *If* $\mathbf{a}, \mathbf{b} \in \mathcal{V}$ *and* $\mathbf{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V})$, *then* $(\mathbf{A}\mathbf{a}) \otimes \mathbf{b} = \mathbf{A}(\mathbf{a} \otimes \mathbf{b})$ *and* $\mathbf{a} \otimes (\mathbf{A}\mathbf{b}) = (\mathbf{a} \otimes \mathbf{b})\,\mathbf{A}^{\mathrm{T}}$.

**Proof.** Let $\mathbf{u} \in \mathcal{V}$. Then

$$(\mathbf{A}\mathbf{a}) \otimes \mathbf{b}\mathbf{u} = \mathbf{b} \cdot \mathbf{u}\,(\mathbf{A}\mathbf{a}) = \mathbf{A}\left((\mathbf{b} \cdot \mathbf{u})\,\mathbf{a}\right) = \mathbf{A}\,(\mathbf{a} \otimes \mathbf{b}\mathbf{u})$$

and since $\mathbf{u}$ is arbitrary, $(\mathbf{A}\mathbf{a}) \otimes \mathbf{b} = \mathbf{A}\,(\mathbf{a} \otimes \mathbf{b})$.

Let $\mathbf{B} = \mathbf{a} \otimes (\mathbf{A}\mathbf{b})$. Then

$$\mathbf{B}^{\mathrm{T}} = (\mathbf{a} \otimes (\mathbf{A}\mathbf{b}))^{\mathrm{T}} = (\mathbf{A}\mathbf{b}) \otimes \mathbf{a} = \mathbf{A}\,(\mathbf{b} \otimes \mathbf{a})$$

using the second property of transposes mentioned above. Taking the transpose again gives

$$\left(\mathbf{B}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{B} = (\mathbf{A}\,(\mathbf{b} \otimes \mathbf{a}))^{\mathrm{T}} = (\mathbf{b} \otimes \mathbf{a})^{\mathrm{T}}\,\mathbf{A}^{\mathrm{T}} = (\mathbf{a} \otimes \mathbf{b})\,\mathbf{A}^{\mathrm{T}}.$$

Thus $\mathbf{a} \otimes (\mathbf{A}\mathbf{b}) = (\mathbf{a} \otimes \mathbf{b})\,\mathbf{A}^{\mathrm{T}}$. ■

**Lemma 4** *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{V}$, *then* $(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = \mathbf{a} \otimes (\mathbf{b} \cdot \mathbf{c})\,\mathbf{d}$.

**Proof.** Let $\mathbf{u} \in \mathcal{V}$. Then

$$\begin{aligned}
(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d})\,\mathbf{u} &= (\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}\mathbf{u}) = (\mathbf{a} \otimes \mathbf{b})(\mathbf{d} \cdot \mathbf{u})\,\mathbf{c} \\
&= (\mathbf{d} \cdot \mathbf{u})(\mathbf{a} \otimes \mathbf{b})\,\mathbf{c} = (\mathbf{d} \cdot \mathbf{u})(\mathbf{a} \otimes \mathbf{b}\mathbf{c}) \\
&= (\mathbf{d} \cdot \mathbf{u})(\mathbf{b} \cdot \mathbf{c})\,\mathbf{a} = (\mathbf{b} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{u})\,\mathbf{a} \\
&= (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d}\mathbf{u}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \otimes \mathbf{d})\,\mathbf{u} \\
&= (\mathbf{a} \otimes (\mathbf{b} \cdot \mathbf{c})\,\mathbf{d})\,\mathbf{u}.
\end{aligned}$$

But since $\mathbf{u}$ is arbitrary, $(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = \mathbf{a} \otimes (\mathbf{b} \cdot \mathbf{c})\,\mathbf{d}$. ■

The trace of $\mathbf{T} \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ is defined by $\mathrm{tr}\mathbf{T} \equiv T^k_k$ where $T^i_j$ are the components of the endomorphism with respect to some basis. Trace can be shown to have the following properties (suppose $\mathbf{A}, \mathbf{B} \in \mathcal{L}(\mathcal{V}, \mathcal{V})$, $\lambda \in \Re$):

1. Trace is independent of the component representation of $\mathbf{A}$ (or $\mathbf{B}$).

2. $\mathrm{tr}(\mathbf{A}+\mathbf{B}) = \mathrm{tr}\mathbf{A}+\mathrm{tr}\mathbf{B}$.

3. $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$.

4. $\mathrm{tr}\lambda\mathbf{A} = \lambda\mathrm{tr}\mathbf{A}$.

5. $\mathrm{tr}\mathbf{A}^{\mathrm{T}} = \mathrm{tr}\mathbf{A}$.

**Lemma 5** *Let $\mathcal{V}$ be a real inner product space. Then for all $\mathbf{a}, \mathbf{b} \in \mathcal{V}$, $\mathbf{a}\cdot\mathbf{b} = \mathrm{tr}(\mathbf{a}\otimes\mathbf{b})$.*

**Proof.** Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_N\}$ denote a basis for $\mathcal{V}$ with corresponding reciprocal basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \ldots, \mathbf{e}^N\}$. If $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathcal{V}$, we write these in terms of the bases as $\mathbf{a} = a^i\mathbf{e}_i$, $\mathbf{b} = b_j\mathbf{e}^j$, $\mathbf{v} = v^l\mathbf{e}_l$ and $\mathbf{u} = u_k\mathbf{e}^k$. Notice $\mathbf{b}$ and $\mathbf{u}$ are written in terms of the reciprocal basis.

Then

$$
\begin{aligned}
\mathbf{a}\otimes\mathbf{b}\mathbf{v} &= (\mathbf{b}\cdot\mathbf{v})\,\mathbf{a} \\
&= \left(b_j\mathbf{e}^j \cdot v^l\mathbf{e}_l\right) a^i\mathbf{e}_i \\
&= a^i b_j v^l \left(\mathbf{e}^j\cdot\mathbf{e}_l\right)\mathbf{e}_i \\
&= a^i b_j v^l \delta^j_l \mathbf{e}_i \\
&= a^i b_j v^j \mathbf{e}_i.
\end{aligned}
$$

Since $\mathbf{v}$ is arbitrary, the component representation for $\mathbf{a}\otimes\mathbf{b}$ is given by $a^i b_j$ for all $i, j \in \{1, 2, 3, \ldots, N\}$. If $\mathbf{T} = \mathbf{a}\otimes\mathbf{b}$, then $\mathbf{T} \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ and $T^i_j = a^i b_j$. By definition, $\mathrm{tr}\mathbf{T} = T^k_k$. Then

$$
\mathrm{tr}\,(\mathbf{a}\otimes\mathbf{b}) = \mathrm{tr}\mathbf{T} = T^k_k = a^k b_k. \tag{2.1}
$$

Now consider the component form of $\mathbf{a}\cdot\mathbf{b}$:

$$
\mathbf{a}\cdot\mathbf{b} = b_j\mathbf{e}^j \cdot a^i\mathbf{e}_i = b_j a^i \mathbf{e}^j\cdot\mathbf{e}_i = a^i b_j \delta^j_i = a^i b_i. \tag{2.2}
$$

Comparing equations (2.1) and (2.2) shows $\mathbf{a}\cdot\mathbf{b} = \mathrm{tr}(\mathbf{a}\otimes\mathbf{b})$. ■

If $\mathcal{V}$ is an inner product space, then an inner product on $\mathcal{L}(\mathcal{V}, \mathcal{V})$ is defined by

$$\mathbf{A} \circ \mathbf{B} \equiv \operatorname{tr}\left(\mathbf{A}\mathbf{B}^{\mathrm{T}}\right). \tag{2.3}$$

It is straightforward to prove this definition satisfies the definition of an inner product.

**Lemma 6** $(\mathbf{a} \otimes \mathbf{b}) \circ (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$.

**Proof.** From equation (2.3),

$$\begin{aligned}
(\mathbf{a} \otimes \mathbf{b}) \circ (\mathbf{c} \otimes \mathbf{d}) &= \operatorname{tr}\left((\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d})^{\mathrm{T}}\right) \\
&= \operatorname{tr}\left((\mathbf{a} \otimes \mathbf{b})(\mathbf{d} \otimes \mathbf{c})\right) \\
&= \operatorname{tr}\left(\mathbf{a} \otimes (\mathbf{b} \cdot \mathbf{d})\, \mathbf{c}\right) \\
&= (\mathbf{b} \cdot \mathbf{d}) \operatorname{tr}(\mathbf{a} \otimes \mathbf{c}) \\
&= (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{c}) \\
&= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})
\end{aligned}$$

using Lemmas 2, 4 and 5. ■

## 2.2 Geometry of a Directed Rod

In this section, the geometric description of the rod in its undeformed and deformed states is given. This description requires introduction of directors and the deformation tensor.

We assume the rod is embedded in a 3-dimensional Euclidean point space which is associated with an inner product space. This inner product space is also known as the translation space. We use the rectangular Cartesian coordinate system with the inner product as the usual scalar product for vectors. The natural basis is denoted $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with dual basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ and in the rectangular Cartesian coordinate system these bases are orthnormal so $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ and $\mathbf{e}^i \cdot \mathbf{e}^j = \delta^{ij}$. In fact we can identify $\mathbf{e}_i$ with $\mathbf{e}^i$ as the same vector so $\mathbf{e}_i = \mathbf{e}^i$ for $i = 1, 2, 3$. Thus we may freely raise and lower indices as required using these bases.

### 2.2.1 The Curve and Curve Parameter

The rod is modelled as a curve with additional structure associated at each point on the curve. The curve is called the rod axis. Since we're interested in deforming the rod, we need to consider two curves and their additional structure and the mappings from one curve with its structure (representing the undeformed rod) to the other (representing the deformed rod).

For the moment, we'll represent the deformation by the notation $F$ without specifying how $F$ is defined. We assume that however $F$ is defined, it will have the appropriate properties of continuity and smoothness.

The directed rod is said to be in a reference configuration prior to deformation. To denote this, various symbols are subscripted with the letter R. The R does not assume any numerical value - it a notational convenience.

Let the rod axis in the reference configuration be denoted by $\mathbf{r}_R : [a_R, b_R] \rightarrow \Re^3$ and the rod axis in the deformed configuration be denoted by $\bar{\mathbf{r}} : [a, b] \rightarrow \Re^3$. We use $s_R$ as the parameter for $\mathbf{r}_R : [a_R, b_R] \rightarrow \Re^3$ and $s$ as the parameter for $\bar{\mathbf{r}} : [a, b] \rightarrow \Re^3$. Then the deformation $F$ includes a mapping from $s_R$ to $s$ denoted by

$$s_R \xrightarrow{F} s.$$

Thus we write

$$\mathbf{r}_R(s_R) \xrightarrow{F} \bar{\mathbf{r}}(s) \tag{2.4}$$

where is it understood that the $s$ on the right side is obtained from the $s_R$ on the left side via the transformation $F$.

Denote the function between $s$ and $s_R$ by $s = f(s_R)$ where the function $f : [a_R, b_R] \rightarrow [a, b]$ is assumed to be continuous, have first derivatives, is monotonically increasing, $f(a_R) = a$ and $f(b_R) = b$. The curve in the deformed state is a function of $s_R$ by

$$\bar{\mathbf{r}}(s) = \bar{\mathbf{r}}(f(s_R))$$

and thus the deformed curve may be expressed in terms of the reference configuration parameter. Define $\mathbf{r} = \bar{\mathbf{r}} \bullet f$ where $\bullet$ denotes function composition. Then

$$\overline{\mathbf{r}}(s) = \overline{\mathbf{r}}(f(s_R)) = \mathbf{r}(s_R) \tag{2.5}$$

so $\overline{\mathbf{r}}(s)$ and $\mathbf{r}(s_R)$ trace out the same curve, but with a different parametrization.

Differentiation with respect to the parameter $s_R$ is denoted by $'$. From elementary differential geometry, $\frac{d\mathbf{r}_R}{ds_R}(s_R) = \mathbf{r}'_R$ is a tangent vector to the reference curve at any point along the curve. In the deformed configuration, a tangent vector to the curve is given by $\frac{d\overline{\mathbf{r}}}{ds}(s)$. The relationship between tangent vector $\frac{d\overline{\mathbf{r}}}{ds}$ and the vector $\frac{d\mathbf{r}}{ds_R}$ is obtained by differentiating equation (2.5) with respect to $s_R$ and applying the chain rule:

$$\mathbf{r}'(s_R) = \frac{d\overline{\mathbf{r}}}{ds_R} = \frac{d\overline{\mathbf{r}}}{ds}\frac{ds}{ds_R} = \lambda\frac{d\overline{\mathbf{r}}}{ds}$$

where $\lambda = \frac{ds}{ds_R} = f'(s)$. Since $f$ is assumed to be monotonically increasing, $\lambda > 0$ for all $s_R$. Notice this implies $\frac{d\mathbf{r}}{ds_R}$ and $\frac{d\overline{\mathbf{r}}}{ds}$ are parallel to each other. Thus a tangent vector at each point on the deformed curve may be obtained by differentiating the curve with respect to the reference parameter. $\mathbf{r}'$ is used as a tangent vector to $\mathbf{r}$ (equivalently $\overline{\mathbf{r}}$) throughout this thesis.

Notice the function $f$ may be selected in infinitely many ways. The simplest way is to choose $f$ is to be linear.

**Lemma 7** *There is a unique linear function satisfying the requirements for $f$.*

**Proof.** The proof consists of demonstrating the function. Since $f : \Re \to \Re$ and $f$ is linear, it must be of the form $f(s_R) = As_R + B$ for some coefficients $A$ and $B$. Since $f(a_R) = a$ and $f(b_R) = b$, the following equations must be simultaneously satisfied:

$$\begin{aligned} Aa_R + B &= a, \\ Ab_R + B &= b. \end{aligned}$$

or

$$\begin{pmatrix} a_R & 1 \\ b_R & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Since $b_R > a_R$,

$$\det\begin{pmatrix} a_R & 1 \\ b_R & 1 \end{pmatrix} = a_R - b_R \neq 0$$

so a solution exists. Since $b > a$, $\begin{pmatrix} a & b \end{pmatrix}^{\mathrm{T}}$ is not the zero vector so the solution

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a_R & 1 \\ b_R & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$

is unique. Completing this calculation gives

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{a-b}{a_R-b_R} \\ -\frac{b_R a - a_R b}{a_R-b_R} \end{pmatrix}$$

so

$$f(s_R) = \frac{a-b}{a_R-b_R} s_R - \frac{b_R a - a_R b}{a_R - b_R}..$$

Finally, note $A > 0$ since $a - b < 0$ and $a_R - b_R < 0$ which means $\lambda = f'(s_R) = A$ is always positive. Thus this unique linear function satisfies the requirements for the transformation between the parameters in the reference and deformed curve. ■

This last lemma demonstrates it is always possible to find a desired transformation between the parameters. The transformation does not have to be linear: it only has to satisfy the requirements for $f$ given above.

### 2.2.2 Directors

At each point of a curve, there is a tangent and cotangent space. The additional structure of the directed rod is defined in terms of these spaces. Specifically, at each point along either curve, we select three linearly independent vectors from the tangent space. These vectors are known as the directors and are denoted as $\mathbf{d}_{R1}, \mathbf{d}_{R2}$ and $\mathbf{d}_{R3}$ in the reference configuration and $\overline{\mathbf{d}}_1, \overline{\mathbf{d}}_2$ and $\overline{\mathbf{d}}_3$ in the deformed configuration. Since they are linearly independent and the dimension of the tangent space is three, the directors form a basis for the tangent space. A directed rod with directors shown at one point is illustrated in Figure 2-1.

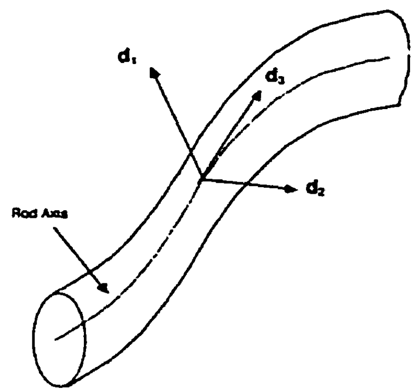


Figure 2-1: Diagram Showing Rod Axis and Directors at a Point

Since there is a tangent space at every point along the curve and we select directors in each tangent space, it follows that each of the directors $\mathbf{d}_{R1}, \mathbf{d}_{R2}$ and $\mathbf{d}_{R3}$ are functions of the parameter $s_R$ and the directors $\overline{\mathbf{d}}_1, \overline{\mathbf{d}}_2$ and $\overline{\mathbf{d}}_3$ are functions of $s$. We assume these functions are continuous and smooth and denote the transformation of directors as

$$\mathbf{d}_{Ri} \xrightarrow{F} \overline{\mathbf{d}}_i,\ i = 1, 2, 3. \tag{2.6}$$

Each of the directors in the deformed configuration is a function of the reference parameter by

$$\overline{\mathbf{d}}_i(s) = \overline{\mathbf{d}}_i(f(s_R)) = \mathbf{d}_i(s_R),\ i = 1, 2, 3$$

and we may write $\mathbf{d}_i = \overline{\mathbf{d}}_i \bullet f$. In order to use the same parameter for all quantities, we use $\mathbf{d}_1, \mathbf{d}_2$ and $\mathbf{d}_3$ as the directors in the deformed configuration.

The collection of all sets of directors at all points along the rod axis defines the additional structure of the rod. Any point in the rod is given as a linear combination of the directors and the curve $\mathbf{r}_R$ (in the reference configuration) or $\mathbf{r}$ (in the deformed configuration).

Although the directors may be any linear independent set from the tangent space, we introduce the constraint that one of the directors is always tangent to the curve at any point along the rod axis. In most cases, we choose the notation so $\mathbf{d}_{R3}$ (or $\mathbf{d}_3$) is the tangent vector to the curve and thus $\mathbf{d}_{R3} = \mathbf{r}'_R$ and $\mathbf{d}_3 = \mathbf{r}'$. This identifies $\mathbf{d}_{R3}$ (or $\mathbf{d}_3$) with the placement of the rod axis. The two dimensional subspace spanned by $\mathbf{d}_{R1}$ and $\mathbf{d}_{R2}$ (or $\mathbf{d}_1$ and $\mathbf{d}_2$) may be thought of a cross-section of the rod at any point along the curve. With this constraint, the directed rod is called a real rod.

For the directors $\{\mathbf{d}_{R1}, \mathbf{d}_{R2}, \mathbf{d}_{R3}\}$, the reciprocal basis is denoted $\{\mathbf{d}_R^1, \mathbf{d}_R^2, \mathbf{d}_R^3\}$. Similarly, $\{\mathbf{d}^1, \mathbf{d}^2, \mathbf{d}^3\}$ is the reciprocal basis for $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$. The following lemma is useful in relating the directors and their derivatives.

**Lemma 8** $\mathbf{d}'_{Ri} \cdot \mathbf{d}_R^j + \mathbf{d}_{Ri} \cdot \left(\mathbf{d}_R^j\right)' = 0$ *and* $\mathbf{d}'_i \cdot \mathbf{d}^j + \mathbf{d}_i \cdot (\mathbf{d}^j)' = 0$ *for all* $i, j \in \{1, 2, 3\}$.

**Proof.** Since $\{\mathbf{d}_{R1}, \mathbf{d}_{R2}, \mathbf{d}_{R3}\}$ and $\{\mathbf{d}_R^1, \mathbf{d}_R^2, \mathbf{d}_R^3\}$ are reciprocal bases, $\mathbf{d}_{Ri} \cdot \mathbf{d}_R^j = \delta_i^j$ for all $i, j \in \{1, 2, 3\}$. Differentiating this equation with respect to $s_R$ gives

$$\left(\mathbf{d}_{Ri} \cdot \mathbf{d}_R^j\right)' = \left(\delta_i^j\right)' = 0$$

and applying the product rule for differentiation gives

$$\mathbf{d}'_{Ri} \cdot \mathbf{d}_R^j + \mathbf{d}_{Ri} \cdot \left(\mathbf{d}_R^j\right)' = 0.$$

The proof for $\mathbf{d}'_i \cdot \mathbf{d}^j + \mathbf{d}_i \cdot (\mathbf{d}^j)' = 0$ is analogous. ■

The following definitions of symbols will also be helpful in dealing with the inner product of directors and derivatives of directors. The idea here is to express derivatives of directors in terms of the basis of the directors:

$$\overline{\mu}_i^j \equiv \mathbf{d}'_i \cdot \mathbf{d}^j,\ \overline{\mu}_{Ri}^j \equiv \mathbf{d}'_{Ri} \cdot \mathbf{d}_R^j,\ \overline{\nu}_j^i \equiv \left(\mathbf{d}^i\right)' \cdot \mathbf{d}_j,\ \overline{\nu}_{Rj}^i \equiv \left(\mathbf{d}_R^i\right)' \cdot \mathbf{d}_{Rj} \tag{2.7}$$

so

$$\mathbf{d}'_i = \overline{\mu}_i^j \mathbf{d}_j,\ \mathbf{d}'_{Ri} = \overline{\mu}_{Ri}^j \mathbf{d}_{Rj},\ \left(\mathbf{d}^i\right)' = \overline{\nu}_j^i \mathbf{d}^j \text{ and } \left(\mathbf{d}_R^i\right)' = \overline{\nu}_{Rj}^i \mathbf{d}_R^j. \tag{2.8}$$

Since $\{\mathbf{d}'_1, \mathbf{d}'_2, \mathbf{d}'_3\}$, $\{\mathbf{d}'_{R1}, \mathbf{d}'_{R2}, \mathbf{d}'_{R3}\}$, $\left\{(\mathbf{d}^1)', (\mathbf{d}^2)', (\mathbf{d}^3)'\right\}$ or $\left\{(\mathbf{d}_R^1)', (\mathbf{d}_R^2)', (\mathbf{d}_R^3)'\right\}$ are linearly independent sets, then the inverse of the quantities defined in equation (2.7) are also useful. Define $\mu_i^j$, $\mu_{Ri}^j$, $\nu_j^i$ and $\nu_{Rj}^i$ such that

$$\overline{\mu}_i^k \mu_k^j \equiv \delta_i^j,\ \overline{\mu}_{Ri}^k \mu_{Rk}^j \equiv \delta_i^j,\ \overline{\nu}_i^k \nu_k^j \equiv \delta_i^j \text{ and } \overline{\nu}_{Ri}^k \nu_{Rk}^j \equiv \delta_i^j \tag{2.9}$$

in which case

$$\mathbf{d}_i = \mu_i^j \mathbf{d}'_j,\ \mathbf{d}_{Ri} = \mu_{Ri}^j \mathbf{d}'_{Rj},\ \mathbf{d}^i = \nu_j^i \left(\mathbf{d}^j\right)' \text{ and } \mathbf{d}_R^i = \nu_{Rj}^i \left(\mathbf{d}_R^j\right)'. \tag{2.10}$$

Using these notations in the result of Lemma 8, we have

$$\overline{\mu}_{Ri}^j + \overline{\nu}_{Ri}^j = 0 \text{ and } \overline{\mu}_i^j + \overline{\nu}_i^j = 0. \tag{2.11}$$

### 2.2.3 Director, Wryness and Deformation Tensors

We introduce three second order tensors which are useful in describing the rod and its behavior under deformation. These tensors are members of $\mathcal{T}_1^1(\mathcal{V})$ where $\mathcal{V}$ is the translation space corresponding to the Euclidean point space. As such, tensors are linear maps between members of the translation space: i.e. if $\mathbf{X} \in \mathcal{T}_1^1(\mathcal{V})$, then $\mathbf{X} : \mathcal{V} \to \mathcal{V}$ and $\mathbf{X}$ is linear. $\mathcal{T}_1^1(\mathcal{V})$ is itself a vector space of dimension nine. A basis for $\mathcal{T}_1^1(\mathcal{V})$ is given by the set $\left\{\mathbf{e}_i \otimes \mathbf{e}^j : i, j \in \{1,2,3\}\right\}$.

The director tensor is defined by

$$\mathbf{D} \equiv \delta_j^i \mathbf{d}_i \otimes \mathbf{e}^j = \mathbf{d}_i \otimes \mathbf{e}^i$$

and describes the distribution of directors along the rod axis. The director tensor maps a vector expressed in the natural basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ into the corresponding vector in the director basis.

**Lemma 9** *If* $\mathbf{v} = v^j \mathbf{e}_j$, *then* $\mathbf{D}\mathbf{v} = v^j \mathbf{d}_j$.

**Proof.**

$$\mathbf{D}\mathbf{v} = \mathbf{d}_i \otimes \mathbf{e}^i v^j \mathbf{e}_j = v^j \left(\mathbf{e}^i \cdot \mathbf{e}_j\right) \mathbf{d}_i = v^j \delta_j^i \mathbf{d}_i = v^j \mathbf{d}_j. \blacksquare$$

**Corollary 1** $\mathbf{D}\mathbf{e}_i = \mathbf{d}_i$.

**Proof.** If $\mathbf{v} = \mathbf{e}_i$, then $v^j = \delta_i^j$ and from the proof of the lemma, $\mathbf{D}\mathbf{e}_i = \delta_i^j \mathbf{d}_j = \mathbf{d}_i$. $\blacksquare$

**Lemma 10** *The director tensor is invertible and the inverse is given by*

$$\mathbf{D}^{-1} = \delta_j^i \mathbf{e}_i \otimes \mathbf{d}^j = \mathbf{e}_i \otimes \mathbf{d}^i. \tag{2.12}$$

**Proof.** The proof that the director tensor is invertible will follow from demonstrating that $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$.

$$\mathbf{D}\mathbf{D}^{-1} = \left(\mathbf{d}_i \otimes \mathbf{e}^i\right)\left(\mathbf{e}_j \otimes \mathbf{d}^j\right) = \mathbf{d}_i \otimes \left(\mathbf{e}^i \cdot \mathbf{e}_j\right)\mathbf{d}^j = \mathbf{d}_i \otimes \delta^i_j \mathbf{d}^j = \mathbf{d}_i \otimes \mathbf{d}^i$$

using Lemma 4. By Lemma 1, $\mathbf{D}\mathbf{D}^{-1} = \mathbf{d}_i \otimes \mathbf{d}^i$ is the identity tensor. Since $\mathbf{D}\mathbf{D}^{-1}$ maps an arbitrary vector to itself, then $\mathbf{D}^{-1}$ is a right inverse to $\mathbf{D}$. Now consider

$$\mathbf{D}^{-1}\mathbf{D} = \left(\mathbf{e}_i \otimes \mathbf{d}^i\right)\left(\mathbf{d}_j \otimes \mathbf{e}^j\right) = \mathbf{e}_i \otimes \left(\mathbf{d}^i \cdot \mathbf{d}_j\right)\mathbf{e}^j = \mathbf{e}_i \otimes \delta^i_j \mathbf{e}^j = \mathbf{e}_i \otimes \mathbf{e}^i$$

using Lemma 4. By Lemma 1, $\mathbf{D}^{-1}\mathbf{D} = \mathbf{e}_i \otimes \mathbf{e}^i$ is the identity tensor. Since $\mathbf{D}^{-1}\mathbf{D}$ maps an arbitrary vector to itself, then $\mathbf{D}^{-1}$ is a left inverse to $\mathbf{D}$. Since $\mathbf{D}^{-1}$ is both a left and right inverse to $\mathbf{D}$, then $\mathbf{D}^{-1}$ is the inverse of $\mathbf{D}$. ∎

The wryness tensor characterizes the differential geometry of the rod describing how the directors are changing along the rod axis. This tensor is defined by

$$\mathbf{W} \equiv \delta^i_j \mathbf{d}'_i \otimes \mathbf{d}^j = \mathbf{d}'_i \otimes \mathbf{d}^i.$$

It has the effect of mapping a director to its first derivative with respect to the curve parameter.

**Lemma 11** *If* $\mathbf{v} = v^j \mathbf{d}_j$, *then* $\mathbf{W}\mathbf{v} = v^j \mathbf{d}'_j$.

**Proof.**

$$\mathbf{W}\mathbf{v} = \mathbf{d}'_i \otimes \mathbf{d}^i v^j \mathbf{d}_j = v^j\left(\mathbf{d}^i \cdot \mathbf{d}_j\right)\mathbf{d}'_i = v^j \delta^i_j \mathbf{d}'_i = v^j \mathbf{d}'_j. \blacksquare$$

**Corollary 2** $\mathbf{W}\mathbf{d}_i = \mathbf{d}'_i$.

**Proof.** If $\mathbf{v} = \mathbf{d}_i$, then $v^j = \delta^j_i$ and from the proof of the lemma, $\mathbf{W}\mathbf{d}_i = \delta^j_i \mathbf{d}'_j = \mathbf{d}'_i$. ∎

**Lemma 12** *The wryness and director tensors are related by*

$$\mathbf{W} = \mathbf{D}'\mathbf{D}^{-1}.$$

**Proof.** From $\mathbf{D} = \mathbf{d}_i \otimes \mathbf{e}^i$,

$$\mathbf{D}' = \left(\mathbf{d}_i \otimes \mathbf{e}^i\right)' = \mathbf{d}'_i \otimes \mathbf{e}^i$$

since $\mathbf{e}^i$ is a constant vector and doesn't depend on $s_R$. Then from equation (2.12) and using Lemma 4,

$$\mathbf{D}'\mathbf{D}^{-1} = \left(\mathbf{d}'_i \otimes \mathbf{e}^i\right)\left(\mathbf{e}_j \otimes \mathbf{d}^j\right) = \mathbf{d}'_i \otimes \left(\mathbf{e}^i \cdot \mathbf{e}_j\right)\mathbf{d}^j = \mathbf{d}'_i \otimes \delta^i_j \mathbf{d}^j = \delta^i_j \mathbf{d}'_i \otimes \mathbf{d}^j$$

which is the definition of $\mathbf{W}$. Hence $\mathbf{W} = \mathbf{D}'\mathbf{D}^{-1}$. ∎

The notation $\mathbf{D}_R$ and $\mathbf{W}_R$ refers to the director and wryness tensors in the reference configuration. In this case, all the directors in the definitions of the director and wryness tensors would be reference directors.

We can now define a second order tensor describing the deformation from the reference configuration to the deformed configuration. The deformation tensor is defined by

$$\mathbf{F} \equiv \mathbf{D}\mathbf{D}_R^{-1}.$$

**Lemma 13** *The deformation tensor is expressed as a tensor product by*

$$\mathbf{F} = \delta^i_j \mathbf{d}_i \otimes \mathbf{d}^j_R = \mathbf{d}_i \otimes \mathbf{d}^i_R. \tag{2.13}$$

**Proof.** From equation (2.12),

$$\mathbf{D}_R^{-1} = \mathbf{e}_i \otimes \mathbf{d}^i_R$$

so using Lemma 4,

$$\mathbf{F} \equiv \mathbf{D}\mathbf{D}_R^{-1} = \left(\mathbf{d}_i \otimes \mathbf{e}^i\right)\left(\mathbf{e}_j \otimes \mathbf{d}^j_R\right) = \mathbf{d}_i \otimes \delta^i_j \mathbf{d}^j_R = \delta^i_j \mathbf{d}_i \otimes \mathbf{d}^j_R = \mathbf{d}_i \otimes \mathbf{d}^i_R$$

which verifies equation (2.13). ∎

Note the important property of $\mathbf{F}$: it maps vectors describing the rod in the reference configuration to the corresponding vectors describing the rod in the deformed configuration.

**Lemma 14** *If* $\mathbf{v} = v^k_R \mathbf{d}_{Rk}$, *then* $\mathbf{F}\mathbf{v} = v^i_R \mathbf{d}_i$.

**Proof.** If $\mathbf{v} = v^k_R \mathbf{d}_{Rk}$,

$$\mathbf{F}\mathbf{v} = \mathbf{d}_i \otimes \mathbf{d}^i_R \mathbf{v} = \left(\mathbf{d}^i_R \cdot \mathbf{v}\right)\mathbf{d}_i = v^k_R\left(\mathbf{d}^i_R \cdot \mathbf{d}_{Rk}\right)\mathbf{d}_i = v^k_R \delta^i_k \mathbf{d}_i = v^i_R \mathbf{d}_i. \blacksquare$$

**Corollary 3** $\mathbf{F}\mathbf{d}_{Rk} = \mathbf{d}_k$.

**Proof.** In particular, if $\mathbf{v} = \mathbf{d}_{Rk}$ so $v^i_R = \delta^i_k$, then $\mathbf{F}\mathbf{v} = \mathbf{F}\mathbf{d}_{Rk} = \delta^i_k\mathbf{d}_i = \mathbf{d}_k$. ■

Thus the deformation tensor maps a reference director into its corresponding director in the deformed configuration.

In the next lemma, we derive expressions for $\mathbf{F}^{\mathrm{T}}$, $\mathbf{F}^{-1}$ and $\left(\mathbf{F}^{\mathrm{T}}\right)^{-1}$which will incidently prove the deformation tensor and its transpose are invertible.

**Lemma 15** *With the deformation tensor defined by equation (2.13),*

$$\mathbf{F}^{\mathrm{T}} = \delta^i_j \mathbf{d}^j_R \otimes \mathbf{d}_i = \mathbf{d}^i_R \otimes \mathbf{d}_i, \tag{2.14}$$

$$\mathbf{F}^{-1} = \delta^i_j \mathbf{d}_{Ri} \otimes \mathbf{d}^j = \mathbf{d}_{Ri} \otimes \mathbf{d}^i \tag{2.15}$$

*and*

$$\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{F}^{-1}\right)^{\mathrm{T}} = \delta^i_j \mathbf{d}^j \otimes \mathbf{d}_{Ri} = \mathbf{d}^i \otimes \mathbf{d}_{Ri}. \tag{2.16}$$

**Proof.** Taking the transpose of both sides of equation (2.13) gives

$$\mathbf{F}^{\mathrm{T}} = \left(\mathbf{d}_i \otimes \mathbf{d}^i_R\right)^{\mathrm{T}} = \mathbf{d}^i_R \otimes \mathbf{d}_i$$

using Lemma 2. This demonstrates equation (2.14).

To prove equation (2.15), evaluate $\mathbf{F}\mathbf{F}^{-1}$ as a tensor product using Lemma 4.

$$\mathbf{F}\mathbf{F}^{-1} = \left(\mathbf{d}_i \otimes \mathbf{d}^i_R\right)\left(\mathbf{d}_{Rj} \otimes \mathbf{d}^j\right) = \mathbf{d}_i \otimes \left(\mathbf{d}^i_R \cdot \mathbf{d}_{Rj}\right)\mathbf{d}^j = \mathbf{d}_i \otimes \delta^i_j \mathbf{d}^j = \mathbf{d}_i \otimes \mathbf{d}^i.$$

By Lemma 1, $\mathbf{F}\mathbf{F}^{-1} = \mathbf{d}_i \otimes \mathbf{d}^i$ is the identity tensor. Since $\mathbf{F}\mathbf{F}^{-1}$ maps an arbitrary vector to itself, then $\mathbf{F}^{-1}$ is a right inverse to $\mathbf{F}$. Now consider

$$\mathbf{F}^{-1}\mathbf{F} = \left(\mathbf{d}_{Ri} \otimes \mathbf{d}^i\right)\left(\mathbf{d}_j \otimes \mathbf{d}^j_R\right) = \mathbf{d}_{Ri} \otimes \left(\mathbf{d}^i \cdot \mathbf{d}_j\right)\mathbf{d}^j_R = \mathbf{d}_{Ri} \otimes \delta^i_j \mathbf{d}^j_R = \mathbf{d}_{Ri} \otimes \mathbf{d}^i_R$$

using Lemma 4. By Lemma 1, $\mathbf{F}^{-1}\mathbf{F} = \mathbf{d}_{Ri} \otimes \mathbf{d}^i_R$ is the identity tensor. Since $\mathbf{F}^{-1}\mathbf{F}$ maps an arbitrary vector to itself, then $\mathbf{F}^{-1}$ is a left inverse to $\mathbf{F}$. Since $\mathbf{F}^{-1}$ is both a left and right inverse to $\mathbf{F}$, then $\mathbf{F}^{-1}$ is the inverse of $\mathbf{F}$.

To prove equation (2.16), take the transpose of equation (2.15):

$$\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} = \left(\mathbf{d}_{Ri} \otimes \mathbf{d}^{i}\right)^{\mathrm{T}} = \mathbf{d}^{i} \otimes \mathbf{d}_{Ri}$$

using Lemma 2. The result $\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} = \left(\mathbf{F}^{\mathrm{T}}\right)^{-1}$ is given as one of the properties of transpose. ∎

**Corollary 4** $\mathbf{F}^{\mathrm{T}}\mathbf{d}^{a} = \mathbf{d}_{R}^{a}$, $\mathbf{F}^{-1}\mathbf{d}_{a} = \mathbf{d}_{Ra}$ *and* $\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{d}_{R}^{a} = \mathbf{d}^{a}$.

**Proof.**

$$\mathbf{F}^{\mathrm{T}}\mathbf{d}^{a} = \mathbf{d}_{R}^{i} \otimes \mathbf{d}_{i}\mathbf{d}^{a} = \left(\mathbf{d}_{i} \cdot \mathbf{d}^{a}\right)\mathbf{d}_{R}^{i} = \delta_{i}^{a}\mathbf{d}_{R}^{i} = \mathbf{d}_{R}^{a},$$

$$\mathbf{F}^{-1}\mathbf{d}_{a} = \mathbf{d}_{Ri} \otimes \mathbf{d}^{i}\mathbf{d}_{a} = \left(\mathbf{d}^{i} \cdot \mathbf{d}_{a}\right)\mathbf{d}_{Ri} = \delta_{a}^{i}\mathbf{d}_{Ri} = \mathbf{d}_{Ra}$$

and

$$\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{d}_{R}^{a} = \mathbf{d}^{i} \otimes \mathbf{d}_{Ri}\mathbf{d}_{R}^{a} = \left(\mathbf{d}_{Ri} \cdot \mathbf{d}_{R}^{a}\right)\mathbf{d}^{i} = \delta_{i}^{a}\mathbf{d}^{i} = \mathbf{d}^{a}$$

which proves the statements of the corollary. ∎

Thus $\mathbf{F}^{\mathrm{T}}$ maps the dual of the directors in the deformed configuration to the corresponding dual of the directors in the reference configuration, $\mathbf{F}^{-1}$ maps a director in the deformed configuration to the corresponding director in the reference configuration and $\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}$ maps the dual of the directors in the reference configuration to corresponding dual of the directors in the deformed configuration.

**Lemma 16** *The relationship between the wryness and deformation tensors is given by*

$$\mathbf{W} = \mathbf{F}'\mathbf{F}^{-1} + \mathbf{F}\mathbf{W}_{R}\mathbf{F}^{-1}. \tag{2.17}$$

**Proof.** Since $\mathbf{F} = \mathbf{d}_{i} \otimes \mathbf{d}_{R}^{i}$, $\mathbf{F}' = \left(\mathbf{d}_{i} \otimes \mathbf{d}_{R}^{i}\right)' = \mathbf{d}_{i}' \otimes \mathbf{d}_{R}^{i} + \mathbf{d}_{i} \otimes \left(\mathbf{d}_{R}^{i}\right)'$. Then

$$\begin{aligned}
\mathbf{F}'\mathbf{F}^{-1} &= \left(\mathbf{d}_{i}' \otimes \mathbf{d}_{R}^{i} + \mathbf{d}_{i} \otimes \left(\mathbf{d}_{R}^{i}\right)'\right)\left(\mathbf{d}_{Rj} \otimes \mathbf{d}^{j}\right) \\
&= \left(\mathbf{d}_{i}' \otimes \mathbf{d}_{R}^{i}\right)\left(\mathbf{d}_{Rj} \otimes \mathbf{d}^{j}\right) + \left(\mathbf{d}_{i} \otimes \left(\mathbf{d}_{R}^{i}\right)'\right)\left(\mathbf{d}_{Rj} \otimes \mathbf{d}^{j}\right) \\
&= \mathbf{d}_{i}' \otimes \left(\mathbf{d}_{R}^{i} \cdot \mathbf{d}_{Rj}\right)\mathbf{d}^{j} + \mathbf{d}_{i} \otimes \left(\left(\mathbf{d}_{R}^{i}\right)' \cdot \mathbf{d}_{Rj}\right)\mathbf{d}^{j} \\
&= \mathbf{d}_{i}' \otimes \delta_{j}^{i}\mathbf{d}^{j} - \mathbf{d}_{i} \otimes \left(\mathbf{d}_{R}^{i} \cdot \mathbf{d}_{Rj}'\right)\mathbf{d}^{j}
\end{aligned}$$

$$= \mathbf{d}'_i \otimes \mathbf{d}^i - \mathbf{d}_i \otimes \overline{\mu}^i_{Rj} \mathbf{d}^j$$

where the second last step uses Lemma 8 and the last step uses the definition given in equation (2.7). The second term evaluates as

$$\begin{aligned}
\mathbf{F}\mathbf{W}_R\mathbf{F}^{-1} &= \left(\mathbf{d}_i \otimes \mathbf{d}^i_R\right)\left(\mathbf{d}'_{Rj} \otimes \mathbf{d}^j_R\right)\left(\mathbf{d}_{Rk} \otimes \mathbf{d}^k\right) \\
&= \left(\mathbf{d}_i \otimes \mathbf{d}^i_R\right)\left(\mathbf{d}'_{Rj} \otimes \left(\mathbf{d}^j_R \cdot \mathbf{d}_{Rk}\right)\mathbf{d}^k\right) \\
&= \left(\mathbf{d}_i \otimes \mathbf{d}^i_R\right)\left(\mathbf{d}'_{Rj} \otimes \left(\delta^j_k\right)\mathbf{d}^k\right) \\
&= \left(\mathbf{d}_i \otimes \mathbf{d}^i_R\right)\left(\mathbf{d}'_{Rj} \otimes \mathbf{d}^j\right) \\
&= \mathbf{d}_i \otimes \left(\mathbf{d}^i_R \cdot \mathbf{d}'_{Rj}\right)\mathbf{d}^j \\
&= \mathbf{d}_i \otimes \overline{\mu}^i_{Rj} \mathbf{d}^j.
\end{aligned}$$

Then

$$\mathbf{F}'\mathbf{F}^{-1} + \mathbf{F}\mathbf{W}_R\mathbf{F}^{-1} = \mathbf{d}'_i \otimes \mathbf{d}^i - \mathbf{d}_i \otimes \overline{\mu}^i_{Rj}\mathbf{d}^j + \mathbf{d}_i \otimes \overline{\mu}^i_{Rj}\mathbf{d}^j = \mathbf{d}'_i \otimes \mathbf{d}^i.$$

But by definition, $\mathbf{W} = \mathbf{d}'_i \otimes \mathbf{d}^i$ and therefore $\mathbf{W} = \mathbf{F}'\mathbf{F}^{-1} + \mathbf{F}\mathbf{W}_R\mathbf{F}^{-1}$. ■

In Lemmas 10 and 15, the tensor product forms of $\mathbf{D}^{-1}$ and $\mathbf{F}^{-1}$ were derived. An analogous result for the wryness tensor is not as simple. In fact, the wryness tensor may not have an inverse. If $\mathbf{W}$ is invertible, the next lemma provides the expression and derivation of $\mathbf{W}^{-1}$.

**Lemma 17** *Using the definitions in equations (2.7) and (2.9),* $\mathbf{W}^{-1} = \mathbf{d}_i \otimes \mu^i_j \mathbf{d}^j$.

**Proof.** Find the tensor product form of $\mathbf{W}\mathbf{W}^{-1}$ and $\mathbf{W}^{-1}\mathbf{W}$ and verify it is the identity tensor.

$$\mathbf{W}\mathbf{W}^{-1} = \left(\mathbf{d}'_i \otimes \mathbf{d}^i\right)\left(\mathbf{d}_j \otimes \mu^j_k \mathbf{d}^k\right) = \mathbf{d}'_i \otimes \delta^i_j \mu^j_k \mathbf{d}^k = \mathbf{d}'_i \otimes \mu^i_k \mathbf{d}^k$$

using Lemma 4. Let $\mathbf{v} = v^j \mathbf{d}_j$. Then

$$\mathbf{W}\mathbf{W}^{-1}\mathbf{v} = \mathbf{d}'_i \otimes \mu^i_k \mathbf{d}^k v^j \mathbf{d}_j = v^j \mu^i_k \left(\mathbf{d}^k \cdot \mathbf{d}_j\right)\ \mathbf{d}'_i = v^j \mu^i_k \delta^k_j \mathbf{d}'_i = v^j \mu^i_j \mathbf{d}'_i = v^j \mathbf{d}_j = \mathbf{v}$$

using equation (2.10). Since $\mathbf{W}\mathbf{W}^{-1}$ maps an arbitrary vector to itself, then $\mathbf{W}^{-1}$ is a right

| $()$ | $()$ Expressed As a Tensor Product | $()^{\mathrm{T}}$ Expressed As a Tensor Product | $()^{-1}$ Expressed As a Tensor Product | $\left(()^{-1}\right)^{\mathrm{T}}$ Expressed As a Tensor Product |
|---|---|---|---|---|
| $\mathbf{D}$ | $\mathbf{d}_i \otimes \mathbf{e}^i$ | $\mathbf{e}^i \otimes \mathbf{d}_i$ | $\mathbf{e}_i \otimes \mathbf{d}^i$ | $\mathbf{d}^i \otimes \mathbf{e}_i$ |
| $\mathbf{W}$ | $\mathbf{d}'_i \otimes \mathbf{d}^i$ | $\mathbf{d}^i \otimes \mathbf{d}'_i$ | $\mathbf{d}_i \otimes \mu^i_j \mathbf{d}^j$ | $\mathbf{d}^j \otimes \mu^i_j \mathbf{d}_i$ |
| $\mathbf{F}$ | $\mathbf{d}_i \otimes \mathbf{d}^i_R$ | $\mathbf{d}^i_R \otimes \mathbf{d}_i$ | $\mathbf{d}_{Ri} \otimes \mathbf{d}^i$ | $\mathbf{d}^i \otimes \mathbf{d}_{Ri}$ |

Table 2.1: Tensor Product Expressions for The Director, Wryness and Deformation Tensors

inverse to $\mathbf{W}$. Now consider

$$\mathbf{W}^{-1}\mathbf{W} = \left(\mathbf{d}_i \otimes \mu^i_k \mathbf{d}^k\right)\left(\mathbf{d}'_j \otimes \mathbf{d}^j\right) = \mathbf{d}_i \otimes \mu^i_k \overline{\mu}^k_j \mathbf{d}^j = \mathbf{d}_i \otimes \delta^i_j \mathbf{d}^j = \mathbf{d}_i \otimes \mathbf{d}^i$$

using Lemma 4. By Lemma 1, $\mathbf{W}^{-1}\mathbf{W} = \mathbf{d}_i \otimes \mathbf{d}^i$ is the identity tensor. Since $\mathbf{W}^{-1}\mathbf{W}$ maps an arbitrary vector to itself, then $\mathbf{W}^{-1}$ is a left inverse to $\mathbf{W}$. Since $\mathbf{W}^{-1}$ is both a left and right inverse to $\mathbf{W}$, then $\mathbf{W}^{-1}$ is the inverse of $\mathbf{W}$. ■

Notice the inverse for the wryness tensor is more complicated than that of the director or deformation tensors. This is due to the presence of the derivative of the director in the definition of the wryness tensor. The wryness tensor is invertible if $\{\mathbf{d}'_1, \mathbf{d}'_2, \mathbf{d}'_3\}$ is a linearly independent set. Equivalently, the wryness tensor is invertible if the matrix of coefficients $\overline{\mu}^i_j$ has a nonzero determinant.

It is possible to derive tensor product expressions for $\mathbf{D}^{\mathrm{T}}$, $\left(\mathbf{D}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{D}^{-1}\right)^{\mathrm{T}}$, $\mathbf{W}^{\mathrm{T}}$ and $\left(\mathbf{W}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{W}^{-1}\right)^{\mathrm{T}}$. The method for proving these tensor product forms is completely analogous to that of Lemma 15. Rather than showing these calculations in detail, the results are summarized in Table 2.1. The notation $()$ refers to any of $\mathbf{D}$, $\mathbf{W}$ or $\mathbf{F}$.

# Chapter 3

# Mathematical Preliminaries to Deriving the Field Equations

In the next two chapters, the field equations for the directed rod model will be derived. These required some sophisticated mathematical tools. The purpose of this chapter is to develop these tools as general results.

## 3.1 Calculus of Real Inner Product Spaces

In this section, we derive the calculus of real inner product spaces. Spivak [12] has done this for maps of the form $f : \Re^n \to \Re^m$ where $n$ and $m$ are any positive integers. A more general treatment for calculus on Banach spaces can be found in Lang [16],[17]. We develop in detail the differential calculus of general vector spaces $\mathcal{V}$ and $\mathcal{U}$ with maps $\phi : \mathcal{V} \to \mathcal{U}$.

In the following definitions and theorems, let $\mathcal{V}, \mathcal{U}$ and $\mathcal{W}$ denote inner product spaces with inner products $\langle , \rangle_1 : \mathcal{V} \times \mathcal{V} \to \Re$ , $\langle , \rangle_2 : \mathcal{U} \times \mathcal{U} \to \Re$ and $\langle , \rangle_3 : \mathcal{W} \times \mathcal{W} \to \Re$ and corresponding norms $|\mathbf{v}|_1 = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_1}$ for any $\mathbf{v} \in \mathcal{V}$, $|\mathbf{u}|_2 = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle_2}$for any $\mathbf{u} \in \mathcal{U}$ and $|\mathbf{w}|_3 = \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle_3}$for any $\mathbf{w} \in \mathcal{W}$.

**Definition 1** *Limit: Let* $\mathbf{a} \in \mathcal{V}$ *and* $\phi : \mathcal{V} \to \mathcal{U}$. *Then* $\lim_{\mathbf{v} \to \mathbf{a}} \phi(\mathbf{v}) = \mathbf{u}$ *where* $\mathbf{u} \in \mathcal{U}$ *is the limit as* $\mathbf{v}$ *approaches* $\mathbf{a}$ *of* $\phi$ *if for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $|\phi(\mathbf{v}) - \mathbf{u}|_2 < \epsilon$ *whenever* $|\mathbf{v} - \mathbf{a}|_1 < \delta$. $\mathbf{u}$ *is called the limit of* $\phi$ *at* $\mathbf{a}$.

**Definition 2** *Continuous: Let* $\mathbf{a} \in \mathcal{V}$ *and* $\phi : \mathcal{V} \to \mathcal{U}$. *If* $\phi(\mathbf{a})$ *is defined,* $\lim_{\mathbf{v}\to\mathbf{a}} \phi(\mathbf{v})$ *exists and* $\lim_{\mathbf{v}\to\mathbf{a}} \phi(\mathbf{v}) = \phi(\mathbf{a})$, *then* $\phi$ *is said to be continuous at* $\mathbf{a}$.

**Definition 3** *Differentiable: Let* $\mathbf{a} \in \mathcal{V}$ *and* $\phi : \mathcal{V} \to \mathcal{U}$. $\phi$ *is said to be differentiable at the point* $\mathbf{a}$ *if there exists a linear transformation* $\lambda : \mathcal{V} \to \mathcal{U}$ *such that*

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\phi(\mathbf{a}+\mathbf{h}) - \phi(\mathbf{a}) - \lambda(\mathbf{h})|_2}{|\mathbf{h}|_1} = 0.$$

**Definition 4** *Derivative: The linear transformation* $\lambda : \mathcal{V} \to \mathcal{U}$ *appearing in the previous definition is called the derivative of* $\phi$ *at the point* $\mathbf{a}$. *This is denoted* $D\phi(\mathbf{a})$.

**Lemma 18** *If* $\phi : \mathcal{V} \to \mathcal{U}$ *is differentiable at* $\mathbf{a} \in \mathcal{V}$, *there is a unique linear transformation* $\lambda : \mathcal{V} \to \mathcal{U}$ *such that*

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\phi(\mathbf{a}+\mathbf{h}) - \phi(\mathbf{a}) - \lambda(\mathbf{h})|_2}{|\mathbf{h}|_1} = 0.$$

**Proof.** By definition, $\lambda$ exists. It remains to prove it is unique. Suppose $\zeta : \mathcal{V} \to \mathcal{U}$ is a linear transformation such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\phi(\mathbf{a}+\mathbf{h}) - \phi(\mathbf{a}) - \zeta(\mathbf{h})|_2}{|\mathbf{h}|_1} = 0.$$

For brevity, denote $\phi(\mathbf{a}+\mathbf{h}) - \phi(\mathbf{a})$ by $d(\mathbf{h})$. Now consider

$$\begin{aligned}
\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\lambda(\mathbf{h}) - \zeta(\mathbf{h})|_2}{|\mathbf{h}|_1} &= \lim_{\mathbf{h}\to\mathbf{0}} \frac{|\lambda(\mathbf{h}) - d(\mathbf{h}) + d(\mathbf{h}) - \zeta(\mathbf{h})|_2}{|\mathbf{h}|_1} \\
&\leq \lim_{\mathbf{h}\to\mathbf{0}} \frac{|\lambda(\mathbf{h}) - d(\mathbf{h})|_2}{|\mathbf{h}|_1} + \lim_{\mathbf{h}\to\mathbf{0}} \frac{|d(\mathbf{h}) - \zeta(\mathbf{h})|_2}{|\mathbf{h}|_1} \\
&= 0.
\end{aligned}$$

If $\mathbf{x} \in \mathcal{V}$ and $t \in \Re$, then $t\mathbf{x} \to \mathbf{0}$ as $t \to 0$. Then if $\mathbf{x} \neq \mathbf{0}$, we may replace $\mathbf{h}$ with $t\mathbf{x}$ to get

$$\begin{aligned}
\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\lambda(\mathbf{h}) - \zeta(\mathbf{h})|_2}{|\mathbf{h}|_1} &= \lim_{t\to 0} \frac{|\lambda(t\mathbf{x}) - \zeta(t\mathbf{x})|_2}{|t\mathbf{x}|_1} \\
&= \lim_{t\to 0} \frac{|t\lambda(\mathbf{x}) - t\zeta(\mathbf{x})|_2}{|t\mathbf{x}|_1} \\
&= \lim_{t\to 0} \frac{|t|\,|\lambda(\mathbf{x}) - \zeta(\mathbf{x})|_2}{|t|\,|\mathbf{x}|_1}
\end{aligned}$$

$$= \frac{|\lambda(\mathbf{x}) - \zeta(\mathbf{x})|}{|\mathbf{x}|}.$$

using the linearity of $\lambda$ and $\zeta$. But

$$\lim_{t\to 0} \frac{|\lambda(t\mathbf{x}) - \zeta(t\mathbf{x})|}{|t\mathbf{x}|} = 0$$

so

$$\frac{|\lambda(\mathbf{x}) - \zeta(\mathbf{x})|}{|\mathbf{x}|} = 0$$

implying

$$\lambda(\mathbf{x}) = \zeta(\mathbf{x}).$$

This is true for all nonzero $\mathbf{x}$ so the functions $\lambda$ and $\zeta$ are the same and thus the derivative is unique. ■

**Lemma 19** *Let $\phi : \mathcal{V} \to \mathcal{U}$ is differentiable at $\mathbf{a} \in \mathcal{V}$. If $\phi$ is linear, then $D\phi(\mathbf{a}) = \phi$.*

**Proof.** Substitute $\phi$ for $\lambda$ in the definition of differentiable to get

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\phi(\mathbf{a}+\mathbf{h}) - \phi(\mathbf{a}) - \phi(\mathbf{h})|_2}{|\mathbf{h}|_1} = \lim_{\mathbf{h}\to\mathbf{0}} \frac{|\phi(\mathbf{a}) + \phi(\mathbf{h}) - \phi(\mathbf{a}) - \phi(\mathbf{h})|_2}{|\mathbf{h}|_1} = 0$$

using the fact $\phi$ is linear. Since the linear transformation called the derivative must be unique, it follows that $\phi = D\phi(\mathbf{a})$. ■

**Theorem 1** *(Chain Rule): Suppose $\phi : \mathcal{V} \to \mathcal{U}$ is differentiable at $\mathbf{a} \in \mathcal{V}$ and $\psi : \mathcal{U} \to \mathcal{W}$ is differentiable at $\phi(\mathbf{a}) \in \mathcal{U}$. Then the composition $\psi \circ \phi : \mathcal{V} \to \mathcal{W}$ is differentiable at $\mathbf{a}$ and $D(\psi \circ \phi)(\mathbf{a}) = D\psi(\phi(\mathbf{a})) \circ D\phi(\mathbf{a})$.*

**Proof.** For brevity, let $\mathbf{u} = \phi(\mathbf{a})$, $\lambda = D\phi(\mathbf{a})$ and $\mu = D\psi(\phi(\mathbf{a})) = D\psi(\mathbf{u})$. Define the following maps:

$$\varphi(\mathbf{x}) = \phi(\mathbf{x}) - \phi(\mathbf{a}) - \lambda(\mathbf{x} - \mathbf{a}), \tag{3.1}$$

$$\zeta(\mathbf{y}) = \psi(\mathbf{y}) - \psi(\mathbf{u}) - \mu(\mathbf{y} - \mathbf{u}), \tag{3.2}$$

$$\rho(\mathbf{x}) = \psi \circ \phi(\mathbf{x}) - \psi \circ \phi(\mathbf{a}) - \mu \circ \lambda(\mathbf{x} - \mathbf{a}). \tag{3.3}$$

Then

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\varphi(\mathbf{x})|_2}{|\mathbf{x}-\mathbf{a}|_1} = \lim_{\mathbf{x}\to\mathbf{a}} \frac{|\phi(\mathbf{x}) - \phi(\mathbf{a}) - \lambda(\mathbf{x}-\mathbf{a})|_2}{|\mathbf{x}-\mathbf{a}|_1} = 0 \tag{3.4}$$

and

$$\lim_{\mathbf{y}\to\mathbf{u}} \frac{|\zeta(\mathbf{y})|_3}{|\mathbf{y}-\mathbf{u}|_2} = \lim_{\mathbf{y}\to\mathbf{u}} \frac{|\psi(\mathbf{y}) - \psi(\mathbf{u}) - \mu(\mathbf{y}-\mathbf{u})|_3}{|\mathbf{y}-\mathbf{u}|_2} = 0. \tag{3.5}$$

We wish to show

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\rho(\mathbf{x})|_3}{|\mathbf{x}-\mathbf{a}|_1} = \lim_{\mathbf{x}\to\mathbf{a}} \frac{|\psi\circ\phi(\mathbf{x}) - \psi\circ\phi(\mathbf{a}) - \mu\circ\lambda(\mathbf{x}-\mathbf{a})|_3}{|\mathbf{x}-\mathbf{a}|_1} = 0.$$

Now

$$\begin{aligned}
\rho(\mathbf{x}) &= \psi(\phi(\mathbf{x})) - \psi(\mathbf{u}) - \mu(\lambda(\mathbf{x}-\mathbf{a})) \\
&= \psi(\phi(\mathbf{x})) - \psi(\mathbf{u}) - \mu(\phi(\mathbf{x}) - \phi(\mathbf{a}) - \varphi(\mathbf{x}))
\end{aligned}$$

from equation (3.1). Then

$$\begin{aligned}
\rho(\mathbf{x}) &= [\psi(\phi(\mathbf{x})) - \psi(\mathbf{u}) - \mu(\phi(\mathbf{x}) - \phi(\mathbf{a}))] + \mu(\varphi(\mathbf{x})) \\
&= \zeta(\phi(\mathbf{x})) + \mu(\varphi(\mathbf{x}))
\end{aligned}$$

since $\mu$ is linear and using equation (3.2). Thus we would like to prove

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\zeta(\phi(\mathbf{x}))|_3}{|\mathbf{x}-\mathbf{a}|_1} = 0 \tag{3.6}$$

and

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\mu(\varphi(\mathbf{x}))|_3}{|\mathbf{x}-\mathbf{a}|_1} = 0. \tag{3.7}$$

Consider equation (3.7) first.

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\mu(\varphi(\mathbf{x}))|_3}{|\mathbf{x}-\mathbf{a}|_1} = \lim_{\mathbf{x}\to\mathbf{a}} \frac{|\mu(\phi(\mathbf{x}) - \phi(\mathbf{a}) - \lambda(\mathbf{x}-\mathbf{a}))|_3}{|\mathbf{x}-\mathbf{a}|_1}$$

where $\mu$ is a linear transformation operating on $\phi(\mathbf{x}) - \phi(\mathbf{a}) - \lambda(\mathbf{x}-\mathbf{a})$. In the theory of vector spaces, it can be shown that $|\mu(\varphi(\mathbf{x}))|_3 \leq M |\varphi(\mathbf{x})|_2$ for some positive real number $M$. Thus

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\mu(\varphi(\mathbf{x}))|_3}{|\mathbf{x}-\mathbf{a}|_1} \leq \lim_{\mathbf{x}\to\mathbf{a}} \frac{M|\varphi(\mathbf{x})|_2}{|\mathbf{x}-\mathbf{a}|_1} = M \lim_{\mathbf{x}\to\mathbf{a}} \frac{|\varphi(\mathbf{x})|_2}{|\mathbf{x}-\mathbf{a}|_1} = 0$$

which establishes equation (3.7).

Now consider equation (3.6). Let $\mathbf{y} = \phi(\mathbf{x})$. From equation (3.5) and by definition of the limit, if $\epsilon > 0$ and

$$\frac{|\zeta(\phi(\mathbf{x}))|_3}{|\phi(\mathbf{x}) - \mathbf{u}|_2} < \epsilon,$$

there exists $\delta > 0$ so $|\phi(\mathbf{x}) - \mathbf{u}|_2 < \delta$ which is true if $|\mathbf{x}-\mathbf{a}|_1 < \delta_1$ for some $\delta_1 > 0$. That is,

$$|\zeta(\phi(\mathbf{x}))|_3 < \epsilon |\phi(\mathbf{x}) - \mathbf{u}|_2 = \epsilon |\varphi(\mathbf{x}) + \phi(\mathbf{a}) + \lambda(\mathbf{x}-\mathbf{a}) - \phi(\mathbf{a})|_2,$$

$$|\zeta(\phi(\mathbf{x}))|_3 < \epsilon |\varphi(\mathbf{x}) + \lambda(\mathbf{x}-\mathbf{a})|_2 < \epsilon |\varphi(\mathbf{x})|_2 + \epsilon |\lambda(\mathbf{x}-\mathbf{a})|_2.$$

But $\lambda$ is a linear operator on $\mathbf{x}-\mathbf{a}$ so $|\lambda(\mathbf{x}-\mathbf{a})|_2 \leq M |\mathbf{x}-\mathbf{a}|_1$ for some positive real number $M$ and thus

$$|\zeta(\phi(\mathbf{x}))|_3 < \epsilon |\varphi(\mathbf{x})|_2 + \epsilon M |\mathbf{x}-\mathbf{a}|_1,$$
$$\frac{|\zeta(\phi(\mathbf{x}))|_3}{|\mathbf{x}-\mathbf{a}|_1} < \epsilon \frac{|\varphi(\mathbf{x})|_2}{|\mathbf{x}-\mathbf{a}|_1} + \epsilon M$$

and taking the limit through this inequality,

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\zeta(\phi(\mathbf{x}))|_3}{|\mathbf{x}-\mathbf{a}|_1} < \epsilon \lim_{\mathbf{x}\to\mathbf{a}} \frac{|\varphi(\mathbf{x})|_2}{|\mathbf{x}-\mathbf{a}|_1} + \epsilon M.$$

But it has already been shown that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\varphi(\mathbf{x})|_2}{|\mathbf{x}-\mathbf{a}|_1} = 0$$

so

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\zeta(\phi(\mathbf{x}))|_3}{|\mathbf{x}-\mathbf{a}|_1} < \epsilon M$$

and since $\epsilon$ may be taken as close to 0 as desired, it follows that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\varsigma(\phi(\mathbf{x}))|_3}{|\mathbf{x}-\mathbf{a}|_1} = 0$$

which establishes equation (3.6).

Since equations (3.6) and (3.7) are true,

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{|\rho(\mathbf{x})|_3}{|\mathbf{x}-\mathbf{a}|_1} = 0$$

which proves the statement $D(\psi \circ \phi)(\mathbf{a}) = D\psi(\phi(\mathbf{a})) \circ D\phi(\mathbf{a})$. ■

The above results are all we need for the model developed in this thesis. We now consider how these results relate to that given in Spivak.

Here is a brief summary of differential calculus for functions of the form $f : \Re^n \to \Re^m$ based on Spivak [12]. If $\mathbf{y} = f(\mathbf{x})$ where $\mathbf{x} \in \Re^n$ and $\mathbf{y} \in \Re^m$, then $\mathbf{x} = (x^1, x^2, x^3, \ldots, x^n)^{\mathrm{T}}$ and $\mathbf{y} = (y^1, y^2, y^3, \ldots, y^m)^{\mathrm{T}}$ represent the components of the vectors with respect to the standard bases for $\Re^n$ and $\Re^m$. The function $f : \Re^n \to \Re^m$ has component functions $f^i : \Re^n \to \Re$ so $y^i = f^i(x^1, x^2, x^3, \ldots, x^n)$. The derivative of $f : \Re^n \to \Re^m$ is defined to be a linear transformation from $\Re^n$ to $\Re^m$. Furthermore this linear transformation is the $m \times n$ matrix of the partial derivatives of the component functions $f^1, f^2, f^3, \ldots, f^m$with respect to the coordinates $x^1, x^2, x^3, \ldots, x^n$, again defined with respect to the standard basis for $\Re^n$ and $\Re^m$. That is,

$$Df(\mathbf{a}) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \frac{\partial f^1}{\partial x^3} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \frac{\partial f^2}{\partial x^3} & \cdots & \frac{\partial f^2}{\partial x^n} \\ \frac{\partial f^3}{\partial x^1} & \frac{\partial f^3}{\partial x^2} & \frac{\partial f^3}{\partial x^3} & \cdots & \frac{\partial f^3}{\partial x^n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1} & \frac{\partial f^m}{\partial x^2} & \frac{\partial f^m}{\partial x^3} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

where $Df(\mathbf{a})$ denotes the derivative of $f$ at some point $\mathbf{a} \in \Re^n$.

The derivative of $f$ or $\phi$ is only defined at a particular point. However, there are many things which are true about derivatives at all points (where the derivative exists) and hence it is convenient to use the notation $Df$ or $D\phi$. It must be understood that these symbols stand for an entire class of linear transformations and $Df(\mathbf{a})$ is not necessarily the same linear transformation as $Df(\mathbf{b})$ if $\mathbf{a}$ and $\mathbf{b}$ are different points.

The connection between calculus on mappings such as $\phi : \mathcal{V} \to \mathcal{U}$ and that given by Spivak occurs when we select bases for $\mathcal{V}$ and $\mathcal{U}$, which is normally what we must do in order to solve specific problems. Suppose $\mathcal{V}$ has dimension $n$ and a basis for $\mathcal{V}$ is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$. Suppose $\mathcal{U}$ has dimension $m$ and a basis for $\mathcal{U}$ is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$. Then for any $\mathbf{v} \in \mathcal{V}$ , $\mathbf{v} = v^i \mathbf{v}_i$ and for any $\mathbf{u} \in \mathcal{U}$, $\mathbf{u} = u^i \mathbf{u}_i$. A mapping $\phi : \mathcal{V} \to \mathcal{U}$ becomes a mapping $\phi^* : \Re^n \to \Re^m$ because the choice of a basis for $\mathcal{V}$ and $\mathcal{U}$ establishes canonical isomorphisms $\mathcal{V} \longleftrightarrow \Re^n$ and $\mathcal{U} \longleftrightarrow \Re^m$ defined by the components of any vector with respect to its basis.

Let $\sigma$ denote this canonical isomorphism from $\mathcal{V}$ to $\Re^n$ (i.e. $\sigma : \mathcal{V} \to \Re^n$) and let $\mu$ denote this canonical isomorphism from $\mathcal{U}$ to $\Re^m$ (i.e. $\mu : \mathcal{U} \to \Re^m$). Then at any point $\mathbf{a} \in \mathcal{V}$, $\sigma(\mathbf{a}) = \left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}}$ where $\mathbf{a} = a^i \mathbf{v}_i$. Similarly if $\mathbf{b} \in \mathcal{U}$, $\mu(\mathbf{b}) = \left(b^1, b^2, b^3, \ldots, b^m\right)^{\mathrm{T}}$ where $\mathbf{b} = b^i \mathbf{u}_i$. The canonical isomorphisms are defined by the bases chosen.

Since $\sigma$ and $\mu$ are isomorphisms, they have inverses $\sigma^{-1}$ and $\mu^{-1}$ and naturally these give

$$\sigma^{-1}\left(\left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}}\right) = \mathbf{a}$$

and

$$\mu^{-1}\left(\left(b^1, b^2, b^3, \ldots, b^m\right)^{\mathrm{T}}\right) = \mathbf{b}.$$

Thus, if $\phi(\mathbf{a}) = \mathbf{b}$, we automatically have $\phi^*(\sigma(\mathbf{a})) = \mu(\mathbf{b})$, or in longer notation,

$$\phi^*\left(\left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}}\right) = \left(b^1, b^2, b^3, \ldots, b^m\right)^{\mathrm{T}}.$$

**Lemma 20** *$\phi = \mu^{-1} \circ \phi^* \circ \sigma$ and $\phi^* = \mu \circ \phi \circ \sigma^{-1}$.*

**Proof.** $\phi^*(\sigma(\mathbf{a})) = \mu(\mathbf{b})$ which can also be written as $\phi^* \circ \sigma(\mathbf{a}) = \mu(\mathbf{b})$. Then

$$\mu^{-1}\left(\phi^* \circ \sigma(\mathbf{a})\right) = \mu^{-1}\left(\mu(\mathbf{b})\right)$$

which can also be written as

$$\mu^{-1} \circ \phi^* \circ \sigma(\mathbf{a}) = \mathbf{b}.$$

But $\phi(\mathbf{a}) = \mathbf{b}$ and since $\mathbf{a}$ is arbitrary, it follows that $\phi = \mu^{-1} \circ \phi^* \circ \sigma$.

Consider $\phi(\mathbf{a}) = \mathbf{b}$ and

$$\mathbf{a} = \sigma^{-1}\left(\left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}}\right)$$

so

$$\phi\left(\sigma^{-1}\left(\left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}}\right)\right) = \mathbf{b}$$

which can also be written as

$$\phi \circ \sigma^{-1}\left(\left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}}\right) = \mathbf{b}.$$

Then

$$\mu\left(\phi \circ \sigma^{-1}\left(\left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}}\right)\right) = \mu(\mathbf{b})$$

which can also be written as

$$\mu \circ \phi \circ \sigma^{-1}\left(\left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}}\right) = \left(b^1, b^2, b^3, \ldots, b^m\right)^{\mathrm{T}}.$$

But

$$\phi^*\left(\left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}}\right) = \left(b^1, b^2, b^3, \ldots, b^m\right)^{\mathrm{T}}$$

so it follows that $\phi^* = \mu \circ \phi \circ \sigma^{-1}$. ■

**Lemma 21** *The isomorphisms $\sigma$ and $\mu$ are linear.*

**Proof.** Let $\mathbf{a}, \mathbf{b} \in \mathcal{V}$. Let $k \in \Re$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{V}$, so $\mathbf{a} = a^i\mathbf{v}_i$ and $\mathbf{b} = b^i\mathbf{v}_i$. Then $k\mathbf{a} = ka^i\mathbf{v}_i$ and

$$\sigma(k\mathbf{a}) = \left(ka^1, ka^2, ka^3, \ldots, ka^n\right)^{\mathrm{T}}.$$

However,

$$k\sigma(\mathbf{a}) = k\left(a^1, a^2, a^3, \ldots, a^n\right)^{\mathrm{T}} = \left(ka^1, ka^2, ka^3, \ldots, ka^n\right)^{\mathrm{T}}$$

so

$$\sigma(k\mathbf{a}) = k\sigma(\mathbf{a}).$$

Now consider

$$\mathbf{a}+\mathbf{b}=a^i\mathbf{v}_i+b^i\mathbf{v}_i=\left(a^i+b^i\right)\mathbf{v}_i.$$

Then

$$\sigma\left(\mathbf{a}+\mathbf{b}\right)=\left(a^1+b^1,a^2+b^2,a^3+b^3,\ldots,a^n+b^n\right)^{\mathrm{T}}.$$

However,

$$\begin{aligned}\sigma\left(\mathbf{a}\right)+\sigma\left(\mathbf{b}\right) &= \left(a^1,a^2,a^3,\ldots,a^n\right)^{\mathrm{T}}+\left(b^1,b^2,b^3,\ldots,b^n\right)^{\mathrm{T}}\\ &= \left(a^1+b^1,a^2+b^2,a^3+b^3,\ldots,a^n+b^n\right)^{\mathrm{T}}\end{aligned}$$

so $\sigma\left(\mathbf{a}+\mathbf{b}\right)=\sigma\left(\mathbf{a}\right)+\sigma\left(\mathbf{b}\right)$.

Since $\sigma\left(k\mathbf{a}\right)=k\sigma\left(\mathbf{a}\right)$ and $\sigma\left(\mathbf{a}+\mathbf{b}\right)=\sigma\left(\mathbf{a}\right)+\sigma\left(\mathbf{b}\right)$, $\sigma$ is linear. The proof for $\mu$ is identical. ■

We now consider the relationship between the derivative $D\phi\left(\mathbf{a}\right)$ and $D\phi^*\left(\left(a^1,a^2,a^3,\ldots,a^n\right)^{\mathrm{T}}\right)$.

**Lemma 22** *$D\phi\left(\mathbf{a}\right)=\mu^{-1}\circ D\phi^*\left(\sigma\left(\mathbf{a}\right)\right)\circ\sigma$ and $D\phi^*\left(\left(a^1,a^2,a^3,\ldots,a^n\right)^{\mathrm{T}}\right)=\mu\circ D\phi\left(\mathbf{a}\right)\circ\sigma^{-1}$.*

**Proof.** Using Lemma 20 and the chain rule gives

$$\begin{aligned}D\phi\left(\mathbf{a}\right) &= D\left(\mu^{-1}\circ\phi^*\circ\sigma\right)\left(\mathbf{a}\right)\\ &= D\mu^{-1}\left(\phi^*\circ\sigma\left(\mathbf{a}\right)\right)\circ D\left(\phi^*\circ\sigma\left(\mathbf{a}\right)\right).\end{aligned}$$

But we can apply the chain rule to $D\left(\phi^*\circ\sigma\left(\mathbf{a}\right)\right)$ and get

$$D\left(\phi^*\circ\sigma\left(\mathbf{a}\right)\right)=D\phi^*\left(\sigma\left(\mathbf{a}\right)\right)\circ D\sigma\left(\mathbf{a}\right)$$

so

$$D\phi\left(\mathbf{a}\right)=D\mu^{-1}\left(\phi^*\circ\sigma\left(\mathbf{a}\right)\right)\circ D\phi^*\left(\sigma\left(\mathbf{a}\right)\right)\circ D\sigma\left(\mathbf{a}\right).$$

But $D\mu^{-1}\left(\phi^*\circ\sigma\left(\mathbf{a}\right)\right)=\mu^{-1}$ since $\mu^{-1}$ is linear and $D\sigma\left(\mathbf{a}\right)=\sigma$ since $\sigma$ is linear so

$$D\phi\left(\mathbf{a}\right)=\mu^{-1}\circ D\phi^*\left(\sigma\left(\mathbf{a}\right)\right)\circ\sigma$$

which is the desired result. Then

$$\mu (D\phi (\mathbf{a})) = \mu \circ D\phi (\mathbf{a}) = D\phi^* (\sigma (\mathbf{a})) \circ \sigma$$

and

$$\mu \circ D\phi (\mathbf{a}) \circ \sigma^{-1} = D\phi^* (\sigma (\mathbf{a}))$$

or

$$D\phi^* \left( \left( a^1, a^2, a^3, \ldots, a^n \right)^{\mathrm{T}} \right) = \mu \circ D\phi (\mathbf{a}) \circ \sigma^{-1}$$

which is the other result to be proved. ■

Thus the derivative in the general vector spaces $\mathcal{V}$ and $\mathcal{U}$ relate to the derivative in the spaces $\Re^n$ and $\Re^m$ exactly the same way as the original functions $\phi$ and $\phi^*$ as described by Lemma 20. Since this result doesn't depend on the specific point $\mathbf{a}$, we may write $D\phi^* = \mu \circ D\phi \circ \sigma^{-1}$ at any point where $\phi$ has a derivative.

Given the mapping $\phi : \mathcal{V} \to \mathcal{U}$, we've defined the corresponding map $\phi^* : \Re^n \to \Re^m$ which depends on the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ chosen for $\mathcal{V}$ and the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ chosen for $\mathcal{U}$. If different bases were chosen, say $\{\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \widehat{\mathbf{v}}_3, \ldots, \widehat{\mathbf{v}}_n\}$ and $\{\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2, \widehat{\mathbf{u}}_3, \ldots, \widehat{\mathbf{u}}_m\}$, this would give rise to a different corresponding map $\widehat{\phi}^* : \Re^n \to \Re^m$. We consider the relationship between the derivative of $\phi^*$ and the derivative of $\widehat{\phi}^*$.

Recall if two different bases are chosen for a vector space $\mathcal{V}$, the members of one basis are related to the other basis by a change of basis matrix. Let $V = \left[V^i_j\right]$ denote the change of basis matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ to $\{\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \widehat{\mathbf{v}}_3, \ldots, \widehat{\mathbf{v}}_n\}$. That is, $\widehat{\mathbf{v}}_j = V^i_j \mathbf{v}_i$. The matrix $V$ is invertible and its inverse is denoted $\widehat{V} = \left[\widehat{V}^i_j\right]$. Of course this is the change of basis matrix from $\{\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \widehat{\mathbf{v}}_3, \ldots, \widehat{\mathbf{v}}_n\}$ to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ so $\mathbf{v}_i = \widehat{V}^k_i \widehat{\mathbf{v}}_k$. Similarly, in the vector space $\mathcal{U}$, the change of basis matrices are $U = \left[U^i_j\right]$ and $\widehat{U} = \left[\widehat{U}^i_j\right]$ defined by $\widehat{\mathbf{u}}_j = U^i_j \mathbf{u}_i$ and $\mathbf{u}_i = \widehat{U}^k_i \widehat{\mathbf{u}}_k$.

