SUBDOMINANT EIGENVALUES OF GRAPHS

BY

JIANWEI LI

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Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
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SUBDOMINANT EIGENVALUES OF GRAPHS

BY

JIANWEI LI

A Thesis submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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SUBDOMINANT EIGENVALUES OF GRAPHS

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ABSTRACT

Consider the set of all undirected simple graphs. The eigenvalues

$$\lambda_1 = \lambda_1(G),\ \lambda_2 = \lambda_2(G), \ldots, \lambda_n, \quad (\lambda_1 \geq \cdots \geq \lambda_n)$$

of a simple graph G are the eigenvalues of its adjacency matrix A. The largest eigenvalue λ_1 is called the index of the graph, and the second largest eigenvalue λ_2 is called the subdominant eigenvalue of the graph. For two graphs G_1 and G_2, the notation $G_1 \gg G_2$ means that G_2 is an induced subgraph of G_1.

The structure of graphs with small subdominant eigenvalues are studied here. It is known that $\lambda_2 \leq 0$ occurs only for complete multipartite graphs. Graphs with λ_2 positive but near 0 are constructed by enlarging certain complete multipartite graphs. These graphs in turn allow graphs with small positive λ_2 to be characterized.

The maximum complete multipartite induced subgraphs of any graph G are considered, and limit methods are used to investigate the structures of the graphs with $0 < \lambda_2 \leq -1 + \sqrt{2}$, or graphs with $0 < \lambda_2 < \frac{-1+\sqrt{5}}{2}$. All the simple graphs G without isolated vertices such that $G \gg K_t$ as a maximum complete multipartite induced subgraph and $0 < \lambda_2(G) < \frac{\sqrt{5}-1}{2}$ are constructed, and they are considered as building blocks for constructing graphs with $0 < \lambda_2 < \frac{\sqrt{5}-1}{2}$. All the minimal simple graphs with $\lambda_2 > -1 + \sqrt{2}$ are determined, as are all simple graphs without isolated vertices such that $0 < \lambda_2 \leq -1 + \sqrt{2}$. The distribution and density of the subdominant eigenvalues in the interval $[0, -1 + \sqrt{2}]$ are investigated. It is proved that the set of the subdominant eigenvalues of simple graphs is nowhere dense in the interval $[0, -1 + \sqrt{2}]$.

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I would like to thank Professor Michael Doob for suggesting this particular topic of study to me, for his invaluable counselling, extensive guidance, and unhesitating willingness to review my manuscript throughout the evolution and culmination of this investigation.

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SECTION 1. INTRODUCTION

Consider the set of all undirected simple graphs $\mathcal{W}$. The eigenvalues

$$\lambda_1 = \lambda_1(G), \lambda_2 = \lambda_2(G), \lambda_3 = \lambda_3(G), \ldots, \lambda_n = \lambda_n(G), \quad (\lambda_1 \geq \ldots \geq \lambda_n)$$

of a graph G belonging to $\mathcal{W}$ are the eigenvalues of its adjacency matrix. The largest eigenvalue λ_1 is called the index of the graph; the second largest eigenvalue λ_2 is called the subdominant eigenvalue of the graph. For two graphs G_1 and G_2, the notation $G_1 \gg G_2$ means that G_2 is an induced subgraph of G_1 and the notation $G_1 \cong G_2$ means that G_2 is isomorphic to G_1. Being roots of a monic polynomial with integral coefficients, all eigenvalues of a graph are either integers or irrational. In this paper, whenever an eigenvalue is expressed as a number in decimal notation, it is understood to be accurate to the number of digits displayed.

Let

$$\mathbf{\Pi_k} = \{\lambda \mid \lambda = \lambda_k(G), \text{ for some } G \in \mathcal{W}\},$$
$$\text{where} \quad 1 \leq k \leq n, \quad n = 1, 2, 3, \ldots,$$

the set of $k\text{-}th$ largest eigenvalues λ_k of G.

First, related to λ_1, it is well known that

(i) $\lambda_1 \geq 0$.

(ii) if $\lambda_1 > 0$, then $\lambda_1 \geq 1$.

(iii) All the graphs for which $\lambda_1 < \sqrt{2+\sqrt{5}}$ are known (see [1] and [5]).

(iv) Let $\tau = (\sqrt{5}+1)/2$ (the golden mean). For $n = 1, 2, \ldots$, let β_n be the positive root of

$$P_n(x) = x^{n+1} - (1 + x + x^2 + \cdots + x^{n-1}).$$

Let $\alpha_n = \beta_n^{1/2} + \beta_n^{-1/2}$. Then

$$2 = \alpha_1 < \alpha_2 < \cdots$$

are all the limit points of $\mathbf{\Pi_1}$ smaller than

$$\tau^{1/2} + \tau^{-1/2} = \sqrt{2+\sqrt{5}} = \lim_{n\to\infty} \alpha_n,$$

(see [13]).

(v) J. B. Shearer (see [18]) proved that any point α with $\sqrt{2+\sqrt{5}} \leq \alpha < +\infty$ is a limit point of $\mathbf{\Pi_1}$, and furthermore M. Doob (see [10]) proved that any such point α is a limit point of $\mathbf{\Pi_k}$, for $k = 1, 2, 3, \ldots$.

Second, related to λ_2, it is easy to see that

(i) $\lambda_2 \geq -1$, and

(ii) if $\lambda_2 > -1$, then $\lambda_2 \geq 0$.
Also

(iii) the limit points of $\mathbf{\Pi_2}$ in the interval $[\sqrt{2+\sqrt{5}}, \infty)$, as a special case of (v) as above, we know that any α with $\sqrt{2+\sqrt{5}} \leq \alpha < +\infty$ is a limit point of $\mathbf{\Pi_2}$. (see[10]).

More and more attention has been paid to the subdominant eigenvalue λ_2 of a graph G. L. Howes (see [15]and [16]) has given conditions to determine whether or not a family of graphs has an upper bound for λ_2. By considering the relation between the eigenvalues λ_i, $i = 1, 2, \ldots, n$ of a graph G and the eigenvalues $\bar{\lambda}_i$, $i = 1, 2, \ldots, n$, of the complementary graph $\bar{G}$ of G, D. Cvetković (see [6]) gave partial characterizations for graphs with $\lambda_2 \leq 1$. The join $G \nabla H$ of graphs G and H is obtained from G and H by joining all vertices of G to all vertices of H. In [6] and [19], D. Cvetković, and S. Simić gave some characterizations of graphs with $\lambda_2 < \frac{\sqrt{5}-1}{2}$ or $\lambda_2 \leq \frac{\sqrt{5}-1}{2}$ by using the join operation, and they proved that the sets $\mathcal{F}$ and $\mathcal{F}^-$ of minimal forbidden graphs with $\lambda_2 \geq \frac{\sqrt{5}-1}{2}$, and $\lambda_2 > \frac{\sqrt{5}-1}{2}$ respectively are finite.

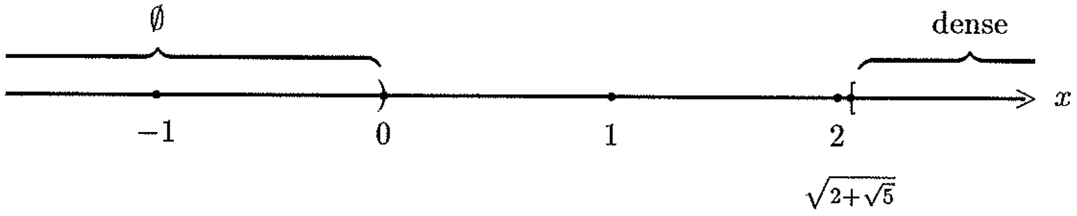
We wish to classify graphs with small second largest eigenvalue. G is a graph without isolated vertices and with $\lambda_2 \leq 0$ if and only if G is a complete multipartite graph, (J. H. Smith, for example, see [4], Theorem 6.7). Note that we include complete graphs as a special case of complete multipartite graphs. In this thesis, graphs with $\lambda_2 > 0$ are constructed by starting from complete multipartite graphs and adding new structures that increase λ_2.

In order to see what constructions are necessary, we consider the induced maximal complete multipartite subgraphs contained in a graph with small positive λ_2 (small will be made more precise later). For a given graph G, those induced maximal complete multipartite subgraphs with the maximum number of vertices are called the maximum complete multipartite subgraphs and denoted as MCM-subgraphs. When there are several MCM-subgraphs of a given graph G, a convention will be adopted to choose one of them. By using limit methods and considering MCM-subgraphs, We are then able to classify graphs with $0 < \lambda_2 \leq -1 + \sqrt{2}$ and graphs with $0 < \lambda_2 < \frac{-1+\sqrt{5}}{2}$ containing complete graph as an MCM-subgraph.

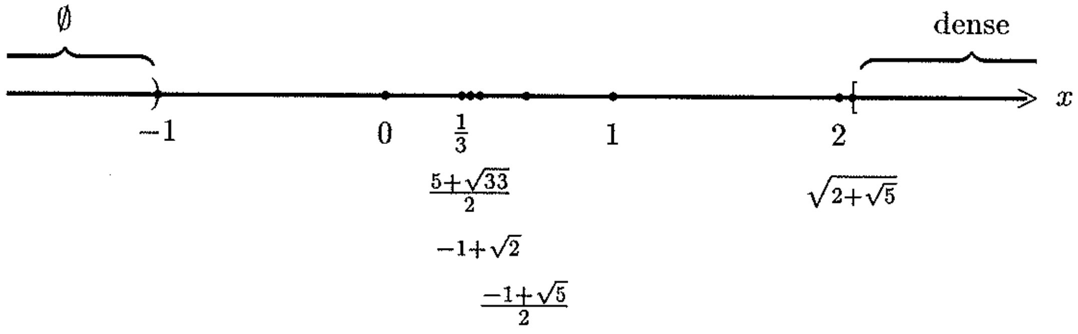
Some prerequisite materials are given in Section 2. In Section 3, an important family of graphs with $0 \leq \lambda_2 < \frac{\sqrt{5}-1}{2}$ is investigated. All the simple graphs without isolated vertices such that $G \gg K_t$ as an MCM-subgraph and $0 < \lambda_2 < \frac{-1+\sqrt{5}}{2}$ are determined in Theorem 3.6. In Section 4, the structure of graphs with $0 < \lambda_2 \leq -1 + \sqrt{2}$ is investigated. All the simple graphs without isolated vertices and with $0 \leq \lambda_2 \leq -1 + \sqrt{2}$ are determined in Theorem 4.2, and all graphs minimal with respect to $\lambda_2 > -1 + \sqrt{2}$ are determined in Theorem 4.3

(when we use the expression that a graph G is minimal with respect to $\lambda_2 > \alpha$, we mean that $\lambda_2(G) > \alpha$ but no proper induced subgraph of G satisfies that property). In Section 5, the distribution and density of subdominant eigenvalues λ_2 in the interval $[0, -1+\sqrt{2}]$ are investigated. The distribution of subdominant eigenvalues λ_2 of the simple graphs without isolated vertices in the interval $(0, 1/3]$ is determined in Theorem 5.1. The distribution of the limit points of subdominant eigenvalues λ_2 of simple graphs without isolated vertices in the interval $\left(0, \frac{-5+\sqrt{33}}{2}\right]$ is determined in Theorem 5.4.

Π_1



Π_2



Let $\mathcal{W}$ be the set of undirected simple graphs. We also wish to consider limit points related to the second largest eigenvalues of graphs in $\mathcal{W}$. To do this, we define $\mathbf{\Pi}_{\mathbf{k}}^{(0)}$ to be $\mathbf{\Pi}_{\mathbf{k}}$, (the set of all possible k-th largest eigenvalues of graphs in $\mathcal{W}$) and $\mathbf{\Pi}_{\mathbf{k}}^{(\mathbf{j})}$ to be the set of limit points of $\mathbf{\Pi}_{\mathbf{k}}^{(\mathbf{j-1})}$, for $j = 1, 2, \ldots$. We determine the smallest value of $\mathbf{\Pi}_{\mathbf{2}}^{(\mathbf{j})}$, that is, we find the the minimal value for the

$$\underbrace{\text{limit points of limits points of} \ldots \text{of limits point of}}_{m \text{ times}}$$

of subdominant eigenvalues of simple graphs without isolated vertices is given in Theorem 5.5. In Theorem 5.7, it is proved that the set $\mathbf{\Pi}_{\mathbf{2}}$ of the subdominant eigenvalues of the simple graphs is nowhere dense in the interval $[0, -1+\sqrt{2}]$.

SECTION 2. PREREQUISITES

In this section, we will present some prerequisite materials.

Theorem 2.1. *(Cauchy Interlacing Theorem)(see [4].) Let A be a Hermitian matrix of order n with eigenvalues $\lambda_n \leq \ldots \leq \lambda_1$. Let B be a principal submatrix of order k with eigenvalues $\mu_k \leq \ldots \leq \mu_1$. Then $\lambda_{n-k+s} \leq \mu_s \leq \lambda_s$, for $s = 1, 2, \ldots, k$.*

Theorem 2.2. *(see [15]) Consider any graph G and any labeling of its n vertices, that is $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let λ be any eigenvalue of $A(G)$, $\lambda \neq -1, 0$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector associated with λ. Consider any two vertices v_i and v_j of G, with*

$$T_i = \{v_k \in V(G) \mid v_k \text{ is adjacent to } v_i \text{ and } v_k \neq v_j\}$$

and

$$T_j = \{v_k \in V(G) \mid v_k \text{ is adjacent to } v_j \text{ and } v_k \neq v_i\}.$$

If $T_i = T_j$, then $x_i = x_j$.

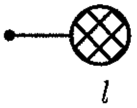
The follow theorem is essentially given in [15].

Theorem 2.3. *Suppose that $p_i(x)$, $i = 1, 2, 3, \ldots, k$, are polynomials and $p_0(x) \neq 0$. Let*

$$P(n, x) = n^k p_0(x) + n^{k-1} p_1(x) + \ldots + p_k(x)$$

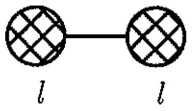
Suppose that $\bar{x}_n$ is a root of $P(n, x)$, $n = 1, 2, 3, \ldots$, and $\lim \bar{x}_n = \bar{x} < \infty$. Then $p_0(\bar{x}) = 0$.

will be a complete graph on l vertices, or clique, abbreviated K_l, where every vertex is adjacent to every other vertex.

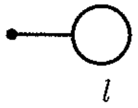


will be a graph formed by K_l and one more vertex adjacent to all the vertices of K_l; that is, K_{l+1}.

will be the independent set of l vertices, abbreviated $\bar{K}_l$, in which no two vertices are adjacent.



will be a graph formed by two cliques on l vertices, where every vertex in each clique is adjacent to all other vertices, that is, K_{2l}.



will be a graph formed by $\bar{K}_l$ and one more vertex adjacent to all the vertices of $\bar{K}_l$, that is $K_{1,l}$.

In short, a solid line joining graphs A and B forms a graph where every vertex in $V(A)$ is adjacent to every vertex in $V(B)$.

Two graphs G and H are said to be "L away from each other" if there exist graphs $\tilde{G}$, and $\tilde{H}$ such that $A(G) + A(\tilde{G}) = A(H) + A(\tilde{H})$ and every vertex of $\tilde{G}$ and $\tilde{H}$ has valence at most L.

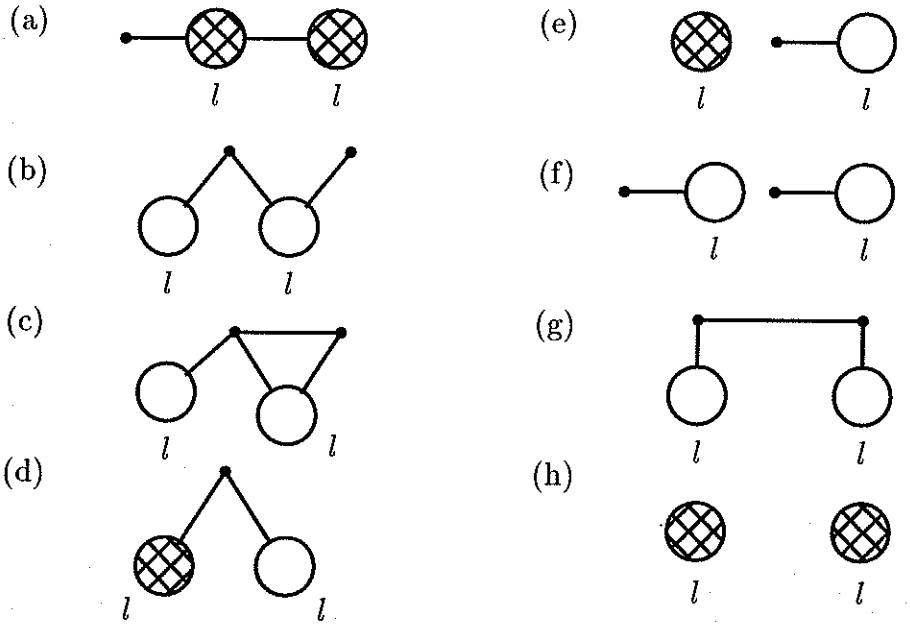


Figure 1.

In [15], L. Howes proved the following:

Theorem2.4. *Let $\mathcal{G}$ be an infinite set of graphs. Then the following statements about $\mathcal{G}$ are equivalent:*

(I) There exists a real number λ such that $\lambda_2(G) \leq \lambda$ for every $G \in \mathcal{G}$.

(II) There exists a positive integer l such that for each $G \in \mathcal{G}$,
the graphs in Figure 1 are not induced subgraphs of G.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$ be the eigenvalues of a graph G and of its complement $\bar{G}$, respectively, both in non-increasing order; Let $A = A(G)$ be the adjacency matrix of the graph G and $\bar{A} = A(\bar{G})$ be the adjacency matrix of the complement graph $\bar{G}$ of the graph G. Then $A + \bar{A} = J - I$, where J is a matrix whose entries are all equal to 1 and I is the identity matrix. Since the largest eigenvalue of $J - I$ is $n - 1$, then Theorem 2.5. implies

Theorem 2.5. *(see [6], Theorem 1.) For any graph G the following inequalities hold:*

$$\lambda_i + \bar{\lambda}_j \geq -1 + n\delta_{2,i+j}, \tag{2.1}$$

$$\lambda_{n-i+1} + \bar{\lambda}_{n-j+1} \leq -1 + n\delta_{n+1,i+j}. \qquad (2.2))$$

where $i, j \geq 1,\ 2 \leq i + j \leq n + 1$, and $\delta_{p,q}$ is the Kronecker δ-symbol.

Corollary. *(see [6].) Putting $i = j = 1$ in (2.1) and putting $i = n - 1, j = 1$ in (2.2), the following inequalities hold.*

$$\lambda_1 + \bar{\lambda}_1 \geq n - 1, \tag{2.3}$$

and

$$\lambda_2 + \bar{\lambda}_n \leq -1. \tag{2.4}$$

Let E, F, Y be subsets of R, the set of real numbers, $E \subseteq Y$, and $F \subseteq Y$. Consider the subspace topology of R on Y, i.e. a subset O_1, (C_1) is open (closed) in Y if and only if there is a open (closed) subset O, (C) of R, such that

$$O_1 = O \cap Y. \ \ (C_1 = C \cup Y.)$$

For the concepts and well-known theorems of topology on the real line R, see for example [11] or [12].

Definitions.

(I). A point $\alpha \in Y$ is called a limit point of E, if every neighborhood of α contains some points of E distinct from α.

(II). A subset E of Y is said to be dense (in Y), if $E \cap U \neq \emptyset$, for all open subsets U of Y.
$E^{(1)} = \{a \mid a$ is a limit point of $E\}$,

(III). A subset F of Y is called nowhere dense (in Y), if for every nonempty open subset U of Y,there exits a nonempty open subset V of U, such that $F \cap V = \emptyset$, or F is not dense in any open subset U of Y.

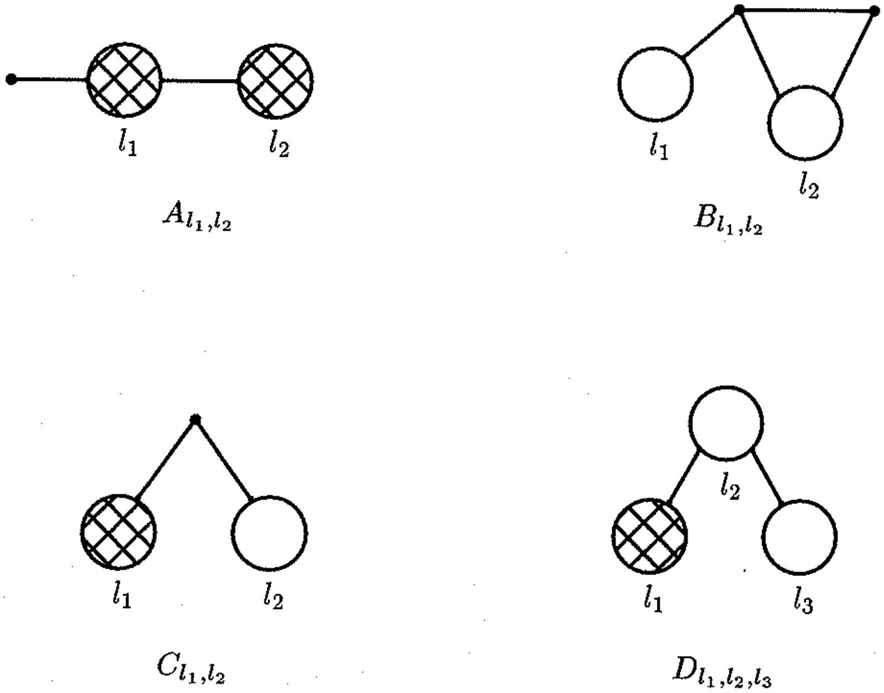
Theorem 2.6. *(Bolzano-Weierstrass Theorem) Each bounded infinite set of real numbers has a limit point in R.*

Theorem 2.7. *(The structure of the open set in R) If $U \subseteq R$, then U is an open set of R if and only if U is a union of countably many disjoint open intervals of R.*

SECTION 3. GRAPHS WITH SUBDOMINANT EIGENVALUES LESS THAN $\frac{-1+\sqrt{5}}{2}$.

In this section, we construct all simple graphs without isolated vertices such that $G \gg K_t$ as an MCM-subgraphs and $0 < \lambda_2 < \frac{-1+\sqrt{5}}{2}$ by starting from a complete graph K_t and adding new structures that increase λ_2 and with a restricted strict upper bound $\lambda_2 < \frac{-1+\sqrt{5}}{2}$ for λ_2. In Theorem 3.6, we will determine all those graphs. This important family of graphs acts as a basis for graphs with $0 < \lambda_2 < \frac{\sqrt{5}-1}{2}$.

First, some families of basic "small" graphs as in *Figure 2* will be considered. These graphs are crucial for the constructions that follow.



A_{l_1,l_2} B_{l_1,l_2} C_{l_1,l_2} D_{l_1,l_2,l_3}

Figure 2.

Lemma 3.1. *For integers l_1, l_2, where $l_1 \geq 1, l_2 \geq 1$, let A_{l_1,l_2} be as in Figure 2. Then*

(I)

$$\lim_{l_2 \to \infty} \lambda_2(A_{l_1,l_2}) = \frac{-1+\sqrt{1+4l_1}}{2}$$

(II)

$$\lim_{l_1 \to \infty} \lambda_2(A_{l_1,l_2}) = -1+\sqrt{l_2}$$

(III)

$$0 < \lambda_2(A_{l_1,l_2}) \leq -1+\sqrt{2}$$

if and only if

$$l_1 \geq 1 \text{ and } l_2 = 2.$$

(IV)

$$0 < \lambda_2(A_{l_1,l_2}) < \frac{-1+\sqrt{5}}{2}$$

if and only if

$$\begin{array}{ll} & l_1 = 2 \text{ and } l_2 = 3; \\ \text{or} & l_1 = 3 \text{ and } l_2 = 3; \\ \text{or} & l_1 \geq 1 \text{ and } l_2 = 2; \\ \text{or} & l_1 = 1 \text{ and } l_2 \geq 2. \end{array}$$

Proof. Because of Theorem 2.2, all the eigenvalues of A_{l_1,l_2} other than -1 or 0 can be found by solving

$$\det \begin{pmatrix} \lambda & -l_1 & 0 \\ -1 & \lambda - l_1 + 1 & -l_2 \\ 0 & -l_1 & \lambda - l_2 + 1 \end{pmatrix} = 0$$

i.e.

$$\lambda^3 + (2 - l_1 - l_2)\lambda^2 - (2l_1 + l_2 - 1)\lambda + l_1(l_2 - 1) = 0. \qquad (3.1)$$

λ_2 l_2 l_1	2	3	4	5	6	7	8	9	10	$\cdots$	$\rightarrow$	∞
1	0.311108	0.428007	0.482696	0.513465	0.532998	0.546445	0.556248	0.563703	0.569561	$\cdots$	$\rightarrow$	$\frac{-1+\sqrt{5}}{2}$
2	0.357926	0.545096	0.653165	0.721557	0.768124	0.801659	0.826873	0.84648	0.862145	$\cdots$	$\rightarrow$	1
3	0.375664	0.597951	0.740489	0.837722	0.907499	0.959672	1.	1.03203	1.05804	$\cdots$	$\rightarrow$	$\frac{-1+\sqrt{13}}{2}$
4	0.384928	0.627719	0.793139	0.911656	1.	1.06802	1.12181	1.16531	1.20116	$\cdots$	$\rightarrow$	$\frac{-1+\sqrt{17}}{2}$
5	0.39061	0.64674	0.828203	0.962703	1.06581	1.14703	1.21246	1.26618	1.31102	$\cdots$	$\rightarrow$	$\frac{-1+\sqrt{21}}{2}$
6	0.394449	0.659922	0.853181	1.	1.11497	1.20718	1.2826	1.34532	1.39823	$\cdots$	$\rightarrow$	2
7	0.397214	0.669589	0.871855	1.02841	1.15306	1.25448	1.33848	1.4091	1.46922	$\cdots$	$\rightarrow$	$\frac{-1+\sqrt{29}}{2}$
8	0.399302	0.676977	0.886336	1.05076	1.18341	1.29264	1.38405	1.4616	1.52816	$\cdots$	$\rightarrow$	$\frac{-1+\sqrt{33}}{2}$
9	0.400933	0.682805	0.897888	1.06878	1.20817	1.32406	1.4219	1.50557	1.57789	$\cdots$	$\rightarrow$	$\frac{-1+\sqrt{37}}{2}$
10	0.402242	0.687521	0.907316	1.08363	1.22873	1.35037	1.45385	1.54293	1.62041	$\cdots$	$\rightarrow$	$\frac{-1+\sqrt{41}}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$	$\searrow$	$\downarrow$
∞	$-1+\sqrt{2}$	$-1+\sqrt{3}$	$-1+\sqrt{4}$	$-1+\sqrt{5}$	$-1+\sqrt{6}$	$-1+\sqrt{7}$	$-1+\sqrt{8}$	$-1+\sqrt{9}$	$-1+\sqrt{10}$	$\cdots$	$\rightarrow$	∞

$$\lambda_2 = \lambda_2(A_{l_1,l_2})$$

where $l_1 \geq 1$, and $l_2 \geq 2$

Table 1.

The left hand side of equation (3.1) is a real polynomial. Since the roots of (3.1) are all eigenvalues of a graph, they are all real numbers.

For $l_2 = 1$,

$$A_{l_1,1} = K_{\underbrace{1,1,1,\ldots,1}_{l_1 \text{ times}},2} \quad l_1 \geq 1$$

are complete multipartite graphs with $\lambda_2(A_{l_1,1}) = 0$. So only $l_1 \geq 1$ and $l_2 > 1$ need to be considered.

For given integers l_1 and l_2, where $l_1 \geq 1$ and $l_2 > 1$, since $A_{l_1,l_2} \gg A_{1,2}$ as an induced subgraph, we have

$$\lambda_2(A_{l_1,l_2}) \geq \lambda_2(A_{1,2}) = 0.311 > 0.$$

So $\lambda_2(A_{l_1,l_2})$ $(\neq -1, 0)$ must be the second largest root of polynomial (3.1).

(I) Let $l_1 \geq 1$ be any fixed positive integer.

$\lambda_2(A_{l_1,l_2})$, $l_2 > 1$, is a sequence of real numbers. By the Cauchy Interlacing Theorem, $\lambda_2(A_{l_1,l_2})$ is monotone increasing in l_2. Apply Theorem 2.4 with $l = l_1+1$. Then the equivalence of part(I) and part(II) in Theorem 2.4 implies that there is a real number λ such that $\lambda_2(A_{l_1,l_2}) \leq \lambda$ for every graph A_{l_1,l_2} with $l_2 > 1$.Therefore $\lim_{l_2\to\infty} \lambda_2(A_{l_1,l_2})$ exists and is a finite real number.

By Theorem 2.3, $\lim_{l_2\to\infty} \lambda_2(A_{l_1,l_2})$ is a root of

$$-\lambda^2 - \lambda + l_1 = 0. \tag{3.2}$$

Because $A_{l_1,l_2} \gg K_{l_2}$ as an induced subgraph,

$$\lim_{l_2\to\infty} \lambda_1(A_{l_1,l_2}) \geq \lim_{l_2\to\infty} \lambda_1(K_{l_2}) = \infty$$

Then $\lim_{l_2\to\infty} \lambda_2(A_{l_1,l_2})$ must be the largest root of (3.2). Therefore

$$\lim_{l_2\to\infty} \lambda_2(A_{l_1,l_2}) = \frac{-1+\sqrt{1+4l_1}}{2}.$$

(II) Let $l_2 \geq 1$ be any fixed integer.

Apply Theorem 2.4 with $l = l_2 + 1$. This implies that

$$\lim_{l_1\to\infty} \lambda_2(A_{l_1,l_2})$$

exists. By Theorem 2.3,

$$\lim_{l_1\to\infty} \lambda_2(A_{l_1,l_2})$$

is a root of

$$-\lambda^2 - 2\lambda + l_2 - 1 = 0. \tag{3.3}$$

Because $A_{l_1,l_2} \gg K_{l_1+1}$ as an induced subgraph,

$$\lim_{l_1 \to \infty} \lambda_1(A_{l_1,l_2}) \geq \lim_{l_1 \to \infty} \lambda_1(K_{l_1+1}) = \infty.$$

Then $\lim_{l_1 \to \infty} \lambda_2(A_{l_1,l_2})$ must be the largest root of (3.3). Therefore

$$\lim_{l_1 \to \infty} \lambda_2(A_{l_1,l_2}) = -1 + \sqrt{l_2}.$$

(III) The result comes from Part(I), Part(II), *Table 1*, and that

$$A_{1,3,} = \hat{F}_3, \ \lambda(A_{1,3}) = \lambda_2(\hat{F}_3) = 0.428007,$$

is a graph minimal with respect to $\lambda_2 > -1 + \sqrt{2}$.

(IV) This follows from Part(I), Part(II), *Table 1*, and that

$$A_{2,4} = F_6, \ \lambda_2(F_6) = 0.653165,$$

and

$$A_{4,3} = F_7, \ \lambda_2(F_7) = 0.627719,$$

are graphs minimal with respect to $\lambda_2 \geq (-1 + \sqrt{5})/2$.

Lemma 3.2. *For integers l_1, l_2, where $l_1 \geq 1$, $l_2 \geq 1$, let B_{l_1,l_2} be as in Figure 2. Then*

(I)

$$\lim_{l_2 \to \infty} \lambda_2(B_{l_1,l_2}) = \frac{-1 + \sqrt{1 + 2l_1}}{2}$$

(II)

$$\lim_{l_1 \to \infty} \lambda_2(B_{l_1,l_2}) = \sqrt{l_2}$$

(III)

$$0 < \lambda_2(B_{l_1,l_2}) \leq -1 + \sqrt{2}$$

if and only if

$$l_1 = 1 \text{ and } l_2 \geq 1.$$

(IV)

$$0 < \lambda_2(B_{l_1,l_2}) < \frac{\sqrt{5} - 1}{2}$$

if and only if

$$\begin{array}{ll} & l_1 = 1 \text{ and } l_2 \geq 1; \\ \text{or} & l_1 = 2 \text{ and } l_2 \geq 1; \\ \text{or} & l_1 = 3 \text{ and } l_2 = 1; \end{array}$$

Proof. Because of Theorem 2.2, all the eigenvalues of B_{l_1,l_2} other than -1 or 0 can be found by solving

$$\det \begin{pmatrix} \lambda & -1 & -l_1 & -l_2 \\ -1 & \lambda & 0 & -l_2 \\ -1 & 0 & \lambda & 0 \\ -1 & -1 & 0 & \lambda \end{pmatrix} = 0$$

i.e.

$$\lambda^4 - (l_1 + 2l_2 + 1)\lambda^2 - 2l_2\lambda + l_1 l_2 = 0. \tag{3.4}$$

For given integers l_1, and l_2, where $l_1 \geq 1$ and $l_2 \geq 1$, $B_{l_1,l_2} \gg B_{1,1}$ as an induced subgraph,

$$\lambda_2(B_{l_1,l_2}) \geq \lambda_2(B_{1,1}) = 0.311 \geq 0$$

and $\lambda_2(B_{l_1,l_2})$ must be the second largest root of polynomial (3.4).

(I) Let $l_1 \geq 1$ be any fixed positive integer.

By the Cauchy Interlacing Theorem, $\lambda_2(B_{l_1,l_2})$ is monotone increasing in l_2. Let $l = l_1 + 1$ in Theorem 2.4, Part(II). Then the equivalence of part(I) and part(II) in Theorem 2.4 implies that there is a real number λ such that $\lambda_2(B_{l_1,l_2}) \leq \lambda$ for every graph B_{l_1,l_2} with $l_2 \geq 1$. Therefore $\lim_{l_2\to\infty} \lambda_2(B_{l_1,l_2})$ exists and is a finite real number.

By Theorem 2.3,

$$\lim_{l_2\to\infty} \lambda_2(B_{l_1,l_2})$$

is a root of

$$-2\lambda^2 - 2\lambda + l_1 = 0. \tag{3.5}$$

Because $B_{l_1,l_2} \gg K_{1,l_2}$ as an induced subgraph, and

$$\lim_{l_2\to\infty} \lambda_1(B_{l_1,l_2}) \geq \lim_{l_2\to\infty} \lambda_1(K_{1,l_2}) = \infty,$$

$\lim_{l_2\to\infty} \lambda_2(B_{l_1,l_2})$ must be the largest root of (3.5). Therefore

$$\lim_{l_2\to\infty} \lambda_2(B_{l_1,l_2}) = \frac{-1 + \sqrt{1 + 2l_1}}{2}.$$

λ_2 l_2	1	2	3	4	5	6	7	8	9	10	$\cdots$	$\to$	∞
l_1													
1	0.311108	0.334904	0.344292	0.349326	0.352465	0.354611	0.35617	0.357354	0.358285	0.359034	$\cdots$	$\to$	$\frac{-1+\sqrt{3}}{2}$
2	0.470683	0.529317	0.55452	0.568568	0.577527	0.583738	0.588298	0.591788	0.594545	0.596777	$\cdots$	$\to$	$\frac{-1+\sqrt{5}}{2}$
3	0.571993	0.664834	0.7075	0.732051	0.748	0.75919	0.767474	0.773853	0.778916	0.783031	$\cdots$	$\to$	$\frac{-1+\sqrt{7}}{2}$
4	0.642074	0.765967	0.826204	0.861876	0.885446	0.902168	0.914643	0.924303	0.932003	0.938283	$\cdots$	$\to$	1
5	0.693225	0.844436	0.921585	0.968498	1.	1.02259	1.03957	1.05279	1.06337	1.07203	$\cdots$	$\to$	$\frac{-1+\sqrt{11}}{2}$
6	0.732051	0.906982	1.	1.05796	1.09748	1.12612	1.1478	1.16478	1.17842	1.18962	$\cdots$	$\to$	$\frac{-1+\sqrt{13}}{2}$
7	0.762433	0.957877	1.06556	1.13417	1.18165	1.2164	1.2429	1.26377	1.2806	1.29447	$\cdots$	$\to$	$\frac{-1+\sqrt{15}}{2}$
8	0.786802	1.	1.1211	1.19985	1.25511	1.29595	1.32733	1.35216	1.37228	1.38891	$\cdots$	$\to$	$\frac{-1+\sqrt{17}}{2}$
9	0.806748	1.03536	1.16869	1.25699	1.31978	1.36663	1.40287	1.43171	1.45517	1.47463	$\cdots$	$\to$	$\frac{-1+\sqrt{19}}{2}$
10	0.823354	1.06542	1.20985	1.30711	1.37711	1.42983	1.4709	1.50374	1.53058	1.55291	$\cdots$	$\to$	$\frac{-1+\sqrt{21}}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$	$\searrow$	$\downarrow$
∞	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$	3	$\sqrt{10}$	$\cdots$	$\to$	∞

$$\lambda_2 = \lambda_2(B_{l_1,l_2})$$

where $l_1 \geq 1$, and $l_2 \geq 1$

Table 2.

(II) Let $l_2 \geq 1$ be any fixed integer.

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(B_{l_1,l_2}),\ l_1 \geq 1$$

is bounded from above. Then

$$\lim_{l_1 \to \infty} \lambda_2(B_{l_1,l_2})$$

exists. By Theorem 2.3, $\lim_{l_1 \to \infty} \lambda_2(B_{l_1,l_2})$ is a root of

$$-\lambda^2 + l_2 = 0. \tag{3.6}$$

Because $B_{l_1,l_2} \gg K_{1,l_1}$ as an induced subgraph,

$$\lim_{l_1 \to \infty} \lambda_1(B_{l_1,l_2}) \geq \lim_{l_1 \to \infty} \lambda_1(K_{1,l_1}) = \infty.$$

Then

$$\lim_{l_1 \to \infty} \lambda_2(B_{l_1,l_2})$$

must be the largest root of (3.6). Therefore

$$\lim_{l_1 \to \infty} \lambda_2(B_{l_1,l_2}) = \sqrt{l_2}.$$

(III) The result comes from Part(I), Part(II), *Table 2*, and that

$$B_{2,1} = \hat{F}_4,\ \lambda(B_{2,1}) = \lambda_2(\hat{F}_4) = 0.470683$$

is a graph minimal with respect to $\lambda_2 > -1 + \sqrt{2}$.

(IV) This follows from Part(I), Part(II), *Table 2*, and the fact that

$$B_{3,2} = F_4, \lambda_2(F_4) = 0.664834$$

and

$$B_{4,1} = F_3, \lambda_2(F_3) = 0.642074,$$

are graphs minimal with respect to $\lambda_2 \geq (-1 + \sqrt{5})/2$.

Lemma 3.3. *For integers l_1, l_2 where $l_1 \geq 1, l_2 \geq 1$, let C_{l_1,l_2} be as in the Figure 2. Then*

(I)

$$\lim_{l_2 \to \infty} \lambda_2(C_{l_1,l_2}) = l_1 - 1.$$

(II)

$$\lim_{l_1 \to \infty} \lambda_2(C_{l_1,l_2}) = \frac{-1+\sqrt{1+4l_2}}{2}.$$

(III)

$$0 < \lambda_2(C_{l_1,l_2}) \le -1+\sqrt{2}$$

if and only if

$$l_1 = 2 \text{ and } l_2 = 1.$$

(IV)

$$0 < \lambda_2(C_{l_1,l_2}) < (\sqrt{5}-1)/2$$

if and only if

$$\begin{array}{ll} & l_1 \ge 2 \text{ and } l_2 = 1; \\ \text{or} & l_1 = 2 \text{ and } l_2 = 2; \\ \text{or} & l_1 = 2 \text{ and } l_2 = 3. \end{array}$$

Proof. Because of Theorem 2.2, all the eigenvalues of C_{l_1,l_2} other than -1 or 0 can be found by solving

$$\det \begin{pmatrix} \lambda & -l_1 & -l_2 \\ -1 & \lambda - l_1 + 1 & 0 \\ -1 & 0 & \lambda \end{pmatrix} = 0$$

i.e.

$$\lambda^3 - (l_1-1)\lambda^2 - (l_1+l_2)\lambda + l_2(l_1-1) = 0. \tag{3.7}$$

For $l_1 = 1, C_{1,l_2} = K_{1,l_2+1}$ are complete bipartite graphs. Then $\lambda_2(C_{1,l_2}) = 0$. So only $l_1 > 1$ and $l_2 \ge 1$ need to be considered.

For given integers l_1 and l_2, where $l_1 > 1$ and $l_2 \ge 1$, $C_{l_1,l_2} \gg C_{2,1}$ as an induced subgraph,

$$\lambda_2(C_{l_1,l_2}) \ge \lambda_2(C_{2,1}) = 0.311 > 0.$$

So $\lambda_2(C_{l_1,l_2})$ $(\neq -1, 0)$ must be the second largest root of polynomial (3.7).

λ_2 l_2	1	2	3	4	5	6	7	8	9	10	$\cdots \to$	∞
l_1												
2	0.311108	0.470683	0.571993	0.642074	0.693225	0.732051	0.762433	0.786802	0.806748	0.823354	$\cdots \to$	1
3	0.428007	0.678363	0.859565	1.	1.11268	1.20507	1.282	1.34683	1.40205	1.44949	$\cdots \to$	2
4	0.482696	0.776538	1.	1.18237	1.33651	1.4695	1.58579	1.68842	1.77962	1.86109	$\cdots \to$	3
5	0.513465	0.830535	1.07651	1.28164	1.45907	1.61589	1.75651	1.88385	2.	2.10652	$\cdots \to$	4
6	0.532998	0.864034	1.12322	1.34156	1.53244	1.7031	1.85797	2.	2.13128	2.25335	$\cdots \to$	5
7	0.546445	0.886667	1.15433	1.38099	1.58025	1.75944	1.92306	2.07409	2.21464	2.34626	$\cdots \to$	6
8	0.556248	0.902923	1.17641	1.4087	1.61355	1.79837	1.96771	2.12458	2.2711	2.40886	$\cdots \to$	7
9	0.563703	0.915141	1.19285	1.42916	1.63795	1.82671	2.	2.16088	2.31148	2.45339	$\cdots \to$	8
10	0.569561	0.924648	1.20554	1.44485	1.65655	1.84818	2.02435	2.18811	2.34162	2.48648	$\cdots \to$	9
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$ $\vdots$	$\vdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$ $\searrow$	$\downarrow$
∞	$\frac{-1+\sqrt{5}}{2}$	1	$\frac{-1+\sqrt{13}}{2}$	$\frac{-1+\sqrt{17}}{2}$	$\frac{-1+\sqrt{21}}{2}$	2	$\frac{-1+\sqrt{29}}{2}$	$\frac{-1+\sqrt{33}}{2}$	$\frac{-1+\sqrt{37}}{2}$	$\frac{-1+\sqrt{41}}{2}$	$\cdots \to$	∞

$$\lambda_2 = \lambda_2(C_{l_1,l_2})$$

where $l_1 \geq 2$, and $l_2 \geq 1$

Table 3.

(I) Let $l_1 > 1$ be any fixed positive integer.

Apply Theorem 2.4 with $l = l_1 + 1$. Then the sequence

$$\lambda_2(C_{l_1,l_2}),\ l_2 \geq 1$$

is bounded from above. Therefore

$$\lim_{l_2\to\infty} \lambda_2(C_{l_1,l_2})$$

exists and is a finite real number.

By Theorem 2.3,

$$\lim_{l_2\to\infty} \lambda_2(C_{l_1,l_2})$$

is a root of

$$-\lambda + l_1 - 1 = 0. \tag{3.8}$$

Because $C_{l_1,l_2} \gg K_{1,l_2+1}$ as an induced subgraph,

$$\lim_{l_2\to\infty} \lambda_1(C_{l_1,l_2}) \geq \lim_{l_2\to\infty} \lambda_1(K_{1,l_2+1}) = \infty.$$

Then $\lim_{l_2\to\infty} \lambda_2(C_{l_1,l_2})$ must be the largest root of (3.8). Therefore

$$\lim_{l_2\to\infty} \lambda_2(C_{l_1,l_2}) = l_1 - 1.$$

(II) Let $l_2 \geq 1$ be any fixed positive integer.

Apply Theorem 2.4 with $l = l_1 + 1$. Then the sequence $\lambda_2(C_{l_1,l_2}), l_1 \geq 1$ is bounded from above. Then $\lim_{l_1\to\infty} \lambda_2(C_{l_1,l_2})$ exists and is a finite real number.

By Theorem 2.3, $\lim_{l_1\to\infty} \lambda_2(C_{l_1,l_2})$ is a root of

$$-\lambda^2 + \lambda + l_2 = 0. \tag{3.9}$$

Because $C_{l_1,l_2} \gg K_{l_1}$ as an induced subgraph,

$$\lim_{l_1\to\infty} \lambda_1(C_{l_1,l_2}) \geq \lim_{l_1\to\infty} \lambda_1(K_{l_1}) = \infty.$$

Then

$$\lim_{l_1\to\infty} \lambda_2(C_{l_1,l_2})$$

must be the largest root of (3.9). Therefore

$$\lim_{l_1\to\infty} \lambda_2(C_{l_1,l_2}) = \frac{-1+\sqrt{1+4l_2}}{2}.$$

(III) The result comes from Part(I), Part(II), *Table 3*, and the fact that

$$C_{2,2} = \hat{F}_4, \lambda(C_{2,2}) = \lambda_2(\hat{F}_4) = 0.470683$$

and

$$C_{3,1} = \hat{F}_3, \lambda(C_{3,1}) = \lambda_2(\hat{F}_3) = 0.428007$$

are graphs minimal with respect to $\lambda_2 > -1 + \sqrt{2}$.

(IV) This follows from Part(I), Part(II), *Table 3*, and the fact that

$$C_{2,4} = F_3, \ \lambda_2(F_3) = 0.642074$$

and

$$C_{3,2} = F_5, \ \lambda_2(F_5) = 0.678363$$

are graphs minimal with respect to $\lambda_2 \geq (-1 + \sqrt{5})/2$.

Lemma 3.4. *For integers l_1, l_2 and l_3, where $l_1 \geq 1$, $l_2 \geq 1$, and $l_3 \geq 1$. let D_{l_1,l_2,l_3} be as in Figure 2. Then*

(I)

$$\lim_{l_2 \to \infty} \lambda_2(D_{l_1,l_2,l_3}) = \frac{(l_1 - 1)l_3}{l_1 + l_3}$$

(II)

$$\lim_{l_1 \to \infty} \lambda_2(D_{l_1,l_2,l_3}) = \frac{-l_2 + \sqrt{{l_2}^2 + 4l_2 l_3}}{2}$$

(III)

$$\lim_{\substack{l_2 \to \infty \\ l_3 \to \infty}} \lambda_2(D_{l_1,l_2,l_3}) = l_1 - 1$$

$$\lim_{\substack{l_2 \to \infty \\ l_1 \to \infty}} \lambda_2(D_{l_1,l_2,l_3}) = l_3$$

(IV)

$$0 < \lambda_2(D_{l_1,l_2}) \leq -1 + \sqrt{2}$$

if and only if

$$l_1 = 2, \qquad l_2 \geq 1 \qquad \text{and } l_3 = 1.$$

(V)

$$0 < \lambda_2(D_{l_1,l_2,l_3}) < \frac{-1 + \sqrt{5}}{2}$$

if and only if

$$\begin{array}{llll} & l_1 = 2, & l_2 \geq 1 & \text{and} \quad l_3 = 1,2,3; \\ \text{or} & l_1 = 3, & l_2 \geq 1 & \text{and} \quad l_3 = 1; \\ \text{or} & l_1 = 4, & l_2 \geq 1 & \text{and} \quad l_3 = 1; \\ \text{or} & l_1 = 5, & l_2 = 1,2,3,4 & \text{and} \quad l_3 = 1; \\ \text{or} & l_1 = 6, & l_2 = 1,2 & \text{and} \quad l_3 = 1; \\ \text{or} & l_1 \geq 7, & l_2 = 1, & \text{and} \quad l_3 = 1. \end{array}$$

Proof. Because of Theorem 2.2, all the eigenvalues of D_{l_1,l_2,l_3} other than -1 or 0 can be found by solving

$$\det \begin{pmatrix} \lambda - l_1 + 1 & -l_2 & 0 \\ -l_1 & \lambda & -l_3 \\ 0 & -l_2 & \lambda \end{pmatrix} = 0.$$

i.e.

$$\lambda^3 - (l_1 - 1)\lambda^2 - l_2(l_1 + l_3)\lambda + (l_1 - 1)l_2 l_3 = 0 \tag{3.10}$$

For $l_1 = 1, D_{1,l_2,l_3} = K_{l_2,l_3+1}$ are complete bipartite graphs. Then

$$\lambda_2(D_{1,l_2,l_3}) = 0.$$

So only $l_1 > 1$, $l_2 \geq 1$ and $l_3 \geq 1$ need to be considered.

