TIME OPTIMAL CONTROL OF LINEAR SYSTEMS WITH NUMERATOR DYNAMICS AND DELAY

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ABSTRACT

This thesis is concerned with linear time-invariant systems with numerator dynamics in their forward transfer functions and a time delay in the feedback path.

Theorems on the Controllability and Stability of such systems are presented. The major part of the thesis investigates the problem of finding a control which will bring the output of the system from some initial state to zero in minimum time and keep it there afterwards. Necessary conditions for such a time optimal control to exist are derived. Based on these conditions, the optimal control can be found by an iterative procedure. The pattern search of Hooke and Jeeves is introduced. Two examples are given to demonstrate the techniques of finding the time optimal control.

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CHAPTER I

INTRODUCTION

I. GENERAL INTRODUCTION

In the mathematical formulation of a physical process, it is often assumed that the future behavior of the system depends only upon the present state, and that the influence of the present state is instantaneous. The latter assumption leads to a system of ordinary differential equations, which satisfactorily describes a great variety of physical processes. For many plants, however, this description is inadequate; and the mathematical models cannot be readily derived unless relations involving time delays are admitted. The time delay may arise as a result of the distributed nature of the process, inherent transport lag, or as a consequence of mathematical approximations[1]. These processes with hereditary effects can be described by a system of differential-difference equations[2].

In recent years, considerable attention has been

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[1] A.T. Fuller,"Optimal Nonlinear Control of Systems with Pure Delay", International Journal of Control, Vol.8, No.2, 1968, pp.145-168.

[2] R. Bellamn and K. Cooke, Differential-Difference Equations, (New York: Academic Press, 1963) 462pp.

devoted to the optimal control of these systems with time delay. Due to the difficulties inherent in the analytic treatment of the differential-difference equations, most of these investigations are on systems with only poles in their transfer function, or on the optimal control of systems whose target set consists of a single point, which is usually the origin of the state space. Very little attention has been directed towards the problem of obtaining the optimal control for systems with zeros in their transfer functions (i.e. systems with numerator dynamics), or of controlling the state of a system to a target set which is more than one single point in the state space. Many physical processes, however, can only be approximated mathematically by system models with numerator dynamics. It is important, therefore, to establish techniques for the optimal control of such systems.

II. THE PROBLEM

This thesis is concerned with the problem of obtaining the time optimal controls for systems with zeros in their forward transfer function and a pure time delay in the feedback path. This type of problem arises quite frequently in feedback control processes, where there is often an inevitable time lag in the transmission of signals between the sensing device and the controller. Figure (I-1) depicts the class of systems that will be considered.

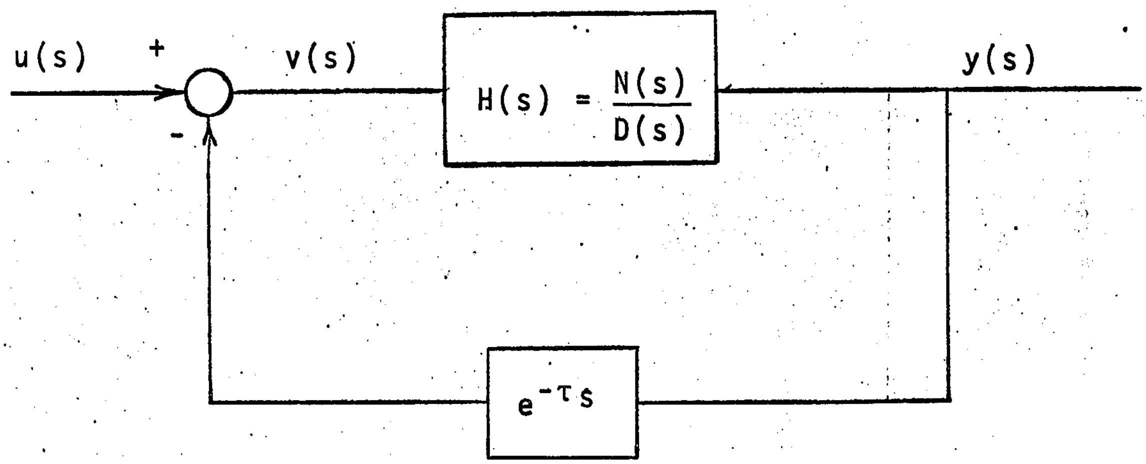

Figure (I-1). Block Diagram of a system with numerator dynamics and time delay in the feedback.

The transfer function, H(s), is of the form

$$H(s) = \frac{N(s)}{D(s)} \qquad \text{I-1}$$

where N(s) and D(s) are polynomials in s with real coefficients; specifically,

$$N(s) = b_k s^k + b_{k-1} s^{k-1} + \ldots + b_1 s + b_0 \qquad \text{I-2}$$

$$\text{and}\quad D(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o \qquad \text{I-3}$$

where a_i and b_j are real constants for all $i = 0,1,2,\ldots,n-1$ and $j = 0,1,2,\ldots,k$, and k is less than n.

Suppose that

$$v(t) = u(t) - y(t-\tau) \qquad \text{I-4}$$

where $u(t)$ is the control force which is assumed bounded, i.e. $|u(t)| \leq 1$; $y(t)$ is the output, and τ, the time delay. Then the transfer function $H(s)$ represents a single-input-single-output plant which can be described by the n-th order linear differential equation:

$$\begin{aligned}[D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_o]y(t) \\ = [b_k D^k + b_{k-1}D^{k-1} + \ldots + b_1 D + b_o]v(t)\end{aligned} \qquad \text{I-5}$$

where $D = d/dt$, $k<n$.

The problem is to find the optimal control strategy which will drive the output $y(t)$ of the system from some initial state

$$y_{t_o}(\sigma) = y(t_o+\sigma), \quad \sigma \in [-\tau, 0] \qquad \text{I-6}$$

to the final state,

$$y_{t^*+\tau}(\sigma) = 0, \qquad \sigma \in [-\tau, 0] \qquad \text{I-7}$$

in minimum time t^*.

Since systems with numerator dynamics involve derivatives of the control variable, it is often more desirable to convert the system state from one co-ordinate system into another in which the derivatives of the control disappear. In so doing, the target set is usually transformed from a one-point set to a smooth k-fold in the new co-ordinate system. Hence the problem of controlling the output state of the plant with numerator dynamics to the zero state is equivalent to the problem of controlling a system from an initial state to a function in the state space.

III. A BRIEF SUMMARY AND PREVIOUS RELATED WORK

The second chapter of this thesis deals with the transformation of the state variables of the plant from one co-ordinate system into another which does not involve derivatives of the control force, u(t). Various transformation techniques for the ordinary systems without time delay have been developed, and treated quite exhaustively in literature; see, for example, the papers by Lee[3], Athanassiades and

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[3] E.B. Lee, "On the Time-optimal Regulation of Plants with Numerator Dynamics", IRE Transaction on Automatic Control, Vol. 6, No. 3, 1961, pp.351-352.

Falb[4], Laning and Battin[5], and Hutchinson[6]. Most of these techniques are also applicable to systems with time delay.

Westdal[7] has taken some of the results from previous studies, for example, from Weiss[8] and Buckalo[9], to establish controllability and stability criteria for the general linear time-delay systems. Chapter III restates, extends and proves some of these theorems presented by Westdal in order to obtain more complete and easily applicable conditions on

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[4] M. Athanassiades and P. Falb, "Time Optimal Control for Plants with Numerator Dynamics", IRE Transaction on Automatic Control, Vol. 7, No. 4, 1962, pp.46-47.

[5] J.H. Laning and R.H. Battin, Random Processes in Automatic Control, (New York: McGraw-Hill, 1963) pp.191-197.

[6] C.E. Hutchinson, "Minimum-Time Control of a Linear Combination of State Variables", Technical Report No. 6311 (Stanford, California: System Theory Laboratory, Stanford University, August, 1963)

[7] John A.S. Westdal, "Time Optimal Control of Linear Systems with Delay" (Unpublished Master's thesis, The University of Manitoba, 1969).

[8] L. Weiss, "On the Controllability of Delay-Differential Sysyems", SIAM Journal on Control, Vol. 5, No. 4, 1967, pp.575-587.

[9] A.F. Buckalo, "Explicit Conditions for Controllability of Linear Systems with Time Lag", IEEE Transactions on Automatic Control, Vol. 13, No. 2, 1968, pp. 193-195.

cancellation, controllability and stability of the time delay system with numerator dynamics. The idea of cancellation and stability for ordinary systems without delay has been investigated by Butman and Sivan[10], as well as by Ogata[11]. It is found that most of these results can be applied to systems with numerator dynamics with or without time delay.

The optimal control problem for time delay systems has been investigated extensively in literature. In many of the studies of the optimal control problem, the delay system was considered to have a distinctive difference from the ordinary differential equation system. However, Repin[12] and Ichikawa[13] showed that the dynamics of a time delay could be approximated by a system of ordinary differential equations, if their dimensionality is allowed to become infinite. Chapter IV

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[10] S. Butman and R. Sivan, "On Cancellations, Controllability and Observability", IEEE Transactions on Automatic Control, Vol.9, July 1964, pp. 317-318.

[11] K. Ogata, State Space Analysis of Control Systems, (Englewood Cliffs: Prentice-Hall Inc., 1967), pp. 388-392.

[12] Iu.M. Repin, "On the Approximate Replacement of Systems with Lag by Ordinary Dynamical Systems", Journal of Applied Mathematics and Mechanics, Vol. 29, No. 2, 1965, pp. 254-264.

[13] K. Ichikawa, "Pontryagin's Maximum Principle in Optimizing Time-Delay Systems", Electrical Engineering in Japan, Vol. 87, No. 12, Dec. 1967, pp. 75-83.

follows Ichikawa's approach to obtain the optimal conditions for the linear time-invariant system with delay. The time optimal problem has been investigated before, see, for example, the papers by Oguztoreli[14], Kharatishvili[15], Ragg and Stapleton[16], Balakirev[17], and Stebbing[18]. These studies were only concerned with the problem of reducing the error to zero in minimum time, without consideration of the problem of keeping it there afterwards. In other words, these studies were on the attainability problem rather than on the settling

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14 M.N. Oguztoreli, "A Time Optimal Control Problem for Systems Described by Differential Difference Equations", SIAM Journal on Control, A-I-3, 1963, pp. 290-310.

15 G.L. Kharatishvili, "The Maximum Principle in the Theory of Optimal Processes with a Delay", Soviet Mathematics - Doklady, Vol. 2, 1961, pp. 28-32.

16 B.C. Ragg and C.A. Stapleton, "Time Optimal Control of Second Order Systems with Transport Lag", International Journal of Control, Vol. 9, No. 3, 1969, pp. 243-257.

17 V.S. Balakirev, "The Principle of the Maximum in the Theory of Second Order Optimal Systems", Automation and Remote Control, Vol. 93, No. 8, Aug. 1962, pp. 948-956.

18 J.D. Stebbing, "An Investigation into the Time Optimal Control of Linear Systems with Delay" (unpublished Master's thesis, The University of Manitoba, 1967).

problem. Westdal[19] considered the settling problem; however, he only dealt with a one-point target set. In systems with numerator dynamics, the target set is usually a function in the state space. The time optimal regulation problem for these systems is equivalent to that of controlling the state of a delay system to a smooth k-fold and keeping it there afterwards. Very little attention has been devoted to this problem. Chung and Lee[20] have considered the optimal control problem of steering the system state to a target set G; however, they used quite a general cost index and specified the final time. Chapter IV, therefore, also investigates the transversality conditions for this type of settling problem, in particular, on plants with n poles and one zero. Chapter V concentrates on the second and third order systems with only one zero; two examples are given here.

Progress in the field of optimal control for systems with delay has been hampered by the difficulties involved in the analysis of differential-difference equations. It is important, therefore, to develop efficient algorithms to handle these equations. In synthesizing the optimal control

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[19] Westdal, loc. cit.

[20] D.H. Chung and E.B. Lee, "Linear Optimal Systems with Time Delays", SIAM Journal on Control, Vol. 4, 1966, pp.548-575.

for time delay systems, especially in the case of systems with numerator dynamics and where usually more than two system parameters are involved, it is almost impossible to obtain a solution unless some sort of multidimensional direct search strategy is used. Kramer[21] has developed a method for solving the problem of optimal control based on the theory of Dynamic Programming, and MacKinnon[22] introduced an algorithm for the numerical calculation of an optimal control by means of a variational calculus approach. However, both methods depend on known terminal time; therefore, they cannot be used for the time optimal control problem. McAulay[23] has recently published a method for finding the control function u(t) which minimizes a function of the final values of the state variables $P[\underline{x}(t_1)]$ by means of a gradient method. Again McAulay's method requires that the terminal time be known. Appendix A introduces a multidimensional direct search strategy,

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[21] J.D.R. Kramer Jr., "On Control of Linear Systems with Time Lag", Information and Control, Vol. 3, 1960, pp. 299-326.

[22] D. MacKinnon, "Optimal Control of Systems with Pure Time Delays using a Variational Programming Approach", IEEE Trnasactions on Automatic Control, Vol. 12, No. 3, June 1967, pp. 255-262.

[23] R.J. McAulay, "A Gradient Method for Systems with Time Delays and its Applications to Waveform Design", IEEE Transactions on Automatic Control, Vol. 14, No. 3, June 1969, pp. 230-237.

the pattern search of Hooke and Jeeves[24]. Differential-difference equations can be solved quite easily on the IBM System 360 digital computer by means of the Continuous System Modelling Program (CSMP s/360). Coupled with CSMP s/360, by adjustment of the system parameters methodically in accordance with the pattern search strategy, the time optimal control for the delay systems can be obtained quite easily. Some of the computer programmes used in this thesis are given in Appendices B and C.

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[24] R. Hooke and T.A. Jeeves, "'Direct Search' Solution of Numerical and Statistical Problems", Journal of the Association for Computing Machinery, Vol. 8, pp. 212-229, April 1961.

CHAPTER II

TRANSFORMATION

Consider the problem given in Section (I.2). In many practical schemes, the time optimal control is usually a relay-type control. Consequently, plants with numerator dynamics may cause discontinuities in some of the variables used to describe the movement of the system; therefore, it is often more desirable to consider the problem in a co-ordinate system which does not involve derivatives of the control variable. Under such a transformation, the target is usually transformed from a one-point set to a smooth k-fold. Chapter II is devoted to the problem of finding such a state representation of the system, and on the resulting target set under such a transformation. The transformation technique used here is completely analogous to that for ordinary systems without time delay[1].

I. TRANSFORMATION

Consider the plant (I-5). The numerator polynomial N(s) in the transfer function is assumed to be of order k = (n-1);

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[1] M. Athans and P.L. Falb, Optimal Control,(New York: McGraw-Hill Book Company, 1966) pp. 182-187.

i.e.

$$H(s) = \frac{y(s)}{v(s)} = \frac{N(s)}{D(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_o}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o} \qquad \text{II-1}$$

In order to eliminate the derivatives in the control force, a state vector $\underline{x}(t)$ is constructed with components $x_1(t)$, $x_2(t)$, ..., $x_n(t)$[2] such that

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \dot{y}(t) - h_1 v(t) \\ x_3(t) &= y^{(2)}(t) - h_1 v(t) - h_2 v(t) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n(t) &= y^{(n-1)}(t) - h_1 v^{(n-2)}(t) - \ldots - h_{n-1} v(t) \end{aligned} \qquad \text{II-2}$$

Or, written in a concise form, for i = 2,3, ... , n

$$\begin{aligned} x_1(t) &= y(t) \\ x_i(t) &= y^{(i-1)}(t) - \sum_{m=1}^{i-1} v^{(m-1)}(t) h_{i-m} \end{aligned} \qquad \text{II-3}$$

Following the derivation technique of Athans and Falb[3],

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[2] Note that this transformation technique can only be applied to the time delay systems with $k \leq (n-1)$. If k=n, then $x_1(t) = y(t) - h_0 u(t) + h_0 y(t-\tau)$. It would be impossible to obtain an explicit relation of y(t) in terms of $\underline{x}(t)$ and u(t).

[3] Athans and Falb, op. cit.

if h_1, h_{n-m} are chosen such that

$$h_1 = b_{n-1} \qquad\qquad \text{II-5}$$

$$h_{n-m} = b_m - \sum_{i=1}^{n-m-1} h_i a_{i+m}$$

for $m = 1,2, \ldots ,(n-2)$

$$h_n = b_o - \sum_{i=1}^{n-1} a_i h_i \qquad\qquad \text{II-6}$$

then it can be proved that

$$\dot{x}_i(t) = x_{i+1}(t) + h_i v(t),$$

for $i = 1,2, \ldots ,(n-1)$ II-7

and $$\dot{x}_n(t) = - \sum_{i=0}^{n-1} a_i x_{i+1}(t) + h_n v(t)$$

Derivatives in $v(t)$ have thereby been eliminated.

Recall that

$$v(t) = u(t) - y(t-\tau) \qquad\qquad \text{II-8}$$

so that the differential-difference equation satisfied by $\underline{x}(t)$ can be written in the form

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdots \\ \dot{x}_{n-1}(t) \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_{n-1}(t) \\ x_n(t) \end{pmatrix}$$

(eq'n. cont'd.)

$$+ \begin{pmatrix} -h_1 & 0 & 0 & \dots & 0 \\ -h_2 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -h_{n-1} & 0 & 0 & \dots & 0 \\ -h_n & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \\ \cdots \\ x_{n-1}(t-\tau) \\ x_n(t-\tau) \end{pmatrix}$$

$$+ \begin{pmatrix} h_1 \\ h_2 \\ \cdots \\ h_{n-1} \\ h_n \end{pmatrix} u(t) \qquad \text{II-9}$$

or, more succinctly, as

$$\underline{\dot{x}}(t) = A_0 \underline{x}(t) + A_1 \underline{x}(t-\tau) + \underline{h}u(t) \qquad \text{II-10}$$

Suppose that the transfer function has only k-zeros and n-poles, with $k<(n-1)$. Substituting in

$$b_{n-1} = b_{n-2} = \dots = b_{k+1} = 0,$$

and $b_k \neq 0$, it can be seen that Eqs. (II-9) and (II-10) still hold, except that

$$h_1 = h_2 = \dots = h_{n-k-1} = 0 \qquad \text{II-11}$$

and $$h_{n-k} = b_k, \qquad \text{II-12}$$

$$h_{n-m} = b_m - \sum_{i=n-k}^{n-m-1} h_i a_{i+m} \qquad \text{II-13}$$

$$m = 1, 2, \ldots, (k-1)$$

$$h_n = b_0 - \sum_{i=n-k}^{n-1} a_i h_i \qquad \text{II-14}$$

Under this transformation of co-ordinate systems, the target set, of course, will also be changed. The following section is concerned with the transformed target set. The case of the dynamical system with k-zeros ($k < (n-1)$) will be discussed. The new target set, under this transformation, is essentially the same as that for the ordinary system without time delay. For the case $k = (n-1)$, the derivation is more straight-forward and is completely analogous. The derivation here follows the method by Athans and Falb[4].

II. TRANSFORMED TARGET SET

Suppose that there exists an admissible time optimal control $u(t)$ (i.e. $|u(t)| \leq 1$) which steers the output $y(t)$ from some initial state $y_{t_0}(\sigma)$, $\sigma \varepsilon [-\tau, 0]$, to zero in minimum time t^*, and keeps it there for all $t \geq t^*$. Then the output vector

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4 Athans and Falb, _op_. _cit_., pp. 647-660.

$$\underline{y}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \cdots\cdot \\ y^{(n-2)}(t) \\ y^{(n-1)}(t) \end{bmatrix} = 0 \qquad \text{II-15}$$

for all $t \geqq t^*$.

Setting $h_1 = h_2 = \ldots = h_{n-k-1} = 0$, the dynamical system with n-poles and k-zeros can be represented by the following system:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = x_3(t)$$

$$\cdots\cdots\cdots$$

$$\dot{x}_{n-k-1}(t) = x_{n-k}(t)$$

$$\dot{x}_{n-k}(t) = x_{n-k+1}(t) - h_{n-k}x_1(t-\tau) + h_{n-k}u(t) \qquad \text{II-16}$$

$$\dot{x}_{n-k+1}(t) = x_{n-k+2}(t) - h_{n-k+1}x_1(t-\tau) + h_{n-k+1}u(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\dot{x}_n(t) = -\sum_{i=0}^{n-1} a_i x_{i+1}(t) - h_n x_1(t-\tau) + h_n u(t)$$

where

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{y}(t)$$

$$\cdots\cdots\cdots$$

$$x_{n-k}(t) = y^{(n-k-1)}(t)$$

$$x_{n-k+1}(t) = y^{(n-k)}(t) + h_{n-k}\, y(t-\tau) - h_{n-k}\, u(t) \qquad \text{II-17}$$

$$x_{n-k+2}(t) = y^{(n-k+1)}(t) + h_{n-k}\, \dot{y}(t-\tau) - h_{n-k}\, \dot{u}(t) + h_{n-k+1}\, y(t-\tau) - h_{n-k+1}\, u(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_n(t) = y^{(n-1)}(t) + h_{n-k}\, y^{(k-1)}(t-\tau) + \ldots + h_{n-1}\, y(t-\tau) - h_{n-k}\, u^{(k-1)}(t) - \ldots - h_{n-1}\, u(t)$$

Setting $y(t) = \dot{y}(t) = \ldots = y^{(n-1)}(t) = 0$, for $t \geq t^*$, implies that

$$x_1(t) = x_2(t) = \ldots = x_{n-k}(t) = 0 \text{ for all } t \geq t^*. \qquad \text{II-18}$$

and

$$x_{n-k+1}(t) = h_{n-k}[x_1(t-\tau) - u(t)] \text{ for all } t \geq t^*$$

II-19

Let $\underline{\eta}(t)$ represent the k-dimensional vector whose components are the last k components of the vector $\underline{x}(t)$; i.e.

$$\underline{\eta}(t) = \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \cdots \\ \eta_k(t) \end{bmatrix} = \begin{bmatrix} x_{n-k+1}(t) \\ x_{n-k+2}(t) \\ \cdots\cdots\cdots \\ x_n(t) \end{bmatrix} \qquad \text{II-20}$$

Substituting

$$x_1(t-\tau) = \frac{x_{n-k+1}(t)}{h_{n-k}} + u(t) \quad \text{for all } t \geq t^*$$

$$= \frac{\eta_1(t)}{h_{n-k}} + u(t) \qquad \text{II-21}$$

into the last k-equations of Eq. (II-20), it can be shown that

$$\dot{\underline{\eta}}(t) = Q\,\underline{\eta}(t) \qquad \text{for all } t \geq t^* \qquad \text{II-22}$$

where

$$Q = \begin{pmatrix} -\frac{h_{n-k+1}}{h_{n-k}} & 1 & 0 & \cdots & 0 \\ -\frac{h_{n-k+2}}{h_{n-k}} & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{h_n}{h_{n-k}} - a_{n-k} & (-a_{n-k+1}) & (-a_{n-k+2}) & \cdots & (-a_{n-1}) \end{pmatrix} \qquad \text{II-23}$$

is a k x k matrix.

The solution to Eq. (II-22) is

$$\underline{\eta}(t) = e^{Q(t-t^*)}\,\underline{\eta}(t^*), \quad \text{for all } t \geq t^* \qquad \text{II-24}$$

where $e^{Q(t-t^*)}$ is the fundamental matrix of Eq. (II-22).

Let

$$\underline{q}^T(t) = [q_1(t) \quad q_2(t) \quad \cdots \quad q_k(t)] \qquad \text{II-25}$$

represent the first row vector of the fundamental matrix $e^{Q(t-t^*)}$.

Then, for $t \geq t^*$,

$$\begin{aligned} \eta_1(t) &= \left\langle \underline{q}(t), \underline{\eta}(t^*) \right\rangle \\ &= \sum_{i=1}^{k} q_i(t)\, \eta_i(t^*) \end{aligned} \qquad \text{II-26}$$

or, $$x_{n-k+1}(t) = \sum_{i=1}^{k} q_i(t)\, x_{n-k+i}(t^*) \qquad \text{II-27}$$

for all $t \geq t^*$.