**Lemma 23** *Suppose $\mathbf{v} \in \mathcal{V}$ with representations $v^i \mathbf{v}_i$ and $\widehat{v}^i \widehat{\mathbf{v}}_i$ in two different bases $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and $\{\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \widehat{\mathbf{v}}_3, \ldots, \widehat{\mathbf{v}}_n\}$. Then the components $v^i$ and $\widehat{v}^i$ are related by $v^i = V^i_j \widehat{v}^j$ and $\widehat{v}^i = \widehat{V}^i_j v^j$.*

**Proof.** Given $\mathbf{v} = v^i \mathbf{v}_i = \widehat{v}^k \widehat{\mathbf{v}}_k$ and using $\mathbf{v}_i = \widehat{V}^k_i \widehat{\mathbf{v}}_k$ as described above, then $v^i \widehat{V}^k_i \widehat{\mathbf{v}}_k = \widehat{v}^k \widehat{\mathbf{v}}_k$ or $v^i \widehat{V}^k_i \widehat{\mathbf{v}}_k - \widehat{v}^k \widehat{\mathbf{v}}_k = \mathbf{0}$ . That is, $\left(v^i \widehat{V}^k_i - \widehat{v}^k\right) \widehat{\mathbf{v}}_k = \mathbf{0}$ and since $\{\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \widehat{\mathbf{v}}_3, \ldots, \widehat{\mathbf{v}}_n\}$ are linearly independent, it follows that $v^i\widehat{V}_i^k - \widehat{v}^k = 0$ and thus $\widehat{v}^k = \widehat{V}_i^k v^i$. On the other hand we could use $\widehat{\mathbf{v}}_k = V_k^i\mathbf{v}_i$ as described above so $v^i\mathbf{v}_i = \widehat{v}^k V_k^i \mathbf{v}_i$ and $v^i\mathbf{v}_i - \widehat{v}^k V_k^i \mathbf{v}_i = \mathbf{0}$. That is, $\left(v^i - \widehat{v}^k V_k^i\right)\mathbf{v}_i = \mathbf{0}$ and since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ are linearly independent, $v^i - \widehat{v}^k V_k^i = 0$ and thus $v^i = V_j^i \widehat{v}^j$. ■

The analogous result for a member of the vector space $\mathcal{U}$ is $u^i = U_j^i \widehat{u}^j$ and $\widehat{u}^i = \widehat{U}_j^i u^j$. Notice these results can be written as the following matrix equations:

$$\begin{aligned}
\left(v^1, v^2, v^3, \cdots, v^n\right)^{\mathrm{T}} &= V\left(\widehat{v}^1, \widehat{v}^2, \widehat{v}^3, \cdots, \widehat{v}^n\right)^{\mathrm{T}}, \\
\left(\widehat{v}^1, \widehat{v}^2, \widehat{v}^3, \cdots, \widehat{v}^n\right)^{\mathrm{T}} &= \widehat{V}\left(v^1, v^2, v^3, \cdots, v^n\right)^{\mathrm{T}}, \\
\left(u^1, u^2, u^3, \cdots, u^n\right)^{\mathrm{T}} &= U\left(\widehat{u}^1, \widehat{u}^2, \widehat{u}^3, \cdots, \widehat{u}^n\right)^{\mathrm{T}} \text{ and} \\
\left(\widehat{u}^1, \widehat{u}^2, \widehat{u}^3, \cdots, \widehat{u}^n\right)^{\mathrm{T}} &= \widehat{U}\left(u^1, u^2, u^3, \cdots, u^n\right)^{\mathrm{T}}.
\end{aligned}$$

**Lemma 24** *Let $\sigma$ denote the canonical isomorphism from $\mathcal{V}$ to $\Re^n$ for the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and let $\widehat{\sigma}$ denote the canonical isomorphism from $\mathcal{V}$ to $\Re^n$ for the basis $\{\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \widehat{\mathbf{v}}_3, \ldots, \widehat{\mathbf{v}}_n\}$. Then $\sigma = V\widehat{\sigma}$ and $\widehat{\sigma} = \widehat{V}\sigma$.*

**Proof.** Let $\mathbf{v} \in \mathcal{V}$ with representations $v^i\mathbf{v}_i$ and $\widehat{v}^i\widehat{\mathbf{v}}_i$. Since

$$\sigma(\mathbf{v}) = \left(v^1, v^2, v^3, \cdots, v^n\right)^{\mathrm{T}} \text{ and } \widehat{\sigma}(\mathbf{v}) = \left(\widehat{v}^1, \widehat{v}^2, \widehat{v}^3, \cdots, \widehat{v}^n\right)^{\mathrm{T}},$$

then

$$\widehat{\sigma}(\mathbf{v}) = \widehat{V}\left(v^1, v^2, v^3, \cdots, v^n\right)^{\mathrm{T}} = \widehat{V}\sigma(\mathbf{v})$$

and

$$\sigma(\mathbf{v}) = V\left(\widehat{v}^1, \widehat{v}^2, \widehat{v}^3, \cdots, \widehat{v}^n\right)^{\mathrm{T}} = V\widehat{\sigma}(\mathbf{v}).$$

Since $\mathbf{v}$ is arbitrary, we may write $\sigma = V\widehat{\sigma}$ and $\widehat{\sigma} = \widehat{V}\sigma$. ■

**Lemma 25** *If $\mu$ denotes the canonical isomorphism from $\mathcal{U}$ to $\Re^m$ under the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ and $\widehat{\mu}$ denotes the canonical isomorphism from $\mathcal{U}$ to $\Re^m$ under the basis $\{\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2, \widehat{\mathbf{u}}_3, \ldots, \widehat{\mathbf{u}}_m\}$, then $\widehat{\mu}^{-1} = \mu^{-1}U$.*

**Proof.** Applying Lemma 24 to the vector space $\mathcal{U}$ gives $\mu = U\widehat{\mu}$ and $\widehat{\mu} = \widehat{U}\mu$. Then

$$\mu \circ \widehat{\mu}^{-1} = U\widehat{\mu} \circ \widehat{\mu}^{-1} = U$$

and

$$\mu^{-1} \circ \mu \circ \widehat{\mu}^{-1} = \mu^{-1}U,$$

that is, $\widehat{\mu}^{-1} = \mu^{-1}U$. ■

**Lemma 26** *Let $\phi : \mathcal{V} \to \mathcal{U}$. Let $\phi^* : \Re^n \to \Re^m$ be the corresponding map which depends on the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ chosen for $\mathcal{V}$ and the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ chosen for $\mathcal{U}$. Let $\widehat{\phi}^* : \Re^n \to \Re^m$ be the corresponding map which depends on the bases $\{\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2, \widehat{\mathbf{v}}_3, \ldots, \widehat{\mathbf{v}}_n\}$ and $\{\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2, \widehat{\mathbf{u}}_3, \ldots, \widehat{\mathbf{u}}_m\}$. Let $V$, $\widehat{V}$, $U$ and $\widehat{U}$ be the change of basis matrices as defined above. Then $D\phi^* = UD\widehat{\phi}^*\widehat{V}$ and $D\widehat{\phi}^* = \widehat{U}D\phi^*V$.*

**Proof.** From Lemmas 20 and 22

$$\phi = \mu^{-1} \circ \phi^* \circ \sigma = \widehat{\mu}^{-1} \circ \widehat{\phi}^* \circ \widehat{\sigma}$$

and

$$D\phi = \mu^{-1} \circ D\phi^* \circ \sigma = \widehat{\mu}^{-1} \circ D\widehat{\phi}^* \circ \widehat{\sigma}.$$

From Lemmas 24 and 25, $\widehat{\sigma} = \widehat{V}\sigma$ and $\widehat{\mu}^{-1} = \mu^{-1}U$ so

$$\mu^{-1} \circ D\phi^* \circ \sigma = \mu^{-1}U \circ D\widehat{\phi}^* \circ \widehat{V}\sigma$$

and it follows that $D\phi^* = UD\widehat{\phi}^*\widehat{V}$. Inverting this procedure gives $D\widehat{\phi}^* = \widehat{U}D\phi^*V$. ■

Note this result agrees with the actual representation as matrices. In the product $UD\widehat{\phi}^*\widehat{V}$, $U$ is an $m$ by $m$ matrix, $D\widehat{\phi}^*$ is an $m$ by $n$ matrix and $\widehat{V}$ is an $n$ by $n$ matrix so the product is an $m$ by $n$ matrix. But $D\phi^*$ is also an $m$ by $n$ matrix. Thus the derivatives $D\phi^*$ and $D\widehat{\phi}^*$ are related by the change of basis matrices. This means that in applying these results to a particular problem, we can choose specific bases to work in. If we need to use different bases, we multiply results appropriately by the change of basis matrices. Of course it is exactly the same derivative for the transformation $\phi$ between the underlying vector spaces $\mathcal{V}$ and $\mathcal{U}$.

We now consider applying these results to a special vector space. Let $\mathcal{V}$ denote some vector space of dimension $n$ and let $\mathcal{L}(\mathcal{V})$ denote the set of linear transformations from $\mathcal{V}$ to $\mathcal{V}$. It is straightforward to show $\mathcal{L}(\mathcal{V})$ is also a vector space having dimension $n^2$. However, unlike an arbitrary vector space, it is possible to unambiguously define a binary operation on $\mathcal{L}(\mathcal{V})$ which, together with $\mathcal{L}(\mathcal{V})$, form a group. This makes $\mathcal{L}(\mathcal{V})$ an associative algebra. Of course, since $\mathcal{L}(\mathcal{V})$ is a vector space, we can apply the vector space calculus to $\mathcal{L}(\mathcal{V})$.

The binary operation on $\mathcal{L}(\mathcal{V})$ is really function composition. If $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{F} \in \mathcal{L}(\mathcal{V})$ and $\mathrm{v} \in \mathcal{V}$, then $\mathbf{F}\mathrm{v}$ and $\mathbf{Q}\mathbf{v}$ are members of $\mathcal{V}$, and so are $\mathbf{RF}\mathbf{v}$ and $\mathbf{FQ}\mathbf{v}$. Thus the products $\mathbf{RF}$ and $\mathbf{FQ}$ are also members of $\mathcal{L}(\mathcal{V})$. Our interest is in relating the derivative of a function $\phi$ at $\mathbf{F}$ to the derivatives of $\phi$ at $\mathbf{RF}$ and $\mathbf{FQ}$. That is, to examine at the effect of left and right multiplication by constants on the derivative.

Let $\phi : \mathcal{L}(\mathcal{V}) \to \mathcal{R}$. Then $D\phi(\mathbf{F})$ is some linear transformation from $\mathcal{L}(\mathcal{V})$ into the real numbers. That is, $D\phi(\mathbf{F}) \in \mathcal{L}(\mathcal{L}(\mathcal{V}), \mathcal{R})$. Notice in this case $D\phi(\mathbf{F})$ is in the dual space of $\mathcal{L}(\mathcal{V})$. Thus $\mathcal{L}(\mathcal{L}(\mathcal{V}), \mathcal{R})$ is isomorphic to $\mathcal{L}(\mathcal{V})$. If we assume an inner product on $\mathcal{L}(\mathcal{V})$, there is a canonical isomorphism between $\mathcal{L}(\mathcal{L}(\mathcal{V}), \mathcal{R})$ and $\mathcal{L}(\mathcal{V})$ and we may naturally identify any member of $\mathcal{L}(\mathcal{L}(\mathcal{V}), \mathcal{R})$ with a member of $\mathcal{L}(\mathcal{V})$. This means the binary operation on $\mathcal{L}(\mathcal{V})$ can be applied to $D\phi(\mathbf{F})$ and members of $\mathcal{L}(\mathcal{V})$.

**Theorem 2** *Let* $\mathbf{R}$*,* $\mathbf{F} \in \mathcal{L}(\mathcal{V})$ *and* $\mathbf{RF}$ *denote their product (function composition). Consider* $\phi : \mathcal{L}(\mathcal{V}) \to \mathcal{R}$ *. If* $\phi$ *is differentiable at the points* $\mathbf{F}$ *and* $\mathbf{RF}$*, then*

$$D\phi(\mathbf{F}) = \mathbf{R}^{\mathrm{T}} D\phi(\mathbf{RF})$$

*where the right side is understood to be the product of the transpose of* $\mathbf{R}$ *with* $D\phi(\mathbf{RF})$.

**Proof.** We assume a basis for $\mathcal{L}(\mathcal{V})$ which establishes a canonical isomorphism $\sigma : \mathcal{L}(\mathcal{V}) \to \mathcal{R}^{n^2}$.

Let $\mathbf{X} = \mathbf{RF}$. Then under the isomorphism $\sigma$, $\mathbf{F}$ has $n^2$ components and forms an $n$ by $n$ matrix:

$$\sigma(\mathbf{F}) = \begin{pmatrix} F_1^1 & F_2^1 & F_3^1 & \cdots & F_n^1 \\ F_1^2 & F_2^2 & F_3^2 & \cdots & F_n^2 \\ F_1^3 & F_2^3 & F_3^3 & \cdots & F_n^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_1^n & F_2^n & F_3^n & \cdots & F_n^n \end{pmatrix}.$$

$\phi^*$ maps this matrix to a real number and we consider $\phi^*$ to be a real valued function of $n^2$ variables. Then $D\phi^*(\sigma(\mathbf{F}))$ is a linear transformation given by a 1 by $n^2$ matrix whose components are the partial derivatives of $\phi^*$ with respect to each component $F_j^i$. However, we may write $D\phi^*(\sigma(\mathbf{F}))$ as an $n$ by $n$ matrix:

$$D\phi^*(\sigma(\mathbf{F})) = \begin{pmatrix} \frac{\partial\phi^*}{\partial F_1^1} & \frac{\partial\phi^*}{\partial F_2^1} & \frac{\partial\phi^*}{\partial F_3^1} & \cdots & \frac{\partial\phi^*}{\partial F_n^1} \\ \frac{\partial\phi^*}{\partial F_1^2} & \frac{\partial\phi^*}{\partial F_2^2} & \frac{\partial\phi^*}{\partial F_3^2} & \cdots & \frac{\partial\phi^*}{\partial F_n^2} \\ \frac{\partial\phi^*}{\partial F_1^3} & \frac{\partial\phi^*}{\partial F_2^3} & \frac{\partial\phi^*}{\partial F_3^3} & \cdots & \frac{\partial\phi^*}{\partial F_n^3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi^*}{\partial F_1^n} & \frac{\partial\phi^*}{\partial F_2^n} & \frac{\partial\phi^*}{\partial F_3^n} & \cdots & \frac{\partial\phi^*}{\partial F_n^n} \end{pmatrix}. \tag{3.8}$$

Notice in this form $D\phi^*(\sigma(\mathbf{F}))$ may be thought of a linear transformation from $\mathcal{V}$ to $\mathcal{V}$. Applying this to the linear transformation $\mathbf{X}$ gives the matrix:

$$D\phi^*(\sigma(\mathbf{X})) = \begin{pmatrix} \frac{\partial\phi^*}{\partial X_1^1} & \frac{\partial\phi^*}{\partial X_2^1} & \frac{\partial\phi^*}{\partial X_3^1} & \cdots & \frac{\partial\phi^*}{\partial X_n^1} \\ \frac{\partial\phi^*}{\partial X_1^2} & \frac{\partial\phi^*}{\partial X_2^2} & \frac{\partial\phi^*}{\partial X_3^2} & \cdots & \frac{\partial\phi^*}{\partial X_n^2} \\ \frac{\partial\phi^*}{\partial X_1^3} & \frac{\partial\phi^*}{\partial X_2^3} & \frac{\partial\phi^*}{\partial X_3^3} & \cdots & \frac{\partial\phi^*}{\partial X_n^3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi^*}{\partial X_1^n} & \frac{\partial\phi^*}{\partial X_2^n} & \frac{\partial\phi^*}{\partial X_3^n} & \cdots & \frac{\partial\phi^*}{\partial X_n^n} \end{pmatrix}. \tag{3.9}$$

But $\mathbf{X} = \mathbf{RF}$ for some $\mathbf{R} \in \mathcal{L}(\mathcal{V})$. Then in component form, we must have $X_j^i = R_k^i F_j^k$. Now we're interested in the relationship between $D\phi^*(\sigma(\mathbf{F}))$ and $D\phi^*(\sigma(\mathbf{X}))$. Since these are both $n$ by $n$ matrices, we consider the relationship between $\frac{\partial\phi^*}{\partial F_j^i}$ and $\frac{\partial\phi^*}{\partial X_j^i}$. $X_j^i$ is a function of the components $F_l^k$ so these partial derivatives can be related. From the chain rule in multivariable calculus (fixing indices $k$ and $l$), we have

$$\frac{\partial \phi^*}{\partial F_l^k} = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \phi^*}{\partial X_j^i} \frac{\partial X_j^i}{\partial F_l^k}$$

Now

$$\frac{\partial X_j^i}{\partial F_l^k} = \frac{\partial}{\partial F_l^k} \left( R_m^i F_j^m \right) = \frac{\partial}{\partial F_l^k} \left( R_1^i F_j^1 + R_2^i F_j^2 + R_3^i F_j^3 + \cdots + R_n^i F_j^n \right) = \delta_l^j R_k^i$$

so

$$\frac{\partial \phi^*}{\partial F_l^k} = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial \phi^*}{\partial X_j^i} \delta_l^j R_k^i = \sum_{i=1}^n R_k^i \frac{\partial \phi^*}{\partial X_l^i}.$$

This result looks like the $(k, l)$ entry of the product of two matrices. $\mathbf{R}$ has the matrix representation

$$\sigma(\mathbf{R}) = \begin{pmatrix} R_1^1 & R_2^1 & R_3^1 & \cdots & R_n^1 \\ R_1^2 & R_2^2 & R_3^2 & \cdots & R_n^2 \\ R_1^3 & R_2^3 & R_3^3 & \cdots & R_n^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_1^n & R_2^n & R_3^n & \cdots & R_n^n \end{pmatrix} \text{ so } \sigma\left(\mathbf{R}^{\mathrm{T}}\right) = \begin{pmatrix} R_1^1 & R_1^2 & R_1^3 & \cdots & R_1^n \\ R_2^1 & R_2^2 & R_2^3 & \cdots & R_2^n \\ R_3^1 & R_3^2 & R_3^3 & \cdots & R_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_n^1 & R_n^2 & R_n^3 & \cdots & R_n^n \end{pmatrix}.$$

Looking at $\sigma\left(\mathbf{R}^{\mathrm{T}}\right) D\phi^*\left(\sigma(\mathbf{X})\right)$ in its matrix form gives

$$\begin{pmatrix} R_1^1 & R_1^2 & R_1^3 & \cdots & R_1^n \\ R_2^1 & R_2^2 & R_2^3 & \cdots & R_2^n \\ R_3^1 & R_3^2 & R_3^3 & \cdots & R_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_n^1 & R_n^2 & R_n^3 & \cdots & R_n^n \end{pmatrix} \begin{pmatrix} \frac{\partial \phi^*}{\partial X_1^1} & \frac{\partial \phi^*}{\partial X_2^1} & \frac{\partial \phi^*}{\partial X_3^1} & \cdots & \frac{\partial \phi^*}{\partial X_n^1} \\ \frac{\partial \phi^*}{\partial X_1^2} & \frac{\partial \phi^*}{\partial X_2^2} & \frac{\partial \phi^*}{\partial X_3^2} & \cdots & \frac{\partial \phi^*}{\partial X_n^2} \\ \frac{\partial \phi^*}{\partial X_1^3} & \frac{\partial \phi^*}{\partial X_2^3} & \frac{\partial \phi^*}{\partial X_3^3} & \cdots & \frac{\partial \phi^*}{\partial X_n^3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \phi^*}{\partial X_1^n} & \frac{\partial \phi^*}{\partial X_2^n} & \frac{\partial \phi^*}{\partial X_3^n} & \cdots & \frac{\partial \phi^*}{\partial X_n^n} \end{pmatrix}$$

whose $(k, l)$ entry is $\sum_{i=1}^n R_k^i \frac{\partial \phi^*}{\partial X_l^i}$. Thus $\frac{\partial \phi^*}{\partial F_l^k}$ is the $(k, l)$ entry of the above matrix multiplication and $D\phi^*\left(\sigma(\mathbf{F})\right) = \sigma\left(\mathbf{R}^{\mathrm{T}}\right) D\phi^*\left(\sigma(\mathbf{X})\right)$. Since $\mathbf{X} = \mathbf{RF}$, then $D\phi^*\left(\sigma(\mathbf{F})\right) = \sigma\left(\mathbf{R}^{\mathrm{T}}\right) D\phi^*\left(\sigma(\mathbf{RF})\right)$. This result does not depend on the basis chosen for $\mathcal{L}(\mathcal{V})$, (i.e. the function $\sigma$ could be any isomorphism), so it follows that

$$D\phi(\mathbf{F}) = \mathbf{R}^{\mathrm{T}} D\phi(\mathbf{RF})$$

relating the derivative of $\phi$ at $\mathbf{F}$ to the derivative of $\phi$ at $\mathbf{RF}$. ■

**Theorem 3** *Let* $\mathbf{Q}$, $\mathbf{F} \in \mathcal{L}(\mathcal{V})$ *and* $\mathbf{FQ}$ *denote their product (function composition). Consider* $\phi : \mathcal{L}(\mathcal{V}) \to \mathcal{R}$ *. If* $\phi$ *is differentiable at the points* $\mathbf{F}$ *and* $\mathbf{FQ}$*, then*

$$D\phi(\mathbf{F}) = D\phi(\mathbf{FQ})\,\mathbf{Q}^{\mathrm{T}}$$

*where the right side is understood to be the product of* $D\phi(\mathbf{FQ})$ *with the transpose of* $\mathbf{Q}$.

**Proof.** We assume a basis for $\mathcal{L}(\mathcal{V})$ which establishes a canonical isomorphism $\sigma : \mathcal{L}(\mathcal{V}) \to \mathcal{R}^{n^2}$. Let $\mathbf{X} = \mathbf{FQ}$. Then equations (3.8) and (3.9) as given in the proof of Theorem 2 still hold. This means $X^i_j = F^i_m Q^m_j$. From the chain rule in multivariable calculus (fixing indices $k$ and $l$), we have

$$\frac{\partial \phi^*}{\partial F^k_l} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \phi^*}{\partial X^i_j}\frac{\partial X^i_j}{\partial F^k_l}$$

Now

$$\frac{\partial X^i_j}{\partial F^k_l} = \frac{\partial}{\partial F^k_l}\left(F^i_m Q^m_j\right) = \frac{\partial}{\partial F^k_l}\left(F^i_1 Q^1_j + F^i_2 Q^2_j + F^i_3 Q^3_j + \cdots + F^i_n Q^n_j\right) = \delta^i_k Q^l_j$$

so

$$\frac{\partial \phi^*}{\partial F^k_l} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \phi^*}{\partial X^i_j}\delta^i_k Q^l_j = \sum_{j=1}^{n} \frac{\partial \phi^*}{\partial X^k_j} Q^l_j.$$

This result looks like the $(k, l)$ entry of the product of two matrices. $\mathbf{Q}$ has the matrix representation

$$\sigma(\mathbf{Q}) = \begin{pmatrix} Q^1_1 & Q^1_2 & Q^1_3 & \cdots & Q^1_n \\ Q^2_1 & Q^2_2 & Q^2_3 & \cdots & Q^2_n \\ Q^3_1 & Q^3_2 & Q^3_3 & \cdots & Q^3_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q^n_1 & Q^n_2 & Q^n_3 & \cdots & Q^n_n \end{pmatrix} \text{ so } \sigma\left(\mathbf{Q}^{\mathrm{T}}\right) = \begin{pmatrix} Q^1_1 & Q^2_1 & Q^3_1 & \cdots & Q^n_1 \\ Q^1_2 & Q^2_2 & Q^3_2 & \cdots & Q^n_2 \\ Q^1_3 & Q^2_3 & Q^3_3 & \cdots & Q^n_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q^1_n & Q^2_n & Q^3_n & \cdots & Q^n_n \end{pmatrix}.$$

Looking at $D\phi^*(\sigma(\mathbf{X}))\,\sigma(\mathbf{Q}^{\mathrm{T}})$ in its component form gives

$$\begin{pmatrix} \frac{\partial\phi^*}{\partial X_1^1} & \frac{\partial\phi^*}{\partial X_2^1} & \frac{\partial\phi^*}{\partial X_3^1} & \cdots & \frac{\partial\phi^*}{\partial X_n^1} \\ \frac{\partial\phi^*}{\partial X_1^2} & \frac{\partial\phi^*}{\partial X_2^2} & \frac{\partial\phi^*}{\partial X_3^2} & \cdots & \frac{\partial\phi^*}{\partial X_n^2} \\ \frac{\partial\phi^*}{\partial X_1^3} & \frac{\partial\phi^*}{\partial X_2^3} & \frac{\partial\phi^*}{\partial X_3^3} & \cdots & \frac{\partial\phi^*}{\partial X_n^3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi^*}{\partial X_1^n} & \frac{\partial\phi^*}{\partial X_2^n} & \frac{\partial\phi^*}{\partial X_3^n} & \cdots & \frac{\partial\phi^*}{\partial X_n^n} \end{pmatrix} \begin{pmatrix} Q_1^1 & Q_1^2 & Q_1^3 & \cdots & Q_1^n \\ Q_2^1 & Q_2^2 & Q_2^3 & \cdots & Q_2^n \\ Q_3^1 & Q_3^2 & Q_3^3 & \cdots & Q_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_n^1 & Q_n^2 & Q_n^3 & \cdots & Q_n^n \end{pmatrix}$$

whose $(k,l)$ entry is $\sum_{j=1}^{n} \frac{\partial\phi^*}{\partial X_j^k} Q_j^l$. Thus $\frac{\partial\phi^*}{\partial F_l^k}$ is the $(k,l)$ entry of the above matrix multiplication and $D\phi^*(\sigma(\mathbf{F})) = D\phi^*(\sigma(\mathbf{X}))\,\sigma(\mathbf{Q}^{\mathrm{T}})$. Since $\mathbf{X} = \mathbf{FQ}$, then $D\phi^*(\sigma(\mathbf{F})) = D\phi^*(\sigma(\mathbf{FQ}))\,\sigma(\mathbf{Q}^{\mathrm{T}})$. This result does not depend on the basis chosen for $\mathcal{L}(\mathcal{V})$, (i.e. the function $\sigma$ could be any isomorphism), so it follows that

$$D\phi(\mathbf{F}) = D\phi(\mathbf{FQ})\,\mathbf{Q}^{\mathrm{T}}$$

relating the derivative of $\phi$ at $\mathbf{F}$ to the derivative of $\phi$ at $\mathbf{FQ}$. ■

The final consideration in this section concerns derivatives of functions of two vectors: that is, $\phi : \mathcal{V} \times \mathcal{V} \to \mathcal{U}$. In particular we are concerned with how to differentiate with respect to one slot, but not the other. To do this, we define $\psi(\mathbf{v}) = \phi(\mathbf{v}, \mathbf{w})$ for any fixed $\mathbf{w}$ and define the derivative of $\phi$ with respect to the first slot to be the derivative of $\psi$. That is, $D_1\phi(\mathbf{v},\mathbf{w}) = D\psi(\mathbf{v})$. Similarly, we define $\varphi(\mathbf{w}) = \phi(\mathbf{v},\mathbf{w})$ for any fixed $\mathbf{v}$ and defined the derivative of $\phi$ with respect to the second slot to be the derivative of $\varphi$. That is, $D_2\phi(\mathbf{v},\mathbf{w}) = D\varphi(\mathbf{w})$. Note that the subscripts on the $D$ indicate which slot is being differentiated. Then all of the above development of the calculus on vector spaces immediately applies to $D_1$ and $D_2$. The derivatives $D_1$ and $D_2$ are called partial derivatives [16], [17].

An important observation needs to be made here. If $\phi : \mathcal{V}\times\mathcal{V} \to \mathcal{U}$, then $D_1\phi$ is independent of whatever is in the second slot providing the second slot is completely independent of the first slot. Thus $D_1\phi(\mathbf{v},\mathbf{w}) = D_1\phi(\mathbf{v},\kappa(\mathbf{w}))$ for any function $\kappa : \mathcal{V} \to \mathcal{V}$ if $\mathbf{w}$ is not a function of $\mathbf{v}$ and vice-versa. Similarly, $D_2\phi(\mathbf{v},\mathbf{w}) = D_2\phi(\eta(\mathbf{v}),\mathbf{w})$ for any function $\eta : \mathcal{V} \to \mathcal{V}$ under the same assumption.

## 3.2 An Integration Theorem

In this section we wish to prove an integration theorem which will fill in an important step in the derivation of the field equations. Two lemmas are proven which lead to the statement and proof of the required theorem. The method of proving Lemma 27 is found in Arnold [13].

**Lemma 27** *Suppose $f^i : [s_1, s_2] \to \Re$, $h^i : [s_1, s_2] \to \Re$, $i = 1, 2, 3, \ldots, n$ are continuous functions for all $i \in \{1, 2, 3, \ldots, n\}$. If*

$$\int_{s_1}^{s_2} \sum_{i=1}^{n} f^i(s)\, h^i(s)\, ds = 0$$

*for all such functions $h^i$, then $f^i \equiv 0$ for all $i \in \{1, 2, 3, \ldots, n\}$.*

**Proof.** Let $s^* \in (s_1, s_2)$. Let $I^- \subset \{1, 2, 3, \ldots, n\}$ be the set of indices such that $f^i(s^*) < 0$ if $i \in I^-$. Let $I^+ \subset \{1, 2, 3, \ldots, n\}$ be the set of indices such that $f^i(s^*) > 0$ if $i \in I^+$. Let $I^0 \subset \{1, 2, 3, \ldots, n\}$ be the set of indices such that $f^i(s^*) = 0$ if $i \in I^0$. Assume $I^- \cup I^+$ is not empty and attempt to derive a contradiction.

Consider the functions $f^i$ for which $i \in I^-$. Since all these functions are continuous, there exists a value $c^- < 0$ in an open interval $s^* - d^- < s < s^* + d^-$ such that $f^i(s) < c^-$ for all $i \in I^-$. Now let $h^i(s)$ be defined in such a way that $h^i(s) = 0$ outside this open interval, $h^i(s) < 0$ inside this interval and $h^i(s) = -1$ in the subinterval $s^* - \frac{1}{2}d^- < s < s^* + \frac{1}{2}d^-$. Then $f^i(s)\, h^i(s) > 0$ for all $i \in I^-$ in the interval $s^* - \frac{1}{2}d^- < s < s^* + \frac{1}{2}d^-$, $f^i(s)\, h^i(s) \geq 0$ everywhere else in $s^* - d^- < s < s^* + d^-$ and $f^i(s)\, h^i(s) = 0$ outside of $s^* - d^- < s < s^* + d^-$.

Consider the functions $f^i$ for which $i \in I^+$. Since all these functions are continuous, there exists a value $c^+ > 0$ in an open interval $s^* - d^+ < s < s^* + d^+$ such that $f^i(s) > c^+$ for all $i \in I^+$. Now let $h^i(s)$ be defined in such a way that $h^i(s) = 0$ outside this open interval, $h^i(s) > 0$ inside this interval and $h^i(s) = 1$ in the subinterval $s^* - \frac{1}{2}d^+ < s < s^* + \frac{1}{2}d^+$. Then $f^i(s)\, h^i(s) > 0$ for all $i \in I^+$ in the interval $s^* - \frac{1}{2}d^- < s < s^* + \frac{1}{2}d^-$, $f^i(s)\, h^i(s) \geq 0$ everywhere else in $s^* - d^+ < s < s^* + d^+$ and $f^i(s)\, h^i(s) = 0$ outside of $s^* - d^+ < s < s^* + d^+$.

Then

$$\int_{s_1}^{s_2} f^i(s)\, h^i(s)\, ds \;\; \geq \;\; d^- \left| c^- \right| > 0 \text{ if } i \in I^-,$$

$$\begin{aligned}\int_{s_1}^{s_2} f^i(s)\, h^i(s)\, ds &\geq d^+c^+ > 0 \text{ if } i \in I^+ \text{ and} \\ \int_{s_1}^{s_2} f^i(s)\, h^i(s)\, ds &= 0 \text{ if } i \in I^0.\end{aligned}$$

It follows that

$$\int_{s_1}^{s_2} \sum_{i=1}^{n} f^i(s)\, h^i(s)\, ds > 0$$

if $I^- \cup I^+$ is not empty. But this contradicts the hypothesis

$$\int_{s_1}^{s_2} \sum_{i=1}^{n} f^i(s)\, h^i(s)\, ds = 0$$

so the assumption $I^- \cup I^+$ is not empty must be incorrect and we conclude $I^- \cup I^+$ is empty. This means $I^0 = \{1, 2, 3, \ldots, n\}$ and $f^i(s^*) = 0$ for all $i \in \{1, 2, 3, \ldots, n\}$. But $s^*$ is an arbitrary point. It follows that $f^i \equiv 0$ for all $i \in \{1, 2, 3, \ldots, n\}$. ■

Notice Lemma 27 can be expressed in terms of the inner product of two vector functions as explained in the following corollary.

**Corollary 5** *Suppose $\mathcal{V}$ is a real inner product space. Let $\mathbf{f} : [s_1, s_2] \to \mathcal{V}$ and $\mathbf{h} : [s_1, s_2] \to \mathcal{V}$ be continuous vector functions. If*

$$\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds = 0$$

*for all such vector functions $\mathbf{h}$, then $\mathbf{f} \equiv \mathbf{0}$.*

**Proof.** Since $\mathcal{V}$ is an inner product space, it is always possible to choose an orthonormal basis for $\mathcal{V}$. Denote this basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n\}$. Then the vector functions $\mathbf{f}(s)$ and $\mathbf{h}(s)$ have component representations $\mathbf{f}(s) = f^i(s)\, \mathbf{e}_i$ and $\mathbf{h}(s) = h^j(s)\, \mathbf{e}_j$. Since $\mathbf{f}(s)$ and $\mathbf{h}(s)$ are continuous, each of their component functions are continuous. Then

$$\mathbf{f}(s) \cdot \mathbf{h}(s) = f^i(s)\, \mathbf{e}_i \cdot h^j(s)\, \mathbf{e}_j = f^i(s)\, h^j(s)\, \mathbf{e}_i \cdot \mathbf{e}_j = f^i(s)\, h^j(s)\, \delta_{ij} = \sum_{i=1}^{n} f^i(s)\, h^i(s)$$

and

$$\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds = \int_{s_1}^{s_2} \sum_{i=1}^{n} f^i(s)\, h^i(s)\, ds = 0.$$

The proof of the Lemma 27 showed this implies $f^i \equiv 0$ for all $i \in \{1, 2, 3, \ldots, n\}$ which means $\mathbf{f} \equiv \mathbf{0}$. ■

**Lemma 28** *Suppose $\mathcal{V}$ is a real inner product space. Let $\mathbf{f} : [s_1, s_2] \to \mathcal{V}$ , $\mathbf{g} : [s_1, s_2] \to \mathcal{V}$ , $\mathbf{h} : [s_1, s_2] \to \mathcal{V}$ be continuous vector functions. If*

$$\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds + \mathbf{g}(s) \cdot \mathbf{h}(s) \big|_{s_1}^{s_2} = 0$$

*for all such vector functions $\mathbf{h}$, then $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$.*

**Proof.** Consider three cases.

1. Suppose $\mathbf{f} \equiv \mathbf{0}$. Then

$$\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds = 0$$

for any choice of the function $\mathbf{h}$. This means $\mathbf{g}(s) \cdot \mathbf{h}(s) \big|_{s_1}^{s_2} = 0$. Since $\mathbf{h}$ may be chosen to be any continuous vector function, then

$$\mathbf{g}(s) \cdot \mathbf{h}(s) \big|_{s_1}^{s_2} = \mathbf{g}(s_2) \cdot \mathbf{h}(s_2) - \mathbf{g}(s_1) \cdot \mathbf{h}(s_1) = 0$$

only if $\mathbf{g}(s_2) = \mathbf{g}(s_1) = \mathbf{0}$.

2. Suppose $\mathbf{f} \neq \mathbf{0}$ and $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$. Then $\mathbf{g}(s) \cdot \mathbf{h}(s) \big|_{s_1}^{s_2} = 0$ for any function $\mathbf{h}$. This means

$$\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds = 0$$

for any function $\mathbf{h}$. But it was shown in Corollary 5 this implies $\mathbf{f} \equiv \mathbf{0}$ contradicting the assumption $\mathbf{f} \neq \mathbf{0}$.

3. Suppose $\mathbf{f} \neq \mathbf{0}$ and $\mathbf{g}(s_1) \neq \mathbf{0}$ or $\mathbf{g}(s_2) \neq \mathbf{0}$. From the proof for Lemma 27, it is always possible to find a function $\mathbf{h}$ such that

$$\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds > 0 \text{ and } \mathbf{h}(s_1) = \mathbf{h}(s_2) = 0.$$

This means

$$\begin{aligned}\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds + \mathbf{g}(s) \cdot \mathbf{h}(s)\, |_{s_1}^{s_2} &= \int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds \\ &\quad + \mathbf{g}(s_2) \cdot \mathbf{h}(s_2) - \mathbf{g}(s_1) \cdot \mathbf{h}(s_1) \\ &= \int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds > 0\end{aligned}$$

which contradicts the hypothesis

$$\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds + \mathbf{g}(s) \cdot \mathbf{h}(s)\, |_{s_1}^{s_2} = 0.$$

Cases 2 and 3 show the hypothesis cannot be satisfied if $\mathbf{f} \neq \mathbf{0}$. Case 1 showed $\mathbf{f} \equiv \mathbf{0}$ implies $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$. From the conclusion of each of these three cases, it follows that

$$\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds + \mathbf{g}(s) \cdot \mathbf{h}(s)\, |_{s_1}^{s_2} = 0$$

implies $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$. ■

**Theorem 4** *Let $\mathcal{U}$ and $\mathcal{V}$ denote inner product spaces. Let $\circ$ and $\cdot$ denote inner products on $\mathcal{U}$ and $\mathcal{V}$ respectively. Let $\mathbf{F} : [s_1, s_2] \to \mathcal{U}$ , $\mathbf{H} : [s_1, s_2] \to \mathcal{U}$ , $\mathbf{f} : [s_1, s_2] \to \mathcal{V}$ , $\mathbf{g} : [s_1, s_2] \to \mathcal{V}$ and $\mathbf{h} : [s_1, s_2] \to \mathcal{V}$ be continuous vector functions. If*

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds + \int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds + \mathbf{g}(s) \cdot \mathbf{h}(s)\, |_{s_1}^{s_2} = 0 \tag{3.10}$$

*for all vector functions $\mathbf{H}$ and $\mathbf{h}$, then $\mathbf{F} \equiv \mathbf{0}$, $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$.*

**Proof.** There are eight cases to consider, but four of these may be handled as one.

1. Suppose $\mathbf{F} \equiv \mathbf{0}$. This leaves

$$\int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds + \mathbf{g}(s) \cdot \mathbf{h}(s)\, |_{s_1}^{s_2} = 0.$$

As shown in Lemma 28, this implies $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$.

2. Suppose $\mathbf{F} \neq \mathbf{0}$, $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$. This leaves

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds = 0.$$

   But from Corollary 5, this implies $\mathbf{F} \equiv \mathbf{0}$ contradicting the assumption $\mathbf{F} \neq \mathbf{0}$. Thus this case cannot happen.

3. Suppose $\mathbf{F} \neq \mathbf{0}$, $\mathbf{f} \neq \mathbf{0}$ and $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$. This leaves

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds + \int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds = 0.$$

   Following the proof of Lemma 27, we can always find $\mathbf{H}$ and $\mathbf{h}$ such that

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds > 0 \text{ and } \int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds > 0$$

   so

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds + \int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds > 0.$$

   This contradicts the hypothesis given in equation (3.10) and thus this case cannot happen.

4. Suppose $\mathbf{F} \neq \mathbf{0}$, $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{g}(s_1) \neq \mathbf{0}$ or $\mathbf{g}(s_2) \neq \mathbf{0}$. This leaves

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds + \mathbf{g}(s) \cdot \mathbf{h}(s)\, \Big|_{s_1}^{s_2} = 0.$$

   We can always choose $\mathbf{h}$ such that $\mathbf{h}(s_1) = \mathbf{h}(s_2) = \mathbf{0}$. This would leave

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds = 0.$$

   But from Corollary 5, this implies $\mathbf{F} \equiv \mathbf{0}$ contradicting the assumption $\mathbf{F} \neq \mathbf{0}$. Thus this case cannot happen.

5. Suppose $\mathbf{F} \neq \mathbf{0}$, $\mathbf{f} \neq \mathbf{0}$ and $\mathbf{g}(s_1) \neq \mathbf{0}$ or $\mathbf{g}(s_2) \neq \mathbf{0}$. From the proof of Lemma 27, we can always find $\mathbf{H}$ and $\mathbf{h}$ such that

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds > 0 \ , \ \int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds > 0$$

with $\mathbf{H}(s_1) = \mathbf{H}(s_2) = \mathbf{0}$ and $\mathbf{h}(s_1) = \mathbf{h}(s_2) = \mathbf{0}$. Then

$$\mathbf{g}(s) \cdot \mathbf{h}(s)\,|_{s_1}^{s_2} = 0$$

and

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds + \int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds > 0$$

contradicting the hypothesis given in equation (3.10). Thus this case cannot happen.

From cases 2 through 5, the hypothesis given in equation (3.10) cannot be satisfied unless $\mathbf{F} \equiv \mathbf{0}$. Case 1 showed $\mathbf{F} \equiv \mathbf{0}$ implies $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$. Thus $\mathbf{F} \equiv \mathbf{0}$, $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{g}(s_1) = \mathbf{g}(s_2) = \mathbf{0}$ in order for

$$\int_{s_1}^{s_2} \mathbf{F}(s) \circ \mathbf{H}(s)\, ds + \int_{s_1}^{s_2} \mathbf{f}(s) \cdot \mathbf{h}(s)\, ds + \mathbf{g}(s) \cdot \mathbf{h}(s)\,|_{s_1}^{s_2} = 0$$

to be true for all $\mathbf{H}$ and $\mathbf{h}$. ■

# Chapter 4

# The Field Equations

The aim of this chapter is to derive the field equations from a variational principle. The derivation combines the approach of Cohen [2] and Cohen and Sun [3]. It makes use of the variational operator $\delta$ for which several useful mathematical properties may be proved [8]. Once the field equations have been derived, they are compared with field equations appearing in the literature, specifically Cohen and Sun [3] and Cohen [1].

## 4.1 Derivation of The Field Equations

In Chapter 2, several quantities were introduced: the curve forming the rod axis, the directors and the deformation tensor. These quantities are of geometric character, used to describe the geometry of the rod and its deformation. To develop the theory of directed rods, we must introduce physical quantities and some relationships between the physical and geometrical quantities defined. The deformation of a directed rod is caused by forces applied to the rod, both externally (surface forces) and internally (body forces).

The physical quantities introduced here are those given by Cohen and Sun [3] with some changes in the notation.

1. Let $\mathbf{f}$ denote the body force density vector.

2. Let $\mathbf{l}^1$, $\mathbf{l}^2$, $\mathbf{l}^3$ denote the director body force density vectors.

3. Let $\mathbf{n}$ denote the contact (axial) force.

4. Let $\mathbf{m}^1$, $\mathbf{m}^2$, $\mathbf{m}^3$ denote the contact (axial) director force vectors.

5. Let $\mathbf{p}^1$, $\mathbf{p}^2$, $\mathbf{p}^3$ denote the intrinsic director force densities.

The quantities listed as items 1 through 4 are used in the derivation of the field equations. The fifth set of vectors will be introduced later when comparing our derived field equations with field equations found in the literature.

Each of these vectors is a function of the parameter $s_R$. We wish to consider the following expressions:

$$\rho\left(\mathbf{f}\cdot\mathbf{r}+\mathbf{l}^1\cdot\mathbf{d}_1+\mathbf{l}^2\cdot\mathbf{d}_2+\mathbf{l}^3\cdot\mathbf{d}_3\right)=\rho\left(\mathbf{f}\cdot\mathbf{r}+\mathbf{l}^i\cdot\mathbf{d}_i\right)$$

and

$$\mathbf{n}\cdot\mathbf{r}+\mathbf{m}^1\cdot\mathbf{d}_1+\mathbf{m}^2\cdot\mathbf{d}_2+\mathbf{m}^3\cdot\mathbf{d}_3=\mathbf{n}\cdot\mathbf{r}+\mathbf{m}^i\cdot\mathbf{d}_i$$

where $\rho$ is the linear mass density along the rod axis. Let $\mathbf{l}^i$ and $\mathbf{m}^i$ be expressed in terms of the basis $\left\{\mathbf{d}^1,\mathbf{d}^2,\mathbf{d}^3\right\}$ as $\mathbf{l}^i=l^i_j\mathbf{d}^j$ and $\mathbf{m}^i=m^i_j\mathbf{d}^j$. The two formulas become

$$\rho\left(\mathbf{f}\cdot\mathbf{r}+\mathbf{l}^i\cdot\mathbf{d}_i\right)=\rho\left(\mathbf{f}\cdot\mathbf{r}+l^i_j\mathbf{d}^j\cdot\mathbf{d}_i\right)=\rho\left(\mathbf{f}\cdot\mathbf{r}+l^i_j\delta^j_i\right)=\rho\left(\mathbf{f}\cdot\mathbf{r}+l^i_i\right)$$

and

$$\mathbf{n}\cdot\mathbf{r}+\mathbf{m}^i\cdot\mathbf{d}_i=\mathbf{n}\cdot\mathbf{r}+m^i_j\mathbf{d}^j\cdot\mathbf{d}_i=\mathbf{n}\cdot\mathbf{r}+m^i_j\delta^j_i=\mathbf{n}\cdot\mathbf{r}+m^i_i.$$

These last expressions look like the traces of some second order endomorphisms. Define such endomorphisms as follows:

$$\overline{\mathbf{L}}\equiv\delta^j_i\mathbf{l}^i\otimes\mathbf{d}_{Rj}=\mathbf{l}^i\otimes\mathbf{d}_{Ri}\text{ and }\overline{\mathbf{M}}\equiv\delta^j_i\mathbf{m}^i\otimes\mathbf{d}_{Rj}=\mathbf{m}^i\otimes\mathbf{d}_{Ri}.$$

Notice $\overline{\mathbf{L}}$ and $\overline{\mathbf{M}}$ are defined in terms of a mixed tensor product. Using $\mathbf{l}^i=l^i_j\mathbf{d}^j$ and $\mathbf{m}^i=m^i_j\mathbf{d}^j$, these definitions are

$$\overline{\mathbf{L}}\equiv l^i_j\mathbf{d}^j\otimes\mathbf{d}_{Ri}\text{ and }\overline{\mathbf{M}}\equiv m^i_j\mathbf{d}^j\otimes\mathbf{d}_{Ri}. \tag{4.1}$$

Then $\overline{\mathbf{L}}$ and $\overline{\mathbf{M}}$ have the components $l^i_j$ and $m^i_j$ respectively. The expressions may be written

$$\rho\left(\mathbf{f}\cdot\mathbf{r}+\mathbf{l}^i\cdot\mathbf{d}_i\right)=\rho\left(\mathbf{f}\cdot\mathbf{r}+l^i_i\right)=\rho\left(\mathbf{f}\cdot\mathbf{r}+\mathrm{tr}\overline{\mathbf{L}}\right) \tag{4.2}$$

and

$$\mathbf{n}\cdot\mathbf{r}+\mathbf{m}^i\cdot\mathbf{d}_i=\mathbf{n}\cdot\mathbf{r}+m^i_i=\mathbf{n}\cdot\mathbf{r}+\mathrm{tr}\overline{\mathbf{M}} \tag{4.3}$$

using the definition of trace for an endomorphism.

**Lemma 29** $\overline{\mathbf{L}}\circ\mathbf{F}=\mathrm{tr}\overline{\mathbf{L}}$ *and* $\overline{\mathbf{M}}\circ\mathbf{F}=\mathrm{tr}\overline{\mathbf{M}}$ *where* $\mathbf{F}$ *is the deformation tensor.*

**Proof.**

$$\begin{aligned}
\overline{\mathbf{L}}\circ\mathbf{F} &= \left(l^i_j\mathbf{d}^j\otimes\mathbf{d}_{Ri}\right)\circ\left(\mathbf{d}_k\otimes\mathbf{d}^k_R\right)\\
&= l^i_j\left(\mathbf{d}^j\otimes\mathbf{d}_{Ri}\right)\circ\left(\mathbf{d}_k\otimes\mathbf{d}^k_R\right)\\
&= l^i_j\left(\mathbf{d}^j\cdot\mathbf{d}_k\right)\left(\mathbf{d}_{Ri}\cdot\mathbf{d}^k_R\right)\\
&= l^i_j\delta^j_k\delta^k_i=l^i_i=\mathrm{tr}\overline{\mathbf{L}}
\end{aligned}$$

using Lemma 6. The steps to prove $\overline{\mathbf{M}}\circ\ \mathbf{F}=\mathrm{tr}\overline{\mathbf{M}}$ are identical. ■

Equations (4.2) and (4.3) become

$$\rho\left(\mathbf{f}\cdot\mathbf{r}+\mathbf{l}^i\cdot\mathbf{d}_i\right)=\rho\left(\mathbf{f}\cdot\mathbf{r}+l^i_i\right)=\rho\left(\mathbf{f}\cdot\mathbf{r}+\overline{\mathbf{L}}\circ\mathbf{F}\right) \tag{4.4}$$

and

$$\mathbf{n}\cdot\mathbf{r}+\mathbf{m}^i\cdot\mathbf{d}_i=\mathbf{n}\cdot\mathbf{r}+m^i_i=\mathbf{n}\cdot\mathbf{r}+\overline{\mathbf{M}}\circ\mathbf{F}. \tag{4.5}$$

Now consider the curve describing the rod axis in some state $c_d$ (not necessarily the reference configuration) and a small virtual displacement of the curve $\delta\mathbf{r}$, $\delta\mathbf{d}_1$, $\delta\mathbf{d}_2$ and $\delta\mathbf{d}_3$ for some arbitrary segment $c$ of the rod. Each point on $c$ is displaced to a corresponding point on the perturbed curve $c^*$. The curve $c$ is described by the function $\mathbf{r}$ with directors $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$. The curve $c^*$ is described by the function $\mathbf{r}+\delta\mathbf{r}$ with directors $\mathbf{d}_1+\delta\mathbf{d}_1$, $\mathbf{d}_2+\delta\mathbf{d}_2$ and $\mathbf{d}_3+\delta\mathbf{d}_3$.

**Postulate 1** The virtual work done in moving a point on $c$ to the corresponding point on $c^*$ is given by

$$\begin{gathered}
\rho\left(\mathbf{f}\cdot(\mathbf{r}+\delta\mathbf{r})-\mathbf{f}\cdot\mathbf{r}+\mathbf{l}^i\cdot(\mathbf{d}_i+\delta\mathbf{d}_i)-\mathbf{l}^i\cdot\mathbf{d}_i\right)\\
+\mathbf{n}\cdot(\mathbf{r}+\delta\mathbf{r})-\mathbf{n}\cdot\mathbf{r}+\mathbf{m}^i\cdot(\mathbf{d}_i+\delta\mathbf{d}_i)-\mathbf{m}^i\cdot\mathbf{d}_i
\end{gathered}$$

$$= \rho\left(\mathbf{f}\cdot\delta\mathbf{r} + \mathbf{l}^i\cdot\delta\mathbf{d}_i\right) + \mathbf{n}\cdot\delta\mathbf{r} + \mathbf{m}^i\cdot\delta\mathbf{d}_i.$$

Integrating over the above expression with respect to $s$ gives the virtual work done on the entire segment of the curve due to the virtual displacements $\delta\mathbf{r}$, $\delta\mathbf{d}_1$, $\delta\mathbf{d}_2$ and $\delta\mathbf{d}_3$. The term $\mathbf{n}\cdot\delta\mathbf{r} + \mathbf{m}^i\cdot\delta\mathbf{d}_i$ is only integrated over the ends of the rod segment while the term $\rho\left(\mathbf{f}\cdot\delta\mathbf{r} + \mathbf{l}^i\cdot\delta\mathbf{d}_i\right)$ is integrated over the entire rod segment. This is expressed as

$$\mathcal{A} = \int_c \rho\ \left(\mathbf{f}\cdot\delta\mathbf{r} + \mathbf{l}^i\cdot\delta\mathbf{d}_i\right)\ ds + \int_{\partial c}\left(\mathbf{n}\cdot\delta\mathbf{r} + \mathbf{m}^i\cdot\delta\mathbf{d}_i\right) ds.$$

If we define

$$\delta\mathbf{F} \equiv \delta\mathbf{d}_i \otimes \mathbf{d}^i_R, \tag{4.6}$$

then from equations (4.4) and (4.5) we obtain

$$\mathcal{A} = \int_c \rho\ \left(\mathbf{f}\cdot\delta\mathbf{r} + \overline{\mathbf{L}}\circ\delta\mathbf{F}\right)\ ds + \int_{\partial c}\left(\mathbf{n}\cdot\delta\mathbf{r} + \overline{\mathbf{M}}\circ\delta\mathbf{F}\right) ds.$$

Noting that $\mathbf{n}$ and $\overline{\mathbf{M}}$ act only at the ends of the rod segment, we get the next postulate.

**Postulate 2** If $\mathcal{A}$ is the virtual work associated with an arbitrary, virtual displacement from the configuration $c$ denoted as $\delta\mathbf{r}$ and $\delta\mathbf{d}_i$:

$$\mathcal{A} = \int_{s_1}^{s_2} \rho\ \left(\mathbf{f}\cdot\delta\ \mathbf{r} + \overline{\mathbf{L}}\circ\delta\mathbf{F}\right)\ ds + \left(\mathbf{n}\cdot\delta\mathbf{r} + \overline{\mathbf{M}}\circ\delta\mathbf{F}\right)\Big|_{s_1}^{s_2} \tag{4.7}$$

where $a \le s_1 < s_2 \le b$ and the interval $[s_1, s_2]$ is the domain of the parameter of the arbitrary segment of curve.

Let $W$ be the energy stored (i.e. work) during deformation.

**Postulate 3** There exists a scalar function $\varepsilon = \varepsilon\left(\mathbf{F}, \mathbf{F}'\right)$ with the property that $W = \int_{s_1}^{s_2} \rho\varepsilon\ ds$.

The function $\varepsilon$ is called the strain energy density function and materials for which a strain energy density is postulated to exist are called hyperelastic. The function $\varepsilon = \varepsilon\left(\mathbf{F}, \mathbf{F}'\right)$ is a constitutive relation or equation. This means its functional form depends upon the material of which the rod is composed.

**Postulate 4** $\mathcal{A} = \delta W$.

This is the key relationship in deriving the field equations from a variational principle. It connects the strain energy density to the interaction between the geometric and physical quantities describing the deformation of the rod.

**Postulate 5** $\delta(\rho\, ds) = 0$.

This postulate is a statement of the principle of mass conservation of the rod under deformation.

$\delta W$ is evaluated using the properties of the variation operator $\delta$:

$$\delta W = \delta \int_{s_1}^{s_2} \rho\varepsilon\, ds = \int_{s_1}^{s_2} \delta(\rho\varepsilon\, ds) = \int_{s_1}^{s_2} (\delta\varepsilon)\, \rho\, ds + \int_{s_1}^{s_2} \varepsilon\, \delta(\rho\, ds) = \int_{s_1}^{s_2} (\delta\varepsilon)\, \rho\, ds$$

using Postulate 5. Furthermore,

$$\delta\varepsilon = D_1\varepsilon \circ \delta\mathbf{F} + D_2\varepsilon \circ \delta\mathbf{F}'$$

using the notation for derivatives discussed in the previous chapter. Thus

$$\delta W = \int_{s_1}^{s_2} \rho\left(D_1\varepsilon \circ \delta\mathbf{F} + D_2\varepsilon \circ \delta\mathbf{F}'\right) ds. \tag{4.8}$$

From Postulate 4, the following integral equation is obtained:

$$\begin{aligned} & \int_{s_1}^{s_2} \rho\left(\mathbf{f}\cdot\delta\mathbf{r} + \overline{\mathbf{L}} \circ \delta\mathbf{F}\right) ds + \left(\mathbf{n}\cdot\delta\mathbf{r} + \overline{\mathbf{M}} \circ \delta\mathbf{F}\right)\Big|_{s_1}^{s_2} \\ = & \int_{s_1}^{s_2} \rho\left(D_1\varepsilon \circ \delta\mathbf{F} + D_2\varepsilon \circ \delta\mathbf{F}'\right) ds. \end{aligned} \tag{4.9}$$

In order to use equation (4.9) to derive the field equations, we need to manipulate both equations (4.7) and (4.8) into other forms.

We wish to absorb the term $\mathbf{n}\cdot\delta\mathbf{r}\,\big|_{s_1}^{s_2}$ into the integral in equation (4.7). To do this, use

$$\mathbf{n}\cdot\delta\mathbf{r}\,\Big|_{s_1}^{s_2} = \int_{s_1}^{s_2} \frac{d(\mathbf{n}\cdot\delta\mathbf{r})}{ds} ds.$$

But

$$\frac{d(\mathbf{n}\cdot\delta\mathbf{r})}{ds} = \frac{d(\mathbf{n}\cdot\delta\mathbf{r})}{ds_R}\frac{ds_R}{ds} = \frac{1}{\lambda}(\mathbf{n}\cdot\delta\mathbf{r})'$$

recalling that $\lambda \equiv \frac{ds}{ds_R}$ and the relationship between $s$ and $s_R$ is assumed to be such that $\lambda \neq 0$. Then

$$\begin{aligned}
\mathbf{n}\cdot\delta\mathbf{r}\,\big|_{s_1}^{s_2} &= \int_{s_1}^{s_2} \frac{1}{\lambda}(\mathbf{n}\cdot\delta\mathbf{r})'\,ds \\
&= \int_{s_1}^{s_2} \left(\frac{1}{\lambda}\mathbf{n}'\cdot\delta\mathbf{r} + \frac{1}{\lambda}\mathbf{n}\cdot(\delta\mathbf{r})'\right) ds \\
&= \int_{s_1}^{s_2} \left(\frac{1}{\lambda}\mathbf{n}'\cdot\delta\mathbf{r} + \frac{1}{\lambda}\mathbf{n}\cdot\delta\mathbf{r}'\right) ds
\end{aligned}$$

where the last step follows from the fact that the operators $\delta$ and $\frac{d}{ds_R}$ commute.

Equation (4.7) becomes

$$\begin{aligned}
\mathcal{A} &= \int_{s_1}^{s_2} \rho\left(\mathbf{f}\cdot\delta\mathbf{r} + \overline{\mathbf{L}}\circ\delta\mathbf{F}\right)\,ds + \left(\mathbf{n}\cdot\delta\mathbf{r} + \overline{\mathbf{M}}\circ\delta\mathbf{F}\right)\big|_{s_1}^{s_2} \\
&= \int_{s_1}^{s_2} \rho\left(\mathbf{f}\cdot\delta\mathbf{r} + \overline{\mathbf{L}}\circ\delta\mathbf{F}\right)\,ds + \int_{s_1}^{s_2} \frac{1}{\lambda}\mathbf{n}'\cdot\delta\mathbf{r}\,ds + \int_{s_1}^{s_2} \frac{1}{\lambda}\mathbf{n}\cdot\delta\mathbf{r}'ds + \overline{\mathbf{M}}\circ\delta\mathbf{F}\,\big|_{s_1}^{s_2} \\
&= \int_{s_1}^{s_2} \left[\left(\rho\,\mathbf{f} + \frac{1}{\lambda}\mathbf{n}'\right)\cdot\delta\mathbf{r} + \frac{1}{\lambda}\mathbf{n}\cdot\delta\mathbf{r}' + \rho\overline{\mathbf{L}}\circ\delta\mathbf{F}\right] ds + \overline{\mathbf{M}}\circ\delta\mathbf{F}\,\big|_{s_1}^{s_2}
\end{aligned}$$

To remove the $\mathbf{n}\cdot\delta\mathbf{r}'$ term in the integral, we use the following theorem.

**Theorem 5** $\mathbf{n}\cdot\delta\mathbf{r}' = (\mathbf{n}\otimes\mathbf{r}'_R)\circ\delta\mathbf{F}$.

**Proof.** The definition of $\delta\mathbf{F}$ is given in equation (4.6) which can also be written as

$$\delta\mathbf{F} = \delta\mathbf{d}_1\otimes\mathbf{d}_R^1 + \delta\mathbf{d}_2\otimes\mathbf{d}_R^2 + \delta\mathbf{r}'\otimes(\mathbf{r}'_R)^*$$

where $(\mathbf{r}'_R)^*$ is the dual vector to $\mathbf{r}'_R$ (i.e. $(\mathbf{r}'_R)^* = \mathbf{d}_R^3$). From the definition given in equation (2.3),

$$\begin{aligned}
(\mathbf{n}\otimes\mathbf{r}'_R)\circ\delta\mathbf{F} &= (\mathbf{n}\otimes\mathbf{r}'_R)\circ(\delta\mathbf{d}_1\otimes\mathbf{d}_R^1 + \delta\mathbf{d}_2\otimes\,\mathbf{d}_R^2 + \delta\mathbf{r}'\otimes(\mathbf{r}'_R)^*) \\
&= (\mathbf{n}\otimes\mathbf{r}'_R)\circ((\delta\mathbf{d}_1\otimes\mathbf{d}_R^1) + (\delta\mathbf{d}_2\otimes\mathbf{d}_R^2) + (\delta\mathbf{r}'\otimes(\mathbf{r}'_R)^*)) \\
&= (\mathbf{n}\otimes\mathbf{r}'_R)\circ(\delta\mathbf{d}_1\otimes\mathbf{d}_R^1) + (\mathbf{n}\otimes\mathbf{r}'_R)\circ(\delta\mathbf{d}_2\otimes\mathbf{d}_R^2)
\end{aligned}$$

$$+ \left(\mathbf{n} \otimes \mathbf{r}'_R\right) \circ \left(\delta\mathbf{r}' \otimes (\mathbf{r}'_R)^*\right).$$

Applying Lemma 6 to each term on the right side of this equation gives:

$$\left(\mathbf{n} \otimes \mathbf{r}'_R\right) \circ \delta\mathbf{F} = \left(\mathbf{n} \cdot \delta\mathbf{d}_1\right)\left(\mathbf{r}'_R \cdot \mathbf{d}^1_R\right) + \left(\mathbf{n} \cdot \delta\mathbf{d}_2\right)\left(\mathbf{r}'_R \cdot \mathbf{d}^2_R\right) + \left(\mathbf{n} \cdot \delta\mathbf{r}'\right)\left(\mathbf{r}'_R \cdot (\mathbf{r}'_R)^*\right).$$

But

$$\mathbf{r}'_R \cdot \mathbf{d}^1_R = 0, \mathbf{r}'_R \cdot \mathbf{d}^2_R = 0 \text{ and } \mathbf{r}'_R \cdot \left(\mathbf{r}'_R\right)^* = 1$$

since $\left\{\mathbf{d}^1_R, \mathbf{d}^2_{R2}, (\mathbf{r}'_R)^*\right\}$ is the reciprocal basis to $\{\mathbf{d}_{R1}, \mathbf{d}_{R2}, \mathbf{r}'_R\}$ and thus

$$\left(\mathbf{n} \otimes \mathbf{r}'_R\right) \circ \delta\mathbf{F} = \mathbf{n} \cdot \delta\mathbf{r}'$$

which is the result to be proved. ■

The formula for the virtual work is now given by

$$\begin{aligned}
\mathcal{A} &= \int_{s_1}^{s_2} \rho\,\left(\mathbf{f} \cdot \delta\mathbf{r} + \overline{\mathbf{L}} \circ \delta\mathbf{F}\right)\, ds + \left(\mathbf{n} \cdot \delta\mathbf{r} + \overline{\mathbf{M}} \circ \delta\mathbf{F}\right)\Big|_{s_1}^{s_2} \\
\mathcal{A} &= \int_{s_1}^{s_2} \left[\left(\rho\mathbf{f} + \frac{1}{\lambda}\mathbf{n}'\right) \cdot \delta\mathbf{r} + \frac{1}{\lambda}\left(\mathbf{n} \otimes \mathbf{r}'_R\right) \circ \delta\mathbf{F} + \rho\overline{\mathbf{L}} \circ \delta\mathbf{F}\right] ds + \overline{\mathbf{M}} \circ \delta\mathbf{F}\,\Big|_{s_1}^{s_2} \\
\mathcal{A} &= \int_{s_1}^{s_2} \left[\left(\rho\mathbf{f} + \frac{1}{\lambda}\mathbf{n}'\right) \cdot \delta\mathbf{r} + \left(\frac{1}{\lambda}\mathbf{n} \otimes \mathbf{r}'_R + \rho\overline{\mathbf{L}}\right) \circ \delta\mathbf{F}\right] ds + \overline{\mathbf{M}} \circ \delta\mathbf{F}\,\Big|_{s_1}^{s_2}.
\end{aligned} \tag{4.10}$$

In equation (4.8), we would like to remove the $(\rho D_2\varepsilon) \circ \delta\mathbf{F}'$ from under the integral on the right side. This can be done with integration by parts after changing the differential $ds$ to $ds_R$.

$$\begin{aligned}
\int_{s_1}^{s_2} \rho D_2\varepsilon \circ \delta\mathbf{F}'\, ds &= \int_{s_1}^{s_2} \rho D_2\varepsilon \circ \delta\mathbf{F}'\, \lambda ds_R \\
&= \lambda\rho D_2\varepsilon \circ \delta\mathbf{F}\,\Big|_{s_1}^{s_2} - \int_{s_1}^{s_2} (\lambda\rho D_2\varepsilon)' \circ \delta\mathbf{F}\, ds_R \\
&= \lambda\rho D_2\varepsilon \circ \delta\mathbf{F}\,\Big|_{s_1}^{s_2} - \int_{s_1}^{s_2} \frac{1}{\lambda}(\lambda\rho D_2\varepsilon)' \circ \delta\mathbf{F}\, ds
\end{aligned}$$

where the switch is made from $ds_R$ back to $ds$ in the last step. So

$$\delta W = \int_{s_1}^{s_2} \rho\left(D_1\varepsilon \circ \delta\mathbf{F} + D_2\varepsilon \circ \delta\mathbf{F}'\right)\, ds$$

$$
\begin{aligned}
\delta W &= \int_{s_1}^{s_2} \rho D_1\varepsilon \circ \delta\mathbf{F}\, ds + \int_{s_1}^{s_2} \rho D_2\varepsilon \circ \delta\mathbf{F}'\, ds \\
\delta W &= \int_{s_1}^{s_2} \rho D_1\varepsilon \circ \delta\mathbf{F}\, ds - \int_{s_1}^{s_2} \frac{1}{\lambda}\left(\lambda\rho D_2\varepsilon\right)' \circ \delta\mathbf{F}\, ds + \lambda\rho D_2\varepsilon \circ \delta\mathbf{F}\,\Big|_{s_1}^{s_2} \\
\delta W &= \int_{s_1}^{s_2} \left[\rho D_1\varepsilon - \frac{1}{\lambda}\left(\lambda\rho D_2\varepsilon\right)'\right] \circ \delta\mathbf{F}\, ds + \lambda\rho D_2\varepsilon \circ \delta\mathbf{F}\,\Big|_{s_1}^{s_2} .
\end{aligned}
\tag{4.11}
$$

Equation (4.9) becomes

$$
\begin{aligned}
&\int_{s_1}^{s_2} \left[\left(\rho\,\mathbf{f} + \frac{1}{\lambda}\mathbf{n}'\right) \cdot \delta\mathbf{r} + \left(\frac{1}{\lambda}\mathbf{n} \otimes \mathbf{r}'_R + \rho\overline{\mathbf{L}}\right) \circ \delta\mathbf{F}\right] ds + \overline{\mathbf{M}} \circ \delta\mathbf{F}\,\Big|_{s_1}^{s_2} \\
= &\int_{s_1}^{s_2} \left[\rho D_1\varepsilon - \frac{1}{\lambda}\left(\lambda\rho D_2\varepsilon\right)'\right] \circ \delta\mathbf{F}\, ds + \lambda\rho D_2\varepsilon \circ \delta\mathbf{F}\,\Big|_{s_1}^{s_2} .
\end{aligned}
\tag{4.12}
$$

which, bringing everything to the left side and combining the integrals, is

$$
\begin{aligned}
\int_{s_1}^{s_2} \left(\rho\mathbf{f} + \frac{1}{\lambda}\mathbf{n}'\right) \cdot \delta\mathbf{r}\, ds \quad &+ \quad \int_{s_1}^{s_2} \left(\frac{1}{\lambda}\left(\lambda\rho D_2\varepsilon\right)' - \rho D_1\varepsilon + \frac{1}{\lambda}\mathbf{n} \otimes \mathbf{r}'_R + \rho\overline{\mathbf{L}}\right) \circ \delta\mathbf{F}\, ds \\
&+ \quad \left(\overline{\mathbf{M}} - \lambda\rho D_2\varepsilon\right) \circ \delta\mathbf{F}\,\Big|_{s_1}^{s_2} = 0
\end{aligned}
$$

Since this integral equation must be true for all possible variations $\delta\mathbf{r}$ and $\delta\mathbf{F}$, it satisfies the conditions of Theorem 4. Invoking this theorem and simplifying, we obtain

$$
\mathbf{n}' + \lambda\rho\mathbf{f} = \mathbf{0}, \tag{4.13}
$$

$$
\left(\lambda\rho D_2\varepsilon\right)' - \lambda\rho D_1\varepsilon + \mathbf{n} \otimes \mathbf{r}'_R + \lambda\rho\overline{\mathbf{L}} = \mathbf{0} \tag{4.14}
$$

and

$$
\left(\overline{\mathbf{M}} - \lambda\rho D_2\varepsilon\right)(s_1) = \left(\overline{\mathbf{M}} - \lambda\rho D_2\varepsilon\right)(s_2) = \mathbf{0}. \tag{4.15}
$$

From equation (4.15),

$$
\overline{\mathbf{M}}\,(s_1) = \left(\lambda\rho D_2\varepsilon\right)(s_1)
$$

and

$$
\overline{\mathbf{M}}\,(s_2) = \left(\lambda\rho D_2\varepsilon\right)(s_2)
$$

so the functions $\overline{\mathbf{M}}$ and $\lambda\rho D_2\varepsilon$ agree at the endpoints of the rod segment. Since $s_1$ and $s_2$ are arbitrary values and equations (4.13), (4.14) and (4.15) would be exactly the same for all choices of $s_1$ and $s_2$, it follows that $\overline{\mathbf{M}}$ and $\lambda\rho D_2\varepsilon$ would agree at all points along the rod and thus

$$\overline{\mathbf{M}} = \lambda\rho D_2\varepsilon. \tag{4.16}$$

Letting

$$\overline{\mathbf{N}} = \lambda\rho D_1\varepsilon \tag{4.17}$$

gives the field equations for the directed rod as equation (4.13) and

$$\overline{\mathbf{M}}' - \overline{\mathbf{N}} + \mathbf{n} \otimes \mathbf{r}'_R + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}. \tag{4.18}$$

The field equations are often called Euler's Laws.

## 4.2 Comparison With Field Equations in the Literature

It is instructive to compare the field equations (4.13) and (4.18) with those found in the literature. Such a comparison provides additional relationships among the various geometrical and physical quantities which are both interesting and useful in applying the field equations to specific problems.

The field equations (4.13) and (4.18) may not appear to be the same as those found in the literature. There are several reasons for this.

1. Different notations for the same geometric or physical quantity may be used.
2. Directed rod theory may be developed as statics (no motion) or dynamics (accounting for motion of the rod). The field equations in a dynamical theory will include terms which are differentiated with respect to time. In the static theory, these terms are zero.
3. Second order tensors in field equations may be expressed in different types of bases. These bases are seen as tensor products of the physical and geometric vectors (always including at least one director). There are three possibilities.

(a) Both vectors in the tensor product are in the reference configuration.

(b) Both vectors in the tensor product are in the deformed configuration.

(c) One vector is taken from the deformed configuration and one vector from the reference configuration.

4. A given field equation may be expressed in terms of its transpose.

5. There is more than one method of handling the parametrization of the reference and the deformed configurations and their relationship.

In this section, fields equations (4.13) and (4.18) will be compared with those given in Cohen and Sun [3] and Cohen [1]. The apparent differences in the field equations will be resolved by observing one or more of the reasons listed above.

Observe that the second order tensors used in this thesis are defined as tensor products of a vector from the reference configuration and a vector from the deformed configuration.

### 4.2.1 The Field Equations of Cohen and Sun [3]

The field equations in [3] were derived for a dynamic theory using balance laws. Here are the field equations of interest (using the equation numbering from that paper prefixed with CS):

$$\mathbf{n}' + \lambda\rho\left(\mathbf{b} - \overset{\bullet}{\mathbf{h}}\right) = \mathbf{0} \tag{CS2.8}$$

and

$$\mathbf{M}' + \mathbf{N} + \lambda\rho\left(\mathbf{B} - \mathbf{P} - \overset{\bullet}{\mathbf{H}} + \mathbf{H}\mathbf{L}^{\mathrm{T}}\right) = \mathbf{0} \tag{CS2.24}$$

where $\bullet$ over a symbol denotes differentiation with respect to time. Another equation of significance is

$$\mathbf{p}^i \otimes \mathbf{d}_i = \mathbf{d}_i \otimes \mathbf{p}^i. \tag{CS2.12}$$

The symbols in CS2.24 are defined by

$$\begin{aligned} \mathbf{M} &\equiv \mathbf{m}^i \otimes \mathbf{d}_i, \mathbf{N} \equiv \mathbf{n} \otimes \mathbf{x}', \mathbf{B} \equiv \mathbf{b}^i \otimes \mathbf{d}_i, \mathbf{P} \equiv \mathbf{p}^i \otimes \mathbf{d}_i, && \text{(CS2.20)} \\ \mathbf{L} &\equiv \overset{\bullet}{\mathbf{d}}_i \otimes \mathbf{d}^i. && \text{(CS2.21)} \end{aligned}$$

| Cohen and Sun [3] | Thesis | Description |
|---|---|---|
| $S$ | $s_R$ | Parameter for curve in reference configuration |
| $s$ | $s$ | Parameter for curve in deformed configuration |
| $\mathbf{X}$ | $\mathbf{r}_R$ | Curve in the reference configuration |
| $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3$ | $\mathbf{d}_{R1}, \mathbf{d}_{R2}, \mathbf{d}_{R3}$ | Directors in the reference configuration |
| $\mathbf{x}$ | $\mathbf{r}$ | Curve in the deformed configuration |
| $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ | $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ | Directors in the deformed configuration |
| $\mathbf{n}$ | $\mathbf{n}$ | Contact force (also called axial force) |
| $\mathbf{b}$ | $\mathbf{f}$ | Body force density |
| $\mathbf{b}^1, \mathbf{b}^2, \mathbf{b}^3$ | $\mathbf{l}^1, \mathbf{l}^2, \mathbf{l}^3$ | Director body force density |
| $\mathbf{p}^1, \mathbf{p}^2, \mathbf{p}^3$ | $\mathbf{p}^1, \mathbf{p}^2, \mathbf{p}^3$ | Intrinsic director force density |
| $\mathbf{m}^1, \mathbf{m}^2, \mathbf{m}^3$ | $\mathbf{m}^1, \mathbf{m}^2, \mathbf{m}^3$ | Director force density |

Table 4.1: Comparison of Scalar and Vector Symbols Between Cohen and Sun and Thesis

Notice the second order tensors are defined as tensor products of vectors both from the deformed configuration. Table 4.1 compares the notation between the scalar and vector quantities in [3] and the corresponding quantities used in this thesis.

To convert equations (CS2.8) and (CS2.24) to field equations for a static theory, set $\overset{\bullet}{\mathbf{h}}= \mathbf{0}$, $\overset{\bullet}{\mathbf{H}}= 0$ and $\mathbf{L} = \mathbf{0}$. Replacing $\mathbf{N}$ with its definition, Equations (CS2.8) and (CS2.24) become

$$\mathbf{n}' + \lambda\rho\mathbf{b} = \mathbf{0} \tag{4.19}$$

and

$$\mathbf{M}' + \mathbf{n} \otimes \mathbf{x}' + \lambda\rho\mathbf{B} - \lambda\rho\mathbf{P} = \mathbf{0} \tag{4.20}$$

respectively. Comparing equations (CS2.12) and (CS2.20) gives

$$sk\mathbf{P} = \mathbf{0}. \tag{4.21}$$

From Table 4.1, it is immediately clear equations (4.13) and (4.19) are identical. Rearranging equation (4.20) slightly gives

$$\mathbf{M}' - \lambda\rho\mathbf{P} + \mathbf{n} \otimes \mathbf{x}' + \lambda\rho\mathbf{B} = \mathbf{0} \tag{4.22}$$

which looks something like equation (4.18). We manipulate equation (4.18) to show these equations are synonymous.

From Corollary 3, $\mathbf{F}\mathbf{d}_{R3} = \mathbf{d}_3$ which is the same as $\mathbf{F}\mathbf{r}'_R = \mathbf{r}'$ since $\mathbf{d}_{R3} = \mathbf{r}'_R$ and $\mathbf{d}_3 = \mathbf{r}'$. Recalling $\mathbf{F}$ is invertible, we may use $\mathbf{r}'_R = \mathbf{F}^{-1}\mathbf{r}'$ in equation (4.18) to get

$$\overline{\mathbf{M}}' - \overline{\mathbf{N}} + \mathbf{n}\otimes\left(\mathbf{F}^{-1}\mathbf{r}'\right) + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}.$$

From Lemma 3, this is the same as

$$\overline{\mathbf{M}}' - \overline{\mathbf{N}} + \mathbf{n}\otimes\mathbf{r}'\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}$$

and from properties of the transpose,

$$\overline{\mathbf{M}}' - \overline{\mathbf{N}} + \mathbf{n}\otimes\mathbf{r}'\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}. \tag{4.23}$$

Multiplying through equation (4.23) on the right by $\mathbf{F}^{\mathrm{T}}$ gives

$$\overline{\mathbf{M}}'\mathbf{F}^{\mathrm{T}} - \overline{\mathbf{N}}\mathbf{F}^{\mathrm{T}} + \mathbf{n}\otimes\mathbf{r}' + \lambda\rho\overline{\mathbf{L}}\mathbf{F}^{\mathrm{T}} = \mathbf{0}. \tag{4.24}$$

From Table 4.1, $\mathbf{r}$ is synonymous with $\mathbf{x}$ so equation (4.24) looks very similar to equation (4.22). The following lemmas establish the relationships between the second order tensors in these two equations.

**Lemma 30** *For the quantities given in equations (4.24) and (4.22),*

$$\mathbf{B} = \overline{\mathbf{L}}\mathbf{F}^{\mathrm{T}} \tag{4.25}$$

*and*

$$\mathbf{M} = \overline{\mathbf{M}}\mathbf{F}^{\mathrm{T}}. \tag{4.26}$$

**Proof.** From the definition given in equation (4.1), $\overline{\mathbf{L}} \equiv \mathbf{l}^i\otimes\mathbf{d}_{Ri}$. From Table 2.1, $\mathbf{F}^{\mathrm{T}} = \mathbf{d}^j_R\otimes\mathbf{d}_j$. Then using Lemma 4

$$\overline{\mathbf{L}}\mathbf{F}^{\mathrm{T}} = \left(\mathbf{l}^i\otimes\mathbf{d}_{Ri}\right)\left(\mathbf{d}^j_R\otimes\mathbf{d}_j\right) = \mathbf{l}^i\otimes\left(\mathbf{d}_{Ri}\cdot\mathbf{d}^j_R\right)\mathbf{d}_j = \mathbf{l}^i\otimes\delta^j_i\mathbf{d}_j = \mathbf{l}^i\otimes\mathbf{d}_i.$$

From equation CS2.20, $\mathbf{B} \equiv \mathbf{b}^i \otimes \mathbf{d}_i$. From Table 4.1, $\mathbf{b}^i$ is synonymous with $\mathbf{l}^i$ so we may write $\mathbf{B} \equiv \mathbf{l}^i \otimes \mathbf{d}_i$. Thus $\mathbf{B} = \overline{\mathbf{L}}\mathbf{F}^{\mathrm{T}}$.

