T I T L E.

A STUDY OF THE INFLEXION POINTS OF THE PLANE CUBIC - WITHOUT SINGULAR POINTS.

(Application of the Relative co-variance of the Hessian to discover all the points of Inflexion, Inflexion triangles, Inflexion Tangents).

By

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OUTLINE:

A. The General Case - The General Ternary Cubic - homogeneous coordinates.

B. Particular examples:

I. The curve $x_1^3 + y_2^3 + x_3^3 + 6x_1x_2x_3 = 0$

i.e. $x^3 + y^3 + 6xy + 1 = 0$

a. Cartesian coordinates - the graph.

b. In homogeneous coordinates -
The problem discussed throughout for this example:
All nine points of inflexion found with the 12 lines, and 4 inflexion triangles, and the Nine Inflexion Tangents.

II. Discussion for such curves as,

$x = y^3. \quad y = x^3 - 3x + 1$

$x^3 + y^3 + x - 1 = 0$

Cartesian
a. In certain coordinates - with graphs.

b. In Homogeneous coordinates where the transformation used is

$$x = \frac{x_1}{x_3} \;. \quad y = \frac{x_2}{x_3}$$

c. In Homogeneous coordinates where the

transformation used is

$$x = \frac{x_1}{x_1 + x_2 - x_3}$$

$$y = \frac{x_2}{x_1 + x_2 - x_3}$$

Illustrated by the curve

$$x = y^3$$

i.e. $x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_1 x_3^2 - x_1^2 x_3$

$$- x_1 x_2 x_3 - x_2^3 = 0$$

THE GENERAL CASE.

The General Ternary Cubic $f(x_1, x_2, x_3) = 0$ may be written in its most general form thus:

$$f = c_1 x_1^3 + c_2 x_2^3 + c_3 x_3^3 + c_4 x_1^2 x_2$$

$$+ c_5 x_1^2 x_3 + c_6 x_2^2 x_1 + c_7 x_2^2 x_3$$

$$+ c_8 x_3^2 x_1 + c_9 x_3^2 x_2 + c_{10} x_1 x_2 x_3 = 0 \quad (1)$$

Since we are here dealing with homogeneous coordinates, it is permissible to so choose the triangle of reference that P (001) is a point on the curve. Then in the equation (1) all the terms except $c_3 x_3^3$ vanish, i.e. $c_3 x_3^3 = 0$. Hence $c_3 = 0$ and the term in x_3^3 is lacking. To effect this change of the triangle of reference, let (a_1, a_2, a_3) be a point on the curve $f = 0$. We then need only subject the variables to the transformation:–

$$x_i = c_{i1}\, y_1 + c_{i2}\, y_2 + c_{i3}\, y_3$$ (note i subscript)

T :

$$x_1 = c_{11}\, y_1 + c_{12}\, y_2 + c_{13}\, y_3$$

$$x_2 = c_{21}\, y_1 + c_{22}\, y_2 + c_{23} y_3$$

$$x_3 = c_{31}\, y_1 + c_{32} + c_{33}\, y_3$$

Where $c_{i3} = a_i$, and c_{i2}, c_{i3}, chosen

so that

$$= \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \neq 0$$

Then Transformation T makes

$(x_1 \;\; x_2 \;\; x_3) = (a_1 \,.\, a_2 \,.\, a_3)$ correspond

to $(y_1, y_2, y_3) = (0\;0\;1)$.

Our f now becomes,

$$f = x_3^2 (c_8\, x_1 + c_9\, x_2) + x_3 (c_5\, x_1^2 + c_7\, x_2^2 + c_{10} x_1 x_2)$$

$$+ (c_1\, x_1^3 + c_2\, x_2^3 + c_4\, x_1^2 x_2 + c_6\, x_2^2\, x_1)$$

$= x_3^2 f_1 + x_3 f_2 + f_3$. Where f_i is a

homeogeneous function of x_1 and x_2 of degree i.

$$f_1 = c_8 x_1 + c_9 x_2$$

$$f_2 = c_5 x_1^2 + c_7 x_2^2 + c_{10} x_1 x_2$$

$$f_3 = c_1 x_1^3 + c_2 x_2^3 + c_4 x_1^2 x_2 + c_6 x_2^2 x_1$$

If f_1 were identically zero, the partial derivatives of f with respect to x_1, x_2, and x_3 would all be zero. But we are here dealing with the non-singular cubic only. Therefore f_1 cannot be identically zero and may be introduced as a new variable in place of x_1.

Let $\bar{x}_1 = c_8 x_1 + c_9 x_2$

whence $x_1 = \dfrac{1}{c_8}(\bar{x}_1 - c_9) = \dfrac{1}{c_8}\bar{x}_1 - \dfrac{c_9}{c_8} x_2$

$$= m\bar{x}_1 + n x_2, \text{ where } m = \frac{1}{c_8}, \ n = -\frac{c_9}{c_8}$$

Then by this transformation, dropping the bars we have :

$$x_3^2 x_1 + x_3 \left\{ c_5 (mx_1 + nx_2)^2 + c_7 x_2^2 + c_{10} (mx_1 + nx_2) x_2 \right\}$$
$$+ c_1 (mx_1 + nx_2)^3 + c_2 x_2^3 + c_4 (mx_1 + nx_2)^2 x_2$$
$$+ c_6 x_2^2 (mx_1 + nx_2).$$

Where a, b, c, d, m, ... m_4 are functions of c_i ($i = 1, 2, 3, 4,$)

The last expression

$$= x_3^2 x_1 + x_3 (ax_1^2 + bx_1x_2 + cx_2^2)$$

$$+ m_1 x_1^3 + m_2 x_1^2 x_2 + m_3 x_2^2 x_1 + m_4 x_2^3$$

$$= x_3^2 x_1 + x_3 (ax_1^2 + bx_1 x_2 + cx_2^2) + \bar{f}_3$$

where $\bar{f} = m_1 x_1^3 + m_2 x_1^2 x_2 + m_3 x_1 x_2^2 + m_4 x_2^3$

If we now replace x_3 by $x_3 - \frac{1}{2} (ax_1 + bx_2)$.

we have:

$$F = x_1 \left\{x_3 - \frac{1}{2} (ax_1 + bx_2)\right\}^2$$

$$+\left\{x_3 - \frac{1}{2} (ax_1 + bx_2)\right\} \left\{ax_1^2 + bx_1 x_2 + cx_1^2\right\} + \bar{f}_3$$

$$= x_3^2 x_1 + c x_3 x_2^2 + a_1 x_1^3 + b_1 x_1^2 x_2$$

$$+ c_1 x_1 x_2^2 + d x_2^3$$

Where a_1, b_1, c_1, d, are functions of a, b, c, Whence

$$F = x_3^2 x_1 + e x_3 x_2^2 + G$$

Where e is written for c to avoid confusion,

and $G = a_1 x_1^3 + b_1 x_1^2 x_2 + c_1 x_1 x_2^2 + d x_2^3$

Let us denote the second partial derivative with respect to xi and xj by Gi j.

Then the Hessian of F,

$$H = \begin{vmatrix} \frac{dF}{dx_1^2}, & \frac{dF}{dx_1 dx_2}, & \frac{dF}{dx_1 dx_3} \\ \frac{dF}{dx_1 dx_2}; & \frac{dF}{dx_2 dx_2}, & \frac{dF}{dx_2 dx_3} \\ \frac{dF}{dx_1 dx_3}, & \frac{dF}{dx_2 dx_3}, & \frac{dF}{dx_3^2} \end{vmatrix}$$

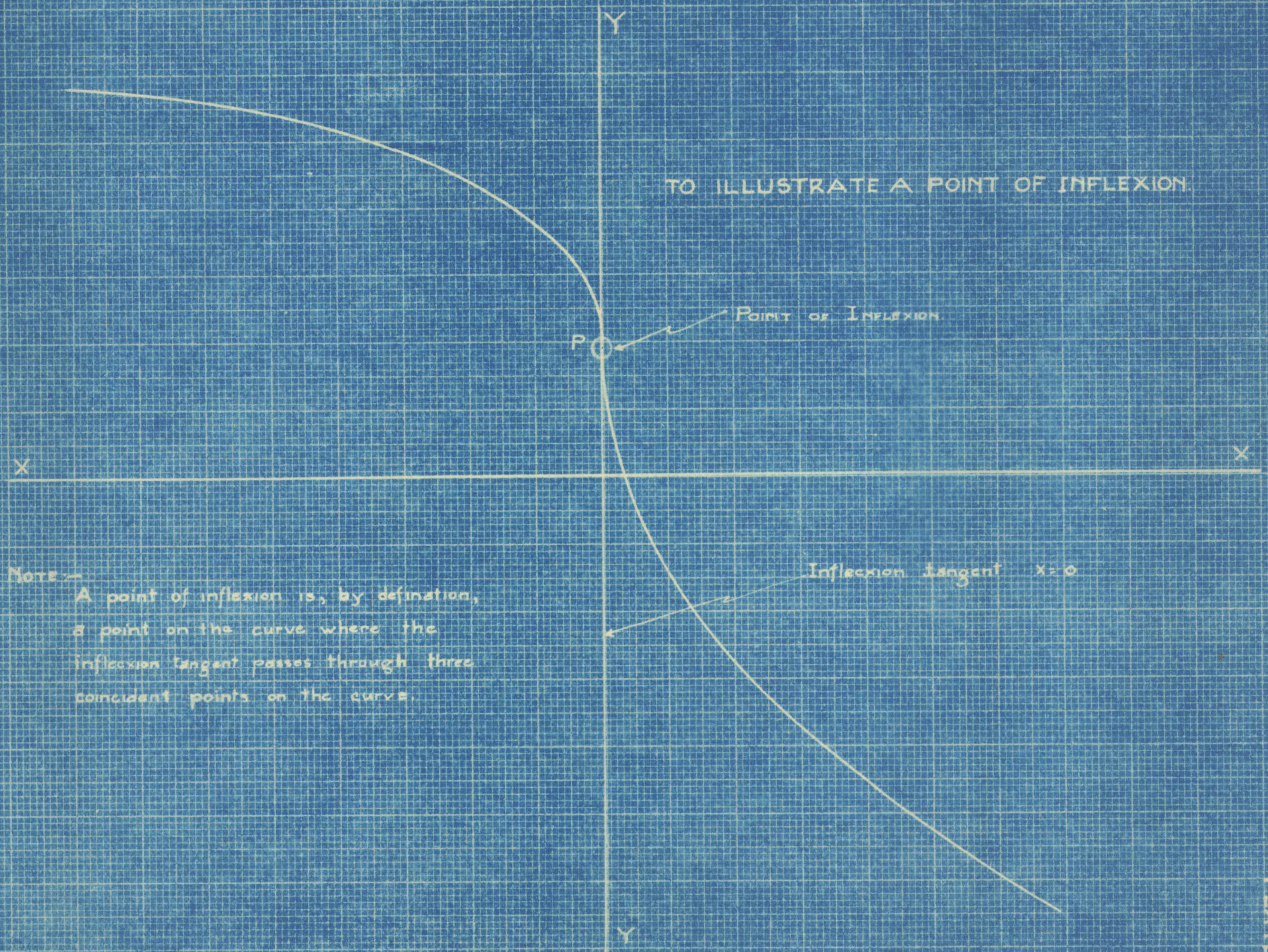
$$= \begin{vmatrix} C_{11} & C_{12} & 2x_3 \\ C_{12} & C_{22} + 2ex_1 & 2ex_2 \\ 2x_3 & 2ex_2 & 2x_1 \end{vmatrix}$$

