

Order Detection and Left-orderability of 3-Manifold Groups

by

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Abstract

The theory of left-orderable groups has deep connections with low-dimensional topology, most notably through the L-space conjecture. This thesis investigates a general framework for order detection, a technique originally introduced by Boyer and Clay as part of their contribution to the proof of the L-space conjecture for graph manifolds, and develops a broader theoretical foundation for its applications.

The theoretical core of this work generalises the framework of order detection to 3-manifolds with multiple torus boundary components and establishes the equivalence between order detection and representation detection. This thesis also develops gluing theorems and applies cofinality to analyse the left-orderability of the resulting manifolds under manifold gluings and Dehn fillings.

As applications, this thesis first studies Dehn surgeries on cable knots in S^3 . Using the developed framework, we obtain explicit conditions under which the resulting manifolds have non-left-orderable fundamental groups. These algebraic results parallel known Heegaard Floer characterisations of L-space surgeries on cable knots, thereby supplying further evidence for the L-space conjecture from the left-orderability perspective. Subsequently, this thesis applies order detection techniques to an infinite family of hyperbolic two-component links, determining explicit slope criteria that produce manifolds with non-left-orderable fundamental groups after Dehn surgery. In the case of the Whitehead link, the classification of non-left-orderable fillings agrees precisely with the predictions of the L-space conjecture. These results highlight the practical applicability of the order detection framework.

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1

Introduction

The theory of left-orderable groups is a vibrant and evolving area of modern algebra, forming a natural bridge between abstract group theory and the broader study of ordered algebraic structures. Emerging in the early 20th century through the foundational works of Dedekind, Hölder, and Hilbert, this field has experienced significant growth, particularly in recent years. A key driver of this progress has been the discovery of new and diverse examples of left-orderable groups arising from geometry, including braid groups, right-angled Artin groups, and, notably, the fundamental groups of closed hyperbolic 3-manifolds. In particular, the fundamental groups of all closed surfaces are left-orderable, except for the sphere and the projective plane.

A particularly intriguing connection between left-orderable groups and topology is encapsulated in the L-space conjecture, which links three seemingly distinct areas: left-orderability of fundamental groups, the existence of taut foliations, and Heegaard Floer homology. This conjecture, which has gained considerable attention in recent years, can be formally stated as follows:

Conjecture (L-space Conjecture). *Let M be an irreducible rational homology 3-sphere. Then the following conditions are equivalent:*

- (1) M is not an L-space;
- (2) $\pi_1(M)$ is left-orderable;
- (3) M admits a co-orientable taut foliation.

The equivalence between conditions (1) and (2) was explicitly conjectured by Boyer, Gordon, and Watson in [8], which also provided an in-depth examination of this relationship. The link between (1) and (3) originated as a question posed by Ozsváth and

Szabó, following their proof that (3) implies (1) [61, 3, 47]. This was later elevated to a conjecture in [45] by Juhász.

Beyond its immediate implications for Heegaard Floer homology, the L-space conjecture provides a powerful algebraic perspective on the geometric and topological properties of 3-manifolds. If it were true, it would offer a fundamental criterion for distinguishing between 3-manifolds that support taut foliations and those that do not, with deep consequences for understanding low-dimensional topology. Recent studies have provided strong evidence for this conjecture, particularly through the examination of Dehn surgeries on knot complements, where left-orderability of the fundamental group frequently correlates with the nontriviality of Heegaard Floer homology. The conjecture has been proven true for significant classes of 3-manifolds, including Seifert manifolds [9] and all graph manifolds [4, 69, 34]. Furthermore, it has been verified for certain non-Seifert spaces, such as many branched covers of knots in the 3-sphere [29] and specific families of Dehn surgeries on fixed manifolds [17]. Computational methods have also confirmed the conjecture for at least 191,089 hyperbolic rational homology spheres, as demonstrated by Dunfield in [23].

An essential tool for investigating the L-space conjecture for 3-manifolds with JSJ decompositions is the concept of detected slopes, introduced in [4] and developed further in [68, 35, 5, 6, 7]. Slope detection records how left-orderings, taut foliations, and Heegaard Floer homology behave relative to the incompressible torus boundary components of a compact, connected, orientable, irreducible 3-manifold, and so slope detection naturally comes in three flavours: order detection, foliation detection, and NLS detection (here, NLS stands for non-L-space). Additionally, representation detection describes the action of the fundamental group of a 3-manifold on the real line via representations into $\text{Homeo}_+(\mathbb{R})$, providing another bridge between algebraic and topological properties.

Each kind of detection comes in two flavours — regular detection (which we will simply call “detection”, only including the word “regular” when needed for emphasis) and strong detection. Detection encodes the boundary information needed to establish left-orderability of the fundamental group of the manifold W obtained by gluing together two 3-manifolds along incompressible torus boundary components, and whether W admits a co-orientable taut foliation or is not a Heegaard Floer L-space. Strong detection is a kind of detection adapted to deal with manifolds that arise from Dehn filling a boundary component and to similarly analyse the structures supported by the resulting manifolds. This thesis extends and generalises the framework of order detection for 3-manifolds with multiple boundary components.

The outline of this thesis is as follows. Chapter 2 develops the algebraic and dynamical foundation of left-orderable groups, establishing the key structural results upon which later chapters build. It introduces the concept of a left-ordering, describes it via positive cones, and studies the topology of the space of left-orderings. The Burns-Hale theorem (Theorem 2.2.2) and the Bludov-Glass theorem (Theorem 2.2.3) provide powerful characterizations of left-orderability, while the dynamic realisation theorem (Theorem 2.3.1) relates algebraic orderings to faithful actions on the real line. The chapter concludes with a study of convex subgroups and cofinal elements, setting the stage for their use in the definition of order detection.

Chapter 3 presents the geometric and homological background motivating the thesis and reviews the L-space conjecture. Foundational results, including Theorem 3.1.3, Theorem 3.1.4 and the Boyer-Rolfsen-Wiest Theorem (Theorem 3.2.4), connect these properties to Heegaard Floer homology and 3-manifold group theory. This chapter also discusses the Prime and JSJ Decomposition Theorems, illustrating how the study of left-orderability can be reduced to individual JSJ pieces. This motivates the development of order detection as a method for tracking orderability through gluings and Dehn fillings. All the materials discussed in Chapter 2 and 3 are well known among researchers in this field.

Chapter 4 forms the theoretical core of the thesis. It introduces the generalised framework of order detection (Definition 4.2.1), which captures how left-orderings on boundary components of 3-manifolds interact under gluings and Dehn fillings. A corresponding concept, representation detection (Definition 4.3.2), is shown to be equivalent to order detection (Theorem 4.4.2). The chapter also establishes the connection between cofinal elements, order detectability, and dynamic realisations in Section 4.5. The gluing theorems in Section 4.6 provide conditions ensuring the left-orderability of the fundamental group of the manifold resulted from gluing or Dehn filling, while the notion of JN-realisability discussed in Section 4.7 serves as a way of computing which tuples of slopes on the boundary of a Seifert fibered manifold are detected. Together, these results generalise the order detection framework to manifolds with multiple torus boundaries and form the theoretical foundation for the applications that follow.

Chapter 5 studies the (non-)left-orderability of the fundamental group of the resulting manifold after performing Dehn surgeries on cable knots under the framework of order detection. After establishing notations and base change conventions, it develops special cases of the gluing theorems adapted to cable spaces in Section 5.3 and introduces methods for computing relatively JN-realisable slopes in Section 5.4. The main application (Corollary 5.7.7) demonstrates that certain Dehn surgeries on (p, q) -cable

knots in S^3 yield manifolds with non-left-orderable fundamental groups. These results parallel known characterizations of L-space surgeries on cable knots, providing new evidence for the L-space conjecture from the perspective of left-orderability.

Chapter 6 extends the use of order detection to an infinite family of hyperbolic two-component links, which includes the Whitehead link as a special case. Using the order detection framework developed earlier, it determines slope conditions under which Dehn surgeries on these links yield manifolds whose fundamental groups are not left-orderable. The principal result, Theorem 6.1.7, gives explicit non-left-orderability criteria for these surgeries, while Theorem 6.2.7 aligns precisely with the non-left-orderability result that is predicted by the L-space conjecture and previously known results. These applications in Chapter 5 and Chapter 6 demonstrate the power and versatility of order detection as a tool for exploring open questions in the L-space conjecture and the order-theoretic dynamics of 3-manifold groups.

1.1 Contributions of authors

This thesis is presented in the grouped manuscript format and is based on two published manuscripts [15, 14] by the author, both coauthored with Adam Clay. This small section is to specify the nature and extent of the author's contribution to these two manuscripts.

The first paper [15] originated from an unpublished draft by Adam Clay and Steven Boyer concerning cable knots. At the time, the notion of order detection in the presence of multiple boundary components had not been fully articulated. The author proposed a practical definition of order detection for such cases and established its connection to representation detection, as detailed in Chapter 4. Under this new framework, the original draft was substantially extended, revised, and corrected by the author, resulting in the finalized version of the paper. The main results appear in Chapter 4 and 5.

The second paper [14] began with a suggestion by Adam Clay to study the non-left-orderability of Dehn surgeries on the Whitehead link. Building upon this idea, the author broadened the scope of investigation to include a wider family of two-component links. The initial calculations made by Clay for the Whitehead link were generalised to this family, and several structural observations were made in the process. All these developments together led to the completion of [14], whose key results are presented in Chapter 6.

2

Left-orderable Groups

This chapter lays the algebraic and dynamic groundwork for the study of left-orderable groups. It introduces left-orderings and their description via positive cones, explains how these orderings behave under group operations, and describes the topology of the space of all left-orderings. This chapter also develops the notions of convex subgroups, cofinal elements, and conjugate orderings, establishing the structural and dynamic framework that underlies the theory of order detection in later chapters.

2.1 Left-orderability

Definition 2.1.1. A *left-ordering* $\circ$ on a non-trivial group G is a total order $\leq_\circ$ on G that is invariant under left multiplication, meaning that for all $a, b, c \in G$, if $a \leq_\circ b$, then $ca \leq_\circ cb$. A group that admits a left-ordering is called *left-orderable*, and a group equipped with a specific left-ordering is referred to as *left-ordered*.

We adopt the convention that the trivial group is not left-orderable. A fundamental example of a left-orderable group is the additive group of integers $\mathbb{Z}$, which admits precisely two left-orderings: the standard order $\leq$ and its opposite order. More generally, any subgroup of the additive real numbers $\mathbb{R}$ inherits a natural left-ordering.

An important example is the uncountable group $\text{Homeo}_+(\mathbb{R})$ of all order-preserving homeomorphisms of the real line. To construct a left-ordering on $\text{Homeo}_+(\mathbb{R})$, pick a countable dense subset $\{x_0, x_1, x_2, \dots\}$ of $\mathbb{R}$. For two distinct maps $f, g \in \text{Homeo}_+(\mathbb{R})$, we declare $f <_\circ g$ if and only if there exists an index j such that $f(x_j) < g(x_j)$ and $f(x_i) = g(x_i)$ for all $i = 0, 1, \dots, j-1$. It is easy to verify that this defines a left-ordering $\circ$ on $\text{Homeo}_+(\mathbb{R})$.

For the remainder of this chapter, let G be a non-trivial group. Each left-ordering $\circ$

on G naturally induces an associated *positive cone*, denoted by $P(\circ)$, defined as

$$P(\circ) = \{g \in G \mid g >_{\circ} \text{id}\}.$$

The set $P(\circ)$ satisfies the following properties:

- (1) $P(\circ) \cdot P(\circ) \subset P(\circ)$,
- (2) $G = \{\text{id}\} \sqcup P(\circ) \sqcup P(\circ)^{-1}$.

Conversely, a subset $P \subset G$ satisfying these two conditions uniquely determines a left-ordering $\circ$ via the prescription

$$a <_{\circ} b \quad \text{if and only if} \quad a^{-1}b \in P.$$

Given a left-ordering $\circ$, we call elements of $P(\circ)$ *positive* and elements of $P(\circ)^{-1}$ *negative*. The *opposite left-ordering* $\circ^{\text{op}}$ is defined by reversing the positive cone, i.e., $P(\circ^{\text{op}}) = P(\circ)^{-1}$, and so $f \leq_{\circ^{\text{op}}} g$ if and only if $g \leq_{\circ} f$.

It is also clear that every non-trivial subgroup of a left-orderable group is itself left-orderable. More generally, if there is a monomorphism $\psi : H \rightarrow G$, then a left-ordering $\circ$ on G induces a left-ordering $\psi^{-1}(\circ)$ on H , whose positive cone is given by

$$P(\psi^{-1}(\circ)) = \psi^{-1}(P(\circ)).$$

We call $\psi^{-1}(\circ)$ the *pullback left-ordering*.

Lemma 2.1.2 (see e.g. [16]). *Let G be a left-orderable group. Then the following are true:*

- (1) *The group G is torsion-free.*
- (2) *If $g, h \in G$ are positive, then gh is also positive.*

Proof. To see part (1), suppose $g \in G$ is a non-trivial torsion element, meaning there exists a positive integer n such that $g^n = \text{id}$. If $\text{id} <_{\circ} g$, then applying the left-ordering property iteratively yields

$$\text{id} <_{\circ} g <_{\circ} \cdots <_{\circ} g^n = \text{id},$$

which is a contradiction. A similar argument applies if $g <_{\circ} \text{id}$, proving that G must be torsion-free.

To see part (b), assume $g >_{\circ} \text{id}$ and $h >_{\circ} \text{id}$. Since $h >_{\circ} \text{id}$, left-multiplication by g preserves the order, giving $gh >_{\circ} g$. Combining this with $g >_{\circ} \text{id}$, we conclude that $gh >_{\circ} \text{id}$, proving the result. $\square$

Given an indexed family of left-orderable groups $\{G_i \mid i \in I\}$, we can define a left-ordering on the direct product $\prod_{i \in I} G_i$ using the *lexicographic ordering*. Let $\prec$ be a well-ordering on I , and let each G_i be left-ordered by σ_i . We define a left-ordering σ on $\prod_{i \in I} G_i$ by setting $(g_i)_{i \in I} <_\sigma (h_i)_{i \in I}$ if and only if there exists $i_0 \in I$ such that $g_i = h_i$ for all $i \prec i_0$ and $g_{i_0} <_{\sigma_{i_0}} h_{i_0}$. It is straightforward to verify that this defines a left-ordering.

Another method of constructing left-orderable groups is via free products. Let G, H be two groups. The free product $G * H$ is the group whose elements are freely generated from those of G and H , subject to the relations given by the group operations in each of these groups.

Theorem 2.1.3 (Vinogradov [75]). *Let G, H be non-trivial groups. Then $G * H$ is left-orderable if and only if G and H are left-orderable.*

In particular, free groups of finite rank are left-orderable. And more generally, left-orderability can be extended through group extensions.

Proposition 2.1.4 ([50, Proposition 1.2.6]). *Let*

$$\{\text{id}\} \longrightarrow N \xrightarrow{i} G \xrightarrow{p} H \longrightarrow \{\text{id}\}$$

be a short exact sequence of groups. If both N and H are left-orderable, then G is left-orderable.

Proof. Let σ_N and σ_H be left-orderings on N and H , respectively. Define a subset of G by

$$P = i(P(\sigma_N)) \cup p^{-1}(P(\sigma_H)).$$

It is straightforward to verify that P satisfies the conditions of a positive cone and so P induces a left-ordering σ_{ext} on G . Explicitly, an element $f \in G$ is σ_{ext} -positive if either $f \in i(P(\sigma_N))$ or $p(f) \in P(\sigma_H)$. $\square$

The left-ordering σ_{ext} constructed above is commonly referred to as the *left-ordering by extension*. Occasionally, we also call σ_{ext} the lexicographic ordering by the short exact sequence.

2.2 Space of left-orderings

By identifying each left-ordering with its corresponding positive cone, it is natural to consider topologizing the set of all left-orderings on a group. This idea was first formalised by Sikora [73], with much of the foundational work focusing on the case of countable groups.

Let G be a left-orderable group. The *space of left-orderings* on G , denoted by $\text{LO}(G)$, is the set of all left-orderings on G , formally given by

$$\text{LO}(G) = \{\circ \mid \circ \text{ is a left-ordering on } G\} = \{P \subset G \mid P \text{ is a positive cone}\}.$$

Thus, we may view $\text{LO}(G)$ as a subset of the power set $\{0, 1\}^G$. This identification allows us to equip $\text{LO}(G)$ with the subspace topology inherited from the product topology on $\{0, 1\}^G$, where $\{0, 1\}$ is given the discrete topology.

For each $x \in G$, define the sets

$$U_x = \{A \subset G \mid x \in A\}, \quad U_x^c = \{A \subset G \mid x \notin A\}.$$

These sets form a basis for the product topology on $\{0, 1\}^G$, which is compact, Hausdorff, and totally disconnected. If G is countable, then $\{0, 1\}^G$ is homeomorphic to the Cantor set and is thus metrisable.

Proposition 2.2.1 ([73, 19, 20]). *Let G be a countable left-orderable group. Then $\text{LO}(G)$ is a closed subset of $\{0, 1\}^G$. Consequently, $\text{LO}(G)$ is compact, Hausdorff, and totally disconnected.*

Proof. We express $\text{LO}(G)$ as the intersection

$$\text{LO}(G) = S \cap C,$$

where

$$S = \{A \subset G \mid A \cdot A \subset A\}, \quad C = \{A \subset G \mid G = \{\text{id}\} \sqcup A \sqcup A^{-1}\}.$$

It suffices to show that both S and C are closed in $\{0, 1\}^G$.

Consider the complement of S ,

$$\begin{aligned} \{0, 1\}^G \setminus S &= \{A \subset G \mid \exists g, h \in A \text{ such that } gh \notin A\} \\ &= \bigcup_{g, h \in G} (U_g \cap U_h \cap U_{gh}^c). \end{aligned}$$

Since this is a union of open sets, S is closed.

Similarly, the complement of C is given by

$$\begin{aligned} \{0, 1\}^G \setminus C &= \{A \subset G \mid G \neq \{\text{id}\} \sqcup A \sqcup A^{-1}\} \\ &= \{A \subset G \mid \text{there is some } g \in G \text{ such that } \{g, g^{-1}\} \in A\} \\ &\quad \cup \{A \subset G \mid \text{there is some } g \in G \setminus \{\text{id}\} \text{ such that } \{g, g^{-1}\} \cap A = \emptyset\} \\ &= \bigcup_{g \in G} (U_g \cap U_{g^{-1}}) \cup \bigcup_{g \in G \setminus \{\text{id}\}} (U_g^c \cap U_{g^{-1}}^c). \end{aligned}$$

Since this is again a union of open sets, C is closed. Thus, $\text{LO}(G)$ is a closed subset of $\{0,1\}^G$, implying that it inherits compactness, Hausdorffness, and total disconnectedness. $\square$

The topology on $\text{LO}(G)$ is called the *Sikora topology*. A sequence of subsets $\{S_n\}$ of G converges to $S \subset G$ in $\{0,1\}^G$ if, for every finite subset $F \subset G$, there exists $n \in \mathbb{N}$ such that

$$S_m \cap F = S \cap F \quad \text{for all } m \geq n.$$

This notion allows us to discuss the convergence of left-orderings. In particular, if G is a countable left-orderable group, then every sequence of left-orderings has a convergent subsequence due to the compactness of $\text{LO}(G)$.

A fundamental consequence of this topological structure is the following characterization of left-orderable groups by Burns and Hale.

Theorem 2.2.2 (Burns-Hale [10]). *Let G be a non-trivial group. Then G is left-orderable if and only if for each non-trivial finitely generated subgroup $H \leq G$, there exists a left-orderable group L and a surjective homomorphism $\varphi : H \rightarrow L$.*

From now on, we assume all groups G in this manuscript are finitely generated and hence countable. Typically, G will be the fundamental group of a compact 3-manifold.

The group G acts naturally on its space of left-orderings by conjugation. For each $g \in G$ and $\sigma \in \text{LO}(G)$, define a new left-ordering $g \cdot \sigma$ by setting

$$P(g \cdot \sigma) = gP(\sigma)g^{-1}.$$

In terms of the order relation, for all $f, h \in G$,

$$h \leq_{g \cdot \sigma} f \quad \text{if and only if} \quad hg \leq_{\sigma} fg.$$

Thus, G acts on $\text{LO}(G)$ from the left.

A subset $N \subset \text{LO}(G)$ is called a *normal family* if it is invariant under this action, i.e., for all $g \in G$ and $\sigma \in N$, we have $g \cdot \sigma \in N$.

A fundamental criterion for deciding the left-orderability of amalgamated free products is due to Bludov and Glass. A more general statement holds for fundamental groups of graphs of groups (due to Chiswell [12]), but we present the special case for two factors.

Theorem 2.2.3 (Bludov-Glass [2]). *Suppose A, G, H are groups, and let $\psi_1 : A \rightarrow G$ and $\psi_2 : A \rightarrow H$ be monomorphisms. The free product with amalgamation $G *_A H$ is left-orderable*

if and only if there exist non-empty normal families $N_1 \subset \text{LO}(G)$ and $N_2 \subset \text{LO}(H)$ such that

$$\forall P \in N_i, \exists Q \in N_j \text{ such that } \psi_i^{-1}(P) = \psi_j^{-1}(Q), \quad (*)$$

for $\{i, j\} = \{1, 2\}$.

With the same setup as in the above theorem, normal families that satisfy the condition $(*)$ for $\{i, j\} = \{1, 2\}$ are called *compatible* with the maps ψ_i .

2.3 Dynamic realisation

For a countable left-orderable group, a left-ordering induces a natural action on the real line $\mathbb{R}$ via the dynamic realisation, which we summarize below.

Let $\circ$ be a left-ordering on a countable group G . A *gap* in G is a pair of elements $\{g, h\} \subset G$ with $g <_\circ h$ such that there is no $f \in G$ satisfying $g <_\circ f <_\circ h$. Gaps exist if and only if the positive cone $P(\circ)$ contains a least element.

An order-preserving embedding $t : G \rightarrow \mathbb{R}$ is called *tight* if for every non-trivial open interval $(a, b) \subset \mathbb{R} \setminus t(G)$, there exists a gap $g <_\circ h$ such that $(a, b) \subset (t(g), t(h))$.

Every countable left-orderable group G admits a tight embedding into $\mathbb{R}$. To construct such an embedding, let $\{g_0 = \text{id}, g_1, g_2, \dots\}$ be an enumeration of G . Define $t : G \rightarrow \mathbb{R}$ recursively as follows. Firstly, set $t(g_0) = 0$. If $t(g_0), t(g_1), \dots, t(g_i)$ have already been defined, then we set

$$t(g_{i+1}) = \begin{cases} \max\{t(g_0), t(g_1), \dots, t(g_i)\} + 1, & \text{if } g_{i+1} >_\circ \max\{g_0, g_1, \dots, g_i\}, \\ \min\{t(g_0), t(g_1), \dots, t(g_i)\} - 1, & \text{if } g_{i+1} <_\circ \min\{g_0, g_1, \dots, g_i\}; \end{cases}$$

otherwise, we set $t(g_{i+1}) = \frac{t(g_j) + t(g_k)}{2}$, if $g_j <_\circ g_{i+1} <_\circ g_k$ and $j, k \in \{0, 1, \dots, i\}$ are chosen so that there is no $l \in \{0, 1, \dots, i\}$ such that $g_j <_\circ g_l <_\circ g_k$.

To verify that t is a tight embedding, suppose there exists an open interval $(a, b) \subset \mathbb{R} \setminus t(G)$. Define

$$L = t(G) \cap (-\infty, a), \quad U = t(G) \cap (b, \infty).$$

Since both L and U are nonempty, set

$$\varepsilon = \inf\{t(h) - t(k) \mid k \in L, h \in U\} \geq b - a > 0.$$

Consider the smallest $n \geq 1$ such that there exist indices $i, j \in \{0, 1, \dots, n\}$ with $g_i \in L$

and $g_j \in U$ satisfying $t(g_j) - t(g_i) < 2\varepsilon$. Assume further that the indices i, j satisfy

$$t(g_i) = \sup(L \cap \{t(g_0), t(g_1), \dots, t(g_n)\}), \quad t(g_j) = \inf(U \cap \{t(g_0), t(g_1), \dots, t(g_n)\}).$$

If $\{g_i, g_j\}$ is not a gap, let r be the smallest integer such that $g_i <_o g_r <_o g_j$. Then $r > n$, and by construction,

$$t(g_r) = \frac{t(g_i) + t(g_j)}{2}.$$

If $g_r \in L$, then

$$\varepsilon \leq t(g_j) - t(g_r) = \frac{t(g_j) - t(g_i)}{2} < \varepsilon,$$

a contradiction. A similar contradiction arises if $g_r \in U$. Thus, $\{g_i, g_j\}$ is a gap, implying that t is a tight embedding.

Given a tight embedding t , we can partition the real line as follows:

$$\mathbb{R} = t(G) \sqcup (\overline{t(G)} \setminus t(G)) \sqcup (\mathbb{R} \setminus \overline{t(G)}).$$

We refer to points in $\overline{t(G)} \setminus t(G)$ as *ideal points* of o (with respect to t). By definition, each ideal point is a two-sided limit point of $t(G)$, and gaps in G correspond bijectively to components of $\mathbb{R} \setminus \overline{t(G)}$, under the identification of a gap $\{h, k\}$ with the interval $(t(h), t(k))$.

Using a tight embedding t , we construct an injective map $\rho_o : G \rightarrow \text{Homeo}_+(\mathbb{R})$ as follows. For each $g \in G$ and $x \in \mathbb{R}$, set $\rho_o(g)(x)$ according to:

- If $x = t(h)$ for some $h \in G$, then define $\rho_o(g)(x) = t(gh)$.
- If $x \in \overline{t(G)} \setminus t(G)$, then choose a sequence $\{t(g_i)\} \subset t(G)$ converging to x and set $\rho_o(g)(x) = \lim_i t(gg_i)$.
- If $x \in \mathbb{R} \setminus \overline{t(G)}$, then there exists a gap $k <_o h$ such that $x \in (t(k), t(h))$, and we set

$$\rho_o(g)(x) = (1 - s)t(gk) + st(gh),$$

where $x = (1 - s)t(k) + st(h)$ and $s \in (0, 1)$.

The resulting map ρ_o is the *dynamic realisation* associated with o (relative to t). It is a well-defined homomorphism, unique up to conjugation by a homeomorphism of $\mathbb{R}$.

Theorem 2.3.1 (see e.g. [39, 27]). *Let o be a left-ordering on a countable group G . Then there exists a faithful representation $\rho_o : G \rightarrow \text{Homeo}_+(\mathbb{R})$, well-defined up to conjugation in $\text{Homeo}_+(\mathbb{R})$, which induces a faithful action of G on $\mathbb{R}$ without a global fixed point.*

Remark 2.3.2. If we insist to use the ρ_o constructed as above the theorem, then ρ_o determines o via

$$g <_o h \quad \text{if and only if} \quad \rho_o(g)(0) < \rho_o(h)(0).$$

And we always assume our dynamic realisations are constructed in this way unless otherwise stated.

For further details, see [46, Section 2.2] and [59].

Corollary 2.3.3. *A non-trivial countable group G is left-orderable if and only if it admits a faithful action on $\mathbb{R}$ without a global fixed point.*

More generally, the following is true, even for uncountable groups.

Proposition 2.3.4 (see e.g. [49, 21]). *A non-trivial group is left-orderable if and only if it admits a faithful action by order-preserving bijections on a totally ordered set.*

2.4 Convexity

The concept of convexity provides a link between left-ordered groups and the dynamics of their actions on totally ordered sets.

Let o be a left-ordering on G . A subset $A \subset G$ is said to be o -convex or *convex relative to o* , if for all $a, b \in A$ and $g \in G$, $a \leq_o g \leq_o b$ implies $g \in A$. In other words, any element of G lying between two elements of A (with respect to the left-ordering o) must also belong to A . For $a, b \in G$, we define o -convex subsets (o -intervals) $(a, b), [a, b], (-\infty, a], (a, \infty)$, etc. in the usual way.

The o -convex hull of a subset $A \subset G$ is defined as

$$C(A) = \{g \in G \mid \text{there are } g_1, g_2 \in A \text{ such that } g_1 \leq_o g \leq_o g_2\}.$$

We say a subset $A \subset G$ is o -cofinal if $C(A) = G$ and an element $g \in G$ is o -cofinal if $C(\langle g \rangle) = G$, where $\langle g \rangle$ denotes the cyclic subgroup generated by g .

When the left-ordering o is clear from context, we may omit it from the notation and simply refer to convex sets, cofinality, and related properties.

Suppose we have a short exact sequence of groups

$$\{\text{id}\} \longrightarrow N \xrightarrow{i} G \xrightarrow{p} H \longrightarrow \{\text{id}\}.$$

Let o_{ext} be the left-ordering by extension as defined in Proposition 2.1.4. Then $i(N)$ is necessarily o_{ext} -convex.

Here are some basic properties of non-trivial proper convex subgroups. Suppose σ is a left-ordering on G with positive cone $P = P(\sigma)$ and C is a proper σ -convex subgroup. Then we have:

- C is closed under taking roots: if $g^n \in C$ for some integer $n \neq 0$, then $g \in C$;
- C is of infinite index: if $g \notin C$, then $g^n \notin C$ for all integers $n \neq 0$;
- C is σ -bounded: if $g \in P \setminus C$, then $C \subset (g^{-1}, g)$;
- the set of σ -convex subgroups is linearly ordered by inclusions: if C, D are σ -convex subgroups and $d \in P \cap (D \setminus C)$, then $C \subset (d^{-1}, d) \subset D$;
- $(P \setminus C) \cdot (P \setminus C) \subset P \setminus C$: if $g, h \in P \setminus C$, then $\text{id} <_{\sigma} g <_{\sigma} gh$ and $gh \notin C$.

Lemma 2.4.1 (see e.g. [5, Lemma 2.1]). *Let (G, σ) be a left-ordered group with positive cone $P = P(\sigma)$, and let C be a subgroup of G .*

(1) *If C is convex, then $P \setminus C$ is a union of right cosets of C .*

(2) *C is convex if and only if $P \setminus C$ is a union of left cosets of C .*

Proof. Suppose first that C is convex, $c \in C$, and $g \in P \setminus C$. If $cg <_{\sigma} \text{id}$, then $\text{id} <_{\sigma} g <_{\sigma} c^{-1}$, which contradicts the convexity of C . Similarly, if $gc <_{\sigma} \text{id}$, then $c <_{\sigma} g^{-1} <_{\sigma} \text{id}$, implying $g^{-1} \in C$, which is also a contradiction. Therefore, $gc, cg \in P \setminus C$ for all $c \in C$, proving (1) and the forward direction of (2).

For the reverse direction of (2), suppose that $P \setminus C$ is a union of left C -cosets, and let $\text{id} <_{\sigma} g <_{\sigma} c$ for some $g \in G$ and $c \in C$. If $g \notin C$, then $g^{-1}c \in P \setminus C$, which implies that $g^{-1}C = (g^{-1}c)C \subset P \setminus C$. This contradicts $g \in P$, since it would require $g^{-1} \in P$. A similar argument holds for $c <_{\sigma} g <_{\sigma} \text{id}$, proving that C is convex. $\square$

Proposition 2.4.2 ([5, Proposition 2.2]). *Suppose G is a left-orderable group, $\sigma_1 \in \text{LO}(G)$, and C is σ_1 -convex.*

(1) *If $\sigma_2 \in \text{LO}(G)$ and $P(\sigma_1) \setminus C = P(\sigma_2) \setminus C$, then C is σ_2 -convex.*

(2) *If $\sigma_3 \in \text{LO}(C)$, then $P(\sigma_3) \sqcup (P(\sigma_1) \setminus C)$ is the positive cone of a left-ordering on G .*

Proof. Part (1) is a direct consequence of Lemma 2.4.1 (2). And part (2) can be verified straightforwardly. $\square$

Definition 2.4.3. If G is left-orderable and C is a subgroup of G , then we say that C is *relatively convex* if there exists a left-ordering σ on G such that C is σ -convex.

Since relatively convex subgroups of G are closed under taking roots, not all subgroups of all left-orderable groups have this property. Conversely, a non-trivial left-orderable group that admits no proper relatively convex subgroup must be an abelian group of rank one; see [50, Proposition 5.1.9].

Proposition 2.4.4 ([50, Proposition 5.1.10]). *Suppose that G is a left-orderable group. Then an arbitrary intersection of relatively convex subgroups of G is relatively convex.*

Definition 2.4.5. Let G be a left-orderable group and $\sigma_1, \sigma_2 \in \text{LO}(G)$. If C is a proper σ_1 -convex subgroup of G and $P(\sigma_1) \setminus C = P(\sigma_2) \setminus C$, then we say that σ_1 is *convexly related* to σ_2 and that σ_1 and σ_2 differ by a *convex swap*.

Proposition 2.4.6 ([50, Propositions 2.1.1 - 2.1.3]). *Suppose C is a convex subgroup of a left-ordered group (G, σ) . Then σ induces a (left) G -invariant total order $<_E$ on the cosets $E = G/C$, where we declare $gC <_E hC$ if and only if $g^{-1}h \notin C$ and $g <_\sigma h$.*

Proof. Let $g, h \in G$. The statement follows if we can show that $g <_\sigma h$ implies $gc_1 <_\sigma hc_2$ for all $c_1, c_2 \in C$. So suppose $g <_\sigma h$ and $g^{-1}h \notin C$. If $c_1, c_2 \in C$, then by Lemma 2.4.1, we have

$$\text{id} <_\sigma (g^{-1}h)c_2, \quad \text{and} \quad \text{id} <_\sigma c_1^{-1}(g^{-1}h)c_2.$$

The latter implies that $gc_1 <_\sigma hc_2$, proving the result. $\square$

Corollary 2.4.7. *If C is a proper convex normal subgroup of a left-ordered group (G, σ) , then the quotient group G/C is left-orderable.*

Since the orbit of a point x under an action of a group G on a set E can be identified with the cosets of $\text{Stab}_G(x)$, the following proposition can be seen as a converse to Proposition 2.4.6.

Proposition 2.4.8 ([50, Corollary 5.1.4]). *Let G be a non-trivial group that acts by order-preserving permutations on a totally ordered set $(E, <_E)$. Let $x \in E$ be a fixed point and denote by $\text{Stab}(x) \leq G$ the stabilizer of x . Given a left-ordering σ_0 on $\text{Stab}(x)$ with positive cone $P_0 = P(\sigma_0)$, there exists a left-ordering σ_x on G extending σ_0 given by*

$$g_1 <_{\sigma_x} g_2 \iff \begin{cases} \text{id} <_{\sigma_0} g_1^{-1}g_2, & \text{if } g_1^{-1}g_2 \in \text{Stab}(x); \\ g_1 \cdot x <_E g_2 \cdot x, & \text{if } g_1^{-1}g_2 \notin \text{Stab}(x). \end{cases}$$

Furthermore, if $g_1, g_2 \in G$, then $g_1 <_{\sigma_x} g_2$ implies $g_1 \cdot x \leq_E g_2 \cdot x$. In particular, $\text{Stab}(x)$ is σ_x -convex.

Proof. Define

$$P = P_0 \sqcup \{g \in G \mid x <_E g \cdot x\}$$

and observe that $g \in P$ implies $g \neq \text{id}$ and $x \leq_E g \cdot x$. We claim that P is a positive cone of a left-ordering on G and denote by σ_x the left-ordering determined by P .

To show that $G = \{\text{id}\} \sqcup P \sqcup P^{-1}$, we proceed as follows. Suppose $g \in P \cap P^{-1}$. Then $x \leq_E g \cdot x$ and $x \leq_E g^{-1} \cdot x$, which implies $g \cdot x = x$. Thus, $g \in P \cap \text{Stab}(x) = P_0$, but this means $g, g^{-1} \in P_0$ and hence $g \in P_0 \cap P_0^{-1}$, which contradicts the definition of a positive cone. Therefore, $P \cap P^{-1} = \emptyset$.

Additionally, for any $g \notin \text{Stab}(x)$, either $x <_E g \cdot x$ or $x <_E g^{-1} \cdot x$. In other words,

$$G \setminus \text{Stab}(x) = (P \setminus P_0) \sqcup (P \setminus P_0)^{-1} = (P \setminus P_0) \sqcup (P^{-1} \setminus P_0^{-1}).$$

Suppose $g, h \in P$. If $g, h \in P_0$, then $gh \in P_0 \subset P$. If $h \in P \setminus P_0$, then $x <_E h \cdot x$ and so

$$x \leq_E g \cdot x <_E g \cdot (h \cdot x) = (gh) \cdot x,$$

which implies $gh \in P$. A similar argument applies when $g \in P \setminus P_0$, showing that $P \cdot P \subset P$. This completes the proof of the claim.

Now, let $g_1, g_2 \in G$. If $g_1 \leq_{\sigma_x} g_2$, then $\text{id} \leq_{\sigma_x} g_1^{-1}g_2$ and so $x \leq_E (g_1^{-1}g_2) \cdot x$. Consequently, $g_1 \cdot x \leq_E g_2 \cdot x$.