From the definition given in equation (4.1), $\overline{\mathbf{M}} \equiv \mathbf{m}^i \otimes \mathbf{d}_{Ri}$. Then using Lemma 4

$$\overline{\mathbf{M}}\mathbf{F}^{\mathrm{T}} = \left(\mathbf{m}^i \otimes \mathbf{d}_{Ri}\right)\left(\mathbf{d}_R^j \otimes \mathbf{d}_j\right) = \mathbf{m}^i \otimes \left(\mathbf{d}_{Ri} \cdot \mathbf{d}_R^j\right)\mathbf{d}_j = \mathbf{m}^i \otimes \delta_i^j \mathbf{d}_j = \mathbf{m}^i \otimes \mathbf{d}_i.$$

But from equation (CS2.20), $\mathbf{M} \equiv \mathbf{m}^i \otimes \mathbf{d}_i$. Thus $\mathbf{M} = \overline{\mathbf{M}}\mathbf{F}^{\mathrm{T}}$. ■

**Lemma 31** *For the quantities given in equations (4.24) and (4.22),*

$$\lambda\rho\mathbf{P} = \overline{\mathbf{N}}\mathbf{F}^{\mathrm{T}} + \overline{\mathbf{M}}\left(\mathbf{F}^{\mathrm{T}}\right)'. \tag{4.27}$$

**Proof.** Comparing equations (4.24) and (4.22) and using $\mathbf{B} = \overline{\mathbf{L}}\mathbf{F}^{\mathrm{T}}$, it follows that

$$\mathbf{M}' - \lambda\rho\mathbf{P} = \overline{\mathbf{M}}'\mathbf{F}^{\mathrm{T}} - \overline{\mathbf{N}}\mathbf{F}^{\mathrm{T}}. \tag{4.28}$$

Differentiating both sides of equation (4.26) with respect to $s_R$ gives

$$\mathbf{M}' = \left(\overline{\mathbf{M}}\mathbf{F}^{\mathrm{T}}\right)' = \overline{\mathbf{M}}'\mathbf{F}^{\mathrm{T}} + \overline{\mathbf{M}}\left(\mathbf{F}^{\mathrm{T}}\right)'.$$

Then

$$\overline{\mathbf{M}}'\mathbf{F}^{\mathrm{T}} + \overline{\mathbf{M}}\left(\mathbf{F}^{\mathrm{T}}\right)' - \lambda\rho\mathbf{P} = \overline{\mathbf{M}}'\mathbf{F}^{\mathrm{T}} - \overline{\mathbf{N}}\mathbf{F}^{\mathrm{T}}$$

and solving for $\lambda\rho\mathbf{P}$ gives equation (4.27). ■

The effect of the $\mathbf{F}^{\mathrm{T}}$ operating on the right in equations (4.25), (4.26) and (4.27) is to adjust for the difference in expressing the tensors in a mixed basis (as derived in the thesis) or deformed basis. Thus the field equations from [3] really are the same as those derived earlier in this chapter.

As explained in [3], the $\mathbf{p}^i$ are intrinsic forces arising from resistance to deformation of the directors. $\mathbf{P}$ is the total Cauchy stress. Substituting equations (4.16) and (4.17) into (4.27), we obtain

$$\lambda\rho\mathbf{P} = \lambda\rho D_1\varepsilon\mathbf{F}^{\mathrm{T}} + \lambda\rho\left(D_2\varepsilon\right)\left(\mathbf{F}^{\mathrm{T}}\right)'$$

or

$$\mathbf{P} = D_1\varepsilon\mathbf{F}^{\mathrm{T}} + (D_2\varepsilon)\left(\mathbf{F}^{\mathrm{T}}\right)' \tag{4.29}$$

which relates the total Cauchy stress to the derivatives of the strain energy density function with respect to both slots. Of course this means we may solve for the derivatives in terms of each other. We obtain

$$D_1\varepsilon\mathbf{F}^{\mathrm{T}} = \mathbf{P} - (D_2\varepsilon)\left(\mathbf{F}^{\mathrm{T}}\right)'$$

or

$$D_1\varepsilon = \mathbf{P}\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} - (D_2\varepsilon)\left(\mathbf{F}^{\mathrm{T}}\right)'\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} \tag{4.30}$$

and

$$(D_2\varepsilon)\left(\mathbf{F}^{\mathrm{T}}\right)' = \mathbf{P} - D_1\varepsilon\mathbf{F}^{\mathrm{T}}$$

or

$$D_2\varepsilon = \mathbf{P}\left(\left(\mathbf{F}^{\mathrm{T}}\right)'\right)^{-1} - D_1\varepsilon\mathbf{F}^{\mathrm{T}}\left(\left(\mathbf{F}^{\mathrm{T}}\right)'\right)^{-1}. \tag{4.31}$$

We can use these results to express $D_1\varepsilon$ and $D_2\varepsilon$ in terms of tensor products. From equation (4.16) and $\overline{\mathbf{M}} \equiv \mathbf{m}^i \otimes \mathbf{d}_{Ri}$,

$$\lambda\rho D_2\varepsilon = \mathbf{m}^i \otimes \mathbf{d}_{Ri}$$

or

$$D_2\varepsilon = \frac{1}{\lambda\rho}\left(\mathbf{m}^i \otimes \mathbf{d}_{Ri}\right). \tag{4.32}$$

Obtaining an expression for $D_1\varepsilon$ is more involved.

**Lemma 32** *Using the definitions found in equation (2.7),*

$$D_1\varepsilon = \left(\mathbf{p}^i + \frac{1}{\lambda\rho}\left(\overline{\mu}^i_{Rj} - \overline{\mu}^i_j\right)\mathbf{m}^j\right) \otimes \mathbf{d}_{Ri}. \tag{4.33}$$

**Proof.** Since $\mathbf{F}^{\mathrm{T}} = \mathbf{d}^j_R \otimes \mathbf{d}_j$, $\left(\mathbf{F}^{\mathrm{T}}\right)' = \left(\mathbf{d}^j_R\right)' \otimes \mathbf{d}_j + \mathbf{d}^j_R \otimes \mathbf{d}'_j$. Using the expression for $\left(\mathbf{F}^{\mathrm{T}}\right)^{-1}$ from Table 2.1 and Lemma 4, we have

$$\left(\mathbf{F}^{\mathrm{T}}\right)'\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} = \left(\left(\mathbf{d}^j_R\right)' \otimes \mathbf{d}_j + \mathbf{d}^j_R \otimes \mathbf{d}'_j\right)\left(\mathbf{d}^i \otimes \mathbf{d}_{Ri}\right)$$

$$
\begin{aligned}
&= \left(\left(\mathbf{d}_R^j\right)' \otimes \mathbf{d}_j\right)\left(\mathbf{d}^i \otimes \mathbf{d}_{Ri}\right) + \left(\mathbf{d}_R^j \otimes \mathbf{d}_j'\right)\left(\mathbf{d}^i \otimes \mathbf{d}_{Ri}\right) \\
&= \left(\mathbf{d}_R^j\right)' \otimes \left(\mathbf{d}_j \cdot \mathbf{d}^i\right)\mathbf{d}_{Ri} + \mathbf{d}_R^j \otimes \left(\mathbf{d}_j' \cdot \mathbf{d}^i\right)\mathbf{d}_{Ri} \\
&= \left(\mathbf{d}_R^j\right)' \otimes \delta_j^i \mathbf{d}_{Ri} + \mathbf{d}_R^j \otimes \overline{\mu}_j^i \mathbf{d}_{Ri} \\
&= \left(\mathbf{d}_R^i\right)' \otimes \mathbf{d}_{Ri} + \mathbf{d}_R^j \otimes \overline{\mu}_j^i \mathbf{d}_{Ri}.
\end{aligned}
$$

From equation (4.32) and using Lemma 4,

$$
\begin{aligned}
D_2\varepsilon\left(\mathbf{F}^{\mathrm{T}}\right)'\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} &= \frac{1}{\lambda\rho}\left(\mathbf{m}^k \otimes \mathbf{d}_{Rk}\right)\left(\left(\mathbf{d}_R^i\right)' \otimes \mathbf{d}_{Ri} + \mathbf{d}_R^j \otimes \overline{\mu}_j^i \mathbf{d}_{Ri}\right) \\
&= \frac{1}{\lambda\rho}\left[\left(\mathbf{m}^k \otimes \mathbf{d}_{Rk}\right)\left(\left(\mathbf{d}_R^i\right)' \otimes \mathbf{d}_{Ri}\right) + \left(\mathbf{m}^k \otimes \mathbf{d}_{Rk}\right)\left(\mathbf{d}_R^j \otimes \overline{\mu}_j^i \mathbf{d}_{Ri}\right)\right] \\
&= \frac{1}{\lambda\rho}\left[\mathbf{m}^k \otimes \left(\mathbf{d}_{Rk} \cdot \left(\mathbf{d}_R^i\right)'\right)\mathbf{d}_{Ri} + \mathbf{m}^k \otimes \left(\mathbf{d}_{Rk} \cdot \mathbf{d}_R^j\right)\overline{\mu}_j^i \mathbf{d}_{Ri}\right] \\
&= \frac{1}{\lambda\rho}\left[\mathbf{m}^k \otimes \overline{\nu}_{Rk}^i \mathbf{d}_{Ri} + \mathbf{m}^k \otimes \delta_k^j \overline{\mu}_j^i \mathbf{d}_{Ri}\right] \\
&= \frac{1}{\lambda\rho}\left[\mathbf{m}^k \otimes \overline{\nu}_{Rk}^i \mathbf{d}_{Ri} + \mathbf{m}^k \otimes \overline{\mu}_k^i \mathbf{d}_{Ri}\right] \\
&= \frac{1}{\lambda\rho}\mathbf{m}^k \otimes \left(\overline{\nu}_{Rk}^i + \overline{\mu}_k^i\right)\mathbf{d}_{Ri} \\
&= \frac{1}{\lambda\rho}\left(\overline{\nu}_{Rk}^i + \overline{\mu}_k^i\right)\mathbf{m}^k \otimes \mathbf{d}_{Ri}.
\end{aligned}
$$

Using $\mathbf{P} \equiv \mathbf{p}^i \otimes \mathbf{d}_i$ and Lemma 4 ,

$$
\begin{aligned}
\mathbf{P}\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} &= \left(\mathbf{p}^i \otimes \mathbf{d}_i\right)\left(\mathbf{d}^j \otimes \mathbf{d}_{Rj}\right) \\
&= \mathbf{p}^i \otimes \left(\mathbf{d}_i \cdot \mathbf{d}^j\right)\mathbf{d}_{Rj} \\
&= \mathbf{p}^i \otimes \delta_i^j \mathbf{d}_{Rj} \\
&= \mathbf{p}^i \otimes \mathbf{d}_{Ri}.
\end{aligned}
$$

Then from equation (4.30),

$$
\begin{aligned}
D_1\varepsilon &= \mathbf{p}^i \otimes \mathbf{d}_{Ri} - \frac{1}{\lambda\rho}\left(\overline{\nu}_{Rk}^i + \overline{\mu}_k^i\right)\ \mathbf{m}^j \otimes \mathbf{d}_{Ri} \\
&= \left(\mathbf{p}^i - \frac{1}{\lambda\rho}\left(\overline{\nu}_{Rk}^i + \overline{\mu}_k^i\right)\mathbf{m}^j\right) \otimes \mathbf{d}_{Ri}
\end{aligned}
$$

and using equation (2.11),

$$\begin{aligned} D_1\varepsilon &= \left(\mathbf{p}^i - \frac{1}{\lambda\rho}\left(-\overline{\mu}^i_{Rk} + \overline{\mu}^i_k\right)\mathbf{m}^j\right) \otimes \mathbf{d}_{Ri} \\ D_1\varepsilon &= \left(\mathbf{p}^i + \frac{1}{\lambda\rho}\left(\overline{\mu}^i_{Rk} - \overline{\mu}^i_k\right)\mathbf{m}^j\right) \otimes \mathbf{d}_{Ri} \end{aligned}$$

which is the result to be proved. ■

### 4.2.2 The Field Equations of Cohen [1]

Cohen [1] states, but does not derive, field equations for static directed rod theory. He does not distinguish between the parameter used in the reference configuration and in the deformed configuration. The symbol $S$ denotes the parameter and is called a material coordinate. In the deformed state, $S$ is viewed as a "convected" material coordinate. As a result of this approach, the stretch factor $\lambda$ does not appear in any of the field equations given in [1]. However, when specific problems to solve are addressed, $\lambda$ is introduced into the deformed configuration. All symbols used for scalars and vectors except for $S$ have the same meaning in [1] and in this thesis. Note that some of the symbols used here (such as $\mathbf{M}$ and $\mathbf{N}$) do not have the same meaning as in the previous subsection.

In Cohen's field equations, the second order tensors are expressed as tensor products in which both vectors are from the deformed configuration. Here are the field equations from [1] using Cohen's equation numbering prefixed with the letter C.

$$\mathbf{n}' + \rho\mathbf{f} = \mathbf{0}, \tag{C2.13}$$

$$\mathbf{M}' - \mathbf{N} + \mathbf{r}' \otimes \mathbf{n} + \rho\mathbf{L} = \mathbf{0}, \tag{C2.14}$$

and

$$sk\mathbf{N} = \mathbf{0}. \tag{C2.15}$$

Equation (C2.15) means $\mathbf{N}$ is symmetric and thus may also be written as

$$\mathbf{N} = \mathbf{N}^{\mathrm{T}} \tag{C2.15A}$$

which will be more useful later. Cohen identifies $\mathbf{M}$ as the cross-sectional tensor moment, $\mathbf{N}$ as the internal force moment and $\mathbf{n}$ as the axial force. The quantities $\mathbf{f}$ and $\mathbf{L}$ are body force distributions. We make the following replacements to the body force distributions to introduce $\lambda$ into the field equations:

$$\mathbf{f} \rightarrow \lambda\mathbf{f}, \mathbf{L} \rightarrow \lambda\mathbf{L}.$$

Equations (C2.13) and (C2.14) now appear as

$$\mathbf{n}' + \lambda\rho\mathbf{f} = \mathbf{0} \tag{4.34}$$

and

$$\mathbf{M}' - \mathbf{N} + \mathbf{r}' \otimes \mathbf{n} + \lambda\rho\mathbf{L} = \mathbf{0}. \tag{4.35}$$

Comparing these field equations with those derived in this thesis, equations (4.13) and (4.34) are identical. Equations (4.18) and (4.35) appear to be quite similar. If we manipulate equation (4.18), we can verify that the two equations are identical and establish the relationships between the various tensor quantities in both equations.

Field equation (4.18) may be converted to the field equation given by Cohen as follows. From Lemma 15, $\mathbf{F}^{-1}$ exists and from Corollary 4, $\mathbf{r}'_R = \mathbf{F}^{-1}\mathbf{r}'$. Thus equation (4.18) may be written

$$\overline{\mathbf{M}}' - \overline{\mathbf{N}} + \mathbf{n}\otimes\left(\mathbf{F}^{-1}\mathbf{r}'\right) + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}.$$

From Lemma 3, we obtain

$$\overline{\mathbf{M}}' - \overline{\mathbf{N}} + \mathbf{n}\otimes\mathbf{r}'\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}. \tag{4.36}$$

Now take the transpose of both sides of equation (4.36) to get

$$\begin{aligned}
\left(\overline{\mathbf{M}}' - \overline{\mathbf{N}} + \mathbf{n}\otimes\mathbf{r}'\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} + \lambda\rho\overline{\mathbf{L}}\right)^{\mathrm{T}} &= \mathbf{0}^{\mathrm{T}} \\
\left(\overline{\mathbf{M}}'\right)^{\mathrm{T}} - \overline{\mathbf{N}}^{\mathrm{T}} + \left(\mathbf{n}\otimes\mathbf{r}'\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\right)^{\mathrm{T}} + \left(\lambda\rho\overline{\mathbf{L}}\right)^{\mathrm{T}} &= \mathbf{0} \\
\left(\overline{\mathbf{M}}'\right)^{\mathrm{T}} - \overline{\mathbf{N}}^{\mathrm{T}} + \left(\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\right)^{\mathrm{T}}\left(\mathbf{n}\otimes\mathbf{r}'\right)^{\mathrm{T}} + \left(\lambda\rho\overline{\mathbf{L}}\right)^{\mathrm{T}} &= \mathbf{0} \\
\left(\overline{\mathbf{M}}'\right)^{\mathrm{T}} - \overline{\mathbf{N}}^{\mathrm{T}} + \mathbf{F}^{-1}\left(\mathbf{r}'\otimes\mathbf{n}\right) + \left(\lambda\rho\overline{\mathbf{L}}\right)^{\mathrm{T}} &= \mathbf{0}
\end{aligned}$$

$$\left(\overline{\mathbf{M}}^{\mathrm{T}}\right)' - \overline{\mathbf{N}}^{\mathrm{T}} + \mathbf{F}^{-1}\left(\mathbf{r}'\otimes\mathbf{n}\right) + \left(\lambda\rho\overline{\mathbf{L}}\right)^{\mathrm{T}} = \mathbf{0}$$

using properties of the transpose including Lemma 2. Multiplying through the last equation by $\mathbf{F}$ on the left will give

$$\mathbf{F}\left(\overline{\mathbf{M}}^{\mathrm{T}}\right)' - \mathbf{F}\overline{\mathbf{N}}^{\mathrm{T}} + \mathbf{r}'\otimes\mathbf{n} + \lambda\rho\mathbf{F}\overline{\mathbf{L}}^{\mathrm{T}} = \mathbf{0}. \tag{4.37}$$

Comparing equations (4.35) and (4.37) suggests the following relationships:

$$\mathbf{M} = \mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}} \tag{4.38}$$

and

$$\mathbf{L} = \mathbf{F}\overline{\mathbf{L}}^{\mathrm{T}}. \tag{4.39}$$

**Lemma 33** *If equations (4.38) and (4.39) hold, then*

$$\mathbf{N} = \mathbf{F}'\overline{\mathbf{M}}^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}^{\mathrm{T}}. \tag{4.40}$$

**Proof.** Comparing equations (4.35) and (4.37) and using equation (4.39) gives

$$\mathbf{M}' - \mathbf{N} = \mathbf{F}\left(\overline{\mathbf{M}}^{\mathrm{T}}\right)' - \mathbf{F}\overline{\mathbf{N}}^{\mathrm{T}}.$$

Differentiating equation (4.38) gives

$$\mathbf{M}' = \left(\mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}}\right)' = \mathbf{F}'\overline{\mathbf{M}}^{\mathrm{T}} + \mathbf{F}\left(\overline{\mathbf{M}}^{\mathrm{T}}\right)'.$$

Then

$$\mathbf{F}'\overline{\mathbf{M}}^{\mathrm{T}} + \mathbf{F}\left(\overline{\mathbf{M}}^{\mathrm{T}}\right)' - \mathbf{N} = \mathbf{F}\left(\overline{\mathbf{M}}^{\mathrm{T}}\right)' - \mathbf{F}\overline{\mathbf{N}}^{\mathrm{T}}$$

and solving for $\mathbf{N}$ gives equation (4.40). ■

**Lemma 34** *If equations (4.38) and (4.39) hold, then*

$$\mathbf{M} = \mathbf{d}_i \otimes \mathbf{m}^i \tag{4.41}$$

*and*

$$\mathbf{L} = \mathbf{l}_i \otimes \mathbf{m}^i \tag{4.42}$$

**Proof.** Using Lemmas 2 and 4,

$$\begin{aligned}
\mathbf{M} &= \mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}} \\
&= \left(\mathbf{d}_i \otimes \mathbf{d}_R^i\right)\left(\mathbf{m}^j \otimes \mathbf{d}_{Rj}\right)^{\mathrm{T}} \\
&= \left(\mathbf{d}_i \otimes \mathbf{d}_R^i\right)\left(\mathbf{d}_{Rj} \otimes \mathbf{m}^j\right) \\
&= \mathbf{d}_i \otimes \left(\mathbf{d}_R^i \cdot \mathbf{d}_{Rj}\right)\mathbf{m}^j \\
&= \mathbf{d}_i \otimes \delta_j^i \mathbf{m}^j \\
&= \mathbf{d}_i \otimes \mathbf{m}^i
\end{aligned}$$

which establishes equation (4.41). Equation (4.42) is derived in exactly the same manner. ■

Thus the transformations given by equations (4.38), (4.39) and (4.40) illustrate that the field equations in the thesis and those given by Cohen [1] are the same. Multiplication by $\mathbf{F}$ on the left is an adjustment from the use of mixed tensor products to vectors only in the deformed configuration and an adjustment to the transpose of the field equations. Comparing the definitions for $\overline{\mathbf{M}}$ and $\overline{\mathbf{L}}$ with equations (4.41) and (4.42) clearly illustrate this.

Cohen includes equation (C2.15) as a field equation although this is often regarded as being a constitutive restriction. In terms of the tensors $\overline{\mathbf{M}}$ and $\overline{\mathbf{N}}$, equation (C2.15A) would be

$$\mathbf{F}'\overline{\mathbf{M}}^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}^{\mathrm{T}} = \left(\mathbf{F}'\overline{\mathbf{M}}^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}^{\mathrm{T}}\right)^{\mathrm{T}}. \tag{4.43}$$

Finally, we note with Cohen [1] that $\mathbf{f}$ and $\mathbf{L}$ are body force distributions on the rod which may occur as a result of boundary tractions and body forces in the case where the rod is viewed as pseudo-rigid [6]. Cohen postulates that force and torque results of these distributions vanish. That is, $\mathbf{f} = \mathbf{0}$ and

$$sk\,\rho\mathbf{L} = \mathbf{0}. \tag{C2.16}$$

From equation (4.13), this implies $\mathbf{n}' = \mathbf{0}$ giving the interesting result that $\mathbf{n}$ is a constant vector along the rod axis. The condition given by equation (C2.16) means $\mathbf{L}$ is symmetric and is equivalent to $\mathbf{L} = \mathbf{L}^{\mathrm{T}}$. From equation (4.39),

$$\mathbf{F}\overline{\mathbf{L}}^{\mathrm{T}} = \left(\mathbf{F}\overline{\mathbf{L}}^{\mathrm{T}}\right)^{\mathrm{T}}$$

or

$$\mathbf{F}\overline{\mathbf{L}}^{\mathrm{T}} = \overline{\mathbf{L}}\mathbf{F}^{\mathrm{T}}.$$

Thus $\mathbf{F}\overline{\mathbf{L}}^{\mathrm{T}}$ is a symmetric tensor. We summarize this as the following postulate.

**Postulate 6** $\mathbf{f} = \mathbf{0}$ and $\mathbf{F}\overline{\mathbf{L}}^{\mathrm{T}} = \overline{\mathbf{L}}\mathbf{F}^{\mathrm{T}}$.

## 4.3 Summary

In this chapter, the field equations of a directed rod were derived using a variation principle. The strain energy density function $\varepsilon = \varepsilon(\mathbf{F}, \mathbf{F}')$ and its derivatives $D_1\varepsilon$ and $D_2\varepsilon$ were introduced. Equations (4.13), (4.18) and (4.43) (with the definitions (4.16) and (4.17)) are the field equations in which the tensors are expressed as mixed tensor products (one vector in the reference configuration and one vector in the deformed configuration).

The field equations of Cohen and Sun [3] given by (CS2.8), (CS2.24) and (CS2.12) are shown to be the same as those derived in this thesis via the relationships (4.25), (4.26) and (4.27). Using these relationships, explicit expressions relating $D_1\varepsilon$ and $D_2\varepsilon$ are given in equation (4.29). Expressions for $D_1\varepsilon$ and $D_2\varepsilon$ in terms of tensor products of vectors are given in equations (4.33) and (4.32).

The field equations of Cohen [1] given by (C2.13), (C2.14) and (C2.15) were also shown to be the same as those derived in this thesis via the relationships (4.38), (4.39) and (4.40). Equation (C2.15) provides the field equation (4.43) using the tensor quantities defined in this thesis.

# Chapter 5

# Constitutive Restrictions

In Chapter 4, the strain energy density function was introduced as a constitutive equation. This means that the nature of this function depends on the material composing the rod. Thus there are constitutive quantities (the derivatives of the strain energy density) built into the field equations. Notice that the field equations themselves are independent of the functional form of these quantities. In this chapter, we impose constraints on the strain energy density function (and consequently its derivatives) which provides additional information with which to solve problems in directed rod theory. These constraints are called the constitutive restrictions.

The constitutive restrictions are physical postulates - properties which the particular rod is believed to possess and depending on the rod material. Constitutive restrictions in a directed rod theory are typically given by expressing the stress tensors as functions of the deformation and derivatives of the deformation with respect to the parameter describing the rod axis (Cohen [1]). If the stress tensor is a function of the deformation, but none of its derivatives, then it is said to describe a first-grade material. First-grade materials are also known as simple materials (Leigh, [7]). If the stress tensor is a function of the deformation and the first derivative of the deformation, it is said to describe second-grade material. Since the strain energy density function is postulated to be a function of both the deformation tensor and its first derivative, the model we are using describes second-grade materials.

We introduce two physical postulates in this directed rod model. Both of these postulates are based on one fundamental idea: strain energy $\varepsilon$ is invariant under a rotation of the rod. The physical meaning of the two postulates is quite different.

The next section reviews matrix groups as these provide a description of the linear transformations required to rotate a vector in space. The remaining two sections discuss two constitutive restrictions introduced into the model.

## 5.1 Review of the Matrix Groups

Before discussing the constitutive restrictions, it is useful to provide a brief review of matrix groups.

Let $\mathcal{V}$ and $\mathcal{L}(\mathcal{V},\mathcal{V})$ denote an $N$-dimensional real inner product space and the set of endomorphisms on that space. Suppose $\mathbf{A} \in \mathcal{L}(\mathcal{V},\mathcal{V})$. Once a basis has been chosen for $\mathcal{V}$, the linear transformation $\mathbf{A}$ has a matrix representation with respect to that basis which will be denoted as $A$. Let $\mathcal{M}_N(\Re)$ denote the set of all $N$ by $N$ real matrices. Then $A \in \mathcal{M}_N(\Re)$. Matrix groups are special subsets of $\mathcal{M}_N(\Re)$ characterized by the fact each subset is a group under matrix multiplication.

The general linear group is denoted $\mathcal{GL}(N,\Re)$ and is the set of all $N$ by $N$ matrices which have an inverse. All the matrix groups are subsets of the general linear group.

The special linear group is denoted $\mathcal{SL}(N,\Re)$ and defined by

$$\mathcal{SL}(N,\Re) = \{A \in \mathcal{M}_N(\Re) \mid \det(A) = 1\}.$$

The orthogonal group is denoted $\mathcal{O}(N,\Re)$ and defined by

$$\mathcal{O}(N,\Re) = \left\{A \in \mathcal{M}_N(\Re) \mid \langle A\mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{y}\rangle \text{ for all } \mathbf{x},\mathbf{y} \in \Re^N\right\}$$

where $\langle , \rangle$ denotes the inner product on the vector space. The following theorems state the most important properties about the orthogonal group.

**Theorem 6** *$\mathcal{O}(N,\Re)$ is a group under matrix multiplication.*

**Theorem 7** *Let $A \in \mathcal{M}_N(\Re)$. The following conditions are equivalent.*

1. $A \in \mathcal{O}(N,\Re)$.

2. $\langle A\mathbf{e}_i, A\mathbf{e}_j\rangle = \delta_{ij}$ where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\ldots,\mathbf{e}_N\}$ is an orthonormal basis for $\mathcal{V}$.

3. $A$ maps orthonormal bases to orthonormal bases.

4. The rows of $A$ form an orthonormal basis.

5. The columns of $A$ form an orthonormal basis.

6. $A^{\mathrm{T}} = A^{-1}$.

**Theorem 8** *Let* $A \in \mathcal{M}_N(\Re)$. $A \in \mathcal{O}(N, \Re)$ *if and only if* $A$ *preserves lengths of vectors.*

**Theorem 9** *If* $A \in \mathcal{O}(N, \Re)$, *then* $\det(A)^2 = 1$ *and thus* $\det(A) = -1$ *or* $\det(A) = 1$.

The special orthogonal group is denoted by $\mathcal{SO}(N, \Re)$ and defined by

$$\mathcal{SO}(N, \Re) = \mathcal{O}(N, \Re) \cap \mathcal{SL}(N, \Re).$$

The special orthogonal group is the subset of the orthogonal group whose determinant is 1. This is sometimes called the rotation group because it consists of the set of all matrices whose effect is to rotate vectors. Members from this matrix group play an important role in defining the constitutive restrictions.

We may apply the terminology of matrix groups to linear transformations and thus, for example, we may speak of a linear transformation $\mathbf{P}$ as a member of the special orthogonal group.

## 5.2 Material Frame Indifference

To motivate the first constitutive restriction, we consider a curve $\mathbf{r} : \Re \rightarrow \Re^3$ whose parameter is denoted as $s_R$ : we may write $\mathbf{r} = \mathbf{r}(s_R)$. Suppose this curve is rotated about a fixed axis. Such a rotation may be done by a linear transformation $\mathbf{P} \in \mathcal{SO}(3, \Re)$ acting on each point on the curve $\mathbf{r}$. The result will be a new curve denoted as $\widetilde{\mathbf{r}}$ and given by $\widetilde{\mathbf{r}} = \mathbf{Pr}$. Since $\mathbf{P}$ is fixed, the result of differentiating both sides with respect to $s_R$ gives $\widetilde{\mathbf{r}}' = \mathbf{Pr}'$. Thus the tangent vector $\widetilde{\mathbf{r}}'$ along the curve is rotated by the same linear transformation as the curve. If we consider the curve $\mathbf{r}$ to have three directors at each point along the curve, the curve $\widetilde{\mathbf{r}}$ will have corresponding directors rotated by the same linear transformation. Thus $\widetilde{\mathbf{d}_a} = \mathbf{Pd}_a$, $a = 1, 2, 3$.

**Lemma 35** *If $\mathbf{P}$ is a rotation of a directed rod following a deformation $\mathbf{F}$, the result is a (new) deformation given by $\mathbf{PF} = \delta^i_j (\mathbf{P}\mathbf{d}_i) \otimes \mathbf{d}^j_R$.*

**Proof.** Since $\mathbf{F} = \delta^i_j \mathbf{d}_i \otimes \mathbf{d}^j_R$, multiplying by $\mathbf{P}$ gives

$$\mathbf{PF} = \mathbf{P}\left(\delta^i_j \mathbf{d}_i \otimes \mathbf{d}^j_R\right) = \delta^i_j \mathbf{P}\left(\mathbf{d}_i \otimes \mathbf{d}^j_R\right) = \delta^i_j (\mathbf{P}\mathbf{d}_i) \otimes \mathbf{d}^j_R \tag{5.1}$$

using Lemma 3. ■

These results suggest the first constitutive postulate called material frame indifference. The principle of material frame indifference states strain energy density is invariant under rotations of the rod following deformation. This is expressed mathematically as

$$\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon(\mathbf{PF}, (\mathbf{PF})') = \varepsilon\left(\widetilde{\mathbf{F}}, \widetilde{\mathbf{F}}'\right) \tag{5.2}$$

for all deformations $\mathbf{F}$ and fixed rotations $\mathbf{P}$. Since $\mathbf{P}$ is fixed (i.e., not a function of $s_R$), then equation (5.2) becomes $\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon(\mathbf{PF}, \mathbf{PF}')$.

We now consider how to express material frame indifference in terms of $D_1\varepsilon(\mathbf{F}, \mathbf{F}')$ and $D_2\varepsilon(\mathbf{F}, \mathbf{F}')$. We consider the effect of differentiating with respect to the first slot on both sides of equation (5.2). This is given by applying Theorem 2 , namely

$$\begin{aligned} D_1\varepsilon(\mathbf{F}, \mathbf{F}') &= \mathbf{P}^{\mathrm{T}} D_1\varepsilon(\mathbf{PF}, \mathbf{PF}') && (5.3) \\ D_2\varepsilon(\mathbf{F}, \mathbf{F}') &= \mathbf{P}^{\mathrm{T}} D_2\varepsilon(\mathbf{PF}, \mathbf{PF}') && (5.4) \end{aligned}$$

for all $\mathbf{P} \in \mathcal{SO}(3, \Re)$. Thus the rotation $\mathbf{P}$ carries through on the derivatives $D_1\varepsilon$ and $D_2\varepsilon$ and equation (5.2) implies equations (5.3) and (5.4).

Let $\mathfrak{F}$ denote the set of all possible deformations that could be done on a directed rod. If $\mathbf{F} \in \mathfrak{F}$, let $\mathbf{G} \sim \mathbf{F}$ mean $\mathbf{G} = \mathbf{PF}$ where $\mathbf{P}$ is any rigid rotation. Of course $\mathbf{G} \in \mathfrak{F}$.

**Theorem 10** $\sim$ *is a equivalence relation.*

**Proof.** Test the axioms for equivalence relations.

1. Identity. If $\mathbf{P} = \mathbf{I}$ (where $\mathbf{I}$ is the identity tensor satisfying $\mathbf{Iv} = \mathbf{v}$ for all vectors $\mathbf{v}$), then $\mathbf{G} = \mathbf{IF} = \mathbf{F}$. This means $\mathbf{F} \sim \mathbf{F}$.

2. Reflexive. Suppose $\mathbf{G} \sim \mathbf{F}$ so there exists $\mathbf{P}$ with $\mathbf{G} = \mathbf{PF}$. Since $\mathbf{P} \in \mathcal{SO}(N, \Re)$, $\mathbf{P}$ must be invertible and $\mathbf{P}^{-1} \in \mathcal{SO}(N, \Re)$. So $\mathbf{P}^{-1}\mathbf{G} = \mathbf{P}^{-1}\mathbf{PF} = \mathbf{IF} = \mathbf{F}$ or $\mathbf{F} = \mathbf{P}^{-1}\mathbf{G}$. Thus if $\mathbf{G} \sim \mathbf{F}$, then $\mathbf{F} \sim \mathbf{G}$.

3. Transitive. Suppose $\mathbf{G} \sim \mathbf{F}$ so there exists $\mathbf{P}_1$ such that $\mathbf{G} = \mathbf{P}_1\mathbf{F}$. Further, suppose $\mathbf{H} \sim \mathbf{G}$ so there exists $\mathbf{P}_2$ such that $\mathbf{H} = \mathbf{P}_2\mathbf{G}$. Then $\mathbf{H} = \mathbf{P}_2\mathbf{P}_1\mathbf{F}$. But $\mathbf{P} = \mathbf{P}_2\mathbf{P}_1$ is also a rotation since the set of rotations forms a group under composition. Thus $\mathbf{H} = \mathbf{PF}$ which means $\mathbf{H} \sim \mathbf{F}$. So if $\mathbf{G} \sim \mathbf{F}$ and $\mathbf{H} \sim \mathbf{G}$, then $\mathbf{H} \sim \mathbf{F}$ .

Since $\sim$ includes identity and is reflexive and transitive, then $\sim$ is a equivalence relation. ■

**Lemma 36** *If* $\mathbf{G} \sim \mathbf{F}$ *then* $\mathbf{G}' \sim \mathbf{F}'$.

**Proof.** Since $\mathbf{G} \sim \mathbf{F}$ there exists $\mathbf{P}$ with $\mathbf{G} = \mathbf{PF}$. Then $\mathbf{G}' = (\mathbf{PF})' = \mathbf{P}'\mathbf{F} + \mathbf{PF}' = \mathbf{PF}'$ (the rotation $\mathbf{P}$ does not depend on the parameter $s_R$). This means $\mathbf{G}' \sim \mathbf{F}'$. ■

**Corollary 6** *If* $\mathbf{G} \sim \mathbf{F}$, *then* $\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon(\mathbf{G}, \mathbf{G}')$.

**Proof.** If $\mathbf{G} \sim \mathbf{F}$ then $\mathbf{G}' \sim \mathbf{F}'$ and there exists rotation $\mathbf{P}$ such that $\mathbf{G} = \mathbf{PF}$ and

$$\mathbf{G}' = \mathbf{PF}' = (\mathbf{PF})'.$$

From the material frame indifference postulate (equation (5.2)),

$$\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon(\mathbf{PF}, (\mathbf{PF})') = \varepsilon(\mathbf{G}, \mathbf{G}')$$

which is the desired result. ■

The property $\sim$ partitions the set $\mathfrak{F}$ into mutually disjoint subsets characterized by the fact $\varepsilon$ is invariant for any two members and their derivatives from the same subset. Any representative member of a subset may be used in evaluating $\varepsilon$. In particular, if the deformation $\mathbf{F}$ is a rotation, then $\mathbf{F} = \mathbf{FI}$ so $\mathbf{F} \sim \mathbf{I}$ and $\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon(\mathbf{I}, \mathbf{I}')$ corresponding to the strain energy for no deformation. Of course this is sensible because a rigid rotation of the rod really doesn't deform the rod: it only changes its orientation in space.

This has further implications for the dependence of strain energy on deformation and its first derivative. We invoke the polar decomposition theorem which states any second order tensor $\mathbf{F}$ with nonzero determinant may be written $\mathbf{F} = \mathbf{PU}$ where $\mathbf{P}$ is an orthogonal tensor and $\mathbf{U}$ is a right stretch tensor (the proof of the polar decomposition theorem is given in Appendix A). Of course $\mathbf{F}$ is really a tensor field, defined along each point of the curve forming the rod axis so the orthogonal tensor $\mathbf{P}$ is now a function of $s_R$ . If the polar decomposition theorem is applied at each point along the curve, the result is

$$\mathbf{F}(s_R) = \mathbf{P}(s_R)\,\mathbf{U}(s_R)$$

and

$$\mathbf{F}'(s_R) = [\mathbf{P}(s_R)\,\mathbf{U}(s_R)]' = \mathbf{P}'(s_R)\,\mathbf{U}(s_R) + \mathbf{P}(s_R)\,\mathbf{U}'(s_R).$$

We'll now assume $\mathbf{P}$ is a member of the special orthogonal group so

$$\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon(\mathbf{PU}, (\mathbf{PU})') = \varepsilon(\mathbf{PU}, \mathbf{P}'\mathbf{U} + \mathbf{PU}'). \tag{5.5}$$

Since $\mathbf{P}$ is a rotation, $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \mathbf{I}$ and differentiating with respect to $s_R$ gives

$$\left(\mathbf{P}^{\mathrm{T}}\right)'\mathbf{P} + \mathbf{P}^{\mathrm{T}}\mathbf{P}' = \left(\mathbf{P}'\right)^{\mathrm{T}}\mathbf{P} + \mathbf{P}^{\mathrm{T}}\mathbf{P}' = \mathbf{0}.$$

Solving for $\mathbf{P}'$ leaves

$$\mathbf{P}' = -\mathbf{P}\left(\mathbf{P}'\right)^{\mathrm{T}}\mathbf{P}.$$

Substituting this into equation (5.5) yields

$$\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon\left(\mathbf{PU}, -\mathbf{P}\left(\mathbf{P}'\right)^{\mathrm{T}}\mathbf{PU} + \mathbf{PU}'\right) = \varepsilon\left(\mathbf{PU}, \mathbf{P}\left(\mathbf{U}' - \left(\mathbf{P}'\right)^{\mathrm{T}}\mathbf{PU}\right)\right).$$

If we consider a particular point on the rod axis, $\varepsilon$ is considered to be invariant under the rotation $\mathbf{P}$ so we obtain

$$\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon\left(\mathbf{U}, \mathbf{U}' - \left(\mathbf{P}'\right)^{\mathrm{T}}\mathbf{PU}\right). \tag{5.6}$$

Notice if the deformation $\mathbf{F}$ happens to be such that $\mathbf{P}(s_R)$ is constant, then equation (5.6) reduces to $\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon(\mathbf{U}, \mathbf{U}')$ agreeing with Corollary 6. We use this kind of analysis later in the thesis to assist in solving deformations of normal uniform rods.

In the case of a simple material, it can be shown material frame indifference implies the strain energy depends only the stretch portion of the deformation tensor (Leigh, [7]). For second-grade materials, equation (5.6) indicates that the dependence of $\varepsilon$ on the deformation tensor and its derivative is not so simple.

## 5.3 Material Symmetry

Material symmetry is the second constitutive restriction to be considered. This constitutive restriction attempts to describe symmetry inherent in the rod material as opposed to its embedding in space. Material symmetry is not a well developed concept for directed rods: there is no general theory to guide the application of this principle (Cohen[1]). This thesis uses a particular form of material symmetry called monotropic symmetry. It should be noted there are other ways to formulate material symmetry as a constitutive restriction.

We could attempt to develop material symmetry along the same lines as material frame indifference. In this case, we would consider the effect of rotating the curve prior to deforming it. As in the previous section, it seems reasonable to suppose the strain energy doesn't depend on the rotation $\mathbf{Q}$ (whether the rotation is done before or after deformation is immaterial - it is no effect on the strain energy). This might suggest we follow the pattern of equation (5.2) and postulate

$$\varepsilon(\mathbf{FQ}, (\mathbf{FQ})') = \varepsilon(\mathbf{FQ}, \mathbf{FQ}') = \varepsilon(\mathbf{F}, \mathbf{F}') \tag{5.7}$$

for all $\mathbf{Q} \in \mathcal{SO}(3, \Re)$, and work out expressions for the derivatives analogous to those of equations (5.3) and (5.4).

However, the constitutive restriction in this case is not as straightforward as this. There is a significant difference *physically* between rotating after deformation and rotating prior to deformation. Rotating after deformation implies the independence of the strain energy from the coordinate system in which the rod is embedded - material frame indifference. But rotating prior to deformation has another implication - a symmetry in the material itself.

It should be noted that material symmetry in *derived* theories is a well defined and understood constitutive restriction. A common way of implementing material symmetry in a derived theory is to view the material as isotropic so there is symmetry in all possible directions. The rotation operator is any member of the special orthogonal group. In this case, equation (5.7) would hold with $\mathbf{Q} \in \mathcal{SO}(3, \Re)$.

In this thesis, we follow Cohen[1] and develops the principle of material symmetry for a directed rod by postulating the need for both symmetry and anti-symmetry transformations. Such transformations are brought about by members of the special linear group. This implies the required transformations for the desired symmetry and anti-symmetry effects must come from one of a pair of subsets of this group. Cohen developed material symmetry in the context of tensor pairs [1]. Unfortunately, this is not appropriate for the variational method employed in this thesis. However, careful investigation of Cohen's approach provides the necessary conditions to duplicate the constitutive restrictions for material symmetry in the variational approach.

Appendix B contains a general development of group theory for pairs of members of the general linear group. Appendix C reviews Cohen's tensor pairs and his development of material symmetry applying the results of Appendix B.

Cohen required the set of tensor pairs for the symmetry transformations to form a group under composition. The union of both sets of tensor pairs (symmetry and anti-symmetry) was also required to be a group. These tensor pairs are built on subsets of the special linear group. The key result from Appendix B is a description of the conditions which must apply to the pairs of subsets required for material symmetry. .

Let $\mathcal{U}$ and $\mathcal{V}$ be these subsets of the special linear group corresponding to the symmetry and anti-symmetry transformations respectively. Then $\mathcal{U}$ and $\mathcal{V}$ must satisfy one of the following conditions in order for the tensor pairs to form groups as described above.

1. $\mathcal{U}$ is a subgroup of the special linear group and $\mathcal{U} = \mathcal{V}$.

2. $\mathcal{U}$ is a subgroup of the special linear group, $\mathcal{U} \cap \mathcal{V} = \emptyset$ and $\mathcal{U} \cup \mathcal{V}$ is a subgroup of the special linear group.

A directed rod is said to be materially symmetric if equation (5.7) holds for all $\mathbf{Q} \in \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ must satisfy one of the conditions listed above.

This definition of materially symmetric implies there may be many types of material symmetry determined by the choice of the sets $\mathcal{U}$ and $\mathcal{V}$. From the principle of material symmetry, Cohen chose a specific case called monotropic symmetry. A rod point is said to be materially monotropic if the set $\mathcal{U}$ contains the transformation effecting a rotation of $\pi$ radians about a fixed axis $\mathbf{e}$. This transformation may be denoted $\mathbf{Q}_e$.

Since $\mathcal{U}$ is a subgroup of the special linear group, $\mathbf{Q}_e \in \mathcal{U}$ implies the inverse of $\mathbf{Q}_e$ is also in $\mathcal{U}$. In fact, $\mathbf{Q}_e$ is its own inverse, since two successive applications of $\mathbf{Q}_e$ to any vector must return the same vector. The minimal such group $\mathcal{U}$ which may be used to impose monotropic symmetry is given by $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_e\}$. The proof of this is straightforward and may be found in Appendix C. Using $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_e\}$ provides monotropic symmetry about one axis. More generally, we can choose $\mathcal{U}$ to impose monotropic symmetry about as many axes as desired provided the group property is retained.

The symmetric form of monotropic symmetry does indeed imply the invariance of the strain energy density after a rotation of $\pi$ radians about the axis. $\mathbf{Q}_e\mathbf{x}$ is a reflection of the point $\mathbf{x}$ though the axis of rotation. This can be written as

$$\mathbf{Q}_e = -\mathbf{I}_e^{\perp} + \mathbf{e} \otimes \mathbf{e} \tag{5.8}$$

where $\mathbf{I}_e^{\perp}$ is the unit tensor in the plane orthogonal to $\mathbf{e}$. In anti-symmetric form, the point is first rotated by $\pi$ radians as in the symmetric form, i.e. $\mathbf{Q}_e\mathbf{x}$, and then reflected through the plane perpendicular to $\mathbf{Q}_e\mathbf{x}$ to give the effect of $-\mathbf{Q}_e\mathbf{x}$. A transformation such as this is known as a roto-inversion. A roto-inversion may be written as

$$-\mathbf{Q}_e = \mathbf{I}_e^{\perp} - \mathbf{e} \otimes \mathbf{e}. \tag{5.9}$$

We still need to specify how to implement monotropic symmetry using the strain energy density function. As already mentioned, we wish to include both symmetric and anti-symmetric properties. Thus there are actually two formulas to define the constitutive restriction of monotropic symmetry - a symmetric form and an anti-symmetric form. Let $\mathbf{Q}_e \in \mathcal{U} \cup \mathcal{V}$.

The symmetric form of monotropic symmetry ($\mathbf{Q}_e \in \mathcal{U}$) is defined by

$$\varepsilon\left(\mathbf{FQ}_e, (\mathbf{FQ}_e)'\right) = \varepsilon\left(\mathbf{F}, \mathbf{F}'\right) \tag{5.10}$$

while the anti-symmetric form of monotropic symmetry ($\mathbf{Q}_e \in \mathcal{V}$) is defined by

$$\varepsilon\left(\mathbf{FQ}_e, -(\mathbf{FQ}_e)'\right) = \varepsilon\left(\mathbf{F}, \mathbf{F}'\right). \tag{5.11}$$

A particular member of $\mathcal{U} \cup \mathcal{V}$ must satisfy one or the other forms, but generally cannot satisfy both (there are special cases in which a member $\mathbf{Q}_e$ does satisfy both forms). Notice that if $\mathcal{U}$ is to be a group, it must contain the identity transformation $\mathbf{I}$. The identity does not cause a rotation, but it does satisfy

$$\varepsilon\left(\mathbf{FI}, (\mathbf{FI})'\right) = \varepsilon\left(\mathbf{F}, \mathbf{F}'\right) \tag{5.12}$$

which is a special case of the definition given by equation (5.10).

Using Theorem 3, the constitutive restrictions for material symmetry can be written in terms of the derivatives of the strain energy density function. Since there are two slots to differentiate and two forms of material symmetry (symmetric and anti-symmetric), there will be four derivative expressions. These are given as

$$D_1\varepsilon\left(\mathbf{FQ}_e, (\mathbf{FQ}_e)'\right)\mathbf{Q}_e^{\mathrm{T}} = D_1\varepsilon\left(\mathbf{F}, \mathbf{F}'\right)$$

and

$$D_2\varepsilon\left(\mathbf{FQ}_e, (\mathbf{FQ}_e)'\right)\mathbf{Q}_e^{\mathrm{T}} = D_2\varepsilon\left(\mathbf{F}, \mathbf{F}'\right)$$

for the symmetric form and

$$D_1\varepsilon\left(\mathbf{FQ}_e, -(\mathbf{FQ}_e)'\right)\mathbf{Q}_e^{\mathrm{T}} = D_1\varepsilon\left(\mathbf{F}, \mathbf{F}'\right)$$

and

$$-D_2\varepsilon\left(\mathbf{FQ}_e, -(\mathbf{FQ}_e)'\right)\mathbf{Q}_e^{\mathrm{T}} = D_2\varepsilon\left(\mathbf{F}, \mathbf{F}'\right)$$

for the anti-symmetric form.

## 5.4 Summary of the Constitutive Restrictions

To summarize the results obtained thus far, we now write the constitutive restrictions as postulates of the theory.

**Postulate 7** Material Frame Indifference. The postulate of material frame indifference states

$$\varepsilon(\mathbf{F}, \mathbf{F}') = \varepsilon(\mathbf{PF}, (\mathbf{PF})') = \varepsilon(\mathbf{PF}, \ \mathbf{PF}') \tag{5.13}$$

for all $\mathbf{P} \in \mathcal{SO}(3, \Re)$. This implies

$$D_1\varepsilon(\mathbf{F}, \mathbf{F}') = \mathbf{P}^{\mathrm{T}} D_1\varepsilon(\mathbf{PF}, \mathbf{PF}') \tag{5.14}$$

and

$$D_2\varepsilon(\mathbf{F}, \mathbf{F}') = \mathbf{P}^{\mathrm{T}} D_2\varepsilon(\mathbf{PF}, \mathbf{PF}') \tag{5.15}$$

or (equivalently)

$$D_1\varepsilon(\mathbf{PF}, \mathbf{PF}') = \mathbf{P} D_1\varepsilon(\mathbf{F}, \mathbf{F}') \tag{5.16}$$

and

$$D_2\varepsilon(\mathbf{PF}, \mathbf{PF}') = \mathbf{P} D_2\varepsilon(\mathbf{F}, \mathbf{F}') \tag{5.17}$$

since $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$.

This postulate is stating that the strain energy function is invariant if the directed rod is rotated following a deformation. Alternatively, it may also be interpreted to mean that rotating the coordinate system in which the rod is embedded does not effect the strain energy.

Equations (5.14) and (5.15) may also be expressed as

$$\overline{\mathbf{N}}(\mathbf{F}, \mathbf{F}') = \mathbf{P}^{\mathrm{T}} \overline{\mathbf{N}}(\mathbf{PF}, \mathbf{PF}') \tag{5.18}$$

and

$$\overline{\mathbf{M}}(\mathbf{F}, \mathbf{F}') = \mathbf{P}^{\mathrm{T}} \overline{\mathbf{M}}(\mathbf{PF}, \mathbf{PF}') \tag{5.19}$$

by multiplying through equations (5.14) and (5.15) by $\lambda\rho$ and using the definitions given by equations (4.16) and (4.17).

An expression for the deformation $\mathbf{F}$ followed by the rotation $\mathbf{P}$ is given by $\delta^i_j\,(\mathbf{P}\mathbf{d}_i)\otimes\mathbf{d}^j_R$.

The postulate of material symmetry considers the effects on strain energy of a rotation prior to deformation.

**Postulate 8** Material Symmetry. Suppose $\mathbf{Q}_e \in \mathcal{U}\cup\mathcal{V}$. Then

$$\varepsilon\left(\mathbf{F},\mathbf{F}'\right) = \varepsilon\left(\mathbf{F}\mathbf{Q}_e,\mathbf{F}'\mathbf{Q}_e\right) \tag{5.20}$$

for the symmetric case ($\mathbf{Q}_e \in \mathcal{U}$) and

$$\varepsilon\left(\mathbf{F},\mathbf{F}'\right) = \varepsilon\left(\mathbf{F}\mathbf{Q}_e,-\mathbf{F}'\mathbf{Q}_e\right) \tag{5.21}$$

for the anti-symmetric case ($\mathbf{Q}_e \in \mathcal{V}$). In the symmetric case, this implies

$$D_1\varepsilon\left(\mathbf{F},\mathbf{F}'\right) = D_1\varepsilon\left(\mathbf{F}\mathbf{Q}_e,\mathbf{F}'\mathbf{Q}_e\right)\mathbf{Q}_e^{\mathrm{T}} \tag{5.22}$$

and

$$D_2\varepsilon\left(\mathbf{F},\mathbf{F}'\right) = D_2\varepsilon\left(\mathbf{F}\mathbf{Q}_e,\mathbf{F}'\mathbf{Q}_e\right)\mathbf{Q}_e^{\mathrm{T}} \tag{5.23}$$

or (equivalently)

$$D_1\varepsilon\left(\mathbf{F}\mathbf{Q}_e,\mathbf{F}'\mathbf{Q}_e\right) = D_1\varepsilon\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_e^{-\mathrm{T}} \tag{5.24}$$

and

$$D_2\varepsilon\left(\mathbf{F}\mathbf{Q}_e,\mathbf{F}'\mathbf{Q}_e\right) = D_2\varepsilon\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_e^{-\mathrm{T}}. \tag{5.25}$$

For the anti-symmetric case, equation (5.21) implies

$$D_1\varepsilon\left(\mathbf{F},\mathbf{F}'\right) = D_1\varepsilon\left(\mathbf{F}\mathbf{Q}_e,-\mathbf{F}'\mathbf{Q}_e\right)\mathbf{Q}_e^{\mathrm{T}} \tag{5.26}$$

and

$$D_2\varepsilon\left(\mathbf{F},\mathbf{F}'\right) = -D_2\varepsilon\left(\mathbf{F}\mathbf{Q}_e,-\mathbf{F}'\mathbf{Q}_e\right)\mathbf{Q}_e^{\mathrm{T}} \tag{5.27}$$

or (equivalently)

$$D_1\varepsilon\left(\mathbf{F}\mathbf{Q}_e,-\mathbf{F}'\mathbf{Q}_e\right)=D_1\varepsilon\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_e^{-\mathrm{T}} \tag{5.28}$$

and

$$D_2\varepsilon\left(\mathbf{F}\mathbf{Q}_e,-\mathbf{F}'\mathbf{Q}_e\right)=-D_2\varepsilon\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_e^{-\mathrm{T}}. \tag{5.29}$$

These may also be expressed as

$$\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{N}}\left(\mathbf{F}\mathbf{Q}_e,\mathbf{F}'\mathbf{Q}_e\right)\mathbf{Q}_e^{\mathrm{T}} \tag{5.30}$$

and

$$\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{M}}\left(\mathbf{F}\mathbf{Q}_e,\mathbf{F}'\mathbf{Q}_e\right)\mathbf{Q}_e^{\mathrm{T}} \tag{5.31}$$

for the symmetric case and

$$\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{N}}\left(\mathbf{F}\mathbf{Q}_e,-\mathbf{F}'\mathbf{Q}_e\right)\mathbf{Q}_e^{\mathrm{T}} \tag{5.32}$$

and

$$\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=-\overline{\mathbf{M}}\left(\mathbf{F}\mathbf{Q}_e,-\mathbf{F}'\mathbf{Q}_e\right)\mathbf{Q}_e^{\mathrm{T}} \tag{5.33}$$

for the anti-symmetric case by multiplying through equations (5.26), (5.27), (5.28) and (5.29) by $\lambda\rho$ and using the definitions given by equations (4.16) and (4.17).

The expressions given in equations (5.18), (5.19), (5.30), (5.31), (5.32) and (5.33) are more useful in the next section and when solving specific deformations in subsequent chapters.

## 5.5 Comparison With Cohen's Constitutive Restrictions [1]

In Chapter 4, we showed the field equations derived in this thesis were equivalent to those given by Cohen [1]. In this section, we show the constitutive restrictions postulated above are equivalent to those given in Cohen's paper. We recall the relationships between the quantities in Cohen [1] and those of this thesis. These are equations (4.38), (4.39 ) and (4.40). For convenience, they are reproduced here (we don't need equation (4.39)).

$$\mathbf{M} = \widetilde{\mathbf{M}}(\mathbf{F}, \mathbf{F}') = \mathbf{F}\overline{\mathbf{M}}(\mathbf{F}, \mathbf{F}')^{\mathrm{T}} \tag{5.34}$$

and

$$\mathbf{N} = \widetilde{\mathbf{N}}(\mathbf{F}, \mathbf{F}') = \mathbf{F}'\overline{\mathbf{M}}(\mathbf{F}, \mathbf{F}')^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}(\mathbf{F}, \mathbf{F}')^{\mathrm{T}} \tag{5.35}$$

where

$$\overline{\mathbf{M}}(\mathbf{F}, \mathbf{F}') = \lambda\rho D_2\varepsilon(\mathbf{F}, \mathbf{F}') \text{ and } \overline{\mathbf{N}}(\mathbf{F}, \mathbf{F}') = \lambda\rho D_1\varepsilon(\mathbf{F}, \mathbf{F}').$$

### 5.5.1 Material Frame Indifference

We consider material frame indifference in terms of the stress functions $\mathbf{N}$ and $\mathbf{M}$ used in Cohen's paper[1]. These quantities are assumed to be functions the deformation tensor and its first derivative:

$$\mathbf{N} = \widetilde{\mathbf{N}}(\mathbf{F}, \mathbf{F}') \text{ and } \mathbf{M} = \widetilde{\mathbf{M}}(\mathbf{F}, \mathbf{F}').$$

Cohen expressed material frame indifference as

$$\widetilde{\mathbf{M}}(\mathbf{PF}, \mathbf{PF}') = \mathbf{P}\widetilde{\mathbf{M}}(\mathbf{F}, \mathbf{F}')\mathbf{P}^{\mathrm{T}} \tag{5.36}$$

and

$$\widetilde{\mathbf{N}}(\mathbf{PF}, \mathbf{PF}') = \mathbf{P}\widetilde{\mathbf{N}}(\mathbf{F}, \mathbf{F}')\mathbf{P}^{\mathrm{T}}. \tag{5.37}$$

for all deformations $\mathbf{F}$ and fixed rotations $\mathbf{P}$.

Let $\mathbf{P} \in \mathcal{SO}(3, \Re)$. Taking the transpose of equation (5.19) gives

$$\overline{\mathbf{M}}(\mathbf{F}, \mathbf{F}')^{\mathrm{T}} = \overline{\mathbf{M}}(\mathbf{PF}, \mathbf{PF}')^{\mathrm{T}}\mathbf{P}$$

and multiplying through by $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$ on the right gives

$$\overline{\mathbf{M}}(\mathbf{F}, \mathbf{F}')^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} = \overline{\mathbf{M}}(\mathbf{PF}, \mathbf{PF}')^{\mathrm{T}}. \tag{5.38}$$

Multiplying through equation (5.38) on the left by $\mathbf{PF}$ yields

$$\mathbf{PF}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} = \mathbf{PF}\overline{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}}.$$

Apply equation (5.34) to the term on the left side of this equation gives

$$\mathbf{P}\widetilde{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{P}^{\mathrm{T}} = \mathbf{PF}\overline{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}}. \tag{5.39}$$

Replacing $\mathbf{F}$ with $\mathbf{PF}$ in equation (5.34) gives

$$\widetilde{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right) = \mathbf{PF}\overline{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}} \tag{5.40}$$

and putting this into equation (5.39) gives

$$\mathbf{P}\widetilde{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{P}^{\mathrm{T}} = \widetilde{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right).$$

This is equation (5.36), one of the constitutive restrictions postulated in Cohen's paper.

Taking the transpose of equation (5.18) gives

$$\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}} = \overline{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}}\mathbf{P}$$

and multiplying through this equation by $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$ on the right and $\mathbf{F}$ on the left results in

$$\mathbf{F}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} = \mathbf{F}\overline{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}}.$$

Add $\mathbf{F}'\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}$ to the left and $\mathbf{F}'\overline{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}}$ to the right sides of this equation. From equation (5.38), these two expressions are equal so the result is

$$\mathbf{F}'\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} = \mathbf{F}'\overline{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}}.$$

Now multiply through this equation by $\mathbf{P}$ on the left to get

$$\mathbf{PF}'\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} + \mathbf{PF}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} = \mathbf{PF}'\overline{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}} + \mathbf{PF}\overline{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}}. \tag{5.41}$$

Replacing $\mathbf{F}$ with $\mathbf{PF}$ in equation (5.35) gives

$$\widetilde{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)=\mathbf{PF}'\overline{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}}+\mathbf{PF}\overline{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)^{\mathrm{T}} \tag{5.42}$$

and substituting this into equation (5.41) gives

$$\widetilde{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)=\mathbf{PF}'\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}+\mathbf{PF}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}.$$

Multiplying through this equation by $\mathbf{P}^{\mathrm{T}}$ on the left and $\mathbf{P}$ on the right gives

$$\mathbf{P}^{\mathrm{T}}\widetilde{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)\mathbf{P}=\mathbf{F}'\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}+\mathbf{F}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}$$

and using equation (5.35) to replace the right side of the last equation yields

$$\mathbf{P}^{\mathrm{T}}\widetilde{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)\mathbf{P}=\widetilde{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right).$$

Multiplying equation (5.35) on the left by $\mathbf{P}$ and on the right by $\mathbf{P}^{\mathrm{T}}$ gives

$$\widetilde{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)=\mathbf{P}\widetilde{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{P}^{\mathrm{T}}$$

which is equation (5.37), the other constitutive restriction for material frame indifference postulated by Cohen.

Thus starting with the constitutive restrictions postulated in this thesis for material frame indifference and using the relationships between Cohen's quantities $\widetilde{\mathbf{N}}(\mathbf{F},\mathbf{F}')$ and $\widetilde{\mathbf{M}}(\mathbf{F},\mathbf{F}')$ and the derivatives $\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')$ and $\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')$, we can derive Cohen's postulated constitutive restrictions. Thus the constitutive restrictions for material frame indifference postulated by this thesis and in Cohen's paper are identical.

### 5.5.2 Material Symmetry - The Symmetric Case

Cohen expressed the symmetric case of material symmetry as

$$\widetilde{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=\widetilde{\mathbf{M}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right) \tag{5.43}$$

and

$$\widetilde{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\widetilde{\mathbf{N}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right) \tag{5.44}$$

for all deformations $\mathbf{F}$ and $\mathbf{Q}\in\mathcal{U}$ to be used as a symmetric transformation.

Suppose $\mathbf{Q}\in\mathcal{U}$. Taking the transpose of equation (5.31) gives

$$\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}=\mathbf{Q}\overline{\mathbf{M}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}} \tag{5.45}$$

and multiplying both sides by $\mathbf{F}$ on the left yields

$$\mathbf{F}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}=\mathbf{FQ}\overline{\mathbf{M}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}}. \tag{5.46}$$

Replacing $\mathbf{F}$ with $\mathbf{FQ}$ in equation (5.34) gives

$$\widetilde{\mathbf{M}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right)= \mathbf{FQ}\overline{\mathbf{M}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}} \tag{5.47}$$

and placing this into equation (5.46) results in

$$\mathbf{F}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}=\widetilde{\mathbf{M}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right).$$

Using equation (5.34) again to replace the term on the left side of this equation results in

$$\widetilde{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=\widetilde{\mathbf{M}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right).$$

But this is equation (5.43), one of Cohen's constitutive restrictions for the symmetric form of material symmetry.

Taking the transpose of equation (5.30) results in

$$\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}=\mathbf{Q}\overline{\mathbf{N}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}}$$

and multiplying by $\mathbf{F}$ on the left gives

$$\mathbf{F}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}=\mathbf{FQ}\overline{\mathbf{N}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}}.$$

Add $\mathbf{F}'\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}}$ to the left side and $\mathbf{F}'\mathbf{Q}\overline{\mathbf{M}}(\mathbf{FQ},\mathbf{F}'\mathbf{Q})^{\mathrm{T}}$ to the right side of this equation. From equation (5.45), these two quantities are equal so the result is

$$\mathbf{F}'\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}}+\mathbf{F}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}}=\mathbf{F}'\mathbf{Q}\overline{\mathbf{M}}(\mathbf{FQ},\mathbf{F}'\mathbf{Q})^{\mathrm{T}}+\mathbf{FQ}\overline{\mathbf{N}}(\mathbf{FQ},\mathbf{F}'\mathbf{Q})^{\mathrm{T}}. \tag{5.48}$$

Replacing $\mathbf{F}$ with $\mathbf{FQ}$ and $\mathbf{F}'$ with $\mathbf{F}'\mathbf{Q}$ in equation (5.35) gives

$$\tilde{\mathbf{N}}(\mathbf{FQ},\mathbf{F}'\mathbf{Q})=\mathbf{F}'\mathbf{Q}\overline{\mathbf{M}}(\mathbf{FQ},\mathbf{F}'\mathbf{Q})^{\mathrm{T}}+\mathbf{FQ}\overline{\mathbf{N}}(\mathbf{FQ},\mathbf{F}'\mathbf{Q})^{\mathrm{T}}$$

and using this result in the right side of equation (5.48) gives

$$\mathbf{F}'\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}}+\mathbf{F}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}}=\tilde{\mathbf{N}}(\mathbf{FQ},\mathbf{F}'\mathbf{Q}).$$

Applying equation (5.35) to the left side of the last equation results in

$$\tilde{\mathbf{N}}(\mathbf{F},\mathbf{F}')=\tilde{\mathbf{N}}(\mathbf{FQ},\mathbf{F}'\mathbf{Q}).$$

This is equation (5.44), the other constitutive restriction postulated by Cohen for the symmetric form of material symmetry.

Thus we are able to derive Cohen's postulated constitutive restrictions for the symmetric form of material symmetry from the corresponding postulated constitutive restrictions for this thesis.

### 5.5.3 Material Symmetry - The Anti-symmetric Case

Cohen expressed the symmetric case of material symmetry as

$$\widetilde{\mathbf{M}}(\mathbf{F},\mathbf{F}')=-\widetilde{\mathbf{M}}(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}) \tag{5.49}$$

and

$$\tilde{\mathbf{N}}(\mathbf{F},\mathbf{F}')=\tilde{\mathbf{N}}(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}) \tag{5.50}$$

for all deformations $\mathbf{F}$ and $\mathbf{Q}\in\mathcal{V}$ to be used as an anti-symmetric transformation.

Taking the transpose of equation (5.33) gives

$$\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}} = -\mathbf{Q}\overline{\mathbf{M}}\left(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}} \tag{5.51}$$

and multiplying through this equation by $\mathbf{F}$ on the left yields

$$\mathbf{F}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}} = -\mathbf{FQ}\overline{\mathbf{M}}\left(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}}. \tag{5.52}$$

Replacing $\mathbf{F}$ with $\mathbf{FQ}$ and $\mathbf{F}'$ with $-\mathbf{F}'\mathbf{Q}$ in equation (5.34) gives

$$\widetilde{\mathbf{M}}\left(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}\right) = \mathbf{FQ}\overline{\mathbf{M}}\left(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}}$$

and replacing the right side of equation (5.52) with this expression results in

$$\mathbf{F}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}} = -\widetilde{\mathbf{M}}\left(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}\right).$$

Applying equation (5.34) to the left side of this last equation yields

$$\widetilde{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right) = -\widetilde{\mathbf{M}}\left(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}\right)$$

which is equation (5.49), one of Cohen's constitutive restrictions for the anti-symmetric form of material symmetry.

Taking the transpose of equation (5.32) gives

$$\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}} = \mathbf{Q}\overline{\mathbf{N}}\left(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}}$$

and multiplying both sides by $\mathbf{F}$ leaves

$$\mathbf{F}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}} = \mathbf{FQ}\overline{\mathbf{N}}\left(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}}.$$

Add $\mathbf{F}'\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)^{\mathrm{T}}$ to the left and $-\mathbf{F}'\mathbf{Q}\overline{\mathbf{M}}\left(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}\right)^{\mathrm{T}}$ to the right side of this equation. From equation (5.51), these two quantities are equal so the result is

$$\mathbf{F}'\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}} = -\mathbf{F}'\mathbf{Q}\overline{\mathbf{M}}(\mathbf{FQ},-\mathbf{F}'\mathbf{Q})^{\mathrm{T}} + \mathbf{FQ}\overline{\mathbf{N}}(\mathbf{FQ},-\mathbf{F}'\mathbf{Q})^{\mathrm{T}}. \tag{5.53}$$

Replacing $\mathbf{F}$ with $\mathbf{FQ}$ and $\mathbf{F}'$ with $-\mathbf{F}'\mathbf{Q}$ in equation (5.35) gives

$$\tilde{\mathbf{N}}(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}) = -\mathbf{F}'\mathbf{Q}\overline{\mathbf{M}}(\mathbf{FQ}^{\bullet},-\mathbf{F}'\mathbf{Q})^{\mathrm{T}} + \mathbf{FQ}\overline{\mathbf{N}}(\mathbf{FQ},-\mathbf{F}'\mathbf{Q})^{\mathrm{T}}$$

and substituting this into the right side of equation (5.53) gives

$$\mathbf{F}'\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}} = \tilde{\mathbf{N}}(\mathbf{FQ},-\mathbf{F}'\mathbf{Q}).$$

Applying equation (5.35) to the left side of this last equation results in

$$\tilde{\mathbf{N}}(\mathbf{F},\mathbf{F}') = \tilde{\mathbf{N}}(\mathbf{FQ},-\mathbf{F}'\mathbf{Q})$$

which is equation (5.50), the other constitutive restriction postulated by Cohen for the anti-symmetric form of material symmetry.

Thus we are able to derive Cohen's postulated constitutive restrictions for the anti-symmetric form of material symmetry from the corresponding postulated constitutive restrictions for this thesis.

## 5.6 Using the Constitutive Restrictions

We can combine the constitutive restrictions given by Postulates 7 and 8 into two rules, one for the symmetric case and one for the anti-symmetric case. We illustrate using $\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')$ and $\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')$.

Observe that the left side of equation (5.18) is the same as the left side of equation (5.30) and the left side of equation (5.19) is the same as the left side of equation (5.31). Thus we may equate the right sides to obtain

$$\mathbf{P}^{\mathrm{T}}\overline{\mathbf{N}}(\mathbf{PF},\mathbf{PF}') = \overline{\mathbf{N}}(\mathbf{FQ},\mathbf{F}'\mathbf{Q})\,\mathbf{Q}^{\mathrm{T}}$$

and

$$\mathbf{P}^{\mathrm{T}}\overline{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right)=\overline{\mathbf{M}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right)\mathbf{Q}^{\mathrm{T}}.$$

Now if it is possible to find $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{PF}=\mathbf{FQ}$, we may define

$$\overline{\mathbf{N}}^{*}\equiv\overline{\mathbf{N}}\left(\mathbf{PF},\mathbf{PF}'\right)=\overline{\mathbf{N}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right)$$

and

$$\overline{\mathbf{M}}^{*}\equiv\overline{\mathbf{M}}\left(\mathbf{PF},\mathbf{PF}'\right)=\overline{\mathbf{M}}\left(\mathbf{FQ},\mathbf{F}'\mathbf{Q}\right)$$

from which $\mathbf{PF}=\mathbf{FQ}$ would imply $\mathbf{P}^{\mathrm{T}}\overline{\mathbf{N}}^{*}=\overline{\mathbf{N}}^{*}\mathbf{Q}^{\mathrm{T}}$ and $\mathbf{P}^{\mathrm{T}}\overline{\mathbf{M}}^{*}=\overline{\mathbf{M}}^{*}\mathbf{Q}^{\mathrm{T}}$. Note we are assuming $\mathbf{P}$ and $\mathbf{Q}$ are constants so $\mathbf{PF}=\mathbf{FQ}$ automatically means $\mathbf{PF}'=\mathbf{F}'\mathbf{Q}$. However, in this case we can use

$$\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\mathbf{P}^{\mathrm{T}}\overline{\mathbf{N}}^{*},$$

$$\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{N}}^{*}\mathbf{Q}^{\mathrm{T}},$$

$$\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=\mathbf{P}^{\mathrm{T}}\overline{\mathbf{M}}^{*}$$

and

$$\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{M}}^{*}\mathbf{Q}^{\mathrm{T}}$$

and arrange these as

$$\overline{\mathbf{N}}^{*}=\mathbf{P}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right),$$

$$\overline{\mathbf{N}}^{*}=\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}^{-\mathrm{T}},$$

$$\overline{\mathbf{M}}^{*}=\mathbf{P}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)$$

and

$$\overline{\mathbf{M}}^{*}=\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}^{-\mathrm{T}}$$

from which we get

$$\mathbf{P}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}^{-\mathrm{T}} \tag{5.54}$$

and

$$\mathbf{P}\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}') = \overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')\,\mathbf{Q}^{-\mathrm{T}}. \tag{5.55}$$

Equations (5.54) and (5.55) will prove to be the more convenient forms to use. They only apply if

$$\mathbf{PF} = \mathbf{FQ}. \tag{5.56}$$

By a similar argument comparing equations (5.18) and (5.32) and equations (5.19) and (5.33), we find

$$\mathbf{P}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}') = \overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')\,\mathbf{Q}^{-\mathrm{T}} \tag{5.57}$$

and

$$\mathbf{P}\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}') = -\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')\,\mathbf{Q}^{-\mathrm{T}} \tag{5.58}$$

for the anti-symmetric case, provided $\mathbf{PF} = \mathbf{FQ}$ and $\mathbf{PF}' = -\mathbf{F}'\mathbf{Q}$. Since $\mathbf{P}$ and $\mathbf{Q}$ are constants, differentiating both sides of $\mathbf{PF} = \mathbf{FQ}$ with respect to $s_R$ gives $\mathbf{PF}' = \mathbf{F}'\mathbf{Q}$. Then $\mathbf{F}'\mathbf{Q} = -\mathbf{F}'\mathbf{Q}$ or $\mathbf{F}'\mathbf{Q} = \mathbf{0}$. Since $\mathbf{Q}$ must be a member of the special linear group, it has determinant one. Thus this case cannot occur unless the determinant of $\mathbf{F}'$ is zero.

Consider the monotropic case in which we choose $\mathbf{Q}$ to be $\mathbf{Q_e}$ where $\mathbf{e}$ is some fixed vector and $\mathbf{Q_e}$ is the rotation of $\pi$ radians about the axis defined by $\mathbf{e}$. Then $\mathbf{Q_e}$ is also a member of the special orthogonal group. We may calculate $\mathbf{P}$ as $\mathbf{P} = \mathbf{FQ_eF}^{-1}$ (recall $\mathbf{F}$ is always invertible) so equation (5.56) is satisfied. If this $\mathbf{P}$ is orthogonal, equations (5.54), (5.55), (5.57) and (5.58) may be replaced with

$$\mathbf{FQ_eF}^{-1}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}') = \overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')\,\mathbf{Q_e}^{-\mathrm{T}},$$

$$\mathbf{FQ_eF}^{-1}\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}') = \overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')\,\mathbf{Q_e}^{-\mathrm{T}},$$

$$\mathbf{FQ_eF}^{-1}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}') = \overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')\,\mathbf{Q_e}^{-\mathrm{T}}$$

and

$$\mathbf{FQ_eF}^{-1}\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}') = -\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')\,\mathbf{Q_e}^{-\mathrm{T}}$$

Since $\mathbf{Q_e}$ is an orthogonal tensor, $\mathbf{Q_e^T} = \mathbf{Q_e}^{-1}$ and thus $\mathbf{Q_e}^{-\mathrm{T}} = \mathbf{Q_e}$ so in fact these equations may be written as

$$\mathbf{F}\mathbf{Q}_{\mathbf{e}}\mathbf{F}^{-1}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_{\mathbf{e}}$$

and

$$\mathbf{F}\mathbf{Q}_{\mathbf{e}}\mathbf{F}^{-1}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_{\mathbf{e}}$$

for the symmetric case and

$$\mathbf{F}\mathbf{Q}_{\mathbf{e}}\mathbf{F}^{-1}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_{\mathbf{e}}$$

and

$$\mathbf{F}\mathbf{Q}_{\mathbf{e}}\mathbf{F}^{-1}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=-\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_{\mathbf{e}}$$

for the anti-symmetric case.

We can generalize this a bit more. If $\mathbf{R}$ is another member of the special orthogonal group (i.e. another rotation) which rotates vectors about the axis defined by the vector $\overline{\mathbf{e}}$ and we define

$$\overline{\mathbf{Q}_{\overline{\mathbf{e}}}}=\mathbf{R}\mathbf{Q}_{\mathbf{e}}\mathbf{R}^{\mathrm{T}},$$

and we may choose $\mathbf{P}$ to be $\overline{\mathbf{Q}_{\overline{\mathbf{e}}}}$, then these four equations may be written as

$$\overline{\mathbf{Q}_{\overline{\mathbf{e}}}}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_{\mathbf{e}} \tag{5.59}$$

and

$$\overline{\mathbf{Q}_{\overline{\mathbf{e}}}}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_{\mathbf{e}} \tag{5.60}$$

for the symmetric case

$$\overline{\mathbf{Q}_{\overline{\mathbf{e}}}}\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)=\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_{\mathbf{e}} \tag{5.61}$$

and

$$\overline{\mathbf{Q}_{\overline{\mathbf{e}}}}\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)=-\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right)\mathbf{Q}_{\mathbf{e}} \tag{5.62}$$

for the anti-symmetric case. Equations (5.59) through (5.62) are valid as an expression of the combined symmetry (using monotropic symmetry) only if

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{F} = \mathbf{F}\mathbf{Q}_{\mathbf{e}}$$

and

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{F}' = \mathbf{F}'\mathbf{Q}_{\mathbf{e}}$$

in the symmetric case and

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{F}' = -\mathbf{F}'\mathbf{Q}_{\mathbf{e}}$$

in the anti-symmetric case. This anti-symmetric case still requires the determinant of $\mathbf{F}'$ to be zero for the reasons discussed earlier. Notice that equation (5.61) is actually redundant - it contains exactly the same information as equation (5.59).

## 5.7 Summary

The field equations of Chapter 4 apply to all hyperelastic rods. However, additional information is required to completely specify the deformation of a directed rod. This information must reflect properties of the material of the rod.

In this thesis, we are concerned with two kinds of constitutive restrictions: Material frame indifference which implies the deformation is independent of the orientation of the rod in space and material symmetry implying a symmetry within the material itself. Postulate 7 gives the constitutive restriction describing material frame indifference. Material symmetry has both a symmetric and anti-symmetric part. Postulate 8 provides the constitutive restriction describing material symmetry.

The constitutive restrictions postulated in this thesis are equivalent to those postulated in Cohen [1].

The constitutive restrictions may be combined into fewer expressions. Furthermore, there are special cases of these combined expressions which are particularly simple to use.

# Chapter 6

# Further Rod Geometry

In this chapter, we look at special deformations which simplify the rod geometry to allow solutions of the differential equations. The solutions are derived in the following chapter.