Since the Hessian is a covariant of index 2, $H = \triangle^2 h$, where $\triangle$ is the determinant of the transformation that replaced f by F. Hence the Hessian curve $H = 0$, is the same curve as $h = 0$. In investigating the properties of $f = 0$, $h = 0$, we may therefore refer them to the triangle of reference for which their equations are $H = 0$ and $F = 0$.

THIS MARGIN RESERVED FOR BINDING.

IF SHEET IS READ THIS WAY (HORIZONTALLY), THIS MUST BE TOP.
IF SHEET IS READ THE OTHER WAY (VERTICALLY), THIS MUST BE LEFT-HAND SIDE.

TO ILLUSTRATE A POINT OF INFLEXION



Note:—
A point of inflexion is, by defination, a point on the curve where the inflexion tangent passes through three coincident points on the curve.

A.R.M. '25

PLATE A

Examining the equation $H = 0$ given on the previous page, we see that the term in x_3^3 is lacking if $e = 0$. i.e. the point P (0, 0, 1) is on the Hessian curve if $e = 0$. Also, the line $x_1 = 0$ meets $F = 0$ at the points given by,

$$e x_3 \; x_2^2 + d \, x_2^3 = 0$$. —since all the other terms contain the factor $x_1 = 0$ -

$$\text{i.e. } x_2^2 (e x_3 + d x_2) = 0 ,$$ and

these points coincide at P if and only if

$e = 0$, i.e. $d x_2^3 = 0$.

P is then a point of Inflexion.

- Points of Inflexion are points that separate arcs concave upwards from arcs concave downwards, or points at which the tangent meets the curve in three coincident points.

See graph on Plate A. -

Hence we may deduce that for a more non-singular cubic curve $f = 0$ every point of ~~inflexion~~ intersection of the curve with its Hessian curve, is a point of inflexion and conversely.

If now we were to eliminate x_3 between $f = 0$, and $h = 0$, the resulting homogeneous relation in x_1, x_2 has at least one set of solutions x_1^1 x_2^1. Hence $f = 0$ and $h = 0$ have at least one common root x_3^1 for the values x_1^1 x_2^1 of x_1 and x_2.

and since all intersections of $f = 0$ and $h = 0$ are in flexion points

(x_1^1 , x_2^1 , x_3^1) is an inflexion point of f. If we subject the variable to the transformation:

$$T_1 : \quad \begin{aligned} x_1 &= c_{11}\, y_1 + c_{12}\, y_2 + c_{13}\, y_3 \\ x_2 &= c_{21}\, y_1 + c_{22}\, y_2 + c_{23}\, y_3 \\ x_3 &= c_{31}\, y_1 + c_{32}\, y_2 + c_{33}\, y_3 \end{aligned}$$

Where $ci_3 = xi^1 \quad (i = 1, 2, 3)$.
$(c_{is} = x_i')$

the point (x_1^1 , x_2^1 , x_3^1) becomes the point $(0\ 0\ 1)$.

If as before we now transform f into F in which e is now zero, we have

$$F = x_3^2\, x_1 + C$$

Where $C = a_1\, x_1^3 + b_1\, x_1^2\, x_2 + c_1\, x_1\, x_2^2 + dx_2^3$

Where d cannot be zero, for then

$$F = x_1 (x_3^2 + a_1\, x_1^3 + b_1 x_1^2 x_2 + c_1 x_1 x_2^2)$$

$$= x_1\, Q \quad \text{and} \quad \frac{dF}{dx_1} = Q + x_1 \frac{dQ}{dx_1}$$

$$\frac{df}{dx_2} = x_1 \frac{dQ}{dx_2}$$

$$\frac{df}{dx_3} = x_1 \frac{dQ}{dx_3}$$

would all vanish at the point for which $x_1 = 0$ $Q = 0$, and the curve would have a singular point.

Replacing x_2 by $d^{-\frac{1}{3}} x_2$ we have

$$F = x_3^{\frac{2}{}} x_1 + G$$

Where $G = a_1 x_1^3 + p x_1^2 x_2 + q x_1 x_2 + x_2^3$

Replacing x_2 by $x_2^1 = x_2 + c\, x_1$.

Where we so determine C that the coefficient of

$x_2^2 x_1$ is zero after the transformation, we have

$F = x_3^2 x_1 + G$, where $G = x_2^3 + 3\, b x_2 x_1^2 + a\, x_1^3$

We now see that the Hessian of F is

$$H = \begin{vmatrix} G_{11} & G_{12} & 2\, x_3 \\ G_{12} & G_{22} & 0 \\ 2\, x_3 & 0 & 2\, x_1 \end{vmatrix}$$

Where Gij is the second partial derivative of G with respect to xi_1 xj

$$= 2\, x_3 \begin{vmatrix} G_{12} & G_{22} \\ 2\, x_3 & 0 \end{vmatrix} + 2\, x_1 \begin{vmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{vmatrix}$$

$$= -4 x_3^2\, G_{22} + 2\, x_1 \begin{vmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{vmatrix}$$

$$= -\, 4 x_3^2 \cdot 6x + 2 x_1 \begin{vmatrix} 6 b x_2 + 6 a x_1 , & 6\, b x_1 \\ 6\, b x_1 & 6\, x_2 \end{vmatrix}$$

$$= -\, 24\, x_3^2 + 72\, x_1 \left(-\, b^2 x_1^2 + a x_1 x_2 + b\, x_2^2 \right)$$

Eliminating x_3^2 between $F = 0$, $H = 0$,

We get from $H = 0$

$$x_3^2 = 3x_1(-b^2 x_1^2 + ax_1 x_2 + bx_2^2)$$

Then substituting in $F = 0$

$$3x_1^2(-b^2 x_1^2 + ax_1 x_2 + bx_2^2) + x_2 (x_2^3 + 3bx_2 x_1^2 + ax_1^3) = 0$$

Whence $x_2^4 + 6bx_2^2 x_1^2 + 4ax_2 x_1^3 - 3b^2 x_1^4 = 0$

This then is the equation satisfied by points of inflexion of the curve $f = 0$. We see immediately that the point (0 0 1) is a point of inflexion.

If for the remaining points we set $x_1 = 1$, we have a quartic in x_2.

$$x_2^4 + 6bx_2^2 + 4ax_2 - 3b^2 = 0 \text{ --------(2)}$$

If ri ($i = 1, 2, 3, 4,$) is a root of (2)
r_i

We obtain from each root ri two points of inflexion ($1, ri + xi_3^1$)

For if $x_3^1 = 0$, then from $F = 0$

$$x_3^2 x_1 + 0 = 0 = 0$$

i.e. (2) would have a multiple root.

$$\therefore \quad r i^4 + 6\,br\,i^2 + 4\,ar\,i - 3\,b^2 = 0$$

and $r^3 i + 3\,br + a = 0$ (i is subscript)

must have a common root.

Dropping the subscript for convenience, we find the condition for this as follows :

$$\begin{array}{r|l}
r^3 + 3\,br + a & r^4 + 6\,br^2 + 4\,ar - 3\,b^2 \\
br^3 + 3\,b^2 r + ab & \underline{r^4 + 3\,br^2 + ar} \\
\underline{br^3 + ar^2 \mp b^2 r} & 3\,br^2 + 3\,ar - 3\,b^2 \\
-\,ar^2 + 4\,b^2 r + ab & br + ar - b^2 \\
-\,abr^2 + 4b^3 r + ab^2 & (a^2 + 4b^3)br^2 + (a^2 + 4b^3)ar - b^2(a^2 + 4b^3) \\
\underline{-\,abr^2 - a^2 r + ab^2} & \underline{(a^2 + 4\,b^3)\,br^2} \\
(a^2 + 4\,b^3)\,r & (a^2 + 4\,b^3)\,ar - b^2\,(a^2 + 4\,b^3) \\
 & \underline{(a^2 + 4\,b^3)\,ar} \\
 & \text{remainder} = -\,b^2\,(a^2 + 4\,b^3)
\end{array}$$

Hence for a double root $b = 0$

or $a^2 + 4\,b^3 = 0$

But $\dfrac{dF}{dx_1} = x_3{}^2 + 6\,bx_2\,x_1 + 3\,ax_1{}^2$

$$\frac{dF}{dx_2} = 3\,x_2{}^2 + 3\,bx_1{}^2$$

$\dfrac{dF}{dx_3} = 2\,x_1\,x_3$ would then all vanish

at $(2\,b, -a, 0)$ or $(1, 0, 0)$ according as

$b \neq 0$ or $b = 0$.