For given integers l_1, l_2 and l_3, where $l_1 \geq 1$, $l_2 \geq 1$ and $l_3 \geq 1$, $D_{l_1,l_2,l_3} \gg D_{2,1,1}$ as an induced subgraph,

$$\lambda_2(D_{l_1,l_2,l_3}) \geq \lambda_2(D_{2,1,1}) = 0.311 > 0.$$

So $\lambda_2(D_{l_1,l_2,l_3})(\neq -1, 0)$ must be the second largest root of the polynomial (3.10).

Let L_1, L_2 and L_3 be sets of some nonnegative integers such that

$$\min\{l_1 \mid l_1 \in L_1\} > 1,$$

$$\min\{l_2 \mid l_2 \in L_2\} \geq 1,$$

and

$$\min\{l_3 \mid l_3 \in L_3\} \geq 1.$$

By Theorem 2.4, in order that the set

$$\{\lambda_2(D_{l_1,l_2,l_3}) \mid l_1 \in L_1, l_2 \in L_2, l_3 \in L_3\}$$

be bounded from above, among the sets L_1 and L_2, at least one must be bounded from above. Otherwise if both L_1 and L_3 are unbounded, then

$$\sup\{\lambda_2(D_{l_1,l_2,l_3}) \mid l_1 \in L_1, l_2 \in L_2, l_3 \in L_3\} \geq \lim_{l\to\infty} \lambda_2(D_{l,1,l}) = \lim_{l\to\infty} \lambda_2(C_{l,l}) = \infty.$$

λ_2^* l_3	1	2	3	4	5	6	7	8	9	10	$\cdots \rightarrow$	∞
l_1												
2	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{7}{9}$	$\frac{4}{5}$	$\frac{9}{11}$	$\frac{5}{6}$	$\cdots \rightarrow$	1
3	$\frac{1}{2}$	$\frac{4}{5}$	1	$\frac{8}{7}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{7}{5}$	$\frac{16}{11}$	$\frac{3}{2}$	$\frac{20}{13}$	$\cdots \rightarrow$	2
4	$\frac{3}{5}$	1	$\frac{9}{7}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{9}{5}$	$\frac{21}{11}$	2	$\frac{27}{13}$	$\frac{15}{7}$	$\cdots \rightarrow$	3
5	$\frac{2}{3}$	$\frac{8}{7}$	$\frac{3}{2}$	$\frac{16}{9}$	2	$\frac{24}{11}$	$\frac{7}{3}$	$\frac{32}{13}$	$\frac{18}{7}$	$\frac{8}{3}$	$\cdots \rightarrow$	4
6	$\frac{5}{7}$	$\frac{5}{4}$	$\frac{5}{3}$	2	$\frac{25}{11}$	$\frac{5}{2}$	$\frac{35}{13}$	$\frac{20}{7}$	3	$\frac{25}{8}$	$\cdots \rightarrow$	5
7	$\frac{3}{4}$	$\frac{4}{3}$	$\frac{9}{5}$	$\frac{24}{11}$	$\frac{5}{2}$	$\frac{36}{13}$	7	$\frac{16}{5}$	$\frac{27}{8}$	$\frac{60}{17}$	$\cdots \rightarrow$	6
8	$\frac{7}{9}$	$\frac{7}{5}$	$\frac{21}{11}$	$\frac{7}{3}$	$\frac{35}{13}$	3	$\frac{49}{15}$	$\frac{7}{2}$	$\frac{63}{17}$	$\frac{35}{9}$	$\cdots \rightarrow$	7
9	$\frac{4}{5}$	$\frac{16}{11}$	2	$\frac{32}{13}$	$\frac{20}{7}$	$\frac{16}{5}$	$\frac{7}{2}$	$\frac{64}{17}$	4	$\frac{80}{19}$	$\cdots \rightarrow$	8
10	$\frac{9}{11}$	$\frac{3}{2}$	$\frac{27}{13}$	$\frac{18}{7}$	3	$\frac{27}{8}$	$\frac{63}{17}$	4	$\frac{81}{19}$	$\frac{9}{2}$	$\cdots \rightarrow$	9
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$ $\vdots$	$\vdots$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\cdots$ $\searrow$	$\downarrow$
∞	1	2	3	4	5	6	7	8	9	10	$\cdots \rightarrow$	∞

$$\lambda_2^* = \lim_{l_2 \to \infty} \lambda_2(D_{l_1,l_2,l_3}) = \frac{(l_1-1)l_3}{l_1+l_3}$$

where $l_1 \geq 2$, and $l_3 \geq 1$

Table 4.

(I) Let $l_1 > 1$ and $l_2 \geq 1$ be any fixed positive integers.

Apply Theorem 2.4 with $l = \max\{l_1+1, l_3+1\}$. Then the equivalence of part(I) and part(II) in Theorem 2.4 implies that there is a real number λ such that

$$\lambda_2(D_{l_1,l_2,l_3}) \leq \lambda$$

for every graph D_{l_1,l_2,l_3} with $l_2 \geq 1$. Therefore $\lim_{l_2\to\infty} \lambda_2(D_{l_1,l_2,l_3})$ exists and is finite.

By Theorem 2.3, $\lim_{l_2\to\infty} \lambda_2(D_{l_1,l_2,l_3})$ is a root of

$$-(l_1 + l_3)\lambda + (l_1 - 1)l_3 = 0. \tag{3.11}$$

Because $D_{l_1,l_2,l_3} \gg K_{1,l_2}$ as an induced subgraph,

$$\lim_{l_2\to\infty} \lambda_1(D_{l_1,l_2,l_3}) \geq \lim_{l_2\to\infty} \lambda_1(K_{1,l_2}) = \infty.$$

So $\lim_{l_2\to\infty} \lambda_2(D_{l_1,l_2,l_3})$ must be the largest root of (3.11). Therefore

$$\lim_{l_2\to\infty} \lambda_2(D_{l_1,l_2,l_3}) = \frac{(l_1-1)l_3}{l_1+l_3}.$$

(II) Let $l_2 \geq 1$ and $l_3 \geq 1$ be any fixed integers.

Apply Theorem 2.4 with $l = l_3 + 1$. Then the sequence

$$\lambda_2(D_{l_1,l_2,l_3}),\ l_1 > 1$$

is bounded from above. Then

$$\lim_{l_1\to\infty} \lambda_2(D_{l_1,l_2,l_3})$$

exists and is finite. By Theorem 2.3,

$$\lim_{l_1\to\infty} \lambda_2(D_{l_1,l_2,l_3})$$

is a root of

$$-\lambda^2 - l_2\lambda + l_2 l_3 = 0. \tag{3.12}$$

Because $D_{l_1,l_2,l_3} \gg K_{l_1+1}$ as an induced subgraph.

$$\lim_{l_1\to\infty} \lambda_1(D_{l_1,l_2,l_3}) \geq \lim_{l_1\to\infty} \lambda_1(K_{l_1+1}) = \infty.$$

Then $\lim_{l_1\to\infty} \lambda_2(D_{l-1,l_2,l_3})$ must be the largest root of (3.12). Therefore

$$\lim_{l_1\to\infty} \lambda_2(D_{l_1,l_2,l_3}) = \frac{-l_2 + \sqrt{l_2^2 + 4l_2l_3}}{2}.$$

(III) The result comes from Part(I) and Part(II).

(IV) The result comes from Part(I), Part(II), *Table 4*, and the fact that

$$D_{3,1,1} = C_{3,1} = \hat{F}_3,\ \lambda_2(D_{3,1,1}) = \lambda_2(\hat{F}_3) = 0.428007$$

and

$$D_{2,1,2} = C_{2,2} = \hat{F}_4,\ \lambda_2(D_{2,1,2}) = \lambda_2(\hat{F}_4) = 0.470683$$

are graphs minimal with respect to $\lambda_2 > -1 + \sqrt{2}$.

(V) This follows from Part(I), Part(II), *Table 4*, and the fact that

$$D_{2,1,4} = C_{2,4} = F_3,\ \lambda_2(D_{2,1,4}) = \lambda_2(F_3) = 0.642074$$

$$D_{3,1,2} = C_{3,2} = F_5,\ \lambda_2(D_{3,1,2}) = \lambda_2(F_5) = 0.678363$$

$$D_{5,5,1} = F_8,\ \lambda_2(D_{5,5,1}) = \lambda_2(F_8) = 0.622979$$

$$D_{6,3,1} = F_9,\ \lambda_2(D_{6,3,1}) = \lambda_2(F_9) = 0.631361$$

and

$$D_{7,2,1} = F_{10},\ \lambda_2(D_{7,2,1}) = \lambda_2(F_{10}) = 0.620535$$

are graphs minimal with respect to $\lambda_2 \geq (-1 + \sqrt{5})/2$, and

$$\lim_{l_1 \to \infty} \lambda_2(D_{l_1,1,1}) = \frac{-1 + \sqrt{5}}{2}$$

where the convergence is monotone and for all $l_1 > 1,\ l_2 = 1$, and $l_3 = 1$.

When $l_2 = l_3 = 1$, (3.10) becomes

$$\lambda^3 - (l_1 - 1)\lambda^2 - (l_1 + 1)\lambda + l_1 - 1 = (\lambda^2 + \lambda - 1)(\lambda - l_1) - 1 = 0,$$

and $(-1+\sqrt{5})/2$ is not a root of this polynomial. Then $\lambda_2(D_{l_1,1,1}) < (-1+\sqrt{5})/2$.

Consider the set of all undirected simple graphs $\mathcal{W}$. For any graph G belonging to $\mathcal{W}$, the inclusion relation is a partial order among the complete multipartite induced subgraphs of the graph G.

Definition. *For $G \in \mathcal{W}$, let $\mathcal{N}(G)$ be the set of all induced maximal complete multipartite subgraphs $K_{r_1,r_2,\ldots,r_t}$ of the graph G. Let*

$$S_{\max}(G) = \max\left\{ \sum_{i=1}^{t} r_i \;\middle|\; K_{r_1,r_2,\ldots,r_t} \in \mathcal{N}(G) \right\},$$

and let

$$\mathcal{M}(G) = \left\{ K_{r_1,r_2,\ldots,r_t} \;\middle|\; \begin{array}{c} K_{r_1,r_2,\ldots,r_t} \in \mathcal{N}(G), \\ \text{and } \sum_{i=1}^{t} r_i = S_{\max}(G). \end{array} \right\}.$$

If

$$K_{r_1,r_2,\ldots,r_t} \in \mathcal{M}(G),$$

then G is said to contain $K_{r_1,r_2,\ldots,r_t}$ as an MCM-subgraph.

When classify all graphs in $\mathcal{W}$ according to their MCM-subgraphs, for a graph $G \in \mathcal{W}$; in the case that $|\mathcal{M}(G)| \geq 2$, the following convention will be adapted to choose one of MCM-subgraph $K_{r_1,r_2,\ldots,r_t}$ of the graph G.

Convention. *Let*

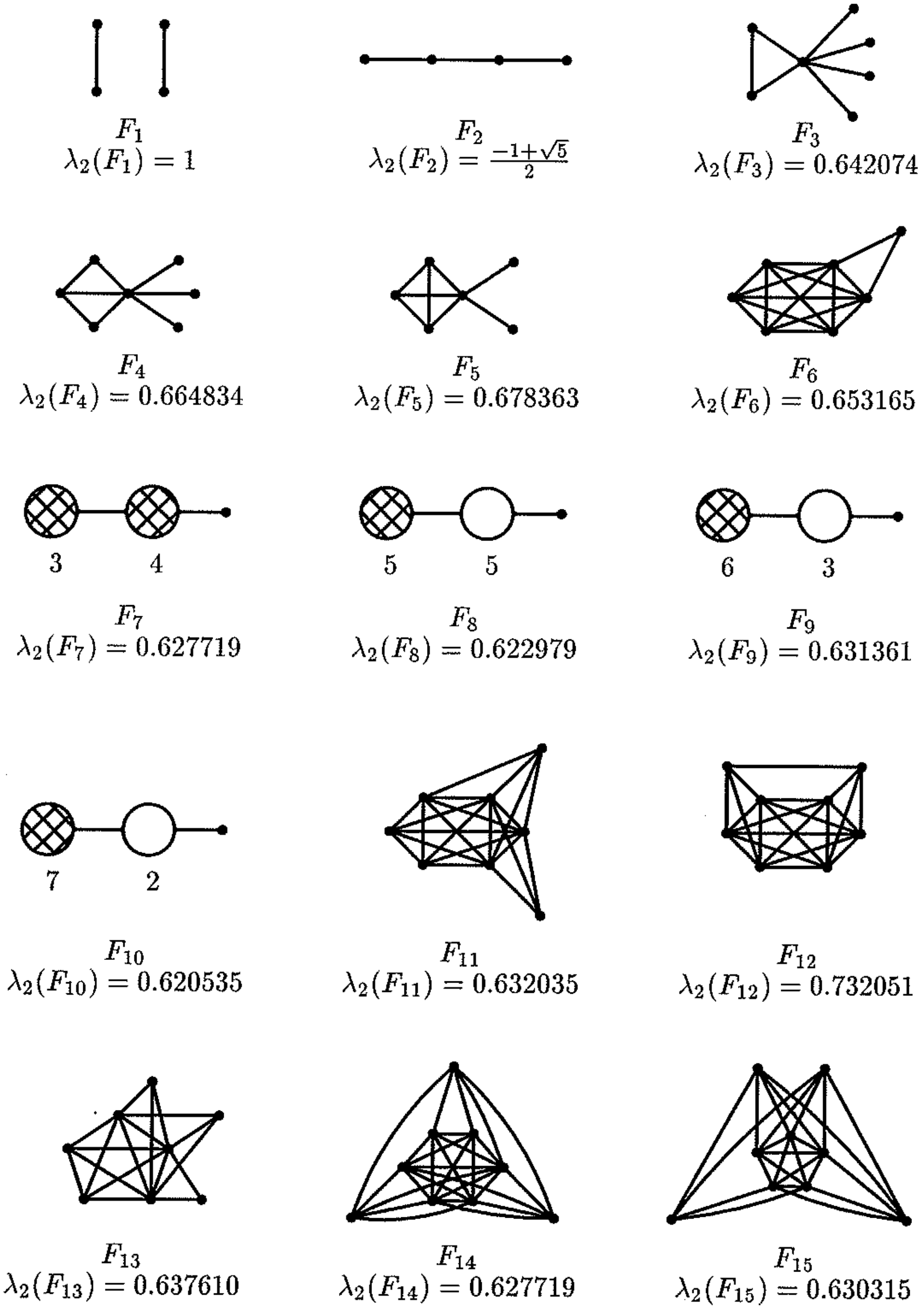
$$T_{\max}(G) = \max\{t \mid K_{r_1,r_2,\ldots,r_t} \in \mathcal{M}(G)\}.$$

An MCM-subgraph $K_{r_1,r_2,\ldots,r_t}$ of the graph G will be chosen at random among all

$$K_{r_1,r_2,\ldots,r_t} \in \mathcal{M}(G)$$
$$\text{such that } t = T_{\max}(G).$$

In this paper, the MCM-subgraph of a graph G (chosen according to the convention if there are more than one) is considered and the limit methods are used to characterize the graphs with $0 < \lambda_2 < \frac{-1+\sqrt{5}}{2}$ or graphs with $0 < \lambda_2 \leq -1+\sqrt{2}$. Once no graph with given bound for λ_2 is missing, the characterization will be complete. Therefore the convention will be adapted without any problem when classifying graphs according to its MCM-subgraphs.

The graphs in the *Figure 3* are minimal with respect to $\lambda_2 \geq \frac{-1+\sqrt{5}}{2}$. This is not an exhaustive list for such graphs, but it is sufficient for the discussion in this paper.



F_1
$\lambda_2(F_1) = 1$

F_2
$\lambda_2(F_2) = \frac{-1+\sqrt{5}}{2}$

F_3
$\lambda_2(F_3) = 0.642074$

F_4
$\lambda_2(F_4) = 0.664834$

F_5
$\lambda_2(F_5) = 0.678363$

F_6
$\lambda_2(F_6) = 0.653165$

F_7
$\lambda_2(F_7) = 0.627719$

F_8
$\lambda_2(F_8) = 0.622979$

F_9
$\lambda_2(F_9) = 0.631361$

F_{10}
$\lambda_2(F_{10}) = 0.620535$

F_{11}
$\lambda_2(F_{11}) = 0.632035$

F_{12}
$\lambda_2(F_{12}) = 0.732051$

F_{13}
$\lambda_2(F_{13}) = 0.637610$

F_{14}
$\lambda_2(F_{14}) = 0.627719$

F_{15}
$\lambda_2(F_{15}) = 0.630315$

Figure 3.

$$A(K_t; K_{l_1,l_2,\cdots,l_k}) =$$

$$
\begin{array}{c|cccc|cccc|c|cccc|cc|cc|c|cc|cccc}
 & \multicolumn{4}{c|}{\overbrace{u_{11}u_{12}\ldots u_{1l_1}}^{l_1}} & \multicolumn{4}{c|}{\overbrace{u_{21}u_{22}\ldots u_{2l_2}}^{l_2}} & \cdots & \multicolumn{4}{c|}{\overbrace{u_{k1}u_{k2}\ldots u_{kl_k}}^{l_k}} & \multicolumn{2}{c|}{\overbrace{v_1v_2}^{2}} & \multicolumn{2}{c|}{\overbrace{v_3v_4}^{2}} & \cdots & \multicolumn{2}{c|}{\overbrace{v_{2k-1}v_{2k}}^{2}} & \multicolumn{4}{c}{\overbrace{v_{2k+1}v_{2k+2}\ldots v_t}^{t-2k}} \\
u_{11} & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
u_{12} & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_{1l_1} & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
u_{21} & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
u_{22} & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_{2l_2} & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_{k1} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 \\
u_{k_2} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_{kl_k} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 \\
v_1 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
v_2 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
v_3 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
v_4 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 0 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{2k-1} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 1 & 1 & 1 & \ldots & 1 \\
v_{2k} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 1 & 0 & 1 & 1 & \ldots & 1 \\
v_{2k+1} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 1 & \ldots & 1 \\
v_{2k+2} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_t & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 0
\end{array}
$$

$$(t + \textstyle\sum_{i=1}^{k} l_i) \times (t + \sum_{i=1}^{k} l_i)$$

Table 5.

$$P_{K_t;K_{l_1,l_2,\cdots,l_k}}(\lambda) =$$

$$\det \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ k-1 \\ k \\ k+1 \\ k+2 \\ \vdots \\ 2k-1 \\ 2k \\ 2k+1 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & \dots & k-1 & k & k+1 & k+2 & \dots & 2k-1 & 2k & 2k+1 \end{array} \\ \left(\begin{array}{ccccccccccc} \lambda & -l_2 & \dots & -l_{k-1} & -l_k & 0 & -2 & \dots & -2 & -2 & -t+2k \\ -l_1 & \lambda & \dots & -l_{k-1} & -l_k & -2 & 0 & \dots & -2 & -2 & -t+2k \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -l_1 & -l_2 & \dots & \lambda & -l_k & -2 & -2 & \dots & 0 & -2 & -t+2k \\ -l_1 & -l_2 & \dots & -l_{k-1} & \lambda & -2 & -2 & \dots & -2 & 0 & -t+2k \\ 0 & -l_2 & \dots & -l_{k-1} & -l_k & \lambda-1 & -2 & \dots & -2 & -2 & -t+2k \\ -l_1 & 0 & \dots & -l_{k-1} & -l_k & -2 & \lambda-1 & \dots & -2 & -2 & -t+2k \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -l_1 & -l_2 & \dots & 0 & -l_k & -2 & -2 & \dots & \lambda-1 & -2 & -t+2k \\ -l_1 & -l_2 & \dots & -l_{k-1} & 0 & -2 & -2 & \dots & -2 & \lambda-1 & -t+2k \\ -l_1 & -l_2 & \dots & -l_{k-1} & -l_k & -2 & -2 & \dots & -2 & -2 & \lambda-t+2k+1 \end{array} \right)_{(2k+1)\times(2k+1)} \end{array}$$

$$= \left\{(\lambda-t+1)-(\lambda-1)^2\sum_{j=1}^{k}\frac{l_j}{\lambda^2+(l_j+1)\lambda-l_j}\right\}\prod_{i=1}^{k}[\lambda^2+(l_i+1)\lambda-l_i]$$

Table 6.

$$A(K_t; K_{l_1,l_2,\cdots,l_k} \cup K_1) =$$

$$\begin{array}{c|cccccccccccccccccccccccccc}
 & \multicolumn{4}{c}{\overbrace{u_{11}\,u_{12}\ldots u_{1l_1}}^{l_1}} & \multicolumn{4}{c}{\overbrace{u_{21}\,u_{22}\ldots u_{2l_2}}^{l_2}} & \cdots & \multicolumn{4}{c}{\overbrace{u_{k1}\,u_{k2}\ldots u_{kl_k}}^{l_k}} & \multicolumn{2}{c}{\overbrace{v_1\,v_2}^{2}} & \multicolumn{2}{c}{\overbrace{v_3\,v_4}^{2}} & \cdots & \multicolumn{2}{c}{\overbrace{v_{2k-1}\,v_{2k}}^{2}} & \multicolumn{4}{c}{\overbrace{v_{2k+1}\,v_{2k+2}\ldots v_{t-1}}^{t-1-2k}} & v_t & u \\
\hline
u_{11} & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
u_{12} & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{1l_1} & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
u_{21} & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
u_{22} & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{2l_2} & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{k1} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 0 \\
u_{k_2} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
u_{kl_k} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 & 1 & 0 \\
v_1 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
v_2 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
v_3 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
v_4 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 0 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
v_{2k-1} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 \\
v_{2k} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 1 & 0 & 1 & 1 & \ldots & 1 & 1 & 0 \\
v_{2k+1} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 1 & \ldots & 1 & 1 & 0 \\
v_{2k+2} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 0 & \ldots & 1 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
v_{t-1} & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 0 & 1 & 0 \\
v_t & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 & \ldots & 1 & 0 & 1 \\
u & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0
\end{array}$$

$$(t + \textstyle\sum_{i=1}^{k} l_i) \times (t + \sum_{i=1}^{k} l_i)$$

Table 7.

$$P_{K_t;K_{l_1,l_2,\ldots,l_k}\cup K_1}(\lambda) =$$

$$\det \begin{array}{c} \\ \end{array}\left(\begin{array}{c|ccccccccccccc}
 & 1 & 2 & \ldots & k-1 & k & k+1 & k+2 & \ldots & 2k-1 & 2k & 2k+1 & 2k+2 & 2k+3 \\
\hline
1 & \lambda & -l_2 & \ldots & -l_{k-1} & -l_k & 0 & -2 & \ldots & -2 & -2 & -t+2k+1 & -1 & 0 \\
2 & -l_1 & \lambda & \ldots & -l_{k-1} & -l_k & -2 & 0 & \ldots & -2 & -2 & -t+2k+1 & -1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
k-1 & -l_1 & -l_2 & \ldots & \lambda & -l_k & -2 & -2 & \ldots & 0 & -2 & -t+2k+1 & -1 & 0 \\
k & -l_1 & -l_2 & \ldots & -l_{k-1} & \lambda & -2 & -2 & \ldots & -2 & 0 & -t+2k+1 & -1 & 0 \\
k+1 & 0 & -l_2 & \ldots & -l_{k-1} & -l_k & \lambda-1 & -2 & \ldots & -2 & -2 & -t+2k+1 & -1 & 0 \\
k+2 & -l_1 & 0 & \ldots & -l_{k-1} & -l_k & -2 & \lambda-1 & \ldots & -2 & -2 & -t+2k+1 & -1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2k-1 & -l_1 & -l_2 & \ldots & 0 & -l_k & -2 & -2 & \ldots & \lambda-1 & -2 & -t+2k+1 & -1 & 0 \\
2k & -l_1 & -l_2 & \ldots & -l_{k-1} & 0 & -2 & -2 & \ldots & -2 & \lambda-1 & -t+2k+1 & -1 & 0 \\
2k+1 & -l_1 & -l_2 & \ldots & -l_{k-1} & -l_k & -2 & -2 & \ldots & -2 & -2 & \lambda-t+2k+2 & -1 & 0 \\
2k+2 & -l_1 & -l_2 & \ldots & -l_{k-1} & -l_k & -2 & -2 & \ldots & -2 & -2 & -t+2k+1 & \lambda & -1 \\
2k+3 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -1 & \lambda
\end{array}\right)_{(2k+3)\times(2k+3)}$$

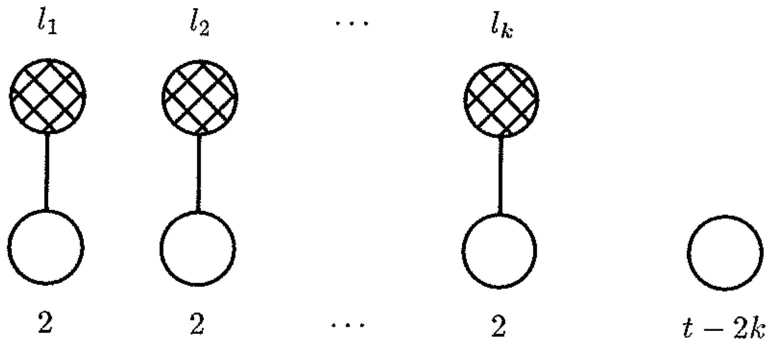
$$= \left\{[\lambda^3-(t-2)\lambda^2-t\lambda+t-2]-(\lambda-1)^2(\lambda^2+\lambda-1)\sum_{j=1}^{k}\frac{l_j}{\lambda^2+(l_j+1)\lambda-l_j}\right\}\prod_{i=1}^{k}[\lambda^2+(l_i+1)\lambda-l_i]$$

Table 8.

Definition. *For integers t, k, $l_1, l_2, \ldots, l_k$, where $t \geq 2k$, $l_i \geq 1$, $i = 1, 2, \ldots, k$, let $K_t; K_{l_1,l_2,\ldots,l_k}$ denote the graph*

$$\big[(K_2 \cup l_1K_1)\nabla(K_2 \cup l_2K_1)\nabla \cdots \nabla(K_2 \cup l_kK_1)\big]\nabla K_{t-2k}.$$

The following figure gives the complement graph of the graph $K_t; K_{l_1,l_2,\ldots,l_k}$.



$\overline{K_t; K_{l_1,l_2,\ldots,l_k}}$

The adjacency matrix $A(K_t; K_{l_1,l_2,\ldots,l_k})$ of the graph $K_t; K_{l_1,l_2,\ldots}$ is given in *Table 5.*

Then by Theorem 2.2, all the eigenvalues of $K_t; K_{l_1,l_2,\ldots,l_k}$ other than -1 or 0 can be found by solving

$$P_{K_t;K_{l_1,l_2,\ldots,l_k}}(\lambda)$$

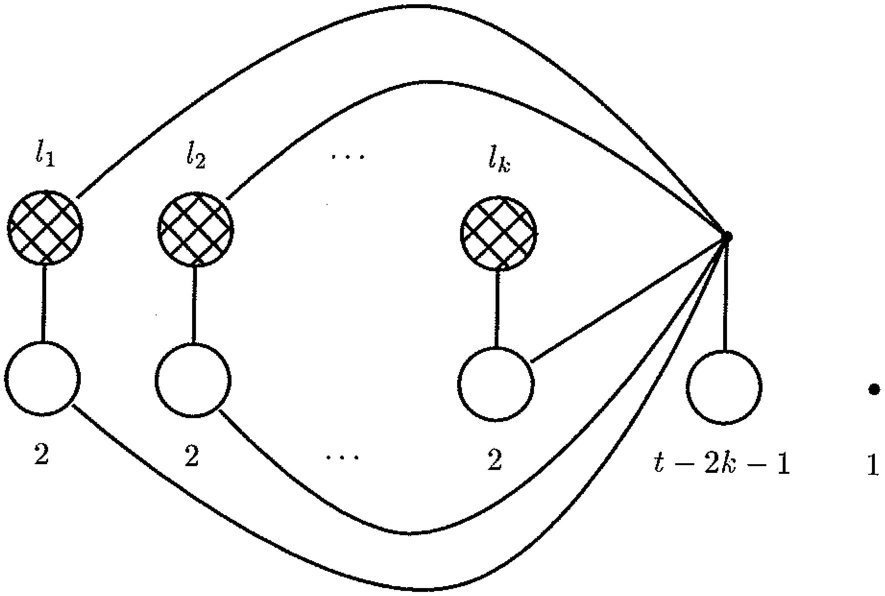
$$= \left\{(\lambda - t + 1) - \sum_{j=1}^{k} \frac{l_j(\lambda-1)^2}{\lambda^2 + (l_j+1)\lambda - l_j}\right\} \cdot \prod_{i=1}^{k}[\lambda^2 + (l_i+1)\lambda - l_i] = 0, \tag{3.13}$$

where $P_{K_t;K_{l_1,l_2,\ldots,l_k}}(\lambda)$ is given by the determinant in the *Table 6.*

Definition. *For integers* $t,\ k,\ l_1, l_2, \ldots, l_k$, *where* $t \geq 2k+1, l_i \geq 1,\ i = 1, 2, \ldots, k$, *let* $K_t; K_{l_1,l_2,\ldots,l_k} \cup K_1$ *denote the graph*

$$\left\{\left\{[(K_2 \cup l_1K_1)\nabla(K_2 \cup l_2K_1)\nabla\cdots\nabla(K_2 \cup l_kK_1)]\nabla K_{t-2k-1}\right\} \cup K_1\right\}\nabla K_1.$$

The following figure gives the complement graph of the graph $K_t; K_{l_1,l_2,\ldots,l_k} \cup K_1$.



$\overline{K_t; K_{l_1,l_2,\ldots,l_k} \cup K_1}$

The adjacency matrix $A(K_t; K_{l_1,l_2,\ldots,l_k} \cup K_1)$ of the graph $K_t; K_{l_1,l_2,\ldots} \cup K_1$ is given in the *Table 7.*

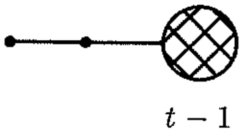
Then by Theorem 2.2, all the eigenvalues of $K_t; K_{l_1,l_2,\ldots,l_k} \cup K_1$ other than -1 or 0 can be found by solving

$$P_{K_t;K_{l_1,l_2,\ldots,l_k}\cup K_1}(\lambda)$$

$$= \left\{[\lambda^3 - (t-2)\lambda^2 - t\lambda + t - 2] - \sum_{j=1}^{k} \frac{l_j(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+(l_j+1)\lambda-l_j}\right\}$$

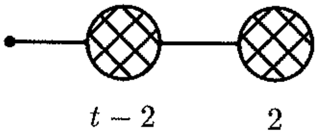
$$\cdot\prod_{i=1}^{k}[\lambda^2+(l_i+1)\lambda-l_i] = 0, \tag{3.14}$$

where $P_{K_t;K_{l_1,l_2,\ldots,l_k}\cup K_1}(\lambda)$ is given by the determinant in *Table 9.*



$A_{1,t-1},\ t \geq 3$

$\lim_{t\to\infty} \lambda_2(A_{1,t-1}) = (-1+\sqrt{5})/2$



$A_{t-2,2},\ t \geq 3$

$\lim_{t\to\infty} \lambda_2(A_{t-2,2}) = -1+\sqrt{2}$

$A_{2,3}$

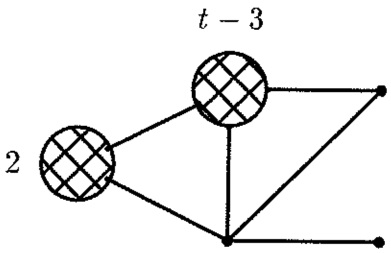
$\lambda_2(A_{2,3}) = 0.545096$

$A_{3,3}$

$\lambda_2(A_{3,3}) = 0.597951$

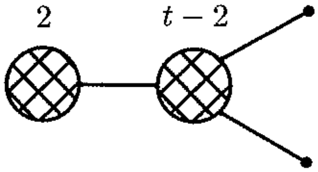
$G|_{V(G)-V(K_t)} = K_1$

Figure 4



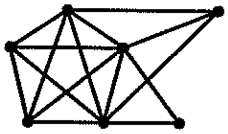
$K_t; K_1 \cup K_1,\ t \geq 3$

$\lim_{t\to\infty} \lambda_2(K_t; K_1 \cup K_1) = (-1 + \sqrt{5})/2$



$K_t; K_2,\ t \geq 3$

$\lim_{t\to\infty} \lambda_2(K_t; K_2) = (-3 + \sqrt{17})/2$

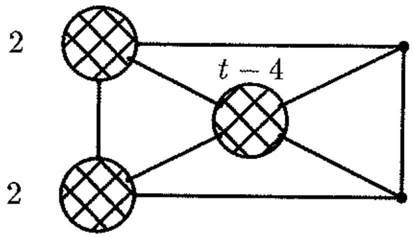


H_1

$\lambda_2(H_1) = 0.586846$

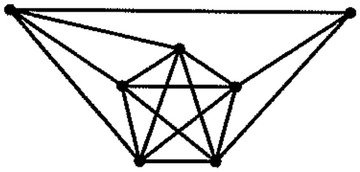
$G|_{V(G)-V(K_t)} = \bar{K}_2$

Figure 5



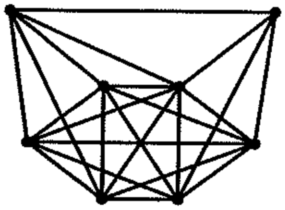
$K_t; K_{1,1},\ t \geq 4$

$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1}) = -1 + \sqrt{2}$



H_2

$\lambda_2(H_2) = 0.570274$

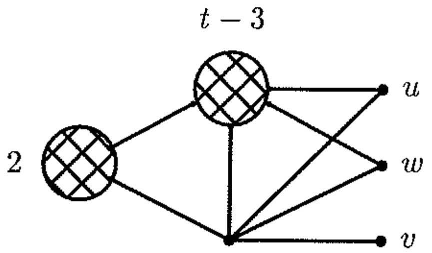


H_3

$\lambda_2(H_3) = 0.607332$

$G|_{V(G)-V(K_t)} = K_2$

Figure 6



$K_t; K_2 \cup K_1,\ t \geq 3$

$$\lim_{t\to\infty} \lambda_2(K_t; K_2 \cup K_1) = (-1+\sqrt{5})/2$$

$G|_{V(G)-V(K_t)} = \bar{K}_3 \cong$

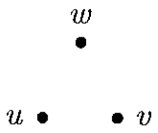
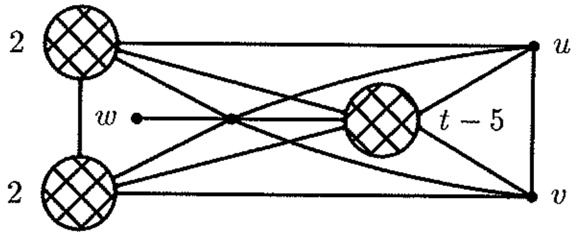


Figure 7



$K_t; K_{1,1} \cup K_1,\ t \geq 5$

$$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1} \cup K_1) = (-1+\sqrt{5})/2$$

$G|_{V(G)-V(K_t)} = K_2 \cup K_1 \cong$

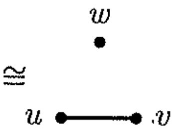
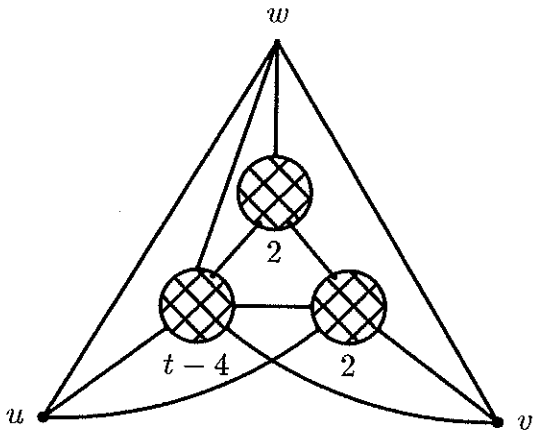
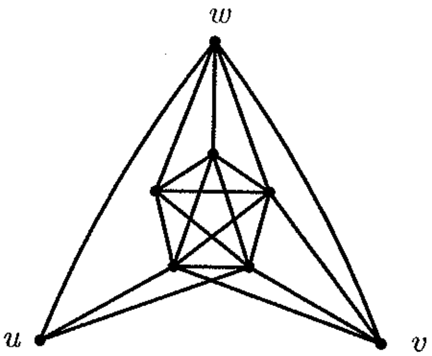


Figure 8.



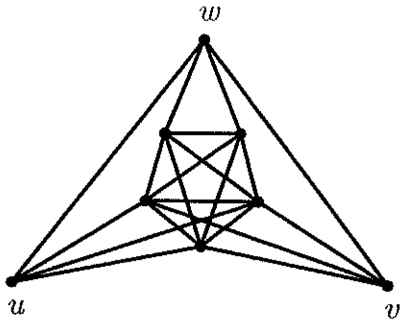
$K_t; K_{2,1},\ t \geq 4$

$\lim_{t\to\infty} \lambda_2(K_t; K_{2,1}) = (-3+\sqrt{17})/2$



H_4

$\lambda_2(H_4) = 0.602501$



H_5

$\lambda_2(H_5) = 0.612279$

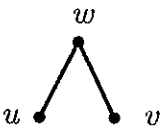
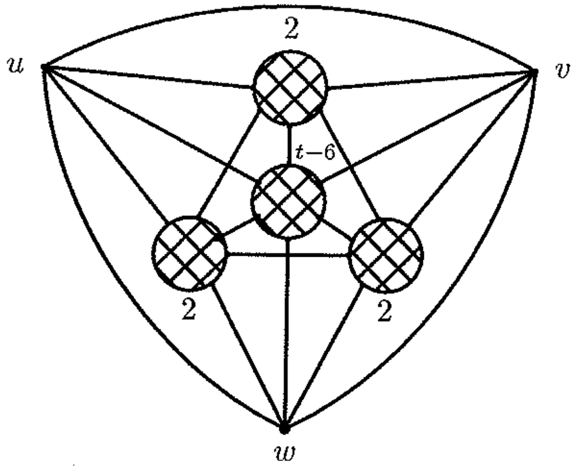
$G|_{V(G)-V(K_t)} = K_{2,1} \cong$ 

Figure 9



$K_t; K_{1,1,1},\ t \geq 6$

$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1,1}) = -1 + \sqrt{2}$

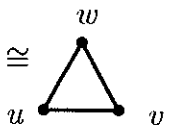
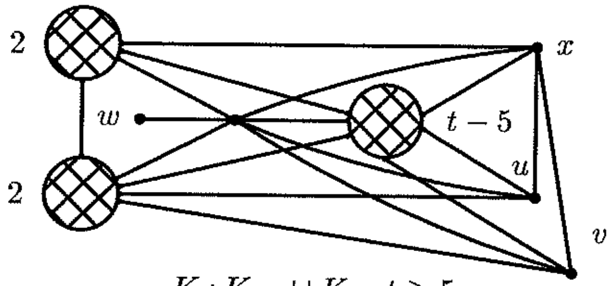
$G|_{V(G)-V(K_t)} = 1,1,1 = K_3 \cong$ 

Figure 10



$K_t; K_{2,1} \cup K_1,\ t \geq 5$

$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1} \cup K_1) = (-1 + \sqrt{5})/2$

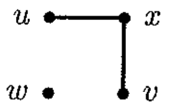
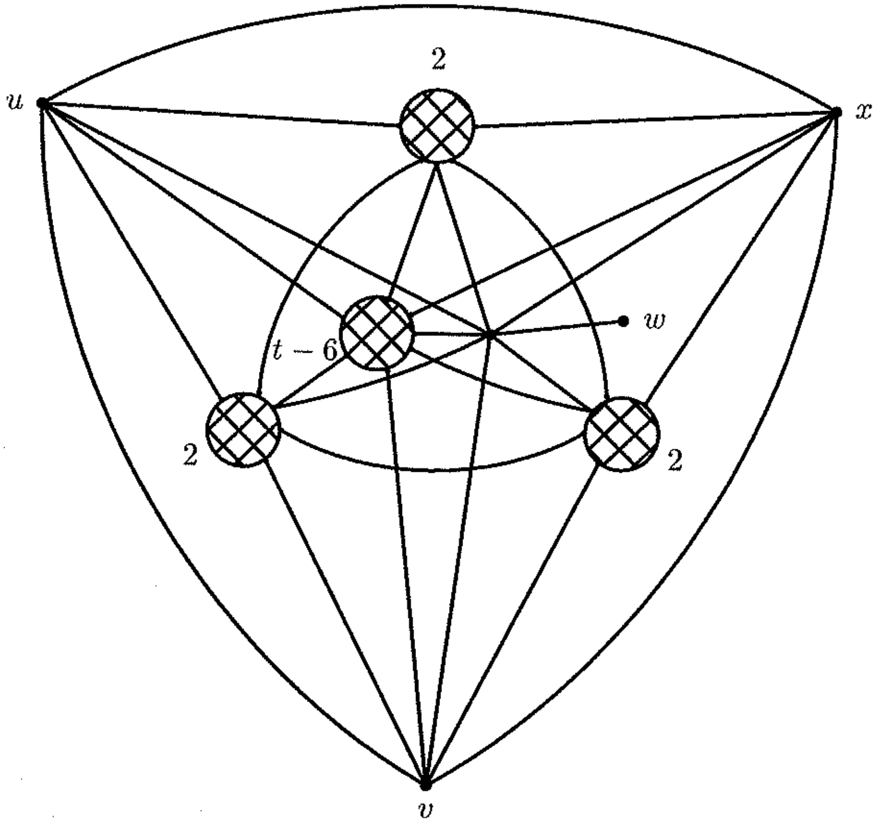
$G|_{V(G)-V(K_t)} = K_{2,1} \cup K_1 \cong$ 

Figure 11.