## 6.1 Normal Uniform Rod Configurations

In this section, we consider the special geometry of normal uniform rods. This geometry leads to deformations which can be solved exactly for the stress tensors.

The geometry to be considered in this chapter is based on Ericksen's results for what he called uniform states [4]. Ericksen discusses special rod configurations in which differential equations of deformation may be replaced by algebraic relationships and shows there are only three shapes of the rod axis for which this possible: straight, circular and helical. The distribution of the directors along the rod axis may have a constant twist rate. Thus in the straight case, the directed rod may have constant directors or the directors may be twisted along the axis.

In this chapter we consider a subset of Ericksen's uniform states. In subsequent chapters, we actually solve the deformation of a straight rod to any of the other four possible configurations.

### 6.1.1 Mathematical Preliminaries

To develop the ideas of uniform rods requires introduction of some special tensors and their properties. These tensors are closely related to the wryness tensor introduced in Chapter 2.

We continue to use $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as the natural basis for Euclidean three dimensional space. In the following definitions, all indices are elements of the set $\{1, 2, 3\}$. Define the following tensors:

$$\mathbf{E}_{ij} \equiv \mathbf{e}_i \otimes \mathbf{e}_j, \tag{6.1}$$

$$\mathbf{I}_k^{\perp} \equiv \mathbf{E}_{ii} + \mathbf{E}_{jj}, i \neq j, i \neq k, j \neq k \tag{6.2}$$

and

$$\mathbf{A}_k^{\perp} \equiv \begin{cases} \mathbf{E}_{23} - \mathbf{E}_{32} \text{ if } k = 1, \\ \mathbf{E}_{13} - \mathbf{E}_{31} \text{ if } k = 2, \\ \mathbf{E}_{21} - \mathbf{E}_{12} \text{ if } k = 3. \end{cases} \tag{6.3}$$

We state and prove several results with respect to these tensors.

**Lemma 37** $\mathbf{E}_{ij}\mathbf{e}_k = \delta_{jk}\mathbf{e}_i$.

**Proof.** $\mathbf{E}_{ij}\mathbf{e}_k = \mathbf{e}_i \otimes \mathbf{e}_j\mathbf{e}_k = (\mathbf{e}_j \cdot \mathbf{e}_k)\,\mathbf{e}_i = \delta_{jk}\mathbf{e}_i$. ■

**Lemma 38** $\mathbf{I}_k^{\perp}\mathbf{e}_l = \delta_{il}\mathbf{e}_i + \delta_{jl}\mathbf{e}_j$ *where* $i \neq j$, $i \neq k$ *and* $j \neq k$.

**Proof.**

$$\begin{aligned} \mathbf{I}_k^{\perp}\mathbf{e}_l &= (\mathbf{E}_{ii} + \mathbf{E}_{jj})\,\mathbf{e}_l \\ &= \mathbf{E}_{ii}\mathbf{e}_l + \mathbf{E}_{jj}\mathbf{e}_l \\ &= \mathbf{e}_i \otimes \mathbf{e}_i\mathbf{e}_l + \mathbf{e}_j \otimes \mathbf{e}_j\mathbf{e}_l \\ &= (\mathbf{e}_i \cdot \mathbf{e}_l)\,\mathbf{e}_i + (\mathbf{e}_j \cdot \mathbf{e}_l)\,\mathbf{e}_j \\ &= \delta_{il}\mathbf{e}_i + \delta_{jl}\mathbf{e}_j. \blacksquare \end{aligned}$$

**Corollary 7** $\mathbf{I}_k^{\perp}\mathbf{e}_k = \mathbf{0}$.

**Proof.** $\mathbf{I}_k^{\perp}\mathbf{e}_k = \delta_{ik}\mathbf{e}_i + \delta_{jk}\mathbf{e}_j = \mathbf{0}$ since $i \neq k$ and $j \neq k$. ■

**Lemma 39**

$$\mathbf{A}_k^{\perp}\mathbf{e}_l = \begin{cases} \delta_{3l}\mathbf{e}_2 - \delta_{2l}\mathbf{e}_3 \text{ } if\, k = 1, \\ \delta_{3l}\mathbf{e}_1 - \delta_{1l}\mathbf{e}_3 \text{ } if\, k = 2, \\ \delta_{1l}\mathbf{e}_2 - \delta_{2l}\mathbf{e}_1 \text{ } if\, k = 3. \end{cases}$$

**Proof.**

$$\begin{aligned} \mathbf{A}_1^{\perp}\mathbf{e}_l &= (\mathbf{E}_{23} - \mathbf{E}_{32})\,\mathbf{e}_l = \mathbf{E}_{23}\mathbf{e}_l - \mathbf{E}_{32}\mathbf{e}_l = \delta_{3l}\mathbf{e}_2 - \delta_{2l}\mathbf{e}_3, \\ \mathbf{A}_2^{\perp}\mathbf{e}_l &= (\mathbf{E}_{13} - \mathbf{E}_{31})\,\mathbf{e}_l = \mathbf{E}_{13}\mathbf{e}_l - \mathbf{E}_{31}\mathbf{e}_l = \delta_{3l}\mathbf{e}_1 - \delta_{1l}\mathbf{e}_3, \\ \mathbf{A}_3^{\perp}\mathbf{e}_l &= (\mathbf{E}_{21} - \mathbf{E}_{12})\,\mathbf{e}_l = \mathbf{E}_{21}\mathbf{e}_l - \mathbf{E}_{12}\mathbf{e}_l = \delta_{1l}\mathbf{e}_2 - \delta_{2l}\mathbf{e}_1. \blacksquare \end{aligned}$$

**Corollary 8** $\mathbf{A}_k^{\perp}\mathbf{e}_k = \mathbf{0}$.

**Proof.**

$$\begin{aligned} \mathbf{A}_1^{\perp}\mathbf{e}_1 &= \delta_{31}\mathbf{e}_2 - \delta_{21}\mathbf{e}_3 = \mathbf{0}, \\ \mathbf{A}_2^{\perp}\mathbf{e}_2 &= \delta_{32}\mathbf{e}_1 - \delta_{12}\mathbf{e}_3 = \mathbf{0}, \\ \mathbf{A}_3^{\perp}\mathbf{e}_l &= \delta_{13}\mathbf{e}_2 - \delta_{23}\mathbf{e}_1 = \mathbf{0}. \blacksquare \end{aligned}$$

**Lemma 40** $(\mathbf{E}_{ij})^{\mathrm{T}} = \mathbf{E}_{ji}$. *In particular,* $(\mathbf{E}_{ii})^{\mathrm{T}} = \mathbf{E}_{ii}$ *so* $\mathbf{E}_{ii}$ *is symmetric.*

**Proof.** Since $\mathbf{E}_{ij} \equiv \mathbf{e}_i \otimes \mathbf{e}_j$, then $(\mathbf{E}_{ij})^{\mathrm{T}} = \mathbf{e}_j \otimes \mathbf{e}_i = \mathbf{E}_{ji}$. $\blacksquare$

**Lemma 41** $(\mathbf{I}_k^{\perp})^{\mathrm{T}} = \mathbf{I}_k^{\perp}$, *that is,* $\mathbf{I}_k^{\perp}$ *is symmetric.*

**Proof.**

$$\left(\mathbf{I}_k^{\perp}\right)^{\mathrm{T}} = (\mathbf{E}_{ii} + \mathbf{E}_{jj})^{\mathrm{T}} = (\mathbf{E}_{ii})^{\mathrm{T}} + (\mathbf{E}_{jj})^{\mathrm{T}} = \mathbf{E}_{ii} + \mathbf{E}_{jj} = \mathbf{I}_k^{\perp}$$

using properties of transpose and Lemma 40. $\blacksquare$

**Lemma 42** $(\mathbf{A}_k^{\perp})^{\mathrm{T}} = -\mathbf{A}_k^{\perp}$, *that is,* $\mathbf{A}_k^{\perp}$ *is skew-symmetric.*

**Proof.**

$$\left(\mathbf{A}_1^{\perp}\right)^{\mathrm{T}} = (\mathbf{E}_{23} - \mathbf{E}_{32})^{\mathrm{T}} = (\mathbf{E}_{23})^{\mathrm{T}} - (\mathbf{E}_{32})^{\mathrm{T}} = \mathbf{E}_{32} - \mathbf{E}_{23} = -\mathbf{A}_1^{\perp},$$

$$\left(\mathbf{A}_2^{\perp}\right)^{\mathrm{T}} = (\mathbf{E}_{13} - \mathbf{E}_{31})^{\mathrm{T}} = (\mathbf{E}_{13})^{\mathrm{T}} - (\mathbf{E}_{31})^{\mathrm{T}} = \mathbf{E}_{31} - \mathbf{E}_{13} = -\mathbf{A}_2^{\perp}$$

and

$$\left(\mathbf{A}_3^{\perp}\right)^{\mathrm{T}} = (\mathbf{E}_{21} - \mathbf{E}_{12})^{\mathrm{T}} = (\mathbf{E}_{21})^{\mathrm{T}} - (\mathbf{E}_{12})^{\mathrm{T}} = \mathbf{E}_{12} - \mathbf{E}_{21} = -\mathbf{A}_3^{\perp}$$

using properties of transpose and Lemma 40. ■

**Lemma 43** $\mathbf{E}_{ij}\mathbf{E}_{kl} = \delta_{jk}\mathbf{E}_{il}$.

**Proof.**

$$\mathbf{E}_{ij}\mathbf{E}_{kl} = (\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_k \otimes \mathbf{e}_l) = \mathbf{e}_i \otimes (\mathbf{e}_j \cdot \mathbf{e}_k)\,\mathbf{e}_l = \delta_{jk}\mathbf{e}_i \otimes \mathbf{e}_l = \delta_{jk}\mathbf{E}_{il}. \blacksquare$$

It is helpful to have a product table of $\mathbf{E}_{ii}$, $\mathbf{I}_k^{\perp}$ and $\mathbf{A}_k^{\perp}$. Using Lemma 43 and the definitions of these quantities, the following table can be created. The calculations of these products are omitted.

| | $\mathbf{E}_{11}$ | $\mathbf{E}_{22}$ | $\mathbf{E}_{33}$ | $\mathbf{I}_1^{\perp}$ | $\mathbf{I}_2^{\perp}$ | $\mathbf{I}_3^{\perp}$ | $\mathbf{A}_1^{\perp}$ | $\mathbf{A}_2^{\perp}$ | $\mathbf{A}_3^{\perp}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{E}_{11}$ | $\mathbf{E}_{11}$ | $\mathbf{0}$ | $\mathbf{0}$ | $\mathbf{0}$ | $\mathbf{E}_{11}$ | $\mathbf{E}_{11}$ | $\mathbf{0}$ | $\mathbf{E}_{13}$ | $-\mathbf{E}_{12}$ |
| $\mathbf{E}_{22}$ | $\mathbf{0}$ | $\mathbf{E}_{22}$ | $\mathbf{0}$ | $\mathbf{E}_{22}$ | $\mathbf{0}$ | $\mathbf{E}_{22}$ | $\mathbf{E}_{23}$ | $\mathbf{0}$ | $\mathbf{E}_{21}$ |
| $\mathbf{E}_{33}$ | $\mathbf{0}$ | $\mathbf{0}$ | $\mathbf{E}_{33}$ | $\mathbf{E}_{33}$ | $\mathbf{E}_{33}$ | $\mathbf{0}$ | $-\mathbf{E}_{32}$ | $-\mathbf{E}_{31}$ | $\mathbf{0}$ |
| $\mathbf{I}_1^{\perp}$ | $\mathbf{0}$ | $\mathbf{E}_{22}$ | $\mathbf{E}_{33}$ | $\mathbf{I}_1^{\perp}$ | $\mathbf{E}_{33}$ | $\mathbf{E}_{22}$ | $\mathbf{A}_1^{\perp}$ | $-\mathbf{E}_{31}$ | $\mathbf{E}_{21}$ |
| $\mathbf{I}_2^{\perp}$ | $\mathbf{E}_{11}$ | $\mathbf{0}$ | $\mathbf{E}_{33}$ | $\mathbf{E}_{33}$ | $\mathbf{I}_2^{\perp}$ | $\mathbf{E}_{11}$ | $-\mathbf{E}_{32}$ | $\mathbf{A}_2^{\perp}$ | $-\mathbf{E}_{12}$ |
| $\mathbf{I}_3^{\perp}$ | $\mathbf{E}_{11}$ | $\mathbf{E}_{22}$ | $\mathbf{0}$ | $\mathbf{E}_{22}$ | $\mathbf{E}_{11}$ | $\mathbf{I}_3^{\perp}$ | $\mathbf{E}_{23}$ | $\mathbf{E}_{13}$ | $\mathbf{A}_3^{\perp}$ |
| $\mathbf{A}_1^{\perp}$ | $\mathbf{0}$ | $-\mathbf{E}_{32}$ | $\mathbf{E}_{23}$ | $\mathbf{A}_1^{\perp}$ | $\mathbf{E}_{23}$ | $-\mathbf{E}_{32}$ | $-\mathbf{I}_1^{\perp}$ | $-\mathbf{E}_{21}$ | $-\mathbf{E}_{31}$ |
| $\mathbf{A}_2^{\perp}$ | $-\mathbf{E}_{31}$ | $\mathbf{0}$ | $\mathbf{E}_{13}$ | $\mathbf{E}_{13}$ | $\mathbf{A}_2^{\perp}$ | $-\mathbf{E}_{31}$ | $-\mathbf{E}_{12}$ | $-\mathbf{I}_2^{\perp}$ | $\mathbf{E}_{32}$ |
| $\mathbf{A}_3^{\perp}$ | $\mathbf{E}_{21}$ | $-\mathbf{E}_{12}$ | $\mathbf{0}$ | $-\mathbf{E}_{12}$ | $\mathbf{E}_{21}$ | $\mathbf{A}_3^{\perp}$ | $-\mathbf{E}_{13}$ | $\mathbf{E}_{23}$ | $-\mathbf{I}_3^{\perp}$ |

Table 6.1

The set $\{\mathbf{E}_{ij}, i, j = 1, 2, 3\}$ serves as a basis for second order tensors. If $\mathbf{T}$ is a second order tensor with components $T^{ij}$ with all possible values $i, j \in \{1, 2, 3\}$, then we may write $\mathbf{T} = T^{ij}\mathbf{E}_{ij}$.

**Lemma 44** *If* $\mathbf{T}$ *is any second order tensor, then* $\mathbf{T}^{\mathrm{T}}\mathbf{A}_3^{\perp}\mathbf{T}$ *is skew-symmetric.*

**Proof.** Let $\mathbf{T} = T^{ij}\mathbf{E}_{ij}$. Then $\mathbf{E}_{ij}\mathbf{E}_{kl} = \delta_{jk}\mathbf{E}_{il}$

$$\begin{aligned}
\mathbf{T}^{\mathrm{T}}\mathbf{A}_3^{\perp}\mathbf{T} &= T^{ji}\mathbf{E}_{ij}\left(\mathbf{E}_{21} - \mathbf{E}_{12}\right)T^{kl}\mathbf{E}_{kl} \\
&= T^{ji}\mathbf{E}_{ij}\mathbf{E}_{21}T^{kl}\mathbf{E}_{kl} - T^{ji}\mathbf{E}_{ij}\mathbf{E}_{12}T^{kl}\mathbf{E}_{kl} \\
&= T^{ji}T^{kl}\delta_{j2}\mathbf{E}_{i1}\mathbf{E}_{kl} - T^{ji}T^{kl}\delta_{j1}\mathbf{E}_{i2}\mathbf{E}_{kl} \\
&= T^{ji}T^{kl}\delta_{j2}\delta_{1k}\mathbf{E}_{il} - T^{ji}T^{kl}\delta_{j1}\delta_{2k}\mathbf{E}_{il} \\
&= \left(T^{2i}T^{1l} - T^{1i}T^{2l}\right)\mathbf{E}_{il}
\end{aligned}$$

using Lemma 43 several times. Since $T^{2i}T^{1l} - T^{1i}T^{2l} = -\left(T^{2l}T^{1i} - T^{1l}T^{2i}\right)$ for any choice of $i$ and $l$ and, in particular, $T^{2i}T^{1l} - T^{1i}T^{2l} = 0$ if $i = l$, then $\mathbf{T}^{\mathrm{T}}\mathbf{A}_3^{\perp}\mathbf{T}$ must be skew-symmetric. ■

We now wish to consider a second skew-symmetric tensor $\mathbf{T}$. A skew-symmetric tensor has components satisfying

$$T^{ij} = -T^{ji}$$

and in particular

$$T^{ii} = 0 \text{ for } i = 1, 2, 3.$$

In this situation, the entire tensor is determined uniquely by only three components. This is nicely summarized in the following lemma.

**Lemma 45** *A skew-symmetric tensor* $\mathbf{T}$ *may be written as*

$$\mathbf{T} = T^{23}\mathbf{A}_1^{\perp} + T^{13}\mathbf{A}_2^{\perp} - T^{12}\mathbf{A}_3^{\perp}. \tag{6.4}$$

**Proof.** If we write the second order tensor in terms of the basis $\mathbf{E}_{ij}$ we obtain

$$\begin{aligned}
\mathbf{T} &= T^{ij}\mathbf{E}_{ij} \\
&= T^{12}\mathbf{E}_{12} + T^{13}\mathbf{E}_{13} - T^{12}\mathbf{E}_{21} + T^{23}\mathbf{E}_{23} - T^{13}\mathbf{E}_{31} - T^{23}\mathbf{E}_{32} \\
&= T^{12}\left(\mathbf{E}_{12} - \mathbf{E}_{21}\right) + T^{13}\left(\mathbf{E}_{13} - \mathbf{E}_{31}\right) + T^{23}\left(\mathbf{E}_{23} - \mathbf{E}_{32}\right) \\
&= T^{23}\left(\mathbf{E}_{23} - \mathbf{E}_{32}\right) + T^{13}\left(\mathbf{E}_{13} - \mathbf{E}_{31}\right) - T^{12}\left(\mathbf{E}_{21} - \mathbf{E}_{12}\right) \\
&= T^{23}\mathbf{A}_1^{\perp} + T^{13}\mathbf{A}_2^{\perp} - T^{12}\mathbf{A}_3^{\perp}
\end{aligned}$$

using the definition in equation (6.3). ■

This suggests that any skew-symmetric tensor may be completely specified by a unique vector. Let $\mathbf{t}$ denote this vector with components $t^1, t^2$ and $t^3$. $\mathbf{t}$ is defined by

$$t^1 \equiv T^{23}, t^2 \equiv T^{13} \text{ and } t^3 \equiv -T^{12}. \tag{6.5}$$

It can be shown that $\mathbf{t}$ is an axial vector and thus $\mathbf{t}$ is called the axial vector of the tensor $\mathbf{T}$. Notice that given an axial vector $\mathbf{t}$, it uniquely determines a skew-symmetric tensor $\mathbf{T}$ by the relationships in equation (6.5). From the definition (6.5), equation (6.4) becomes

$$\mathbf{T} = t^1\mathbf{A}_1^{\perp} + t^2\mathbf{A}_2^{\perp} + t^3\mathbf{A}_3^{\perp} = t^i\mathbf{A}_i^{\perp}. \tag{6.6}$$

The magnitude of the vector $\mathbf{t}$ is denoted $t$ and defined by $t = \sqrt{\mathbf{t}\cdot\mathbf{t}}$ where $\cdot$ denotes inner product. With the natural basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$,

$$\mathbf{t}\cdot\mathbf{t} = t^1t^1 + t^2t^2 + t^3t^3.$$

The magnitude of the second order tensor $\mathbf{T}$ is denoted $T$ and defined by $T = \sqrt{\frac{1}{2}\mathrm{tr}\left(\mathbf{T}\mathbf{T}^{\mathrm{T}}\right)}$. For the next lemma, it is easier to use $t^2 = \mathbf{t}\cdot\mathbf{t}$ and $T^2 = \frac{1}{2}\mathrm{tr}\left(\mathbf{T}\mathbf{T}^{\mathrm{T}}\right)$.

**Lemma 46** *If $\mathbf{t}$ is the axial vector of the skew-symmetric tensor $\mathbf{T}$, then $T = t$.*

**Proof.** Compare the expressions for $T^2$ and $t^2$ to see if they're equal.

$$T^2 = \frac{1}{2}\mathrm{tr}\left(\mathbf{T}\mathbf{T}^{\mathrm{T}}\right) = \frac{1}{2}\mathrm{tr}\left(-\mathbf{T}\mathbf{T}\right) = -\frac{1}{2}\mathrm{tr}\left(\mathbf{T}\mathbf{T}\right).$$

Evaluate $\mathbf{TT}$ to get

$$\begin{aligned}
\mathbf{TT} &= \left(T^{23}\mathbf{A}_1^{\perp} + T^{13}\mathbf{A}_2^{\perp} - T^{12}\mathbf{A}_3^{\perp}\right)\left(T^{23}\mathbf{A}_1^{\perp} + T^{13}\mathbf{A}_2^{\perp} - T^{12}\mathbf{A}_3^{\perp}\right) \\
&= T^{23}T^{23}\mathbf{A}_1^{\perp}\mathbf{A}_1^{\perp} + T^{13}T^{23}\mathbf{A}_1^{\perp}\mathbf{A}_2^{\perp} - T^{12}T^{23}\mathbf{A}_1^{\perp}\mathbf{A}_3^{\perp} \\
&\quad + T^{13}T^{23}\mathbf{A}_2^{\perp}\mathbf{A}_1^{\perp} + T^{13}T^{13}\mathbf{A}_2^{\perp}\mathbf{A}_2^{\perp} - T^{12}T^{13}\mathbf{A}_2^{\perp}\mathbf{A}_3^{\perp} \\
&\quad - T^{12}T^{23}\mathbf{A}_3^{\perp}\mathbf{A}_1^{\perp} - T^{12}T^{13}\mathbf{A}_3^{\perp}\mathbf{A}_2^{\perp} + T^{12}T^{12}\mathbf{A}_3^{\perp}\mathbf{A}_3^{\perp}.
\end{aligned}$$

$$
\begin{aligned}
\mathbf{TT} &= \left(T^{23}\mathbf{A}_1^{\perp} + T^{13}\mathbf{A}_2^{\perp} - T^{12}\mathbf{A}_3^{\perp}\right)\left(T^{23}\mathbf{A}_1^{\perp} + T^{13}\mathbf{A}_2^{\perp} - T^{12}\mathbf{A}_3^{\perp}\right) \\
&= -T^{23}T^{23}\mathbf{I}_1^{\perp} - T^{13}T^{23}\mathbf{E}_{21} + T^{12}T^{23}\mathbf{E}_{31} \\
&\quad -T^{13}T^{23}\mathbf{E}_{12} - T^{13}T^{13}\mathbf{I}_2^{\perp} - T^{12}T^{13}\mathbf{E}_{32} \\
&\quad +T^{12}T^{23}\mathbf{E}_{13} - T^{12}T^{13}\mathbf{E}_{23} - T^{12}T^{12}\mathbf{I}_3^{\perp}.
\end{aligned}
$$

Using Table 6.1 and rearranging gives

$$
\begin{aligned}
\mathbf{TT} &= -T^{23}T^{23}\mathbf{I}_1^{\perp} - T^{13}T^{13}\mathbf{I}_2^{\perp} - T^{12}T^{12}\mathbf{I}_3^{\perp} \\
&\quad -T^{13}T^{23}\left(\mathbf{E}_{21} + \mathbf{E}_{12}\right) + T^{12}T^{23}\left(\mathbf{E}_{31} + \mathbf{E}_{13}\right) - T^{12}T^{13}\left(\mathbf{E}_{32} + \mathbf{E}_{23}\right) \\
&= -T^{23}T^{23}\left(\mathbf{E}_{22} + \mathbf{E}_{33}\right) - T^{13}T^{13}\left(\mathbf{E}_{11} + \mathbf{E}_{33}\right) - T^{12}T^{12}\left(\mathbf{E}_{11} + \mathbf{E}_{22}\right) \\
&\quad -T^{13}T^{23}\left(\mathbf{E}_{21} + \mathbf{E}_{12}\right) + T^{12}T^{23}\left(\mathbf{E}_{31} + \mathbf{E}_{13}\right) - T^{12}T^{13}\left(\mathbf{E}_{32} + \mathbf{E}_{23}\right) \\
&= -\left(T^{12}T^{12} + T^{13}T^{13}\right)\mathbf{E}_{11} - \left(T^{12}T^{12} + T^{23}T^{23}\right)\mathbf{E}_{22} - \left(T^{13}T^{13} + T^{23}T^{23}\right)\mathbf{E}_{33} \\
&\quad -T^{13}T^{23}\mathbf{E}_{12} + T^{12}T^{23}\mathbf{E}_{13} - T^{13}T^{23}\mathbf{E}_{21} - T^{12}T^{13}\mathbf{E}_{23} - T^{12}T^{23}\mathbf{E}_{31} + T^{12}T^{13}\mathbf{E}_{32}.
\end{aligned}
$$

To evaluate the trace of $\mathbf{TT}$, we need only consider the coefficients of $\mathbf{E}_{11}$, $\mathbf{E}_{22}$ and $\mathbf{E}_{33}$. We get

$$
\begin{aligned}
T^2 &= -\frac{1}{2}\operatorname{tr}\left(\mathbf{TT}\right) \\
&= -\frac{1}{2}\operatorname{tr}(-\left(T^{12}T^{12} + T^{13}T^{13}\right)\mathbf{E}_{11} - \left(T^{12}T^{12} + T^{23}T^{23}\right)\mathbf{E}_{22} \\
&\quad - \left(T^{13}T^{13} + T^{23}T^{23}\right)\mathbf{E}_{33}) \\
&= -\frac{1}{2}\left(-T^{12}T^{12} - T^{13}T^{13} - T^{12}T^{12} - T^{23}T^{23} - T^{13}T^{13} - T^{23}T^{23}\right) \\
&= T^{12}T^{12} + T^{13}T^{13} + T^{23}T^{23}.
\end{aligned}
$$

Now consider the expression for $t^2 = \mathbf{t} \cdot \mathbf{t}$:

$$
\begin{aligned}
t^2 &= \mathbf{t} \cdot \mathbf{t} \\
&= t^1t^1 + t^2t^2 + t^3t^3. \\
&= \left(-T^{23}\right)\left(-T^{23}\right) + \left(T^{12}\right)\left(T^{12}\right) + \left(-T^{13}\right)\left(-T^{13}\right)
\end{aligned}
$$

$$= T^{12}T^{12} + T^{13}T^{13} + T^{23}T^{23}.$$

Comparing the expressions for $T^2$ and $t^2$ shows $T^2 = t^2$. Since the magnitude of both the tensor and the vector must be nonnegative, it follows that $T = t$. ■

This is not a surprising result since the vector $\mathbf{t}$ contains the same information as the second order tensor $\mathbf{T}$.

The next lemma indicates the effect of $\mathbf{T}$ acting on the vector $\mathbf{t}$.

**Lemma 47** *If* $\mathbf{t}$ *is the axial vector of the skew-symmetric tensor* $\mathbf{T}$, *then* $\mathbf{t}$ *is in the kernel of* $\mathbf{T}$.

**Proof.** Recall

$$\ker \mathbf{T} = \{\mathbf{x} \in \mathcal{V} \text{ such that } \mathbf{Tx} = \mathbf{0}\}.$$

Thus we consider solving the equation $\mathbf{Tx} = \mathbf{0}$. This has nontrivial solutions if and only if $\det \mathbf{T} = 0$. Evaluating $\det \mathbf{T}$ gives

$$\begin{aligned} \det \mathbf{T} &= T^{11}\left(T^{22}T^{33} - T^{32}T^{23}\right) \\ &\quad -T^{12}\left(T^{21}T^{33} - T^{31}T^{23}\right) \\ &\quad +T^{13}\left(T^{21}T^{32} - T^{31}T^{22}\right). \end{aligned}$$

Using $T^{ii} = 0$ and $T^{ij} = -T^{ji}$ gives

$$\det \mathbf{T} = -T^{12}T^{13}T^{23} + T^{12}T^{13}T^{23} = 0$$

and thus $\mathbf{T}$ is singular so nontrivial solutions to $\mathbf{Tx} = \mathbf{0}$ exist. That is, $\ker \mathbf{T}$ is not the trivial set $\{\mathbf{0}\}$. To determine this set, solve the system of linear equations given by

$$T^{12}x^2 + T^{13}x^3 = 0, \tag{6.7}$$

$$-T^{12}x^1 + T^{23}x^3 = 0 \tag{6.8}$$

and

$$-T^{13}x^1 - T^{23}x^2 = 0. \tag{6.9}$$

From equation (6.7)

$$x^2 = -\frac{T^{13}}{T^{12}}x^3.$$

From equation (6.8)

$$x^1 = \frac{T^{23}}{T^{12}}x^3.$$

From equation (6.9)

$$x^1 = -\frac{T^{23}}{T^{13}}x^2 = \left(-\frac{T^{23}}{T^{13}}\right)\left(-\frac{T^{13}}{T^{12}}x^3\right) = \frac{T^{23}}{T^{12}}x^3.$$

Thus equation (6.9) is redundant. The result is

$$\ker \mathbf{T} = \left\{\frac{k}{T^{12}}\left(T^{23}\mathbf{e}_1 - T^{13}\mathbf{e}_2 + T^{12}\mathbf{e}_3\right) \text{ for all } k \in \Re\right\}.$$

This can be verified by multiplying $\mathbf{T}$ by $\frac{k}{T^{12}}\left(T^{23}\mathbf{e}_1 - T^{13}\mathbf{e}_2 + T^{12}\mathbf{e}_3\right)$. Thus the kernel of $\mathbf{T}$ is a one-dimensional subspace of $\mathcal{V}$. Choosing $k = -T^{12}$ yields

$$\begin{aligned} \frac{-T^{12}}{T^{12}}\left(T^{23}\mathbf{e}_1 - T^{13}\mathbf{e}_2 + T^{12}\mathbf{e}_3\right) &= -\left(T^{23}\mathbf{e}_1 - T^{13}\mathbf{e}_2 + T^{12}\mathbf{e}_3\right) \\ &= -T^{23}\mathbf{e}_1 + T^{13}\mathbf{e}_2 - T^{12}\mathbf{e}_3 \\ &= t^1\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3 \\ &= \mathbf{t} \end{aligned}$$

using the definitions given in equation (6.5). Thus $\mathbf{t} \in \ker \mathbf{T}$ which is the result to be proved. ∎

Thus the kernel of $\mathbf{T}$ may be thought of as all vectors parallel and anti-parallel to the axial vector $\mathbf{t}$. This lemma implies $\mathbf{Tt} = \mathbf{0}$, $(\mathbf{Tt})^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$ or, using properties of the transpose, $\mathbf{t}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} = \mathbf{0}$. But since $\mathbf{T}$ is skew-symmetric, $\mathbf{T}^{\mathrm{T}} = -\mathbf{T}$ so $-\mathbf{t}^{\mathrm{T}}\mathbf{T} = \mathbf{0}$ or $\mathbf{t}^{\mathrm{T}}\mathbf{T} = \mathbf{0}$.

**Lemma 48** *Let* $\mathbf{t}$ *be the axial vector of the skew-symmetric tensor* $\mathbf{T}$. *If* $\mathbf{x} \in \mathcal{V}$ *then* $\mathbf{Tx} = \mathbf{t} \times \mathbf{x}$ *where* $\times$ *denotes the vector cross product.*

**Proof.** Let $\mathbf{x} = x^i\mathbf{e}_i$. Then

$$\mathbf{Tx} = t^i \mathbf{A}_i^{\perp} x^j \mathbf{e}_j$$

$$
\begin{aligned}
&= t^i x^j \mathbf{A}_i^{\perp} \mathbf{e}_j \\
&= t^1 x^1 \mathbf{A}_1^{\perp} \mathbf{e}_1 + t^1 x^2 \mathbf{A}_1^{\perp} \mathbf{e}_2 + t^1 x^3 \mathbf{A}_1^{\perp} \mathbf{e}_3 \\
&\quad + t^2 x^1 \mathbf{A}_2^{\perp} \mathbf{e}_1 + t^2 x^2 \mathbf{A}_2^{\perp} \mathbf{e}_2 + t^2 x^3 \mathbf{A}_2^{\perp} \mathbf{e}_3 \\
&\quad + t^3 x^1 \mathbf{A}_3^{\perp} \mathbf{e}_1 + t^3 x^2 \mathbf{A}_3^{\perp} \mathbf{e}_2 + t^3 x^3 \mathbf{A}_3^{\perp} \mathbf{e}_3.
\end{aligned}
$$

Applying the results of Lemma 39 gives

$$
\begin{aligned}
\mathbf{Tx} &= t^1 x^2 \mathbf{e}_3 - t^1 x^3 \mathbf{e}_2 - t^2 x^1 \mathbf{e}_3 + t^2 x^3 \mathbf{e}_1 + t^3 x^1 \mathbf{e}_2 - t^3 x^2 \mathbf{e}_1 \\
&= \left(t^2 x^3 - t^3 x^2\right) \mathbf{e}_1 - \left(t^1 x^3 - t^3 x^1\right) \mathbf{e}_2 + \left(t^1 x^2 - t^2 x^1\right) \mathbf{e}_3.
\end{aligned}
$$

Now consider $\mathbf{t} \times \mathbf{x}$. Using the determinant method of evaluating a cross product yields

$$
\begin{aligned}
\mathbf{t} \times \mathbf{x} &= \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ t^1 & t^2 & t^3 \\ x^1 & x^2 & x^3 \end{pmatrix} \\
&= \left(t^2 x^3 - t^3 x^2\right) \mathbf{e}_1 - \left(t^1 x^3 - t^3 x^1\right) \mathbf{e}_2 + \left(t^1 x^2 - t^2 x^1\right) \mathbf{e}_3.
\end{aligned}
$$

Comparing the result of $\mathbf{Tx}$ and $\mathbf{t} \times \mathbf{x}$ shows $\mathbf{Tx} = \mathbf{t} \times \mathbf{x}$. ■

### 6.1.2 Normal Uniform Rods

A rod configuration is defined to be isometric if the director distribution is given by

$$\mathbf{d}_i = \mathbf{O}\mathbf{d}_{i0}$$

where $\mathbf{O}$ is a proper orthogonal tensor function of the curve parameter and $\mathbf{d}_{i0} = \mathbf{d}_i\left(s_{R0}\right)$ is the director basis at a fixed, arbitrary point $s_{R0}$. Notice that this defines all directors in terms of a single set of directors at a fixed point along the rod. Several interesting relationships arise from this type of rod configuration.

**Lemma 49** *For an isometric rod configuration,* $\mathbf{d}_i' = \mathbf{O}'\mathbf{d}_{i0}$.

**Proof.** Since $\mathbf{d}_i = \mathbf{O}\mathbf{d}_{i0}$,

$$\mathbf{d}_i' = (\mathbf{O}\mathbf{d}_{i0})' = \mathbf{O}'\mathbf{d}_{i0} + \mathbf{O}\mathbf{d}_{i0}' = \mathbf{O}'\mathbf{d}_{i0}$$

since $\mathbf{d}_{i0}$ is a fixed vector (not a function of $s_R$) and hence $\mathbf{d}_{i0}' = \mathbf{0}$. ■

**Lemma 50** *For an isometric rod configuration,* $\mathbf{W} = \mathbf{O}'\mathbf{O}^{-1}$ *where* $\mathbf{W}$ *is the wryness tensor.*

**Proof.** From Corollary 2, $\mathbf{W}\mathbf{d}_i = \mathbf{d}_i'$ - that is, $\mathbf{W}$ maps a director to its derivative. Consider the effect of $\mathbf{O}'\mathbf{O}^{-1}$ on a director.

$$\mathbf{O}'\mathbf{O}^{-1}\mathbf{d}_i = \mathbf{O}'\mathbf{d}_{i0} = \mathbf{d}_i'$$

using the result of the Lemma 49. Since $\mathbf{W}$ and $\mathbf{O}'\mathbf{O}^{-1}$ have the same effect on an arbitrary director, it follows that $\mathbf{W} = \mathbf{O}'\mathbf{O}^{-1}$. ■

**Lemma 51** *If a rod configuration is isometric,* $\mathbf{W}$ *is skew symmetric.*

**Proof.** $\mathbf{W} = \mathbf{O}'\mathbf{O}^{-1}$ from Lemma 50. Taking transpose on both sides gives

$$\mathbf{W}^{\mathrm{T}} = \left(\mathbf{O}'\mathbf{O}^{-1}\right)^{\mathrm{T}} = \left(\mathbf{O}^{\mathrm{T}}\right)^{\mathrm{T}}\left(\mathbf{O}'\right)^{\mathrm{T}} = \mathbf{O}\left(\mathbf{O}'\right)^{\mathrm{T}} = \mathbf{O}\left(\mathbf{O}^{\mathrm{T}}\right)'$$

using the fact $\mathbf{O}$ is orthogonal so $\mathbf{O}^{-1} = \mathbf{O}^{\mathrm{T}}$ and properties of transpose. Then

$$\mathbf{W} + \mathbf{W}^{\mathrm{T}} = \mathbf{O}'\mathbf{O}^{\mathrm{T}} + \mathbf{O}\left(\mathbf{O}^{\mathrm{T}}\right)' = \left(\mathbf{O}\mathbf{O}^{\mathrm{T}}\right)' = \mathbf{I}' = \mathbf{0}$$

where $\mathbf{I}$ is the identity tensor. Then $\mathbf{W} = -\mathbf{W}^{\mathrm{T}}$ and thus $\mathbf{W}$ is skew symmetric. ■

**Lemma 52** *For an isometric rod configuration,* $\mathbf{O}' = \mathbf{W}\mathbf{O}$.

**Proof.** From Lemma 50, $\mathbf{W} = \mathbf{O}'\mathbf{O}^{-1}$. Multiplying on the right by $\mathbf{O}$ gives the desired result. ■

A rod configuration is defined to be uniform if it is isometric and the wryness tensor is constant. To emphasize this, we write $\mathbf{W}_0$ for $\mathbf{W}$ when discussing uniform rod configurations. Since $\mathbf{W}_0$ is a constant, the rotation must have a fixed axis along the axial vector of $\mathbf{W}_0$. We denote the axial vector as $\mathbf{w}_0$. Of course this vector is also fixed. Denote the components of $\mathbf{w}_0$ as $w_0^i$, the magnitude of $\mathbf{w}_0$ as $w_0$ and the magnitude of $\mathbf{W}_0$ as $W_0$. From Lemma 46, $W_0 = w_0$.

**Lemma 53** *Let* $\mathbf{O}$ *be a proper orthogonal tensor function whose axis of rotation is the axial vector of a constant wryness tensor* $\mathbf{W}_0$. *Then* $\mathbf{W}_0\mathbf{O} = \mathbf{O}\mathbf{W}_0$.

**Proof.** The proof of this lemma is based on the fact that the axial vector of $\mathbf{W}_0$ is the axis of rotation for the orthogonal tensor $\mathbf{O}$. It can be shown ([5],[14],[15]) that any proper orthogonal operator $\mathbf{O}$ may be written in the form

$$\mathbf{O} = (\cos\theta)\,\mathbf{I} + (1-\cos\theta)\,\mathbf{t}\mathbf{t}^{\mathrm{T}} + (\sin\theta)\,\mathbf{T}$$

where $\theta$ is the angle of rotation, $\mathbf{I}$ is the identity tensor, $\mathbf{t}$ is the unit vector along the axis of rotation of $\mathbf{O}$ and $\mathbf{T}$ is a tensor which has the effect of the cross product by $\mathbf{t}$; that is, $\mathbf{T}\mathbf{x} = \mathbf{t}\times\mathbf{x}$. From Lemma 48, we know this means $\mathbf{T}$ is the skew-symmetric tensor whose axial vector is $\mathbf{t}$. Note that

$$\mathbf{O}\mathbf{x} = (\cos\theta)\,\mathbf{x} + (1-\cos\theta)\,\mathbf{t}\,(\mathbf{t}\cdot\mathbf{x}) + (\sin\theta)\,\mathbf{t}\times\mathbf{x}.$$

Now consider a proper orthogonal tensor whose axis of rotation is $\mathbf{w}_0/w_0$ where $\mathbf{w}_0$ is the axial vector of $\mathbf{W}_0$. Then

$$\mathbf{O} = (\cos\theta)\,\mathbf{I} + \frac{(1-\cos\theta)}{w_0^2}\mathbf{w}_0\mathbf{w}_0^{\mathrm{T}} + \frac{(\sin\theta)}{w_0}\mathbf{W}_0.$$

Then

$$\mathbf{W}_0\mathbf{O} = (\cos\theta)\,\mathbf{W}_0 + \frac{(1-\cos\theta)}{w_0^2}\mathbf{W}_0\mathbf{w}_0\mathbf{w}_0^{\mathrm{T}} + \frac{(\sin\theta)}{w_0}\mathbf{W}_0\mathbf{W}_0.$$

But from Lemma 47, $\mathbf{W}_0\mathbf{w}_0 = \mathbf{0}$ so

$$\mathbf{W}_0\mathbf{O} = (\cos\theta)\,\mathbf{W}_0 + \frac{(\sin\theta)}{w_0}\mathbf{W}_0\mathbf{W}_0. \tag{6.10}$$

Now

$$\mathbf{O}\mathbf{W}_0 = (\cos\theta)\,\mathbf{W}_0 + \frac{(1-\cos\theta)}{w_0^2}\mathbf{w}_0{\mathbf{w}_0}^{\mathrm{T}}\mathbf{W}_0 + \frac{(\sin\theta)}{w_0}\mathbf{W}_0\mathbf{W}_0.$$

But from Lemma 47, $\mathbf{w}_0^T\mathbf{W}_0 = \mathbf{0}$ so

$$\mathbf{O}\mathbf{W}_0 = (\cos\theta)\,\mathbf{W}_0 + \frac{(\sin\theta)}{w_0}\mathbf{W}_0\mathbf{W}_0. \tag{6.11}$$

Comparing equations (6.10) and (6.11) shows $\mathbf{W}_0\mathbf{O} = \mathbf{O}\mathbf{W}_0$. ∎

The next few lemmas provide a specific example of the relationship between $\mathbf{O}$ and $\mathbf{W}_0$. This particular case is one which will be used in later chapters.

**Lemma 54** *If* $\mathbf{w}_0 = w_0\mathbf{e}_3$, *then* $\mathbf{W}_0 = W_0\mathbf{A}_3^\perp$ *and*

$$\mathbf{O} = \cos(w_0 s_R)\,\mathbf{I}_3^\perp + \sin(w_0 s_R)\,\mathbf{A}_3^\perp + \mathbf{E}_{33}. \tag{6.12}$$

**Proof.** $\mathbf{w}_0 = w_0\mathbf{e}_3$ means $w_0^1 = w_0^2 = 0$ and $w_0^3 = w_0$. Then

$$\mathbf{W}_0 = w_0^1\mathbf{A}_1^\perp + w_0^2\mathbf{A}_2^\perp + w_0^3\mathbf{A}_3^\perp = w_0^3\mathbf{A}_3^\perp = w_0\mathbf{A}_3^\perp = W_0\mathbf{A}_3^\perp$$

using equation (6.6) and Lemma 46.

To show equation (6.12) is true, we show (6.12) satisfies the differential equation $\mathbf{O}' = \mathbf{W}_0\mathbf{O}$. Evaluate both sides of the differential equation and compare the results.

From equation (6.12)

$$\mathbf{O}' = -w_0\sin(w_0 s_R)\,\mathbf{I}_3^\perp + w_0\cos(w_0 s_R)\,\mathbf{A}_3^\perp.$$

Evaluating the right side of the differential equation gives

$$\begin{aligned}
\mathbf{W}_0\mathbf{O} &= w_0\mathbf{A}_3^\perp\left(\cos(w_0 s_R)\,\mathbf{I}_3^\perp + \sin(w_0 s_R)\,\mathbf{A}_3^\perp + \mathbf{E}_{33}\right)\\
&= w_0\cos(w_0 s_R)\,\mathbf{A}_3^\perp\mathbf{I}_3^\perp + w_0\sin(w_0 s_R)\,\mathbf{A}_3^\perp\mathbf{A}_3^\perp + w_0\mathbf{A}_3^\perp\mathbf{E}_{33}\\
&= -w_0\sin(w_0 s_R)\,\mathbf{I}_3^\perp + w_0\cos(w_0 s_R)\,\mathbf{A}_3^\perp
\end{aligned}$$

using Table 6.1. Comparing the expressions for $\mathbf{O}'$ and $\mathbf{W}_0\mathbf{O}$ shows the differential equation is satisfied. It follows that equation (6.12) is true. ∎

**Lemma 55** $\mathbf{O}\mathbf{w}_0 = \mathbf{w}_0$.

**Proof.**

$$\begin{aligned}
\mathbf{O}\mathbf{w}_0 &= \left(\cos(w_0 s_R)\,\mathbf{I}_3^{\perp} + \sin(w_0 s_R)\,\mathbf{A}_3^{\perp} + \mathbf{E}_{33}\right) w_0\mathbf{e}_3 \\
&= w_0\cos(w_0 s_R)\,\mathbf{I}_3^{\perp}\mathbf{e}_3 + w_0\sin(w_0 s_R)\,\mathbf{A}_3^{\perp}\mathbf{e}_3 + w_0\mathbf{E}_{33}\mathbf{e}_3 \\
&= w_0\delta_{33}\mathbf{e}_3 \\
&= w_0\mathbf{e}_3 \\
&= \mathbf{w}_0
\end{aligned}$$

using Lemma 37 and Corollaries 7 and 8. ■

A rod configuration is said to be normal if it satisfies the condition

$$\mathbf{d}_i \cdot \mathbf{d}_3 = 0 \text{ for } i \in \{1, 2\}.$$

Thus in a normal rod configuration, the two dimensional subspace spanned by $\mathbf{d}_1$ and $\mathbf{d}_2$ is always perpendicular to the rod axis. This two dimensional subspace may be thought of as a cross-section to the rod.

### 6.1.3 Deformations and Normal Uniform Rods

The previous section defined and described a normal uniform rod configuration. In this section, we consider how this might affect a deformation tensor if we assume any rod configuration before and after deformation is normal and uniform. Such a deformation is described as normal uniform. It turns out normal uniform rods are severely restricted with respect to the possible geometries they may assume.

Let $\mathbf{O}$ denote the orthogonal tensor in the reference configuration so $\mathbf{d}_{Ri} = \mathbf{O}\mathbf{d}_{Ri_0}$. Let $\mathbf{W}_{R_0}$ denote the wryness tensor. Of course $\mathbf{W}_{R_0}$ is fixed along the rod axis. Let $\mathbf{W}_0$ denote the wryness tensor in the deformed configuration. Then $\mathbf{d}_i = \mathbf{F}\mathbf{d}_{Ri}$ from Corollary 3 and since $\mathbf{d}_{Ri} = \mathbf{O}\mathbf{d}_{Ri_0}$, we obtain

$$\mathbf{d}_i = \mathbf{F}\mathbf{O}\mathbf{d}_{Ri_0}.$$

This means that in the case of normal uniform rods, any director in the deformed configuration is completely specified by the deformation tensor, the orthogonal tensor and one director in the reference configuration. Recall that the director tensor $\mathbf{D} \equiv \delta^i_j \mathbf{d}_i \otimes \mathbf{e}^j = \mathbf{d}_i \otimes \mathbf{e}^i$ was the tensor describing the distribution of the directors along the rod axis. In the case of a normal uniform rod, the distribution of the directors is also described by the orthogonal tensor $\mathbf{O}$. If we adjust the reference director $\mathbf{d}_{Ri_0}$ to coincide with the vector $\mathbf{e}_i$, $\mathbf{D} = \mathbf{O}$ where we now use $\mathbf{D}$ as the director tensor in the reference configuration. $\mathbf{FD}$ is the director tensor in the deformed configuration.

If we consider the special case in which $\mathbf{W}_{R_0} = \mathbf{0}$, then from Lemma 16 we have

$$\mathbf{W}_0 = \mathbf{F}'\mathbf{F}^{-1} + \mathbf{F}\mathbf{W}_{R_0}\mathbf{F}^{-1} = \mathbf{F}'\mathbf{F}^{-1}$$

and since $\mathbf{F}$ is always invertible, we obtain

$$\mathbf{F}' = \mathbf{W}_0\mathbf{F}. \tag{6.13}$$

This particular case corresponds to a configuration in which the directors in the reference configuration remain parallel along the entire rod axis. Another way of saying this is that the directors in the reference configuration are constants, i.e. they do not depend on the parameter $s_R$. We obtain further simplification by choosing our directors in this configuration to be the standard basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

### 6.1.4 Four Types of Configurations For Normal Uniform Rods

We now indicate the four types of normal uniform rod configurations. In each of these, note that the parameter $s_R$ is now arc length and summation over the indices $\alpha$ and $\beta$ is from one to two. All scalar quantities appearing are constants (independent of $s_R$ unless otherwise specified).

1. Straight. In this case, the rod axis is given by $\mathbf{r} = s_R\mathbf{e}_3$, the director distribution is given by $\mathbf{d}_i = \mathbf{D}\mathbf{e}_i$ where

$$\mathbf{D} = D^{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta + \mathbf{E}_{33} \text{ and } \mathbf{W}_0 = \mathbf{0}.$$

This describes a straight rod in which a given director is parallel along the entire rod axis.

2. Straight Twisted. In this case, the rod axis is given by $\mathbf{r} = s_R \mathbf{e}_3$, the director distribution is given by $\mathbf{d}_i = \mathbf{O}\mathbf{D}\mathbf{e}_i$ where

$$\begin{aligned} \mathbf{D} &= D^{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta + \mathbf{E}_{33}, \\ \mathbf{O} &= \cos(w_0 s_R)\,\mathbf{I}_3^\perp + \sin(w_0 s_R)\,\mathbf{A}_3^\perp + \mathbf{E}_{33}, \\ \mathbf{W}_0 &= W_0 \mathbf{A}_3^\perp. \end{aligned}$$

This describes a straight rod, but the directors are twisting about the rod axis at a constant rate given by $W_0$.

3. Circular. In this case, the rod axis is a circular arc with equation $\mathbf{r} = a\mathbf{O}\mathbf{e}_1$, the director distribution is given by $\mathbf{d}_i = \mathbf{O}\mathbf{D}\mathbf{e}_i$ where

$$\begin{aligned} \mathbf{D} &= \mathbf{E}_{22} + D^{11}\mathbf{E}_{11} + D^{13}\mathbf{E}_{13} + D^{31}\mathbf{E}_{31} + D^{33}\mathbf{E}_{33}, \\ \mathbf{O} &= \cos(w_0 s_R)\,\mathbf{I}_3^\perp + \sin(w_0 s_R)\,\mathbf{A}_3^\perp + \mathbf{E}_{33}, \\ \mathbf{W}_0 &= W_0 \mathbf{A}_3^\perp. \end{aligned}$$

The geometry for this case is modified so $\mathbf{d}_2 = \mathbf{r}'$ and $\mathbf{d}_1$ and $\mathbf{d}_3$ span the cross-section of the rod. The radius of the circle is given by $a$ and this is related to the twist by $w_0 = \frac{1}{a}$.

4. Helical. In this case, the rod axis is a circular helix with radius $a$ and pitch $b$. Let $c = \sqrt{a^2 + b^2}$. The equation of the rod axis is given by $\mathbf{r} = a\mathbf{O}\mathbf{e}_1 + b s_R \mathbf{e}_3$, the director distribution is given by $\mathbf{d}_i = \mathbf{O}\mathbf{O}_1\mathbf{D}\mathbf{e}_i$ where

$$\begin{aligned} \mathbf{D} &= D^{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta + \mathbf{E}_{33}, \\ \mathbf{O} &= \cos(w_0 s_R)\,\mathbf{I}_3^\perp + \sin(w_0 s_R)\,\mathbf{A}_3^\perp + \mathbf{E}_{33}, \\ \mathbf{W}_0 &= W_0 \mathbf{A}_3^\perp, \\ \mathbf{O}_1 &= \frac{b}{c}(\mathbf{E}_{22} + \mathbf{E}_{33}) + \frac{a}{c}(\mathbf{E}_{23} - \mathbf{E}_{32}) + \mathbf{E}_{11}. \end{aligned}$$

Notice this case is made more difficult by the fact the axial vector of the wryness tensor is not the axis of rotation of $\mathbf{O}\mathbf{O}_1\mathbf{D}$.

Our concern will be with deformations from straight configuration to each of the four configurations listed above. This particular case means the twist in the reference configuration is zero, that is, $\mathbf{W}_{R_0} = \mathbf{0}$, and so equation (6.13) applies. We now consider the effects of this on the field equations.

## 6.2 The Field Equations of Normal Uniform Rods

In this section, we derive the special form of the field equations for normal uniform rods.

### 6.2.1 Tensor Form of The Field Equations of Normal Uniform Rods

To see the effects of the conditions of normality and uniformity on the field equations, we use the polar decomposition theorem. The statement and proof of this theorem are given in Appendix A. The theorem states that we can decompose any nonsingular second order tensor as follows:

$$\mathbf{F} = \mathbf{VR} = \mathbf{RU}$$

where $\mathbf{V}$ and $\mathbf{U}$ are symmetric tensors and $\mathbf{R}$ is orthogonal. Furthermore this decomposition is unique. Since $\mathbf{R}$ is orthogonal, it does not change the length of any vector it operates on, but only rotates or reflects the vector. $\mathbf{V}$ and $\mathbf{U}$ are called the left and right stretch tensors respectively. Their effect is to lengthen any vector operated on by $\mathbf{F}$. Since the deformation tensor $\mathbf{F}$ is always invertible, the polar decomposition theorem always applies to $\mathbf{F}$.

Of course in general, $\mathbf{F}$ is a function of the parameter $s_R$ and so we would need to write

$$\mathbf{F}(s_R) = \mathbf{V}(s_R)\,\mathbf{R}(s_R) = \mathbf{R}(s_R)\,\mathbf{U}(s_R).$$

For normal uniform rods, this general statement may be simplified. In a deformation from a straight configuration to any of the above four configurations, the rod must remain normal. That is, its cross-section must stay normal to the rod axis. This means there can be no shearing effects of the rod material under deformation. This, in turn, means the right stretch tensor $\mathbf{U}$ must be constant with respect to the parameter $s_R$. Thus $\mathbf{U}(s_R) = \mathbf{U}_0$ where $\mathbf{U}_0$ denotes a constant symmetric right stretch tensor. Thus we may write $\mathbf{F}(s_R) = \mathbf{R}(s_R)\,\mathbf{U}_0$ for the type of deformations we are interested in which will more briefly be written as

$$\mathbf{F} = \mathbf{R}\mathbf{U}_0. \tag{6.14}$$

From equation (6.13), we obtain $(\mathbf{R}\mathbf{U}_0)' = \mathbf{W}_0\mathbf{R}\mathbf{U}_0$ or, upon applying the product rule to the left side, $\mathbf{R}'\mathbf{U}_0 + \mathbf{R}\mathbf{U}_0' = \mathbf{R}'\mathbf{U}_0 = \mathbf{W}_0\mathbf{R}\mathbf{U}_0$. If $\mathbf{U}_0$ is invertible (as we will choose it to be),

$$\mathbf{R}' = \mathbf{W}_0\mathbf{R}. \tag{6.15}$$

Notice if we specify $\mathbf{R}$ and $\mathbf{U}_0$, we completely determine the deformation tensor $\mathbf{F}$ and the wryness tensor $\mathbf{W}_0$. If we specify the wryness tensor, we can use the differential equations $\mathbf{R}' = \mathbf{W}_0\mathbf{R}$ and $\mathbf{F}' = \mathbf{W}_0\mathbf{F}$ to determine $\mathbf{R}$ and $\mathbf{F}$ to within arbitrary constants. The property $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ further reduces the possible values for the constants, but do not uniquely determine them. To determine the constants exactly, we need to specify the axis of rotation of $\mathbf{R}$ (in the case where $\mathbf{R}$ is a member of the special orthogonal group).

We now define a quantity $\mathbf{Z}$ as

$$\mathbf{Z} \equiv \mathbf{R}^T\mathbf{W}_0\mathbf{R}$$

so $\mathbf{W}_0\mathbf{R} = \mathbf{R}\mathbf{Z}$. Then equation (6.15) becomes

$$\mathbf{R}' = \mathbf{R}\mathbf{Z} \tag{6.16}$$

and

$$\mathbf{F}' = \mathbf{R}'\mathbf{U}_0 = \mathbf{W}_0\mathbf{R}\mathbf{U}_0 = \mathbf{R}\mathbf{Z}\mathbf{U}_0. \tag{6.17}$$

We are interested in an expression such as equation (6.17) in order to apply the constitutive restriction of material frame indifference.

We now prove that $\mathbf{Z}$ is skew-symmetric and constant.

**Lemma 56** *If $\mathbf{Z} \equiv \mathbf{R}^T\mathbf{W}_0\mathbf{R}$, then $\mathbf{Z}$ is skew-symmetric.*

**Proof.** We know $\mathbf{W}_0$ is skew-symmetric. Since $\mathbf{W}_0 = \mathbf{R}\mathbf{Z}\mathbf{R}^T$, then $\mathbf{R}\mathbf{Z}\mathbf{R}^T$ is skew-symmetric. This means

$$\mathbf{R}\mathbf{Z}\mathbf{R}^T = -\left(\mathbf{R}\mathbf{Z}\mathbf{R}^T\right)^T$$

and applying the rules of transpose gives

$$\mathbf{RZR}^{\mathrm{T}} = -\left(\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}} \mathbf{Z}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} = -\mathbf{RZ}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}.$$

Multiplying through this equation by $\mathbf{R}^{\mathrm{T}}$ on the left and $\mathbf{R}$ on the right leaves

$$\mathbf{Z} = -\mathbf{Z}^{\mathrm{T}}$$

which means $\mathbf{Z}$ is skew-symmetric. ■

**Lemma 57** *If* $\mathbf{Z} \equiv \mathbf{R}^{\mathrm{T}}\mathbf{W}_0\mathbf{R}$, *then* $\mathbf{Z}$ *is a constant.*

**Proof.** We start with $\mathbf{W}_0 = \mathbf{RZR}^{\mathrm{T}}$ and differentiate both sides with respect to $s_R$. Since $\mathbf{W}_0$ is independent of $s_R$, this leaves

$$\mathbf{0} = \left(\mathbf{RZR}^{\mathrm{T}}\right)' = \mathbf{R}'\mathbf{ZR}^{\mathrm{T}} + \mathbf{RZ}'\mathbf{R}^{\mathrm{T}} + \mathbf{RZ}\left(\mathbf{R}^{\mathrm{T}}\right)'$$

and rearranging and using the fact $\left(\mathbf{R}^{\mathrm{T}}\right)' = \left(\mathbf{R}'\right)^{\mathrm{T}}$ gives

$$\mathbf{RZ}'\mathbf{R}^{\mathrm{T}} = -\mathbf{R}'\mathbf{ZR}^{\mathrm{T}} - \mathbf{RZ}\left(\mathbf{R}'\right)^{\mathrm{T}}.$$

Multiplying through this equation by $\mathbf{R}^{\mathrm{T}}$ on the left and $\mathbf{R}$ on the right gives

$$\mathbf{Z}' = -\mathbf{R}^{\mathrm{T}}\mathbf{R}'\mathbf{ZR}^{\mathrm{T}}\mathbf{R} - \mathbf{R}^{\mathrm{T}}\mathbf{RZ}\left(\mathbf{R}'\right)^{\mathrm{T}}\mathbf{R} = -\mathbf{R}^{\mathrm{T}}\mathbf{R}'\mathbf{Z} - \mathbf{Z}\left(\mathbf{R}'\right)^{\mathrm{T}}\mathbf{R}$$

using the fact $\mathbf{R}^{\mathrm{T}}\mathbf{R}$ is the identity tensor. From equation (6.15), $\mathbf{R}' = \mathbf{W}_0\mathbf{R}$ so

$$\mathbf{R}^{\mathrm{T}}\mathbf{R}' = \mathbf{R}^{\mathrm{T}}\mathbf{W}_0\mathbf{R} = \mathbf{Z}.$$

Then taking the transpose of this gives

$$\left(\mathbf{R}^{\mathrm{T}}\mathbf{R}'\right)^{\mathrm{T}} = \mathbf{Z}^{\mathrm{T}}$$

and since $\mathbf{Z}$ is skew-symmetric as proved in the previous lemma and using properties of transpose, we obtain

$$(\mathbf{R}')^{\mathrm{T}}\,\mathbf{R} = -\mathbf{Z}.$$

Putting this information into the equation for $\mathbf{Z}'$ gives

$$\mathbf{Z}' = -\mathbf{Z}\mathbf{Z} - \mathbf{Z}\,(-\mathbf{Z}) = -\,\mathbf{Z}\mathbf{Z} + \mathbf{Z}\mathbf{Z} = \mathbf{0}.$$

Since $\mathbf{Z}' = \mathbf{0}$, it follows that $\mathbf{Z}$ is independent of $s_R$, that is, $\mathbf{Z}$ is a constant. ■

To emphasize $\mathbf{Z}$ is a constant, we now write it as $\mathbf{Z}_0$. Thus we have $\mathbf{F} = \mathbf{R}\mathbf{U}_0$ and $\mathbf{F}' = \mathbf{R}\mathbf{Z}_0\mathbf{U}_0$ so both $\mathbf{F}$ and $\mathbf{F}'$ can be expressed as the product of the orthogonal tensor $\mathbf{R}$ and some constant tensor.

There is an important special case of the above results. If $\mathbf{R}$ is a member of the special orthogonal group, it has the effect of rotating any vector it operates on. If the axis of rotation of $\mathbf{R}$ is parallel to the axial vector of $\mathbf{W}_0$, then from Lemma 53, $\mathbf{W}_0\mathbf{R} = \mathbf{R}\mathbf{W}_0$ and thus $\mathbf{R}' = \mathbf{R}\mathbf{W}_0$ and $\mathbf{R}^{\mathrm{T}}\mathbf{R}' = \mathbf{W}_0$. This means $\mathbf{F}' = \mathbf{W}_0\mathbf{R}\mathbf{U}_0 = \mathbf{R}\mathbf{W}_0\mathbf{U}_0$. Since $\mathbf{R}' = \mathbf{R}\mathbf{Z}_0$, this implies $\mathbf{Z}_0 = \mathbf{W}_0$.

We may apply this to the field equations using the constitutive restrictions. Recall the field equations are given as

$$\mathbf{n}' + \lambda\rho\mathbf{f} = \mathbf{0}, \tag{6.18}$$

$$\overline{\mathbf{M}}\,(\mathbf{F},\mathbf{F}')' - \overline{\mathbf{N}}\,(\mathbf{F},\mathbf{F}') + \;\mathbf{n}\otimes\mathbf{r}'_R + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}, \tag{6.19}$$

$$\mathbf{F}'\overline{\mathbf{M}}\,(\mathbf{F},\mathbf{F}')^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}\,(\mathbf{F},\mathbf{F}')^{\mathrm{T}} = \left(\mathbf{F}'\overline{\mathbf{M}}\,(\mathbf{F},\mathbf{F}')^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}\,(\mathbf{F},\mathbf{F}')^{\mathrm{T}}\right)^{\mathrm{T}} \tag{6.20}$$

and Postulate 6 states

$$\mathbf{F}\overline{\mathbf{L}}^{\mathrm{T}} = \overline{\mathbf{L}}\mathbf{F}^{\mathrm{T}}. \tag{6.21}$$

Notice we have chosen to use $\overline{\mathbf{M}}\,(\mathbf{F},\mathbf{F}') = \lambda\rho D_2\varepsilon\,(\mathbf{F},\mathbf{F}')$ and $\overline{\mathbf{N}}\,(\mathbf{F},\mathbf{F}') = \lambda\rho D_2\varepsilon\,(\mathbf{F},\mathbf{F}')$ in the field equations. This turns out be more convenient since it means we do not have to worry about the $\lambda\rho$ factor.

We now consider equations (6.19) to (6.21) in turn. Replace $\mathbf{F}$ with $\mathbf{R}\mathbf{U}_0$ in equation (6.19) to give

$$\overline{\mathbf{M}}\,(\mathbf{R}\mathbf{U}_0,(\mathbf{R}\mathbf{U}_0)')' - \overline{\mathbf{N}}\,(\mathbf{R}\mathbf{U}_0,(\mathbf{R}\mathbf{U}_0)') + \mathbf{n}\otimes\mathbf{r}'_R + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}$$

and simplifying this yields

$$\overline{\mathbf{M}}\left(\mathbf{R}\mathbf{U}_0, \mathbf{R}\mathbf{Z}_0\mathbf{U}_0\right)' - \overline{\mathbf{N}}\left(\mathbf{R}\mathbf{U}_0, \mathbf{R}\mathbf{Z}_0\mathbf{U}_0\right) + \mathbf{n} \otimes \mathbf{r}'_R + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}$$

using equation (6.17) replacing the symbol $\mathbf{Z}$ with $\mathbf{Z}_0$ as discussed above. Applying the constitutive restrictions given by equation (5.18) gives

$$\left(\mathbf{R}\overline{\mathbf{M}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right)\right)' - \mathbf{R}\overline{\mathbf{N}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right) + \mathbf{n} \otimes \mathbf{r}'_R + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}. \tag{6.22}$$

Notice that $\overline{\mathbf{M}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right)$ and $\overline{\mathbf{N}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right)$ are constant tensors, independent of the parameter $s_R$ since $\mathbf{U}_0$ and $\mathbf{Z}_0\mathbf{U}_0$ are independent of $s_R$. This means $\overline{\mathbf{M}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right)' = \mathbf{0}$ and applying the product rule to the first term of equation (6.22) gives

$$\mathbf{R}'\overline{\mathbf{M}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right) - \mathbf{R}\overline{\mathbf{N}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right) + \mathbf{n} \otimes \mathbf{r}'_R + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}.$$

Notice that if we know $\mathbf{R}$, this is no longer a differential equation! This is the benefit of considering normal uniform rods - we turn differential equations into algebraic relationships.

Let $\overline{\mathbf{M}}_0$ and $\overline{\mathbf{N}}_0$ denote $\overline{\mathbf{M}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right)$ and $\overline{\mathbf{N}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right)$ respectively. Then

$$\mathbf{R}'\overline{\mathbf{M}}_0 - \mathbf{R}\overline{\mathbf{N}}_0 + \mathbf{n} \otimes \mathbf{r}'_R + \lambda\rho\overline{\mathbf{L}} = \mathbf{0}$$

or

$$\lambda\rho\overline{\mathbf{L}} = \mathbf{R}\overline{\mathbf{N}}_0 - \mathbf{R}'\overline{\mathbf{M}}_0 - \mathbf{n} \otimes \mathbf{r}'_R.$$

Multiplying through by $\mathbf{R}^{\mathrm{T}}$ gives

$$\lambda\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \mathbf{R}^{\mathrm{T}}\mathbf{R}\overline{\mathbf{N}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{R}'\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R$$

which becomes

$$\lambda\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{Z}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R.$$

This is the basic field equation we use to solve the special deformations of normal uniform rods. In the special case where the axis of rotation of $\mathbf{R}$ is parallel to the axial vector of $\mathbf{W}_0$, we can replace this equation with

$$\lambda\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R$$

and we may use this as our basic field equation.

Equation (6.20) may be written as

$$\mathbf{F}'\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}} = \left(\mathbf{F}'\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}}\right)^{\mathrm{T}} + \left(\mathbf{F}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}}\right)^{\mathrm{T}}$$

which becomes

$$\mathbf{F}'\overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}} + \mathbf{F}\overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')^{\mathrm{T}} = \overline{\mathbf{M}}(\mathbf{F},\mathbf{F}')(\mathbf{F}')^{\mathrm{T}} + \overline{\mathbf{N}}(\mathbf{F},\mathbf{F}')\mathbf{F}^{\mathrm{T}}.$$

Replacing $\mathbf{F}$ and $\mathbf{F}'$ with $\mathbf{R}\mathbf{U}_0$ and $\mathbf{R}\mathbf{Z}_0\mathbf{U}_0$ respectively yields the following:

$$\begin{aligned}&\mathbf{R}\mathbf{Z}_0\mathbf{U}_0\overline{\mathbf{M}}(\mathbf{R}\mathbf{U}_0,\mathbf{R}\mathbf{Z}_0\mathbf{U}_0)^{\mathrm{T}} + \mathbf{R}\mathbf{U}_0\overline{\mathbf{N}}(\mathbf{R}\mathbf{U}_0,\mathbf{R}\mathbf{Z}_0\mathbf{U}_0)^{\mathrm{T}}\\ = \;&\overline{\mathbf{M}}(\mathbf{R}\mathbf{U}_0,\mathbf{R}\mathbf{Z}_0\mathbf{U}_0)(\mathbf{R}\mathbf{Z}_0\mathbf{U}_0)^{\mathrm{T}} + \overline{\mathbf{N}}(\mathbf{R}\mathbf{U}_0,\mathbf{R}\mathbf{Z}_0\mathbf{U}_0)(\mathbf{R}\mathbf{U}_0)^{\mathrm{T}}.\end{aligned}$$

Using Postulate 7 (material frame indifference), this becomes

$$\begin{aligned}&\mathbf{R}\mathbf{Z}_0\mathbf{U}_0\left(\mathbf{R}\overline{\mathbf{M}}(\mathbf{U}_0,\mathbf{Z}_0\mathbf{U}_0)\right)^{\mathrm{T}} + \mathbf{R}\mathbf{U}_0\left(\mathbf{R}\overline{\mathbf{N}}(\mathbf{U}_0,\mathbf{Z}_0\mathbf{U}_0)\right)^{\mathrm{T}}\\ = \;&\mathbf{R}\overline{\mathbf{M}}(\mathbf{U}_0,\mathbf{Z}_0\mathbf{U}_0)(\mathbf{R}\mathbf{Z}_0\mathbf{U}_0)^{\mathrm{T}} + \mathbf{R}\overline{\mathbf{N}}(\mathbf{U}_0,\mathbf{Z}_0\mathbf{U}_0)(\mathbf{R}\mathbf{U}_0)^{\mathrm{T}}\end{aligned}$$

and introducing the notation $\overline{\mathbf{M}}_0$ and $\overline{\mathbf{N}}_0$ defined above and multiplying through the equation by $\mathbf{R}^{\mathrm{T}}$ on the left gives

$$\mathbf{Z}_0\mathbf{U}_0\left(\mathbf{R}\overline{\mathbf{M}}_0\right)^{\mathrm{T}} + \mathbf{U}_0\left(\mathbf{R}\overline{\mathbf{N}}_0\right)^{\mathrm{T}} = \overline{\mathbf{M}}_0(\mathbf{R}\mathbf{Z}_0\mathbf{U}_0)^{\mathrm{T}} + \overline{\mathbf{N}}_0(\mathbf{R}\mathbf{U}_0)^{\mathrm{T}}.$$

Further simplification gives

$$\mathbf{Z}_0\mathbf{U}_0\overline{\mathbf{M}}_0^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} + \mathbf{U}_0\overline{\mathbf{N}}_0^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} = \overline{\mathbf{M}}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{Z}_0^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} + \overline{\mathbf{N}}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}.$$

But $\mathbf{U}_0 = \mathbf{U}_0^T$ and $\mathbf{Z}_0 = -\mathbf{Z}_0^T$. Placing these in the equation and multiplying by $\mathbf{R}$ through the equation on the right gives

$$\mathbf{Z}_0\mathbf{U}_0\overline{\mathbf{M}}_0^T + \mathbf{U}_0\overline{\mathbf{N}}_0^T = \overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{Z}_0.$$

In the special case where the axis of rotation of $\mathbf{R}$ is parallel to the axial vector of $\mathbf{W}_0$, we can replace this equation with

$$\mathbf{W}_0\mathbf{U}_0\overline{\mathbf{M}}_0^T + \mathbf{U}_0\overline{\mathbf{N}}_0^T = \overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{W}_0.$$

Replace $\mathbf{F}$ with $\mathbf{RU}_0$ in equation (6.21) to give

$$\mathbf{RU}_0\overline{\mathbf{L}}^T = \overline{\mathbf{L}}\,(\mathbf{RU}_0)^T$$

which simplifies to

$$\mathbf{RU}_0\overline{\mathbf{L}}^T = \overline{\mathbf{L}}\mathbf{U}_0^T\mathbf{R}^T$$

which becomes

$$\mathbf{RU}_0\overline{\mathbf{L}}^T = \overline{\mathbf{L}}\mathbf{U}_0\mathbf{R}^T$$

using the fact that $\mathbf{U}_0$ is symmetric.

### 6.2.2 Summary

The geometry of normal uniform rods is very restricted - the rod axis can only assume the shapes of straight lines, circles or circular helices. Furthermore, the wryness tensor must be a constant. Combining this information with the polar decomposition theorem gives rise to special field equations for deformations of normal uniform rods. The right stretch tensor $\mathbf{U}$ must be a constant $\mathbf{U}_0$. Specifying the right deformation tensor $\mathbf{U}_0$ and the orthogonal tensor $\mathbf{R}$ completely specifies the deformation tensor $\mathbf{F}$ and the wryness tensor $\mathbf{W}_0$ (which is a constant skew-symmetric tensor). These field equations are given as

$$\mathbf{n}' + \lambda\rho\mathbf{f} = \mathbf{0}, \tag{6.23}$$

$$\lambda\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{Z}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R, \tag{6.24}$$

$$\mathbf{Z}_0\mathbf{U}_0\overline{\mathbf{M}}_0^{\mathrm{T}} + \mathbf{U}_0\overline{\mathbf{N}}_0^{\mathrm{T}} = \overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{Z}_0, \tag{6.25}$$

and

$$\mathbf{R}\mathbf{U}_0\overline{\mathbf{L}}^{\mathrm{T}} = \overline{\mathbf{L}}\mathbf{U}_0\mathbf{R}^{\mathrm{T}}. \tag{6.26}$$

In the special case where the axis of rotation of $\mathbf{R}$ is parallel to the axial vector of $\mathbf{W}_0$, we can replace equations (6.24) and (6.25) with

$$\lambda\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R \tag{6.27}$$

and

$$\mathbf{W}_0\mathbf{U}_0\overline{\mathbf{M}}_0^{\mathrm{T}} + \mathbf{U}_0\overline{\mathbf{N}}_0^{\mathrm{T}} = \overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{W}_0 \tag{6.28}$$

respectively.

Notice that the constitutive restriction of material frame indifference has been incorporated into the field equations (6.24) and (6.25) or (6.27) and (6.28).

# Chapter 7

# Solving Four Special Deformations

In this chapter, we develop a scheme for solving the four special deformations described in the previous chapter. This plan of attack is based on exploiting the symmetry properties of various quantities arising from the field equations and the nature of the stress-tensor as a cross-section tensor.

By solving the deformations, we have two goals:

1. Reduce the tensors $\overline{\mathbf{M}}_0 = \overline{\mathbf{M}}(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0)$ and $\overline{\mathbf{N}}_0 = \overline{\mathbf{N}}(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0)$ to as few nonzero components as possible;

2. Express the components of $\mathbf{n}$ and $\overline{\mathbf{L}}$ in terms of the components of $\overline{\mathbf{M}}_0$ and $\overline{\mathbf{N}}_0$.

Essentially we are characterizing the tensor $\overline{\mathbf{L}}$ and the vector $\mathbf{n}$ in terms of the components of $\overline{\mathbf{M}}_0$ and $\overline{\mathbf{N}}_0$ which are the derivatives of the strain energy density function evaluated at the special point $(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0)$.

In fact, we actually do step 2 first as far as it is possible. This is done by simply manipulating the field equations to get as much information as possible about the relationships among the components of these tensors. We then finish each problem by introducing the appropriate transformations for monotropic symmetry. This accomplishes step 1 and completes step 2.

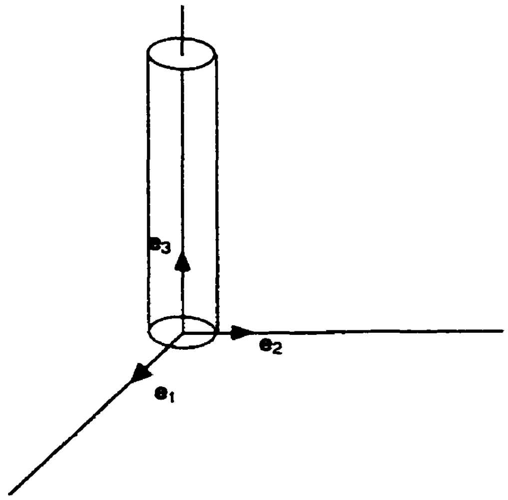


Figure 7-1: Reference Configuration For $q = 3$

## 7.1 The Reference Configuration

The reference configuration will be the same in all four cases, namely the straight configuration described in detail in the previous chapter. However, the axis along which the straight reference configuration is orientated varies from case to case. This will be designated by a parameter $q \in \{1, 2, 3\}$. Once $q$ is specified, it means the curve of the reference configuration is parallel to $\mathbf{e}_q$. In the reference configuration, the curve parameter $s_R$ will be the arc length of the curve defining the rod axis. The curve is expressed by $\mathbf{r}_R(s_R) = s_R\mathbf{e}_q$. Then $\mathbf{r}'_R(s_R) = \mathbf{e}_q$ and thus is a constant in all cases. The directors in the reference configuration will be given by $\mathbf{d}_{R\alpha} = \mathbf{e}_\alpha$. The reference configuration for $q = 3$ is shown in Figure 7-1.

## 7.2 Component Form of the Tensors

This section provides the notation for the component form of the tensors. To solve the four special deformations, it is necessary to use component equations. The objective of this chapter is determine all the relationships among the components of the various tensors which hold for any problem. The additional information about a specific problem may be used to simplify these relationships.

In all four deformations, the problem will be solved with respect to the natural basis with orthonormal vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and dual basis $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$. It will be necessary in a couple of problems to employ other bases to describe the problem. The basis for the second order tensors will either be $\mathbf{e}_i \otimes \mathbf{e}^j$ or $\mathbf{E}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j$. In view of the equivalence $\mathbf{e}_i = \mathbf{e}^i$, there is no distinction between these bases and we use them interchangeably.