Finally, suppose $g_1 <_{\sigma_x} g <_{\sigma_x} g_2$, where $g \in G$ and $g_1, g_2 \in \text{Stab}(x)$. Then

$$x = g_1 \cdot x \leq_E g \cdot x \leq_E g_2 \cdot x = x.$$

It follows that $g \cdot x = x$, so $g \in \text{Stab}(x)$. We conclude that $\text{Stab}(x)$ is σ_x -convex. □

Let $\rho_o : G \rightarrow \text{Homeo}_+(\mathbb{R})$ be the dynamic realization associated with a left-ordering o as constructed above Theorem 2.3.1, which gives us an action of G on the reals $\mathbb{R}$.

Definition 2.4.9. We denote by $\text{Stab}_{\rho_o}(x)$ the stabilizer

$$\{g \in G \mid \rho_o(g)(x) = x\}$$

of $x \in \mathbb{R}$ under the ρ_o -action. We denote by σ_x the left-ordering constructed in Proposition 2.4.8 via the restriction $o|_{\text{Stab}_{\rho_o}(x)}$ and the natural order on $\mathbb{R}$.

We noted in Remark 2.3.2 that a left-ordering $\sigma \in \text{LO}(G)$ is determined by the natural order on the orbit of $0 \in \mathbb{R}$ under the dynamic realization ρ_o of σ . Here, we show how the orbit of each $x \in \mathbb{R}$ under ρ_o determines a convexly related family $\mathcal{D}_x$ of left-orderings.

Lemma 2.4.10 ([5, Lemma 3.2]). *If $x \in \mathbb{R}$ and $\text{Stab}_{\rho_o}(x) \neq \{\text{id}\}$, then x is an ideal point of $\mathfrak{o}$.*

Proof. If $x \in t(G)$, then $\text{Stab}_{\rho_o}(x) = \{\text{id}\}$. On the other hand, if x is contained in a gap $(t(g), t(h))$ of $\mathfrak{o}$, then by construction, any $k \in \text{Stab}_{\rho_o}(x)$ acts as the identity on $(t(g), t(h))$; in particular, $k \in \text{Stab}_{\rho_o}(t(g)) = \{\text{id}\}$. So x must be an ideal point. $\square$

The restriction of $\mathfrak{o}$ to $\text{Stab}_{\rho_o}(x)$ determines a left-ordering $\mathfrak{o}_x \in \text{LO}(G)$ by Proposition 2.4.8, relative to which $\text{Stab}_{\rho_o}(x)$ is convex. Furthermore, each orbit

$$\mathcal{O}_{\rho_o}(x) = \{\rho_o(g)(x) \mid g \in G\} \subset \mathbb{R}$$

is a left G -set canonically identified with the cosets $E_x = G/\text{Stab}_{\rho_o}(x)$ under the map $g\text{Stab}_{\rho_o}(x) \mapsto \rho_o(g)(x)$. This identification preserves the total order $<_E$ on E_x , given by Proposition 2.4.6, mapping it to the natural order on $\mathcal{O}_{\rho_o}(x)$ induced from $\mathbb{R}$.

Definition 2.4.11. We use $\mathcal{D}_x$ to denote the set of left-orderings on G obtained, via Proposition 2.4.8, from the action of G on E_x equipped with the natural total order $<_E$ and a choice of left-orderings on $\text{Stab}_{\rho_o}(x)$.

From Proposition 2.4.8, we see that $\text{Stab}_{\rho_o}(x)$ is $\mathfrak{o}'$ -convex for each $\mathfrak{o}' \in \mathcal{D}_x$, forming a family of convexly related left-orderings. Furthermore, $\mathcal{D}_x$ always contains the left-ordering $\mathfrak{o}_x$, determined by $\mathfrak{o}|_{\text{Stab}_{\rho_o}(x)}$ and $<_E$ (i.e., $\mathfrak{o}_x$ is the left-ordering defined in Definition 2.4.9). If $\text{Stab}_{\rho_o}(x)$ is trivial, then $\mathcal{D}_x = \{\mathfrak{o}_x\}$. Proposition 2.4.8 also shows that for any $\mathfrak{o}' \in \mathcal{D}_x$,

$$g \leq_{\mathfrak{o}'} h \implies \rho_o(g)(x) \leq \rho_o(h)(x),$$

with equality on the right-hand side if and only if $g^{-1}h \in \text{Stab}_{\rho_o}(x)$, meaning that the comparison $g \leq_{\mathfrak{o}'} h$ depends on the chosen left-ordering on $\text{Stab}_{\rho_o}(x)$.

Recall that we have a canonical action of G on $\text{LO}(G)$. For $\mathfrak{o} \in \text{LO}(G)$, we write $\mathcal{O}(\mathfrak{o})$ to denote its orbit:

$$\mathcal{O}(\mathfrak{o}) = \{g \cdot \mathfrak{o} \mid g \in G\}.$$

Lemma 2.4.12 ([5, Lemma 3.5]). (1) *For each $g \in G$, we have $\mathfrak{o}_{t(g)} = g \cdot \mathfrak{o}$.*

(2) *If x lies in a gap $(t(g), t(h))$ of $\mathfrak{o}$, then $\mathfrak{o}_x = g \cdot \mathfrak{o}$.*

Proof. We first prove part (1). Let $h \in G$ be arbitrary. Since the stabilizer of $t(g)$ is trivial, we have $\text{id} <_{\mathfrak{o}_{t(g)}} h$ if and only if

$$\rho_o(\text{id})(t(g)) = t(g) < \rho_o(h)(t(g)) = t(hg).$$

The latter holds if and only if $g <_o hg$, or equivalently, $\text{id} <_o g^{-1}hg$. Thus, we conclude that

$$P(\mathfrak{o}_{t(g)}) = gP(\mathfrak{o})g^{-1}.$$

This proves part (1).

For part (2), note that the orbit of the interval $(t(g), t(h))$ under the action of G is the collection $\{(t(kg), t(kh)) \mid k \in G\}$, which consists of a family of disjoint open intervals in bijection with G , whose union is $\mathbb{R} \setminus \overline{t(G)}$. It follows that $\mathfrak{o}_x = \mathfrak{o}_{t(g)}$ by noting that

$$\rho_o(\text{id})(x) < \rho_o(k)(x) \quad \text{if and only if} \quad \rho_o(\text{id})(t(g)) < \rho_o(k)(t(g))$$

for any $k \in G$. So the result follows from part (1). $\square$

Lemma 2.4.13 ([5, Lemma 3.6]). *Suppose that x is an ideal point of $\mathfrak{o}$, and that $\{g_n\} \subset G$ is a sequence with $\lim_n t(g_n) = x$. Then,*

$$P(\mathfrak{o}_x) \setminus \text{Stab}_{\rho_o}(x) = \lim_n (P(\mathfrak{o}_{t(g_n)}) \setminus \text{Stab}_{\rho_o}(x)).$$

Proof. Let $g \in G \setminus \text{Stab}_{\rho_o}(x)$. Then, by construction, we have:

$$\begin{cases} g <_{\mathfrak{o}_x} \text{id} & \Rightarrow \rho_o(g)(x) < x, \\ g >_{\mathfrak{o}_x} \text{id} & \Rightarrow \rho_o(g)(x) > x. \end{cases}$$

So there exists an integer $N_g > 0$, depending on g , such that for all $n \geq N_g$,

$$\begin{cases} g <_{\mathfrak{o}_x} \text{id} & \Rightarrow \rho_o(g)(t(g_n)) < t(g_n) \text{ and } g <_{\mathfrak{o}_{t(g_n)}} \text{id}, \\ g >_{\mathfrak{o}_x} \text{id} & \Rightarrow \rho_o(g)(t(g_n)) > t(g_n) \text{ and } g >_{\mathfrak{o}_{t(g_n)}} \text{id}. \end{cases}$$

Fix a finite generating set of G and let B_j be the ball of radius j in G in the associated word metric. Choose an arbitrary j and set $N = \max\{N_g \mid g \in B_j \setminus \text{Stab}_{\rho_o}(x)\}$. Then for all $n \geq N$, we have

$$B_j \cap (P(\mathfrak{o}_x) \setminus \text{Stab}_{\rho_o}(x)) = B_j \cap (P(\mathfrak{o}_{t(g_n)}) \setminus \text{Stab}_{\rho_o}(x)).$$

This completes the proof. $\square$

Lemma 2.4.14 ([5, Lemma 3.7]). *Suppose that x is an ideal point of $\mathfrak{o}$. Then, $\mathcal{D}_x \cap \overline{\mathcal{O}(\mathfrak{o})} \neq \emptyset$.*

Proof. Let $\{g_n\} \subset G$ be a sequence such that $\lim_n t(g_n) = x$. Since $\text{LO}(G)$ is a compact metrisable space, we can extract a subsequence $\{g_{n_j}\}$ such that $\mathfrak{o}_{t(g_{n_j})}$ converges to some

$\mathfrak{o}' \in \text{LO}(G)$. Then, Lemma 2.4.13 implies that

$$P(\mathfrak{o}') \setminus \text{Stab}_{\rho_{\mathfrak{o}}}(x) = P(\mathfrak{o}_x) \setminus \text{Stab}_{\rho_{\mathfrak{o}}}(x).$$

This shows that $\mathfrak{o}' \in \mathcal{D}_x \cap \overline{\mathcal{O}(\mathfrak{o})}$. □

Combining Lemmas 2.4.12, 2.4.13 and 2.4.14, we conclude the following result.

Proposition 2.4.15 ([5, Proposition 3.4]). *Let $x \in \mathbb{R}$, $\mathfrak{o} \in \text{LO}(G)$. Suppose $\rho_{\mathfrak{o}}$ is the associated dynamic realization. Then we have the following:*

- (1) *If x is not an ideal point of $\mathfrak{o}$, then $\mathcal{D}_x = \{\mathfrak{o}\} \subset \mathcal{O}(\mathfrak{o})$.*
- (2) *If x is an ideal point of $\mathfrak{o}$, then $\mathcal{D}_x \cap \overline{\mathcal{O}(\mathfrak{o})} \neq \emptyset$.*

3

L-space Conjecture

This chapter introduces the topological and homological background underlying the L-space conjecture, which asserts that for an irreducible rational homology 3-sphere M , the properties of supporting a co-orientable taut foliation, having a left-orderable fundamental group, and not being a Heegaard Floer homology L-space are equivalent. It begins with a concise review of foliations and the geometric meaning of tautness and co-orientability, followed by a summary of key results linking these properties to Heegaard Floer theory. The chapter then surveys the structure of 3-manifold groups, covering the Prime Decomposition and JSJ Decomposition Theorems, which underpin the decomposition-based approach used later in order detection. Finally, it explains how slope detection arises naturally from the interplay between manifold gluings and the behaviour of left-orderings on boundary tori, motivating the theoretical framework that follows. Throughout this thesis, we focus primarily on the left-orderability aspect of 3-manifold groups, where a 3-manifold group refers to the fundamental group of a compact connect orientable 3-manifold. We always assume that 3-manifolds are compact, connected and orientable.

3.1 Foliations and Heegaard Floer homology

This section provides a brief discussion of foliations and Heegaard Floer homology to offer some background to the L-space conjecture.

A *foliation* $\mathcal{F}$ of an n -manifold M is a decomposition into a disjoint union of immersed submanifolds:

$$M = \bigsqcup_{\lambda \in \Lambda} L_{\lambda},$$

where each L_{λ} is a connected k -dimensional submanifold with $0 \leq k < n$. These sub-

manifolds L_λ , called *leaves*, vary continuously across M in the sense that M is covered by local coordinate charts $\phi : U \rightarrow \mathbb{R}^n$, where for each leaf L_λ intersecting U , we have

$$\phi(L_\lambda \cap U) = \mathbb{R}^k \times \{x_\lambda\},$$

for some $x_\lambda \in \mathbb{R}^{n-k}$.

The number $n - k$ is called the *codimension* of the foliation. If M has a boundary, we require either that ∂M is a leaf or that the foliation restricts to a foliation of ∂M . A simple example is the foliation of $\mathbb{R}^2$ by vertical lines.

Foliations play a fundamental role in the study of 3-manifolds, providing important geometric and dynamic information. The following classical result illustrates the relationship between foliations and the topology of 3-manifolds.

Theorem 3.1.1 (Rosenberg-Sondow [70]). *If a closed 3-manifold M admits a foliation in which all leaves are homeomorphic to $\mathbb{R}^2$, then $M \cong S^1 \times S^1 \times S^1$.*

Let M be a closed, connected, orientable 3-manifold, and let $\mathcal{F}$ be a codimension-1 foliation on M . A *transverse loop* is a loop in M that intersects every leaf and each intersection is transverse. The foliation $\mathcal{F}$ is *taut* if there exists a transverse loop γ that intersects every leaf. We say the foliation $\mathcal{F}$ is *co-orientable* if there is a consistent transverse orientation to the leaves. While every 3-manifold admits a codimension-one foliation, the additional conditions of being taut and co-orientable impose significant restrictions on the topology of M (see e.g. [74, 11]). For example, the following result holds.

Proposition 3.1.2 ([55, Section 3.1]). *A transverse loop to a taut foliation of M represents an element of infinite order in $\pi_1(M)$.*

As an immediate consequence, a spherical 3-manifold cannot admit a taut foliation. On the other hand, the following classical theorem provides a sufficient condition for the existence of a taut foliation.

Theorem 3.1.3 ([26, Theorem 5.5]). *If $b_1(M) > 0$ (i.e. M has a positive first Betti number), then M admits a taut foliation.*

Heegaard Floer homology, introduced by Ozsváth and Szabó [62], is a powerful invariant of 3-manifolds, constructed using Heegaard diagrams. A Heegaard diagram consists of a closed surface Σ of genus g along with two sets of g curves, $\{\alpha_1, \dots, \alpha_g\}$ and $\{\beta_1, \dots, \beta_g\}$, which intersect transversely in pairs. Heegaard Floer homology is defined as the homology of a chain complex derived from these Heegaard diagrams,

making use of techniques from symplectic geometry, particularly the study of pseudo-holomorphic curves.

There are three main versions of Heegaard Floer homology: the hat version $\widehat{HF}$, the plus version HF^+ , and the minus version HF^- . These invariants are interrelated through exact triangles and are isomorphic when appropriately graded. In particular, the hat version provides a powerful invariant for distinguishing between different 3-manifolds and has proven to be particularly useful in practice.

A *rational (resp. integral) homology sphere* is a closed 3-manifold M having the same rational (resp. integral) homology as S^3 . Note that by Poincaré duality, a homology sphere is necessarily closed and orientable. In general, for any closed 3-manifold M , the rank of Heegaard Floer homology satisfies $\text{rank}(\widehat{HF}(M)) \geq |H_1(M; \mathbb{Z})|$. A rational homology sphere M is called an *L-space* if its Heegaard Floer homology satisfies the rank condition:

$$\text{rank}(\widehat{HF}(M)) = |H_1(M; \mathbb{Z})|.$$

This means that Heegaard Floer homology is “as simple as possible”. The term “L-space” originates from the primary class of examples: *lens spaces*, which are among the simplest nontrivial 3-manifolds.

One direction of the equivalence in the L-space conjecture, namely the implication from the existence of a co-orientable taut foliation to the property of not being an L-space, is already established.

Theorem 3.1.4 ([61, 3, 47]). *If M admits a co-orientable taut foliation, then M is not an L-space.*

3.2 Basics of 3-manifold groups and left-orderability

To study the left-orderability of 3-manifold groups, it is essential to understand some fundamental concepts in 3-manifold theory, which is discussed in this section. After introducing some basic concepts used in 3-manifold theory, an important result of determining the left-orderability of the fundamental group of a prime 3-manifold is presented. Finally, this section concludes with a discussion about the origin of the idea of slope detection.

A fundamental result in the theory of 3-manifold is the following.

Theorem 3.2.1 (Moise [56, 57]; see also [33, 72]). *Every topological 3-manifold admits a unique smooth structure (up to diffeomorphism) and a unique piecewise-linear structure (up to piecewise-linear homeomorphism).*

This allows us to work interchangeably in the smooth and piecewise-linear categories without ambiguity.

Let M_1 and M_2 be oriented 3-manifolds, and let $B_i \subset \text{int}(M_i)$ for $i = 1, 2$ be embedded 3-balls. The connected sum of M_1 and M_2 is defined by:

$$M_1 \# M_2 = (M_1 - \text{int}(B_1)) \cup_h (M_2 - \text{int}(B_2)),$$

where $h : \partial B_1 \rightarrow \partial B_2$ is an orientation-reversing homeomorphism. It can be shown that the connected sum is well-defined up to orientation-preserving homeomorphism. This operation is both associative and commutative, and $M \# S^3 = M$ for any 3-manifold M .

A 3-manifold M is called *prime* if the decomposition $M = M_1 \# M_2$ implies either $M_1 = S^3$ or $M_2 = S^3$. A 3-manifold is said to be *irreducible* if every embedded 2-sphere bounds a 3-ball. Clearly, every irreducible manifold is prime, and the only prime manifold that is not irreducible is $S^1 \times S^2$. Let $S \subset M$ be a properly embedded surface, meaning that $\partial S \subset \partial M$. We say that S is *incompressible* if the map $\pi_1(S) \rightarrow \pi_1(M)$ induced by inclusion is injective, and S does not bound a 3-ball. It can be shown that a 3-manifold is irreducible if and only if embedded every 2-sphere is compressible [36].

Remarkably, the fundamental group of a closed prime 3-manifold encodes deep geometric information about the manifold itself.

Theorem 3.2.2 ([64, 65, 66]). *Let M and M' be closed 3-manifolds. If M is prime and not a lens space, and $\pi_1(M)$ and $\pi_1(M')$ are isomorphic, then M and M' are homeomorphic.*

The following theorem is the classical Prime Decomposition Theorem, which gives a unique decomposition of a 3-manifold into prime pieces.

Theorem 3.2.3 (Prime Decomposition Theorem [48, 54]). *Let M be a compact, connected, and orientable 3-manifold with no spherical boundary components. Then there is a decomposition*

$$M = M_1 \# \cdots \# M_n,$$

where each M_i is prime. Moreover, this decomposition is unique up to order and orientation-preserving homeomorphism.

By the Seifert-van Kampen theorem, the fundamental group of a connected sum decomposes as a free product:

$$\pi_1(M) = \pi_1(M_1) * \cdots * \pi_1(M_n).$$

Together with Vinogradov's Theorem (Theorem 2.1.3), this implies that $\pi_1(M)$ is left-orderable if and only if each $\pi_1(M_i)$ is left-orderable. If M_i is reducible for some i ,

then $\pi_1(M_i) = \mathbb{Z}$ is left-orderable. Thus, determining whether a 3-manifold group is left-orderable reduces to studying the left-orderability of its irreducible factors.

If $\pi_1(M)$ is left-orderable (resp. not left-orderable), we also say M is left-orderable (resp. not left-orderable). A key result concerning the left-orderability of irreducible 3-manifold groups is the following theorem.

Theorem 3.2.4 (Boyer, Rolfsen, and Wiest [9]). *Let M be a compact, connected, orientable, irreducible 3-manifold. Then $\pi_1(M)$ is left-orderable if and only if it has a left-orderable quotient. Consequently, if $b_1(M) > 0$ (i.e. M has a positive first Betti number), then $\pi_1(M)$ is left-orderable.*

This implies that prime 3-manifolds with a non-spherical boundary component are left-orderable, as shown by the following lemma.

Lemma 3.2.5 ([9, Lemma 3.3]). *Let M be a compact 3-manifold and $\mathbb{F}$ a field. Then*

$$\dim H_1(M; \mathbb{F}) \geq \frac{1}{2} \dim H_1(\partial M; \mathbb{F}).$$

In particular, if M has a boundary component not S^2 , then $H_1(M; \mathbb{Z})$ is infinite.

So the only case left is where $b_1(M) = 0$, where the L-space conjecture aims to classify the left-orderability of the fundamental group of irreducible 3-manifolds.

A compact, connected 3-manifold M is called *Seifert fibered* if it admits a decomposition into disjoint simple closed curves, called *Seifert fibers*, such that each fiber has a tubular neighbourhood homeomorphic to a *standard fibered torus*. The standard fibered torus associated to a pair of coprime integers (p, q) with $p > 0$ is the surface bundle of the automorphism of a disk given by a rotation by an angle of $2\pi q/p$, equipped with the natural fibering by circles. A classical result of Epstein [25] shows that a 3-manifold is Seifert fibered if and only if it admits a foliation by circles.

Following [41], a compact, connected, orientable 3-manifold M is called *atoroidal* if every map $T \rightarrow M$ from a torus T that induces an injective homomorphism $\pi_1(T) \rightarrow \pi_1(M)$ can be homotoped into ∂M . The following fundamental structure theorem was proved independently by Jaco–Shalen [41] and Johannson [44], and was first announced earlier by Waldhausen [76].

Theorem 3.2.6 (JSJ Decomposition). *Let M be a compact, orientable, irreducible 3-manifold. Then there exists a finite collection of incompressible tori in M such that each connected component of the complement is either a Seifert fibered space, or an atoroidal 3-manifold. Moreover, this decomposition is unique up to isotopy.*

In the context of left-orderability, the JSJ decomposition allows us to reduce the problem of determining the left-orderability of the fundamental group of an irreducible 3-manifold to studying the left-orderability of the fundamental groups of its JSJ components. If M is orientable, irreducible and closed, then the JSJ decomposition of M provides a unique (up to isotopy) minimal collection $\mathcal{T}$ of embedded disjoint incompressible tori. This decomposition allows one to realise $\pi_1(M)$ as the fundamental group of groups, whose vertex groups are $\pi_1(M_i)$'s and whose edge groups are $\pi_1(T) = \mathbb{Z}^2$, where T ranges over all tori in the collection $\mathcal{T}$. As no obstruction to left-orderability arises from considering these groups independently, the key to understanding left-orderability of $\pi_1(M)$ therefore lies in an analysis of the gluing maps used to reassemble M from the pieces M_i , and the behaviour of the left-orderings of each $\pi_1(M)$ restricted to the components of ∂M_i with respect to these gluing maps. This is the very place where the ideas of order detection come from.

3.3 Results from Heegaard Floer homology

This section provides some basic results from Heegaard Floer homology together with the definitions of L-space knots and L-space links.

From now on, we denote by M , unless otherwise defined, a compact, connected, orientable 3-manifold whose boundary is a union of incompressible tori, denoted by $\partial M = T_1 \cup T_2 \cup \dots \cup T_n$. If $n = 1$, we call such M a *knot manifold* as well. We assume that along each boundary torus T_i , a distinguished basis $\{\mu_i, \lambda_i\}$ of $\pi_1(T_i) = H_1(T_i; \mathbb{Z})$ is chosen. The manifold obtained by performing Dehn surgeries along each T_i with surgery coefficients $r_i \in \mathbb{Q}$ is denoted by $M(r_1, r_2, \dots, r_n)$. More precisely, if $r_i = p_i/q_i$ in lowest terms with $q_i \neq 0$, then

$$M(r_1, r_2, \dots, r_n) = M \cup \bigcup_{i=1}^n (S^1 \times D_i^2), \quad \text{where } \{1\} \times \partial D_i^2 \text{ is glued to } p_i \mu_i + q_i \lambda_i.$$

Here D_i^2 denotes the 2-disk, and each $S^1 \times D_i^2$ is a solid torus. As we will see in the next chapter, the notion of surgery coefficients naturally overlaps with that of slopes.

Let $L = L_1 \cup L_2 \cup \dots \cup L_n$ be a link in S^3 and denote by M its link complement, which is the manifold obtained by removing an open regular neighbourhood of L from S^3 . For each component L_i of the link, the corresponding boundary torus is denoted by $T_i \subset \partial M$. In this context, the basis of $\pi_1(T_i)$ consists of the canonical meridian and longitude along T_i and the manifold resulting from Dehn surgeries along each component L_i with slope r_i is also written as $S_{r_1, r_2, \dots, r_n}^3(L)$. It was shown by Lickorish

[51] and Wallace [77] that any closed 3-manifold can be obtained by Dehn surgery on a link in S^3 with all surgery coefficients being integral.

For a two-component link $L = L_1 \cup L_2$, the notation $S^3_{r_1,*}(L)$ refers to the 3-manifold obtained by performing Dehn filling along L_1 with slope r_1 while leaving L_2 unfilled. Similarly, $S^3_{*,r_2}(L)$ denotes the manifold obtained by filling only along L_2 with slope r_2 . Both of these resulting manifolds have boundary consisting of a single torus.

Definition 3.3.1. A knot K in S^3 is called an *L-space knot* if there exists a positive integer n such that the result of Dehn surgery $S^3_n(K)$ is an L-space.

The following theorem provides a criterion for determining whether a surgery on a knot produces an L-space.

Theorem 3.3.2 ([63, Proposition 9.6]). *Let K be an L-space knot. Then $S^3_{p/q}(K)$ is an L-space if and only if*

$$p/q \geq 2g(K) - 1,$$

where $g(K)$ is the genus of K .

All torus knots are L-space knots, and as shown by Ni in [60], every L-space knot is a fibered knot. If the L-space conjecture holds, a similar characterization should apply to the left-orderability of the fundamental group of the 3-manifold resulted from Dehn surgery on knots. However, the left-orderability aspect presents significant challenges, and progress in this direction remains ongoing.

A motivating result for applying order detection to cable knots is the behaviour of L-space knot under cabling.

Theorem 3.3.3 ([37, 40]). *The (p, q) -cable of a knot $K \subset S^3$ is an L-space knot if and only if K is an L-space knot and*

$$p/q \geq 2g(K) - 1.$$

We will revisit the case of cable knots in Chapter 5, where we establish an analogous result in the context of left-orderability.

For Dehn fillings of link complements, the situation is complicated by the simultaneous filling of multiple torus boundary components. In analogy with the case of knots, we call an n -component link L an *L-space link* if any Dehn surgery on L with sufficiently large integral surgery coefficients always yields an L-space. In the Thistlethwaite Link Table and up to mirror images, the following are some example of L-space links: $L5a1$ (the Whitehead link), $L6a5$, $L6n1$, $L7a7$, and $L7n1$.

Many other examples of L-space links are also discussed in [53], and progress towards characterising the possible shapes of regions of surgery coefficients that yield L-spaces can be found in [32, 31, 52]. In particular, they studied two-component links having linking number zero or with unknotted components, and showed that the region where pairs of integral surgery coefficients yield L-spaces may not be a product of intervals. Focusing on the case of 2-component links, the pairs of slopes (r_1, r_2) for which the corresponding Dehn filled manifolds are not L-space, admit a co-orientable taut foliation, or has left-orderable fundamental groups, define regions in the plane. Santoro established in [71] that (r_1, r_2) -surgery on the mirror image of the Whitehead link is not an L-space if and only if $r_1 < 1$ or $r_2 < 1$; admits a co-orientable taut foliation when $r_1 < 1$ or $r_2 < 1$; and when one of r_1, r_2 is an integer, has a left-orderable fundamental group precisely when $r_1 < 1$ or $r_2 < 1$.

Later, in Chapter 6, we discuss the non-left-orderability of Dehn surgeries on the Whitehead link and a class of two-component links.

4

Order Detection

This chapter forms the theoretical core of the thesis, developing a general framework for order detection and its equivalent formulation, representation detection. These concepts capture how left-orderings on boundary components of 3-manifolds interact under gluings and Dehn fillings. The chapter also explores the role of cofinal elements in the theory of order detection. The gluing theorems in Section 4.6 provide conditions ensuring the left-orderability of the fundamental group of the manifold resulted from gluing or Dehn filling, while the notion of JN-realisability discussed in Section 4.7 serves as a way of computing which tuples of slopes on the boundary of a Seifert fibered manifold are detected. Together, these results form the theoretical foundation for the applications in Chapters 5 and 6. While the materials in Section 4.1 are well known among researchers in this field, the materials in Section 4.5 are taken from [14] and all the rest are from [15].

4.1 Left-orderings on $\mathbb{Z}^2$

The case $G = \mathbb{Z}^2$ serves as an important and illustrative example in the study of order detection, both here and throughout the remainder of this chapter.

Firstly, we observe that $\mathbb{Z}^2$ admits uncountably many left-orderings. Consider the standard embedding of $\mathbb{Z}^2$ as a lattice in $\mathbb{R}^2$ via the identification

$$\mathbb{R}^2 = H_1(\mathbb{Z}^2; \mathbb{R}) = \mathbb{Z}^2 \otimes \mathbb{R}.$$

Let $L \subset \mathbb{R}^2$ be the line of slope k , defined by $L = \{(x, kx) \mid x \in \mathbb{R}\}$. If k is irrational, then the map

$$\psi : \mathbb{Z}^2 \rightarrow \mathbb{R}, \quad (m, n) \mapsto n - km$$

is injective. This allows us to define a left-ordering σ_k on $\mathbb{Z}^2$ by declaring

$$(m, n) \leq_{\sigma_k} (m', n') \quad \text{if and only if} \quad n - km \leq n' - km'.$$

Geometrically, elements of $\mathbb{Z}^2$ are ordered according to their projection onto a line orthogonal to L . See Figure 4.1 for a geometric depiction. It is straightforward to verify that σ_k is a left-ordering on $\mathbb{Z}^2$. Therefore, each irrational slope k leads to two distinct left-orderings, σ_k and its opposite σ_k^{op} . It follows that there are uncountably many left-orderings on $\mathbb{Z}^2$.

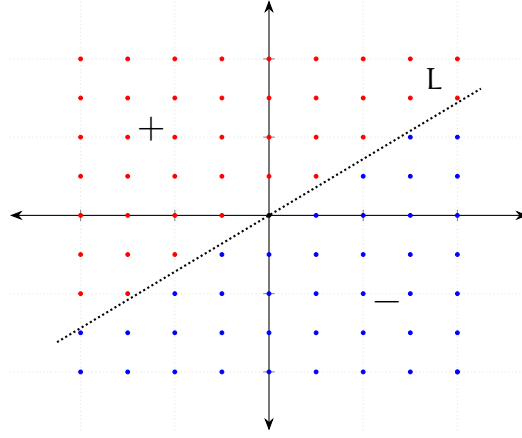


Figure 4.1: A left-ordering σ_k determined by a line with an irrational slope k

When the slope k is rational, the intersection $L \cap \mathbb{Z}^2$ is a subgroup of $\mathbb{Z}^2$ isomorphic to $\mathbb{Z}$, which corresponds to integral points on the line. To construct a left-ordering, we must also decide which half of the subgroup $L \cap \mathbb{Z}^2$ to declare as positive. This additional choice leads to four distinct left-orderings arising from the following short exact sequence:

$$0 \longrightarrow L \cap \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 / (L \cap \mathbb{Z}^2) \longrightarrow 0.$$

The four left-orderings come from the choice of direction along $L \cap \mathbb{Z}^2$ (two possibilities) and the left-orderings on the quotient group $\mathbb{Z}^2 / (L \cap \mathbb{Z}^2)$ (which is isomorphic to $\mathbb{Z}$). See Figure 4.2 for a visual representation.

Conversely, let σ be a left-ordering on $\mathbb{Z}^2$. Then $P(\sigma)^{-1} = -P(\sigma)$ is disjoint from $P(\sigma)$. If $a, b \in P(\sigma)$, then $a + b \in P(\sigma)$. If $na \in P(\sigma)$ for some $1 \leq n \in \mathbb{Z}$, then $a \in P(\sigma)$. These properties imply that $P(\sigma)$ and $-P(\sigma)$ correspond to two disjoint sectors¹ of $\mathbb{Z}^2$ in the plane. Moreover, because the group decomposes as $\mathbb{Z}^2 \setminus \{(0, 0)\} = P(\sigma) \sqcup -P(\sigma)$, we see

¹A sector refers to an angular region of rays.

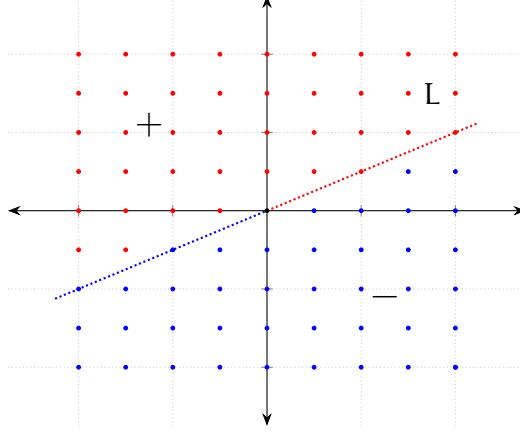


Figure 4.2: A left-ordering σ_k determined by a line with a rational slope k

that there must exist a unique line $L(\sigma) \subset \mathbb{R}^2$ through the origin that separates these two sectors. All elements of $\mathbb{Z}^2$ lying on one side of $L(\sigma)$ are positive, while those on the other side are negative.

If we equip the set of lines through the origin in $\mathbb{R}^2$ with the standard topology and write $[L]$ for the image of such a line $L \subset \mathbb{R}^2$ in the resulting copy of $\mathbb{RP}^1 \cong S^1$, then the map $\mathcal{L} : \text{LO}(\mathbb{Z}^2) \rightarrow \mathbb{RP}^1, \sigma \mapsto [L(\sigma)]$ is continuous.

Lemma 4.1.1 ([13, Chapter 6]). *The map $\mathcal{L} : \text{LO}(\mathbb{Z}^2) \rightarrow \mathbb{RP}^1, \sigma \mapsto [L(\sigma)]$ is continuous.*

Proof. Let $\{\sigma_i\}_{i=1}^\infty$ be a convergent sequence of left-orderings converging to σ , meaning that for any finite subset $S \subset P(\sigma)$, there exists N such that for all $n > N$ we have $S \subset P(\sigma_n)$. It suffices to show that the sequence $\{[L(\sigma_i)]\}_{i=1}^\infty$ converges to $[L(\sigma)]$.

After identifying of $\mathbb{Z}^2$ with the integer lattice points in $\mathbb{R}^2$, we assume the slope of the line $L(\sigma)$ is k . Further assume $k \neq \pm\infty$ as the case of a vertical line can be treated analogously, and let $\varepsilon > 0$ be arbitrary. It is possible to find a rational number $\frac{p}{q}$ in the lowest form such that $k - \varepsilon < \frac{p}{q} < k < \frac{p+1}{q} < k + \varepsilon$.

Therefore, (p, q) and $(p+1, q)$, regarded as elements of $\mathbb{Z}^2$, have opposite signs with respect to σ . Hence, there exists an $N > 0$ such that (p, q) and $(p+1, q)$ have opposite signs with respect to σ_n for all $n > N$. Thus for all $n > N$, the slope of the line $L(\sigma_n)$ lies in $(\frac{p}{q}, \frac{p+1}{q}) \subset (k - \varepsilon, k + \varepsilon)$. In other words, the slopes of $\{L(\sigma_i)\}_{i=1}^\infty$ converge to k , in other words, $\{[L(\sigma_i)]\}_{i=1}^\infty$ converges to $[L(\sigma)]$. $\square$

4.2 Definition of order detection

This section introduces the formal definition of order detection, generalising prior notions of order detection for knot manifolds and provides the algebraic foundation for later equivalence results.

For the remainder of this section and chapter, let M be a compact, connected, orientable, irreducible 3-manifold that is not homeomorphic to $S^1 \times S^1 \times I$ or $S^1 \times D^2$. We also assume that the boundary of M consists of a disjoint union of incompressible tori, denoted $\partial M = T_1 \cup T_2 \cup \dots \cup T_n$. Since the fundamental group of a torus is $\pi_1(T_i) \cong \mathbb{Z}^2$, we define a *slope* on T_i as an element $[\alpha]$ in the projective space $\mathbb{P}H_1(T_i; \mathbb{R})$, where $\alpha \in H_1(T_i; \mathbb{R})$ represents a homology class. By the incompressibility of the boundary tori, the inclusion-induced maps $\pi_1(T_i) \rightarrow \pi_1(M)$ are injective. This allows us to identify each $\pi_1(T_i)$ with a subgroup of $\pi_1(M)$ that is isomorphic to $\mathbb{Z}^2$. We fix such an identification for each boundary torus and, from this point forward, will simply write $\pi_1(T_i) \subset \pi_1(M)$ without explicit mention of the inclusion.

Denote by $\mathcal{S}(T_i)$ the set of slopes on T_i . And the set of slopes on M is defined as $\mathcal{S}(M) = \mathcal{S}(T_1) \times \dots \times \mathcal{S}(T_n)$. Let $r_i : \text{LO}(\pi_1(M)) \rightarrow \text{LO}(\pi_1(T_i))$ denote the restriction map of left-orderings. The *slope map* $s : \text{LO}(\pi_1(M)) \rightarrow \mathcal{S}(M)$ is then given by

$$s(\mathfrak{o}) = (\mathcal{L}(r_1(\mathfrak{o})), \dots, \mathcal{L}(r_n(\mathfrak{o}))),$$

where $\mathcal{L}$ is the map defined in Lemma 4.1.1. Since all r_i 's and the map $\mathcal{L}$ are continuous, the slope map s is continuous as well. We also define $s_i : \text{LO}(\pi_1(M)) \rightarrow \mathcal{S}(T_i)$ as the composition of s with the projection onto the i -th factor, which is equivalently given by $s_i = \mathcal{L} \circ r_i$. When there is no risk of confusion, we abbreviate a slope tuple $([\alpha_1], \dots, [\alpha_n]) \in \mathcal{S}(M)$ as $[\alpha_*]$.