Hence there must be exactly nine points of inflexion on any non-singular cubic.

We see, moreover, that the three points $P\ (0,\ 0,\ 1),\ (1,\ ri_3 \pm x_3^{1})$ lie on the straight line $x_2 = r\,i\,x_1$ and P is any of the points of inflexion so that we must have $\frac{9 \times 4}{3}$ straight lines, each passing through three points of inflexion. Any three of these form a triangle. Hence there are four inflexion triangles.

- See pages 88, 89, for a particular example where all these lines are found -

We now proceed to reduce the equation to what has been called the "Canonical form" and thence find the inflexion triangles etc.

$$\frac{1}{24} H + riF = - x_3^2 + 3 x_1 (- b^2 x_1^2 + ax_1 x_2 + b x_2^2)$$
$$+ ri (x_3^2 x_1 + x_2^3 + 3 b x_2 x_1^2 + a x_1^3)$$

Using equation (2) this reduces to

$$(r i x_1 - x_2) \left[x_3^2 - \frac{1}{r} \left\{ ri x_2 + \frac{1}{2} (r^2 + 3 b) x_1 \right\}^2 \right]$$

Thus we have the inflexion triangle

$$(ri - x_2) \left(\sqrt{ri}\, x_3 + ri x_2 + \frac{ri^2 + 3 b}{2} x_1\right) \left(\sqrt{ri}\, x_3 - rx_2 \frac{ri^2 + 3b}{2} x_1\right)$$

$$= (ri - x_2) (\sqrt{ri}\, x_3 + ri x_2 + kx_1) (\sqrt{ri}\, x_3 - rx_2 - k x_1)$$

Using now the inflexion triangle

$y_1 \quad y_2 \quad y_3 = 0$, where

$ri x_1 - x_2 = y_1$ ----------------(1)

$\sqrt{ri}\, x_3 + ri x_2 + kx_1 = 2 y_2$-----(2)

$\sqrt{ri}\, x_3 + ri x_2 - kx_1 = 2 y_3$-----(3)

(Note: Hereafter we shall write r for ri) ri

Adding (2) and (3) we get

$$2 \sqrt{r}\, x_3 = 2 (y_2 + y_3)$$

$$\sqrt{r}\, x_3 = y_2 + y_3$$

Subtracting (3) from (2)

$r x_2 + kx_1 = y_2 - y_3 \equiv D$ (Where $D \equiv y_2 - y_3$)

Adding $x_1 \cdot (1)$ to this,

$(r^2 + k)x_1 = ry_1 + D$ ------------(5)

Similarly $(r^2 + k) x_2 = - ky_1 + rD$-------(6)

Now $F = x_3^2 x_1 + x_2^3 + 3\,bx_2\,x_1^2 + ax_1^3$ ------(I)

And we have also the equation

$$r^4 + 6\,br^2 + 4\,ar - 3\,b^2 = 0 \text{ ------------(II)}$$

Whence $a = \dfrac{3\,b^2 - 6\,br^2 - r^4}{4\,r}$

$$\therefore \quad F = x_3^2 x_1 + x_2^3 + 3bx_2x_1^2 + \left\{\frac{3b^2 - 6br^2 - r^4}{4\,r}\right\} x_1^3$$

$$\frac{9}{8}(r^2 + b)\left[x_3^2 x_1 + x_2^3 + 3bx_2x_1^2 + \left\{\frac{3b^2 - 6br^2 - r^4}{4r}\right\} x_1^3\right.$$

$$= \frac{9}{8}(r^2 + b)\,F.$$

If in this expression we now substitute

for x_1, x_2, x_3, their values

$$x_1 = \frac{ry_1 + D}{r^2 + k} = \frac{ry_1 + y_2 - y_3}{r^2 + k}$$

$$x_2 = \frac{-ky_1 + rD}{r^2 + k} = \frac{-ky_1 + r(y_2 - y_3)}{r^2 + k}$$

$$x_3 = \frac{y_2 + y_3}{\sqrt{r}}$$

We get:

$$\frac{9}{8}(r^2 + b)\left[\left(\frac{y_2 + y_3}{r}\right)^2 \cdot \frac{ry_1 + y_2 - y_3}{r^2 + k} + \left\{\frac{-ky_1 + r(y_2 - y_3)}{r^2 + k}\right\}^3\right.$$

$$+ 3b\left\{\frac{-ky_1 + r(y_2 - y_3)}{r^2 + k}\right\}\left\{\frac{ry_1 + y_2 - y_3}{r^2 + k}\right\}^2$$

$$+ \left\{\frac{3b^2 - 6\,br^2 - r^4}{4\,r}\right\}\left\{\frac{ry_1 + y_2 - y_3}{r^2 + k}\right\}^3$$

Where as above, $k = \dfrac{(r^2 + 3\,b)}{2}$

whence $r^2 + k = \frac{3}{2}(r^2 + b)$

This simplifies down to

$$\frac{1}{r}(y_2^3 - y_3^3) + 3\,y_1\,y_2\,y_3 - \frac{1}{8}(r^2 + 9b)\,y_1^3$$

Hence

$$F = -\frac{(r^2+9b)}{9(r^2+b)} y^3 + \frac{8}{9(r)(r^2+b)} y_2^3 + (-)\frac{8}{9(r)(r^2+b)} y_3^3$$

since $r^2+b \neq 0$

$$+\frac{8}{3(r^2+b)} y_1 y_2 y_3$$

If the transformation

$$z_1 = \sqrt[3]{-(r^2-9b)} \quad y_1$$

$$z_2 = \sqrt[3]{\frac{8}{r}} \quad y_2$$

$$z_3 = \sqrt{-\frac{8}{r}} \quad y_3$$

replaces y_1 y_2 y_3 by z_1 z_2 z_3 respectively

we have $f = \alpha(z_1^3 + z_2^3 + z_3^3) + 6B \; z_1 \; z_2 \; z_3$

which is the Canonical form of the Ternary Cubic.

To find the Hessian of this canonocial form

we have

$$\frac{df}{dz_1} = 3d\,z_1^2 + 6B\,z_1z_3) \qquad \frac{df}{dz_2} = 3dz_2^2 + 6B\,z_1z_3$$

$$\frac{d^2f}{dz_1^2} = 6dz_1 \qquad \frac{d^2f}{dz^2} = 6d\,z_2$$

$$\frac{d^2f}{dz_1dz_2} = 6B\,z_3 \qquad \frac{d^2f}{dz_2dz_3} = 6B\,z_1$$

$$\frac{df}{dz_3} = 3d\,z_3^3 + 6B\,z_1\,z_2$$

$$\frac{d^2f}{dz_3^2} = 6d\,z_3$$

$$\therefore \quad h = 6^3 \begin{vmatrix} d z_1 , & B z_3 , & B z_2 \\ B z_3 , & d z_2 , & B z_1 \\ B z_2 , & B z_1 , & d z_3 \end{vmatrix}$$

$$= d^3 z_1 z_2 z_3 + B^3 z_1 z_2 z_3 + B^3 z_1 z_2 z_3$$

$$- dB^2 z_2^3 - d B^2 z_1^3 - d B^2 z_3^3$$

$$= - dB^2 \left\{ z_1^3 + z_2^3 + z_3^3 \right\} + \left\{ d^3 + 2B^3 \right\} z_1 z_2 z_3$$

Since then

$$f = d \left(z_1^3 + z_2^3 + z_3^3 \right) + 6 B z_1 z_2 z_3$$

$$h = - dB^2 \left(z_1^3 + z_2^3 + z_3^3 \right) + \left\{ d^3 + 2 B^3 \right\} z_1 z_2 z_3$$

The equation to the points of inflexion may be taken as

$$0 = h + B^2 f$$

$$= \left(8 B^3 + d^3 \right) z_1 z_2 z_3$$

And one inflexion triangle is

$$z_1 z_2 z_3 = 0 \text{ ----------------- } (1)$$

Again from the intersection of $f = 0$, $h = 0$

We see that

$$\sum_{z_1 z_2 z_3} z_1^3 = \frac{-6B}{d} = \frac{d^3 + 2 B^3}{- d B^2}$$

$$\therefore d^3 + 8 B^3 = 0$$

$$2 B = - l d \text{ where } l = 1, w, w^2$$

w being an imaginary cube root of unity.

Hence

$$\frac{\sum z_1^3}{z_1\ z_2\ z_3} = \frac{-6\ B}{d} = \frac{3\ L\ d}{d} = 3\ L$$

$$\sum z_1^3 - 3\ L\ z_1\ z_2\ z_3 = 0$$

i.e. We have then the four inflexion triangles

$$z_1\ z_2\ z_3 = 0 \text{ ------------------------------ } (1)$$

$$z_1^3 + z_2^3 + z_3^3 - 3\ z_1\ z_2\ z_3 = 0 \text{ -------- } (2)$$

$$z_1^3 + z_2^3 + z_3^3 - 3\ w\ z_1\ z_2\ z_3 = 0 \text{ ------- } (3)$$

and $z_1^3 + z_2^3 + z_3^3 - 3\ w^2\ z_1\ z_2\ z_3 = 0$ ------- (4)

From, say the first inflexion triangle

$z_1\ z_2\ z_3 = 0$, we may then discover all the nine points of inflexion, and find the equations to the 12 lines on which they lie by threes.