Here are the component forms of all the second order tensors in the $\mathbf{e}_i \otimes \mathbf{e}^j$ basis. The stretch tensor $\mathbf{U}_0$ is given by

$$\mathbf{U}_0 = \lambda_{(i)} \mathbf{e}_i \otimes \mathbf{e}^i = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}^1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}^2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^3, \tag{7.1}$$

the orthogonal tensor $\mathbf{R}$ is given by

$$\mathbf{R} = R^i_j \mathbf{e}_i \otimes \mathbf{e}^j \tag{7.2}$$

so

$$\mathbf{R}^{\mathrm{T}} = \sum_{ij} R^j_i \mathbf{e}_i \otimes \mathbf{e}^j. \tag{7.3}$$

The stress tensor $\overline{\mathbf{L}}$ is given by

$$\overline{\mathbf{L}} = \overline{L}^i_j \mathbf{e}_i \otimes \mathbf{e}^j \tag{7.4}$$

and

$$\overline{\mathbf{L}}^{\mathrm{T}} = \sum_{ij} \overline{L}^j_i \mathbf{e}_i \otimes \mathbf{e}^j.$$

The wryness tensor $\mathbf{W}_0$ is defined by

$$\mathbf{W}_0 = w_0 \left(\mathbf{e}_2 \otimes \mathbf{e}^1 - \mathbf{e}_1 \otimes \mathbf{e}^2\right). \tag{7.5}$$

The tensors $\overline{\mathbf{M}}_0$ and $\overline{\mathbf{N}}_0$ will have component forms

$$\overline{\mathbf{M}}_0 = \overline{M}^i_j \mathbf{e}_i \otimes \mathbf{e}^j \tag{7.6}$$

and

$$\overline{\mathbf{N}}_0 = \overline{N}^i_j \mathbf{e}_i \otimes \mathbf{e}^j \tag{7.7}$$

respectively. Notice $F^i_j$, $R^i_j$ and $\overline{L^i_j}$ are functions of $s_R$ whereas $\overline{M^i_j}$ and $\overline{N^i_j}$ are constants. The vector $\mathbf{n}$ is expressed as $\mathbf{n} = n^i\mathbf{e}_i$ where, from Postulate 6, the $n^i$ are constants. The second order tensor $\mathbf{n} \otimes \mathbf{r}'_R$ may be written as $n^i\mathbf{e}_i \otimes \mathbf{e}^q$ where $q$ was defined earlier or in the equivalent form

$$\mathbf{n} \otimes \mathbf{r}'_R = n^i\delta^q_j\mathbf{e}_i \otimes \mathbf{e}^j. \tag{7.8}$$

In all cases $\mathbf{W}_0 = W_0\mathbf{A}^{\perp}_3$ ($W_0$ is a constant which may or may not be zero). Thus in all cases where there is an axis of rotation, the axis is always $\mathbf{e}_3$.

The right stretch tensor $\mathbf{U}_0$ will is given by equation (7.1) (the index in parenthesis indicates it is to be included with the summation). Notice this may also be written as

$$\mathbf{U}_0 = \lambda_1\mathbf{E}_{11} + \lambda_2\mathbf{E}_{22} + \lambda_3\mathbf{E}_{33}.$$

The quantities $\lambda_{(i)}$ represent stretch factors in their respective directions. The stretch factors are constants. In every case, two of stretch factors will be equal, but which stretch factors are equal varies from case to case.

This scheme of finding solutions leans heavily on the polar decomposition theorem, in particular $\mathbf{F} = \mathbf{R}\mathbf{U}_0$. In some cases, $\mathbf{R}$ and $\mathbf{U}_0$ will be specified and thus $\mathbf{F}$ is automatically determined. In other cases, the directors will be specified, thus defining $\mathbf{F}$. Then a polar decomposition $\mathbf{R}\mathbf{U}_0$ for this deformation tensor will be given. In all cases, the right stretch tensor $\mathbf{U}_0$ will be that as given above. The tensor $\mathbf{R}$ will vary from case to case.

The work developed in this chapter applies to the first three deformations considered in subsequent chapters. In these three deformations, $\mathbf{W}_0$ is either zero or the axis of rotation of $\mathbf{R}$ is parallel to the axial vector of $\mathbf{W}_0$. This means we can apply equations (6.27) and (6.28). These are the equations used in the following sections. The final deformation must use equations (6.24) and (6.25) and hence the results from many of the remaining sections in this chapter will not apply.

## 7.3 The Cross-Sectional Stress Tensor

There is one more piece of information regarding the stress tensor $\mathbf{T} = \rho\mathbf{L}$ in Cohen's paper [1] which has not been discussed. This is related to the quantity $\overline{\mathbf{L}}$ appearing in the field equations. The stress tensor $\rho\mathbf{L}$ is a cross-sectional tensor which means it is nonzero only in the subspace spanned by the two directors defining the cross-section. In the case of normal rods, this means $\mathbf{L}$ is nonzero only in the two dimensional subspace perpendicular to the director acting as the tangent vector in the deformed configuration. Recall equation (4.39)

$$\mathbf{L} = \mathbf{F}\overline{\mathbf{L}}^{\mathrm{T}}$$

or

$$\mathbf{L}^{\mathrm{T}} = \overline{\mathbf{L}}\mathbf{F}^{\mathrm{T}}$$

giving

$$\overline{\mathbf{L}} = \mathbf{L}^{\mathrm{T}}\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} = \mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}.$$

Let the components of $\mathbf{F}^{-\mathrm{T}}$ be given as

$$\mathbf{F}^{-\mathrm{T}} = X^i_j \mathbf{e}_i \otimes \mathbf{e}^j$$

(we don't actually care about the values of $X^i_j$ for this calculation). If the cross-section of the deformed rod is parallel to one of the principal planes (that is, a plane spanned by $\mathbf{e}_i$ and $\mathbf{e}_j$ for some distinct $i, j \in \{1, 2, 3\}$), then five of the components of $\mathbf{L}$ will be zero in the Cartesian basis. The value $q$ discussed earlier denotes the direction in which $\mathbf{L}$ does not act. That is, $q \neq i$ and $q \neq j$. Then all components of the form $L^k_q$ and $L^q_k$ are zero where $k \in \{1, 2, 3\}$.

As an example, suppose the tangent vector to the curve in the deformed configuration is parallel to $\mathbf{e}_3$. Then the components of $\mathbf{L}$ satisfy

$$L^1_3 = L^2_3 = L^3_3 = L^3_1 = L^3_2 = 0. \tag{7.9}$$

Calculating $\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$ gives

$$\begin{aligned}
\mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}} &= \left(\sum_{kl} L^l_k \left(1-\delta^l_3\right)\left(1-\delta^3_k\right)\mathbf{e}_k\otimes\mathbf{e}^l\right)\left(X^i_j\mathbf{e}_i\otimes\mathbf{e}^j\right) \\
&= \sum_{kl} L^l_k X^i_j \left(1-\delta^l_3\right)\left(1-\delta^3_k\right)\left(\mathbf{e}_k\otimes\mathbf{e}^l\right)\left(\mathbf{e}_i\otimes\mathbf{e}^j\right) \\
&= \sum_{kl} L^l_k X^i_j \delta^l_i \left(1-\delta^l_3\right)\left(1-\delta^3_k\right)\mathbf{e}_k\otimes\mathbf{e}^j \\
&= \sum_{kl} L^l_k X^l_j \left(1-\delta^l_3\right)\left(1-\delta^3_k\right)\mathbf{e}_k\otimes\mathbf{e}^j.
\end{aligned}$$

When $l=3$ or $k=3$, the coefficient is zero. It follows that the coefficients of $\mathbf{e}_3\otimes\mathbf{e}^1$, $\mathbf{e}_3\otimes\mathbf{e}^2$ and $\mathbf{e}_3\otimes\mathbf{e}^3$ are all zero which shows

$$\overline{L}^3_3=\overline{L}^3_1=\overline{L}^3_2=0. \tag{7.10}$$

More generally, the cross-section spanned by the two non-tangent directors is not parallel to a standard plane, and in fact the orientation of the plane of the cross-section may vary from point to point. In that case, it is necessary to express the tensor $\mathbf{L}$ in some coordinate system in which five components of $\mathbf{L}$ are zero. Then a transformation to the Cartesian basis must be made followed by the transformation to $\overline{\mathbf{L}}$. The objective of this exercise is to see if there are components of $\overline{\mathbf{L}}$ which are zero in the Cartesian basis. Zero components significantly simplify the computations.

## 7.4 Component Equations From Symmetry

In this section, we consider several symmetric equations and derive their component forms.

### 7.4.1 Symmetry Equations for $\overline{\mathbf{L}}$

We wish to consider the symmetry of $\overline{\mathbf{L}}$ for any tensor $\mathbf{R}$ and the stretch tensor $\mathbf{U}_0$ given by equation (7.1). The symmetry of $\overline{\mathbf{L}}$ is expressed in equation (6.26). However, for our purposes it is easier to express this symmetry by

$$\mathbf{R}\mathbf{U}_0\overline{\mathbf{L}}^{\mathrm{T}} = \left(\mathbf{R}\mathbf{U}_0\overline{\mathbf{L}}^{\mathrm{T}}\right)^{\mathrm{T}}$$

or, equivalently,

$$\overline{\mathbf{L}}\mathbf{U}_0\mathbf{R}^{\mathrm{T}} = \left(\overline{\mathbf{L}}\mathbf{U}_0\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}}.$$

We use the definitions given in equations (7.1) to (7.4).

Then the quantity $\overline{\mathbf{L}}\mathbf{U}_0$ evaluates as

$$\begin{aligned}
\overline{\mathbf{L}}\mathbf{U}_0 &= \left(\overline{L}^i_j \mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\lambda_1 \mathbf{e}_1 \otimes \mathbf{e}^1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}^2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^3\right) \\
&= \lambda_1 \overline{L}^i_j \delta^j_1 \mathbf{e}_i \otimes \mathbf{e}^1 + \lambda_2 \overline{L}^i_j \delta^j_2 \mathbf{e}_i \otimes \mathbf{e}^2 + \lambda_3 \overline{L}^i_j \delta^j_3 \mathbf{e}_i \otimes \mathbf{e}^3 \\
&= \lambda_1 \overline{L}^i_1 \mathbf{e}_i \otimes \mathbf{e}^1 + \lambda_2 \overline{L}^i_2 \mathbf{e}_i \otimes \mathbf{e}^2 + \lambda_3 \overline{L}^i_3 \mathbf{e}_i \otimes \mathbf{e}^3 \\
&= \lambda_{(j)} \overline{L}^i_j \mathbf{e}_i \otimes \mathbf{e}^j
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mathbf{L}}\mathbf{U}_0\mathbf{R}^{\mathrm{T}} &= \left(\lambda_{(j)} \overline{L}^i_j \mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\sum_{kl} R^l_k \mathbf{e}_k \otimes \mathbf{e}^l\right) \\
&= \sum_{kl} \lambda_{(j)} \overline{L}^i_j R^l_k \delta^j_k \mathbf{e}_i \otimes \mathbf{e}^l \\
&= \sum_{kl} \lambda_{(k)} \overline{L}^i_k R^l_k \mathbf{e}_i \otimes \mathbf{e}^l \\
&= \sum_{kj} \lambda_{(k)} \overline{L}^i_k R^j_k \mathbf{e}_i \otimes \mathbf{e}^j.
\end{aligned}$$

The quantity $\left(\overline{\mathbf{L}}\mathbf{U}_0\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}}$ evaluates as

$$\left(\overline{\mathbf{L}}\mathbf{U}_0\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}} = \sum_{kj} \lambda_{(k)} \overline{L}^j_k R^i_k \mathbf{e}_i \otimes \mathbf{e}^j.$$

The symmetry equation is satisfied if

$$\sum_{kj} \lambda_{(k)} \overline{L}^j_k R^i_k \mathbf{e}_i \otimes \mathbf{e}^j = \sum_{kj} \lambda_{(k)} \overline{L}^i_k R^j_k \mathbf{e}_i \otimes \mathbf{e}^j.$$

Thus corresponding components can be set equal to each other to give

$$\sum_k \lambda_{(k)} \overline{L}^j_k R^i_k = \sum_k \lambda_{(k)} \overline{L}^i_k R^j_k$$

for all possible choices of $i$ and $j$. This gives rise to nine equations. In fact there are only three independent equations since swapping the roles of a given $i$ and $j$ produces the same equation and in the case $i = j$, the equation is trivially true. Hence we need only consider the cases $(i,j) = (1,2)$, $(i,j) = (1,3)$ and $(i,j) = (2,3)$.

For $(i,j) = (1,2)$ we obtain

$$\sum_k \lambda_{(k)} \overline{L}^2_k R^1_k = \sum_k \lambda_{(k)} \overline{L}^1_k R^2_k$$

or

$$\lambda_1 \overline{L}^2_1 R^1_1 + \lambda_2 \overline{L}^2_2 R^1_2 + \lambda_3 \overline{L}^2_3 R^1_3 = \lambda_1 \overline{L}^1_1 R^2_1 + \lambda_2 \overline{L}^1_2 R^2_2 + \lambda_3 \overline{L}^1_3 R^2_3 \tag{7.11}$$

For $(i,j) = (1,3)$ we obtain

$$\sum_k \lambda_{(k)} \overline{L}^3_k R^1_k = \sum_k \lambda_{(k)} \overline{L}^1_k R^3_k$$

or

$$\lambda_1 \overline{L}^3_1 R^1_1 + \lambda_2 \overline{L}^3_2 R^1_2 + \lambda_3 \overline{L}^3_3 R^1_3 = \lambda_1 \overline{L}^1_1 R^3_1 + \lambda_2 \overline{L}^1_2 R^3_2 + \lambda_3 \overline{L}^1_3 R^3_3 \tag{7.12}$$

For $(i,j) = (2,3)$ we obtain

$$\sum_k \lambda_{(k)} \overline{L}^3_k R^2_k = \sum_k \lambda_{(k)} \overline{L}^2_k R^3_k$$

or

$$\lambda_1 \overline{L}^3_1 R^2_1 + \lambda_2 \overline{L}^3_2 R^2_2 + \lambda_3 \overline{L}^3_3 R^2_3 = \lambda_1 \overline{L}^2_1 R^3_1 + \lambda_2 \overline{L}^2_2 R^3_2 + \lambda_3 \overline{L}^2_3 R^3_3. \tag{7.13}$$

We now prove $\mathbf{R}^T\overline{\mathbf{L}}\mathbf{U}_0$ is symmetric using the symmetry of $\mathbf{L}$. This is shown in the following lemma.

**Lemma 58** *If* $\mathbf{L}$ *and* $\overline{\mathbf{L}}$ *are second order tensors related by* $\mathbf{L} = \mathbf{R}\mathbf{U}_0\overline{\mathbf{L}}^T$ *where* $\mathbf{R}$ *is a special orthogonal tensor,* $\mathbf{U}_0$ *is a symmetric tensor and* $\mathbf{L}$ *is also symmetric, then* $\mathbf{R}^T\overline{\mathbf{L}}\mathbf{U}_0$ *is symmetric.*

**Proof.** From $\mathbf{L} = \mathbf{R}\mathbf{U}_0\overline{\mathbf{L}}^{\mathrm{T}}$ we obtain

$$\mathbf{L}^{\mathrm{T}} = \overline{\mathbf{L}}\mathbf{U}_0^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} = \overline{\mathbf{L}}\mathbf{U}_0\mathbf{R}^{\mathrm{T}}$$

using the fact $\mathbf{U}_0$ is a symmetric tensor. Multiplying on the right by $\mathbf{R}$ and on the left by $\mathbf{R}^{\mathrm{T}}$ gives

$$\mathbf{R}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{R} = \mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0$$

which uses the fact $\mathbf{R}$ is an orthogonal tensor. Since $\mathbf{L}$ is symmetric, this may be written as

$$\mathbf{R}^{\mathrm{T}}\mathbf{L}\mathbf{R} = \mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0.$$

But

$$\left(\mathbf{R}^{\mathrm{T}}\mathbf{L}\mathbf{R}\right)^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\left(\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{L}\mathbf{R}$$

since $\mathbf{L}$ is symmetric. Thus $\mathbf{R}^{\mathrm{T}}\mathbf{L}\mathbf{R}$ must also be symmetric. But if $\mathbf{R}^{\mathrm{T}}\mathbf{L}\mathbf{R}$ is symmetric then $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0$ is symmetric. ■

Now derive the component forms of this symmetry. We continue to use the definitions given in equations (7.1) to (7.4).

Evaluate $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ as

$$\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \left(\sum_{il} R^l_i \mathbf{e}_i \otimes \mathbf{e}^l\right)\left(\overline{L}^k_j \mathbf{e}_k \otimes \mathbf{e}^j\right) = \sum_{il} R^l_i \overline{L}^k_j \delta^l_k \mathbf{e}_i \otimes \mathbf{e}^j = \sum_{ik} R^k_i \overline{L}^k_j \mathbf{e}_i \otimes \mathbf{e}^j \tag{7.14}$$

and evaluate $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0$ as

$$\begin{aligned}
\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0 &= \left(\sum_{ik} R^k_i \overline{L}^k_l \mathbf{e}_i \otimes \mathbf{e}^l\right)\left(\lambda_{(j)}\mathbf{e}_j \otimes \mathbf{e}^j\right) \\
&= \sum_{ik} R^k_i \overline{L}^k_l \lambda_{(j)} \delta^l_j \mathbf{e}_i \otimes \mathbf{e}^j \\
&= \sum_{ik} R^k_i \overline{L}^k_j \lambda_{(j)} \mathbf{e}_i \otimes \mathbf{e}^j.
\end{aligned}$$

Since $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0$ is symmetric, then

$$\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0 = \left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0\right)^{\mathrm{T}}.$$

The expression $\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0\right)^{\mathrm{T}}$ is given by

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\mathbf{U}_0\right)^{\mathrm{T}} = \sum_{ik} R_j^k \overline{L}_i^k \lambda_{(i)} \mathbf{e}_i \otimes \mathbf{e}^j.$$

We may now equate components. By the same argument as above, it is sufficient to consider only the cases $(i, j) = (1, 2)$ , $(i, j) = (1, 3)$ and $(i, j) = (2, 3)$. The component equation for $(i, j) = (1, 2)$ is given as

$$\left(R_1^1\overline{L}_2^1 + R_1^2\overline{L}_2^2 + R_1^3\overline{L}_2^3\right)\lambda_2 = \left(R_2^1\overline{L}_1^1 + R_2^2\overline{L}_1^2 + R_2^3\overline{L}_1^3\right)\lambda_1, \tag{7.15}$$

the component equation for $(i, j) = (1, 3)$ is given as

$$\left(R_1^1\overline{L}_3^1 + R_1^2\overline{L}_3^2 + R_1^3\overline{L}_3^3\right)\lambda_3 = \left(R_3^1\overline{L}_1^1 + R_3^2\overline{L}_1^2 + R_3^3\overline{L}_1^3\right)\lambda_1 \tag{7.16}$$

and the component equation for $(i, j) = (2, 3)$ is given as

$$\left(R_2^1\overline{L}_3^1 + R_2^2\overline{L}_3^2 + R_2^3 L_3^3\right)\lambda_3 = \left(R_3^1\overline{L}_2^1 + R_3^2\overline{L}_2^2 + R_3^3\overline{L}_2^3\right)\lambda_2. \tag{7.17}$$

### 7.4.2 Symmetry Equations Of $\overline{\mathbf{N}}_0$ and $\overline{\mathbf{M}}_0$

We now derive component equations of equation (6.28) which is repeated here for convenience:

$$\mathbf{W}_0\mathbf{U}_0\overline{\mathbf{M}}_0^{\mathrm{T}} + \mathbf{U}_0\overline{\mathbf{N}}_0^{\mathrm{T}} = \overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{W}_0.$$

For our purposes, it is more convenient to express this as

$$\mathbf{W}_0\mathbf{U}_0\overline{\mathbf{M}}_0^{\mathrm{T}} + \mathbf{U}_0\overline{\mathbf{N}}_0^{\mathrm{T}} = \left(\mathbf{W}_0\mathbf{U}_0\overline{\mathbf{M}}_0^{\mathrm{T}} + \mathbf{U}_0\overline{\mathbf{N}}_0^{\mathrm{T}}\right)^{\mathrm{T}}$$

or, equivalently

$$\overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{W}_0 = \left(\overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{W}_0\right)^{\mathrm{T}}.$$

The various quantities are those as defined in equations (7.1) and (7.5) through (7.7).

The two terms in the expression evaluate as

$$\begin{aligned}
\overline{\mathbf{N}}_0\mathbf{U}_0 &= \left(\overline{N}^i_j \mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\lambda_{(k)}\delta^k_l \mathbf{e}_k \otimes \mathbf{e}^l\right) \\
&= \overline{N}^i_j \lambda_{(k)}\delta^k_l \left(\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\mathbf{e}_k \otimes \mathbf{e}^l\right) \\
&= \overline{N}^i_j \lambda_{(k)}\delta^k_l \delta^j_k \mathbf{e}_i \otimes \mathbf{e}^l \\
&= \overline{N}^i_j \lambda_{(l)}\delta^j_l \mathbf{e}_i \otimes \mathbf{e}^l \\
&= \overline{N}^i_j \lambda_{(j)} \mathbf{e}_i \otimes \mathbf{e}^j
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{W}_0 &= W_0\left(\overline{M}^i_j \mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\lambda_2 \mathbf{e}_2 \otimes \mathbf{e}^1 - \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}^2\right) \\
&= W_0\lambda_2 \overline{M}^i_j \left(\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\mathbf{e}_2 \otimes \mathbf{e}^1\right) - W_0\lambda_1 \overline{M}^i_j \left(\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\mathbf{e}_1 \otimes \mathbf{e}^2\right) \\
&= W_0\lambda_2 \overline{M}^i_j \delta^j_2 \mathbf{e}_i \otimes \mathbf{e}^1 - W_0\lambda_1 \overline{M}^i_j \delta^j_1 \mathbf{e}_i \otimes \mathbf{e}^2 \\
&= W_0\lambda_2 \overline{M}^i_2 \mathbf{e}_i \otimes \mathbf{e}^1 - W_0\lambda_1 \overline{M}^i_1 \mathbf{e}_i \otimes \mathbf{e}^2 \\
&= W_0\lambda_2 \overline{M}^i_2 \delta^1_j \mathbf{e}_i \otimes \mathbf{e}^j - W_0\lambda_1 \overline{M}^i_1 \delta^2_j \mathbf{e}_i \otimes \mathbf{e}^j \\
&= \left(W_0\lambda_2 \overline{M}^i_2 \delta^1_j - W_0\lambda_1 \overline{M}^i_1 \delta^2_j\right) \mathbf{e}_i \otimes \mathbf{e}^j
\end{aligned}$$

so

$$\overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{W}_0 = \left(\overline{N}^i_j \lambda_{(j)} + W_0\lambda_1 \overline{M}^i_1 \delta^2_j - W_0\lambda_2 \overline{M}^i_2 \delta^1_j\right) \mathbf{e}_i \otimes \mathbf{e}^j.$$

Then $\left(\overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{W}_0\right)^{\mathrm{T}}$ evaluates as

$$\left(\overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{W}_0\right)^{\mathrm{T}} = \sum_{ij}\left(\overline{N}^j_i \lambda_{(i)} + W_0\lambda_1 \overline{M}^j_1 \delta^2_i - W_0\lambda_2 \overline{M}^j_2 \delta^1_i\right) \mathbf{e}_i \otimes \mathbf{e}^j.$$

Equating components from the two expressions gives

$$\lambda_{(i)}\overline{N}^j_i + W_0\lambda_1 \overline{M}^j_1 \delta^2_i - W_0\lambda_2 \overline{M}^j_2 \delta^1_i = \overline{N}^i_j \lambda_{(j)} + W_0\lambda_1 \overline{M}^i_1 \delta^2_j - W_0\lambda_2 \overline{M}^i_2 \delta^1_j$$

for $i, j = 1, 2, 3$. We expect that if $i = j$, the equations are trivial and no information is obtained. Also, switching the values of a given $i$ and $j$ will result in the same equation. Thus there will be only three independent equations, corresponding to $(i, j) = (1, 2)$, $(i, j) = (1, 3)$ and $(i, j) = (2, 3)$. For $(i, j) = (1, 2)$, the equation is

$$\lambda_1 \overline{N}_1^2 - W_0 \lambda_2 \overline{M}_2^2 = \overline{N}_2^1 \lambda_2 + W_0 \lambda_1 \overline{M}_1^1 \tag{7.18}$$

or

$$\overline{M}_1^1 = \frac{1}{W_0} \overline{N}_1^2 - \frac{\lambda_2}{W_0 \lambda_1} \overline{N}_2^1 - \frac{\lambda_2}{\lambda_1} \overline{M}_2^2, \tag{7.19}$$

for $(i, j) = (1, 3)$, the equation is

$$\lambda_1 \overline{N}_1^3 - W_0 \lambda_2 \overline{M}_2^3 = \lambda_3 \overline{N}_3^1 \tag{7.20}$$

or

$$\overline{M}_2^3 = \frac{\lambda_1}{W_0 \lambda_2} \overline{N}_1^3 - \frac{\lambda_3}{W_0 \lambda_2} \overline{N}_3^1 \tag{7.21}$$

and for $(i, j) = (2, 3)$, the equation is

$$\lambda_2 \overline{N}_2^3 + W_0 \lambda_1 \overline{M}_1^3 = \overline{N}_3^2 \lambda_3 \tag{7.22}$$

or

$$\overline{M}_1^3 = \frac{\lambda_3}{W_0 \lambda_1} \overline{N}_3^2 - \frac{\lambda_2}{W_0 \lambda_1} \overline{N}_2^3. \tag{7.23}$$

If we multiply through the field equation (6.27) on the right by $\mathbf{U}_0$, we get

$$\lambda \rho \mathbf{R}^{\mathrm{T}} \overline{\mathbf{L}} \mathbf{U}_0 = \left( \overline{\mathbf{N}}_0 - \mathbf{W}_0 \overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}} \mathbf{n} \otimes \mathbf{r}'_R \right) \mathbf{U}_0.$$

Since the quantity $\mathbf{R}^{\mathrm{T}} \overline{\mathbf{L}} \mathbf{U}_0$ is symmetric, then $\left( \overline{\mathbf{N}}_0 - \mathbf{W}_0 \overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}} \mathbf{n} \otimes \mathbf{r}'_R \right) \mathbf{U}_0$ must also be symmetric. We now compute the component form of this expression.

First we evaluate $\overline{\mathbf{N}}_0 - \mathbf{W}_0 \overline{\mathbf{M}}_0$ in component form. The expression $\overline{\mathbf{N}}_0 - \mathbf{W}_0 \overline{\mathbf{M}}_0$ computes as

$$
\begin{aligned}
\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 &= \overline{N}^i_j \mathbf{e}_i \otimes \mathbf{e}^j - W_0\left(\mathbf{e}_2 \otimes \mathbf{e}^1 - \mathbf{e}_1 \otimes \mathbf{e}^2\right)\left(\overline{M}^i_j \mathbf{e}_i \otimes \mathbf{e}^j\right) \\
&= \overline{N}^i_j \mathbf{e}_i \otimes \mathbf{e}^j - W_0\overline{M}^i_j\left(\mathbf{e}_2 \otimes \mathbf{e}^1\right)\left(\mathbf{e}_i \otimes \mathbf{e}^j\right) + W_0\overline{M}^i_j\left(\mathbf{e}_1 \otimes \mathbf{e}^2\right)\left(\mathbf{e}_i \otimes \mathbf{e}^j\right) \\
&= \overline{N}^i_j \mathbf{e}_i \otimes \mathbf{e}^j - W_0\overline{M}^i_j\delta^1_i \mathbf{e}_2 \otimes \mathbf{e}^j + W_0\overline{M}^i_j\delta^2_i \mathbf{e}_1 \otimes \mathbf{e}^j \\
&= \overline{N}^i_j \mathbf{e}_i \otimes \mathbf{e}^j - W_0\overline{M}^1_j \mathbf{e}_2 \otimes \mathbf{e}^j + W_0\overline{M}^2_j \mathbf{e}_1 \otimes \mathbf{e}^j \\
&= \overline{N}^i_j \mathbf{e}_i \otimes \mathbf{e}^j - W_0\overline{M}^1_j\delta^i_2 \mathbf{e}_i \otimes \mathbf{e}^j + W_0\overline{M}^2_j\delta^i_1 \mathbf{e}_i \otimes \mathbf{e}^j \\
&= \left(\overline{N}^i_j - W_0\overline{M}^1_j\delta^i_2 + W_0\overline{M}^2_j\delta^i_1\right)\mathbf{e}_i \otimes \mathbf{e}^j.
\end{aligned}
$$

Now consider the expression $\mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R$. The term $\mathbf{r}'_R$ is always $\mathbf{e}^q$ for some fixed $q$ (either one or three). Equation (7.8) gives the component form of $\mathbf{n} \otimes \mathbf{r}'_R$. Then $\mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R$ evaluates as

$$
\begin{aligned}
\mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R &= \left(\sum_{il} R^l_i \mathbf{e}_i \otimes \mathbf{e}^l\right)\left(n^k\delta^q_j \mathbf{e}_k \otimes \mathbf{e}^j\right) \\
&= \sum_{il} R^l_i n^k \delta^q_j \delta^l_k \mathbf{e}_i \otimes \mathbf{e}^j \\
&= \sum_{il} R^l_i n^k \delta^q_j \delta^l_k \mathbf{e}_i \otimes \mathbf{e}^j \\
&= \sum_{ik} R^k_i n^k \delta^q_j \mathbf{e}_i \otimes \mathbf{e}^j.
\end{aligned}
$$

Combining these last two results together, the quantity $\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R$ evaluates as

$$
\begin{aligned}
\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R &= \left(\overline{N}^i_j - W_0\overline{M}^1_j\delta^i_2 + W_0\overline{M}^2_j\delta^i_1\right)\mathbf{e}_i \otimes \mathbf{e}^j - \sum_{ik} R^k_i n^k \delta^q_j \mathbf{e}_i \otimes \mathbf{e}^j \\
&= \left(\overline{N}^i_j - W_0\overline{M}^1_j\delta^i_2 + W_0\overline{M}^2_j\delta^i_1 - \sum_{ik} R^k_i n^k \delta^q_j\right)\mathbf{e}_i \otimes \mathbf{e}^j.
\end{aligned}
$$

Then $\mathbf{X} = \left(\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R\right)\mathbf{U}_0$ is evaluated as

$$
\mathbf{X} = \left(\left(\overline{N}^i_l - W_0\overline{M}^1_l\delta^i_2 + W_0\overline{M}^2_l\delta^i_1 - \sum_{ik} R^k_i n^k \delta^q_l\right)\mathbf{e}_i \otimes \mathbf{e}^l\right)\left(\lambda_{(j)}\mathbf{e}_j \otimes \mathbf{e}^j\right)
$$

$$= \left(\overline{N}^i_l - W_0\overline{M}^1_l\delta^i_2 + W_0\overline{M}^2_l\delta^i_1 - \sum_{ik} R^k_i n^k \delta^q_l\right)\lambda_{(j)}\delta^l_j \mathbf{e}_i \otimes \mathbf{e}^j$$

$$= \left(\overline{N}^i_l\lambda_{(j)}\delta^l_j - W_0\overline{M}^1_l\delta^i_2\lambda_{(j)}\delta^l_j + W_0\overline{M}^2_l\delta^i_1\lambda_{(j)}\delta^l_j - \sum_{ik} R^k_i n^k\lambda_{(j)}\delta^l_j\delta^q_l\right)\mathbf{e}_i \otimes \mathbf{e}^j$$

$$= \left(\overline{N}^i_j\lambda_{(j)} - W_0\overline{M}^1_j\delta^i_2\lambda_{(j)} + W_0\overline{M}^2_j\delta^i_1\lambda_{(j)} - \sum_{ik} R^k_i n^k\lambda_{(j)}\delta^q_j\right)\mathbf{e}_i \otimes \mathbf{e}^j.$$

The quantity $\mathbf{X}$ is symmetric so $\mathbf{X} = \mathbf{X}^{\mathrm{T}}$. Calculate $\mathbf{X}^{\mathrm{T}}$ as

$$\mathbf{X}^{\mathrm{T}} = \sum_{ij}\left(\overline{N}^j_i\lambda_{(i)} - W_0\overline{M}^1_i\delta^j_2\lambda_{(i)} + W_0\overline{M}^2_i\delta^j_1\lambda_{(i)} - \sum_{k} R^k_j n^k\lambda_{(i)}\delta^q_i\right)\mathbf{e}_i \otimes \mathbf{e}^j.$$

We can now equate components. Following the same arguments given earlier, it is sufficient to equate only the cases $(i,j) = (1,2)$, $(i,j) = (1,3)$ and $(i,j) = (2,3)$.

The component equation for $(i,j) = (1,2)$ is given as

$$\overline{N}^1_2\lambda_2 + W_0\overline{M}^2_2\lambda_2 - \sum_k R^k_1 n^k\lambda_2\delta^q_2 = \overline{N}^2_1\lambda_1 - W_0\overline{M}^1_1\lambda_1 - \sum_k R^k_2 n^k\lambda_1\delta^q_1, \tag{7.24}$$

$$q = 1: \left(\overline{N}^1_2 + W_0\overline{M}^2_2\right)\lambda_2 = \left(\overline{N}^2_1 - W_0\overline{M}^1_1 - R^1_2n^1 - R^2_2n^2 - R^3_2n^3\right)\lambda_1,$$

$$q = 3: \left(\overline{N}^1_2 + W_0\overline{M}^2_2\right)\lambda_2 = \left(\overline{N}^2_1 - W_0\overline{M}^1_1\right)\lambda_1,$$

the component equation for $(i,j) = (1,3)$ is given as

$$\overline{N}^1_3\lambda_3 + W_0\overline{M}^2_3\lambda_3 - \sum_k R^k_1 n^k\lambda_3\delta^q_3 = \overline{N}^3_1\lambda_1 - \sum_k R^k_3 n^k\lambda_1\delta^q_1, \tag{7.25}$$

$$q = 1: \left(\overline{N}^1_3 + W_0\overline{M}^2_3\right)\lambda_3 = \left(\overline{N}^3_1 - R^1_3n^1 - R^2_3n^2 - R^3_3n^3\right)\lambda_1,$$

$$q = 3: \left(\overline{N}^1_3 + W_0\overline{M}^2_3 - R^1_1n^1 - R^2_1n^2 - R^3_1n^3\right)\lambda_3 = \overline{N}^3_1\lambda_1,$$

and the component equation for $(i,j) = (2,3)$ is given as

$$\overline{N}^2_3\lambda_3 - W_0\overline{M}^1_3\lambda_3 - \sum_k R^k_2 n^k\lambda_3\delta^q_3 = \overline{N}^3_2\lambda_2 - \sum_k R^k_3 n^k\lambda_2\delta^q_2, \tag{7.26}$$

$$q = 1 : \left(\overline{N}_3^2 - W_0\overline{M}_3^1\right)\lambda_3 = \overline{N}_2^3\lambda_2,$$

$$q = 3 : \left(\overline{N}_3^2 - W_0\overline{M}_3^1 - R_2^1 n^1 - R_2^2 n^2 - R_2^3 n^3\right)\lambda_3 = \overline{N}_2^3\lambda_2.$$

In several cases, it can be shown that $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ is symmetric. From equation (6.27), this implies $\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R$ is symmetric. We can derive component equations for this situation which are simpler than those derived above.

We've already shown

$$\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R = \left(\overline{N}_j^i - W_0\overline{M}_j^1\delta_2^i + W_0\overline{M}_j^2\delta_1^i - \sum_{ik} R_i^k n^k \delta_j^q\right)\mathbf{e}_i\otimes\mathbf{e}^j$$

so

$$\left(\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R\right)^{\mathrm{T}} = \sum_{ij}\left(\overline{N}_i^j - W_0\overline{M}_i^1\delta_2^j + W_0\overline{M}_i^2\delta_1^j - \sum_k R_j^k n^k \delta_i^q\right)\mathbf{e}_i\otimes\mathbf{e}^j.$$

When this quantity is symmetric, this means

$$\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R = \left(\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R\right)^{\mathrm{T}}$$

or as a component equation

$$\overline{N}_j^i - W_0\overline{M}_j^1\delta_2^i + W_0\overline{M}_j^2\delta_1^i - \sum_k R_i^k n^k \delta_j^q = \overline{N}_i^j - W_0\overline{M}_i^1\delta_2^j + W_0\overline{M}_i^2\delta_1^j - \sum_k R_j^k n^k \delta_i^q$$

for $i, j = 1, 2, 3$. Writing the component equations for $(i, j) = (1, 2)$ gives

$$\overline{N}_2^1 + W_0\overline{M}_2^2 - \sum_k R_1^k n^k \delta_2^q = \overline{N}_1^2 - W_0\overline{M}_1^1 - \sum_k R_2^k n^k \delta_1^q, \tag{7.27}$$

for $(i, j) = (1, 3)$ gives

$$\overline{N}_3^1 + W_0\overline{M}_3^2 - \sum_k R_1^k n^k \delta_3^q = \overline{N}_1^3 - \sum_k R_3^k n^k \delta_1^q \tag{7.28}$$

and for $(i,j)=(2,3)$ gives

$$\overline{N}_3^2 - W_0\overline{M}_3^1 - \sum_k R_2^k n^k \delta_3^q = \overline{N}_2^3 - \sum_k R_3^k n^k \delta_2^q. \tag{7.29}$$

Equations (7.27) through (7.29) are a bit simpler than those of equations (7.24) through (7.26). However, equations (7.24) through (7.26) apply in all cases: equations (7.27) through (7.29 ) only apply if it has been shown that $\mathbf{R}^T\overline{\mathbf{L}}$ is symmetric.

## 7.5 Transformations for Monotropic Symmetry

In each of the four deformations, we introduce monotropic symmetry. The only rotations we consider are those about the axes $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. These rotations are denoted as $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ respectively. From equation (5.8), we must have

$$\mathbf{Q}_i = -\mathbf{I}_i + \mathbf{e}_i \otimes \mathbf{e}_i$$

for $i=1,2,3$. Since $\mathbf{e}_i \otimes \mathbf{e}_i = \mathbf{E}_{ii}$ and $\mathbf{I}_i = \mathbf{E}_{jj} + \mathbf{E}_{kk}$, where $i \neq j$, $i \neq k$ and $j \neq k$, we conclude

$$\mathbf{Q}_1 = \mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33},$$

$$\mathbf{Q}_2 = -\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}$$

and

$$\mathbf{Q}_3 = -\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}.$$

We now consider how to monotropic symmetry to normal uniform rods. We have

$$\overline{\mathbf{N}}\left(\mathbf{F},\mathbf{F}'\right) = \mathbf{R}\overline{\mathbf{N}}\left(\mathbf{U}_0,\mathbf{Z}_0\mathbf{U}_0\right) = \mathbf{R}\overline{\mathbf{N}}_0$$

and

$$\overline{\mathbf{M}}\left(\mathbf{F},\mathbf{F}'\right) = \mathbf{R}\overline{\mathbf{M}}\left(\mathbf{U}_0,\mathbf{Z}_0\mathbf{U}_0\right) = \mathbf{R}\overline{\mathbf{M}}_0.$$

Applying these to equations (5.59) through (5.62) gives

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{R}\overline{\mathbf{N}}_0 = \mathbf{R}\overline{\mathbf{N}}_0\mathbf{Q}_{\mathbf{e}}$$

and

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{R}\overline{\mathbf{M}}_0 = \mathbf{R}\overline{\mathbf{M}}_0\mathbf{Q}_{\mathbf{e}}$$

for the symmetric case and

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{R}\overline{\mathbf{N}}_0 = \mathbf{R}\overline{\mathbf{N}}_0\mathbf{Q}_{\mathbf{e}}$$

and

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{R}\overline{\mathbf{M}}_0 = -\mathbf{R}\overline{\mathbf{M}}_0\mathbf{Q}_{\mathbf{e}}$$

for the anti-symmetric case where

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}} = \mathbf{R}^*\mathbf{Q}_{\mathbf{e}}\mathbf{R}^{*\mathrm{T}}$$

and $\mathbf{R}^*$ is some orthogonal tensor. These were valid as an expression of the combined symmetry (using monotropic symmetry) only if

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{R}\mathbf{U}_0 = \mathbf{R}\mathbf{U}_0\mathbf{Q}_{\mathbf{e}}$$

and

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{R}\mathbf{Z}_0\mathbf{U}_0 = \mathbf{R}\mathbf{Z}_0\mathbf{U}_0\mathbf{Q}_{\mathbf{e}}$$

in the symmetric case and

$$\overline{\mathbf{Q}}_{\bar{\mathbf{e}}}\mathbf{R}\mathbf{Z}_0\mathbf{U}_0 = -\mathbf{R}\mathbf{Z}_0\mathbf{U}_0\mathbf{Q}_{\mathbf{e}}$$

in the anti-symmetric case. An obvious choice for $\mathbf{R}^*$ is $\mathbf{R}$ in which case the left side of the equations reduce to

$$\mathbf{R}\mathbf{Q}_{\mathbf{e}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\overline{\mathbf{N}}_0 = \mathbf{R}\mathbf{Q}_{\mathbf{e}}\overline{\mathbf{N}}_0$$

and

$$\mathbf{R}\mathbf{Q}_{\mathbf{e}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\overline{\mathbf{M}}_0 = \mathbf{R}\mathbf{Q}_{\mathbf{e}}\overline{\mathbf{M}}_0$$

and the combined symmetry equations are

$$\mathbf{Q_e}\overline{\mathbf{N}}_0 = \overline{\mathbf{N}}_0\mathbf{Q_e} \tag{7.30}$$

and

$$\mathbf{Q_e}\overline{\mathbf{M}}_0 = \overline{\mathbf{M}}_0\mathbf{Q_e} \tag{7.31}$$

for the symmetric case

$$\mathbf{Q_e}\overline{\mathbf{N}}_0 = \overline{\mathbf{N}}_0\mathbf{Q_e} \tag{7.32}$$

and

$$\mathbf{Q_e}\overline{\mathbf{M}}_0 = -\overline{\mathbf{M}}_0\mathbf{Q_e} \tag{7.33}$$

for the anti-symmetric case. The condition for this to hold is

$$\mathbf{Q_e}\mathbf{U}_0 = \mathbf{U}_0\mathbf{Q_e} \tag{7.34}$$

and

$$\mathbf{Q_e}\mathbf{Z}_0\mathbf{U}_0 = \mathbf{Z}_0\mathbf{U}_0\mathbf{Q_e} \tag{7.35}$$

in the symmetric case and

$$\mathbf{Q_e}\mathbf{Z}_0\mathbf{U}_0 = -\mathbf{Z}_0\mathbf{U}_0\mathbf{Q_e} \tag{7.36}$$

in the anti-symmetric case. We base our application of monotropic symmetry for normal uniform rods on equations (7.30) through (7.36). In the case where the axial vector of the wryness tensor $\mathbf{W}_0$ is parallel to the axis of rotation of $\mathbf{R}$, we may substitute $\mathbf{W}_0$ for $\mathbf{Z}_0$ in equations (7.35) and (7.36).

Since all of the problems we will consider use the same tensor $\mathbf{U}_0$ defined in equation (7.1) and three of the problems use the same $\mathbf{W}_0$ defined in equation (7.5) (the only exception to this has $\mathbf{W}_0 = \mathbf{0}$), we will establish equations (7.34) and (7.35) or (7.36) for the monotropic transformations $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ defined above. We will use $\mathbf{Z}_0 = \mathbf{W}_0$.

The quantity $\mathbf{Q}_1\mathbf{U}_0$ computes as

$$\mathbf{Q}_1\mathbf{U}_0 \quad = \quad (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33})(\lambda_1\mathbf{E}_{11} + \lambda_2\mathbf{E}_{22} + \lambda_3\mathbf{E}_{33})$$

$$
\begin{aligned}
&= \lambda_1 (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) \mathbf{E}_{11} + \lambda_2 (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) \mathbf{E}_{22} + \lambda_3 (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) \mathbf{E}_{33} \\
&= \lambda_1 \mathbf{E}_{11} - \lambda_2 \mathbf{E}_{22} - \lambda_3 \mathbf{E}_{33}.
\end{aligned}
$$

The quantity $\mathbf{U}_0\mathbf{Q}_1$ computes as

$$
\begin{aligned}
\mathbf{U}_0\mathbf{Q}_1 &= (\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}) (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= \lambda_1 \mathbf{E}_{11} (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) + \lambda_2 \mathbf{E}_{22} (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) + \lambda_3 \mathbf{E}_{33} (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= \lambda_1 \mathbf{E}_{11} - \lambda_2 \mathbf{E}_{22} - \lambda_3 \mathbf{E}_{33}.
\end{aligned}
$$

Thus $\mathbf{Q}_1\mathbf{U}_0 = \mathbf{U}_0\mathbf{Q}_1$ so equation (7.34) is true for the monotropic transformation $\mathbf{Q}_1$.

The quantity $\mathbf{Q}_2\mathbf{U}_0$ computes as

$$
\begin{aligned}
\mathbf{Q}_2\mathbf{U}_0 &= (-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) (\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}) \\
&= \lambda_1 (-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) \mathbf{E}_{11} + \lambda_2 (-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) \mathbf{E}_{22} \\
&\quad + \lambda_3 (-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) \mathbf{E}_{33} \\
&= -\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} - \lambda_3 \mathbf{E}_{33}.
\end{aligned}
$$

The quantity $\mathbf{U}_0\mathbf{Q}_2$ computes as

$$
\begin{aligned}
\mathbf{U}_0\mathbf{Q}_2 &= (\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}) (-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= \lambda_1 \mathbf{E}_{11} (-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) + \lambda_2 \mathbf{E}_{22} (-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&\quad + \lambda_3 \mathbf{E}_{33} (-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= -\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} - \lambda_3 \mathbf{E}_{33}.
\end{aligned}
$$

Thus $\mathbf{Q}_2\mathbf{U}_0 = \mathbf{U}_0\mathbf{Q}_2$ so equation (7.34) is true for the monotropic transformation $\mathbf{Q}_2$.

The quantity $\mathbf{Q}_3\mathbf{U}_0$ computes as

$$
\begin{aligned}
\mathbf{Q}_3\mathbf{U}_0 &= (-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) (\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}) \\
&= \lambda_1 (-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) \mathbf{E}_{11} + \lambda_2 (-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) \mathbf{E}_{22} \\
&\quad + \lambda_3 (-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) \mathbf{E}_{33}
\end{aligned}
$$

$$= \quad -\lambda_1\mathbf{E}_{11} - \lambda_2\mathbf{E}_{22} + \lambda_3\mathbf{E}_{33}.$$

The quantity $\mathbf{U}_0\mathbf{Q}_3$ computes as

$$\begin{aligned}
\mathbf{U}_0\mathbf{Q}_3 &= (\lambda_1\mathbf{E}_{11} + \lambda_2\mathbf{E}_{22} + \lambda_3\mathbf{E}_{33})(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) \\
&= \lambda_1\mathbf{E}_{11}(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) + \lambda_2\mathbf{E}_{22}(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) \\
&\quad +\lambda_3\mathbf{E}_{33}(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) \\
&= -\lambda_1\mathbf{E}_{11} - \lambda_2\mathbf{E}_{22} + \lambda_3\mathbf{E}_{33}.
\end{aligned}$$

Thus $\mathbf{Q}_3\mathbf{U}_0 = \mathbf{U}_0\mathbf{Q}_3$ so equation (7.34) is true for the monotropic transformation $\mathbf{Q}_3$.

The quantity $\mathbf{W}_0\mathbf{U}_0$ computes as

$$\begin{aligned}
\mathbf{W}_0\mathbf{U}_0 &= w_0(\mathbf{E}_{21} - \mathbf{E}_{12})(\lambda_1\mathbf{E}_{11} + \lambda_2\mathbf{E}_{22} + \lambda_3\mathbf{E}_{33}) \\
&= w_0(\mathbf{E}_{21}(\lambda_1\mathbf{E}_{11} + \lambda_2\mathbf{E}_{22} + \lambda_3\mathbf{E}_{33}) - \mathbf{E}_{12}(\lambda_1\mathbf{E}_{11} + \lambda_2\mathbf{E}_{22} + \lambda_3\mathbf{E}_{33})) \\
&= w_0(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}).
\end{aligned}$$

The quantity $\mathbf{Q}_1\mathbf{W}_0\mathbf{U}_0$ computes as

$$\begin{aligned}
\mathbf{Q}_1\mathbf{W}_0\mathbf{U}_0 &= (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33})w_0(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) \\
&= w_0(\mathbf{E}_{11}(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) - \mathbf{E}_{22}(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) - \mathbf{E}_{33}(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12})) \\
&= w_0(-\lambda_2\mathbf{E}_{12} - \lambda_1\mathbf{E}_{21}) \\
&= -w_0(\lambda_2\mathbf{E}_{12} + \lambda_1\mathbf{E}_{21}).
\end{aligned}$$

The quantity $\mathbf{W}_0\mathbf{U}_0\mathbf{Q}_1$ computes as

$$\begin{aligned}
\mathbf{W}_0\mathbf{U}_0\mathbf{Q}_1 &= w_0(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12})(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= w_0(\lambda_1\mathbf{E}_{21}(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) - \lambda_2\mathbf{E}_{12}(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33})) \\
&= w_0(\lambda_1\mathbf{E}_{21} + \lambda_2\mathbf{E}_{12}).
\end{aligned}$$

Thus $\mathbf{Q}_1\mathbf{W}_0\mathbf{U}_0 = -\mathbf{W}_0\mathbf{U}_0\mathbf{Q}_1$ so equation (7.36) is true for the monotropic transformation $\mathbf{Q}_1$.

Thus $\mathbf{Q}_1$ may be used as an anti-symmetric monotropic transformation for this choice of $\mathbf{U}_0$ and $\mathbf{W}_0$.

The quantity $\mathbf{Q}_2\mathbf{W}_0\mathbf{U}_0$ computes as

$$\begin{aligned}
\mathbf{Q}_2\mathbf{W}_0\mathbf{U}_0 &= (-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33})\, w_0 \,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) \\
&= w_0\,(-\mathbf{E}_{11}\,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) + \mathbf{E}_{22}\,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) - \mathbf{E}_{33}\,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12})) \\
&= w_0\,(\lambda_2\mathbf{E}_{12} + \lambda_1\mathbf{E}_{21})
\end{aligned}$$

The quantity $\mathbf{W}_0\mathbf{U}_0\mathbf{Q}_2$ computes as

$$\begin{aligned}
\mathbf{W}_0\mathbf{U}_0\mathbf{Q}_2 &= w_0\,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12})\,(-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= w_0\,(\lambda_1\mathbf{E}_{21}\,(-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) - \lambda_2\mathbf{E}_{12}\,(-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33})) \\
&= w_0\,(-\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) \\
&= -w_0\,(\lambda_1\mathbf{E}_{21} + \lambda_2\mathbf{E}_{12})
\end{aligned}$$

Thus $\mathbf{Q}_2\mathbf{W}_0\mathbf{U}_0 = -\mathbf{W}_0\mathbf{U}_0\mathbf{Q}_2$ so equation (7.36) is true for the monotropic transformation $\mathbf{Q}_2$. Thus $\mathbf{Q}_2$ may be used as an anti-symmetric monotropic transformation for this choice of $\mathbf{U}_0$ and $\mathbf{W}_0$.

The quantity $\mathbf{Q}_3\mathbf{W}_0\mathbf{U}_0$ computes as

$$\begin{aligned}
\mathbf{Q}_3\mathbf{W}_0\mathbf{U}_0 &= (-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33})\, w_0 \,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) \\
&= w_0\,(-\mathbf{E}_{11}\,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) - \mathbf{E}_{22}\,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12}) + \mathbf{E}_{33}\,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12})) \\
&= w_0\,(\lambda_2\mathbf{E}_{12} - \lambda_1\mathbf{E}_{21})
\end{aligned}$$

The quantity $\mathbf{W}_0\mathbf{U}_0\mathbf{Q}_3$ computes as

$$\begin{aligned}
\mathbf{W}_0\mathbf{U}_0\mathbf{Q}_3 &= w_0\,(\lambda_1\mathbf{E}_{21} - \lambda_2\mathbf{E}_{12})\,(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) \\
&= w_0\,(\lambda_1\mathbf{E}_{21}\,(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) - \lambda_2\mathbf{E}_{12}\,(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33})) \\
&= w_0\,(-\lambda_1\mathbf{E}_{21} + \lambda_2\mathbf{E}_{12}) \\
&= w_0\,(\lambda_2\mathbf{E}_{12} - \lambda_1\mathbf{E}_{21})
\end{aligned}$$

Thus $\mathbf{Q}_3\mathbf{W}_0\mathbf{U}_0 = \mathbf{W}_0\mathbf{U}_0\mathbf{Q}_3$ so equation (7.35) is true for the monotropic transformation $\mathbf{Q}_3$. Thus $\mathbf{Q}_3$ may be used as a symmetric monotropic transformation for this choice of $\mathbf{U}_0$ and $\mathbf{W}_0$.

Finally, we consider what sets $\mathcal{U}$ and $\mathcal{V}$ may be formed for this choice of $\mathbf{U}_0$ and $\mathbf{W}_0$. Obviously $\mathcal{U}$ can contain only $\mathbf{I}$ and $\mathbf{Q}_3$ where $\mathbf{I}$ is the identity tensor and $\mathcal{V}$ can only contain $\mathbf{Q}_1$ and $\mathbf{Q}_2$. It is straightforward to see

$$\mathbf{Q}_1\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{Q}_2 = \mathbf{Q}_3\mathbf{Q}_3 = \mathbf{I}$$

so each of these monotropic elements is its only inverse, as it must be geometrically (two rotations of $\pi$ radians about a fixed axis must return a vector to itself). We now compute $\mathbf{Q}_i\mathbf{Q}_j$ for $i \neq j$ to see what values these give. We illustrate the calculation for $(i,j) = (1,2)$ and give the other products in a table. The quantity $\mathbf{Q}_1\mathbf{Q}_2$ computes as

$$\begin{aligned}
\mathbf{Q}_1\mathbf{Q}_2 &= (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33})(-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= \mathbf{E}_{11}(-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) - \mathbf{E}_{22}(-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) - \mathbf{E}_{33}(-\mathbf{E}_{11} + \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= -\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33} \\
&= \mathbf{Q}_3.
\end{aligned}$$

Using analogous calculations, the following table shows all the products among the three elements $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$:

| | $\mathbf{I}$ | $\mathbf{Q}_1$ | $\mathbf{Q}_2$ | $\mathbf{Q}_3$ |
|---|---|---|---|---|
| $\mathbf{I}$ | $\mathbf{I}$ | $\mathbf{Q}_1$ | $\mathbf{Q}_2$ | $\mathbf{Q}_3$ |
| $\mathbf{Q}_1$ | $\mathbf{Q}_1$ | $\mathbf{I}$ | $\mathbf{Q}_3$ | $\mathbf{Q}_2$ |
| $\mathbf{Q}_2$ | $\mathbf{Q}_2$ | $\mathbf{Q}_3$ | $\mathbf{I}$ | $\mathbf{Q}_1$ |
| $\mathbf{Q}_3$ | $\mathbf{Q}_3$ | $\mathbf{Q}_2$ | $\mathbf{Q}_1$ | $\mathbf{I}$ |

Table 6.2

By examining the products in the Table 6.2, we can see that in order to keep $\mathcal{U}$ and $\mathcal{U} \cup \mathcal{V}$ as groups, the only way of defining these sets is as follows:

$$\mathcal{U} = \{\mathbf{I}\} \text{ and } \mathcal{V} \text{ empty},$$

$$\mathcal{U} = \{\mathbf{I}\} \text{ and } \mathcal{V} = \{\mathbf{Q}_1\},$$

$$\mathcal{U} = \{\mathbf{I}\} \text{ and } \mathcal{V} = \{\mathbf{Q}_2\}$$

$$\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_3\} \text{ and } \mathcal{V} \text{ empty}$$

and

$$\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_3\} \text{ and } \mathcal{V} = \{\mathbf{Q}_1, \mathbf{Q}_2\}.$$

With this information, we can calculate all the possible effects of monotropic symmetry on the components of $\overline{\mathbf{N}}_0$ and $\overline{\mathbf{M}}_0$ using the monotropic transformations $\mathbf{Q}_1, \mathbf{Q}_2$ and $\mathbf{Q}_3$ and continuing to use the same choice of $\mathbf{U}_0$ and $\mathbf{W}_0$.

To facilitate the calculations, define the three indexed quantities $\xi_1$, $\xi_2$ and $\xi_3$ whose values depend on which monotropic element is being considered. We can express an monotropic transformation $\mathbf{Q}_p$ as

$$\mathbf{Q}_p = \xi_{(i)} \mathbf{e}_i \otimes \mathbf{e}^i$$

where $p \in \{1, 2, 3\}$ . The values of $\xi_{(i)}$ depend on the value of $p$.

Computing $\mathbf{Q}_p \overline{\mathbf{N}}_0$ gives

$$\mathbf{Q}_p \overline{\mathbf{N}}_0 = \left(\xi_{(i)} \mathbf{e}_i \otimes \mathbf{e}^i\right) \left(\overline{N}^k_j \mathbf{e}_k \otimes \mathbf{e}^j\right) = \overline{N}^k_j \xi_{(i)} \delta^i_k \mathbf{e}_i \otimes \mathbf{e}^j = \overline{N}^i_j \xi_{(i)} \mathbf{e}_i \otimes \mathbf{e}^j$$

and computing $\overline{\mathbf{N}}_0 \mathbf{Q}_p$ gives

$$\overline{\mathbf{N}}_0 \mathbf{Q}_p = \left(\overline{N}^i_k \mathbf{e}_i \otimes \mathbf{e}^k\right) \left(\xi_{(j)} \mathbf{e}_j \otimes \mathbf{e}^j\right) = \overline{N}^i_k \xi_{(j)} \delta^k_j \mathbf{e}_i \otimes \mathbf{e}^j = \overline{N}^i_j \xi_{(j)} \mathbf{e}_i \otimes \mathbf{e}^j.$$

Then the monotropic equation is

$$\overline{N}^i_j \xi_{(i)} \mathbf{e}_i \otimes \mathbf{e}^j = \overline{N}^i_j \xi_{(j)} \mathbf{e}_i \otimes \mathbf{e}^j$$

and equating components gives

$$\overline{N}^i_j \xi_{(i)} = \overline{N}^i_j \xi_{(j)}$$

for all values $i, j = 1, 2, 3$. Obviously when $i = j$, the equations contain no information. The remaining six component equations are

$$\overline{N}_2^1\xi_1 = \overline{N}_2^1\xi_2,\ \overline{N}_3^1\xi_1 = \overline{N}_3^1\xi_3,\ \overline{N}_1^2\xi_2 = \overline{N}_1^2\xi_1,$$

$$\overline{N}_3^2\xi_2 = \overline{N}_3^2\xi_3,\ \overline{N}_1^3\xi_3 = \overline{N}_1^3\xi_1 \text{ and } \overline{N}_2^3\xi_3 = \overline{N}_2^3\xi_2.$$

The calculation for $\mathbf{Q}_p\overline{\mathbf{M}}_0 = \overline{\mathbf{M}}_0\mathbf{Q}_p$ is identical to that for $\overline{\mathbf{N}}_0$. This occurs when $p = 3$ since $\mathbf{Q}_3$ is the only monotropic transformation for the symmetric case. The resulting component equations must be

$$\overline{M}_2^1\xi_1 = \overline{M}_2^1\xi_2,\ \overline{M}_3^1\xi_1 = \overline{M}_3^1\xi_3,\ \overline{M}_1^2\xi_2 = \overline{M}_1^2\xi_1,$$

$$\overline{M}_3^2\xi_2 = \overline{M}_3^2\xi_3,\ \overline{M}_1^3\xi_3 = \overline{M}_1^3\xi_1 \text{ and } \overline{M}_2^3\xi_3 = \overline{M}_2^3\xi_2.$$

The calculation for $\mathbf{Q}_p\overline{\mathbf{M}}_0 = -\overline{\mathbf{M}}_0\mathbf{Q}_p$ is identical to that for $\overline{\mathbf{N}}_0$ except for the presence of the negative sign on the right side. This occurs when $p = 1$ or $p = 2$ since $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the monotropic transformations for the anti-symmetric case. The resulting component equation must be

$$\overline{M}_j^i\xi_{(i)} = -\overline{M}_j^i\xi_{(j)}$$

which gives the six component equations

$$\overline{M}_2^1\xi_1 = -\overline{M}_2^1\xi_2,\ \overline{M}_3^1\xi_1 = -\overline{M}_3^1\xi_3,\ \overline{M}_1^2\xi_2 = -\overline{M}_1^2\xi_1,$$

$$\overline{M}_3^2\xi_2 = -\overline{M}_3^2\xi_3,\ \overline{M}_1^3\xi_3 = -\overline{M}_1^3\xi_1,\ \overline{M}_2^3\xi_3 = -\overline{M}_2^3\xi_2$$

However, we also must consider the case when $i = j$ and this gives

$$\overline{M}_1^1 = -\overline{M}_1^1,\ \overline{M}_2^2 = -\overline{M}_2^2 \text{ and } \overline{M}_3^3 = -\overline{M}_3^3$$

which implies $\overline{M}_1^1 = \overline{M}_2^2 = \overline{M}_3^3 = 0$.

For $p = 1$, that is, using monotropic element $\mathbf{Q}_1$, the values of $\xi_{(i)}$ are assigned as $\xi_1 = 1$, $\xi_2 = -1$ and $\xi_3 = -1$. This gives the six component equations of $\overline{\mathbf{N}}_0$ as

$$\overline{N}_2^1 = -\overline{N}_2^1,\ \overline{N}_3^1 = -\overline{N}_3^1,\ -\overline{N}_1^2 = \overline{N}_1^2\ ,\ -\overline{N}_3^2 = -\overline{N}_3^2,\ -\overline{N}_1^3 = \overline{N}_1^3,\ -\overline{N}_2^3 = -\overline{N}_2^3$$

from which we can conclude

$$\overline{N}_2^1 = \overline{N}_3^1 = \overline{N}_1^2 = \overline{N}_1^3 = 0. \tag{7.37}$$

The component equations of $\overline{\mathbf{M}}_0$ are

$$\overline{M}_2^1 = \overline{M}_2^1,\ \overline{M}_3^1 = \overline{M}_3^1,\ -\overline{M}_1^2 = -\overline{M}_1^2,\ -\overline{M}_3^2 = \overline{M}_3^2,\ -\overline{M}_1^3 = -\overline{M}_1^3,\ -\overline{M}_2^3 = \overline{M}_2^3,$$

$$\overline{M}_1^1 = -\overline{M}_1^1,\ \overline{M}_2^2 = -\overline{M}_2^2 \text{ and } \overline{M}_3^3 = -\overline{M}_3^3$$

from which we can conclude

$$\overline{M}_3^2 = \overline{M}_2^3 = \overline{M}_1^1 = \overline{M}_2^2 = \overline{M}_3^3 = 0. \tag{7.38}$$

For $p = 2$, that is, using monotropic element $\mathbf{Q}_2$, the values of $\xi_{(i)}$ are assigned as $\xi_1 = -1$, $\xi_2 = 1$ and $\xi_3 = -1$. This gives the six component equations of $\overline{\mathbf{N}}_0$ as

$$-\overline{N}_2^1 = \overline{N}_2^1,\ -\overline{N}_3^1 = -\overline{N}_3^1,\ \overline{N}_1^2 = -\overline{N}_1^2\ ,\ \overline{N}_3^2 = -\overline{N}_3^2,\ -\overline{N}_1^3 = -\overline{N}_1^3 \text{ and } -\overline{N}_2^3 = \overline{N}_2^3$$

from which we can conclude

$$\overline{N}_2^1 = \overline{N}_3^2 = \overline{N}_1^2 = \overline{N}_2^3 = 0. \tag{7.39}$$

The component equations of $\overline{\mathbf{M}}_0$ are

$$-\overline{M}_2^1 = -\overline{M}_2^1,\ -\overline{M}_3^1 = \overline{M}_3^1,\ \overline{M}_1^2 = \overline{M}_1^2\ ,\ \overline{M}_3^2 = \overline{M}_3^2,\ -\overline{M}_1^3 = \overline{M}_1^3,\ -\overline{M}_2^3 = -\overline{M}_2^3$$

$$\overline{M}_1^1 = -\overline{M}_1^1,\ \overline{M}_2^2 = -\overline{M}_2^2 \text{ and } \overline{M}_3^3 = -\overline{M}_3^3$$

from which we can conclude

$$\overline{M}_3^1 = \overline{M}_1^3 = \overline{M}_1^1 = \overline{M}_2^2 = \overline{M}_3^3 = 0. \tag{7.40}$$

For $p = 3$, that is, using monotropic element $\mathbf{Q}_3$, the values of $\xi_{(i)}$ are assigned as $\xi_1 = -1$,

$\xi_2 = -1$ and $\xi_3 = 1$. This gives the six component equations of $\overline{\mathbf{N}}_0$ as

$$-\overline{N}_2^1 = -\overline{N}_2^1, \ -\overline{N}_3^1 = \overline{N}_3^1, \ -\overline{N}_1^2 = -\overline{N}_1^2, \ -\overline{N}_3^2 = \overline{N}_3^2, \overline{N}_1^3 = -\overline{N}_1^3 \text{ and } \overline{N}_2^3 = -\overline{N}_2^3$$

from which we can conclude

$$\overline{N}_3^1 = \overline{N}_3^2 = \overline{N}_1^3 = \overline{N}_2^3 = 0. \tag{7.41}$$

The component equations of $\overline{\mathbf{M}}_0$ are

$$-\overline{M}_2^1 = -\overline{M}_2^1, \ -\overline{M}_3^1 = \overline{M}_3^1, \ -\overline{M}_1^2 = -\overline{M}_1^2, \ -\overline{M}_3^2 = \overline{M}_3^2, \overline{M}_1^3 = -\overline{M}_1^3 \text{ and } \overline{M}_2^3 = -\overline{M}_2^3$$

from which we can conclude

$$\overline{M}_3^1 = \overline{M}_3^2 = \overline{M}_1^3 = \overline{M}_2^3 = 0. \tag{7.42}$$

# Chapter 8

# Solving Straight to Straight and Straight Twisted

In this chapter, the simplest two deformation problems for normal uniform rods are solved. These are deformations from straight to straight and from straight to straight twisted.

## 8.1 Straight to Straight

The first case is a deformation from a straight configuration to a straight configuration without twisting the rod. This is a particularly simple case. The deformation is illustrated in Figure 8-1.

The curve in the deformed state is defined by $\mathbf{r}\left(s_R\right) = \lambda_3 s_R \mathbf{e}_3$ where $\lambda_3$ is a constant quantity called the stretch (the scalar quantity appearing in the definition of $\mathbf{U}_0$). The orthogonal tensor is given by $\mathbf{R} = \mathbf{I}$ where $\mathbf{I}$ is the identity tensor. It immediately follows that $\mathbf{W}_0 = \mathbf{Z}_0 = \mathbf{0}$. For monotropic symmetry, we use $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_3\}$ and $\mathcal{V} = \{\mathbf{I}, \mathbf{Q}_3\}$.

Equations (6.24) to (6.26) become

$$\lambda_3 \rho \overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{n} \otimes \mathbf{r}'_R, \tag{8.1}$$

$$\overline{\mathbf{N}}_0 \mathbf{U}_0 = \mathbf{U}_0 \overline{\mathbf{N}}_0^T$$

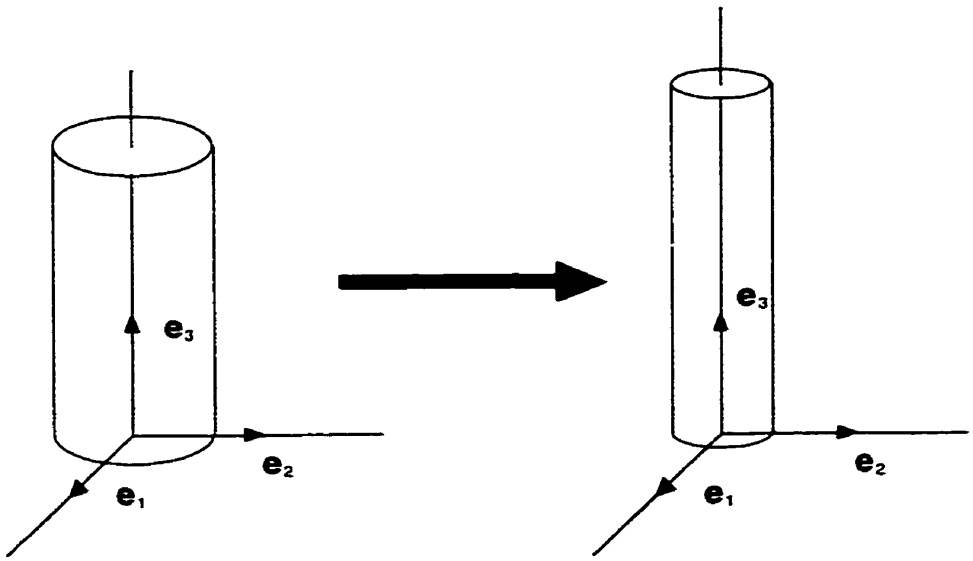


Figure 8-1: Straight to Straight

and

$$\overline{\mathbf{L}}\mathbf{U}_0 = \mathbf{U}_0\overline{\mathbf{L}}^{\mathrm{T}}$$

respectively. Since $\mathbf{U}_0$ is symmetric, these last two equations may be written as

$$\overline{\mathbf{N}}_0\mathbf{U}_0 = \left(\overline{\mathbf{N}}_0\mathbf{U}_0\right)^{\mathrm{T}}$$

and

$$\overline{\mathbf{L}}\mathbf{U}_0 = \left(\overline{\mathbf{L}}\mathbf{U}_0\right)^{\mathrm{T}}.$$

Thus $\overline{\mathbf{N}}_0\mathbf{U}_0$ and $\overline{\mathbf{L}}\mathbf{U}_0$ are both symmetric quantities. This means

$$\left(\overline{L}^i_j\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\lambda_{(k)}\mathbf{e}_k \otimes \mathbf{e}^k\right) = \left(\left(\overline{L}^i_j\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\lambda_{(k)}\mathbf{e}_k \otimes \mathbf{e}^k\right)\right)^{\mathrm{T}},$$

$$\overline{L}^i_j\lambda_{(k)}\left(\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\mathbf{e}_k \otimes \mathbf{e}^k\right) = \left(\overline{L}^i_j\lambda_{(k)}\left(\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\mathbf{e}_k \otimes \mathbf{e}^k\right)\right)^{\mathrm{T}},$$

$$\overline{L}^i_j\lambda_{(k)}\delta^j_k\mathbf{e}_i \otimes \mathbf{e}^k = \left(\overline{L}^i_j\lambda_{(k)}\delta^j_k\mathbf{e}_i \otimes \mathbf{e}^k\right)^{\mathrm{T}},$$

$$\overline{L}^i_j\lambda_{(j)}\mathbf{e}_i \otimes \mathbf{e}^j = \left(\overline{L}^i_j\lambda_{(j)}\mathbf{e}_i \otimes \mathbf{e}^j\right)^{\mathrm{T}}$$

and

$$\overline{L}^i_j \lambda_{(j)} \mathbf{e}_i \otimes \mathbf{e}^j = \overline{L}^j_i \lambda_{(i)} \mathbf{e}_i \otimes \mathbf{e}^j.$$

Equating components on both sides of this equation gives three nontrivial equations; namely

$$\overline{L}^1_2 \lambda_2 = \overline{L}^2_1 \lambda_1,$$

$$\overline{L}^1_3 \lambda_3 = \overline{L}^3_1 \lambda_1 \tag{8.2}$$

and

$$\overline{L}^2_3 \lambda_3 = \overline{L}^3_2 \lambda_2. \tag{8.3}$$

Similarly

$$\overline{N}^1_2 \lambda_2 = \overline{N}^2_1 \lambda_1,$$

$$\overline{N}^1_3 \lambda_3 = \overline{N}^3_1 \lambda_1 \tag{8.4}$$

and

$$\overline{N}^2_3 \lambda_3 = \overline{N}^3_2 \lambda_2. \tag{8.5}$$

If we use $\lambda_1 = \lambda_2 = \lambda$, we obtain $\overline{L}^1_2 = \overline{L}^2_1$ and $\overline{N}^1_2 = \overline{N}^2_1$.

Cohen's stress tensor $\mathbf{L}$ operates only in the plane spanned by $\mathbf{d}_1$ and $\mathbf{d}_2$. In this case, the same plane is spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. This means

$$L^1_3 = L^2_3 = L^3_3 = L^3_1 = L^3_2 = 0$$

which was shown in the previous chapter to imply

$$\overline{L}^3_3 = \overline{L}^3_1 = \overline{L}^3_2 = 0.$$

From equations (8.2) and (8.3), we get $\overline{L}^1_3 = \overline{L}^2_3 = 0$. This means $\overline{\mathbf{L}}$ is symmetric in this case. Then $\overline{\mathbf{N}}_0 - \mathbf{n} \otimes \mathbf{r}'_R$ must also be symmetric.

The value of $q$ is 3 so applying the symmetry equations (7.27) to (7.29) gives

$$\overline{N}^1_2 = \overline{N}^2_1,$$

$$\overline{N}_3^1 - n^1 = \overline{N}_1^3 \tag{8.6}$$

and

$$\overline{N}_3^2 - n^2 = \overline{N}_2^3. \tag{8.7}$$

Also note that

$$\mathbf{R}^T\mathbf{n} \otimes \mathbf{r}'_R = \sum_{ik} \delta_i^k n^k \delta_j^3 \mathbf{e}_i \otimes \mathbf{e}^j = n^1\mathbf{e}_1 \otimes \mathbf{e}^3 + n^2\mathbf{e}_2 \otimes \mathbf{e}^3 + n^3\mathbf{e}_3 \otimes \mathbf{e}^3.$$

The field equation (8.1) can now be written in component form as the following nine equations:

$$\lambda_3 \rho \overline{L}_1^1 = \overline{N}_1^1, \tag{8.8}$$

$$\lambda_3 \rho \overline{L}_2^1 = \overline{N}_2^1, \tag{8.9}$$

$$0 = \overline{N}_3^1 - n^1, \tag{8.10}$$

$$\lambda_3 \rho \overline{L}_1^2 = \overline{N}_1^2, \tag{8.11}$$

$$\lambda_3 \rho \overline{L}_2^2 = \overline{N}_2^2, \tag{8.12}$$

$$0 = \overline{N}_3^2 - n^2, \tag{8.13}$$

$$0 = \overline{N}_1^3, \tag{8.14}$$

$$0 = \overline{N}_2^3 \tag{8.15}$$

and

$$0 = \overline{N}_3^3 - n^3. \tag{8.16}$$

Combining equations (8.4) with (8.14) and (8.5) with (8.15 ), it follows that $\overline{N}_3^1 = \overline{N}_3^2 = 0$. From equations (8.6) and (8.7), this implies $n^1 = n^2 = 0$. Since $n^1 = n^2 = 0$, then $\mathbf{n} = n^3\mathbf{e}_3 = n_0\mathbf{e}_3$ where $n_0$ is the length of $\mathbf{n}$. Then

$$\mathbf{n} \otimes \mathbf{r}'_R = n_0 \mathbf{e}_3 \otimes \mathbf{e}^3.$$

We can now write equation (8.1) as

$$\lambda\rho\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - n_0\mathbf{e}_3 \otimes \mathbf{e}^3$$

and can immediately conclude the following:

$$\overline{L}_1^1 = \frac{\overline{N}_1^1}{\lambda_3\rho},\ \overline{L}_2^1 = \frac{\overline{N}_2^1}{\lambda_3\rho} = \overline{L}_1^2 = \frac{\overline{N}_1^2}{\lambda_3\rho},\ \overline{L}_2^2 = \frac{\overline{N}_2^2}{\lambda_3\rho},$$

$$\overline{N}_3^1 = \overline{N}_3^2 = \overline{N}_1^3 = \overline{N}_2^3 = 0$$

and

$$\overline{N}_3^3 = n_0.$$

Notice up to this point, we have no information about $\overline{\mathbf{M}}_0$ and indeed, up until now, it is completely arbitrary.

To complete the problem, we now introduce monotropic symmetry about the $\mathbf{e}_3$ axis. The condition given as equation (7.34) becomes

$$\mathbf{Q}_3\mathbf{U}_0 = \mathbf{U}_0\mathbf{Q}_3$$

and this condition has been shown to be true in Chapter 7. We now consider the conditions under which

$$\overline{\mathbf{M}}_0\mathbf{Q}_3 = \mathbf{Q}_3\overline{\mathbf{M}}_0, \tag{8.17}$$

$$\overline{\mathbf{M}}_0\mathbf{Q}_3 = -\mathbf{Q}_3\overline{\mathbf{M}}_0 \tag{8.18}$$

and

$$\overline{\mathbf{N}}_0\mathbf{Q}_3 = \mathbf{Q}_3\overline{\mathbf{N}}_0. \tag{8.19}$$

Notice that since $\mathbf{F}' = \mathbf{0}$, all three of these conditions must be satisfied. Summing equations (8.17) and (8.18) gives $\overline{\mathbf{M}}_0\mathbf{Q}_3 = \mathbf{0}$. Evaluating $\overline{\mathbf{M}}_0\mathbf{Q}_3$ gives

$$\begin{aligned}\overline{\mathbf{M}}_0\mathbf{Q}_3 &= \left(\overline{M}_j^i\mathbf{e}_i \otimes \mathbf{e}^j\right)(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}) \\ &= \overline{M}^{ij}\mathbf{E}_{ij}(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33})\end{aligned}$$

$$
\begin{aligned}
&= \overline{M}^{ij}\left(-\mathbf{E}_{ij}\mathbf{E}_{11} - \mathbf{E}_{ij}\mathbf{E}_{22} + \mathbf{E}_{ij}\mathbf{E}_{33}\right) \\
&= \overline{M}^{ij}\left(-\delta_{j1}\mathbf{E}_{i1} - \delta_{j2}\mathbf{E}_{i2} + \delta_{j3}\mathbf{E}_{i3}\right) \\
&= -\overline{M}^{i1}\mathbf{E}_{i1} - \overline{M}^{i2}\mathbf{E}_{i2} + \overline{M}^{i3}\mathbf{E}_{i3}. \qquad (8.20)
\end{aligned}
$$

Equating components on both sides of this equation, we find $\overline{M}^{ij} = 0$. Thus monotropic symmetry about the $\mathbf{e}_3$ axis implies $\overline{\mathbf{M}}_0 \equiv \mathbf{0}$.

Using equation (8.20), we see

$$
\overline{\mathbf{N}}_0\mathbf{Q}_3 = -\overline{N}^{i1}\mathbf{E}_{i1} - \overline{N}^{i2}\mathbf{E}_{i2} + \overline{N}^{i3}\mathbf{E}_{i3}.
$$

Now compute $\mathbf{Q}_3\overline{\mathbf{N}}_0$ as

$$
\begin{aligned}
\mathbf{Q}_3\overline{\mathbf{N}}_0 &= \left(-\mathbf{E}_{11} - \mathbf{E}_{22} + \mathbf{E}_{33}\right)\overline{N}^{ij}\mathbf{E}_{ij} \\
&= -\overline{N}^{ij}\mathbf{E}_{11}\mathbf{E}_{ij} - \overline{N}^{ij}\ \mathbf{E}_{22}\mathbf{E}_{ij} + \overline{N}^{ij}\mathbf{E}_{33}\mathbf{E}_{ij} \\
&= -\overline{N}^{ij}\delta_{1i}\mathbf{E}_{1j} - \overline{N}^{ij}\delta_{2i}\mathbf{E}_{2j} + \overline{N}^{ij}\delta_{3i}\mathbf{E}_{3j} \\
&= -\overline{N}^{1j}\mathbf{E}_{1j} - \overline{N}^{2j}\mathbf{E}_{2j} + \overline{N}^{3j}\mathbf{E}_{3j}.
\end{aligned}
$$

Then equation (8.19) implies

$$
-\overline{N}^{i1}\mathbf{E}_{i1} - \overline{N}^{i2}\mathbf{E}_{i2} + \overline{N}^{i3}\mathbf{E}_{i3} = -\overline{N}^{1j}\mathbf{E}_{1j} - \overline{N}^{2j}\mathbf{E}_{2j} + \overline{N}^{3j}\mathbf{E}_{3j}.
$$

Equating components from both sides of the equation gives four nontrivial equations: $-\overline{N}^{31} = \overline{N}^{31}$, $-\overline{N}^{32} = \overline{N}^{32}$, $\overline{N}^{13} = -\overline{N}^{13}$ and $\overline{N}^{23} = -\overline{N}^{23}$. It follows that $\overline{N}^3_1 = \overline{N}^3_2 = \overline{N}^1_3 = \overline{N}^2_3 = 0$. Notice this is exactly what has already been determined from the field equations.