Identifying $H_1(T_i; \mathbb{Z}) = \pi_1(T_i)$ with the integer lattice points in $H_1(T_i; \mathbb{R})$, we say that a slope $[\alpha]$ is *rational* if $\alpha \in H_1(T_i; \mathbb{Z})$, and *irrational* otherwise. A tuple $([\alpha_1], \dots, [\alpha_n])$ of slopes is rational if each $[\alpha_i]$ is rational, and we denote by $\mathcal{S}_{\text{rat}}(M)$ the subset of $\mathcal{S}(M)$ consisting of all rational tuples. If $[\alpha]$ is rational, we always assume that α is a primitive element in $H_1(T_i; \mathbb{Z}) = \pi_1(T_i)$. In the context of slopes induced by left-orderings on $\mathbb{Z}^2$, a slope $[\mathcal{L}(r_i(\mathfrak{o}))]$ is rational if $\mathcal{L}(r_i(\mathfrak{o})) \cap H_1(T_i; \mathbb{Z}) \cong \mathbb{Z}$; otherwise, the slope is irrational. This terminology is consistent with our discussions of rational and irrational slopes made in Section 4.1.

Since $\mathcal{S}(T_i)$ is homeomorphic to $\mathbb{RP}^1$, we may also identify $\mathcal{S}(T_i)$ with $\mathbb{R} \cup \{\infty\}$ once a distinguished basis of $\pi_1(T_i)$ has been chosen. More precisely, let $\{\mu_i, \lambda_i\}$ be a chosen basis of $\pi_1(T_i)$, which is also referred to as the *peripheral system* along T_i . For $r = p/q \in \mathbb{Q}$

in lowest terms, define $\tau = \mu_i^p \lambda_i^q$, viewed as an element of $H_1(T_i; \mathbb{Z})$. This provides an identification between $\mathbb{Q} \cup \{\infty\}$ and the subset of rational slopes in $\mathcal{S}(T_i)$ via $r \mapsto [\tau]$ and $\infty \mapsto [\mu_i]$, where we call $[\tau]$ the slope corresponding to r . Note that the surgery coefficients $p_i/q_i \in \mathbb{Q}$ in Dehn surgery can also be interpreted in this way, where the basis consists of the canonical meridian and longitude.

Definition 4.2.1. Assume σ is a left-ordering on $\pi_1(M)$, and suppose $J \subset K \subset \{1, \dots, n\}$ and $([\alpha_1], \dots, [\alpha_n]) \in \mathcal{S}(M)$. We say that $(J, K; [\alpha_1], \dots, [\alpha_n])$ is *order-detected* by σ if the following conditions hold:

- O1. The slope map gives $s(\sigma) = ([\alpha_1], \dots, [\alpha_n])$.
- O2. For all $g \in \pi_1(M)$, the slope of the conjugate left-ordering satisfies

$$s(g \cdot \sigma) = ([\beta_1], \dots, [\beta_n]),$$

where $[\beta_i] = [\alpha_i]$ for all $i \in K$.

- O3. There exists an σ -convex normal subgroup C such that for each $i \in \{1, 2, \dots, n\}$:
 - (a) If $[\alpha_i]$ is rational, then $\pi_1(T_i) \cap C \leq \langle \alpha_i \rangle$, and furthermore, if $i \in J$, then $\pi_1(T_i) \cap C = \langle \alpha_i \rangle$.
 - (b) If $[\alpha_i]$ is irrational, then $\pi_1(T_i) \cap C = \{\text{id}\}$.

When the conditions above are satisfied, we also say that $(J, K; [\alpha_1], \dots, [\alpha_n])$ is *σ -detected*, or that σ (order-)detects $(J, K; [\alpha_1], \dots, [\alpha_n])$. If $J = K = \emptyset$, we are allowed to take $C = \{\text{id}\}$.

Remark 4.2.2. It is clear that if $J' \subset J, K' \subset K, J' \subset K'$ and $(J, K; [\alpha_*])$ is σ -detected, then $(J', K'; [\alpha_*])$ is also σ -detected. And to say σ order-detects $(\emptyset, \emptyset; [\alpha_*])$ is to say $s(\sigma) = [\alpha_*]$.

If $(J, K; [\alpha_1], \dots, [\alpha_n])$ is order-detected, we say that $[\alpha_i]$ is *weakly order-detected* for $i = 1, \dots, n$, is *strongly order-detected* if $i \in J$, and is *(regularly) order-detected* if $i \in K$. We denote by $\mathcal{D}_{\text{ord}}(J, K; M) \subset \mathcal{S}(M)$ the set of tuples $[\alpha_*]$ for which $(J, K; [\alpha_*])$ is order-detected. If M is a knot manifold, the terminology introduced here (strong detection, weak detection, detection) aligns with the definitions given in [5].

4.3 Definition of representation detection

This section discusses representation detection, describing how slope data arises from representations of the fundamental group into the orientation-preserving homeomorphism group of the real line.

All representations of groups in our discussions will be assumed to have images in $\text{Homeo}_+(\mathbb{R})$ unless otherwise stated. A *pointed representation* of a group G is a pair (ρ, x_0) where ρ is a representation $\rho : G \rightarrow \text{Homeo}_+(\mathbb{R})$ and $x_0 \in \mathbb{R}$ is a choice of basepoint that is not a global fixed point for the action of G on $\mathbb{R}$ determined by ρ . We use the notation $\mathcal{R}(G)$ to denote the set of all pointed representations of G .

Lemma 4.3.1. *A pointed representation (ρ, x_0) of $G = \mathbb{Z}^2$ determines a line $L(\rho, x_0)$ in $\mathbb{R}^2 \cong \mathbb{Z}^2 \otimes \mathbb{R}$ according to the prescription that all elements g of $\mathbb{Z}^2$ lying to one side of $L(\rho, x_0)$ satisfy $\rho(g)(x_0) > x_0$, and those on the other side satisfy $\rho(g)(x_0) < x_0$.*

Proof. First, consider the case where ρ is injective. Let $\{y_0 = x_0, y_1, y_2, \dots\}$ be an enumeration of a dense subset of $\mathbb{R}$. This enumeration determines a left-ordering $\circ$ on $\text{Homeo}_+(\mathbb{R})$, namely, $f <_\circ g$ if there exists an index i such that $f(y_i) < g(y_i)$ and $f(y_j) = g(y_j)$ for all $j = 0, 1, \dots, i-1$. Let $\rho^{-1}(\circ)$ be the pullback of $\circ$. Then $\rho^{-1}(\circ)$ is a left-ordering on G and determines a line $L(\rho^{-1}(\circ)) = L(\rho, x_0)$. By definition, for an element $g \in G$, we have $\text{id} <_{\rho^{-1}(\circ)} g$ if $\rho(\text{id})(x_0) = x_0 < \rho(g)(x_0)$, and $g <_{\rho^{-1}(\circ)} \text{id}$ if $\rho(g)(x_0) < x_0$. The line $L(\rho, x_0)$ is completely determined by these prescriptions and is independent of the choice of the dense countable subset of $\mathbb{R}$.

If ρ is not injective, then $\ker(\rho) \cong \mathbb{Z}$, which determines a line $L(\rho, x_0) \subset \mathbb{R}^2$ in a natural way. It follows that there exists an injective map $\rho' : G/\ker(\rho) \cong \mathbb{Z} \rightarrow \text{Homeo}_+(\mathbb{R})$ with x_0 not being a fixed point of any element in $G/\ker(\rho)$ except the identity. Let s be a generator of $G/\ker(\rho)$. Then either $\rho'(s)(x_0) > 0$ or $\rho'(s)(x_0) < 0$. If $\rho'(s)(x_0) > 0$, then $\rho'(s^n)(x_0) > 0$ for $n > 0$ and $\rho'(s^n)(x_0) < 0$ for $n < 0$. The elements of the cosets $s^n \ker(G)$ exhibit the same property. The same applies to the case $\rho'(s)(x_0) < 0$. The statement of this lemma follows. $\square$

Definition 4.3.2. Suppose ρ_1, ρ_2 are representations of a group G . Then ρ_1 is said to be *semi-conjugate* to ρ_2 if there is a proper² non-decreasing map $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$h \circ \rho_1(g) = \rho_2(g) \circ h$$

for all $g \in G$.

It turns out that semi-conjugacy is an equivalence relation [21, Proposition 1.1.16].

Lemma 4.3.3. *Let $\circ$ be a left-ordering of a group G and C an $\circ$ -convex normal subgroup of G . If $t : G \rightarrow \mathbb{R}$ is a tight embedding coming from an enumeration $\{g_0 = \text{id}, g_1, g_2, \dots\}$ of G and ρ its associated dynamic realisation, then there is a dynamic realisation $\eta : G/C \rightarrow \text{Homeo}_+(\mathbb{R})$*

²Here, properness means that the preimage of any bounded set is bounded.

such that ρ is semi-conjugate to $\eta \circ p$, where $p : G \rightarrow G/C$ is the quotient map. Moreover, the proper non-decreasing map $\nu : \mathbb{R} \rightarrow \mathbb{R}$ demonstrating the semi-conjugacy satisfies $\nu(0) = 0$.

Proof. Since C is convex, $P(o) \setminus C$ is a union of left C -cosets by Lemma 2.4.1. In other words, given two cosets gC and hC , we have either $gc <_o hc'$ for all $c', c \in C$ or $hc' <_o gc$ for all $c', c \in C$. Therefore, $t(gC)$ is bounded for each coset gC and so $\inf\{t(gC)\}$ and $\sup\{t(gC)\}$ exist.

Since C is convex and normal, G/C inherits a canonical left-ordering from G by Corollary 2.4.7. Fix a tight embedding $\omega : G/C \rightarrow \mathbb{R}$ with $\omega(C) = 0$ and η its dynamic realisation. We define the map $\nu : \mathbb{R} \rightarrow \mathbb{R}$ by the following prescription:

- if $r = t(g)$ for some $g \in G$, then $\nu(r) = \omega(gC)$;
- if $r \in (t(g), t(h))$ for some gap $g <_o h$ in G , then $gC = hC$ as $g^{-1}h$ is the least positive element in G and $\nu(r) = \omega(hC) = \omega(gC)$;
- if $r \in \overline{t(G)} \setminus t(G)$, then $\nu(r) = \sup\{\nu(g) : t(g) < r\}$.

Note that the image of ν is unbounded and ν is a well-defined non-decreasing function, and by definition it satisfies $\nu(0) = 0$. We check that it serves as the required function to prove the semi-conjugacy relation claimed.

For any nonempty bounded subset I of $\mathbb{R}$, choose cosets gC and hC such that $\omega(gC)$ and $\omega(hC)$ are a lower bound and an upper bound of I respectively. Choose g' and h' such that $\omega(g'C) < \omega(gC)$ and $\omega(hC) < \omega(h'C)$. Then $\nu^{-1}(I)$ is bounded by $t(g')$ and $t(h')$, since $g' <_o gc$ and $hc <_o h'$ for any $c \in C$. Therefore, ν is proper.

Fix an arbitrary element $g \in G$. It is left to check that $\nu \circ \rho(g)(x) = \eta \circ p(g) \circ \nu(x)$ for all $x \in \mathbb{R}$. Firstly, note $\nu \circ \rho(g)(t(h)) = \nu(t(gh)) = \omega(ghC)$ and $\eta \circ p(g) \circ \nu(t(h)) = \eta(gC)\omega(hC) = \omega(ghC)$ for any $g, h \in G$. In other words, $\nu \circ \rho(g)$ and $\eta \circ p(g) \circ \nu$ agree on $t(G)$. Secondly, if $x \notin \overline{t(G)}$, then $t(k) < x < t(h)$ and $x = (1-s)t(k) + st(h)$ for some gap $k <_o h$ and $s \in (0, 1)$. It follows that $k, h \in kC = hC$ and so $\eta \circ p(g) \circ \nu(x) = \eta(gC)(\omega(hC)) = \omega(ghC)$. On the other hand, we have $\nu \circ \rho(g)(x) = \nu((1-s)t(gk) + st(gh)) = \omega(ghC)$. So $\nu \circ \rho(g)$ and $\eta \circ p(g) \circ \nu$ also agree on $\mathbb{R} \setminus \overline{t(G)}$. Finally, consider the case $x \in \overline{t(G)} \setminus t(G)$. We have

$$\begin{aligned}
\eta \circ p(g) \circ \nu(x) &= \eta(gC)(\sup\{\omega(hC) : t(h) < x\}) \\
&= \sup\{\omega(ghC) : t(h) < x\} \\
&= \sup\{\nu(t(gh)) : t(h) < x\} \\
&= \sup\{\nu(t(h)) : t(g^{-1}h) < x\} \\
&= \nu \circ \rho(g)(x).
\end{aligned}$$

This proves that ρ is semi-conjugate to $\eta \circ p$ via the map v . $\square$

For each $i \in \{1, \dots, n\}$, if $\pi_1(T_i) \not\subset \text{Stab}_\rho(x_0)$ for some $(\rho, x_0) \in \mathcal{R}(\pi_1(M))$, then (ρ, x_0) determines an element $(\rho|_{\pi_1(T_i)}, x_0) \in \mathcal{R}(\pi_1(T_i))$ via the restriction of ρ to the subgroup $\pi_1(T_i)$. Thus, we focus on the subset $\mathcal{R}^*(\pi_1(M)) \subset \mathcal{R}(\pi_1(M))$, defined by

$$\mathcal{R}^*(\pi_1(M)) = \{(\rho, x_0) \in \mathcal{R}(\pi_1(M)) \mid \pi_1(T_i) \not\subset \text{Stab}_\rho(x_0) \text{ for } i = 1, \dots, n\}.$$

Definition 4.3.4. Let $(\rho, x_0) \in \mathcal{R}^*(\pi_1(M))$, $J \subset K \subset \{1, \dots, n\}$ and $([\alpha_1], \dots, [\alpha_n]) \in \mathcal{S}(M)$. We say that $(J, K; [\alpha_1], \dots, [\alpha_n])$ is *representation-detected* by (ρ, x_0) if we have:

R1. The slopes match the lines determined by ρ , i.e.,

$$([L(\rho|_{\pi_1(T_1)}, x_0)], \dots, [L(\rho|_{\pi_1(T_n)}, x_0)]) = ([\alpha_1], \dots, [\alpha_n]).$$

R2. For every $g \in \pi_1(M)$, we have $(\rho, \rho(g)(x_0)) \in \mathcal{R}^*(\pi_1(M))$ with

$$([L(\rho|_{\pi_1(T_1)}, \rho(g)(x_0))], \dots, [L(\rho|_{\pi_1(T_n)}, \rho(g)(x_0))]) = ([\beta_1], \dots, [\beta_n]),$$

where $[\beta_i] = [\alpha_i]$ for all $i \in K$.

R3. The representation ρ is semi-conjugate to another representation $\varphi : \pi_1(M) \rightarrow \text{Homeo}_+(\mathbb{R})$ via a proper non-decreasing map $v : \mathbb{R} \rightarrow \mathbb{R}$ such that $(\varphi, v(x_0)) \in \mathcal{R}^*(\pi_1(M))$, and

$$[L(\varphi|_{\pi_1(T_i)}, v(x_0))] = [\alpha_i],$$

with $\varphi(\alpha_i) = \text{id}$ whenever $i \in J$ and $[\alpha_i]$ is rational.

When the conditions above are satisfied, we also say that (ρ, x_0) *representation-detects* $(J, K; [\alpha_1], \dots, [\alpha_n])$.

We write $\mathcal{D}_{\text{rep}}(J, K; M) \subset \mathcal{S}(M)$ to denote the set of tuples $[\alpha_*]$ for which $(J, K; [\alpha_*])$ is representation-detected by some $(\rho, x_0) \in \mathcal{R}^*(\pi_1(M))$. As in the case of order detection, if $(J, K; [\alpha_1], \dots, [\alpha_n])$ is representation-detected, we say that $[\alpha_i]$ is *weakly representation-detected*; it is *strongly representation-detected* if $i \in J$ and (regularly) *representation-detected* if $i \in K$.

4.4 Equivalence between order and representation detections

This section establishes the connection between order detection and representation detection. We begin with a lemma that allows us to alter the third condition of order detection.

Lemma 4.4.1. *Let $J \subset K \subset \{1, \dots, n\}$ and $([\alpha_1], \dots, [\alpha_n]) \in \mathcal{S}(M)$. Then $(J, K; [\alpha_1], \dots, [\alpha_n])$ is order-detected by some left-ordering if and only if there exists a left-ordering σ' of $\pi_1(M)$ satisfying O1 and O2 and*

O3'. there exists an σ' -convex subgroup H such that for all $i \in \{1, 2, \dots, n\}$

- (1) if $[\alpha_i]$ is rational, then $\pi_1(T_i) \cap H \leq \langle \alpha_i \rangle$, with $\langle \langle \alpha_i \rangle \rangle \leq H$ whenever $i \in J$;*
- (2) if $[\alpha_i]$ is irrational, then $\pi_1(T_i) \cap H = \{\text{id}\}$.*

Proof. Assume that σ is a left-ordering of $\pi_1(M)$ that order-detects $(J, K; [\alpha_1], \dots, [\alpha_n])$. Then O1 and O2 are satisfied, and the subgroup C from O3 automatically satisfies the properties required of H in O3'. Thus, the conclusion holds in this direction.

Conversely, suppose there exists a left-ordering σ' satisfying O1, O2, and O3'. Since H is σ' -convex, one can check that the conjugate gHg^{-1} is convex with respect to $g \cdot \sigma'$ for every $g \in \pi_1(M)$.

For convenience, set $G = \pi_1(M)$ and define

$$C = \bigcap_{g \in G} gHg^{-1}.$$

By Proposition 2.4.4, C is normal and σ -convex for some left-ordering σ ; we give the explicit construction of the left-ordering σ and verify that it satisfies O1, O2, and O3.

Since H is σ' -convex, the set of left cosets G/H inherits a natural total ordering $\prec$, preserved by the left G -action, with kernel C . Choose a complete set of coset representatives

$$E = \{g_0 = \text{id}, g_1, g_2, \dots\}$$

and define the left-ordering σ by the rule:

$$g <_{\sigma} h \quad \text{if} \quad gg_iH \prec hg_iH,$$

where g_iH is the first element in the enumeration for which $gg_iH \neq hg_iH$; or if $gg_iH = hg_iH$ for all i and $g <_{\sigma'} h$.

Now we check that $\circ$ satisfies the required conditions and complete the proof.

Suppose $i \in \{1, 2, \dots, n\}$ and $\gamma \in \pi_1(T_i)$ with $\gamma \notin \langle \alpha_i \rangle$ if $[\alpha_i]$ is rational. By O3', we have $\gamma \notin H$, so $H \neq \gamma H$. By construction, $\text{id} <_{\circ} \gamma$ if and only if $H \prec \gamma H$, which is equivalent to $\text{id} <_{\circ'} \gamma$. Thus, $\circ$ satisfies O1.

Suppose $i \in K$ and $\gamma \in \pi_1(T_i)$ with $\gamma \notin \langle \alpha_i \rangle$ if $[\alpha_i]$ is rational. Then

$$\text{id} <_{g \cdot \circ} \gamma \quad \text{if and only if} \quad \text{id} <_{\circ} g^{-1} \gamma g.$$

If $g^{-1} \gamma g$ lies in C , then $\text{id} <_{\circ} g^{-1} \gamma g$ if and only if $\text{id} <_{\circ'} g^{-1} \gamma g$, which is equivalent to $\text{id} <_{g \cdot \circ'} \gamma$. So in this case,

$$\text{id} <_{g \cdot \circ} \gamma \quad \text{if and only if} \quad \text{id} <_{g \cdot \circ'} \gamma.$$

If $g^{-1} \gamma g$ does not lie in C , then $\text{id} <_{\circ} g^{-1} \gamma g$ if and only if there exists an index j such that:

$$g_j H \prec g^{-1} \gamma g g_j H,$$

and $g_s H = g^{-1} \gamma g g_s H$ for all $s < j$.

Note that $g_j H \prec g^{-1} \gamma g g_j H$ if and only if:

$$g_j <_{\circ'} g^{-1} \gamma g g_j,$$

which is true if and only if $\text{id} <_{gg_j \cdot \circ'} \gamma$. Combined with the property O2 of $\circ'$, these two cases allow us to conclude that:

$$\text{id} <_{g \cdot \circ} \gamma \quad \text{if and only if} \quad \text{id} <_{\circ'} \gamma,$$

showing that $\circ$ satisfies O2.

Finally, we observe that O3 follows immediately from the construction of C and the property O3' of H . $\square$

Theorem 4.4.2. *Let $J \subset K \subset \{1, \dots, n\}$ and $[\alpha_*] \in \mathcal{S}(M)$. Then $(J, K; [\alpha_*])$ is order-detected if and only if it is representation-detected.*

Proof. First, we show that order detection implies representation detection. Assume $(J, K; [\alpha_*])$ is order-detected by $\circ \in \text{LO}(\pi_1(M))$. Let $t : \pi_1(M) \rightarrow \mathbb{R}$ be a tight embedding associated with an enumeration $\{g_0 = \text{id}, g_1, g_2, \dots\}$ of $\pi_1(M)$. In this setup, $0 = t(g_0)$ is not a fixed point of $\rho(g)$ for any non-identity element $g \in \pi_1(M)$. We claim that $(J, K; [\alpha_*])$ is representation-detected by $(\rho_{\circ}, 0)$, where $\rho_{\circ}$ is the dynamic realisation associated with t . Notice that we have $[L(\circ|_{\pi_1(T_i)})] = [L(\rho_{\circ}|_{\pi_1(T_i)}, 0)]$ by O1 and Lemma 4.3.1, that is, R1 is satisfied.

If $x = t(g) = \rho(g)(0)$ for some $g \in \pi_1(M)$, then x is not a fixed point of $\rho(h)$ for any non-identity element $h \in \pi_1(M)$. Therefore, we have $(\rho, \rho(g)(0)) \in \mathcal{R}^*(\pi_1(M))$ for every $g \in G$. Moreover, $\rho(g_1)(x) < \rho(g_2)(x)$ if and only if $t(g_1g) < t(g_2g)$. Since ρ is constructed from the tight embedding t , this is equivalent to saying $g_1g <_o g_2g$, in other words, $g_1 <_{g \cdot o} g_2$. Then by O2, we have

$$[L(\rho_o|_{\pi_1(T_j)}, x)] = [L((g \cdot o)|_{\pi_1(T_j)})] = [L(o|_{\pi_1(T_j)})] = [\alpha_j]$$

for all $j \in K$. Hence, R2 is fulfilled.

Applying Lemma 4.3.3 to the convex subgroup $C \leq \pi_1(M)$ using the tight embedding t , we obtain a representation $\varphi = \eta \circ p : \pi_1(M) \rightarrow \text{Homeo}_+(\mathbb{R})$, and ρ is semi-conjugate to this representation by $v : \mathbb{R} \rightarrow \mathbb{R}$, where $v(0) = 0$. For any $i \in \{1, \dots, n\}$, let $\gamma \in \pi_1(T_i)$ be given, with $\gamma \notin \langle \alpha_i \rangle$ if $[\alpha_i]$ is rational. Then we see that $\gamma \notin C$ by O3. Because of this, $\text{id} <_o \gamma$ if and only if $0 < t(\gamma)$ and also $\varphi(\gamma)(0) = \varphi(\gamma)v(0) = v\rho(\gamma)(0) = v(t(\gamma)) > 0$ if and only if $t(\gamma) > 0$, by the construction of v in Lemma 4.3.3. Therefore, $\text{id} <_o \gamma$ if and only if $\varphi(\gamma)(0) > 0$. It follows that $(\varphi, v(0)) \in \mathcal{R}^*(\pi_1(M))$ and, in fact, $[\mathcal{L}(\varphi|_{\pi_1(T_i)}, v(0))] = [\alpha_i]$. This proves R3 and finishes the proof in the first direction.

For the other direction, assume $(J, K; [\alpha_*])$ is representation-detected by (ρ, x_0) with φ the representation whose existence is guaranteed by R3, and with $v : \mathbb{R} \rightarrow \mathbb{R}$ the proper, non-decreasing map demonstrating the semiconjugacy from ρ to φ . Since M is compact connected irreducible and orientable, a nontrivial representation $\rho : \pi_1(M) \rightarrow \text{Homeo}_+(\mathbb{R})$ ensures that $\pi_1(M)$ is left-orderable by Theorem 3.2.4. By choosing a countable dense subset $E = \{r_0 = x_0, r_1, r_2, \dots\}$ of $\mathbb{R}$, we define a left-ordering o' on $\text{Homeo}_+(\mathbb{R})$ in the usual way. If $\ker(\rho)$ is trivial, then $\rho^{-1}(o')$ is a left-ordering on $\pi_1(M)$. If $\ker(\rho)$ is nontrivial, then we can give a left-ordering on $\pi_1(M)$ via the short exact sequence

$$\{\text{id}\} \longrightarrow \ker(\rho) \longrightarrow \pi_1(M) \longrightarrow \rho(\pi_1(M)) \longrightarrow \{\text{id}\},$$

where we take any left-ordering of $\ker(\rho)$ and the restriction left-ordering $o'|_{\rho(\pi_1(M))}$ on $\rho(\pi_1(M))$. By abuse of notation, we also denote this left-ordering of $\pi_1(M)$ as o' . By the construction of o' , we have $g <_{o'} f$ for $g, f \in \pi_1(M)$, if there is an index i such that $\rho(g)(r_i) < \rho(f)(r_i)$ and $\rho(g)(r_j) = \rho(f)(r_j)$ for all $j = 0, 1, \dots, i-1$.

Set $H = \text{Stab}_\varphi(v(x_0))$ and order the cosets of H according to

$$gH < fH \text{ if } \varphi(g)(v(x_0)) < \varphi(f)(v(x_0)).$$

Now we create the desired left-ordering o on $\pi_1(M)$ by declaring $g <_o f$ if either $gH <$

fH or $gH = fH$ and $\text{id} <_{\circ'} g^{-1}f \in H$. We claim that $\circ$ together with H satisfies O1, O2 and O3'. Then by Lemma 4.4.1, this direction is done. It remains to prove the claim.

To show O1, we fix $i \in \{1, 2, \dots, n\}$ and let $g \in \pi_1(T_i)$ such that if $[\alpha_i]$ is rational, then $g \notin \langle \alpha_i \rangle$. We have $g \notin \ker(\rho)$ by the definition of $\mathcal{R}^*(\pi_1(M))$. It suffices to show that $\text{id} <_{\circ} g$ if and only if $\rho(g)(x_0) > x_0$. There are two cases depending on whether g lies in H or not. If $g \in H$, then $\text{id} <_{\circ} g$ if and only if $\text{id} <_{\circ'} g$. Since $g \notin \ker(\rho)$, the latter is true if there is an index j such that $\rho(g)(r_j) > r_j$ and $\rho(g)(r_s) = r_s$ for all $s = 0, 1, \dots, j-1$. By R1, the first r_i where they differ is $r_0 = x_0$ and so indeed $\rho(g)(x_0) > x_0$ if and only if $\text{id} <_{\circ} g$. This finishes the first case. For the second, suppose $g \notin H$, then by the construction of $\circ$, we have $\text{id} <_{\circ} g$ if and only if $H \prec gH$, which is equivalent to $\varphi(g)(\nu(x_0)) > \nu(x_0)$ or equivalently $\nu(\rho(g)(x_0)) > \nu(x_0)$. Now if $\rho(g)(x_0) \leq x_0$, we would have $\nu(\rho(g)(x_0)) \leq \nu(x_0)$ since ν is non-decreasing, a contradiction. Therefore, it must be the case that $\rho(g)(x_0) > x_0$. Putting these two cases together proves O1.

To show O2, fix an $i \in K$ and take $g \in \pi_1(T_i)$ in such a way that if $[\alpha_i]$ is rational, then g does not lie in $\langle \alpha_i \rangle$. It suffices to show that for any $f \in \pi_1(M)$, we have $\text{id} <_{\circ} g$ if and only if $\text{id} <_{f \cdot \circ} g$. By the definition of $f \cdot \circ$, the latter is equivalent to $\text{id} <_{\circ} f^{-1}gf$. Again, we divide into two cases depending on whether g is in H or not.

Firstly, consider the case where $g \notin H$. Then $\text{id} <_{\circ} g$ if and only if $\varphi(g)(\nu(x_0)) > \nu(x_0)$. By semiconjugacy, the latter is equivalent to $\nu(\rho(g)(x_0)) > \nu(x_0)$. By the same argument as in the last paragraph, this implies $\rho(g)(x_0) > x_0$. By R2, it follows that $\rho(g)(\rho(f)(x_0)) > \rho(f)(x_0)$, that is, $\rho(f^{-1}gf)(x_0) > x_0$. If $f^{-1}gf \in H$, then $\rho(f^{-1}gf)(x_0) > x_0$ implies $\text{id} <_{\circ'} f^{-1}gf$, and so $\text{id} <_{\circ} f^{-1}gf$. If $f^{-1}gf \notin H$, then $\rho(f^{-1}gf)(x_0) > x_0$ implies $\nu(\rho(f^{-1}gf)(x_0)) > \nu(x_0)$. It follows that $\varphi(f^{-1}gf)(\nu(x_0)) > \nu(x_0)$ and therefore $\text{id} <_{\circ} f^{-1}gf$.

Now consider the second case $g \in H$. Then $\text{id} <_{\circ} g$ if and only if $\text{id} <_{\circ'} g$. Since $g \notin \ker(\rho)$, this is equivalent to $\rho(g)(x_0) > x_0$. By R2, it follows that $\rho(g)(\rho(f)(x_0)) > \rho(f)(x_0)$ for any $f \in \pi_1(M)$. The second case now follows from an argument identical to the final steps of the first case. Therefore, the property O2 is satisfied.

To show O3', we first note that H is $\circ$ -convex by construction. Next we observe that $\ker(\varphi) < H$ and $\alpha_i \in \ker(\varphi)$ for any $i \in J$ with $[\alpha_i]$ rational, which implies $\langle \langle \alpha_i \rangle \rangle \leq H$. Since $(\varphi, \nu(x_0)) \in \mathcal{R}^*(\pi_1(M))$, $H \cap \pi_1(T_i)$ is at most of rank 1, and since $[\mathcal{L}(\varphi|_{\pi_1(T_i)}, \nu(x_0))] = [\alpha_i]$ we know that $H \cap \pi_1(T_i) \leq \langle \alpha_i \rangle$ when $[\alpha_i]$ is rational and $H \cap \pi_1(T_i) = \{\text{id}\}$ when $[\alpha_i]$ is irrational. It follows that O3' is satisfied. $\square$

This result also allows us to rework our definition of representation detection as follows.

Lemma 4.4.3. *Let $J \subset K \subset \{1, \dots, n\}$ and $([\alpha_1], \dots, [\alpha_n]) \in \mathcal{S}(M)$. Then $(J, K; [\alpha_1], \dots, [\alpha_n])$ is representation-detected if and only if there exists $(\rho, x_0) \in \mathcal{R}^*(\pi_1(M))$ satisfying R1, R3 and*

R2'. for all $x \in \mathbb{R}$, if $(\rho, x) \in \mathcal{R}^(\pi_1(M))$, then*

$$([L(\rho|_{\pi_1(T_1)}, x)], \dots, [L(\rho|_{\pi_1(T_n)}, x)]) = ([\beta_1], \dots, [\beta_n]),$$

with $[\beta_i] = [\alpha_i]$ for all $i \in K$.

Proof. Given (ρ, x_0) satisfying R1, R2', and R3, it is clear that R1, R2, and R3 are satisfied by noting that $(\rho, \rho(g)(x_0)) \in \mathcal{R}^*(\pi_1(M))$.

For the other direction, by Theorem 4.4.2 and its proof, it suffices to show that if $\circ$ order-detects $(J, K; [\alpha_*])$, then $(\rho_\circ, 0)$ satisfies R1, R2', and R3, where $\rho_\circ$ is the dynamic realisation. That R1 and R3 are satisfied is contained in the proof of Theorem 4.4.2, so we only need to show that the dynamic realisation $\rho_\circ$ satisfies R2'.

Let $t : \pi_1(M) \rightarrow \mathbb{R}$ be the tight embedding used to construct $\rho_\circ$ satisfying $t(\text{id}) = 0$, and choose $x \in \mathbb{R}$ such that $(\rho, x) \in \mathcal{R}^*(\pi_1(M))$. We consider three cases.

- Case 1: $x = t(g)$ for some $g \in \pi_1(M)$.

Since $t(\text{id}) = 0$, we have $x = \rho_\circ(g)(0)$, and R2' in this case simply reduces to R2.

- Case 2: $x \in \overline{t(\pi_1(M))}$.

Suppose that there exists $i \in K$ such that $[L(\rho_\circ|_{\pi_1(T_i)}, x)] \neq [\alpha_i]$. In this case, there must exist $g \in \pi_1(T_i) \setminus \langle \alpha_i \rangle$ such that $g >_\circ \text{id}$ and $\rho_\circ(g)(x) < x$.

Since $x \in \overline{t(\pi_1(M))}$, we may choose a sequence $\{t(g_i)\} \subset t(\pi_1(M))$ that converges to x . However, since $\rho_\circ(g)(x) < x$, the continuity of $\rho_\circ(g)$ ensures that there exists j such that $\rho_\circ(g)(t(g_j)) < t(g_j)$. But then $gg_j <_\circ g_j$, which contradicts O2, since $s_i(g_j \cdot \circ) = [\alpha_i]$ implies $g <_\circ \text{id}$.

- Case 3: $x \in (a, b) \subset \mathbb{R} \setminus t(\pi_1(M))$.

Assume $[L(\rho_\circ|_{\pi_1(T_i)}, x)] = [\alpha_i]$, so there is $g \in \pi_1(T_i) \setminus \langle \alpha_i \rangle$ such that $g >_\circ \text{id}$ and $\rho_\circ(g)(x) < x$. There is a gap in $h, k \in \pi_1(M)$ such that $(a, b) \subset (t(h), t(k))$. Then

$$x = (1 - s)t(h) + st(k)$$

for some $s \in (0, 1)$. Then by definition, we have

$$\rho_\circ(g)(x) = (1 - s)t(gh) + st(gk),$$

Since $\rho_o(g)(x) < x$ implies

$$(1-s)(\rho_o(g)(t(h)) - t(h)) + s(\rho_o(g)(t(k)) - t(k)) < 0,$$

it means that at least one of $t(gh) - t(h) < 0$ or $t(gk) - t(k) < 0$ holds. In other words, either $gh <_o h$ or $gk <_o k$, which contradicts O2 as in the previous case.

Therefore, R2' holds in all cases, completing the proof. $\square$

4.5 Cofinality and order detection

Cofinality and convexity have already been discussed briefly in Section 2.4. Now we want to understand them in the context of order detection. Recall that M is compact, connected, orientable, and irreducible, with boundary consisting of a disjoint union of incompressible tori, denoted as $\partial M = T_1 \cup \dots \cup T_n$.

Definition 4.5.1 (cf. [5, Definition 5.7]). If $o \in \text{LO}(\pi_1(M))$ and every $\pi_1(T_i)$ is o -cofinal, then we say that o is *boundary-cofinal*.

Some of the results below generalise those in [5], where only knot manifolds are considered.

Lemma 4.5.2. *Let $J \subset K \subset \{1, \dots, n\}$ and $([\alpha_1], \dots, [\alpha_n]) \in \mathcal{S}(M)$. Suppose o is a left-ordering of $\pi_1(M)$ that order-detects $(J, K; [\alpha_1], \dots, [\alpha_n])$. If $\pi_1(T_1)$ is not o -cofinal, then there exists a left-ordering o' of $\pi_1(M)$ and a proper subgroup $C \subset \pi_1(M)$ such that C is o' -convex, $\pi_1(T_1) \subset C$, and o' order-detects $(\emptyset, \emptyset; [\beta_1], \dots, [\beta_n])$, where $[\beta_1] = [\alpha_1]$ and $[\beta_i] = [\alpha_i]$ for all $i \in K$.*

Proof. We follow the proof of [5, Lemma 5.9]. Suppose that $\pi_1(T_1)$ is not o -cofinal and choose a positive element $\gamma \in \pi_1(T_1)$ that is $o|_{\pi_1(T_1)}$ -cofinal. Choose a tight order-preserving embedding $t : \pi_1(M) \rightarrow \mathbb{R}$ and use it to construct the dynamic realisation $\rho_o : \pi_1(M) \rightarrow \text{Homeo}_+(\mathbb{R})$.

Since γ is not o -cofinal, the limit $\lim_n t(\gamma^n) = x_0$ exists. Since $\rho_o(\gamma)(x_0) = x_0$ and γ is $o|_{\pi_1(T_1)}$ -cofinal, one can show that $\rho_o(h)(x_0) = x_0$ for all $h \in \pi_1(T_1)$.