If $u(t)$ steers the output $y(t)$ to zero and keeps it there afterwards, then

$$x_1(t-\tau) = 0, \text{ for all } t \geq t^* + \tau \qquad \text{II-28}$$

From Eq. (II-19) and the constraint $|u(t)| \leq 1$, it can be deduced that

$$x_{n-k+1}(t) = -h_{n-k}\, u(t) \quad \text{for all } t \geq t^* + \tau \qquad \text{II-29}$$

and hence that

$$\left| -\frac{1}{h_{n-k}}\, x_{n-k+1}(t) \right| \leq 1 \quad \text{for all } t \geq t^* + \tau \qquad \text{II-30}$$

Eq. (II-30), in turn, implies that

$$\left| -\frac{1}{h_{n-k}} \left\langle \underline{q}(t), \underline{n}(t^*) \right\rangle \right| \leq 1 \quad \text{for all } t \geq t^* + \tau \qquad \text{II-31}$$

From Eq. (II-12), it is known that

$$h_{n-k} = b_k \qquad \text{II-32}$$

Therefore, Eq. (II-31) may be written as

$$\left| \frac{1}{b_k} \left\langle \underline{q}(t), \underline{n}(t^*) \right\rangle \right| \leqq 1 \quad \text{for all } t \geqq t^* + \tau \tag{II-33}$$

or

$$\left| \frac{1}{b_k} \sum_{i=1}^{k} q_i(t+\tau)\, x_{n-k+i}(t^*) \right| \leqq 1 \tag{II-34}$$

$$\text{for all} \quad t \geqq t^*.$$

Let G denote the set of states of $\underline{x}$ given by

$$G = \{\, \underline{x}\colon x_1 = x_2 = \ldots = x_{n-k} = 0;$$

$$\left| \frac{1}{b_k} \sum_{i=1}^{k} q_i(t+\tau)\, x_{n-k+i}(t^*) \right| \leqq 1; \text{ for all} \quad t \geqq t^* \,\} \tag{II-35}$$

The time optimal problem of steering the output $y(t)$ to zero in minimum time t^*, and keeping it there afterwards, therefore, requires that the state $\underline{x}_{t+\tau}(\sigma)$, $\sigma \in [-\tau, 0]$, belong to the set G for all $t \geqq t^*$; hence G is called the target set.

When the plant has n-poles and k-zeros, but no time delay, it is known that[5] the target set

$$G' = \{\underline{x}\colon x_1 = x_2 = \ldots = x_{n-k} = 0; \left| \frac{1}{b_k} \left\langle \underline{q}(t), \underline{n}(t^*) \right\rangle \right| \leqq 1;$$

$$\text{for all} \quad t \geqq t^*\} \tag{II-36}$$

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[5] Ibid.

Except for the term $\underline{q}(t)$ and $\underline{q}(t+\tau)$, the two sets G and G' are identical. Therefore, rather than reproducing proofs which are completely analogous to those for the systems without delay, three theorems are now stated without proof. The interested reader is asked to refer to Athans and Falb[6] for details of the verifications.

Theorem II-1

The eigenvalues of the matrix Q are the zeros of the system (II-1).

Theorem II-2

The target set G, defined by Eq. (II-35), is closed and convex, and the orgin $\underline{x} = \underline{0}$ is an element of G.

Theorem II-3

If any of the zeros of the system (II-1) lie in the right-hand complex plane, then the target set G consists of a single point, the origin of the state space; i.e.

$$G = \{\underline{x}: \underline{x} = \underline{0}\} \qquad \text{II-37}$$

From Theorem II-1, the vector $\underline{q}(t+\tau)$, and hence the

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[6] Ibid.

set G for the system (II-1) can be constructed if the zeros are known.

From Theorem II-2, it can be established that if the target set G is closed and convex, then the optimal control must satisfy Pontryagin's Maximum Principle[7]. Hence for the linear problem, the time optimal control must be bang-bang in nature, until the trajectory intersects the target set G. After intersection, the control u(t) must be changed so as to keep the state $\underline{x}_{t+\tau}(\sigma)$, $\sigma \in [-\tau, 0]$, in G for all $t \geqq t^*$. For $\underline{x}_t(\sigma)$ in G, u(t) can be found from Eq. (II-19).

From Theorem II-3, it is seen that if the system (II-1) contains zeros in the right hand plane, the optimal control problem is essentially the same as that for systems without numerator dynamics. This topic has been investigated by Westdal[8]; therefore, the case of right hand plane zeros will not be considered in this thesis.

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[7] L.I. Rozonoer, "L.S. Pontryagin's Principle of Maximum in the Theory of Optimal Systems", Avtomatika i Telemakhanika, Vol. 20, Nos. 10-12, 1959, pp. 1441-1458; E.B. Lee, loc. cit.

[8] J.A.S. Westdal, "Time Optimal Control of Linear Systems with Delay" (unpublished Master's thesis, the University of Manitoba, 1969).

CHAPTER III

CONTROLLABILITY AND STABILITY

The properties of stability and controllability play an important role in the control theory. Most of the criteria used for testing the controllability of a dynamical system with zeros can also be applied to the system with time delay in the feedback. This chapter develops some properties of controllability and stability.

I. CANCELLATION AND CONTROLLABILITY

Consider the system

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-\tau) + \underline{h}u(t) \qquad \text{III-1}$$

where A_0, A_1 are constant n x n matrices,

$\underline{h}$ = a constant n x 1 vector,

$\underline{x}(t)$ = an n-dimensional vector,

$u(t)$ = the control input,

and τ is a positive constant representing the time delay.

Let E_{t_0} be the initial time segment such that

$$E_{t_0} = \{t: t \in [t_0-\tau, t_0]\} \qquad \text{III-2}$$

Let Ψ represent the Banach space of real n-vector-valued continuous functions, defined on E_{t_0}, and Γ, the linear space of functions, defined on the interval $[t^*, t^*+\tau]$. The system (III-1) is said to be controllable to a function $\underline{g}(t) \in \Gamma$ with respect to the space of initial function Ψ if, for any given $\underline{\phi} \in \Psi$, there exists a time t^*, $t_0 < t^* < \infty$, and an admissible control segment $u_{[t_0, t^*+\tau]}$[1], such that $\underline{x}(t;t_0,\underline{\phi},u) = \underline{g}(t)$, $t \in [t^*, t^*+\tau]$, where $\underline{x}(t;t_0,\underline{\phi},u)$ is the solution of (III-1), starting at t_0, with initial function $\underline{\phi}$ and control $u(t)$.

The system is said to be completely state controllable to $\underline{g}(t)$ if, for any initial set E_{t_0}, it is possible to construct an admissible control $u(t) \in U$, which will transfer all given continuous initial functions $\underline{\phi}(t)$ to $\underline{g}(t)$ in the finite time interval $[t_0, t^*+\tau]$.

The following theorem establishes a necessary and sufficient condition for the controllability of a general linear time delay system with a scalar input. Proof of this theorem follows that given by Westdal.[2]

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[1] $u_{[t_0, t^*+\tau]}$ denotes the segment of the function $u(t) \in U$ over the interval $[t_0, t^*+\tau]$. U is the set of admissible controls.

[2] Westdal, op. cit., pp. 5-9.

Theorem III-1

The system

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-\tau) + \underline{h}u(t) \qquad \text{III-3}$$

is completely state controllable if and only if no constant nonzero row vector $\underline{H}$ exists, such that

$$\underline{H}(sI - A_0 - A_1e^{-\tau s})^{-1}\,\underline{h} = 0 \qquad \text{III-4}$$

where I represents the n x n matrix.

Proof. Suppose that there exists a mn x 1 system such that

$$\begin{aligned}
\dot{\underline{x}}_1(t) &= \frac{1}{\tau/m}[\underline{x}(t) - \underline{x}_1(t)]\\
\dot{\underline{x}}_2(t) &= \frac{1}{\tau/m}[\underline{x}_1(t) - \underline{x}_2(t)]\\
&\cdots\cdots\cdots\cdots\cdots\\
\dot{\underline{x}}_m(t) &= \frac{1}{\tau/m}[\underline{x}_{m-1}(t) - \underline{x}_m(t)]
\end{aligned} \qquad \text{III-5}$$

where m = some arbitrarily large integer

$\underline{x}(t)$ = the n-dimensional vector of the system (III-3)

and $\underline{x}_i(t)$ = an n-dimensional vector, i=1,2, ... ,m.

Now

$$e^{-\tau s} = \lim_{m\to\infty}\left\{\frac{1}{1 + \tau s/m}\right\}^m . \qquad \text{III-6}$$

It has been shown by Repin[3] that $\underline{x}_m(t)$ converges to $\underline{x}(t-\tau)$ as $m\to\infty$ for any function $\underline{x}(t)$ which fulfills the requirements of continuity in the interval $[t_o-\tau,\ t^*]$; i.e.

$$\underline{x}_m(t) \approx \underline{x}(t-\tau) \quad \text{as } m \to \infty. \tag{III-7}$$

Hence the system (III-3) can be approximated by the following system of ordinary differential equations of order $(m+1)n$:

$$\begin{aligned} \dot{\underline{x}}(t) &= A_0\underline{x}(t) + A_1\underline{x}_m(t) + \underline{h}u(t) \\ \dot{\underline{x}}_i(t) &= \frac{1}{\tau/m}[\underline{x}_{i-1}(t) - \underline{x}_i(t)] \end{aligned} \tag{III-8}$$

where $i = 1, 2, \ldots, m$, and $\underline{x}_0(t) = \underline{x}(t)$.

Let

$$\underline{X}(t) = \begin{pmatrix} \underline{x}(t) \\ \underline{x}_1(t) \\ \cdots\cdot \\ \underline{x}_m(t) \end{pmatrix}, \quad \text{a } (m+1)n \text{ vector} \tag{III-9}$$

$$\underline{B} = \begin{pmatrix} \underline{h} \\ \underline{0} \\ \cdots \\ \underline{0} \end{pmatrix}, \quad \text{a } (m+1)n \text{ vector} \tag{III-10}$$

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[3] Repin, *op*. *cit*. pp. 254-261.

and

$$A = \begin{bmatrix} A_0 & \underline{0} & \underline{0} & \cdots & \underline{0} & A_1 \\ \frac{m}{\tau} I_n & \frac{m}{\tau} I_n & \underline{0} & \cdots & \underline{0} & \underline{0} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \underline{0} & \underline{0} & \underline{0} & \cdots & \frac{m}{\tau} I_n & \frac{m}{\tau} I_n \end{bmatrix} \qquad \text{III-11}$$

a $(m+1)n \times (m+1)n$ matrix, where I_n is the $n \times n$ identity matrix. Then the system (III-8) can be represented by the system

$$\dot{\underline{X}}(t) = A\,\underline{X}(t) + \underline{B}\,u(t) \qquad \text{III-12}$$

It is well known that the system (III-12) is completely state controllable if and only if the composite $(m+1)n \times (m+1)n$ matrix P', where

$$P' = [\ \underline{B} \ \vdots\ A\underline{B} \ \vdots\ \ldots \ \vdots\ A^{n-1}\underline{B}\ ] \qquad \text{III-13}$$

is of rank n[4]. It has been proven by Brockett and Mesarovic[5]

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[4] K. Ogata, State Space Analysis of Control Systems, (Englewood Cliffs: Prentice-Hall Inc., 1967), pp. 385.

[5] R.W. Brockett and M.D. Mesarovic, "The Reproducibility of Multivariable Systems", Journal of Mathematical Analysis and Applications, Vol. 11, July 1965, pp. 548-563.

that the matrix P' will have rank n if and only if no constant nonzero row vector $\underline{H}$ exists, such that

$$\underline{H}\,(sI - A)^{-1}\,\underline{B} = 0 \qquad \text{III-14}$$

Westdal[6] has shown that in the limit as $m \to \infty$,

$$(sI - A)^{-1}\,\underline{B} = (sI - A_0 - A_1 e^{-\tau s})\,\underline{h} \qquad \text{III-15}$$

Therefore, the system (III-3) will be completely state controllable if no nonzero row vector $\underline{H}$ exists, such that

$$\underline{H}\,(sI - A_0 - A_1 e^{-\tau s})\,\underline{h} = 0$$

Q.E.D.

Suppose that the system (III-1) represents a system with numerator dynamics and the transfer function is

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_o}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o} \qquad \text{III-16}$$

where $D(s)$ = the denominator polynomial of $H(s)$

$$= s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o$$

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[6] Westdal, *op. cit.* pp. 7-9.

and $N(s)$ = the numerator polynomial of $H(s)$

$$= b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_1 s + b_o$$

Then

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_o & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \qquad \text{III-17}$$

$$A_1 = \begin{bmatrix} -h_1 & 0 & 0 & \dots & 0 \\ -h_2 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -h_{n-1} & 0 & 0 & \dots & 0 \\ -h_n & 0 & 0 & \dots & 0 \end{bmatrix} \qquad \text{III-18}$$

and

$$\underline{h} = \begin{bmatrix} h_1 \\ h_2 \\ \dots \\ h_n \end{bmatrix} \qquad \text{III-19}$$

where $\quad h_1 = b_{n-1}$

$$h_{n-m} = b_m - \sum_{i=1}^{n-m-1} h_i a_{i+m} \qquad \text{III-20}$$

$$m = 0, 1, 2, \dots, n-2.$$

If N(s) is of order $k < (n-1)$, then

$$h_1 = h_2 = \ldots = h_{n-k-1} = 0$$

$$h_{n-k} = b_k \qquad \text{III-21}$$

$$h_{n-m} = b_m - \sum_{i=n-k}^{n-m-1} h_i a_{i+m} \qquad m = 0,1,2, \ldots , k-1$$

From Eq. (III-17), using repeated row operations, it can be deduced that

$$(sI - A_0)^{-1} \underline{h} = \frac{1}{|sI - A_0|} \begin{pmatrix} w_1(s) \\ w_2(s) \\ \cdots\cdots \\ w_n(s) \end{pmatrix}$$

$$= \frac{1}{D(s)} \begin{pmatrix} N(s) \\ sN(s) - h_1 D(s) \\ \cdots\cdots\cdots\cdots \\ s^{n-2}N(s) - \sum_{i=1}^{n-2} s^{n-i-2} h_i D(s) \\ s^{n-1}N(s) - \sum_{i=1}^{n-1} s^{n-i-1} h_i D(s) \end{pmatrix} \qquad \text{III-22}$$

The matrix $(sI - A_0)^{-1} \underline{h}$ is said to have no cancellation if and only if the polynomials $w_1(s)$, $w_2(s)$, ... , $w_n(s)$ and $|sI - A_0|$ have no common factor. If $(sI - A_0)^{-1} \underline{h}$ has a cancellation, then the system cannot be controlled in the direction of the cancelled mode. The following lemma is introduced to provide a necessary and sufficient condition for cancellation in the transfer matrix. Proof of Lemma III-1 for the delay system is completely analogous to that given by Ogata[7] for systems without delay.

Lemma III-1

The matrix $(sI - A_0)^{-1}\ \underline{h}$ has a cancellation if and only if the rank of the matrix

$$P = [\ \underline{h} \vdots A_0\underline{h} \vdots \ldots \vdots A_0^{n-1}\underline{h}\ ] \qquad \text{III-23}$$

is less than n.

As a consequence of the lemma, the following theorem is established.

Theorem III-2

Consider the system with numerator dynamics,

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-\tau) + \underline{h}\ u(t) \qquad \text{III-24}$$

where A_0, A_1 and $\underline{h}$ are as given by Eqs. (III-17), (III-18) and (III-19) respectively. A necessary and sufficient

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[7] Ogata, op. cit., p. 389

condition for the system to be completely state controllable is that $(sI - A_0)^{-1}\underline{h}$ has no cancellations.

Proof. If $(sI - A_0)^{-1}$ has no cancellations, then the rank of

$$P = [\ \underline{h} \vdots A_0\underline{h} \vdots \ldots \vdots A_0^{n-1}\underline{h}\ ]$$

is equal to n. From Brockett and Mesarovic[8], it is known that the matrix P will have rank n if and only if no constant nonzero row vector $\underline{H}$ exists such that

$$\underline{H}(sI - A_0)^{-1}\underline{h} = 0 \qquad \text{III-25}$$

It is known that

$$(sI - A_0)^{-1}\underline{h} = \frac{1}{D(s)}\begin{bmatrix} N(s) \\ sN(s) - h_1D(s) \\ \cdots\cdots\cdots\cdots \\ s^{n-2}N(s) - \sum_{i=1}^{n-2} s^{n-i-2}h_iD(s) \\ s^{n-1}N(s) - \sum_{i=1}^{n-1} s^{n-i-1}h_iD(s) \end{bmatrix} \qquad \text{III-26}$$

where N(s) and D(s) represent the numerator and denominator polynomials of the transfer function H(s) as given in Eq. (III-16).

From Eqs. (III-17) to (III-19), it can be deduced that

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[8] Brockett and Mesarovic, loc. cit.

$$(sI-A_0-A_1e^{-\tau s})^{-1}\underline{h} = \frac{1}{D(s)+N(s)e^{-\tau s}}\begin{bmatrix} N(s) \\ sN(s) - h_1D(s) \\ \cdots\cdots\cdots\cdots \\ s^{n-2}N(s) - \sum_{i=1}^{n-2} s^{n-i-2}h_iD(s) \\ s^{n-1}N(s) - \sum_{i=1}^{n-1} s^{n-i-1}h_iD(s) \end{bmatrix} \qquad \text{III-27}$$

Therefore,

$$(sI - A_0 - A_1e^{-\tau s})^{-1}\underline{h} = \frac{D(s)}{D(s) + N(s)e^{-\tau s}}(sI - A_0)^{-1}\underline{h} \qquad \text{III-28}$$

From Theorem III-1, it is proved that the system (III-24) is completely state controllable if and only if no constant nonzero row vector $\underline{H}$ exists, such that

$$\underline{H}(sI - A_0 - A_1e^{-\tau s})^{-1}\underline{h} = 0 \qquad \text{III-29}$$

Hence it can be concluded that if no constant nonzero row vector $\underline{H}$ exists such that

$$\underline{H}(sI - A_0)^{-1}\underline{h} = 0 \qquad \text{III-30}$$

then, no constant nonzero row vector $\underline{H}$ can make

$$\underline{H}(sI - A_0 - A_1e^{-\tau s})^{-1}\underline{h} = 0$$

In other words, the system (III-24) is completely state controllable if $(sI - A_0)^{-1}\underline{h}$ has no cancellations.

Conversely, if the system (III-24) is not completely state controllable, then there exists a constant nonzero row vector $\underline{H}$, such that

$$\underline{H}(sI - A_0 - A_1 e^{-\tau s})^{-1}\underline{h} = 0 \qquad \text{III-31}$$

which implies that

$$\underline{H}(sI - A_0)^{-1}\underline{h} = 0 \qquad \text{III-32}$$

Hence P has rank less than n, indicating that $(sI - A_0)^{-1}\underline{h}$ has a cancellation.

Q.E.D.

From Theorem III-2, it can thus be concluded that the delay system with zeros is completely state controllable if and only if the rank of the matrix P, where

$$P = [\ \underline{h} : A_0\underline{h} : \ldots : A_0^{n-1}\underline{h}\ ]$$

is equal to n.

II. STABILITY

Consider the homogeneous linear time-invariant system with delay:

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-\tau) \qquad \text{III-33}$$

which represents the dynamical plant with numerator dynamics and $u(t) = 0$. A_0 and A_1 are as given in Eqs. (III-17) and (III-18). Figure (III-1) depicts such a homogeneous system.

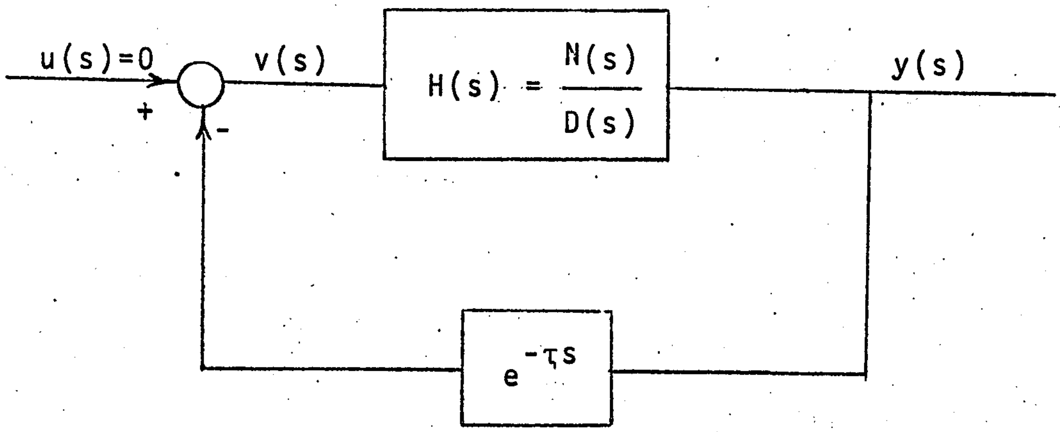

Figure (III-1). Block Diagram of the Homogeneous System with Time Delay.

<u>Theorem III-3</u>

The system (III-33) is stable, in the sense of bounded-input-bounded-output, if and only if all the roots of the polynomial

$$\psi(s) = D(s) + N(s)e^{-\tau s} = 0 \qquad \text{III-34}$$

lie in the open left-hand plane.

$$D(s) = \text{the denominator polynomial in } H(s)$$

$$= s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o$$

III-35

$$\text{and} \quad N(s) = \text{the numerator polynomial in } H(s)$$

$$= b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_1 s + b_o$$

Proof. From Eqs.(III-3) to (III-12), it has been shown that the system (III-3) can be represented by a system of ordinary differential equations. Setting $u(t) = 0$, therefore, the system (III-33) can be represented by the following system of order $(m+1)n$, in the limit as $m \to \infty$:

$$\dot{\underline{X}}(t) = A\,\underline{X}(t) \qquad \text{III-36}$$

where the matrices $\underline{X}(t)$, and A are as defined in Eqs. (III-9) and (III-11).

It is well known[9] that the system (III-36) is stable if and only if every eigenvalue λ of A satisfies $Re\{\lambda\} < 0$. Now, the eigenvalues of A can be obtained by setting

$$\det.\ (\lambda I - A) = 0 \qquad \text{III-37}$$

But

$$\det.\ (\lambda I - A) = (\lambda + \frac{m}{\tau})^{mn} \left\{ D(\lambda) + \frac{1}{(1 + \frac{\lambda\tau}{m})} N(\lambda) \right\} \qquad \text{III-38}$$

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[9] Athans and Falb, op. cit., p. 149

In the limit as $m \to \infty$,

$$\det. (\lambda I - A) = \lim_{m\to\infty} (\lambda + \frac{m}{\tau})^{mn} [D(\lambda) + e^{-\tau\lambda} N(\lambda)] \qquad \text{III-39}$$

Since the roots of $(\lambda + \frac{m}{\tau})$ always lie in the left-hand plane for $m > 0$ and $\tau > 0$, therefore the system (III-33) will be stable if and only if the eigenvalues of the polynomial

$$D(\lambda) + e^{-\lambda\tau} N(\lambda) = 0 \qquad \text{III-40}$$

lie in the left hand plane.

Q.E.D.

Various graphical techniques are available for determining the roots of the polynomial (III-40). For example, the Satche Diagram[10] can be applied to obtain the plots of $f(s) = D(s)/N(s)$, and the unit circle $g(s) = -e^{-\tau s}$, as s encircles the right half complex plane. The system will be stable if the two plots are completely disjoint; or, if the two intersect, then if $Re[g(s)] > Re[f(s)]$ when $f(s)$ lies within the unit circle. Other methods that can be applied are, just to mention a few, the Direct Nyquist plot, the D-decomposition and the root-locus method.

Appendix C gives a FORTRAN IV computer programme which

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[10] Bellman and Cooke, op. cit., pp. 452-453.

can be used to determine the stability of the system (III-33) using the root Locus method, as developed by Huang and Li[11]. This method considers the equation:

$$D(s) + KN(s)e^{-\tau s} = 0 \qquad \text{III-41}$$

where K is the forward gain of the system. If the complex variable

$$s = p + j\omega \qquad \text{III-42}$$

then setting the real and imaginary parts of Eq. (III-41) equal to zero, it can be deduced that

$$\cot \omega\tau = \frac{Im[D(\omega,p)].Im[N(\omega,p)] + Re[D(\omega,p)].Re[N(\omega,p)]}{Re[D(\omega,p)].Im[N(\omega,p)] - Im[D(\omega,p)].Re[N(\omega,p)]}$$

$$= F(\omega,p) \qquad \text{III-43}$$

and

$$K = \left| -\frac{D(\omega,p)}{N(\omega,p)} \right| e^{\tau p} \qquad \text{III-44}$$

From Eq. (III-43), using p as a parameter, solutions of ω can be obtained graphically by plotting cot $(\omega\tau)$ and $F(\omega,p)$ versus ω. From the solutions in ω and p, the complete

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[11] I. Huang and L.L. Li, "Root Locus Determination of Linear Systems with Transport Lag", IEEE Transactions on Automatic Control, Vol. 12, No. 3, Oct. 1967, pp. 632-634.

root locus, and hence the gain curve, can be plotted. With $K = 1$, the corresponding characteristic values of p of the system (III-33) can be determined from the gain curve. The system will be stable if all these p's are less than zero.