$K_t; K_{1,1,1} \cup K_1,\ t \geq 6$

$\lim_{t \to \infty} \lambda_2(K_t; K_{1,1,1} \cup K_1) = (\sqrt{5} - 1)/2$

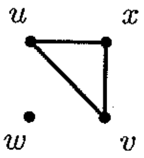
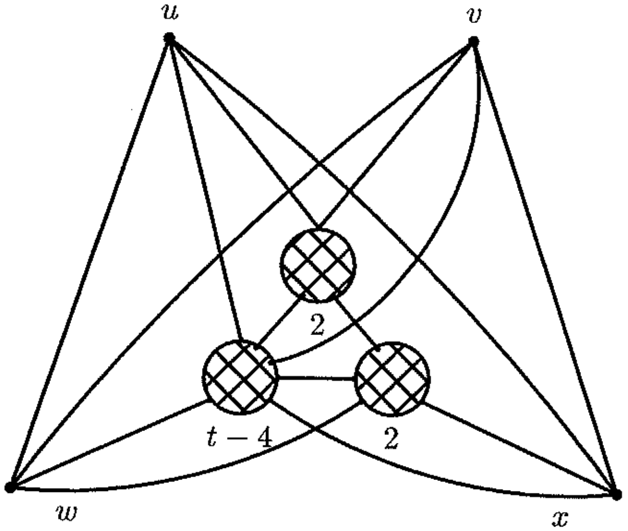
$G|_{V(G)-V(K_t)} = K_{1,1,1} \cup K_1 = K_3 \cup K_1 \cong$ 

Figure 12



$K_t; K_{2,2},\ t \geq 4$

$\lim_{t\to\infty} \lambda_2(K_t; K_{2,2}) = (-3+\sqrt{17})/2$

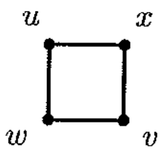
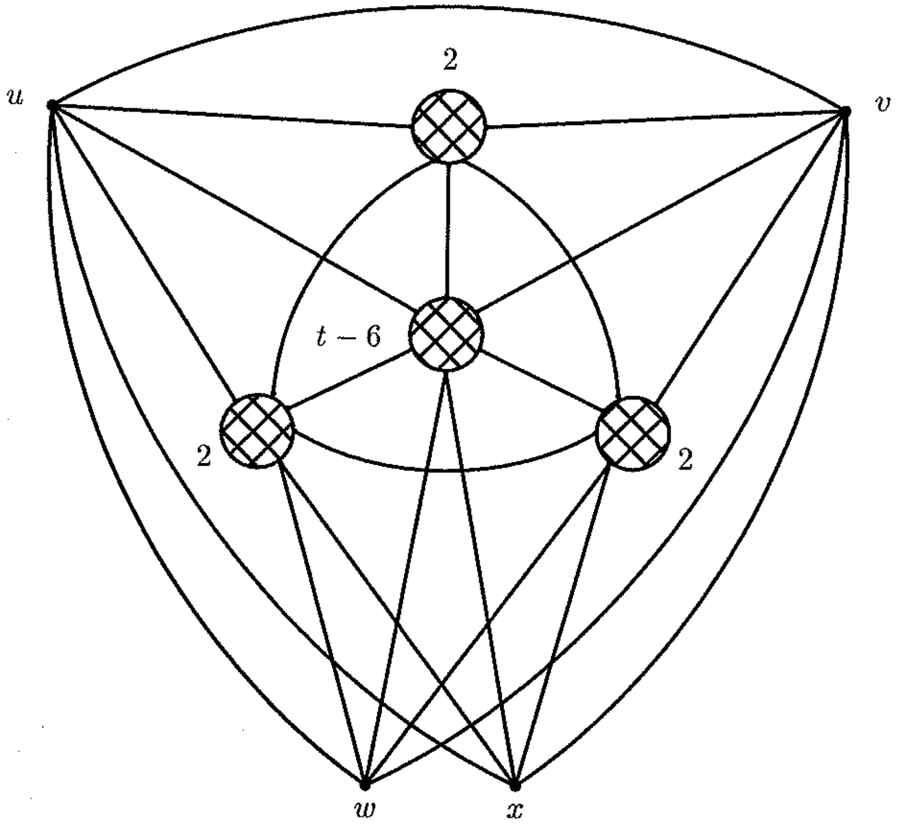
$G|_{V(G)-V(K_t)} = K_{2,2} \cong$ 

Figure 13.



$$K_t; K_{2,1,1}, \quad t \geq 6$$

$$\lim_{t \to \infty} \lambda_2(K_t; K_{2,1,1}) = (-3 + \sqrt{17})/2$$

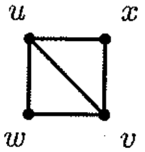
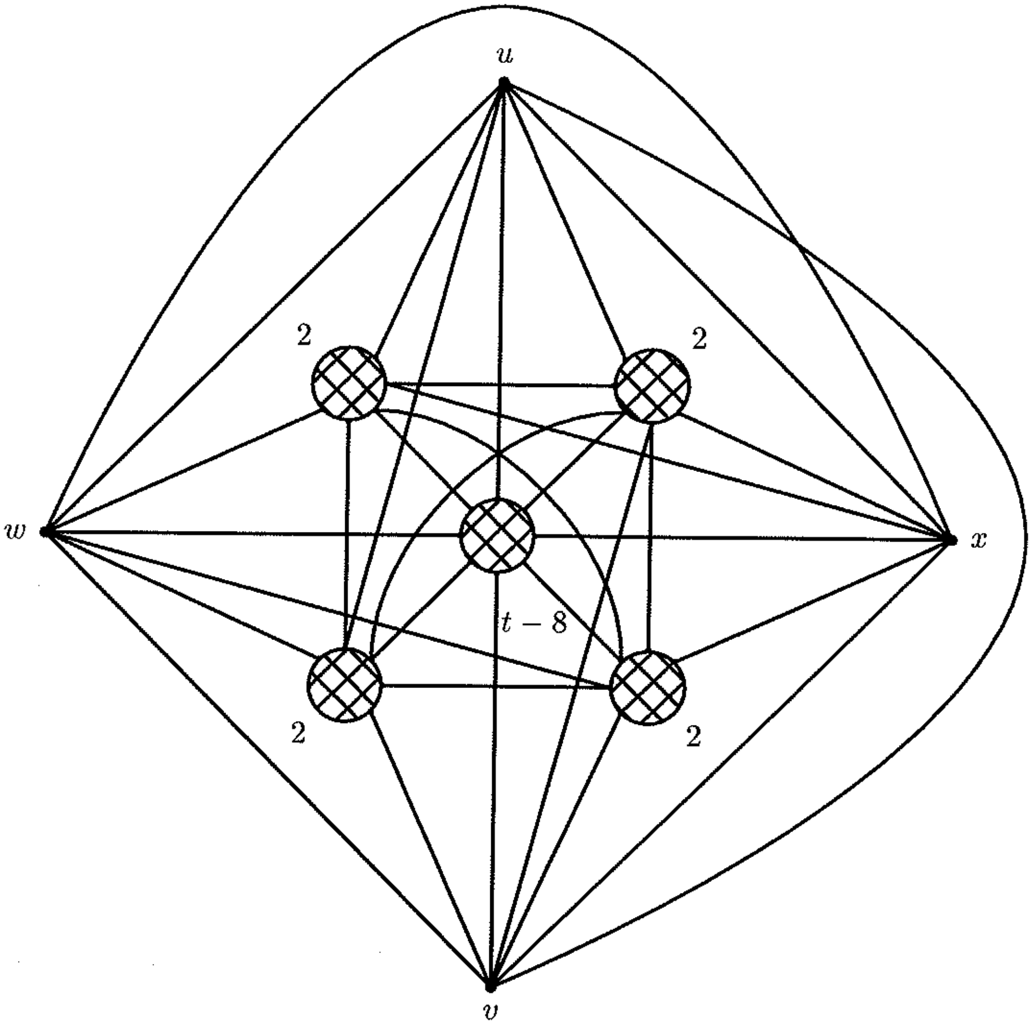
$G|_{V(G)-V(K_t)} = K_{2,1,1} \cong$ 

Figure 14



$K_t; K_{1,1,1,1}, \quad t \geq 8$

$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1,1,1}) = -1 + \sqrt{2}$

$G|_{V(G)-V(K_t)} = K_{1,1,1,1} = K_4 \cong$

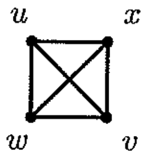
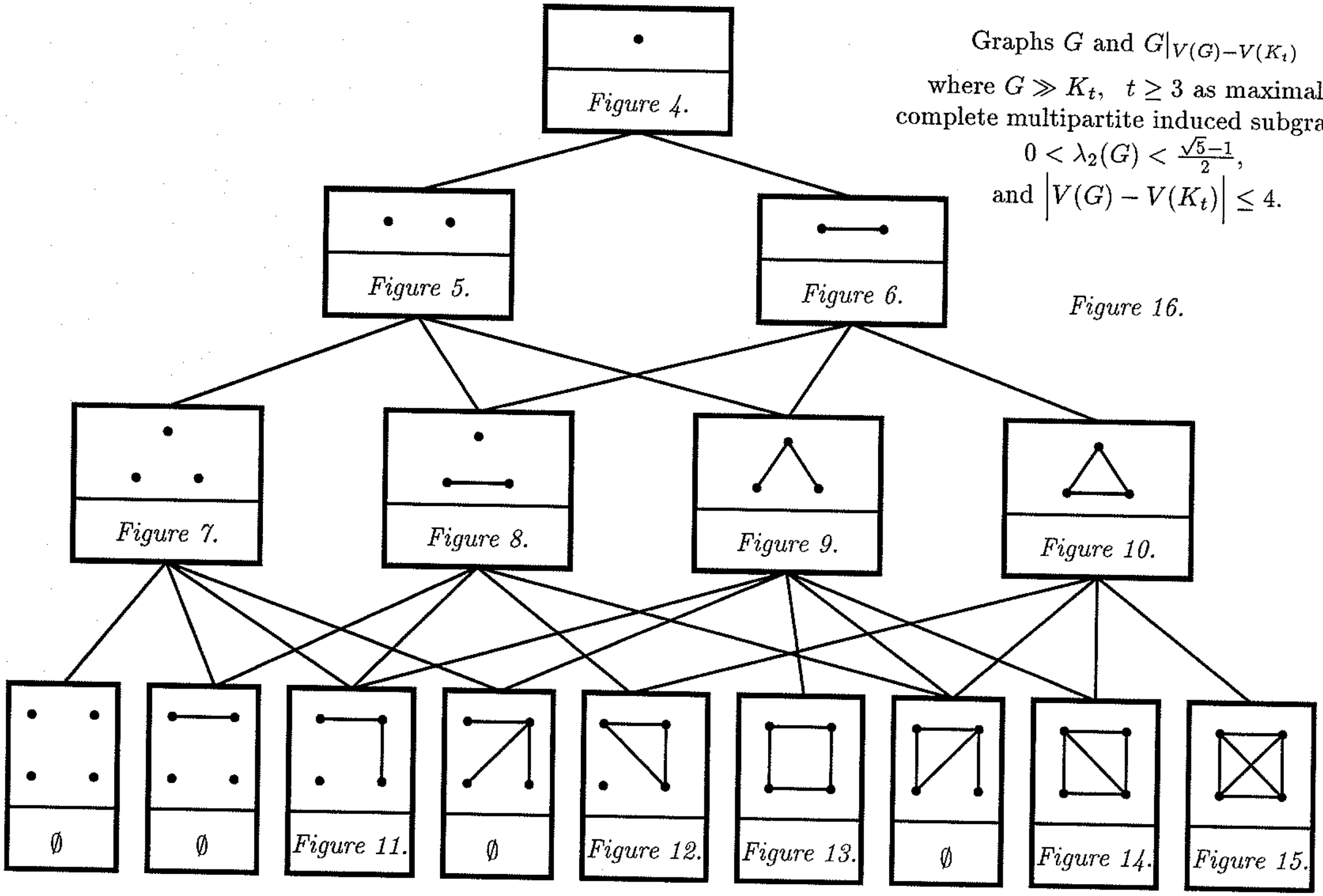


Figure 15



Graphs G and $G|_{V(G)-V(K_t)}$
where $G \gg K_t$, $t \geq 3$ as maximal
complete multipartite induced subgraph,
$0 < \lambda_2(G) < \frac{\sqrt{5}-1}{2}$,
and $\left|V(G) - V(K_t)\right| \leq 4$.

Figure 16.

Definition. *For $u \in V(G)$, define $N(u) = \{w \mid w \in V(G), w \sim u\}$.*

Lemma 3.5. *Let G be a simple graph without isolated vertices, and suppose $G \gg K_t$, as the MCM-subgraph. If $0 < \lambda_2(G) < (-1+\sqrt{5})/2$, then G satisfies the following conditions:*

(I) Each vertex $v \in V(G) - V(K_t)$ is adjacent to exactly q vertices in $V(K_t)$, where q and t must be one of the following:

1.1) $q = 1$, and $t \geq 3$;
1.2) $q = 2$, and $t = 5$;
1.3) $q = 3$, and $t = 6$;
1.4) $q = t-2$, and $t \geq 4$.

So

$$1 \leq q = \left|N(v) \cap V(K_t)\right| \leq t-2. \tag{3.15}$$

In particular, if $G|_{V(G)-V(K_t)} = K_1$, then G is one of the graphs in Figure 4.

(II) Suppose $u \in V(G) - V(K_t)$ and $v \in V(G) - V(K_t)$.

2.1) If $u \not\sim v$, then

either
$$\big(N(u) \cap V(K_t)\big) \subseteq \big(N(v) \cap V(K_t)\big), \tag{3.16}$$
or
$$\big(N(v) \cap V(K_t)\big) \subseteq \big(N(u) \cap V(K_t)\big). \tag{3.17}$$

Without lost of generality, suppose (3.17) holds, then

$$\left|N(u) \cap V(K_t)\right| = t-2. \tag{3.18}$$

In particular, if $G|_{V(G)-V(K_t)} = \bar{K}_2$, then G is one of the graphs in Figure 5.

2.2) If $u \sim v$, then

$$\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) = V(K_t). \tag{3.19}$$

In particular, if $G|_{V(G)-V(K_t)} = K_2$, then G is one of the graphs in *Figure 6.*

(III) If $\left|G|_{V(G)-V(K_t)}\right| = 3$, then G is one of the graphs in *Figure 7, Figure 8, Figure 9* or *Figure 10.*

(VI) If $\left|G|_{V(G)-V(K_t)}\right| = 4$, then $G|_{V(G)-V(K_t)}$ is one of the graphs $K_{2,1} \cup K_1$, $K_3 \cup K_1$, $K_{2,2}$, $K_{2,1,1}$ and K_4. and G is one of the graphs in *Figure 11, Figure 12, Figure13, Figure 14*, or *Figure 15.*

(V) Each component of $G|_{V(G)-V(K_t)}$ is a complete multipartite graph and only one component of $G|_{V(G)-V(K_t)}$ contains edges.

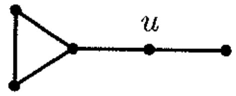
Proof. For any graph G with $\lambda_2 > 0$, G is not a complete multipartite graph. Then G must contains one of graphs $2K_2 = F_1$, $P_4 = F_2$ and $A_{1,2}$ (see [4], Theorem 6.7). So if $0 < \lambda_2 < (-1+\sqrt{5})/2$, then $G \gg A_{1,2}$ as an induced subgraph. Therefore if $G \gg K_t$ as an MCM-subgraph, then $t \geq 3$.

If G is a graph without isolated vertices and $0 < \lambda_2(G) < (-1+\sqrt{5})/2$, then G is connected. Otherwise $G \gg 2K_2 = F_1$, $\lambda_2(2K_2) = 1$ a contradiction.

For (I), let $v \in V(G) - V(K_t)$.

If v is not adjacent to any vertices in K_t, then there is a vertex u and a path which joins u and v, where u is adjacent to q vertices in $V(K_t)$, with $1 \leq q \leq t-2$ and $t \geq 3$. Then

$G \gg G_1 \cong$



$\gg P_4 = F_2$,

a contradiction.

If v is adjacent to exactly q vertices, $1 \leq q \leq t$, in $V(K_t)$, then $1 \leq q \leq t-2$. Otherwise if $q = t$, then

$$G \gg K_{t+1}$$

or if $q = t-1$ then

$$G \gg K_{\underbrace{1,1,\ldots,1,}_{t-1 \text{ times}}\, 2},$$

which contradicts that K_t is an MCM-subgraph.

If

$$G|_{V(G)-V(K_t)} = K_1,$$

then

$$G|_{\{v\}\cup K_t}$$

is of the form $A_{q,t-q}$ with v adjacent to q vertices in $V(K_t)$. By Lemma 3.1, q must lie in one of the cases 1.1), 1.2), 1.3) and 1.4). So G is one of the graphs in *Figure 4*.

For (II), let u and v be a pair of vertices belonging to $V(G) - V(K_t)$.

Proof 2.1). If $u \not\sim v$, and neither

$$\big(N(u) \cap V(K_t)\big) \subseteq \big(N(v) \cap V(K_t)\big),$$

nor

$$\big(N(v) \cap V(K_t)\big) \subseteq \big(N(u) \cap V(K_t)\big),$$

then take

$$u' \in \big(N(u) \cap V(K_t)\big) - \big(N(v) \cap V(K_t)\big),$$

and

$$v' \in \big(N(v) \cap V(K_t)\big) - \big(N(u) \cap V(K_t)\big).$$

Then $G \gg P_4 = G|_{\{u,u',v',v\}} = F_2$, a contradiction.

Without loss of generality, suppose (3.17) holds, i.e.

$$\big(N(v) \cap V(K_t)\big) \subseteq \big(N(u) \cap V(K_t)\big).$$

By Part (I), (3.15),

$$1 \leq \big|N(u) \cap V(K_t)\big| \leq t-2.$$

Suppose

$$1 \leq \big|N(u) \cap V(K_t)\big| < t-2. \tag{3.20}$$

For t=3, (3.20) implies

$$1 \leq \big|N(u) \cap V(K_t)\big| < 3-2 = 1,$$

a contradiction. For $t \geq 4$, let

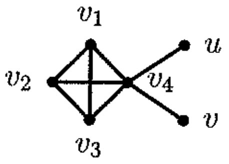
$$\{v_1, v_2, v_3\} \subseteq \big(K_t - N(u)\big)$$

and

$$v_4 \in \big(N(v) \cap V(K_t)\big) \subseteq \big(N(u) \cap V(K_t)\big).$$

Then

$$G \gg F_5 = G_{\{v_1,v_2,v_3,v_4,u,v\}} \cong$$



,

which is a contradiction. Therefore

$$\big|N(u) \cap V(K_t)\big| = t-2,$$

and so (3.18) holds.

If

$$G|_{V(G)-V(K_t)} = \bar{K}_2,$$

without loss of generality, suppose (3.15), (3.17) and (3.18) hold. i.e.

$$\big|N(v) \cap V(K_t)\big| = q,\ 1 \leq q \leq t-2$$

$$\big(N(v) \cap V(K_t)\big) \subseteq \big(N(u) \cap V(K_t)\big),$$

and

$$\big|N(u) \cap V(K_t)\big| = t-2.$$

Here as a pairs (q, t) must correspond to one of the cases in Part (I), 1.1), 1.2), 1.3) and 1.4).

Then G is of the form

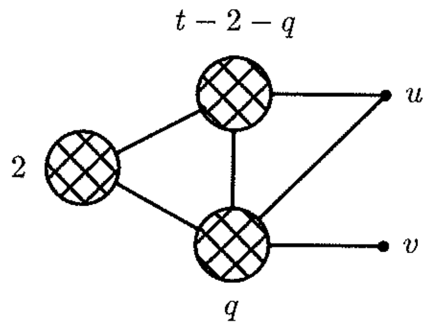


Figure 17.

If 1.1) holds with $q = 1$, and $t \geq 3$, then (3.15), (3.17) and (3.18) imply that,

$$G = K_t; K_1 \cup K_1, \ t \geq 3$$

as in *Figure 5.* Then in (3.14), let $k = 1, l_1 = 1$. The eigenvalues of $G = K_t; K_1 \cup K_1$ other than -1 and 0 must be roots of

$$\begin{aligned} & P_{K_t;K_1 \cup K_1}(\lambda) \\ &= \left\{ [\lambda^3 - (t-2)\lambda^2 - t\lambda + t - 2] - \frac{(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+2\lambda-1} \right\} \cdot [\lambda^2 + 2\lambda - 1] \\ &= \lambda^5 - (t-3)\lambda^4 - (3t-4)\lambda^3 - 2\lambda^2 + (3t-7)\lambda - t + 3 \\ &= (\lambda^2+\lambda-1)[\lambda^3 + (-t+2)\lambda^2 + (-2t+3)\lambda + t - 3] - \lambda = 0. \end{aligned} \tag{3.21}$$

Apply Theorem 2.4, Part (II) with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_1 \cup K_1), \ t \geq 3$$

is monotone increasing and bounded from above. So

$$\lim_{t \to \infty} \lambda_2(K_t; K_1 \cup K_1)$$

exists and is the largest root of

$$-\lambda^4 - 3\lambda^3 + 3\lambda - 1 = -(\lambda^2+\lambda-1)(\lambda^2+2\lambda-1) = 0.$$

Then

$$\lim_{t \to \infty} \lambda_2(K_t; K_1 \cup K_1) = (-1+\sqrt{5})/2.$$

From (3.21)

$$\lambda_2(K_t; K_1 \cup K_1) \neq (-1+\sqrt{5})/2, \ t \geq 3.$$

Therefore

$$0 < \lambda_2(G) = \lambda_2(K_t; K_1 \cup K_1) < (-1+\sqrt{5})/2, \ t \geq 3.$$

If 1.2) holds, then $q = 2$, and $t = 5$. Because $G - u$ must be one of the graphs in *Figure 4*, $G - u = A_{2,3}$. Then (3.15), (3.17) and (3.18) imply that $G = H_1$ as in the *Figure 5.* with

$$\lambda_2(H_1) = 0.586846 < (-1+\sqrt{5})/2.$$

If 1.3) holds, then $q = 3$, and $t = 6$. Because $G - u$ must be one of the graphs in *Figure 4*, $G - u = A_{3,3}$, Then (3.15), (3.17) and (3.18) imply that $G = F_{11}$, a contradiction.

If 1.4) holds with $q = t - 2$, and $t \geq 4$, then (3.15), (3.17) and (3.18) imply that $G = K_t; K_2$, $t \geq 4$ as in the *Figure 5.* Then in (3.13), let $k = 1, l_1 = 2$. The eigenvalues of $G = K_t; K_2$ other than -1 and 0 must be roots of

$$\begin{aligned} &P_{K_t;K_2}(\lambda) \\ &= \left\{(\lambda - t + 1) - \frac{2(\lambda-1)^2}{\lambda^2+3\lambda-2}\right\} \cdot \left[\lambda^2+3\lambda-2\right] \\ &= \lambda^3 - (t-2)\lambda^2 - (3t-5)\lambda + 2t - 4 = 0. \end{aligned}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_2), \ t \geq 3$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty} \lambda_2(K_t; K_2)$$

exists and is the largest root of

$$\lambda^2 + 3\lambda - 2 = 0.$$

Then

$$\lim_{t\to\infty} \lambda_2(K_t; K_2) = (-3+\sqrt{17})/2 = 0.561553.$$

Therefore

$$0 < \lambda_2(G) = \lambda_2(K_t; K_1 \cup K_1) \leq (-3+\sqrt{17})/2 < (-1+\sqrt{5})/2, \ t \geq 3.$$

Proof 2.2). If $u \sim v$, but (3.19) is not true, then

$$\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) \neq V(K_t),$$

so

$$\big|\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big)\big| \le t - 1.$$

If

$$\big|\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big)\big| \le t - 2,$$

take

$$x, y \in V(K_t) - \big[N(u) \cup N(v)\big].$$

Then

$$G|_{\{u,v,x,y\}} = 2K_2 = F_1,$$

a contradiction.

If

$$\big|\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big)\big| = t - 1. \tag{3.22}$$

let

$$\{w\} = V(K_t) - \big[N(u) \cup N(v)\big].$$

Because of (3.15),

$$\big|N(v) \cap N(K_t)\big| \le t - 2.$$

Together with (3.22), this implies that there is a u', such that

$$u' \in \big(N(u) \cap N(K_t)\big) - \big(N(v) \cap N(K_t)\big).$$

Then

$$G|_{\{w,u',u,v\}} = P_4 = F_2,$$

a contradiction.

In particular, if

$$G|_{V(G)-V(K_t)} = K_2,$$

let q_1, q_2 be integers such that

$$\big|N(v) \cap V(K_t)\big| = q_1, \quad \text{where } 1 \le q_1 \le t - 2,$$

and

$$\big|N(u) \cap V(K_t)\big| = q_2, \quad \text{where } 1 \le q_2 \le t - 2.$$

Without loss of generality, suppose $q_1 \le q_2$, where q_1 and q_2 are in one of the cases 1.1), 1.2), 1.3) and 1.4)

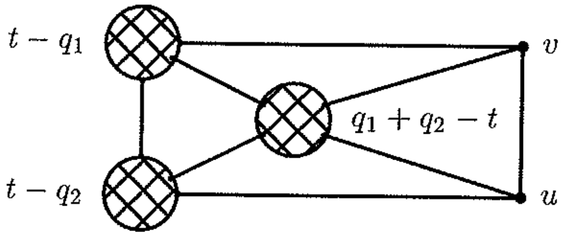


Figure 18.

If 1.1) holds with $q_1 = 1$ and $t \geq 3$, then

$$\begin{aligned}\left|\left(N(u)\cap N(K_t)\right)\cup\left(N(v)\cap N(K_t)\right)\right| \leq & \left|N(u)\cap N(K_t)\right| + \left|N(v)\cap N(K_t)\right| \\ = & q_1 + q_2 \leq 1 + t - 2 = t - 1,\end{aligned}$$

contradicting (3.19).

If 1.2) holds with $q_1 = 2$, and $t = 5$, then (3.15) and (3.19) imply that

$$q_2 = \left|N(u)\cap V(K_t)\right| = 3,$$

and $G = H_2$ as in *Figure 6*, with $\lambda_2(H_2) = 0.570274 < (-1+\sqrt{5})/2$.

If 1.3) holds with $q_1 = 3$, and $t = 6$, then (3.15) and (3.19) implies that

$$q_2 = \left|N(u)\cap V(K_t)\right| = 3 \text{ or } 4.$$

If

$$q_2 = \left|N(u)\cap V(K_t)\right| = 3,$$

then (3.15) and (3.19) imply that

$$G = F_{12}, \text{ with } \lambda_2(F_{12}) = 0.732051 > (-1+\sqrt{5})/2,$$

a contradiction.

If

$$q_2 = \left|N(u)\cap V(K_t)\right| = 4,$$

then (3.15) and (3.16) imply that $G = H_3$ as in *Figure 6*, with

$$\lambda_2(H_3) = 0.607332 < (-1+\sqrt{5})/2.$$

If 1.4) holds with $q_1 = t-2,\ t \geq 3$, then $q_1 = q_2 = t-2$, and (3.19) imply that

$$G = K_t; K_{1,1},\ t \geq 4$$

as in *Figure 6.* Then take $k = 2, l_1 = l_2 = 1$ in (3.13); all the eigenvalues of $G = K_t; K_{1,1}$ other than -1 or 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{1,1}}(\lambda) \\
&= \left\{(\lambda - t + 1) - 2\frac{(\lambda-1)^2}{\lambda^2+2\lambda-1}\right\} \cdot \left[\lambda^2 + 2\lambda - 1\right]^2 \\
&= \lambda^5 + (-t+3)\lambda^4 + (-4t+6)\lambda^3 + (-2t+6)\lambda^2 + (4t-11)\lambda - t + 3 \\
&= (\lambda^2 + 2\lambda - 1)\left[\lambda^3 + (-t+1)\lambda^2 + (-2t+5)\lambda + t - 3\right] = 0.
\end{aligned}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{1,1}),\ t \geq 4$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1})$$

exists and is the largest root of

$$\lambda^4 + 4\lambda^3 + 2\lambda^2 - 4\lambda + 1 = (\lambda^2 + 2\lambda - 1)^2 = 0.$$

Then

$$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1}) = -1 + \sqrt{2},$$

and

$$0 < \lambda_2(G) = \lambda_2(K_t; K_{1,1}) \leq -1 + \sqrt{2} < (-1 + \sqrt{5})/2,\ t \geq 4.$$

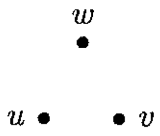
For (III), if

$$\left|G|_{V(G)-V(K_t)}\right| = 3,$$

let $\{u, v, w\} = V(G) - V(K_t)$. Then $G|_{V(G)-V(K_t)} = G|_{\{u,v,w\}}$ is one of the graphs from $\bar{K}_3, K_2 \cup K_1, K_{2,1}$ and K_3.

Case 1. If

$$G|_{V(G)-V(K_t)} = \bar{K}_3 \cong$$



then G must contain one of the graphs in *Figure 5* as an induced subgraph,

By Part(II), (3.17) and (3.18), without loss of generality, suppose

$$\big(N(v)\cap V(K_t)\big)\subseteq\big(N(w)\cap V(K_t)\big)\subseteq\big(N(u)\cap V(K_t)\big), \tag{3.23}$$

$$1\le\big|N(v)\cap V(K_t)\big|=q\le t-2, \tag{3.24}$$

and

$$\big|N(w)\cap V(K_t)\big|=\big|N(u)\cap V(K_t)\big|=t-2. \tag{3.25}$$

Because G must contain one of the graphs in *Figure 5* as an induced subgraph, only q in one of the cases 1.1), 1.2), and 1.4) need to be considered.

By (3.23), (3.24) and (3.25), G is of the form

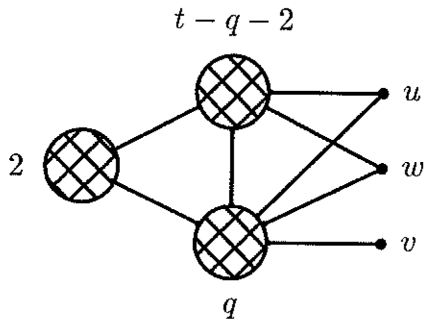


Figure 19

If 1.1) holds with $q=1$ and $t\ge 3$, then (3.23),(3.24) and (3.25) imply that

$$G=K_t;K_2\cup K_1,\ t\ge 3$$

as in the *Figure 7*.

In (3.14),take $k=1, l_1=2$.Then all the eigenvalues of

$$G=K_t;K_2\cup K_1,\ t\ge 3$$

other than -1 or 0 can be found by solving

$$\begin{aligned}&P_{K_t;K_2\cup K_1}(\lambda)\\&=\left\{[\lambda^3-(t-2)\lambda^2-t\lambda+t-2]-\frac{2(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+3\lambda-2}\right\}\cdot[\lambda^2+3\lambda-2]\\&=\lambda^5-(t-3)\lambda^4-(4t-6)\lambda^3-2\lambda^2+(5t-12)\lambda-(2t-6)\\&=(\lambda^2+\lambda-1)\big[\lambda^3+(-t+2)\lambda^2+(-3t+5)\lambda+(2t-5)\big]-2\lambda+1\\&=0.\end{aligned} \tag{3.26}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_2 \cup K_1),\ t \geq 3$$

is monotone increasing in t and bounded from above. Then

$$\lim_{t \to \infty} \lambda_2(K_t; K_2 \cup K_1)$$

exists and is the largest root of

$$\lambda^4 + 4\lambda^3 - 5\lambda + 2 = (\lambda^2 + 3\lambda - 2)(\lambda^2 + \lambda - 1) = 0.$$

Then

$$\lim_{t \to \infty} \lambda_2(K_t; K_2 \cup K_1) = (-1 + \sqrt{5})/2.$$

From (3.26),

$$0 < \lambda_2(G) = \lambda_2(K_t; K_2 \cup K_1) \neq (-1 + \sqrt{5})/2,\ t \geq 3.$$

So

$$0 < \lambda_2(G) = \lambda_2(K_t; K_2 \cup K_1) < (-1 + \sqrt{5})/2,\ t \geq 3.$$

If 1.2) holds with $q = 2$ and $t = 5$. Then (3.23),(3.24) and (3.25) imply that $G = F_{13}$, with $\lambda_2(F_{13}) = 0.637610 > (-1 + \sqrt{5})/2$, a contradiction.

If 1.4) holds with $q = t - 2$ and $t \geq 4$, as in *Figure 20.* Then (3.23),(3.24) and (3.25) imply that

$$G \gg K_{\underbrace{1,1,\ldots,1,}_{t-2 \text{ times}}\ 3}$$

as in *Figure 19*, with

$$\left|K_{\underbrace{1,1,\ldots,1,}_{t-2 \text{ times}}\ 3}\right| = t + 1 > t.$$

This contradicts the fact that K_t is the MCM-subgraph in G.

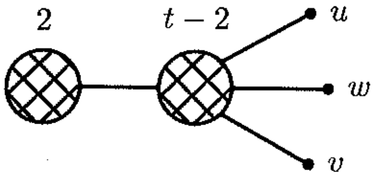
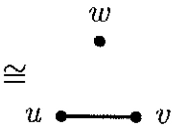


Figure 20

Case 2. If

$$G|_{V(G)-V(K_t)} = K_2 \cup K_1 \cong$$



then G contains one of the graphs in *Figure 5* and one of the graphs in *Figure 6* as an induced subgraph.

By (3.16), (3.17), (3.18),(3.19) and symmetry, only 3 subcases need to be considered.

Subcase 1. Suppose

$$\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) = V(K_t),$$

$$\big(N(w) \cap V(K_t)\big) \supseteq \big(N(v) \cap V(K_t)\big),$$

$$\big(N(w) \cap V(K_t)\big) \supseteq \big(N(u) \cap V(K_t)\big),$$

and

$$\big|N(w) \cap V(K_t)\big| = t - 2.$$

Then (3.19) implies that

$$\big(N(w) \cap V(K_t)\big) \supseteq \big[\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big)\big] = V(K_t).$$

But then

$$t - 2 = \big|N(w) \cap V(K_t)\big| \geq |V(K_t)| = t,$$

a contradiction.

Subcase 2. Suppose

$$\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) = V(K_t),$$

$$\big(N(w) \cap V(K_t)\big) \supseteq \big(N(v) \cap V(K_t)\big),$$

$$\big|N(w) \cap V(K_t)\big| = t - 2,$$

$$\big(N(w) \cap V(K_t)\big) \subseteq \big(N(u) \cap V(K_t)\big),$$

and

$$\big|N(u) \cap V(K_t)\big| = t - 2.$$

Then

$$\big(N(v) \cap V(K_t)\big) \subseteq \big(N(w) \cap V(K_t)\big) = \big(N(u) \cap V(K_t)\big),$$

and (3.19) implies that

$$V(K_t) = \big(N(v) \cap V(K_t)\big) \cup \big(N(u) \cap V(K_t)\big) = N(u) \cap V(K_t).$$

But then

$$t = \big|V(K_t)\big| = \big|N(u) \cap V(K_t)\big| = t - 2,$$

a contradiction.

Subcase 3. Suppose

$$\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) = V(K_t),$$

$$\big(N(w) \cap V(K_t)\big) \subseteq \big(N(v) \cap V(K_t)\big),$$

$$\big|N(v) \cap V(K_t)\big| = t - 2,$$

$$\big(N(w) \cap V(K_t)\big) \subseteq \big(N(u) \cap V(K_t)\big),$$

and

$$\big|N(u) \cap V(K_t)\big| = t - 2.$$

Then

$$\big(N(w) \cap V(K_t)\big) \subseteq \big[\big(N(v) \cap V(K_t)\big) \cap \big(N(u) \cap N(K_t)\big)\big]. \tag{3.27}$$

Let

$$q = \big|N(w) \cap V(K_t)\big|,$$

where q is in one of the cases 1.1), 1.2), 1.3) and 1.4).

From (3.27),

$$\begin{aligned} q &= \big|N(w) \cap V(K_t)\big| \leq \big|\big(N(v) \cap V(K_t)\big) \cap \big(N(u) \cap N(K_t)\big)\big| \\ &= \big|N(v) \cap V(K_t)\big| + \big|N(u) \cap N(K_t)\big| - \big|\big(N(v) \cap V(K_t)\big) \cup \big(N(u) \cap N(K_t)\big)\big| \\ &= t - 2 + t - 2 - t = t - 4. \end{aligned} \tag{3.28}$$

If 1.2) holds with $q = 2$ and $t = 5$, then $q = 2 > t - 4 = 1$, and (3.28) is false.

If 1.3) holds with $q = 3$ and $t = 6$, then $q = 3 > t - 4 = 2$, and (3.28) is false.

If 1.4) holds with $q = t - 2, t \geq 4$, then (3.28) is false.

So only 1.1) is possible, with $q = 1,\ t \geq 3$.

If 1.1) Holds then $q = 1, t \geq 3$. Take $k = 2, l_1 = l_2 = 1$. Then by (3.14), all the eigenvalues of $G = K_t; K_{1,1} \cup K_1$ other than -1 or 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{1,1}\cup K_1}(\lambda)\\
&= \left\{ [\lambda^3 - (t-2)\lambda^2 - t\lambda + t - 2] - \sum_{j=1}^{2} \frac{(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+2\lambda-1} \right\} \cdot \prod_{i=1}^{2} [\lambda^2 + 2\lambda - 1]\\
&= \lambda^7 - (t-4)\lambda^6 - (5t-8)\lambda^5 - (5t-8)\lambda^4 + (6t-15)\lambda^3 + (5t-16)\lambda^2 - (5t-18)\lambda + t - 4\\
&= (\lambda^2+2\lambda-1)[\lambda^5 - (t-2)\lambda^4 - (3t-5)\lambda^3 + (3t-10)\lambda - t + 4]\\
&= (\lambda^2+2\lambda-1)\Big\{(\lambda^2+\lambda-1)[\lambda^3 - (t-1)\lambda^2 - (2t-5)\lambda + t - 4] + \lambda\Big\}\\
&= 0.
\end{aligned} \tag{3.29}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{1,1} \cup K_1), \ t \geq 5$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1} \cup K_1)$$

exists and is the largest root of

$$\lambda^6 + 5\lambda^5 + 5\lambda^4 - 6\lambda^3 - 5\lambda^2 + 5\lambda - 1 = (\lambda^2+\lambda-1)(\lambda^2+2\lambda-1)^2 = 0.$$

Then

$$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1} \cup K_1) = (-1+\sqrt{5})/2.$$

From (3.29),

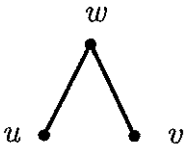
$$\lambda_2(G) = \lambda_2(K_t; k_{1,1} \cup K_1) \neq (-1+\sqrt{5})/2.$$

Then

$$0 < \lambda_2(G) = \lambda_2(K_t; K_{1,1} \cup K_1) < (-1+\sqrt{5})/2, \ t \geq 5.$$

Case 3. If

$$G|_{V(G)-V(K_t)} = K_{2,1} \cong$$



By (3.15), (3.16), (3.17), (3.18) and (3.19), without loss of generality, suppose

$$q_1 = \big|N(v) \cap V(K_t)\big|,\ 1 \le q_1 \le t-2, \tag{3.30}$$

$$q_2 = \big|N(w) \cap V(K_t)\big|,\ 1 \le q_2 \le t-2, \tag{3.31}$$

$$\big(N(v) \cap V(K_t)\big) \subseteq \big(N(u) \cap V(K_t)\big), \tag{3.32}$$

$$\big|N(u) \cap V(K_t)\big| = t-2, \tag{3.33}$$

$$\big(N(v) \cap N(K_t)\big) \cup \big(N(w) \cap N(K_t)\big) = V(K_t), \tag{3.34}$$

and

$$\big(N(u) \cap N(K_t)\big) \cup \big(N(w) \cap N(K_t)\big) = V(K_t). \tag{3.35}$$

where q_1 and q_2 are in one of the cases 1.1), 1.2), 1.3), and 1.4).

Then (3.34) implies that

$$q_1 + q_2 \ge t. \tag{3.36}$$

If 1.1) holds with $q_1 = 1,\ t \ge 3$. Then

$$\begin{aligned}\big|\big(N(v) \cap N(K_t)\big) \cup \big(N(w) \cap N(K_t)\big)\big| \le & \big|\big(N(v) \cap N(K_t)\big)\big| + \big|\big(N(w) \cap N(K_t)\big)\big| \\ \le & q_1 + t - 2 = t - 1,\end{aligned}$$

which contradicts (3.34).

Similarly, if 1.1) holds with $q_2 = 1,\ t \ge 3$, then

$$\begin{aligned}\big|\big(N(u) \cap N(K_t)\big) \cup \big(N(w) \cap N(K_t)\big)\big| \le & \ \big|\big(N(u) \cap N(K_t)\big)\big| + \big|\big(N(w) \cap N(K_t)\big)\big| \\ = & t - 2 + q_2 = t - 1,\end{aligned}$$

which contradicts (3.35).

If 1.2) holds with $q_1 = 2,\ t = 5$, then (3.31) and (3.36) imply that $1 \le q_2 \le t - 2 = 3$ and $q_1 + q_2 = 2 + q_2 \ge t = 5$, which implies that $q_2 = 3$. Then (3.32), (3.33), (3.34) and (3.35) imply that $G = H_4$, with

$$\lambda_2(H_4) = 0.602501 < (-1 + \sqrt{5})/2.$$

If 1.2) holds with $q_2 = 2,\ t = 5$, then

$$\big|N(u) \cap V(K_t)\big| = t - 2 = 4.$$

From (3.30) and (3.36), $1 \le q_1 \le t - 2 = 3$ and $q_1 + q_2 = q_1 + 2 \ge t = 5$, which implies that $q_1 = 3$. Then (3.32), (3.33), (3.34) and (3.35) imply that $G = H_5$, with

$$\lambda_2(H_5) = 0.612279 < (-1 + \sqrt{5})/2.$$

If 1.3) holds with $q_1 = 3,\ t = 6$, then (3.33) implies that

$$\left|N(u) \cap V(K_t)\right| = t - 2 = 4.$$

Then (3.32), (3.33), (3.34) and (3.35) imply that

$$G - w = F_{11}, \text{ with } \lambda_2(F_{11}) = 0.632035 > (-1 + \sqrt{5})/2,$$

a contradiction.

If 1.3) holds with $q_2 = 3,\ t = 6$ and $q_1 \neq 3$, then (3.30) and (3.36) imply that

$$1 \leq q_1 \leq t - 2 = 4, \text{ and } q_1 + q_2 = q_1 + 3 \geq t = 6.$$

which implies that $q_1 = 4$.
From (3.33),

$$\left|N(u) \cap V(K_t)\right| = t - 2 = 4.$$

Then (3.32), (3.33), (3.34) and (3.35) imply that

$$G = F_{14}, \text{ with } \lambda_2(F_{14}) = 0.627719 > (-1 + \sqrt{5})/2,$$

a contradiction.

If 1.4) holds with $q_1 = q_2 = t - 2,\ t \geq 4$, then (3.32), (3.33), (3.34) and (3.35) imply that

$$G = K_t; K_{2,1}$$

as in *Figure 9.* Take $k = 2, l_1 = 2$ and $l_2 = 1$. Then by (3.13) all the eigenvalues of $K_t; K_{2,1}$ are the roots of

$$\begin{aligned} &P_{K_t;K_{2,1}}(\lambda) \\ &= \left\{(\lambda - t + 1) - \frac{2(\lambda - 1)^2}{\lambda^2 + 3\lambda - 2} - \frac{(\lambda - 1)^2}{\lambda^2 + 2\lambda - 1}\right\} \cdot \left[\lambda^2 + 3\lambda - 2\right] \cdot \left[\lambda^2 + 2\lambda - 1\right] \\ &= \lambda^5 - (t - 3)\lambda^4 - (5t - 7)\lambda^3 - (3t - 11)\lambda^2 + (7t - 20)\lambda - 2t + 6 = 0. \end{aligned}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{2,1}),\ t \geq 4$$

is monotone increasing in t and bounded from above. So

$$\lim_{t \to \infty} \lambda_2(K_t; K_{2,1})$$

exists and is the largest root of

$$\lambda^4 + 5\lambda^3 + 3\lambda^2 - 7\lambda + 2 = (\lambda^2 + 2\lambda - 1)(\lambda^2 + 3\lambda - 2) = 0.$$

So

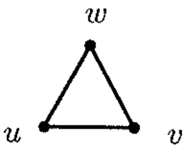
$$\lim_{t\to\infty} \lambda_2(K_t; K_{2,1}) = (-3+\sqrt{17})/2,$$

and

$$0 < \lambda_2(G) = \lambda_2(K_t; K_{2,1}) \le (-3+\sqrt{17})/2 < (-1+\sqrt{5})/2.$$

Case 4. If

$$G|_{V(G)-V(K_t)} = K_{1,1,1} = K_3 \cong$$



By (3.15), (3.19) and symmetry, without loss of generality, suppose

$$|N(v) \cap V(K_t)| = q_1, \ 1 \le q_1 \le t-2,$$
$$|N(u) \cap V(K_t)| = q_2, \ 1 \le q_2 \le t-2,$$
$$|N(w) \cap V(K_t)| = q_3, \ 1 \le q_3 \le t-2, \tag{3.37}$$
$$\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) = V(K_t), \tag{3.38}$$
$$\big(N(v) \cap N(K_t)\big) \cup \big(N(w) \cap N(K_t)\big) = V(K_t), \tag{3.39}$$
$$\big(N(w) \cap N(K_t)\big) \cup \big(N(u) \cap N(K_t)\big) = V(K_t), \tag{3.40}$$
$$1 \le q_1 \le q_2 \le q_3 \le t-2 \tag{3.41}$$

where $q_i, i = 1, 2, 3$ is in one of the cases 1.1), 1.2) 1.3) and 1.4).

If 1.1) holds with $q_1 = 1, \ t \ge 3$, then $q_1 + q_2 \le 1 + t - 2 = t - 1$, and (3.38) is false.

Suppose 1.2) holds with $q_1 = 2, \ t = 5$. Because $G - w$ is one of graphs in the *Figure 6*, $G - w$ must be H_2 with $q_2 = 3$. Then (3.39) and (3.40) imply that

$$\big(N(w) \cap N(K_t)\big) \supseteq \big(V(K_t) - (N(v) \cap N(K_t)\big),$$

and

$$\big(N(w) \cap N(K_t)\big) \supseteq \big(V(K_t) - (N(u) \cap N(K_t)\big).$$

Then

$$\big(N(w) \cap N(K_t)\big) = V(K_t),$$

and

$$q_3 = |N(w) \cap V(K_t)| = t = 5 > t - 2 = 3,$$

which contradicts (3.37).

Suppose 1.3) holds with $q_1 = 3,\ t = 6$. Because $G - w$ is one of graphs in *Figure 6*, $G - w$ must be H_3 with $q_2 = 4$ and

$$N(v) \cap N(u) \cap N(K_t) = 1.$$

Then (3.39) and (3.40) imply that

$$\big(N(w) \cap N(K_t)\big) \supseteq \big(V(K_t) - (N(v) \cap N(K_t)\big),$$

and

$$\big(N(w) \cap N(K_t)\big) \supseteq \big(V(K_t) - (N(u) \cap N(K_t)\big).$$

Therefore

$$\begin{aligned}\big(N(w) \cap N(K_t)\big) &\supseteq \Big\{\big[V(K_t) - \big(N(v) \cap N(K_t)\big)\big] \cup \big[V(K_t) - \big(N(u) \cap N(K_t)\big)\big]\Big\}\\ &= V(K_t) - \big(N(v) \cap N(u) \cap N(K_t)\big),\end{aligned}$$

and

$$\begin{aligned} q_3 &= \big|N(w) \cap V(K_t)\big|\\ &= \big|V(K_t) - \big(N(v) \cap N(u) \cap N(K_t)\big)\big|\\ &\geq 6 - 1 = 5 > t - 2 = 4,\end{aligned}$$

which contradicts (3.37).

If 1.4) holds with $q_1 = q_2 = q_3 = t - 2$, then (3.38), (3.39) and (3.40) imply that $t \geq 6$ and $G = K_t; K_{1,1,1}$ as in *Figure 10*. Take $k = 3, l_1 = l_2 = l_3 = 1$ in (3.13). Then by (3.13), all the eigenvalues of $K_t; K_{1,1,1}$ other than -1 or 0 can be found by solving

$$\begin{aligned} &P_{K_t;K_{1,1,1}}(\lambda)\\ &= \Big\{(\lambda - t + 1) - \sum_{j=1}^{3} \frac{(\lambda-1)^2}{\lambda^2 + 2\lambda - 1}\Big\} \cdot \prod_{i=1}^{3} [\lambda^2 + 2\lambda - 1]\\ &= \lambda^7 - (t-4)\lambda^6 - (6t-9)\lambda^5 - (9t-20)\lambda^4 + (4t-1)\lambda^3\\ &\qquad\qquad + (9t-36)\lambda^2 - (6t-23)\lambda + t - 4\\ &= (\lambda^2 + 2\lambda - 1)^2(\lambda^3 - t\lambda^2 - 2t\lambda + 7\lambda + t - 4)\\ &= 0.\end{aligned}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{1,1,1}),\ t \geq 4$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1,1})$$

exists and is the largest root of

$$\lambda^6 + 6\lambda^5 + 9\lambda^4 - 4\lambda^3 - 9\lambda^2 + 6\lambda - 1 = (\lambda^2 + 2\lambda - 1)^2 = 0.$$

Then,

$$\lim_{t\to\infty} \lambda_2(G) = \lim_{t\to\infty} \lambda_2(K_t; K_{1,1,1}) = -1 + \sqrt{2},\ t \geq 6,$$

and

$$0 < \lambda_2(G) = \lambda_2(K_t; K_{1,1,1}) \leq -1 + \sqrt{2} < (-1 + \sqrt{5})/2,\ t \geq 6.$$

For (VI), if

$$\left| G|_{V(G)-V(K_t)} \right| = 4,$$

then G contains at least one of the graphs in *Figure 7*, *Figure 8*, *Figure 9* or *Figure 10* as an induced subgraph. Because $0 < \lambda_2(G) < (-1+\sqrt{5})/2$, $2K_2$ and P_4 are not induced subgraphs of G, and $G|_{V(G)-V(K_t)}$ must be one of the graphs in *Figure 21.*

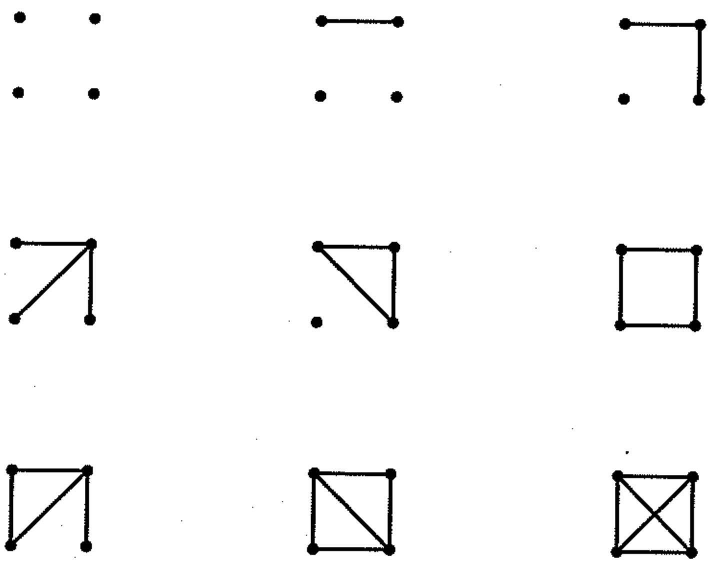
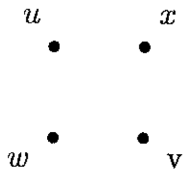


Figure 21.