To summarize,

$$
\overline{\mathbf{M}}_0 = \mathbf{0} \text{ or, equivalently } \overline{M}^i_j{=}0 \text{ for } i, j = 1, 2, 3
$$

and

$$
sk\left(\overline{\mathbf{N}}_0\right) = \mathbf{0} \text{ with } \overline{N}^3_1 = \overline{N}^3_2 = \overline{N}^1_3 = \overline{N}^2_3 = 0.
$$

In terms of $\overline{\mathbf{M}}_0$ and $\overline{\mathbf{N}}_0$,

$$\begin{aligned}\overline{L}_1^1 &= \frac{\overline{N}_1^1}{\lambda_3\rho}, \overline{L}_2^1 = \overline{L}_1^2 = \frac{\overline{N}_2^1}{\lambda_3\rho}, \overline{L}_2^2 = \frac{\overline{N}_2^2}{\lambda_3\rho} \text{ and} \\ \overline{L}_1^3 &= \overline{L}_2^3 = \overline{L}_3^1 = \overline{L}_3^2 = \overline{L}_3^3 = 0\end{aligned}$$

describes the components of the tensor $\overline{\mathbf{L}}$. The vector $\mathbf{n}$ is given by

$$\mathbf{n} = \overline{N}_3^3\mathbf{e}_3.$$

## 8.2 Straight to Straight Twisted

In this deformation, the deformed state is straight, but with a twist which is constant along the rod axis. This case uses $\mathbf{r}_R(s_R) = s_R\mathbf{e}_3$ so $\mathbf{r}'_R(s_R) = \mathbf{e}_3$. Then the stretch factors $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfy $\lambda_1 = \lambda_2 = \lambda$ and the field equation (6.24) is

$$\lambda_3\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{Z}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R.$$

The sets for monotropic symmetry are $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_3\}$ and $\mathcal{V} = \{\mathbf{Q}_1, \mathbf{Q}_2\}$. The orthonogal tensor is given by

$$\begin{aligned}\mathbf{R} &= \cos w_0 s_R \mathbf{e}_1\otimes\mathbf{e}^1 - \sin w_0 s_R\mathbf{e}_1\otimes\mathbf{e}^2 + \sin w_0 s_R\mathbf{e}_2\otimes\mathbf{e}^1 + \cos w_0 s_R\mathbf{e}_2\otimes\mathbf{e}^2 \\ &\quad + \mathbf{e}_3\otimes\mathbf{e}^3\end{aligned}$$

so the quantity $\mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R$ will be

$$\begin{aligned}\mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R &= \left(n^1\cos w_0 s_R + n^2\sin w_0 s_R\right)\mathbf{e}_1\otimes\mathbf{e}^3 \\ &\quad + \left(n^2\cos w_0 s_R - n^1\sin w_0 s_R\right)\mathbf{e}_2\otimes\mathbf{e}^3 + n^3\mathbf{e}_3\otimes\mathbf{e}^3.\end{aligned}$$

Figure 8-2 illustrates this deformation.

We note that the rotation axis of $\mathbf{R}$ (the line along the $\mathbf{e}_3$ vector) is parallel to the axial

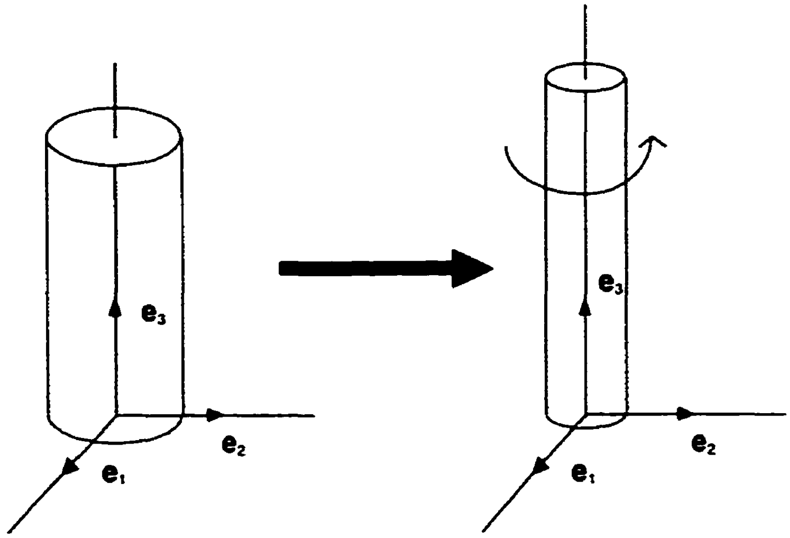


Figure 8-2: Straight to Straight Twisted

vector of $\mathbf{W}_0$. Hence we can use equations (6.27) and (6.28) so the basic field equation becomes

$$\lambda_3 \rho \mathbf{R}^T \overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{W}_0 \overline{\mathbf{M}}_0 - \mathbf{R}^T \mathbf{n} \otimes \mathbf{r}'_R.$$

For this geometry, Cohen's cross-sectional tensor $\mathbf{L}$ satisfies equation (7.9). This was shown to imply equation (7.10), that is, $\overline{L}_3^i = 0$ for $i = 1, 2, 3$. Combining this information with the fact $\lambda_1 = \lambda_2 = \lambda$ and using equations (7.11) to (7.13) and equations (7.15) through (7.17) gives the following component equations:

$$\overline{L}_1^2 \cos w_0 s_R - \overline{L}_2^2 \sin w_0 s_R = \overline{L}_1^1 \sin w_0 s_R + \overline{L}_2^1 \cos w_0 s_R, \tag{8.21}$$

$$\overline{L}_1^3 \cos w_0 s_R - \overline{L}_2^3 \sin w_0 s_R = 0,$$

$$\overline{L}_1^3 \sin w_0 s_R + \overline{L}_2^3 \cos w_0 s_R = 0,$$

$$\overline{L}_2^1 \cos w_0 s_R + \overline{L}_2^2 \sin w_0 s_R = -\overline{L}_1^1 \sin w_0 s_R + \overline{L}_1^2 \cos w_0 s_R, \tag{8.22}$$

$$0 = \overline{L}_1^3 \lambda$$

and

$$0 = \overline{L}_2^3 \lambda.$$

Equations (8.21) and (8.22) are identical. From the remaining equations we have

$$\overline{L}_3^1 = \overline{L}_3^2 = \overline{L}_3^3 = \overline{L}_1^3 = \overline{L}_2^3 = 0,$$

indicating $\overline{\mathbf{L}}$ is a cross-sectional tensor.

Now consider the components of $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$. We are interested to see if $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ is symmetric. We now compute $\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_j^i$ with $i \neq j$ and look for symmetry. For $(i, j) = (1, 2)$, we obtain

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_2^1 = \overline{L}_2^1 \cos w_0 s_R + \overline{L}_2^2 \sin w_0 s_R \text{ and } \left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_1^2 = -\overline{L}_1^1 \sin w_0 s_R + \overline{L}_1^2 \cos w_0 s_R.$$

From equation (8.22), the right sides of these two equations are equal so

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_2^1 = \left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_1^2.$$

For $(i, j) = (1, 3)$ we obtain

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_3^1 = 0 \text{ and } \left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_1^3 = 0$$

so

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_3^1 = \left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_1^3.$$

For $(i, j) = (2, 3)$ we obtain

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_3^2 = 0 \text{ and } \left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_2^3 = 0$$

so

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_3^2 = \left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_2^3.$$

Thus $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ is a symmetric tensor in this case.

Since $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ is symmetric, this implies $\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R$ must be symmetric. This means equations (7.27) to (7.29) may be applied. Using $q = 3$, we get the following component equations:

$$\overline{N}_2^1 + W_0\overline{M}_2^2 = \overline{N}_1^2 - W_0\overline{M}_1^1, \tag{8.23}$$

$$\overline{N}_3^1 + W_0\overline{M}_3^2 - n^1 \cos w_0 s_R - n^2 \sin w_0 s_R = \overline{N}_1^3 \tag{8.24}$$

and

$$\overline{N}_3^2 - W_0\overline{M}_3^1 + n^1 \sin w_0 s_R - n^2 \cos w_0 s_R = \overline{N}_2^3. \tag{8.25}$$

Equation (8.23) is exactly equation (7.18). Now rearrange equation (8.24) as

$$\overline{N}_3^1 + W_0\overline{M}_3^2 - \overline{N}_1^3 = n^1 \cos w_0 s_R + n^2 \sin w_0 s_R.$$

Since everything on the left is constant and $n^1$ and $n^2$ are constants, the only possible values for $n^1$ and $n^2$ which satisfy this equation are $n^1 = n^2 = 0$. This implies

$$\overline{N}_3^1 + W_0\overline{M}_3^2 - \overline{N}_1^3 = 0$$

or

$$\overline{M}_3^2 = \frac{1}{W_0}\left(\overline{N}_1^3 - \overline{N}_3^1\right). \tag{8.26}$$

Consider equation (8.25). Since we've already determined $n^1 = n^2 = 0$, this leaves

$$\overline{N}_3^2 - W_0\overline{M}_3^1 - \overline{N}_2^3 = 0$$

or

$$\overline{M}_3^1 = \frac{1}{W_0}\left(\overline{N}_3^2 - \overline{N}_2^3\right). \tag{8.27}$$

The fact $n^1 = n^2 = 0$ means $\mathbf{n} = n^3\mathbf{e}_3 = n_0\mathbf{e}_3$ where $n_0$ is the length of $\mathbf{n}$ which implies $\mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R = n_0\mathbf{E}_{33}$.

We now write the equation

$$\lambda_3\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - n_0\mathbf{e}_3 \otimes \mathbf{e}^3$$

in component form. Since we know this equation is symmetric, we need only consider the terms indexed $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(2,3)$ and $(3,3)$. These six component equations are as follows.

$$\overline{L}_1^1 \cos w_0 s_R + \overline{L}_1^2 \sin w_0 s_R = \overline{N}_1^1 + W_0\overline{M}_2^2 \tag{8.28}$$

$$\lambda_3\rho\left(\overline{L}_2^1\cos w_0 s_R + \overline{L}_2^2\sin w_0 s_R\right) = \overline{N}_2^1 + W_0\overline{M}_2^2, \tag{8.29}$$

$$0 = \overline{N}_3^1 + W_0\overline{M}_3^2 \tag{8.30}$$

$$\lambda_3\rho\left(\overline{L}_2^2\cos w_0 s_R - \overline{L}_2^1\sin w_0 s_R\right) = \overline{N}_2^2 - W_0\overline{M}_2^1, \tag{8.31}$$

$$0 = \overline{N}_3^2 - W_0\overline{M}_3^1 \tag{8.32}$$

and

$$0 = \overline{N}_3^3 - n^3. \tag{8.33}$$

Using equation (8.26) in equation (8.30) gives

$$0 = \overline{N}_3^1 + W_0\left(\frac{1}{W_0}\left(\overline{N}_1^3 - \overline{N}_3^1\right)\right)$$

which gives

$$\overline{N}_1^3 = 0.$$

Putting this result into equation (8.26) results in

$$\overline{M}_3^2 = -\frac{\overline{N}_3^1}{W_0}.$$

Using equation (8.27) in equation (8.32) gives

$$0 = \overline{N}_3^2 - W_0\left(\frac{1}{W_0}\left(\overline{N}_3^2 - \overline{N}_2^3\right)\right)$$

which gives

$$\overline{N}_2^3 = 0.$$

Putting this result into equation (8.27) gives

$$\overline{M}_3^1 = \frac{\overline{N}_3^2}{W_0}.$$

Since $n_0 = n^3$, then equation (8.33) becomes

$$0 = \overline{N}_3^3 - n_0$$

or

$$n_0 = \overline{N}_3^3. \tag{8.34}$$

From equation (8.34), we have $\mathbf{n} = \overline{N}_3^3 \mathbf{e}_3$ and $\mathbf{R}^T\mathbf{n} \otimes \mathbf{r}'_R = \overline{N}_3^3 \mathbf{e}_3 \otimes \mathbf{e}^3$. The field equation is

$$\lambda_3 \rho \mathbf{R}^T \overline{\mathbf{L}} = \left(\overline{N}_1^1 + W_0 \overline{M}_1^2\right) \mathbf{E}_{11} + \overline{N}_2^1 \mathbf{E}_{12} + \overline{N}_2^1 \mathbf{E}_{21} + \left(\overline{N}_2^2 - W_0 \overline{M}_2^1\right) \mathbf{E}_{22}.$$

At this point, we introduce monotropic symmetry. As in the straight to straight case, we consider the rotation of $\pi$ radians about the $\mathbf{e}_3$ axis, namely $\mathbf{Q}_3$. From equations (7.41) and (7.42), we know

$$\overline{N}_1^3 = \overline{N}_2^3 = \overline{N}_3^1 = \overline{N}_3^2 = 0$$

and

$$\overline{M}_3^1 = \overline{M}_3^2 = \overline{M}_1^3 = \overline{M}_2^3 = 0.$$

Notice this agrees with the result for the components $\overline{N}_1^3$ and $\overline{N}_2^3$ and is consistent with the relationships of $\overline{M}_3^1$ with $\overline{N}_3^2$ and $\overline{M}_3^2$ with $\overline{N}_3^1$ from consideration of the field equations.

In addition to monotropic symmetry about the $\mathbf{e}_3$ axis, we also include monotropic anti-symmetry using the element $\mathbf{Q}_1$ representing rotation of $\pi$ radians about the $\mathbf{e}_1$ axis. From equations (7.37) and (7.38), we know

$$\overline{N}_2^1 = \overline{N}_3^1 = \overline{N}_1^2 = \overline{N}_1^3 = 0$$

and

$$\overline{M}_3^2 = \overline{M}_2^3 = \overline{M}_1^1 = \overline{M}_2^2 = \overline{M}_3^3 = 0.$$

In order to keep the symmetry set a group under composition, we must include the element

$\mathbf{Q}_2$ which represents the rotation about the $\mathbf{e}_2$ axis. From equations (7.39) and (7.40) we know

$$\overline{N}_2^1 = \overline{N}_3^2 = \overline{N}_1^2 = \overline{N}_2^3 = 0$$

and

$$\overline{M}_3^1 = \overline{M}_1^3 = \overline{M}_1^1 = \overline{M}_2^2 = \overline{M}_3^3 = 0,$$

results we have already obtained from looking at the other symmetries. Thus including $\mathbf{Q}_2$ as an anti-symmetry element adds no information, but nevertheless is included to force the combined material symmetry set to be a group.

Combining all these monotropic results, we conclude

$$\overline{N}_2^1 = \overline{N}_3^1 = \overline{N}_1^2 = \overline{N}_3^2 = \overline{N}_1^3 = \overline{N}_2^3 = 0$$

and

$$\overline{M}_3^1 = \overline{M}_3^2 = \overline{M}_1^3 = \overline{M}_2^3 = \overline{M}_1^1 = \overline{M}_2^2 = \overline{M}_3^3 = 0.$$

The components of $\lambda_3\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ are now

$$\lambda_3\rho\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_1^1 = \overline{N}_1^1 + W_0\overline{M}_1^2,$$

$$\lambda_3\rho\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_2^1 = 0,$$

$$\lambda_3\rho\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_1^2 = 0$$

and

$$\lambda_3\rho\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_2^2 = \overline{N}_2^2 - W_0\overline{M}_2^1$$

so

$$\lambda_3\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \left(\overline{N}_1^1 + W_0\overline{M}_1^2\right)\mathbf{E}_{11} + \left(\overline{N}_2^2 - W_0\overline{M}_2^1\right)\mathbf{E}_{22}.$$

To find an expression for $\overline{\mathbf{L}}$, multiply through this equation by $\mathbf{R}$. We omit this calculation here.

# Chapter 9

# Solving Straight to Circular and Helical

This chapter describes the solutions of the deformations from straight to circular and straight to helical. These deformations are more difficult to solve than those of the previous chapter.

## 9.1 Straight to Circular

In this deformation, the deformed state is a circular arc centered at the origin. The rod axis lies in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ having radius $a$. The tensors $\mathbf{W}_0$, $\mathbf{U}_0$ and $\mathbf{R}$ are exactly those given in the previous problem. The value of $W_0 = w_0$ is chosen so that

$$w_0 = \frac{\lambda_2}{a}.$$

This case uses $\mathbf{r}_R(s_R) = s_R\mathbf{e}_1$ so $\mathbf{r}'_R(s_R) = \mathbf{e}_1$. In the deformed configuration, $\mathbf{d}_2$ is chosen as the director to be the tangent vector to the curve (this is different from the other three cases). Then the stretch factors $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfy $\lambda_1 = \lambda_3 = \lambda$ and the field equation (6.24 ) is

$$\lambda_2 \rho \mathbf{R}^{\mathrm{T}} \overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{Z}_0 \overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}} \mathbf{n} \otimes \mathbf{r}'_R.$$

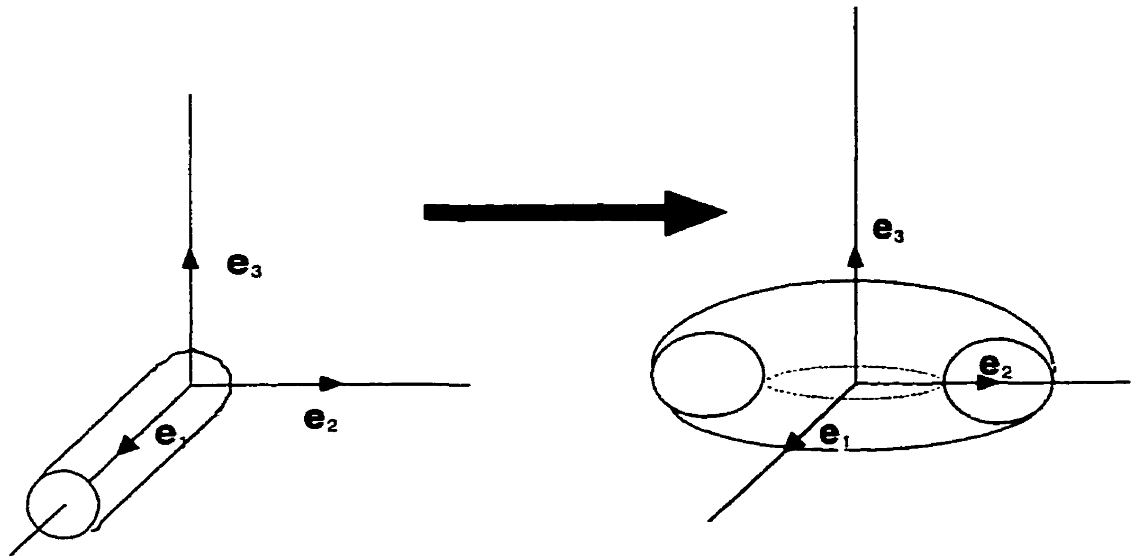


Figure 9-1: Straight to Circular

The sets for monotropic symmetry are $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_3\}$ and $\mathcal{V} = \{\mathbf{Q}_1, \mathbf{Q}_2\}$. The quantity $\mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R$ will be

$$\begin{aligned}\mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R &= \left(n^1\cos w_0 s_R + n^2 \sin w_0 s_R\right)\mathbf{E}_{11} \\ &\quad + \left(n^2\cos w_0 s_R - n^1 \sin w_0 s_R\right)\mathbf{E}_{21} + n^3\mathbf{E}_{31}.\end{aligned}$$

This deformation is illustrated in Figure 9-1.

We note that the rotation axis of $\mathbf{R}$ (the line along the $\mathbf{e}_3$ vector) is parallel to the axial vector of $\mathbf{W}_0$. Hence we can use equations (6.27) and (6.28) so the basic field equation becomes

$$\lambda_2\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R.$$

For some aspects of this case, it is simpler to use cylindrical polar coordinates $(r, \theta, z)$ with basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$. The relationship between the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$ is given by (Cohen [1])

$$\mathbf{e}_r = (\cos w_0 s_R)\,\mathbf{e}_1 + (\sin w_0 s_R)\,\mathbf{e}_2,$$

$$\mathbf{e}_\theta = -(\sin w_0 s_R)\,\mathbf{e}_1 + (\cos w_0 s_R)\,\mathbf{e}_2$$

and

$$\mathbf{e}_3 = \mathbf{e}_3.$$

These are easily inverted to give

$$\mathbf{e}_1 = (\cos w_0 s_R)\,\mathbf{e}_r - (\sin w_0 s_R)\,\mathbf{e}_\theta,$$

$$\mathbf{e}_2 = (\sin w_0 s_R)\,\mathbf{e}_r + (\cos w_0 s_R)\,\mathbf{e}_\theta$$

and

$$\mathbf{e}_3 = \mathbf{e}_3.$$

Note that $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$ is an orthonormal set, so the normality conditions for normal uniform rods are still be described by this set.

In the deformed configuration, the equation of the rod axis is given by

$$\mathbf{r}\,(s_R) = a\mathbf{R}\mathbf{e}_1$$

so

$$\mathbf{r}'\,(s_R) = a\mathbf{R}'\mathbf{e}_1.$$

The term $\mathbf{R}\mathbf{e}_1$ evaluates as

$$\begin{aligned}
\mathbf{R}\mathbf{e}_1 &= \left((\cos w_0 s_R)\,\mathbf{I}_3^\perp + (\sin w_0 s_R)\,\mathbf{A}_3^\perp + \mathbf{E}_{33}\right)\mathbf{e}_1 \\
&= (\cos w_0 s_R)\,\mathbf{I}_3^\perp\mathbf{e}_1 + (\sin w_0 s_R)\,\mathbf{A}_3^\perp\mathbf{e}_1 + \mathbf{E}_{33}\mathbf{e}_1 \\
&= (\cos w_0 s_R)\,\mathbf{e}_1 + (\sin w_0 s_R)\,\mathbf{e}_2 \\
&= \mathbf{e}_r
\end{aligned}$$

so in fact $\mathbf{r}\,(s_R) = a\mathbf{e}_r$. Then $\mathbf{r}'\,(s_R) = a\mathbf{R}'\mathbf{e}_1 = a\mathbf{e}_r' = aw_0\mathbf{e}_\theta = \lambda_2\mathbf{e}_\theta$ (using $w_0 = \lambda_2/a$) so the tangent vector to the rod axis in the deformed state is $\mathbf{e}_\theta$.

In terms of the directors, this means

$$\mathbf{d}_1 = \lambda_1\mathbf{e}_r, \mathbf{d}_2 = \lambda_2\ \ \mathbf{e}_\theta, \mathbf{d}_3 = \lambda_3\mathbf{e}_3.$$

Then the deformation tensor is

$$\begin{aligned}\mathbf{F} &= \mathbf{d}_1 \otimes \mathbf{d}_R^1 + \mathbf{d}_2 \otimes \mathbf{d}_R^2 + \mathbf{d}_3 \otimes \mathbf{d}_R^3 \\ &= \lambda_1 \mathbf{e}_r \otimes \mathbf{e}^1 + \lambda_2 \mathbf{e}_\theta \otimes \mathbf{e}^2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^3\end{aligned}$$

and using the above relationships between $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, this gives

$$\begin{aligned}\mathbf{F} &= \lambda_1 (\cos w_0 s_R \mathbf{e}_1 + \sin w_0 s_R \mathbf{e}_2) \otimes \mathbf{e}^1 + \lambda_2 (-\sin w_0 s_R \mathbf{e}_1 + \cos w_0 s_R \mathbf{e}_2) \otimes \mathbf{e}^2 \\ &\quad + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^3 \\ &= \lambda_1 \cos w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^1 + \lambda_1 \sin w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^1 - \lambda_2 \sin w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^2 \\ &\quad + \lambda_2 \cos w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^3\end{aligned}$$

The polar decomposition of this tensor is $\mathbf{F} = \mathbf{R}\mathbf{U}_0$ where $\mathbf{R}$ is given as previously and $\mathbf{U}_0$ is as defined in equation (7.1).

Evidently we can also establish a cylindrical polar coordinate basis for the second order tensors. The members of this basis are denoted

$$\mathcal{CPC} = \{\mathbf{E}_{rr}, \mathbf{E}_{r\theta}, \mathbf{E}_{r3}, \mathbf{E}_{\theta r}, \mathbf{E}_{\theta\theta}, \mathbf{E}_{\theta 3}, \mathbf{E}_{3r}, \mathbf{E}_{3\theta}, \mathbf{E}_{33}\}$$

and are defined by

$$\mathbf{E}_{rr} = \mathbf{e}_r \otimes \mathbf{e}_r, \mathbf{E}_{r\theta} = \mathbf{e}_r \otimes \mathbf{e}_\theta, \mathbf{E}_{r3} = \mathbf{e}_r \otimes \mathbf{e}_3,$$

$$\mathbf{E}_{\theta r} = \mathbf{e}_\theta \otimes \mathbf{e}_r, \mathbf{E}_{\theta\theta} = \mathbf{e}_\theta \otimes \mathbf{e}_\theta, \mathbf{E}_{\theta 3} = \mathbf{e}_\theta \otimes \mathbf{e}_3,$$

$$\mathbf{E}_{3r} = \mathbf{e}_3 \otimes \mathbf{e}_r, \mathbf{E}_{3\theta} = \mathbf{e}_3 \otimes \mathbf{e}_\theta, \mathbf{E}_{33} = \mathbf{e}_3 \otimes \mathbf{e}_3.$$

Using the relationships between $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$, it is possible to establish the relationships between

$$\mathcal{RC} = \{\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{13}, \mathbf{E}_{21}, \mathbf{E}_{22}, \mathbf{E}_{23}, \mathbf{E}_{31}, \mathbf{E}_{32}, \mathbf{E}_{33}\}$$

(rectangular coordinates) and $\mathcal{CPC}$ (cylindrical polar coordinates). The calculations are tedious but straightforward. Here are the relationships establishing the members in the set $\mathcal{RC}$ in terms of the members of the set $\mathcal{CPC}$ :

$$\mathbf{E}_{11} = \cos^2 w_0 s_R \mathbf{E}_{rr} - \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{r\theta} - \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{\theta r} + \sin^2 w_0 s_R \mathbf{E}_{\theta\theta}, \quad (9.1)$$

$$\mathbf{E}_{12} = \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{rr} + \cos^2 w_0 s_R \mathbf{E}_{r\theta} - \sin^2 w_0 s_R \mathbf{E}_{\theta r} - \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{\theta\theta}, \quad (9.2)$$

$$\mathbf{E}_{13} = \cos w_0 s_R \mathbf{E}_{r3} - \sin w_0 s_R \mathbf{E}_{\theta 3}, \quad (9.3)$$

$$\mathbf{E}_{21} = \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{rr} - \sin^2 w_0 s_R \mathbf{E}_{r\theta} + \cos^2 w_0 s_R \mathbf{E}_{\theta r} - \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{\theta\theta}, \quad (9.4)$$

$$\mathbf{E}_{22} = \sin^2 w_0 s_R \mathbf{E}_{rr} + \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{r\theta} + \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{\theta r} + \cos^2 w_0 s_R \mathbf{E}_{\theta\theta}, \quad (9.5)$$

$$\mathbf{E}_{23} = \sin w_0 s_R \mathbf{E}_{r3} + \cos w_0 s_R \mathbf{E}_{\theta 3}, \quad (9.6)$$

$$\mathbf{E}_{31} = \cos w_0 s_R \mathbf{E}_{3r} - \sin w_0 s_R \mathbf{E}_{3\theta}, \quad (9.7)$$

$$\mathbf{E}_{32} = \sin w_0 s_R \mathbf{E}_{3r} + \cos w_0 s_R \mathbf{E}_{3\theta}, \quad (9.8)$$

and

$$\mathbf{E}_{33} = \mathbf{E}_{33}. \quad (9.9)$$

It is important to realize that in the notation for these basis vectors, $r$ and $\theta$ are not indices which may assume numerical values - they are just symbols designating a member of the cylindrical polar basis.

The tensor $\mathbf{L}$ from Cohen's paper is still a cross-sectional tensor, but now the cross-sections are not parallel as $s_R$ varies as they were in the previous two cases. However, the normality condition implies that the cross-sections will be normal to the tangent vector $\mathbf{e}_\theta$. This means that if the components of $\mathbf{L}$ are expressed with respect to the cylindrical polar basis, several of the components will be zero. That is, if $\mathbf{L}$ is expressed in terms of the basis $\mathcal{CPC}$ say by

$$\mathbf{L} = L^r_r \mathbf{E}_{rr} + L^r_\theta \mathbf{E}_{r\theta} + L^r_3 \mathbf{E}_{r3} + L^\theta_r \mathbf{E}_{\theta r} + L^\theta_\theta \mathbf{E}_{\theta\theta} + L^\theta_3 \mathbf{E}_{\theta 3} + L^3_r \mathbf{E}_{3r} + L^3_\theta \mathbf{E}_{3\theta} + L^3_3 \mathbf{E}_{33},$$

then only components acting the plane normal to $\mathbf{e}_\theta$ are nonzero. This means all components with $\theta$ as an index, either upper or lower, are zero. That is

$$L^r_\theta = L^\theta_r = L^\theta_\theta = L^\theta_3 = L^3_\theta = 0 \tag{9.10}$$

and

$$\mathbf{L} = L^r_r \mathbf{E}_{rr} + L^r_3 \mathbf{E}_{r3} + L^3_r \mathbf{E}_{3r} + L^3_3 \mathbf{E}_{33}.$$

Despite the advantages of working in the cylindrical polar coordinate system, it is easier to solve this case in rectangular coordinates. This means it is necessary to express $\mathbf{L}$ in terms of the basis $\mathcal{RC}$ and then convert this to the tensor $\overline{\mathbf{L}}$. The objective for doing this is to determine which components of $\overline{\mathbf{L}}$ are zero.

The tensor $\mathbf{L}$ is expressed in terms of the basis $\mathcal{RC}$ by the following:

$$\begin{aligned}
\mathbf{L} &= L^r_r \mathbf{e}_r \otimes \mathbf{e}^r + L^r_3 \ \mathbf{e}_r \otimes \mathbf{e}^3 + L^3_r \mathbf{e}_3 \otimes \mathbf{e}^r + L^3_3 \mathbf{e}_3 \otimes \mathbf{e}^3 \\
&= L^r_r \left(\cos w_0 s_R \mathbf{e}_1 + \sin w_0 s_R \mathbf{e}_2\right) \otimes \left(\cos w_0 s_R \mathbf{e}^1 + \sin w_0 s_R \mathbf{e}^2\right) \\
&\quad + L^r_3 \left(\cos w_0 s_R \mathbf{e}_1 + \sin w_0 s_R \mathbf{e}_2\right) \otimes \mathbf{e}^3 \\
&\quad + L^3_r \mathbf{e}_3 \otimes \left(\cos w_0 s_R \mathbf{e}^1 + \sin w_0 s_R \mathbf{e}^2\right) + L^3_3 \mathbf{e}_3 \otimes \mathbf{e}^3 \\
&= L^r_r \cos^2 w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^1 + L^r_r \sin w_0 s_R \cos w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^2 \\
&\quad + L^r_r \sin w_0 s_R \cos w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^1 + L^r_r \sin^2 w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^2 \\
&\quad + L^r_3 \cos w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^3 + L^r_3 \sin w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^3 \\
&\quad + L^3_r \cos w_0 s_R \mathbf{e}_3 \otimes \mathbf{e}^1 + L^3_r \sin w_0 s_R \mathbf{e}_3 \otimes \mathbf{e}^2 + L^3_3 \mathbf{e}_3 \otimes \mathbf{e}^3.
\end{aligned}$$

Now relate this to the tensor $\overline{\mathbf{L}}$ via the relationship $\overline{\mathbf{L}} = \mathbf{L}^{\mathrm{T}} \left(\mathbf{F}^{\mathrm{T}}\right)^{-1}$. Since

$$\begin{aligned}
\mathbf{F} &= \lambda_1 \cos w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^1 + \lambda_1 \sin w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^1 - \lambda_2 \sin w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^2 \\
&\quad + \lambda_2 \cos w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^3
\end{aligned}$$

then

$$\begin{aligned}
\mathbf{F}^{\mathrm{T}} &= \lambda_1 \cos w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^1 - \lambda_2 \sin w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^1 + \lambda_1 \sin w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^2 \\
&\quad + \lambda_2 \cos w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^3
\end{aligned}$$

and

$$
\begin{aligned}
\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} &= \frac{\cos w_0 s_R}{\lambda_1}\mathbf{e}_1\otimes\mathbf{e}^1 - \frac{\sin w_0 s_R}{\lambda_2}\mathbf{e}_1\otimes\mathbf{e}^2 + \frac{\sin w_0 s_R}{\lambda_1}\mathbf{e}_2\otimes\mathbf{e}^1 \\
&\quad + \frac{\cos w_0 s_R}{\lambda_2}\mathbf{e}_2\otimes\mathbf{e}^2 + \frac{1}{\lambda_3}\mathbf{e}_3\otimes\mathbf{e}^3
\end{aligned}
$$

as can be verified by multiplying the expressions for $\mathbf{F}^{\mathrm{T}}$ and $\left(\mathbf{F}^{\mathrm{T}}\right)^{-1}$ to get the identity tensor.

After a lengthy computation, the tensor $\overline{\mathbf{L}}$ is evaluated as

$$
\begin{aligned}
\overline{\mathbf{L}} &= \mathbf{L}^{\mathrm{T}}\left(\mathbf{F}^{\mathrm{T}}\right)^{-1} \\
&= \frac{L_r^r}{\lambda_1}\cos w_0 s_R \mathbf{e}_1\otimes\mathbf{e}^1 + \frac{L_r^3}{\lambda_3}\cos w_0 s_R \mathbf{e}_1\otimes\mathbf{e}^3 + \frac{L_r^r}{\lambda_1}\sin w_0 s_R \mathbf{e}_2\otimes\mathbf{e}^1 \\
&\quad + \frac{L_r^3}{\lambda_3}\sin w_0 s_R \mathbf{e}_2\otimes\mathbf{e}^3 + \frac{L_3^r}{\lambda_1}\mathbf{e}_3\otimes\mathbf{e}^1 + \frac{L_3^3}{\lambda_3}\mathbf{e}_3\otimes\mathbf{e}^3.
\end{aligned}
$$

The important thing to observe here is that

$$
\overline{L}_2^1 = \overline{L}_2^2 = \overline{L}_2^3 = 0
$$

(we're really not interested in how $\overline{L}_j^i$ depend on $L_r^r, L_\theta^r, L_3^r$ etc., except for possibly checking that our solution agrees with Cohen's solution).

Equations (7.11) through (7.13) become

$$
\overline{L}_1^2 \cos w_0 s_R = \overline{L}_1^1 \sin w_0 s_R, \tag{9.11}
$$

$$
\overline{L}_1^3 \cos w_0 s_R = \overline{L}_3^1 \tag{9.12}
$$

and

$$
\overline{L}_1^3 \sin w_0 s_R = \overline{L}_3^2. \tag{9.13}
$$

Equations (7.15) to (7.17) become

$$
0 = -\overline{L}_1^1 \sin w_0 s_R + \overline{L}_1^2 \cos w_0 s_R, \tag{9.14}
$$

$$
\overline{L}_3^1 \cos w_0 s_R + \overline{L}_3^2 \sin w_0 s_R = \overline{L}_1^3 \tag{9.15}
$$

and

$$-\overline{L}_3^1 \sin w_0 s_R + \overline{L}_3^2 \cos w_0 s_R = 0. \tag{9.16}$$

Obviously equations (9.11) and (9.14) are identical. If we consider equations (9.12) through (9.16) to be a system of linear equations in the components $\overline{L}_1^1$, $\overline{L}_3^1$, $\overline{L}_1^2$, $\overline{L}_3^2$ and $\overline{L}_1^3$, we obtain the following matrix equation:

$$\begin{pmatrix} 0 & -1 & 0 & 0 & \cos w_0 s_R \\ 0 & 0 & 0 & -1 & \sin w_0 s_R \\ -\sin w_0 s_R & 0 & \cos w_0 s_R & 0 & 0 \\ 0 & \cos w_0 s_R & 0 & \sin w_0 s_R & -1 \\ 0 & -\sin w_0 s_R & 0 & \cos w_0 s_R & 0 \end{pmatrix} \begin{pmatrix} \overline{L}_1^1 \\ \overline{L}_3^1 \\ \overline{L}_1^2 \\ \overline{L}_3^2 \\ \overline{L}_1^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This has only trivial solutions unless

$$\det \begin{pmatrix} 0 & -1 & 0 & 0 & \cos w_0 s_R \\ 0 & 0 & 0 & -1 & \sin w_0 s_R \\ -\sin w_0 s_R & 0 & \cos w_0 s_R & 0 & 0 \\ 0 & \cos w_0 s_R & 0 & \sin w_0 s_R & -1 \\ 0 & -\sin w_0 s_R & 0 & \cos w_0 s_R & 0 \end{pmatrix}$$

is zero. Evaluating this determinant shows in fact it is zero so there are nontrivial solutions. Solving the system of equations gives

$$\overline{L}_1^1 \sin w_0 s_R - \overline{L}_1^2 \cos w_0 s_R = 0$$

and

$$\overline{L}_3^1 = \overline{L}_3^2 = \overline{L}_1^3 = 0.$$

Using equation (7.14), we can write the components of $\mathbf{R}^T\overline{\mathbf{L}}$ for this case. Since we're interested in whether $\mathbf{R}^T\overline{\mathbf{L}}$ is symmetric, we consider only components with $i \neq j$. These give

$$(\mathbf{R}^T\overline{\mathbf{L}})_2^1 = \overline{L}_2^1 \cos w_0 s_R + \overline{L}_2^2 \sin w_0 s_R = 0$$

and

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_1^2 = -\overline{L}_1^1 \sin w_0 s_R + \overline{L}_1^2 \cos w_0 s_R = 0$$

so $(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_2^1 = (\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_1^2$,

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_3^1 = 0$$

and

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_1^3 = 0$$

so $(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_3^1 = (\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_1^3$ and

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_3^2 = 0$$

and

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_2^3 = 0$$

so $(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_3^2 = (\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_2^3$. This means $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ is symmetric.

Since $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ is symmetric, this implies $\overline{\mathbf{N}}_0 - \mathbf{W}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R$ must be symmetric and we may apply equations (7.27) through (7.29). We obtain the following three component equations:

$$\overline{N}_2^1 + W_0\overline{M}_2^2 = \overline{N}_1^2 - W_0\overline{M}_1^1 + n^1 \sin w_0 s_R - n^2 \cos w_0 s_R, \tag{9.17}$$

$$\overline{N}_3^1 + W_0\overline{M}_3^2 = \overline{N}_1^3 - n^3 \tag{9.18}$$

and

$$\overline{N}_3^2 - W_0\overline{M}_3^1 = \overline{N}_2^3. \tag{9.19}$$

Equations (7.18), (7.20) and (7.22) become

$$\lambda_1\overline{N}_1^2 - W_0\lambda_2\overline{M}_2^2 = \overline{N}_2^1\lambda_2 + W_0\lambda_1\overline{M}_1^1 \tag{9.20}$$

$$\lambda_1\overline{N}_1^3 - W_0\lambda_2\overline{M}_2^3 = \lambda_3\overline{N}_3^1 \tag{9.21}$$

and

$$\lambda_2\overline{N}_2^3 + W_0\lambda_1\overline{M}_1^3 = \overline{N}_3^2\lambda_3. \tag{9.22}$$

From equation (9.17) we have

$$n^1 \sin w_0 s_R + n^2 \cos w_0 s_R = \overline{N}_2^1 + W_0 \overline{M}_2^2 - \overline{N}_1^2 + W_0 \overline{M}_1^1.$$

Since everything on the right side of this equation is constant and $n^1$ and $n^2$ are constants, the only possible values for $n^1$ and $n^2$ which satisfy this equation are $n^1 = n^2 = 0$. This implies

$$\overline{N}_2^1 + W_0 \overline{M}_2^2 - \overline{N}_1^2 + W_0 \overline{M}_1^1 = 0. \tag{9.23}$$

The fact $n^1 = n^2 = 0$ means $\mathbf{n} = n^3 \mathbf{e}_3 = n_0 \mathbf{e}_3$ where $n_0$ is the length of $\mathbf{n}$ which implies $\mathbf{R}^{\mathrm{T}} \mathbf{n} \otimes \mathbf{r}'_R = n_0 \mathbf{e}_3 \otimes \mathbf{e}^1$.

From equation (9.18) we obtain

$$n^3 = \overline{N}_1^3 - \overline{N}_3^1 - W_0 \overline{M}_3^2. \tag{9.24}$$

From equation (9.19) we get

$$\overline{M}_3^1 = \frac{1}{W_0} \left( \overline{N}_3^2 - \overline{N}_2^3 \right). \tag{9.25}$$

We now form the component equations of

$$\lambda_2 \rho \mathbf{R}^{\mathrm{T}} \overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{W}_0 \overline{\mathbf{M}}_0 - n_0 \mathbf{e}_3 \otimes \mathbf{e}^1.$$

using the information derived above. The component equations are

$$\lambda_2 \rho \left( \overline{L}_1^1 \cos w_0 s_R + \overline{L}_1^2 \sin w_0 s_R \right) = \overline{N}_1^1 + W_0 \overline{M}_1^2, \tag{9.26}$$

$$0 = \overline{N}_2^1 + W_0 \overline{M}_2^2, \tag{9.27}$$

$$0 = \overline{N}_3^1 + W_0 \overline{M}_3^2, \tag{9.28}$$

$$0 = \overline{N}_1^2 - W_0 \overline{M}_1^1, \tag{9.29}$$

$$0 = \overline{N}_2^2 - W_0 \overline{M}_2^1, \tag{9.30}$$

$$0 = \overline{N}_3^2 - W_0 \overline{M}_3^1, \tag{9.31}$$

$$0 = \overline{N}_1^3 - n_0, \tag{9.32}$$

$$0 = \overline{N}_2^3 \tag{9.33}$$

and

$$\lambda_2 \rho \overline{L}_3^3 = \overline{N}_3^3. \tag{9.34}$$

From equation (9.27) we get

$$\overline{M}_2^2 = -\frac{1}{W_0}\overline{N}_2^1$$

and combining with equation (9.23) gives

$$-\overline{N}_1^2 + W_0 \overline{M}_1^1 = 0$$

or

$$\overline{M}_1^1 = \frac{1}{W_0}\overline{N}_1^2.$$

This result also follows from equation (9.29).

From equations (9.28), (9.30) and (9.31) we get

$$\overline{M}_3^2 = -\frac{1}{W_0}\overline{N}_3^1,$$

$$\overline{M}_2^1 = \frac{1}{W_0}\overline{N}_2^2$$

and

$$\overline{M}_3^1 = \frac{1}{W_0}\overline{N}_3^2.$$

Equation (9.33) is

$$\overline{N}_2^3 = 0;$$

combining it with equation (9.25) gives

$$\overline{M}_3^1 = \frac{1}{W_0}\overline{N}_3^2,$$

consistent with the result from equation (9.31). From equation (9.32), we have

$$n_0 = \overline{N}_1^3$$

which is consistent with equations (9.24) and (9.28).

The monotropic symmetry to be applied for this deformation is identical to that of the previous case. Thus we can immediately say monotropic symmetry implies

$$\overline{N}_2^1 = \overline{N}_3^1 = \overline{N}_1^2 = \overline{N}_3^2 = \overline{N}_1^3 = \overline{N}_2^3 = 0$$

and

$$\overline{M}_3^1 = \overline{M}_3^2 = \overline{M}_1^3 = \overline{M}_2^3 = \overline{M}_1^1 = \overline{M}_2^2 = \overline{M}_3^3 = 0.$$

Notice this means $\mathbf{n} = \mathbf{0}$. The components of $\lambda_2 \rho \mathbf{R}^T \overline{\mathbf{L}}$ are now

$$\lambda_2 \rho \left(\mathbf{R}^T \overline{\mathbf{L}}\right)_1^1 = \overline{N}_1^1 + W_0 \overline{M}_1^2,$$

$$\lambda_2 \rho \left(\mathbf{R}^T \overline{\mathbf{L}}\right)_3^3 = \overline{N}_3^3$$

and

$$\lambda_2 \rho \left(\mathbf{R}^T \overline{\mathbf{L}}\right)_j^i = 0$$

for all other choices of $i$ and $j$. In tensor form, this result is expressed as

$$\lambda_2 \rho \mathbf{R}^T \overline{\mathbf{L}} = \left(\overline{N}_1^1 + W_0 \overline{M}_1^2\right) \mathbf{E}_{11} + \overline{N}_3^3 \mathbf{E}_{33}.$$

We could find $\overline{\mathbf{L}}$ by multiplying through this equation by $\mathbf{R}$, but we omit this calculation.

## 9.2 Straight to Helical

In this deformation, the deformed state is a piece of circular helix with axis parallel to $\mathbf{e}_3$ and centered at the origin. The helix has radius $a$ and pitch $b$. The tensors $\mathbf{W}_0$ and $\mathbf{U}_0$ are exactly those given in the previous problem. The value of $W_0 = w_0$ is chosen so that

$$w_0 = \frac{\lambda_3}{c} \tag{9.35}$$

where

$$c = \sqrt{a^2 + b^2}. \tag{9.36}$$

The equation of the rod axis in the reference configuration is given by

$$\mathbf{r}_R(s_R) = s_R\mathbf{e}^3$$

so $\mathbf{d}_R^3 = \mathbf{r}_R{}'(s_R) = \mathbf{e}^3$. The other directors are chosen so $\mathbf{d}_R^1 = \mathbf{e}^1$ and $\mathbf{d}_R^2 = \mathbf{e}^2$. Then the stretch factors $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfy $\lambda_1 = \lambda_2 = \lambda$ and the field equation (6.24) is

$$\lambda_3\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{Z}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R.$$

The equation of the rod axis in the deformed configuration is given by

$$\mathbf{r}(s_R) = a\cos w_0 s_R\mathbf{e}_1 + a\sin w_0 s_R\mathbf{e}_2 + bw_0 s_R\mathbf{e}_3.$$

The sets for monotropic symmetry are $\mathcal{U} = \{\mathbf{I}\}$ and $\mathcal{V} = \{\mathbf{Q}_1\}$. An illustration of this deformation is given as Figure 9-2.

The directors in the deformed configuration are chosen to be parallel to the Frenet-Serret frame $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ given by unit vectors

$$(\mathbf{e}_T, \mathbf{e}_N, \mathbf{e}_B) := (\mathbf{T}, -\mathbf{N}, -\mathbf{B})$$

when the helix is parameterized with respect to arc length. The quantities $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ are given by [18]

$$\mathbf{T} = -\frac{w_0 a}{\lambda_3}\sin w_0 s_R\mathbf{e}_1 + \frac{w_0 a}{\lambda_3}\cos w_0 s_R\mathbf{e}_2 + \frac{bw_0}{\lambda_3}\mathbf{e}_3,$$

$$\mathbf{N} = -\cos w_0 s_R\mathbf{e}_1 - \sin w_0 s_R\mathbf{e}_2$$

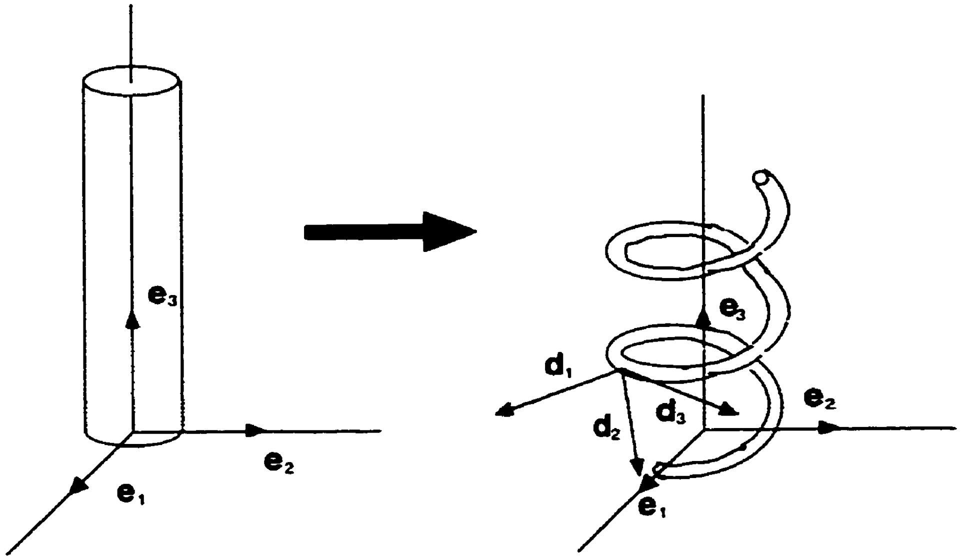


Figure 9-2: Straight to Helical

and

$$\mathbf{B} = \frac{bw_0}{\lambda_3}\sin w_0 s_R \mathbf{e}_1 - \frac{bw_0}{\lambda_3}\cos w_0 s_R \mathbf{e}_2 + \frac{aw_0}{\lambda_3}\mathbf{e}_3.$$

The curvature and torsion of the helix are given by

$$\kappa = \frac{a}{c^2}$$

and

$$\tau = \frac{b}{c^2}.$$

Note these are both constants (a special property of a few curves including the helix). Then

$$a = \kappa c^2 \tag{9.37}$$

and

$$b = \tau c^2. \tag{9.38}$$

Note that

$$a^2 + b^2 = \kappa^2 c^4 + \tau^2 c^4 = c^2$$

so

$$\left(\kappa^2+\tau^2\right)c^2=1. \tag{9.39}$$

It is also convenient to define

$$\overline{\kappa}=cw_0\kappa \text{ and } \overline{\tau}=cw_0\tau. \tag{9.40}$$

Thus

$$\mathbf{T}=-\kappa c\sin w_0s_R\mathbf{e}_1+\kappa c\cos w_0s_R\mathbf{e}_2+\tau c\mathbf{e}_3,$$

$$\mathbf{N}=-\cos w_0s_R\mathbf{e}_1-\sin w_0s_R\mathbf{e}_2$$

and

$$\mathbf{B}=\tau c\sin w_0s_R\mathbf{e}_1-\tau c\cos w_0s_R\mathbf{e}_2+\kappa c\mathbf{e}_3$$

give the Frenet-Serret frame for the circular helix.

The basis $\{\mathbf{e}_T,\mathbf{e}_N,\mathbf{e}_B\}$ is a convenient one in this case. We relate these to the standard basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_2\}$ by the following expressions.

$$\mathbf{e}_T=\mathbf{T}=-\kappa c\sin w_0s_R\mathbf{e}_1+\kappa c\cos w_0s_R\mathbf{e}_2+\tau c\mathbf{e}_3,$$

$$\mathbf{e}_N=-\mathbf{N}=\cos w_0s_R\mathbf{e}_1+\sin w_0s_R\mathbf{e}_2$$

and

$$\mathbf{e}_B=-\mathbf{B}=-\tau c\sin w_0s_R\mathbf{e}_1+\tau c\cos w_0s_R\mathbf{e}_2-\kappa c\mathbf{e}_3.$$

It is easy to see the set $\{\mathbf{e}_T,\mathbf{e}_N,\mathbf{e}_B\}$ is orthonormal.

The directors are now expressed as

$$\mathbf{d}_1=\lambda_1\mathbf{e}_N=-\lambda_1\mathbf{N}=\lambda_1\cos w_0s_R\mathbf{e}_1+\lambda_1\sin w_0s_R\mathbf{e}_2,$$

$$\mathbf{d}_2=\lambda_2\mathbf{e}_B=-\lambda_2\mathbf{B}=-\tau c\lambda_2\sin w_0s_R\mathbf{e}_1+\tau c\lambda_2\cos w_0s_R\mathbf{e}_2-\kappa c\lambda_2\mathbf{e}_3$$

and

$$\mathbf{d}_3=\lambda_3\mathbf{e}_T=\lambda_3\mathbf{T}=-\kappa c\lambda_3\sin w_0s_R\mathbf{e}_1+\kappa c\lambda_3\cos w_0s_R\mathbf{e}_2+\tau c\lambda_3\mathbf{e}_3.$$

Then the deformation tensor is given by

$$
\begin{aligned}
\mathbf{F} &= \mathbf{d}_1 \otimes \mathbf{d}_R^1 + \mathbf{d}_2 \otimes \mathbf{d}_R^2 + \mathbf{d}_3 \otimes \mathbf{d}_R^3 \\
&= (\lambda_1 \cos w_0 s_R \mathbf{e}_1 + \lambda_1 \sin w_0 s_R \mathbf{e}_2) \otimes \mathbf{e}^1 \\
&\quad + (-\tau c \lambda_2 \sin w_0 s_R \mathbf{e}_1 + \tau c \lambda_2 \cos w_0 s_R \mathbf{e}_2 - \kappa c \lambda_2 \mathbf{e}_3) \otimes \mathbf{e}^2 \\
&\quad + (-\kappa c \lambda_3 \sin w_0 s_R \mathbf{e}_1 + \kappa c \lambda_3 \cos w_0 s_R \mathbf{e}_2 + \tau c \lambda_3 \mathbf{e}_3) \otimes \mathbf{e}^3 \\
&= \lambda_1 \cos w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^1 - \tau c \lambda_2 \sin w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^2 - \kappa c \lambda_3 \sin w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^3 \\
&\quad + \lambda_1 \sin w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^1 + \tau c \lambda_2 \cos w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^2 + \kappa c \lambda_3 \cos w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^3 \\
&\quad - \kappa c \lambda_2 \mathbf{e}_3 \otimes \mathbf{e}^2 + \tau c \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^3
\end{aligned}
$$

As usual, the polar decomposition of $\mathbf{F}$ is $\mathbf{F} = \mathbf{R}\mathbf{U}_0$ where $\mathbf{U}_0$ is given by equation (7.1). In this case, the rotation tensor is more complicated. The following lemma shows this tensor may be decomposed into the rotation tensor of the previous two deformations (straight to straight twisted and straight to circular) and another orthogonal tensor.

**Lemma 59** *In the decomposition* $\mathbf{F} = \mathbf{R}\mathbf{U}_0$, $\mathbf{R} = \mathbf{R}_3\mathbf{R}_1$ *where*

$$
\mathbf{R}_3 = \cos w_0 s_R \mathbf{I}_3^{\perp} + \sin w_0 s_R \mathbf{A}_3^{\perp} + \mathbf{E}_{33}
$$

*and*

$$
\mathbf{R}_1 = \tau c \mathbf{I}_1^{\perp} + \kappa c \mathbf{A}_1^{\perp} + \mathbf{E}_{11}.
$$

**Proof.** The proof is done by showing $\mathbf{R}\mathbf{U}_0 = \mathbf{R}_3\mathbf{R}_1\mathbf{U}_0$ gives the expression for $\mathbf{F}$ given above. The component form of $\mathbf{R}$ is given by

$$
\begin{aligned}
\mathbf{R} &= \left(\cos w_0 s_R \mathbf{I}_3^{\perp} + \sin w_0 s_R \mathbf{A}_3^{\perp} + \mathbf{E}_{33}\right)\left(\tau c \mathbf{I}_1^{\perp} + \kappa c \mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right) \\
&= \tau c \cos w_0 s_R \mathbf{I}_3^{\perp}\mathbf{I}_1^{\perp} + \tau c \sin w_0 s_R \mathbf{A}_3^{\perp}\mathbf{I}_1^{\perp} + \tau c \mathbf{E}_{33}\mathbf{I}_1^{\perp} \\
&\quad + \kappa c \cos w_0 s_R \mathbf{I}_3^{\perp}\mathbf{A}_1^{\perp} + \kappa c \sin w_0 s_R \mathbf{A}_3^{\perp}\mathbf{A}_1^{\perp} + \kappa c \mathbf{E}_{33}\mathbf{A}_1^{\perp} \\
&\quad + \cos w_0 s_R \mathbf{I}_3^{\perp}\mathbf{E}_{11} + \sin w_0 s_R \mathbf{A}_3^{\perp}\mathbf{E}_{11} + \mathbf{E}_{33}\mathbf{E}_{11}.
\end{aligned}
$$

Using Table 6.1 and rearranging the order in which the terms are written, we obtain

$$\begin{aligned}\mathbf{R} &= \cos w_0 s_R \mathbf{E}_{11} - \tau c \sin w_0 s_R \mathbf{E}_{12} - \kappa c \sin w_0 s_R \mathbf{E}_{13} \\ &\quad + \sin w_0 s_R \mathbf{E}_{21} + \tau c \cos w_0 s_R \mathbf{E}_{22} + \kappa c \cos w_0 s_R \mathbf{E}_{23} - \kappa c \mathbf{E}_{32} + \tau c \mathbf{E}_{33}.\end{aligned}$$

Now multiplying this by $\mathbf{U}_0$ gives

$$\begin{aligned}\mathbf{R}\mathbf{U}_0 &= \cos w_0 s_R \mathbf{E}_{11}\mathbf{U}_0 - \tau c \sin w_0 s_R \mathbf{E}_{12}\mathbf{U}_0 - \kappa c \sin w_0 s_R \mathbf{E}_{13}\mathbf{U}_0 \\ &\quad + \sin w_0 s_R \mathbf{E}_{21}\mathbf{U}_0 + \tau c \cos w_0 s_R \, \mathbf{E}_{22}\mathbf{U}_0 + \kappa c \cos w_0 s_R \mathbf{E}_{23}\mathbf{U}_0 - \kappa c \mathbf{E}_{32}\mathbf{U}_0 + \tau c \mathbf{E}_{33}\mathbf{U}_0 \\ &= \cos w_0 s_R \mathbf{E}_{11}\left(\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}\right) - \tau c \sin w_0 s_R \mathbf{E}_{12}\left(\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}\right) \\ &\quad - \kappa c \sin w_0 s_R \mathbf{E}_{13}\left(\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}\right) + \sin w_0 s_R \mathbf{E}_{21}\left(\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}\right) \\ &\quad + \tau c \cos w_0 s_R \mathbf{E}_{22}\left(\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}\right) + \kappa c \cos w_0 s_R \mathbf{E}_{23}\left(\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}\right) \\ &\quad - \kappa c \mathbf{E}_{32}\left(\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}\right) + \tau c \mathbf{E}_{33}\left(\lambda_1 \mathbf{E}_{11} + \lambda_2 \mathbf{E}_{22} + \lambda_3 \mathbf{E}_{33}\right) \\ &= \lambda_1 \cos w_0 s_R \mathbf{E}_{11} - \lambda_2 \tau c \sin w_0 s_R \mathbf{E}_{12} - \lambda_3 \kappa c \sin w_0 s_R \mathbf{E}_{13} + \lambda_1 \sin w_0 s_R \mathbf{E}_{21} \\ &\quad + \lambda_2 \tau c \cos w_0 s_R \mathbf{E}_{22} + \lambda_3 \kappa c \cos w_0 s_R \mathbf{E}_{23} - \lambda_2 \kappa c \mathbf{E}_{32} + \lambda_3 \tau c \mathbf{E}_{33}.\end{aligned}$$

Comparing this with the expression for $\mathbf{F}$ given above, shows the expressions are identical. ∎

It is useful to have $\mathbf{R}^{\mathrm{T}} = (\mathbf{R}_3\mathbf{R}_1)^{\mathrm{T}} = \mathbf{R}_1^{\mathrm{T}}\mathbf{R}_3^{\mathrm{T}}$. Calculate $\mathbf{R}^{\mathrm{T}}$ as

$$\begin{aligned}\mathbf{R}^{\mathrm{T}} &= \cos w_0 s_R \mathbf{E}_{11} + \sin w_0 s_R \mathbf{E}_{12} - \tau c \sin w_0 s_R \mathbf{E}_{21} + \tau c \cos w_0 s_R \mathbf{E}_{22} - \kappa c \mathbf{E}_{23} \\ &\quad - \kappa c \sin w_0 s_R \mathbf{E}_{31} + \kappa c \cos w_0 s_R \mathbf{E}_{32} + \tau c \mathbf{E}_{33}.\end{aligned}$$

Then

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \left(\mathbf{R}_1^{\mathrm{T}}\mathbf{R}_3^{\mathrm{T}}\right)\left(\mathbf{R}_3\mathbf{R}_1\right) = \mathbf{R}_1^{\mathrm{T}}\mathbf{R}_3^{\mathrm{T}}\mathbf{R}_3\mathbf{R}_1 = \mathbf{R}_1^{\mathrm{T}}\mathbf{I}\mathbf{R}_1 = \mathbf{R}_1^{\mathrm{T}}\mathbf{R}_1$$

where $\mathbf{I}$ denotes the identity tensor. Computing $\mathbf{R}_1^{\mathrm{T}}\mathbf{R}_1$ gives

$$\begin{aligned}\mathbf{R}_1^{\mathrm{T}}\mathbf{R}_1 &= \left(\tau c \mathbf{I}_1^{\perp} + \kappa c \mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right)^{\mathrm{T}}\left(\tau c \mathbf{I}_1^{\perp} + \kappa c \mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right) \\ &= \left(\tau c \mathbf{I}_1^{\perp} - \kappa c \mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right)\left(\tau c \mathbf{I}_1^{\perp} + \kappa c \mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right) \\ &= \tau^2 c^2 \mathbf{I}_1^{\perp}\mathbf{I}_1^{\perp} + \tau\kappa c^2 \mathbf{I}_1^{\perp}\mathbf{A}_1^{\perp} + \tau c \mathbf{I}_1^{\perp}\mathbf{E}_{11} - \tau\kappa c^2 \mathbf{A}_1^{\perp}\mathbf{I}_1^{\perp} - \kappa^2 c^2 \mathbf{A}_1^{\perp}\mathbf{A}_1^{\perp} - \kappa c \mathbf{A}_1^{\perp}\mathbf{E}_{11}\end{aligned}$$

$$+\tau c\mathbf{E}_{11}\mathbf{I}_1^{\perp} + \kappa c\mathbf{E}_{11}\mathbf{A}_1^{\perp} + \mathbf{E}_{11}\mathbf{E}_{11}.$$

Using Table 6.1, we obtain

$$\begin{aligned}
\mathbf{R}_1^{\mathrm{T}}\mathbf{R}_1 &= \tau^2 c^2 \mathbf{I}_1^{\perp} + \tau\kappa c^2\mathbf{A}_1^{\perp} - \tau\kappa c^2\mathbf{A}_1^{\perp} + \kappa^2 c^2\mathbf{I}_1^{\perp} + \mathbf{E}_{11} \\
&= \left(\tau^2 + \kappa^2\right) c^2 \mathbf{I}_1^{\perp} + \mathbf{E}_{11} \\
&= \mathbf{I}_1^{\perp} + \mathbf{E}_{11} \\
&= \mathbf{E}_{11} + \mathbf{E}_{22} + \mathbf{E}_{33} \\
&= \mathbf{I}
\end{aligned}$$

using equation (9.39). Since $\mathbf{R}_1^{\mathrm{T}}\mathbf{R}_1 = \mathbf{I}$, then $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ which confirms $\mathbf{R}$ is orthogonal.

It should be noted that in this deformation, the axis of rotation of the rotation tensor $\mathbf{R}$ is not parallel to the axial vector of $\mathbf{W}_0$. Thus we need to compute

$$\mathbf{Z}_0 = \mathbf{R}^{\mathrm{T}}\mathbf{W}_0\mathbf{R} = \mathbf{R}_1^{\mathrm{T}}\mathbf{R}_3^{\mathrm{T}}\mathbf{W}_0\mathbf{R}_3\mathbf{R}_1.$$

Since the axis of rotation of the rotation tensor $\mathbf{R}_3$ is parallel to the axial vector of $\mathbf{W}_0$, we have from Lemma 53 that $\mathbf{W}_0\mathbf{R}_3 = \mathbf{R}_3\mathbf{W}_0$ so

$$\mathbf{Z}_0 = \mathbf{R}_1^{\mathrm{T}}\mathbf{R}_3^{\mathrm{T}}\mathbf{R}_3\mathbf{W}_0\mathbf{R}_1 = \mathbf{R}_1^{\mathrm{T}}\mathbf{W}_0\mathbf{R}_1$$

using the fact $\mathbf{R}_3^{\mathrm{T}}\mathbf{R}_3 = \mathbf{I}$. The computation of $\mathbf{Z}_0$ is as follows:

$$\begin{aligned}
\mathbf{Z}_0 &= \mathbf{R}_1^{\mathrm{T}}\mathbf{W}_0\mathbf{R}_1 \\
&= \left(\tau c\mathbf{I}_1^{\perp} - \kappa c\mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right)\mathbf{W}_0\left(\tau c\mathbf{I}_1^{\perp} + \kappa c\mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right) \\
&= W_0\left(\tau c\mathbf{I}_1^{\perp}\mathbf{A}_3^{\perp} - \kappa c\mathbf{A}_1^{\perp}\mathbf{A}_3^{\perp} + \mathbf{E}_{11}\mathbf{A}_3^{\perp}\right)\left(\tau c\mathbf{I}_1^{\perp} + \kappa c\mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right) \\
&= W_0\left(\tau c\mathbf{E}_{21} + \kappa c\mathbf{E}_{31} - \mathbf{E}_{12}\right)\left(\tau c\mathbf{I}_1^{\perp} + \kappa c\mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right) \\
&= W_0(\tau^2 c^2\mathbf{E}_{21}\mathbf{I}_1^{\perp} + \kappa\tau c^2\ \mathbf{E}_{31}\mathbf{I}_1^{\perp} - \tau c\mathbf{E}_{12}\mathbf{I}_1^{\perp} + \kappa\tau c^2\mathbf{E}_{21}\mathbf{A}_1^{\perp} + \kappa^2 c^2\mathbf{E}_{31}\mathbf{A}_1^{\perp} - \kappa c\mathbf{E}_{12}\mathbf{A}_1^{\perp} \\
&\quad + \tau c\mathbf{E}_{21}\mathbf{E}_{11} + \kappa c\mathbf{E}_{31}\mathbf{E}_{11} - \mathbf{E}_{12}\mathbf{E}_{11})
\end{aligned}$$

using $\mathbf{W}_0 = W_0\mathbf{A}_3^{\perp}$ and the information in Table 6.1 The various tensors remaining in the expression for $\mathbf{Z}_0$ evaluate as follows:

$$\mathbf{E}_{21}\mathbf{I}_1^{\perp} = \mathbf{E}_{21}\left(\mathbf{E}_{22} + \mathbf{E}_{33}\right) = \mathbf{E}_{21}\mathbf{E}_{22} + \mathbf{E}_{21}\mathbf{E}_{33} = \mathbf{0},$$

$$\mathbf{E}_{31}\mathbf{I}_1^{\perp} = \mathbf{E}_{31}\left(\mathbf{E}_{22} + \mathbf{E}_{33}\right) = \mathbf{E}_{31}\mathbf{E}_{22} + \mathbf{E}_{31}\mathbf{E}_{33} = \mathbf{0},$$

$$\mathbf{E}_{12}\mathbf{I}_1^{\perp} = \mathbf{E}_{12}\left(\mathbf{E}_{22} + \mathbf{E}_{33}\right) = \mathbf{E}_{12}\mathbf{E}_{22} + \mathbf{E}_{12}\mathbf{E}_{33} = \mathbf{E}_{12},$$

$$\mathbf{E}_{21}\mathbf{A}_1^{\perp} = \mathbf{E}_{21}\left(\mathbf{E}_{23} - \mathbf{E}_{32}\right) = \mathbf{E}_{21}\mathbf{E}_{23} - \mathbf{E}_{21}\mathbf{E}_{32} = \mathbf{0},$$

$$\mathbf{E}_{31}\mathbf{A}_1^{\perp} = \mathbf{E}_{31}\left(\mathbf{E}_{23} - \mathbf{E}_{32}\right) = \mathbf{E}_{31}\mathbf{E}_{23} - \mathbf{E}_{31}\mathbf{E}_{32} = \mathbf{0},$$

$$\mathbf{E}_{12}\mathbf{A}_1^{\perp} = \mathbf{E}_{12}\left(\mathbf{E}_{23} - \mathbf{E}_{32}\right) = \mathbf{E}_{12}\mathbf{E}_{23} - \mathbf{E}_{12}\mathbf{E}_{32} = \mathbf{E}_{13},$$

$$\mathbf{E}_{21}\mathbf{E}_{11} = \mathbf{E}_{21},$$

$$\mathbf{E}_{31}\mathbf{E}_{11} = \mathbf{E}_{31}$$

and

$$\mathbf{E}_{12}\mathbf{E}_{11} = \mathbf{0}.$$

Then

$$\begin{aligned}
\mathbf{Z}_0 &= W_0\left(-\tau c\mathbf{E}_{12} - \kappa c\mathbf{E}_{13} + \tau c\mathbf{E}_{21} + \kappa c\mathbf{E}_{31}\right) \\
&= W_0\left(\tau c\left(\mathbf{E}_{21} - \mathbf{E}_{12}\right) - \kappa c\left(\mathbf{E}_{13} - \mathbf{E}_{31}\right)\right) \\
&= W_0\tau c\mathbf{A}_3^{\perp} - W_0\kappa c\mathbf{A}_2^{\perp} \\
&= \overline{\tau}\mathbf{A}_3^{\perp} - \overline{\kappa}\mathbf{A}_2^{\perp}
\end{aligned}$$

using the definitions in equation (9.40) and the field equation

$$\lambda_3\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \mathbf{Z}_0\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R$$

becomes

$$\lambda_3\rho\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \overline{\tau}\mathbf{A}_3^{\perp}\overline{\mathbf{M}}_0 + \overline{\kappa}\mathbf{A}_2^{\perp}\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R. \tag{9.41}$$

The quantity $\mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R$ needs to be calculated. This evaluates as

$$\mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R = \mathbf{R}_1^{\mathrm{T}}\mathbf{R}_3^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R.$$

From the results on straight to straight twisted,

$$\begin{aligned} \mathbf{R}_3^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R &= \left(n^1 \cos w_0 s_R + n^2 \sin w_0 s_R\right) \mathbf{E}_{13} \\ &\quad + \left(n^2 \cos w_0 s_R - n^1 \sin w_0 s_R\right) \mathbf{E}_{23} + n^3 \mathbf{E}_{33} \end{aligned}$$

so

$$\begin{aligned} \mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}'_R &= \left(n^1 \cos w_0 s_R + n^2 \sin w_0 s_R\right) \mathbf{R}_1^{\mathrm{T}}\mathbf{E}_{13} \\ &\quad + \left(n^2 \cos w_0 s_R - n^1 \sin w_0 s_R\right) \mathbf{R}_1^{\mathrm{T}}\mathbf{E}_{23} + n^3 \mathbf{R}_1^{\mathrm{T}}\mathbf{E}_{33} \end{aligned}$$

and it necessary to compute $\mathbf{R}_1^{\mathrm{T}}\mathbf{E}_{13}$, $\mathbf{R}_1^{\mathrm{T}}\mathbf{E}_{23}$ and $\mathbf{R}_1^{\mathrm{T}}\mathbf{E}_{33}$. The quantity $\mathbf{R}_1^{\mathrm{T}}\mathbf{E}_{13}$ evaluates as

$$\begin{aligned} \mathbf{R}_1^{\mathrm{T}}\mathbf{E}_{13} &= \left(\tau c \mathbf{I}_1^{\perp} - \kappa c \mathbf{A}_1^{\perp} + \mathbf{E}_{11}\right) \mathbf{E}_{13} \\ &= \tau c \mathbf{I}_1^{\perp}\mathbf{E}_{13} - \kappa c \mathbf{A}_1^{\perp}\mathbf{E}_{13} + \mathbf{E}_{11}\mathbf{E}_{13}. \end{aligned}$$

These tensor components evaluate as

$$\mathbf{I}_1^{\perp}\mathbf{E}_{13} = (\mathbf{E}_{22} + \mathbf{E}_{33})\,\mathbf{E}_{13} = \mathbf{E}_{22}\mathbf{E}_{13} + \mathbf{E}_{33}\mathbf{E}_{13} = \delta_{21}\mathbf{E}_{23} + \delta_{31}\mathbf{E}_{33} = \mathbf{0},$$

$$\mathbf{A}_1^{\perp}\mathbf{E}_{13} = (\mathbf{E}_{23} - \mathbf{E}_{32})\,\mathbf{E}_{13} = \mathbf{E}_{23}\mathbf{E}_{13} - \mathbf{E}_{32}\mathbf{E}_{13} = \delta_{31}\mathbf{E}_{23} - \delta_{21}\mathbf{E}_{33} = \mathbf{0}$$

and

$$\mathbf{E}_{11}\mathbf{E}_{13} = \delta_{11}\mathbf{E}_{13} = \mathbf{E}_{13}$$

so

$$\mathbf{R}_1^{\mathrm{T}}\mathbf{E}_{13} = \mathbf{E}_{13}.$$

The quantity $\mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{E}_{23}$ evaluates as

$$\begin{aligned}\mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{E}_{23} &= \left(\tau c\mathbf{I}_\mathrm{I}^\perp - \kappa c\mathbf{A}_\mathrm{I}^\perp + \mathbf{E}_{11}\right)\mathbf{E}_{23} \\ &= \tau c\mathbf{I}_\mathrm{I}^\perp\mathbf{E}_{23} - \kappa c\mathbf{A}_\mathrm{I}^\perp\mathbf{E}_{23} + \mathbf{E}_{11}\mathbf{E}_{23}.\end{aligned}$$

These tensor components evaluate as

$$\mathbf{I}_\mathrm{I}^\perp\mathbf{E}_{23} = (\mathbf{E}_{22}+\mathbf{E}_{33})\,\mathbf{E}_{23} = \mathbf{E}_{22}\mathbf{E}_{23} + \mathbf{E}_{33}\mathbf{E}_{23} = \delta_{22}\mathbf{E}_{23} + \delta_{32}\mathbf{E}_{33} = \mathbf{E}_{23},$$

$$\mathbf{A}_\mathrm{I}^\perp\mathbf{E}_{23} = (\mathbf{E}_{23}-\mathbf{E}_{32})\,\mathbf{E}_{23} = \mathbf{E}_{23}\mathbf{E}_{23} - \mathbf{E}_{32}\mathbf{E}_{23} = \delta_{32}\mathbf{E}_{23} - \delta_{22}\mathbf{E}_{33} = -\mathbf{E}_{33}$$

and

$$\mathbf{E}_{11}\mathbf{E}_{23} = \delta_{12}\mathbf{E}_{13} = \mathbf{0}$$

so

$$\mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{E}_{23} = \tau c\mathbf{E}_{23} + \kappa c\mathbf{E}_{33}.$$

The quantity $\mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{E}_{33}$ evaluates as

$$\begin{aligned}\mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{E}_{33} &= \left(\tau c\mathbf{I}_\mathrm{I}^\perp - \kappa c\mathbf{A}_\mathrm{I}^\perp + \mathbf{E}_{11}\right)\mathbf{E}_{33} \\ &= \tau c\mathbf{I}_\mathrm{I}^\perp\mathbf{E}_{33} - \kappa c\mathbf{A}_\mathrm{I}^\perp\mathbf{E}_{33} + \mathbf{E}_{11}\mathbf{E}_{33} \\ &= \tau c\mathbf{E}_{33} - \kappa c\mathbf{E}_{23}.\end{aligned}$$

using Table 6.1. Then

$$\begin{aligned}\mathbf{R}^\mathrm{T}\mathbf{n}\otimes\mathbf{r}'_R &= \left(n^1\cos w_0 s_R + n^2\sin w_0 s_R\right)\mathbf{E}_{13} \\ &\quad + \left(n^2\cos w_0 s_R - n^1\sin w_0 s_R\right)(\tau c\mathbf{E}_{23} + \kappa c\mathbf{E}_{33}) + n^3(\tau c\mathbf{E}_{33} - \kappa c\mathbf{E}_{23}) \\ &= \left(n^1\cos w_0 s_R + n^2\sin w_0 s_R\right)\mathbf{E}_{13} \\ &\quad + \left(\tau c n^2\cos w_0 s_R - \tau c n^1\sin w_0 s_R - \kappa c n^3\right)\mathbf{E}_{23} \\ &\quad + \left(\kappa c n^2\cos w_0 s_R - \kappa c n^1\sin w_0 s_R + \tau c n^3\right)\mathbf{E}_{33}.\end{aligned}$$

Now define a basis for second order tensors by

$$\mathcal{FSC} = \{\mathbf{E}_{TT}, \mathbf{E}_{TN}, \mathbf{E}_{TB}, \mathbf{E}_{NT}, \mathbf{E}_{NN}, \mathbf{E}_{NB}, \mathbf{E}_{BT}, \mathbf{E}_{BN}, \mathbf{E}_{BB}\}$$

where

$$\mathbf{E}_{TT} = \mathbf{e}_T \otimes \mathbf{e}_T,\ \mathbf{E}_{TN} = \mathbf{e}_T \otimes \mathbf{e}_N,\ \mathbf{E}_{TB} = \mathbf{e}_T \otimes \mathbf{e}_B,$$

$$\mathbf{E}_{NT} = \mathbf{e}_N \otimes \mathbf{e}_T,\ \mathbf{E}_{NN} = \mathbf{e}_N \otimes \mathbf{e}_N,\ \mathbf{E}_{NB} = \mathbf{e}_N \otimes \mathbf{e}_B,$$

$$\mathbf{E}_{BT} = \mathbf{e}_B \otimes \mathbf{e}_T,\ \mathbf{E}_{BN} = \mathbf{e}_B \otimes \mathbf{e}_N \text{ and } \mathbf{E}_{BB} = \mathbf{e}_B \otimes \mathbf{e}_B.$$

Using the relationships between $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_N, \mathbf{e}_B, \mathbf{e}_T\}$, it is possible to establish the relationships between members of $\mathcal{FSC}$ (Frenet-Serret coordinates) and members of

$$\mathcal{RC} = \{\mathbf{E}_{11}, \mathbf{E}_{12}, \mathbf{E}_{13}, \mathbf{E}_{21}, \mathbf{E}_{22}, \mathbf{E}_{23}, \mathbf{E}_{31}, \mathbf{E}_{32}, \mathbf{E}_{33}\}$$

(rectangular coordinates). The calculations to do this are straightforward, but tedious and are omitted. The results are as follows:

$$\begin{aligned}
\mathbf{E}_{TT} &= \kappa^2 c^2 \sin^2 w_0 s_R \mathbf{E}_{11} - \kappa^2 c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{12} - \kappa\tau c^2 \sin w_0 s_R \mathbf{E}_{13} \\
&\quad -\kappa^2 c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{21} + \kappa^2 c^2 \cos^2 w_0 s_R \mathbf{E}_{22} + \kappa\tau c^2 \cos w_0 s_R \mathbf{E}_{23} \\
&\quad -\kappa\tau c^2 \sin w_0 s_R \mathbf{E}_{31} + \kappa\tau c^2 \cos w_0 s_R \mathbf{E}_{32} + \tau^2 c^2 \mathbf{E}_{33},
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}_{TN} &= -\kappa c \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{11} - \kappa c \sin^2 w_0 s_R \mathbf{E}_{12} + \kappa c \cos^2 w_0 s_R \mathbf{E}_{21} \\
&\quad +\kappa c \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{22} + \tau c \cos w_0 s_R \mathbf{E}_{31} + \tau c \sin w_0 s_R \mathbf{E}_{32},
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}_{TB} &= \kappa\tau c^2 \sin^2 w_0 s_R \mathbf{E}_{11} - \kappa\tau c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{12} + \kappa^2 c^2 \sin w_0 s_R \mathbf{E}_{13} \\
&\quad -\kappa\tau c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{21} + \kappa\tau c^2 \cos^2 w_0 s_R \mathbf{E}_{22} - \kappa^2 c^2 \cos w_0 s_R \mathbf{E}_{23} \\
&\quad -\tau^2 c^2 \sin w_0 s_R \mathbf{E}_{31} + \tau^2 c^2 \cos w_0 s_R \mathbf{E}_{32} - \kappa\tau c^2 \mathbf{E}_{33}
\end{aligned}$$

$$\begin{aligned}\mathbf{E}_{NT} &= -\kappa c\sin w_0 s_R \cos w_0 s_R \mathbf{E}_{11} + \kappa c\cos^2 w_0 s_R \mathbf{E}_{12} + \tau c\cos w_0 s_R \mathbf{E}_{13}\\ &\quad -\kappa c\sin^2 w_0 s_R \mathbf{E}_{21} + \kappa c\cos w_0 s_R \sin w_0 s_R \mathbf{E}_{22} + \tau c\sin w_0 s_R \mathbf{E}_{23},\end{aligned}$$

$$\mathbf{E}_{NN} = \cos^2 w_0 s_R \mathbf{E}_{11} + \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{12} + \cos w_0 s_R \sin w_0 s_R \mathbf{E}_{21} + \sin^2 w_0 s_R \mathbf{E}_{22},$$

$$\begin{aligned}\mathbf{E}_{NB} &= -\tau c\sin w_0 s_R \cos w_0 s_R \mathbf{E}_{11} + \tau c\cos^2 w_0 s_R \mathbf{E}_{12} - \kappa c\cos w_0 s_R \mathbf{E}_{13}\\ &\quad -\tau c\sin^2 w_0 s_R \mathbf{E}_{21} + \tau c\cos w_0 s_R \sin w_0 s_R \mathbf{E}_{22} - \kappa c\sin w_0 s_R \mathbf{E}_{23},\end{aligned}$$

$$\begin{aligned}\mathbf{E}_{BT} &= \kappa\tau c^2 \sin^2 w_0 s_R \mathbf{E}_{11} - \kappa\tau c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{12} - \tau^2 c^2 \sin w_0 s_R \mathbf{E}_{13}\\ &\quad -\kappa\tau c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{21} + \kappa\tau c^2 \cos^2 w_0 s_R \mathbf{E}_{22} + \tau^2 c^2 \cos w_0 s_R \mathbf{E}_{23}\\ &\quad +\kappa^2 c^2 \sin w_0 s_R \mathbf{E}_{31} - \kappa^2 c^2 \cos w_0 s_R \mathbf{E}_{32} - \kappa\tau c^2 \mathbf{E}_{33},\end{aligned}$$

$$\begin{aligned}\mathbf{E}_{BN} &= -\tau c\sin w_0 s_R \cos w_0 s_R \mathbf{E}_{11} - \tau c\sin^2 w_0 s_R \mathbf{E}_{12} + \tau c\cos^2 w_0 s_R \mathbf{E}_{21}\\ &\quad +\tau c\sin w_0 s_R \cos w_0 s_R \mathbf{E}_{22} - \kappa c\cos w_0 s_R \mathbf{E}_{31} - \kappa c\sin w_0 s_R \mathbf{E}_{32}\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_{BB} &= \tau^2 c^2 \sin^2 w_0 s_R \mathbf{E}_{11} - \tau^2 c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{12} + \kappa\tau c^2 \sin w_0 s_R \mathbf{E}_{13}\\ &\quad -\tau^2 c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{21} + \tau^2 c^2 \cos^2 w_0 s_R \mathbf{E}_{22} - \kappa\tau c^2 \cos w_0 s_R \mathbf{E}_{23}\\ &\quad +\kappa\tau c^2 \sin w_0 s_R \mathbf{E}_{31} - \kappa\tau c^2 \cos w_0 s_R \mathbf{E}_{32} + \kappa^2 c^2 \mathbf{E}_{33}.\end{aligned}$$

The significant thing about using the coordinates with respect to the basis given by $\mathcal{FSC}$ is that Cohen's stress tensor $\mathbf{L}$ has several zero components in this basis. Since this stress tensor is cross-sectional, all components referring to the tangential direction (i.e. with index $T$) must be zero. Thus

$$\mathbf{L} = L^N_N \mathbf{E}_{NN} + L^N_B \mathbf{E}_{NB} + L^B_N \mathbf{E}_{BN} + L^B_B \mathbf{E}_{BB}.$$

For purposes of this thesis, we need the zero components of $\overline{\mathbf{L}}$ in the basis $\mathcal{RC}$. To find these, we express $\mathbf{L}$ in the basis $\mathcal{RC}$ and then convert to $\overline{\mathbf{L}}$ using equation (4.39). Calculating $\mathbf{L}$ in the basis $\mathcal{RC}$ gives

$$
\begin{aligned}
\mathbf{L} &= L_N^N \mathbf{E}_{NN} + L_B^N \mathbf{E}_{NB} + L_N^B \mathbf{E}_{BN} + L_B^B \mathbf{E}_{BB} \\
&= L_N^N \cos^2 w_0 s_R \mathbf{E}_{11} + L_N^N \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{12} + L_N^N \cos w_0 s_R \sin w_0 s_R \mathbf{E}_{21} \\
&\quad + L_N^N \sin^2 w_0 s_R \mathbf{E}_{22} \\
&\quad - L_B^N \tau c \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{11} + L_B^N \tau c \cos^2 w_0 s_R \mathbf{E}_{12} - L_B^N \kappa c \cos w_0 s_R \mathbf{E}_{13} \\
&\quad - L_B^N \tau c \sin^2 w_0 s_R \mathbf{E}_{21} + L_B^N \tau c \cos w_0 s_R \sin w_0 s_R \mathbf{E}_{22} - L_B^N \kappa c \sin w_0 s_R \mathbf{E}_{23} \\
&\quad - L_N^B \tau c \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{11} - L_N^B \tau c \sin^2 w_0 s_R \mathbf{E}_{12} + L_N^B \tau c \cos^2 w_0 s_R \mathbf{E}_{21} \\
&\quad + L_N^B \tau c \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{22} - L_N^B \kappa c \cos w_0 s_R \mathbf{E}_{31} - L_N^B \kappa c \sin w_0 s_R \mathbf{E}_{32} \\
&\quad + L_B^B \tau^2 c^2 \sin^2 w_0 s_R \mathbf{E}_{11} - L_B^B \tau^2 c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{12} + L_B^B \kappa \tau c^2 \sin w_0 s_R \mathbf{E}_{13} \\
&\quad - L_B^B \tau^2 c^2 \sin w_0 s_R \cos w_0 s_R \mathbf{E}_{21} + L_B^B \tau^2 c^2 \cos^2 w_0 s_R \mathbf{E}_{22} - L_B^B \kappa \tau c^2 \cos w_0 s_R \mathbf{E}_{23} \\
&\quad + L_B^B \kappa \tau c^2 \sin w_0 s_R \mathbf{E}_{31} - L_B^B \kappa \tau c^2 \cos w_0 s_R \mathbf{E}_{32} + L_B^B \kappa^2 c^2 \mathbf{E}_{33}
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\mathbf{L} &= \left(L_N^N \cos^2 w_0 s_R - \left(L_B^N + L_N^B\right) \tau c \sin w_0 s_R \cos w_0 s_R + L_B^B \tau^2 c^2 \sin^2 w_0 s_R\right) \mathbf{E}_{11} \\
&\quad + \left(\left(L_N^N - L_B^B \tau^2 c^2\right) \sin w_0 s_R \cos w_0 s_R + L_B^N \tau c \cos^2 w_0 s_R - L_N^B \tau c \sin^2 w_0 s_R\right) \mathbf{E}_{12} \\
&\quad + \left(L_B^B \kappa \tau c^2 \sin w_0 s_R - L_B^N \kappa c \cos w_0 s_R\right) \mathbf{E}_{13} \\
&\quad + \left(\left(L_N^N - L_B^B \tau^2 c^2\right) \sin w_0 s_R \cos w_0 s_R - L_B^N \tau c \sin^2 w_0 s_R + L_N^B \tau c \cos^2 w_0 s_R\right) \mathbf{E}_{21} \\
&\quad + \left(L_N^N \sin^2 w_0 s_R + \left(L_B^N + L_N^B\right) \tau c \sin w_0 s_R \cos w_0 s_R + L_B^B \tau^2 c^2 \cos^2 w_0 s_R\right) \mathbf{E}_{22} \\
&\quad - \left(L_B^N \kappa c \sin w_0 s_R + L_B^B \kappa \tau c^2 \cos w_0 s_R\right) \mathbf{E}_{23} \\
&\quad + \left(L_B^B \kappa \tau c^2 \sin w_0 s_R - L_N^B \kappa c \cos w_0 s_R\right) \mathbf{E}_{31} \\
&\quad - \left(L_N^B \kappa c \sin w_0 s_R + L_B^B \kappa \tau c^2 \cos w_0 s_R\right) \mathbf{E}_{32} \\
&\quad + L_B^B \kappa^2 c^2 \mathbf{E}_{33}.
\end{aligned}
$$

This is the tensor $\mathbf{L}$ with respect to the basis $\mathcal{RC}$.