Set $C = \text{Stab}_{\rho_o}(x_0)$ and so $\pi_1(T_1) \subset C$. Note that C is proper since dynamic realisations do not have global fixed points. Then we can use Proposition 2.4.8 to construct a left-ordering o' such that C is o' -convex and $o'|_C = o|_C$. Namely, we declare $g <_{o'} h$ if $\rho_o(g)(x_0) < \rho_o(h)(x_0)$ or $g^{-1}h \in C$ and $g <_o h$. It follows that $[L(o'|_{\pi_1(T_1)})] = [\alpha_1]$. Now

it remains to verify that the left-ordering $\mathfrak{o}'$ order-detects the tuple $(\emptyset, \emptyset; [\beta_1], \dots, [\beta_n])$ with $[\beta_i] = [\alpha_i]$ for all $i \in K$. To do this, take an arbitrary $i \in K$ and consider three cases.

Case 1. $\pi_1(T_i) \subset C$. In this case $\mathfrak{o}'|_{\pi_1(T_i)} = \mathfrak{o}|_{\pi_1(T_i)}$, so $[L(\mathfrak{o}'|_{\pi_1(T_i)})] = [\alpha_i]$.

Cases 2 and 3. $\pi_1(T_i) \cap C \cong \mathbb{Z}$, or $\pi_1(T_i) \cap C = \{\text{id}\}$. In either case, it suffices to show that $(\pi_1(T_i) \cap P(\mathfrak{o}')) \setminus C = Q \setminus C$, where Q is a positive cone in $\pi_1(T_i)$, whose corresponding left-ordering order-detects $[\alpha_i]$.

Since the space of left-orderings $\text{LO}(\pi_1(M))$ is compact, we can find a convergent subsequence $\{\gamma^{n_l} \cdot \mathfrak{o}\}_{l \in \mathbb{N}}$ of $\{\gamma^k \cdot \mathfrak{o}\}_{k \in \mathbb{N}}$. Now by the proof of Proposition 2.4.15, we have

$$\left(\lim_{l \rightarrow \infty} P(\gamma^{n_l} \cdot \mathfrak{o}) \right) \setminus C = P(\mathfrak{o}') \setminus C.$$

Next, note that

$$\left(\lim_{l \rightarrow \infty} P(\gamma^{n_l} \cdot \mathfrak{o}) \right) \setminus C = \lim_{l \rightarrow \infty} (P(\gamma^{n_l} \cdot \mathfrak{o}) \setminus C),$$

and since the restriction map $r_i : \{0, 1\}^{\pi_1(M)} \rightarrow \{0, 1\}^{\pi_1(T_i)}$, $P \mapsto P \cap \pi_1(T_i)$ is continuous for all i , the limit above gives

$$r_i(P(\mathfrak{o}') \setminus C) = \lim_{l \rightarrow \infty} r_i(P(\gamma^{n_l} \cdot \mathfrak{o}) \setminus C).$$

And therefore

$$(\pi_1(T_i) \cap P(\mathfrak{o}')) \setminus C = \left(\lim_{l \rightarrow \infty} \pi_1(T_i) \cap P(\gamma^{n_l} \cdot \mathfrak{o}) \right) \setminus C.$$

Since $[L((\gamma^{n_l} \cdot \mathfrak{o})|_{\pi_1(T_i)})] = [\alpha_i]$ for all $n_l \geq 0$, and there are precisely four (resp. two) positive cones in $\pi_1(T_i)$ corresponding to left-orderings detecting $[\alpha_i]$ when $[\alpha_i]$ is rational (resp. irrational), we can choose a subsequence $\{m_k\}_{k \in \mathbb{N}}$ of $\{n_l\}_{l \in \mathbb{N}}$ such that $P(\gamma^{m_k} \cdot \mathfrak{o}) \cap \pi_1(T_i) = Q$ is constant for all k .

Then the limit becomes

$$(\pi_1(T_i) \cap P(\mathfrak{o}')) \setminus C = Q \setminus C,$$

showing that $\mathfrak{o}'$ order-detects $(\emptyset, \emptyset; [\beta_1], \dots, [\beta_n])$, where $[\beta_i] = [\alpha_i]$ for all $i \in K$. □

Theorem 4.5.3. *Let $\mathfrak{o}$ be a left-ordering of $\pi_1(M)$, $I \subset \{2, \dots, n\}$ and $([\alpha_1], \dots, [\alpha_n]) \in \mathcal{S}(M)$. Suppose that $\mathfrak{o}$ order-detects $(\emptyset, \emptyset; [\alpha_1], \dots, [\alpha_n])$ and C is an $\mathfrak{o}$ -convex subgroup of $\pi_1(M)$. If $\pi_1(T_1) \subset C$ and $\pi_1(T_i) \not\subset C$ for all $i \in I$, then for all $[\beta] \in \mathcal{S}(T_1)$ there exists a left-ordering $\mathfrak{o}'$ of $\pi_1(M)$ that order-detects $(\emptyset, \emptyset; [\beta_1], \dots, [\beta_n])$ with $[\beta_1] = [\beta]$ and $[\beta_i] = [\alpha_i]$ for all $i \in I$.*

Proof. The proof is a slight modification of the proof of [5, Proposition 5.3]. Firstly, note that C is of infinite index because it is $\mathfrak{o}$ -convex. Let $W \rightarrow M$ be a covering such that $\pi_1(W) = C$. Then W is non-compact, and moreover, $T_1 \subset \partial M$ lifts to a torus $T \subset \partial W$

since $\pi_1(T_1) \subset C$. Recall that we denote by $\mathcal{S}_{\text{rat}}(M)$ the subset of $\mathcal{S}(M)$ consisting of all rational tuples. Set

$$Z = \{[\gamma] \in \mathcal{S}_{\text{rat}}(T) \mid \text{the inclusion-induced map } H_1(T; \mathbb{Z}) \rightarrow H_1(W(\gamma); \mathbb{Z}) \text{ is zero}\},$$

where $W(\gamma)$ is the manifold obtained by Dehn filling T with slope γ .

Claim 1. The set Z is a discrete subset of $\mathcal{S}(T)$.

Proof of Claim 1. Denote by i the inclusion map $T \rightarrow W$. Suppose the map $H_1(T; \mathbb{Q}) \rightarrow H_1(W; \mathbb{Q})$ induced by i is zero. Then there is a compact $W_0 \subset W$ containing T such that $H_1(T; \mathbb{Q}) \rightarrow H_1(W_0; \mathbb{Q})$ is zero. But then we can attach compact 3-manifolds to W_0 along $\partial W_0 \setminus T$ to create a compact, orientable 3-manifold W_1 with boundary T such that $H_1(T; \mathbb{Q}) \rightarrow H_1(W_0; \mathbb{Q})$ is zero; however, this is impossible by [36, Lemma 3.5]. Thus the image of $H_1(T; \mathbb{Q}) \rightarrow H_1(W; \mathbb{Q})$ is at least of dimension 1.

If the image of $H_1(T; \mathbb{Q}) \rightarrow H_1(W; \mathbb{Q})$ is $\mathbb{Q}^2$, then for each primitive $\gamma \in H_1(T)$, the image of $H_1(T; \mathbb{Q})$ in $H_1(W(\gamma); \mathbb{Q})$ has dimension 1, so that $Z = \emptyset$ and the claim holds.

If the image of $H_1(T; \mathbb{Q}) \rightarrow H_1(W; \mathbb{Q})$ is $\mathbb{Q}$, then the kernel of $H_1(T; \mathbb{Q}) \rightarrow H_1(W; \mathbb{Q})$ is a summand of rank 1. Let $\gamma_0 \in H_1(T)$ be a primitive element that spans the kernel and $\gamma_1 \in H_1(T)$ a primitive element dual to γ_0 . Then the map $i_* : H_1(T) \rightarrow H_1(W)$ induced by inclusion is nontrivial and $i_*(\gamma_1)$ is of infinite order. It follows that for a primitive element $\gamma \in H_1(T)$, the order of γ_1 in $H_1(W(\gamma))$ is a non-zero multiple of $\Delta(\gamma, \gamma_0) = |\gamma \cdot \gamma_0|$. In particular, $H_1(W(\gamma))$ is nontrivial if $\Delta(\gamma, \gamma_0) \neq 1$. It follows that Z is a subset of $\{[\gamma_1 + n\gamma_0] \mid n \in \mathbb{Z}\}$, which is a discrete subset of $\mathcal{S}(T)$. $\square$

Let Z^* be the union of Z and the set of rational slopes $[\gamma] \in \mathcal{S}(T)$ such that $W(\gamma)$ is reducible. Then by a result of Gordon and Luecke [30], there are only finitely many slopes of the latter type, and so Z^* is a nowhere dense subset of $\mathcal{S}(T)$. It follows that the subset of rational slopes in $\mathcal{S}(T) \setminus Z^*$ is dense in $\mathcal{S}(T)$.

Claim 2. If $[\gamma] \in \mathcal{S}_{\text{rat}}(T) \setminus Z^*$, then $\pi_1(W(\gamma))$ is infinite and torsion-free.

Proof of Claim 2. Since $[\gamma] \notin Z^*$, $W(\gamma)$ is irreducible. Claim 1 implies that $\pi_1(W(\gamma))$ is nontrivial. Claim 2 follows immediately; see [1, Section 3.2 (C.1) and (C.2)]. $\square$

Claim 3. If $[\gamma] \in \mathcal{S}_{\text{rat}}(T) \setminus Z^*$, then the image of $\pi_1(T)$ in $\pi_1(W(\gamma))$ is infinite cyclic.

Proof of Claim 3. The map $\pi_1(T) \rightarrow \pi_1(W(\gamma))$ induced by inclusion factors through $\pi_1(T)/\langle \gamma \rangle$, which is infinite cyclic, and so the image is cyclic. The proof of Claim 1 indicates that the image of $H_1(T) \rightarrow H_1(W(\gamma))$ is non-zero when $[\gamma] \notin Z$. It follows from Claim 2 that the image of $\pi_1(T)$ in $\pi_1(W(\gamma))$ is infinite cyclic. $\square$

Claim 4. If $[\gamma] \in \mathcal{S}_{\text{rat}}(T) \setminus Z^*$, then $\pi_1(W(\gamma))$ is left-orderable.

Proof of Claim 4. Indeed, we show that $\pi_1(W(\gamma))$ is locally indicable, and hence left-orderable by Theorem 2.2.2. Now assume H is a nontrivial finitely generated subgroup of $\pi_1(W(\gamma))$ and $V \rightarrow W(\gamma)$ is the associated cover.

If $V \rightarrow W(\gamma)$ is a finite cover, then $\pi_1(W(\gamma))$ is finitely generated. We claim that at least one boundary component of a compact core N of the non-compact manifold $W(\gamma)$ has genus 1 or more. Otherwise, each is a 2-sphere, so the irreducibility of $W(\gamma)$ couples with the nontriviality of its fundamental group to show that each boundary component of N bounds a 3-ball contained in $W(\gamma) \setminus \text{int}(N)$, which is impossible as it would imply that $W(\gamma)$ is compact. It follows that the first Betti number of N , and hence $W(\gamma)$, is positive [9, Lemma 3.5], so the same is true of any of its finite degree covers. In particular, there is a surjection $H = \pi_1(V) \rightarrow \mathbb{Z}$.

If $V \rightarrow W(\gamma)$ is an infinite cover, an argument of Howie and Short shows that the first Betti number of V is positive (cf. the proof of [9, Theorem 1.1]). Thus $\pi_1(W(\gamma))$ is locally indicable. $\square$

Then the series of claims shows that Z^* is a nowhere dense subset of $\mathcal{S}(T)$ and for each $[\gamma] \in \mathcal{S}_{\text{rat}}(T) \setminus Z^*$, $\pi_1(W(\gamma))$ is left-orderable.

Now take an arbitrary $[\gamma] \in \mathcal{S}_{\text{rat}}(T) \setminus Z^*$. Note that $C = \pi_1(W)$ and $C/\langle\langle\gamma\rangle\rangle_C = \pi_1(W(\gamma))$. The short exact sequence

$$\{\text{id}\} \longrightarrow \langle\langle\gamma\rangle\rangle_C \longrightarrow C \longrightarrow C/\langle\langle\gamma\rangle\rangle_C \longrightarrow \{\text{id}\}$$

gives rise to a left-ordering $\mathfrak{o}_C$ on C by extension, where $\langle\langle\gamma\rangle\rangle_C$ is a proper $\mathfrak{o}_C$ -convex subgroup. Since $\langle\langle\gamma\rangle\rangle_C \cap \pi_1(T) = \langle\gamma\rangle$ by Claim 3, $\langle\gamma\rangle$ is a proper $\mathfrak{o}_C|_{\pi_1(T)}$ -convex subgroup of $\pi_1(T_1) \leq C$. It follows from Proposition 2.4.2 that $P(\mathfrak{o}_C) \sqcup (P(\mathfrak{o}) \setminus C)$ is a positive cone and determines a left-ordering $\mathfrak{o}_\gamma$ on $\pi_1(M)$. Now it is clear that $[L(\mathfrak{o}_\gamma|_{\pi_1(T_1)})] = [L(\mathfrak{o}_C|_{\pi_1(T_1)})] = [\gamma]$ by their constructions. Also note that for $i \in I$, since $\pi_1(T_i) \not\subset C$ and $P(\mathfrak{o}_\gamma) \setminus C = P(\mathfrak{o}) \setminus C$, we have $[L(\mathfrak{o}_\gamma|_{\pi_1(T_i)})] = [L(\mathfrak{o}|_{\pi_1(T_i)})]$. Hence, $\mathfrak{o}_\gamma$ order-detects $(\emptyset, \emptyset; [\beta_1], \dots, [\beta_n])$ with $[\beta_1] = [\gamma]$ and $[\beta_i] = [\alpha_i]$ for all $i \in I$. This shows that the conclusion of this theorem holds for all $[\beta] \in \mathcal{S}_{\text{rat}}(T) \setminus Z^*$.

It remains to show this theorem for $[\beta] \in \mathcal{S}(T)$ with $[\beta] \notin \mathcal{S}_{\text{rat}}(T) \setminus Z^*$. Since $\mathcal{S}_{\text{rat}}(T) \setminus Z^*$ is dense in $\mathcal{S}(M)$, we can pick a sequence $\{[\gamma_l]\} \subset \mathcal{S}_{\text{rat}}(T) \setminus Z^*$ that converges to $[\beta]$. Moreover, since $\text{LO}(\pi_1(M))$ is compact, the sequence $\{\mathfrak{o}_{\gamma_l}\}$, where each left-ordering $\mathfrak{o}_{\gamma_l}$ is constructed as in the last paragraph, admits a convergent subsequence $\{\mathfrak{o}_{\gamma_{l_t}}\}$, say converging to $\mathfrak{o}_\beta$. Since $\mathfrak{o}_{\gamma_{l_t}}$ order-detects $(\emptyset, \emptyset; [\beta_1], \dots, [\beta_n])$ with $[\beta_1] = [\gamma_{l_t}]$ and

$[\beta_i] = [\alpha_i]$ for all $i \in I$, we see that σ_β order-detects $(\emptyset, \emptyset; [\beta_1], \dots, [\beta_n])$ with $[\beta_1] = [\beta]$ and $[\beta_i] = [\alpha_i]$ for all $i \in I$. $\square$

Remark 4.5.4. We can enumerate the index of boundary tori and alter the index in Lemma 4.5.2 and Theorem 4.5.3, making them to be statements on $\pi_1(T_j)$ for any fixed $j \in \{1, \dots, n\}$ instead of $\pi_1(T_1)$.

The following results will allow us to “enlarge” intervals of non-detected slopes and we will use them in Chapter 6.

Theorem 4.5.5. *Let $L = L_1 \cup L_2$ be a hyperbolic two-component link in S^3 with M being its link complement. Suppose that no proper, relatively convex subgroup of $\pi_1(M)$ contains both $\pi_1(T_1)$ and $\pi_1(T_2)$ and σ is a left-ordering on $\pi_1(M)$. If $(\emptyset, \{1\}; [\alpha_1], [\alpha_2])$ is order-detected by σ and $\pi_1(T_2)$ is σ -bounded, then $(\emptyset, \emptyset; [\alpha_1], [\beta])$ is order-detected for all $[\beta] \in \mathcal{S}(T_2)$.*

Proof. By Lemma 4.5.2, there is a proper subgroup C and a left-ordering σ' of $\pi_1(M)$ such that C is σ' -convex and $\pi_1(T_2) \subset C$, and σ' order-detects $(\emptyset, \emptyset; [\alpha_1], [\beta'])$ for some $[\beta'] \in \mathcal{S}(T_2)$. Since $\pi_1(T_1) \not\subset C$ by assumption, we can apply Theorem 4.5.3 to σ' , and conclude that for every $\beta \in \mathcal{S}(T_2)$ there exists a left-ordering σ'' that order-detects $(\emptyset, \emptyset; [\alpha_1], [\beta])$. $\square$

Theorem 4.5.6. *Suppose $L = L_1 \cup L_2$ is a hyperbolic two-component link in S^3 with M being its link complement. For $i \in \{1, 2\}$, let a_i/b_i be a rational number in lowest terms and $\{m_i, l_i\}$ the peripheral system consisting of a meridian and a longitude along T_i . Further assume that $\pi_1(T_1) \not\subset \langle\langle m_1^{a_1} l_1^{b_1} \rangle\rangle$ and $\pi_1(T_2) \not\subset \langle\langle m_2^{a_2} l_2^{b_2} \rangle\rangle$. If $\pi_1(S_{a_1/b_1, a_2/b_2}^3(L))$ is left-orderable, then one of $(\{1\}, \{1, 2\}; [m_1^{a_1} l_1^{b_1}], [m_2^{a_2} l_2^{b_2}])$ or $(\{2\}, \{1, 2\}; [m_1^{a_1} l_1^{b_1}], [m_2^{a_2} l_2^{b_2}])$ is order-detected, and in general, $(\emptyset, \{1, 2\}; [m_1^{a_1} l_1^{b_1}], [m_2^{a_2} l_2^{b_2}])$ is always order-detected.*

Proof. Set $C = \langle\langle m_1^{a_1} l_1^{b_1}, m_2^{a_2} l_2^{b_2} \rangle\rangle$. Note that $\pi_1(S_{a_1/b_1, a_2/b_2}^3(L))$ is nontrivial since it is left-orderable. Since $\pi_1(M)$ is generated by the conjugates of $\{m_1, m_2\}$, we cannot have $\pi_1(T_1) \subset C$ and $\pi_1(T_2) \subset C$ simultaneously. So there are two cases to consider.

As a first case, suppose that $\pi_1(T_i) \not\subset C$ for $i = 1, 2$. Construct a left-ordering σ of $\pi_1(M)$ by extension using the short exact sequence

$$\{\text{id}\} \longrightarrow C \longrightarrow \pi_1(M) \longrightarrow \pi_1(S_{a_1/b_1, a_2/b_2}^3(L)) \longrightarrow \{\text{id}\},$$

so that C is σ -convex. Observe that $C \cap \pi_1(T_i)$ is a proper subgroup of $\pi_1(T_i)$ that contains $m_i^{a_i} l_i^{b_i}$, and the quotient $\pi_1(M)/C$ is torsion-free. This forces $C \cap \pi_1(T_i) = \langle m_i^{a_i} l_i^{b_i} \rangle$ for $i = 1, 2$. Because C is normal, C is $g \cdot \sigma$ -convex for all $g \in \pi_1(M)$. It follows

that $C \cap \pi_1(T_i) = \langle m_i^{a_i} l_i^{b_i} \rangle$ is $(g \cdot o)|_{\pi_1(T_i)}$ -convex in $\pi_1(T_i)$ for $i = 1, 2$. This means that $s(g \cdot o) = ([m_1^{a_1} l_1^{b_1}], [m_2^{a_2} l_2^{b_2}])$ and thus o order-detects $(\{1, 2\}, \{1, 2\}; [m_1^{a_1} l_1^{b_1}], [m_2^{a_2} l_2^{b_2}])$.

On the other hand, suppose that one of $\pi_1(T_1)$ or $\pi_1(T_2)$ is contained in C and the other is not, say $\pi_1(T_1) \subset C$ and $\pi_1(T_2) \not\subset C$. Construct a left-ordering o' of C by extension using the short exact sequence

$$\{\text{id}\} \longrightarrow \langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle \longrightarrow C \longrightarrow C / \langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle \longrightarrow \{\text{id}\}.$$

Since $\pi_1(T_1) \not\subset \langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle$, we have $C \neq \langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle$. Note that $\pi_1(S_{a_1/b_1,*}^3(L))$ is left-orderable by Theorem 3.2.4, since $S_{a_1/b_1,*}^3(L)$ is an irreducible manifold with infinite first homology. It follows that $C / \langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle$ is left-orderable, because it is a subgroup of $\pi_1(S_{a_1/b_1,*}^3(L))$. Next, consider the short exact sequence

$$\{\text{id}\} \longrightarrow C \longrightarrow \pi_1(M) \longrightarrow \pi_1(S_{a_1/b_1, a_2/b_2}^3(L)) \longrightarrow \{\text{id}\}$$

and construct a left-ordering o'' of $\pi_1(M)$ by extension using o' on the subgroup C . By our construction, both C and $\langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle$ are normal and o'' -convex. Arguing as above, $\langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle$ is $g \cdot o''$ -convex for all $g \in \pi_1(M)$, and so $\langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle \cap \pi_1(T_1) = \langle m_1^{a_1} l_1^{b_1} \rangle$ is $(g \cdot o'')|_{\pi_1(T_1)}$ -convex in $\pi_1(T_1)$ (here we use that $\pi_1(T_1) \not\subset \langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle$). We can similarly analyse the restriction of $g \cdot o''$ to $\pi_1(T_2)$ and conclude that $(\{2\}, \{1, 2\}; [m_1^{a_1} l_1^{b_1}], [m_2^{a_2} l_2^{b_2}])$ is order-detected by o'' .

In the case where $\pi_1(T_2) \subset C$ and $\pi_1(T_1) \not\subset C$, we proceed similarly and deduce that $(\{1\}, \{1, 2\}; [m_1^{a_1} l_1^{b_1}], [m_2^{a_2} l_2^{b_2}])$ is o'' -detected (here we use that $\pi_1(T_2) \not\subset \langle \langle m_2^{a_2} l_2^{b_2} \rangle \rangle$).

Finally, we note that regardless of cases, $(\emptyset, \{1, 2\}; [m_1^{a_1} l_1^{b_1}], [m_2^{a_2} l_2^{b_2}])$ is always order-detected by Remark 4.2.2. $\square$

Remark 4.5.7. The condition that $\pi_1(T_1) \not\subset \langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle$ and $\pi_1(T_2) \not\subset \langle \langle m_2^{a_2} l_2^{b_2} \rangle \rangle$ holds generally for homological reasons. Note that $\pi_1(M) / \langle \langle m_1^{a_1} l_1^{b_1} \rangle \rangle = \pi_1(S_{a_1/b_1,*}^3(L))$, but the meridian and/or the longitude of $\pi_1(T_i)$ survive inside $H_1(S_{a_1/b_1,*}^3(L); \mathbb{Z})$ in general. For example, see Lemma 6.1.6.

4.6 Gluing theorems

The goal of this section is to develop theorems that allow one to analyse the boundary behaviour of left-orderings of the fundamental group of a 3-manifold M in terms of tuples of slopes which are order-detected on the boundary tori of the JSJ pieces of M . We already have at our disposal various gluing theorems from the literature that deal with

special cases of left-orderable groups and detection. We review these results and offer improvements, with a later focus on special cases which apply to cable knots. We also remark that while this section is written in the language of orderings of fundamental groups, each “orderability gluing theorem” has a representation-theoretic counterpart, owing to our work in the previous sections. Our focus will be on generalising the following result.

Theorem 4.6.1 ([5, Theorem 7.10]). *For $i = 1, 2$, let M_i be a compact, connect, irreducible and orientable 3-manifolds such that ∂M_i is a nonempty union of incompressible tori $\partial M_i = T_{i,1} \sqcup T_{i,2} \sqcup \cdots \sqcup T_{i,r_i}$. Suppose $\mathfrak{o}_i \in \text{LO}(\pi_1(M_i))$ is a left-ordering on $\pi_1(M_i)$ that order-detects $(\emptyset, \{1, 2, \dots, r_i\}; [\alpha_{i,1}], [\alpha_{i,2}], \dots, [\alpha_{i,r_i}])$ and $f : T_{1,1} \rightarrow T_{2,1}$ is a homeomorphism that identifies $[\alpha_{1,1}]$ with $[\alpha_{2,1}]$. Reindex the boundary components $T_{1,2}, T_{1,3}, \dots, T_{1,r_1}, T_{2,2}, T_{2,3}, \dots, T_{2,r_2}$ of $M_1 \cup_f M_2$ as $T_1, \dots, T_{r_1+r_2-2}$, respectively. Then $\pi_1(M_1 \cup_f M_2)$ is left-orderable and admits a left-ordering $\mathfrak{o}'$ order-detecting*

$$(\emptyset, \{1, 2, \dots, r_1 + r_2 - 2\}; [\alpha_{1,2}], \dots, [\alpha_{1,r_1}], [\alpha_{2,2}], \dots, [\alpha_{2,r_2}]),$$

whose restriction to $\pi_1(M_i)$ is $\mathfrak{o}_i$.

We begin our efforts with some preparatory lemmas. Let M be a compact connected orientable 3-manifold with incompressible torus boundary components $T_1, \dots, T_n$, $n \geq 1$, and fix a choice of peripheral subgroup $\pi_1(T_i) \subset \pi_1(M)$ for each i . Recall that the map $s_i : \text{LO}(\pi_1(M)) \rightarrow \mathcal{S}(T_i)$ is the composition $\mathcal{L} \circ r_i$, where r_i is the restriction map to $\pi_1(T_i)$ and $\mathcal{L}(\mathfrak{o}) = [L(\mathfrak{o})]$ is the map defined in Lemma 4.1.1.

Definition 4.6.2 ([4]). A set $\mathcal{O}$ of left-orderings of $\pi_1(M)$ is called *ready to glue* on T_i , or ready to glue along $s_i(\mathcal{O})$ on T_i , if $\mathcal{O}$ is normal and for all $[\alpha] \in s_i(\mathcal{O})$ we have $\mathcal{L}^{-1}([\alpha]) \subset r_i(\mathcal{O})$. More generally, suppose that G is a left-orderable group with a subgroup $H \cong \mathbb{Z}^2$. A set $\mathcal{O}$ of left-orderings of G is called *ready to glue* on H if $\mathcal{O}$ is normal and for all $[\alpha] \in s(\mathcal{O})$ we have $\mathcal{L}^{-1}([\alpha]) \subset r(\mathcal{O})$, where s is the composition

$$\text{LO}(G) \xrightarrow{r} \text{LO}(H) \xrightarrow{\mathcal{L}} S^1.$$

The following is a slight generalisation of [4, Proposition 11.5], the proof being nearly identical (and so it is omitted). In the following, we use the map $s_i : \text{LO}(G_i) \rightarrow \mathcal{S}(\phi_i(A))$ to denote the composition $\mathcal{L} \circ r_i$, where r_i is the restriction map.

Proposition 4.6.3. *Suppose that A, G_1 and G_2 are groups equipped with injective homomorphisms $\phi_1 : A \rightarrow G_1, \phi_2 : A \rightarrow G_2$ and that $A \cong \mathbb{Z}^2$. Suppose that $N_1 \subset \text{LO}(G_1)$ and $N_2 \subset \text{LO}(G_2)$ are normal families containing orderings $\mathfrak{o}_1$ and $\mathfrak{o}_2$ respectively, and that each*

N_i is ready to glue along $\phi_i(A)$. If $s_1(N_1) = s_2(N_2)$ and $\phi_1^{-1}(P(o_1)) = \phi_2^{-1}(P(o_2))$, then $G_1 *_{\phi_i} G_2$ admits a left-ordering o whose restriction to G_i is o_i .

The next lemma and proposition are essential for creating families of orderings that are ready to glue.

Lemma 4.6.4. *Let $J \subset K \subset \{1, \dots, n\}$ and $[\alpha_*] \in \mathcal{S}(M)$. Suppose that o is a left-ordering of $\pi_1(M)$ that order-detects $(J, K; [\alpha_*])$ and C is an o -convex subgroup of $\pi_1(M)$. Then $Q = (P(o) \cap C)^{-1} \cup (P(o) \setminus C)$ is the positive cone of a left-ordering o' that also detects $(J, K; [\alpha_*])$.*

Proof. That Q is a positive cone is a standard check. We verify that the ordering o' has the required properties. Fix $i \in \{1, \dots, r\}$ and consider $C \cap \pi_1(T_i)$. Since C is a convex subgroup, there are three possibilities.

First, if $C \cap \pi_1(T_i) = \{\text{id}\}$ then $\pi_1(T_i) \cap Q = \pi_1(T_i) \cap P(o)$ so that $s_i(o') = [\alpha_i]$. Second, if $C \cap \pi_1(T_i) \cong \mathbb{Z}$, then $[\alpha_i]$ must be rational with $C \cap \pi_1(T_i) = \langle \alpha_i \rangle$. It follows that $(\pi_1(T_i) \setminus \langle \alpha_i \rangle) \cap Q = P(o) \cap (\pi_1(T_i) \setminus \langle \alpha_i \rangle)$ so that again $s_i(o') = [\alpha_i]$. In the final case, if $\pi_1(T_i) \subset C$, then $Q \cap \pi_1(T_i) = P(o)^{-1} \cap \pi_1(T_i)$ so that again $s_i(o') = [\alpha_i]$. This shows O1.

Next, assume $i \in K$ and let $g \in \pi_1(M)$. Consider $g \cdot o'$ whose positive cone is gQg^{-1} , there are three cases as in the previous paragraph. First, if $\pi_1(T_i) \cap gCg^{-1} = \{\text{id}\}$, then $gQg^{-1} \cap \pi_1(T_i) = gP(o)g^{-1} \cap \pi_1(T_i)$ and so $s_i(g \cdot o') = s_i(g \cdot o) = s_i(o) = s_i(o')$. Second, if $\pi_1(T_i) \cap gCg^{-1} \cong \mathbb{Z}$, then since gCg^{-1} is also $g \cdot o$ -convex and $s_i(g \cdot o) = [\alpha_i]$ since $i \in K$, we know that $\pi_1(T_i) \cap gCg^{-1} = \langle \alpha_i \rangle$. Then observe that $gQg^{-1} = (P(g \cdot o) \setminus gCg^{-1}) \cup (P(g \cdot o) \cap gCg^{-1})^{-1}$, so that

$$gQg^{-1} \cap (\pi_1(T_i) \setminus \langle \alpha_i \rangle) = (P(g \cdot o) \setminus gCg^{-1}) \cap (\pi_1(T_i) \setminus \langle \alpha_i \rangle),$$

from which it follows that $s_i(g \cdot o') = s_i(g \cdot o) = s_i(o) = s_i(o')$. Finally, if $\pi_1(T_i) \subset gCg^{-1}$, then repeating an argument that is nearly identical to the first case proves O2.

Lastly, if H is the normal subgroup of $\pi_1(M)$ that is o -convex and satisfies O3, then either $H \subset C$ or $C \subset H$. In either case, it is easy to verify that H is also o' -convex and satisfies the required properties. $\square$

Proposition 4.6.5. *Let $J \subset K \subset \{1, \dots, n\}$ and $[\alpha_*] \in \mathcal{S}(M)$. Suppose that o is a left-ordering of $\pi_1(M)$ that order-detects $(J, K; [\alpha_*])$, and that $[\alpha_1]$ is rational. Then there exists a left-ordering o' of $\pi_1(M)$ such that:*

- (1) $P(o) \cap (\pi_1(T_1) \setminus \langle \alpha_1 \rangle) = P(o') \cap (\pi_1(T_1) \setminus \langle \alpha_1 \rangle)$,
- (2) $s_j(g \cdot o') = s_j(o') = s_j(o)$ for all $j \in K$ and $g \in \pi_1(M)$,
- (3) if $\alpha_1 >_o \text{id}$, then $\alpha_1 <_{o'} \text{id}$; and if $\alpha_1 <_o \text{id}$, then $\alpha_1 >_{o'} \text{id}$.

Proof. Let $\rho_o : \pi_1(M) \rightarrow \text{Homeo}_+(\mathbb{R})$ be the dynamic realisation of o defined from the tight embedding $t : G \rightarrow \mathbb{R}$ satisfying $t(\text{id}) = 0$. Set

$$x_0 = \sup\{t(\alpha_1^k) : k \in \mathbb{Z}\}$$

and set $C = \text{Stab}_{\rho_o}(x_0)$. Note that x_0 is an ideal point of the dynamic realisation; see Section 2.3.

By Lemma 2.4.14, there exists $o' \in \overline{\{h \cdot o : h \in \pi_1(M)\}}$, the closure of the orbit of o in $\text{LO}(\pi_1(M))$, such that C is o' -convex.

By Lemma 2.4.13, the left-ordering o' satisfies (1). To show (2), note that $g \cdot o' \in \overline{\{h \cdot o : h \in \pi_1(M)\}}$ for all $g \in \pi_1(M)$. Since the slope map $s_j : \text{LO}(\pi_1(M)) \rightarrow \mathcal{S}(T_j)$ is continuous and takes the constant value $s_j(o)$ on $\{h \cdot o : h \in \pi_1(M)\}$ for all $j \in K$, (2) holds.

Lastly, by Lemma 4.6.4 we can choose the sign of α_1 so that (3) is also satisfied. $\square$

Remark 4.6.6. Although Proposition 4.6.5 is stated in a way that deals only with 3-manifold groups, an analogous proposition holds if we replace $\pi_1(M)$ with an arbitrary group G and $\pi_1(T_i)$ with subgroups $H_i \subset G$ each isomorphic to $\mathbb{Z}^2$.

Theorem 4.6.7. *Let M_1 and M_2 be 3-manifolds such that ∂M_i is a union of incompressible tori $T_{i,1} \sqcup T_{i,2} \sqcup \dots \sqcup T_{i,r_i}$. Suppose that for $i = 1, 2$, $(J_i, K_i; [\alpha_{i,1}], [\alpha_{i,2}], \dots, [\alpha_{i,r_i}])$ is order-detected by $o_i \in \text{LO}(\pi_1(M_i))$ and that $f : T_{1,1} \rightarrow T_{2,1}$ is a homeomorphism that identifies $[\alpha_{1,1}]$ with $[\alpha_{2,1}]$. Re-index the boundary components $T_{1,2}, T_{1,3}, \dots, T_{1,r_1}, T_{2,2}, T_{2,3}, \dots, T_{2,r_2}$ of the manifold $M_1 \cup_f M_2$ as $T_1, \dots, T_{r_1+r_2-2}$, respectively. Set $J'_1 = \{n-1 : n \in J_1, n \geq 2\}$, $K'_1 = \{n-1 : n \in K_1, n \geq 2\}$, $J'_2 = \{n+r_1-2 : n \in J_2, n \geq 2\}$ and $K'_2 = \{n+r_1-2 : n \in K_2, n \geq 2\}$. Suppose that $1 \in K_i$ for $i = 1, 2$, and let $C_i \leq \pi_1(M_i)$ be the o_i -convex normal subgroup guaranteed by O3. Suppose that if $[\alpha_{i,1}]$ is rational and there exists $i \in \{1, 2\}$ such that $C_i \cap \pi_1(T_{i,1}) = \langle \alpha_{i,1} \rangle$, then either $(J_i \cup \{\text{id}\}, K_i; [\alpha_{i,1}], [\alpha_{i,2}], \dots, [\alpha_{i,r_i}])$ is order-detected for $i = 1, 2$ or $C_i \cap \pi_1(T_{i,j}) = \{\text{id}\}$ for all $j \neq 1$.*

Then $\pi_1(M_1 \cup_f M_2)$ is left-orderable and admits a left-ordering detecting

$$(J'_1 \cup J'_2, K'_1 \cup K'_2; [\alpha_{1,2}], \dots, [\alpha_{1,r_1}], [\alpha_{2,2}], \dots, [\alpha_{2,r_2}]).$$

Proof. We can largely mirror the proof of [5, Theorem 7.10], making ad hoc changes. Observe as in [5] that if either manifold is a product $T^2 \times [0, 1]$ then the result holds trivially. Therefore, for each i , $\pi_1(T_{i,1})$ is a proper subgroup of $\pi_1(M_i)$, and we consider two cases.

For the first case, suppose each $[\alpha_{i,1}]$ is irrational. Let $C_i \subset \pi_1(M_i)$ be the o_i -convex subgroup guaranteed by O3, and $q_i : \pi_1(M_i) \rightarrow \pi_1(M_i)/C_i$ the quotient map. The maps

q_1, q_2 induce a map

$$q : \pi_1(M_1 \cup_f M_2) \rightarrow \pi_1(M_1)/C_1 *_{f_i} \pi_1(M_2)/C_2$$

where $f_i : \mathbb{Z}^2 \rightarrow q_i(\pi_1(T_{i,1}))$ are isomorphisms that satisfies $f_* \circ f_1 = f_2$ where we denote by $f_* : q_1(\pi_1(T_{1,1})) \rightarrow q_2(\pi_1(T_{2,1}))$ the isomorphism induced by f . We will write $\Gamma = \pi_1(M_1)/C_1 *_{f_i} \pi_1(M_2)/C_2$ for convenience in the rest of the proof.