III. DOMAIN OF CONTROLLABILITY

Consider again the control process

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-\tau) + \underline{h}u(t) \qquad \text{III-45}$$

which represents the system with numerator dynamics. The output of the system is $y(t) = x_1(t)$.

The domain of G-controllability is defined as the set of continuous functions $\underline{x}_{t_0}(\sigma) = \phi(\sigma)$[12], $\sigma \varepsilon [-\tau, 0]$, which can be steered to the target set G in the n-dimensional Banach space of real continuous functions by an admissible control $u(t)$ on some finite interval $t \varepsilon [t_0, t^*+\tau]$. The control $u(t)$ is admissible if it satisfies the constraint $|u(t)| \leq 1$. If the system has k zeros, the target set G is defined by Eq. (II-35); i.e.

$$G = \{\underline{x}: x_1 = x_2 = \dots = x_{n-k} = 0; \left|\frac{1}{b_k}\sum_{i=1}^{k} q_i(t+\tau)x_{n-k+i}(t^*)\right| \leq 1; \text{ for all } t \geq t^*\} \qquad \text{III-46}$$

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[12] $\underline{x}_{t_0}(\sigma) = \underline{x}(t_0+\sigma), \quad -\tau \leq \sigma \leq 0$

This definition is similar to that for the domain of Null Controllability given by Westdal[13].

Theorem III-4

Consider the system (III-45)

$$\underline{\dot{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-\tau) + \underline{h}u(t) \qquad \text{III-45}$$

with control constraint

$$|u(t)| \leqq 1 \qquad \text{III-47}$$

A_0, A_1 and $\underline{h}$ are as defined in Eqs. (III-17), (III-18) and (III-19) respectively. If

(a) the system is completely state controllable, and

(b) every eigenvalue λ of

$$\psi(s) = D(s) + N(s)e^{-\tau s} \qquad \text{III-48}$$

satisfies $Re[\lambda] < 0$,

then the domain of G-controllability is the set of all real functions $\underline{x}(t)$ in Ψ, the Banach space of real continuous n-dimensional functions on the interval $[t_0-\tau, t_0]$.

Proof. It has been proved by Westdal[14] that if the

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[13] Westdal, op. cit., p. 13.

[14] Westdal, op. cit., pp. 13-14.

system (III-45) satisfies conditions (a) and (b), then all $\underline{x}_t(\sigma) \in C^0[t_0-\tau, t_0]$[15] can be steered to the origin of the state space.

Now, it has also been established in Theorem II-2 that the set G, defined by (III-46), is closed and convex, having the origin $\underline{x} = \underline{0}$ as an element. Since all $\underline{x}_t(\sigma) \in C^0[t_0-\tau, t_0]$ can be steered to a point within G, therefore, the domain of G-controllability is all $\underline{x}_t(\sigma) \in C^0[t_0-\tau, t_0]$.

Q.E.D.

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[15] $C^0[t_0-\tau, t_0]$ represents the set of real functions continuous for $t \in [t_0-\tau, t_0]$.

CHAPTER IV

OPTIMALITY CONDITIONS

It has been shown by Ichikawa[1] and Repin[2] that the dynamics of a time delay system described by a set of differential-difference equations can be expressed in terms of an infinite-dimensional system of ordinary differential equations. It has also been established in Chapter II that the target set G of the delay system with numerator dynamics is closed and convex so that the optimal control force must satisfy Pontryagin's Maximum Principle. This chapter is concerned with the investigation of the optimal criteria and the transversality conditions of the costate of the time optimal regulation problem. The results obtained, except for the transversality conditions, are similar to the optimality conditions given by Kharatishvili[3] and Westdal[4].

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[1] Ichikawa, loc. cit.

[2] Repin, loc. cit.

[3] L.S. Pontryagin et. al. The Mathematical Theory of Optimal Processes, (New York: Interscience Publishers, 1962), pp. 213-226.

[4] Westdal, op. cit., pp. 15-27.

I. THE ADJOINT SYSTEM

Problem IV-1

Consider the system

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-\tau) + \underline{h}u(t) \qquad \text{IV-1}$$

with continuous initial function

$$\underline{x}(t_0+\sigma) = \underline{x}_{t_0}(\sigma) = \underline{\phi}(\sigma) \ , \ \sigma \ \varepsilon \ [-\tau, 0] \qquad \text{IV-2}$$

where $\tau > 0$ is the time delay,

A_0, A_1 are nxn constant matrices,

$\underline{h}$ is a nx1 constant column matrix,

$\underline{x}(t)$, the n-dimensional state vector,

and $u(t)$, the control with constraint

$$|u(t)| \leqq 1 \qquad \text{IV-3}$$

The problem is to find an optimal control segment $u_{(t_0,\ t^*+\tau]}$ that will steer the state $\underline{x}(t)$ from the initial state $\underline{x}_{t_0}(\sigma) = \underline{\phi}(\sigma)$, $\sigma \ \varepsilon \ [-\tau, 0]$, to the target set G in minimum time t*, and to keep it there afterwards; i.e.

$$\underline{x}_{t+\tau}(\sigma) \ \varepsilon \ G, \text{ for all } t \geqq t^*. \qquad \text{IV-4}$$

Such a problem is called a time optimal regulation problem, or a settling problem.

Theorem IV-1

Let $u(t)$ be a time optimal control for Problem IV-1 which transfers the initial state $x_{t_0}(\sigma)$, $\sigma \varepsilon [-\tau,0]$, to the target set G. Let $\underline{x}_t(\sigma)$ denote the state of the system (IV-1) on the time optimal trajectory corresponding to $u(t)$. Let t^* be the minimum time. Then there exists a costate vector $\underline{p}(t)$ such that

(a) $$\dot{\underline{p}}(t) = -A_0{}^T\underline{p}(t) - A_1{}^T\underline{p}(t+\tau) \qquad \text{IV-5}$$

where $A_i{}^T$ denotes the transpose of A_i.

(b) $$u(t) = -\operatorname{sgn}\{\underline{h}^T\underline{p}(t)\} \qquad t\ \varepsilon\ [t_0, t^*] \qquad \text{IV-6}$$

where $u(t) = -\operatorname{sgn}\{0\}$ implies that $u(t)$ is undefined.

Proof.

The system (IV-1) can be represented mathematically by the ordinary vector differential equations

$$\begin{aligned}
\dot{\underline{x}}(t) &= A_0\underline{x}(t) + A_1\underline{x}_m(t) + \underline{h}u(t) \\
\dot{\underline{x}}_1(t) &= \frac{1}{\tau/m}[\underline{x}(t) - \underline{x}_1(t)] \\
\dot{\underline{x}}_2(t) &= \frac{1}{\tau/m}[\underline{x}_1(t) - \underline{x}_2(t)] \qquad \text{IV-7} \\
&\cdots\cdots\cdots\cdots \\
\dot{\underline{x}}_m(t) &= \frac{1}{\tau/m}[\underline{x}_{m-1}(t) - \underline{x}_m(t)]
\end{aligned}$$

where $\underline{x}_m(t) = \underline{x}(t-\tau)$ as $m \to \infty$, IV-8

and $\underline{x}_r(t) = \underline{x}(t - \frac{r\tau}{m})$ is an n-dimensional vector

$$r = 1, 2, \ldots , m$$

Eq. (IV-7) can be written more succinctly as

$$\dot{\underline{X}}(t) = A\underline{X}(t) + \underline{B}u(t) \qquad \text{IV-9}$$

where $\underline{X}(t)$, A and $\underline{B}$ are as defined in Eqs. (III-9), (III-11) and (III-10) respectively. The initial state $\underline{x}_{t_0}(\sigma) = \underline{\phi}(\sigma)$, $\sigma \varepsilon [-\tau, 0]$, implies an initial condition

$$\underline{X}(t_o) = \underline{X}_o \qquad \text{IV-10}$$

and the target set $\underline{x}_{t+\tau}(\sigma) \varepsilon G$, for $t \geqq t^*$, implies a target set

$$\underline{X}(t) \varepsilon G' , \quad \text{for all } t \geqq t^*+\tau \qquad \text{IV-11}$$

where G' represents the target set in the (m+1)n-dimensional space; i.e.

$$G' = \{\underline{X}: \underline{x}, \underline{x}_1, \underline{x}_2, \ldots , \underline{x}_m \varepsilon G, \text{ for all } t \geqq t^*+\tau\} \qquad \text{IV-12}$$

so that G' is closed and convex if G is closed and convex.

Suppose that the costate vector corresponding to the state is

$$\underline{W}(t)^T = [\underline{w}^T(t) \quad \underline{w}_1^T(t) \ \ldots \ \underline{w}_m^T(t)] \qquad \text{IV-13}$$

where $\underline{w}_i(t)$ is an n-vector, $i = 1, 2, \ldots, m$, and $\underline{w}_0(t) = \underline{w}(t)$.

The Hamiltonian for the system (IV-9) is

$$H = 1 + \left\langle A\underline{X}(t), \underline{W}(t) \right\rangle + \left\langle \underline{B}u(t), \underline{W}(t) \right\rangle \qquad \text{IV-14}$$

$$= 1 + \left\langle A_0\underline{x}(t), \underline{w}(t) \right\rangle + \left\langle A_1\underline{x}_m(t), \underline{w}(t) \right\rangle + \left\langle \underline{h}u(t), \underline{w}(t) \right\rangle$$

$$+ \left\langle \frac{1}{\tau/m}[\underline{x}(t)-\underline{x}_1(t)], \underline{w}_1(t) \right\rangle + \left\langle \frac{1}{\tau/m}[\underline{x}_1(t)-\underline{x}_2(t)], \underline{w}_2(t) \right\rangle$$

$$+ \ldots + \left\langle \frac{1}{\tau/m}[\underline{x}_{m-1}(t)-\underline{x}_m(t)], \underline{w}_m(t) \right\rangle$$

From Pontryagin's Principle, it can be deduced that[5]

$$u(t) = -\text{sgn}\,\{\underline{B}^T\underline{W}(t)\} \qquad \text{IV-15}$$

and

$$\dot{\underline{W}}(t) = -\frac{\partial H}{\partial \underline{X}} \qquad \text{IV-16}$$

Eqs. (IV-15) and (IV-16) imply that

$$u(t) = -\text{sgn}\,\{\underline{h}^T\underline{w}(t)\} \qquad \text{IV-17}$$

and

$$\dot{\underline{w}}(t) = -A_0^T\underline{w}(t) - \frac{1}{\tau/m}\,\underline{w}_1(t)$$

$$\dot{\underline{w}}_j(t) = \frac{1}{\tau/m}\,[\underline{w}_j(t) - \underline{w}_{j+1}(t)] \qquad \text{IV-18}$$

$$j = 1, 2, \ldots, m-1$$

$$\dot{\underline{w}}_m(t) = -A_1^T\underline{w}(t) + \frac{1}{\tau/m}\,\underline{w}_m(t)$$

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[5] Athans and Falb, op. cit., p. 383.

Let $\underline{p}(t) = \underline{w}(t)$

and $$\underline{p}_j(t) = \frac{1}{\tau/m}\,\underline{w}_j(t) \qquad j = 1, 2, \ldots, m \qquad \text{IV-19}$$

Then the following relations can be obtained:

$$\dot{\underline{p}}(t) = -A_0{}^T\underline{p}(t) - \underline{p}_1(t)$$

$$\dot{\underline{p}}_j(t) = \frac{1}{\tau/m}\,[\underline{p}_j(t) - \underline{p}_{j+1}(t)] \qquad j = 1, 2, \ldots, m-1 \qquad \text{IV-20}$$

$$\dot{\underline{p}}_m(t) = \frac{1}{\tau/m}\,[-A_1{}^T\underline{p}(t) + \underline{p}_m(t)]$$

and $$u(t) = -\mathrm{sgn}\,\{\underline{h}^T\underline{p}(t)\} \qquad \text{IV-21}$$

Taking the Laplace transform of Eq. (IV-20)[6], it can be shown that

$$\underline{p}_1(s) = \frac{1}{(1 - s\tau/m)^m}\,A_1{}^T\underline{p}(s) \qquad \text{IV-22}$$

In the limit as $m \to \infty$,

$$\underline{p}_1(s) = A_1{}^T\underline{p}(s)e^{\tau s} \qquad \text{IV-23}$$

which implies that

$$\underline{p}_1(t) = A_1{}^T\underline{p}(t+\tau) \qquad \text{IV-24}$$

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[6] Ichikawa, <u>loc</u>. <u>cit</u>.

Therefore,

$$\dot{\underline{p}}(t) = -A_0^T \underline{p}(t) - A_1^T \underline{p}(t+\tau) \qquad \text{IV-25}$$

and
$$u(t) = -\mathrm{sgn}\,\{\underline{h}^T \underline{p}(t)\} \qquad \text{for } t \in [t_0,\ t^*] \qquad \text{IV-26}$$

Q.E.D.

For the system (II-9) with k zeros in the forward transfer function,

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_o & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \qquad \text{IV-27}$$

and

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 0 \\ -h_{n-k} & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -h_n & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \text{IV-28}$$

From Eqs. (IV-25) and (IV-26), the following corollary can be formed:

Corollary IV-1

Consider the system with forward transfer function,

$$H(s) = \frac{y(s)}{v(s)} = \frac{b_k s^k + b_{k-1} s^{k-1} + \ldots + b_1 s + b_o}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_o} \qquad \text{IV-29}$$

and time delay in the feedback path. The control $u(t)$ has constraint $|u(t)| \leq 1$. For $u(t)$ to be time optimal, it is necessary that

(a)
$$\dot{p}_1(t) = a_o p_n(t) - \sum_{i=n-k}^{n} h_i p_i(t+\tau)$$
$$\dot{p}_i(t) = a_{i-1} p_n(t) - p_{i-1}(t), \qquad \text{IV-30}$$
$$i = 2, 3, \ldots, n.$$

and (b)
$$u(t) = -\text{sgn}\left\{ \sum_{i=n-k}^{n} h_i p_i(t) \right\} \qquad \text{IV-31}$$

where

$$\underline{p}^T(t) = [p_1(t) \quad p_2(t) \quad \ldots \quad p_n(t)] \qquad \text{IV-32}$$

is the adjoint corresponding to the state vector $\underline{x}(t)$ describing the system (IV-29);

$$h_{n-k} = b_k$$

and
$$h_{n-m} = b_m - \sum_{i=n-k}^{n-m-1} h_i a_{i+m} \qquad \text{IV-33}$$

$$m = 0, 1, 2, \ldots, k-1$$

For the system with k zeros, from Eq. (II-35), the target set is

$$G = \{\underline{x}: x_1 = x_2 = \ldots = x_{n-k} = 0; \left| \frac{1}{b_k} \sum_{i=1}^{k} q_i(t+\tau) x_{n-k+i}(t^*) \right| \leqq 1; \text{ for all } t \geqq t^*\} \qquad \text{IV-34}$$

Let ∂G denote the boundary of the set G. Then

$$\partial G = \{\underline{x}: x_1 = x_2 = \ldots = x_{n-k} = 0; \left| \frac{1}{b_k} \sum_{i=1}^{k} q_i(t+\tau) x_{n-k+i}(t^*) \right| = 1; \text{ for all } t \geqq t^*\} \qquad \text{IV-35}$$

∂G is a smooth (k-1)-fold.

In general, the target set is in such a form that no easy algebraic expression can be written for ∂G. Rather than spending a lot of effort in search for an algebraic expression, only dynamical systems with one zero will be considered from now on.

II. TRANSVERSALITY CONDITIONS

Consider the dynamical system whose transfer function has only one zero, i.e.

$$H(s) = \frac{b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_o} \tag{IV-36}$$

where a_i, b_j = real constants.

The control process with a time delay in the feedback can be represented by the following systems of differential-difference equations:

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-\tau) + \underline{h}u(t) \tag{IV-37}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \ldots. & 0 \\ 0 & 1 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ -a_o & -a_1 & -a_2 & \ldots & -a_{n-1} \end{pmatrix} \tag{IV-38}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 \\ -h_{n-1} & 0 & 0 & \ldots & 0 \\ -h_n & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{IV-39}$$

$$\underline{h} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ h_{n-1} \\ h_n \end{pmatrix} \qquad \text{IV-40}$$

$$h_{n-1} = b_1$$

and $$h_n = b_o - a_{n-1}b_1 \qquad \text{IV-41}$$

Suppose that

$$x_1(t) = x_2(t) = \ldots = x_{n-1}(t) = 0, \quad \text{for all } t \geqq t^* \qquad \text{IV-42}$$

Then

$$u(t) = x_1(t-\tau) - \frac{x_n(t)}{b_1}, \qquad \text{for all } t \geqq t^* \qquad \text{IV-43}$$

and $$x_n(t) = x_n(t^*)\ e^{-\beta(t-t^*)} \qquad \text{for all } t \geqq t^* \qquad \text{IV-44}$$

where $$\beta = \frac{b_o}{b_1} \qquad \text{IV-45}$$

From Eqs. (II-20) to (II-35), it can be deduced that

$$-b_1u(t) = x_n(t) = x_n(t^*)e^{-\beta(t-t^*)} \qquad \text{for all } t \geqq t^*+\tau \qquad \text{IV-46}$$

Hence the target set G is defined by the relation:

$$G = \{\underline{x}: x_1 = x_2 = \ldots = x_{n-1} = 0; \quad |x_n(t^*)| \leq |b_1| e^{\beta\tau};$$

$$\text{for all } t \geq t^*\} \qquad \text{IV-47}$$

Futhermore,

$$x_1(t-\tau) = \frac{x_n(t)}{b_1} + u(t) \qquad \text{for all } t \geq t^* \qquad \text{IV-48}$$

Since both $|x_n(t)|$ and $|u(t)|$ are bounded for all $t \geq t^*$, therefore $|x_1(t-\tau)|$ is also bounded for all $t \geq t^*$. Using Schwarz's inequality, it can be deduced that

$$|x_1(t)| \leq [1 + e^{-\beta(t-t^*)}] \quad \text{for all } t \in [t^*-\tau, t^*]$$

However, this is only a crude upper bound for $|x_1(t)|$. Derivation of the exact boundary for $|x_1(t)|$ in the interval, $t \in [t^*-\tau, t^*]$, is quite involved, and is beyond the scope of this thesis. At this point, it is sufficient to realize that a least upper bound does exist for $|x_1(t)|$ in $t \in [t^*-\tau, t^*]$. This least upper bound can be described by a bounded positive real decreasing function $\gamma(t)$ defined on $t \in [t^*-\tau, t^*]$ such that

$$|x_1(t)| \leq \gamma(t) \qquad \text{for all } t \in [t^*-\tau, t^*] \qquad \text{IV-49}$$

Let the delay system be represented by the equivalent infinite-dimensional system

$$\dot{\underline{X}}(t) = A\underline{X}(t) + \underline{B}u(t) \tag{IV-50}$$

where $\underline{B}$ and A are as defined in Eqs. (III-10) and (III-11). Then the target set G' in the (m+1)n-dimensional space is specified by the equations:

$$x_1(t) = x_2(t) = \ldots = x_{n-1}(t) = 0 \tag{IV-51}$$

$$|x_n(t)| \leqq |b_1| e^{\beta\tau} \tag{IV-52}$$

$$\left.\begin{array}{l} |x_{11}(t)| \leqq \gamma_1(t^*) \\ |x_{12}(t)| \leqq \gamma_2(t^*) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ |x_{1m}(t)| \leqq \gamma_m(t^*) \end{array}\right\} \quad \text{for } t \geqq t^* \tag{IV-53}$$

where $x_{1r}(t)$ represents the first component of the state vector

$$\underline{x}_r(t) = \underline{x}(t-\frac{r\tau}{m})$$

$$\text{and} \qquad \gamma_r(t) = \gamma(t-\frac{r\tau}{m}); \tag{IV-54}$$

$$\gamma_r(t^*) \geqq \gamma_{r-1}(t^*) \geqq \ldots \geqq \gamma(t^*)$$

If there exists a time optimal control which steers the output $y(t) = x_1(t)$ from some initial state $y_{t_0}(\sigma)$, $\sigma \varepsilon [-\tau, 0]$, to zero, and keeps it there, at $t = t^*$, one of the following considerations applies:

(a) The trajectory hits the end point of the target set G, i.e. at $|x_n(t^*)| = |b_1|e^{\beta\tau}$, and none of the equalities in Eq. (IV-53) is satisfied.

(b) The trajectory intersects the target set G at a point in the interior of G, and none of the equalities in Eq. (IV-53) is satisfied; i.e. the trajectory hits the interior of the target set G'.

(c) The trajectory hits the end point of G and some equalities in (IV-53) are satisfied.

(d) The trajectory hits the interior of G, and some of the equalities in (IV-53) are satisfied.

Figures (IV-1) and (IV-2) illustrate the time response of $x_1(t)$ for cases (a) and (b) and cases (c) and (d) respectively.

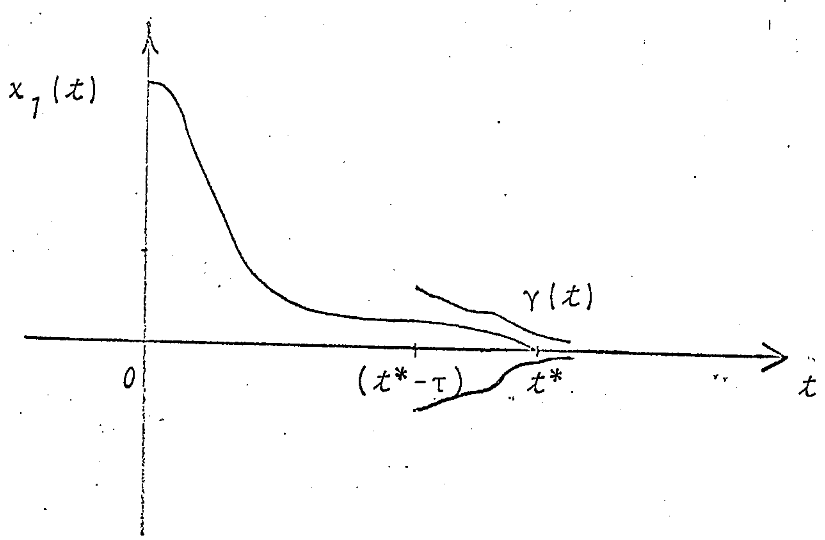

Figure (IV-1). $x_1(t)$ is steered to the target $x_1(t^*)=0$ and none of the equalities in (IV-53) is satisfied.

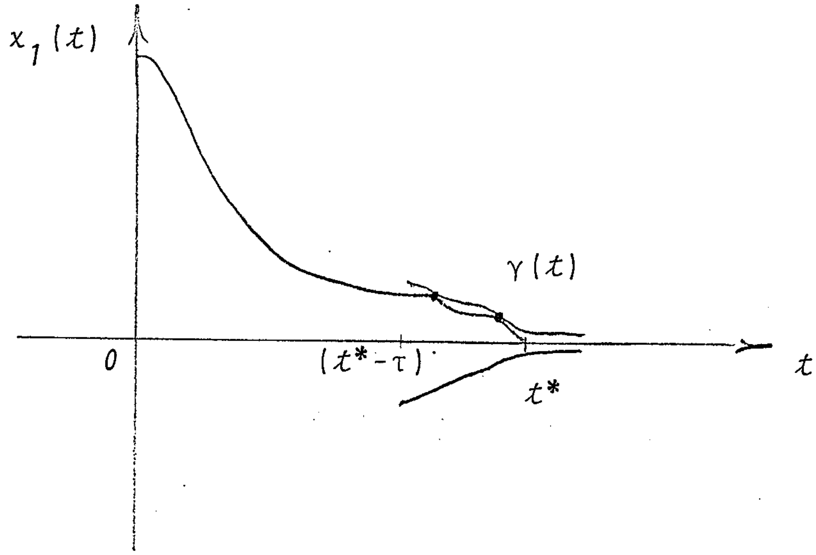

Figure (IV-2). $x_1(t)$ reaches the target $x_1=0$ at $t=t^*$ and some of the equalities in (IV-53) are satisfied.