See pages 26 - 37 for a complete discussion of these points, inflexion tangents etc.

We may now turn our attention to some particular examples.

Let us then first study the curve which we obtain by letting $A = 1$, $B = 1$, in the above canonical form, i.e. the curve

$$z_1^3 + z_2^3 + z_3^3 + 6\, z_1\, z_2\, z_3 = 0$$

Let us use the variables $x_1 \quad x_2 \quad x_3$.

our equation then is

$$x_1^3 + x_2^3 + x_3^3 + 6\, x_1\, x_2\, x_3 = 0$$

We wish first to study the curve in Cartesian Coordinates. The general transformation is given by

$$x = \frac{A_1 X_1 + A_2 X_2 + A_3 X_3}{C_1 X_1 + C_2 X_2 + C_3 X_3}$$

$$y = \frac{B_1 X_1 + B_2 X_2 + B_3 X_3}{C_1 X_1 + C_2 X_2 + C_3 X_3}$$

By setting

$$A_2 = A_3 = C_1 = C_2 = 0$$

$$\text{and } A_1 = C_3 = 1$$

in the expression for x_1' , and

$$B_1 = B_3 = C_1 = C_2 = 0$$

$$B_2 = C_3 = 1$$

in the expression for y, we have the particular form

$$x = \frac{x_1}{x_3} \quad , \qquad y = \frac{x_2}{x_3}$$

This is equivalent to taking one side of the triangle of reference for the y - axis, another as the x - axis, and letting the third side recede to infinity.

If, we use ~~using~~ this particular transformation which, being very simple, is common, we get the Cartesian equation

$$x^3 + y^3 + 6\,xy + 1 = 0$$

(a) The question of singular points

$$\frac{df}{dx} = 3\,x^2 + 6\,x \qquad \frac{df}{dy} = 3\,y^2 + 6\,y$$

For singular points

$$3\,x(\,x+2)=0 \qquad 3\,y\,(y+2)=0$$

simultaneously. But there are no points on the curve for which this can be true. Hence there are no singular points.

(b) We may then proceed to obtain values and graph the curve.

x.	y.
$-1\ (w_1 - w^2)$	---- 0
$-\ .3275$	---- 1
$-\ .7$	---- 2
$-\ 1.4$	---- 3
$-\ 2.24$	---- 4
$-\ 3.2$	---- 5
0 . 2.4 - 2.4	------ -1
3.7 - .6	------ -2
4.8 - 1.7	------ -3
- .4	------ $\frac{1}{2}$
.3 - 1.9	..... $-\frac{1}{2}$

We have, then, certain data but need more in the neighborhood of the origin, and more negative values.

But for every pair of values of the variable we must solve a cubic equation, and the work is extremely lengthy. To enable us to study this curve more easily we now rotate the axes through 45^{0} since the curve is evidently symmetrical with respect to the line $x = y$.

Using the transformation formulas

$$x = x^{1} \cos \theta - y^{1} \sin \theta$$

$$y = x^{1} \sin \theta + y^{1} \cos \theta$$

For $\theta = 45^{0}$

$$x = \frac{x^{1} - y^{1}}{2} \qquad y = \frac{x^{1} + y^{1}}{2}$$

Hence our equation becomes

$$\frac{(x^{1} - y^{1})^{3}}{2^{\frac{3}{2}}} + \frac{(x^{1} + y^{1})^{3}}{2^{\frac{3}{2}}} + 6 \left(\frac{x^{1} - y^{1}}{\sqrt{2}}\right) \left(\frac{x^{1} + y^{1}}{\sqrt{2}}\right) = 0$$

On simplifying and dropping the accents we have

$$x^{3} + 3 xy^{2} + 3\sqrt{2}\, x^{2} - 3\sqrt{2}\, y^{2} + 1 = 0$$

or $x^{3} + 3\sqrt{2}\, x^{2} + y^{2}(3x - 3\sqrt{2}) + 1 = 0$

Solving for y^2 we get.

$$y^2 = - \frac{x^3 + 3\sqrt{2} + 1}{3x - 3\sqrt{2}}$$

We are now more easily able to obtain the requisite points on the curve. The results are as follows:

x.	y.
0	$\pm$ .685
1	$\pm$ 2.24
$\sqrt{2}$ = 1.4142	∞
2	Imaginary
- 1	$\pm$ .8
- 2	$\pm$ 1.0029
- 3	$\pm$ .985
- 4	$\pm$ .149
- 4.29	0
- 4.29 +	Imaginary

Hence the curve is entirely confined to the portion of the plane lying between

$x^1 = \sqrt{2} = 1.4142$

and $x^1 = -4.29$

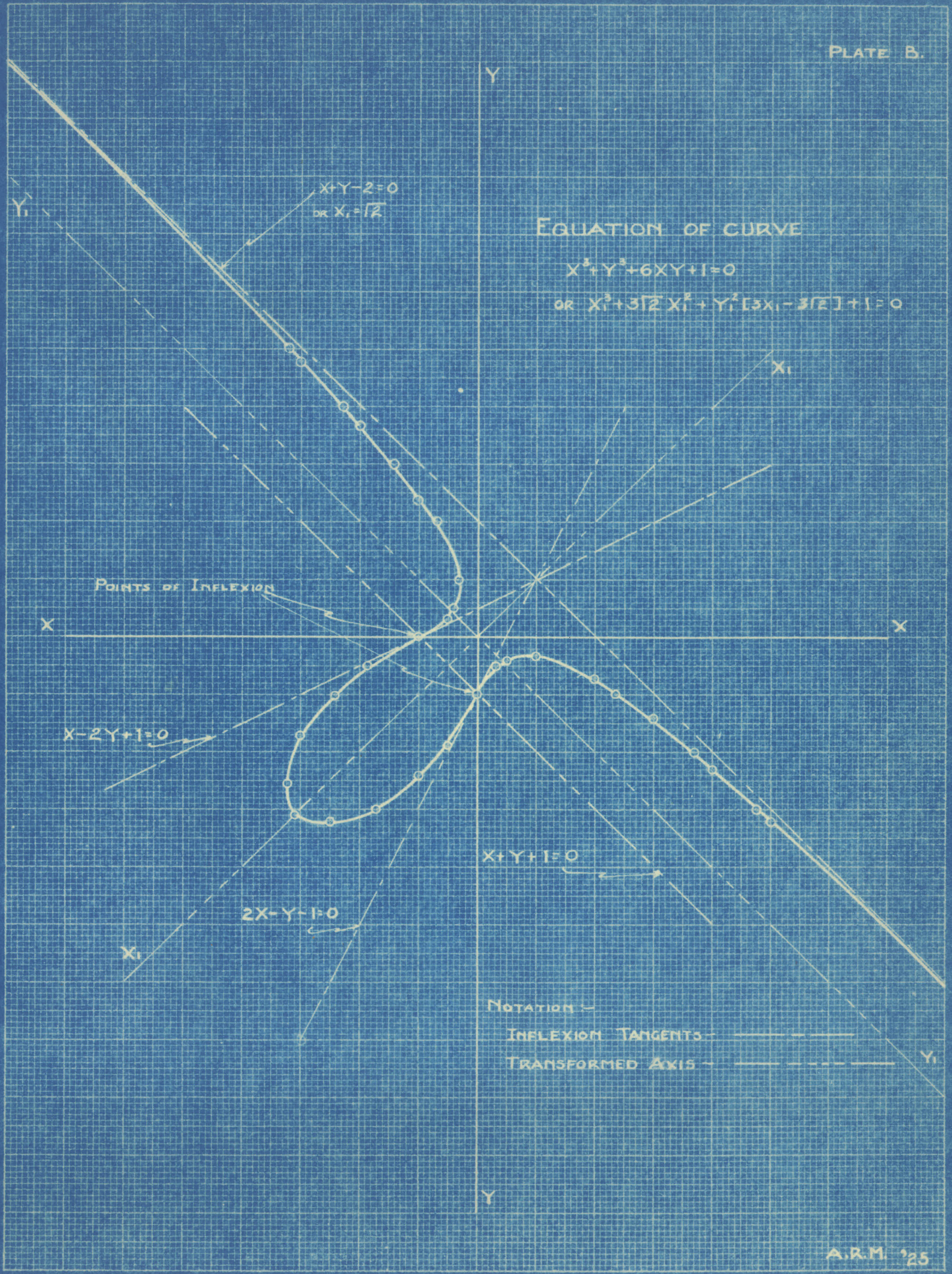


PLATE B.
EQUATION OF CURVE
$X^3+Y^3+6XY+1=0$
OR $X_1^3+3\sqrt{2}\,X_1^2+Y_1^2[3X_1-3\sqrt{2}]+1=0$
$X+Y-2=0$
OR $X_1=\sqrt{2}$
POINTS OF INFLEXION
$X-2Y+1=0$
$X+Y+1=0$
$2X-Y-1=0$
X
Y
X_1
Y_1
NOTATION –
INFLEXION TANGENTS –
TRANSFORMED AXIS –
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We are now in a position to graph the curve. See graph on Plate B. The new axes are x^1, y^1. The graph also shows the curve clearly with reference to the axes x, y. It is with regard to the latter axes, i.e., for the equation

$$x^3 + y^3 + 6xy + 1 = 0$$

Or $x_1^3 + x_2^3 + x_3^3 + 6 x_1 x_2 x_3 = 0$

that this curve is discussed.