Let $\hat{o}_i$ denote the natural left-ordering of the quotient $\pi_1(M_i)/C_i$ induced by σ_i , and consider the normal families

$$N_i = \{g \cdot \hat{o}_i : g \in \pi_1(M_i)/C_i\} \cup \{g \cdot \hat{o}_i^{\text{op}} : g \in \pi_1(M_i)/C_i\} \subset \text{LO}(\pi_1(M_i)/C_i)$$

for $i = 1, 2$. Since $1 \in K_i$ for $i = 1, 2$, and therefore $s_{i,1}(g \cdot \sigma_i) = [\alpha_{i,1}]$ for all $g \in \pi_1(M_i)$, we conclude that the image of N_i under the composition

$$\pi_1(M_i)/C_i \xrightarrow{r_{i,1}} q_i(\pi_1(T_{i,1})) \xrightarrow{\mathcal{L}} \mathcal{S}(\pi_1(T_{i,1}))$$

is the singleton $\{[\alpha_{i,1}]\}$, here $r_{i,1}$ is the restriction map and $s_{i,1} : \text{LO}(\pi_1(M_i)) \rightarrow \mathcal{S}(T_{i,1})$ the slope map. Moreover, $\mathcal{L}^{-1}([\alpha_{i,1}]) \subset r_{i,1}(N_i)$. Therefore, the families N_1, N_2 are compatible with the maps f_i , in fact, they are ready to glue. So, by Theorem 2.2.3 there exists a left-ordering $\hat{o}$ of Γ whose restriction to $\pi_1(M_i)/C_i$ is $\hat{o}_i$.

Construct a left-ordering σ of $\pi_1(M_1 \cup_f M_2)$ lexicographically from the short exact sequence

$$\{\text{id}\} \longrightarrow \ker(q) \longrightarrow \pi_1(M_1 \cup_f M_2) \xrightarrow{q} \Gamma \longrightarrow \{\text{id}\}$$

using $\hat{o}$ on the quotient and an arbitrary left-ordering of the kernel (note that, indeed, the kernel is left-orderable since it is a subgroup of $\pi_1(M_1 \cup_f M_2)$ which is left-orderable by Theorem 4.6.1.) We verify that this ordering satisfies the required properties.

Firstly, $s(\sigma) = ([\alpha_{1,2}], \dots, [\alpha_{1,r_1}], [\alpha_{2,2}], \dots, [\alpha_{2,r_2}])$ by construction, so that O1 holds. Moreover, from the proof of [2, Theorem A], the restriction of the left-ordering $\hat{o}$ of Γ satisfies $\hat{o}|_{h\pi_1(M_i)/C_i h^{-1}} \in N_i$ for all $h \in \Gamma$. Therefore, the restriction of $\hat{o}$ to $h\pi_1(M_i)/C_i h^{-1}$ is of the form $g \cdot \hat{o}_i$ or $g \cdot \hat{o}_i^{\text{op}}$ for some $g \in \pi_1(M_i)/C_i$ for all $h \in \Gamma$. It follows that $s(h \cdot \sigma) = ([\beta_1], \dots, [\beta_{r_1+r_2-2}])$ where $[\beta_j] = s_j(\sigma)$ for all $j \in K'_1 \cup K'_2$. Thus, O2 holds. Finally, the kernel $C = \ker(q)$ is the normal, σ -convex subgroup required by O3, so we conclude that

$$(J'_1 \cup J'_2, K'_1 \cup K'_2; [\alpha_{1,2}], \dots, [\alpha_{1,r_1}], [\alpha_{2,2}], \dots, [\alpha_{2,r_2}])$$

is detected by σ . This proves the theorem in the first case, where $[\alpha_{i,1}]$ is irrational.

Next, we modify the proof to handle the case where $[\alpha_{i,1}]$ is rational.

First, suppose that $(J_t \cup \{1\}, K_t; [\alpha_{t,1}], [\alpha_{t,2}], \dots, [\alpha_{t,r_t}])$ is order-detected for $t = 1, 2$ and that there exists i such that $C_i \cap \pi_1(T_{i,1}) = \langle \alpha_{i,1} \rangle$. Then perhaps after replacing one of the orderings σ_i , we may assume that $(J_i \cup \{1\}, K_i; [\alpha_{i,1}], [\alpha_{i,2}], \dots, [\alpha_{i,r_i}])$ is σ_i -detected for $i = 1, 2$.

In this case, q_1, q_2 induce a map

$$q : \pi_1(M_1 \cup_f M_2) \rightarrow \Gamma$$

where $f_i : \mathbb{Z} \rightarrow q_i(\pi_1(T_{i,1}))$ are isomorphisms satisfying $f_* \circ f_1 = f_2$ where $f_* : \mathbb{Z} \cong q_1(\pi_1(T_{1,1})) \rightarrow q_2(\pi_1(T_{2,1})) \cong \mathbb{Z}$ is the isomorphism induced by f . Since the image of q is an amalgam of left-orderable groups along a cyclic subgroup, it is left-orderable, with any two compatible orderings of the factors extending to a left-ordering of the amalgam.

Denoting the natural left-orderings of the quotients $\pi_1(M_i)/C_i$ again by $\hat{\sigma}_i$, we fix a left-ordering $\hat{\sigma}$ of Γ that restricts to $\hat{\sigma}_i$ on $\pi_1(M_i)/C_i$. As before, construct a left-ordering σ of $\pi_1(M_1 \cup_f M_2)$ lexicographically from the short exact sequence

$$\{\text{id}\} \longrightarrow \ker(q) \longrightarrow \pi_1(M_1 \cup_f M_2) \xrightarrow{q} \Gamma \longrightarrow \{\text{id}\}$$

using $\hat{\sigma}$ on the quotient and an arbitrary left-ordering of the kernel. Arguing exactly as in the previous case, we see that σ satisfies O1, O2, and O3, so that

$$(J'_1 \cup J'_2, K'_1 \cup K'_2; [\alpha_{1,2}], \dots, [\alpha_{1,r_1}], [\alpha_{2,2}], \dots, [\alpha_{2,r_2}])$$

is order-detected in this case.

Last, suppose that $[\alpha_{i,1}]$ is rational and that for each i , either $C_i \cap \pi_1(T_{i,1}) = \{\text{id}\}$ or $C_i \cap \pi_1(T_{i,1}) = \langle \alpha_{i,1} \rangle$ and $C_i \cap \pi_1(T_{i,j}) = \{\text{id}\}$ for all $j \neq 1$. Then, as in the previous cases, we begin with convex subgroups C_1, C_2 arising from O3, but we replace C_i with $C_i = \{\text{id}\}$ if $C_i \cap \pi_1(T_{i,1}) = \langle \alpha_{i,1} \rangle$ and $C_i \cap \pi_1(T_{i,j}) = \{\text{id}\}$ for all $j \neq 1$. Then again we have quotient maps q_1, q_2 , and

$$q : \pi_1(M_1 \cup_f M_2) \rightarrow \Gamma$$

where $f_i : \mathbb{Z}^2 \rightarrow q_i(\pi_1(T_{i,1}))$ are isomorphisms satisfying $f_2 \circ f_* = f_1$ and f_* is the isomorphism $q_1(\pi_1(T_{1,1})) \rightarrow q_2(\pi_1(T_{2,1}))$ induced by f . Note that our maps f_i are isomorphisms precisely because we chose C_i such that $\alpha_{i,1} \notin C_i$ for $i = 1, 2$. Let $\hat{\sigma}_i$ denote the natural ordering of the quotient $\pi_1(M_i)/C_i$ induced by σ_i . By Remark 4.6.6 and Proposition 4.6.5 there is a left-ordering $\hat{\sigma}'_i$ of each quotient $\pi_1(M_i)/C_i$ such that:

$$(1) \ P(\hat{\sigma}_i) \cap (q_i(\pi_1(T_{i,1})) \setminus \langle q(\alpha_{i,1}) \rangle) = P(\hat{\sigma}'_i) \cap (q_i(\pi_1(T_{i,1})) \setminus \langle q(\alpha_{i,1}) \rangle),$$

- (2) when restricted to $q_i(\pi_1(T_{i,j}))$ where $j \in K$, the orderings $g \cdot \hat{\sigma}'_i$ and $\hat{\sigma}_i$ determine the same slope, and
- (3) if $\alpha_{i,1} >_{\hat{\sigma}_i} \text{id}$ then $\alpha_{i,1} <_{\hat{\sigma}'_i} \text{id}$ and if $\alpha_{i,1} <_{\hat{\sigma}_i} \text{id}$ then $\alpha >_{\hat{\sigma}'_i} \text{id}$.

Now define

$$N_i = \{g \cdot \hat{\sigma}_i : g \in \pi_1(M_i)/C_i\} \cup \{g \cdot \hat{\sigma}_i^{\text{op}} : g \in \pi_1(M_i)/C_i\} \\ \cup \{g \cdot \hat{\sigma}'_i : g \in \pi_1(M_i)/C_i\} \cup \{g \cdot (\hat{\sigma}'_i)^{\text{op}} : g \in \pi_1(M_i)/C_i\}$$

By construction, the families N_i are compatible with the maps f_i and are ready to glue along $f_i(\pi_1(T_{i,1}))$. So by Proposition 4.6.3 there is a left-ordering $\hat{\sigma}$ of Γ whose restriction to $\pi_1(M_i)/C_i$ is $\hat{\sigma}_i$, and we can construct σ of $\pi_1(M_1 \cup_f M_2)$ lexicographically as in the first case. By arguments identical to the first case, the ordering σ detects

$$(J'_1 \cup J'_2, K'_1 \cup K'_2; [\alpha_{1,2}], \dots, [\alpha_{1,r_1}], [\alpha_{2,2}], \dots, [\alpha_{2,r_2}]),$$

as required. $\square$

We can also glue, in some circumstances, along slopes that are weakly detected. However, in doing so, we cannot control the boundary behaviour of the resulting left-ordering to the same degree as in Theorem 4.6.7. We begin with lemmas that allow us to construct families of orderings which are ready to glue. The next lemma can be viewed as a special case of Proposition 4.6.5 which is sufficient for our purposes here.

Lemma 4.6.8. [4, Lemma 11.10] *Suppose that σ is a left-ordering of G and $g, f \in G$. If $\{g^k : k \in \mathbb{Z}\}$ is bounded above by f in the left-ordering σ , then there is a left-ordering σ' of G and a proper σ' -convex subgroup C of G that contains g but not f .*

For the statement of the next lemma, recall that $r_i : \text{LO}(\pi_1(M)) \rightarrow \text{LO}(\pi_1(T_i))$ denotes the restriction map, and $\mathcal{L}_i$ the map that associates a slope with each ordering of $\pi_1(T_i)$; the map s_i denotes their composition.

Lemma 4.6.9. *Suppose that $\mathcal{O}$ is a family of left-orderings of $\pi_1(M)$. Then for each fixed $i = 1, \dots, n$, there is a family $R_i(\mathcal{O})$ of left-orderings of $\pi_1(M)$ that contains $\mathcal{O}$ and satisfies $\mathcal{L}^{-1}([\alpha]) \subset r_i(R_i(\mathcal{O}))$ for all $[\alpha] \in s_i(R_i(\mathcal{O}))$.*

Proof. This is a restatement of [5, Lemma 7.8] in the case where M has multiple boundary components; where the original lemma is stated for knot manifolds. We proceed as follows:

For every slope $[\gamma] \in s_i(\mathcal{O})$, define a set $R([\gamma])$ as follows. If $[\gamma]$ is irrational, choose $\mathfrak{o} \in \mathcal{O}$ with $s_i(\mathfrak{o}) = [\gamma]$ and set $R([\gamma]) = \{\mathfrak{o}, \mathfrak{o}^{\text{op}}\}$ so that $r_i(R([\gamma])) = \mathcal{L}^{-1}([\gamma])$. If $[\gamma]$ is rational, choose $\mathfrak{o} \in \mathcal{O}$ with $s_i(\mathfrak{o}) = [\gamma]$. By Lemma 4.6.8, there exists a left-ordering $\mathfrak{o}'$ and a proper $\mathfrak{o}'$ -convex subgroup C of $\pi_1(M)$ such that $\pi_1(T_i) \cap C = \langle \gamma \rangle$. So in this case, we can create a set $R([\gamma])$ of four left-orderings of $\pi_1(M)$ satisfying $r_i(R([\gamma])) = \mathcal{L}^{-1}([\gamma])$ as in Lemma 4.6.4. Finally, set

$$R_i(\mathcal{O}) = \mathcal{O} \cup \left(\bigcup_{[\gamma] \in s_i(\mathcal{O})} R([\gamma]) \right)$$

and by construction, the set $R_i(\mathcal{O})$ satisfies the required condition. $\square$

Proposition 4.6.10. *Fix a left-ordering $\mathfrak{o}$ of $\pi_1(M)$ and $i \in \{1, \dots, n\}$. Then there exists a family $\mathcal{O}$ of left-orderings of $\pi_1(M)$ containing $\mathfrak{o}$ that is ready to glue along $s_i(\mathcal{O})$ on T_i .*

Proof. Given a set S of left-orderings of $\pi_1(M)$, write $N(S)$ for the set $\{g \cdot \mathfrak{o} : \mathfrak{o} \in S \text{ and } g \in \pi_1(M)\}$. Set $X_0 = \{\mathfrak{o}\}$ and for $j \geq 0$ set $X_{j+1} = N(R_i(X_j))$, where $R_i(S)$ is defined as in Lemma 4.6.9. Set $\mathcal{O} = \bigcup_{j=0}^{\infty} X_j$. By construction, $\mathcal{O}$ is both normal and ready to glue along $s_i(\mathcal{O})$ on T_i . $\square$

Theorem 4.6.11. *Let M_1 and M_2 be 3-manifolds such that ∂M_i is a union of incompressible tori $T_{i,1} \sqcup T_{i,2} \sqcup \dots \sqcup T_{i,r_i}$. Suppose that for each manifold M_i , $(J_i, K_i; [\alpha_{i,1}], [\alpha_{i,2}], \dots, [\alpha_{i,r_i}])$ is order-detected by $\mathfrak{o}_i \in \text{LO}(\pi_1(M_i))$ and that $f : T_{1,1} \rightarrow T_{2,1}$ is a homeomorphism that identifies $[\alpha_{1,1}]$ with $[\alpha_{2,1}]$. Re-index the boundary components $T_{1,2}, T_{1,3}, \dots, T_{1,r_1}, T_{2,2}, T_{2,3}, \dots, T_{2,r_2}$ of the manifold $M_1 \cup_f M_2$ as $T_1, \dots, T_{r_1+r_2-2}$ respectively. Suppose that $K_2 = \{1, \dots, r_2\}$, and for every $[\alpha] \in \mathcal{S}(T_{2,1})$, there exists a left-ordering $\mathfrak{o}_{[\alpha]}$ of $\pi_1(M_2)$ detecting some $(J'_2, K'_2; [\alpha_*])$ with $1 \in K'_2$ and $s_{2,1}(\mathfrak{o}) = [\alpha]$.*

Then $\pi_1(M_1 \cup_f M_2)$ is left-orderable and admits a left-ordering $\mathfrak{o}$ detecting

$$(\emptyset, \emptyset; [\alpha_{1,2}], \dots, [\alpha_{1,r_1}], [\alpha_{2,2}], \dots, [\alpha_{2,r_2}]);$$

moreover, the restriction of $\mathfrak{o}$ to $\pi_1(M_1)$ is $\mathfrak{o}_1$.

Proof. By Proposition 4.6.10, there exists $N_1 \subset \text{LO}(\pi_1(M_1))$ containing $\mathfrak{o}_1$ that is ready to glue along $T_{1,1}$.

Set

$$S = \{[\alpha] \in \mathcal{S}(T_{2,1}) : \exists [\gamma] \in s_{1,1}(N_1) \text{ such that } f_*([\gamma]) = [\alpha]\}$$

where $s_{1,1} : \text{LO}(\pi_1(M_1)) \rightarrow \mathcal{S}(T_{1,1})$ is the slope map. For each $[\alpha] \in S$ with $[\alpha] \neq f_*(s_{1,1}(\mathfrak{o}_1))$, choose a left-ordering $\mathfrak{o}_{[\alpha]}$ of $\pi_1(M_2)$ that order-detects some $(J, K; [\alpha_*])$ with

$s_{2,1}(\mathfrak{o}_{[\alpha]}) = [\alpha]$. When $[\alpha] = f_*(s_{1,1}(\mathfrak{o}_1))$, choose $\mathfrak{o}_{[\alpha]} = \mathfrak{o}_2$. Using Proposition 4.6.5, for each $\mathfrak{o}_{[\alpha]}$ choose $\hat{\mathfrak{o}}_{[\alpha]}$ that satisfies the conditions of Proposition 4.6.5 and set

$$N_{[\alpha]} = \{g \cdot \mathfrak{o}_{[\alpha]} : g \in \pi_1(M_2)\} \cup \{g \cdot \mathfrak{o}_{[\alpha]}^{\text{op}} : g \in \pi_1(M_2)\} \\ \cup \{g \cdot \hat{\mathfrak{o}}_{[\alpha]} : g \in \pi_1(M_2)\} \cup \{g \cdot (\hat{\mathfrak{o}}_{[\alpha]})^{\text{op}} : g \in \pi_1(M_2)\},$$

by construction $N_{[\alpha]}$ is ready to glue along $T_{2,1}$. Now set

$$N_2 = \bigcup_{[\alpha] \in S} N_{[\alpha]},$$

by construction N_2 is ready to glue along $T_{2,1}$. Choosing maps $f_i : \mathbb{Z}^2 \rightarrow \pi_1(M_i)$ for $i = 1, 2$ such that $f_i(\mathbb{Z}^2) = \pi_1(T_{i,1})$ and $f_* \circ f_1 = f_2$, the normal families N_1, N_2 are compatible with the maps f_i by construction, and so $\pi_1(M_1 \cup_f M_2) = \pi_1(M_1) *_{f_i} \pi_1(M_2)$ is left-orderable by Proposition 4.6.3.

Moreover, there exists $\mathfrak{o}'_2 \in \{\mathfrak{o}_{s_{1,1}(\mathfrak{o}_1)}, \mathfrak{o}_{s_{1,1}(\mathfrak{o}_1)}^{\text{op}}, \hat{\mathfrak{o}}_{s_{1,1}(\mathfrak{o}_1)}, (\hat{\mathfrak{o}}_{s_{1,1}(\mathfrak{o}_1)})^{\text{op}}\}$ satisfying $f_1^{-1}(P(\mathfrak{o}_1)) = f_2^{-1}(P(\mathfrak{o}'_2))$, so that we may choose a left-ordering $\mathfrak{o}$ of $\pi_1(M_1 \cup_f M_2)$ that restricts to $\mathfrak{o}_1$ on $\pi_1(M_1)$ and $\mathfrak{o}'_2$ on $\pi_1(M_2)$. Since $K_2 = \{1, \dots, r_2\}$ and $\mathfrak{o}'_2$ arises from an application of Proposition 4.6.5, the ordering $\mathfrak{o}$ detects

$$(\emptyset, \emptyset; [\alpha_{1,2}], \dots, [\alpha_{1,r_1}], [\alpha_{2,2}], \dots, [\alpha_{2,r_2}]).$$

□

4.7 JN-realisability and representation detection

In this section, we introduce the notion of JN-realisability, following [4]. This serves as a way of computing which tuples of slopes on the boundary of a Seifert fibered manifold are representation-detected.

Let P be a punctured 2-sphere with r boundary components and $p_1, p_2, \dots, p_n$ distinct points on P . The cyclic group $\mathbb{Z}/a_i\mathbb{Z}$ acts on the open regular neighbourhood of p_i , which is homeomorphic to a disk, in a natural way. Denote by $P(a_1, a_2, \dots, a_n)$ the orbifold $(P, \mathbb{Z}/a_1\mathbb{Z}, \mathbb{Z}/a_2\mathbb{Z}, \dots, \mathbb{Z}/a_n\mathbb{Z})$. Throughout this section, we use M to denote an orientable Seifert manifold with base orbifold $P(a_1, a_2, \dots, a_n)$, whose boundary is a nonempty collection of incompressible tori $T_1, \dots, T_r$. We use $h \in \pi_1(M)$ to denote the class of a regular Seifert fiber of M . Suppose that the Seifert invariants of the exceptional fibers of M are given by $(a_1, b_1), \dots, (a_n, b_n)$ with $0 < b_i < a_i$ for all $i = 1, 2, \dots, n$.

Set

$$\gamma_i = \frac{b_i}{a_i} \in (0, 1).$$

The fundamental group of M has a presentation

$$\pi_1(M) = \langle y_1, \dots, y_n, x_1, \dots, x_r, h \mid h \text{ central}, y_i^{a_i} = h^{b_i}, y_1 y_2 \dots y_n x_1 x_2 \dots x_r = 1 \rangle.$$

In this presentation, x_j is a dual class to h on each T_j for $1 \leq j \leq r$. We call $[\alpha_*] \in \mathcal{S}(M)$ *horizontal* if no $[\alpha_i]$ coincides with the slope of the fiber class $[h]$.

For $\gamma \in \mathbb{R}$, denote by $\text{sh}(\gamma)$ the translation by γ , that is, $\text{sh}(\gamma)(x) = x + \gamma$ for all $x \in \mathbb{R}$. The universal cover $\widetilde{\text{Homeo}}_+(S^1)$ of $\text{Homeo}_+(S^1)$ is canonically isomorphic to the subgroup of $\text{Homeo}_+(\mathbb{R})$ which consists of homeomorphisms that commute with $\text{sh}(1)$:

$$\widetilde{\text{Homeo}}_+(S^1) = \{f \in \text{Homeo}_+(\mathbb{R}) : f(x+1) = f(x) + 1, \forall x \in \mathbb{R}\}.$$

For each $f \in \widetilde{\text{Homeo}}_+(S^1)$, we define the translation number $\tau : \widetilde{\text{Homeo}}_+(S^1) \rightarrow \mathbb{R}$ by

$$\tau(f) = \lim_{n \rightarrow \infty} \frac{f^n(0)}{n},$$

which satisfies $\tau(\text{sh}(\gamma)) = \gamma$. The translation number map τ is invariant under conjugation and becomes a homomorphism when restricted to any abelian subgroup of $\widetilde{\text{Homeo}}_+(S^1)$; moreover, $\tau(f) = 0$ if and only if f has a fixed point.

Definition 4.7.1 ([4, Section 3.2]). For a subset $J \subset \{1, \dots, r\}$ and an r -tuple $(\tau_1, \dots, \tau_r)$ of real numbers, we say $(J; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_r)$ is *JN-realisable* if there is a homomorphism $\rho : \pi_1(M) \rightarrow \widetilde{\text{Homeo}}_+(S^1)$ such that

- (1) $\rho(h) = \text{sh}(1)$;
- (2) $\tau_j = \tau(\rho(x_j))$ for $1 \leq j \leq r$;
- (3) $\rho(x_j)$ is conjugate to $\text{sh}(\tau_j)$ for each $j \in J$.

More generally, $(J; b; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_r)$ is *JN-realisable* if there are elements f_i, g_j from $\widetilde{\text{Homeo}}_+(S^1)$ such that

- (1) f_i is conjugate to $\text{sh}(\gamma_i)$ for all $i = 1, 2, \dots, n$;
- (2) $\tau_j = \tau(g_j)$ for $1 \leq j \leq r$;
- (3) g_j is conjugate to $\text{sh}(\tau_j)$ for each $j \in J$;
- (4) $f_1 \circ \dots \circ f_n \circ g_1 \circ \dots \circ g_r = \text{sh}(b)$.

For each $\tau_j = \tau(\rho(x_j))$, let $[\alpha_j] = [x_j - \tau_j h] \in H_1(T_j; \mathbb{R})$ and write $[\tau_*] = (\tau_1, \dots, \tau_r)$. Note that $[\alpha_*]$ is horizontal.

Proposition 4.7.2. *If $(J; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_r)$ is JN-realisable, then $(J, \{1, \dots, r\}; [\alpha_*])$ is representation-detected.*

Proof. Suppose $(J; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_r)$ is JN-realisable with the homomorphism $\rho : \pi_1(M) \rightarrow \widetilde{\text{Homeo}}_+(S^1)$ satisfying the conditions of JN-realisability. We will show that $(J, \{1, \dots, r\}; [\alpha_*])$ is representation-detected by $(\rho, 0)$.

Note that $(\rho, 0) \in \mathcal{R}^*(\pi_1(M))$, since $\rho(h) = \text{sh}(1)$. Since the translation number is a group homomorphism on abelian subgroups, the map

$$(\tau \circ \rho|_{\pi_1(T_i)}) \otimes \mathbf{1}_{\mathbb{R}} : \pi_1(T_i) \otimes \mathbb{R} = H_1(T_i; \mathbb{R}) \rightarrow \mathbb{R}$$

is a nontrivial linear map for all $i = 1, 2, \dots, r$. The kernel of this map contains $[\alpha_i]$ and divides $H_1(T_i; \mathbb{R}) \setminus \ker((\tau \circ \rho|_{\pi_1(T_i)}) \otimes \mathbf{1}_{\mathbb{R}})$ into a disjoint union $H_+ \cup H_-$, with $\tau(\rho(\gamma)) > 0$ for every $\gamma \in \pi_1(T_i) \cap H_+$ and $\tau(\rho(\gamma)) < 0$ for every $\gamma \in \pi_1(T_i) \cap H_-$. Then $\tau(\rho(\gamma)) > 0$ leads to $\rho(\gamma)(x) > x$ for all $x \in \mathbb{R}$, and $\rho(\gamma)(x) < x$ when $\tau(\rho(\gamma)) < 0$. In particular, this shows that $[\mathcal{L}(\rho|_{\pi_1(T_i)}, 0)] = [\alpha_i]$ for all $i = 1, \dots, r$, and so R1 holds.

The observation that $\rho(\gamma)(x) > x$ for all $x \in \mathbb{R}$ or $\rho(\gamma)(x) < x$ for all $x \in \mathbb{R}$ whenever $\gamma \in H_+ \cup H_- \subset H_1(T_i; \mathbb{R})$ also shows that $(\rho, \rho(g)(0)) \in \mathcal{R}^*(\pi_1(M))$ for all $g \in \pi_1(M)$. With this fact in hand, the fact that conjugation leaves the translation number invariant implies that R2 is true.

Finally, given that $[\mathcal{L}(\rho|_{\pi_1(T_i)}, 0)] = [\alpha_i]$ we know that if $i \in J$ and $\tau_i = \frac{p}{q}$ is rational, then $\rho(x_i)$ is conjugate to $\text{sh}(\tau_i)$, and thus $\rho(qx_i - ph)$ is the identity in $\widetilde{\text{Homeo}}_+(S^1)$. This implies $\ker(\rho) \cap \pi_1(T_i) = \langle \alpha_i \rangle$. Consequently, R3 is true with $v : \mathbb{R} \rightarrow \mathbb{R}$ the identity and $\phi = \rho$. $\square$

The converse of the previous proposition also holds, provided we place a restriction on our representations.

Proposition 4.7.3.

(1) *Assume that $(J, \{1, \dots, r\}; [\alpha_*])$ is representation-detected and $[\alpha_*]$ is horizontal. Then $(J; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_r)$ is JN-realisable.*

(2) *If $(\emptyset, \emptyset; [\alpha_*])$ is representation-detected and $[\alpha_*]$ is horizontal, then $(\emptyset, \{1, \dots, r\}; [\alpha_*])$ is representation-detected.*

Proof. To prove (1), apply Theorem 4.4.2 and let $\circ$ be a left-ordering of $\pi_1(M)$ detecting $(J, \{1, \dots, r\}; [\alpha_*])$. The result now follows from [4, Proposition 5.4].

For (2), apply Theorem 4.4.2 and suppose that $\mathfrak{o}$ order-detects $(\emptyset, \emptyset; [\alpha_*])$. Then $h \in \pi_1(M)$ is $\mathfrak{o}$ -cofinal due to the relators $y_i^{a_i} = h^{b_i}$ and the assumption $[\alpha_j] \neq [h]$ which implies that h is $\mathfrak{o}|_{\pi_1(T_j)}$ -cofinal in $\pi_1(T_j)$ for each $i = 1, \dots, r$. Since h is $\mathfrak{o}$ -cofinal, we conclude $\mathfrak{o}$ is boundary-cofinal, from which it follows that $\mathfrak{o}$ order-detects $(\emptyset, \{1, \dots, r\}; [\alpha_*])$ by [5, Proposition 7.2]. Therefore $(\emptyset, \{1, \dots, r\}; [\alpha_*])$ is representation-detected by Theorem 4.4.2. $\square$

It is possible to calculate when $(J; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_r)$ is JN-realisable, and the technical tools required to do so are in Appendix A. We will need them in the next chapter.

5

Applications to Cable Knots

This chapter studies the (non-)left-orderability of 3-manifold groups obtained from Dehn surgeries on cable knots under the framework of order detection. After establishing conventions for bases and slope parameters, the chapter develops special cases of the gluing theorems adapted to cable spaces and introduces methods for computing relatively JN-realizable slopes. Recall that the result of Hedden and Hom (Theorem 3.3.3) completely describes how L-space knots behave with respect to cabling in the aspect of Heegaard Floer homology. The results of this chapter parallel known characterisations of L-space surgeries, providing new algebraic evidence supporting the L-space conjecture. All the materials in this chapter are taken from [15].

5.1 Notations and conventions

This section establishes some notations and convention that will be used throughout this chapter.

Let K be a knot in S^3 , and let $N(K) \subset S^3$ denote a closed tubular neighbourhood of K , which is homeomorphic to a solid torus. Consider another unknotted solid torus V embedded in a separate copy of S^3 , and let $P \subset V$ be a knot such that P is not contained in any 3-ball within V . A homeomorphism $h : V \rightarrow N(K)$ that maps a meridian of the solid torus $S^3 - V$ to a longitude of K sends the knot P to a new knot $K' = h(P)$ in S^3 . The knot K' is called a *satellite knot* with *companion* K , and the pair (V, P) is referred to as a *pattern* for K' . See Figure 5.1 for an illustration.

If the knot P is ambient isotopic to a simple closed curve T on the boundary ∂V , that is, if there is a continuous map $H : P \times [0, 1] \rightarrow V$ such that the map

$$H(*, t) : P \rightarrow V, x \mapsto H(x, t)$$

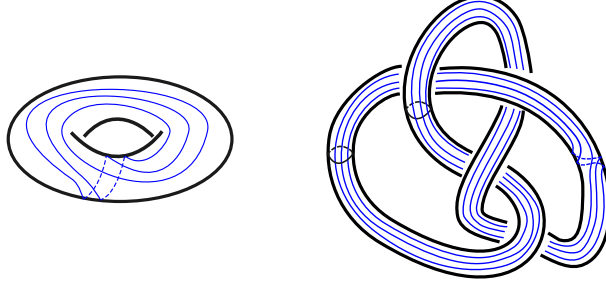


Figure 5.1: A satellite knot (right) with its pattern a torus knot (left)

is a homeomorphism from P onto its image $H(*, t)(P)$ for each $t \in [0, 1]$ with $H(*, 0)$ being the identity map and $H(*, 1)(P) = T$, then the resulting satellite knot $K' = h(P)$ is also called a *cable knot* on K . In particular, if P is a (p, q) -torus knot lying on ∂V , we denote the corresponding cable knot by $C_{p,q}(K)$. Figure 5.1 also demonstrates a cable knot. To determine the fundamental group of a cable knot, we proceed as follows.

Consider a fibered solid torus $V \cong D^2 \times S^1$, which can be obtained by taking the cylinder $D^2 \times I$ and identifying each $(x, 0)$ with $(xe^{2\pi i \frac{p}{q}}, 1)$, where $p \geq 1$ and $q > 1$ are coprime numbers. Let h denote a regular fiber in the interior of the fibered solid torus, and let $n(h)$ be an open regular neighbourhood around h . The complement of $n(h)$ in V is denoted by $C_{p,q}$, namely,

$$C_{p,q} = V - n(h).$$

To calculate the fundamental group $\pi_1(C_{p,q})$, we employ the following strategy. Let $\frac{1}{2}D^2$ be a concentric disk within D^2 with half the radius of D^2 . We set $V_0 = \frac{1}{2}D^2 \times S^1$ and choose a regular fiber h that lies on ∂V_0 . In this context, $C_{p,q}$ can be viewed as the gluing of $V_0 - n(h)$ and $\overline{(V - V_0)} - n(h)$ along an annulus, whose central curve is a regular fiber.

Note that $\overline{(V - V_0)} - n(h)$, being homotopy equivalent to a thickened torus, has the fundamental group $\mathbb{Z} \oplus \mathbb{Z}$. This group is generated by the standard meridian and longitude basis μ, λ . Here, μ is the homotopy class of the curve $\partial D^2 \times \{*\}$ and λ is the homotopy class of $\{*\} \times S^1$. On the other hand, the space V_0 is homotopy equivalent to a solid torus with its fundamental group being infinite cyclic, generated by t , the homotopy class of the core curve of V_0 . Applying the Seifert-van Kampen theorem, we deduce that the fundamental group of $C_{p,q}$ is given by

$$\pi_1(C_{p,q}) = (\mathbb{Z} \oplus \mathbb{Z}) *_{\mu^p \lambda^q = t^q} \mathbb{Z}.$$

Let K be a given knot in S^3 . Consider the cable knot $C_{p,q}(K)$. Its knot complement

is constructed by gluing $C_{p,q}$ to $S^3 - n(K)$ along their respective boundaries, ∂V and $\partial(S^3 - n(K))$. This gluing operation is performed by identifying the generators μ and λ of the fundamental group of $C_{p,q}$ with the canonical meridian and longitude of $\partial(S^3 - n(K))$, respectively.

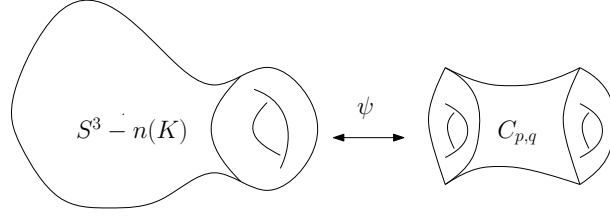


Figure 5.2: Cable knot complement obtained by gluing $C_{p,q}$ to $S^3 - n(K)$

In other words, the knot complement of the cable knot $C_{p,q}(K)$ in S^3 can be expressed as

$$S^3 - n(C_{p,q}(K)) = (S^3 - n(K)) \cup_{\psi} C_{p,q},$$

where ψ denotes the identification map. Refer to Figure 5.2 for an illustrative depiction. We can compute the meridian and longitude of the cable knot $C_{p,q}(K)$, denoted by μ_C and λ_C respectively. They are given by the formulas

$$\mu_C = t^s \mu^r \lambda^{-s} \text{ and } \lambda_C = \mu_C^{-pq} t^q,$$

where $ps + qr = 1$ and $-q < s < 0 < r \leq p$.

Remark 5.1.1. The numbers s and r are known as Bézout's coefficients. The restriction $-q < s < 0 < r \leq p$ is not mandatory to obtain a correct expressions for μ_C and λ_C in terms of μ, λ and t , but it is for computational convenience in later sections. Note that $r = p$ occurs if and only if $p = 1$.

Note that the subspace $C_{p,q}$ is a Seifert-fibered space with incompressible tori as its boundary components. Therefore, the results of Section 4.7 apply, and there exists an alternative construction for the cable space $C_{p,q}$ that provides a different presentation of $\pi_1(C_{p,q})$ that agrees with Section 4.7. This alternative approach is more useful when we calculate sets of order-detected slopes.

We start with the same fibered solid torus as before and define $U = V - (\frac{1}{2}D^2 \times S^1)$. Let μ and λ denote the meridional and longitudinal classes in $\pi_1(U)$. Suppose A is an annulus such that U is homeomorphic to $A \times S^1$. Here, the S^1 factor corresponds to a regular fiber, $\mu^p \lambda^q$. The boundary ∂A consists of curves x_1 and y' , representing $\mu^r \lambda^{-s}$ and $\mu^{-r} \lambda^s$ respectively, with $-q < s < 0 < r \leq p$ and $ps + qr = 1$.