<u>Case (a)</u>

Consider first the case when the trajectory hits the end point of G, so that G' is specified by the equations:

$$x_\alpha(t^*) = 0, \quad \alpha = 1, 2, \ldots, n-1$$

$$|x_n(t^*)| = |b_1| e^{\beta\tau} \qquad \text{IV-55}$$

$$\underline{x}_r(t^*) = \text{free}, \quad r = 1, 2, \ldots, m$$

Consider the costate $\underline{W}(t)$ again. From Pontryagin's Maximum Principle, it can be deduced that

$$w_\alpha(t^*) = k_\alpha, \quad \alpha = 1, 2, \ldots, n$$

$$\text{IV-56}$$

and $\underline{w}_r(t^*) = \underline{0}, \quad r = 1, 2, \ldots, m$

Recall that

$$\underline{p}(t) = \underline{w}(t)$$

and
$$\underline{p}_j(t) = \frac{1}{\tau/m}\,\underline{w}_j(t) \quad , \quad j = 1, 2, \ldots , m \qquad \text{IV-57}$$

Therefore,

$$p_\alpha(t^*) = k_\alpha \quad , \qquad \alpha = 1, 2, \ldots , n$$

and
$$\underline{p}_r(t^*) = \underline{0} \quad , \qquad r = 1, 2, \ldots , m \qquad \text{IV-58}$$

From Eqs. (IV-20) to (IV-24), it can be established that

$$\underline{p}_1(t) = A_1{}^T \underline{p}(t+\tau)$$

$$\underline{p}_2(t) = \underline{p}_1(t - \frac{\tau}{m}) = A_1{}^T\ \underline{p}(t + \frac{(m-1)}{m}\tau)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad \text{IV-59}$$

$$\underline{p}_m(t) = \underline{p}_1(t - \frac{(m-1)}{m}\tau) = A_1{}^T\ \underline{p}(t + \frac{\tau}{m})$$

Since $\underline{p}_1(t^*) = \underline{p}_2(t^*) = \ldots = \underline{p}_m(t^*) = \underline{0}$

Therefore in the limit as $m \to \infty$,

$$A_1{}^T\ \underline{p}(t+\tau) = \underline{0} \qquad \text{for } t \in (t^*-\tau,\ t^*] \qquad \text{IV-60}$$

Hence if the trajectory hits the boundary of the target set G, the costate must satisfy the condition

$$\dot{\underline{p}}(t) = - A_0^T \underline{p}(t) - A_1^T \underline{p}(t+\tau) \quad \text{for } t \in [t_o, t^*]$$

where

$$A_1^T \underline{p}(t+\tau) = \underline{0} \qquad \text{for } t \in (t^*-\tau, t^*] \qquad \text{IV-61}$$

or, more simply, the conditions

$$\dot{p}_1(t) = a_o p_n(t) + b_1 p_{n-1}(t+\tau) + (b_o - a_{n-1} b_1) p_n(t+\tau)$$

$$\text{for } t \in [t_o, t^*-\tau]$$

and IV-62

$$\dot{p}_1(t) = a_o p_n(t) \qquad \text{for } t \in (t^*-\tau, t^*]$$

and

$$\dot{p}_i(t) = a_{i-1} p_n(t) - p_{i-1}(t) \quad \text{for } t \in [t_o, t^*] \qquad \text{IV-63}$$

$$i = 2, 3, \ldots, n$$

with the terminal conditions

$$p_\alpha(t^*) = k_\alpha \;, \quad \alpha = 1, 2, \ldots, n \qquad \text{IV-64}$$

<u>Case (b)</u>

If the trajectory hits the interior of the target set G, the necessary conditions that the costate must satisfy are the same as those in case (a), i.e. Eqs. (IV-61) to (IV-63) still hold, except that the terminal condition is now

$$p_\alpha(t^*) = k_\alpha \,, \quad \alpha = 1, 2, \ldots , n-1$$

and $$p_n(t^*) = 0. \qquad \text{IV-65}$$

From cases (a) and (b), it can be seen that the costate $\underline{p}(t)$ must satisfy the conditions:

$$\dot{\underline{p}}(t) = - A_0{}^T \underline{p}(t) - A_1{}^T \underline{p}(t+\tau) \quad \text{for } t \in [t_o, t^*]$$

with $$A_1{}^T \underline{p}(t+\tau) = \underline{0} \quad \text{for } t \in (t^*-\tau, t^*] \qquad \text{IV-66}$$

Condition (IV-66) is exactly the same as that given by Kharatishvili[6].

Case (c)

Suppose that the trajectory hits the end point of the target set G, and at some point $x_{1,r+1}(t^*)$ on the boundary defined by the equalities of Eq. (IV-53), such that

$$x_1(t^*) = x_2(t^*) = \ldots = x_{n-1}(t^*) = 0$$

$$|x_n(t^*)| = |b_1| e^{\beta^\tau} \qquad \text{IV-67}$$

$$\underline{x}_j(t^*) = \text{free}, \quad j = 1, 2, \ldots , r, r+2, r+3, \ldots, m$$

(eq'n cont'd)

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[6] Pontryagin et. al., loc. cit.

and $$|x_{1,r+1}(t^*)| = \gamma_{r+1}(t^*)$$

$$x_{i,r+1} = \text{free}, \quad i = 2, 3, \ldots, n$$

Then, in addition to the terminal conditions

$$p_\alpha(t^*) = k_\alpha, \quad \alpha = 1, 2, \ldots, n \qquad \text{IV-68}$$

and $$\underline{p}_j(t^*) = \underline{0}, \quad j = 1, 2, \ldots, r, r+2, r+3, \ldots, m$$

there may be a

$$\underline{p}_{r+1}(t^*) = \begin{pmatrix} k_r \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix} \neq \underline{0} \qquad \text{IV-69}$$

Now from Eqs. (IV-21) to (IV-24), it is known that

$$\underline{p}_{r+1}(t) = \underline{p}_r(t - \frac{\tau}{m}) = \underline{p}_1(t - \frac{r}{m}\tau) \qquad \text{IV-70}$$

or that

$$\underline{p}_1(t) = \underline{p}_{r+1}(t + \frac{r}{m}\tau) \qquad \text{IV-71}$$

Let

$$\tau' = \frac{r}{m}\tau, \quad \tau \geqq \tau' \geqq 0 \qquad \text{IV-72}$$

then $\underline{p}_1(t) = \underline{0}$ for $t \in (t^*-\tau, t^*]$ IV-73

except at $t = t^* - \tau'$

In Section (IV-1), it has been shown that

$$\underline{p}_1(t) = A_1^T \underline{p}(t+\tau) \qquad \text{IV-24}$$

Therefore

$$A_1^T \underline{p}(t+\tau) = \underline{0} \quad \text{for} \quad t \in (t^*-\tau, t^*] \qquad \text{IV-74}$$

except at $t = t^* - \tau'$.

Hence if the trajectory hits the end point of G, and if, at some point

$$|x_{1,r+1}(t^*)| = \gamma_{r+1}(t^*)$$

is satisfied, then the costate must satisfy the necessary conditions

$$\dot{\underline{p}}(t) = - A_0^T \underline{p}(t) - A_1^T \underline{p}(t+\tau) \quad \text{for } t \in [t_0, t^*-\tau] \qquad \text{IV-75a}$$

and $\dot{\underline{p}}(t) = - A_0^T \underline{p}(t)$ for $t \in (t^*-\tau, t^*]$ IV-75b

with $A_1^T \underline{p}(t^*+\tau-\tau') \neq \underline{0}$ IV-75c

and $p_\alpha(t^*) = k_\alpha \quad \alpha = 1, 2, \ldots, n$ IV-75d

Since Eq. (IV-75) is a system of linear differential-difference equations, it can be represented by a linear combination of the equations

$$\dot{\underline{p}}(t) = - A_0^T \underline{p}(t) - A_1^T \underline{p}(t+\tau) \quad \text{for } t \varepsilon [t_0, t^*] \qquad \text{IV-76a}$$

with $$A_1^T \underline{p}(t+\tau) = \underline{0} \quad \text{for } t \varepsilon (t^*-\tau, t^*] \qquad \text{IV-76b}$$

and $$p_\alpha(t^*) = k_\alpha \; , \quad \alpha = 1, 2, \ldots , n \qquad \text{IV-76c}$$

and the equations

$$\dot{\underline{p}}(t) = - A_0^T \underline{p}(t) - A_1^T \underline{p}(t+\tau) \quad \text{for } t \varepsilon [t_0, t^*-\tau] \qquad \text{IV-77a}$$

$$\dot{\underline{p}}(t) = -A_0^T \underline{p}(t) \quad \text{for } t \varepsilon (t^*-\tau, t^*] \qquad \text{IV-77b}$$

with $$A_1^T \underline{p}(t^*+\tau-\tau') \neq \underline{0} \qquad \text{IV-77c}$$

and $$\underline{p}(t^*) = \underline{0} \; . \qquad \text{IV-77d}$$

If the trajectory intersects the target set G, and more than one of the equality conditions in Eq. (IV-53) are satisfied, then the costate $\underline{p}(t)$ can be described by a linear combination of the solutions of Eq. (IV-76), and of Eq. (IV-77), each of which satisfies the corresponding terminal conditions

$$A_1^T \underline{p}(t+\tau) \neq \underline{0} \quad \text{at } t = t^*-\tau' + \frac{\tau}{m} \; , \; t^*-\tau'' + \frac{\tau}{m} \; , \ldots \qquad \text{IV-78}$$

where $t^*-\tau'$, $t^*-\tau''$, ... , are the instants in which the equalities in Eq. (IV-53) are satisfied.

Substituting in A_0^T and A_1^T from Eqs. (IV-38) and (IV-39) into Eqs. (IV-76) and (IV-77), the costate $\underline{p}(t)$ can be expressed as a linear combination of the solutions to the following equations

$$\dot{p}_1(t) = a_o p_n(t) + b_1 p_{n-1}(t+\tau) + (b_o - b_1 a_{n-1}) p_n(t+\tau) \quad \text{for } t \in [t_o, t^*-\tau] \qquad \text{IV-79a}$$

$$\dot{p}_1(t) = a_o p_n(t) \quad \text{for } t \in (t^*-\tau, t^*] \qquad \text{IV-79b}$$

$$\dot{p}_i(t) = a_{i-1} p_n(t) - p_{i-1}(t) \quad \text{for } t \in [t_o, t^*] \qquad \text{IV-79c}$$

$$i = 2, 3, \ldots, n$$

with $\quad p_\alpha(t^*) = k_\alpha \, , \quad \alpha = 1, 2, \ldots, n \qquad$ IV-80

and

$$\dot{p}_1(t) = a_o p_n(t) + b_1 p_{n-1}(t+\tau) + (b_o - b_1 a_{n-1}) p_n(t+\tau) \quad \text{for } t \in [t_o, t^*-\tau] \qquad \text{IV-81a}$$

$$\dot{p}_1(t) = a_o p_n(t) \quad \text{for } t \in (t^*-\tau, t^*] \qquad \text{IV-81b}$$

$$\dot{p}_i(t) = a_{i-1} p_n(t) - p_{i-1}(t) \quad \text{for } t \in [t_o, t^*] \qquad \text{IV-81c}$$

$$i = 2, 3, \ldots, n$$

with $\quad b_1 p_{n-1}(t+\tau) + (b_o - b_1 a_{n-1}) p_n(t+\tau) \neq 0 \qquad$ IV-82a

$$\text{for } t = t^* - \tau'$$

and $\underline{p}(t^*) = \underline{0}$ IV-82b

It is interesting to note that the results obtained here agree exactly with those given by Westdal[7] who used a reversed time approach.

Case (d)

When the trajectory hits the interior of G, and the boundary point

$$|x_{1,r+1}(t^*)| = \gamma_{r+1}(t^*) \qquad \text{IV-83}$$

following the derivation as in case (c), it can be shown that Eqs. (IV-76) to (IV-82) still hold, except that the terminal conditions in Eqs. (IV-76c) and (IV-80) now becomes

$$p_\alpha(t^*) = k_\alpha \; , \quad \alpha = 1, 2, \ldots , n-1 \qquad \text{IV-84}$$

and $p_n(t^*) = 0.$

To recapitulate, it can be concluded that if there exists a time optimal control which steers the output $y(t)$ of the n^{th}-order dynamical system with only one zero from some initial state to zero in minimum time t^*, and keeps it there afterwards, then the following conditions must be satisfied:

---

[7] Westdal, op. cit., pp. 15-27.

(1) The costate $\underline{p}(t)$ must be a solution to the system of differential-difference equations

$$\dot{p}_1(t) = a_o p_n(t) + h_{n-1} p_{n-1}(t+\tau) + h_n p_n(t+\tau)$$

$$\text{for} \quad t \in [t_o, t^*-\tau] \qquad \text{IV-85a}$$

$$\dot{p}_1(t) = a_o p_n(t) \qquad \text{for} \quad t \in (t^*-\tau, t^*] \qquad \text{IV-85b}$$

$$\dot{p}_i(t) = a_{i-1} p_n(t) - p_{i-1}(t) \quad \text{for } t \in [t_o, t^*] \qquad \text{IV-85c}$$

$$i = 2, 3, \ldots, n$$

(2) If the trajectory hits the interior of the target set G, then the costate $\underline{p}(t)$, as defined by Eqs. (IV-85), must satisfy the terminal conditions

$$\begin{aligned} p_\alpha(t^*) &= k_\alpha , \quad \alpha = 1, 2, \ldots, n-1 \\ p_n(t^*) &= 0 \end{aligned} \qquad \text{IV-86}$$

If the trajectory hits the boundary of G, then

$$p_\alpha(t^*) = k_\alpha , \quad \alpha = 1, 2, \ldots, n \qquad \text{IV-87}$$

(3) If the trajectory intersects the target set G, and if at some point

$$|x_1(t^*-\tau'-\frac{\tau}{m})| = \gamma(t^* - \tau' - \frac{\tau}{m}) \qquad \text{IV-88}$$

$$\text{where} \quad \tau \geq \tau' \geq 0$$

is satisfied, then the costate $\underline{p}(t)$ is a linear combination of the solution of Eq. (IV-85), with terminal condition as described in (2), depending on the situation, and the solution of the following systems of differential-difference equations:

$$\dot{p}_1(t) = a_o p_n(t) + h_{n-1} p_{n-1}(t+\tau) + h_n p_n(t+\tau) \quad \text{for } t \varepsilon [t_o, t^*-\tau] \qquad \text{IV-89a}$$

$$\dot{p}_1(t) = a_o p_n(t) \quad \text{for } t \varepsilon (t^*-\tau, t^*] \qquad \text{IV-89b}$$

$$\dot{p}_i(t) = a_{i-1} p_n(t) - p_{i-1}(t) \quad \text{for } t \varepsilon [t_o, t^*] \quad i = 2, 3, \ldots, n \qquad \text{IV-89c}$$

with $\quad h_{n-1} p_{n-1}(t^*+\tau-\tau') + h_n p_n(t^*+\tau-\tau') \neq 0 \qquad$ IV-90a

and $\quad \underline{p}(t^*) = \underline{0} \qquad$ IV-90b

(4) If more than one equality conditions in Eq. (IV-53) are satisfied in the process as the output is steered to the zero state, say at $t = t^* - \tau' - \tau/m$, $t^* - \tau'' - \tau/m$, $t^* - \tau''' - \tau/m$, and so on, then the costate $\underline{p}(t)$ is the resultant vector of the superposition of the solutions of equations similar to Eq. (IV-89), with terminal conditions satisfying

$$h_{n-1} p_{n-1}(t+\tau) + h_n p_n(t+\tau) \neq 0 \qquad \text{IV-91}$$

$$\text{at } t = t^* - \tau'' - \frac{\tau}{m}, \quad t^* - \tau''' - \frac{\tau}{m}, \text{ and so on,}$$

and $\underline{p}(t^*) = \underline{0}$,

upon the linear combination of the solutions to Eqs. (IV-85) and (IV-89).

(5) If the control has constraint $|u(t)| \leq 1$ for all $t \geq 0$, then

$$u(t) = -\text{sgn}\,\{\underline{h}^T \underline{p}(t)\} \qquad \text{for} \quad t \in [t_o, t^*] \qquad \text{IV-92}$$

and

$$u(t) = x_1(t-\tau) - \frac{1}{b_1} x_n(t) \qquad \text{for} \quad t \in (t^*, t^*+\tau] \qquad \text{IV-93}$$

$$u(t) = -\frac{1}{b_1} x_n(t^*)\, e^{-\beta(t-t^*)} \qquad \text{for} \quad t \geq t^* + \tau \qquad \text{IV-94}$$

CHAPTER V

EXAMPLES

In Chapter IV, necessary conditions have been given for the existence of a time optimal control, which brings the output $y(t)$ of the n^{th}-order dynamical plant with one zero from some initial state $y_{t_0}(\sigma) = \phi(\sigma)$, $\sigma \in [-\tau, 0]$ to the zero state and maintains it there for all $t \geq t^*$. Based on these necessary conditions, the optimal control can be found quite easily by means of an iterative procedure. To demonstrate the technique, two examples are given here.

From Eq. (IV-75), it has been shown that the costate of the system (IV-37) can be described by a system of differential-difference equations with advanced arguments, which implies that the adjoint system may only be solved for decreasing time. The costate can be obtained by means of the method of steps; but this method can be quite tedious. It is more desirable to implement the problem on analog or digital computers. However, most computers can only work in increasing time; therefore, to tackle this synthesis problem of optimal control, it is necessary to convert the costate to a system of linear differential-difference equations with retarded arguments, and solve the adjoint in reversed time. A state $\underline{z}(t)$ is defined such that

$$\underline{z}(t) = \underline{p}(t^*-t) \qquad \text{V-1}$$

Since the system considered is time invariant, to simplify mathematical derivations, from now on, it will be assumed that $t_o = 0$.

In Eqs. (IV-85) to (IV-94), let $\underline{p}(t)$ be replaced by $\underline{z}(t^*-t)$. Then if there exists a time optimal control for the dynamical system (IV-37), the following conditions must be satisfied:

(1) $\underline{z}(t)$ must be a solution to the equations

$$\dot{z}_1(t) = -a_o z_n(t) - h_{n-1} z_{n-1}(t-\tau) - h_n z_n(t-\tau) \qquad \text{for} \quad t \in [\tau, t^*] \qquad \text{V-2a}$$

$$\dot{z}_1(t) = -a_o z_n(t) \qquad \text{for} \quad t \in [0, \tau) \qquad \text{V-2b}$$

and

$$\dot{z}_i(t) = -a_{i-1} z_n(t) + z_{i-1}(t) \qquad \text{for} \quad t \in [0, t^*] \qquad \text{V-2c}$$

$$i = 2, 3, \ldots, n$$

(2) If the trajectory hits the interior of the target set G, then $\underline{z}(t)$, as defined by Eq. (V-2), must satisfy the terminal conditions

$$z_\alpha(0) = k_\alpha, \quad \alpha = 1, 2, \ldots, n-1$$
$$z_n(0) = 0 \qquad \text{V-3}$$

If the trajectory hits the boundary ∂G, then

$$z_\alpha(0) = k_\alpha \ , \qquad \alpha = 1, 2, \ldots, n \qquad \text{V-4}$$

(3) If the trajectory intersects G, and at some point

$$|x_1(t^* - \tau' - \frac{\tau}{m})| = \gamma(t^* - \tau' - \frac{\tau}{m}) \qquad \text{V-5}$$

then $\underline{z}(t)$ is the superposition of the solutions of Eqs. (V-2) and the equations:

$$\dot{z}_1(t) = - a_o z_n(t) \qquad \text{for } t \in [0, \tau) \qquad \text{V-6a}$$

$$\dot{z}_1(t) = -a_o z_n(t) - h_{n-1} z_{n-1}(t-\tau) - h_n z_n(t-\tau)$$
$$\text{for } t \in [\tau, t^*] \qquad \text{V-6b}$$

$$\dot{z}_i(t) = -a_{i-1} z_n(t) + z_{i-1}(t) \qquad \text{for } t \in [0, t^*] \qquad \text{V-7}$$
$$i = 2, 3, \ldots, n$$

with

$$h_{n-1} z_{n-1}(\tau' - \tau) + h_n z_n(\tau' - \tau) \neq 0 \qquad \text{V-8}$$

and $\underline{z}(0) = \underline{0}$ where $\tau \geq \tau' \geq 0$

(4) If the control u(t) has constraint $|u(t)| \leq 1$, then

$$u(t) = -\text{sgn}\,\{\underline{h}^T \underline{z}(t^*-t)\} \qquad \text{for } t \in [0, t^*] \qquad \text{V-9}$$

and

$$u(t) = x_1(t-\tau) - \frac{1}{b_1} x_n(t) \qquad \text{for} \quad t \in (t^*, t^*+\tau]$$

$$u(t) = -\frac{1}{b_1} x_n(t^*)\, e^{-\beta(t-t^*)} \qquad \text{for} \quad t \geq t^*+\tau \qquad \text{V-10}$$

Eqs. (V-2) and (V-6) are differential-difference equations with retarded arguments and can be simulated quite easily on computers. To distinguish the two states in Eqs. (V-2) and (V-6) from one another, the notations $\underline{Z}(t)$ and $\underline{Y}(t)$, respectively, will be used.

Based on these necessary conditions, the time optimal control can be synthesized by using the following procedure:

(1) Approximate a minimum time t^* [TF]; it is assumed that the output can be brought from an initial state to zero and maintained there in the time interval $t \in [0, t^*+\tau]$.

(2) To simplify the problem, it is also assumed that the trajectory hits the envelope of $x_1(t)$, as defined by Eq. (IV-53), at $t = t^* - \tau'' - \tau/m$, and slides along the envelope so that

$$|x_1(t-\tau)| = \gamma(t) \qquad \text{for} \quad t \in [t^* - \tau'' - \frac{\tau}{m},\ t^* - \tau' - \frac{\tau}{m}] \qquad \text{V-11}$$

$$\text{where} \quad \tau \geq \tau'' \geq \tau' \geq 0$$

τ' and τ'' are denoted as TPI and TII respectively in the computer program. It can be seen from the relation

$$h_{n-1}z_{n-1}(t-\tau) + h_n z_n(t-\tau) \neq 0 \qquad \text{V-12}$$

$$\text{for} \quad t \in [\tau', \tau'']$$

that any shape of the terminal condition for the superimposed costate $\underline{Y}(t)$ can be approximated by varying τ' and τ''.

(3) To generate the control $u_{(0,\ t^*+\tau]}$ [U], the terminal conditions for the adjoint system Z10, Z20, Y10, Y20, etc. are also approximated. Together with TF, τ' and τ'' the adjoint system is simulated in reversed time to obtain SIG, where

$$\text{SIG} = \underline{h}^T (\underline{Z} + \underline{Y}) \quad \text{for} \quad t \in [0, t^*] \qquad \text{V-13}$$

(4) From SIG, the switching instants TA and TB are obtained, and hence the control segment [U].

(5) Applying the control force u(t) as described by Eqs. (V-9) and (V-10), the system is simulated. The time in which the output reaches the zero state is denoted as t_1 = TEND - τ, where TEND is the settling time. Let ERRTF stand for the difference between t_1 and TF, the final time [TF] will be equal exactly to the minimum time t^* if ERRTF = 0.

(6) By adjustment of the parameters TF, TPI, Z10, Z20, Y10, Y20, etc., the minimum time t^*, and hence the control

segment $u_{(0,\ t^*+\tau]}$, can be found, by which the output can be steered from an initial state to zero and kept there afterwards.

The system (IV-37), of course, can be simulated on an analog computer, and the parameters adjusted by hand after each run. However this procedure can be rather tedious, if done manually. Use of the digital computer is more desirable.