Passing then over to homogeneous coordinates, we have

$$f = x_1^3 + y_2^3 + x_3^3 + 6 x_1 x_2 x_3 = 0$$

as above. If we again examine for a singular point, we see that

$$\frac{df}{dx_1} = 3 x_1^2 + 6 x_2 x_3$$

$$\frac{df}{dx_2} = 3 x_2^2 + 6 x_1 x_3$$

$$\frac{df}{dx_3} = 3 x_3^2 + 6 x_1 x_2$$

If these are all zero, we have

$$x_1^2 + 2\,x_2\,x_3 = 0$$

$$x_2^2 + 2\,x_1\,x_3 = 0$$

$$x_3^2 + 2\,x_1\,x_2 = 0$$

which is true only for (0, 0,0) which cannot be considered as a point. Hence there can be no singular point on this curve.

We may then enquire as to the points of inflexion.

The Hessian of $f = x_1^3 + x_2^3 + x_3^3 + 6\,x_1\,x_2\,x_3 = 0$

is $h = -(x_1^3 + x_2^3 + x_3^3) + 3\,x_1\,x_2\,x_3$

Hence $0 = f + h = 9\,x_1\,x_2\,x_3$ gives

the inflexion triangle $x_1\ x_2\ x_3 = 0$.

Now $x_1 = 0$, meets the cubic in the points for which

$$x_2^3 + x_3^3 = 0$$

i.e. in the points for which

$$(x_2 + x_3)\ (x_2 + w\,x_3)\ (x_2 + w^2\,x_3) = 0$$

On $x_1 = 0$ we therefore have the three points of inflexion.

$$(x_1) = (0, 1, -1), (x_1^{1}) = (0, w, -1), (x_1^{11}) = (0, w^2, -1)$$

The straight line $x_2 = 0$ meets the curve in the points for which

$$x_1^3 + x_3^3 = 0 \,. \quad (x_3 + x_1)(x_3 + w\,x_1)(x_3 + w^2\,x_1) = 0$$

The points of inflexion

$$(x_2) = (1, 0, -1), \quad (x_2^{1}) = (1, 0, -w), (x_2^{11}) = (1, 0, -w^2)$$

lie on the line $x_2 = 0$

The straight line $x_3 = 0$, meets the curve in the points for which

$$(x_1^3 + x_2^3) = (x_1 + x_2)(x_1 + w\,x_2)(x_1 + w^2\,x^2) = 0$$

i.e. in the points.

$$(x_3) = (1, -1, 0) \quad (x_3^{1}) = (w, -1, 0) \; (x_3^{11}) = (w^2, -1, 0)$$

which shows that the straight line $x_3 = 0$ meets the curve in three inflexion points at infinity. Only one of them is real, namely $(1, -1, 0)$.

The Twelve Lines composing the Four Inflexion Triangles.

We saw that the points

(x_1), (x_1^1), (x_1^{11}) lie on $x_1 = 0$ ----------------(1)

(x_2) (x_2^1) (x_2^{11}) lie on $x_2 = 0$ ----------------(2)

(x_3) (x_3^1) (x_3^{11}) lie on $x_3 = 0$ ----------------(3)

We find also that,

$(x_1) = (0, 1, -1)$, $(x_2) = (1, 0, -1)$ $(x_3) = (1, -1, 0)$

lie on the line $x_1 + x_2 + x_3 = 0$ ----------------(4)

$(x_1^1) = (0, w, -1)$ $(x_2^1) = (1, 0, -w)$, $(x_3^1) = (w, -1, 0)$

lie on the line $x_1 + w x_2 + w^2 x_3 = 0$ ----------------(5)

$(x_1^{11}) = (0, w^2, -1)$ $(x_2^{11}) = (1, 0, -w^2)$, $(x_3^{11}) = (w^2, -1, 0)$

lie on the line $x_1 + w^2 x_2 + w x_3 = 0$ ------------(6)

$(x_1) = (0, 1, -1)$ $(x_2^1) = (1, 0, -w)$ $(x_3^{11}) = (w^2, -1, 0)$

lie on $wx + x_2 + x_3 = 0$ ------------------------(7)

$(x_1) = (0, 1, -1)$, $(x_2^{11}) = (1, 0, -w^2)$ $(x_3^1) = (w, -1, 0)$

on the line $w^2 x_1 + x_2 + x_3 = 0$ ----------------(8)

$(x_1^{11}) = (0,\ w^2,\ -1),\ (x_2) = (1,\ 0,\ -1)\ (x_3^{1}) = (w,\ -1,\ 0)$

lie on the line $x_1 + w\,x_2 + x_3 = 0$ ------------(9)

$(x_1^{1}) = (0,\ w,\ -1)\ \ (x_2) = (1,\ 0,\ -1)\ \ (x_3^{11}) = (w^2,\ -1,\ 0)$

lie on the line $x_1 + w^2 x_2 + x_3 = 0$ -------------(10)

$(x_1^{1}) = (0,\ w,\ -1)\ \ (x_2^{11}) = (1,\ 0,\ -w^2)\ (x_3) = (1,\ -1,\ 0)$

lie on the line $x_1 + x_2 + wx_3 = 0$ -----------(11)

$(x_1^{11}) = (0,\ w^2,\ -1)\ \ (x_2^{1}) = (1,\ 0,\ -w)\ (x_3) = (1,\ -1,\ 0)$

lie on the line $x_1 + x_2 + w^2 x_3 = 0$ ------------(12)

We have, then, found twelve straight lines each passing through three points of inflexion, and in all we have shown nine points of inflexion, (x_1), (x_2), (x_3), (x_1^{1}), (x_2^{1}) (x_3^{1}), (x_1^{11}), (x_2^{11}), (x_3^{11}), lying by threes on twelve straight lines. Any three of these twelve straight lines form an Inflexion Triangle. Hence there are four Inflexion Triangles. Evidently, the triangle of reference is the inflexion triangle $x_1\ x_2\ x_3 = 0$

Two of the lines making up this triangle, namely, $x_1 = 0$ and $x_2 = 0$, i.e., the y-axis and the x-axis, lie in the finite part of the plane - i.e. in part -

The Third line $x_3 = 0$ is the side of the triangle of reference which by this transformation

--- $x = \frac{x_1}{x_3}$ $y = \frac{x_2}{x_3}$ ------

receded to infinity. In addition to these three, only one of the twelve lines, is real, i. e. the line $x_1 + x_2 + x_3 = 0$

This is the line that passes through the three real points of inflexion. Two of these inflexion points lie in the finite part of the plane.

i.e. $(x_1) = (0, 1, -1)$ $(x_2) = (1, 0, -1)$

The third $(x_3) = (1, -1, 0)$ lies "at infinity."

In cartesian coordinates the line $x_1 + x_2 + x_3 = 0$ is the line $x + y + 1 = 0$. It passes through the inflexion points $(0, -1)$, $(-1, 0)$ in the finite part of the plane. See graph on Plate B.

The four inflexion triangles as seen for the general case, Page 19, are given by

$$x_1 \; x_2 \; x_3 = 0 \text{ ------------------------ } (1)$$

$$x_1^3 + x_2^3 + x_3^3 - 3\, x_1 \, x_2 \, x_3 = 0 \text{ -------------- } (2)$$

$$x_1^3 + x_2^3 + x_3^3 - 3\, w\, x_1 \, x_2 \, x_3 = 0 \text{ -------------- } (3)$$

$$x_1^3 + x_2^3 + x_3^3 - 3\, w^2\, x_1 \, x_2 \, x_3 = 0 \text{ ------------- } (4)$$

Where w is an imaginary cube root of unity. Only the first of these triangles is entirely real. And even this triangle cannot be completed in the finite part of the plane. (See Plate B).

That the twelve lines given above correspond in pairs of three to the inflexion triangles whose equations have just been given, may easily be verified, thus for example:

$$0 = (x_1 + x_2 + x_3)\,(x_1 + wx_2 + w^2x_3)\,(x_1 + w^2x_2 + w\,x_3)$$

$$= x_1^3 + x_2^3 + x_3^3 - 3\,x_1\,x_2\,x_3$$

by straight multiplication.

INFLEXION TANGENTS.

Let the tangent at the points

$(x_1^1) = (0, w, -1)$ be

$$x_2 + w\,x_3 = m\,x_1$$

substituting for x_1 in the equation of the curve, we see that the equation

$$(x_2 + w\,x_3)^3 - m^3\,(x_2^3 + x_3^3) + 6\,m^2(x_2 + w\,x_3)x_2\,x_3 = 0$$

must have two equal roots

$$(x_2 + w\,x_3) = 0$$

Hence $(x_2 + w\,x_3)$ must satisfy the equation.

$$(x_2 + w\,x_3)^2 + m^3 (x_2^2 - w\,x_2\,x_3 + w^2\,x_3^2)$$
$$+ 6\,m^2\,x_2\,x_3 = 0$$

In that case $x_2 = -w\,x_3$ must satisfy the equation.

$$m\,x_2^2 - m\,w\,x_2\,x_3 + m\,w^2\,x_3^2 + 6\,x_2\,x_3 = 0$$

$$m\,w^2\,x_3^2 + m\,w^2\,x_3^2 + m\,w^2\,x_3^2 - 6\,w\,x_3^2 = 0$$

$$3\,m\,w^2 = 6\,w.$$

$$m = \frac{2}{w} = 2\,\frac{w^3}{w} = 2\,w^2$$

Hence our equation.

$$(x_2 + w\,x_3)^2 + m^3(x_2^2 - w\,x_2\,x_3 + w^2\,x_3^2)$$
$$+ 6\,m^2\,x_2\,x_3 = 0$$