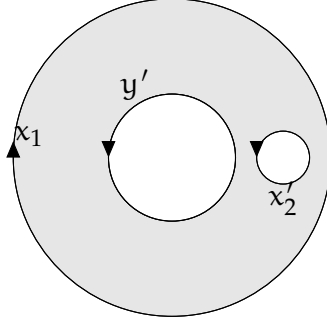


Figure 5.3: The planar surface P

By removing an open regular neighbourhood of a regular fiber from U , we obtain a space homeomorphic to $P \times S^1$, where P is a planar surface with three boundary components. See Figure 5.3 for an illustration. The fundamental group of $U - n(h)$ is then given by

$$\pi_1(U - n(h)) = \langle x_1, x_2', y', h \mid h \text{ central}, y' x_1 x_2' = 1 \rangle,$$

where h is the class of a regular fiber. The cable space $C_{p,q}$ can be obtained by attaching the solid torus $\frac{1}{2}D^2 \times S^1$ back to $U - n(h)$ through mapping $\partial(\frac{1}{2}D^2) \times \{*\}$ to $(y')^{-q}h^s$. According to the Seifert-van Kampen theorem, the fundamental group of $C_{p,q}$ is

$$\pi_1(C_{p,q}) = \langle x_1, x_2', y', h \mid h \text{ central}, (y')^q = h^s, y' x_1 x_2' = 1 \rangle.$$

This presentation is different from the one discussed in Section 4.7, since $\frac{s}{q} < 0$. To align it with that section, we define $y = y'h$ and $x_2 = x_2'h^{-1}$ to obtain

$$\pi_1(C_{p,q}) = \langle x_1, x_2, y, h \mid h \text{ central}, y^q = h^{q+s}, y x_1 x_2 = 1 \rangle.$$

A concrete isomorphism ϕ linking this presentation of $\pi_1(C_{p,q})$ and the previously computed presentation $\pi_1(C_{p,q}) = (\mathbb{Z} \oplus \mathbb{Z}) *_{\mu^p \lambda^q = t^q} \mathbb{Z}$ is specified by

$$\phi(x_1) = \mu^r \lambda^{-s}, \phi(h) = \mu^p \lambda^q, \phi(y) = t^{q+s}, \phi(x_2) = \mu^{-r} \lambda^s t^{-(s+q)}.$$

Consider $C_{p,q}$ as a subspace of the knot complement of the cable knot $C_{p,q}(K)$. Each peripheral subgroup of $\pi_1(C_{p,q})$ now naturally has two bases, arising from our two presentations of the fundamental group above. The change of basis, computed via ϕ , allows us to translate between slopes expressed relative to one basis and slopes relative to the other.

For the “inner” boundary torus T_1 of $C_{p,q}$, we have two bases arising from the

identification of $\partial(S^3 - n(K))$ with T_1 , namely $\mathcal{B}_K = \{\mu, \lambda\}$ and $\mathcal{B}_1 = \{h, -x_1\}$. The change of basis matrix from $\{h, -x_1\}$ to $\{\mu, \lambda\}$, computed from ϕ , is

$$\begin{bmatrix} p & -r \\ q & s \end{bmatrix}.$$

The “outer” boundary torus T_2 of $C_{p,q}$ has bases $\mathcal{B}_2 = \{h, -x_2\}$ and $\mathcal{B}_C = \{\mu_C, \lambda_C\}$, with the corresponding change of basis matrix

$$\begin{bmatrix} pq & pq+1 \\ 1 & 1 \end{bmatrix}.$$

To discuss order detection, we fix the convention that $[\alpha_*] = ([\alpha_1], [\alpha_2]) \in \mathcal{S}(C_{p,q})$ means that $[\alpha_1]$ is slope on the “inner” boundary torus, and $[\alpha_2]$ on the “outer” boundary torus.

5.2 Transition between bases and non-horizontal slopes

In this section, we show how to translate the slopes between different bases mentioned at the end of last section and deal with the non-horizontal slopes.

Use the notation developed above. In particular, there are integers p, q, r, s with $p \geq 1$ and $q > 1$ being coprime, $-q < s < 0 < r \leq p$ and $ps + qr = 1$. Each of the bases $\mathcal{B}_K, \mathcal{B}_C, \mathcal{B}_1, \mathcal{B}_2$ gives rise to a homeomorphism $\mathcal{S}(T_i) \rightarrow \mathbb{R} \cup \{\infty\} \cong S^1$ in the usual way. Specifically, for a distinguished basis $\mathcal{B} = \{b_1, b_2\}$, we have the map $[nb_1 + mb_2] \mapsto \frac{n}{m}$ (see Section 4.2). Therefore, the change of basis matrix from $\mathcal{B}_K$ to $\mathcal{B}_1$ yields a homeomorphism $f : \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}$ given by $f(x) = \frac{sx+r}{-qx+p}$, where $f(p/q) = \infty$, $f(-r/s) = 0$ and $f(\infty) = -s/q \in (0, 1)$. It is easy to verify that f is strictly increasing on $(-\infty, p/q)$ and on $(p/q, \infty)$ and so that

$$f((-\infty, p/q)) = (-s/q, \infty) \text{ and } f((p/q, \infty)) = (-\infty, -s/q).$$

Thus we have the following result.

Lemma 5.2.1. *For $b \in \mathbb{R}$, we have*

$$f((-\infty, b)) = \begin{cases} (-s/q, f(b)) & \text{if } b \leq p/q, \\ [-\infty, f(b)) \cup (-s/q, \infty] & \text{if } b > p/q. \end{cases}$$

Similarly, the change of basis matrix from $\mathcal{B}_2$ to $\mathcal{B}_C$ yields a homeomorphism $g : \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}$ given by $g(x) = pq + \frac{1}{x+1}$, where $g(-1) = \infty$ and $g(\infty) = pq$.

Analogously, we have the following.

Lemma 5.2.2. *For $b \in \mathbb{R}$ with $p - qb > 0$, we have*

$$g((-1 - \frac{1}{p - qb}, -1)) = (-\infty, pq - p + qb);$$

if $p - qb < 0$, then

$$g((-1, -1 - \frac{1}{p - qb})) = (pq - p + qb, \infty).$$

In order to implement the results concerning JN-realizability, recall that we require $[\alpha_*] = ([\alpha_1], [\alpha_2])$ to be horizontal, that is, $[\alpha_i] \neq [h]$ for $i = 1, 2$, meaning $[\alpha_i] \neq [p\mu + q\lambda]$ relative to the basis $\mathcal{B}_K$. We will need the following to deal with the non-horizontal cases separately.

Lemma 5.2.3. *For the cable space $C_{p,q}$, $(\{1, 2\}, \{1, 2\}; [h], [h])$ is order-detected.*

Proof. Consider the short exact sequence

$$\{\text{id}\} \longrightarrow K \longrightarrow \pi_1(C_{p,q}) \longrightarrow \mathbb{Z} \longrightarrow \{\text{id}\},$$

where the map $\pi_1(C_{p,q}) \rightarrow \mathbb{Z}$ is the result of killing the fiber class $h \in \pi_1(C_{p,q})$ and any resulting torsion. In other words, $K = \langle \langle h, y \rangle \rangle$. Since $K \leq \pi_1(C_{p,q})$, K is left-orderable. Now any left-ordering $\circ$ obtained lexicographically from the short exact sequence above order-detects $(\{1, 2\}, \{1, 2\}; [h], [h])$. $\square$

Lemma 5.2.4. *Suppose that for the cable space $C_{p,q}$, $(\emptyset, \emptyset; [\alpha_1], [\alpha_2])$ is order-detected. Then either $[\alpha_1] = [\alpha_2] = [h]$ or $[h] \notin \{[\alpha_1], [\alpha_2]\}$.*

Proof. Let $\circ$ be a left-ordering of $\pi_1(C_{p,q})$ that order-detects $(\emptyset, \emptyset; [\alpha_1], [\alpha_2])$. By the previous lemma, we need only to prove that if $[h] \in \{[\alpha_1], [\alpha_2]\}$ then $[\alpha_1] = [\alpha_2] = [h]$. To this end, we prove that if $[h] = [\alpha_1]$ then $[\alpha_2] = [h]$, with the case of $[h] = [\alpha_2]$ implying that $[h] = [\alpha_1]$ is similar.

We do this by showing the contrapositive. Suppose that $[h] \neq [\alpha_2]$ and recall the generators x_1, x_2, y, h of $\pi_1(C_{p,q})$ from Section 5.1. Then observe that h is cofinal with respect to $\circ|_{\pi_1(T_2)}$, and that

$$H = \{g \in \pi_1(C_{p,q}) : \exists k \in \mathbb{Z} \text{ such that } h^{-k} <_{\circ} g <_{\circ} h^k\}$$

is therefore a subgroup containing x_2 . However, H also contains y , and hence x_1 , and thus $H = \pi_1(C_{p,q})$. It follows that h is $\circ$ -cofinal, so that $[\alpha_1]$ cannot be equal to $[h]$. $\square$

Note that the two lemmas above ensure that when translating slopes from one basis to the other, we are able to take care of the slopes which are expressed as ∞ relative to the given bases.

5.3 Special cases of the gluing theorems for use when cabling

In later Section 5.5, we are interested in the case of gluing $C_{p,q}$ to ∂M where M is a knot manifold. For that purpose, we adopt the results from Section 4.6 and highlight this special case of our gluing theorems in this section.

Corollary 5.3.1. *Let M be a knot manifold with boundary torus $T_{1,1}$ and $C_{p,q}$ a cable space with two torus boundary components $T_{2,1}$ and $T_{2,2}$. Fix a choice of peripheral subgroup $\pi_1(T_{i,j})$ for each boundary torus. Let $f : T_{1,1} \rightarrow T_{2,1}$ be a homeomorphism that identifies the slope $[\alpha_1] \in \mathcal{S}(\pi_1(T_{1,1}))$ with the slope $[\alpha_2] \in \mathcal{S}(\pi_1(T_{2,1}))$.*

- (1) *Suppose that $\mathfrak{o} \in \text{LO}(\pi_1(M))$ detects $(\emptyset, \emptyset; [\alpha_1])$ and that $\mathfrak{o}' \in \text{LO}(\pi_1(C_{p,q}))$ detects $(\emptyset, \{2\}; [\alpha_2], [\alpha_3])$. Then $M \cup_f C_{p,q}$ is left-orderable and $(\emptyset, \emptyset; [\alpha_3])$ is order-detected by some left-ordering in $\text{LO}(\pi_1(M \cup_f C_{p,q}))$.*
- (2) *Suppose that $\mathfrak{o} \in \text{LO}(\pi_1(M))$ detects $(\emptyset, \{1\}; [\alpha_1])$ and that $\mathfrak{o}' \in \text{LO}(\pi_1(C_{p,q}))$ detects $(\emptyset, \{2, 3\}; [\alpha_2], [\alpha_3])$. Then $M \cup_f C_{p,q}$ is left-orderable and $(\emptyset, \{1\}; [\alpha_3])$ is order-detected by some left-ordering in $\text{LO}(\pi_1(M \cup_f C_{p,q}))$.*
- (3) *Suppose that $\mathfrak{o} \in \text{LO}(\pi_1(M))$ detects $(\{1\}, \{1\}; [\alpha_1])$ and that $\mathfrak{o}' \in \text{LO}(\pi_1(C_{p,q}))$ detects $(\{2, 3\}, \{2, 3\}; [\alpha_2], [\alpha_3])$. Then $M \cup_f C_{p,q}$ is left-orderable and $(\{1\}, \{1\}; [\alpha_3])$ is order-detected by some left-ordering in $\text{LO}(\pi_1(M \cup_f C_{p,q}))$.*

Proof. Case (1) is a special application of Theorem 4.6.11. First, we note that $\mathfrak{o}'$ can be replaced by an ordering detecting $(\emptyset, \{1, 2\}; [\alpha_2], [\alpha_3])$ by Proposition 4.7.3 and Theorem 4.4.2. Next, we verify that $C_{p,q}$ satisfies the conditions required of M_2 in that theorem as follows. To this end, let $C_{p,q}(\alpha)$ denote the Dehn filling of $C_{p,q}$ obtained by attaching a solid torus to $T_{2,1}$ in such a way that the meridian is identified with the slope $[\alpha] \in \mathcal{S}(T_{2,1})$. Now let $[\alpha] \in \mathcal{S}(T_{2,1})$ be an arbitrary rational slope other than the fiber slope. By [38, Proposition 5], $C_{p,q}(\alpha)$ is an irreducible Seifert fibered manifold with one torus boundary component. It follows that $\pi_1(C_{p,q}(\alpha))$ is left-orderable by the Boyer-Rolfson-Weist Theorem (Theorem 3.2.4), since it surjects onto $\mathbb{Z}$. Therefore,

a short exact sequence argument, using

$$\{\text{id}\} \longrightarrow \langle\langle\alpha\rangle\rangle \longrightarrow \pi_1(C_{p,q}) \longrightarrow \pi_1(C_{p,q}(\alpha)) \longrightarrow \{\text{id}\}$$

to lexicographically order $\pi_1(C_{p,q})$, shows that $(\{1\}, \{1\}; [\alpha], [\beta])$ is order-detected for some $[\beta] \in \mathcal{S}(T_{2,2})$. When $[\alpha]$ is the slope of the fiber, we may use Lemma 5.2.3. Finally, we observe that $\text{LO}(\pi_1(C_{p,q}))$ is compact and that the slope map $s_{1,1}$ is continuous, having all rational slopes in its image. As rational slopes form a dense set in $\mathcal{S}(T_{1,2})$, the map $s_{1,1}$ must be surjective. Thus, whenever $[\alpha]$ is irrational, $(\emptyset, \{1\}; [\alpha], [\beta])$ is order-detected for some $[\beta] \in \mathcal{S}(T_{2,2})$ by arguments analogous to [5, Proposition 7.6]. Cases (2) and (3) are applications of Theorem 4.6.7. $\square$

5.4 Computations of relatively JN-realisable slopes in cable spaces

In this section, we apply the computational theorems in Appendix A to find out the interval of order detected slopes of $C_{p,q}$.

We adopt the notation introduced in Sections 4.7 and 5.1. Consider a Seifert fibered manifold as described in Section 4.7, and let J be a subset of $\{1, \dots, r-1\}$ and $\tau_* = (\tau_1, \dots, \tau_{r-1}) \in \mathbb{R}^{r-1}$. Define

$$\mathcal{T}(M; J; \tau_*) = \{\tau' \in \mathbb{R} : (J; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_{r-1}, \tau') \text{ is JN-realisable}\}$$

and

$$\mathcal{T}_{\text{str}}(M; J; \tau_*) = \{\tau' \in \mathbb{R} : (J \cup \{r\}; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_{r-1}, \tau') \text{ is JN-realisable}\}.$$

For this section, and in the Appendix, we follow [4] (also see Appendix A) and for a fixed tuple $\tau_* = (\tau_1, \dots, \tau_{r-1})$, we set

- $r_1 = |\{j : \tau_j \notin \mathbb{Z}, 1 \leq j \leq r-1\}|$, the number of non-integral τ_j ;
- $s_0 = |\{j : \tau_j \in \mathbb{Z} \text{ and } j \in \{1, 2, \dots, r-1\} \setminus J\}|$, the number of integral τ_j whose indices are not in J ;
- $b_0 = -(\lfloor \tau_1 \rfloor + \dots + \lfloor \tau_{r-1} \rfloor)$;
- $m_0 = b_0 - (n + r_1 + s_0 - 1)$;
- $m_1 = b_0 + s_0 - 1$.

We can explicitly calculate how $\mathcal{T}(C_{p,q}; \emptyset; \tau)$ changes for different values of τ . Recall that $p \geq 1$ and $q > 1$ are coprime and r, s are chosen so that $ps + qr = 1$ and $-q < s < 0 < r \leq p$. For $\tau \in \mathbb{R}$, let $\bar{\tau}$ denote its fractional part, that is, $\bar{\tau} = \tau - \lfloor \tau \rfloor$. Then $(J; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_r)$ is JN-realisable if and only if $(J; b; \gamma_1, \dots, \gamma_n; \bar{\tau}_1, \dots, \bar{\tau}_r)$ is JN-realisable, where

$$b = -(\lfloor \tau_1 \rfloor + \dots + \lfloor \tau_r \rfloor).$$

Proposition 5.4.1. *For $\tau \in \mathbb{R}$, we have $\mathcal{T}(C_{p,q}; \emptyset; \tau) \subset (-\lfloor \tau \rfloor - 2, -\lfloor \tau \rfloor]$. More precisely, there are two rational numbers $\eta(\tau) \in (-\lfloor \tau \rfloor - 2, -\lfloor \tau \rfloor - 1)$ and $\xi(\tau) \in (-\lfloor \tau \rfloor - 1, -\lfloor \tau \rfloor)$ such that*

$$\mathcal{T}(C_{p,q}; \emptyset; \tau) = \begin{cases} [-\tau - 1, -\tau] & \text{if } \bar{\tau} = 0, \\ [-\lfloor \tau \rfloor - 1, \xi(\tau)] & \text{if } \bar{\tau} > 0 \text{ and } \gamma + \bar{\tau} < 1, \\ \{-\lfloor \tau \rfloor - 1\} & \text{if } \bar{\tau} > 0 \text{ and } \gamma + \bar{\tau} = 1, \\ [\eta(\tau), -\lfloor \tau \rfloor - 1] & \text{if } \bar{\tau} > 0 \text{ and } \gamma + \bar{\tau} > 1. \end{cases}$$

Proof. Note that we have $n = 1$, $\gamma = \frac{q+s}{q}$, $r_1 + s_0 = 1$, $b_0 = -\lfloor \tau \rfloor$ and $m_0 = -\lfloor \tau \rfloor - 1$. If τ is an integer, then $\bar{\tau} = 0$ and Theorem A.0.1 applies. So $(\emptyset; 0; \frac{q+s}{q}; \tau, \tau')$ is JN-realisable if and only if $(\emptyset; -\lfloor \tau \rfloor - \lfloor \tau' \rfloor; \frac{q+s}{q}; 0, \bar{\tau}')$ is JN-realisable, which happens only when $\tau = \lfloor \tau \rfloor = -\lfloor \tau' \rfloor$ yielding $\tau' \in [-\tau - 1, -\tau]$.

If τ is not an integer, then $n + r_1 = 2$ and we can resolve this case by appealing to Theorems A.0.2 and A.0.3(2)(b), A.0.3(3)(b), A.0.3(4)(b), considering the cases $\bar{\tau} + \gamma = 1$, $\bar{\tau} + \gamma > 1$, and $\bar{\tau} + \gamma < 1$. Observing that $\mathcal{T}(C_{p,q}; \emptyset; \tau) \subset (-\lfloor \tau \rfloor - 2, -\lfloor \tau \rfloor]$, we consider the following cases.

First, if $\bar{\tau} + \gamma = 1$ then Theorem A.0.3(4)(b) applies and we find $\mathcal{T}(C_{p,q}; \emptyset; \tau) = \{m_0\} = \{-\lfloor \tau \rfloor - 1\}$.

Next, if $\gamma + \bar{\tau} < 1$, then the conditions of Theorem A.0.3(3)(b)(i) are met while the conditions of Theorem A.0.3(2)(b)(i) are not. Therefore, we choose $\xi(\tau) \in (-\lfloor \tau \rfloor - 1, -\lfloor \tau \rfloor)$ that is maximal subject to the property that there are coprime integers $0 < a < m$ and a permutation $\{a_1, a_2, a_3\}$ of $\{a, m - a, 1\}$ such that $\gamma < \frac{a_1}{m}$, $\bar{\tau} \leq \frac{a_2}{m}$, and $\xi(\tau) - \lfloor \xi(\tau) \rfloor = \xi(\tau) + \lfloor \tau \rfloor + 1 \leq \frac{a_3}{m}$. Using Theorem A.0.2(4), we conclude that $\mathcal{T}(C_{p,q}; \emptyset; \tau) = [-\lfloor \tau \rfloor - 1, \xi(\tau)]$.

If $\bar{\tau} \neq 0$ and $\gamma + \bar{\tau} > 1$ then the conditions of Theorem A.0.3(2)(b)(i) are met and the conditions of Theorem A.0.3(2)(3)(i) are not, so we choose $\eta(\tau) \in (-\lfloor \tau \rfloor - 2, -\lfloor \tau \rfloor - 1)$ that is minimal subject to the property that there are coprime integers $0 < a < m$ and a permutation $\{a_1, a_2, a_3\}$ of $\{a, m - a, 1\}$ such that $1 - \gamma < \frac{a_1}{m}$, $1 - \bar{\tau} \leq \frac{a_2}{m}$, and $1 - (\eta(\tau) - \lfloor \eta(\tau) \rfloor) = -\eta(\tau) - \lfloor \tau \rfloor - 1 \leq \frac{a_3}{m}$. Then by Theorem A.0.2(4) we conclude that $\mathcal{T}(C_{p,q}; \emptyset; \tau) = [\eta(\tau), -\lfloor \tau \rfloor - 1]$. $\square$

Recall that $-q < s < 0$, and note $1 - \gamma = \frac{-s}{q}$. To visualize how the sets change with τ , we suppose that $n \leq \tau \leq n + 1$ for some integer n . Proposition 5.4.1 tells us that:

$$\mathcal{T}(C_{p,q}; \emptyset; \tau) = \begin{cases} [-(n+1), -n] & \text{if } \tau = n, \\ [-(n+1), \xi(\tau)] & \text{if } n < \tau < n + \frac{-s}{q}, \\ \{-(n+1)\} & \text{if } \tau = n + \frac{-s}{q}, \\ [\eta(\tau), -(n+1)] & \text{if } n + \frac{-s}{q} < \tau < n + 1, \\ [-(n+2), -(n+1)] & \text{if } \tau = n + 1, \end{cases}$$

where $\eta(\tau) \in (-n-2, -n-1)$ and $\xi(\tau) \in (-n-1, -n)$ are some rational numbers depending on the value of τ .

Proposition 5.4.2. *Assume $n \leq \tau_1 \leq \tau_2 \leq n + 1$ for some integer n .*

(1) *If $0 \leq \overline{\tau}_1 \leq \overline{\tau}_2 \leq 1 - \gamma$, then*

$$\{-(n+1)\} \subset \mathcal{T}(C_{p,q}; \emptyset; \tau_2) \subset \mathcal{T}(C_{p,q}; \emptyset; \tau_1) \subset [-(n+1), -n].$$

(2) *If $1 - \gamma \leq \overline{\tau}_1 \leq \overline{\tau}_2 < 1$, then*

$$\{-(n+1)\} \subset \mathcal{T}(C_{p,q}; \emptyset; \tau_1) \subset \mathcal{T}(C_{p,q}; \emptyset; \tau_2) \subset [-(n+2), -(n+1)].$$

Proof. First, consider the case where $0 \leq \overline{\tau}_1 \leq \overline{\tau}_2 \leq 1 - \gamma$. It follows immediately from Proposition 5.4.1 that

$$\begin{aligned} \{-(n+1)\} &\subset \mathcal{T}(C_{p,q}; \emptyset; \tau_1) \cap \mathcal{T}(C_{p,q}; \emptyset; \tau_2) \\ &\subset \mathcal{T}(C_{p,q}; \emptyset; \tau_1) \cup \mathcal{T}(C_{p,q}; \emptyset; \tau_2) \subset [-(n+1), -n]. \end{aligned}$$

It remains to show that $\mathcal{T}(C_{p,q}; \emptyset; \tau_2) \subset \mathcal{T}(C_{p,q}; \emptyset; \tau_1)$.

Suppose $x \in \mathcal{T}(C_{p,q}; \emptyset; \tau_2)$. If x is an integer, then either $x = -(n+1) \in \mathcal{T}(C_{p,q}; \emptyset; \tau_1)$ or $x = -n$ with $\tau_2 = n$. In the latter case, since $n \leq \tau_1 \leq \tau_2 = n$, we have $\tau_1 = \tau_2 = n$ and so $x \in \mathcal{T}(C_{p,q}; \emptyset; \tau_1)$. Now, assuming x is non-integral, then by definition, $(\emptyset; 0; \gamma; \tau_2, x)$ is JN-realizable which implies that $(\emptyset; -(\lfloor \tau_2 \rfloor + \lfloor x \rfloor); \gamma; \overline{\tau}_2, \bar{x})$ is also JN-realizable. Note that $n \leq \tau_1 \leq \tau_2$ and $-(n+1) < x < -n$. However, when $\tau_2 = n + 1$, $\mathcal{T}(C_{p,q}; \emptyset; \tau_2) = [-(n+2), -(n+1)]$ so that $-(n+1) < x < -n$ is not possible. Therefore, $-(\lfloor \tau_2 \rfloor + \lfloor x \rfloor) = -(n - (n+1)) = 1$.

Therefore, $-(\lfloor \tau_2 \rfloor + \lfloor x \rfloor) = 1$ and there are coprime integers $0 < a < m$ and a permutation $\{a_1, a_2, a_3\}$ of $\{a, m - a, 1\}$ such that $\gamma < \frac{a_1}{m}$, $\overline{\tau}_2 \leq \frac{a_2}{m}$, and $\bar{x} \leq \frac{a_3}{m}$, according

to Theorem A.0.2(4). Because $\bar{\tau}_1 \leq \bar{\tau}_2 \leq \frac{a_2}{m}$, $(\emptyset; 1; \gamma; \bar{\tau}_1, \bar{x})$ is JN-realisable and so $x \in \mathcal{T}(C_{p,q}; \emptyset; \tau_1)$. It follows that (1) is proved.

Now, suppose $1 - \gamma \leq \bar{\tau}_1 \leq \bar{\tau}_2 < 1$. Proposition 5.4.1 tells us that

$$\begin{aligned} \{-(n+1)\} &\subset \mathcal{T}(C_{p,q}; \emptyset; \tau_1) \cap \mathcal{T}(C_{p,q}; \emptyset; \tau_2) \\ &\subset \mathcal{T}(C_{p,q}; \emptyset; \tau_1) \cup \mathcal{T}(C_{p,q}; \emptyset; \tau_2) \subset [-(n+2), -(n+1)]. \end{aligned}$$

To complete the proof, we need to show $\mathcal{T}(C_{p,q}; \emptyset; \tau_1) \subset \mathcal{T}(C_{p,q}; \emptyset; \tau_2)$.

Consider an element $x \in \mathcal{T}(C_{p,q}; \emptyset; \tau_1)$. If x is an integer, then $x = -(n+1) \in \mathcal{T}(C_{p,q}; \emptyset; \tau_2)$, since τ_1 cannot be an integer by assumption. Now, suppose that x is not integral. By definition, $(\emptyset; 0; \gamma; \tau_1, x)$ is JN-realizable and, therefore, equivalently, $(\emptyset; -(\lfloor \tau_1 \rfloor + \lfloor x \rfloor); \gamma; \bar{\tau}_1, \bar{x})$ is JN-realisable. But $\lfloor \tau_1 \rfloor = 1$ and $\lfloor x \rfloor = -n-2$, so the latter condition is that $(\emptyset; 2; \gamma; \bar{\tau}_1, \bar{x})$ is JN-realisable. By Theorem A.0.2(3), this happens if and only if $(\emptyset; 1; 1-\gamma; 1-\bar{\tau}_1, 1-\bar{x})$ is JN-realisable, and so we can find coprime integers $0 < a < m$ and a permutation $\{a_1, a_2, a_3\}$ of $\{a, m-a, 1\}$ so that $1-\gamma < \frac{a_1}{m}$, $1-\bar{\tau}_1 \leq \frac{a_2}{m}$, and $1-\bar{x} \leq \frac{a_3}{m}$ by Theorem A.0.2(4). Given $1-\gamma \leq \bar{\tau}_1 \leq \bar{\tau}_2 < 1$, we have $1-\bar{\tau}_2 \leq 1-\bar{\tau}_1 \leq \frac{a_2}{m}$. It follows that $(\emptyset; 2; \gamma; \bar{\tau}_2, \bar{x})$ is JN-realisable, and so $x \in \mathcal{T}(C_{p,q}; \emptyset; \tau_2)$. This completes our proof. $\square$

Remark 5.4.3. According to the previous two propositions, the sets $\mathcal{T}(C_{p,q}; \emptyset; \tau)$ act as an “inch-worm” moving towards $-\infty$ and ∞ as τ increases and decreases to $\pm\infty$ respectively. To see this, fix $n \in \mathbb{Z}$. Starting from $\tau = n$ to $\tau = n + \frac{-s}{q}$ the inchworm pulls up its tail from $-n$ to $-(n+1)$, while keeping its head fixed at $-(n+1)$. Then, as τ increases from $n + \frac{-s}{q}$ to $n+1$ the inchworm moves its head forward from $-(n+1)$ to $-(n+2)$.

Corollary 5.4.4. *Let $\mathcal{T}(C_{p,q}; \emptyset; \tau) = [f(\tau), g(\tau)]$. The functions $\tau \mapsto f(\tau)$ and $\tau \mapsto g(\tau)$ are non-increasing. Moreover, f decreases only over intervals where g is constant and vice versa.*

For each integer n , we have that

$$[-(n+1), -n] = \mathcal{T}(C_{p,q}; \emptyset; n) = \bigcup_{n-1+\frac{-s}{q} \leq \tau \leq n+\frac{-s}{q}} \mathcal{T}(C_{p,q}; \emptyset; \tau),$$

and it follows that

$$\bigcup_{n \leq \tau} \mathcal{T}(C_{p,q}; \emptyset; \tau) = (-\infty, -n] \quad \text{and} \quad \bigcup_{\tau \leq n} \mathcal{T}(C_{p,q}; \emptyset; \tau) = [-(n+1), \infty).$$

And for general $\tau_1, \tau_2 \in \mathbb{R}$, we have

$$\bigcup_{\tau \geq \tau_1} \mathcal{T}(C_{p,q}; \emptyset; \tau) = \begin{cases} (-\infty, -\lfloor \tau_1 \rfloor] & \text{if } \overline{\tau_1} = 0, \\ (-\infty, -\lfloor \tau_1 \rfloor - 1] \cup \mathcal{T}(C_{p,q}; \emptyset; \tau_1) & \text{if } 0 < \overline{\tau_1} < 1 - \gamma, \\ (-\infty, -\lfloor \tau_1 \rfloor - 1] & \text{if } \overline{\tau_1} \geq 1 - \gamma, \end{cases}$$

$$\bigcup_{\tau \leq \tau_2} \mathcal{T}(C_{p,q}; \emptyset; \tau) = \begin{cases} [-\lfloor \tau_2 \rfloor - 1, \infty) & \text{if } \overline{\tau_2} \leq 1 - \gamma, \\ \mathcal{T}(C_{p,q}; \emptyset; \tau_2) \cup [-\lfloor \tau_2 \rfloor - 1, \infty) & \text{if } \overline{\tau_2} > 1 - \gamma. \end{cases}$$

Analogues of these results can be obtained for $\mathcal{T}_{\text{str}}(C_{p,q}; J; \tau)$. If $J = \emptyset$, then it follows from Theorem A.0.3 that when $\mathcal{T}(C_{p,q}; \emptyset; \tau)$ is a nondegenerate interval, $\mathcal{T}_{\text{str}}(C_{p,q}; \emptyset; \tau)$ is its interior, and when $\mathcal{T}(C_{p,q}; \emptyset; \tau)$ is a degenerate interval, $\mathcal{T}_{\text{str}}(C_{p,q}; \emptyset; \tau)$ coincides with it. If $J = \{1\}$, then Theorem A.0.3 still applies but some special attention is needed, for example, $\mathcal{T}_{\text{str}}(C_{p,q}; \{1\}; \tau) = \{-\tau - \gamma\}$ is no longer an interval, if τ is an integer.

5.5 Detected slopes and cabling

In this section, we recall some notations that will be used in later sections of this chapter. Let M be a 3-manifold with incompressible torus boundary components $T_1, \dots, T_n$ and fixed choices of peripheral subgroups $\pi_1(T_i) \subset \pi_1(M)$. We recall our notation from earlier sections and fix the new notation needed to complete our computations relative to certain choices of bases for the peripheral subgroups $\pi_1(T_i)$.

Recall from Sections 4.2 and 4.3 that our sets of order-detected and representation-detected slopes are written $\mathcal{D}_{\text{ord}}(J, K; M)$ and $\mathcal{D}_{\text{rep}}(J, K; M)$ respectively, with each of these being subsets of $\mathcal{S}(M) = \mathcal{S}(T_1) \times \dots \times \mathcal{S}(T_n)$.

Fix bases $\mathcal{B}_j = \{h_j, h_j^*\}$ for $\pi_1(T_j)$, $j = 1, \dots, n$; and let $\mathcal{B} = \{\mathcal{B}_j : j = 1, \dots, n\}$ and subsets $J \subset K \subset \{1, \dots, n\}$. Then we define

$$\mathcal{T}_{\text{rep}}(J, K; M; \mathcal{B}) = \{(\tau_1, \dots, \tau_n) \in (\mathbb{R} \cup \{\infty\})^n : ([\tau_1 h_1 + h_1^*], \dots, [\tau_n h_n + h_n^*]) \in \mathcal{D}_{\text{rep}}(J, K; M)\},$$

where ∞ appears in the i -th coordinate in place of the slope $[h_i]$. We similarly define $\mathcal{T}_{\text{ord}}(J, K; M; \mathcal{B})$, and note that in our new notation, Theorem 4.4.2 says precisely that $\mathcal{T}_{\text{rep}}(J, K; M; \mathcal{B}) = \mathcal{T}_{\text{ord}}(J, K; M; \mathcal{B}')$ as long as $\mathcal{B} = \mathcal{B}'$; however, these sets could differ if $\mathcal{B} \neq \mathcal{B}'$.

5.6 Attaching cable spaces to knot manifolds

This section shows how, with appropriate changes of bases, we can combine the notion of JN-realisability with that of representation and order-detection to calculate how detected slopes on the boundary of a knot manifold behave with respect to gluing on a copy of the cable space $C_{p,q}$. We first deal with the slope ∞ separately.

Proposition 5.6.1. *Assume that $p \geq 1$ and $q > 1$ are coprime integers. Suppose that $M' = M \cup C_{p,q}$ is a knot manifold that can be expressed as a union of a knot manifold M and a cable space $C_{p,q}$ whose inner boundary torus (see Section 5.1 for our inner and outer conventions) is identified with ∂M . Fix bases $\mathcal{B}_1 = \{h, -x_1\}$ and $\mathcal{B}_2 = \{h, -x_2\}$ of $\pi_1(\partial M)$ and $\pi_1(\partial M')$ as described in Section 5.1. Then $\infty \in \mathcal{T}_{\text{ord}}(J, K; M; \mathcal{B}_1)$ if and only if $\infty \in \mathcal{T}_{\text{ord}}(J, K; M'; \mathcal{B}_2)$, where (J, K) is any one of $(\emptyset, \emptyset), (\emptyset, \{1\}), (\{1\}, \{1\})$.*

Proof. Begin with $\infty \in \mathcal{T}_{\text{ord}}(J, K; M; \mathcal{B}_1)$ and let σ be a left-ordering of $\pi_1(M)$ that order-detects $(J, K; [h])$. Lemma 5.2.3 tells us that there is a left-ordering σ' of $\pi_1(C_{p,q})$ that order-detects $(\{1, 2\}, \{1, 2\}; [h], [h])$. By Corollary 5.3.1, $(J, K; [h])$ is order-detected by some left-ordering of $\pi_1(M')$. $\square$

The next theorem deals with the remaining slopes via JN-realisability.

Theorem 5.6.2. *Assume that $p \geq 1$ and $q > 1$ are coprime integers. Suppose that $M' = M \cup C_{p,q}$ is a knot manifold that can be expressed as a union of a knot manifold M and a cable space $C_{p,q}$ whose inner boundary torus (see Section 5.1 for our inner and outer conventions) is identified with ∂M . Fix bases $\mathcal{B}_1 = \{h, -x_1\}$ and $\mathcal{B}_2 = \{h, -x_2\}$ of $\pi_1(\partial M)$ and $\pi_1(\partial M')$ as described in Section 5.1. Then, we have*

- (1)
$$\bigcup_{\tau \in \mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) \setminus \{\infty\}} \mathcal{T}(C_{p,q}; \emptyset; \tau) = \mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M'; \mathcal{B}_2) \setminus \{\infty\},$$
- (2)
$$\bigcup_{\tau \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1) \setminus \{\infty\}} \mathcal{T}(C_{p,q}; \emptyset; \tau) \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2) \setminus \{\infty\},$$
- (3)
$$\bigcup_{\tau \in \mathcal{T}_{\text{ord}}(\{1\}, \{1\}; M; \mathcal{B}_1) \setminus \{\infty\}} \mathcal{T}_{\text{str}}(C_{p,q}; \{1\}; \tau) \subset \mathcal{T}_{\text{ord}}(\{1\}, \{1\}; M'; \mathcal{B}_2) \setminus \{\infty\}.$$

Proof. Choose s, r such that $ps + qr = 1$ with $-q < s < 0 < r \leq p$. First, we prove (1). Let $\tau' \in \bigcup_{\tau \in \mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) \setminus \{\infty\}} \mathcal{T}(C_{p,q}; \emptyset; \tau)$ be given. Then we can find some $\tau \in \mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) \setminus \{\infty\}$ such that $\tau' \in \mathcal{T}(C_{p,q}; \emptyset; \tau)$. In other words, $(\emptyset; 0; \frac{q+s}{q}; \tau, \tau')$ is JN-realisable. By Proposition 4.7.2 and Theorem 4.4.2, $(\emptyset, \{1, 2\}; [\tau h - x_1], [\tau' h - x_2])$ is order-detected in $\pi_1(C_{p,q})$. Since $(\emptyset, \emptyset; [\tau h - x_1])$ is also order-detected in $\pi_1(M)$, $(\emptyset, \emptyset; [\tau' h - x_2])$ is order-detected in $\pi_1(M')$ by Corollary 5.3.1(1). Therefore, $\tau' \in \mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M'; \mathcal{B}_2) \setminus \{\infty\}$.