Simulations can be set up quite easily on an IBM System/360 computer by means of the Continuous System Modeling Program (s/360 CSMP)[1]. Coupled with a multi-dimensional direct search subroutine, the system can be simulated on a digital computer, and the parameter adjusted in accordance to the search strategy to obtain the solution. The pattern search of Hooke and Jeeves[2] is used in the examples presented here. A brief description of the search method can be found in Appendix A. The search routine was originally written for use with FORTRAN IV[3]. To suit the

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[1] System/360 Continuous System Modeling Program (360-CX-16X) User's Manual (H20-0367-2). IBM Corporation, Technical Publications Department, White Plains, New York, 1968.

[2] Hooke and Jeeves, loc. cit.

[3] MacDonald, loc. cit.

the purpose here, the routine is rewritten for use with s/360 CSMP.

The search strategy will be most effective if an appropriate objective function can be found to represent the system. The objective function to be minimized, in the time optimal problem here, is ERR, where

$$ERR = E1 + m_1 ISEU + m_2 ISEX2 + \ldots \qquad \text{V-14}$$

$$E1 = |ERRTF| + (ERRTF)^2$$
$$ERRTF = t_1 - TF \qquad \text{V-15}$$

and ISE stands for integral square error, so that ISEU is the integral square error in $u(t)$, and ISEX2 is the integral square error in $x_2(t)$ and so on for $t \geqq t^*$. m_1, m_2 ... are arbitrary large constant multipliers, so that a run may be penalized for any deviations of the system state from the target set G, i.e. $\underline{x}(t) \notin G$, and for any inadmissible control, i.e. $|u(t)| \geqq 1$, for any $t \geqq t_1$. Figure (V-1) gives the general flow diagram of the problem (IV-1).

A SECOND ORDER EXAMPLE

Consider a second order dynamical system with forward transfer function

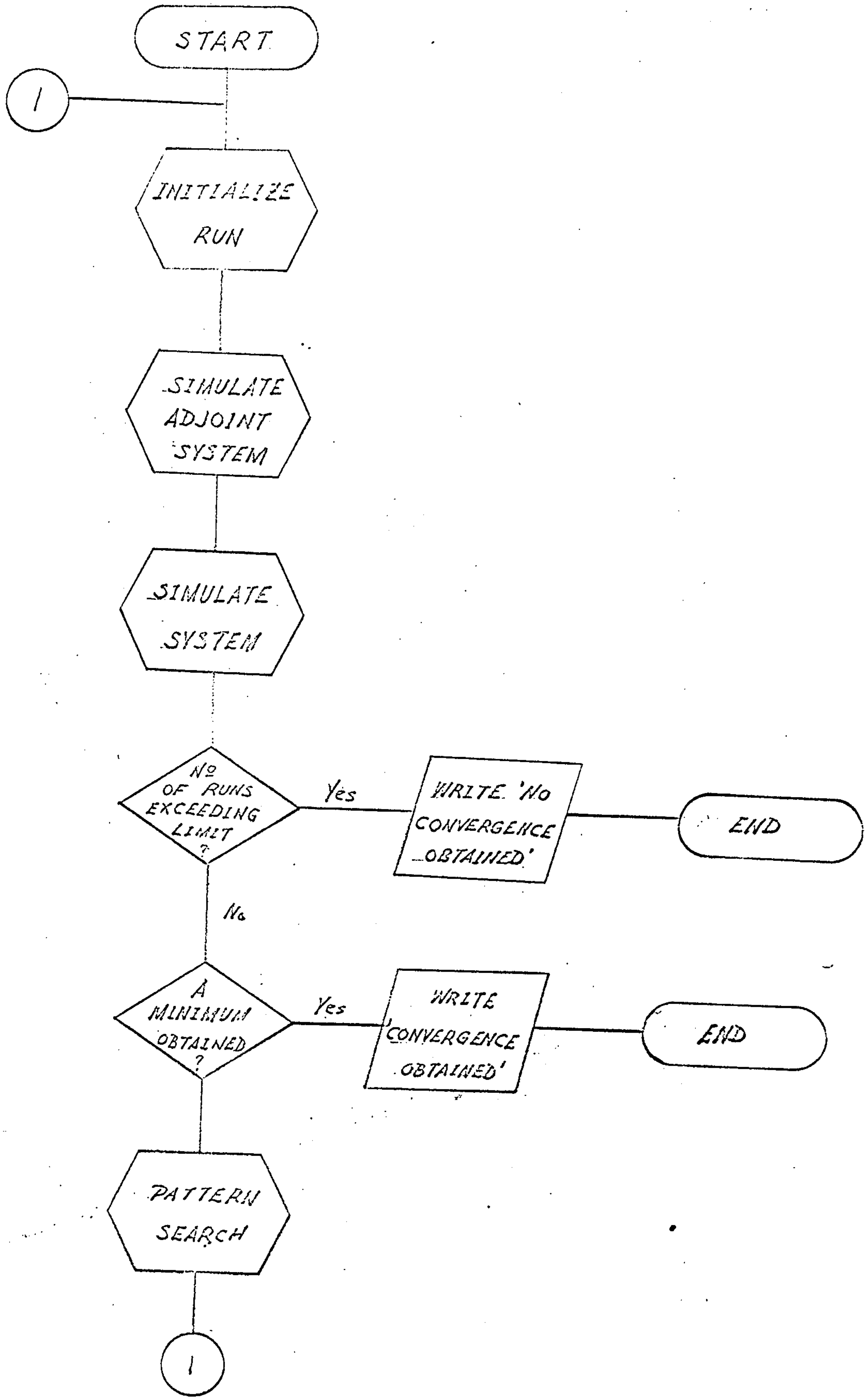

Figure (V-1). General flow diagram of Problem (IV-1).

$$H(s) = \frac{s+1}{s(s+2)} \qquad \text{V-17}$$

There is a time delay of 1 second in the feedback path, and a control force $u(t)$ satisfying the constraint $|u(t)| \leq 1$ for all $t \geq 0$. Figure (V-2) depicts such a system.

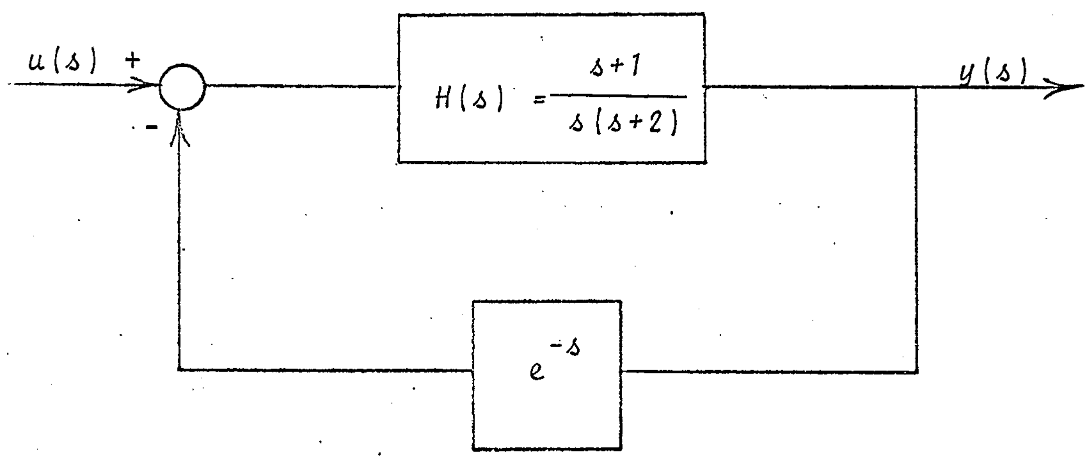

Figure (V-2). A Second Order System.

From Eqs. (II-3) to (II-10), letting

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= \dot{y}(t) + y(t-1) - u(t) \end{aligned} \qquad \text{V-18}$$

the control process can be described by the system

$$\dot{\underline{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-1) + \underline{h}u(t) \qquad \text{V-19}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \qquad A_1 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{V-20}$$

$$\underline{h} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \qquad \text{V-21}$$

From Eq. (IV-47), the target set

$$G = \{\underline{x}: x_1 = 0; \quad |x_2(t^*)| \leqq e \text{ for all } \quad t \geqq t^*\} \qquad \text{V-22}$$

The costate described in reversed time, is the linear combination of the solutions of the following equations:

$$\begin{aligned} \dot{Z}_1(t) &= 0 && \text{for} \quad t \in [0, 1) \\ \dot{Z}_1(t) &= -Z_1(t-1) + Z_2(t-1) && \text{for} \quad t \in [1, t^*] \\ \dot{Z}_2(t) &= Z_1(t) - 2Z_2(t) && \text{for} \quad t \in [0, t^*] \\ \underline{Z}(0) &= \underline{k} \end{aligned} \qquad \text{V-23}$$

and

$$\begin{aligned} \dot{Y}_1(t) &= 0 && \text{for} \quad t \in [0, 1) \\ \dot{Y}_1(t) &= -Y_1(t-1) + Y_2(t-1) && \text{for} \quad t \in [1, t^*] \\ \dot{Y}_2(t) &= Y_1(t) - 2Y_2(t) && \text{for} \quad t \in [0, t^*] \\ \underline{Y}(0) &= \underline{0} \end{aligned} \qquad \text{V-24}$$

with $\quad Y_1(t-1) - Y_2(t-1) \neq 0 \quad$ for $\quad t \in [\tau', \tau'']$

where $\; 1 \geqq \tau'' \geqq \tau' \geqq 0$

The system (V-19) is completely state controllable because the matrix P, where

$$P = [\underline{h} : A_0\underline{h}] = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \qquad \text{V-25}$$

has rank 2.

From Theorem (III-3), the system (V-19) is stable in the bounded-input-bounded-output sense because the roots of the equation

$$\psi(s) = s^2 + 2s + e^{-\tau s}(s+1) = 0$$

lie in the open left-hand plane. Hence the domain of G controllability is the entire two dimensional Banach space of real continuous functions.

The system (V-19) was simulated on an IBM System/360 Model 65 computer. The minimum time t* and control segment $u_{(0,\, t^*]}$ for various initial states were found by means of the pattern search. Figure (V-3), (V-4) and (V-5) show the system responses and control segments required to steer the output from three different initial states to zero in minimum time t* and maintain it there afterwards. The dashed lines in the figures indicate the natural responses of the system (V-19) when $u(t) = 0$. Figure (V-6) depicts the trajectories on the (x_1-x_2) plane. The computer program used for simulating the second order system can be found in Appendix B.

Contruction of the optimal control segment for the system (V-19) was first attempted using only the solution of Eq. (V-23) as the costate. In other words, keeping Z_{1o} and Z_{2o} fixed, the optimal control was synthesized by a one dimensional search on TF. However, in so doing, it was discovered that for certain initial states, it was impossible to maintain the system states in the target set G without violating the control constraint $|u(t)| \leqq 1$ for $t \geqq TF$. The linear combination of Eqs. (V-23) and (V-24) was then used as the adjoint system for the system (V-19). Keeping Z_{1o} and Z_{2o} fixed as before, this became a five dimensional search: namely on TF, τ', τ'', Y_{1o} and Y_{2o}. It was found that all initial states could now be steered to G and maintained there. Tables (V-1) and (V-2) show some of the data obtained.

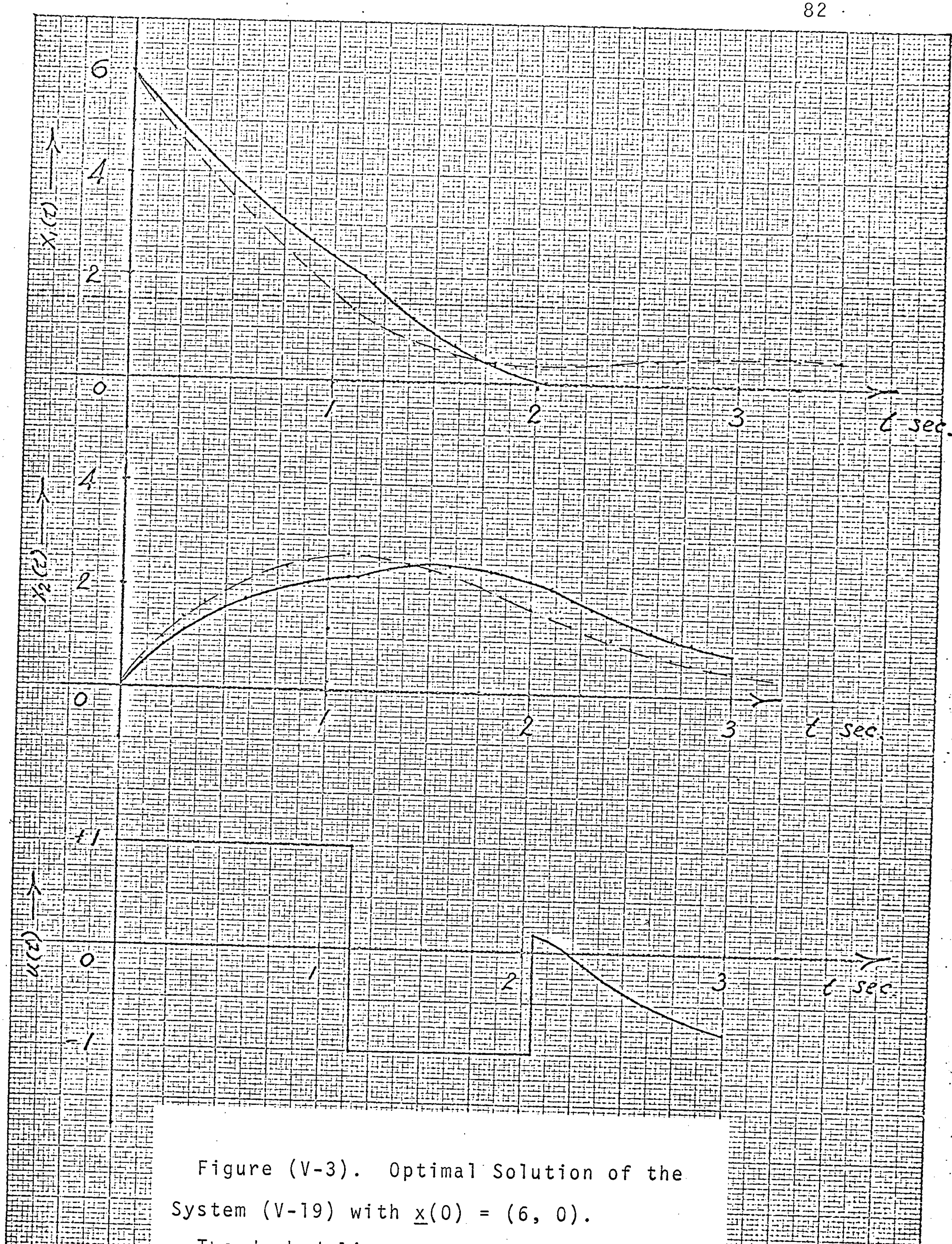

Figure (V-3). Optimal Solution of the System (V-19) with $\underline{x}(0) = (6, 0)$.

The dashed lines represent the natural reponse of the system.

Figure (V-4). Optimal Solution for the System (V-19) with $\underline{x}(0) = (10, 0)$. The dashed lines represent the natural response of the system.

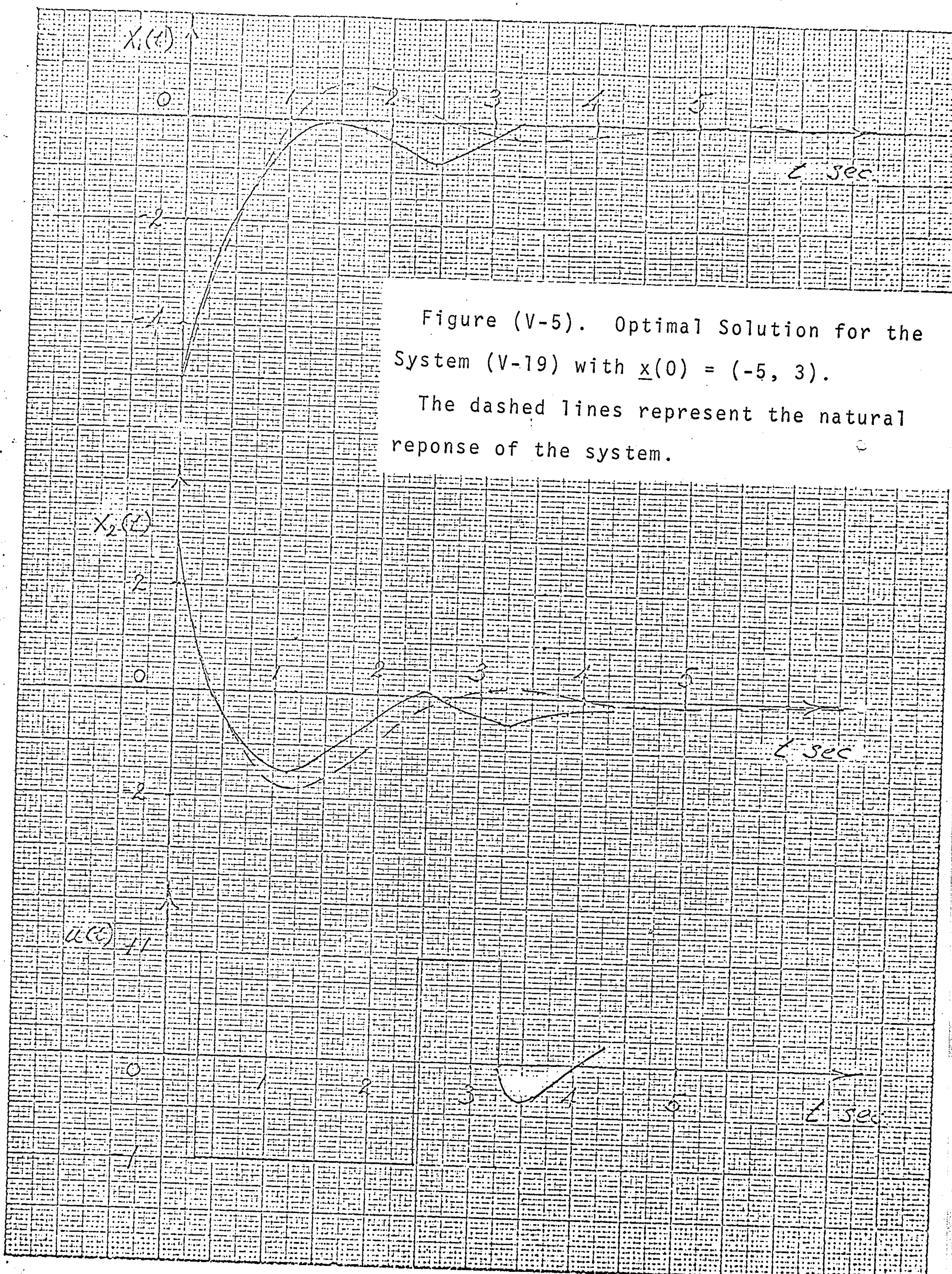

Figure (V-5). Optimal Solution for the System (V-19) with $\underline{x}(0) = (-5, 3)$. The dashed lines represent the natural reponse of the system.

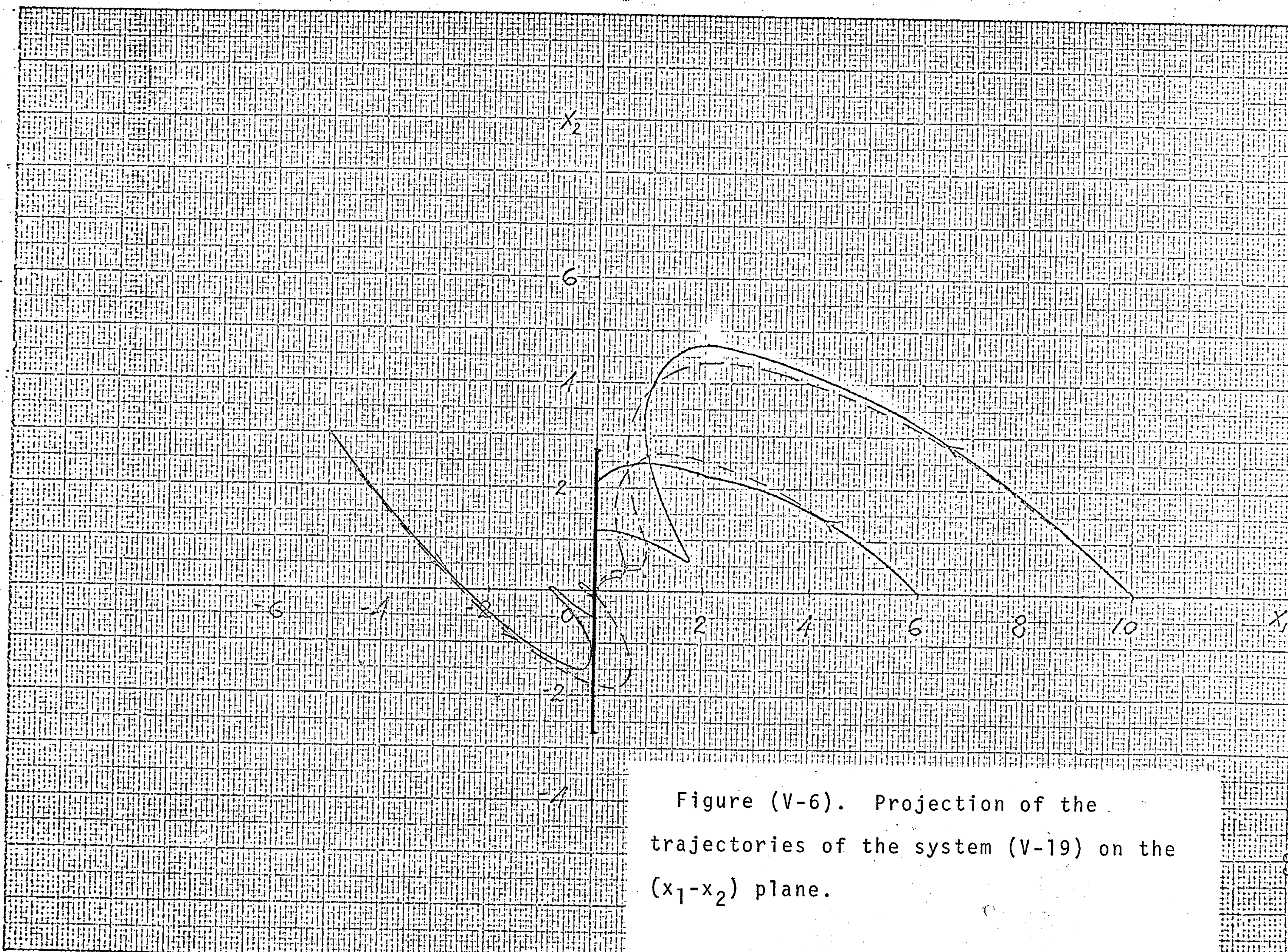

Figure (V-6). Projection of the trajectories of the system (V-19) on the $(x_1 - x_2)$ plane.

x_{1o}	4	6	8	10	12	15
x_{2o}	0	0	0	0	0	0
z_{1o}	5	5	5	5	5	5
z_{2o}	0	0	0	0	0	0
TF	1.17	***	***	***	3.76	4.17
ERRTF	0	***	***	***	.004	.003

Table (V-1). A one dimensional search on TF. *** indicates that the trajectory originating from this initial state could not be maintained in G.

x_{1o}	4	6	8	10	12	15
x_{2o}	0	0	0	0	0	0
y_{1o}	1	4.52	6	5	4.95	12
y_{2o}	-0.012	0	-1	0.1	0.075	5
z_{1o}	5	5	5	5	5	5
z_{2o}	0	0	0	0	0	0
τ'	0.023	0	0.46	0.09	0.51	0.83
τ''	1	1	0.96	1	0.68	1
TF	1.176	2.05	2.0	3.95	3.76	4.18
ERRTF	0.001	0.006	0.004	0.001	0.004	0.009

Table (V-2). A five dimensional search.