Case 1. If

$$G|_{V(G)-V(K_t)} = \bar{K}_4 \cong$$



then G must contain a graph in *Figure 7* as an induced subgraph.

Because any pair of vertices in $\{v, u, w, x\}$ are not adjacent, by (3.16),(3.17) and(3.18), the set

$$\{N(v) \cap V(K_t), N(u) \cap N(K_t), N(w) \cap N(K_t), N(x) \cap N(K_t)\}$$

must be a linearly ordered set with respect to the inclusion relation between its elements. Without loss of generality suppose

$$\big(N(v) \cap V(K_t)\big) \subseteq \big(N(u) \cap N(K_t)\big) \subseteq \big(N(w) \cap N(K_t)\big) \subseteq \big(N(x) \cap N(K_t)\big), \quad (3.42)$$

$$q = \big|N(v) \cap V(K_t)\big|, \quad 1 \leq q \leq t-2, \qquad (3.43)$$

and

$$\big|N(u) \cap V(K_t)\big| = \big|N(w) \cap N(K_t)\big| = \big|N(x) \cap N(K_t)\big| = t-2. \qquad (3.44)$$

Because $G - u$ must be the graph $K_t; K_2 \cup K_1$ as in the *Figure 7*, q must be 1. From (3.42),(3.43) and (3.44), G is as in *Figure 22.* But then

$$G \gg K_{\underbrace{1,1,\ldots,1,}_{t-2 \text{ times}}\,3},$$

with

$$t - 2 + 3 = t + 1 > t,$$

which contradicts the fact that $G \gg K_t$ is an MCM-subgraph in G.

Therefore, in this case, no graph G satisfies the conditions.

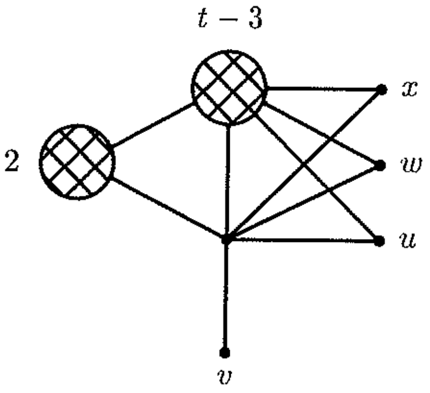
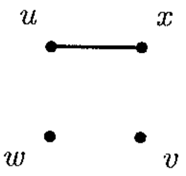


Figure 22.

Case 2. If

$$G|_{V(G)-V(K_t)} = K_2 \cup \bar{K}_2 \cong$$



then G must contain at least one of the graphs in *Figure 7* and at least one of the graphs in *Figure 8* as an induced subgraph.

Thus $G - w$ and $G - v$ must be graphs $K_t; K_{1,1} \cup K_1,\ t \geq 5$ as in *Figure 8*, with

$$\left|N(w) \cap V(K_t)\right| = 1,$$

and

$$\left|N(v) \cap V(K_t)\right| = 1. \tag{3.45}$$

Because $w \not\sim v$ and by the symmetry of w and v in G, using (3.16) (3.17) and (3.18), without loss of generality, suppose that

$$\left(N(w) \cap V(K_t)\right) \subseteq \left(N(v) \cap V(K_t)\right);$$

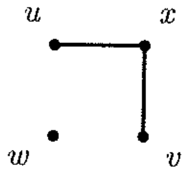
$$\left|N(v) \cap V(K_t)\right| = t - 2. \tag{3.46}$$

But (3.46) contradicts (3.45) for $t \geq 5$.

Therefore, in this case, no graph G satisfies the conditions.

Case 3. If

$$G|_{V(G)-V(K_t)} = K_{2,1} \cup K_1 \cong$$



then G must contain at least one of the graphs in *Figure 7*, at least one of the graphs in *Figure 8*, and at least one graph in *Figure 9* as an induced subgraph.

Thus $G-u$ must be the graph $K_t; K_{1,1} \cup K_1,\ t \geq 5$ as in *Figure 8* with

$$\left|N(w) \cap V(K_t)\right| = 1, \tag{3.47}$$

$$\left|N(x) \cap V(K_t)\right| = t-2, \tag{3.48}$$

$$\left|N(v) \cap V(K_t)\right| = t-2, \tag{3.49}$$

$$\big(N(x) \cap V(K_t)\big) \cup \big(N(v) \cap V(K_t)\big) = V(K_t), \tag{3.50}$$

and

$$\big(N(w) \cap V(K_t)\big) \subseteq \big[\big(N(v) \cap V(K_t)\big) \cap \big(N(x) \cap V(K_t)\big)\big]. \tag{3.51}$$

So $G-x$ must be the graph $K_t; K_2 \cup K_1$ as in *Figure 7*, with

$$N(u) \cap V(K_t) = N(v) \cap V(K_t). \tag{3.52}$$

Also $G-w$ must be one of the graphs in *Figure 9*. Then (3.47)–(3.52) imply that $G-w$ must be the graph $K_t; K_{2,1},\ t \geq 4$ as in *Figure 9*, and

$$G = K_t; K_{2,1} \cup K_1,\ t \geq 5$$

is the graph of *Figure 11*.

Take $k=2, l_1=2$ and $l_2=1$. Then by (3.14), all the eigenvalues of $G = K_t; K_{2,1} \cup K_1$ other than -1 or 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{2,1}\cup K_1}(\lambda) \\
&= \left\{\left[\lambda^3-(t-2)\lambda^2-t\lambda+t-2\right] - \frac{2(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+3\lambda-2} - \frac{(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+2\lambda-1}\right\} \\
&\qquad\qquad \cdot\left[\lambda^2+3\lambda-2\right]\cdot\left[\lambda^2+2\lambda-1\right] \\
&= \lambda^7-(t-4)\lambda^6-(6t-9)\lambda^5-(7t-14)\lambda^4+(9t-21)\lambda^3 \\
&\qquad\qquad +(8t-28)\lambda^2-(9t-33)\lambda+2t-8 \\
&= (\lambda^2+\lambda-1)\left[\lambda^5-(t-3)\lambda^4-(5t-7)\lambda^3-(3t-10)\lambda^2-(7t+24)\lambda-2t+6\right]+3\lambda-2 \\
&= 0.
\end{aligned} \tag{3.53}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{2,1} \cup K_1),\ t \geq 4$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty} \lambda_2(K_t; K_{2,1} \cup K_1)$$

exists and is the largest root of

$$\begin{aligned}&\lambda^6 + 6\lambda^5 + 7\lambda^4 - 9\lambda^3 - 8\lambda^2 + 9\lambda - 2\\ &\qquad = (\lambda^2 + \lambda - 1)(\lambda^2 + 2\lambda - 1)(\lambda^2 + 3\lambda - 2) = 0.\end{aligned}$$

Then

$$\lim_{t\to\infty} \lambda_2(G) = \lim_{t\to\infty} \lambda_2(K_t; K_{2,1} \cup K_1) = (-1 + \sqrt{5})/2.$$

From (3.53),

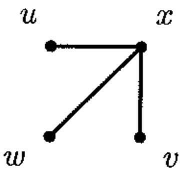
$$\lambda_2(G) = \lambda_2(K_t; K_{2,1} \cup K_1) \neq (-1 + \sqrt{5})/2.$$

So

$$0 < \lambda_2(G) = \lambda_2(K_t; K_{2,1} \cup K_1) < (-1 + \sqrt{5})/2.$$

Case 4. If

$$G|_{V(G)-V(K_t)} = K_{3,1} \cong$$



then G must contain at least one of the graphs in *Figure 7* and at least one graph in *Figure 9* as an induced subgraph.

$G - x$ must be of the graph $K_t; K_2 \cup K_1$ as in *Figure 7.* Because the vertices u, v, w are symmetric in G, without loss of generality, suppose

$$|N(u) \cap V(K_t)| = t - 2,$$

$$|N(v) \cap V(K_t)| = t - 2,$$

and

$$|N(w) \cap V(K_t)| = 1. \tag{3.54}$$

Because $w \sim x$, from (3.19)

$$\big(N(w) \cap N(K_t)\big) \cup \big(N(x) \cap N(K_t)\big) = V(K_t). \tag{3.55}$$

But then (3.15), (3.54) and (3.55) imply that

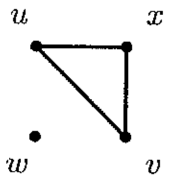
$$\begin{aligned} t &= \big|\big(N(w) \cap N(K_t)\big) \cup \big(N(x) \cap N(K_t)\big)\big| \\ &\leq \big|N(w) \cap N(K_t)\big| + \big|N(x) \cap N(K_t)\big| \\ &\leq 1 + t - 2 = t - 1, \end{aligned}$$

a contradiction.

Therefore, in this case, no graph G satisfies the conditions.

Case 5. If

$$G|_{V(G)-V(K_t)} = K_{1,1,1} \cup K_1 = K_3 \cup K_1 \cong$$



then $G - x$ must be the graph $K_t; K_{1,1} \cup K_1,\ t \geq 5$ as in *Figure 8* with

$$\big|N(w) \cap V(K_t)\big| = 1, \tag{3.56}$$

$$\big(N(w) \cap V(K_t)\big) \subseteq \big(N(v) \cap V(K_t)\big), \tag{3.57}$$

and

$$\big(N(w) \cap V(K_t)\big) \subseteq \big(N(u) \cap V(K_t)\big). \tag{3.58}$$

$G - u$ must be the graph $K_t; K_{1,1} \cup K_1,\ t \geq 5$ as in *Figure 8* with

$$\big(N(w) \cap V(K_t)\big) \subseteq \big(N(x) \cap V(K_t)\big). \tag{3.59}$$

$G - w$ must be the graph $K_t; K_{1,1,1},\ t \geq 6$ as in *Figure 10* with

$$\big|N(x) \cap V(K_t)\big| = t - 2, \tag{3.60}$$

$$\big|N(u) \cap V(K_t)\big| = t - 2, \tag{3.61}$$

$$\big|N(v) \cap V(K_t)\big| = t - 2, \tag{3.62}$$

$$\big(N(u) \cap N(K_t)\big) \cup \big(N(x) \cap N(K_t)\big) = V(K_t), \tag{3.63}$$

$$\big(N(x) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) = V(K_t), \tag{3.64}$$

and

$$\big(N(v) \cap N(K_t)\big) \cup \big(N(u) \cap N(K_t)\big) = V(K_t). \tag{3.65}$$

Then (3.56)–(3.65) imply that

$$G = K_t; K_{1,1,1} \cup K_1, \ t \geq 7$$

as in *Figure 12.*

Then take $k = 3, l_1 = l_2 = l_3 = 1$ in (3.14). All the eigenvalues of $G = K_t; K_{1,1,1} \cup K_1$ other than -1 and 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{1,1,1}\cup K_1}(\lambda) \\
&= \left\{ \left[\lambda^3 - (t-2)\lambda^2 - t\lambda + t - 2\right] - 3\frac{(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+2\lambda-1} \right\} \cdot (\lambda^2+2\lambda-1)^3 \\
&= (\lambda^2+2\lambda-1)^2\left[\lambda^5 - (t-1)\lambda^4 - (3t-6)\lambda^3 + 2\lambda^2 + (3t-13)\lambda - t + 5\right] \\
&= (\lambda^2+2\lambda-1)^2\left\{(\lambda^2+\lambda-1)\left[\lambda^3 - t\lambda^2 - (2t-7)\lambda + t + 5\right] - \lambda\right\} \qquad (3.66) \\
&= \lambda^9 - (t-5)\lambda^8 - (7t-12)\lambda^7 - (14t-24)\lambda^6 + (t+4)\lambda^5 \\
&\qquad + (22t-66)\lambda^4 - (t+8)\lambda^3 - (14t-64)\lambda^2 + (7t-33)\lambda - t + 5 \\
&= 0.
\end{aligned}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{1,1,1} \cup K_1), \ t \geq 7$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty} \lambda_2(G) = \lim_{t\to\infty} \lambda_2(K_t; K_{1,1,1} \cup K_1)$$

exists and is the largest root of

$$\begin{aligned}
&\lambda^8 + 7\lambda^7 + 14\lambda^6 - \lambda^5 - 22\lambda^4 + \lambda^3 + 14\lambda^2 - 7\lambda + 1 \\
&= (\lambda^2+\lambda-1)(\lambda^2+2\lambda-1)^3 = 0.
\end{aligned}$$

Then

$$\lim_{t\to\infty} \lambda_2(K_t; K_{1,1,1} \cup K_1) = (-1+\sqrt{5})/2.$$

From (3.66),

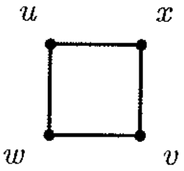
$$0 < \lambda_2(K_t; K_{1,1,1} \cup K_1) \neq (-1+\sqrt{5})/2, \ t \geq 7$$

So

$$0 < \lambda_2(K_t; K_{1,1,1} \cup K_1) < (-1+\sqrt{5})/2,\ t \geq 7.$$

Case 6. If

$$G|_{V(G)-V(K_t)} = K_{2,2} \cong$$



then G must contain a graph in *Figure 9* as an induced subgraph and $G|_{V(G)-V(K_t)}$ is symmetric in $u, v, w,$ and x.

Thus $G - x$ must be one of the graphs H_4, H_5 and $K_t; K_{2,1}$ as in *Figure 9.*

Because $x \not\sim w$, then (3.16) and (3.17) imply that

either

$$\big(N(x) \cap V(K_t)\big) \subseteq \big(N(w) \cap V(K_t)\big), \tag{3.67}$$

or

$$\big(N(w) \cap V(K_t)\big) \subseteq \big(N(x) \cap V(K_t)\big); \tag{3.68}$$

Because $x \sim u$ and $x \sim v$, then (3.19) imply that

$$\big(N(x) \cap N(K_t)\big) \cup \big(N(u) \cap N(K_t)\big) = V(K_t), \tag{3.69}$$

and

$$\big(N(x) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) = V(K_t). \tag{3.70}$$

Subcase 1. If $G - x = H_4$, then $t = 5$. Without loss of generality, suppose

$$\big|N(u) \cap V(K_t)\big| = 3, \tag{3.71}$$

$$\big|N(v) \cap V(K_t)\big| = 2, \tag{3.72}$$

and

$$\big|N(w) \cap V(K_t)\big| = 3. \tag{3.73}$$

Then(3.15), (3.67), (3.68), (3.70), (3.72) imply that

$$\big(N(w) \cap V(K_t)\big) = \big(N(x) \cap V(K_t)\big), \tag{3.74}$$

and

$$\big|\big(N(w) \cap V(K_t)\big)\big| = \big|\big(N(x) \cap V(K_t)\big)\big| = t - 2 = 3. \tag{3.75}$$

Then (3.69)–(3.75) imply that $G = F_{15}$ with

$$\lambda_2(F_{15}) = 0.630315 > (-1+\sqrt{5})/2,$$

a contradiction.

Subcase 2. If $G - x = H_5$, then $t = 5$ and

$$N(u) \cap V(K_t) = N(v) \cap V(K_t). \tag{3.76}$$

Then

$$\left|N(u) \cap V(K_t)\right| = \left|N(v) \cap V(K_t)\right| = t - 2 = 3, \tag{3.77}$$

and

$$\left|N(w) \cap V(K_t)\right| = 2. \tag{3.78}$$

From (3.15), let

$$q = \left|N(x) \cap V(K_t)\right|, \text{ then } 1 \le q \le t - 2 = 3.$$

If $q = 1$, by (3.15) and (3.69),

$$\begin{aligned} t &= \left|\left(N(x) \cap N(K_t)\right) \cup \left(N(u) \cap N(K_t)\right)\right| \\ &\le \left|N(x) \cap V(K_t)\right| + \left|N(u) \cap V(K_t)\right| \\ &\le q + t - 2 = 1 + t - 2 = t - 1, \end{aligned}$$

a contradiction.

If $q = 2$, then (3.67), (3.68), and (3.78) imply that

$$N(x) \cap V(K_t) = N(w) \cap V(K_t). \tag{3.79}$$

So

$$\left|N(x) \cap V(K_t)\right| = \left|N(w) \cap V(K_t)\right| = q = 2. \tag{3.80}$$

Then by (3.69),(3.70), (3.76), (3.77), (3.79), and (3.80), G is a graph as in the *Figure 23*. But then $G \gg F_5$ as an induced subgraph with

$$\lambda_2(F_5) = 0.678363 > (-1+\sqrt{5})/2,$$

a contradiction.

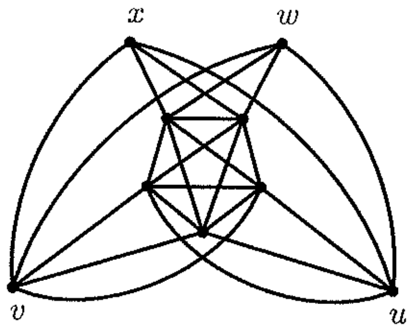


Figure 23

If $q = 3$, then (3.67), (3.68), and (3.78) imply that

$$\big(N(w) \cap V(K_t)\big) \subseteq \big(N(x) \cap V(K_t)\big). \tag{3.81}$$

Then (3.69), (3.70) and (3.81) imply that $G = F_{15}$, a contradiction.

Subcase 3. If $G - x = K_t; K_{2,1},\ t \geq 4$, then

$$\big|N(w) \cap V(K_t)\big| = t - 2. \tag{3.82}$$

$G - w$ must be one of graphs H_4, H_5, and $K_t; K_{2,1}$ as in *Figure 9.*

Because x and w are symmetric in $G|_{V(G)-V(K_T)}$, just as in Subcase 1 and Subcase 2, if $G - w = H_4$ or $G - w = H_5$, then there is no graph that satisfies the conditions.

If $G - w = K_t; K_{2,1},\ t \geq 4$, then

$$\big|N(x) \cap V(K_t)\big| = t - 2. \tag{3.83}$$

By (3.67), (3.68), (3.82) and (3.83),

$$N(x) \cap V(K_t) = N(w) \cap V(K_t).$$

Then $G - x = K_t; K_{2,1}\ t \geq 4$ and $G - w = K_t; K_{2,1},\ \ t \geq 4$ imply that $G = K_t; K_{2,2}$ as *Figure 13.*

Take $k = 2, l_1 = l_2 = 2$. Then by (3.13), all the eigenvalues of $K_t; K_{2,2}$ other than -1 or 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{2,2}}(\lambda)\\
&= \left\{(\lambda - t + 1) - \frac{4(\lambda-1)^2}{\lambda^2 + 3\lambda - 2}\right\} \cdot \left[\lambda^2 + 3\lambda - 2\right]^2\\
&= (\lambda^2 + 3\lambda - 2)\left[\lambda^3 - t\lambda^2 - (3t - 9)\lambda + 2t - 6\right]\\
&= \lambda^5 - (t-3)\lambda^4 - (6t-7)\lambda^3 - (5t-21)\lambda^2 + (12t-36)\lambda - 4t + 12 = 0.
\end{aligned}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{2,2}),\ t \geq 4$$

is monotone increasing in t and bounded from above. So

$$\lim_{t \to \infty} \lambda_2(K_t; K_{2,2})$$

exists and is the largest root of

$$\lambda^4 + 6\lambda^3 + 5\lambda^2 - 12\lambda + 4 = (\lambda^2 + 3\lambda - 2)^2 = 0.$$

So

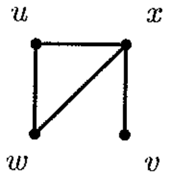
$$\lim_{t \to \infty} \lambda_2(K_t; K_{2,2}) = (-3 + \sqrt{17})/2,$$

and

$$0 < \lambda_2(K_t; K_{2,2}) \leq (-3 + \sqrt{17})/2 < (-1 + \sqrt{5})/2.$$

Case 7. If

$$G|_{V(G)-V(K_t)} = A_{1,2} \cong$$



then G must contain at least one of the graphs in *Figure 8*, at least one of the graphs in *Figure 9* and at least one graph in *Figure 10* as an induced subgraph.

The fact that $G - x$ must be a graph in *Figure 8* implies that

$$G - x = K_t; K_{1,1} \cup K_1,\ t \geq 5$$

with

$$\left|N(v) \cap V(K_t)\right| = 1. \tag{3.84}$$

Because $x \sim v$, (3.19) implies that

$$\big(N(x) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) = V(K_t). \tag{3.85}$$

But then (3.15), (3.84) and (3.85) imply that

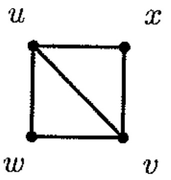
$$\begin{aligned} t &= \left|\big(N(x) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big)\right| \\ &= \left|N(x) \cap V(K_t)\right| + \left|N(v) \cap V(K_t)\right| \\ &\leq t - 2 + 1 = t - 1, \end{aligned}$$

a contradiction.

Therefore, in this case, no graph G satisfies the conditions.

Case 8. If

$$G|_{V(G)-V(K_t)} = K_{2,1,1} \cong$$



then G must contain at least one of the graphs in *Figure 9* and at least one of the graphs in *Figure 10* as an induced subgraph.

$G-x$ must be the graph $K_t; K_{1,1,1}$ $t \geq 6$ as in *Figure 10* implying that

$$\left|N(v) \cap V(K_t)\right| = t-2,$$

$$\left|N(u) \cap V(K_t)\right| = t-2, \tag{3.86}$$

and

$$\left|N(w) \cap V(K_t)\right| = t-2. \tag{3.87}$$

$G-v$ is a graph in *Figure 9* imply that

$$G - v = K_t; K_{2,1}$$

with

$$N(w) \cap V(K_t) = N(x) \cap V(K_t).$$

Then G must be the graph $K_t; K_{2,1,1}$ as in *Figure 14*.

Take $K=3, l_1=2, l_2=l_3=1$ in (3.13), then all the eigenvalues of $K_t; K_{2,1,1}$ other than -1 or 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{2,1,1}}(\lambda)\\
&= \left\{(\lambda - t + 1) - \frac{2(\lambda-1)^2}{\lambda^2+3\lambda-2} - \frac{2(\lambda-1)^2}{\lambda^2+2\lambda-1}\right\} \cdot (\lambda^2+3\lambda-2)\cdot(\lambda^2+2\lambda-1)^2\\
&= (\lambda^2+2\lambda-1)\left[\lambda^5-(t-2)\lambda^4-(5t-6)\lambda^3-(3t-18)\lambda^2+(7t-27)\lambda-2t+8\right]\\
&= \lambda^7-(t-4)\lambda^6-(7t-9)\lambda^5-(12t-28)\lambda^4+(6t+3)\lambda^3\\
&\qquad\qquad +(15t-64)\lambda^2-(11t-43)\lambda+2t-8=0.
\end{aligned}$$

Apply Theorem 2.4 with $l=3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{2,1,1}),\ t \geq 4$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty} \lambda_2(K_t; K_{2,1,1})$$

exists and is the largest root of

$$\begin{aligned}\lambda^6 + 7\lambda^5 + 12\lambda^4 - 6\lambda^3 - 15\lambda^2 + 11\lambda - 2 \\ = (\lambda^2 + 2\lambda - 1)^2(\lambda^2 + 3\lambda - 2) = 0.\end{aligned}$$

So

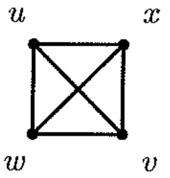
$$\lim_{t\to\infty} \lambda_2(G) = \lim_{t\to\infty} \lambda_2(K_t; K_{2,1,1}) = (-3 + \sqrt{17})/2.$$

Therefore

$$0 < \lambda_2(G) = \lim_{t\to\infty} \lambda_2(K_t; K_{2,1,1}) \leq (-3 + \sqrt{17})/2 < (-1 + \sqrt{5})/2.$$

Case 9. If

$$G|_{V(G)-V(K_t)} = K_{1,1,1,1} = K_4 \cong$$



then G must contain at least one of the graphs in *Figure 10*, and $G|_{V(G)-V(K_t)}$ is symmetric about u, v, w, and x.

The fact that $G - x$ must be a graph $K_t; K_{1,1,1}$, $t \geq 6$, as in *Figure 10* implies that

$$|N(u) \cap V(K_t)| = t - 2, \tag{3.88}$$

$$|N(v) \cap V(K_t)| = t - 2, \tag{3.89}$$

$$|N(w) \cap V(K_t)| = t - 2, \tag{3.90}$$

$$\big(N(u) \cap N(K_t)\big) \cup \big(N(v) \cap N(K_t)\big) = V(K_t), \tag{3.91}$$

$$\big(N(v) \cap N(K_t)\big) \cup \big(N(w) \cap N(K_t)\big) = V(K_t), \tag{3.92}$$

and

$$\big(N(w) \cap N(K_t)\big) \cup \big(N(u) \cap N(K_t)\big) = V(K_t). \tag{3.93}$$

The fact that $G - w$ must be a graph $K_t; K_{1,1,1}$, $t \geq 6$ as in the it Figure 10 implies that

$$|N(x) \cap V(K_t)| = t - 2, \tag{3.94}$$

$$\big(N(u) \cap N(K_t)\big) \cup \big(N(x) \cap N(K_t)\big) = V(K_t), \tag{3.95}$$

$$\big(N(v) \cap N(K_t)\big) \cup \big(N(x) \cap N(K_t)\big) = V(K_t), \tag{3.96}$$

and

$$\big(N(w) \cap N(K_t)\big) \cup \big(N(x) \cap N(K_t)\big) = V(K_t). \tag{3.97}$$

Then (3.88)–(3.97) imply that G must be a graph $K_t; K_{1,1,1,1}$, $t \geq 8$ as in *Figure 15.*

Take $k = 4, l_1 = l_2 = l_3 = l_4 = 1$ in (3.13). Then all the eigenvalues of $K_t; K_{l_1,l_2,\ldots,l_k}$ other than -1 or 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{1,1,1,1}}(\lambda)\\
&= \left\{(\lambda - t + 1) - \frac{4(\lambda-1)^2}{\lambda^2 + 2\lambda - 1}\right\} \cdot (\lambda^2 + 2\lambda - 1)^4\\
&= (\lambda^2 + 2\lambda - 1)^3 \left[\lambda^3 - (t+1)\lambda^2 - (2t-9)\lambda + t - 5)\right]\\
&= \lambda^9 - (t-5)\lambda^8 - (8t-12)\lambda^7 - (20t-36)\lambda^6 - (8t-46)\lambda^5 + (26t-66)\lambda^4\\
&\qquad +(8t-68)\lambda^3 - (20t-100)\lambda^2 + (8t-39)\lambda - t + 5 = 0.
\end{aligned}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{1,1,1,1}), \ t \geq 8$$

is monotone increasing in t and bounded from above. So

$$\lim_{t \to \infty} \lambda_2(K_t; K_{1,1,1,1})$$

exists and is the largest root of

$$\begin{aligned}
&\lambda^8 + 8\lambda^7 + 20\lambda^6 + 8\lambda^5 - 26\lambda^4 - 8\lambda^3 + 20\lambda^2 - 8\lambda + 1\\
&= (\lambda^2 + 2\lambda - 1)^4 = 0.
\end{aligned}$$

Then

$$\lim_{t \to \infty} \lambda_2(K_t; K_{1,1,1,1}) = -1 + \sqrt{2},$$

and

$$0 < \lambda_2(G) = \lambda_2(K_t; K_{1,1,1,1}) \leq -1 + \sqrt{2} < (-1 + \sqrt{5})/2.$$

For (V), if $G \gg K_t$ as an MCM-subgraph and $0 < \lambda_2(G) < (-1 + \sqrt{5})/2$, then

$$G \not\gg 2K_2 = F_1, \ G \not\gg P_4 = F_2$$

as induced subgraphs, and from the Part (IV), *Case 7,*

$$G|_{V(G)-V(K_t)} \not\gg A_{1,2}$$

as an induced subgraph. So each component of $G|_{V(G)-V(K_t)}$ must be a complete multipartite graph. If there are two components of $G|_{V(G)-V(K_t)}$ containing edges, then

$$G \gg F_2 = 2K_2, \text{ with } \lambda_2(F_2) = 1,$$

a contradiction. So only one component of $G|_{V(G)-V(K_t)}$ contains edges.

Theorem 3.6. *Let G be a simple graph without isolated vertices and such that $G \gg K_t$ a complete graph on t vertices as an MCM-subgraph. Then*

$$0 < \lambda_2(G) < (-1+\sqrt{5})/2$$

if and only if G is one of the following graphs:

(I) $A_{2,3}$, $A_{3,3}$, H_1, H_2, H_3, H_4, *and* H_5.

(II) An induced subgraphs of one of the graphs $K_t; K_{l_1,l_2,\ldots,l_k}$, where $t \geq 3$, $1 \leq k \leq t/2$, and $1 \leq l_i \leq 2$, for $i = 1, 2, \ldots, k$,

(III) An induced subgraphs of one the of graphs $K_t; K_{l_1,l_2,\ldots,l_k} \cup K_1$. where $t \geq 3$, $1 \leq k \leq (t-1)/2$, and $1 \leq l_i \leq 2$, for $i = 1, 2, \ldots, k$.

Proof. Let G be a simple graph without isolated vertices and such that $G \gg K_t$ a complete graph on t vertices as MCM-subgraph such that

$$0 < \lambda_2(G) < (-1+\sqrt{5})/2.$$

Then the fact that G must contain $A_{1,2}$ implies that $t \geq 3$.

From Lemma 3.5. Part (IV), *Case 1*, $G|_{V(G)-V(K_t)} \not\gg \bar{K}_4$. So if $G|_{V(G)-(V(K_t)}$ only contains isolated vertices, then

$$\left|G|_{V(G)-V(K_t)}\right| \leq 3.$$

If

$$\left|G|_{V(G)-V(K_t)}\right| \leq 4,$$

then from Lemma 3.5, G is a graph in Part (I), Part (II), or Part (III).

From now on, suppose

$$\left|G|_{V(G)-V(K_t)}\right| > 4.$$

In this situation, $G|_{V(G)-(V(K_t)}$ must contains at least one edge.

From Lemma 3.5. Part (V), each component of $G|_{V(G)-V(K_t)}$ is a complete multipartite graph and only one component of $G|_{V(G)-V(K_t)}$ contains edges.

From Lemma 3.5. Part (IV), *Case 4*,

$$G|_{V(G)-V(K_t)} \not\gg K_{3,1}.$$

So if

$$G|_{V(G)-(V(K_t)} \gg K_{l_1,l_2,\ldots,l_k},$$
$$\text{where } k \geq 2, \text{ and } l_i \geq 1, i = 1, 2, \ldots, k,$$

then

$$1 \leq l_i \leq 2, \ i = 1, 2, \ldots, l_k. \tag{3.98}$$

From Lemma 3.5. Part (IV), *Case 2*,

$$G|_{V(G)-V(K_t)} \not\gg K_2 \cup \bar{K}_2.$$

So $G|_{V(G)-V(K_t)}$ contains at most two components. According to the number of components in $G|_{V(G)-V(K_t)}$, there are two cases.

Case 1. $G|_{V(G)-V(K_t)}$ only has one component.

Then the facts that

$$G|_{V(G)-V(K_t)}$$

contains edges, each component of

$$G|_{V(G)-V(K_t)}$$

is a complete multipartite and (3.98) imply

$$\begin{gathered} G|_{V(G)-V(K_t)} = K_{l_1, l_2, \ldots, l_k}, \\ \text{where } k \geq 2, \ 1 \leq l_1 \leq l_2 \leq \cdots \leq l_k \leq 2. \end{gathered} \tag{3.99}$$

If $k = 2$, then (3.99) implies that

$$\left| G|_{V(G)-V(K_t)} \right| = \left| K_{l_1, l_2} \right| \leq 4.$$

The structure of G is already clear from the Lemma 3.5.

Let $k \geq 3$.

Subcase 1. Let $k \geq 3, \ l_i = 1, \ i = 1, 2, \ldots, k$.

$$G|_{V(G)-V(K_t)} = K_{l_1, l_2, \ldots, l_k}, = K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}} = K_k.$$

Mathematical induction can be used on k to prove that

$$G = K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}}, \text{ where } 1 \leq k \leq t/2.$$

For $k = 3$, if

$$G|_{V(G)-V(K_t)} = K_{1,1,1} = K_3,$$

then from Lemma 3.5, Part (III), *Case 4*, $G = K_t; K_{1,1,1}$.

For $k = 4$, if

$$G|_{V(G)-V(K_t)} = K_{1,1,1,1} = K_4,$$

then from Lemma 3.5, Part (IV), *Case 9*, $G = K_t; K_{1,1,1,1}$.

For $k > 3$, suppose that if

$$G|_{V(G)-V(K_t)} = K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}} = K_k,$$

then

$$G = K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}}, \text{ where } 1 \leq k \leq t/2. \tag{3.100}$$

For $k+1,\ k > 3$, if

$$G|_{V(G)-V(K_t)} = K_{\underbrace{1,1,\ldots,1}_{k+1 \text{ times}}} = K_{k+1},$$

let

$$V\Big(G|_{V(G)-V(K_t)}\Big) = V\big(K_{\underbrace{1,1,\ldots,1}_{k+1 \text{ times}}}\big) = \{v_1, v_2, \ldots, v_{k+1}\}.$$

Then

$$G - v_{i+1}|_{V(G-v_{i+1})-V(K_t)} = K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}} = K_k.$$

By the inductive hypothesis (3.100),

$$\begin{gathered} G - \{v_{k+1}\} = K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}}, \\ \text{where } 1 \leq k \leq t/2, \\ \text{and } 1 \leq l_i \leq 2, \text{ for } i = 1, 2, \ldots, k. \end{gathered} \tag{3.101}$$

The structure of the graph

$$K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}}$$

and (3.101) imply that

$$\left| V(K_t) - \bigcap_{j=1}^{k} \big(N(v_j) \cap N(K_t)\big) \right| = 2k. \tag{3.102}$$

Because $v_{k+1} \sim v_j,\ j = 1, 2, \ldots, k$, from (3.19)

$$\big(N(v_{k+1}) \cap N(K_t)\big) \cup \big(N(v_j) \cap N(K_t)\big) = V(K_t), \text{ for } j = 1, 2, \ldots, k.$$

Then

$$\big(N(v_{k+1}) \cap N(K_t)\big) \supseteq \Big(V(K_t) - \big(N(v_j) \cap N(K_t)\big)\Big), \text{ for } j = 1, 2, \ldots, k. \quad (3.103)$$

This in turn implies that

$$\begin{aligned}\big(N(v_{k+1}) \cap N(K_t)\big) &\supseteq \bigcup_{j=1}^{k} \big[V(K_t) - \big((N(v_j) \cap N(K_t)\big)\big] \\ &= V(K_t) - \bigcap_{j=1}^{k} \big[N(v_j) \cap N(K_t)\big]. \quad (3.104)\end{aligned}$$

Also by the inductive hypothesis (3.100),

$$G - v_k = K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}}. \quad (3.105)$$

(3.105) implies that

$$\big|N(v_{k+1}) \cap V(K_t)\big| = t - 2. \quad (3.106)$$

Then (3.104) and (3.106) imply that, except for two vertices in

$$\bigcap_{j=1}^{k} \big[N(v_j) \cap N(K_t)\big],$$

v_{k+1} is adjacent to all the remaining vertices in $V(K_t)$. Then from (3.102),

$$\begin{aligned}&\Big|V(K_t) - \bigcap_{j=1}^{k+1} \big[N(v_j) \cap N(K_t)\big]\Big| \\ =&\Big|V(K_t) - \bigcap_{j=1}^{k} \big[N(v_j) \cap N(K_t)\big]\Big| + 2 \\ =&2k + 2 = 2(k+1) \leq \big|V(K_t)\big| = t. \quad (3.107)\end{aligned}$$

This implies that $k + 1 < t/2$.

Therefore

$$G = K_t; K_{\underbrace{1,1,\ldots,1}_{k+1 \text{ times}}}, \text{ where } t \geq 3,\ 3 \leq k + 1 \leq t/2.$$

This ends the induction.

For $k \geq 1$, let $, l_1 = l_2 = \cdots = l_k = 1$. By (3.13), all the eigenvalues of

$$G = K_t; K_{\underbrace{1,1,\ldots,1,}_{k \text{ times}}}$$

other than -1 or 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{\underbrace{1,1,\ldots,1,}_{k\text{ times}}}}(\lambda) \\
&= \left\{(\lambda - t + 1) - k\frac{(\lambda-1)^2}{\lambda^2+2\lambda-1}\right\} \cdot (\lambda^2 + 2\lambda - 1)^k \\
&\qquad = (\lambda^2 + 2\lambda - 1)^{k-1}\left[\lambda^3 - (t+k-3)\lambda^2 - (2t-2k-1)\lambda + (t-k-1)\right] \\
&= 0.
\end{aligned}$$

For $3 \leq k \leq t/2$, apply Theorem 2.4 with $l = 3$. Then

$$\lambda_2(G) = \lambda_2(K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}})$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty} \lambda_2(K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}})$$

exists and is the largest root of

$$(\lambda^2 + 2\lambda - 1)^k = 0.$$

Therefore

$$\lim_{t\to\infty} \lambda_2(K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}}) = -1 + \sqrt{2},$$

and

$$0 < \lambda_2(K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}}) \leq -1 + \sqrt{2} < (-1 + \sqrt{5})/2.$$

Subcase 2. Let

$$k \geq 3,\ l_1 = l_2 = \cdots = l_g = 1,\ l_{g+1} = l_{g+2} = \cdots = l_k = 2,\ 0 \leq g < k,$$

$$G|_{V(G)-V(K_t)} = K_{l_1,l_2,\ldots,l_k} = K_{\underbrace{1,1,\ldots,1,}_{g \text{ times}}\ \underbrace{2,2,\ldots,2}_{k-g \text{ times}}}\ .$$

Let

$$V_i = \begin{cases} \{v_{i,1}\}, & \text{if } 1 \le i \le g; \\ \{v_{i,1}, v_{i,2}\}. & \text{if } g+1 \le i \le k. \end{cases}$$

the V_i, $i = 1, 2, \ldots, k$, are maximum independent sets in $G|_{V(G)-V(K_t)}$, and

$$V\Big(G|_{V(G)-V(K_t)}\Big) = \bigcup_{i=1}^{k} V_i.$$

Let

$$G_1 = G - \bigcup_{i=g+1}^{k} \{v_{i,2}\}. \tag{3.108}$$

Then

$$G_1|_{V(G_1)-V(K_t)} = K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}} = K_k.$$

G_1 is a graph in Subcase 1. Then the results in Subcase 1 imply that

$$G_1 = K_t; K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}}, \text{ where } 3 \le k \le t/2. \tag{3.109}$$

Let

$$1 \le i_1 < i_2 < i_3 \le k, \text{ and } g < i_3 \le k. \tag{3.110}$$

Because $k \ge 3$ and $g < k$, i_1, i_2 and i_3 exist.

Let

$$G_2 = G|_{K_t \cup \{v_{i_1,1}, v_{i_2,1}, v_{i_3,1}, v_{i_3,2}\}}.$$

Then

$$G_2|_{V(G_2)-V(K_t)} = G|_{\{v_{i_1,1}, v_{i_2,1}, v_{i_3,1}, v_{i_3,2}\}} = K_{2,1,1}.$$

Lemma 3.5. Part (IV), *Case 8* implies that

$$G_2 = K_t; K_{2,1,1}.$$

Then (3.110) and the structure of $K_t; K_{2,1,1}$ imply that

$$N(v_{i_3,1}) \cap V(K_t) = N(v_{i_3,2}) \cap V(K_t),\ g < i_3 \le k. \tag{3.111}$$

Because $G \gg G_1 \gg K_t$, it follows that (3.111) holds both in G and G_1.

From (3.108), (3.109) and (3.111)

$$G = K_t; K_{\underbrace{1,1,\ldots,1}_{g \text{ times}},\ \underbrace{2,2,\ldots,2}_{k-g \text{ times}}} \quad \text{where } 3 \le k \le t/2,\ 0 \le g < k.$$

If $l_1 = l_2 = \cdots = l_k = 2,\ g = 0$, then from (3.13) all the eigenvalues of

$$G = K_t; K_{\underbrace{2,2,\ldots,2}_{k \text{ times}}} \text{ where } 1 \leq k \leq t/2$$

other than -1 and 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{\underbrace{2,2,\ldots,2,}_{k \text{ times}}}}(\lambda) \\
&= \left\{(\lambda - t + 1) - \frac{2k(\lambda-1)^2}{\lambda^2+3\lambda-2}\right\}(\lambda^2+3\lambda-2)^k \\
&\quad = (\lambda^2+3\lambda-2)^{k-1}\left[\lambda^3 - (t+2k-4)\lambda^2 - (3t-4k-1)\lambda + (2t-2k-2)\right]. \\
&= 0.
\end{aligned}$$

Apply Theorem 2.4 with $l = 3$. Then the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{\underbrace{2,2,\ldots,2}_{k \text{ times}}})$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty} \lambda_2(K_t; K_{\underbrace{2,2,\ldots,2}_{k \text{ times}}})$$

exists and is the largest root of

$$(\lambda^2+3\lambda-2)^k = 0.$$

So

$$\lim_{t\to\infty} \lambda_2(K_t; K_{\underbrace{2,2,\ldots,2}_{k \text{ times}}}) = (-3+\sqrt{17})/2.$$

Therefore

$$0 < \lambda_2(K_t; K_{\underbrace{2,2,\ldots,2}_{k \text{ times}}}) \leq (-3+\sqrt{17})/2 < (-1+\sqrt{5}/2).$$

If

$$l_1 = l_2 = \cdots = l_g = 1,\ l_{g+1} = l_{g+2} = \cdots = l_k = 2,\ 1 \leq g < k,$$

then by (3.13), all the eigenvalues of

$$G = K_t; K_{\underbrace{1,1,\ldots,1,}_{g \text{ times}}\ \underbrace{2,2,\ldots,2,}_{k-g \text{ times}}} \quad \text{where } 1 \leq k \leq t/2, 1 \leq g < k.$$

other than -1 or 0 can be found by solving

$$\begin{aligned}
&P_{K_t;K_{\underbrace{1,1,\ldots,1,}_{g\text{ times}}\underbrace{2,2,\ldots,2,}_{k-g\text{ times}}}}(\lambda) \\
&= \left\{(\lambda-t+1) - \frac{g(\lambda-1)^2}{\lambda^2+2\lambda-1} - \frac{2(k-g)(\lambda-1)^2}{\lambda^2+3\lambda-2}\right\} \\
&\qquad\qquad\qquad \cdot(\lambda^2+2\lambda-1)^g\cdot(\lambda^2+3\lambda-2)^{k-g} \\
&= (\lambda^2+2\lambda-1)^{g-1}(\lambda^2+3\lambda-2)^{k-g-1} \\
&\quad \cdot\big[\lambda^5-(t+2k-g-6)\lambda^4-(5t+g-8)\lambda^3-(3t-8k+g+4)\lambda^2 \\
&\qquad\qquad\qquad\qquad +(7t-8k+g-5)-2t+2k+2\big] \\
&= 0.
\end{aligned}$$

Apply Theorem 2.4 with $l=3$. Then for fixed k and g the sequence

$$\lambda_2(G) = \lambda_2(K_t;K_{\underbrace{1,1,\ldots,1,}_{g\text{ times}}\ \underbrace{2,2,\ldots,2}_{k-g\text{ times}}})$$

is monotone increasing in t and bounded from above. So

$$\lim_{t\to\infty}\lambda_2(K_t;K_{\underbrace{1,1,\ldots,1,}_{g\text{ times}}\ \underbrace{2,2,\ldots,2}_{k-g\text{ times}}})$$

exists and is the largest root of

$$(\lambda^2+2\lambda-1)^g(\lambda^2+3\lambda-2)^{k-g}=0.$$

Then

$$\lim_{t\to\infty}\lambda_2(K_t;K_{\underbrace{1,1,\ldots,1,}_{g\text{ times}}\ \underbrace{2,2,\ldots,2}_{k-g\text{ times}}}) = (-3+\sqrt{17})/2.$$

So

$$0<\lambda_2(K_t;K_{\underbrace{1,1,\ldots,1,}_{g\text{ times}}\ \underbrace{2,2,\ldots,2}_{k-g\text{ times}}}) \le (-3+\sqrt{17})/2 < (-1+\sqrt{5})/2.$$

Case 2. $G|_{V(G)-V(K_t)}$ has two components. Then one component must be an isolated vertex $\{y\}$, and the other component which contains edges must be a complete multipartite graph.

Let $G_3 = G-y$. Without loss of generality, let

$$G_3|_{V(G_3)-V(K_t)} = K_{l_1,l_2,\ldots,l_k}\ ,$$
$$\text{where } k\ge 2,\ 1\le l_1\le l_2\le\cdots\le l_k\le 2. \tag{3.112}$$

Then

$$G|_{V(G)-V(K_t)} = K_{l_1,l_2,\ldots,l_k} \cup K_1,$$
$$\text{where } k \geq 2,\ 1 \leq l_1 \leq l_2 \leq \cdots \leq l_k \leq 2. \tag{3.113}$$

Let $k = 2$. From (3.112),

$$G_3|_{V(G_3)-V(K_t)} = K_{l_1,l_2},\quad 1 \leq l_1 \leq l_2 \leq 2. \tag{3.114}$$

From (3.113),

$$G|_{V(G)-V(K_t)} = K_{l_1,l_2} \cup K_1,\quad 1 \leq l_1 \leq l_2 \leq 2, \tag{3.115}$$

and then

$$\left|G|_{V(G)-V(K_t)}\right| = \left|K_{l_1,l_2} \cup K_1\right| \leq 5.$$

If $l_1 = 1$ or $l_2 = 1$, from (3.115),

$$\left|G|_{V(G)-V(K_t)}\right| = \left|K_{l_1,l_2} \cup K_1\right| \leq 4.$$

The structure of G is already known from Lemma 3.5.

If $l_1 = l_2 = 2$. From (3.114),

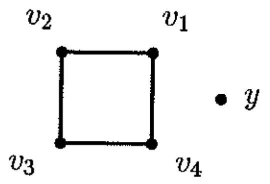
$$G_3|_{V(G_3)-V(K_t)} = K_{2,2}.$$

Then from Lemma 3.5. Part (IV), *Case 6*,

$$G - y = G_3 = K_t; K_{2,2},\ t \geq 4. \tag{3.116}$$

Let

$$G|_{V(G)-V(K_t)} = K_{2,2} \cup K_1 \cong$$



The structure of the graph $K_t; K_{2,2}$ and (3.116) imply that

$$\left|V(K_t) - \bigcap_{i=1}^{4}[N(v_i) \cap V(K_t)]\right| = 4. \tag{3.117}$$

Then

$$G - \{v_3, v_4\} \gg K_t,$$

and

$$G - \{v_3, v_4\}|_{\{v_1, v_2, y\}} = K_{1,1} \cup K_1. \tag{3.118}$$

From (3.118) and Lemma 3.5. Part (III), *Case 2*,

$$G - \{v_3, v_4\} = K_t; K_{1,1} \cup K_1, \tag{3.119}$$

$$\left|N(y) \cap V(K_t)\right| = 1, \tag{3.120}$$

$$\big(N(y) \cap N(K_t)\big) \subseteq \big(N(v_1) \cap V(K_t)\big), \tag{3.121}$$

$$\big(N(y) \cap N(K_t)\big) \subseteq \big(N(v_2) \cap V(K_t)\big). \tag{3.122}$$

Because G is symmetric in v_1, v_2, v_3, v_4, the following also holds:

$$\big(N(y) \cap N(K_t)\big) \subseteq \big(N(v_3) \cap V(K_t)\big), \tag{3.123}$$

$$\big(N(y) \cap N(K_t)\big) \subseteq \big(N(v_4) \cap V(K_t)\big). \tag{3.124}$$

From (3.121)-(3.124),

$$\big[N(y) \cap N(K_t)\big] \subseteq \bigcap_{i=1}^{4} \big[N(v_i) \cap V(K_t)\big]. \tag{3.125}$$

Now (3.117), (3.120) and (3.125) imply that

$$t = \left|V(K_t)\right| \geq 4 + 1 = 5,$$

and so $\qquad 2 = k \leq (t-1)/2$ holds. $\qquad$ (3.126)

From (3.116), (3.119), (3.120), (3.125) and (3.126),

$$G = K_t; K_{2,2} \cup K_1, \ t \geq 5.$$

Let $k \geq 3$.