For the inclusion in the other direction in part (1), take $\tau' \in \mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M'; \mathcal{B}_2) \setminus \{\infty\}$. Then there exists a left-ordering $\circ$ of $\pi_1(M')$ such that $[\mathcal{L}(\circ|_{\pi_1(\partial M')})] = [\tau'h - x_2]$. Then the restriction $\circ|_{\pi_1(\partial M)}$ determines a slope $[\alpha] \in \mathcal{S}(M)$. By Lemma 5.2.4, $[\alpha] = [\tau h - x_1]$ for some $\tau \in \mathbb{R}$. Thus, $(\emptyset, \emptyset; \tau') \in \mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) \setminus \{\infty\}$ and $(\emptyset, \emptyset; [\tau h - x_1], [\tau'h - x_2])$ is order-detected in $\pi_1(C_{p,q})$ by $\circ|_{\pi_1(C_{p,q})}$. By Propositions 4.4.2 and 4.7.3(2), $(\emptyset, \{1, 2\}; [\tau h - x_1], [\tau'h - x_2])$ is representation-detected. But then by Proposition 4.7.3(1), $(\emptyset; 0; \frac{q+s}{q}; \tau, \tau')$ is JN-realisable. Hence $\tau' \in \mathcal{T}(C_{p,q}; \emptyset; \tau)$.

To show part (2), let $\tau' \in \bigcup_{\tau \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1) \setminus \{\infty\}} \mathcal{T}(C_{p,q}; \emptyset; \tau)$. Then we can find some $\tau \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1) \setminus \{\infty\}$ such that $\tau' \in \mathcal{T}(C_{p,q}; \emptyset; \tau)$. In other words, $(\emptyset; 0; \frac{q+s}{q}; \tau, \tau')$ is JN-realisable. By Proposition 4.7.2 and Theorem 4.4.2, $(\emptyset, \{1, 2\}; [\tau h - x_1], [\tau'h - x_2])$ is order-detected in $\pi_1(C_{p,q})$. Since $(\emptyset, \{1\}; [\tau h - x_1])$ is also order-detected in $\pi_1(M)$, $(\emptyset, \{1\}; [\tau'h - x_2])$ is order-detected in $\pi_1(M')$ by Corollary 5.3.1(2). Therefore, $\tau' \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2) \setminus \{\infty\}$.

For part (3), we proceed as in the previous case. Take $\tau' \in \mathcal{T}_{\text{str}}(C_{p,q}; \{1\}; \tau)$ for some $\tau \in \mathcal{T}_{\text{ord}}(\{1\}, \{1\}; M; \mathcal{B}_1) \setminus \{\infty\}$. Then $(\{1\}, \{1\}, [\tau h - x_1])$ is order-detected in $\pi_1(M)$ and by Proposition 4.7.2 and Theorem 4.4.2 $(\{1, 2\}, \{1, 2\}; [\tau h - x_1], [\tau'h - x_2])$ is order-detected in $\pi_1(C_{p,q})$. Then apply Corollary 5.3.1(3) to conclude. $\square$

Under certain assumptions, we can improve the second containment in the previous theorem so as to become an equality.

Corollary 5.6.3. *With the same assumptions as in Theorem 5.6.2, if $\mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) \neq \mathbb{R} \cup \{\infty\}$ or if $\mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1)$, then*

$$\bigcup_{\tau \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1) \setminus \{\infty\}} \mathcal{T}(C_{p,q}; \emptyset; \tau) = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2) \setminus \{\infty\}.$$

Proof. Note that $\mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) \neq \mathbb{R} \cup \{\infty\}$ implies

$$\mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1)$$

by [5, Theorem 1.2]. Thus, we assume that $\mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1)$ in order to complete the proof. As Theorem 5.6.2(2) holds, the proof will be finished if we show the reverse inclusion.

Let $\tau' \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2) \setminus \{\infty\}$. Then there is a left-ordering $\circ$ of $\pi_1(M')$ with $[\mathcal{L}(\circ|_{\pi_1(\partial M')})] = [\tau'h - x_2]$. The restriction $\circ|_{\pi_1(M)}$, regarding $\pi_1(M)$ as a subgroup of $\pi_1(M')$, order-detects $(\emptyset, \emptyset; [\tau h - x_1])$ for some τ by Lemma 5.2.4. By our assumption that $\mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_1) = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1)$, we know that $(\emptyset, \{1\}; [\tau h - x_1])$ is also order-detected. Similarly, the restriction $\circ|_{\pi_1(C_{p,q})}$, regarding $\pi_1(C_{p,q})$ as a subgroup

of $\pi_1(M')$, order-detects $(\emptyset, \emptyset; [\tau h - x_1], [\tau' h - x_2])$. By Proposition 4.7.3, $(\emptyset; 0; \frac{q+s}{q}; \tau, \tau')$ is JN-realizable. Therefore, $\tau' \in \mathcal{T}(C_{p,q}; \emptyset; \tau)$ with $\tau \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1) \setminus \{\infty\}$. $\square$

In particular, if $\mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1)$ is known to be a proper subinterval of $\mathbb{R} \cup \{\infty\}$, then we are able to compute the union $\bigcup_{\tau \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1)} \mathcal{T}(C_{p,q}; \emptyset; \tau)$ from the results of Section 5.4 and thus the set of detected slopes on the boundary of the manifold resulting from attaching a cable space.

5.7 Cable knots in S^3

This section gives much more precise results in the case of cable knots in S^3 . First, we require several technical computations of the intervals $\mathcal{T}(C_{p,q}; \emptyset; \tau)$ for some specific values of τ .

Lemma 5.7.1. *Suppose that $p \geq 1$ and $q > 1$ are coprime and $ps + qr = 1$ with $-q < s < 0 < r \leq p$. If b is an integer satisfying $0 \leq b \leq \frac{p}{q}$, then*

$$0 < \min\left\{\frac{bs+r}{p-qb}, \frac{q+s}{q}\right\} \leq \frac{1}{2} < \max\left\{\frac{bs+r}{p-qb}, \frac{q+s}{q}\right\} \leq 1.$$

Moreover, $\frac{-s}{q} < \frac{(b-1)s+r}{p-q(b-1)} < \frac{bs+r}{p-qb}$.

Proof. If $p = 1$, then we have $s = 1 - q$ and $r = 1$. The only possible choice of b is $b = 0$. The statements can be easily verified by direct calculations. So we may assume $p, q \geq 2$ and $-q < s < 0 < r < p$.

Since $q(bs+r) + s(p-qb) = 1$, $bs+r$ and $p-qb$ are coprime. Let us show the first set of inequalities. The assumption $-q < s < 0$ gives $0 < \frac{q+s}{q} < 1$ immediately. Since $p, q \geq 2$ are coprime, $\frac{p}{q}$ is not integral and so $0 \leq b \leq \frac{p-1}{q} < \frac{p}{q}$ and $p-qb > 0$. It follows that $\frac{bs+r}{p-qb} > \frac{(\frac{p}{q})s+r}{p-qb} = \frac{1}{q(p-qb)} > 0$. Therefore, we have $0 < \min\{\frac{bs+r}{p-qb}, \frac{q+s}{q}\}$. Observe that $\frac{bs+r}{p-qb} + \frac{q+s}{q} = 1 + \frac{1}{q(p-qb)} > 1$. And hence $\frac{1}{2} < \max\{\frac{bs+r}{p-qb}, \frac{q+s}{q}\}$.

Next, suppose that $\frac{bs+r}{p-qb} > \frac{1}{2}$. Then $2(bs+r) - (p-qb) > 0$ and so that $\frac{bs+r}{p-qb} - \frac{1}{2} = \frac{2(bs+r) - (p-qb)}{2(p-qb)} \geq \frac{1}{2(p-qb)} \geq \frac{1}{q(p-qb)}$. Therefore, we have $\frac{q+s}{q} = 1 + \frac{1}{q(p-qb)} - \frac{bs+r}{p-qb} \leq 1 + \frac{1}{q(p-qb)} - (\frac{1}{2} + \frac{1}{q(p-qb)}) = \frac{1}{2}$. It follows that $\min\{\frac{bs+r}{p-qb}, \frac{q+s}{q}\} \leq \frac{1}{2}$. Finally, note that $\frac{bs+r}{p-qb} \leq 1$ if and only if $b \leq \frac{p-r}{s+q}$. The latter is true since $b \leq \frac{p-1}{q}$ and $\frac{p-1}{q} \leq \frac{p-r}{s+q}$ if and only if $1 \leq q+s$. This completes the proof of the first set of inequalities.

For the first part of the last inequality, note that $\frac{-s}{q} < \frac{(b-1)s+r}{p-q(b-1)}$ is equivalent to $(-s)(p-q(b-1)) < q((b-1)s+r)$, which reduces to $0 < ps+qr = 1$ and therefore holds. The second part follows from observing that the function $f(x) = \frac{xs+r}{p-qx}$ is increasing on $(\infty, \frac{p}{q})$. $\square$

Proposition 5.7.2. Assume that $p \geq 1$ and $q > 1$ are coprime integers and r, s are chosen such that $ps + qr = 1$ and $-q < s < 0 < r \leq p$. Let b be an integer with $0 \leq b \leq \frac{p}{q}$ and $C_{p,q}$ be a cable space as defined in Section 5.1. Then $0 < \frac{bs+r}{p-qb} \leq 1$ and

$$\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1 - \frac{1}{p-qb}, -1].$$

Moreover, if $p, q \geq 2$, then

$$\mathcal{T}(C_{p,q}; \{1\}; \frac{bs+r}{p-qb}) = \begin{cases} [-1 - \frac{1}{p-q(b-1)}, -1], & \frac{bs+r}{p-qb} < 1 \\ \{-\frac{2q+s}{q}\}, & \frac{bs+r}{p-qb} = 1. \end{cases}$$

Proof. If $p = 1$, then we have $s = 1 - q$ and $r = 1$. The only possible choice of b is $b = 0$ and so $\frac{bs+r}{p-qb} = 1$. And by Proposition 5.4.1, we see that $\mathcal{T}(C_{p,q}; \emptyset; 1) = [-2, -1]$. So the claim holds. Now, we assume $p, q \geq 2$ and so $-q < s < 0 < r < p$.

From Lemma 5.7.1, we see that $0 < \frac{bs+r}{p-qb} \leq 1$ and that $bs+r$ and $p-qb$ are coprime. First, we consider the case $\frac{bs+r}{p-qb} = 1$. It follows that $bs+r = p-qb = 1$. By Proposition 5.4.1, we have $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-2, -1]$, which is consistent with the formula $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1 - \frac{1}{p-qb}, -1]$. On the other hand, by the discussion at the beginning of Appendix A.0.1, $(\{1\}; 0; \frac{q+s}{q}; \frac{bs+r}{p-qb}, \tau')$ is JN-realizable if and only if $(\emptyset; -1; \frac{q+s}{q}; \tau')$ is JN-realizable, which happens if and only if $\tau' + \frac{q+s}{s} = -1$, meaning $\tau' = -\frac{2q+s}{q}$, that is, $\mathcal{T}(C_{p,q}; \{1\}; \frac{bs+r}{p-qb}) = \{-\frac{2q+s}{q}\}$. So, the proposition holds in this case.

By Lemma 5.7.1, the remaining cases are $0 < \frac{q+s}{q} \leq \frac{1}{2} < \frac{bs+r}{p-qb} < 1$ and $0 < \frac{bs+r}{p-qb} \leq \frac{1}{2} < \frac{q+s}{q} < 1$. Note that regardless of whether $J = \{1\}$ or $J = \emptyset$ we have $\{-1\} \subset \mathcal{T}(C_{p,q}; J; \frac{bs+r}{p-qb}) \subset (-2, 0)$ by Theorem A.0.3(1). Furthermore, since $\frac{bs+r}{p-qb} + \frac{q+s}{q} = 1 + \frac{1}{q(p-qb)} > 1$, we have

$$\mathcal{T}(C_{p,q}; J; \frac{bs+r}{p-qb}) \cap (-1, 0) = \emptyset \text{ and } \mathcal{T}(C_{p,q}; J; \frac{bs+r}{p-qb}) \cap (-2, -1) \neq \emptyset$$

by Theorem A.0.3(2)(b) and (3)(b). Therefore, we have $\mathcal{T}(C_{p,q}; J; \frac{bs+r}{p-qb}) \subset (-2, -1]$.

Next for $J = \{1\}$ or $J = \emptyset$, we need to determine all possible τ' with $-2 < \tau' < -1$ such that $(J; 0; \frac{q+s}{q}; \frac{bs+r}{p-qb}, \tau')$ is JN-realizable. Since $-2 < \tau' < -1$, $(J; 0; \frac{q+s}{q}; \frac{bs+r}{p-qb}, \tau')$ is JN-realizable if and only if $(J; 2; \frac{q+s}{q}; \frac{bs+r}{p-qb}, 2 + \tau')$ is JN-realizable, which is equivalent to JN-realizability of $(J; 1; 1 - \frac{q+s}{q}; 1 - \frac{bs+r}{p-qb}, -1 - \tau')$ by Theorem A.0.2(3). Moreover, by Theorem A.0.2(4), $(J; 1; 1 - \frac{q+s}{q}; 1 - \frac{bs+r}{p-qb}, -1 - \tau')$ is JN-realizable if and only if there are coprime A, N with $0 < A < N$ and a permutation $\{A_1, A_2, A_3\}$ of $\{A, N - A, 1\}$ such that

$$1 - \frac{q+s}{q} < \frac{A_1}{N}, 1 - \frac{bs+r}{p-qb} \leq \frac{A_2}{N} \text{ and } -1 - \tau' \leq \frac{A_3}{N}$$

if $J = \emptyset$; if $J = \{1\}$, then the second inequality is strict. We rewrite these inequalities as

$$\frac{q+s}{q} > 1 - \frac{A_1}{N}, \quad \frac{bs+r}{p-qb} \geq 1 - \frac{A_2}{N} \quad (\text{resp. } \frac{bs+r}{p-qb} > 1 - \frac{A_2}{N}) \quad \text{and } \tau' \geq -1 - \frac{A_3}{N}$$

to prepare for the following case-by-case analysis.

Case 1: $0 < \frac{q+s}{q} \leq \frac{1}{2} < \frac{bs+r}{p-qb} < 1$.

Since $0 < \frac{q+s}{q} \leq \frac{1}{2}$ and $\frac{q+s}{q} > 1 - \frac{A_1}{N}$, we must have $A_1 \neq 1$. So without loss of generality by replacing A with $N - A$, we may assume $A_1 = N - A$. It follows that $\frac{q+s}{q} > \frac{A}{N}$. We claim that $A_2 = A$ and $A_3 = 1$. To see this, suppose not, and proceed as follows:

From $\frac{q+s}{q} > \frac{A}{N}$ and $\frac{bs+r}{p-qb} \geq 1 - \frac{1}{N}$, we have $s > q(\frac{A-N}{N})$ and $bs+r \geq (p-qb)(\frac{N-1}{N})$. If $A \geq 2$, then

$$\begin{aligned} 1 &= (p-qb)s + q(bs+r) > q(p-qb)(\frac{A-N}{N}) + q(p-qb)(\frac{N-1}{N}) \\ &= q(p-qb)(\frac{A-1}{N}) \\ &\geq q(p-qb)\frac{1}{N} \\ &\geq q(p-qb)(1 - \frac{bs+r}{p-qb}) \\ &= q((p-qb) - (bs+r)) > 0. \end{aligned}$$

Since $q((p-qb) - (bs+r))$ is an integer, and we have $1 > q((p-qb) - (bs+r)) > 0$, this is a contradiction. Therefore, $A = 1$. If $A = 1$, then $\frac{bs+r}{p-qb} \geq 1 - \frac{1}{N}$ becomes $\frac{bs+r}{p-qb} \geq 1 - \frac{A}{N}$ as claimed (with strict inequality if $J = \{1\}$).

Subcase 1(i): $J = \emptyset$. In this case, whether or not $\tau' \in \mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb})$ is determined by whether or not we can find two coprime integers A, N with $0 < A < N$ and

$$\frac{q+s}{q} > \frac{A}{N}, \quad \frac{bs+r}{p-qb} \geq 1 - \frac{A}{N} \quad \text{and } \tau' \geq -1 - \frac{1}{N}.$$

Therefore $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1 - \frac{1}{M}, -1]$, where $M \geq 2$ is the smallest N for which such a coprime pair A, N exist. We claim $M = p - qb$ to complete the proof in this case. Recall that $0 < \frac{q+s}{q} \leq \frac{1}{2} < \frac{bs+r}{p-qb} < 1$ and $0 \leq b \leq \frac{p-1}{q} < \frac{p}{q}$.

Taking $A = (p-qb) - (bs+r) > 0$ and $M = p - qb$ gives $\frac{bs+r}{p-qb} = 1 - \frac{A}{M} \geq 1 - \frac{A}{M}$. Also note $\frac{q+s}{q} > \frac{A}{M}$ if and only if $\frac{-s}{q} < \frac{bs+r}{p-qb}$, which is true by Lemma 5.7.1. We also have $0 < A < M$ since $M - A = bs+r > 0$, and we know that A, M are coprime since $(s+q)M - qA = 1$. To see any such N satisfies $N \geq M = p - qb$ so that M is minimal, we proceed as follows.

The inequalities $\frac{q+s}{q} > \frac{A}{N}$, $\frac{bs+r}{p-qb} \geq 1 - \frac{A}{N}$ are equivalent to $\frac{q+s}{q} > \frac{A}{N} \geq \frac{(p-qb)-(bs+r)}{p-qb}$, which is also equivalent to

$$(p-qb)(q+s)N > Aq(p-qb) \geq ((p-qb)-(bs+r))qN.$$

It follows that $0 < (p-qb)(q+s)N - Aq(p-qb) = (p-qb)((q+s)N - Aq)$ and thus $p-qb \leq (p-qb)(q+s)N - Aq(p-qb)$. Next note that $0 \leq Aq(p-qb) + ((bs+r)-(p-qb))qN$, adding this inequality to the previous one we arrive at

$$(Aq(p-qb) + ((bs+r)-(p-qb))qN) + ((p-qb)(q+s)N - Aq(p-qb)) \geq p-qb.$$

Since $(p-qb)s + q(bs+r) = 1$, the left-hand side of the inequality is actually equal to N . This completes the proof of minimality.

Subcase 1(ii): $J = \{1\}$. In this case, we investigate pairs of coprime integers A, N with $0 < A < N$ and

$$\frac{q+s}{q} > \frac{A}{N}, \quad \frac{bs+r}{p-qb} > 1 - \frac{A}{N} \quad \text{and} \quad \tau' \geq -1 - \frac{1}{N}.$$

As above, it follows that $\mathcal{T}(C_{p,q}; \{1\}; \frac{bs+r}{p-qb}) = [-1 - \frac{1}{M}, -1]$, where $M \geq 2$ is the smallest choice of N for which such a coprime pair A, N exists. We claim $M = p - q(b-1)$.

Firstly, take $A = (p - q(b-1)) - ((b-1)s + r) = ((p-qb) - (bs+r)) + (q+s) > 0$ and $M = p - q(b-1) = (p-qb) + q > 0$. Since $M - A = (b-1)s + r > 0$ and $(s+q)M - qA = 1$, we have $0 < A < M$ and M, A are coprime. Furthermore, we observe that $\frac{bs+r}{p-qb} > 1 - \frac{A}{M}$ and $\frac{q+s}{q} > \frac{A}{M}$ are equivalent to $\frac{bs+r}{p-qb} > 1 - \frac{A}{M} > \frac{-s}{q}$, and since $1 - \frac{A}{M} = \frac{(b-1)s+r}{p-q(b-1)}$ these inequalities hold by Lemma 5.7.1.

To see that this choice of M is minimal, we proceed as the same as in Subcase 1(i) and suppose that A, N with $0 < A < N$ are another coprime pair satisfying the inequalities $\frac{q+s}{q} > \frac{A}{N}$ and $\frac{bs+r}{p-qb} > 1 - \frac{A}{N}$. These are equivalent to $\frac{q+s}{q} > \frac{A}{N} > \frac{(p-qb)-(bs+r)}{p-qb}$, which is also equivalent to

$$(p-qb)(q+s)N > Aq(p-qb) > ((p-qb)-(bs+r))qN.$$

It follows that $0 < (p-qb)(q+s)N - Aq(p-qb) = (p-qb)((q+s)N - Aq)$ and $0 < Aq(p-qb) + ((bs+r)-(p-qb))qN = q(A(q-qb) + ((bs+r)-(p-qb))N)$. We conclude $p-qb \leq (p-qb)(q+s)N - Aq(p-qb)$ and $q \leq Aq(q-qb) + ((bs+r)-(p-qb))qN$. Adding these inequalities, we see that

$$(Aq(p-qb) + ((bs+r)-(p-qb))qN) + ((p-qb)(q+s)N - Aq(p-qb)) \geq p-qb+q.$$

By using $(p-qb)s + q(bs+r) = 1$, we see that the left-hand side is equal to N , so our

choice of M above is minimal.

Case 2: $0 < \frac{bs+r}{p-qb} \leq \frac{1}{2} < \frac{q+s}{q} < 1$.

Since $\frac{1}{2} \geq \frac{bs+r}{p-qb}$ and $\frac{bs+r}{p-qb} \geq 1 - \frac{A_2}{N}$ with the inequality being strict if $J = \{1\}$, we must have $A_2 \neq 1$. We may assume $A_2 = N - A$ by replacing A by $N - A$ if necessary. Using exactly the same reasoning as in Case 1, we can assume $A_1 = N$ and $A_3 = 1$. So, the problem reduces to determining the existence of coprime A, N with $0 < A < N$ such that

$$\frac{q+s}{q} > 1 - \frac{A}{N}, \quad \frac{bs+r}{p-qb} \geq \frac{A}{N} \quad \text{and} \quad \tau' \geq -1 - \frac{1}{N},$$

with the second inequality being strict if $J = \{1\}$. To complete the proof, we mimic Case 1.

Subcase 2(i): $J = \emptyset$. In this case, we aim to find two coprime integers A, N with $0 < A < N$ and

$$\frac{q+s}{q} > 1 - \frac{A}{N}, \quad \frac{bs+r}{p-qb} \geq \frac{A}{N} \quad \text{and} \quad \tau' \geq -1 - \frac{1}{N},$$

and with $N \geq 2$ minimal, so that $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1 - \frac{1}{N}, -1]$. For clarity, we write M for the minimal such integer and claim $M = p - qb$ to complete the proof in this case.

Let $M = p - qb > 0$ and $A = bs + r > 0$. We see that $0 < A < M$ and A, M are coprime. Moreover $\frac{bs+r}{p-qb} = \frac{A}{M}$ and $\frac{q+s}{q} > 1 - \frac{A}{M}$ by direct calculations and an application of Lemma 5.7.1. To see that M is minimal, we proceed as follows.

The inequalities $\frac{q+s}{q} > 1 - \frac{A}{N}$ and $\frac{bs+r}{p-qb} \geq \frac{A}{N}$ are equivalent to $-\frac{s}{q} < \frac{A}{N} \leq \frac{bs+r}{p-qb}$, which is also equivalent to $-(p-qb)sN < Aq(p-qb) \leq q(bs+r)N$. It follows that $0 < (p-qb)sN + Aq(p-qb)$, which factors as $(p-qb)(sN + qA)$, and thus $p-qb \leq (p-qb)sN + Aq(p-qb)$. Similarly, $0 \leq q(bs+r)N - Aq(p-qb)$. Also note that $(p-qb)s + q(bs+r) = 1$ and we finish the proof in this case by calculating

$$N = (q(bs+r)N - Aq(p-qb)) + ((p-qb)sN + Aq(p-qb)) \geq p - qb.$$

Subcase 2(ii): $J = \{1\}$. In this case, we aim to find two coprime integers A, N with $0 < A < N$ and

$$\frac{q+s}{q} > 1 - \frac{A}{N}, \quad \frac{bs+r}{p-qb} > \frac{A}{N} \quad \text{and} \quad \tau' \geq -1 - \frac{1}{N},$$

and with $N \geq 2$ minimal, so that $\mathcal{T}(C_{p,q}; \{1\}; \frac{bs+r}{p-qb}) = [-1 - \frac{1}{N}, -1]$. As before, we write M for the minimal such integer and claim that $M = p - q(b-1)$.

We proceed as in Subcase 2(i), however, $0 < q(bs+r)N - Aq(p-qb)$ is now a strict inequality. Since it factors as $q((bs+r)N - A(p-qb))$, we have $q \leq (bs+r)qN - Aq(p-qb)$

qb). Recalling that $(p - qb)s + q(bs + r) = 1$, we compute

$$N = (q(bs + r)N - A(q(p - qb))) + ((p - qb)sN + Aq(p - qb)) \geq p - qb + q.$$

□

Lemma 5.7.3. *Suppose that $p \geq 1$ and $q > 1$ are coprime and $ps + qr = 1$ with $-q < s < 0 < r \leq p$. If b is an integer with $b \geq \frac{p}{q}$, then $b \geq \frac{r}{-s} > \frac{p}{q}$ and*

$$0 \leq \min\left\{\frac{bs + r}{p - qb}, \frac{q + s}{q}\right\} < \frac{1}{2} \leq \max\left\{\frac{bs + r}{p - qb}, \frac{q + s}{q}\right\} < 1.$$

Moreover, if $\frac{bs+r}{p-qb} \neq 0$, then $\frac{-s}{q} > \frac{bs+r}{p-qb} \geq \frac{-s}{q+1}$ and $b > \frac{r}{-s}$.

Proof. It is clear that $0 < \frac{q+s}{q} < 1$. From $ps + qr = 1$, we have $\frac{r}{-s} = \frac{p}{q} + \frac{1}{q(-s)}$. But $b \geq \frac{p}{q}$ is an integer, so $b \geq \frac{p}{q} + \frac{1}{q} \geq \frac{p}{q} + \frac{1}{q(-s)} = \frac{r}{-s}$, since $p \geq 1$ and $q > 1$ are coprime integers and $-q < s < 0 < r \leq p$. Also note that $bs + r$ and $p - qb$ are coprime.

Observe that $\frac{bs+r}{p-qb} < \frac{-s}{q}$ holds, as it is equivalent to $0 < s(p - qb) + q(bs + r)$, the right-hand side of which is equal to 1. Thus $\frac{q+s}{q} + \frac{bs+r}{p-qb} < \frac{q+s}{q} + \frac{-s}{q} = 1$ and it follows that

$$0 \leq \min\left\{\frac{bs + r}{p - qb}, \frac{q + s}{q}\right\} < \frac{1}{2} \quad \text{and} \quad \max\left\{\frac{bs + r}{p - qb}, \frac{q + s}{q}\right\} < 1.$$

Note that $b \geq \frac{r}{-s}$ is an integer. If $b = \frac{r}{-s}$, then $s = -1$; it follows that $\frac{q+s}{q} \geq \frac{1}{2}$ and this is the only case where $\frac{bs+r}{p-qb} = 0$, so the claimed inequalities of the lemma hold in this case. On the other hand, if $b > \frac{r}{-s}$, then $b \geq \frac{r+1}{-s}$. The inequality $\frac{bs+r}{p-qb} \geq \frac{-s}{q+1}$ reduces to $bs + r \leq -1$ upon using the identity $ps + qr = 1$, and thus it holds. Therefore

$$\max\left\{\frac{bs + r}{p - qb}, \frac{q + s}{q}\right\} \geq \max\left\{\frac{-s}{q + 1}, \frac{q + s}{q}\right\}.$$

If $-s \leq \frac{q}{2}$, then $\frac{q+s}{q} \geq \frac{1}{2}$; if $\frac{q}{2} < -s$, then $\frac{q}{2} + \frac{1}{2} \leq -s$ and so $\frac{-s}{q+1} \geq \frac{1}{2}$. It follows that $\max\left\{\frac{q+s}{q}, \frac{-s}{q+1}\right\} \geq \frac{1}{2}$ and hence

$$\frac{1}{2} \leq \max\left\{\frac{bs + r}{p - qb}, \frac{q + s}{q}\right\}.$$

□

Proposition 5.7.4. *Assume $p \geq 1$ and $q > 1$ are coprime integers and r, s satisfy $ps + qr = 1$ and $-q < s < 0 < r \leq p$. Let b be an integer with $b \geq \frac{p}{q}$, and $C_{p,q}$ be the cable space defined in Section 5.1. Then*

$$\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1, -1 + \frac{1}{bq-p}].$$

Proof. Lemma 5.7.3 applies here. We see that $0 \leq \frac{bs+r}{p-qb} < 1$. Moreover, $\frac{bs+r}{p-qb} = 0$ occurs only when $s = -1$ and $b = \frac{r}{-s} = r$. If this is the case, then $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1, 0]$ by Proposition 5.4.1 and so the statement holds since $bq - p = ps + qr = 1$.

By Lemma 5.7.3 we are left to consider the cases $0 < \frac{bs+r}{p-qb} < \frac{1}{2} \leq \frac{q+s}{q} < 1$ and $0 < \frac{q+s}{q} < \frac{1}{2} \leq \frac{bs+r}{p-qb} < 1$. In either case, $bs + r$ and $p - qb$ are coprime and $\{-1\} \subset \mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) \subset [-1, 0]$ by Proposition 5.4.1, since $\frac{bs+r}{p-qb} < \frac{-s}{q}$ by Lemma 5.7.3. Hence, we need to determine all possible $-1 < \tau' < 0$ such that $(\emptyset; 0; \frac{q+s}{q}; \frac{bs+r}{p-qb}, \tau')$ is JN-realizable by the definition of $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb})$. Since $-1 < \tau' < 0$, $(\emptyset; 0; \frac{q+s}{q}; \frac{bs+r}{p-qb}, \tau')$ is JN-realizable if and only if $(\emptyset; 1; \frac{q+s}{q}; \frac{bs+r}{p-qb}, 1 + \tau')$ is JN-realizable from the discussion at the beginning of the Appendix. Moreover, by Theorem A.0.2(4), $(\emptyset; 1; \frac{q+s}{q}; \frac{bs+r}{p-qb}, 1 + \tau')$ is JN-realizable if and only if there are coprime numbers A, N with $0 < A < N$ and a permutation $\{A_1, A_2, A_3\}$ of $\{A, N - A, 1\}$ such that

$$\frac{q+s}{q} < \frac{A_1}{N}, \quad \frac{bs+r}{p-bq} \leq \frac{A_2}{N} \quad \text{and} \quad 1 + \tau' \leq \frac{A_3}{N}.$$

Case 1: $0 < \frac{bs+r}{p-qb} < \frac{1}{2} \leq \frac{q+s}{q} < 1$.

Since $\frac{1}{2} \leq \frac{q+s}{q} < \frac{A_1}{N}$, we must have $A_1 \neq 1$ and $N > 2$. We may assume $A_1 = N - A$ by replacing A by $N - A$ if necessary.

Subcase 1(i): $A_2 = 1$. In this case, we aim to find two coprime integers A, N with $0 < A < N$ and

$$\frac{q+s}{q} < \frac{N-A}{N}, \quad \frac{bs+r}{p-bq} \leq \frac{1}{N} \quad \text{and} \quad 1 + \tau' \leq \frac{A}{N}.$$

The first inequality $\frac{q+s}{q} < \frac{N-A}{N}$ is equivalence to $\frac{-s}{q} > \frac{A}{N}$. By Lemma 5.7.3, we have $\frac{1}{N} \geq \frac{bs+r}{p-bq} \geq \frac{-s}{q+1}$. Therefore, $\frac{-s}{q} > \frac{-s}{q+1}A$ and so $A < \frac{q+1}{q}$. The only possible choice is $A = 1$ and so the next subcase will finish the proof of Case 1.

Subcase 1(ii): $A_3 = 1$. In this case, we aim to find two coprime integers A, N with $0 < A < N$ and

$$\frac{q+s}{q} < \frac{N-A}{N}, \quad \frac{bs+r}{p-bq} \leq \frac{A}{N} \quad \text{and} \quad 1 + \tau' \leq \frac{1}{N},$$

and with N minimal, so $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1, -1 + \frac{1}{N}]$. As before we write M for the minimal such integer and prove $M = qb - p$ to complete the proof in this case. Recall that from Lemma 5.7.3, $0 < \frac{-s}{q+1} \leq \frac{bs+r}{p-bq} < \frac{-s}{q}$.

Firstly, note taking $A = -(bs + r)$ and $M = qb - p$ gives $\frac{bs+r}{p-bq} = \frac{A}{M} \leq \frac{A}{N}$ and $\frac{q+s}{q} < \frac{M-A}{M}$, since $\frac{bs+r}{p-bq} < \frac{-s}{q}$. Also $0 < A < M$ and A, M are coprime. To see that any such N satisfies $N \geq M = qb - p$ so that M is minimal, we proceed as follows.

The inequalities $\frac{q+s}{q} < \frac{N-A}{N}$ and $\frac{bs+r}{p-bq} \leq \frac{A}{N}$ are equivalent to $\frac{bs+r}{p-bq} \leq \frac{A}{N} < \frac{-s}{q}$. Observing that both $bs + r$ and $p - qb$ are negative, we arrive at $Nq(bs + r) \geq Aq(p - qb)$ and

$sN(p - qb) > -Aq(p - qb)$. Since both $sN(p - qb)$ and $-Aq(p - qb)$ are multiples of $(p - qb)$, we have $sN(p - qb) \geq -Aq(p - qb) + (qb - p)$. Since $(p - qb)s + q(bs + r) = 1$, we see that

$$N = Ns(p - qb) + Nq(bs + r) \geq -Aq(p - qb) + (qb - p) + Aq(p - qb) = qb - p.$$

Case 2: $0 < \frac{q+s}{q} < \frac{1}{2} \leq \frac{bs+r}{p-qb} < 1$.

If $A_2 = 1$, then $\frac{1}{2} \leq \frac{bs+r}{p-qb} \leq \frac{1}{N}$ implies $N = 2$. Moreover, $N = 2$ and $A = 1$ is a possible pair if and only if $\frac{bs+r}{p-qb} = \frac{1}{2}$. If this is the case, $bs + r = -1$ and $p - qb = -2$, since $bs + r$ and $p - qb$ are coprime. Moreover $N = 2$ is the minimal N satisfying the desired inequalities, so $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1, -\frac{1}{2}]$. This agrees with the formula $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1, -1 + \frac{1}{bq-p}]$.

Having dealt with $A_2 = 1$, we may assume $A_2 = A$ by replacing A by $N - A$ if necessary.

Subcase 2(i): $A_1 = 1$. In this case, we aim to find two coprime integers A, N with $0 < A < N$ and

$$\frac{q+s}{q} < \frac{1}{N'}, \quad \frac{bs+r}{p-bq} \leq \frac{A}{N} \quad \text{and} \quad 1 + \tau' \leq \frac{N-A}{N}.$$

Since $-q < s < 0$, we have $\frac{1}{q} \leq \frac{q+s}{q} < \frac{1}{N}$. Therefore, $N < q$. Lemma 5.7.3 says $\frac{bs+r}{p-bq} \geq \frac{-s}{q+1}$, which gives us

$$\frac{q}{q+1} = \frac{q+s}{q+1} + \frac{-s}{q+1} < \frac{q+s}{q} + \frac{bs+r}{p-qb} < \frac{1}{N} + \frac{A}{N}.$$

So if $A \leq N - 2$, then we have

$$N < q \quad \text{and} \quad \frac{q}{q+1} < \frac{A+1}{N} \leq \frac{N-1}{N},$$

which cannot be simultaneously true. The only possibility is that $A = N - 1$ and so we can resolve Case 2 by addressing the following subcase.

Subcase 2(ii): $A_1 = N - A$. In this case, we aim to find two coprime integers A, N with $0 < A < N$ and

$$\frac{q+s}{q} < \frac{N-A}{N}, \quad \frac{bs+r}{p-bq} \leq \frac{A}{N} \quad \text{and} \quad 1 + \tau' \leq \frac{1}{N},$$

and with N minimal, so that $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1, -1 + \frac{1}{N}]$. The rest of the proof is exactly the same as Subcase 1(ii).