A THIRD ORDER EXAMPLE

Consider a third order system with a time delay of 0.2 second in the feedback path and a forward transfer function

$$H(s) = \frac{s+0.3}{(s+1)(s+2)(s+5)} \qquad \text{V-26}$$

The control force u(t) must satisfy the contraint

$$|u(t)| \leq 1 \qquad \text{V-27}$$

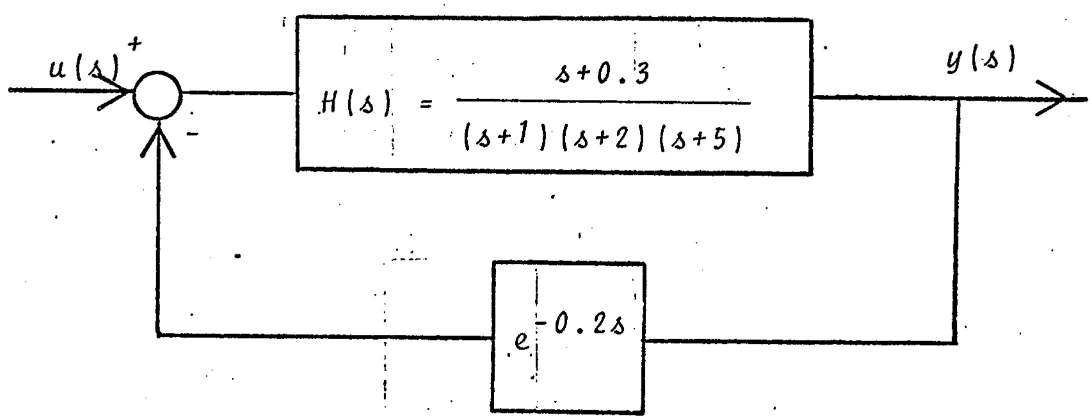

Figure (V-7). A third order system with numerator dynamics and delay in the feedback.

The process can be represented by the system

$$\underline{\dot{x}}(t) = A_0\underline{x}(t) + A_1\underline{x}(t-0.2) + \underline{h}u(t) \qquad \text{V-28}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 7.7 & 0 & 0 \end{pmatrix} \qquad \text{V-29}$$

$$\underline{h} = \begin{pmatrix} 0 \\ 1 \\ -7.7 \end{pmatrix} \qquad \text{V-30}$$

with

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y^{(1)}(t) \\ x_3(t) &= y^{(2)}(t) + y(t-0.2) - u(t) \end{aligned} \qquad \text{V-31}$$

To bring the output $y(t)$ to zero and keep it there is equivalent to bringing the state $\underline{x}(t)$ to the target set G, where

$$G = \{\underline{x}:\ x_1 = x_2 = 0;\quad |x_3(t^*)| \leq 1.068 \quad \text{for all } t \geq t^*\} \qquad \text{V-32}$$

$$P = [\ \underline{h} \ \vdots\ A_0\underline{h} \ \vdots\ A_0{}^2\underline{h}\ ]$$

is given by

$$P = \begin{pmatrix} 0 & 1 & -7.7 \\ 1 & -7.7 & 44.6 \\ -7.7 & 44.6 & -235.9 \end{pmatrix} \qquad \text{V-33}$$

From Theorem (III-2), the system is completely state controllable because P has rank 3.

The adjoint system, in reversed time, can be described by a linear combination of the following equations:

$$\dot{Z}_1(t) = -10Z_3(t) \qquad \text{for } t \in [0, 0.2)$$

$$\dot{Z}_1(t) = -10Z_3(t) - Z_2(t-0.2) + 7.7Z_3(t-0.2) \quad \text{for } t \in [0.2, t^*]$$

$$\left.\begin{array}{l}\dot{Z}_2(t) = Z_1(t) - 17Z_3(t) \\ \dot{Z}_3(t) = Z_2(t) - 8Z_3(t)\end{array}\right\} \text{for } t \in [0, t^*] \qquad \text{V-34}$$

with $\underline{Z}(0) = \underline{k}$

and

$$\dot{Y}_1(t) = -10Y_3(t) \qquad \text{for } t \in [0, 0.2)$$

$$\dot{Y}_1(t) = -10Y_3(t) - Y_2(t-0.2) + 7.7Y_3(t-0.2) \quad \text{for } t \in [0.2, t^*]$$

$$\left.\begin{array}{l}\dot{Y}_2(t) = Y_1(t) - 17Y_3(t) \\ \dot{Y}_3(t) = Y_2(t) - 8Y_3(t)\end{array}\right\} \text{for } t \in [0, t^*] \qquad \text{V-35}$$

with $Y_2(t-0.2) - 7.7Y_3(t-0.2) \neq 0$ for $t \in [\tau', \tau'']$

where $0.2 \geqq \tau'' \geqq \tau' \geqq 0$.

From Eqs. (V-9), (V-10) and (V-13),

$$u(t) = -\text{sgn}\{SIG\} \quad \text{for} \quad t \in (0, t^*] \qquad \text{V-36}$$

where $$SIG = \{Z_2(t) + 7.7Z_3(t)\} + \{Y_2(t) - 7.7Y_3(t)\} \qquad \text{V-37}$$

and $$u(t) = x_1(t-0.2) - x_3(t) \quad \text{for} \quad t \in [t^*, t^*+0.2]$$

$$= x_3(t^*)\, e^{-0.3(t-t^*)} \quad \text{for} \quad t \geqq t^*+0.2 \qquad \text{V-38}$$

Using $u(t) = 0$, for all $t \geqq 0$, the system (V-28) is found to be stable, because simulations show that the system state $\underline{x}(t)$ eventually settles down to the origin from any arbitrary initial state.

Construction of the time optimal control $u(t)$ was attempted, at first, using only Eq. (V-34) as the costate. However, it was discovered that for the third order system (V-28), it was impossible maintain the system state in the target set G without violating the control constraint $|u(t)| \leqq 1$, for $t \geqq TF$. The synthesis problem was then tackled again using the superposition of Eqs.(V-34) and (V-35) as the adjoint system. The problem, using pattern search, now became a seven-dimensional search problem: namely on TF, Z_{1o}, Z_{2o}, τ', τ'', Y_{1o} and Y_{2o}. Figure (V-8) and (V-9) show the subsequent system responses from two different initial states and the corresponding optimal control segment $u_{(0,\ t^*+0.2]}$. Figure (V-10) is the projection of the trajectories on the

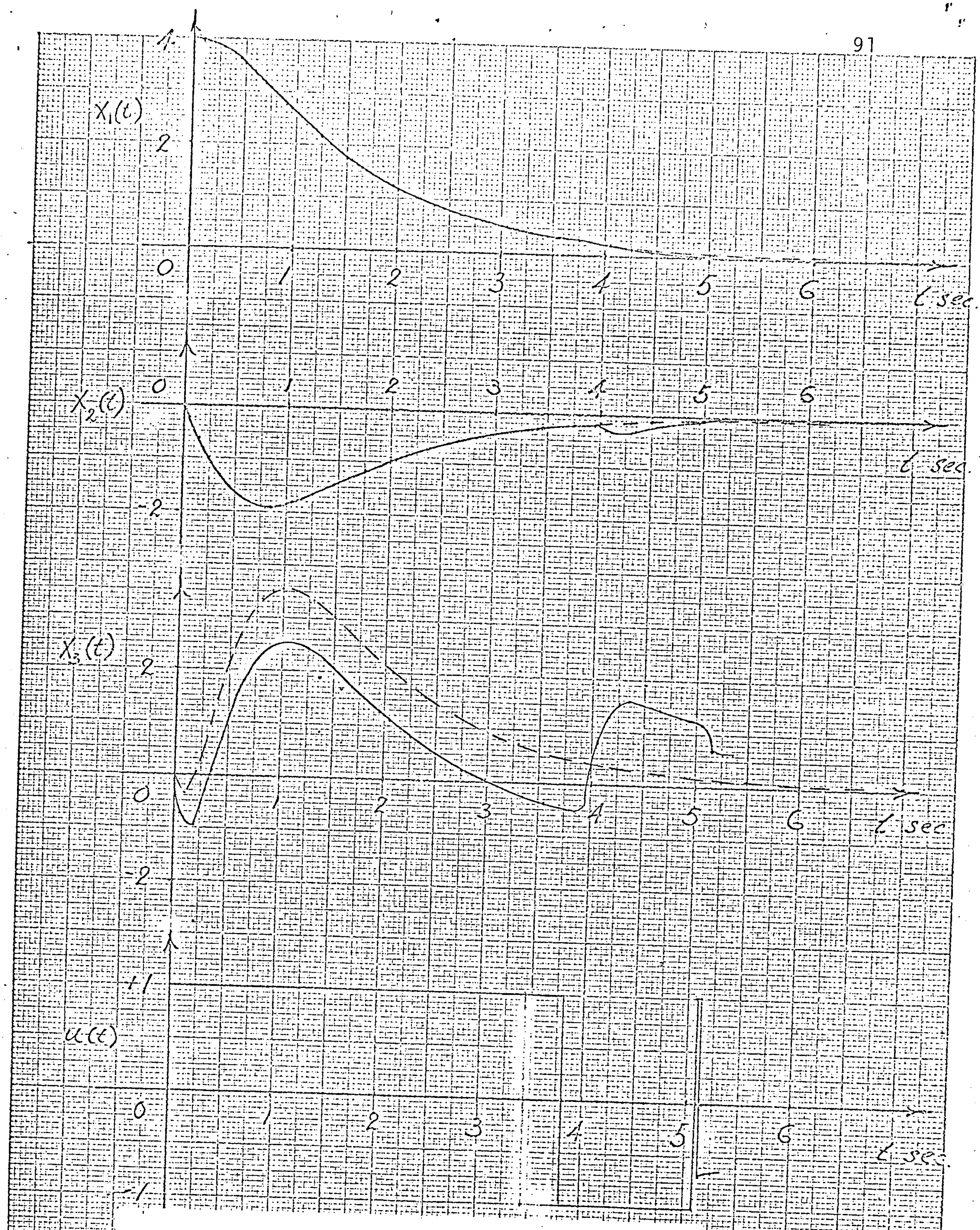

Figure (V-8). Optimal Solution of the System (V-28) with $\underline{x}(0) = (4, 0, 0)$.

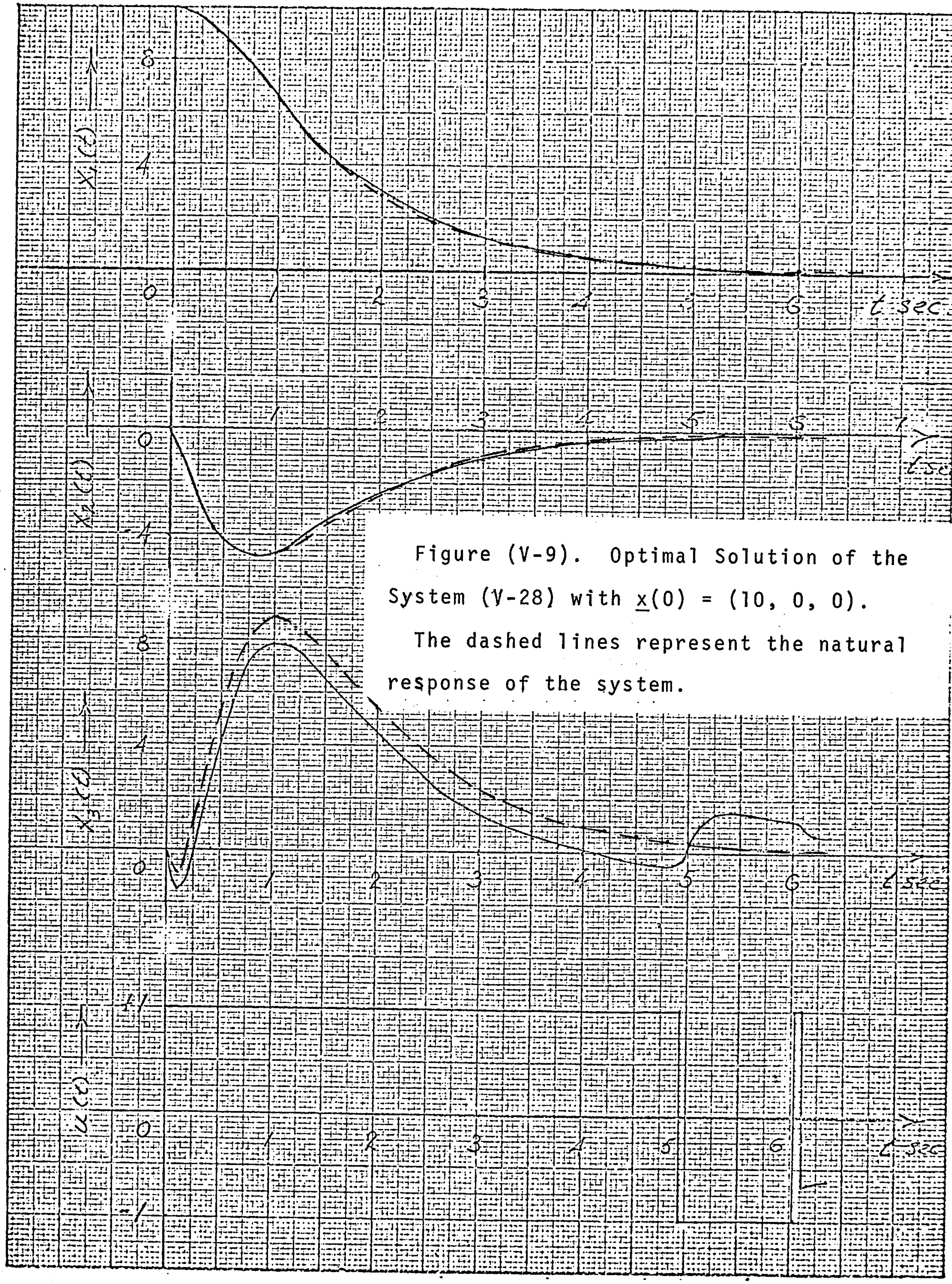

Figure (V-9). Optimal Solution of the System (V-28) with $\underline{x}(0) = (10, 0, 0)$. The dashed lines represent the natural response of the system.

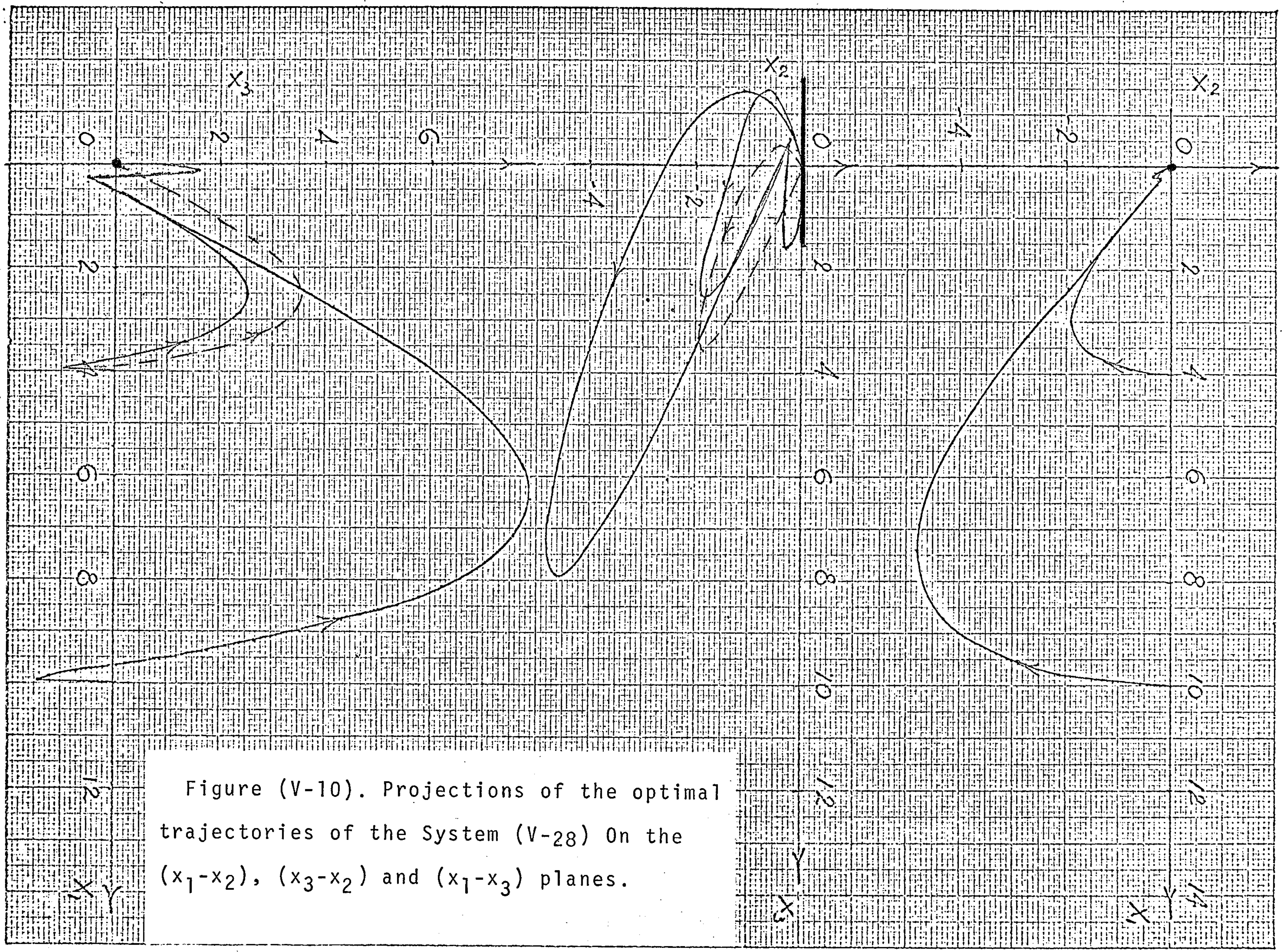

Figure (V-10). Projections of the optimal trajectories of the System (V-28) On the (x_1-x_2), (x_3-x_2) and (x_1-x_3) planes.

(x_1-x_2), (x_3-x_2) and (x_1-x_3) planes. The computer program used to simulate the system (V-28) can be found in Appendix B.

In the examples, it was assumed that the trajectories will hit the interior of the target set G, rather than the boundary ∂G. Attempts have been made to force the trajectory to the boundary ∂G of the target set. However, these were successful only for certain initial states in the state space; and the amount of computer time required prohibited a detail mapping of the initial states from which the system state can be steered to ∂G. Derivation of the conditions, which determines when the trajectory would hit the interior of G, and when it would hit ∂G, is rather involved, and not as straightforward as in the case of dynamical systems without time delay, and is beyond the scope of this thesis. However, it can be mentioned that in the case of the second order system (V-19), those system states, that could not be maintained in the target set G after hitting the target at an interior point without using a superposition on the adjoint system, could not be steered to ∂G as well. From the examples considered, it can be concluded that a boundary region does exist for $x_1(t-\tau)$, for $t \in [t^*, t^*+\tau]$; therefore, to bring the output of the system from an initial state to zero in minimum time and keep it there afterwards, it is required that the adjoint must be a linear combination of the solutions of Eqs. (V-2), (V-6) and (V-7).

To give an approximate idea of the computing time, assuming that a reasonably good initial approximation has been made on the values of the system parameters, a typical run on the second order system requires approximately nine minutes, and on the third order system, approximately twenty minutes of Central Processing Unit (CPU) Time on the computer in order to obtain the optimal solution. The number of iterations and computer time required to obtain the time optimal control, of course, are highly dependent on the initial guess of the values of the parameters to be adjusted.

CHAPTER VI

CONCLUSIONS AND PROBLEMS FOR FURTHER STUDIES

The movement of a control process with numerator dynamics can be described in a co-ordinate system which does not involve derivatives of the control function u(t). The transformation technique used is applicable to systems with or without time lag. In the regulation problem, the target set is transformed from a one-point set to a smooth k-fold.

It has been proved that the system with numerator dynamics is completely state controllable if and only if no cancellations occur. This condition is similar to the controllability criterion for systems without delay. From controllability, it has been further derived that if the system is stable, all initial states in the Banach space of real continuous functions can be steered to the target set G by using a bounded control $|u(t)| \leq 1$.

The optimality conditions for the delay system with only one zero have also been given. In working out the examples, it was discovered that, for some cases, the system state could not be brought to the target set G, and kept there, unless the transversality conditions were applied. It would certainly be desirable to extend these conditions to systems with more than one zero.

The delay system with numerator dynamics only brings out a particular case in the optimal control of system state to a function in the Banach space. When the target set consists of more than one single point in the state space, the general attainability and settling problems are practically left untouched in literature. More research on time delay systems could be directed to this area. It is believed that extra necessary conditions can be obtained by a study of the target set and the set of attainability.

From the investigation of the delay system with only one zero, it appears that no easy conclusion can be drawn on the condition which determines when the trajectory would hit the interior of the target set G, or when it would hit the boundary. It would be desirable if some complete theory could be developed in this area.

The IBM System/360 digital computer provides facilities for solving the time delay problem in the form of Continuous System Modeling Program (s/360 CSMP). Coupled with some direct search strategy, solutions to the delay system can be obtained fairly easily by trial and error. The pattern search of Hooke and Jeeves has been rewritten for use with s/360 CSMP. The search strategy handles the problem quite efficiently even when the system involves three or more parameters that require adjusting. By suitable representation of the objective function to be optimized, the search strategy should be able to handle problems with any general cost functional.

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APPENDIX A

PATTERN SEARCH

I. INTRODUCTION

The synthesis problem for a linear dynamical control process with time delay can be solved by simulating the system on a computer and obtain the optimal control by trial and error. If the plant involves two or more system parameters that require adjusting, the optimal control will not be readily determined unless some scheme of sequential search strategy is used. Simulation of the system can be set up on an IBM System/360 digital computer with the help of the Continuous System Modeling Program. The direct search strategy used in this thesis for synthezing the time optimal control for the delay system with numerator dynamics is the pattern search method of Hooke and Jeeves[1]. It is felt that if an objective function can be set up properly, the pattern search

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[1] R. Hooke and T.A. Jeeves, "'Direct Search' Solution of Numerical and Statistical Problems", Journal of the Association for Computing Machinery, Vol. 8, April 1961, pp. 212-229; D.J. Wilde and C.S. Beightler, Foundations of Optimization, (Englewood Cliffs: Prentice-Hall Inc., 1967) pp. 307-313; J.W. Bandler, "Optimization Methods for Computer-Aided Design", Department of Electrical Engineering Publications (Winnipeg: University of Manitoba, February, 1969) pp. 27-28.

should be able to handle the linear time delay problem with any cost functional. When the program is set up to search for the optimal control, care should be taken in handling the parameter constraints of the system. The interested reader is asked to refer to Bandler[2] and Box[3] for a detail discussion on parameter constraints.

II. PATTERN SEARCH

Pattern search is a direct search routine for minimizing an objective function $E(\underline{w})$ of several variables, where $\underline{w}^T = (w_1 \quad w_2 \quad \ldots \quad w_k)$. The argument $\underline{w}$ is varied until the minimum $E(\underline{w})$ is obtained. The pattern search routine determines the sequence of values for $\underline{w}$, while an independent routine computes the function values of $E(\underline{w})$. Figure (A-1) shows a two dimensional example of pattern search.

A point $\underline{w}^1$ is arbitrarily selected to be the first "base point", which is denoted as $\underline{t}^1$, with $E(\underline{w}^1) = E(\underline{t}^1) = E^1$. Suppose that an exploratory move made from $\underline{t}^1$ in the $+w_1$ direction to $\underline{w}^2$ with increment s_1 results in an objective

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[2] Bandler, op. cit., pp. 14-22.