From (3.112)

$$G_3|_{V(G_3) - V(K_t)} = K_{l_1, l_2, \ldots, l_k},$$
$$\text{where } k \geq 3, \ 1 \leq l_1 \leq l_2 \leq \cdots \leq l_k \leq 2. \tag{3.127}$$

From (3.113)

$$G|_{V(G) - V(K_t)} = K_{l_1, l_2, \ldots, l_k} \cup K_1,$$
$$\text{where } k \geq 3, \ 1 \leq l_1 \leq l_2 \leq \cdots \leq l_k \leq 2. \tag{3.128}$$

Then from (3.127), $G_3 = G - y$ is a graph satisfying the conditions in *Case 1*, so

$$G - y = G_3 = K_t; K_{l_1,l_2,\ldots,l_k}, \\ \text{where } 3 \le k \le t/2,\ 1 \le l_1 \le l_2 \le \cdots \le l_k \le 2. \tag{3.129}$$

For $3 \le k \le (t-1)/2$ and $0 \le g \le k$, let

$$l_i = \begin{cases} 1, & \text{if } 1 \le i \le g; \\ 2, & \text{if } g+1 \le i \le k. \end{cases}$$

Then

$$G|_{V(G)-V(K_t)} = K_{l_1,l_2,\ldots,l_k} \cup K_1 = K_{\underbrace{1,1,\ldots,1,}_{g \text{ times}}\ \underbrace{2,2,\ldots,2}_{k-g \text{ times}}} \cup K_1.$$

Let

$$V_i = \begin{cases} \{v_{i,1}\}, & \text{if } 0 \le i \le g; \\ \{v_{i,1}, v_{i,2}\}, & \text{if } g+1 \le i \le k. \end{cases}$$

The $V_i, i = 1, 2, \ldots, k$ are maximum independent sets in $G_3|_{V(G_3)-V(K_t)}$, and

$$V\big(G_3|_{V(G_3)-V(K_t)}\big) = \bigcup_{i=1}^{k} V_i.$$

Then (3.127) and the structure of the graph $K_t; K_{l_1,l_2,\ldots,l_k}$ imply that

$$\left| V(K_t) - \bigcap_{i=1}^{k} [N(v_{i,1}) \cap N(K_t)] \right| = 2k. \tag{3.130}$$

For any $i \ne j$,

$$G|_{\{v_{i,1},v_{j,1},y\}} = K_{1,1} \cup K_1.$$

Then Lemma 3.5. Part (III), Case 2 implies that

$$G|_{K_t \cup \{v_{i,1},v_{j,1},y\}} = K_t; K_{1,1} \cup K_1, \\ \big|N(y) \cap V(K_t)\big| = 1, \\ \big(N(y) \cup N(K_t)\big) \subseteq \big(N(v_{i,1}) \cup V(K_t)\big), \tag{3.131}$$

and

$$\big(N(y) \cup N(K_t)\big) \subseteq \big(N(v_{j,1}) \cup V(K_t)\big).$$

By symmetry, the following also holds:

$$\big(N(y) \cup N(K_t)\big) \subseteq \big(N(v_{i,1}) \cup V(K_t)\big),\ i = 1, 2, \ldots, k.$$

Then

$$\big(N(y) \cup N(K_t)\big) \subseteq \bigcap_{i=1}^{k} \big[N(v_{i,1}) \cup V(K_t)\big]. \tag{3.132}$$

Also (3.130), (3.131) and (3.132) imply that

$$2k + 1 \leq \big|V(K_t)\big| \leq t, \quad \text{and so} \quad k \leq (t-1)/2. \tag{3.133}$$

From (3.129), (3.131) (3.132) and (3.133),

$$G = K_t; K_{\underbrace{1,1,\ldots,1,}_{g\text{ times}}\ \underbrace{2,2,\ldots,2}_{k-g\text{ times}}} \cup K_1, \text{ where } 3 \leq k \leq (t-1)/2.$$

Let

$$l_i = \begin{cases} 1, & \text{if } 1 \leq i \leq g; \\ 2, & \text{if } g+1 \leq i \leq k. \end{cases},$$
$$\text{where } 0 \leq g \leq k,\ 2 \leq k \leq (t-1)/2.$$

From (3.14), all the eigenvalues of

$$G = K_t; K_{\underbrace{1,1,\ldots,1,}_{g\text{ times}}\ \underbrace{2,2,\ldots,2,}_{k-g\text{ times}}} \cup K_1, \text{ where } \ 2 \leq k \leq (t-1)/2,\ 0 \leq g \leq k,$$

other than -1 or 0 can be found by solving

$$\begin{aligned} &P_{K_t; K_{\underbrace{1,1,\ldots,1,}_{g\text{ times}}\ \underbrace{2,2,\ldots,2}_{k-g\text{ times}}} \cup K_1}(\lambda) \\ &= \Big\{ \big[\lambda^3 - (t-2)\lambda^2 - t\lambda + t - 2\big] - \frac{g(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+2\lambda-1} \\ &\qquad\qquad - \frac{2(k-g)(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+3\lambda-2} \Big\} \\ &\qquad \cdot (\lambda^2+2\lambda-1)^g \cdot (\lambda^2+3\lambda-2)^{k-g} = 0. \end{aligned} \tag{3.134}$$

If $g = k$, (3.134) becomes

$$
\begin{aligned}
&P_{K_t;K_{\underbrace{1,1,\ldots,1}_{k \text{ times}}} \cup K_1}(\lambda)\\
&= \left\{ \left[\lambda^3 - (t-2)\lambda^2 - t\lambda + t - 2\right] - \frac{k(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+2\lambda-1} \right\} (\lambda^2+2\lambda-1)^k\\
&= (\lambda^2+2\lambda-1)^{k-1}\left[\lambda^5-(t+k-4)\lambda^4-(3t-k-3)\lambda^3+(2k-4)\lambda^2+(3t-3k-4)\lambda-t+k+2\right]\\
&= \left\{ (\lambda^2+\lambda-1)\left[\lambda^3-(t+k-3)\lambda^2-(2t-2k-1)\lambda+t-k-2\right]-\lambda \right\}\\
&\qquad \cdot(\lambda^2+2\lambda-1)^{k-1} = 0. \qquad (3.135)
\end{aligned}
$$

If $g = 0$, (3.134) becomes

$$
\begin{aligned}
&P_{K_t;K_{\underbrace{2,2,\ldots,2}_{k \text{ times}}} \cup K_1}(\lambda)\\
&= \left\{ \left[\lambda^3 - (t-2)\lambda^2 - t\lambda + t - 2\right] - \frac{2k(\lambda-1)^2(\lambda^2+\lambda-1)}{\lambda^2+3\lambda-2} \right\} (\lambda^2+3\lambda-2)^k\\
&= \left[\lambda^5-(t+2k-5)\lambda^4-(4t-2k-4)\lambda^3+(4k-6)\lambda^2+(5t-6k-6)\lambda-2t+2k+4\right]\\
&\qquad \cdot(\lambda^2+3\lambda-2)^{k-1}\\
&= \left\{ (\lambda^2+\lambda-1)\left[\lambda^3-(t+2k-4)\lambda^2-(3t-4k-1)\lambda+2t-2k-3\right]-2\lambda+1 \right\}\\
&\qquad \cdot(\lambda^2+3\lambda-2)^{k-1} = 0. \qquad (3.136)
\end{aligned}
$$

If $1 < g < k$, (3.134) becomes

$$
\begin{aligned}
&P_{K_t;K_{\underbrace{1,1,\ldots,1,}_{g \text{ times}}\underbrace{2,2,\ldots,2}_{k-g \text{ times}}} \cup K_1}(\lambda)\\
&= \Big\{ \left[\lambda^3 - (t-2)\lambda^2 - t\lambda + t - 2\right](\lambda^2+2\lambda-1)(\lambda^2+3\lambda-2)\\
&-g(\lambda-1)^2(\lambda^2+\lambda-1)(\lambda^2+3\lambda-2) - 2(k-g)(\lambda-1)^2(\lambda^2+\lambda-1)(\lambda^2+2\lambda-1) \Big\}\\
&\qquad \cdot(\lambda^2+2\lambda-1)^{g-1}(\lambda^2+3\lambda-2)^{k-g-1}\\
&= \big[\lambda^7-(t+2k-g-7)\lambda^6-(6t+2k-13)\lambda^5-(7t-10k+3g+3)\lambda^4\\
&\qquad +(9t+g-22)\lambda^3+(8t-14k+2g-2)\lambda^2-(9t-10k+g-14)\lambda+2t-2k-4\big]\\
&\qquad \cdot(\lambda^2+2\lambda-1)^{g-1}\cdot(\lambda^2+3\lambda-2)^{k-g-1}\\
&= \Big\{ (\lambda^2+\lambda-1)\big[\lambda^5-(t+2k-g-6)\lambda^4-(5t-g+8)\lambda^3-(3t-8k+g+5)\lambda^2\\
&\qquad +(7t+g-8k-9)\lambda+(-2t+2k+2)\big]+3\lambda-2 \Big\}\\
&\qquad \cdot(\lambda^2+2\lambda-1)^{g-1}(\lambda^2+3\lambda-2)^{k-g-1} = 0. \qquad (3.137)
\end{aligned}
$$

Apply Theorem 2.4 with $l = 3$. Then for fixed k and g the sequence

$$\lambda_2(G) = \lambda_2(K_t; K_{\underbrace{1,1,\ldots,1,}_{g \text{ times}} \underbrace{2,2,\ldots,2}_{k-g \text{ times}}} \cup K_1),$$

where $t \geq 3,\ 2 \leq k \leq (t-1)/2, 0 \leq g \leq k.$

is monotone increasing in t and bounded from above. So

$$\lim_{t \to \infty} \lambda_2(K_t; K_{\underbrace{1,1,\ldots,1,}_{g \text{ times}} \underbrace{2,2,\ldots,2}_{k-g \text{ times}}} \cup K_1)$$

. where $t \geq 3,\ 2 \leq k \leq (t-1)/2, 0 \leq g \leq k,$

exists and (from (3.135), (3.136), and (3.137)) is the largest root of

$$(\lambda^2 + 2\lambda - 1)^g (\lambda^2 + 3\lambda - 2)^{k-g} (\lambda^2 + \lambda - 1) = 0.$$

Then

$$\lim_{t \to \infty} \lambda_2(K_t; K_{\underbrace{1,1,\ldots,1,}_{g \text{ times}} \underbrace{2,2,\ldots,2}_{k-g \text{ times}}} \cup K_1) = (-1 + \sqrt{5})/2,$$

where $t \geq 3,\ 2 \leq k \leq (t-1)/2,\ 0 \leq g \leq k.$

From (3.135), (3.136), and (3.137),

$$\lambda_2(K_t; K_{\underbrace{1,1,\ldots,1,}_{g \text{ times}} \underbrace{2,2,\ldots,2}_{k-g \text{ times}}} \cup K_1) \neq (-1 + \sqrt{5})/2,$$

where $t \geq 3,\ 2 \leq k \leq (t-1)/2,\ 0 \leq g \leq k.$

Therefore

$$0 < \lambda_2(K_t; K_{\underbrace{1,1,\ldots,1,}_{g \text{ times}} \underbrace{2,2,\ldots,2}_{k-g \text{ times}}} \cup K_1) < (-1 + \sqrt{5})/2,$$

where $t \geq 3,\ 2 \leq k \leq (t-1)/2,\ 0 \leq g \leq k.$

SECTION 4. GRAPHS WITH SUBDOMINANT EIGENVALUE NOT EXCEEDING $-1+\sqrt{2}$

In this section, we will investigate the structure of graphs with $0 < \lambda_2 \leq -1 + \sqrt{2}$. We will construct those graphs by starting from MCM-subgraphs $K_{r_1,r_2,\ldots,r_t}$ and adding new structures that increase λ_2 but with restricted upper bound $\lambda_2 \leq -1 + \sqrt{2}$ for λ_2. In Theorem 4.2, we will determine all graphs without isolated vertices and with $0 < \lambda_2 \leq -1+\sqrt{2}$ and the proof of Theorem 4.2 also give us all graphs minimal with respect to $\lambda_2 > -1 + \sqrt{2}$. These are listed in Theorem 4.3.

Let G be a connected simple graph. Then there is a complete multipartite graph $K_{r_1,r_2,\ldots,r_t}$ such that

$$G \gg K_{r_1,r_2,\ldots,r_t},$$
$$\text{where } \; r_i \geq 1, \; i = 1,2,\ldots,t, \; t \geq 2,$$

as an MCM-subgraph. Here we take the convention to choose an MCM-subgraph $K_{r_1,r_2,\ldots,r_t}$ as in Section 3. For any

$$u \in V(G) - V(K_{r_1,r_2,\ldots,r_t}),$$

if u adjacent to some vertex in $V(K_{r_1,r_2,\ldots,r_t})$, then u is not adjacent to all vertices in $V(K_{r_1,r_2,\ldots,r_t})$.
This is true since otherwise

$$G \gg K_{r_1,r_2,\ldots,r_t,1},$$

which contradicts the fact that $K_{r_1,r_2,\ldots,r_t}$ is maximum.

Let

$$V_i = \{v_{i,j} \mid 1 \leq j \leq r_i \}, \; 1 \leq i \leq t,$$

be maximum independent sets in $K_{r_1,r_2,\ldots,r_t}$.

Then

$$G|_{\cup_1^t V_i} = K_{r_1,r_2,\ldots,r_t}.$$

Definition. *Let $u \in G$, $\widetilde{V} \subseteq V(G)$. Then the notation $u \sim \widetilde{V}$ means u adjacent to all the vertices in $\widetilde{V}$.*

Definition. *Let*

$$K_{r_1,r_2,\ldots,r_q;r_{q+1},\ldots,r_t}; K_1,$$
$$\text{where } t \geq 3, \; 1 \leq q \leq t-2, \; r_i \geq 1, \; i = 1,2,\ldots,t, \tag{4.1}$$

denote a graph such that

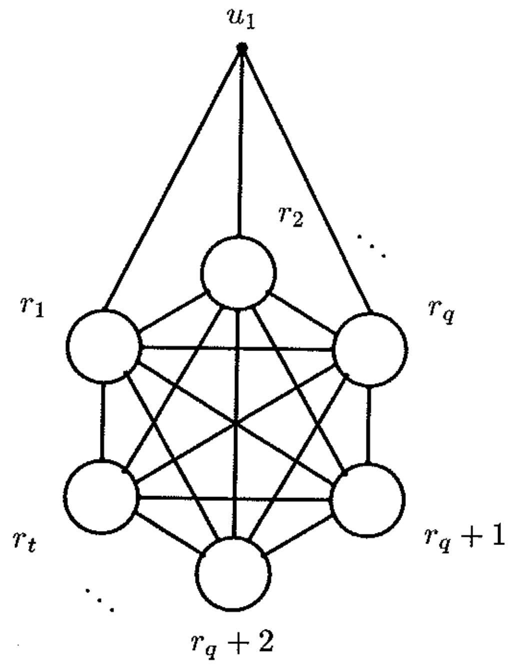
$$K_{r_1,r_2,\ldots,r_q;r_{q+1},\ldots,r_t}; K_1 \gg K_{r_1,r_2,\ldots,r_t}$$

as an MCM-subgraph and there is a vertex u_1 *of the graph which satisfies the conditions*

$$\{u_1\} = V(K_{r_1,r_2,\ldots,r_q;r_q+1,\ldots,r_t}; K_s) - V(K_{r_1,r_2,\ldots,r_t}),$$

and

$$\begin{cases} u_1 \sim V_i, & \text{for } i = 1,2,\ldots,q; \\ u_1 \not\sim V_i, & \text{for } i = q+1, q+2, \ldots, t-1, t. \end{cases}$$



$K_{r_1,r_2,\ldots,r_q;r_{q+1},\ldots,r_t}; K_1$

where $t \geq 3,\ 1 \leq q \leq t-2.\ \ r_i \geq 1,\ 1 \leq i \leq t.$

Figure 24

$$P_{K_{r_1,r_2,\cdots,r_t};K_s} =$$

$$\det \begin{array}{c|ccccccccccccccccc}
 & u_1 & u_2 & \dots & u_{s-1} & u_s & V_1 & V_2 & \dots & V_{t-2s} & V_{t-2s+1} & V_{t-2s+2} & \dots & V_{t-3} & V_{t-2} & V_{t-1} & V_t \\
\hline
u_1 & \lambda & -1 & \dots & -1 & -1 & -r_1 & -r_2 & \dots & -r_{t-2s} & -r_{t-2s+1} & -r_{t-2s+2} & \dots & -r_{t-3} & -r_{t-2} & 0 & 0 \\
u_2 & -1 & \lambda & \dots & -1 & -1 & -r_1 & -r_2 & \dots & -r_{t-2s} & -r_{t-2s+1} & -r_{t-2s+2} & \dots & 0 & 0 & -r_{t-1} & -r_t \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
u_{s-1} & -1 & -1 & \dots & \lambda & -1 & -r_1 & -r_2 & \dots & -r_{t-2s} & -r_{t-2s+1} & -r_{t-2s+2} & \dots & -r_{t-3} & -r_{t-2} & -r_{t-1} & -r_t \\
u_s & -1 & -1 & \dots & -1 & \lambda & -r_1 & -r_2 & \dots & -r_{t-2s} & 0 & 0 & \dots & -r_{t-3} & -r_{t-2} & -r_{t-1} & -r_t \\
V_1 & -1 & -1 & \dots & -1 & -1 & \lambda & -r_2 & \dots & -r_{t-2s} & -r_{t-2s+1} & -r_{t-2s+2} & \dots & -r_{t-3} & -r_{t-2} & -r_{t-1} & -r_t \\
V_2 & -1 & -1 & \dots & -1 & -1 & -r_1 & \lambda & \dots & -r_{t-2s} & -r_{t-2s+1} & -r_{t-2s+2} & \dots & -r_{t-3} & -r_{t-2} & -r_{t-1} & -r_t \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
V_{t-2s} & -1 & -1 & \dots & -1 & -1 & -r_1 & -r_2 & \dots & \lambda & -r_{t-2s+1} & -r_{t-2s+2} & \dots & -r_{t-3} & -r_{t-2} & -r_{t-1} & -r_t \\
V_{t-2s+1} & -1 & -1 & \dots & -1 & 0 & -r_1 & -r_2 & \dots & -r_{t-2s} & \lambda & -r_{t-2s+2} & \dots & -r_{t-3} & -r_{t-2} & -r_{t-1} & -r_t \\
V_{t-2s+2} & -1 & -1 & \dots & -1 & 0 & -r_1 & -r_2 & \dots & -r_{t-2s} & -r_{t-2s+1} & \lambda & \dots & -r_{t-3} & -r_{t-2} & -r_{t-1} & -r_t \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
V_{t-3} & -1 & 0 & \dots & -1 & -1 & -r_1 & -r_2 & \dots & -r_{t-2s} & -r_{t-2s+1} & -r_{t-2s+2} & \dots & \lambda & -r_{t-2} & -r_{t-1} & -r_t \\
V_{t-2} & -1 & 0 & \dots & -1 & -1 & -r_1 & -r_2 & \dots & -r_{t-2s} & -r_{t-2s+1} & -r_{t-2s+2} & \dots & -r_{t-3} & \lambda & -r_{t-1} & -r_t \\
V_{t-1} & 0 & -1 & \dots & -1 & -1 & -r_1 & -r_2 & \dots & -r_{t-2s} & -r_{t-2s+1} & -r_{t-2s+2} & \dots & -r_{t-3} & -r_{t-2} & \lambda & -r_t \\
V_t & 0 & -1 & \dots & -1 & -1 & -r_1 & -r_2 & \dots & -r_{t-2s} & -r_{t-2s+1} & -r_{t-2s+2} & \dots & -r_{t-3} & -r_{t-2} & -r_{t-1} & \lambda
\end{array}_{(t+s)\times(t+s)}$$

$$= \left\{ s\lambda - s + 1 - \sum_{i=1}^{t-2s} \left(\frac{r_i}{\lambda + r_i} \right) - \sum_{j=1}^{s} \frac{\lambda^2}{\lambda + 1 - \frac{r_{t-2j+1}}{\lambda + r_{t-2j+1}} - \frac{r_{t-2j+2}}{\lambda + r_{t-2j+2}}} \right\} \cdot \prod_{i=1}^{t} (\lambda + r_i) \cdot \prod_{i=1}^{s} \left[\lambda + 1 - \frac{r_{t-2j+1}}{\lambda + r_{t-2j+1}} - \frac{r_{t-2j+2}}{\lambda + r_{t-2j+2}} \right]$$

Table 9.

Definition. *Let*

$$K_{r_1,r_2,\ldots,r_t};K_s,$$
$$\text{where } t \geq 3,\ 1 \leq s \leq t/2,\ r_i \geq 1, i = 1,2,\ldots,t,$$

denote a graph such that

$$K_{r_1,r_2,\ldots,r_t};K_s \gg K_{r_1,r_2,\ldots,r_t}$$

as an MCM-subgraph and there is a subset V_0 *of the vertex set* $V(G)$ *with*

$$V_0 = V(K_{r_1,r_2,\ldots,r_t};K_s) - V(K_{r_1,r_2,\ldots,r_t}) = \{u_1, u_2, \ldots, u_s\},$$

$$K_{r_1,r_2,\ldots,r_t};K_s\big|_{V_0} = K_s,$$

where

$$\begin{cases} u_j \not\sim V_i, & \text{for } i = t-2j+1, t-2j+2; \\ u_j \sim V_i, & \text{for } i = 1,2,\ldots,t-2j, t-2j+3, t-2j+4, \ldots, t. \end{cases}$$
$$j = 1,2,\ldots.s.$$

In particular, if in (4.1), $q = t-2$, *then*

$$\begin{aligned} &K_{r_1,r_2,\ldots,r_q;r_{q+1},\ldots,r_t};K_1 \\ =&K_{r_1,r_2,\ldots,r_{t-2};r_{t-1},r_t};K_1 \\ =&K_{r_1,r_2,\ldots,r_t};K_1. \end{aligned}$$

Lemma 4.1. *The eigenvalues of the graph* $K_{r_1,r_2,\ldots,r_t};K_s$ *other than* -1 *and* 0 *can be found by solving*

$$\begin{aligned} &P_{K_{r_1,r_2,\ldots,r_t};K_s}(\lambda) \\ &\quad = \left\{ s\lambda - s + 1 - \sum_{i=1}^{t-2s}\left(\frac{r_i}{\lambda + r_i}\right) - \sum_{j=1}^{s} \frac{\lambda^2}{\lambda + 1 - \frac{r_{t-2j+1}}{\lambda + r_{t-2j+1}} - \frac{r_{t-2j+2}}{\lambda + r_{t-2j+2}}} \right\} \\ &\qquad \cdot \prod_{i=1}^{t}(\lambda + r_i) \cdot \prod_{i=1}^{s}\left[\lambda + 1 - \frac{r_{t-2j+1}}{\lambda + r_{t-2j+1}} - \frac{r_{t-2j+2}}{\lambda + r_{t-2j+2}}\right] = 0, \end{aligned} \tag{4.2}$$

where $P_{K_{r_1,r_2,\ldots,r_t};K_s}$ *is the determinant of the matrix in the Table 9.*

Proof. The result comes from applying Theorem 2.2 to the graph $K_{r_1,r_2,\ldots,r_t};K_s$.

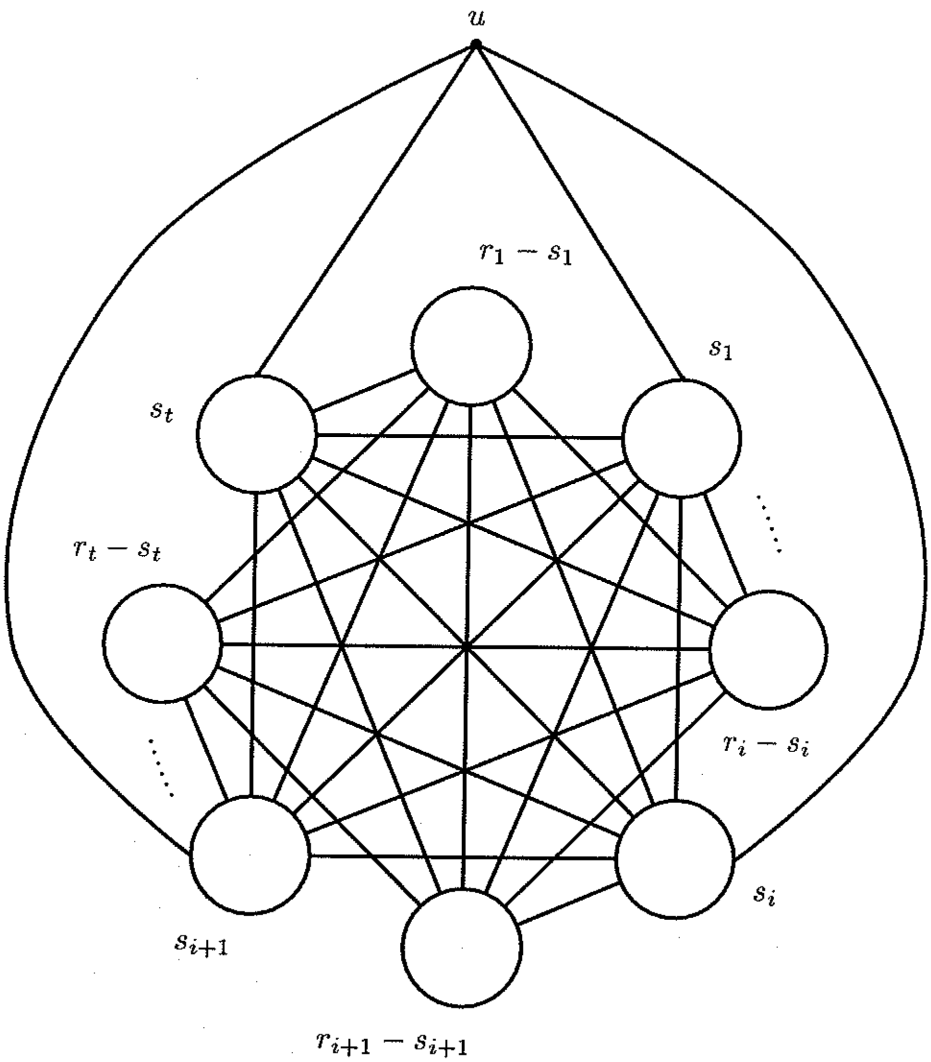


Figure 25

In this section, simple graphs without isolated vertices with subdominant eigenvalue greater than zero and not exceeding $-1+\sqrt{2}$ will be determined.

First, consider those simple graphs G without isolated vertices which satisfy the condition

$$0 < \lambda_2(G) < (-1+\sqrt{5})/2. \tag{4.3}$$

If there is a vertex u, $u \in V(G) - V(K_{r_1,r_2,\ldots,r_t})$, which is not adjacent to any vertex in $K_{r_1,r_2,\ldots,r_t}$, then the fact that G is connected implies that there is a vertex $v \in V(G) - V(K_{r_1,r_2,\ldots,r_t})$, v adjacent to some but not all vertices in $K_{r_1,r_2,\ldots,r_t}$, and there is a path of length greater than or equal to 1 in

$$G|_{V(G)-V(K_{r_1,r_2,\ldots,r_t})}$$

which joins u and v. Then $G \gg P_4 = F_2$, which contradicts (4.3).

Let

$$\begin{gathered} u \in V(G) - V(K_{r_1,r_2,\ldots,r_t}), \\ u \sim v_{i,j},\ j = 1,2,\ldots,s_i,\ 0 \le s_i \le r_i, i = 1,2,\ldots,t \end{gathered} \tag{4.4}$$

as in *Figure 25.*

Because u must be adjacent to some vertex in $K_{r_1,r_2,\ldots,r_t}$, some s_i is nonzero. There exists i_0, therefore,

$$1 \le i_0 \le t, \text{ such that } 1 \le s_i \le r_{i_0}, \tag{4.5}$$

where u is not adjacent to all the vertices in $K_{r_1,r_2,\ldots,r_t}$, since otherwise

$$G \gg K_{r_1,r_2,\ldots,r_t,1},$$

which contradicts the fact that $K_{r_1,r_2,\ldots,r_t}$ is maximum. Therefore, there exists i_1,

$$1 \le i_1 \le t, \text{ such that } 0 \le s_{i_1} \le r_{i_1} - 1. \tag{4.6}$$

Case 1. Suppose

$$\text{for all } i,\ 1 \le i \le t,\ s_i \ne 0,\ 1 \le s_i \le r_i. \tag{4.7}$$

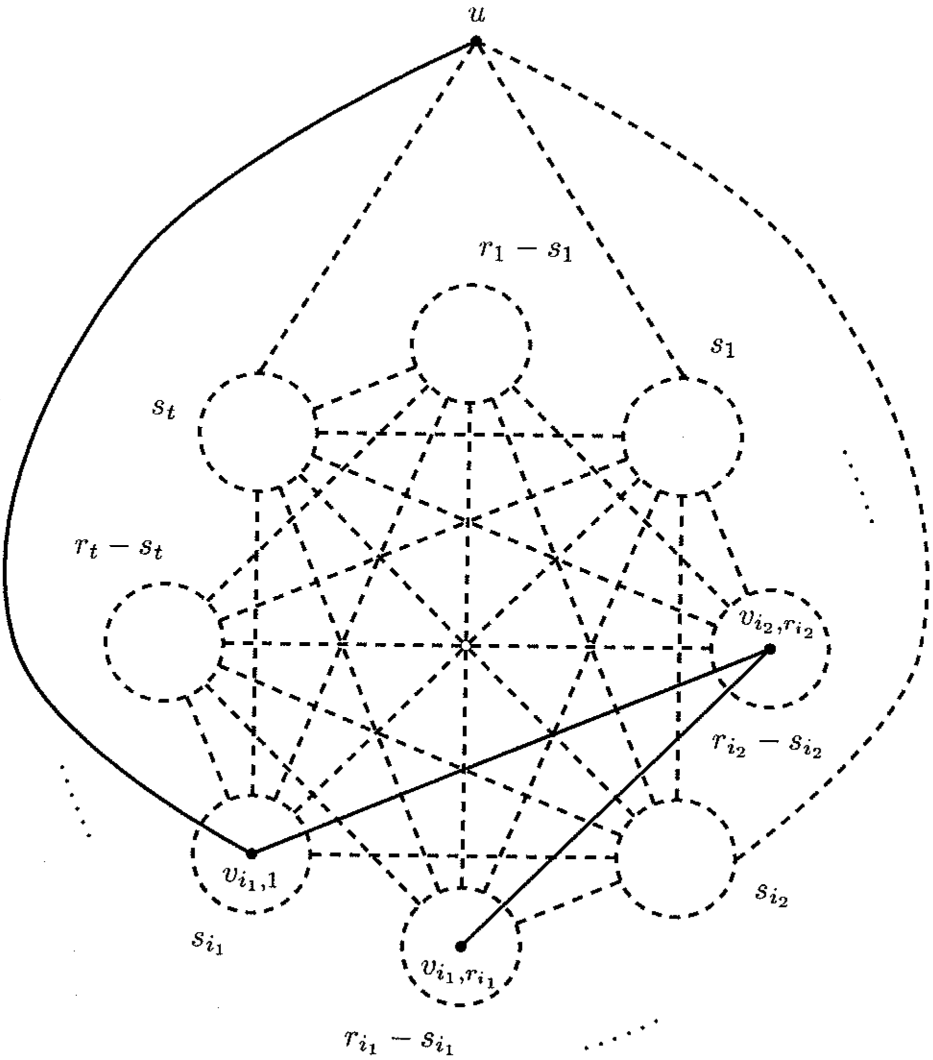
Then

$$s_i = r_i, \quad \text{for all } i \ne i_1,\ 1 \le i \le t. \tag{4.8}$$

Otherwise, if there is $i_2 \ne i_1$, such that $s_{i_2} \ne r_{i_2}$, then from (4.7), $1 \le s_{i_2} \le r_{i_2} - 1$, which implies that $r_{i_2} \ge 2$, and, as in *Figure 26,*

$$G|_{\{u,v_{i_1,1},v_{i_2,r_{i_2}},v_{i_1,r_{i_1}}\}} = P_4 = F_2,$$

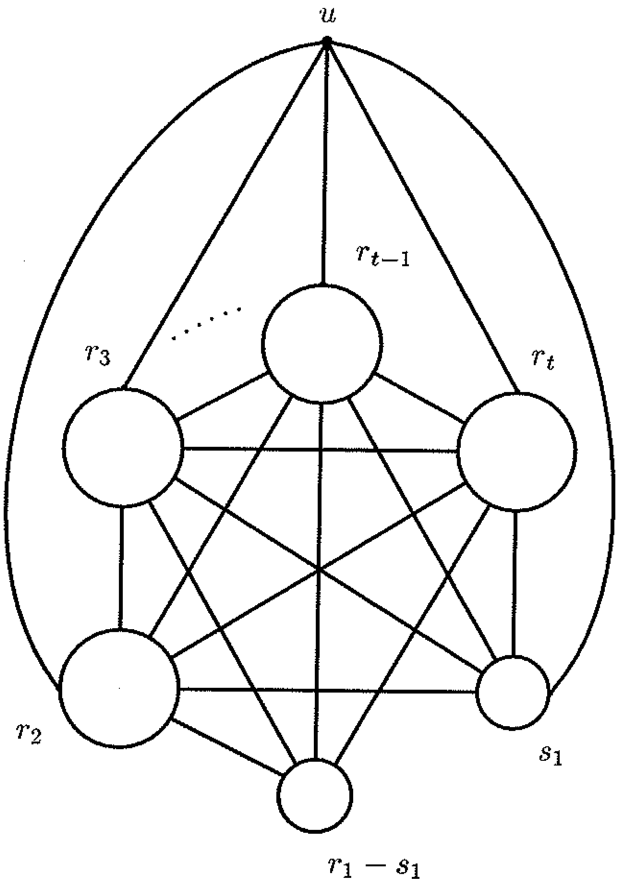
which contradicts the assumption (4.3).



where $s_{i_1} \geq 1, s_{i_2} \geq 1,$

$r_{i_1} - s_{i_1} \geq 1,$ and $r_{i_2} - s_{i_2} \geq 1.$

Figure 26



where $1 \leq s_1 \leq r_1 - 2,\ r_1 \geq 3,$

and $1 \leq s_i = r_i$, for $2 \leq i \leq t$

Figure 27

If $s_{i_1} = r_{i_1} - 1$, then (4.8) implies that

$$G - v_{i_1, r_{i_1}} \gg K_{r_1, r_2, \ldots, r_{i_1} - 1, \ldots, r_t, 1}.$$

Then

$$G \gg K_{r_1, r_2, \ldots, r_{i_1} - 1, \ldots, r_t, 1},$$

and

$$G \gg K_{r_1, r_2, \ldots r_{i_1}, \ldots, r_t},$$

where

$$r_1 + r_2 + \cdots + r_{i_1} - 1 + \cdots + r_t + 1 = r_1 + r_2 + \cdots + r_{i_1} + \cdots + r_t.$$

By convention, $K_{r_1, \ldots, r_{i_1} - 1, \ldots, r_t, 1}$ should have been taken as an MCM-subgraph in G.

So in this case, without loss of generality, suppose

$$\begin{cases} 1 \le s_1 \le r_1 - 2, \ r_1 \ge 3; \\ 1 \le s_i = r_i, \ \text{ for }, \ 2 \le i \le t. \end{cases} \tag{4.9}$$

Therefore

$$G|_{K_{r_1, r_2, \ldots, r_t} \cup \{u\}}$$

is a graph as in *Figure 27*, where the parameters $s_1, r_1, r_2, \cdot, \ldots, r_t$, must be chosen such that $0 < \lambda_2(G) < (-1 + \sqrt{5})/2$. It is not very hard to determine those parameters.

Case 2. Suppose there exists i_3,

$$1 \le i_3 \le t, \text{ such that } s_{i_3} = 0. \tag{4.10}$$

Then

$$\text{for any } i \ne i_3, \ i = 1, 2, \ldots, t, \text{ either } s_i = r_i, \text{ or } s_i = 0. \tag{4.11}$$

Otherwise, if there is a $i_4, i_4 \ne i_3$, and $0 < s_{i_4} < r_{i_4}$, then as in *Figure 28*,

$$G|_{\{u, v_{i_4,1}, v_{i_3,1}, v_{i_4, r_{i_4}}\}} = P_4 = F_2.$$

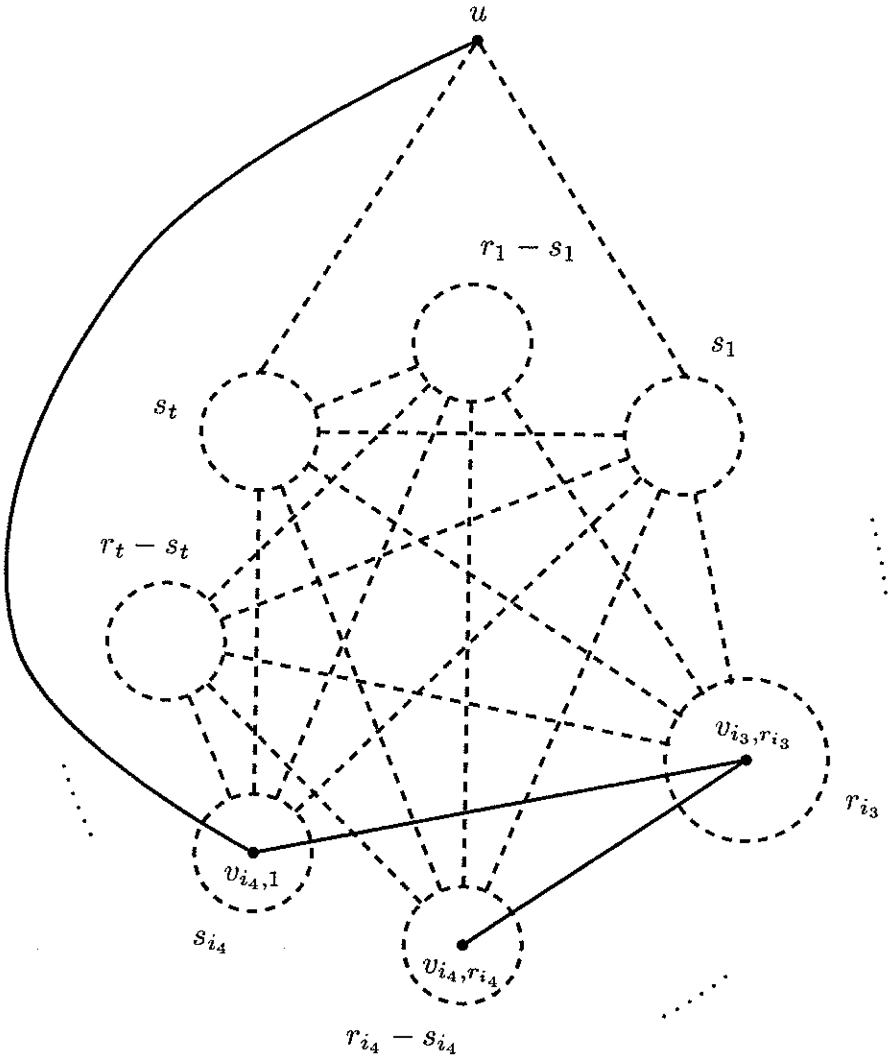
This contradicts (4.3).

From (4.6), (4.10) and (4.11), without loss of generality, suppose

$$u \sim v_{i,j}, \ 1 \le i \le q \le t - 1, \ 1 \le j \le r_i.$$

If $q = t - 1$, then $G \gg K_{r_1, r_2, \ldots, r_t + 1}$, which contradicts the fact that $K_{r_1, r_2, \ldots, r_t}$ is maximum. So

$$u \sim v_{i,j}, \ 1 \le i \le q \le t - 2, \ 1 \le j \le r_i.$$



where $s_{i_3} = 0,\ 0 < s_{i_4} < r_{i_4}$,

$1 \leq i_3,\ i_4 \leq t,\ i_3 \neq i_4$

and $r_i \geq 1,\ i = 1, 2, 3, \ldots, t$.

Figure 28

Because

$$G|_{\{u,v_{1,1},v_{2,1},\ldots,v_{r_t,1}\}}$$

is of the form $A_{q,t-q}$, as in Lemma 3.1, q and t should fall into one of the cases (3.1), (3.2), (3.3), and (3.4).

Therefore

$$G|_{K_{r_1,r_2,\ldots,r_t}\cup\{u\}}$$

is a graph of the form

$$K_{r_1,r_2,\ldots,r_q;r_{q+1},\ldots,r_t};K_1$$

with $1 \leq q \leq t-2$, $t \geq 3$, as in *Figure 24.* There the parameters $q, r_1, r_2, \ldots, r_t$, must be chosen such that $0 \leq \lambda_2(G) < (-1+\sqrt{5})/2$.

It turn out to be a difficult task to determine those parameters $r_1, r_2, \ldots, r_t$, which satisfy $0 < \lambda_2(G) < (-1+\sqrt{5})/2$.

For example: Let $t=4$, $q=1$. Consider the graphs $K_{r_1;r_2,r_3,r_4};K_1$ given in the *Figure 29.* From Theorem 2.3, the eigenvalues of the graphs $K_{r_1;r_2,r_3,r_4};K_1$ other than 0 and -1 can be found by solving

$$\det\begin{pmatrix} \lambda & -r_1 & 0 & 0 & 0 \\ -1 & \lambda & -r_2 & -r_3 & -r_4 \\ 0 & -r_1 & \lambda & -r_3 & -r_4 \\ 0 & -r_1 & -r_2 & \lambda & -r_4 \\ 0 & -r_1 & -r_2 & -r_3 & \lambda \end{pmatrix} = 0.$$

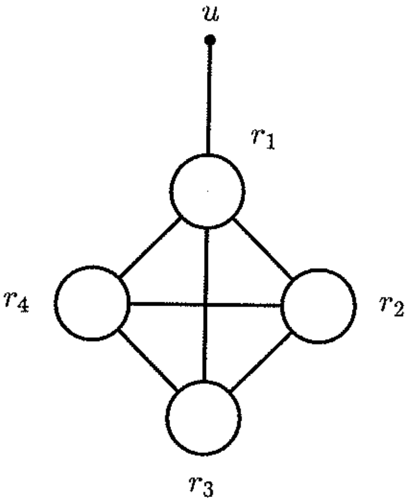
i.e.

$$\begin{aligned} &\lambda^5 + (-r_1 - r_1r_2 - r_1r_3 - r_1r_4 - r_2r_3 - r_2r_4 - r_3r_4)\lambda^3 \\ &\qquad +(-r_1r_2r_3 - r_1r_2r_4 - r_1r_3r_4 - r_2r_3r_4)\lambda^2 \\ &\qquad\qquad +(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 - 3r_1r_2r_3r_4)\lambda + 2r_1r_2r_3r_4 = 0. \end{aligned}$$

But then, for the parameters $r_1 = 6$, $r_2 = 244,224,577$, $r_3 = 4035$, and $r_4 = 10$, the corresponding graph

$$K_{6;244224577,4035,10};K_1$$

as in *Figure 29* is a graph minimal with respect to $\lambda_2(G) \geq (-1+\sqrt{5})/2$.



$$K_{r_1;r_2,r_3,r_4};K_1$$

where $r_i \geq 1,\ i = 1,2,3,4.$

Figure 29

Second, consideration will be limited to the graphs with

$$0 < \lambda_2(G) \leq -1 + \sqrt{2}. \tag{4.12}$$

Definition. *For fixed*

$$j_g,\ g = 1,2,\ldots,k,$$
$$\text{where1} \leq j_1 < j_2 < \cdots < j_k \leq t,$$

and for fixed

$$r_i,\quad i \notin \{j_g \mid 1 \leq g \leq k\},\ 1 \leq i \leq t,$$

define

$$K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_2-1},\infty,r_{j_2+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t};K_s$$

to be

$$\{K_{r_1,r_2,\ldots,r_{j_1-1},r_{j_1},r_{j_1+1},\ldots,r_{j_k-1},r_{j_k},r_{j_k+1},\ldots,r_{t-1},r_t};K_s \mid r_{j_g} \geq 1, 1 \leq g \leq k\},$$

which is an infinite family of the graphs of the form $K_{r_1,r_2,\ldots,r_t};K_s$,

and define

$$\lambda_2(K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_2-1},\infty,r_{j_2+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t};K_s)$$

to be

$$\lim_{\substack{r_{j_1}\to\infty\\ r_{j_2}\to\infty\\ \vdots\\ r_{j_k}\to\infty}} \lambda_2(K_{r_1,r_2,\ldots,r_t};K_s) \tag{4.13}$$

if the right hand side limit exists.

Definition. *Let $\alpha > 0$ be a real number. For fixed integers $j_1, j_2, \cdots, j_k$, where*

$$1 \le j_1 < j_2 < \cdots < j_k \le t,$$

and for fixed integers r_i, where

$$i \notin \{j_g \mid 1 \le g \le k\}, 1 \le i \le t, \text{ and } r_i \ge 1$$

$$K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_2-1},\infty,r_{j_2+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t};K_s$$

is called a critical family for $\lambda_2 > \alpha$, if

$$\lambda_2(K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_2-1},\infty,r_{j_2+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t};K_s) > \alpha,$$

and for all i such that

$$i \notin \{j_g \mid 1 \le g \le k\}, 1 \le i \le t,$$

either $r_i = 1,$
or

$$r_i \ge 2 \text{ and}$$
$$\lambda_2(K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{i-1},r_i-1,r_{i+1},\ldots,r_{j_k},\infty,r_{j_k+1},\ldots,r_{t-1},r_t};K_s) \le \alpha.$$

If there is no confusion in the context, the statement "for $\lambda_2 > \alpha$" in the Definition will be omitted.