To calculate $\overline{\mathbf{L}}$ from this, rearrange equation (4.39) as

$$\overline{\mathbf{L}} = \mathbf{L}^{\mathrm{T}} \mathbf{F}^{-\mathrm{T}}.$$

The most straightforward way to compute $\mathbf{F}^{-\mathrm{T}}$ is as follows. From

$$\mathbf{F} = \mathbf{R}\mathbf{U}_0 = \mathbf{R}_3\mathbf{R}_1\mathbf{U}_0$$

we get

$$\mathbf{F}^{\mathrm{T}} = (\mathbf{R}_3\mathbf{R}_1\mathbf{U}_0)^{\mathrm{T}} = \mathbf{U}_0^{\mathrm{T}}\mathbf{R}_1^{\mathrm{T}}\mathbf{R}_3^{\mathrm{T}} = \mathbf{U}_0\mathbf{R}_1^{-1}\mathbf{R}_3^{-1}$$

using the facts that $\mathbf{U}_0$ is symmetric and $\mathbf{R}_1$ and $\mathbf{R}_3$ are orthogonal. Then

$$\mathbf{F}^{-\mathrm{T}} = \left(\mathbf{U}_0\mathbf{R}_1^{-1}\mathbf{R}_3^{-1}\right)^{-1} = \left(\mathbf{R}_3^{-1}\right)^{-1}\left(\mathbf{R}_1^{-1}\right)^{-1}\mathbf{U}_0^{-1} = \mathbf{R}_3\mathbf{R}_1\mathbf{U}_0^{-1}.$$

Since $\mathbf{U}_0$ is diagonal, it is straightforward to calculate its inverse as

$$\mathbf{U}_0^{-1} = \frac{1}{\lambda_1}\mathbf{E}_{11} + \frac{1}{\lambda_2}\mathbf{E}_{22} + \frac{1}{\lambda_3}\mathbf{E}_{33}.$$

The rotation tensor $\mathbf{R}_3\mathbf{R}_1$ has been calculated earlier. Thus

$$\begin{aligned}
\mathbf{F}^{-\mathrm{T}} &= \mathbf{R}_3\mathbf{R}_1\mathbf{U}_0^{-1} \\
&= \cos w_0 s_R \mathbf{E}_{11}\mathbf{U}_0^{-1} - \tau c \sin w_0 s_R \mathbf{E}_{12}\mathbf{U}_0^{-1} - \kappa c \sin w_0 s_R \mathbf{E}_{13}\mathbf{U}_0^{-1} \\
&\quad + \sin w_0 s_R \mathbf{E}_{21}\mathbf{U}_0^{-1} + \tau c \cos w_0 s_R \mathbf{E}_{22}\mathbf{U}_0^{-1} + \kappa c \cos w_0 s_R \mathbf{E}_{23}\mathbf{U}_0^{-1} \\
&\quad - \kappa c \mathbf{E}_{32}\mathbf{U}_0^{-1} + \tau c \mathbf{E}_{33}\mathbf{U}_0^{-1}.
\end{aligned}$$

Now calculate the term $\mathbf{E}_{ij}\mathbf{U}_0^{-1}$ :

$$\begin{aligned}
\mathbf{E}_{ij}\mathbf{U}_0^{-1} &= \mathbf{E}_{ij}\left(\frac{1}{\lambda_1}\mathbf{E}_{11} + \frac{1}{\lambda_2}\mathbf{E}_{22} + \frac{1}{\lambda_3}\mathbf{E}_{33}\right) \\
&= \frac{1}{\lambda_1}\mathbf{E}_{ij}\mathbf{E}_{11} + \frac{1}{\lambda_2}\mathbf{E}_{ij}\mathbf{E}_{22} + \frac{1}{\lambda_3}\mathbf{E}_{ij}\mathbf{E}_{33} \\
&= \frac{1}{\lambda_1}\delta_{j1}\mathbf{E}_{i1} + \frac{1}{\lambda_2}\delta_{j2}\mathbf{E}_{i2} + \frac{1}{\lambda_3}\delta_{j3}\mathbf{E}_{i3}.
\end{aligned}$$

Substituting this into the expression for $\mathbf{F}^{-\mathrm{T}}$ gives

$$\mathbf{F}^{-\mathrm{T}} = \frac{1}{\lambda_1}\cos w_0 s_R \mathbf{E}_{11} - \frac{\tau c}{\lambda_2}\sin w_0 s_R \mathbf{E}_{12} - \frac{\kappa c}{\lambda_3}\sin w_0 s_R \mathbf{E}_{13}$$

$$
\begin{aligned}
&+\frac{1}{\lambda_1}\sin w_0 s_R \mathbf{E}_{21} + \frac{\tau c}{\lambda_2}\cos w_0 s_R \mathbf{E}_{22} + \frac{\kappa c}{\lambda_3}\cos w_0 s_R \mathbf{E}_{23} \\
&-\frac{\kappa c}{\lambda_2}\mathbf{E}_{32} + \frac{\tau c}{\lambda_3}\mathbf{E}_{33}.
\end{aligned}
$$

Now calculate $\overline{\mathbf{L}} = \mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}$ by

$$
\begin{aligned}
\overline{\mathbf{L}} &= \mathbf{L}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}} \\
&= \frac{1}{\lambda_1}\cos w_0 s_R \mathbf{L}^{\mathrm{T}}\mathbf{E}_{11} - \frac{\tau c}{\lambda_2}\sin w_0 s_R \mathbf{L}^{\mathrm{T}}\mathbf{E}_{12} - \frac{\kappa c}{\lambda_3}\sin w_0 s_R \mathbf{L}^{\mathrm{T}}\mathbf{E}_{13} \\
&\quad +\frac{1}{\lambda_1}\sin w_0 s_R \mathbf{L}^{\mathrm{T}}\mathbf{E}_{21} + \frac{\tau c}{\lambda_2}\cos w_0 s_R \mathbf{L}^{\mathrm{T}}\mathbf{E}_{22} + \frac{\kappa c}{\lambda_3}\cos w_0 s_R \mathbf{L}^{\mathrm{T}}\mathbf{E}_{23} \\
&\quad -\frac{\kappa c}{\lambda_2}\mathbf{L}^{\mathrm{T}}\mathbf{E}_{32} + \frac{\tau c}{\lambda_3}\mathbf{L}^{\mathrm{T}}\mathbf{E}_{33}.
\end{aligned}
$$

It is necessary to calculate $\mathbf{L}^{\mathrm{T}}\mathbf{E}_{ij}$ done by

$$
\begin{aligned}
\mathbf{L}^{\mathrm{T}}\mathbf{E}_{ij} &= \left(L_N^N \cos^2 w_0 s_R - \left(L_B^N + L_N^B\right)\tau c \sin w_0 s_R \cos w_0 s_R + L_B^B \tau^2 c^2 \sin^2 w_0 s_R\right)\mathbf{E}_{11}\mathbf{E}_{ij} \\
&\quad +(\left(L_N^N - L_B^B \tau^2 c^2\right)\sin w_0 s_R \cos w_0 s_R \\
&\quad +L_B^N \tau c \cos^2 w_0 s_R - L_N^B \tau c \sin^2 w_0 s_R)\mathbf{E}_{12}\mathbf{E}_{ij} \\
&\quad +\left(L_B^B \kappa \tau c^2 \sin w_0 s_R - L_B^N \kappa c \cos w_0 s_R\right)\mathbf{E}_{13}\mathbf{E}_{ij} \\
&\quad +(\left(L_N^N - L_B^B \tau^2 c^2\right)\sin w_0 s_R \cos w_0 s_R \\
&\quad -L_B^N \tau c \sin^2 w_0 s_R + L_N^B \tau c \cos^2 w_0 s_R)\mathbf{E}_{21}\mathbf{E}_{ij} \\
&\quad +(L_N^N \sin^2 w_0 s_R + \left(L_B^N + L_N^B\right)\tau c \sin w_0 s_R \cos w_0 s_R \\
&\quad +L_B^B \tau^2 c^2 \cos^2 w_0 s_R)\mathbf{E}_{22}\mathbf{E}_{ij} \\
&\quad -\left(L_B^N \kappa c \sin w_0 s_R + L_B^B \kappa \tau c^2 \cos w_0 s_R\right)\mathbf{E}_{23}\mathbf{E}_{ij} \\
&\quad +\left(L_B^B \kappa \tau c^2 \sin w_0 s_R - L_N^B \kappa c \cos w_0 s_R\right)\mathbf{E}_{31}\mathbf{E}_{ij} \\
&\quad -\left(L_N^B \kappa c \sin w_0 s_R + L_B^B \kappa \tau c^2 \cos w_0 s_R\right)\mathbf{E}_{32}\mathbf{E}_{ij} \\
&\quad +L_B^B \kappa^2 c^2 \mathbf{E}_{33}\mathbf{E}_{ij}
\end{aligned}
$$

which becomes

$$
\mathbf{L}^{\mathrm{T}}\mathbf{E}_{ij} = \left(L_N^N \cos^2 w_0 s_R - \left(L_B^N + L_N^B\right)\tau c \sin w_0 s_R \cos w_0 s_R + L_B^B \tau^2 c^2 \sin^2 w_0 s_R\right)\delta_{1i}\mathbf{E}_{1j}
$$

$$
\begin{aligned}
&+((L_N^N - L_B^B\tau^2c^2)\sin w_0s_R\cos w_0s_R \\
&+L_B^N\tau c\cos^2 w_0s_R - L_N^B\tau c\sin^2 w_0s_R)\delta_{2i}\mathbf{E}_{1j} \\
&+\left(L_B^B\kappa\tau c^2\sin w_0s_R - L_B^N\kappa c\cos w_0s_R\right)\delta_{3i}\mathbf{E}_{1j} \\
&+((L_N^N - L_B^B\tau^2c^2)\sin w_0s_R\cos w_0s_R \\
&-L_B^N\tau c\sin^2 w_0s_R + L_N^B\tau c\cos^2 w_0s_R)\delta_{1i}\mathbf{E}_{2j} \\
&+(L_N^N\sin^2 w_0s_R + \left(L_B^N + L_N^B\right)\tau c\sin w_0s_R\cos w_0s_R \\
&+L_B^B\tau^2c^2\cos^2 w_0s_R)\delta_{2i}\mathbf{E}_{2j} \\
&-\left(L_B^N\kappa c\sin w_0s_R + L_B^B\kappa\tau c^2\cos w_0s_R\right)\delta_{3i}\mathbf{E}_{2j} \\
&+\left(L_B^B\kappa\tau c^2\sin w_0s_R - L_N^B\kappa c\cos w_0s_R\right)\delta_{1i}\mathbf{E}_{3j} \\
&-\left(L_N^B\kappa c\sin w_0s_R + L_B^B\kappa\tau c^2\cos w_0s_R\right)\delta_{2i}\mathbf{E}_{3j} \\
&+L_B^B\kappa^2c^2\delta_{3i}\mathbf{E}_{3j}.
\end{aligned}
$$

Then, for example,

$$
\begin{aligned}
\mathbf{L}^T\mathbf{E}_{11} &= \left(L_N^N\cos^2 w_0s_R - \left(L_B^N + L_N^B\right)\tau c\sin w_0s_R\cos w_0s_R + L_B^B\tau^2c^2\sin^2 w_0s_R\right)\mathbf{E}_{11} \\
&\quad +((L_N^N - L_B^B\tau^2c^2)\sin w_0s_R\cos w_0s_R \\
&\quad -L_B^N\tau c\sin^2 w_0s_R + L_N^B\tau c\cos^2 w_0s_R)\mathbf{E}_{21} \\
&\quad +\left(L_B^B\kappa\tau c^2\sin w_0s_R - L_N^B\kappa c\cos w_0s_R\right)\mathbf{E}_{31}.
\end{aligned}
$$

The other calculations are done analogously. The calculations are tedious and the rest of this calculation for $\overline{\mathbf{L}}$ are omitted. The final result is

$$
\begin{aligned}
\overline{\mathbf{L}} &= \left(\frac{1}{\lambda_1}L_N^N\cos w_0s_R - \frac{\tau c}{\lambda_1}L_B^N\sin w_0s_R\right)\mathbf{E}_{11} \\
&\quad +\left(\frac{1}{\lambda_2}L_N^B\cos w_0s_R - \frac{\tau c}{\lambda_2}L_B^B\sin w_0s_R\right)\mathbf{E}_{12} \\
&\quad +\left(\frac{1}{\lambda_1}L_N^N\sin w_0s_R + \frac{\tau c}{\lambda_1}L_B^N\cos w_0s_R\right)\mathbf{E}_{21} \\
&\quad +\left(\frac{\tau c}{\lambda_2}L_B^B\cos w_0s_R + \frac{1}{\lambda_2}L_N^B\sin w_0s_R\right)\mathbf{E}_{22} \\
&\quad -\frac{\kappa c}{\lambda_1}L_B^N\mathbf{E}_{31} - \frac{\kappa c}{\lambda_2}L_B^B\mathbf{E}_{32}.
\end{aligned}
$$

Note

$$\overline{L}_3^1 = \overline{L}_3^2 = \overline{L}_3^3 = 0.$$

For brevity, we now define $\Gamma \equiv \cos w_0 s_R$ and $\Sigma \equiv \sin w_0 s_R$. Equations (7.11) to (7.13) and (7.15) to (7.17) are given as follows:

$$\lambda_1 \Gamma \overline{L}_1^2 - \lambda_2 \tau c \Sigma \overline{L}_2^2 - \lambda_3 \kappa c \Sigma \overline{L}_3^2 = \lambda_1 \Sigma \overline{L}_1^1 + \lambda_2 \tau c \Gamma \overline{L}_2^1 + \lambda_3 \kappa c \Gamma \overline{L}_3^1,$$

$$\lambda_1 \Gamma \overline{L}_1^3 - \lambda_2 \tau c \Sigma \overline{L}_2^3 - \lambda_3 \kappa c \Sigma \overline{L}_3^3 = -\lambda_2 \kappa c \overline{L}_2^1 + \lambda_3 \tau c \overline{L}_3^1,$$

$$\lambda_1 \Sigma \overline{L}_1^3 + \lambda_2 \tau c \Gamma \overline{L}_2^3 + \lambda_3 \kappa c \Gamma \overline{L}_3^3 = -\lambda_2 \kappa c \overline{L}_2^2 + \lambda_3 \tau c \overline{L}_3^2,$$

$$\Gamma \overline{L}_2^1 + \Sigma \overline{L}_2^2 = -\tau c \Sigma \overline{L}_1^1 + \tau c \Gamma \overline{L}_1^2 - \kappa c \overline{L}_1^3,$$

$$\lambda_3 \Gamma \overline{L}_3^1 + \lambda_3 \Sigma \overline{L}_3^2 = -\lambda_1 \kappa c \Sigma \overline{L}_1^1 + \lambda_1 \kappa c \Gamma \overline{L}_1^2 + \lambda_1 \tau c \overline{L}_1^3$$

and

$$-\lambda_3 \tau c \Sigma \overline{L}_3^1 + \lambda_3 \tau c \Gamma \overline{L}_3^2 - \lambda_3 \kappa c L_3^3 = -\lambda_2 \kappa c \Sigma \overline{L}_2^1 + \lambda_2 \kappa c \Gamma \overline{L}_2^2 + \lambda_2 \tau c \overline{L}_2^3.$$

This is a system of six linear equations in the nine unknowns $\overline{L}_1^1$, $\overline{L}_2^1$, $\overline{L}_3^1$, $\overline{L}_1^2$, $\overline{L}_2^2$, $\overline{L}_3^2$, $\overline{L}_1^3$, $\overline{L}_2^3$ and $\overline{L}_3^3$. Solving this system gives

$$\overline{L}_1^1 \sin w_0 s_R - \overline{L}_1^2 \cos w_0 s_R = 0,$$

$$\overline{L}_2^1 = \frac{\tau}{\kappa} \overline{L}_2^3 \sin w_0 s_R + \frac{\lambda_3}{\lambda_2} \overline{L}_3^3 \sin w_0 s_R,$$

$$\overline{L}_3^1 = 0,$$

$$\overline{L}_2^2 = -\frac{\tau}{\kappa} \overline{L}_2^3 \cos w_0 s_R - \frac{\lambda_3}{\lambda_2} \overline{L}_3^3 \cos w_0 s_R,$$

$$\overline{L}_3^2 = 0$$

and

$$\overline{L}_1^3 = 0.$$

Using the information

$$\overline{L}_3^1 = \overline{L}_3^2 = \overline{L}_3^3 = 0,$$

the components of $\overline{\mathbf{L}}$ satisfy

$$\overline{L}_1^1 \sin w_0 s_R - \overline{L}_1^2 \cos w_0 s_R = 0,$$

$$\overline{L}_2^1 = \frac{\tau}{\kappa} \overline{L}_2^3 \sin w_0 s_R,$$

$$\overline{L}_2^2 = -\frac{\tau}{\kappa} \overline{L}_2^3 \cos w_0 s_R$$

and

$$\overline{L}_3^1 = \overline{L}_1^3 = \overline{L}_3^2 = \overline{L}_3^3 = 0$$

with no restriction on the value of $\overline{L}_2^3$ (i.e. $\overline{L}_2^3$ is a parameter).

Now calculate $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ using equation (7.14). We're particularly interested to see if this quantity is symmetric. Thus we first consider the components in which $i \neq j$. These components of $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ are as follows:

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_2^1 = \frac{\tau}{\kappa} \overline{L}_2^3 \cos w_0 s_R \sin w_0 s_R - \frac{\tau}{\kappa} \overline{L}_2^3 \cos w_0 s_R \sin w_0 s_R = 0,$$

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_1^2 = -\tau c \overline{L}_1^1 \sin w_0 s_R + \tau c \overline{L}_1^2 \cos w_0 s_R = 0,$$

so

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_2^1 = (\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_1^2,$$

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_3^1 = 0$$

and

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_1^3 = -\kappa c \overline{L}_1^1 \sin w_0 s_R + \kappa c \overline{L}_1^2 \cos w_0 s_R = 0$$

so

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_3^1 = (\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_1^3$$

and

$$(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}})_3^2 = 0$$

and

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_2^3 = -c\tau\overline{L}_2^3 \sin w_0 s_R \sin w_0 s_R - c\tau\overline{L}_2^3 \cos w_0 s_R \cos w_0 s_R + \tau c\overline{L}_2^3 = 0$$

so

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_3^2 = \left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_2^3.$$

Thus the quantity $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ is symmetric and is, in fact, diagonal. Now compute the diagonal elements of $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ as

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_1^1 = \overline{L}_1^1 \cos w_0 s_R + \overline{L}_1^2 \sin w_0 s_R,$$

$$\begin{aligned}
\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_2^2 &= -\tau c\frac{\tau}{\kappa}\overline{L}_2^3 \sin w_0 s_R \sin w_0 s_R - \tau c\frac{\tau}{\kappa}\overline{L}_2^3 \cos w_0 s_R \cos w_0 s_R - \kappa c\overline{L}_2^3 \\
&= -\left(c\frac{\tau^2}{\kappa}\sin^2 w_0 s_R + c\frac{\tau^2}{\kappa}\cos^2 w_0 s_R + \kappa c\right)\overline{L}_2^3 \\
&= -\left(c\frac{\tau^2}{\kappa} + \kappa c\right)\overline{L}_2^3 \\
&= -\left(\frac{\tau^2}{\kappa} + \frac{\kappa^2}{\kappa}\right)c\overline{L}_2^3 \\
&= -\frac{\left(\tau^2+\kappa^2\right)c^2}{\kappa c}\overline{L}_2^3 \\
&= -\frac{1}{\kappa c}\overline{L}_2^3
\end{aligned}$$

using equation (9.39) and

$$\left(\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}\right)_3^3 = 0.$$

Since the left side of equation (9.41) (that is, $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$) is symmetric, then the right side of equation (9.41) must also be symmetric. Thus,

$$\mathbf{X} = \overline{\mathbf{N}}_0 - \overline{\tau}\mathbf{A}_3^{\perp}\overline{\mathbf{M}}_0 + \overline{\kappa}\mathbf{A}_2^{\perp}\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}_R'$$

is symmetric. From equation (6.25), we also have

$$\left(\overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{Z}_0\right)^{\mathrm{T}} = \overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{Z}_0$$

and substituting for the value of $\mathbf{Z}_0$, this becomes

$$\left(\overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\left(\overline{\tau}\mathbf{A}_3^{\perp} - \overline{\kappa}\mathbf{A}_2^{\perp}\right)\right)^{\mathrm{T}} = \overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\mathbf{M}}_0\mathbf{U}_0\left(\overline{\tau}\mathbf{A}_3^{\perp} - \overline{\kappa}\mathbf{A}_2^{\perp}\right)$$

or

$$\left(\overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\tau}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_3^{\perp} + \overline{\kappa}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_2^{\perp}\right)^{\mathrm{T}} = \overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\tau}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_3^{\perp} + \overline{\kappa}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_2^{\perp}.$$

We need to calculate component representations of both of these tensor equations to establish relationships among the components of $\overline{\mathbf{N}}_0$, $\overline{\mathbf{M}}_0$ and $\mathbf{n}$.

We first consider the expression

$$\overline{\mathbf{N}}_0 - \overline{\tau}\mathbf{A}_3^{\perp}\overline{\mathbf{M}}_0 + \overline{\kappa}\mathbf{A}_2^{\perp}\overline{\mathbf{M}}_0 - \mathbf{R}^{\mathrm{T}}\mathbf{n}\otimes\mathbf{r}'_R.$$

The quantity $\mathbf{A}_3^{\perp}\overline{\mathbf{M}}_0$ is calculated as follows:

$$\begin{aligned}
\mathbf{A}_3^{\perp}\overline{\mathbf{M}}_0 &= \left(\mathbf{e}_2\otimes\mathbf{e}^1 - \mathbf{e}_1\otimes\mathbf{e}^2\right)\left(\overline{M}^i_j\mathbf{e}_i\otimes\mathbf{e}^j\right) \\
&= \overline{M}^i_j\left(\mathbf{e}_2\otimes\mathbf{e}^1\right)\left(\mathbf{e}_i\otimes\mathbf{e}^j\right) - \overline{M}^i_j\left(\mathbf{e}_1\otimes\mathbf{e}^2\right)\left(\mathbf{e}_i\otimes\mathbf{e}^j\right) \\
&= \overline{M}^i_j\delta^1_i\mathbf{e}_2\otimes\mathbf{e}^j - \overline{M}^i_j\delta^2_i\mathbf{e}_1\otimes\mathbf{e}^j \\
&= \overline{M}^1_j\mathbf{e}_2\otimes\mathbf{e}^j - \overline{M}^2_j\mathbf{e}_1\otimes\mathbf{e}^j \\
&= \overline{M}^1_j\delta^i_2\mathbf{e}_i\otimes\mathbf{e}^j - \overline{M}^2_j\delta^i_1\mathbf{e}_i\otimes\mathbf{e}^j \\
&= \left(\overline{M}^1_j\delta^i_2 - \overline{M}^2_j\delta^i_1\right)\mathbf{e}_i\otimes\mathbf{e}^j.
\end{aligned}$$

The quantity $\mathbf{A}_2^{\perp}\overline{\mathbf{M}}_0$ is calculated as follows:

$$\begin{aligned}
\mathbf{A}_2^{\perp}\overline{\mathbf{M}}_0 &= \left(\mathbf{e}_1\otimes\mathbf{e}^3 - \mathbf{e}_3\otimes\mathbf{e}^1\right)\left(\overline{M}^i_j\mathbf{e}_i\otimes\mathbf{e}^j\right) \\
&= \overline{M}^i_j\left(\mathbf{e}_1\otimes\mathbf{e}^3\right)\left(\mathbf{e}_i\otimes\mathbf{e}^j\right) - \overline{M}^i_j\left(\mathbf{e}_3\otimes\mathbf{e}^1\right)\left(\mathbf{e}_i\otimes\mathbf{e}^j\right) \\
&= \overline{M}^i_j\delta^3_i\mathbf{e}_1\otimes\mathbf{e}^j - \overline{M}^i_j\delta^1_i\mathbf{e}_3\otimes\mathbf{e}^j \\
&= \overline{M}^3_j\mathbf{e}_1\otimes\mathbf{e}^j - \overline{M}^1_j\mathbf{e}_3\otimes\mathbf{e}^j \\
&= \overline{M}^3_j\delta^i_1\mathbf{e}_i\otimes\mathbf{e}^j - \overline{M}^1_j\delta^i_3\mathbf{e}_i\otimes\mathbf{e}^j \\
&= \left(\overline{M}^3_j\delta^i_1 - \overline{M}^1_j\delta^i_3\right)\mathbf{e}_i\otimes\mathbf{e}^j.
\end{aligned}$$

The quantity $\mathbf{R}^T\mathbf{n} \otimes \mathbf{r}'_R$ was calculated earlier. Putting all this together, the component representation of $\mathbf{X}$ is given as

$$
\begin{aligned}
\mathbf{X} &= \overline{N}^i_j \mathbf{e}_i \otimes \mathbf{e}^j - \overline{\tau}\left(\overline{M}^1_j \delta^i_2 - \overline{M}^2_j \delta^i_1\right) \mathbf{e}_i \otimes \mathbf{e}^j + \overline{\kappa}\left(\overline{M}^3_j \delta^i_1 - \overline{M}^1_j \delta^i_3\right) \mathbf{e}_i \otimes \mathbf{e}^j \\
&\quad - \left(n^1 \cos w_0 s_R + n^2 \sin w_0 s_R\right) \mathbf{e}_1 \otimes \mathbf{e}^3 \\
&\quad - \left(\tau c n^2 \cos w_0 s_R - \tau c n^1 \sin w_0 s_R - \kappa c n^3\right) \mathbf{e}_2 \otimes \mathbf{e}^3 \\
&\quad - \left(\kappa c n^2 \cos w_0 s_R - \kappa c n^1 \sin w_0 s_R + \tau c n^3\right) \mathbf{e}_3 \otimes \mathbf{e}^3 \\
&= \left(\overline{N}^i_j - \overline{\tau}\left(\overline{M}^1_j \delta^i_2 - \overline{M}^2_j \delta^i_1\right) + \overline{\kappa}\left(\overline{M}^3_j \delta^i_1 - \overline{M}^1_j \delta^i_3\right)\right) \mathbf{e}_i \otimes \mathbf{e}^j \\
&\quad - \left(n^1 \cos w_0 s_R + n^2 \sin w_0 s_R\right) \mathbf{e}_1 \otimes \mathbf{e}^3 \\
&\quad - \left(\tau c n^2 \cos w_0 s_R - \tau c n^1 \sin w_0 s_R - \kappa c n^3\right) \mathbf{e}_2 \otimes \mathbf{e}^3 \\
&\quad - \left(\kappa c n^2 \cos w_0 s_R - \kappa c n^1 \sin w_0 s_R + \tau c n^3\right) \mathbf{e}_3 \otimes \mathbf{e}^3.
\end{aligned}
$$

Since $\mathbf{X}$ is symmetric, we must have $X^i_j = X^j_i$ for all $i, j = 1, 2, 3$. When $i = j$, this is trivial and contains no information. We need only consider the three cases $X^1_2 = X^2_1$, $X^1_3 = X^3_1$ and $X^2_3 = X^3_2$. For $X^1_2 = X^2_1$ the component equation is

$$\overline{N}^1_2 + \overline{\tau}\overline{M}^2_2 + \overline{\kappa}\overline{M}^3_2 = \overline{N}^2_1 - \overline{\tau}\overline{M}^1_1. \tag{9.42}$$

For $X^1_3 = X^3_1$ the component equation is

$$\overline{N}^1_3 + \overline{\tau}\overline{M}^2_3 + \overline{\kappa}\overline{M}^3_3 - \left(n^1 \cos w_0 s_R + n^2 \sin w_0 s_R\right) = \overline{N}^3_1 - \overline{\kappa}\overline{M}^1_1. \tag{9.43}$$

For $X^2_3 = X^3_2$ the component equation is

$$\overline{N}^2_3 - \overline{\tau}\overline{M}^1_3 - \left(\tau c n^2 \cos w_0 s_R - \tau c n^1 \sin w_0 s_R - \kappa c n^3\right) = \overline{N}^3_2 - \overline{\kappa}\overline{M}^1_2. \tag{9.44}$$

Now rearrange equation (9.43) as

$$n^1 \cos w_0 s_R + n^2 \sin w_0 s_R = \overline{N}^1_3 - \overline{N}^3_1 + \overline{\kappa}\overline{M}^1_1 + \overline{\tau}\overline{M}^2_3 + \overline{\kappa}\overline{M}^3_3.$$

In this form, the right side consists entirely of constants. Since $n^1$ and $n^2$ are also constants, this equation can only be true for all values of $s_R$ if $n^1 = n^2 = 0$. Thus equations (9.43) and (9.44) reduce to

$$\overline{N}_3^1 - \overline{N}_1^3 + \overline{\kappa}\overline{M}_1^1 + \overline{\tau}\overline{M}_3^2 + \overline{\kappa}\overline{M}_3^3 = 0$$

and

$$\overline{N}_3^2 - \overline{\tau}\overline{M}_3^1 + \kappa c n^3 = \overline{N}_2^3 - \overline{\kappa}\overline{M}_2^1$$

respectively. This means $\mathbf{n} = n^3\mathbf{e}_3 = n_0\mathbf{e}_3$ and

$$\mathbf{R}^{\mathrm{T}}\mathbf{n} \otimes \mathbf{r}_R' = -\kappa c n_0 \mathbf{E}_{23} + \tau c n_0 \mathbf{E}_{33}$$

and the field equation becomes

$$\lambda_3 \rho \mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}} = \overline{\mathbf{N}}_0 - \overline{\tau}\mathbf{A}_3^{\perp}\overline{\mathbf{M}}_0 + \overline{\kappa}\mathbf{A}_2^{\perp}\overline{\mathbf{M}}_0 + \kappa c n_0 \mathbf{E}_{23} - \tau c n_0 \mathbf{E}_{33}. \tag{9.45}$$

The quantity $n_0$ is the length of the vector $\mathbf{n}$. Solving for $n^3 = n_0$ in equation (9.44) and using $n^1 = n^2 = 0$ gives

$$n_0 = \frac{\overline{N}_2^3 - \overline{N}_3^2 + \overline{\tau}\overline{M}_3^1 - \overline{\kappa}\overline{M}_2^1}{\kappa c}. \tag{9.46}$$

We now consider the symmetry of

$$\mathbf{Y} = \overline{\mathbf{N}}_0\mathbf{U}_0 - \overline{\tau}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_3^{\perp} + \overline{\kappa}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_2^{\perp}.$$

We first calculate a component expression for $\mathbf{Y}$. The quantity $\overline{\mathbf{N}}_0\mathbf{U}_0$ evaluates as

$$\begin{aligned}
\overline{\mathbf{N}}_0\mathbf{U}_0 &= \left(\overline{N}_k^i \mathbf{e}_i \otimes \mathbf{e}^k\right)\left(\lambda_{(j)}\mathbf{e}_j \otimes \mathbf{e}^j\right) \\
&= \overline{N}_k^i \delta_j^k \lambda_{(j)} \mathbf{e}_i \otimes \mathbf{e}^j \\
&= \lambda_{(j)} \overline{N}_j^i \mathbf{e}_i \otimes \mathbf{e}^j.
\end{aligned}$$

The quantity $\mathbf{U}_0\mathbf{A}_3^{\perp}$ evaluates as

$$\begin{aligned}
\mathbf{U}_0\mathbf{A}_3^{\perp} &= \left(\lambda_{(k)}\mathbf{e}_k \otimes \mathbf{e}^k\right)\left(\mathbf{e}_2 \otimes \mathbf{e}^1 - \mathbf{e}_1 \otimes \mathbf{e}^2\right) \\
&= \lambda_{(k)}\left(\mathbf{e}_k \otimes \mathbf{e}^k\right)\left(\mathbf{e}_2 \otimes \mathbf{e}^1\right) - \lambda_{(k)}\left(\mathbf{e}_k \otimes \mathbf{e}^k\right)\left(\mathbf{e}_1 \otimes \mathbf{e}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_{(k)}\delta_2^k \mathbf{e}_k \otimes \mathbf{e}^1 - \lambda_{(k)}\delta_1^k \mathbf{e}_k \otimes \mathbf{e}^2 \\
&= \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}^1 - \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}^2
\end{aligned}
$$

so the quantity $\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_3^{\perp}$ evaluates as

$$
\begin{aligned}
\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_3^{\perp} &= \left(\overline{M}^i_j \mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\lambda_2 \mathbf{e}_2 \otimes \mathbf{e}^1 - \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}^2\right) \\
&= \lambda_2 \overline{M}^i_j \left(\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\mathbf{e}_2 \otimes \mathbf{e}^1\right) - \lambda_1 \overline{M}^i_j \left(\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\mathbf{e}_1 \otimes \mathbf{e}^2\right) \\
&= \lambda_2 \overline{M}^i_j \delta^j_2 \mathbf{e}_i \otimes \mathbf{e}^1 - \lambda_1 \overline{M}^i_j \delta^j_1 \mathbf{e}_i \otimes \mathbf{e}^2 \\
&= \lambda_2 \overline{M}^i_2 \mathbf{e}_i \otimes \mathbf{e}^1 - \lambda_1 \overline{M}^i_1 \mathbf{e}_i \otimes \mathbf{e}^2 \\
&= \lambda_2 \overline{M}^i_2 \delta^1_j \mathbf{e}_i \otimes \mathbf{e}^j - \lambda_1 \overline{M}^i_1 \delta^2_j \mathbf{e}_i \otimes \mathbf{e}^j \\
&= \left(\lambda_2 \overline{M}^i_2 \delta^1_j - \lambda_1 \overline{M}^i_1 \delta^2_j\right) \mathbf{e}_i \otimes \mathbf{e}^j .
\end{aligned}
$$

The quantity $\mathbf{U}_0\mathbf{A}_2^{\perp}$ evaluates as

$$
\begin{aligned}
\mathbf{U}_0\mathbf{A}_2^{\perp} &= \left(\lambda_{(k)} \mathbf{e}_k \otimes \mathbf{e}^k\right)\left(\mathbf{e}_1 \otimes \mathbf{e}^3 - \mathbf{e}_3 \otimes \mathbf{e}^1\right) \\
&= \lambda_{(k)}\left(\mathbf{e}_k \otimes \mathbf{e}^k\right)\left(\mathbf{e}_1 \otimes \mathbf{e}^3\right) - \lambda_{(k)}\left(\mathbf{e}_k \otimes \mathbf{e}^k\right)\left(\mathbf{e}_3 \otimes \mathbf{e}^1\right) \\
&= \lambda_{(k)}\delta_1^k \mathbf{e}_k \otimes \mathbf{e}^3 - \lambda_{(k)}\delta_3^k \mathbf{e}_k \otimes \mathbf{e}^1 \\
&= \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}^3 - \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^1
\end{aligned}
$$

so the quantity $\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_2^{\perp}$ evaluates as

$$
\begin{aligned}
\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{A}_2^{\perp} &= \left(\overline{M}^i_j \mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\lambda_1 \mathbf{e}_1 \otimes \mathbf{e}^3 - \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}^1\right) \\
&= \lambda_1 \overline{M}^i_j \left(\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\mathbf{e}_1 \otimes \mathbf{e}^3\right) - \lambda_3 \overline{M}^i_j \left(\mathbf{e}_i \otimes \mathbf{e}^j\right)\left(\mathbf{e}_3 \otimes \mathbf{e}^1\right) \\
&= \lambda_1 \overline{M}^i_j \delta^j_1 \mathbf{e}_i \otimes \mathbf{e}^3 - \lambda_3 \overline{M}^i_j \delta^j_3 \mathbf{e}_i \otimes \mathbf{e}^1 \\
&= \lambda_1 \overline{M}^i_1 \mathbf{e}_i \otimes \mathbf{e}^3 - \lambda_3 \overline{M}^i_3 \mathbf{e}_i \otimes \mathbf{e}^1 \\
&= \lambda_1 \overline{M}^i_1 \delta^3_j \mathbf{e}_i \otimes \mathbf{e}^j - \lambda_3 \overline{M}^i_3 \delta^1_j \mathbf{e}_i \otimes \mathbf{e}^j \\
&= \left(\lambda_1 \overline{M}^i_1 \delta^3_j - \lambda_3 \overline{M}^i_3 \delta^1_j\right) \mathbf{e}_i \otimes \mathbf{e}^j .
\end{aligned}
$$

The expression represented by $\mathbf{Y}$ is now expressed in component from as

$$\begin{aligned}\mathbf{Y} &= \lambda_{(j)}\overline{N}^i_j \mathbf{e}_i \otimes \mathbf{e}^j - \overline{\tau}\left(\lambda_2 \overline{M}^i_2 \delta^1_j - \lambda_1 \overline{M}^i_1 \delta^2_j\right) \mathbf{e}_i \otimes \mathbf{e}^j + \overline{\kappa}\left(\lambda_1 \overline{M}^i_1 \delta^3_j - \lambda_3 \overline{M}^i_3 \delta^1_j\right) \mathbf{e}_i \otimes \mathbf{e}^j \\ &= \left(\lambda_{(j)}\overline{N}^i_j - \overline{\tau}\left(\lambda_2 \overline{M}^i_2 \delta^1_j - \lambda_1 \overline{M}^i_1 \delta^2_j\right) + \overline{\kappa}\left(\lambda_1 \overline{M}^i_1 \delta^3_j - \lambda_3 \overline{M}^i_3 \delta^1_j\right)\right) \mathbf{e}_i \otimes \mathbf{e}^j.\end{aligned}$$

Since $\mathbf{Y}$ is symmetric, we must have $Y^i_j = Y^j_i$ for all $i, j = 1, 2, 3$. When $i = j$, this is trivial and contains no information. We need only consider the three cases $Y^1_2 = Y^2_1$, $Y^1_3 = Y^3_1$ and $Y^2_3 = Y^3_2$. For $Y^1_2 = Y^2_1$ the component equation is

$$\lambda_2 \overline{N}^1_2 + \overline{\tau}\lambda_1 \overline{M}^1_1 = \lambda_1 \overline{N}^2_1 - \overline{\tau}\lambda_2 \overline{M}^2_2 - \overline{\kappa}\lambda_3 \overline{M}^2_3. \tag{9.47}$$

For $Y^1_3 = Y^3_1$ the component equation is

$$\lambda_3 \overline{N}^1_3 + \overline{\kappa}\lambda_1 \overline{M}^1_1 = \lambda_1 \overline{N}^3_1 - \overline{\tau}\lambda_2 \overline{M}^3_2 - \overline{\kappa}\lambda_3 \overline{M}^3_3. \tag{9.48}$$

For $Y^2_3 = Y^3_2$ the component equation is

$$\lambda_3 \overline{N}^2_3 + \overline{\kappa}\lambda_1 \overline{M}^2_1 = \lambda_2 \overline{N}^3_2 + \overline{\tau}\lambda_1 \overline{M}^3_1. \tag{9.49}$$

Solving for $\lambda_3 \overline{N}^2_3$ in equation (9.49) gives

$$\lambda_3 \overline{N}^2_3 = \lambda_2 \overline{N}^3_2 + \overline{\tau}\lambda_1 \overline{M}^3_1 - \overline{\kappa}\lambda_1 \overline{M}^2_1.$$

Multiply through equation (9.46) by $\lambda_3$ to get

$$\lambda_3 n_0 = \frac{\lambda_3 \overline{N}^3_2 - \lambda_3 \overline{N}^2_3 + \overline{\tau}\lambda_3 \overline{M}^1_3 - \overline{\kappa}\lambda_3 \overline{M}^1_2}{\kappa c}$$

and substituting for $\lambda_3 \overline{N}^2_3$ gives

$$\lambda_3 n_0 = \frac{\lambda_3 \overline{N}^3_2 - \left(\lambda_2 \overline{N}^3_2 + \overline{\tau}\lambda_1 \overline{M}^3_1 - \overline{\kappa}\lambda_1 \overline{M}^2_1\right) + \overline{\tau}\lambda_3 \overline{M}^1_3 - \overline{\kappa}\lambda_3 \overline{M}^1_2}{\kappa c}$$

which simplifies to

$$n_0 = \frac{\frac{(\lambda_3-\lambda_2)}{\lambda_3}\overline{N}_2^3 + \tau\left(\lambda_3\overline{M}_3^1 - \lambda_1\overline{M}_1^3\right) + \kappa\left(\lambda_1\overline{M}_1^2 - \lambda_3\overline{M}_2^1\right)}{\kappa c} \tag{9.50}$$

using equations (9.35) and (9.40)

We can now write the component form of the field equation (9.45). The components of the left side of this equation have already been calculated (when it was shown that $\mathbf{R}^{\mathrm{T}}\overline{\mathbf{L}}$ was symmetric and, in fact, diagonal). Equating these with the components of the right side of equation (9.45), the component equations are as follows:

$$\overline{L}_1^1 \cos w_0 s_R + \overline{L}_1^2 \sin w_0 s_R = \overline{N}_1^1 + \overline{\tau}\overline{M}_1^2 + \overline{\kappa}\overline{M}_1^3, \tag{9.51}$$

$$0 = \overline{N}_2^1 + \overline{\tau}\overline{M}_2^2 + \overline{\kappa}\overline{M}_2^3, \tag{9.52}$$

$$0 = \overline{N}_3^1 + \overline{\tau}\overline{M}_3^2 + \overline{\kappa}\overline{M}_3^3, \tag{9.53}$$

$$0 = \overline{N}_1^2 - \overline{\tau}\overline{M}_1^1, \tag{9.54}$$

$$-\frac{1}{\kappa c}\overline{L}_2^3 = \overline{N}_2^2 - \overline{\tau}\overline{M}_2^1, \tag{9.55}$$

$$0 = \overline{N}_3^2 - \overline{\tau}\overline{M}_3^1 + \kappa c n_0, \tag{9.56}$$

$$0 = \overline{N}_1^3 - \overline{\kappa}\overline{M}_1^1, \tag{9.57}$$

$$0 = \overline{N}_2^3 - \overline{\kappa}\overline{M}_2^1 \tag{9.58}$$

and

$$0 = \overline{N}_3^3 - \overline{\kappa}\overline{M}_3^1 - \tau c n_0. \tag{9.59}$$

From equation (9.58), we have $\overline{N}_2^3 = \overline{\kappa}\overline{M}_2^1$ and thus

$$\frac{(\lambda_3-\lambda_2)}{\lambda_3}\overline{N}_2^3 = \frac{(\lambda_3-\lambda_2)}{\lambda_3}\overline{\kappa}\overline{M}_2^1 = (\lambda_3-\lambda_2)\,\kappa\overline{M}_2^1$$

using equations (9.35) and (9.40). Then equation (9.50) becomes

$$n_0 = \frac{(\lambda_3 - \lambda_2)\,\kappa\overline{M}_2^1 + \tau\left(\lambda_3\overline{M}_3^1 - \lambda_1\overline{M}_1^3\right) + \kappa\left(\lambda_1\overline{M}_1^2 - \lambda_3\overline{M}_2^1\right)}{\kappa c}$$

which simplifies to

$$n_0 = \frac{\tau\left(\lambda_3\overline{M}_3^1 - \lambda_1\overline{M}_1^3\right) + \kappa\left(\lambda_1\overline{M}_1^2 - \lambda_2\overline{M}_2^1\right)}{\kappa c}. \tag{9.60}$$

If we multiply through the sum of equations (9.56) and (9.58) by $\lambda_3$ we obtain

$$\lambda_3\overline{N}_2^3 + \lambda_3\overline{N}_3^2 - \overline{\kappa}\lambda_3\overline{M}_2^1 - \overline{\tau}\lambda_3\overline{M}_3^1 + \kappa c\lambda_3 n_0 = 0.$$

Rearranging equation (9.49) gives

$$\lambda_3\overline{N}_3^2 + \overline{\kappa}\lambda_1\overline{M}_1^2 - \lambda_2\overline{N}_2^3 - \overline{\tau}\lambda_1\overline{M}_1^3 = 0$$

and adding this to the previous equation gives

$$\lambda_3\overline{N}_3^2 + \overline{\kappa}\lambda_1\overline{M}_1^2 - \lambda_2\overline{N}_2^3 - \overline{\tau}\lambda_1\overline{M}_1^3 + \lambda_3\overline{N}_2^3 + \lambda_3\overline{N}_3^2 - \overline{\kappa}\lambda_3\overline{M}_2^1 - \overline{\tau}\lambda_3\overline{M}_3^1 + \overline{\kappa}cn_0 = 0$$

which, upon simplifying gives

$$2\lambda_3\overline{N}_3^2 + \overline{\kappa}\lambda_3\overline{M}_2^1 - \overline{\kappa}\lambda_2\overline{M}_2^1 + \overline{\kappa}\lambda_1\overline{M}_1^2 - \overline{\kappa}\lambda_3\overline{M}_2^1 - \overline{\tau}\lambda_1\overline{M}_1^3 - \overline{\tau}\lambda_3\overline{M}_3^1 + \overline{\kappa}cn_0 = 0$$

which, in turn, gives

$$2\lambda_3\overline{N}_3^2 - \overline{\kappa}\lambda_2\overline{M}_2^1 + \overline{\kappa}\lambda_1\overline{M}_1^2 - \overline{\tau}\lambda_1\overline{M}_1^3 - \overline{\tau}\lambda_3\overline{M}_3^1 + \overline{\kappa}cn_0 = 0.$$

Thus the field equation (9.45) becomes

$$\lambda_3\rho\mathbf{R}^T\overline{\mathbf{L}} = \left(\overline{N}_1^1 + \overline{\tau}\overline{M}_1^2 + \overline{\kappa}\overline{M}_1^3\right)\mathbf{E}_{11} + \left(\overline{N}_2^2 - \overline{\tau}\overline{M}_2^1\right)\mathbf{E}_{22}.$$

We now apply monotropic symmetry with the identity as the symmetric element and $\mathbf{Q}_1$ as the anti-symmetric element. Hence we use equations (7.32) and (7.33). In order to use this, we need the conditions to be satisfied:

$$\mathbf{Q}_1\mathbf{U}_0 = \mathbf{U}_0\mathbf{Q}_1$$

and

$$\mathbf{Q}_1\mathbf{Z}_0\mathbf{U}_0 = -\mathbf{Z}_0\mathbf{U}_0\mathbf{Q}_1.$$

The first of these conditions was shown to be true in Chapter 7. We now show the second condition is true.

The quantity $\mathbf{Z}_0\mathbf{U}_0$ is calculated as

$$\begin{aligned}
\mathbf{Z}_0\mathbf{U}_0 &= \left(\overline{\tau}\mathbf{A}_3^{\perp} - \overline{\kappa}\mathbf{A}_2^{\perp}\right)(\lambda_1\mathbf{E}_{11} + \lambda_2\mathbf{E}_{22} + \lambda_3\mathbf{E}_{33}) \\
&= \overline{\tau}\lambda_1\mathbf{A}_3^{\perp}\mathbf{E}_{11} + \overline{\tau}\lambda_2\mathbf{A}_3^{\perp}\mathbf{E}_{22} + \overline{\tau}\lambda_3\mathbf{A}_3^{\perp}\mathbf{E}_{33} - \overline{\kappa}\lambda_1\mathbf{A}_2^{\perp}\mathbf{E}_{11} - \overline{\kappa}\lambda_2\mathbf{A}_2^{\perp}\mathbf{E}_{22} - \overline{\kappa}\lambda_3\mathbf{A}_2^{\perp}\mathbf{E}_{33} \\
&= \overline{\tau}\lambda_1\mathbf{E}_{21} - \overline{\tau}\lambda_2\mathbf{E}_{12} + \overline{\kappa}\lambda_1\mathbf{E}_{31} - \overline{\kappa}\lambda_3\mathbf{E}_{13}
\end{aligned}$$

using Table 6.1.

The quantity $\mathbf{Q}_1\mathbf{Z}_0\mathbf{U}_0$ is calculated as

$$\begin{aligned}
\mathbf{Q}_1\mathbf{Z}_0\mathbf{U}_0 &= (\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33})(\overline{\tau}\lambda_1\mathbf{E}_{21} - \overline{\tau}\lambda_2\mathbf{E}_{12} + \overline{\kappa}\lambda_1\mathbf{E}_{31} - \overline{\kappa}\lambda_3\mathbf{E}_{13}) \\
&= \overline{\tau}\lambda_1(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33})\mathbf{E}_{21} - \overline{\tau}\lambda_2(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33})\mathbf{E}_{12} \\
&\quad + \overline{\kappa}\lambda_1(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33})\mathbf{E}_{31} - \overline{\kappa}\lambda_3(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33})\mathbf{E}_{13} \\
&= -\overline{\tau}\lambda_1\mathbf{E}_{21} - \overline{\tau}\lambda_2\mathbf{E}_{12} - \overline{\kappa}\lambda_1\mathbf{E}_{31} - \overline{\kappa}\lambda_3\mathbf{E}_{13}.
\end{aligned}$$

The quantity $\mathbf{Z}_0\mathbf{U}_0\mathbf{Q}_1$ is calculated as

$$\begin{aligned}
\mathbf{Z}_0\mathbf{U}_0\mathbf{Q}_1 &= (\overline{\tau}\lambda_1\mathbf{E}_{21} - \overline{\tau}\lambda_2\mathbf{E}_{12} + \overline{\kappa}\lambda_1\mathbf{E}_{31} - \overline{\kappa}\lambda_3\mathbf{E}_{13})(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= \overline{\tau}\lambda_1\mathbf{E}_{21}(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) - \overline{\tau}\lambda_2\mathbf{E}_{12}(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&\quad + \overline{\kappa}\lambda_1\mathbf{E}_{31}(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) - \overline{\kappa}\lambda_3\mathbf{E}_{13}(\mathbf{E}_{11} - \mathbf{E}_{22} - \mathbf{E}_{33}) \\
&= \overline{\tau}\lambda_1\mathbf{E}_{21} + \overline{\tau}\lambda_2\mathbf{E}_{12} + \overline{\kappa}\lambda_1\mathbf{E}_{31} + \overline{\kappa}\lambda_3\mathbf{E}_{13}.
\end{aligned}$$

Thus $\mathbf{Q}_1\mathbf{Z}_0\mathbf{U}_0 = -\mathbf{Z}_0\mathbf{U}_0\mathbf{Q}_1$ and the second condition is satisfied.

Since both conditions are satisfied, we may use equations (7.32) and (7.33) to describe the monotropic symmetry:

$$\mathbf{Q}_1\overline{\mathbf{N}}_0 = \overline{\mathbf{N}}_0\mathbf{Q}_1$$

and

$$\mathbf{Q}_1\overline{\mathbf{M}}_0 = -\overline{\mathbf{M}}_0\mathbf{Q}_1.$$

From equations (7.37) and (7.38), we know

$$\overline{N}_2^1 = \overline{N}_3^1 = \overline{N}_1^2 = \overline{N}_1^3 = 0$$

and

$$\overline{M}_3^2 = \overline{M}_2^3 = \overline{M}_1^1 = \overline{M}_2^2 = \overline{M}_3^3 = 0.$$

These results for monotropic symmetry are consistent with satisfying equations (9.52), (9.54) and (9.57). They do not effect any of the other equations for this case.

## 9.3 Calculation of the Wrench

In this section, we further consider the straight to helical deformation by computing the resulting torque on the ends of the rod. We then see that the déformation may be accomplished by the application of a wrench. The wrench is defined to be the pair $(\mathbf{n}, \mathbf{C}_0)$ where $\mathbf{n} = n_0\mathbf{e}_3$ and the quantity $\mathbf{C}_0$ is the torque and is given by

$$\mathbf{C}_0 \equiv sk\left(\mathbf{M} + \mathbf{r} \otimes \mathbf{n}\right)$$

where $\mathbf{M}$ is the tensor in Cohen's paper. The quantity $\mathbf{C}_0$ represents the resulting torque acting on the ends of the rod to produce the deformation. Note

$$\mathbf{C}_0' \equiv sk\left(\mathbf{M} + \mathbf{r} \otimes \mathbf{n}\right)' = sk\left(\mathbf{M}' + \mathbf{r}' \otimes \mathbf{n}\right).$$

From equations (C2.14), (C2.15) and (C2.16),

$$sk\left(\mathbf{M}' + \mathbf{r}'\otimes\mathbf{n}\right) = \mathbf{0}$$

so $\mathbf{C}_0' = \mathbf{0}$. Thus $\mathbf{C}_0$ is a constant skew-symmetric tensor. Then $\mathbf{C}_0$ must have an axial vector. The objective of this section is to express $\mathbf{C}_0$ in terms of the tensor $\mathbf{A}_3^{\perp}$ and show the axial vector of $\mathbf{C}_0$ is parallel to $\mathbf{e}_3$.

In this thesis, $\mathbf{M}$ is related to $\overline{\mathbf{M}}$ by $\mathbf{M} = \mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}}$. Thus we could write

$$\mathbf{C}_0 \equiv sk\left(\mathbf{M} + \mathbf{r}\otimes\mathbf{n}\right) = sk\left(\mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}} + (\mathbf{n}\otimes\mathbf{r})^{\mathrm{T}}\right).$$

This evaluates as

$$\begin{aligned}
\mathbf{C}_0 &= \frac{1}{2}\left(\mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}} + (\mathbf{n}\otimes\mathbf{r})^{\mathrm{T}} - \left(\mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}} + (\mathbf{n}\otimes\mathbf{r})^{\mathrm{T}}\right)^{\mathrm{T}}\right) \\
&= \frac{1}{2}\left(\mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}} + (\mathbf{n}\otimes\mathbf{r})^{\mathrm{T}} - \left(\mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}}\right)^{\mathrm{T}} - \left((\mathbf{n}\otimes\mathbf{r})^{\mathrm{T}}\right)^{\mathrm{T}}\right) \\
&= \frac{1}{2}\left(\mathbf{F}\overline{\mathbf{M}}^{\mathrm{T}} - \overline{\mathbf{M}}\mathbf{F}^{\mathrm{T}} + (\mathbf{n}\otimes\mathbf{r})^{\mathrm{T}} - \mathbf{n}\otimes\mathbf{r}\right)
\end{aligned}$$

using the definition of skew-symmetric and properties of transpose.

The quantity $\overline{\mathbf{M}}$ is given by

$$\overline{\mathbf{M}} = \mathbf{R}\overline{\mathbf{M}}\left(\mathbf{U}_0, \mathbf{Z}_0\mathbf{U}_0\right) = \mathbf{R}\overline{\mathbf{M}}_0$$

and $\mathbf{F}$ is given by the polar decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U}_0$$

and thus

$$\begin{aligned}
\mathbf{C}_0 &= \frac{1}{2}\left(\mathbf{R}\mathbf{U}_0\left(\mathbf{R}\overline{\mathbf{M}}_0\right)^{\mathrm{T}} - \mathbf{R}\overline{\mathbf{M}}_0\left(\mathbf{R}\mathbf{U}_0\right)^{\mathrm{T}} + (\mathbf{n}\otimes\mathbf{r})^{\mathrm{T}} - \mathbf{n}\otimes\mathbf{r}\right) \\
&= \frac{1}{2}\left(\mathbf{R}\mathbf{U}_0\overline{\mathbf{M}}_0^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} - \mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}} + (\mathbf{n}\otimes\mathbf{r})^{\mathrm{T}} - \mathbf{n}\otimes\mathbf{r}\right)
\end{aligned}$$

using properties of transpose and the fact $\mathbf{U}_0$ is symmetric. The objective of this section is to compute a component form representation of $\mathbf{C}_0$.

The equation of the deformed rod axis is given by

$$\mathbf{r}(s_R) = a \cos w_0 s_R \mathbf{e}_1 + a \sin w_0 s_R \mathbf{e}_2 + b w_0 s_R \mathbf{e}_3.$$

We showed earlier that $\mathbf{n} = n_0 \mathbf{e}^3 = n_0 \mathbf{e}_3$. Thus the quantity $\mathbf{n} \otimes \mathbf{r}$ calculates as

$$\begin{aligned} \mathbf{n} \otimes \mathbf{r} &= n_0 \mathbf{e}_3 \otimes \left(a \cos w_0 s_R \mathbf{e}^1 + a \sin w_0 s_R \mathbf{e}^2 + b w_0 s_R \mathbf{e}^3\right) \\ &= a n_0 \cos w_0 s_R \mathbf{e}_3 \otimes \mathbf{e}^1 + a n_0 \sin w_0 s_R \mathbf{e}_3 \otimes \mathbf{e}^2 + b w_0 n_0 s_R \mathbf{e}_3 \otimes \mathbf{e}^3 \end{aligned}$$

and its transpose is given by

$$(\mathbf{n} \otimes \mathbf{r})^{\mathrm{T}} = a n_0 \cos w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^3 + a n_0 \sin w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^3 + b w_0 n_0 s_R \mathbf{e}_3 \otimes \mathbf{e}^3.$$

Then

$$\begin{aligned} (\mathbf{n} \otimes \mathbf{r})^{\mathrm{T}} - \mathbf{n} \otimes \mathbf{r} &= a n_0 \cos w_0 s_R \mathbf{e}_1 \otimes \mathbf{e}^3 + a n_0 \sin w_0 s_R \mathbf{e}_2 \otimes \mathbf{e}^3 \\ &\quad - a n_0 \cos w_0 s_R \mathbf{e}_3 \otimes \mathbf{e}^1 - a n_0 \sin w_0 s_R \mathbf{e}_3 \otimes \mathbf{e}^2. \end{aligned}$$

We now calculate $\mathbf{R}\overline{\mathbf{M}}_0 \mathbf{U}_0 \mathbf{R}^{\mathrm{T}}$. We include the results of monotropic symmetry so

$$\overline{\mathbf{M}}_0 = \overline{M}_2^1 \mathbf{E}_{12} + \overline{M}_3^1 \mathbf{E}_{13} + \overline{M}_1^2 \mathbf{E}_{21} + \overline{M}_1^3 \mathbf{E}_{31}.$$

Then

$$\begin{aligned} \mathbf{R}\overline{\mathbf{M}}_0 &= \cos w_0 s_R \mathbf{E}_{11} \overline{\mathbf{M}}_0 - \tau c \sin w_0 s_R \mathbf{E}_{12} \overline{\mathbf{M}}_0 - \kappa c \sin w_0 s_R \mathbf{E}_{13} \overline{\mathbf{M}}_0 \\ &\quad + \sin w_0 s_R \mathbf{E}_{21} \overline{\mathbf{M}}_0 + \tau c \cos w_0 s_R \mathbf{E}_{22} \overline{\mathbf{M}}_0 + \kappa c \cos w_0 s_R \mathbf{E}_{23} \overline{\mathbf{M}}_0 \\ &\quad - \kappa c \mathbf{E}_{32} \overline{\mathbf{M}}_0 + \tau c \mathbf{E}_{33} \overline{\mathbf{M}}_0. \end{aligned}$$

Computing these terms gives

$$\mathbf{E}_{11}\overline{\mathbf{M}}_0 = \overline{M}_2^1\mathbf{E}_{11}\mathbf{E}_{12} + \overline{M}_3^1\mathbf{E}_{11}\mathbf{E}_{13} + \overline{M}_1^2\mathbf{E}_{11}\mathbf{E}_{21} + \overline{M}_1^3\mathbf{E}_{11}\mathbf{E}_{31} = \overline{M}_2^1\mathbf{E}_{12} + \overline{M}_3^1\mathbf{E}_{13},$$

$$\mathbf{E}_{12}\overline{\mathbf{M}}_0 = \overline{M}_2^1\mathbf{E}_{12}\mathbf{E}_{12} + \overline{M}_3^1\mathbf{E}_{12}\mathbf{E}_{13} + \overline{M}_1^2\mathbf{E}_{12}\mathbf{E}_{21} + \overline{M}_1^3\mathbf{E}_{12}\mathbf{E}_{31} = \overline{M}_1^2\mathbf{E}_{11},$$

$$\mathbf{E}_{13}\overline{\mathbf{M}}_0 = \overline{M}_2^1\mathbf{E}_{13}\mathbf{E}_{12} + \overline{M}_3^1\mathbf{E}_{13}\mathbf{E}_{13} + \overline{M}_1^2\mathbf{E}_{13}\mathbf{E}_{21} + \overline{M}_1^3\mathbf{E}_{13}\mathbf{E}_{31} = \overline{M}_1^3\mathbf{E}_{11},$$

$$\mathbf{E}_{21}\overline{\mathbf{M}}_0 = \overline{M}_2^1\mathbf{E}_{21}\mathbf{E}_{12} + \overline{M}_3^1\mathbf{E}_{21}\mathbf{E}_{13} + \overline{M}_1^2\mathbf{E}_{21}\mathbf{E}_{21} + \overline{M}_1^3\mathbf{E}_{21}\mathbf{E}_{31} = \overline{M}_2^1\mathbf{E}_{22} + \overline{M}_3^1\mathbf{E}_{23},$$

$$\mathbf{E}_{22}\overline{\mathbf{M}}_0 = \overline{M}_2^1\mathbf{E}_{22}\mathbf{E}_{12} + \overline{M}_3^1\mathbf{E}_{22}\mathbf{E}_{13} + \overline{M}_1^2\mathbf{E}_{22}\mathbf{E}_{21} + \overline{M}_1^3\mathbf{E}_{22}\mathbf{E}_{31} = \overline{M}_1^2\mathbf{E}_{21},$$

$$\mathbf{E}_{23}\overline{\mathbf{M}}_0 = \overline{M}_2^1\mathbf{E}_{23}\mathbf{E}_{12} + \overline{M}_3^1\mathbf{E}_{23}\mathbf{E}_{13} + \overline{M}_1^2\mathbf{E}_{23}\mathbf{E}_{21} + \overline{M}_1^3\mathbf{E}_{23}\mathbf{E}_{31} = \overline{M}_1^3\mathbf{E}_{21},$$

$$\mathbf{E}_{32}\overline{\mathbf{M}}_0 = \overline{M}_2^1\mathbf{E}_{32}\mathbf{E}_{12} + \overline{M}_3^1\mathbf{E}_{32}\mathbf{E}_{13} + \overline{M}_1^2\mathbf{E}_{32}\mathbf{E}_{21} + \overline{M}_1^3\mathbf{E}_{32}\mathbf{E}_{31} = \overline{M}_1^2\mathbf{E}_{31}$$

and

$$\mathbf{E}_{33}\overline{\mathbf{M}}_0 = \overline{M}_2^1\mathbf{E}_{33}\mathbf{E}_{12} + \overline{M}_3^1\mathbf{E}_{33}\mathbf{E}_{13} + \overline{M}_1^2\mathbf{E}_{33}\mathbf{E}_{21} + \overline{M}_1^3\mathbf{E}_{33}\mathbf{E}_{31} = \overline{M}_1^3\mathbf{E}_{31}.$$

Thus

$$\begin{aligned}\mathbf{R}\overline{\mathbf{M}}_0 \;=\;& \overline{M}_2^1\cos w_0 s_R\mathbf{E}_{12} + \overline{M}_3^1\cos w_0 s_R\mathbf{E}_{13} - \tau c\overline{M}_1^2\sin w_0 s_R\mathbf{E}_{11} - \kappa c\overline{M}_1^3\sin w_0 s_R\mathbf{E}_{11}\\ &+\overline{M}_2^1\sin w_0 s_R\mathbf{E}_{22} + \overline{M}_3^1\sin w_0 s_R\mathbf{E}_{23} + \tau c\overline{M}_1^2\cos w_0 s_R\mathbf{E}_{21}\\ &+\kappa c\overline{M}_1^3\cos w_0 s_R\mathbf{E}_{21} - \kappa c\overline{M}_1^2\mathbf{E}_{31} + \tau c\overline{M}_1^3\mathbf{E}_{31}\end{aligned}$$

and upon simplifying

$$\begin{aligned}\mathbf{R}\overline{\mathbf{M}}_0 \;=\;& -\left(\tau c\overline{M}_1^2 + \kappa c\overline{M}_1^3\right)\sin w_0 s_R\mathbf{E}_{11} + \overline{M}_2^1\cos w_0 s_R\mathbf{E}_{12} + \overline{M}_3^1\cos w_0 s_R\mathbf{E}_{13}\\ &+\left(\tau c\overline{M}_1^2 + \kappa c\overline{M}_1^3\right)\cos w_0 s_R\mathbf{E}_{21} + \overline{M}_2^1\sin w_0 s_R\mathbf{E}_{22} + \overline{M}_3^1\sin w_0 s_R\mathbf{E}_{23}\\ &+\left(\tau c\overline{M}_1^3 - \kappa c\overline{M}_1^2\right)\mathbf{E}_{31}.\end{aligned}$$

The quantity $\mathbf{U}_0\mathbf{R}^{\mathrm{T}}$ is $\mathbf{F}^{\mathrm{T}}$. Using the expression for $\mathbf{F}$ given earlier, $\mathbf{F}^{\mathrm{T}}$ can be expressed in component form by

$$\begin{aligned}
\mathbf{F}^{\mathrm{T}} &= \lambda_1 \cos w_0 s_R \mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{12} - \tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{21} + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{22} \\
&\quad -\kappa c \lambda_2 \mathbf{E}_{23} - \kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{31} + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{32} + \tau c \lambda_3 \mathbf{E}_{33}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}} &= -\left(\tau c \overline{M}_1^2 + \kappa c \overline{M}_1^3\right) \sin w_0 s_R \mathbf{E}_{11}\mathbf{F}^{\mathrm{T}} + \overline{M}_2^1 \cos w_0 s_R \mathbf{E}_{12}\mathbf{F}^{\mathrm{T}} \\
&\quad + \overline{M}_3^1 \cos w_0 s_R \mathbf{E}_{13}\mathbf{F}^{\mathrm{T}} + \left(\tau c \overline{M}_1^2 + \kappa c \overline{M}_1^3\right) \cos w_0 s_R \mathbf{E}_{21}\mathbf{F}^{\mathrm{T}} \\
&\quad + \overline{M}_2^1 \sin w_0 s_R \mathbf{E}_{22}\mathbf{F}^{\mathrm{T}} + \overline{M}_3^1 \sin w_0 s_R \mathbf{E}_{23}\mathbf{F}^{\mathrm{T}} + \left(\tau c \overline{M}_1^3 - \kappa c \overline{M}_1^2\right) \mathbf{E}_{31}\mathbf{F}^{\mathrm{T}}.
\end{aligned}$$