[3] M.J. Box, "A Comparison of Several Current Optimization Methods, and the Use of Transformations in Constraint Problems", Computer Journal, Vol. 9, May 1966, pp. 67-77.
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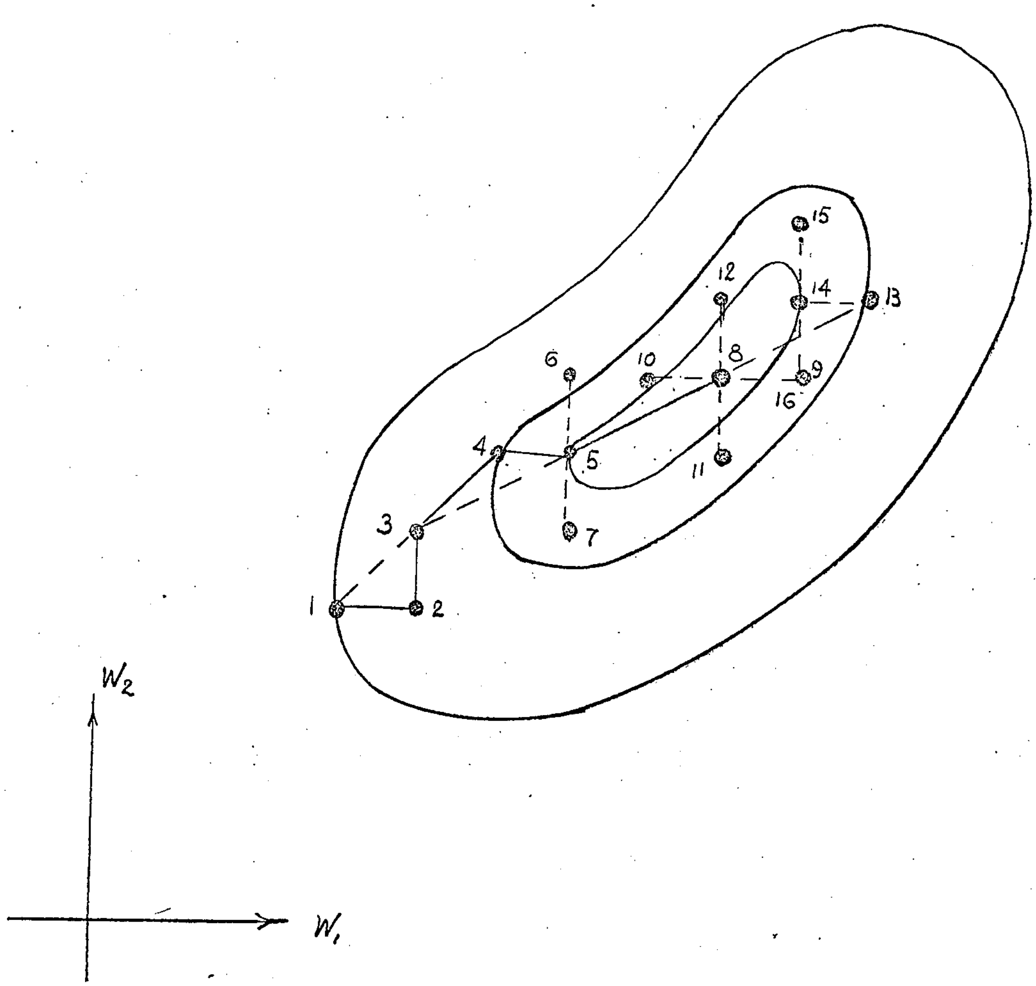

Figure A-1. Example of a 2-dimensional pattern search.*

* Bandler, op. cit., p. 61.

function value $E^2 < E^1$; then $\underline{w}^2$ is retained. The exploration is continued by incrementing w_2 to $\underline{w}^3$. Suppose that $E(\underline{w}^3) = E^3 < E^2$, then $\underline{w}^3$ is retained in place of $\underline{w}^2$, and a set of "exploratory moves" is said to have been completed $\underline{w}^3$ is designated as the second base point $\underline{t}^2$.

A "pattern move" is now made from $\underline{t}^2$ to $\underline{w}^4$, so that

$$\underline{w}^4 = 2\,\underline{t}^2 - \underline{t}^1 \qquad \text{A-1}$$

E^4 is not immediately compared with E^3. Instead, a local exploration is carried out about $\underline{w}^4$ to correct the tentative second pattern. Suppose that a move in the previous successful $+w_1$ direction to $\underline{w}^5$ is a success (i.e. $E^5 < E^3$), and subsequent moves in the $+w_2$ direction is a failure, then $\underline{w}^5$ becomes the third base point $\underline{t}^3$. The pattern search continues with another pattern move to $\underline{w}^8$ so that

$$\underline{w}^8 = 2\,\underline{t}^3 - \underline{t}^2 \qquad \text{A-2}$$

If the pattern move and subsequent exploratory moves fail, the pattern is destroyed, and local exploratory moves are started at the previous base point $\underline{t}^3$. If this scouting expedition locates a better point, then a new pattern is started again. But if, as in Figure (A-1), no better point is found, the parameter increments are reduced, and the whole procedure is restarted again at the previous base point, in this case $\underline{t}^3$. The search is terminated when the parameter increments fall below a preselected value.

A flow diagram of pattern search is given in Figure (A-2). The sequence following the label 1 is for an initial set of exploratory moves from a base point when a new pattern must be established. The sequence following 2 is the basic iterative loop consisting of a pattern followed by a set of exploratory moves. The sequence 3 controls the reduction of step size and termination of the search.

In the time optimal control problem for the systems with delay, the objective function is chosen as the difference between the estimated minimum time t_f, and the actual minimum time t*, based on this initial guess, for the state to be steered to the target set G. The independent routine for calculating the value of the objective function is a simulation of the system in this case. A computer program of pattern search rewritten for use with s/360 CSMP is presented in Appendix B. This program is based on another computer program written in FORTRAN IV.[4]

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[5] P.A. MacDonald, "Locating the Minimum of a Function by Pattern Search", Department of Electrical Engineering Publications, 69-TR-12, NAG-011, (Winnipeg: University of Manitoba, April 1969)

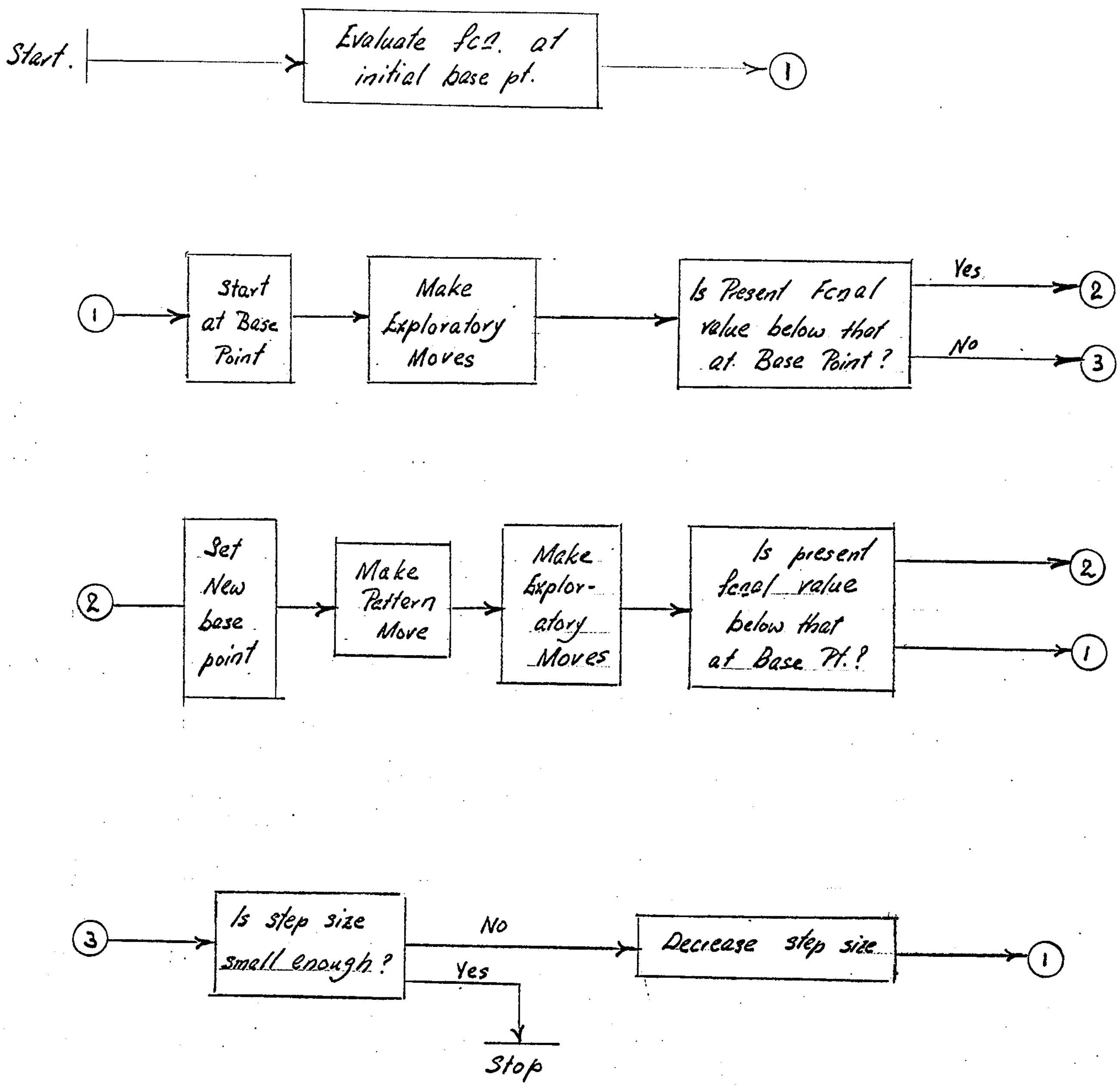

Figure A-2. Descriptive flow diagram for pattern search.*

* Hooke and Jeeves, <u>op</u>. <u>cit</u>., p. 225.

APPENDIX B

s/360 CSMP PROGRAMMES

Both the second and third order examples given in Chapter V were solved by simulating the system on the digital computer and constructing the optimal control by means of pattern search. Figure (B-1) gives the computer program used for simulating the second order example with an initial condition of $x_{1o} = 6$ and $x_{2o} = 0$. The s/360 CSMP program used in the third order example with the initial condition $\underline{x}(0) = (4, 0, 0)$ is given in Figure (B-2). Both the second and third order examples used the Pattern Search Package, which can be found in Figure (B-3).

CONTINUOUS SYSTEM MODELING PROGRAM*

PROBLEM INPUT STATEMENTS

```
*       ****  OPTIMAL STRATEGY FOR (S+1)/S(S+2)   ****
*----     A 5-D SEARCH

        INITIAL
        FIXED N,M,NB,LIMIT,IS,I,IT,J,K
        STORAGE S(8), M(8), W(8), T(8)
        CONST NB=0, N=1, J=0
        CONST IS=1, I=1, IT=0

*---- USER SUPPLIED INITIAL CONDITIONS

        CONST K=5
        CONST SL=0.1, RHO=0.5, LAMBDA=0.0001, LIMIT=100, H=10000.0
        CONST Z10=5.0, Z20=0.
        CONST X10=6.0, X20=0.
        CONST TF=5.0, Y10=5.0, Y20=0.
        TABLE W(2)=0.5, W(3)=0.6
        CONST D3=0
        NOSORT
        IF (IS .NE. 1) GO TO 61
        DELTA =SL

*---- CHANGE SYSTEM PARAMETER INTO CO-ORDINATE W(I)

        W(1)=ALOG(TF)
        W(4)=0.1*Y10
        W(5)=0.1*Y20

*---- INITIALIZE BASE POINT, INCREMENT, CONTROL NO.

        DO 62 J=1,K
        T(J)=W(J)
        S(J)=DELTA
62      M(J)=0
61      CONTINUE

*---- CHANGE CO-ORDINATE VALUE BACK TO SYSTEM PARAMETER VALUE

        TF=EXP(W(1))
        WT=4*W(2)
        WI=4*W(3)
        TPI=1.0*SIN(WT)*SIN(WT)
        TII =(1.0-TPI)*COS(WI)*COS(WI)+TPI
```

Figure (B-1). s/360 CSMP Program for the Second Order Example.

```
      Y10=10*W(4)
      Y20=10*W(5)

*==== USER SUPPLIED SYSTEM SIMULATION

      SORT
      TENCO=1.0-TF
      SZ=(Z10-Z20) + (Y10-Y20)
      UP=INSW(SZ,-1.0,1.0)
      D1=INSW(X10,-1.0,1.0)
      D2=INSW(X20,-1.0,1.0)
      Z10=5.0*D1*D2
      CYY=IOR(TPI,K-5)
DYNAMIC
*      **** THE ADJOINT SYSTEM ****
      A1=Z2D-Z1D
      Z1=INTGRL(Z10,A2)
      Z1D=DELAY(100,1.0,Z1)
      Z2D=DELAY(100,1.0,Z2)
      A3=COMPAR(TIME,0.999)
      A2=A1*A3
      Z2=REALPL(Z20,0.5,A)
      A=0.5*Z1
      Y2D=Y20*C1Y*C2Y*(1.0-A3)*CYY+A3*DELAY(100,1.0,Y2)
      Y1D=Y10*C1Y*C2Y*(1.0-A3)+A3*DELAY(100,1.0,Y1)
      C1Y=COMPAR(TIME,TPI-1.0E-5)
      C2Y=COMPAR(TII+1.0E-5,TIME)
      Y1=INTGRL(0.,Y1N)
      Y2=INTGRL(0.,Y2N)
      Y1N=(Y2D-Y1D)*COMPAR(TIME,TPI)
      Y2N=Y1-2.0*Y2
      ZSZ=Z1-Z2
      YSY=Y1-Y2
      SIG=ZSZ+YSY
      TAF=(AND(AA,TIME-2.5))*L
      TBF=(AND(AB,2.5-TIME))*L
      AA=-SIG*UP
      AB=SIG*UP
      L=COMPAR(TF,TIME)
      GA=INTGRL(2.5,TAF)
      FA=COMPAR(TF,2.5)
      TA=(TF-GA)*FA
      GB=INTGRL(0.,TBF)
      TB=TF-GB
*      **** THE SYSTEM ****
      X1D=DELAY(100,1.0,X1)
      V1=(X2-X1D+U)*COMPAR(TIME,TF-1.0E-5)
```

Figure (B-1) cont'd.

```
      V2=(X1D-2.0*X2-U)*COMPAR(TIME,TF-1.0E-5)
      X1=INTGRL(X10,V1)
      X2=INTGRL(X20,V2)
      UA=COMPAR(TIME-TF,TA)
      UB=COMPAR(TIME-TF,TB)
      Q=D1*X1
      TMF=COMPAR(Q,0.001)
      UTF=(X1D-X2)*(1.0-TMF)
      U=UP*TMF*(1.0-L)*(-1.0+2.0*UA-2.0*UB)+UTF
      EX2=ABS(X2)-2.7183
      C1=COMPAR(EX2,0.)
      C2=C1+(1.0-C1)*D3
      SUE=UE*UE*(1.0-TMF)
      SX2=EX2*EX2*C2*(1.0-TMF)
      EU=COMPAR(ABS(UTF),1.001)
      UE=(ABS(UTF)-1.0)*EU
      ISEX2=INTGRL(0.,SX2)
      ISEU=INTGRL(0.,SUE)
      TEND=INTGRL(TENCO,TMF)
      TIMA=TIME-TF
      TIMER DELT=0.002, FINTIM=14.0
      FINISH TEND=TIMA
TERMINAL

*---- SET APPROPIATE OBJECTIVE FUNCTION, ERR
      ERRTF=TEND-TF-1.0
      E1=ERRTF*ERRTF+ABS(ERRTF)
      ERR=E1+ISEX2+100.0*ISEU

*---- USER SUPPLIED DESIRED PRINT OUT
      WRITE (6,31) X10,X20
      WRITE (6,32) X1,X2
      WRITE (6,33) Z10, Z20
      WRITE (6,45) Y10, Y20
      WRITE (6,44) TF, TPI , TII
      WRITE (6,34) TF,TEND,TA,TB
      WRITE (6,35) ERRTF, ISEU, ISEX2
31    FORMAT (' X10 = ' F10.5, 5X, ' X20 = ' F10.5)
32    FORMAT (' X1 = ' F10.5, 5X, ' X2= ' F10.5)
33    FORMAT (' Z10 = ' F10.5, 5X, ' Z20 = ' F10.5)
34    FORMAT (' TF='F10.5,5X,' TEND='F10.5,5X,' TA='F10.5,5X,'TB='F10.5)
35    FORMAT (' ERRTF = 'F10.5,5X,' ISEU = 'F10.5,5X,' ISEX2 = 'F10.5)
44    FORMAT (/' TF = ' F10.5,10X,' TPI = 'F10.5,10X,' TII = 'F10.5//)
45    FORMAT (' Y10 = 'F10.5, 5X, ' Y20 = 'F10.5)

*---- PATTERN SEARCH ROUTINE
```

Figure (B-1) cont'd.

PROBLEM INPUT STATEMENTS

```
*       ******  A THIRD ORDER SYSTEM  ******
*       ******  T = 0.2  ******
*       ******  TARGET IS    X1=0, X2=0, X3 .LE. 1.0618   ******
INITIAL
        STORAGE S(8), M(8), W(8), T(8)
        FIXED N,M,NB,LIMIT,IS,I,IT,J,K
        CONST X10=4.0, X20=0., X30=0.
        CONST Z10=20.0, Z20=0., Z30=0.
        CONST TF=5.0, Y10=0., Y20=-26.0, Y30=16.0
        TABLE W(4)=0.4, W(5)=0.3
        CONST SL=0.1, RHO=0.5, LAMBDA=0.0001, LIMIT=100, H=1000.0
        CONST K=7
        CONST NB=0, N=1, J=0, IS=1, I=1, IT=0
        NOSORT
        IF (IS .NE. 1) GO TO 61
        DELTA=SL
        W(1)=ALOG(TF)
        W(2)=0.1*Z10
        W(3)=0.1*Z20
        W(6)=0.1*Y20
        W(7)=0.1*Y30
        DO 62 J=1,K
        T(J)=W(J)
        S(J)=DELTA
    62  M(J)=0
    61  CONTINUE
        TF=EXP(W(1))
        Z10=10.0*W(2)
        Z20=10.0*W(3)
        TPI=0.2*SIN(W(4))*SIN(W(4))
        TII=(0.2-TPI)*COS(W(5))*COS(W(5))+TPI
        Y20=10.0*W(6)
        Y30=10.0*W(7)
        CYY=IOR(TPI, K-7)
        SORT
        TENCO =0.2-TF
        SZ=(Z20-7.7*Z30)+(Y20-7.7*Y30)
        UP=INSW(SZ,-1.0,1.0)

DYNAMIC
```

Figure (B-2). s/360 CSMP Program for the Third Order Example.

```
*        ****** THE ADJOINT  ******

         Z1=INTGRL(Z10,DZ1)
         Z2=INTGRL(Z20,DZ2)
         Z3=INTGRL(Z30,DZ3)
         DZ1=-10.0*Z3+(7.7*Z3D-Z2D)*CZ
         DZ2=Z1-17.0*Z3
         DZ3=Z2-8.0*Z3
         CZ=COMPAR(TIME,0.199)
         ZSZ=Z2-7.7*Z3
         Z2D=Z20*(1.0-CZ)+DELAY(50,0.2,Z2)*CZ
         Z3D=Z30*(1.0-CZ)+DELAY(50,0.2,Z3)*CZ
         Y1=INTGRL(Y10,DY1)
         Y2=INTGRL(0.,DY2)
         Y3=INTGRL(0.,DY3)
         DY1=-10.0*Y3+(7.7*Y3D-Y2D)
         DY2=Y1-17.0*Y3
         DY3=Y2-8.0*Y3
         C1Y=COMPAR(TIME,TPI-1.0E-5)
         C2Y=COMPAR(TII+1.0E-5,TIME)
         Y2D=Y20*C1Y*C2Y*(1.0-CZ)+CZ*DELAY(50,0.2,Y2)
         Y3D=Y30*C1Y*C2Y*CYY*(1.0-CZ)+CZ*DELAY(50,0.2,Y3)
         YSY=Y2-7.7*Y3
         SIG=ZSZ+YSY
         TAF=(AND(AA,TIME-1.2))*L
         TBF=(AND(AB,1.2-TIME))*L
         AA=-SIG*UP
         AB=SIG*UP
         L=COMPAR(TF,TIME)
         GB=INTGRL(0.,TBF)
         FA=COMPAR(TF,1.2)
         GA=INTGRL(1.2,TAF)
         TA=(TF-GA)*FA
         TB=TF-GB

*        ****** THE SYSTEM  ******

         X1=INTGRL(X10,V1)
         X2=INTGRL(X20,V2)
         X3=INTGRL(X30,V3)
         V1=X2*CX*TMF
         V2=(X3-X1D+U)*CX
         V3=((-10.0*X1-17.0*X2)*TMF-8.0*X3+7.7*(X1D-U))*CX
         X1D=DELAY(50,0.2,X1)
         CX=COMPAR(TIME,TF-1.0E-5)
         UA=COMPAR(TIME-TF,TA)
         UB=COMPAR(TIME-TF,TB)
         TMF=COMPAR(Q,0.001)
```

Figure (B-2) cont'd.

```
         Q=-X2*(1.0-QQ)+QQ
         QQ=COMPAR(TF+0.3,TIME)
         UTF=(X1D-X3)*(1.0-TMF)
         U=UP*TMF*CX*(-1.0+2.0*UA-2.0*UB)+UTF
         EX3=ABS(X3)-1.0618
         C1=COMPAR(EX3,0.)
         EU=COMPAR(ABS(UTF),1.001)
         UE=(ABS(UTF)-1.0)*EU
         SUE=UE*UE*(1.0-TMF)
         SX3=EX3*EX3*C1*(1.0-TMF)
         C2=COMPAR(ABS(X2),0.001)
         XE2=(ABS(X2))*C2
         SX2=XE2*XE2*(1.0-TMF)
         EX1=COMPAR(ABS(X1),0.001)
         XE1=(ABS(X1))*EX1
         SX1=XE1*XE1*(1.0-TMF)
         ISEU=INTGRL(0.,SUE)
         ISEX1=INTGRL(0.,SX1)
         ISEX2=INTGRL(0.,SX2)
         ISEX3=INTGRL(0.,SX3)
         TEND=INTGRL(TENCO,TMF)
         TIMA=TIME-TF
         FINISH TEND=TIMA
         TIMER DELT=0.01, FINTIM=25.0
TERMINAL
         ERRTF=TEND-TF-0.2
         E1=ERRTF*ERRTF+ABS(ERRTF)
         ERR=E1+500*ISEU+100*(ISEX1+ISEX2+ISEX3)
         WRITE (6,31) X10,X20, X30
         WRITE (6,32) X1,X2, X3
         WRITE (6,45) TF,TPI,TII
         WRITE (6,33) Z10, Z20 , Z30
         WRITE (6,44) Y10, Y20, Y30
         WRITE (6,34) TF,TEND,TA,TB
         WRITE (6,35) ERRTF, ISEU
         WRITE (6,46) ISEX2, ISEX3
 31      FORMAT (' X10 = 'F10.5,5X,' X20 = 'F10.5,5X,'X30 = 'F10.5)
 32      FORMAT (' X1 = ' F10.5, 5X, ' X2= ' F10.5,5X,' X3 = 'F10.5)
 33      FORMAT (' Z10 = 'F10.5,5X,' Z20 = 'F10.5,5X,' Z30 = 'F10.5)
 34      FORMAT (' TF='F10.5,5X,' TEND='F10.5,5X,' TA='F10.5,5X,'TB='F10.5)
 35      FORMAT (' ERRTF = 'F10.5,5X, ' ISEU = 'F10.5)
 44      FORMAT(' Y10 = 'F10.5, 5X, ' Y20 = 'F10.5, 5X, ' Y30 = ' F10.5)
 45      FORMAT (' TF = 'F10.5,5X,' TPI = 'F10.5,5X,' TII = 'F10.5//)
 46      FORMAT (' ISEX2 = 'F10.5,4X,' ISEX3 = 'F10.5)


*---  PATTERN SEARCH ROUTINE
```

Figure (B-2) cont'd.

```
*--- THE PATTERN SEARCH PACKAGE

*---- BUILT-IN OUTPUT

      WRITE (6,36) (W(J),J=1,K)
      WRITE (6,37) (S(J),J=1,K)
      WRITE (6,38) H,ERR
      WRITE (6,39) N,NB, (M(J),J=1,K)
      IF(IT .EQ. 1) GO TO 96
*---- USER CAN SUPPLY DESIRED PRECISION FACTOR HERE FOR
*---- PREMATURE STOP OF RUN
      IF (ERR .LT. 0.01) GO TO 97
      IF (IS .EQ. 1) H=ERR
      IS=0
      IF (N .GT. LIMIT) GO TO 98
      IF (NB .LT. 2) GO TO 1
      NB=1
      I=1
      H=ERR
      GO TO 6
*---- START EXPLORATORY MOVES
1     IF (M(1) .NE. 0) GO TO 2
      HH=H
      GO TO 6
2     IF (H .GT. ERR) GO TO 4
      IF (M(I) .EQ.3) GO TO 3
      S(I)=-S(I)
      W(I)=W(I) + 2.0*S(I)
      M(I)=3
      GO TO 99
3     W(I)=W(I)-S(I)
      M(I)=4
      GO TO 8
4     H=ERR
5     IF (M(I) .EQ. 2 .OR. M(I) .EQ. 3) GO TO 7
6     W(I)=W(I)+S(I)
      M(I)=2
      GO TO 99
7     M(I)=1
8     IF (I .GE. K) GO TO 9
      I=I+1
      GO TO 6
9     IF (NB .GE. 1) GO TO 13
```