Becomes,

$$(x_2 + w\,x_3)^2 + 8\,(x_2^2 - w\,x_2\,x_3 + w^2\,x_3^2)$$
$$+ 24\,w\,x_2\,x_3 = 0$$

$$(x_2 + w\,x_3)^2 + 8\,(x_2^2 + w^2\,x_3^2) + 16wx_2\,x_3 = 0$$

$$(x_2 + wx_3)^2 + 8\,x_2(x_2 + w\,x_3) + 8\,w\,x_3(x_2 + w\,x_3) = 0$$

i.e. $9\,(x_2 + w\,x_3)^2 = 0$

Therefore our original equation

$$(x_2 + w\,x_3)^3 + m^3\,(x_2^3 + x_3^3) + 6\,m^2\,(x_2 + w\,x_3)\,x_2\,x_3 = 0$$

has $(x_2 + w\,x_3)$ as factor thrice, i.e. reduces to

$$(x_2 + w\,x_3)^3 = 0$$

Thus the straight line $x_2 + w\,x_3 = 2\,w^2\,x_1$

meets the curve at three coincident points at

$(0, w, -1)$. It follows that

$x_2 + w\,x_3 = 2\,w^2\,x_1$ is the inflexion tangent at

the point of inflexion $(x_1{}^1) = (0, w, -1)$

Similarly we may show that the other

points of inflexion have inflexion tangents as

follows:

Point:	Inflexion Tangent:
$(x_1) = (0, 1, -1)$	$x_2 + x_3 = 2\,x_1$
$(x_1{}^{11}) = (0, w^2, -1)$	$x_2 + w^2x_3 = 2\,wx_1$
$(x_2) = (1, 0, -1)$	$x_3 + x_1 = 2\,x_2$
$(x_2{}^1) = (1, 0, -w)$	$x_3 + wx_1 = 2\,w^2x_2$

Point:	Inflexion Tangent:
$(x_2^{11}) = (1,\ 0,\ -w^2)$	$x_3 + w^2x_1 = 2\ w\ x_2$
$(x_3) = (1,\ -1,\ 0)$	$x_1 + x_2 = 2\ x_3$
$(x_3^{1}) = (w,\ -1,\ 0)$	$x_1 + wx_2 = 2\ w^2x_3$
$(x_3^{11}) = (w^2,\ -1,\ 0)$	$x_1 + w^2x_2 = 2\ w\ x_3$

Thus we have found the nine inflexion tangents, three of which are real, namely:

$x_2 + x_3 = 2\ x_1$ at the inflexion point $(x_1) = (0,\ 1,\ -1)$

$x_3 + x_1 = 2\ x_2$ at the inflexion point $(x_2) = (1,\ 0,\ -1)$

$x_1 + x_2 = 2\ x_3$ at the inflexion point $(x_3) = (1,\ -1,\ 0)$

(See Plate B).

For our transformation

$$x = \frac{x_1}{x_3} \qquad y = \frac{x_2}{x_3}$$

the last inflexion tangent is evidently tangent to the curve at infinity and the inflexion point

$x_3 = (1,\ -1,\ 0)$ (or $[\infty\ -\infty]$ in cartesians),

is a point at infinity, according to the convention adopted by Bocher, "Introduction to Higher Algebra," Sect. 4, Page 12.

Now reverting to our Cartesian coordinates, we see that the points of inflexion on the finite part of the curve are:

$$(x_1) = (0, 1, -1) \text{ i.e. } (0, -1)$$

and $(x_2) = (1, 0, -1)$ i.e. $(-1, 0)$

and the inflexion tangent at $(0, -1)$ is $y - 1 = 2x$ ---- a line inclined at an angle of $63^\circ 20'$ to the x - axis, and the inflexion tangent at $(-1, 0)$ is

$1 + x = 2y$

---- inclined at an angle of $26^\circ\ 40'$ to x - axis. See graph on Plate B.

The third real inflexion tangent i.e. $x + y = 2$ clearly coincides with the line $x^1 = \sqrt{2}$ which we discovered when we rotated the axes through 45°.

For by our transformation,

$$x = \frac{x_1^{1} - y_1^{1}}{\sqrt{2}} \qquad y = \frac{x_1^{1} + y_1^{1}}{\sqrt{2}}$$

For the line $x^1 = \sqrt{2}$

$$x = \frac{\sqrt{2} - y^1}{\sqrt{2}} \qquad y = \frac{\sqrt{2} + y^1}{\sqrt{2}}$$

$$x - y = \frac{\sqrt{2}}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}} = 2$$

See graph on Plate B.

We have then supplemented the general case by fully discussing the canonical form for the values $\alpha = 1$, $B = 1$ of the constants.

We shall now study briefly the curves

$x = y^3, \quad y = x^3 - 3x + 1$ and

$x^3 + y^3 + x + 1 = 0$, and apply the preceding theory to the corresponding ternary cubics in homogeneous coordinates.

If we examine the curve $x = y^3$ for singular points we see that,

$$\frac{df}{dx} = 1 \neq 0 \qquad \frac{df}{dy} = 3y^2$$

Hence there can be no singular point on this curve.

Points of Inflexion.

$$\frac{dx}{dy} = 3y^2 \qquad \frac{d^2x}{dy^2} = 6y$$

Thus we have

y	$\frac{d^2x}{dy^2}$	$\frac{dx}{dy}$	Curve
< 0	$-$	decreasing	concave upward.
> 0	$+$	increasing	concave downward.
$= 0$	0		$x = 0, y = 0$ Point of inflexion.

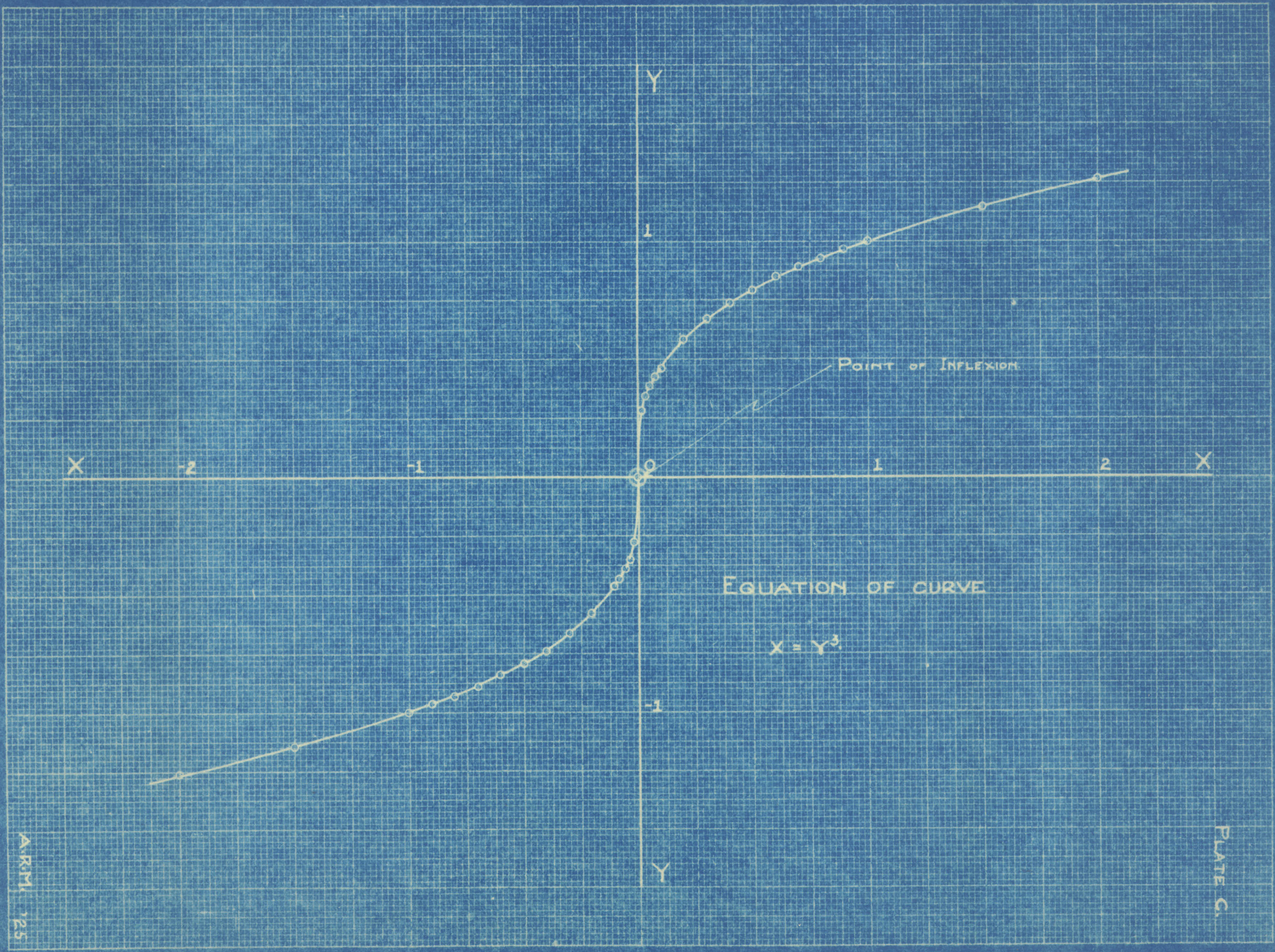
Y
1
Point of Inflexion
X
-2
-1
0
1
2
X
Equation of curve
X = Y³
-1
Y
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Plate C.

Let the inflexion tangent be $x = m\,y$.

$$\text{Here } m = \frac{dx}{dy}_{(x = 0,\ y = 0)} = 0$$

i.e. $x = 0$ is the inflexion tangent at the point $(0, 0)$ - the y-axis.