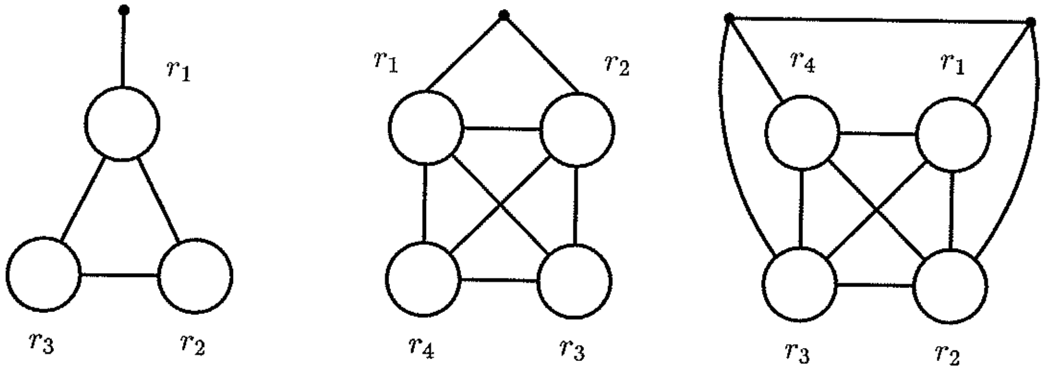
From the Definition above, if

$$K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_2-1},\infty,r_{j_2+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_t};K_s$$

is a critical family for $\lambda_2 > \alpha$, then in order that the condition

$$\lambda_2(K_{r_1,\ldots,r_{j_1-1},r_{j_1},r_{j_1+1},\ldots,r_{j_2-1},r_{j_2},r_{j_2+1},\ldots,r_{j_k-1},r_{j_k},r_{j_k+1},\ldots,r_t};K_s) \le \alpha \le l + \sqrt{2}$$

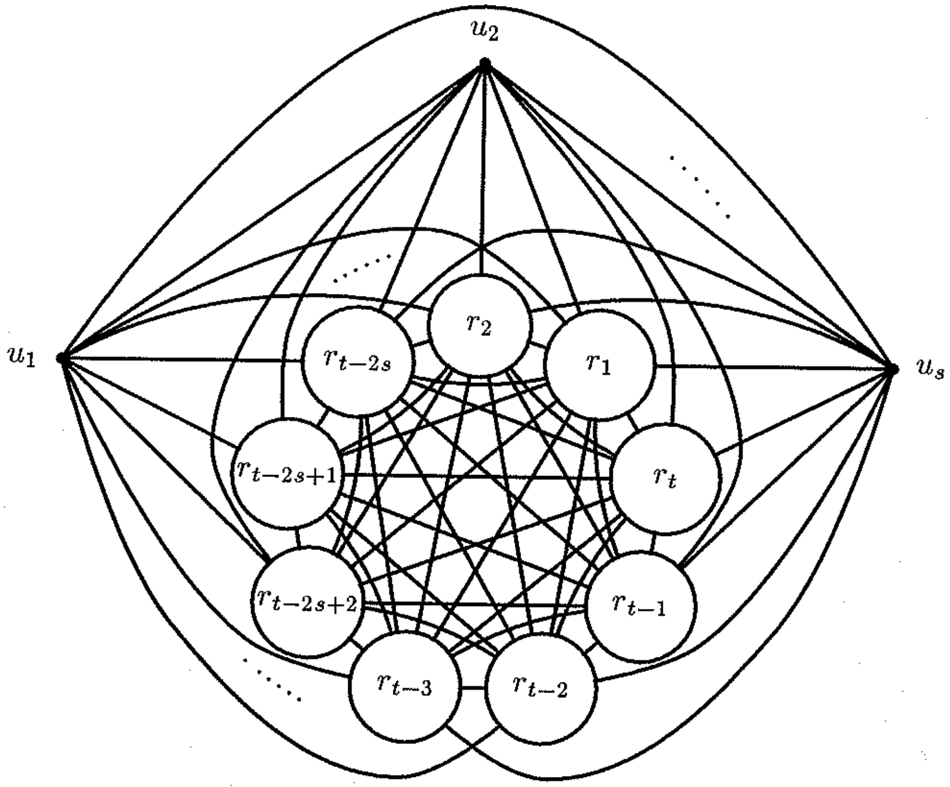
be satisfied, at least one of the parameters $r_{j_g}, 1 \le g \le k$ must be bounded from above.



$K_{r_1,r_2,r_3};K_1$
where $r_1 \geq 1$,
and $r_2 \geq r_3 \geq 1$,

$K_{r_1,r_2,r_3,r_4};K_1$
where $r_1 \geq 1$,
and $r_2 \geq r_3 \geq r_4 \geq 1$,

$K_{r_1,r_2,r_3,r_4};K_2$
where $r_1 \geq r_2 \geq 1$,
and $r_3 \geq r_4 \geq 1$.



$K_{r_1,r_2,\cdots,r_t};K_s$,
where $t \geq 3, 1 \leq s \leq [t/2]$

Figure 30

Theorem 4.2. *Let G be a simple graph without isolated vertices. Then G satisfies $0 < \lambda_2(G) \le -1 + \sqrt{2}$, if and only if G is one of the following graphs:*

(I) $K_{r_1,r_2,r_3};K_1$, where $r_1 \ge 1, r_2 \ge r_3 \ge 1$,

$$\begin{array}{lllll}
\text{for} & r_1 \ge 1, & r_2 \ge 1, & \text{and} & r_3 = 1; \\
 & r_1 = 1, & r_2 \ge r_3, & \text{and} & r_3 \ge 1; \\
 & r_1 \ge 1, & r_2 = 2, & \text{and} & r_3 = 2; \\
 & r_1 \ge 1, & r_2 = 3, & \text{and} & r_3 = 2; \\
 & r_1 = 2, & r_2 \ge r_3, & \text{and} & r_3 = 2; \\
 & r_1 = 3\text{–}7, & r_2 = 4, & \text{and} & r_3 = 2; \\
 & r_1 = 3\text{–}4, & r_2 = 5, & \text{and} & r_3 = 2; \\
 & r_1 = 3, & r_2 = 6\text{–}8, & \text{and} & r_3 = 2; \\
 & r_1 = 2\text{–}4, & r_2 = 3, & \text{and} & r_3 = 3; \\
 & r_1 = 2, & r_2 = 4\text{–}7, & \text{and} & r_3 = 3; \\
 & r_1 = 2, & r_2 = 4, & \text{and} & r_3 = 4.
\end{array}$$

(II) $K_{r_1,r_2,r_3,r_4};K_1$, where $r_1 \ge r_2, \quad r_3 \ge r_4 \ge 1$,

$$\begin{array}{llllll}
\text{for} & r_1 \ge 1, & r_2 = 1, & r_3 = 2, & \text{and} & r_4 = 1; \\
 & r_1 \ge r_2, & r_2 = 2, & r_3 = 2, & \text{and} & r_4 = 1: \\
 & r_1 = 3\text{–}7, & r_2 = 3, & r_3 = 2, & \text{and} & r_4 = 1; \\
 & r_1 = 4, & r_2 = 4, & r_3 = 2, & \text{and} & r_4 = 1; \\
 & r_1 = 1, & r_2 = 1, & r_3 = 3, & \text{and} & r_4 = 1.
\end{array}$$

(III) $K_{r_1,r_2,\ldots,r_t};K_s$,

where $t \ge 4$ and $1 \le s \le [t/2]$,
for $r_1 \ge r_2 \ge r_3 \ge \cdots \ge r_{t-2s} \ge 1$
and $r_{t-2s+1} = 1, r_{t-2s+2} = 1, \ldots, r_t = 1$.

Proof.

$$\text{Let } u \in V(G) - V(K_{r_1,r_2,\ldots,r_t}),$$
$$u \sim v_{i,j},\ j = 1,2,\ldots,s_i,\ 0 \le s_i \le r_i, i = 1,2,\ldots,t.$$

as in *Figure 25*. Because the assumption in (4.12) is stronger than that in (4.3), the formulas (4.4)–(4.11) hold.

Case 1. Suppose

$$\text{for all } i,\ 1 \le i \le t,\ s_i \ne 0,\ 1 \le s_i \le r_i. \tag{4.14}$$

Then from (4.9), without loss of generality, suppose

$$\begin{cases} 1 \le s_1 \le r_1 - 2,\ r_1 \ge 3; \\ 1 \le s_i = r_i, \quad \text{for } \ 2 \le i \le t. \end{cases}$$

Then

$$G|_{K_{r_1,r_2,\ldots,r_t} \cup \{u\}}$$

is a graph as in *Figure 27.* But then

$$G|_{\{u,v_{1,1},v_{1,r_1-1},v_{1,r_1}\}} = C_{2,2} = \hat{F}_4,$$

with

$$\lambda_2(\hat{F}_4) = 0.470683,$$

which contradicts $0 < \lambda_2(G) \leq -1 + \sqrt{2}$.

Therefore under the assumption (4.14), there is no graph satisfying the conditions $0 < \lambda_2(G) \leq -1 + \sqrt{2}$.

Case 2. Suppose there exists i_3,

$$1 \leq i_3 \leq t, \text{ such that } s_{i_3} = 0.$$

Then the same discussion as in (4.10) implies that

$$\text{for any } i \neq i_3,\ i = 1, 2, \ldots, t, \text{ either } s_i = r_i, \text{ or } s_i = 0.$$

Then, without loss of generality, suppose $u \in V(G) - V(K_{r_1,r_2,\ldots,r_t})$

$$u \sim v_{i,j},\ 1 \leq i \leq q \leq t-2,\ 1 \leq j \leq r_i.$$

Because

$$G|_{\{u,v_{1,1},v_{2,1},\ldots,v_{r_t,1}\}}$$

is of the form $A_{q,t-q}$, Lemma 3.1, Part (III) implies that $q \geq 1,\ t - q = 2$, i.e. $q = t-2,\ t \geq 3$

Therefore

$$G|_{K_{r_1,r_2,\ldots,r_t} \cup \{u\}} \tag{4.15}$$

is of the form $K_{r_1,r_2,\ldots,r_t}; K_1,\ t \geq 3$. Here the parameters $r_1, r_2, \ldots, r_t$ must be chosen such that $0 < \lambda_2(G) \leq -1 + \sqrt{2}$.

Let

$$\{u, v\} \subseteq V(G) - V(K_{r_1,r_2,\ldots,r_t}),\ t \geq 3.$$

Then from discussion above, both $G|_{K_{r_1,r_2,\ldots,r_t} \cup \{u\}}$ and $G|_{K_{r_1,r_2,\ldots,r_t} \cup \{v\}}$ are isomorphic to the graph $K_{r_1,r_2,\ldots,r_t}; K_1,\ t \geq 3$ with

$$N(u) \cap N(K_{r_1,r_2,\ldots,r_t}) = \bigcup_{k=1}^{t-2} V_{i_k}, \tag{4.16}$$

and

$$N(v) \cap N(K_{r_1,r_2,\ldots,r_t}) = \bigcup_{k=1}^{t-2} V_{j_k}. \tag{4.17}$$

Suppose

$$u \not\sim v. \tag{4.18}$$

Then

$$G_1 = G|_{\{u,v,v_{1,1},v_{2,1},\ldots,v_{t,1}\}} \supseteq G|_{\{v_{1,1},v_{2,1},\ldots,v_{t,1}\}} = K_t \tag{4.19}$$

as an MCM-subgraph, and

$$V(G_1) - V(K_t) = \{u, v\} \text{ and } u \not\sim v. \tag{4.20}$$

Then from (3.18), without loss of generality, suppose

$$(N(v) \cap V(K_t)) \subseteq (N(v) \cap V(K_t)).$$

From (4.16)–(4.20), without loss of generality, suppose

$$N(u) \cap N(K_{r_1,r_2,\ldots,r_t}) = N(v) \cap N(K_{r_1,r_2,\ldots,r_t}) = \bigcup_{i=1}^{t-2} V_i, \ t \geq 3.$$

But then

$$G|_{\{u,v,v_{1,1},v_{t-1,1},v_{t,1}\}} = C_{2,2} = \widehat{F}_4.$$

which contradicts the assumption (4.12).

Therefore,

$$\text{for any } \{u, v\} \subseteq \left[V(G) - V(K_{r_1,r_2,\ldots,r_t})\right],$$

u must be adjacent to v, and

$$G|_{V(G)-V(K_{r_1,r_2,\ldots,r_t})} = K_s. \tag{4.21}$$

From (3.19), (4.16), (4.17), and (4.21)

$$\left[N(u) \cap V(K_{r_1,r_2,\ldots,r_t})\right] \cup \left[N(v) \cap V(K_{r_1,r_2,\ldots,r_t})\right] = K_{r_1,r_2,\ldots,r_t}. \tag{4.22}$$

From (4.16), (4.17), (4.21), and (4.22),

$$G|_{\{u,v,V(K_{r_1,r_2,\ldots,r_t})\}} = K_{r_1,r_2,\ldots,r_t}; K_2, \ t \geq 4.$$

Then mathematical induction can be used to prove that

$$G = K_{r_1,r_2,\ldots,r_t}; K_s,$$
$$\text{where } \ t \geq 3, \ 1 \leq s \leq [t/2], r_i \geq 1, i = 1, 2, \ldots, t, \tag{4.23}$$

and the parameters $r_1, r_2, \ldots, r_t$ should be chosen such that $0 < \lambda_2(G) \leq -1 + \sqrt{2}$.

For different size t, to determine the parameters in (4.23), symmetry leads to the following cases:

Case I. Let

$$t = 3,\ 1 = s \le [t/2],\ G = K_{r_1,r_2,r_3}; K_1,$$
$$\text{with } r_1 \ge 1, r_2 \ge r_3 \ge 1.$$

as in *Figure 30.*

Then from Lemma 4.1, the eigenvalues of $K_{r_1,r_2,r_3}; K_1$ other than -1 and 0 can be found by solving

$$P_{K_{r_1,r_2,r_3};K_1} = \det \begin{array}{c} \\ u_1 \\ V_1 \\ V_2 \\ V_3 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & V_1 & V_2 & V_3 \end{array} \\ \begin{pmatrix} \lambda & -r_1 & 0 & 0 \\ -1 & \lambda & -r_2 & -r_3 \\ 0 & -r_1 & \lambda & -r_3 \\ 0 & -r_1 & -r_2 & \lambda \end{pmatrix} \end{array}$$
$$= \lambda^4 + (-r_1r_2 - r_1r_3 - r_2r_3 - r_1)\lambda^2 - 2r_1r_2r_3\lambda + r_1r_2r_3 = 0.$$

Apply Theorem 2.4 with $l = 2$. Then the

$$\lambda_2(K_{r_1,r_2,r_3}; K_1), r_1 \ge 1, r_2 \ge r_3 \ge 1$$

are monotone increasing in $r_i, i = 1, 2, 3$, and bounded from above.

Then by considering all possible combinations and using Theorem 2.3 on r_i, $1 \le i \le 3$, the following can be proven:

(I.1)

$$\lambda_2(K_{\infty,\infty,\infty}; K_1) = \lim_{\substack{r_1\to\infty \\ r_2\to\infty \\ r_3\to\infty}} \lambda_2(K_{r_1,r_2,r_3,}; K_1)$$

exists and is the largest root of

$$2\lambda - 1 = 0.$$

Therefore

$$\lambda_2(K_{\infty,\infty,\infty}; K_1) = 1/2 > -1 + \sqrt{2} = 0.414214. \tag{4.24}$$

From (4.24), the condition

$$0 < \lambda_2(K_{r_1,r_2,r_3}; K_1) \le -1 + \sqrt{2}$$

can not be satisfied for all possible nonnegative integer values of r_1, r_2 and r_3.

(I.2)

$$\lambda_2(K_{\infty,\infty,r_3}; K_1) = \lim_{\substack{r_1\to\infty \\ r_2\to\infty}} \lambda_2(K_{r_1,r_2,r_3,}; K_1)$$

exists and is the largest root of

$$\lambda^2 + 2r_3\lambda - r_3 = 0.$$

Therefore

$$\lambda_2(K_{\infty,\infty,r_3}; K_1) = -r_3 + \sqrt{r_3}\sqrt{r_3+1}. \tag{4.25}$$

Let $r_3 = 1$. From (4.25)

$$\lambda_2(K_{\infty,\infty,1}; K_1) = -1 + \sqrt{2}.$$

Therefore

$$\text{for } r_1 \geq 1, r_2 \geq 1, r_3 = 1,$$
$$0 < \lambda_2(K_{r_1,r_2,1}; K_1) \leq -1 + \sqrt{2}.$$

Let $r_3 = 2$. From (4.25)

$$\lambda_2(K_{\infty,\infty,2}; K_1) = -2 + \sqrt{6} = 0.44949 > -1 + \sqrt{2}.$$

Therefore, $K_{\infty,\infty,2}; K_1$ is a critical family for $\lambda_2 > -1 + \sqrt{2}$.

(I.3)

$$\lambda_2(K_{r_1,\infty,\infty}; K_1) = \lim_{\substack{r_2\to\infty \\ r_3\to\infty}} \lambda_2(K_{r_1,r_2,r_3,}; K_1)$$

exists and is the largest root of

$$\lambda^2 + 2r_1\lambda - r_1 = 0. \tag{4.26}$$

Therefore

$$\lambda_2(K_{r_1,\infty,\infty}; K_1) = -r_1 + \sqrt{r_1}\sqrt{r_1+1}.$$

Let $r_1 = 1$,

$$\lambda_2(K_{1,\infty,\infty}; K_1) = -1 + \sqrt{2}.$$

Therefore

$$\text{for } r_1 = 1, r_2 \geq r_3 \geq 1,$$
$$0 < \lambda_2(K_{1,r_2,r_3}; K_1) \leq -1 + \sqrt{2}.$$

Let $r_1 = 2$,

$$\lambda_2(K_{2,\infty,\infty}; K_1) = -2 + \sqrt{6} = 0.449490 > -1 + \sqrt{2}.$$

Therefore, $K_{2,\infty,\infty}; K_1$ is a critical family for $\lambda_2 > -1 + \sqrt{2}$.

(I.4)

$$\lambda_2(K_{\infty,r_2,r_3};K_1) = \lim_{r_1\to\infty} \lambda_2(K_{r_1,r_2,r_3,};K_1)$$

exists and is the largest root of

$$(r_2+r_3+1)\lambda^2 + 2r_2r_3\lambda - r_2r_3 = 0.$$

Therefore

$$\begin{aligned}\lambda_2(K_{\infty,r_2,r_3};K_1) =& \frac{-r_2r_3 + \sqrt{r_2^2r_3^2 + r_2r_3(r_2+r_3+1)}}{r_2+r_3+1}\\ =& \frac{-r_2r_3 + \sqrt{(r_2+1)(r_3+1)}\sqrt{r_2r_3}}{r_2+r_3+1}.\end{aligned} \tag{4.27}$$

For the parameters $r_2 \geq r_3 \geq 1$, the infinite families satisfy the inclusion relation

$$K_{\infty,r_2,r_3};K_1 \subseteq K_{\infty,\infty,r_3};K_1.$$

In order that

$$0 < \lambda_2(K_{\infty,r_2,r_3};K_1) \leq -1+\sqrt{2},$$

one only needs to start from the critical family $K_{\infty,\infty,2};K_1$ for $\lambda_2 > -1+\sqrt{2}$ as in the (I.2),

Let $r_2 = r_3 = 2$. From (4.27)

$$\lambda_2(K_{\infty,2,2};K_1) = 2/5 < -1+\sqrt{2}.$$

Therefore

$$\text{for } r_1 \geq 1, r_2 = 2, r_3 = 2,$$
$$0 < \lambda_2(K_{r_1,2,2};K_1) \leq -1+\sqrt{2}.$$

Let $r_2 = 3 \geq r_3 = 2$,. From (4.27)

$$\lambda_2(K_{\infty,3,2};K_1) = -1+\sqrt{2}.$$

Therefore

$$\text{for } r_1 \geq 1, r_2 = 3, r_3 = 2,$$
$$0 < \lambda_2(K_{r_1,3,2};K_1) \leq -1+\sqrt{2}.$$

Let $r_2 = 4 \geq r_3 = 2$,. From (4.27)

$$\lambda_2(K_{\infty,4,2};K_1) = \frac{-8+2\sqrt{30}}{7} = 0.422064 > -1+\sqrt{2}.$$

Therefore, $K_{\infty,4,2}; K_1$ is a critical family for $\lambda_2 > -1+\sqrt{2}$.

(I.5)

$$\lambda_2(K_{r_1,\infty,r_3}; K_1) = \lim_{r_2\to\infty} \lambda_2(K_{r_1,r_2,r_3,}; K_1)$$

exists and is the largest root of

$$(r_1+r_3)\lambda^2 + 2r_1r_3\lambda - r_1r_3 = 0.$$

Therefore

$$\lambda_2(K_{r_1,\infty,r_3}; K_1) = \frac{-r_1r_3 + \sqrt{r_1^2r_3^2 + r_1r_3(r_1+r_3)}}{r_1+r_3}. \tag{4.28}$$

The formula (4.28) is symmetric with respect to r_1 and r_3. For the independent parameters $r_1 \geq 1, r_3 \geq 1$, because the infinite families satisfy the inclusion relations

$$K_{r_1,\infty,r_3}; K_1 \subseteq K_{\infty,\infty,r_3}; K_1,$$

and

$$K_{r_1,\infty,r_3}; K_1 \subseteq K_{r_1,\infty,\infty,}; K_1,$$

to obtain

$$0 < \lambda_2(K_{r_1,\infty,r_3}; K_1) \leq -1+\sqrt{2},$$

one only needs to start from the critical family $K_{\infty,\infty,2}; K_1$ for $\lambda_2 > -1+\sqrt{2}$ as in the (I.2), and the critical family $K_{2,\infty,\infty}; K_1$ for $\lambda_2 > -1+2$ as in the (I.3). Only $r_1 \geq 2$ and $r_3 \geq 2$ need to be considered.

Let $r_1 = 2, r_3 = 2$.. From (4.28)

$$\lambda_2(K_{2,\infty,2}; K_1) = -1+\sqrt{2}.$$

Therefore

$$\begin{gathered} \text{for } r_1 = 2, r_2 \geq r_3 = 2, r_3 = 2, \\ 0 < \lambda_2(K_{2,r_2,2}; K_1) \leq -1+\sqrt{2}. \end{gathered} \tag{4.29}$$

Let $r_1 = 3, r_3 = 2$,. From (4.28)

$$\lambda_2(K_{\infty,3,2}; K_1) = \frac{-6+\sqrt{66}}{5} = 0.424808 > -1+\sqrt{2}. \tag{4.30}$$

Therefore, $K_{3,\infty,2}; K_1$ is a critical family for $\lambda_2 \geq -1+\sqrt{2}$. By the symmetry of r_1 and r_3 in (4.28), (4.29) and (4.30), we see that $K_{2,\infty,3}; K_1$ is also a critical family for $\lambda_2 \leq -1+\sqrt{2}$.

r_1	r_2	r_3	$\lambda_2(K_{r_1,r_2,r_3};K_1)$	
			$\lambda_2 \leq -1+\sqrt{2}$	$\lambda_2 > -1+\sqrt{2}$
3	4	2	0.402613	
7	4	2	0.413319	
8	4	2		0.414376
3	5	2	0.406718	
4	5	2	0.411471	
5	5	2		0.414422
3	6	2	0.409541	
4	6	2		0.414430
3	6	2	0.409541	
3	8	2	0.413170	
3	9	2		0.414407

$\lambda_2(K_{r_1,r_2,r_3};K_1)$
where $r_1 \geq 3,\ r_2 \geq 4,\ r_3 = 2$

Table 10.

r_1	r_2	r_3	$\lambda_2(K_{r_1,r_2,r_3};K_1)$	
			$\lambda_2 \leq -1+\sqrt{2}$	$\lambda_2 > -1+\sqrt{2}$
2	3	3	0.398995	
4	3	3	0.412824	
5	3	3		0.415806
2	4	3	0.404918	
3	4	3		0.414503
2	4	3	0.404918	
2	7	3	0.413022	
2	8	3		0.414430
2	4	4	0.410795	
2	5	4		0.414474

$\lambda_2(K_{r_1,r_2,r_3};K_1)$
where $r_1 \geq 2,\ r_2 \geq r_3 = 3.$

Table 11.

Now in order to find all minimal graphs of the form $K_{r_1,r_2,r_3};K_1$ with respect to $\lambda_2 > -1+\sqrt{2}$, we only need to start from the critical families $K_{\infty,4,2};K_1$ as in (I.4), and $K_{3,\infty,2};K_1$ and $K_{2,\infty,3};K_1$ as in (I.5).

For $r_3 = 2$, because $K_{\infty,4,2};K_1$ and $K_{3,\infty,2};K_1$ are critical families for $\lambda > -1+\sqrt{2}$, only $r_1 \geq 3$ and $r_2 \geq 4$ need to be considered.

From the *Table 10.* For

$r_1 = 3$–7, $r_2 = 4$, and $r_3 = 2$;
$r_1 = 3$–4 $r_2 = 5$, and $r_3 = 2$;
$r_1 = 3$ $r_2 = 6$–8, and $r_3 = 2$,

$$0 < \lambda_2(K_{r_1,r_2,2};K_1) \leq -1+\sqrt{2}.$$

All the minimal graphs of the form of $K_{r_1,r_2,2};K_1$ with respect to $\lambda_2 > -1+\sqrt{2}$ are

$\widehat{F}_5 = K_{8,4,2};K_1$ with $\lambda_2(\widehat{F}_5) = \lambda_2(K_{8,4,2};K_1) = 0.414376$;
$\widehat{F}_6 = K_{5,5,2};K_1$ with $\lambda_2(\widehat{F}_6) = \lambda_2(K_{5,5,2};K_1) = 0.414422$;
$\widehat{F}_7 = K_{4,6,2};K_1$ with $\lambda_2(\widehat{F}_7) = \lambda_2(K_{4,6,2};K_1) = 0.414430$;
$\widehat{F}_8 = K_{3,9,2};K_1$ with $\lambda_2(\widehat{F}_8) = \lambda_2(K_{3,9,2};K_1) = 0.414407$.

For $r_3 \geq 3$, because $K_{2,\infty,3};K_1$ is a critical family for $\lambda_2 > -1+\sqrt{2}$, only $r_1 \geq 2$ and $r_2 \geq r_3 \geq 3$ need to be considered.

From the *Table 11*, for

$r_1 = 2$–4, $r_2 = 3$, and $r_3 = 3$;
$r_1 = 2$ $r_2 = 4$–7, and $r_3 = 3$,

$$0 < \lambda_2(K_{r_1,r_2,3};K_1) \leq -1+\sqrt{2}.$$

All the minimal graph of the form of $K_{r_1,r_2,3};K_1$ with respect to $\lambda_2 > -1+\sqrt{2}$ are

$\widehat{F}_9 = K_{5,3,3};K_1$ with $\lambda_2(\widehat{F}_9) = \lambda_2(K_{5,3,3};K_1) = 0.415806$;
$\widehat{F}_{10} = K_{3,4,3};K_1$ with $\lambda_2(\widehat{F}_{10}) = \lambda_2(K_{3,4,3};K_1) = 0.414503$;
$\widehat{F}_{11} = K_{2,8,3};K_1$ with $\lambda_2(\widehat{F}_{11}) = \lambda_2(K_{2,8,3};K_1) = 0.414430$.

From the *Table 11*, for

$$r_1 = 2, r_2 = 4, \text{ and } r_3 = 4,$$
$$0 < \lambda_2(K_{2,4,4}; K_1) \leq -1 + \sqrt{2},$$

and

$$\widehat{F}_{12} = K_{2,5,4}; K_1 \qquad \text{with } \lambda_2(\widehat{F}_{12}) = \lambda_2(K_{2,5,4}; K_1) = 0.414474$$

is the only minimal graph of the form $K_{r_1,r_2,4}; K_1$ with respect to $\lambda_2 > -1 + \sqrt{2}$.

For $r_1 \geq 2, r_2 \geq r_3 \geq 5$,

$$K_{r_1,r_2,r_3}; K_1 \gg K_{2,5,4}; K_1 = \widehat{F}_{12}.$$

There is no graph of the form of $K_{r_1,r_2,r_3}; K_1$ with $r_1 \geq 2, r_2 \geq r_3 \geq 5$ which satisfies the condition, $0 < \lambda_2(K_{r_1,r_2,r_3}; K_1) \leq -1 + \sqrt{2}$.

Case II. Let

$$t = 4, \; 1 = s \leq t/2,$$
$$i.e. \; s = 1, \text{ or } 2,$$
$$G = K_{r_1,r_2,r_3,r_4}; K_s,$$
$$\text{with } r_1 \geq r_2 \geq 1, r_3 \geq r_4 \geq 1.$$

as in *Figure 30.*

For $s = 1$, from Lemma 4.1, the eigenvalues of $K_{r_1,r_2,r_3,r_4}; K_1$ other than -1 and 0 can be found out by solving

$$P_{K_{r_1,r_2,r_3,r_4};K_1} = \det \begin{array}{c} \\ u_1 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{pmatrix} \lambda & -r_1 & -r_2 & 0 & 0 \\ -1 & \lambda & -r_2 & -r_3 & -r_4 \\ -1 & -r_1 & \lambda & -r_3 & -r_4 \\ 0 & -r_1 & -r_2 & \lambda & -r_4 \\ 0 & -r_1 & -r_2 & -r_3 & \lambda \end{pmatrix} \end{array}$$
$$= \lambda^5 + (-r_1r_2 - r_1r_3 - r_1r_4 - r_2r_3 - r_2r_4 - r_3r_4 - r_1 - r_2)\lambda^3$$
$$+(-2r_1r_2r_3 - 2r_1r_2r_4 - 2r_1r_3r_4 - 2r_2r_3r_4 - 2r_1r_2)\lambda^2$$
$$(-3r_1r_2r_3r_4 + r_1r_3r_4 + r_2r_3r_4)\lambda + 2r_1r_2r_3r_4$$
$$= 0.$$

Apply Theorem 2.4 with $l = 2$. Then the

$$\lambda_2(K_{r_1,r_2,r_3,r_4}; K_1), r_1 \geq r_2 \geq 1, r_3 \geq r_4 \geq 1$$

are monotone increasing in $r_i, r_i = 1, 2, 3, 4$ and bounded from above.

Then by considering all possible combinations and applying Theorem 2.3 to combinations of $r_i, 1 \leq i \leq 4$, the following hold:

(II.1)

$$\lambda_2(K_{\infty,\infty,\infty,\infty}; K_1) = \lim_{\substack{r_1 \to \infty \\ r_2 \to \infty \\ r_3 \to \infty \\ r_4 \to \infty}} \lambda_2(K_{r_1,r_2,r_3,r_4}; K_1)$$

exists and is the largest root of

$$-3\lambda + 2 = 0.$$

Therefore

$$\lambda_2(K_{\infty,\infty,\infty,\infty}; K_1) = 2/3 > -1 + \sqrt{2}.$$

Since for $r_2 \geq 1$

$$K_{\infty,\infty,\infty,\infty}; K_1 \supseteq \ K_{\infty,r_2,\infty,\infty}; K_1 \supseteq \ K_{\infty,\infty,\infty}; K_1,$$

we have

$$\lambda_2(K_{\infty,\infty,\infty,\infty}; K_1) \geq \lambda_2(K_{\infty,r_2,\infty,\infty}; K_1) \geq \lambda_2(K_{\infty,\infty,\infty}; K_1).$$

From (4.24)

$$\lambda_2(K_{\infty,\infty,\infty}; K_1) = 1/2 > -1 + \sqrt{2}.$$

Therefore in order that

$$0 < \lambda_2(K_{r_1,r_2,r_3,r_4}; K_1) < -1 + \sqrt{2},$$

at least one of the parameters r_1, r_3, r_4 must be bounded from above and $K_{\infty,1,\infty,\infty}; K_1$ is a critical family for $-1 + \sqrt{2}$.

(II.2)

$$\lambda_2(K_{\infty,\infty,\infty,r_4}; K_1) = \lim_{\substack{r_1 \to \infty \\ r_2 \to \infty \\ r_3 \to \infty}} \lambda_2(K_{r_1,r_2,r_3,r_4}; K_1)$$

exists and is the largest root of

$$-2\lambda^2 - 3r_4\lambda_3 + 2r_4 = 0.$$

Therefore

$$\lambda_2(K_{\infty,\infty,\infty,r_4}; K_1) = \frac{-3r_4 + \sqrt{r_4}\sqrt{9r_4 + 16}}{4}. \tag{4.31}$$

Let $r_4 = 1$. From (4.31)

$$\lambda_2(K_{\infty,\infty,\infty,1}; K_1) = 1/2 > -1 + \sqrt{2}.$$

Therefore $K_{\infty,\infty,\infty,1}; K_1$ is a critical family for $-1+\sqrt{2}$.

(II.3)

$$\lambda_2(K_{\infty,\infty,r_3,r_4}; K_1) = \lim_{\substack{r_1\to\infty \\ r_2\to\infty}} \lambda_2(K_{r_1,r_2,r_3,r_4}; K_1)$$

exists and is the largest root of

$$-\lambda^3 + (-2r_3 - 2r_4 - 2)\lambda^2 - 3r_3r_4\lambda + 2r_3r_4 = 0.$$

Let $r_3 = r_4 = 1$,

$$\lambda_2(K_{\infty,\infty,1,1}; K_1) = \frac{-5+\sqrt{33}}{2} = 0.372281 < -1+\sqrt{2}.$$

Therefore

$$\text{for } \ r_1 \geq r_2 \geq 1, r_3 = 1, r_4 = 1,$$
$$0 < \lambda_2(K_{r_1,r_2,1,1}; K_1) \leq -1+\sqrt{2}.$$

Let $r_3 = 2, r_4 = 1$.

$$\lambda_2(K_{\infty,\infty,2,1}; K_1) = 0.419601 > -1+\sqrt{2}.$$

Therefore, $K_{\infty,\infty,2,1}; K_1$ is a critical family for $\lambda_2 > -1+\sqrt{2}$.

(II.4)

$$\lambda_2(K_{\infty,r_2,\infty,r_4}; K_1) = \lim_{\substack{r_1\to\infty \\ r_3\to\infty}} \lambda_2(K_{r_1,r_2,r_3,r_4}; K_1)$$

exists and is the largest root of

$$-\lambda^3 + (-2r_4 - 2r_2)\lambda^2 + (-3r_2r_4 + r_4)\lambda + 2r_2r_4 = 0. \tag{4.32}$$

Let $r_2 = 1$, and $r_4 = 1$. From (4.32)

$$\lambda_2(K_{\infty,1,\infty,1}; K_1) = 0.481194 > -1+\sqrt{2}.$$

Therefore $K_{\infty,1,\infty,1}; K_1$ is a critical family for $-1+\sqrt{2}$.

(II.5)

$$\lambda_2(K_{r_1,r_2,\infty,\infty}; K_1) = \lim_{\substack{r_1\to\infty \\ r_2\to\infty}} \lambda_2(K_{r_1,r_2,r_3,r_4}; K_1)$$

exists and is the largest root of

$$-\lambda^3 + (-2r_1 - 2r_2)\lambda^2 + (r_1 + r_2 - 3r_1r_2)\lambda + 2r_1r_2 = 0. \tag{4.33}$$

Let $r_1 = r_2 = 1$. From (4.33)

$$\lambda_2(K_{1,1,\infty,\infty}; K_1) = \frac{-3+\sqrt{17}}{2} = 0.561553 > -1+\sqrt{2}.$$

Therefore, $K_{1,1,\infty,\infty}; K_1$ is a critical family for $\lambda_2 > -1+\sqrt{2}$.

(II.6)

$$\lambda_2(K_{\infty,r_2,r_3,r_4}; K_1) = \lim_{r_1 \to \infty} \lambda_2(K_{r_1,r_2,r_3,r_4}; K_1)$$

exists and is the largest root of

$$\begin{aligned}(-r_2 - r_3 - r_4 - 1)\lambda^3 + (-2r_2r_3 - 2r_2r_4 - 2r_3r_4 - 2r_2)\lambda^2 \\ +(-3r_2r_3r_4 + r_3r_4)\lambda + 2r_2r_3r_4 = 0.\end{aligned} \tag{4.34}$$

In particular, if $r_3 = r_4 = 1$, then (4.34) becomes

$$\begin{aligned}&(-r_2 - 3)\lambda^3 + (-6r_2 - 2)\lambda^2 + (-3r_2 + 1)\lambda + 2r_2 \\ =&(\lambda + 1)\left[(-r_2 - 3)\lambda^2 + (-5r_2 + 1) + 2r_2\right] = 0.\end{aligned}$$

Then

$$\lambda_2(K_{\infty,r_2,1,1;K_1}) = \frac{-5r_2 + 1 + \sqrt{(1-5r_2)^2 + 8r_2(r_2+3)}}{2(r_2+3)}.$$

Because the infinite families satisfy the inclusion relations

$$K_{\infty,r_2,r_3,r_4}; K_1 \subseteq K_{\infty,\infty,r_3,r_4}; K_1,$$

and

$$K_{\infty,r_2,r_3,r_4}; K_1 \subseteq K_{\infty,r_2,\infty,r_4}; K_1,$$

arguing as in the (II.3),

$$K_{\infty,\infty,2,1}; K_1$$

is a critical family; and as in (II.4),

$$K_{\infty,1,\infty,1}; K_1$$

is a critical family. So in order that

$$0 < \lambda_2(K_{\infty,r_2,r_3,r_4}; K_1) \le -1+\sqrt{2},$$

only $r_2 \ge 1, r_3 \ge 2, r_4 \ge 1$ and $r_3 \ge r_4$ need be considered.

Let $r_2 = 1, r_3 = 2, r_4 = 1$. From (4.34),

$$\lambda_2(K_{\infty,1,2,1}; K_1) = 0.409890 < -1+\sqrt{2}.$$

Therefore, for $r_1 \geq 1, r_2 = 1, r_3 = 2, r_4 = 1$,

$$0 < \lambda_2(K_{r_1,1,2,1}; K_1) \leq -1 + \sqrt{2}.$$

Let $r_2 = 2, r_3 = 2, r_4 = 1$. From (4.34),

$$\lambda_2(K_{\infty,2,2,1}; K_1) = -1 + \sqrt{2}$$

Therefore, for $r_1 \geq 1, r_2 = 2, r_3 = 2, r_4 = 1$,

$$0 < \lambda_2(K_{r_1,2,2,1}; K_1) \leq -1 + \sqrt{2}.$$

Let $r_2 = 3, r_3 = 2, r_4 = 1$. From (4.34),

$$\lambda_2(K_{\infty,3,2,1}; K_1) = 0.415872 > -1 + \sqrt{2}.$$

From (4.105), (4.115) and (4.116), $K_{\infty,3,2,1}; K_1$ is a critical family for $-1 + \sqrt{2}$.

Let $r_2 = 1, r_3 = 3, r_4 = 1$. From (4.34),

$$\lambda_2(K_{\infty,1,3,1}; K_1) = 0.429331 > -1 + \sqrt{2}.$$

From (4.113), $K_{\infty,1,3,1}; K_1$ is a critical family for $-1 + \sqrt{2}$.

Let $r_2 = 1, r_3 = 2, r_4 = 2$. From (4.34),

$$\lambda_2(K_{\infty,1,2,2}; K_1) = 0.457427 > -1 + \sqrt{2}.$$

From the assumption $r_3 \geq r_4 \geq 1$, which comes from the symmetry of r_3 and r_4 in $K_{r_1,r_2,r_3,r_4}; K_1$ and (4.113), $K_{\infty,1,2,2}; K_1$ is a critical family for $-1 + \sqrt{2}$.

(II.7)

$$\lambda_2(K_{r_1,r_2,\infty,r_4}; K_1) = \lim_{r_3 \to \infty} \lambda_2(K_{r_1,r_2,r_3,r_4}; K_1)$$

exists and is the largest root of

$$\begin{aligned}&(-r_1 - r_2 - r_4)\lambda^3 + (-2r_1r_2 - 2r_1r_4 - 2r_2r_4)\lambda^2 \\ &+(-3r_1r_2r_4 + r_1r_4 + r_2r_4)\lambda + 2r_1r_3r_4 = 0. \qquad (4.35)\end{aligned}$$

Because the infinite families satisfy the inclusion relations

$$K_{r_1,r_2,\infty,r_4}; K_1 \subseteq K_{\infty,r_2,\infty,r_4}; K_1,$$

and

$$K_{r_1,r_2,\infty,r_4}; K_1 \subseteq K_{r_1,r_2,\infty,\infty}; K_1,$$

arguing as in (II.4), $K_{\infty,1,\infty,1}; K_1$ is a critical family; and as in the (II.5), $K_{1,1,\infty,\infty}; K_1$ is a critical family. So in order that

$$0 < \lambda_2(K_{r_1,r_2,\infty,r_4}; K_1) \leq -1 + \sqrt{2},$$

only parameters $r_1 \geq r_2 \geq 1$, and $r_4 \geq 1$ need to be considered.

Let $r_1 = r_2 = r_4 = 1$. From (4.35),

$$\lambda_2(K_{1,1,\infty,1}; K_1) = 0.457427 > -1 + \sqrt{2}.$$

Therefore $K_{1,1,\infty,1} : K_1$ is a critical family for $-1 + \sqrt{2}$.

(II.8) Now in order to find all minimum graphs of the form $K_{r_1,r_2,r_3,r_4}; K_1$ where $r_1 \geq r_2 \geq 1, r_3 \geq r_4 \geq 1$, with respect to $\lambda > -1 + \sqrt{2}$, one only need to start from the critical families

$$K_{\infty,3,2,1}; K_1,$$

$$K_{\infty,1,3,1}; K_1$$

$$K_{\infty,1,2,2}; K_1$$

as in (II.6); and

$$K_{1,1,\infty,1}; K_1$$

as in (II.7).

From *Table 12,*

if $r_1 = 3–7, \quad r_2 = 3, \quad r_3 = 2, \quad$ and $\quad r_4 = 1;$
$r_1 = 4, \quad r_2 = 4, \quad r_3 = 2, \quad$ and $\quad r_4 = 1;$
or $r_1 = 1, \quad r_2 = 1, \quad r_3 = 3, \quad$ and $\quad r_4 = 1,$

then

$$0 < \lambda_2(K_{r_1,r_2,r_3,r_4}; K_1) \leq -1 + \sqrt{2}.$$

Therefore, all the minimum graphs of the form $K_{r_1,r_2,r_3,r_4}; K_1$ with respect to $\lambda_2 > -1 + \sqrt{2}$ are

$\widehat{F}_{13} = K_{8,3,2,1}; K_1$ with $\lambda_2(\widehat{F}_{13}) = \lambda_2(K_{8,3,2,1}; K_1) = 0.414248;$
$\widehat{F}_{14} = K_{5,4,2,1}; K_1$ with $\lambda_2(\widehat{F}_{14}) = \lambda_2(K_{5,4,2,1}; K_1) = 0.414255;$
$\widehat{F}_{15} = K_{2,1,3,1}; K_1$ with $\lambda_2(\widehat{F}_{15}) = \lambda_2(K_{2,1,3,1}; K_1) = 0.420936;$
$\widehat{F}_{16} = K_{1,1,2,2}; K_1$ with $\lambda_2(\widehat{F}_{16}) = \lambda_2(K_{1,1,2,2}; K_1) = 0.438447;$
$\widehat{F}_{17} = K_{1,1,4,1}; K_1$ with $\lambda_2(\widehat{F}_{17}) = \lambda_2(K_{1,1,4,1}; K_1) = 0.423677.$

r_1	r_2	r_3	r_4	$\lambda_2(K_{r_1,r_2,r_3,r_4};K_1)$	
				$\lambda_2 \le -1+\sqrt{2}$	$\lambda_2 > -1+\sqrt{2}$
3	3	2	1	0.411746	
7	3	2	1	0.414024	
8	3	2	1		0.414248
4	4	2	1	0.413668	
5	4	2	1		0.414255
1	1	3	1	$-1+\sqrt{2}$	
2	1	3	1		0.420936
1	1	2	2		0.438447
1	1	3	1	$-1+\sqrt{2}$	
2	1	4	1		0.423677

$\lambda_2(K_{r_1,r_2,r_3,r_4};K_1)$
where $r_1 \ge r_2 \ge 1,\ r_3 \ge r_4 = 1,$
and the following conditions are satisfied:
if $r_4 = 1$, then $r_1 \ge r_2 \ge 3,\ r_3 \ge 2,\ r_4 = 1.$
or if $r_4 \ge 2$, then $r_1 \ge r_2 \ge 1,\ r_3 \ge r_4 \ge 2.$

Table 12.

For $s = 2$. From lemma 4.1, the eigenvalues of $K_{r_1,r_2,r_3,r_4}; K_2$, where $r_1 \geq r_2 \geq 1, r_3 \geq r_4 \geq 1$, other than -1 and 0 can be found by solving

$$P_{K_{r_1,r_2,r_3,r_4};K_2} = \det \begin{array}{c} \\ u_1 \\ u_2 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \!\! \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{pmatrix} \lambda & -1 & -r_1 & -r_2 & 0 & 0 \\ -1 & \lambda & 0 & 0 & -r_3 & -r_4 \\ -1 & 0 & \lambda & -r_2 & -r_3 & -r_4 \\ -1 & 0 & -r_1 & \lambda & -r_3 & -r_4 \\ 0 & -1 & -r_1 & -r_2 & \lambda & -r_4 \\ 0 & -1 & -r_1 & -r_2 & -r_3 & \lambda \end{pmatrix} \end{array}$$

$$\begin{aligned} &= \lambda^6 + (-r_1r_2 - r_1r_3 - r_1r_4 - r_2r_3 - r_2r_4 - r_3r_4 - r_1 - r_2 - r_3 - r_4 - 1)\lambda^4 \\ &\quad +(-2r_1r_2r_3 - 2r_1r_2r_4 - 2r_1r_3r_4 - 2r_2r_3r_4 - 2r_1r_2 - 2r_3r_4)\lambda^3 \\ &\quad +(-3r_1r_2r_3r_4 + r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 + r_1r_2 + r_3r_4)\lambda^2 \\ &\quad +4r_1r_2r_3r_4\lambda - r_1r_2r_3r_4 \\ &= 0. \end{aligned} \tag{4.36}$$

Because both $K_{r_1,r_2,r_3,r_4}; K_2 - u_1$ and $K_{r_1,r_2,r_3,r_4}; K_2 - u_2$ are isomorphic to $K_{r_1,r_2,r_3,r_4}; K_1$, and from (4.124) $K_{1,1,4,1}$ is a minimum graph with respect to $\lambda_2 > -1 + \sqrt{2}$, it follows that both r_1 and r_3 must bounded from above in order that

$$K_{r_1,r_2,r_3,r_4}; K_2 \leq -1 + \sqrt{2}.$$

(II.9) From (4.36), for

$$r_1 = 1, \qquad r_2 = 1, \qquad r_3 = 1, \qquad \text{and} \quad r_4 = 1;$$

$$0 < \lambda_2(K_{1,1,1,1}; K_1) = -1 + \sqrt{2}.$$

The only minimum graph of the form $K_{r_1,r_2,r_3,r_4}; K_2$ with respect to $\lambda_2 > -1 + \sqrt{2}$ is

$\widehat{F}_{18} = K_{2,1,1,1}; K_2$ with $\lambda_2(\widehat{F}_{18}) = \lambda_2(K_{2,1,1,1}; K_2) = 0.446658$.

Case III. Let

$$t \geq 5, \ 1 = s \leq [t/2],$$
$$G = K_{r_1,r_2,\ldots,r_t}; K_s.$$

The following cases will be considered.