Figure (B-3). The s/360 CSMP Pattern Search Routine.

```
      IF (HH .GT. H) GO TO 11
      WRITE (6,41) HH
      GO TO 17
*---- PATTERN MOVE
11    DO 12 J=1,K
      IF (W(J) .GT. T(J) .AND. S(J) .LT. 0. ...
      .OR. W(J) .LE. T(J) .AND. S(J) .GE. 0.) S(J)=-S(J)
      P=T(J)
      T(J)=W(J)
12    W(J)=2.0*W(J) - P
      WRITE (6,42) (T(J),J=1,K)
      HB=H
      NB=2
      I=1
      GO TO 95
13    NB=0
      IF (H .LT. HB) GO TO 15
      WRITE (6,43) HB
      H=HB
      DO 14 J=1,K
      M(J)=0
14    W(J)=T(J)
      I=1
      GO TO 1
15    DO 16 J=1,K
      IF (ABS(W(J)-T(J)) .GT. 0.5*DELTA) GO TO 11
16    CONTINUE
17    IF (DELTA .LT. LAMBDA) GO TO 97
      DELTA = RHO * DELTA
      DO 19 J=1,K
      M(J)=0
      S(J)=RHO*S(J)
19    W(J)=T(J)
      I=1
      GO TO 1
97    WRITE (6,21)N
21    FORMAT (' CONVERGENCE OBTAINED AFTER ' I3, ' EVALUATIONS')
      IT=1
      GO TO 96
98    WRITE (6,23) N
23    FORMAT (' NO CONCLUSION AFTER ' I3, ' EVALUATIONS')
      GO TO 96
95    DO 91 J=1,K
91    M(J)=0
99    N=N+1
      CALL RERUN
```

Figure (B-3) cont'd.

```
96      CONTINUE
36      FORMAT (' W(J) = ' 8F10.5)
37      FORMAT (' S(J) = ' 8F10.6/)
38      FORMAT (' H = ' F10.5, 5X, ' ERR =' F10.5/)
39      FORMAT (' N = 'I4,5X,' NB = 'I4,5X,' M(J) = ' 8I4//)
41      FORMAT (' EXPLORATION MOVE UNSUCCESSFUL,  HH = ' F10.5//)
42      FORMAT (' T(J) = ' 8F10.5)
43      FORMAT (' PATTERN MOVE UNSUCCESSFUL,  HB = ' F10.5//)
        END
        PRTPLT X1(X2,TIMA,U)
        TIMER OUTDEL=0.05
        END
        STOP

PUT VARIABLE SEQUENCE
005 DELTA   TF      WT      WI      TPI     TII     Y10     Y20     TENCO
    D1      Z10     SZ      UP      CYY     A3      Z1D     Z2D     A1
    Z1      A       ZZ0008  Z2      C2Y     C1Y     Y1D     Y2D     Y1N
    Y2N     Y2      L       YSY     ZSZ     SIG     AA      TAF     GA
    TBF     GB      Q       TMF     X1D     UTF     TB      UB      FA
    UA      U       V1      X1      V2      X2      EX2     C1      C2
    ISEX2   EU      UE      SUE     ISEU    TEND    TIMA    ERRTF   E1
    ZZ0024  H       IS      NB      I       H       HH      H       I
    HB      NB      I       NB      H       I       DELTA   I       IT

TPUTS     INPUTS     PARAMS   INTEGS + MEM BLKS   FORTRAN   DATA CDS
(500)   171(1400)   23(400)   11+   5= 16(300)   183(600)      15

      ENDJOB
```

Figure (B-3) cont'd.

APPENDIX C

STABILITY

The stability of a system with time delay can be determined by means of graphical methods. However, to do a point by point plotting manually can be rather tedious and time consuming. Use of the digital computer and CALCOMP Pen Plotter offers an easier approach to this problem. In the case of the dynamical system with numerator dynamics and a time delay in the feedback path, the method of Huang and Li[1] can be used to determine the stability of the system Let

$$s = p + j\,\omega \qquad \text{C-1}$$

and τ be the time delay. Using p as a parameter, the solution to Eq. (III-43) can be found from the graphs of cot $(\omega\tau)$ and $F(\omega, p)$ which are plotted against ω. Figure (C-1) shows the FORTRAN program used in computing and calling the CALCOMP subroutines[2] to plot these functions.

---

[1] Huang and Li, loc. cit.

[2] Programming CALCOMP Pen Plotters, (Anaheim: California Computer Products Inc., 1968) 24pp.

The system used here is the second order example. The intersection points of the function $F(\omega, p)$ and $\cot(\omega\tau)$ can then be used as the input data for the second computer program given in Figure (C-2). From this second program, the root locus and gain curve (Eq. (III-44)) of the system can be obtained. From the gain curve of the system, with K = 1, the characteristic values of p, and hence the stability and domain of G controllability can be found. Figures (C-3), (C-4) and (C-5) demonstrate the graphical technique that can be used to determine the stability of the system (V-19).

```
      C
      C
      C
      C==== TO FIND THE CHARACTERISTIC ROOTS OF A SYSTEM WITH OPEN-LOOP
      C==== TRANSFER FUNCTION H(S)=N(S)/D(S) BY MEANS OF A GRAPHICAL METHOD.
      C==== DEVELOPED BY HUANG AND LI
            DIMENSION IBUF(1000),X(500),Y(500),Z(500),V(500)
      C==== INITIALIZING THE PLOTTER
            CALL PLOTS(IBUF,1000,6)
      C==== SETTING OF THE ORIGIN
            CALL PLOT(0.,0.,-3)
            REAL IMD,IMN
            COMPLEX S,D,AN
            P=-25.0
            M=0
      C==== CALCULATE AND PLOT COT(W)
            W=0.
            I=0
            PI=3.142
          4 W=W+0.2
            I=I+1
            FCOT=COS(W)/SIN(W)
            AFC=ABS(FCOT)
            IF(AFC .GT. 6.0) FCOT=6.0*FCOT/AFC
            X(I)=W
            Y(I)=FCOT
            WW=PI-W
            IF (WW .LE. 0.17) GO TO 3
            WRITE(6,101) FCOT,W,I
        101 FORMAT (' FCOT = ' F10.4,5X,' W= 'F6.3,5X,' I = ' I3)
            GO TO 4
          3 W=PI+0.01
            PI=PI+3.142
            IF (W .LT. 20.0) GO TO 4
      C==== SCALE THE RANGE OF DATA
            CALL SCALE (X,10.0, I,1)
            CALL SCALE (Y,10.0, I,1)
      C==== THE X-AXIS
            CALL AXIS(0.,10.0,6HW-AXIS,+6,10.0,0.,X(I+1),X(I+2))
      C==== THE Y-AXIS
            CALL AXIS(-0.5,0.,6HF-AXIS,+6,10.0,90.0,Y(I+1),Y(I+2))
            CALL LINE(X,Y,I,1,0,0)
      C==== CALCULATE AND PLOT F(P,W)
         11 P=P+5.0
         13 W=0.
            J=0
```

Figure (C-1). Computer Program for plotting $F(\omega, p)$ and Cot $(\omega\tau)$ versus ω.

```
C
C
C
   12 W=W+0.2
      J=J+1
      S=CMPLX(P,W)
C====  H(S) = (S+1)/S(S+2)
C====   D = THE DENOMINATOR POLYNOMIAL IN H(S)
C====  AN = THE NUMERATOR POLYNOMIAL IN H(S)
      AN=S+CMPLX(1.0,0.)
      D=S*S+2.0*S
      RED=REAL(D)
```

Figure (C-1) cont'd.

```
C
C
C
      IMD=AIMAG(D)
      REN=REAL(AN)
      IMN=AIMAG(AN)
      FPW=(IMD*IMN+RED*REN)/(RED*IMN-IMD*REN)
      AF=ABS(FPW)
      IF(AF.GT.6.0) FPW=6.0*FPW/AF
      WRITE (6,102) FPW,P,W,J
  102 FORMAT (' FPW = ' F10.4, 5X, ' P= 'F10.4, 5X, ' W = ' F10.4,5X,
     1' J = ' I3)
      V(J)=W
      Z(J)=FPW
      IF (W .LE. 20.0) GO TO 12
      V(J+1)=X(I+1)
      V(J+2)=X(I+2)
      Z(J+1)=Y(I+1)
      Z(J+2)=Y(I+2)
C==== PLOT F(W,P) VS. W USING P AS A PARAMETER
      CALL LINE(V,Z,J,1,0,0)
      IF (M .NE. 1) GO TO 14
      P=P+0.5
      IF (P .EQ. 0.) P=0.5
      IF (P .EQ. 5.0) GO TO 15
      GO TO 13
   14 IF (P .LT. 20.0) GO TO 11
      M=1
      P=-4.5
      GO TO 13
C==== ENDING THE PLOTTING PROCEDURE
   15 CALL PLOT (12.0,0.,999)
      CALL EXIT
      END
```

Figure (C-1) cont'd.

```
      C
      C
      C
      C==== TO PLOT THE ROOT LOCUS AND GAIN-CURVE FOR THE SYSTEM
      C==== (S+1)/S(S+2)
            DIMENSION P(15), W(4,15),X(20),Y(20),Z(20),IBUF(1000)
            COMPLEX D,AN,S
      C====  S = P + JW
            REAL K
            REAL IMN,IMD
            N=15
            M=3
      C===  THE VALUES OF P AND W (OMEGA) ARE OBTAINED FROM THE COT(W) AN
      C===  F(P,W) PLOT
            READ (5,101) (P(I), I=1,N)
            READ (5,1) ((W(I,J),J=1,N),I=1,M)
          1 FORMAT (15F5.2)
        101 FORMAT (15F5.2)
      C===  INITIALIZE PLOT ROUTINE
            CALL PLOTS(IBUF,1000)
            CALL PLOT(0.,0.,-3)
            I=0
            N1=M+1
          2 N1=N1-1
          3 I=I+1
            X(I)=P(I)
            Y(I)=W(N1,I)
            A=EXP(X(I))
            S=CMPLX(X(I),Y(I))
      C===  AN = NUMERATOR PLOYNOMIAL
      C          = S+1
            AN=S+CMPLX(1.0,0.)
      C===  D = DENMINATOR POLYNOMIAL
      C         = S(S+2)
            D=S*S+2.0*S
            RED=REAL(D)
            IMD=AIMAG(D)
            REN=REAL(AN)
            IMN=AIMAG(AN)
            RS=REN*COS(Y(I))+IMN*SIN(Y(I))
            IF (ABS(RS) .LE. 1.0E-4) GO TO 7
            R=RED*A/RS
            GO TO 8
          7 R=10.0
          8 K=ABS(R)
            IF (K .GT. 20.0) K=20.0
```

Figure (C-2). Computer Program for plotting the root locus and gain curve.

```
C
C
C
      Z(I)=K
      WRITE (6,4) X(I),Y(I),Z(I)
    4 FORMAT (' P = 'F10.4,5X, ' W = ' F10.4,5X,' K = ' F10.4)
      IF (I .LT. N) GO TO 3
      IF (N1 .LT. M) GO TO 5
      J=I+1
      Y(J)=W(1,N)
      CALL SCALE(Y,6.0,J,1)
      Y(I+1)=Y(J+1)
```

Figure (C-2) cont'd.

```
C
C
C
      Y(I+2)=Y(J+2)
      CALL SCALE(X,6.0,I,1)
      CALL SCALE(Z,6.0,I,1)
      P1=X(I+1)
      P2=X(I+2)
      Q1=Y(I+1)
      Q2=Y(I+2)
      S1=Z(I+1)
      S2=Z(I+2)
      CALL AXIS(0.,0.,6HP-AXIS,-6,8.0,0.,X(I+1),X(I+2))
      CALL AXIS(0.,0.,6HW-AXIS,6,6.0,90.0,Y(I+1),Y(I+2))
    5 X(I+1)=P1
      X(I+2)=P2
      Y(I+1)=Q1
      Y(I+2)=Q2
      CALL LINE(X,Y,I,1,1,2)
      CALL PLOT(12.0,0.,-3)
      IF (N1 .LT. M) GO TO 6
      CALL AXIS(0.,0.,6HP-AXIS,-6,8.0,0.,X(I+1),X(I+2))
      CALL AXIS(0.,0.,6HK-AXIS,6,6.0,90.0,Z(I+1),Z(I+2))
    6 Z(I+1)=S1
      Z(I+2)=S2
      CALL LINE(X,Z,I,1,1,2)
      CALL PLOT(-12.0,0.,-3)
      I=0
      IF (N1 .GT. 1) GO TO 2
      CALL PLOT(24.0,0.,999)
      CALL EXIT
      END
```

Figure (C-2) cont'd.

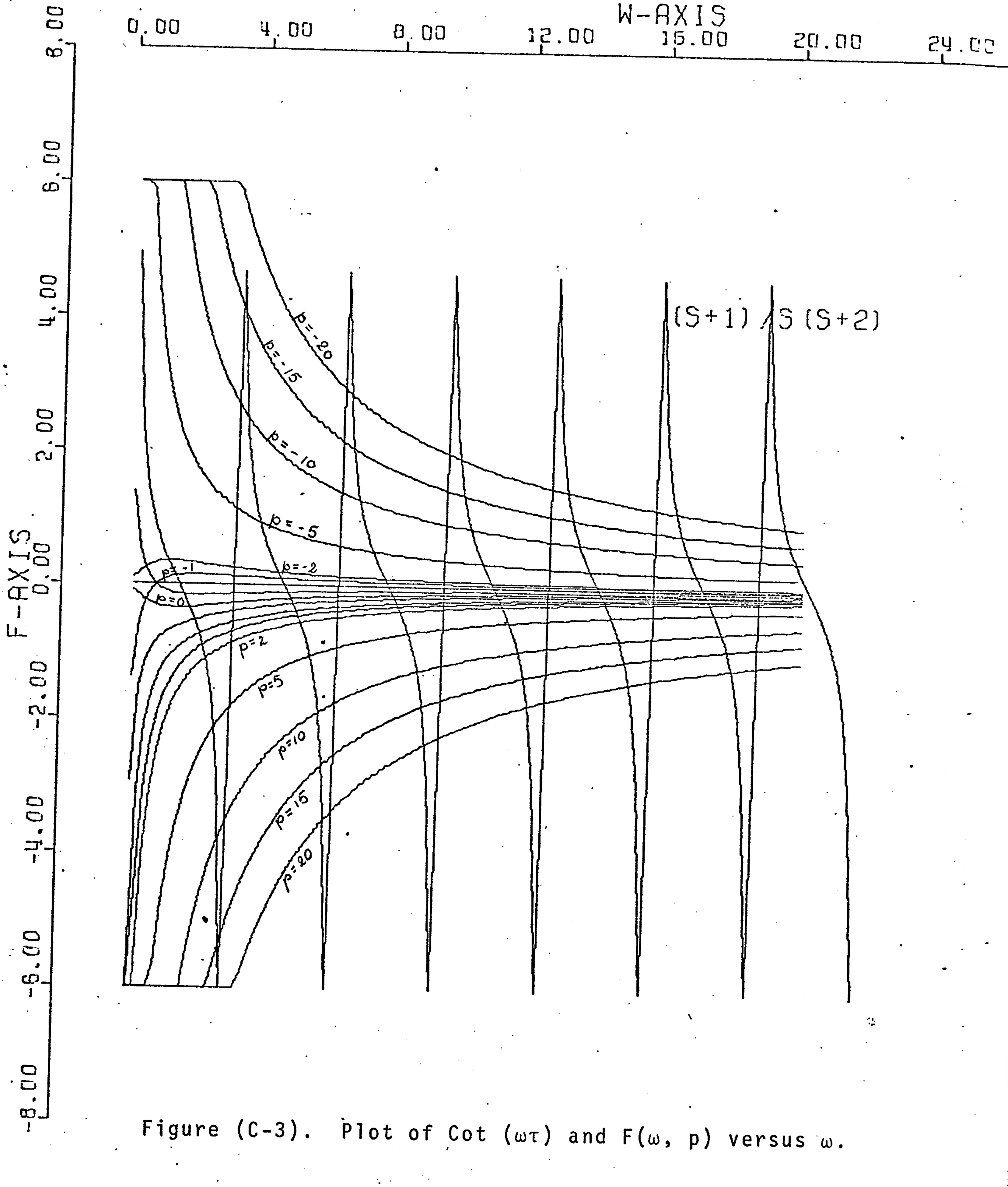

Figure (C-3). Plot of Cot $(\omega\tau)$ and $F(\omega, p)$ versus ω.

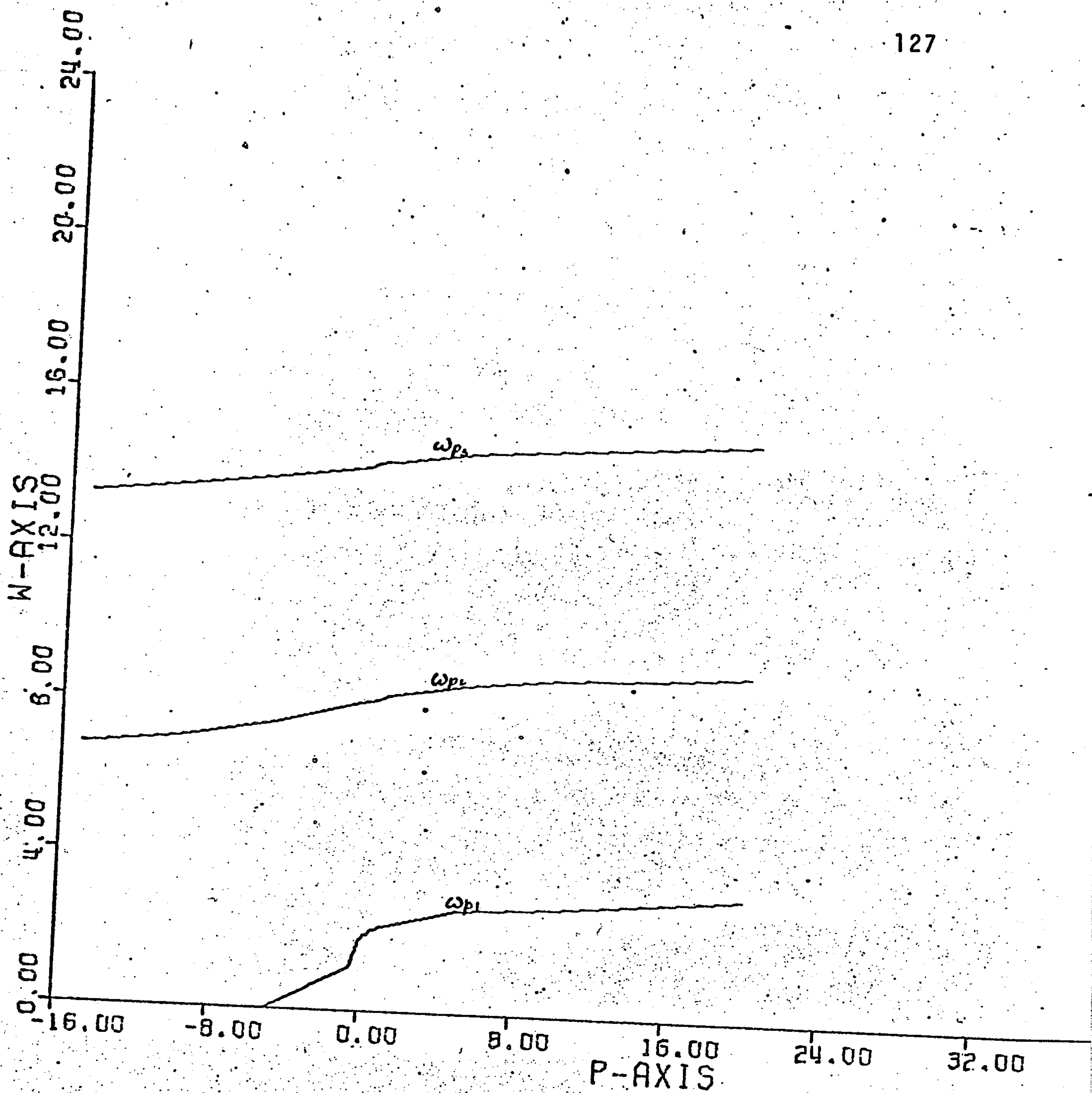

Figure (C-4). Root Locus of the system (V-19).

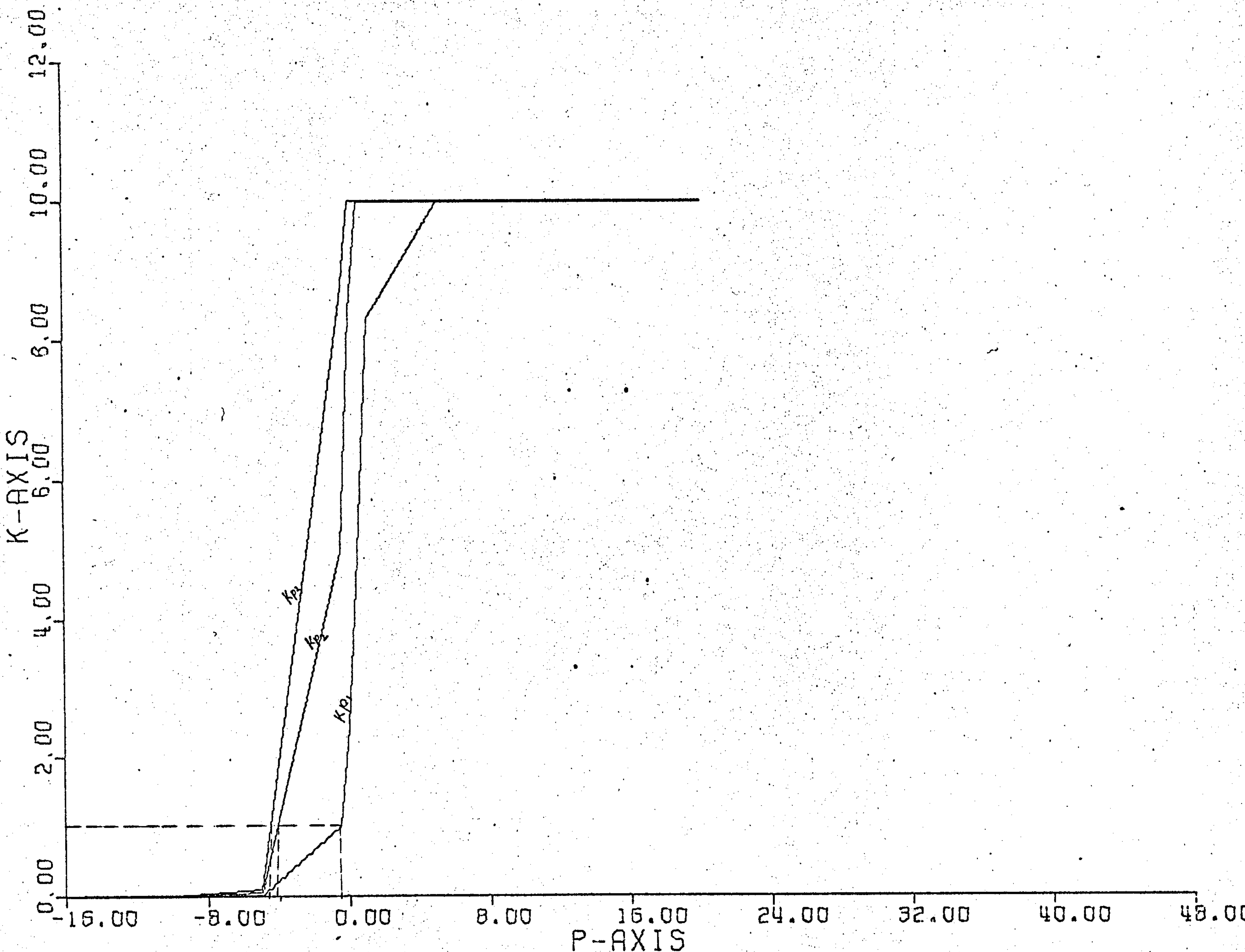

Figure (C-5). Gain Curve of the System (V-19).