To plot the curve we have the values:

$\pm x$	$\pm y$	$\pm x$	$\pm y$
2	1.26	.3	0.670
1.5	1.14	.2	0.585
1	1	.1	0.464
.9	0.965	.08	0.43
.8	0.928	.06	0.39
.7	0.848	.04	0.34
.6	0.843	.02	0.27
.5	0.794	0	0
.4	0.737		

See graph on Plate C.

If we now pass to homogeneous coordinates, for the particular transformation

$$x = \frac{x_1}{x_3} \qquad\qquad y = \frac{x_2}{x_3}$$

our equation becomes

$$\left(\frac{x_1}{x_3}\right) = \left(\frac{x_2}{x_3}\right)^3 \quad \text{or} \quad x_1\ x_3^{\,2} - x_2^{\,3} = 0$$

It now appears that we are no longer dealing with a non-singular cubic for

$$\frac{df}{dx_1} = x_3^2 \qquad \frac{df}{dx_2} = -2x_2 \cdot \qquad \frac{df}{dx_3} = 2x_1 x_3$$

and it is clear that the point (1, 0, 0) is a singular point. This seeming contradiction comes from the fact that our transformation,

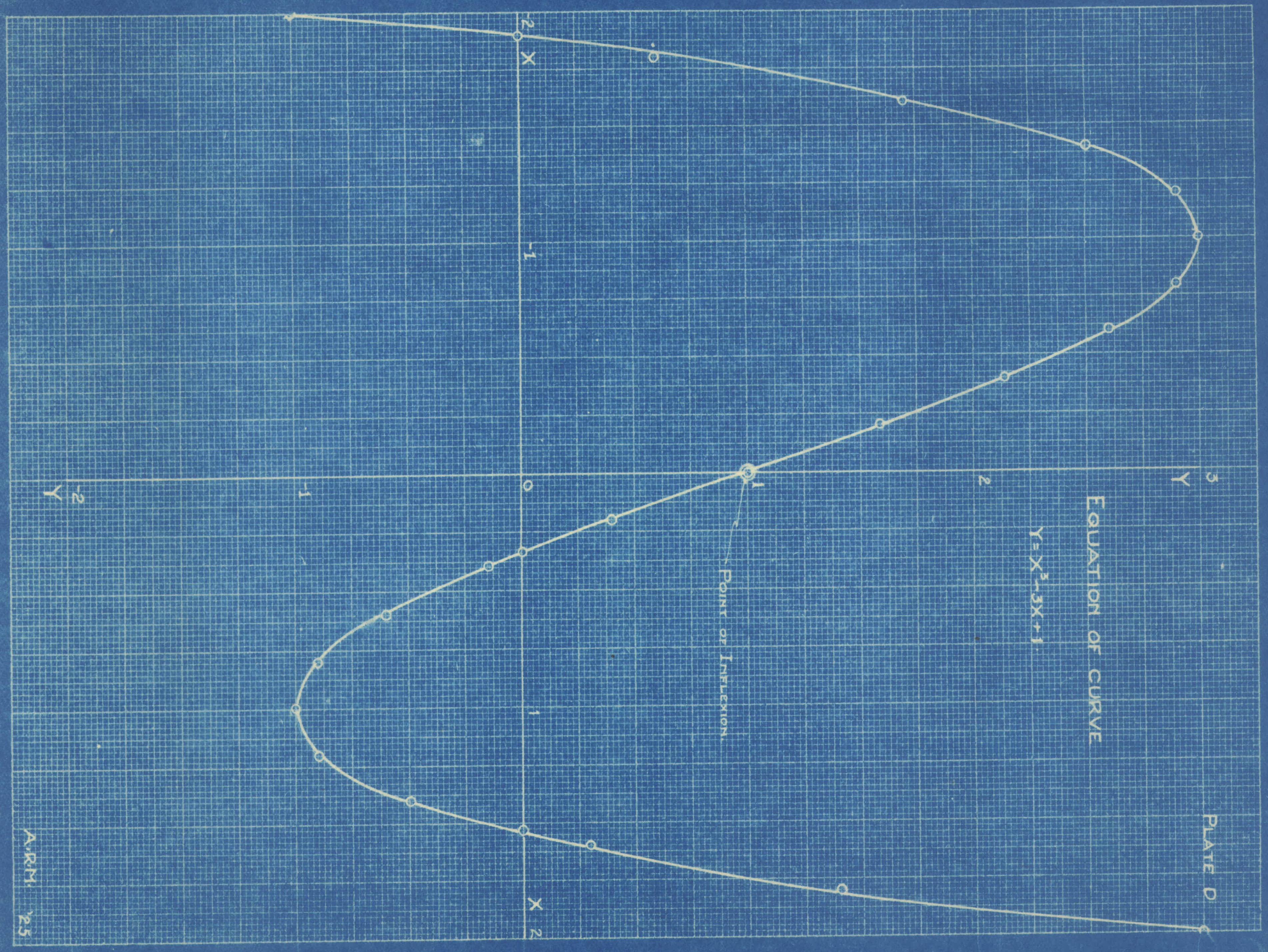
$$x = \frac{x_1}{x_3} \qquad y = \frac{x_2}{x_3}$$

has, strictly speaking, no meaning for $x_3 = 0$. When we speak of the point $(x_1\ x_2\ 0)$ we are speaking of quantities as a "point," which are not the coordinates of any point. (See Boche "Introduction to Higher Algebra," Sec. 4.). Hence for this transformation we need not carry our investigation of the curve $x = y^3$ any further.

THIS MARGIN RESERVED FOR BINDING.

IF SHEET IS READ THIS WAY (HORIZONTALLY), THIS MUST BE TOP.

IF SHEET IS READ THE OTHER WAY (VERTICALLY), THIS MUST BE LEFT-HAND SIDE.



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Let us then examine the curve

$$y = x^3 - 3x + 1$$

It has no singular points for

$$\frac{df}{dy} = 1 = 0$$

To discover points of inflexion we have

$$\frac{dy}{dx} = 3x^2 - 3 \qquad \frac{d^2y}{dx^2} = 6x$$

$$= 0, \text{ if } x = 0$$

And when we examine the curve in the neighborhood of (0, 1) we see that (0, 1) is a point of inflexion.

For the graph of this curve see Plate D. We find the values:

x	y	x	y
2	3	- .4	2.14
1.8	1.4	- .6	2.6
1.6	.3	- .8	2.9
1.4	- .5	- 1.0	3.0
1.2	- .9	- 1.2	2.9
1.0	- 1.0	- 1.4	2.5
.8	- .9	- 1.6	1.7
.6	- .6	- 1.8	.6
.4	- .14	- 2.0	- 1
.2	- .4	-	-
0	1	-	-
- .2	1.59	-	-

In homogeneous coordinates this equation becomes

$$\frac{x_2}{x_3} = \left(\frac{x_1}{x_3}\right)^3 - \frac{x_1}{x_3} + 1 = 0$$

or $x_2 x_3^2 - x_1^3 + 3 x_1 x_3^2 - x_3^3 = 0$

Here $\frac{df}{dx_1} = -a\, 3 x_1^2 + 3 x_3^2$

$$\frac{df}{dx_2} = x_3^2$$

$$\frac{df}{dx_3} = 2 x_2 x_3 + 6 x_1 x_3 - 3 x_3^2$$

And again we have a singular point (o, 1, 0) and since we do not have a point on the Cartesian curve corresponding to this we drop our investigation here.

Finally, let us examine the curve

$$x^3 + y^3 + x + 1 = 0$$

The question of singular points.

$$\frac{df}{dx} = 3x^2 + 1, \qquad \frac{df}{dy} = 3y^2$$

The coordinates of the points

$(\pm \frac{1}{\sqrt{3}}, 0)$ do not satisfy the equation, hence there are no singular points on the curve.

Points of inflexion.

$$3x^2 \frac{dx}{dy} + 3y^2 + \frac{dx}{dy} = 0$$

$$\frac{dx}{dy} = - \frac{3y^2}{3x^2 + 1}$$

$$\frac{d^2x}{dy^2} = - \left\{ \frac{(3x^2 + 1)\, 6y - 3y^2 \; 6x \frac{dx}{dy}}{(3x^2 + 1)^2} \right\}$$