III.1 For $t = 5, s = 1, G = K_{r_1,r_2,r_3,r_4,r_5}; K_1$ from Lemma 4.1, the eigenvalues of $K_{r_1,r_2,r_3,r_4,r_5}; K_1$ where $r_1 \geq r_2 \geq r_3 \geq 1, r_4 \geq r_5 \geq 1$, other than -1 and 0 can be found by solving

$$P_{K_{r_1,r_2,r_3,r_4,r_5};K_1}(\lambda) = \det \begin{array}{c} \\ u_1 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} u_1 & V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{pmatrix} \lambda & -r_1 & -r_2 & -r_3 & 0 & 0 \\ -1 & \lambda & -r_2 & -r_3 & -r_4 & -r_5 \\ -1 & -r_1 & \lambda & -r_3 & -r_4 & -r_5 \\ -1 & -r_1 & -r_2 & \lambda & -r_4 & -r_5 \\ 0 & -r_1 & -r_2 & -r_3 & \lambda & -r_5 \\ 0 & -r_1 & -r_2 & -r_3 & -r_4 & \lambda \end{pmatrix} \end{array}$$

$$\begin{aligned} &= \lambda^6 + (-r_1r_2 - r_1r_3 - r_1r_4 - r_1r_5 - r_2r_3 - r_2r_4 - r_2r_5 - r_3r_4 - r_3r_5 - r_4r_5 - r_1 - r_2 - r_3)\lambda^4 \\ &\quad + (-2r_1r_2r_3 - 2r_1r_2r_4 - 2r_1r_2r_5 - 2r_1r_3r_4 - 2r_1r_3r_5 - 2r_1r_4r_5 - 2r_2r_3r_4 \\ &\qquad -2r_2r_3r_5 - 2r_2r_4r_5 - 2r_3r_4r_5 - 2r_1r_2 - 2r_1r_3 - 2r_2r_3)\lambda^3 \\ &\qquad + (-3r_1r_2r_3r_4 - 3r_1r_2r_3r_5 - 3r_1r_2r_4r_5 - 3r_1r_3r_4r_5 \\ &\qquad -3r_2r_3r_4r_5 + r_1r_4r_5 + r_2r_4r_5 + r_3r_4r_5 - 3r_1r_2r_3)\lambda^2 \\ &\qquad + (-4r_1r_2r_3r_4r_5 + 2r_1r_2r_4r_5 + 2r_1r_3r_4r_5 + 2r_2r_3r_4r_5)\lambda \\ &\qquad + 3r_1r_2r_3r_4r_5 \\ &= 0. \end{aligned}$$

Because

$$K_{r_1,r_2,r_3,r_4,r_5};K_1 \supseteq K_{r_1,r_2,r_3,r_4};K_1,$$

and $K_{1,1,4,1};K_1$ is a minimal graph with respect to $\lambda_2 > -1+\sqrt{2}$, $r_4 \geq r_5 \geq 1$ must be bounded from above in order that

$$K_{r_1,r_2,r_3,r_4,r_5};K_2 \leq -1+\sqrt{2}.$$

For fixed r_4 and r_5, apply Theorem 2.4 with $l=2$. Then the

$$\lambda_2(K_{r_1,r_2,r_3,r_4,r_5};K_1), r_1 \geq 1, r_2 \geq r_3 \geq 1$$

are monotone increasing in $r_i, r = 1,2,3$ and bounded from above.

$$\lambda_2(K_{\infty,\infty,\infty,r_4,r_5};K_1) = \lim_{\substack{r_1\to\infty \\ r_2\to\infty \\ r_3\to\infty}} \lambda_2(K_{r_1,r_2,r_3,r_4,r_5};K_1)$$

exists and is the largest root of

$$-2\lambda^3 + (-3r_4 - 3r_5 - 3)\lambda^2 - 4r_4r_5\lambda_3 + 3r_4r_5 = 0. \tag{4.37}$$

Let $r_4 = r_5 = 1$. From (4.37)

$$\lambda_2(K_{\infty,\infty,\infty,1,1};K_1) = (-7+\sqrt{73})/4 = 0.386001 < -1+\sqrt{2}.$$

Therefore, for $r_1 \geq r_2 \geq r_3 \geq 1, r_4 = r_5 = 1$,

$$0 < \lambda_2(K_{r_1,r_2,r_3,1,1}; K_1) \leq -1 + \sqrt{2}. \qquad (4.38)$$

The only minimal graph of the form of $K_{r_1,r_2,r_3,r_4,r_5}; K_1$ with respect to $\lambda_2 > -1 + \sqrt{2}$ is
$\widehat{F}_{19} = K_{1,1,1,2,1}; K_1$ with $\lambda_2(\widehat{F}_{19}) = \lambda_2(K_{1,1,1,2,1}; K_1) = 0.422594.$

(4.39)

(III.2) For the general situation, by symmetry one may suppose without loss of generality that,

$$\begin{gathered} t \geq 5,\ 1 = s \leq [t/2], \\ G = K_{r_1,r_2,\ldots,r_t}; K_s, \\ \text{where } r_1 \geq r_2 \geq \ldots \geq r_{t-2s}, \\ \text{and } r_{t-2s+2i-1} \geq r_{t-2s+2i} \geq 1, \\ \text{for } i = 1, 2, \ldots, s. \end{gathered} \qquad (4.40)$$

Because

$$\begin{gathered} K_{r_1,r_2,\ldots,r_t}; K_s - \{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t\} \cong K_{r_1,r_2,\ldots,r_t}; K_1, \\ \text{for } 1 \leq i \leq s, \end{gathered} \qquad (4.41)$$

It follows from (4.38), (4.39), (4.40) and (4.41) that if

$$0 < \lambda_2(G) = \lambda_2(K_{r_1,r_2,\ldots,r_t}; K_s) \leq -1 + \sqrt{2},$$

then

$$r_i = 1, \quad t - 2s + 1 \leq i \leq t$$

and

$$G = K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s \text{ times}}}; K_s.$$

From (4.2), all the eigenvalues of the graph

$$G = K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s \text{ times}}}; K_s$$

other than -1 and 0 can be found by solving

$$P_{K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s \text{ times}};K_s}}$$

$$= \left\{ \left[\lambda^2 - (3s-2)\lambda + s - 1\right] - (\lambda^2 + \lambda - 1)\sum_{j=1}^{t-2s} \frac{r_j}{\lambda + r_j} \right\} \cdot (\lambda^2 + 2\lambda - 1)^{s-1} \prod_{i=1}^{t-2s} (\lambda + r_i)$$

$$= 0. \tag{4.42}$$

Apply Theorem 2.4 with $l = 2$. Then the

$$\lambda_2(K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s \text{ times}}}; K_s)$$

are monotone increasing in $r_i, i = 1, 2, \ldots, t-2s$ and bounded from above. Then in turn use Theorem 2.3 on $r_i, 1 \le i \le t-2s$.

From (4.42),

$$\lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{t-2s \text{ times}},\underbrace{1,1,\ldots,1}_{2s \text{ times}}}; K_s) = \lim_{\substack{r_1 \to \infty \\ r_2 \to \infty \\ \vdots \\ r_{t-2s} \to \infty}} \lambda_2(K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s \text{ times}}}; K_s)$$

exist and is the largest root of

$$\left\{ \left[\lambda^2 - (3s-2)\lambda + s - 1\right] - (t-2s)(\lambda^2 + 2\lambda - 1) \right\} (\lambda^2 + 2\lambda - 1)^{s-1} = 0. \tag{4.43}$$

Also

$$\lim_{t \to \infty} \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{t-2s \text{ times}},\underbrace{1,1,\ldots,1}_{2s \text{ times}}}; K_s) = \lim_{\substack{r_1 \to \infty \\ r_2 \to \infty \\ \vdots \\ r_{t-2s} \to \infty \\ t \to \infty}} \lambda_2(K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s \text{ times}}}; K_s)$$

exist and is the largest root of

$$(\lambda^2 + 2\lambda - 1)^s = 0. \tag{4.44}$$

Then

$$\lim_{t \to \infty} \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{t-2s \text{ times}},\underbrace{1,1,\ldots,1}_{2s \text{ times}}}; K_s) = -1 + \sqrt{2}. \tag{4.45}$$

Therefore, (4.39) (4.44) and (4.45) imply that

$$\text{for } t \geq 5, 1 \leq s \leq [t/2],$$
$$0 < \lambda_2(G) = \lambda_2(K_{r_1,r_2,\ldots,r_t}; K_s) \leq -1+\sqrt{2}$$

if and only if

$$r_1 \geq 1,\ \ 1 \leq i \leq t-2s,$$
$$\text{and } r_i = 1,\ \ t-2s+1 \leq i \leq t.$$

In particular, for $s = 1$. From (4.43), (4.44) and (4.45),

$$\lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{t-2 \text{ times}},1,1}; K_1) = \lim_{\substack{r_1\to\infty\\ r_2\to\infty\\ \vdots\\ r_{t-2}\to\infty}} \lambda_2(K_{r_1,r_2,\ldots,r_{t-2s},1,1}; K_1)$$

exist and is the largest root of

$$(\lambda^2-\lambda)-(t-2)(\lambda^2+2\lambda-1) = (3-t)\lambda^2+(3-2t)\lambda+t-2 = 0.$$

Then

$$\lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{t-2 \text{ times}},1,1}; K_1) = \frac{-2t+3+\sqrt{8t^2-32t+33}}{2(t-3)} < -1+\sqrt{2},$$

and

$$\lim_{t\to\infty} \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{t-2 \text{ times}},1,1}; K_1) = -1+\sqrt{2}.$$

In (4.42), let $s = 1, r_2 = r_3 = \cdots = r_t = 1$. Then the eigenvalues of the graph

$$G = K_{r_1,\underbrace{1,1,\ldots,1}_{t-1 \text{ times}}}; K_1$$

other than -1 and 0 can be found by solving

$$\begin{aligned}
&P_{K_{r_1,\underbrace{1,1,\ldots,1}_{t-1 \text{ times}}};K_1}\\
&= \Big\{(\lambda^2-\lambda)-(\lambda^2+2\lambda-1)\Big[\frac{r_1}{\lambda+r_1}+(t-3)\frac{1}{\lambda+1}\Big]\Big\}(\lambda+r_1)(\lambda+1)^{t-3}\\
&= \Big\{(\lambda+r_1)(\lambda^2-\lambda)(\lambda+1)-(\lambda^2+2\lambda-1)\big[r_1(\lambda+1)+(t-3)(\lambda+r_1)\big]\Big\}(\lambda+1)^{t-4}\\
&= 0.
\end{aligned}$$

So

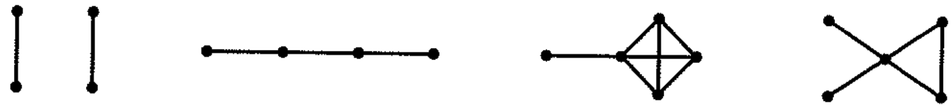
$$\lambda_2(K_{\infty,\underbrace{1,1,\ldots,1}_{t-1\text{ times}}};K_1)$$

is the largest root of

$$t\lambda^2 + 2(t-2)\lambda - (t-2) = 0.$$

Then

$$\lambda_2(K_{\infty,\underbrace{1,1,\ldots,1,}_{t-1\text{ times}}};K_1) = \frac{-2(t-2)+\sqrt{2t^2-6t+4}}{t}.$$

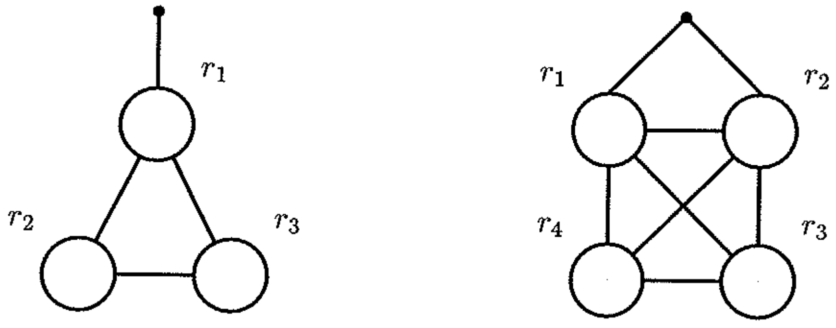


$\widehat{F}_1 = 2K_2,$ $\lambda_2(\widehat{F}_1) = 1.$

$\widehat{F}_2 = P_4,$ $\lambda_2(\widehat{F}_2) = \frac{-1+\sqrt{5}}{2}.$

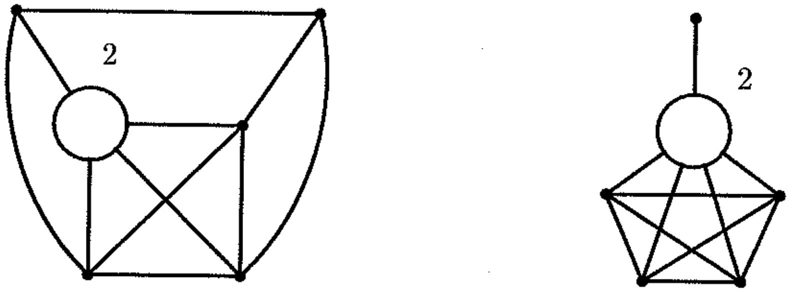
$\widehat{F}_3 = A_{1,3},$ $\lambda_2(\widehat{F}_3) = 0.428007.$

$\widehat{F}_4 = B_{2,1},$ $\lambda_2(\widehat{F}_4) = 0.470683.$



$\widehat{F}_5 - \widehat{F}_{12} \simeq K_{r_1,r_2,r_3}; K_1$ where $r_i,\ i = 1,2,3$ are given in Theorem 4.3 Part (II) .

$\widehat{F}_{13} - \widehat{F}_{17} \simeq K_{r_1,r_2,r_3,r_4}; K_1$ where $r_i,\ i = 1,2,3,4$ are given in Theorem 4.3 Part (III) .



$\widehat{F}_{18} = K_{2,1,1,1}; K_2$ $\lambda_2(\widehat{F}_{18}) = 0.446658.$

$\widehat{F}_{19} = K_{2,1,1,1,1}; K_1$ $\lambda_2(\widehat{F}_{19}) = 0.422594.$

Minimal graphs $\hat{F}$ with respect to $\lambda_2(\hat{F}) > -1 + \sqrt{2}$

Figure 31

Theorem 4.3. *Let G be a simple graph. Then G is a graph minimal with respect to*

$$\lambda_2(G) > -1 + \sqrt{2}$$

if and only if G is one of the following graphs :

(I) $\widehat{F}_1 = 2K_2,\ \widehat{F}_2 = P_4,\ \widehat{F}_3 = A_{1,3},\ \widehat{F}_4 = C_{2,2}.$

(II)

$\widehat{F}_5 = K_{8,4,2};K_1$ with $\lambda_2(\widehat{F}_5) = \lambda_2(K_{8,4,2};K_1) = 0.414376;$
$\widehat{F}_6 = K_{5,5,2};K_1$ with $\lambda_2(\widehat{F}_6) = \lambda_2(K_{5,5,2};K_1) = 0.414422;$
$\widehat{F}_7 = K_{4,6,2};K_1$ with $\lambda_2(\widehat{F}_7) = \lambda_2(K_{4,6,2};K_1) = 0.414430;$
$\widehat{F}_8 = K_{3,9,2};K_1$ with $\lambda_2(\widehat{F}_8) = \lambda_2(K_{3,9,2};K_1) = 0.414407;$
$\widehat{F}_9 = K_{5,3,3};K_1$ with $\lambda_2(\widehat{F}_9) = \lambda_2(K_{5,3,3};K_1) = 0.415806;$
$\widehat{F}_{10} = K_{3,4,3};K_1$ with $\lambda_2(\widehat{F}_{10}) = \lambda_2(K_{3,4,3};K_1) = 0.414503;$
$\widehat{F}_{11} = K_{2,8,3};K_1$ with $\lambda_2(\widehat{F}_{11}) = \lambda_2(K_{2,8,3};K_1) = 0.414430;$
$\widehat{F}_{12} = K_{2,5,4};K_1$ with $\lambda_2(\widehat{F}_{12}) = \lambda_2(K_{2,5,4};K_1) = 0.414474,$

(III)

$\widehat{F}_{13} = K_{8,3,2,1};K_1$ with $\lambda_2(\widehat{F}_{13}) = \lambda_2(K_{8,3,2,1};K_1) = 0.414248;$
$\widehat{F}_{14} = K_{5,4,2,1};K_1$ with $\lambda_2(\widehat{F}_{14}) = \lambda_2(K_{5,4,2,1};K_1) = 0.414255;$
$\widehat{F}_{15} = K_{2,1,3,1};K_1$ with $\lambda_2(\widehat{F}_{15}) = \lambda_2(K_{2,1,3,1};K_1) = 0.420936;$
$\widehat{F}_{16} = K_{1,1,2,2};K_1$ with $\lambda_2(\widehat{F}_{16}) = \lambda_2(K_{1,1,2,2};K_1) = 0.438447;$
$\widehat{F}_{17} = K_{1,1,4,1};K_1$ with $\lambda_2(\widehat{F}_{17}) = \lambda_2(K_{1,1,4,1};K_1) = 0.423677;$
$\widehat{F}_{18} = K_{2,1,1,1};K_2$ with $\lambda_2(\widehat{F}_{18}) = \lambda_2(K_{2,1,1,1};K_2) = 0.446658,$

(IV)

$\widehat{F}_{19} = K_{1,1,1,2,1};K_1$ with $\lambda_2(\widehat{F}_{19}) = \lambda_2(K_{1,1,1,2,1};K_1) = 0.422594,$

Proof. Let G be a graph minimal with respect to $\lambda_2(G) > -1 + \sqrt{2}$.

If G is disconnected, because G is minimal, G contains no isolated vertices, and so G must be $\widehat{F}_1 = 2K_2$.

Suppose G is connected and $G \neq F_2 = P_4$. Let

$$G \gg K_{r_1,r_2,\ldots,r_t}$$

be the MCM-subgraph in G. Then G must be one of the graphs of the form $\widehat{F}_3$–$\widehat{F}_{19}$ as in the proof of Theorem 4.2.

SECTION 5. THE DISTRIBUTION AND DENSITY OF THE SUBDOMINANT EIGENVALUES OF GRAPHS IN THE INTERVAL $[0, \; -1+\sqrt{2}]$

In this section, the distribution and density of subdominant eigenvalues of simple graphs in the interval $[0, -1+\sqrt{2}]$ will be determined. The reasons for investigating this distribution are given in the Introduction.

It is well known that for a simple connected graph G, the following hold:

(I) $\lambda_2(G) \geq -1$. $\lambda_2(G) = -1$ if and only if G is a complete graph.

(II) If $\lambda_2(G) > -1$, then $\lambda_2 \geq 0$. $\lambda_2 = 0$ if and only if G is a complete multipartite graph.

If λ_2 is a subdominant eigenvalue of simple graph G and $0 < \lambda_2 \leq -1+\sqrt{2}$, then the fact that $G \not\succ 2K_2$ implies that only one component G_1 of G contains edges, and $\lambda_2(G) = \lambda_2(G_1)$.

Although as the sets of graphs

$$\{G \mid G \text{ is a simple graph and } 0 < \lambda_2(G) \leq -1+\sqrt{2}\}$$
$$\neq \{G \mid G \text{ is a simple graph without isolated vertices, and } 0 < \lambda_2(G) \leq -1+\sqrt{2}\},$$

as the sets of real numbers

$$\begin{aligned}
&\{\lambda_2 \mid \lambda_2 = \lambda_2(G) \text{ is the subdominant eigenvalue of a simple graph } G, \\
&\qquad\qquad\qquad\qquad \text{and } 0 < \lambda_2 \leq -1+\sqrt{2}\} \\
=&\{\lambda_2 \mid \lambda_2 = \lambda_2(G) \text{ is the subdominant eigenvalue of a simple graph } G, \\
&\qquad\qquad\qquad\qquad \text{without isolated vertices and } 0 < \lambda_2 \leq -1+\sqrt{2}\} \\
=&\{\lambda_2 \mid \lambda_2 = \lambda_2(G) \text{ is the subdominant eigenvalue of a connected simple graph } G, \\
&\qquad\qquad\qquad\qquad \text{and } 0 < \lambda_2 < -1+\sqrt{2}\}.
\end{aligned}$$

We do not consider the set of simple graphs G with $0 < \lambda_2(G) \leq -1+\sqrt{2}$, but only the distribution and density of the set of the subdominant eigenvalues λ_2 of simple graphs in the interval $(0, -1+\sqrt{2}]$; without loss of generality, suppose λ_2 is the subdominant eigenvalue of a connected simple graph.

Theorem 5.1. *In the interval $(0, \frac{1}{3}]$, among the subdominant eigenvalues of simple graphs,*

(I) The smallest one is

$$(\lambda_2)_{\min} = \lambda_2(K_{1,1,1}; K_1) = \lambda_2(A_{1,2}) = 0.311108.$$

(II) The second smallest one is

$$(\lambda_2)_{2nd} = \lambda_2(K_{2,1,1}; K_1) = 0.321637.$$

(III) The third smallest one is

$$(\lambda_2)_{3rd} = \lambda_2(K_{3,1,1}; K_1) = 0.325397.$$

(IV) For $r \geq 1$,the rth smallest one is

$$(\lambda_2)_{r\text{-}th} = \lambda_2(K_{r,1,1}; K_1).$$

(V)

$$\lim_{r\to\infty}(\lambda_2)_{r\text{-}th} = \lim_{r\to\infty}\lambda_2(K_{r,1,1}; K_1) = \frac{1}{3}.$$

Proof. Check all the graphs with $0 < \lambda_2 \leq -1+\sqrt{2}$ listed in Theorem 4.2.

For $t = 3$, consider the graphs of the form $K_{r_1,r_2,r_3}; K_1$, with $r_1 \geq 1$, $r_2 \geq r_3 \geq 1$.

If $r_2 = r_3 = 1$, the graphs are

$$K_{r_1,1,1}; K_1, \quad r_1 = 1, 2, 3, \ldots,$$

From (4.60), the eigenvalue for $K_{r_1,r_2,r_3}; K_1$ can be found by solving

$$P_{K_{r_1,1,1};K_1}(\lambda) = \lambda^4 + (-3r_1 - 1)\lambda_2 - 2r_1\lambda + r_1 = 0.$$

The formulas for the roots of (5.1) are complicated; the first three numerical values for λ_2 are

$$\begin{aligned}(\lambda_2)_{\min} &= \lambda_2(K_{1,1,1}; K_1) = 0.311108,\\ (\lambda_2)_{2nd} &= \lambda_2(K_{2,1,1}; K_1) = 0.321637,\\ (\lambda_2)_{3rd} &= \lambda_2(K_{3,1,1}; K_1) = 0.325397.\end{aligned} \tag{5.1}$$

From (4.74),

$$\lim_{r_1\to\infty}\lambda_2(K_{r_1,1,1}; K_1) = 1/3. \tag{5.2}$$

If $r_2 > 1$, then

$$K_{r_1,r_2,r_3}; K_1 \gg K_{1,2,1}; K_1.$$

But

$$\lambda_2(K_{1,2,1}; K_1) = 0.334904 > 1/3. \tag{5.3}$$

For $t \geq 4$,

$$K_{r_1,r_2,\ldots,r_t}; K_1 \gg K_{1,1,1,1}; K_1,$$

but

$$\lambda_2(K_{1,1,1,1}; K_1) = 0.357926 > 1/3. \tag{5.4}$$

Then (5.1)–(5.4) imply Theorem 5.1.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$ be the eigenvalues of a graph G and of its complement $\bar{G}$, respectively, both in non-increasing order.

Definition. *(see [6]). An eigenvalue λ of a graph G is called a main eigenvalue if its eigenspace S_λ contains a vector X_λ with the coordinate sum $\sum_{i=1}^n x_i$ different from zero.*

From

$$A = A(G) = J - I - A(\bar{G}) = J - I - \bar{A},$$

where J is the $n \times n$ matrix with all entries 1, and I is the $n \times n$ identity matrix, if $\bar{\lambda}$ is an eigenvalue of $\bar{G}$ which is not main, then G contains $-\bar{\lambda} - 1$ as an eigenvalue of the same multiplicity as $\bar{G}$.

Let

$$W = \left\{ X = \{x_1, x_2, \ldots, x_n\} \middle| \sum_{i=1}^{n} x_i = 0 \right\};$$

$$\text{Then } \dim(W) = n - 1.$$

Also

$$\dim(S_{\bar{\lambda}} \cup W) = \begin{cases} n-1, & \text{if } S_{\bar{\lambda}} \subseteq W \text{ (i.e.} \bar{\lambda} \text{ is not a main eigenvalue of } \bar{G}); \\ n, & \text{if } S_{\bar{\lambda}} \not\subseteq W \text{ (i.e.} \bar{\lambda} \text{ is a main eigenvalue of } \bar{G}). \end{cases}$$

Therefore

$$\begin{aligned} \dim(S_{\bar{\lambda}} \cap W) &= \dim(S_{\bar{\lambda}}) + \dim W - \dim(S_{\bar{\lambda}} \cup W) \\ &= \begin{cases} \dim(S_{\bar{\lambda}}), & \text{if } \bar{\lambda} \text{ is not a main eigenvalue of } \bar{G}; \\ \dim(S_{\bar{\lambda}}) - 1, & \text{if } \bar{\lambda} \text{ is a main eigenvalue of } \bar{G}. \end{cases} \end{aligned}$$

If X is an eigenvector corresponding to $\bar{\lambda}$ for $\bar{G}$ such that $\sum_{i=1}^n x_i = 0$, then X is also an eigenvector corresponding to $-\bar{\lambda}-1$ for G. From (5.6), if $\bar{\lambda}$ is an eigenvalue of $\bar{G}$ of multiplicity $p \geq 2$, then $-\bar{\lambda} - 1$ is an eigenvalue of G of multiplicity at least $p - 1$.

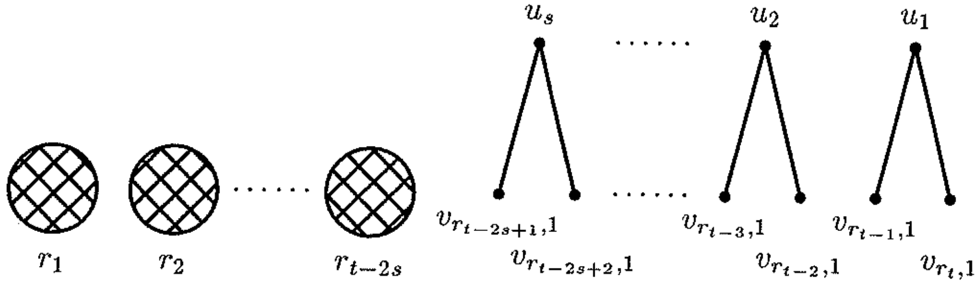
Theorem 5.2. *If*

$$2 \leq s \leq [t/2],$$

then for any

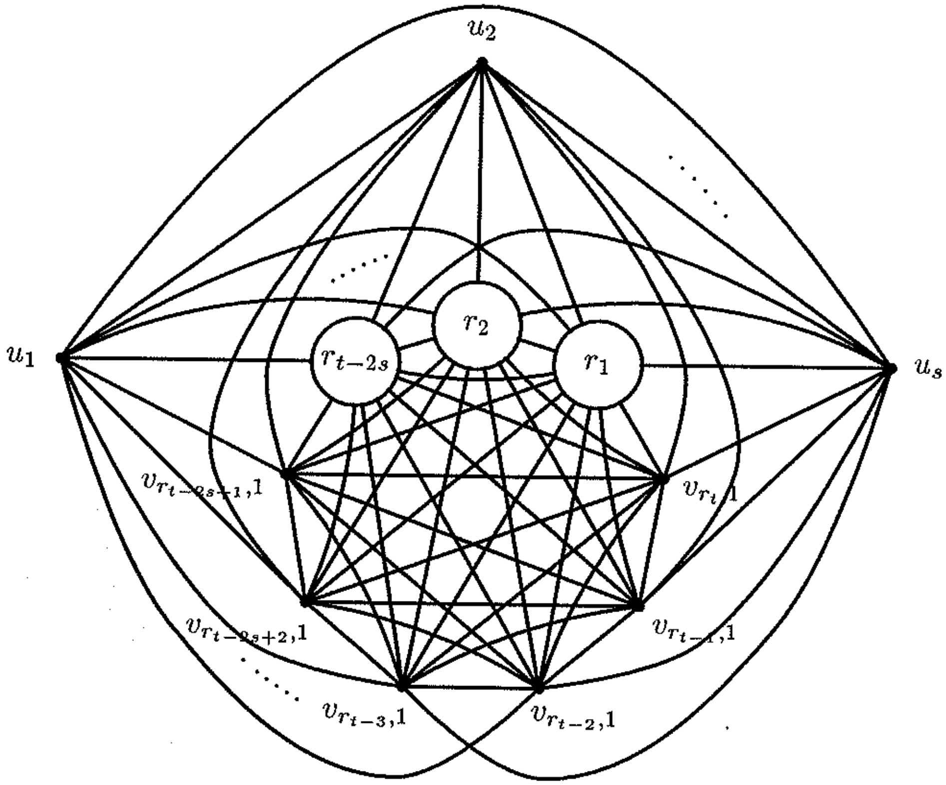
$$r_i \geq 1, \ i = 1, 2, \ldots, t - 2s,$$

$$\lambda_2(K_{r_1, r_2, \ldots, r_{t-2s}, \underbrace{1,1,\ldots,1}_{2s \text{ times}}}; K_s) = -1 + \sqrt{2}.$$



$$\overline{K_{r_1,r_2,\cdots,r_{t-2s}\underbrace{1,1,\cdots,1}_{2s\text{ times}};K_s} = \left(\bigcup_{i=1}^{t-2s} K_{r_i}\right)\bigcup\left(sP_3\right),$$

where $t \geq 3,\ r_i \geq 1,\ i = 1,\cdots,t-2s,\ 1 \leq s \leq [t/2]$.



$$K_{r_1,r_2,\cdots,r_{t-2s},\underbrace{1,1,\cdots,1}_{2s\text{ times}};K_s}$$

where $t \geq 3,\ r_i \geq 1, i = 1,\cdots,t-2s,\ 1 \leq s \leq [t/2]$.

Figure 32

Proof. The complement graph

$$\overline{K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s\text{ times}}};K_s} = sP_2 \cup \left(\bigcup_{i=1}^{t-2s} K_{r_i}\right)$$

as in *Figure 32.*

If $2 \leq s \leq [t/2]$, then from (5.13), $-\sqrt{2}$ is the smallest eigenvalue of

$$\overline{K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s\text{ times}}};K_s}$$

of multiplicity $s \geq 2$. So $-1+\sqrt{2}$ is an eigenvalue of

$$K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s\text{ times}}};K_s$$

with multiplicity at least $s-1 \geq 1$.

From formula (2.4) in the Corollary to Theorem 2.5,

$$\lambda_2(K_{r_1,r_2,\ldots,r_{t-2s}\ \underbrace{1,1,\ldots,1}_{2s\text{ times}}};K_s) \leq -1+\sqrt{2}.$$

Therefore

$$\lambda_2(K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s\text{ times}}};K_s) = -1+\sqrt{2}$$

where $r_i \geq 1,\ i = 1,2,3,\ldots,t-2s,\ 2 \leq s \leq [t/2]$.

Theorem 5.2 can also be proved directly from (5.1) and (4.42), because the eigenvalues of the graph

$$K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s\text{ times}}};K_s$$

other than -1 and 0 can be found by solving

$$P_{K_{r_1,r_2,\ldots,r_{t-2s},\underbrace{1,1,\ldots,1}_{2s\text{ times}};K_s}}$$

$$= \left\{\left[\lambda^2-(3s-2)\lambda+s-1\right]-(\lambda^2+2\lambda-1)\sum_{j=1}^{t-2s}\frac{r_j}{\lambda+r_j}\right\}$$

$$\cdot(\lambda^2+2\lambda-1)^{s-1}\prod_{i=1}^{t-2s}(\lambda+r_i)=0.$$

Theorem 5.3. *If λ_2^* is a limit point of subdominant eigenvalues of simple graphs and $0 < \lambda_2^* < -1 + \sqrt{2}$. Then there are fixed integers*

$$t \geq 3,\ t \geq k \geq 1,$$

fixed integers

$$1 \leq j_1 < j_2 < \ldots < j_k \leq t,$$

and fixed integers

$$r_i \geq 1, i \notin \{j_g \mid 1 \leq g \leq k\}, \text{ and } 1 \leq i \leq t,$$

such that

$$K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t}; K_1$$

is one of the infinite families of graphs listed in Table 13, and

$$\lambda_2^* = \lambda_2(K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t}; K_1)$$

is the limit value for the subdominant eigenvalues for the corresponding infinite families of graphs.

Proof. Let $\{G_n\}$ be a sequence of connected simple graphs such that

$$0 < \lambda_2^* = \lim_{n\to\infty} \lambda_2(G_n) < -1 + \sqrt{2}. \tag{5.5}$$

By considering subsequences, without loss of generality, suppose

$$\begin{gathered} G_n = K_{r_{1,n},r_{2,n},\ldots,r_{t_n,n}}; K_s, \\ \text{where } t_n \geq 3,\ n = 1, 2, 3, \ldots, \end{gathered} \tag{5.6}$$

are graphs in the list of Theorem 4.2. which satisfy the condition

$$0 < \lambda_2(G_n) < -1 + \sqrt{2}. \tag{5.7}$$

From Theorem 5.3, without loss of generality, suppose that in (5.6) $s = 1$,

$$\begin{gathered} G_n = K_{r_{1,n},r_{2,n},\ldots,r_{t_n,n}}; K_1, \\ t_n \geq 3,\ n = 1, 2, 3, \ldots, \end{gathered} \tag{5.8}$$

are graphs listed in Theorem 4.2, with $s = 1$ and (5.7) satisfied.

(I) $t = 3$	
$K_{\infty,1,1};K_1$	$\lambda_2(K_{\infty,1,1};K_1) = 1/3 = 0.333333$
(II) $t = 4$	
$K_{\infty,r_2,1,1};K_1$	$\lambda_2(K_{\infty,r_2,1,1};K_1) = \frac{-5r_2+1+\sqrt{(1-5r_2)^2+8r_2(r_2+3)}}{2(r_2+3)}$
$K_{\infty,\infty,1,1};K_1$	$\lambda_2(K_{\infty,\infty,1,1};K_1) = \frac{-5+\sqrt{33}}{2} = 0.372281$
$K_{\infty,1,2,1};K_1$	$\lambda_2(K_{\infty,1,2,1};K_1) = 0.40989$
(III) $t \geq 5$	
$K_{\infty,\underbrace{1,1,\ldots,1,1}_{t-1 \text{ times}}};K_1$	$\lambda_2(K_{\infty,\underbrace{1,1,\ldots,1,1}_{t-1 \text{ times}}};K_1) = \frac{-(t-2)+\sqrt{2t^2-6t+4}}{t} < -1+\sqrt{2}$
$K_{\underbrace{\infty,\infty,\ldots,\infty,}_{t-2 \text{ times}}1,1};K_1$	$\lambda_2(K_{\underbrace{\infty,\ldots,\infty,}_{t-2 \text{ times}}1,1};K_1) = \frac{-2t+3+\sqrt{8t^2-32t+33}}{2(t-3)} < -1+\sqrt{2}$
$K_{\underbrace{\infty,\ldots,\infty,}_{h \text{ times}}r_{h+1},\ldots,r_{t-2},1,1} \cup K_1$	

$$\lambda_2(K_{\underbrace{\infty,\ldots,\infty,}_{h \text{ times}}r_{h+1},\ldots,r_{t-2},1,1} \cup K_1) < -1+\sqrt{2}$$

where $1 \leq h \leq t-2$,

and $\frac{-(t-2)+\sqrt{2t^2-6t+4}}{t} \leq \lambda_2(K_{\underbrace{\infty,\ldots,\infty,}_{h \text{ times}}r_{h+1},\ldots,r_{t-2},1,1} \cup K_1) \leq \frac{-2t+3+\sqrt{8t^2-32t+33}}{2(t-3)}$

$$\lambda_2^* = \lim_{n\to\infty}(\lambda_2)_n$$

where $0 < \lambda_2^* < -1+\sqrt{2}$

$t \geq 3, r_1 \geq r_2 \geq \cdots \geq r_{t-2} \geq 1,$ and $r_{t-1} \geq r_t \geq 1.$

Table 13

(I) $t = 3$

$K_{\infty,\infty,1};K_1$	$\lambda_2(K_{\infty,\infty,1};K_1) = -1+\sqrt{2}$
$K_{\infty,r_2,1};K_1$	$\lambda_2(K_{\infty,r_2,1};K_1) = \frac{-r_2+\sqrt{r_2^2+r_2(r_2+2)}}{r_2+2} < -1+\sqrt{2}$
$K_{r_1,\infty,1};K_1$	$\lambda_2(K_{r_1,\infty,1};K_1) = \frac{-r_1+\sqrt{r_1^2+r_1(r_1+1)}}{r_1+1} < -1+\sqrt{2}$
$K_{1,\infty,\infty};K_1$	$\lambda_2(K_{1,\infty,\infty};K_1) = -1+\sqrt{2}$
$K_{1,\infty,r_3};K_1$	$\lambda_2(K_{1,\infty,r_3};K_1) = \frac{-r_3+\sqrt{r_3^2+r_3(r_3+1)}}{r_3+1} < -1+\sqrt{2}$
$K_{\infty,2,2};K_1$	$\lambda_2(K_{\infty,2,2};K_1) = 2/5 = 0.4$
$K_{\infty,3,2};K_1$	$\lambda_2(K_{\infty,3,2};K_1) = -1+\sqrt{2}$
$K_{2,\infty,2};K_1$	$\lambda_2(K_{2,\infty,2};K_1) = -1+\sqrt{2}$

(II) $t = 4$

$K_{\infty,\infty,1,1};K_1$	$\lambda_2(K_{\infty,\infty,1,1};K_1) = \frac{-5+\sqrt{33}}{2} = 0.372281 < -1+\sqrt{2}$
$K_{\infty,r_2,1,1};K_1$	$\lambda_2(K_{\infty,r_2,1,1};K_1) = \frac{-5r_2+1+\sqrt{(1-5r_2)^2+8r_2(r_2+3)}}{2(r_2+3)} < -1+\sqrt{2}$
$K_{\infty,1,2,1};K_1$	$\lambda_2(K_{\infty,1,2,1};K_1) = 0.409890 < -1+\sqrt{2}$
$K_{\infty,2,2,1};K_1$	$\lambda_2(K_{\infty,2,2,1};K_1) = -1+\sqrt{2}$

(III) $t \geq 5$

$K_{\infty,\underbrace{1,1,\ldots,1,1}_{t-1 \text{ times}}};K_1$	$\lambda_2(K_{\infty,\underbrace{1,1,\ldots,1,1}_{t-1 \text{ times}}};K_1) = \frac{-(t-2)+\sqrt{2t^2-6t+4}}{t} < -1+\sqrt{2}$
$K_{\underbrace{\infty,\infty,\ldots,\infty,}_{t-2 \text{ times}}1,1};K_1$	$\lambda_2(K_{\underbrace{\infty,\ldots,\infty,}_{t-2 \text{ times}}1,1};K_1) = \frac{-2t+3+\sqrt{8t^2-32t+33}}{2(t-3)} < -1+\sqrt{2}$

$K_{\underbrace{\infty,\ldots,\infty}_{h \text{ times}}r_{h+1},\ldots,r_{t-2},1,1} \cup K_1$

$$\lambda_2(K_{\underbrace{\infty,\ldots,\infty}_{h \text{ times}}r_{h+1},\ldots,r_{t-2},1,1} \cup K_1) < -1+\sqrt{2}$$

where $1 \leq h \leq t-2$,

and $\frac{-(t-2)+\sqrt{2t^2-6t+4}}{t} \leq \lambda_2(K_{\underbrace{\infty,\ldots,\infty,}_{h \text{ times}}r_{h+1},\ldots,r_{t-2},1,1} \cup K_1) \leq \frac{-2t+3+\sqrt{8t^2-32t+33}}{2(t-3)}$

$$K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t};K_s$$

where $0 < \lambda_2(K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t};K_s) \leq -1+\sqrt{2}$

$t \geq 3, r_1 \geq r_2 \geq \ldots \geq r_{t-2} \geq 1,$ and $r_{t-1} \geq r_t \geq 1$

Table 14.

Furthermore, for the purpose of determining

$$\{\lambda_2^* \mid 0 < \lambda_2^* < -1 + \sqrt{2}\},$$

the limit points of $\mathbf{\Pi}_2$ in the interval $(0, -1 + \sqrt{2})$, for the graphs of the form $K_{r_1,r_2,\ldots,r_t}; K_s$ with $0 < \lambda_2(G) \leq -1 + \sqrt{2}$ listed in Theorem 4.2, only $s = 1$ needs to be considered and finitely many graphs of the form $K_{r_1,r_2,\ldots,r_t}; K_1$ in the cases $t = 3$, or $t = 4$ may be dropped. Therefore, without loss of generality, suppose that G_n, $n = 1, 2, 3, \ldots$ in (5.8) are graphs that come from the infinite families of the form of $K_{r_1,r_2,\ldots,r_t}; K_1$ listed in the following *Table 14*. *Table 14* is compiled from the Proof of Theorem 4.2.

If in (5.8)

$$\sup_n \{t_n\} = \infty,$$

by considering subsequences, one may suppose without loss of generality that,

$$\lim_{n\to\infty} t_n = \infty.$$

Let

$$\{u_n\} = V(G_n) - V(K_{r_{1,n},r_{2,n},\ldots,r_{t_n,n}}),$$
$$n = 1, 2, 3, \ldots. \qquad (5.9)$$

Then from (5.8) and (5.9),

$$G_n|_{\{u_n,v_{1,1},v_{2,1},\ldots,v_{t_n,1}\}} = A_{t_n-2,2},$$

and

$$\lambda_2^* = \lim_{n\to\infty} \lambda_2(G_n) \geq \lim_{n\to\infty} \lambda_2(A_{t_n-2,2}) = -1 + \sqrt{2}.$$

which contradicts (5.5). Therefore

$$\sup_n \{t_n\} = t_{\max} < \infty,$$

where $t_{\max}$ must be a positive integer.

Then the sequence

$$G_n = K_{r_{1,n},r_{2,n},\ldots,r_{t_n,n}}; K_1,\ t_n \geq 3,\ n = 1, 2, 3, \ldots$$

contains infinitely many different graphs and

$$3 \leq \sup_n \{t_n\} = t_{\max} < \infty.$$

Since $\sup_n\{t_n\}$ is finite, by the Pigeonhole Principle, there is a positive integer t,

$$3 \le t \le t_{\max}$$

such that there is a subsequence $\{G_{n_l}\}$ of $\{G_n\}$ which satisfies

$$G_{n_l} = K_{r_{1,n_l}, r_{2,n_l}, \ldots, r_{t,n_l}}; K_1, \; l = 1, 2, 3, \ldots, \tag{5.10}$$

where t is fixed for all graphs $\{G_{n_l}\}$, $l = 1, 2, 3, \ldots$.

The graph sequence

$$\{G_{n_l} \mid l = 1, 2, 3, \ldots\}$$

contains graphs with infinitely many different $\lambda_2(G_n)$.

Consider the infinite sequences

$$\begin{aligned} &\{r_{1,n_l}\}, \; l = 1, 2, 3, \ldots, \\ &\{r_{2,n_l}\}, \; l = 1, 2, 3, \ldots, \\ &\{r_{3,n_l}\}, \; l = 1, 2, 3, \ldots, \\ &\qquad \vdots \\ &\{r_{t,n_l}\}, \; l = 1, 2, 3, \ldots. \end{aligned}$$

For the sequence

$$\{r_{1,n_l}\}, \; l = 1, 2, 3, \ldots,$$

there are two cases.

Case 1. If

$$\sup\{r_{1,n_l} \mid l = 1, 2, 3, \ldots\} = \infty, \tag{5.11}$$

let

$$\{r_{1,n_{l_p}}\}, \; p = 1, 2, 3, \ldots$$

be a subsequence of

$$\{r_{1,n_l}\}, \; l = 1, 2, 3, \ldots,$$

such that

$$\lim_{p \to \infty} r_{1,n_{l_p}} = \infty. \tag{5.12}$$

Case 2. If

$$1 \le \sup\{r_{1,n_l} \mid l = 1, 2, 3, \ldots\} = r_{1,\max} < \infty, \tag{5.13}$$

then $\{r_{1,n_l}\}$, $l = 1, 2, 3, \ldots\}$ is an infinite sequence of positive integers and it is bounded from above. By the Pigeonhole Principle, there is a fixed positive integer r_1,

$$1 \le r_1 \le r_{1,\max}, \tag{5.14}$$

and a subsequence

$$\{r_{1,n_{l_p}}\},\ p = 1, 2, 3, \ldots$$

of

$$\{r_{1,n_l}\},\ l = 1, 2, 3, \ldots,$$

such that

$$r_{1,n_{l_p}} = r_1 \text{ for } p = 1, 2, 3, \ldots. \tag{5.15}$$

For the subsequence

$$\{r_{2,n_{l_p}}\},\ p = 1, 2, 3, \ldots \tag{5.16}$$

of the sequence

$$\{r_{2,n_l}\},\ l = 1, 2, 3, \ldots,$$

there are two cases.