We compute each of these terms separately. Then

$$\begin{aligned}
\mathbf{E}_{11}\mathbf{F}^{\mathrm{T}} &= \lambda_1 \cos w_0 s_R \mathbf{E}_{11}\mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{11}\mathbf{E}_{12} - \tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{11}\mathbf{E}_{21} \\
&\quad + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{11}\mathbf{E}_{22} - \kappa c \lambda_2 \mathbf{E}_{11}\mathbf{E}_{23} - \kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{11}\mathbf{E}_{31} \\
&\quad + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{11}\mathbf{E}_{32} + \tau c \lambda_3 \mathbf{E}_{11}\mathbf{E}_{33} \\
&= \lambda_1 \cos w_0 s_R \mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{12},
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}_{12}\mathbf{F}^{\mathrm{T}} &= \lambda_1 \cos w_0 s_R \mathbf{E}_{12}\mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{12}\mathbf{E}_{12} - \tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{12}\mathbf{E}_{21} \\
&\quad + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{12}\mathbf{E}_{22} - \kappa c \lambda_2 \mathbf{E}_{12}\mathbf{E}_{23} - \kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{12}\mathbf{E}_{31} \\
&\quad + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{12}\mathbf{E}_{32} + \tau c \lambda_3 \mathbf{E}_{12}\mathbf{E}_{33} \\
&= -\tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{11} + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{12} - \kappa c \lambda_2 \mathbf{E}_{13},
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}_{13}\mathbf{F}^{\mathrm{T}} &= \lambda_1 \cos w_0 s_R \mathbf{E}_{13}\mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{13}\mathbf{E}_{12} - \tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{13}\mathbf{E}_{21} \\
&\quad + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{13}\mathbf{E}_{22} - \kappa c \lambda_2 \mathbf{E}_{13}\mathbf{E}_{23} - \kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{13}\mathbf{E}_{31} \\
&\quad + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{13}\mathbf{E}_{32} + \tau c \lambda_3 \mathbf{E}_{13}\mathbf{E}_{33} \\
&= -\kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{11} + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{12} + \tau c \lambda_3 \mathbf{E}_{13},
\end{aligned}$$

$$
\begin{aligned}
\mathbf{E}_{21}\mathbf{F}^{\mathrm{T}} &= \lambda_1 \cos w_0 s_R \mathbf{E}_{21}\mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{21}\mathbf{E}_{12} - \tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{21}\mathbf{E}_{21} \\
&\quad + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{21}\mathbf{E}_{22} - \kappa c \lambda_2 \mathbf{E}_{21}\mathbf{E}_{23} - \kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{21}\mathbf{E}_{31} \\
&\quad + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{21}\mathbf{E}_{32} + \tau c \lambda_3 \mathbf{E}_{21}\mathbf{E}_{33} \\
&= \lambda_1 \cos w_0 s_R \mathbf{E}_{21} + \lambda_1 \sin w_0 s_R \mathbf{E}_{22},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}_{22}\mathbf{F}^{\mathrm{T}} &= \lambda_1 \cos w_0 s_R \mathbf{E}_{22}\mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{22}\mathbf{E}_{12} - \tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{22}\mathbf{E}_{21} \\
&\quad + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{22}\mathbf{E}_{22} - \kappa c \lambda_2 \mathbf{E}_{22}\mathbf{E}_{23} - \kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{22}\mathbf{E}_{31} \\
&\quad + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{22}\mathbf{E}_{32} + \tau c \lambda_3 \mathbf{E}_{22}\mathbf{E}_{33} \\
&= -\tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{21} + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{22} - \kappa c \lambda_2 \mathbf{E}_{23},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}_{23}\mathbf{F}^{\mathrm{T}} &= \lambda_1 \cos w_0 s_R \mathbf{E}_{23}\mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{23}\mathbf{E}_{12} - \tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{23}\mathbf{E}_{21} \\
&\quad + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{23}\mathbf{E}_{22} - \kappa c \lambda_2 \mathbf{E}_{23}\mathbf{E}_{23} - \kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{23}\mathbf{E}_{31} \\
&\quad + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{23}\mathbf{E}_{32} + \tau c \lambda_3 \mathbf{E}_{23}\mathbf{E}_{33} \\
&= -\kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{21} + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{22} + \tau c \lambda_3 \mathbf{E}_{23}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{E}_{31}\mathbf{F}^{\mathrm{T}} &= \lambda_1 \cos w_0 s_R \mathbf{E}_{31}\mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{31}\mathbf{E}_{12} - \tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{31}\mathbf{E}_{21} \\
&\quad + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{31}\mathbf{E}_{22} - \kappa c \lambda_2 \mathbf{E}_{31}\mathbf{E}_{23} - \kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{31}\mathbf{E}_{31} \\
&\quad + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{31}\mathbf{E}_{32} + \tau c \lambda_3 \mathbf{E}_{31}\mathbf{E}_{33} \\
&= \lambda_1 \cos w_0 s_R \mathbf{E}_{31} + \lambda_1 \sin w_0 s_R \mathbf{E}_{32}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}} &= -\left(\tau c \overline{M}_1^2 + \kappa c \overline{M}_1^3\right) \sin w_0 s_R \left(\lambda_1 \cos w_0 s_R \mathbf{E}_{11} + \lambda_1 \sin w_0 s_R \mathbf{E}_{12}\right) \\
&\quad + \overline{M}_2^1 \cos w_0 s_R \left(-\tau c \lambda_2 \sin w_0 s_R \mathbf{E}_{11} + \tau c \lambda_2 \cos w_0 s_R \mathbf{E}_{12} - \kappa c \lambda_2 \mathbf{E}_{13}\right) \\
&\quad + \overline{M}_3^1 \cos w_0 s_R \left(-\kappa c \lambda_3 \sin w_0 s_R \mathbf{E}_{11} + \kappa c \lambda_3 \cos w_0 s_R \mathbf{E}_{12} + \tau c \lambda_3 \mathbf{E}_{13}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\tau c\overline{M}_1^2+\kappa c\overline{M}_1^3\right)\cos w_0 s_R\left(\lambda_1\cos w_0 s_R\mathbf{E}_{21}+\lambda_1\sin w_0 s_R\mathbf{E}_{22}\right)\\
&+\overline{M}_2^1\sin w_0 s_R\left(-\tau c\lambda_2\sin w_0 s_R\mathbf{E}_{21}+\tau c\lambda_2\cos w_0 s_R\mathbf{E}_{22}-\kappa c\lambda_2\mathbf{E}_{23}\right)\\
&+\overline{M}_3^1\sin w_0 s_R\left(-\kappa c\lambda_3\sin w_0 s_R\mathbf{E}_{21}+\kappa c\lambda_3\cos w_0 s_R\mathbf{E}_{22}+\tau c\lambda_3\mathbf{E}_{23}\right)\\
&+\left(\tau c\overline{M}_1^3-\kappa c\overline{M}_1^2\right)\left(\lambda_1\cos w_0 s_R\mathbf{E}_{31}+\lambda_1\sin w_0 s_R\mathbf{E}_{32}\right)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}} = & -\left(\tau c\overline{M}_1^2\lambda_1+\kappa c\overline{M}_1^3\lambda_1+\tau c\overline{M}_2^1\lambda_2+\kappa c\overline{M}_3^1\lambda_3\right)\sin w_0 s_R\cos w_0 s_R\mathbf{E}_{11}\\
&+(\left(\tau c\overline{M}_2^1\lambda_2+\kappa c\overline{M}_3^1\lambda_3\right)\cos^2 w_0 s_R\\
&-\left(\tau c\overline{M}_1^2\lambda_1+\kappa c\overline{M}_1^3\lambda_1\right)\sin^2 w_0 s_R)\mathbf{E}_{12}\\
&+\left(\tau c\overline{M}_3^1\lambda_3-\kappa c\overline{M}_2^1\lambda_2\right)\cos w_0 s_R\mathbf{E}_{13}\\
&+(\left(\tau c\overline{M}_1^2\lambda_1+\kappa c\overline{M}_1^3\lambda_1\right)\cos^2 w_0 s_R\\
&-\left(\tau c\overline{M}_2^1\lambda_2+\kappa c\overline{M}_3^1\lambda_3\right)\sin^2 w_0 s_R)\mathbf{E}_{21}\\
&+\left(\tau c\overline{M}_1^2\lambda_1+\kappa c\overline{M}_1^3\lambda_1+\tau c\overline{M}_2^1\lambda_2+\kappa c\overline{M}_3^1\lambda_3\right)\sin w_0 s_R\cos w_0 s_R\mathbf{E}_{22}\\
&+\left(\tau c\overline{M}_3^1\lambda_3-\kappa c\overline{M}_2^1\lambda_2\right)\sin w_0 s_R\mathbf{E}_{23}\\
&+\left(\tau c\overline{M}_1^3\lambda_1-\kappa c\overline{M}_1^2\lambda_1\right)\cos w_0 s_R\mathbf{E}_{31}\\
&+\left(\tau c\overline{M}_1^3\lambda_1-\kappa c\overline{M}_1^2\lambda_1\right)\sin w_0 s_R\mathbf{E}_{32}.
\end{aligned}
$$

Now we can calculate the components of

$$
\mathbf{C}_0=\frac{1}{2}\left(\mathbf{R}\mathbf{U}_0\overline{\mathbf{M}}_0^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}-\mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}}+(\mathbf{n}\otimes\mathbf{r})^{\mathrm{T}}-\mathbf{n}\otimes\mathbf{r}\right).
$$

Since this is a skew-symmetric tensor, we need only calculate three components, namely $(2,1)$, $(1,3)$ and $(2,3)$ (we choose to calculate $(2,1)$ instead of $(1,2)$ for a slight convenience later). Then

$$
\begin{aligned}
2\left(\mathbf{C}_0\right)_1^2 &= \left(\mathbf{R}\,\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}}\right)_2^1-\left(\mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}}\right)_1^2+(\mathbf{r}\otimes\mathbf{n})_1^2-(\mathbf{n}\otimes\mathbf{r})_1^2\\
&= \left(\tau c\overline{M}_2^1\lambda_2+\kappa c\overline{M}_3^1\lambda_3\right)\cos^2 w_0 s_R-\left(\tau c\overline{M}_1^2\lambda_1+\kappa c\overline{M}_1^3\lambda_1\right)\sin^2 w_0 s_R\\
&\quad -\left(\left(\tau c\overline{M}_1^2\lambda_1+\kappa c\overline{M}_1^3\lambda_1\right)\cos^2 w_0 s_R-\left(\tau c\overline{M}_2^1\lambda_2+\kappa c\overline{M}_3^1\lambda_3\right)\sin^2 w_0 s_R\right)
\end{aligned}
$$

$$\begin{aligned}
&= \left(\tau c\overline{M}^1_2\lambda_2 + \kappa c\overline{M}^1_3\lambda_3\right)\cos^2 w_0 s_R - \left(\tau c\overline{M}^2_1\lambda_1 + \kappa c\overline{M}^3_1\lambda_1\right)\sin^2 w_0 s_R \\
&\quad - \left(\tau c\overline{M}^2_1\lambda_1 + \kappa c\overline{M}^3_1\lambda_1\right)\cos^2 w_0 s_R + \left(\tau c\overline{M}^1_2\lambda_2 + \kappa c\overline{M}^1_3\lambda_3\right)\sin^2 w_0 s_R \\
&= \left(\tau c\overline{M}^1_2\lambda_2 + \kappa c\overline{M}^1_3\lambda_3\right)\cos^2 w_0 s_R + \left(\tau c\overline{M}^1_2\lambda_2 + \kappa c\overline{M}^1_3\lambda_3\right)\sin^2 w_0 s_R \\
&\quad - \left(\tau c\overline{M}^2_1\lambda_1 + \kappa c\overline{M}^3_1\lambda_1\right)\sin^2 w_0 s_R - \left(\tau c\overline{M}^2_1\lambda_1 + \kappa c\overline{M}^3_1\lambda_1\right)\cos^2 w_0 s_R \\
&= \left(\tau c\overline{M}^1_2\lambda_2 + \kappa c\overline{M}^1_3\lambda_3\right)\left(\cos^2 w_0 s_R + \sin^2 w_0 s_R\right) \\
&\quad - \left(\left(\tau c\overline{M}^2_1\lambda_1 + \kappa c\overline{M}^3_1\lambda_1\right)\left(\sin^2 w_0 s_R + \cos^2 w_0 s_R\right)\right) \\
&= \tau c\overline{M}^1_2\lambda_2 + \kappa c\overline{M}^1_3\lambda_3 - \tau c\overline{M}^2_1\lambda_1 - \kappa c\overline{M}^3_1\lambda_1,
\end{aligned}$$

$$\begin{aligned}
2\left(\mathbf{C}_0\right)^1_3 &= \left(\mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}}\right)^3_1 - \left(\mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}}\right)^1_3 + \left(\mathbf{r}\otimes\mathbf{n}\right)^3_1 - \left(\mathbf{n}\otimes\mathbf{r}\right)^3_1 \\
&= \left(\tau c\overline{M}^3_1\lambda_1 - \kappa c\overline{M}^2_1\lambda_1\right)\cos w_0 s_R - \left(\tau c\overline{M}^1_3\lambda_3 - \kappa c\overline{M}^1_2\lambda_2\right)\cos w_0 s_R \\
&\quad + a n_0 \cos w_0 s_R \\
&= \left(\tau c\overline{M}^3_1\lambda_1 - \kappa c\overline{M}^2_1\lambda_1 - \tau c\overline{M}^1_3\lambda_3 + \kappa c\overline{M}^1_2\lambda_2 + a n_0\right)\cos w_0 s_R
\end{aligned}$$

and

$$\begin{aligned}
2\left(\mathbf{C}_0\right)^2_3 &= \left(\mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}}\right)^3_2 - \left(\mathbf{R}\overline{\mathbf{M}}_0\mathbf{U}_0\mathbf{R}^{\mathrm{T}}\right)^2_3 + \left(\mathbf{r}\otimes\mathbf{n}\right)^3_2 - \left(\mathbf{n}\otimes\mathbf{r}\right)^3_2 \\
&= \left(\tau c\overline{M}^3_1\lambda_1 - \kappa c\overline{M}^2_1\lambda_1\right)\sin w_0 s_R - \left(\tau c\overline{M}^1_3\lambda_3 - \kappa c\overline{M}^1_2\lambda_2\right)\sin w_0 s_R \\
&\quad + a n_0 \sin w_0 s_R \\
&= \left(\tau c\overline{M}^3_1\lambda_1 - \kappa c\overline{M}^2_1\lambda_1 - \tau c\overline{M}^1_3\lambda_3 + \kappa c\overline{M}^1_2\lambda_2 + a n_0\right)\sin w_0 s_R.
\end{aligned}$$

Since $\mathbf{C}_0$ must be a constant tensor and components $(1,3)$, $(2,3)$, $(3,1)$ and $(3,2)$ are functions of $s_R$, the following equation must be satisfied:

$$\tau c\overline{M}^3_1\lambda_1 - \kappa c\overline{M}^2_1\lambda_1 - \tau c\overline{M}^1_3\lambda_3 + \kappa c\overline{M}^1_2\lambda_2 + a n_0 = 0.$$

This means

$$\mathbf{C}_0 = \frac{1}{2}\left(\tau c\overline{M}^1_2\lambda_2 + \kappa c\overline{M}^1_3\lambda_3 - \tau c\overline{M}^2_1\lambda_1 - \kappa c\overline{M}^3_1\lambda_1\right)\left(\mathbf{E}_{21} - \mathbf{E}_{12}\right)$$

which can be expressed succinctly as

$$\mathbf{C}_0 = \frac{1}{2}\left(\tau c\overline{M}^1_2\lambda_2 + \kappa c\overline{M}^1_3\lambda_3 - \tau c\overline{M}^2_1\lambda_1 - \kappa c\overline{M}^3_1\lambda_1\right)\mathbf{A}_3^{\perp}.$$

We can write this as

$$\mathbf{C}_0 = \frac{1}{2}\left(\tau c\left(\overline{M}^1_2\lambda_2 - \overline{M}^2_1\lambda_1\right) + \kappa c\left(\overline{M}^1_3\lambda_3 - \overline{M}^3_1\lambda_1\right)\right)\mathbf{A}_3^{\perp}. \tag{9.61}$$

Recall equation (9.60) which is reproduced here for convenience:

$$n_0 = \frac{\tau\left(\lambda_3\overline{M}^1_3 - \lambda_1\overline{M}^3_1\right) + \kappa\left(\lambda_1\overline{M}^2_1 - \lambda_2\overline{M}^1_2\right)}{\kappa c}.$$

Solving for $\tau\left(\lambda_3\overline{M}^1_3 - \lambda_1\overline{M}^3_1\right)$ gives

$$\tau\left(\lambda_3\overline{M}^1_3 - \lambda_1\overline{M}^3_1\right) = \kappa c n_0 - \kappa\left(\lambda_1\overline{M}^2_1 - \lambda_2\overline{M}^1_2\right)$$

and multiplying through this last equation by $\frac{\kappa c}{\tau}$ gives

$$\kappa c\left(\lambda_3\overline{M}^1_3 - \lambda_1\overline{M}^3_1\right) = \frac{\kappa^2c^2n_0}{\tau} - \frac{\kappa^2 c}{\tau}\left(\lambda_1\overline{M}^2_1 - \lambda_2\overline{M}^1_2\right).$$

Substituting this into equation (9.61) yields

$$\mathbf{C}_0 = \frac{1}{2}\left(\tau c\left(\overline{M}^1_2\lambda_2 - \overline{M}^2_1\lambda_1\right) + \frac{\kappa^2c^2n_0}{\tau} - \frac{\kappa^2 c}{\tau}\left(\lambda_1\overline{M}^2_1 - \lambda_2\overline{M}^1_2\right)\right)\mathbf{A}_3^{\perp}.$$

Then

$$\mathbf{C}_0 = \frac{1}{2}\left(\frac{\tau^2 c}{\tau}\left(\lambda_2\overline{M}^1_2 - \lambda_1\overline{M}^2_1\right) + \frac{\kappa^2 c}{\tau}\left(\lambda_2\overline{M}^1_2 - \lambda_1\overline{M}^2_1\right) + \frac{\kappa^2c^2n_0}{\tau}\right)\mathbf{A}_3^{\perp}$$

so

$$\mathbf{C}_0 = \frac{1}{2}\left(\left(\frac{\tau^2 c}{\tau} + \frac{\kappa^2 c}{\tau}\right)\left(\lambda_2\overline{M}^1_2 - \lambda_1\overline{M}^2_1\right) + \frac{\kappa^2c^2n_0}{\tau}\right)\mathbf{A}_3^{\perp}.$$

Multiplying through this by $\frac{c}{c}$ gives

$$\mathbf{C}_0 = \frac{1}{2}\left(\left(\frac{\tau^2 c^2}{\tau c} + \frac{\kappa^2 c^2}{\tau c}\right)\left(\lambda_2\overline{M}^1_2 - \lambda_1\overline{M}^2_1\right) + \frac{\kappa^2c^3n_0}{\tau c}\right)\mathbf{A}_3^{\perp}$$

and from equation (9.39), we get

$$\mathbf{C}_0 = \frac{1}{2}\left(\frac{1}{\tau c}\left(\lambda_2\overline{M}_2^1 - \lambda_1\overline{M}_1^2\right) + \frac{\kappa^2 c^3 n_0}{\tau c}\right)\mathbf{A}_3^{\perp} = \frac{1}{2\tau c}\left(\lambda_2\overline{M}_2^1 - \lambda_1\overline{M}_1^2 + \kappa^2 c^3 n_0\right)\mathbf{A}_3^{\perp}.$$

Finally, consider

$$\kappa^2 c^3 = \frac{a^2}{c^4}c^3 = \frac{a^2}{c}$$

so

$$\mathbf{C}_0 = \frac{1}{2\tau c}\left(\lambda_2\overline{M}_2^1 - \lambda_1\overline{M}_1^2 + \frac{a^2 n_0}{c}\right)\mathbf{A}_3^{\perp}. \tag{9.62}$$

This expression is analogous to the one given in Cohen [1] and can be shown to be identical to it.

Notice $\mathbf{C}_0$ is parallel to $\mathbf{A}_3^{\perp}$ so the axial vector of $\mathbf{C}_0$ is parallel to $\mathbf{e}_3$. Thus the deformation of the directed rod for the straight to helical case can be effected by the vector $\mathbf{n}$ and the torque given as equation (9.62), both acting at the ends of the rod and both parallel to $\mathbf{e}_3$.

# Appendix A

# Polar Decomposition Theorem

This appendix states and supplies a proof of the Polar Decomposition Theorem. This theorem and its proof are restricted to second order tensors.

**Theorem 11** *Suppose* $\mathbf{F}$ *is a second order nonsingular tensor. Then there exist second order tensors* $\mathbf{R}$, $\mathbf{U}$ *and* $\mathbf{V}$ *such that* $\mathbf{F} = \mathbf{RU}$ *(right decomposition) and* $\mathbf{F} = \mathbf{VR}$ *(left decomposition) in which* $\mathbf{R}$ *is orthogonal (i.e.* $\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$*) and* $\mathbf{U}$ *and* $\mathbf{V}$ *are symmetric and positive.*

**Proof.** Let $\mathbf{F}$ be a nonsingular tensor. Then if $\mathbf{v}$ is any vector and $\mathbf{w} = \mathbf{F}\mathbf{v}$, $\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v} = \mathbf{0}$. In component form, $w_i = F_{ij}v_j$.

Then $w_i w_i = (F_{ij}v_j)(F_{ik}v_k) = F_{ij}F_{ik}v_j v_k$ which must be a positive definite quadratic form. This form can be associated with a symmetric matrix $C$ via $c_{jk} \equiv \left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)_{jk}$ or $C = \mathbf{F}^{\mathrm{T}}\mathbf{F}$.

Since the form is positive definite, $C$ has positive eigenvalues. So along the principal axes, $C$ is represented by a diagonal matrix with positive entries.

Now define $\mathbf{U}$ to be the tensor whose representation with respect to these axes is a diagonal matrix consisting of the positive square roots of corresponding entries for $C$. In this basis, and hence in all bases, $\mathbf{U}$ has the property that $\mathbf{U}^{\mathrm{T}} = \mathbf{U}$ and $\mathbf{UU} = C$.

Since $\mathbf{U}$ has positive eigenvalues, it must be nonsingular and hence invertible. Thus we may define $\mathbf{R}$ by $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. Then $\mathbf{F} = \mathbf{RU}$.

It remains to show $\mathbf{R}$ is orthogonal. Consider

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \left(\mathbf{U}^{-1}\right)^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}C\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}\mathbf{U}\mathbf{U}^{-1} = \mathbf{I},$$

the identity tensor.

Thus given $\mathbf{F}$, we can construct $\mathbf{U}$ and $\mathbf{F}$ with the indicated properties. Now consider whether this decomposition is unique. Suppose $\mathbf{F} = \mathbf{RU}$ and $\mathbf{F} = \overline{\mathbf{R}}\overline{\mathbf{U}}$ for two possibly different decompositions. Then

$$\mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{U} = \mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{U}\mathbf{U}.$$

Similarly,

$$\mathbf{F}^{\mathrm{T}}\mathbf{F} = \overline{\mathbf{U}}^{\mathrm{T}}\overline{\mathbf{R}}^{\mathrm{T}}\overline{\mathbf{R}}\overline{\mathbf{U}} = \overline{\mathbf{U}}^{\mathrm{T}}\overline{\mathbf{U}} = \overline{\mathbf{U}}\overline{\mathbf{U}}.$$

Hence $\mathbf{U}\mathbf{U} = \overline{\mathbf{U}}\overline{\mathbf{U}}$. From uniqueness of positive square root, it follows that $\mathbf{U} = \overline{\mathbf{U}}$. Then $\mathbf{R} = \overline{\mathbf{R}}$ from $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$ and $\overline{\mathbf{R}} = \mathbf{F}\overline{\mathbf{U}}^{-1}$.

Now suppose $\mathbf{F} = \mathbf{VP}$ provides a left decomposition. Since $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$, we can write

$$\mathbf{F} = \mathbf{P}\mathbf{P}^{\mathrm{T}}\mathbf{V}\mathbf{P} = \mathbf{P}\left(\mathbf{P}^{\mathrm{T}}\mathbf{V}\mathbf{P}\right).$$

But $\mathbf{P}$ is a rotation tensor and by the uniqueness of $\mathbf{F} = \mathbf{RU}$, $\mathbf{RU} = \mathbf{P}\left(\mathbf{P}^{\mathrm{T}}\mathbf{V}\mathbf{P}\right)$ or $\mathbf{R} = \mathbf{P}$ and $\mathbf{P}^{\mathrm{T}}\mathbf{V}\mathbf{P} = \mathbf{U}$. Consequently, the decomposition $\mathbf{F} = \mathbf{VR}$ exists if $\mathbf{F} = \mathbf{RU}$ exits. Since the latter has been shown to uniquely exist, then the statement of the theorem is true. ■

# Appendix B

# Algebra of Tensor Pairs

This appendix develops the algebra of tensor pairs of the general linear group. It is purely mathematical in character - its application to directed rod theory is given in Appendix C.

## B.1 Preliminary Definitions and Lemmas

The general linear group on $\Re^N$ will be denoted $\mathcal{GL}(N, \Re)$ throughout this appendix.

**Definition 5** *<u>Pair Set $\mathcal{PGL}(N, \Re)$</u>: The pair set on the general linear group over $\Re^N$ consists of pairs of members of $\mathcal{GL}(N, \Re)$. Thus $\mathcal{PGL}(N, \Re) := \{(\mathbf{G}, \mathbf{H}) \mid \mathbf{G}, \mathbf{H} \in \mathcal{GL}(N, \Re)\}$.*

**Definition 6** *<u>Pair Product</u>: Let $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{H}_1$, $\mathbf{H}_2 \in \mathcal{GL}(N, \Re)$. If we form the pairs $(\mathbf{G}_1, \mathbf{G}_2)$, $(\mathbf{H}_1, \mathbf{H}_2) \in \mathcal{PGL}(N, \Re)$, the pair product is the binary operation*

$$* : \mathcal{PGL}(N, \Re) \times \mathcal{PGL}(N, \Re) \to \mathcal{PGL}(N, \Re)$$

*defined by $(\mathbf{G}_1, \mathbf{G}_2) * (\mathbf{H}_1, \mathbf{H}_2) := (\mathbf{G}_1\mathbf{H}_1, \mathbf{G}_2\mathbf{H}_2)$.*

**Lemma 60** *Let $\mathbf{I}$ be the identity member of $\mathcal{GL}(N, \Re)$. Then $(\mathbf{I}, \mathbf{I})$ will act as the identity element under the operation $*$ in any subset of $\mathcal{PGL}(N, \Re)$ including $\mathcal{PGL}(N, \Re)$.*

**Proof.** Let $\mathcal{X}(N,\Re)$ be any subset of $\mathcal{PGL}(N,\Re)$. By definition, $(\mathbf{I},\mathbf{I}) \in \mathcal{PGL}(N,\Re)$. Let $(\mathbf{G}_1,\mathbf{G}_2) \in \mathcal{X}(N,\Re)$. Then

$$\begin{aligned}(\mathbf{G}_1,\mathbf{G}_2)*(\mathbf{I},\mathbf{I}) &= (\mathbf{G}_1\mathbf{I},\mathbf{G}_2\mathbf{I}) = (\mathbf{G}_1,\mathbf{G}_2)\\ (\mathbf{I},\mathbf{I})*(\mathbf{G}_1,\mathbf{G}_2) &= (\mathbf{I}\mathbf{G}_1,\mathbf{I}\mathbf{G}_2) = (\mathbf{G}_1,\mathbf{G}_2)\end{aligned}$$

and thus $(\mathbf{I},\mathbf{I})$ is an identity for any member of $\mathcal{X}(N,\Re)$. In particular, this must be true for $\mathcal{X}(N,\Re) = \mathcal{PGL}(N,\Re)$. ■

**Lemma 61** *$(\mathcal{PGL}(N,\Re),*)$ is a group.*

**Proof.** Since $*$ is a binary operation on $\mathcal{PGL}(N,\Re)$, this set is closed under this operation. It remains to prove the axioms of a group are satisfied. For purposes of this proof, let $(\mathbf{G}_1,\mathbf{G}_2)$, $(\mathbf{H}_1,\mathbf{H}_2),(\mathbf{J}_1,\mathbf{J}_2) \in \mathcal{PGL}(N,\Re)$. Let $\mathbf{G}_1^{-1}$ and $\mathbf{G}_2^{-1}$ be the inverses of $\mathbf{G}_1$ and $\mathbf{G}_2$ respectively so $\left(\mathbf{G}_1^{-1},\mathbf{G}_2^{-1}\right) \in \mathcal{PGL}(N,\Re)$.

1. Associativity.

$$\begin{aligned}((\mathbf{G}_1,\mathbf{G}_2)*(\mathbf{H}_1,\mathbf{H}_2))*(\mathbf{J}_1,\mathbf{J}_2) &= (\mathbf{G}_1\mathbf{H}_1,\mathbf{G}_2\mathbf{H}_2)*(\mathbf{J}_1,\mathbf{J}_2)\\ &= (\mathbf{G}_1\mathbf{H}_1\mathbf{J}_1,\mathbf{G}_2\mathbf{H}_2\mathbf{J}_2)\\ &= (\mathbf{G}_1,\mathbf{G}_2)*(\mathbf{H}_1\mathbf{J}_1,\mathbf{H}_2\mathbf{J}_2)\\ &= (\mathbf{G}_1,\mathbf{G}_2)*((\mathbf{H}_1,\mathbf{H}_2)*(\mathbf{J}_1,\mathbf{J}_2)).\end{aligned}$$

   Thus $*$ is associative.

2. Identity. As shown in Lemma 60, $(\mathbf{I},\mathbf{I}) \in \mathcal{PGL}(N,\Re)$ and acts as an identity for $*$ in $\mathcal{PGL}(N,\Re)$.

3. Inverse.

$$\begin{aligned}(\mathbf{G}_1,\mathbf{G}_2)*\left(\mathbf{G}_1^{-1},\mathbf{G}_2^{-1}\right) &= \left(\mathbf{G}_1\mathbf{G}_1^{-1},\ \mathbf{G}_2\mathbf{G}_2^{-1}\right) = (\mathbf{I},\mathbf{I}).\\ \left(\mathbf{G}_1^{-1},\mathbf{G}_2^{-1}\right)*(\mathbf{G}_1,\mathbf{G}_2) &= \left(\mathbf{G}_1^{-1}\mathbf{G}_1,\mathbf{G}_2^{-1}\mathbf{G}_2\right) = (\mathbf{I},\mathbf{I}).\end{aligned}$$

Thus if $(\mathbf{G}_1, \mathbf{G}_2) \in \mathcal{PGL}(N, \Re)$, there exists a member of $\mathcal{PGL}(N, \Re)$, namely $\left(\mathbf{G}_1^{-1}, \mathbf{G}_2^{-1}\right)$, which is the inverse of $(\mathbf{G}_1, \mathbf{G}_2)$. Every member of $\mathcal{PGL}(N, \Re)$ has an inverse.

Since the group axioms are satisfied for $*$ in the set $\mathcal{PGL}(N, \Re)$, then $(\mathcal{PGL}(N, \Re), *)$ is a group. ■

Our interest is in three special classes of subsets of $\mathcal{PGL}(N, \Re)$.

**Definition 7** $\underline{\mathcal{G}_R^+()}$: *Let* $\mathcal{U} \subset \mathcal{GL}(N, \Re)$. *Then* $\mathcal{G}_R^+(\mathcal{U}) := \{(\mathbf{G}, \mathbf{G}) \mid \mathbf{G} \in \mathcal{U}\}$.

**Definition 8** $\underline{\mathsf{G}^+}$: *If* $(\mathbf{G}, \mathbf{G}) \in \mathcal{G}_R^+(\mathcal{U})$ *for some* $\mathcal{U} \subset \mathcal{GL}(N, \Re)$, *the member* $(\mathbf{G}, \mathbf{G})$ *is denoted* $\mathsf{G}^+$. *That is,* $\mathsf{G}^+ := (\mathbf{G}, \mathbf{G})$.

**Definition 9** $\underline{\mathcal{G}_R^-()}$: *Let* $\mathcal{V} \subset \mathcal{GL}(N, \Re)$. *Then* $\mathcal{G}_R^-(\mathcal{V}) := \{(\mathbf{G}, -\mathbf{G}) \mid \mathbf{G} \in \mathcal{V}\}$.

**Definition 10** $\underline{\mathsf{G}^-}$: *If* $(\mathbf{G}, -\mathbf{G}) \in \mathcal{G}_R^-(\mathcal{V})$ *for some* $\mathcal{V} \subset \mathcal{GL}(N, \Re)$, *the member* $(\mathbf{G}, -\mathbf{G})$ *is denoted* $\mathsf{G}^-$. *That is,* $\mathsf{G}^- := (\mathbf{G}, -\mathbf{G})$.

**Definition 11** $\underline{\mathsf{G}^\pm}$: *The notation* $\mathsf{G}^\pm$ *denotes either* $\mathsf{G}^+$ *or* $\mathsf{G}^-$ .

**Definition 12** $\underline{\mathcal{G}_R()}$: *Let* $\mathcal{U}, \mathcal{V} \subset \mathcal{GL}(N, \Re)$. *Then* $\mathcal{G}_R(\mathcal{U}, \mathcal{V}) := \mathcal{G}_R^+(\mathcal{U}) \cup \mathcal{G}_R^-(\mathcal{V})$.

Note that the sets $\mathcal{U}$ and $\mathcal{V}$ may be identical, different or even disjoint. Quite often, they are understood to be present and the notation is simplified so $\mathcal{G}_R^+ = \mathcal{G}_R^+(\mathcal{U})$, $\mathcal{G}_R^- = \mathcal{G}_R^-(\mathcal{V})$ and $\mathcal{G}_R = \mathcal{G}_R(\mathcal{U}, \mathcal{V})$. In this simplified notation, we could write $\mathcal{G}_R := \mathcal{G}_R^+ \cup \mathcal{G}_R^-$.

We also note that $\mathcal{G}_R^+(\mathcal{U}), \mathcal{G}_R^-(\mathcal{V}), \mathcal{G}_R(\mathcal{U}, \mathcal{V}) \subset \mathcal{PGL}(N, \Re)$. In particular, Lemma 60 applies to $\mathcal{G}_R^+(\mathcal{U})$, $\mathcal{G}_R^-(\mathcal{V})$ and $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$.

The purpose of this appendix is to characterize the relationship between $\mathcal{U}$ and $\mathcal{V}$ so that $\mathcal{G}_R^+(\mathcal{U})$ and $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ are groups under $*$. To begin, we note the five distinct ways nonempty $\mathcal{U}$ and $\mathcal{V}$ may be related.

1. $\mathcal{U} = \mathcal{V}$ (same set),
2. $\mathcal{V} \subset \mathcal{U}$, $\mathcal{V} \neq \mathcal{U}$ ($\mathcal{V}$ is a proper subset of $\mathcal{U}$),
3. $\mathcal{U} \subset \mathcal{V}$, $\mathcal{U} \neq \mathcal{V}$ ($\mathcal{U}$ is a proper subset of $\mathcal{V}$),

4. $\mathcal{U} \cap \mathcal{V} \neq \emptyset, \mathcal{V} \not\subseteq \mathcal{U}, \mathcal{V} \not\subseteq \mathcal{U}$ ($\mathcal{U}$ and $\mathcal{V}$ have nonempty intersection, neither is a subset of the other) and

5. $\mathcal{U} \cap \mathcal{V} = \emptyset$ ($\mathcal{U}$ and $\mathcal{V}$ are disjoint).

In this appendix, it will be shown $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ can be a group only if case one or five holds.

For purposes of brevity, the expression "$\mathcal{U}$ is closed" will mean $\mathcal{U}$ is closed under the binary operation of composition of members of the general linear group. Similarly, "$\mathcal{U}$ is a subgroup" will mean $\mathcal{U}$ is a subgroup of the general linear group.

## B.2 Condition Under Which $\mathcal{G}_R^+(\mathcal{U})$ Is A Group

**Lemma 62** *$(\mathcal{G}_R^+(\mathcal{U}), *)$ is closed under $*$ if and only if $\mathcal{U}$ is closed.*

**Proof.**

1. Suppose $\mathcal{U}$ is closed. Let $\mathbf{G}, \mathbf{H} \in \mathcal{U}$. Then $(\mathbf{G}, \mathbf{G})$ and $(\mathbf{H}, \mathbf{H})$ are members of $(\mathcal{G}_R^+(\mathcal{U}), *)$. $(\mathbf{G}, \mathbf{G}) * (\mathbf{H}, \mathbf{H}) = (\mathbf{GH}, \mathbf{GH})$. But $\mathcal{U}$ is closed so $\mathbf{GH} \in \mathcal{U}$. Then $(\mathbf{GH}, \mathbf{GH}) \in (\mathcal{G}_R^+(\mathcal{U}), *)$ which shows $(\mathcal{G}_R^+(\mathcal{U}), *)$ is closed under $*$.

2. Suppose $(\mathcal{G}_R^+(\mathcal{U}), *)$ is closed under $*$. If $(\mathbf{G}, \mathbf{G})$ and $(\mathbf{H}, \mathbf{H})$ are members of $(\mathcal{G}_R^+(\mathcal{U}), *)$, then $\mathbf{G}, \mathbf{H} \in \mathcal{U}$. But $(\mathbf{G}, \mathbf{G}) * (\mathbf{H}, \mathbf{H}) = (\mathbf{GH}, \mathbf{GH}) \in (\mathcal{G}_R^+(\mathcal{U}), *)$ so $\mathbf{GH} \in \mathcal{U}$ showing $\mathcal{U}$ is closed.

Thus the statement of the lemma is proven. ■

**Lemma 63** *$(\mathcal{G}_R^+(\mathcal{U}), *)$ is a subgroup of $(\mathcal{PGL}(N, \Re), *)$ if and only if $\mathcal{U}$ is a subgroup.*

**Proof.**

1. Suppose $\mathcal{U}$ is a subgroup. The previous lemma showed if $\mathcal{U}$ is closed, $\mathcal{G}_R^+(\mathcal{U})$ must also be closed under the operation $*$. Now demonstrate the three axioms of a group.

   (a) Associative. It has already been shown that $\mathcal{PGL}(N, \Re)$ is associative under $*$ and since $\mathcal{G}_R^+(\mathcal{U}) \subset \mathcal{PGL}(N, \Re)$, it follows that $\mathcal{G}_R^+(\mathcal{U})$ is associative under $*$.

(b) Identity. Since $\mathcal{U}$ is a subgroup, it must contain the identity element $\mathbf{I}$. Then $(\mathbf{I},\mathbf{I}) \in \mathcal{G}_R^+(\mathcal{U})$. But it was shown in Lemma 60 that $(\mathbf{I},\mathbf{I})$ is the identity element for any subset of $\mathcal{PGL}(N,\Re)$ and since $\mathcal{G}_R^+(\mathcal{U}) \subset \mathcal{PGL}(N,\Re)$, $(\mathbf{I},\mathbf{I})$ will also serve as an identity for $\mathcal{G}_R^+(\mathcal{U})$ under $*$.

(c) Inverse. If $\mathbf{G} \in \mathcal{U}$, then $\mathbf{G}^{-1} \in \mathcal{U}$ since $\mathcal{U}$ is a subgroup. This implies $(\mathbf{G},\mathbf{G}) \in \mathcal{G}_R^+(\mathcal{U})$ and $(\mathbf{G}^{-1},\mathbf{G}^{-1}) \in \mathcal{G}_R^+(\mathcal{U})$ . As shown earlier, $(\mathbf{G}^{-1},\mathbf{G}^{-1})$ is the inverse for $(\mathbf{G},\mathbf{G})$ (and vice-versa) for the operation $*$. Thus every member of $\mathcal{G}_R^+(\mathcal{U})$ has its inverse in $\mathcal{G}_R^+(\mathcal{U})$.

Since the three axioms of a group are satisfied for the subset $\mathcal{G}_R^+(\mathcal{U})$, it follows $(\mathcal{G}_R^+(\mathcal{U}),*)$ is a subgroup of $(\mathcal{PGL}(N,\Re),*)$.

2. Suppose $(\mathcal{G}_R^+(\mathcal{U}),*)$ is a subgroup of $(\mathcal{PGL}(N,\Re),*)$. Let $(\mathbf{G},\mathbf{G}),(\mathbf{H},\mathbf{H}),(\mathbf{K},\mathbf{K}) \in \mathcal{G}_R^+(\mathcal{U})$. Look at the three axioms of group.

(a) Associative. $(\mathcal{G}_R^+(\mathcal{U}),*)$ is associative so

$$((\mathbf{G},\mathbf{G})*(\mathbf{H},\mathbf{H}))*(\mathbf{K},\mathbf{K}) = (\mathbf{G},\mathbf{G})*((\mathbf{H},\mathbf{H})*(\mathbf{K},\mathbf{K}))$$

or

$$(\mathbf{GH},\mathbf{GH})*(\mathbf{K},\mathbf{K}) = (\mathbf{G},\mathbf{G})*(\mathbf{HK},\mathbf{HK}).$$

That is,

$$((\mathbf{GH})\mathbf{K},(\mathbf{GH})\mathbf{K}) = (\mathbf{G}(\mathbf{HK}),\mathbf{G}(\mathbf{HK})).$$

But this means $(\mathbf{GH})\mathbf{K} = \mathbf{G}(\mathbf{HK})$, so $\mathbf{G},\mathbf{H},\mathbf{K} \in \mathcal{U}$ are associative.

(b) Identity. $(\mathcal{G}_R^+(\mathcal{U}),*)$ contains the identity $(\mathbf{I},\mathbf{I})$. Thus $(\mathbf{G},\mathbf{G})*(\mathbf{I},\mathbf{I}) = (\mathbf{G},\mathbf{G})$ and $(\mathbf{I},\mathbf{I})*(\mathbf{G},\mathbf{G}) = (\mathbf{G},\mathbf{G})$ for all $(\mathbf{G},\mathbf{G}) \in \mathcal{G}_R^+(\mathcal{U})$. That is, $\mathbf{GI} = \mathbf{G}$ and $\mathbf{IG} = \mathbf{G}$ for all $\mathbf{G} \in \mathcal{U}$. Then $\mathbf{I} \in \mathcal{U}$ is the identity element on $\mathcal{U}$.

(c) Inverse. Suppose $(\mathbf{G}^{-1},\mathbf{G}^{-1})$ is the inverse of $(\mathbf{G},\mathbf{G})$, so $(\mathbf{G}^{-1},\mathbf{G}^{-1})*(\mathbf{G},\mathbf{G}) = (\mathbf{I},\mathbf{I})$ and $(\mathbf{G},\mathbf{G})*(\mathbf{G}^{-1},\mathbf{G}^{-1}) = (\mathbf{I},\mathbf{I})$. Then $\mathbf{G}^{-1}\mathbf{G} = \mathbf{I}$ and $\mathbf{GG}^{-1} = \mathbf{I}$ where $\mathbf{I}$ is the identity element in $\mathcal{U}$. This means every $\mathbf{G} \in \mathcal{U}$ had an inverse $\mathbf{G}^{-1} \in \mathcal{U}$.

Since the three axioms of a group are satisfied for the subset $\mathcal{U}$ , it follows $\mathcal{U}$ is a subgroup of the general linear group.

Thus the statement of the lemma is proven. ■

Although $\mathcal{V}$ may be a group and contain the identity element $\mathbf{I}$, $(\mathcal{G}_R^-(\mathcal{V}), *)$ can never be a group, since $\mathcal{G}_R^-(\mathcal{V})$ never contains the identity element $(\mathbf{I}, \mathbf{I})$. Is it possible to make $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ a group? The remainder of this appendix investigates this question.

## B.3 Conditions Under Which $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ Is A Group

This section examines the conditions under which $(\mathcal{G}_R(\mathcal{U}, \mathcal{V}), *)$ is a group given $(\mathcal{G}_R^+(\mathcal{U}), *)$ is a subgroup of $(\mathcal{PGL}(N, \Re), *)$ (our interest in only in the case where $(\mathcal{G}_R^+(\mathcal{U}), *)$ is a subgroup). From Lemma 63 this means we're working under the hypothesis $\mathcal{U}$ is a subgroup of the general linear group. The subsequent investigation will show $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ may be a group for only two of the five ways the sets $\mathcal{U}$ and $\mathcal{V}$ may be related to each other.

This section is divided into two subsections. The first subsection deals with the case $\mathcal{U} = \mathcal{V}$. This special case can be completely characterized with regard to the group property of $(\mathcal{G}_R(\mathcal{U}, \mathcal{V}), *)$ - both necessary and sufficient conditions can be demonstrated. The second subsection addresses the remaining cases. Only necessary conditions are given for $(\mathcal{G}_R(\mathcal{U}, \mathcal{V}), *)$ to be a group.

The first concern in both subsections is to determine the conditions under which $(\mathcal{G}_R(\mathcal{U}, \mathcal{V}), *)$ is closed under $*$. The following lemma will be very useful. It shows the results of the products of members of $\mathcal{U}$ and $\mathcal{V}$ if $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ is closed.

**Lemma 64** *Suppose $\mathcal{U}, \mathcal{V} \subset \mathcal{GL}(N, \Re)$ and suppose $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ is closed under $*$. Then the following holds:*

1. $\mathcal{U}$ is closed.
2. The product of a member of $\mathcal{U}$ with a member of $\mathcal{V}$ is a member of $\mathcal{V}$.
3. The product of a member of $\mathcal{V}$ with a member of $\mathcal{U}$ is a member of $\mathcal{V}$.
4. The product of two members of $\mathcal{V}$ is a member of $\mathcal{U}$.

Thus $\mathcal{U} \cup \mathcal{V}$ is closed.

**Proof.** Look at each of the four cases.

1. Suppose $(\mathbf{G}, \mathbf{G}), (\mathbf{H}, \mathbf{H}) \in \mathcal{G}_R(\mathcal{U}, \mathcal{V})$. Then

$$(\mathbf{G}, \mathbf{G}) * (\mathbf{H}, \mathbf{H}) = (\mathbf{GH}, \mathbf{GH}) \in \mathcal{G}_R(\mathcal{U}, \mathcal{V}).$$

   This means $\mathbf{G}, \mathbf{H} \in \mathcal{U}$ and $\mathbf{GH} \in \mathcal{U}$. Thus $\mathcal{U}$ is closed.

2. Suppose $(\mathbf{G}, \mathbf{G}), (\mathbf{H}, -\mathbf{H}) \in \mathcal{G}_R(\mathcal{U}, \mathcal{V})$. Then

$$(\mathbf{G}, \mathbf{G}) * (\mathbf{H}, -\mathbf{H}) = (\mathbf{GH}, -\mathbf{GH}) \in \mathcal{G}_R(\mathcal{U}, \mathcal{V}).$$

   This means $\mathbf{G} \in \mathcal{U}, \mathbf{H} \in \mathcal{V}$ and $\mathbf{GH} \in \mathcal{V}$.

3. Suppose $(\mathbf{G}, -\mathbf{G}), (\mathbf{H}, \mathbf{H}) \in \mathcal{G}_R(\mathcal{U}, \mathcal{V})$. Then

$$(\mathbf{G}, -\mathbf{G}) * (\mathbf{H}, \mathbf{H}) = (\mathbf{GH}, -\mathbf{GH}) \in \mathcal{G}_R(\mathcal{U}, \mathcal{V}).$$

   This means $\mathbf{G} \in \mathcal{V}$, $\mathbf{H} \in \mathcal{U}$ and $\mathbf{GH} \in \mathcal{V}$.

4. Suppose $(\mathbf{G}, -\mathbf{G}), (\mathbf{H}, -\mathbf{H}) \in \mathcal{G}_R(\mathcal{U}, \mathcal{V})$. Then

$$(\mathbf{G}, -\mathbf{G}) * (\mathbf{H}, -\mathbf{H}) = (\mathbf{GH}, \mathbf{GH}) \in \mathcal{G}_R(\mathcal{U}, \mathcal{V}).$$

   This means $\mathbf{G} \in \mathcal{V}$, $\mathbf{H} \in \mathcal{V}$ and $\mathbf{GH} \in \mathcal{U}$.

Thus all four cases have been established. Since the product of any two members of either set is in one of the sets, $\mathcal{U} \cup \mathcal{V}$ is closed. ■

Note the last lemma is true, regardless of how $\mathcal{U}$ and $\mathcal{V}$ are related to each other.

### B.3.1 The Case $\mathcal{U} = \mathcal{V}$

**Lemma 65** *If $\mathcal{V} \subset \mathcal{U} \subset \mathcal{GL}(N, \Re)$, $\mathcal{U}$ is closed and $\mathsf{G}^-, \mathsf{H}^- \in \mathcal{G}_R^-(\mathcal{V})$, then $\mathsf{G}^- * \mathsf{H}^- \in \mathcal{G}_R^+(\mathcal{U})$.*

**Proof.** $\mathsf{G}^- = (\mathbf{G}, -\mathbf{G})$ and $\mathsf{H}^- = (\mathbf{H}, -\mathbf{H})$. Then

$$\mathsf{G}^- * \mathsf{H}^- = (\mathbf{G}, -\mathbf{G}) * (\mathbf{H}, -\mathbf{H}) = (\mathbf{GH}, (-\mathbf{G})(-\mathbf{H})) = (\mathbf{GH}, \mathbf{GH}).$$

Since $\mathcal{V} \subset \mathcal{U}$, $\mathbf{G}, \mathbf{H} \in \mathcal{U}$ and thus $(\mathbf{G}, \mathbf{G}), (\mathbf{H}, \mathbf{H}) \in \mathcal{G}_R^+(\mathcal{U})$. Since $\mathcal{U}$ is closed, $(\mathcal{G}_R^+(\mathcal{U}), *)$ is closed under $*$ as shown in Lemma 62 and thus $(\mathbf{GH}, \mathbf{GH}) \in \mathcal{G}_R^+(\mathcal{U})$, i.e. $\mathsf{G}^- * \mathsf{H}^- \in \mathcal{G}_R^+(\mathcal{U})$. ∎

**Lemma 66** *$\mathcal{G}_R(\mathcal{U}, \mathcal{U})$ is closed under $*$ if and only if $\mathcal{U}$ is closed.*

**Proof.**

1. Assume $\mathcal{U}$ is closed and attempt to demonstrate $\mathcal{G}_R(\mathcal{U}, \mathcal{U})$ is closed. By definition, $\mathcal{G}_R(\mathcal{U}, \mathcal{U}) := \mathcal{G}_R^+(\mathcal{U}) \cup \mathcal{G}_R^-(\mathcal{U})$. Thus there are four cases to consider.

   (a) If $\mathsf{G}, \mathsf{H} \in \mathcal{G}_R^+(\mathcal{U}) \subset \mathcal{G}_R(\mathcal{U}, \mathcal{U})$, then $\mathsf{G} * \mathsf{H} \in \mathcal{G}_R^+(\mathcal{U}) \subset \mathcal{G}_R(\mathcal{U}, \mathcal{U})$ since $(\mathcal{G}_R^+(\mathcal{U}), *)$ is closed under $*$.

   (b) If $\mathsf{G}, \mathsf{H} \in \mathcal{G}_R^-(\mathcal{U}) \subset \mathcal{G}_R(\mathcal{U}, \mathcal{U})$, then $\mathsf{G} * \mathsf{H} \in \mathcal{G}_R^+(\mathcal{U}) \subset \mathcal{G}_R(\mathcal{U}, \mathcal{U})$ as shown in Lemma 65 (use $\mathcal{V} = \mathcal{U}$).

   (c) Suppose $\mathsf{G}^+ \in \mathcal{G}_R^+(\mathcal{U})$ and $\mathsf{H}^- \in \mathcal{G}_R^-(\mathcal{U})$, i.e. $(\mathbf{G}, \mathbf{G}) \in \mathcal{G}_R^+(\mathcal{U})$and $(\mathbf{H}, -\mathbf{H}) \in \mathcal{G}_R^-(\mathcal{U})$. Then

   $$\mathsf{G}^+ * \mathsf{H}^- = (\mathbf{G}, \mathbf{G}) * (\mathbf{H}, -\mathbf{H}) = (\mathbf{GH}, -\mathbf{GH}).$$

   Since $\mathbf{G}, \mathbf{H} \in \mathcal{U}$ and $\mathcal{U}$ is closed, then $\mathbf{GH} \in \mathcal{U}$ and hence $(\mathbf{GH}, -\mathbf{GH}) \in \mathcal{G}_R^-(\mathcal{U})$. That is, $\mathsf{G}^+ * \mathsf{H}^- \in \mathcal{G}_R^-(\mathcal{U}) \subset \mathcal{G}_R(\mathcal{U}, \mathcal{U})$.

   (d) Suppose $\mathsf{G}^- \in \mathcal{G}_R^-(\mathcal{U})$ and $\mathsf{H}^+ \in \mathcal{G}_R^+(\mathcal{U})$, i.e. $(\mathbf{G}, -\mathbf{G}) \in \mathcal{G}_R^-(\mathcal{U})$ and $(\mathbf{H}, \mathbf{H}) \in \mathcal{G}_R^+(\mathcal{U})$. Then

   $$\mathsf{G}^- * \mathsf{H}^+ = (\mathbf{G}, -\mathbf{G}) * (\mathbf{H}, \mathbf{H}) = (\mathbf{GH}, -\mathbf{GH}).$$

   Using the same argument as the previous case, $\mathsf{G}^- * \mathsf{H}^+ \in \mathcal{G}_R^-(\mathcal{U}) \subset \mathcal{G}_R(\mathcal{U}, \mathcal{U})$.

   Since the product of any two members of $\mathcal{G}_R(\mathcal{U}, \mathcal{U})$ is another member of $\mathcal{G}_R(\mathcal{U}, \mathcal{U})$, this set must be closed under the binary operation $*$.

2. Now assume $\mathcal{G}_R(\mathcal{U}, \mathcal{U})$ is closed. By Lemma 64 $\mathcal{U}$ is closed.

This proves the statement of the lemma. ■

Thus if $\mathcal{U}$ is closed, then we may legitimately write

$$*: \mathcal{G}_R(\mathcal{U},\mathcal{U}) \times \mathcal{G}_R(\mathcal{U},\mathcal{U}) \to \mathcal{G}_R(\mathcal{U},\mathcal{U})$$

or, suppressing the $\mathcal{U}$ from the notation as described earlier,

$$*: \mathcal{G}_R \times \mathcal{G}_R \to \mathcal{G}_R.$$

**Theorem 12** *$(\mathcal{G}_R(\mathcal{U},\mathcal{U}), *)$ is a group if and only if $\mathcal{U}$ is a subgroup.*

**Proof.** The previous lemma demonstrated $\mathcal{G}_R(\mathcal{U},\mathcal{U})$ is closed under $*$ if $\mathcal{U}$ is closed and vice-versa. It remains to show the group axioms hold.

1. Assume $\mathcal{U}$ is a subgroup.

   (a) Associative. It was shown in Lemma 61 that $(\mathcal{PGL}(N,\Re), *)$ is a group. Since $\mathcal{G}_R(\mathcal{U},\mathcal{U}) \subset \mathcal{PGL}(N,\Re)$, the associative property of $*$ must also hold for $\mathcal{G}_R(\mathcal{U},\mathcal{U})$.

   (b) Identity. Since $(\mathbf{I},\mathbf{I}) \in \mathcal{G}_R^+(\mathcal{U}) \subset \mathcal{G}_R(\mathcal{U},\mathcal{U})$, $(\mathbf{I},\mathbf{I})$ is the identity element.

   (c) Inverse. It was demonstrated earlier that $\mathcal{G}_R^+(\mathcal{U})$ contains inverses for each of its members. Suppose $\mathsf{G}^- \in \mathcal{G}_R^-(\mathcal{U})$, i.e. $(\mathbf{G},-\mathbf{G}) \in \mathcal{G}_R^-(\mathcal{U})$. Then $\mathbf{G} \in \mathcal{U}$ and since $\mathcal{U}$ is a subgroup, $\mathbf{G}^{-1} \in \mathcal{U}$. Then $(\mathbf{G}^{-1},-\mathbf{G}^{-1}) \in \mathcal{G}_R^-(\mathcal{U})$. We try this as the inverse of $\mathsf{G}^-$ in the set $\mathcal{G}_R(\mathcal{U},\mathcal{U})$.

   $$(\mathbf{G},-\mathbf{G}) * (\mathbf{G}^{-1},-\mathbf{G}^{-1}) = (\mathbf{G}\mathbf{G}^{-1},(-\mathbf{G})(-\mathbf{G}^{-1})) = (\mathbf{G}\mathbf{G}^{-1},\mathbf{G}\mathbf{G}^{-1}) = (\mathbf{I},\mathbf{I})$$

   and

   $$(\mathbf{G}^{-1},-\mathbf{G}^{-1}) * (\mathbf{G},-\mathbf{G}) = (\mathbf{G}^{-1}\mathbf{G},(-\mathbf{G}^{-1})(-\mathbf{G})) = (\mathbf{G}^{-1}\mathbf{G},\mathbf{G}^{-1}\mathbf{G}) = (\mathbf{I},\mathbf{I}).$$

   Thus $(\mathbf{G}^{-1},-\mathbf{G}^{-1})$ is the inverse for $(\mathbf{G},-\mathbf{G})$. Therefore, every member of $\mathcal{G}_R(\mathcal{U},\mathcal{U})$ has an inverse with the identity element $(\mathbf{I},\mathbf{I})$.

Since all the axioms of a group are satisfied, $(\mathcal{G}_R(\mathcal{U},\mathcal{U}), *)$ is a group.

2. Assume $(\mathcal{G}_R(\mathcal{U},\mathcal{U}),*)$ is a group. Recall $\mathcal{G}_R(\mathcal{U},\mathcal{U}) = \mathcal{G}_R^+(\mathcal{U}) \cup \mathcal{G}_R^-(\mathcal{U})$. Then $\mathcal{G}_R^+(\mathcal{U})$ must be associative, must contain the identity element since $(\mathbf{I},\mathbf{I}) \in \mathcal{G}_R(\mathcal{U},\mathcal{U})$ and every member of $\mathcal{G}_R^+(\mathcal{U})$ must have its inverse since $(\mathbf{G}^{-1},\mathbf{G}^{-1}) \in \mathcal{G}_R(\mathcal{U},\mathcal{U})$ if $(\mathbf{G},\mathbf{G}) \in \mathcal{G}_R(\mathcal{U},\mathcal{U})$. Thus $(\mathcal{G}_R^+(\mathcal{U}),*)$ is a group. But by Lemma 63, this means $\mathcal{U}$ is a subgroup.

Thus the statement of the theorem is proved. ■

### B.3.2 The General Case

There is second way $\mathcal{U}$ and $\mathcal{V}$ may be related to each other which may allow $\mathcal{G}_R(\mathcal{U},\mathcal{V})$ to be closed under $*$. Throughout this section, $\mathcal{U}$ is a subgroup.

**Theorem 13** *$\mathcal{G}_R(\mathcal{U},\mathcal{V})$ is closed under $*$ only if $\mathcal{U}=\mathcal{V}$ or $\mathcal{U}\cap\mathcal{V}=\emptyset$ and $\mathcal{U}\cup\mathcal{V}$ is a monoid.*

**Proof.** Note this theorem is providing necessary, but not sufficient conditions for $\mathcal{G}_R(\mathcal{U},\mathcal{V})$ to be closed. The objective of the theorem is to show that of the five ways $\mathcal{U}$ and $\mathcal{V}$ may be related, only the two mentioned in the statement of the theorem are possible if $\mathcal{G}_R(\mathcal{U},\mathcal{V})$ is to be closed. Thus this proof is done by considering the five ways $\mathcal{U}$ and $\mathcal{V}$ may be related and showing that in cases two through four, this leads to a contradiction. In all of these cases, the hypothesis $\mathcal{G}_R(\mathcal{U},\mathcal{V})$ is closed under $*$ plays an important role. We use $\mathbf{G}\in\mathcal{U}$ and $\mathbf{H}\in\mathcal{V}$.

1. The case $\mathcal{U}$ is closed (since $\mathcal{U}$ is a subgroup) and $\mathcal{U}=\mathcal{V}$ has been handled in Lemma 66. Indeed, if $\mathcal{U}=\mathcal{V}$, this lemma demonstrated this is a sufficient condition for $\mathcal{G}_R(\mathcal{U},\mathcal{V})$ to be closed.

2. Assume $\mathcal{V}\subset\mathcal{U}$, $\mathcal{V}\neq\mathcal{U}$ and attempt to derive a contradiction. Then $\mathbf{GH}\in\mathcal{V}$ for all $\mathbf{G}\in\mathcal{U}$ and $\mathbf{H}\in\mathcal{V}$ based on Lemma 64. Since $\mathcal{V}\subset\mathcal{U}$, $\mathcal{V}\neq\mathcal{U}$, there must exist $\mathbf{K}\in\mathcal{U}$ with $\mathbf{K}\notin\mathcal{V}$. Since $\mathcal{V}\subset\mathcal{U}$, $\mathbf{H}\in\mathcal{U}$ and it has an inverse $\mathbf{H}^{-1}\in\mathcal{U}$. Since $\mathbf{K},\mathbf{H}^{-1}\in\mathcal{U}$, $\mathbf{L}=\mathbf{KH}^{-1}$ is also a member of $\mathcal{U}$. Then $\mathbf{LH}=\mathbf{KH}^{-1}\mathbf{H}=\mathbf{KI}=\mathbf{K}$, that is $\mathbf{K}=\mathbf{LH}$. But $\mathbf{L}\in\mathcal{U}$ and $\mathbf{H}\in\mathcal{V}$ and thus $\mathbf{K}\in\mathcal{V}$. This contradicts the earlier assertion $\mathbf{K}\notin\mathcal{V}$. Thus the assumption $\mathcal{V}\subset\mathcal{U}$, $\mathcal{V}\neq\mathcal{U}$ must be false and $\mathcal{G}_R(\mathcal{U},\mathcal{V})$ cannot be closed under $*$ with this relationship between $\mathcal{U}$ and $\mathcal{V}$.

3. Assume $\mathcal{U}\subset\mathcal{V}$, $\mathcal{U}\neq\mathcal{V}$ and attempt to derive a contradiction. In this case, $\mathbf{KH}\in\mathcal{U}$ for all $\mathbf{H}\in\mathcal{V}$ and $\mathbf{K}\in\mathcal{V}$ based on Lemma 64. Since $\mathcal{U}\subset\mathcal{V}$, $\mathcal{U}\neq\mathcal{V}$, it is possible to pick

$\mathbf{K} \in \mathcal{U}$ and $\mathbf{H} \notin \mathcal{U}$. Then $\mathbf{G} = \mathbf{KH} \in \mathcal{U}$ since $\mathbf{K}$ and $\mathbf{H}$ are both members of $\mathcal{V}$. $\mathbf{K}^{-1}$ exists since $\mathcal{U}$ is a subgroup so $\mathbf{K}^{-1}\mathbf{G} = \mathbf{K}^{-1}\mathbf{KH} = \mathbf{IH} = \mathbf{H}$ , that is $\mathbf{H} = \mathbf{K}^{-1}\mathbf{G}$. But $\mathbf{K}^{-1}, \mathbf{G} \in \mathcal{U}$ and thus $\mathbf{H} \in \mathcal{U}$. But this contradicts the earlier assertion $\mathbf{H} \notin \mathcal{U}$. Thus the assumption $\mathcal{U} \subset \mathcal{V}, \mathcal{U} \neq \mathcal{V}$ must be false and $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ cannot be closed under $*$ with this relationship between $\mathcal{U}$ and $\mathcal{V}$.

4. Assume $\mathcal{U} \cap \mathcal{V} \neq \emptyset, \mathcal{U} \neq \mathcal{V}, \mathcal{V} \not\subseteq \mathcal{U}, \mathcal{V} \not\subseteq \mathcal{U}$ and attempt to derive a contradiction. If $\mathbf{K}, \mathbf{H} \in \mathcal{V}$, then $\mathbf{KH} \in \mathcal{U}$ based on Lemma 64. Now pick $\mathbf{K} \in \mathcal{U} \cap \mathcal{V}$ and $\mathbf{H} \in \mathcal{V}$ with $\mathbf{H} \notin \mathcal{U}$. Let $\mathbf{G} = \mathbf{KH}$ so $\mathbf{G} \in \mathcal{U}$ since $\mathbf{K}$ and $\mathbf{H}$ are both members of $\mathcal{V}$. Then $\mathbf{K}^{-1}$ exists since $\mathcal{U}$ is a subgroup. $\mathbf{K}^{-1}\mathbf{G} = \mathbf{K}^{-1}\,\mathbf{KH} = \mathbf{IH} = \mathbf{H}$, that is $\mathbf{H} = \mathbf{K}^{-1}\mathbf{G}$ . But $\mathbf{K}^{-1}\mathbf{G} \in \mathcal{U}$ since $\mathcal{U}$ is a subgroup and thus $\mathbf{H} \in \mathcal{U}$. But this contradicts the earlier assertion $\mathbf{H} \notin \mathcal{U}$. Thus the assumption $\mathcal{U} \cap \mathcal{V} \neq \emptyset, \mathcal{U} \neq \mathcal{V}, \mathcal{V} \not\subseteq \mathcal{U}, \mathcal{V} \not\subseteq \mathcal{U}$ must be false and $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ cannot be closed under $*$ with this relationship between $\mathcal{U}$ and $\mathcal{V}$.

5. Assume $\mathcal{U} \cap \mathcal{V} = \emptyset$. By Lemma 64, $\mathcal{U} \cup \mathcal{V}$ is closed. Since $\mathcal{U}$ is a subgroup, it contains the identity so $\mathcal{U} \cup \mathcal{V}$ contains the identity. Since $\mathcal{G}_R(\mathcal{U}, \mathcal{V})$ is associative under $*$ (by inheritance from $(\mathcal{PGL}(N, \Re), *)$), the group product in $\mathcal{U} \cup \mathcal{V}$ is associative (this is proved in detail in the next theorem). This defines $\mathcal{U} \cup \mathcal{V}$ as a monoid.

This completes the proof of the theorem. ■

**Corollary 9** *Let $\mathcal{U}, \mathcal{V} \in \mathcal{GL}(N, \Re)$. $(\mathcal{G}_R(\mathcal{U}, \mathcal{V}), *)$ is a group only if $\mathcal{U} = \mathcal{V}$ or $\mathcal{U} \cap \mathcal{V} = \emptyset$ and $\mathcal{U} \cup \mathcal{V}$ is a subgroup.*

**Proof.** From Theorem 12, $\mathcal{U} = \mathcal{V}$ provides both necessary and sufficient conditions for $(\mathcal{G}_R(\mathcal{U}, \mathcal{V}), *)$ to be a group.

Consider the case in which $\mathcal{U} \cap \mathcal{V} = \emptyset$. Look at the three axioms of a group.

1. Associative. Suppose $\mathsf{G}^{\pm}, \mathsf{H}^{\pm}, \mathsf{K}^{\pm} \in (\mathcal{G}_R(\mathcal{U}, \mathcal{V}), *)$. Let $\mathsf{G}^{\pm} = (\mathbf{G}, \mathbf{G}^*)$, $\mathsf{H}^{\pm} = (\mathbf{H}, \mathbf{H}^*)$ and $\mathsf{K}^{\pm} = (\mathbf{K}, \mathbf{K}^*)$ where $\mathbf{G}^* = \mathbf{G}$ in the case of the plus sign and $\mathbf{G}^* = -\mathbf{G}$ in the case of the minus sign and similarly for the other letters. Since $(\mathcal{G}_R(\mathcal{U}, \mathcal{V}), *)$ is associative, we must have

$$((\mathbf{G}, \mathbf{G}^*) * (\mathbf{H}, \mathbf{H}^*)) * (\mathbf{K}, \mathbf{K}^*) = (\mathbf{G}, \mathbf{G}^*) * ((\mathbf{H}, \mathbf{H}^*) * (\mathbf{K}, \mathbf{K}^*))$$

$$(\mathbf{GH}, \mathbf{G}^*\mathbf{H}^*) * (\mathbf{K}, \mathbf{K}^*) = (\mathbf{G}, \mathbf{G}^*) * (\mathbf{HK}, \mathbf{H}^*\mathbf{K}^*)$$

and

$$((\mathbf{GH})\,\mathbf{K}, (\mathbf{G}^*\mathbf{H}^*)\,\mathbf{K}^*) = (\mathbf{G}\,(\mathbf{HK})\,, \mathbf{G}^*\,(\mathbf{H}^*\mathbf{K}^*))\,.$$

From this we see $(\mathbf{G}^*\mathbf{H}^*)\,\mathbf{K}^* = \mathbf{G}^*\,(\mathbf{H}^*\mathbf{K}^*)$ which shows $\mathcal{U} \cup \mathcal{V}$ is associative.

2. Identity. $(\mathcal{G}_R\,(\mathcal{U}, \mathcal{V})\,, *)$ is a group and thus contains $(\mathbf{I}, \mathbf{I})$. Thus $\mathbf{I} \in \mathcal{U} \cup \mathcal{V}$.

3. Inverse. Suppose $(\mathbf{G}, \mathbf{G}) \in (\mathcal{G}_R\,(\mathcal{U}, \mathcal{V})\,, *)$. Then $(\mathbf{G}^{-1}, \mathbf{G}^{-1}) \in (\mathcal{G}_R\,(\mathcal{U}, \mathcal{V})\,, *)$ since $(\mathcal{G}_R\,(\mathcal{U}, \mathcal{V})\,, *)$ is a group. Since $\mathbf{G}, \mathbf{G}^{-1} \in \mathcal{U}$, this means $\mathcal{U}$ is a subgroup, consistent with our hypothesis. Now consider $(\mathbf{H}, -\mathbf{H}) \in (\mathcal{G}_R\,(\mathcal{U}, \mathcal{V})\,, *)$. Then $(\mathbf{H}^{-1}, -\mathbf{H}^{-1}) \in (\mathcal{G}_R\,(\mathcal{U}, \mathcal{V})\,, *)$ since $(\mathcal{G}_R\,(\mathcal{U}, \mathcal{V})\,, *)$ is a group. Thus $\mathbf{H}, \mathbf{H}^{-1} \in \mathcal{V}$. We note $\mathbf{HH}^{-1} = \mathbf{I} \in \mathcal{U}$ which is consistent with Lemma 64. Thus if $\mathbf{G}, \mathbf{H} \in \mathcal{U} \cup \mathcal{V}$, then $\mathbf{G}^{-1}, \mathbf{H}^{-1} \in \mathcal{U} \cup \mathcal{V}$.

Since the group property of $(\mathcal{G}_R\,(\mathcal{U}, \mathcal{V})\,, *)$ implies all the group axioms for $\mathcal{U} \cup \mathcal{V}$, then $\mathcal{U} \cup \mathcal{V}$ is a subgroup of the general linear group. ■

It should be emphasized that we have provided only necessary conditions for $(\mathcal{G}_R\,(\mathcal{U}, \mathcal{V})\,, *)$ to be a group. In the case $\mathcal{U} = \mathcal{V}$, these are also sufficient conditions. For the case $\mathcal{U} \cap \mathcal{V} = \emptyset$ and $\mathcal{U} \cup \mathcal{V}$ is a subgroup, we would have to provide some additional hypothesis regarding how members of $\mathcal{U} \cup \mathcal{V}$ multiply together (we could use Lemma 64 as a guide). The above results are sufficient for purposes of this thesis.

# Appendix C

# Tensor Pairs in Directed Rod Theory

Throughout this appendix, let $\mathbf{F}$, $\mathbf{M}$ and $\mathbf{N}$ denote the tensor quantities in the main body of the thesis. Let $'$ continue to denote differentiation with respect to the parameter for curvature. We make the following definitions of tensor pairs: $\mathsf{F} = (\mathbf{F}, \mathbf{F}')$, $\mathsf{T} = (\mathbf{N}, \mathbf{M})$, $\mathsf{G} = (\mathbf{G}, \mathbf{G})$ and $\mathsf{Q} = (\mathbf{Q}, \mathbf{Q})$ where $\mathbf{G}$ is any member of the special linear group $\mathcal{SL}(3, \Re)$ and $\mathbf{Q}$ is any member of the special orthogonal group. Note $\mathsf{F}, \mathsf{T}, \mathsf{G}, \mathsf{Q} \in \mathcal{PGL}(3, \Re)$ and $\mathsf{G} \in \mathcal{G}_R^+(\mathcal{U})$ for some $\mathcal{U} \subset \mathcal{GL}(3, \Re)$ as defined in Appendix B (notice we use $N = 3$). We need to consider the binary operation on tensor pairs as defined in Appendix B and denoted $*$. We drop the notation $*$ so if $\mathsf{G}, \mathsf{H} \in \mathcal{PGL}(3, \Re)$, $\mathsf{G} * \mathsf{H} \equiv \mathsf{GH}$.

In this appendix, we wish to briefly review Cohen's use of tensor pairs [1], particularly in the formulation of monotropic symmetry. Tensor pairs provide a compact way of expressing constitutive restrictions.

In Cohen's paper, the constitutive equations are expressed by stress tensors as a function of deformation and its first derivative. The quantity $\mathsf{F}$ is called the generalized deformation and the quantity $\mathsf{T}$ is called the generalized stress. The constitutive equation is

$$\mathsf{T} = \tilde{\mathsf{T}}(\mathsf{F}) \tag{C.1}$$

which is an abbreviated way of writing

$$\mathbf{N} = \widetilde{\mathbf{N}}\left(\mathbf{F}, \mathbf{F}'\right), \mathbf{M} = \widetilde{\mathbf{M}}\left(\mathbf{F}, \mathbf{F}'\right).$$

We now consider how frame indifference and monotropic symmetry are expressed in the notation of tensor pairs. Frame indifference is straightforward; the expression is

$$\widetilde{\mathsf{T}}\left(\mathsf{QF}\right) = \mathsf{Q}\widetilde{\mathsf{T}}\left(\mathsf{F}\right)\mathsf{Q}^{\mathrm{T}}$$

where $\mathsf{Q}^{\mathrm{T}}$ denotes the transpose of Q. Since $\mathbf{Q}$ is any member of the special orthogonal group, it represents a rotation and thus $\mathsf{Q} = (\mathbf{Q}, \mathbf{Q})$ represents a pair of identical rotations. Naturally $\mathsf{Q}^{\mathrm{T}} = \left(\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}^{\mathrm{T}}\right)$. This expression indicates a rotation of the rod following deformation has no effect on the stress tensors.

Material symmetry is developed by Cohen in terms of material symmetry and material anti-symmetry sets. The underlying sets $\mathcal{U}$ and $\mathcal{V}$ are restricted to subsets of the special linear group $\mathcal{SL}\left(N, \Re\right)$. These are defined as follows.

**Definition 13** *Material Symmetry Set: A material symmetry set is $\mathcal{G}_R^+\left(\mathcal{U}\right)$ for $\mathcal{U} \subset \mathcal{SL}\left(3, \Re\right)$ where $\mathcal{G}_R^+\left(\mathcal{U}\right)$ is defined in Appendix B.*

**Definition 14** *Material Anti-symmetry Set: A material anti-symmetry set is $\mathcal{G}_R^-\left(\mathcal{V}\right)$ for $\mathcal{V} \subset \mathcal{SL}\left(3, \Re\right)$ where $\mathcal{G}_R^-\left(\mathcal{V}\right)$ is defined in Appendix B.*

**Definition 15** *Combined Material Symmetry Set: A combined material symmetry set is $\mathcal{G}_R\left(\mathcal{U}, \mathcal{V}\right)$ for some $\mathcal{U}, \mathcal{V} \subset \mathcal{SL}\left(3, \Re\right)$ where $\mathcal{G}_R\left(\mathcal{U}, \mathcal{V}\right)$ is defined in Appendix B.*

Of course, once a material symmetry set and material anti-symmetry set have been defined, the combined material symmetry set is immediately defined. Recall from Appendix B the notation $\mathsf{G}^{\pm}$ indicates either $\mathsf{G}^{+}$ or $\mathsf{G}^{-}$ where $\mathsf{G}^{+}$ denotes a member of the material symmetry set and $\mathsf{G}^{-}$ denotes a member of the material anti-symmetry set.

**Definition 16** *Principle of Combined Material Symmetry: The principle of combined material symmetry states that the constitutive restriction C.1 satisfies the transformation rules*

$$\widetilde{\mathsf{T}}\left(\mathsf{F}\right) = \widetilde{\mathsf{T}^{\pm}}\left(\mathsf{F}\mathsf{G}^{\pm}\right)$$

*for all admissible generalized deformations* $\mathsf{F}$ *and all admissible* $\mathsf{G}^{\pm}$ *in the combined material symmetry set.*

This is a way of expressing that the effect of the deformation on the stress tensors remains invariant if the rod is first rotated (but only by members of $\mathcal{U}$ and $\mathcal{V}$) and then deformed.

The question then is how to choose the sets $\mathcal{U}$ and $\mathcal{V}$. Cohen suggested the material symmetry and combined material symmetry sets be groups under the binary operation on tensor pairs. In Appendix B it was demonstrated that this is only possible if $\mathcal{U}$ is a subgroup of the general linear group and $\mathcal{U} = \mathcal{V}$ or $\mathcal{U} \cap \mathcal{V} = \emptyset$ and $\mathcal{U} \cup \mathcal{V}$ is a subgroup. This places some restrictions on the sets $\mathcal{U}$ and $\mathcal{V}$. Cohen defines monotropic symmetry by making special choices for the set $\mathcal{U}$ and $\mathcal{V}$ with $\mathcal{U} \cap \mathcal{V} = \emptyset$.

**Definition 17** *Materially Monotropic: A rod point is said to be materially monotropic if its combined material symmetry set contains at least one of the elements* $\mathsf{Q}_{\mathbf{e}}^{\pm}$ *where* $\mathsf{Q}_{\mathbf{e}} = (\mathbf{Q}_e, \mathbf{Q}_e)$ *in which* $\mathbf{Q}_e$ *is a rotation of angle* $\pi$ *about a fixed axis* $\mathbf{e}$ *(*$\mathbf{e}$ *is a element of the tangent space at some point* $P$ *on the rod axis).*

We now wish to look at this special case of $\mathcal{U}$ and $\mathcal{G}_R(\mathcal{U},\mathcal{U})$, namely when $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_e\}$ where $\mathbf{Q}_e$ is the rotation described above.

**Theorem 14** *If* $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_e\}$ *with* $\mathbf{Q}_e$ *defined as above, then* $\mathcal{U}$ *is a subgroup of the special linear group* $\mathcal{SL}(3, \Re)$.

**Proof.** Tensors which have the effect of rotating a vector about a fixed axis belong to the special orthogonal group $\mathcal{SO}(3, \Re)$. $\mathcal{SO}(3, \Re)$ is defined as $\mathcal{SL}(3, \Re) \cap \mathcal{O}(3, \Re)$ where $\mathcal{O}(3, \Re)$ is the orthogonal group. Thus $\mathcal{SO}(3, \Re) \subset \mathcal{SL}(3, \Re)$. Since $\mathbf{Q}_e \in \mathcal{SO}(3, \Re)$, $\mathbf{Q}_e \in \mathcal{SL}(3, \Re)$. Since $\mathbf{I} \in \mathcal{SL}(3, \Re)$, it follows that $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_e\} \subset \mathcal{SL}(3, \Re)$. It remains to show $\mathcal{U}$ is closed under the group product and satisfies the group axioms.

Now $\mathbf{Q}_e$ is a rotation of angle $\pi$ radians and thus $\mathbf{Q}_e\mathbf{Q}_e$ must be a rotation of $2\pi$ radians. If $\mathbf{v} \in \Re^3$, $\mathbf{Q}_e\mathbf{Q}_e\mathbf{v} = \mathbf{v}$, the original vector. That is, $\mathbf{Q}_e\mathbf{Q}_e = \mathbf{I}$. Of course $\mathbf{IQ}_e = \mathbf{Q}_e\mathbf{I} = \mathbf{Q}_e$ and $\mathbf{II} = \mathbf{I}$ which shows $\mathcal{U}$ is closed under the binary operation on $\mathcal{SL}(3, \Re)$.

1. Associative. This is obvious since $\mathcal{U} \subset \mathcal{SL}(3, \Re)$.

2. Identity. $\mathcal{U}$ contains the identity $\mathbf{I}$.

3. Inverse. $\mathbf{Q}_e\mathbf{Q}_e = \mathbf{I}$ so $\mathbf{Q}_e$ is its own inverse. Of course $\mathbf{I}$ is its own inverse. Then every member of $\mathcal{U}$ has its inverse in $\mathcal{U}$.

Since $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_e\} \subset \mathcal{SL}(3, \Re) \subset \mathcal{GL}(3, \Re)$, $\mathcal{U}$ is closed under the group product and satisfies the group axioms, then $\mathcal{U}$ is a subgroup of the special linear group $\mathcal{SL}(3, \Re)$. ■

The set $\mathcal{U} = \{\mathbf{I}, \mathbf{Q}_e\}$ becomes the defining set for monotropic symmetry for the symmetric case for this thesis, as explained in Chapter 5. Then we must make a careful choice of $\mathcal{V}$ such that $\mathcal{U} \cap \mathcal{V} = \emptyset$, $\mathcal{U} \cup \mathcal{V}$ is a subgroup and $\mathcal{V}$ contains members of $\mathcal{SL}(3, \Re)$ representing rotations of $\pi$ radians about an axis. Alternatively, we may allow $\mathcal{V}$ to be empty which will still satisfy the conditions $\mathcal{U} \cap \mathcal{V} = \emptyset$ and $\mathcal{U} \cup \mathcal{V}$ is a subgroup.

If $\mathcal{V}$ is empty, then the monotropic symmetry imposed has only a symmetric case, corresponding to rotation of $\pi$ radians. If $\mathcal{V}$ is not empty, the monotropic symmetry also includes an antisymmetric case which are actually reflections in the plane orthogonal to the direction of the axis of rotation.

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