$$= -6 \left\{ \frac{(3x^2 + 1)^2 y + 9xy^4}{(3x^2 + 1)^3} \right\}$$

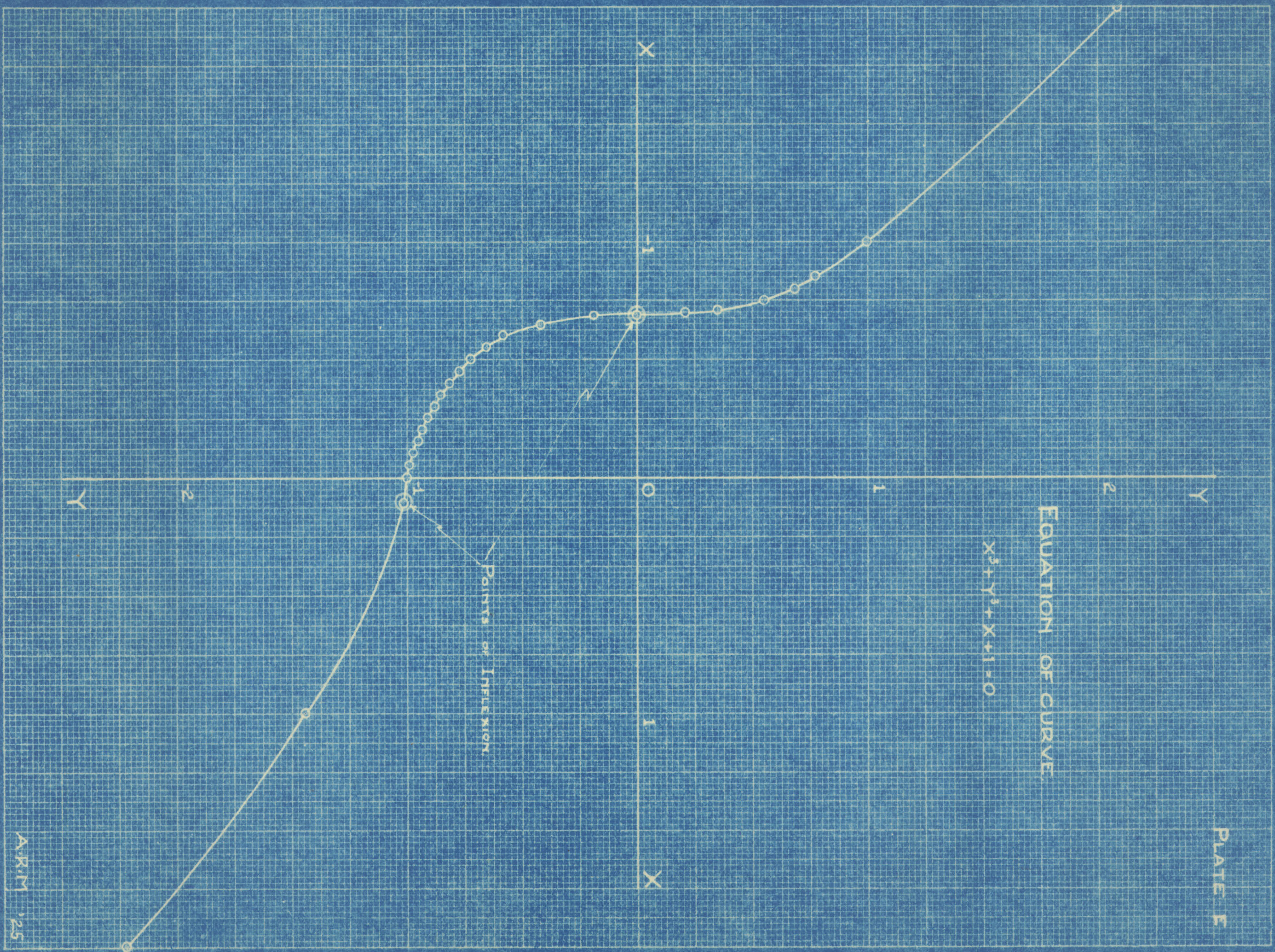
Thus the second derivative passes through the value 0, for $y = 0$, which on closer examination gives the point of inflexion (- .6835, 0).

(See graph on Plate E.)

THIS MARGIN RESERVED FOR BINDING.

IF SHEET IS READ THIS WAY (HORIZONTALLY), THIS MUST BE TOP.

IF SHEET IS READ THE OTHER WAY (VERTICALLY), THIS MUST BE LEFT-HAND SIDE.



The second derivative is also zero for

$$y^3 = -\frac{(3x^2+1)^2}{9x}$$

Or substituting in the equation we see that the second derivative vanishes if

$$x^3 - \frac{(3x^2+1)^2}{9x} + x + 1 = 0$$

$$\text{if} \quad 3x^2 + 9x - 1 = 0$$

Whence, on further examination we find the point of inflexion (.1073, - 1.03). See graph on Plate B.

In order to graph this curve we obtain the values:

x	y	x	y
0	-1	- .70	.350
- .05	- .983	- .75	.556
- .1	- .965	- .80	.678
- .15	- .946	- .85	.774
- .20	- .925	- 1.	1.
- .25	- .902	- 2.	2.
- .30	- .876	- 3.	3.07
- .35	- .847	- 4.	4.06
- .40	- .812		
- .45	- .771	1.	- 1.44
- .50	- .721	2.	- 2.22
- .55	- .657		
- .60	- .579	3.	- 3.14
- .65	- .422	4	- 4.10

Reverting now to homogeneous coordinates under the transformation

$$x = \frac{x_1}{x_3} \qquad y = \frac{x_2}{x_3}\text{, our equation}$$

$$x^3 + y^3 + x + 1 = 0\text{, now becomes}$$

$$x_1^{\,3} + x_2^{\,3} + x_1\,x_3^{\,2} + x_3^{\,3} = 0$$

$$\frac{df}{dx} = 3\,x_1^{\,2} + x_3^{\,2}\,, \qquad \frac{df}{dx_2} = 3\,x_2^{\,2}$$

$$\frac{df}{dx_3} = 2\,x_1\,x_3 + 3\,x_3^{\,2}$$

And the curve has no singular point. In this case it is then evident that we might proceed as in the general case, by a change of the triangle of reference, reducing the equation to the canonical form thus discovering the nine points of inflexion, the inflexion triangles and inflexion tangents.

We have seen that the point $(-.6835, 0, 1)$ lies on the curve. Hence following the procedure in the general case we could change the triangle of reference by the transformation,

$$x_1 = -.6835\ y_3$$
$$x_2 = y_2$$
$$x_3 = y_1 + y_3$$

$$\text{Where } \triangle = \begin{vmatrix} 0 & 0 & -.6835 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \neq 0$$

so that for the new triangle of reference $y_1\ y_2\ y_3 = 0$, the term in $x_3^{\,3}$ would be lacking.

Since the method is identical with that used in the general case, and the calculation laborious, we proceed now to the further study of the curve $x = y^3$ which our particular transformation failed to throw any light on.

We examined the curves $x = y^3$ and $y = x^3 - 3x + 1$ in cartesian coordinates and due to the nature of the transformation

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}$$

failed to apply to them the preceding theory. This was due to the fact that for (x_1 x_2 0) our transformation has really no meaning. We need not, however, in homogeneous coordinates confine ourselves to the triangle of reference composed of the y - axis, the x - axis, and a line at infinity.

Let us take for our triangle of reference, the y - axis, the x - axis, and the line $x + y = 1$ which gives us the transformation

$$P x_1 = k_1 x, \qquad P x_2 = k_2 y$$

$$P x_3 = \frac{k_3 x + y - 1}{\sqrt{2}}$$

Let $k_1 = k_2 = 1, \qquad k_3 = \sqrt{2}$

Then we have $\rho x_1 = x, \quad \rho x_2 = y$

$$\rho x_3 = x + y - 1$$

The determinant of this transformation is

$$\triangle = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} \neq 0$$

Hence

$$x = \frac{\underline{Pw \quad Ai \ xi}}{Pw \quad Ci \ xi} = \frac{-x_1}{-x_1 - x_2 + x_3} = \frac{x_1}{x_1 + x_2 - x_3}$$

$$y = \frac{\underline{w \ Bi \quad xi}}{w \ Ci \ xi} = \frac{-x_2}{-x_1 - x_2 + x_3} = \frac{x_2}{x_1 + x_2 - x_3}$$

Let us now study the curve $x = y^3$ in homogeneous coordinates for the above transformation.

Our equation $x = y^3$, becomes

$$\frac{x_1}{x_1 + x_2 - x_3} = \frac{x_2^3}{(x_1 + x_2 - x_3)^3}$$

$$x_1 = \frac{x_2^3}{(x_1 + x_2 - x_3)^2} \qquad (\text{for } x_1 + x_2 - x_3 \neq 0)$$

$$x_1 (x_1 + x_2 - x_3)^2 = x_2^3$$

$$x_1^3 + x_1 x_2^2 + x_1 x_3^2 + 2 x_1^2 x_2 - 2 x_1^2 x_3 - 2 x_1 x_2 x_3 - x_2^3 = 0$$

$$\frac{df}{dx_1} = 3x_1^2 + x_2^2 + x_3^2 + 4x_1 x_2 - 4x_1 x_3 - 2x_2 x_3$$

$$\frac{df}{dx_2} = 2x_1 x_2 + 2x_1^2 - 2x_1 x_3 - 3x_2^2$$

$$\frac{df}{dx_3} = 2x_1 x_3 - 2x_1^2 - 2x_1 x_2$$

$$= 2x_1 (x_3 - x_1 - x_2)$$

$$= -2x_1 (x_1 + x_2 - x_3)$$

$\frac{df}{dx_3} = 0$, gives $x_1 = 0$

Since our transformation would be meaningless for

$x_1 + x_2 - x_3 = 0$

For this value of x_1.

$$\frac{df}{dx_1} = x_2^2 + x_3^2 - 2x_1 x_2 = (x_2 - x_3)^2$$

$$\frac{df}{dx_2} = - 3x_2^2$$

cannot both vanish. And we see again that the Cartesian curve in question has no singular point. But we also see that even this transformation has given us an equation which we cannot call non-singular since (1 0 1) is evidently a singular point.

For,

$$\frac{df}{dx_3} = 0, \text{ if } x_1 + x_2 = x_3$$

$\frac{df}{dx_2}$ then becomes

$$2\,x_1\,x_2 + 2\,x_1^2 - 2\,x_1\,(\,x_1 + x_2) - 3\,x_2^2$$

$$= 2\,x_1\,x_2 + 2\,x_1^2 - 2\,x_1^2 - 2\,x_1\,x_2 - 3\,x_2^2$$

$$= 0, \text{ if } x_2 = 0$$

But if $x_2 = 0$, we have $x_1 = x_3$

And if we take $x_1 = x_3 = 1$, we have

$$\frac{df}{dx_1} = 3 + 1 - 4 = 0$$

This point satisfies the homogeneous equation of the curve for

$$x_1^3 + x_1\,x_2^2 + x_1\,x_3^2 + 2\,x_1^2\,x_2 - 2\,x_1^2\,x_3$$

$$- 2\,x_1\,x_2\,x_3 - x_2^3 \quad (= 0)$$

for the point, (1, 0, 1), vanishes.

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