Case 1. If

$$\sup\{r_{2,n_{l_p}} \mid l = 1, 2, 3, \ldots\} = \infty, \tag{5.17}$$

let

$$\{r_{2,n_{l_{p_q}}}\},\ q = 1, 2, 3, \ldots$$

be a subsequence of

$$\{r_{2,n_{l_p}}\},\ p = 1, 2, 3, \ldots,$$

such that

$$\lim_{q\to\infty} r_{1,n_{l_{p_q}}} = \infty. \tag{518}$$

Case 2. If

$$1 \leq \sup\{r_{2,n_{l_q}} \mid l = 1, 2, 3, \ldots\} = r_{2,\max} < \infty, \tag{5.19}$$

then $\{r_{2,n_{l_q}}\},\ q = 1, 2, 3, \ldots$ is an infinite sequence of positive integers and it is bounded from above. By the Pigeonhole Principle, there is a fixed positive integer r_2,

$$1 \leq r_2 \leq r_{2,\max}, \tag{5.20}$$

and a subsequence

$$\{r_{1,n_{l_{p_q}}}\},\ q = 1, 2, 3, \ldots$$

of

$$\{r_{1,n_{l_p}}\},\ p = 1, 2, 3, \ldots,$$

such that

$$r_{1,n_{l_{p_q}}} = r_2 \text{ for } q = 1, 2, 3, \ldots. \tag{5.21}$$

We repeat this procedure to get t times altogether. Without loss of generality, we may suppose for $1 \leq i \leq t$, either

$$\lim_{l\to\infty} r_{i,n_l} = \infty,$$

or there is a fixed positive integer r_i, such that

$$r_{i,n_l} = r_i \text{ for } l = 1, 2, 3, \ldots .$$

Let

$$1 \le j_1 < j_2 < \cdots < j_k \le t, \tag{5.22}$$

be all subscripts which satisfy the condition

$$\lim_{l\to\infty} r_{j_g,n_l} = \infty, \text{ for } 1 \le g \le k. \tag{5.23}$$

Let

$$\begin{gathered}\{i_1, i_2, \ldots, i_{t-k}\} = \{i \mid i \notin \{j_g \mid 1 \le g \le k\}, \text{ and } 1 \le i \le t\} \\ \text{where } 1 \le i_1 \le i_2 \ldots \le i_{t-k} \le t.\end{gathered} \tag{5.24}$$

be all subscripts which satisfy the condition that there is a fixed positive integers

$$r_{i_h}, 0 \le h \le t-k,$$

such that

$$r_{i_h,n_l} = r_{i_h}, \text{ for } l = 1, 2, 3, \ldots . \tag{5.25}$$

From (5.10), and (5.22)–(5.25),

$$\begin{gathered}G_{n_l} = K_{r_{1,n_l},\ldots,r_{i_1-1,n_l},r_{i_1},r_{i_1+1,n_l},\ldots,r_{t-k-1,n_l},r_{t-k},r_{t-k+1,n_l},\ldots,r_{t,n_l}};K_1, \\ l = 1, 2, 3, \ldots .\end{gathered} \tag{5.26}$$

From (5.5), (5.7) and (5.22)–(5.26),

$$\begin{aligned}0 < \lambda_2^* &= \lim_{n\to\infty} \lambda_2(G_n) \\ &= \lim_{l\to\infty} K_{r_{1,n_l},\ldots,r_{i_1-1,n_l},r_{i_1},r_{i_1+1,n_l},\ldots,r_{t-k-1,n_l},r_{t-k},r_{t-k+1,n_l},\ldots,r_{t-1,n_l},r_{t,n_l}};K_1 \\ &= \lim_{\substack{r_{j_1}\to\infty \\ r_{j_2}\to\infty \\ \vdots \\ r_{j_k}\to\infty}} \lambda_2(K_{r_1,r_2,\ldots,r_t};K_1) \\ &\overset{\text{def}}{=} \lambda_2(K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_2-1},\infty,r_{j_2+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t};K_s) \\ &< -1 + \sqrt{2}.\end{aligned} \tag{5.27}$$

Observe the infinite families and their corresponding limit values for the subdominant eigenvalues in *Table 14*. There are some values λ_2^* appearing repeatedly as limit values for subdominant eigenvalues for the the different infinite families.

$$\begin{aligned}\{\lambda_2(K_{r_1,\infty,1,};K_1) \mid r_1 \geq 1\} &= \{\frac{-r+\sqrt{r^2+r(r+1)}}{r+1} \mid r \geq 1\}\\ &= \{\lambda_2(K_{1,\infty,r_3,};K_1) \mid r_3 \geq 1\}.\end{aligned} \tag{5.28}$$

Let $r_2 = 2r$, in $\lambda_2(K_{\infty,r_2,1};K_1)$; then

$$\begin{aligned}\lambda_2(K_{\infty,r_2,1};K_1) &= \frac{-r_2+\sqrt{r_2^2+r_2(r_2+2)}}{r_2+2}\\ &= \frac{-2r+\sqrt{(2r)^2+2r(2r+2)}}{2r+2}\\ &= \frac{-r+\sqrt{r^2+r(r+1)}}{r+1}\\ &= \lambda_2(K_{r,\infty,1};K_1),\end{aligned} \tag{5.29}$$

$$\{\lambda_2(K_{\infty,r_2,1};K_1) \mid r_2 \geq 1\} \supseteq \{\lambda_2(K_{r,\infty,1};K_1) \mid r \geq 1\}. \tag{5.30}$$

Let $r_2 = t-2, r_2 \geq 1, t \geq 3$; then

$$\begin{aligned}\lambda_2(K_{\infty,r_2,1};K_1) &= \frac{-r_2+\sqrt{r_2^2+r_2(r_2+2)}}{r_2+2}\\ &= \frac{-(t-2)+\sqrt{(t-2)^2+t(t-2)}}{t}\\ &= \frac{-(t-2)+\sqrt{2t^2-6t+4}}{t}\\ &= \lambda_2(K_{\infty,\underbrace{1,1,\ldots,1,1}_{t-1 \text{ times}}};K_1),\end{aligned} \tag{5.31}$$

$$\lambda_2(K_{\infty,2,2};K_1) = 2/5 = 0.4 = \lambda_2(K_{\infty,\underbrace{1,1,\ldots,1,1}_{9 \text{ times}}};K_1). \tag{5.32}$$

Table 15 lists the first several numerical values of the infinite families

$$\lambda_2(K_{\infty,r,1};K-1) \text{ and } \lambda_2(K_{r,\infty,1};K_1),$$

and *Table 16* lists the first several numerical values of the infinite families

$$\lambda_2(K_{\infty,r,1,1};K_1), \lambda_2(K_{\infty,\underbrace{1,1,\ldots,1,1}_{t-1 \text{ times}}};K_1) \text{ and } \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{t-2 \text{ times}},1,1};K_1).$$

$r \geq 1$	$\lambda_2(K_{\infty,r,1};K_1)$	$\lambda_2(K_{r,\infty,1};K_1)$
r	$\frac{-r+\sqrt{r^2+r(r+2)}}{r+2}$	$\frac{-r+\sqrt{r^2+r(r+1)}}{r+1}$
1	$1/3 = 0.333333$	$(-1+\sqrt{3})/2 = 0.366025$
2	$(-1+\sqrt{3})/2 = 0.366025$	$(-2+\sqrt{10})/2 = 0.387426$
3	$(-3+\sqrt{24})/5 = 0.379796$	$(-3+\sqrt{21})/4 = 0.395644$
4	$(-2+\sqrt{10})/3 = 0.387426$	$2/5 = 0.4$
5	$(-5+2\sqrt{15})/7 = 0.392281$	$(-5+\sqrt{55})/6 = 0.4027$
6	$(-3+\sqrt{21})/4 = 0.395644$	$(-6+\sqrt{78})/7 = 0.404537$
7	$(-7+4\sqrt{7})/9 = 0.398112$	$(-7+\sqrt{105})/8 = 0.405869$
8	$2/5 = 0.4$	$(-8+\sqrt{136})/9 = 0.406878$
9	$(-9+6\sqrt{5})/11 = 0.401492$	$(-9+3\sqrt{19})/10 = 0.40767$
10	$(-5+\sqrt{55})/6 = 0.4027$	$(-10+\sqrt{210})/11 = 0.408307$
$\vdots$	$\vdots$	$\vdots$

Table 15.

$r \geq 1$	$\lambda_2(K_{\infty,r,1,1};K_1)$
r	$\frac{-5r+1+\sqrt{(1-5r)^2+8r(r+3)}}{2(r+3)}$
1	$(-1+\sqrt{3})/2 = 0.366025$
2	$(-9+\sqrt{161})/10 = 0.368858$
3	$(-7+\sqrt{85})/6 = 0.369924$
4	$(-19+3\sqrt{65})/14 = 0.370484$
5	$(-3+\sqrt{14})/2 = 0.370829$
6	$(-29+\sqrt{1273})/18 = 0.371063$
7	$(-17+\sqrt{429})/10 = 0.371232$
8	$(-39+5\sqrt{89})/22 = 0.371359$
9	$(-11+5\sqrt{7})/6 = 0.371459$
10	$(-49+\sqrt{3441})/26 = 0.37154$
$\vdots$	$\vdots$

$t \geq 3$	$\lambda_2(K_{\infty,\underbrace{1,1,\ldots,1,1}_{t-1 \text{ times}}};K_1)$	$\lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{t-2 \text{ times}},1,1};K_1)$
t	$\frac{-(t-2)+\sqrt{2t^2-6t+4}}{t}$	$\frac{-2t+3+\sqrt{8t^2-32t+33}}{2(t-3)}$, for $t \geq 4$
3	$1/3 = 0.333333$	$1/3 = 0.333333$
4	$(-1+\sqrt{3})/2 = 0.366025$	$(-5+\sqrt{33})/2 = 0.372281$
5	$(-3+\sqrt{24})/5 = 0.379796$	$(-7+\sqrt{73})/4 = 0.386001$
6	$(-2+\sqrt{10})/3 = 0.387426$	$(-9+\sqrt{129})/6 = 0.392969$
7	$(-5+2\sqrt{15})/7 = 0.392281$	$(-11+\sqrt{201})/8 = 0.397181$
8	$(-6+2\sqrt{21})/8 = 0.395644$	$2/5 = 0.4$
9	$(-7+4\sqrt{7})/9 = 0.398112$	$(-15+\sqrt{393})/12 = 0.402019$
10	$2/5 = 0.4$	$(-17+\sqrt{513})/14 = 0.403536$
11	$(-9+6\sqrt{5})/11 = 0.401492$	$(-19+\sqrt{649})/16 = 0.404717$
12	$(-5+\sqrt{55})/6 = 0.4027$	$(-7+\sqrt{89})/6 = 0.405664$
13	$(-11+\sqrt{264})/13 = 0.403698$	$(-23+\sqrt{969})/20 = 0.406438$
14	$(-6+\sqrt{78})/7 = 0.403698$	$(-25+\sqrt{1153})/22 = 0.407084$
$\vdots$	$\vdots$	$\vdots$

Table 16.

Because of (5.28)–(5.32), all the subdominant eigenvalues that appear in *Table 15* also appear in *Table 16.*

From (5.27), by checking the infinite families and their corresponding limit values in *Table 13* and avoiding the repeating graphs with the same subdominant eigenvalues in (5.28)–(5.32), one see that Theorem 5.3 holds.

Corollary. *If* λ_2^* *is a limit point of subdominant eigenvalues of simple graphs and* $0 < \lambda_2^* < -1 + \sqrt{2}$, *then there are fixed integers*

$$t \geq 3,\ t-2 \geq k \geq 1,$$

and fixed integers

$$r_i \geq 1, k+1 \leq i \leq t,$$

where

$$r_{k+1} \geq r_{k+2} \geq \cdots \geq r_{t-2} \geq 1, r_{t-1} \geq r_t \geq 1,$$

such that

$$K_{\underbrace{\infty,\infty,\ldots,\infty}_{k\ \text{times}},r_{k+1},\ldots,r_{t-1},r_t}; K_1$$

is one of the infinite families of graphs listed in Table 13, and

$$\lambda_2^* = \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{k\ \text{times}},r_{j_k+1},\ldots,r_{t-1},r_t}; K_1)$$

is the limit value for the subdominant eigenvalues for the corresponding infinite families of graphs.

Proof. For all the infinite families of graphs of the form

$$K_{r_1,r_2,\ldots,r_{j_1-1},\infty,r_{j_1+1},\ldots,r_{j_k-1},\infty,r_{j_k+1},\ldots,r_{t-1},r_t}; K_1,$$

listed in *Table 13*, the conditions

$$t \geq 3,\ t-2 \geq k \geq 1$$

are satisfied; the subscripts may be arranged such that

$$j_1 = 1, j_2 = 2, \ldots, j_k = k;$$

and

$$r_{k+1} \geq r_{k+2} \geq \ldots \geq r_{t-2}, r_{t-1} \geq r_t.$$

Then Theorem 5.3 implies that the Corollary holds.

Theorem 5.4. *In the interval* $\left(0, \frac{-5+\sqrt{33}}{2}\right]$*, among the limit points of subdominant eigenvalues of simple graphs*

(I) the smallest one is

$$(\lambda_2^*)_{\min} = 1/3 = \lambda_2(K_{\infty,1,1}; K_1),$$

(II) the second smallest one is

$$(\lambda_2^*)_{2nd} = \frac{-1+\sqrt{3}}{2}(= 0.366025) = \lambda_2(K_{\infty,1,1,1}; K_1),$$

(III) the third smallest one is

$$(\lambda_2^*)_{3rd} = \frac{9+\sqrt{161}}{10}(= 0.368858) = \lambda_2(K_{\infty,2,1,1}; K_1),$$

(IV) for $r \geq 1$, the $(r+1)$-st smallest one is

$$(\lambda_2^*)_{(r+1)\text{-}st} = \frac{-5r+1+\sqrt{(1-5r)^2+8r(r+3)}}{2(r+3)} = \lambda_2(K_{\infty,r,1,1}; K_1),$$

(V)

$$\begin{aligned}\lim_{r\to\infty}(\lambda_2^*)_{r\text{-}th} &= \lim_{r\to\infty}\frac{-5r+1+\sqrt{(1-5r)^2+8r(r+3)}}{2(r+3)} \\ &= \frac{-5+\sqrt{33}}{2}(= 0.372281) = \lambda_2(K_{\infty,\infty,1,1}; K_1).\end{aligned}$$

Proof. Check the list in Theorem 5.3, *Table 13* and *Table 16.*

The smallest limit point is given by $t = 3, \lambda_2(K_{\infty,1,1}; K_1) = 1/3$.

For $t \geq 5$, among the limit values of subdominant eigenvalues of simple graphs given by the infinite families of graphs of the form $K_{r_1,r_2,\ldots,r_t}; K_1$,

$$\lambda_2(K_{\infty,1,1,1,1}; K_1) = \frac{-3+\sqrt{24}}{5} = 0.379796$$

gives the smallest one for $t \geq 5$; but

$$\frac{-3+\sqrt{24}}{5} = 0.379796 > 0.372281 = \frac{-5+\sqrt{33}}{2}.$$

For $t = 4$, among the limit values of subdominant eigenvalues of simple graphs given by the infinite families of graphs of the form $K_{r_1,r_2,r_3,r_4}; K_1$,

$$\lambda_2(K_{\infty,1,1,1}; K_1) = \frac{-1+\sqrt{3}}{2}$$
$$\leq \lambda_2(K_{\infty,r,1,1}; K_1) = \frac{-5r+1+\sqrt{(1-5r)^2+8r(r+3)}}{2(r+3)}$$
$$\leq \lambda_2(K_{\infty,\infty,1,1}; K_1) = \frac{-5+\sqrt{33}}{2},$$
$$\text{where } r \geq 1,$$

and

$$\lambda_2(K_{\infty,1,2,1}; K_1) = 0.40989 > \frac{-5+\sqrt{33}}{2}.$$

Then

$$r = 1,\ \lambda_2(K_{\infty,1,1,1}; K_2) = \frac{-1+\sqrt{3}}{2} = 0.366025$$

gives the second smallest one,

$$r = 2,\ \lambda_2(K_{\infty,2,1,1}; K_2) = \frac{-9+\sqrt{161}}{2} = 0.368858$$

gives the third smallest one, and

$$r \geq 1,\ \lambda_2(K_{\infty,r,1,1}; K_1) = \frac{-5r+1+\sqrt{(1-5r)^2+8r(r+3)}}{2(r+3)}$$

gives the $(r+1)$-st smallest one.

Then

$$\lim_{r\to\infty} (\lambda_2^*)_{r\text{-}th} = \lim_{r\to\infty} \frac{-5r+1+\sqrt{(1-5r)^2+8r(r+3)}}{2(r+3)}$$
$$= \lim_{r\to\infty} \lambda_2(K_{\infty,r,1,1}; K_1) = \lambda_2(K_{\infty,\infty,1,1}; K_1) = \frac{-5+\sqrt{33}}{2} = 0.372281.$$

Recall that $\mathbf{\Pi}_{\mathbf{k}}^{(\mathbf{j})}$ has been defined in the Introduction.

Definition. *Let*

$$\begin{aligned}
\mathbf{\Pi_k} = \mathbf{\Pi_k^{(0)}} &= \{\lambda_k \mid \lambda_k = \lambda_k(G),\ \textit{for } G \in \mathcal{W}\},\ \textit{for } k = 1,2,3,\dots,\\
\mathbf{\Pi_k^{(1)}} &= \{\lambda_k^{(1)} \mid \lambda_k^{(1)} \textit{ is a limit point of } \mathbf{\Pi_k^{(0)}}\},\ \textit{for } k = 1,2,3,\dots,\\
\mathbf{\Pi_k^{(2)}} &= \{\lambda_k^{(2)} \mid \lambda_k^{(2)} \textit{ is a limit point of } \mathbf{\Pi_k^{(1)}}\},\ \textit{for } k = 1,2,3,\dots,\\
&\ \vdots\\
\mathbf{\Pi_k^{(m+1)}} &= \{\lambda_k^{(m+1)} \mid \lambda_k^{(m+1)} \textit{ is a limit point of } \mathbf{\Pi_k^{(m)}}\},\\
&\qquad\qquad\qquad\qquad \textit{for } k = 1,2,3,\dots,\\
&\ \vdots
\end{aligned}$$

Theorem 5.5. *For* $m \geq 1$,

(I) If

$$\lambda_2^{*^m} \in \left[\mathbf{\Pi_2^{(m)}} \cap (0, -1+\sqrt{2})\right],$$

then there are fixed integers t *and* k,

$$t \geq 3,\ t-2 \geq k \geq m,$$

and fixed integers r_i,

$$r_i \geq 1, k+1 \leq i \leq t,$$

where

$$r_{k+1} \geq r_{k+2} \geq \cdots \geq r_{t-2} \geq 1,\ r_{t-1} \geq r_t \geq 1,$$

such that

$$K_{\underbrace{\infty,\infty,\dots,\infty,}_{k \text{ times}} r_{k+1},\dots,r_{t-1},r_t}$$

is one of the infinite families of graphs listed in Table 13, and

$$\lambda_2^{*^m} = \lambda_2(K_{\underbrace{\infty,\infty,\dots,\infty,}_{k \text{ times}} r_{k+1},\dots,r_{t-1},r_t}; K_1).$$

is the limit value for the subdominant eigenvalues for the corresponding infinite families of graphs.

(II) For $m \geq 1$,

$$\inf[\mathbf{\Pi_2^{(m)}} \cap (0, -1+\sqrt{2})]$$

$$= \begin{cases} \frac{1}{3} = \lambda_2(K_{\infty,1,1}; K_1), & \textit{if } m = 1;\\[2ex] \frac{-2(m+2)+3+\sqrt{8(m+2)^2-32(m+2)+33}}{2[(m+2)-3]} = \lambda_2(K_{\underbrace{\infty,\infty,\dots,\infty,}_{m \text{ times}}1,1}; K_1), & \textit{if } m \geq 2. \end{cases}$$

Proof. For (I), use mathematical induction on m.

For $m = 1$, the Corollary to Theorem 5.3 implies the results hold.

Suppose for $m \geq 1$, Theorem 5.5. Part(I) holds.

For $m + 1$, let

$$\lambda_2^{*^{m+1}} \in \mathbf{\Pi}_{\mathbf{2}}^{\mathbf{(m+1)}},$$

where

$$0 < \lambda_2^{*^{m+1}} < -1 + \sqrt{2}. \tag{5.33}$$

Let

$$\{(\lambda_2^{*^{m}})_n\} \subseteq \mathbf{\Pi}_{\mathbf{2}}^{\mathbf{(m)}},$$

where

$$0 < (\lambda_2^{*^{m+1}})_n < -1 + \sqrt{2},$$

and

$$0 < \lambda_2^{*^{m+1}} = \lim_{n \to \infty} (\lambda_2^{*^{m}})_n < -1 + \sqrt{2}.$$

For

$$(\lambda_2^{*^{m}})_n, \ n = 1, 2, 3, \ldots,$$

by the inductive assumption, there are fixed integers

$$t_n \geq 3, \ t_n - 2 \geq k_n \geq m,$$

and fixed integers

$$r_{i,n} \geq 1, k_n + 1 \leq i \leq t_n,$$

where

$$r_{k_n+1,n} \geq r_{k_n+2,n} \geq \cdots \geq r_{t_n-2,n} \geq 1, \ r_{t_n-1,n} \geq r_{t_n,n} \geq 1,$$

such that

$$K_{\underbrace{\infty,\infty,\ldots,\infty}_{k_n \text{ times}}, r_{k_n+1,n}, \ldots, r_{t_n-1,n}, r_{t_n,n}}; K_1$$

is one of the infinite families of graphs listed in the the *Table 13*, and

$$(\lambda_2^{*^{m}})_n = \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{k_n \text{ times}}, r_{k_n+1,n}, \ldots, r_{t_n-1,n}, r_{t_n,n}}; K_1) \tag{5.34}$$

is the limit value for the subdominant eigenvalues of the corresponding infinite families of graphs listed in *Table 13.*

If in (5.34)

$$\sup_n \{t_n\} = \infty, \tag{5.35}$$

by considering subsequences, one may without loss of generality suppose

$$\lim_{n\to\infty} t_n = \infty. \tag{5.36}$$

Then from (5.34)–(5.36),

$$(\lambda_2^{*^m})_n \geq \lambda_2(K_{\underbrace{1,1,\ldots,1}_{t_n \text{ times}}}; K_1) = \lambda_2(A_{t_{n-2},2}).$$

Then

$$\lambda_2^{*^{m+1}} = \lim_{n\to\infty} (\lambda_2^{*^m})_n \geq \lim_{n\to\infty} \lambda_2(A_{t_n-2,2}) = -1+\sqrt{2},$$

which contradicts (5.33).

Therefore

$$\sup_n \{t_n\} = t_{\max} < \infty,$$

where $t_{\max}$ is a positive integer.

Then the sequence

$$(\lambda_2^{*^m})_n = \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty,}_{k_n \text{ times}} r_{k_n+1,n},\ldots,r_{t_n-1,n},r_{t_n,n}}; K_1)$$

contains infinitely many different real numbers and

$$3 \leq \sup_n \{t_n\} = t_{\max} < \infty,$$

$\sup_n\{t_n\}$ is finite.

By the Pigeonhole Principle, there is a fixed integer t,

$$3 \leq t \leq t_{\max}$$

such that there is a subsequence

$$\{(\lambda_2^{*^m})_{n_l}\},\ l = 1,2,3,\ldots$$

of

$$\{(\lambda_2^{*^m})_n\},\ n = 1,2,3,\ldots,$$

such that

$$(\lambda_2^{*^m})_{n_l} = \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty,}_{k_{n_l} \text{ times}} r_{k_{n_l}+1,n_l},\ldots,r_{t-1,n_l},r_{t,n_l}}; K_1), \tag{5.37}$$

where t is a fixed positive integer for all

$$(\lambda_2^{*^m})_{n_l}, l = 1,2,3,\ldots.$$

Because

$$\{k_{n_l}\},\ l = 1, 2, 3, \dots$$

is an infinite sequence, and

$$m \leq k_{n_l} \leq t - 2,\ l = 1, 2, 3, \dots,$$

where $t \geq 3$ is a fixed integer, by the Pigeonhole Principle, there is a fixed integer

$$\tilde{k},\ m \leq \tilde{k} \leq t - 2,$$

and a subsequence

$$\{k_{n_{l_p}}\},\ p = 1, 2, 3, \dots$$

of

$$\{k_{n_l}\},\ l = 1, 2, 3, \dots,$$

such that

$$k_{n_{l_p}} = \tilde{k},\ p = 1, 2, 3, \dots.$$

Then from (5.37) and (5.38),

$$(\lambda_2^{*^m})_{n_{l_p}} = \lambda_2(K_{\underbrace{\infty,\infty,\dots,\infty,}_{\tilde{k}\text{ times}} r_{\tilde{k}+1,n_{l_p}},\dots,r_{t-1,n_{l_p}},r_{t,n_{l_p}}}; K_1),$$
$$p = 1, 2, 3, \dots, \tag{5.39}$$

where

$$t \geq 3,\ \text{and}\ \ m \leq \tilde{k} \leq t - 2 \tag{5.40}$$

are fixed integers for all

$$\{(\lambda_2^{*^m})_{n_{l_p}}\},\ p = 1, 2, 3, \dots,$$

and

$$r_{\tilde{k}+1,n_{l_p}} \geq \dots \geq r_{t-2,n_{l_p}} \geq 1,\ r_{t-1,n_{l_p}} \geq r_{t,n_{l_p}} \geq 1. \tag{5.41}$$

Because

$$\{(\lambda_2^{*^m})_{n_{l_p}} \mid p = 1, 2, 3, \dots\}$$

contains infinitely many different real numbers and both t and $\tilde{k}$ are fixed positive integers in (5.46), by the Pigeonhole Principle, among the infinite sequences

$$\begin{gathered}
\{r_{\tilde{k}+1,n_{l_p}}\},\ p = 1, 2, 3, \dots, \\
\{r_{\tilde{k}+2,n_{l_p}}\},\ p = 1, 2, 3, \dots, \\
\vdots \\
\{r_{t,n_{l_p}}\},\ p = 1, 2, 3, \dots,
\end{gathered} \tag{5.42}$$

at least one must be unbounded from above.

Because

$$r_{\tilde{k}+1,n_{l_p}} \geq r_{\tilde{k}+2,n_{l_p}} \geq \cdots \geq r_{t,n_{l_p}}, \tag{5.43}$$

(5.42) and (5.43) imply that

$$\sup\{r_{\tilde{k}+1,n_{l_p}} \mid p = 1, 2, 3, \ldots\} = \infty.$$

Let

$$\{r_{\tilde{k}+1,n_{l_{p_\mu}}}\},\ \mu = 1, 2, 3, \ldots$$

be a subsequence of

$$\{r_{\tilde{k}+1,n_{l_p}}\},\ p = 1, 2, 3, \ldots,$$

such that

$$\lim_{\mu \to \infty} r_{\tilde{k}+1,n_{l_{p_\mu}}} = \infty,$$

Consider the sequence

$$\{r_{\tilde{k}+2,n_{l_{p_\mu}}}\}\ \mu = 1, 2, 3, \ldots.$$

There are two cases.

Case 1. If

$$\sup\{r_{\tilde{k}+2,n_{l_{p_\mu}}} \mid \mu = 1, 2, 3, \ldots\} = \infty, \tag{5.44}$$

let

$$\{r_{\tilde{k}+2,n_{l_{p_{\mu_\nu}}}}\},\ \nu = 1, 2, 3, \ldots$$

be a subsequence of

$$\{r_{\tilde{k}+2,n_{l_{p_\mu}}}\}\ \mu = 1, 2, 3, \ldots,$$

such that

$$\lim_{\nu \to \infty} r_{\tilde{k}+2,n_{l_{p_{\mu_\nu}}}} = \infty, \tag{5.45}$$

Case 2. If

$$\sup\{r_{\tilde{k}+2,n_{l_{p_\mu}}} \mid \mu = 1, 2, 3, \ldots\} = r_{\tilde{k}+2,\max} < \infty, \tag{5.46}$$

Then by Pigeonhole Principle, there is a subsequence

$$\{r_{\tilde{k}+2,n_{l_{p_{\mu_\nu}}}}\},\ \nu = 1, 2, 3, \ldots$$

of

$$\{r_{\tilde{k}+2,n_{l_{p_\mu}}}\}\ \mu = 1, 2, 3, \ldots,$$

and a fixed integer

$$r_{\tilde{k}+2}, \text{ where } , 1 \leq r_{\tilde{k}+2} \leq r_{\tilde{k}+2,\max}$$

such that

$$r_{\tilde{k}+2,n_{l_{p_{\mu_\nu}}}} = r_{\tilde{k}+2}, \text{for} \nu = 1,2,3,\ldots. \tag{5.47}$$

In this case, (5.43) and (5.47) imply that,

$$\infty > r_{\tilde{k}+2} = r_{\tilde{k}+2,n_{l_{p_{\mu_\nu}}}} \geq r_{\tilde{k}+t,n_{l_{p_{\mu_\nu}}}} \geq \cdots r_{\tilde{t},n_{l_{p_{\mu_\nu}}}}, \text{ for } \nu = 1,2,3,\ldots \tag{5.48}$$

We repeat this procedure to get $t - \tilde{k}$ times altogether. Without loss of generality, we may suppose

$$\begin{aligned} \lim_{p\to\infty} r_{\tilde{k}+1,n_{l_p}} &= \infty, \\ \lim_{p\to\infty} r_{\tilde{k}+2,n_{l_p}} &= \infty, \\ &\vdots \\ \lim_{p\to\infty} r_{k,n_{l_p}} &= \infty. \end{aligned} \tag{5.49}$$

and

$$\begin{aligned} r_{k+1,n_{l_p}} &= r_{\tilde{k}+1}, \text{ for } p = 1,2,3,\ldots \\ r_{k+2,n_{l_p}} &= r_{\tilde{k}+2}, \text{ for } p = 1,2,3,\ldots \\ &\vdots \\ r_{t,n_{l_p}} &= r_{\tilde{t}}, \text{ for } p = 1,2,3,\ldots \end{aligned} \tag{5.50}$$

where

$$m \leq \tilde{k} < k \leq t, \tag{5.51}$$

and $r_{\tilde{k}+1}, r_{\tilde{k}+2}, \ldots, r_t$ are fixed positive integers for all $p = 1,2,3,\ldots$.

Therefore, from (5.39) and (5.49)–(5.51),

$$\begin{aligned} 0 &< {\lambda_2^*}^{m+1} \\ &= \lim_{q\to\infty} ({\lambda_2^*}^m)_{n_{l_{p_q}}} \\ &= \lim_{q\to\infty} \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty,}_{\tilde{k} \text{ times}} r_{\tilde{k}+1,n_{l_{p_q}}},\ldots,r_{k,n_{l_{p_q}}},r_{k+1,n_{l_{p_q}}},\ldots r_{t-1,n_{l_{p_q}}},r_{t,n_{l_{p_q}}}}; K_1) \\ &= \lim_{q\to\infty} \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty,}_{\tilde{k} \text{ times}} r_{\tilde{k}+1,n_{l_{p_q}}},\ldots,r_{k,n_{l_{p_q}}},r_{k+1},\ldots,r_{t-1},r_t}; K_1) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\substack{r_{\tilde{k}+1,n_{l_{p_q}}} \to\infty \\ r_{\tilde{k}+2,n_{l_{p_q}}} \to\infty \\ \vdots \\ r_{k,n_{l_{p_q}}} \to\infty}} \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty,}_{\tilde{k}\text{ times}} r_{\tilde{k}+1,n_{l_{p_q}}},\ldots,r_{k,n_{l_{p_q}}},r_{k+1},\ldots,r_{t-1},r_t}; K_1) \\
&\overset{\text{def}}{=} \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty,}_{k\text{ times}} r_{k+1},\ldots,r_{t-1},r_t}; K_1) \\
&< -1+\sqrt{2}.
\end{aligned} \tag{5.52}
$$

Then (5.111) implies that

$$t \geq 3,\ m+1 \leq k \leq t. \tag{5.53}$$

From the Corollary to Theorem 5.3, the infinite families in (5.52) can be chosen from *Table 13* such that

$$\lambda_2^{*^{m+1}}$$

is the limit value for the subdominant eigenvalue for the corresponding infinite family of the graphs and $k \leq t-2$ in (5.53). Then (5.53) becomes

$$t \geq 3, m+1 \leq k \leq t-2. \tag{5.54}$$

For Part (II).

If $m = 1$, from Theorem 5.4. Part (I),

$$\inf[\mathbf{\Pi}_2^{(1)} \cap (0, -1+\sqrt{2})] = (\lambda_2^*)_{\min} = 1/3 = \lambda_2(K_{\infty,1,1}; K_1).$$

If $m \geq 2$, from Part (I),

$$
\begin{aligned}
&\inf[\in \mathbf{\Pi}_2^{(\mathbf{m})} \cap (0,-1+\sqrt{2})] \\
&= \inf\left\{ \lambda_2^{*^m} \;\middle|\; \lambda_2^{*^m} \in [\mathbf{\Pi}_2^{(\mathbf{m})} \cap (0,-1+\sqrt{2})] \right\} \\
&= \inf\left\{ \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty,}_{k\text{ times}} r_{k+1},\ldots,r_{t-1},r_t}; K_1) \;\middle|\; \begin{array}{l} t \geq 3,\ m \leq k \leq t-2, \\ 0 < \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty,}_{k\text{ times}} r_{k+1},\ldots,r_{t-1},r_t}; K_1) < -1+\sqrt{2}. \end{array} \right\} \\
&= \inf\left\{ \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty,}_{k\text{ times}} r_{k+1},\ldots,r_{t-1},r_t}; K_1) \;\middle|\; k = m,\ t = m+2, \right.
\end{aligned}
$$

$$\left. 0 < \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{k \text{ times}},r_{k+1},\ldots,r_{t-1},r_t};K_1) < -1+\sqrt{2}. \right\}$$

$$= \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{m \text{ times}},1,1};K_1)$$

(from *Table 13*)

$$= \frac{-2(m+2)+3+\sqrt{8(m+2)^2-32(m+2)+33}}{2[(m+2)-3]}.$$

Theorem 5.6. *Let $p \geq 0$ be any nonnegative integer. If U_p is an open subinterval of $Y = [0,-1+\sqrt{2}]$, such that*

$$U_p \neq \emptyset,\ U_p \subseteq Y = [0,-1+\sqrt{2}],\ \textit{and}\ U_p \cap \mathbf{\Pi_2^{(p)}} = \emptyset, \tag{5.55}$$

then there is an open subinterval V_p of U_p, such that

$$V_p \neq \emptyset,\ V_p \subseteq U_p \subseteq Y = [0,-1+\sqrt{2}],\ \textit{and}\ V_p \cap \mathbf{\Pi_2} = \emptyset. \tag{5.56}$$

Proof. Use mathematical induction on p.

For $p = 0$, (5.55) becomes

$$U_0 \neq \emptyset,\ U_0 \subset U,\ \text{and}\ U_0 \cap \mathbf{\Pi_2^{(0)}} = U_0 \cap \mathbf{\Pi_2} = \emptyset.$$

With $V_0 = U_0$, The Theorem holds.

For $p = 1$, suppose U_1 is an open subinterval of $Y = [0,-1+\sqrt{2}]$, such that

$$U_1 \neq \emptyset, U_1 \subseteq Y = [0,-1+\sqrt{2}],\ \text{and}\ U_1 \cap \mathbf{\Pi_2^{(1)}} = \emptyset, \tag{5.57}$$

where $\mathbf{\Pi_2^{(1)}}$ is the set of all limit points λ_2^* of subdominant eigenvalues of simple graphs.

Let W_1 be a closed interval such that

$$W = [a_1, b_1],\ W_1 \subseteq U_1,$$

where $a_1 < b_1$, a_1 and b_1 are real numbers. (5.58)

If $W_1 \cap \mathbf{\Pi_2}$ is an infinite set, then because W_1 is bounded, so is $W_1 \cap \mathbf{\Pi_2}$. By the Bolzano-Weierstrass Theorem, there is a limit point λ_2^* of $W_1 \cap \mathbf{\Pi_2} \subseteq \mathbf{\Pi_2}$. Then λ_2^* is a limit point of $\mathbf{\Pi_2}$,

$$\lambda_2^* \in \mathbf{\Pi_2}^{(1)} \tag{5.59}$$

and λ_2^* is also a limit point of W_1. The fact that W_1 is a closed interval implied that λ_2^* must belong to W_1. Then from (5.58),

$$\lambda_2^* \in W_1 \subseteq U_1. \tag{5.60}$$

But then (5.59) and (5.60) imply that

$$\lambda_2^* \in U_1 \cap \mathbf{\Pi_2}^{(1)},$$

which contradicts the assumption (5.57). Therefore $W_1 \cap \mathbf{\Pi_2}$ must be a finite set.

Because $W_1 \cap \mathbf{\Pi_2}$ is a finite set and from (5.58), W_1 is a closed interval with positive length. Thus there is an open subinterval V_1 of W_1 such that

$$V_1 \neq \emptyset,\ V_1 \subseteq W_1 \subseteq U_1 \subseteq Y = [0, -1+\sqrt{2}],\ \text{and}\ V_1 \cap \mathbf{\Pi_2} = \emptyset.$$

Suppose for $p \geq 2$, that Theorem 5.6. holds, i.e., if U_p is an open subinterval of $Y = [0, -1+\sqrt{2}]$, such that

$$U_p \neq \emptyset,\ U_p \subseteq Y = [0, -1+\sqrt{2}],\ \text{and}\ U_p \cap \mathbf{\Pi_2}^{(p)} = \emptyset,$$

then there is an open subinterval V_p of U_p, such that

$$V_p \neq \emptyset,\ V_p \subseteq U_p \subseteq Y = [0, -1+\sqrt{2}],\ \text{and}\ V_p \cap \mathbf{\Pi_2} = \emptyset.$$

For $p+1$, suppose U_{p+1} is an open subinterval of $Y = [0, -1+\sqrt{2}]$ such that

$$U_{p+1} \neq \emptyset,\ U_{p+1} \subseteq Y = [0, -1+\sqrt{2}],\ \text{and}\ U_{p+1} \cap \mathbf{\Pi_2}^{(p+1)} = \emptyset, \tag{5.61}$$

where $\mathbf{\Pi_2}^{(p+1)}$ is the set of all limit points $\lambda_2^{*^{p+1}}$ of $\mathbf{\Pi_2^{(p)}}$.

Let W_{p+1} be a closed interval such that

$$W = [a_{p+1}, b_{p+1}],\ \text{and}\ W_{p+1} \subseteq U_{p+1},$$

where $a_{p+1} < b_{p+1}, a_{p+1}$ and b_{p+1} are real numbers. (5.62)

If $W_{p+1} \cap \mathbf{\Pi_2^{(p)}}$ is an infinite set, then because W_{p+1} is bounded, so is the set $W_{p+1} \cap \mathbf{\Pi_2^{(p)}}$. By the Bolzano-Weierstrass Theorem, there is a limit point $\lambda_2^{*^{p+1}}$ of $W_{p+1} \cap \mathbf{\Pi_2^{(p)}} \subseteq \mathbf{\Pi_2^{(p)}}$, Then $\lambda_2^{*^{p+1}}$ is a limit point of $\mathbf{\Pi_2^{(p)}}$,

$$\lambda_2^{*^{p+1}} \in \mathbf{\Pi_2^{(p+1)}}, \tag{5.63}$$

and $\lambda_2^{*^{p+1}}$ is also a limit point of W_{p+1}. The fact that W_{p+1} is a closed interval implies that $\lambda_2^{*^{p+1}}$ must belong to W_{p+1}. Then from (5.62),

$$\lambda_2^{*^{p+1}} \in W_{p+1} \subseteq U_{p+1}. \tag{5.64}$$

But then (5.63) and (5.64) imply that

$$\lambda_2^{*^{p+1}} \in U_{p+1} \cap \mathbf{\Pi}_2^{(\mathbf{p+1})},$$

which contradicts the assumption (5.61). Therefore $W_{p+1} \cap \mathbf{\Pi}_2^{(\mathbf{m})}$ must be a finite set.

Because $W_{p+1} \cap \mathbf{\Pi}_2^{(\mathbf{p})}$ is a finite set and from (5.62), W_{p+1} is a closed interval with positive length. So there is an open subinterval U_p of W_{p+1},

$$U_p \subseteq W_{p+1} \subseteq U_{p+1} \subseteq Y = [0, -1+\sqrt{2}],$$

such that

$$U_p \neq \emptyset,\ U_p \subseteq Y = [0, -1+\sqrt{2}],\ \text{and } U_p \cap \mathbf{\Pi}_2^{(\mathbf{p})} = \emptyset.$$

Then by the induction assumption, there is an open subinterval V_p of U_p, such that

$$V_p \neq \emptyset,\ V_p \subseteq U_p \subseteq Y = [0, -1+\sqrt{2}],\ \text{and } V_p \cap \mathbf{\Pi_2} = \emptyset. \tag{5.65}$$

Take $V_{p+1} = V_p$. Then

$$V_{p+1} = V_p \subseteq U_p \subseteq W_{p+1} \subseteq U_{p+1} \subseteq Y = [0, -1+\sqrt{2}]. \tag{5.66}$$

From (5.65) and (5.66), there is an open subinterval V_{p+1} of U_{p+1}, such that

$$V_{p+1} \neq \emptyset,\ V_{p+1} \subseteq U_{p+1} \subseteq Y = [0, -1+\sqrt{2}],\ \text{and } V_{p+1} \cap \mathbf{\Pi_2} = \emptyset.$$

Theorem 5.7. *The set of all subdominant eigenvalues of simple graphs is nowhere dense in the interval* $[0, -1+\sqrt{2}]$.

Proof. Let

$$Y = [0, -1+\sqrt{2}].$$

The theorem claims that

$$\mathbf{\Pi_2} = \{\lambda_2 \mid \lambda_2 = \lambda_2(G),\ G \text{ is a simple graph } \}$$

is nowhere dense in the interval $Y = [0, -1+\sqrt{2}]$.

For $m = 0$, let

$$(\lambda_2^{*^0})_{\min} = (\lambda_2)_{\min} = 0.311108. \tag{5.67}$$

For $m \geq 1$, from Theorem 5.5, let

$$(\lambda_2^{*^m})_{\min} = \inf[\mathbf{\Pi}_2^{(\mathbf{m})} \cap (0, -1+\sqrt{2})]$$

$$= \begin{cases} \frac{1}{3} = \lambda_2(K_{\infty,1,1}; K_1), & \text{if } m = 1; \\ \frac{-2(m+2)+3+\sqrt{8(m+2)^2-32(m+2)+33}}{2[(m+2)-3]} = \lambda_2(K_{\underbrace{\infty,\infty,\ldots,\infty}_{m \text{ times}},1,1}; K_1), & \text{if } m \geq 2. \end{cases}$$

(5.68)

Then $(\lambda_2^{*^m})_{\min}$ is monotone increasing in m and

$$\lim_{m\to\infty}(\lambda_2^{*^m})_{\min} = -1+\sqrt{2}.$$

Let

$$\begin{aligned}
I_0 &= (0,\ (\lambda_2)_{\min}) = (0,\ \lambda_2(K_{1,1,1};K_1)) = (0,\ 0.311108),\\
I_1 &= ((\lambda_2)_{\min},\ (\lambda_2^*)_{\min}) = (0.311108,\ 1/3),\\
I_2 &= \left((\lambda_2^*)_{\min},\ (\lambda_2^{*^2})_{\min}\right) = \left(\frac{1}{3},\ \frac{-5+\sqrt{33}}{2}\right),\\
&\vdots\\
I_m &= \left((\lambda_2^{*^{m-1}})_{\min},\ (\lambda_2^{*^m})_{\min}\right)\\
&= \left(\frac{-2(m+1)+3+\sqrt{8(m+1)^2-32(m+1)+33}}{2[(m+1)-3]},\right.\\
&\qquad\left.\frac{-2(m+2)+3+\sqrt{8(m+2)^2-32(m+2)+33}}{2[(m+2)-3]}\right),\\
&\vdots
\end{aligned} \tag{5.69}$$

Then

$$\begin{aligned}
Y &= [0,-1+\sqrt{2}] = A\cup B\\
&= \left(\bigcup_{m=0}^{\infty} I_m\right)\bigcup\left[\left(\bigcup_{m=0}^{\infty}(\lambda_2^{*^m})_{\min}\right)\bigcup\left\{0,-1+\sqrt{2}\right\}\right],
\end{aligned} \tag{5.70}$$

where

$$A = \left(\bigcup_{m=0}^{\infty} I_m\right) \tag{5.71}$$

is a union of countably many disjoint bounded open intervals, and

$$B = \left(\bigcup_{m=0}^{\infty}(\lambda_2^{*^m})_{\min}\right)\bigcup\left\{0,-1+\sqrt{2}\right\} \tag{5.72}$$

is a union of countable infinite many isolated points,

$$\text{and } A\cap B=\emptyset. \tag{5.73}$$

From (5.67), (5.68), and (5.69), for $m \geq 0$,

$$\mathbf{\Pi}_{\mathbf{2}}^{(\mathrm{m})} \cap I_m = \emptyset. \tag{5.74}$$

Let $U \neq \emptyset$ be any open subset of $Y = [0,\ -1+\sqrt{2}]$. Then from Theorem 2.7, U is a union of countably many disjoint open subintervals of Y. From (5.76),

$$U = U \cap Y = U \cap (A \cup B) = (U \cap A) \cup (U \cap B). \tag{5.75}$$

Because B is a union of infinitely many isolated points, (5.69)–(5.75) imply that

$$\bigcup_{m=0}^{\infty} (U \cap I_m) = U \cap \left(\bigcup_{m=0}^{\infty} I_m \right) = U \cap A \neq \emptyset. \qquad (5.76).$$

From (5.76), there is a fixed integer $p \geq 0$, such that

$$U \cap I_p \neq \emptyset. \tag{5.77}$$

Because I_p is an open interval and U is a union of finitely or countably many disjoint open intervals in Y, from (5.77), there is an open interval U_p, such that

$$U_p \neq \emptyset, \text{ and } U_p \subseteq \big(U \cap I_p\big). \tag{5.78}$$

Then (5.75), (5.77) and (5.78) imply that

$$U_P \neq \emptyset, U_p \subset U \subseteq Y = [0, -1+\sqrt{2}], \text{and } U_p \cap \mathbf{\Pi}_{\mathbf{2}}^{(\mathrm{p})} = \emptyset. \tag{5.79}$$

From (5.170) and Theorem 5.6, there is an open subinterval V_p of U_p, such that

$$V_P \neq \emptyset,\ V_P \subseteq U_p \subseteq Y = [0, -1+\sqrt{2}],\ \text{and } V_p \cap \mathbf{\Pi}_{\mathbf{2}} = \emptyset.$$

Then because U is any open subset of $[0, -1+\sqrt{2}]$, the open subinterval V_p satisfies

$$V_p \subseteq U_p \subseteq U \subset [0, -1+\sqrt{2}], \tag{5.80}$$

hence (5.79) and (5.80) imply that $\mathbf{\Pi}_{\mathbf{2}}$, the set of subdominant eigenvalues of simple graphs is nowhere dense in the interval $[0, -1+\sqrt{2}]$.

BIBLIOGRAPHY

[1] A. E. Brouwer and A. Neumaier, The graphs with spectral radius $\sqrt{2+\sqrt{5}}$, Linear Algebra and Appl. 114/115 (1989), 273–276.

[2] J.A. Bondy and U. S. R. Murty, Graph Theory with Applications. North-Holland, New York, Amsterdam , Oxford, 1976.

[3] Dasong Cao and Hong Yuan, Graphs Characterized by the Second Eigenvalue. Journal of Graph Theory, vol. 17, No.3, 325–331 (1993).

[4] D. Cvetković, M. Doob and H. Sachs, Spectra of Graphs -Theory and application, second edition, Deutscher Verlag det Wissenschaften, Berlin, Academic Press, New York, 1982.

[5] D. Cvetković, M. Doob, and I. Gutman, On graphs whose eigenvalues do not exceed $\sqrt{2+\sqrt{5}}$. Ars Combinatoria 14(1982) 225–239.

[6] D. Cvetković, On graphs whose second largest eigenvalue does not exceed 1, Publications de L'Institut Mathématique, Nouvelle série. Tome 31 (45) 1982, 15-20.

[7] D. Cvetković, M. Doob, I. Gutman, Torgašev A, Recent Results in The Theory of Graph Spectra, North Holland, Amsterdam, 1988.

[8] D. Cvetković and P Rowlinson. The largest eigenvalue of a graph, a survey. Linear and Multilinear Algebra, 28(1990), 3–33.

[9] D. Cvetković, S. Simić, On graphs whose second largest eigenvalue does not exceed $(\sqrt{5}-1)/2$, to appear.

[10] M. Doob. The limit points of eigenvalues of graphs. Linear Algebra and Appl. 114/115 (1989) 659–662.

[11] Murray Eisenberg, Topology, Holt, Rinehart and Winston, Inc. 1973,

[12] George F. Simmons, Introduction to Topology and Modern Analysis. McGraw-Hill Book Company, Inc. 1963.

[13] A. J. Hoffman, On limit points of spectral radii of non-negative symmetric integral matrices, Y.Alavi et al.(eds). Lecture Notes Math. 303, Spring-Verlag, Berlin-Heidelberg-New York, 1972, 165–172.

[14] A. J. Hoffman, On limit point of the least eigenvalue of a graph, Ars Combinatoria, Vol. 3 (1977), pp.3–14.

[15] Howes L., On subdominantly bounded graphs, Ph.D. Thesis, City Univ. of New York, 1970.

[16] Howes L., On subdominantly bounded graphs-summary of results, Recent trends in Graph Theory, Proc. of the First New York City Graph Theory Conf. Held on Jun. 11–13, 1970, Capobianco M., Frechen J.B., Krolik M.,

Springer-Verlag, Berlin, 1971, 181–183.

[17] Miroslav Petrović, On graphs with exactly one eigenvalue less than -1, Journal of Combinatorial, Series B 52, 102–112 (1991).

[18] J. B. Shearer. On the distribution of the maximum eigenvalue of graphs, Linear Algebra and Appl. 114/115 (1989) 17–20.

[19] Slobodan K. Simić, Some note on graphs whose second largest eigenvalue is less Than $(\sqrt{5}-1)/2$, to appear.