□

Theorem 5.7.5. Assume $p \geq 1$ and $q > 1$ are coprime integers and K is a knot in S^3 . Let $M = S^3 - n(K)$ and $M' = S^3 - n(C_{p,q}(K))$ denote the knot complements of K and its cable

$C_{p,q}(K)$, respectively. Fix bases $\mathcal{B}_K = \{\mu, \lambda\}$ and $\mathcal{B}_C = \{\mu_C, \lambda_C\}$ of $\pi_1(\partial M)$ and $\pi_1(\partial M')$, respectively, as explained in Section 5.1. Then, we have:

(1) If K is the unknot and $p, q \geq 2$, then $C_{p,q}(K)$ is the torus knot $T_{p,q}$ and we have

$$\mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C) = [-\infty, pq - p - q]$$

and

$$\mathcal{T}_{\text{ord}}(\{1\}, \{1\}; M'; \mathcal{B}_C) = (-\infty, pq - p - q).$$

(2) If K is a nontrivial knot, then $[-\infty, pq - p] \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C)$. Moreover,

(a) If $2g(K) - 1 < \frac{p}{q}$, then

$$[-\infty, pq - p + (2g(K) - 1)q] \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C);$$

(b) if $2g(K) - 1 > \frac{p}{q}$, then

$$[pq - p + (2g(K) - 1)q, \infty] \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C).$$

Proof. For part (1), choose r and s such that $ps + qr = 1$ and $-q < s < 0 < r < p$. Note that in this case M' is the knot complement of the torus knot $T_{p,q}$, which can be viewed as the result of gluing a solid torus to $C_{p,q}$ by identifying the canonical meridian of the solid torus to λ in $\mathcal{B}_K$, or to the slope $f(0) = \frac{r}{q}$ with respect to the basis $\mathcal{B}_1$. Here f is the transition map between the bases $\mathcal{B}_K$ and $\mathcal{B}_1$ discussed in Section 5.2. Now M' is a Seifert manifold and $\tau' \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2)$ if and only if $(\emptyset; 0; \frac{q+s}{q}, \frac{r}{p}; \tau')$ is JN-realizable, which is also equivalent to saying $(\{1\}; 0; \frac{q+s}{q}, \frac{r}{p}; \tau')$ is JN-realizable. Therefore, we have

$$\mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2) = \mathcal{T}(C_{p,q}; \{1\}; \frac{r}{p}) = [-\frac{p+q+1}{p+q}, -1]$$

by Proposition 5.7.2 with $b = 0$. According to the change of basis described in Lemma 5.2.2, we get

$$\mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C) = [-\infty, pq - p - q].$$

Similarly, $\mathcal{T}_{\text{ord}}(\{1\}, \{1\}; M'; \mathcal{B}_C) = (-\infty, pq - p - q)$ by Theorem A.0.3(4)(b).

To see part (2), suppose that K is a nontrivial knot and choose r and s such that $ps + qr = 1$ and $-q < s < 0 < r \leq p$. Then $0 \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_K)$ by [6, Example 6.3], and so $f(0) = \frac{r}{q} \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1)$. By Theorem 5.6.2 and Proposition 5.7.2 with $b = 0$, we have $[-1 - \frac{1}{p}, -1] = \mathcal{T}(C_{p,q}; \emptyset; \frac{r}{q}) \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2)$. By Lemma 5.2.2, we have

$$[-\infty, pq - p] \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C).$$

In addition, note that $2g(K) - 1 \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_K)$ by [6, Corollary 1.4]. For ease of writing, set $b = 2g(K) - 1$. Then $f(b) = \frac{bs+r}{p-qb} \in \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1)$ and hence $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2)$ by Theorem 5.6.2. If $1 \leq b \leq \frac{p}{q}$, then Proposition 5.7.2 applies and $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1 - \frac{1}{p-qb}, -1]$; if $b \geq \frac{p}{q}$, then Proposition 5.7.4 applies and $\mathcal{T}(C_{p,q}; \emptyset; \frac{bs+r}{p-qb}) = [-1, -1 + \frac{1}{qb-q}]$. Lemma 5.2.2 then gives us that

$$[-\infty, pq - p - q + 2g(K)q] \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C),$$

when $1 \leq 2g(K) - 1 \leq \frac{p}{q}$; and

$$[pq - p - q + 2g(K)q, \infty] \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C),$$

when $2g(K) - 1 > \frac{p}{q}$. □

Remark 5.7.6. The previous result is based on [6, Corollary 1.4], which shows that the slope $2g(K) - 1$ is always order-detected. In fact, [6, Corollary 1.4] shows that every rational slope whose distance from the longitudinal slope divides $2g(K) - 1$ is order-detected. Because of this, the conclusion of Theorem 5.7.5(2)(b) can be improved in cases where $2g(K) - 1$ is not prime.

With these results, we are able to mirror the result of Hedden and Hom (Theorem 3.3.3), which completely describes how L-space knots behave with respect to cabling. Recall from Section 3.3 that a knot $K \subset S^3$ is an L-space knot if and only if

$$\{r \in \mathbb{Q} \mid S^3_r(K) \text{ is not an L-space}\} = (-\infty, 2g(K) - 1) \cap \mathbb{Q};$$

otherwise $S^3_r(L)$ is not an L-space for all $r \in \mathbb{Q}$. Theorem 3.3.3 therefore has a natural interpretation in terms of the interval of slopes along which the result of Dehn surgery is not an L-space and their behaviour with respect to cabling. We have an analogous result for the behaviour of order-detected slopes, and their behaviour with respect to cabling.

Corollary 5.7.7. *Assume $p \geq 1$ and $q > 1$ are coprime integers. Let $M = S^3 - n(K)$ and $M' = S^3 - n(C_{p,q}(K))$ be the knot complements of a nontrivial knot K and its cable $C_{p,q}(K)$, respectively. Fix bases $\mathcal{B}_K = \{\mu, \lambda\}$ and $\mathcal{B}_C = \{\mu_C, \lambda_C\}$ of $\pi_1(\partial M)$ and $\pi_1(\partial M')$ as explained in Section 5.1.*

(1) *If $[-\infty, 2g(K) - 1] = \mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_K)$ and $2g(K) - 1 < \frac{p}{q}$, then*

$$\mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C) = [-\infty, pq - p - q + 2g(K)q].$$

(2) If $[-\infty, 2g(K) - 1] = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_K)$ and $2g(K) - 1 > \frac{p}{q}$, then

$$\mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C) = \mathbb{R} \cup \{\infty\}.$$

(3) If $\mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_K) = \mathbb{R} \cup \{\infty\}$, then

$$\mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C) = \mathbb{R} \cup \{\infty\}.$$

Proof. For convenience, write $b = 2g(K) - 1$.

To show part (1), we argue as follows. Choose integers r, s with $ps + qr = 1$ and $-q < s < 0 < r \leq p$. Since $\mathcal{T}(\emptyset, \emptyset; M; \mathcal{B}_K) = [-\infty, b] \neq \mathbb{R}$, we have $\mathcal{T}_{\text{ord}}(\emptyset, \emptyset; M; \mathcal{B}_K) = \mathcal{T}(\emptyset, \{1\}; M; \mathcal{B}_K) = [-\infty, b]$ by [5, Theorem 1.2]. From $[-\infty, b] = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_K)$ and $b < \frac{p}{q}$, we have

$$f([-\infty, b]) = [-\frac{s}{q}, \frac{sb+r}{p-qb}] = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1)$$

by Lemma 5.2.1, where f is the map arising from the change of basis. It follows from Corollary 5.6.3 that

$$\bigcup_{\tau \in [-\frac{s}{q}, \frac{sb+r}{p-qb}]} \mathcal{T}(C_{p,q}; \emptyset; \tau) = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2).$$

Since $b < \frac{p}{q}$, Proposition 5.7.2 applies and gives us

$$\mathcal{T}(C_{p,q}; \emptyset; \frac{sb+r}{p-qb}) = [-1 - \frac{1}{p-qb}, -1].$$

From Proposition 5.4.1, we have $\mathcal{T}(C_{p,q}; \emptyset; \frac{-s}{q}) = \{-1\}$. By Corollary 5.4.4 and its accompanying discussion,

$$\bigcup_{\tau \in [-\frac{s}{q}, \frac{sb+r}{p-qb}]} \mathcal{T}(C_{p,q}; \emptyset; \tau) = [-1 - \frac{1}{p-qb}, -1],$$

and so $[-1 - \frac{1}{p-qb}, -1] = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2)$. It follows from Lemma 5.2.2 that

$$[-\infty, pq - p - q + 2g(K)] = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C).$$

Part (2) is argued in a similar way. Since $[-\infty, b] = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_K)$ and $b > \frac{p}{q}$, Lemma 5.2.1 says

$$f([-\infty, b]) = [-\infty, \frac{sb+r}{p-qb}] \cup [-\frac{s}{q}, \infty] = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M; \mathcal{B}_1),$$

where f is the map arising from the change of basis. It follows from Theorem 5.6.2 that

$$\bigcup_{\tau \in (-\infty, \frac{sb+r}{p-qb}] \cup [\frac{-s}{q}, \infty)} \mathcal{T}(C_{p,q}; \emptyset; \tau) \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2).$$

We apply Corollary 5.4.4 and its accompanying discussion as above. We find that

$$\bigcup_{\tau \in [\frac{-s}{q}, \infty)} \mathcal{T}(C_{p,q}; \emptyset; \tau) = (-\infty, -1].$$

Since $b > \frac{p}{q}$, Proposition 5.7.4 gives us $\mathcal{T}(C_{p,q}; \emptyset; \frac{sb+r}{p-qb}) = [-1, -1 + \frac{1}{qb-p}]$. It follows that

$$\bigcup_{\tau \in (-\infty, \frac{sb+r}{p-qb}]} \mathcal{T}(C_{p,q}; \emptyset; \tau) = [-1, \infty).$$

Since the meridional slope (i.e., the slope ∞) is always order-detected [6, Corollary 1.4], we conclude that $\mathbb{R} \cup \{\infty\} = \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2)$ and so $\mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_C) = \mathbb{R} \cup \{\infty\}$.

Part (3) follows immediately from the fact

$$\bigcup_{\tau \in \mathbb{R}} \mathcal{T}(C_{p,q}; \emptyset; \tau) = \mathbb{R} \subset \mathcal{T}_{\text{ord}}(\emptyset, \{1\}; M'; \mathcal{B}_2),$$

together with the same observations as above. □

Remark 5.7.8. In a similar manner, one can analyse the set of strongly order-detected slopes: if some interval is contained in $\mathcal{T}_{\text{ord}}(\{1\}, \{1\}; M; \mathcal{B})$, then $\mathcal{T}_{\text{ord}}(\{1\}, \{1\}; M'; \mathcal{B}_C)$ contains a corresponding interval. While there are many special cases where intervals of strongly order-detected slopes are shown to exist, there are no general results similar to [6, Corollary 1.4] that allow for a clean statement like Theorem 5.6.2 which applies to all knots in S^3 . Since our results would only be piecemeal, we do not pursue an explicit computation of strongly-detected intervals of slopes on the boundary of cable knots in this manuscript.

6

Applications to a Family of Two-component Links

This chapter applies the framework of order detection to an infinite family of hyperbolic two-component links. Building on the representation-theoretic perspective developed earlier, it identifies slope conditions under which Dehn surgeries on these links produce manifolds with non-left-orderable fundamental groups. The family includes the Whitehead link, and in that case we produce a collection of non-left-orderable Dehn fillings that precisely matches the prediction of the L-space conjecture. All the materials in this chapter are taken from [14].

6.1 An infinite family of hyperbolic links

In this section, we apply the theory of slope detection to an infinite family of hyperbolic links. After explicitly determining the presentation of the fundamental group, we use the techniques developed in Section 4.5 to conclude the (non)-left-orderability of the fundamental group of the result of Dehn surgeries.

Let $\mathbb{L}' = L_0 \cup L_1 \cup L_2$ be the three-component link as shown in Figure 6.1. The fundamental group of the link complement of $\mathbb{L}'$ is given by SnapPy [18] as

$$\Gamma = \langle x, y, z \mid xz^2y^2x^{-1}z^{-1}y^{-2}z^{-1}, x^2zy^{-1}z^{-2}x^{-2}yz \rangle,$$

together with the peripheral systems of L_0, L_1 and L_2 respectively given as

$$\begin{aligned} m_0 &= z^{-1}x^{-2}y, & l_0 &= zx^2z; \\ m_1 &= z^{-1}x^{-1}, & l_1 &= xy^{-2}z^{-1}(xz)^3; \end{aligned}$$

$$m_2 = xzy,$$

$$l_2 = yzxy^{-1}z^{-1}x^{-1}.$$

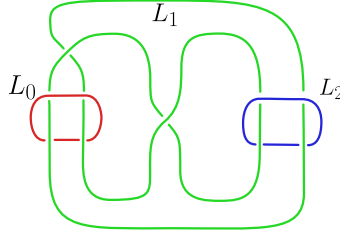


Figure 6.1: The three-component link $\mathbb{L}'$

For $n \in \mathbb{N}$, denote by $\mathbb{L}_n = L_1 \cup L_2$ the link as shown in Figure 6.2, where the first component L_1 is the torus knot $T(2, 2n+1)$ and L_2 is the unknot. In the Thistlethwaite Link Table and up to mirror images, $\mathbb{L}_0$ is L5a1 (the Whitehead link); $\mathbb{L}_1$ is L7a3; $\mathbb{L}_2$ is L9a14; and $\mathbb{L}_3$ is L11a110.

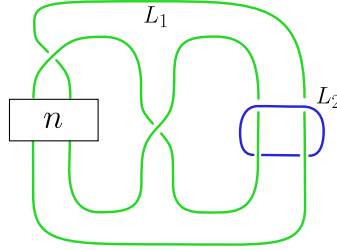


Figure 6.2: The two-component link $\mathbb{L}_n$; the block represents n full twists

From here forward, we denote by M_n the link complement of $\mathbb{L}_n$. Since $\mathbb{L}_n$ can be obtained by a Rolfsen twist along the first component L_0 of $\mathbb{L}'$, the fundamental group $\pi_1(M_n)$ admits the following presentation:

$$\pi_1(M_n) = \langle x, y, z \mid xz^2y^2x^{-1}z^{-1}y^{-2}z^{-1}, x^2zy^{-1}z^{-2}x^{-2}yz, m_0^{-1}l_0^n \rangle.$$

Write $r_1 = xz^2y^2x^{-1}z^{-1}y^{-2}z^{-1}$ and $r_2 = x^2zy^{-1}z^{-2}x^{-2}yz$. From $m_0^{-1}l_0^n = 1$, we see that $y = (x^2z^2)^n x^2 z$. Substituting $y = (x^2z^2)^n x^2 z$ into r_2 , we see that r_2 is killed automatically. Substituting $y = (x^2z^2)^n x^2 z$ into r_1 , we obtain the following presentation:

$$\pi_1(M_n) = \langle x, z \mid x(z^2x^2)^{n+1}z(x^2z^2)^{n+1}z^{-1}x^{-1}(z^{-2}x^{-2})^{n+1}z^{-1}(x^{-2}z^{-2})^{n+1}z \rangle.$$

Replacing x with a and z with $a^{-2}b$, $\pi_1(M_n)$ admits the following presentation

$$\pi_1(M_n) = \langle a, b \mid (a^2b^{-2})^n a^2 b^{-1} a^{-1} (b^2 a^{-2})^n b^3 = b^3 (a^{-2} b^2)^n a^{-1} b^{-1} a^2 (b^{-2} a^2)^n \rangle.$$

Under this replacement, the peripheral elements become:

$$\begin{aligned} m_1 &= b^{-1}a, & l_1 &= a(b^{-2}a^2)^n b^{-3} (a^2b^{-2})^n a^2 (a^{-1}b)^3; \\ m_2 &= a^{-1}b^2(a^{-2}b^2)^n, & l_2 &= b(a^{-2}b^2)^n a^{-2}ba(b^{-2}a^2)^n b^{-2}a. \end{aligned}$$

However, our new peripheral systems must take into account the change of framing when performing the Rolfsen twist along L_0 of $\mathbb{L}'$ (see e.g. [67, §16]). It follows that the peripheral systems along L_1 and L_2 in M_n are given respectively by

$$\begin{aligned} m &= b^{-1}a, & l &= a(b^{-2}a^2)^n b^{-3} (a^2b^{-2})^n a^2 (a^{-1}b)^{4n+3}; \\ \mu &= a^{-1}b^2(a^{-2}b^2)^n, & \lambda &= b(a^{-2}b^2)^n a^{-2}ba(b^{-2}a^2)^n b^{-2}a, \end{aligned}$$

where $\{m, l\}$ and $\{\mu, \lambda\}$ serve as generators for the subgroups $\pi_1(T_1)$ and $\pi_1(T_2)$ respectively. Note that for clarity in the arguments that follow, we have renamed the meridian and longitude pairs to avoid subscripts. We note the following formulas, to be used in the computations below:

$$\begin{aligned} m^{4n+2}l &= a(b^{-2}a^2)^n b^{-3} (a^2b^{-2})^n ab, & \mu\lambda &= b(a^{-2}b^2)^n a^{-2}ba = ba^{-1}\mu m, \\ m^{4n+3}l &= a(b^{-2}a^2)^n b^{-3} (a^2b^{-2})^n a^2, & \mu^2\lambda &= ba^{-2}(b^2a^{-2})^n b^3 (a^{-2}b^2)^n. \end{aligned}$$

Theorem 6.1.1. *If $([\alpha_1], [\alpha_2]) \in (4n+2, \infty) \times (2, \infty) \subset \mathcal{S}(M_n)$, then $(\emptyset, \emptyset; [\alpha_1], [\alpha_2])$ is not order-detected.*

Proof. Assume that $([\alpha_1], [\alpha_2]) \in (4n+2, \infty) \times (2, \infty) \subset \mathcal{S}(M_n)$, and that $(\emptyset, \emptyset; [\alpha_1], [\alpha_2])$ is order-detected by a left-ordering $\circ$ of $\pi_1(M_n)$. Then for all sufficiently large integers N , $m^{4n+2}l$ and $m^N l$ are of opposite signs under $\circ$ and $\mu^2\lambda$ and $\mu^N\lambda$ are also of opposite signs. Replacing $\circ$ with its opposite if necessary, we can further assume $m^{4n+2}l <_\circ 1 <_\circ m^N l$ and therefore $m^{4n+2}l <_\circ 1 <_\circ m$. It follows that a and b are of the same sign, for otherwise $m^{4n+2}l$ and m would have the same sign. We consider cases based on the signs of a, b and $\mu^2\lambda$, and show there is a contradiction in each case.

Case 1: $\mu^2\lambda >_\circ 1$.

If $\mu >_\circ 1$, then it follows immediately that $\mu^{t+2}\lambda >_\circ 1$ for all $t \in \mathbb{N}$, which is a contradiction. So $\mu <_\circ 1$ in this case.

Subcase 1(i): a, b are positive.

Observe that one of $a^{-1}b^2 >_o 1$ or $ba^{-1} >_o 1$ must hold; for if they are both negative, then the expression

$$\begin{aligned}\mu^2\lambda &= ba^{-2}(b^2a^{-2})^nb^3(a^{-2}b^2)^n \\ &= (ba^{-1})[(a^{-1}b^2)a^{-1}]^n(a^{-1}b^2)(ba^{-1})[(a^{-1}b^2)a^{-1}]^{n-1}(a^{-1}b^2)\end{aligned}$$

would imply that $\mu^2\lambda <_o 1$, a contradiction.

Suppose that $ba^{-1} >_o 1$. We see that

$$\mu = a^{-1}b^2(a^{-2}b^2)^n = a^{-1}b^{-1}(a^2b^{-2})^n ab(m^{4n+2}l)^{-1}a.$$

Hence, for all $2 < t \in \mathbb{N}$, we have

$$\begin{aligned}(\mu\lambda)\mu^t &= b(a^{-2}b^2)^na^{-2}ba(a^{-1}b^{-1}(a^2b^{-2})^n ab(m^{4n+2}l)^{-1}a)^t \\ &= b(a^{-2}b^2)^na^{-2}(a^2b^{-2})^n ab((m^{4n+2}l)^{-1}b^{-1}(a^2b^{-2})^n ab)^{t-1}(m^{4n+2}l)^{-1}a \\ &= ba^{-1}b((m^{4n+2}l)^{-1}b^{-1}(a^2b^{-2})^n ab)^{t-1}(m^{4n+2}l)^{-1}a.\end{aligned}$$

But now $b^{-1}(a^2b^{-2})^n ab = b((b^{-2}a^2)^nb^{-2}a)b = b\mu^{-1}b$ is positive, as are ba^{-1} , a , b and $(m^{4n+2}l)^{-1}$. This implies $\mu^{t+1}\lambda$ is positive for all $2 < t \in \mathbb{N}$, which is a contradiction.

On the other hand, suppose $a^{-1}b^2 >_o 1$. Making use of the relator, we see that

$$\mu = a^{-1}b^2(a^{-2}b^2)^n = a^{-1}b^{-1}(a^2b^{-2})^na^2(m^{4n+2}l)^{-1}a^{-1}ba.$$

Then for $2 < t \in \mathbb{N}$ we can write

$$\begin{aligned}(\mu\lambda)\mu^t &= b(a^{-2}b^2)^na^{-2}ba(a^{-1}b^{-1}(a^2b^{-2})^na^2(m^{4n+2}l)^{-1}a^{-1}ba)^t \\ &= b(a^{-2}b^2)^na^{-2}(a^2b^{-2})^na^2((m^{4n+2}l)^{-1}a^{-1}(a^2b^{-2})^na^2)^{t-1}(m^{4n+2}l)^{-1}a^{-1}ba \\ &= b((m^{4n+2}l)^{-1}a^{-1}(a^2b^{-2})^na^2)^{t-1}(m^{4n+2}l)^{-1}a^{-1}ba \\ &= b((m^{4n+2}l)^{-1}(a^{-1}b^2)(b^{-2}a^2)^nb^{-2}a^2)^{t-1}(m^{4n+2}l)^{-1}(a^{-1}b^2)(b^{-1}a).\end{aligned}$$

But then $(b^{-2}a^2)^nb^{-2}a^2 = \mu^{-1}a$ is positive, as are $a^{-1}b^2$, a , b , $b^{-1}a = m$, and $(m^{4n+2}l)^{-1}$, which implies $\mu^{t+1}\lambda$ is positive for all $2 < t \in \mathbb{N}$ in this case. So we have reached a contradiction in this case.

Subcase 1(ii): a, b are negative.

First suppose that $a^{-2}b^2 <_o 1$. Note that

$$m = b^{-1}a = a(\mu\lambda)^{-1}b(a^{-2}b^2)^na^{-1}.$$

For $2 < t \in \mathbb{N}$, we have

$$\begin{aligned}
m^t(m^{4n+2}l) &= (a(\mu\lambda)^{-1}b(a^{-2}b^2)^na^{-1})^t(a(b^{-2}a^2)^nb^{-3}(a^2b^{-2})^nab) \\
&= a((\mu\lambda)^{-1}b(a^{-2}b^2)^n)^{t-1}(\mu\lambda)^{-1}b(a^{-2}b^2)^na^{-1}(a(b^{-2}a^2)^nb^{-3}(a^2b^{-2})^nab) \\
&= a((\mu\lambda)^{-1}b(a^{-2}b^2)^n)^{t-1}(\mu\lambda)^{-1}b^{-2}(a^2b^{-2})^nab \\
&= a((\mu\lambda)^{-1}b(a^{-2}b^2)^n)^{t-1}(\mu\lambda)^{-1}\mu^{-1}b \\
&= a((\mu\lambda)^{-1}b(a^{-2}b^2)^n)^{t-1}(\mu^2\lambda)^{-1}b.
\end{aligned}$$

Since $(a^{-2}b^2)^n <_o 1$ and $a, b, (\mu\lambda)^{-1}, (\mu^2\lambda)^{-1}$ are also negative, $m^t(m^{4n+3}l)$ is negative. But this gives us a contradiction.

Next suppose that $b^{-2}a^2 <_o 1$. Note that

$$m = b^{-1}a = b((b^{-2}a^2)(a^{-1}b))b^{-1}$$

and therefore if $t \geq 2$ then

$$\begin{aligned}
m^t &= (b((b^{-2}a^2)(a^{-1}b))b^{-1})^t \\
&= b((b^{-2}a^2)(a^{-1}b))^{t-1}((b^{-2}a^2)(a^{-1}b))b^{-1} \\
&= b((b^{-2}a^2)(a^{-1}b))^{t-1}(b^{-2}a^2)a^{-1}.
\end{aligned}$$

Now, since $m^{4n+3}l = a(\mu^2\lambda)^{-1}b$, we can write

$$m^t m^{4n+3}l = b((b^{-2}a^2)(a^{-1}b))^{t-1}(b^{-2}a^2)(\mu^2\lambda)^{-1}b.$$

Since $b, b^{-2}a^2, a^{-1}b$ and $(\mu^2\lambda)^{-1}$ are negative, $m^t m^{4n+3}l$ is also negative for all $t \geq 2$, a contradiction.

Case 2: $\mu^2\lambda <_o 1$.

If $\mu <_o 1$, it follows that $\mu^{t+2}\lambda <_o 1$ for all $t \in \mathbb{N}$, which is a contradiction. So $\mu >_o 1$ in this case. Since $\mu\lambda = ba^{-1}\mu m <_o 1$ and both μ and m are positive, $ba^{-1} <_o 1$. Also observe that

$$\begin{aligned}
m\mu m &= (b^{-1}a)(a^{-1}b^2(a^{-2}b^2)^n)(b^{-1}a) \\
&= b(a^{-2}b^2)^nb^{-1}a \\
&= ((ba^{-1})(a^{-1}b))^na.
\end{aligned}$$

Since $ba^{-1} <_o 1, a^{-1}b = m^{-1} <_o 1$ and $m\mu m >_o 1$, we must have $a >_o 1$, and therefore $b >_o 1$ as well. Since $((ba^{-1})(a^{-1}b))^na = ((ba^{-1})(a^{-1}b))^{n-1}(ba^{-1})a^{-1}ba >_o 1$, we also conclude that $a^{-1}ba >_o 1$ and so $a^{-1}b^{-1}a <_o 1$.

Note there is an equality $a^{-1}(m^{4n+3}l)b^{-1}(\mu^2\lambda) = 1$ that can be rewritten as

$$(a^{-1}b^{-1}a)(m^{4n+2}l)b^{-1}(\mu^2\lambda) = 1.$$

However $a^{-1}b^{-1}a, b^{-1}, m^{4n+2}l$ and $\mu^2\lambda$ are negative, while the right-hand side is the identity. This leads to a contradiction. □

Lemma 6.1.2. *No proper subgroup of $\pi_1(M_n)$ contains both $\pi_1(T_1)$ and $\pi_1(T_2)$.*

Proof. Suppose $H \subset \pi_1(M_n)$ contains $\pi_1(T_1)$ and $\pi_1(T_2)$. Then H contains $\mu^{-1}m^{-1}\mu\lambda = ba^{-1}$. From the identity $m\mu m = ((ba^{-1})(a^{-1}b))^n a$, we see that $a \in H$, from which it follows easily that $b \in H$, so that $H = \pi_1(M_n)$. □

As a result, we have the following corollary.

Corollary 6.1.3. (1) *If $[\alpha_1] \in (4n+2, \infty)$ and $(\emptyset, \{1\}; [\alpha_1], [\alpha_2])$ is order-detected by a left-ordering σ of $\pi_1(M_n)$, then $\pi_1(T_2)$ is σ -cofinal.*

(2) *If $[\alpha_2] \in (2, \infty)$ and $(\emptyset, \{2\}; [\alpha_1], [\alpha_2])$ is order-detected by a left-ordering σ of $\pi_1(M_n)$, then $\pi_1(T_1)$ is σ -cofinal.*

Proof. We prove only (1), with the argument for (2) being similar.

Suppose that $[\alpha_1] \in (4n+2, \infty)$ and $(\emptyset, \{1\}; [\alpha_1], [\alpha_2])$ is order-detected by σ , and that $\pi_1(T_2)$ is σ -bounded. By Lemma 6.1.2 we may apply Theorem 4.5.5, concluding $(\emptyset, \emptyset; [\alpha_1], [\beta])$ is order-detected for all $\beta \in \mathcal{S}(T_2)$. But when $[\beta] \in (2, \infty)$, we know that $(\emptyset, \emptyset; [\alpha_1], [\beta])$ is not order-detected by Theorem 6.1.1, a contradiction. □

Lemma 6.1.4. *If $[\alpha_1] \in (2n+2, \infty)$ and $(\emptyset, \emptyset; [\alpha_1], [\alpha_2])$ is order-detected by a left-ordering σ of $\pi_1(M_n)$, then $\pi_1(T_1)$ is not σ -cofinal.*

Proof. Suppose $[\alpha_1] \in (2n+2, \infty)$ and $(\emptyset, \emptyset; [\alpha_1], [\alpha_2])$ is order-detected by σ , and that $\pi_1(T_1)$ is σ -cofinal. In particular, $m = b^{-1}a$ is σ -cofinal, we may assume that m is positive and $m^{2n+2}l$ is negative.

Using the fact (see [5]) that the products and conjugates of positive, σ -cofinal elements are also positive and σ -cofinal, we see that $b^{-2}ab = b^{-1}mb$ and $ab^{-1} = ama^{-1}$ are positive and σ -cofinal. Also note that

$$\prod_{i=0}^{n-1} (a^{-1}b)^{n-i} (ab^{-1})(b^{-1}a)^{n-i} = (a^{-1}b)^n [(ab^{-1})(b^{-1}a)]^n$$

and since the left-hand side is a product of conjugates of m , the right-hand side is positive and σ -cofinal. Similarly, the equality

$$\prod_{i=0}^{n-1} (a^{-1}b)^{n-i} (b^{-2}ab)(b^{-1}a)^{n-i} = (a^{-1}b)^n [(b^{-2}ab)(b^{-1}a)]^n$$

allows us to conclude that the right-hand side is positive and σ -cofinal. It follows that $[(b^{-2}ab)(b^{-1}a)]^n (b^{-2}ab)(a^{-1}b)^n$ is positive and σ -cofinal as well, since it is a product of the two positive σ -cofinal terms $[(b^{-2}ab)(b^{-1}a)]^n (a^{-1}b)^n$, $(a^{-1}b)^{-n} (b^{-2}ab)(a^{-1}b)^n$. Now our final observation is that

$$m^{4n+2}l = [(ab^{-1})(b^{-1}a)]^n (ab^{-1}) [(b^{-2}ab)(b^{-1}a)]^n (b^{-2}ab)$$

and therefore, adding a power of m^{-2n} to both sides, we get

$$m^{2n+2}l = (a^{-1}b)^n [(ab^{-1})(b^{-1}a)]^n (ab^{-1}) [(b^{-2}ab)(b^{-1}a)]^n (b^{-2}ab) (a^{-1}b)^n.$$

But this last expression is a product of positive terms, so $m^{2n+2}l$ is positive. This contradicts the fact that $m^{2n+2}l$ is negative. $\square$

Combining Lemma 6.1.4 and Corollary 6.1.3, we have the following.

Corollary 6.1.5. *If $([\alpha_1], [\alpha_2]) \in (2n+2, \infty) \times (2, \infty)$, then $(\emptyset, \{2\}; [\alpha_1], [\alpha_2])$ is not order-detected by a left-ordering σ of $\pi_1(M_n)$.*

Lemma 6.1.6. *Suppose that $(a_1/b_1, a_2/b_2) \in (2n+2, \infty) \times (2, \infty)$ are rational numbers written in lowest terms with $a_i, b_i > 0$, then $\pi_1(T_1) \not\subset \langle\langle m^{a_1}l^{b_1} \rangle\rangle$ and $\pi_1(T_2) \not\subset \langle\langle \mu^{a_2}\lambda^{b_2} \rangle\rangle$.*

Proof. We only argue that $\pi_1(T_1) \not\subset \langle\langle m^{a_1}l^{b_1} \rangle\rangle$, the other case being similar. Note that $H_1(M_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, with the copies of $\mathbb{Z}$ generated by m and μ . Therefore, if $\pi_1(T_1) \subset \langle\langle m^{a_1}l^{b_1} \rangle\rangle$, then the factor generated by m is killed when one appropriately Dehn fills L_1 , meaning $H_1(S_{a_1/b_1, a_2/b_2}^3(L_n); \mathbb{Z})$ is a cyclic group generated by μ whose order divides a_2 . On the other hand, since the linking number of L_1 and L_2 is zero, we have $|H_1(S_{a_1/b_1, a_2/b_2}^3(L_n); \mathbb{Z})| = a_1 a_2$ by [28, Proposition 5.3.11]. This is a contradiction since $a_1, a_2 > 1$. $\square$

Theorem 6.1.7. *If $(r_1, r_2) \in (2n+2, \infty) \times (2, \infty)$ is a pair of rational numbers, then (r_1, r_2) -surgery on $\mathbb{L}_n$ produces a manifold whose fundamental group is not left-orderable.*

Proof. If $S_{r_1, r_2}^3(\mathbb{L}_n)$ is left-orderable for some $(r_1, r_2) \in (2n+2, \infty) \times (2, \infty)$, then by Lemma 6.1.6 and Theorem 4.5.6, we know that $(\emptyset, \{1, 2\}; [m^{a_1}l^{b_1}], [\mu^{a_2}\lambda^{b_2}])$ is order-detected. By Remark 4.2.2, we have a contradiction to Corollary 6.1.5, which finishes the proof. $\square$

Remark 6.1.8. Assuming the truth of the L-space conjecture, [22, Lemma 3.8] suggests that the manifolds arising from (r_1, r_2) -surgery on $\mathbb{L}_n$ should have non-left-orderable fundamental groups if $(r_1, r_2) \in [2n + 1, \infty) \times [1, \infty)$.

6.2 Whitehead link

In this section, we use the simplicity of the presentation of the fundamental group of the Whitehead link complement to improve the Dehn filling results of the previous section in the case $n = 0$.

Denote by Wh the mirror image of the Whitehead link in S^3 and by M its link complement. Setting $n = 0$ in the presentation for $\pi_1(M_n)$ of the previous section, we have

$$\pi_1(M) = \langle a, b \mid a^2 b^{-1} a^{-1} b^3 = b^3 a^{-1} b^{-1} a^2 \rangle,$$

with meridians and longitudes given by

$$\begin{aligned} m &= b^{-1}a, & l &= ab^{-3}a^2(a^{-1}b)^3; \\ \mu &= a^{-1}b^2, & \lambda &= ba^{-2}bab^{-2}a. \end{aligned}$$

Given a group G and a non-identity element $g \in G$, in the discussion below we use $N(g)$ to denote the root-closed, conjugacy-closed subsemigroup of G generated by g .

Theorem 6.2.1. *Suppose that M is a knot manifold with peripheral subgroup $\pi_1(T)$ generated by $\{\mu, \lambda\}$, and there exist coprime integers $p, q > 0$ such that $N(\mu^p \lambda^q) \cap N(\mu) \neq \emptyset$. If there exists a slope $[\alpha] \in \mathcal{S}(M)$ that is not weakly order-detected, then no $[\beta] \in (p/q, \infty)$ is weakly order-detected.*

Proof. Let σ be a left-ordering of $\pi_1(M)$ and $[\beta] \in (p/q, \infty)$. Suppose that σ order-detects $(\emptyset, \emptyset; [\beta])$. By [5, Theorem 1.7], since there exists a slope which is not weakly order-detected, $\pi_1(T)$ is σ -cofinal. Then in particular, $\mu^p \lambda^q$ and μ are of opposite signs under σ , and each is cofinal in $\pi_1(T)$ and thus cofinal in $\pi_1(M)$ as well. Without loss of generality, we may assume that $\mu^p \lambda^q >_\sigma 1$ and $\mu <_\sigma 1$.

The set of positive (resp. negative) σ -cofinal elements forms a root-closed, conjugacy-closed subsemigroup of $\pi_1(M)$; see [5]. Let us denote by N_+ (resp. N_-) this subsemigroup. Then as $\mu^p \lambda^q$ is positive and cofinal, $N(\mu^p \lambda^q) \subset N_+$ and similarly $N(\mu) \subset N_-$. Yet $N_+ \cap N_- = \emptyset$, while $N(\mu^p \lambda^q) \cap N(\mu)$ is assumed to be nonempty, a contradiction. $\square$

Next we confirm that certain fillings of M satisfy the hypotheses of Theorem 6.2.1.

Lemma 6.2.2. *If M denotes the Whitehead link complement with peripheral systems as above, then $N(ml) \cap N(m) \neq \emptyset$.*

Proof. Using $ml = a^{-1}bab^{-3}ab$, one can verify that

$$\begin{aligned} (a^{-1}b^{-1}amla^{-1}ba)ml &= (a^{-1}b^{-1}a)(a^{-1}bab^{-3}ab)(a^{-1}ba)(a^{-1}bab^{-3}ab) \\ &= b^{-3}a(ba^{-1}b^2ab^{-3}ab^{-2}a)a^{-1}b^3 \\ &= b^{-3}a(ba^{-1}b^3(b^{-1}a)b^{-3}ab^{-1}(b^{-1}a))a^{-1}b^3. \end{aligned}$$

Clearly $(a^{-1}b^{-1}amla^{-1}ba)ml \in N(ml)$, while the right-hand side of the equation above, being a product of conjugates of $m = b^{-1}a$, lies in $N(m)$. $\square$

Proposition 6.2.3. *Let p, q be coprime with $0 < p \leq q$. If $[\beta] \in (3, 4) \subset \mathcal{S}(S_{*, 1+\frac{p}{q}}^3(Wh))$, then $[\beta]$ is not weakly order-detected.*

Proof. The special case $p = q = 1$ will occasionally require a slightly different computation. Whenever necessary, we will note this exceptional case.

Firstly, we note that

$$\pi_1(S_{*, 1+\frac{p}{q}}^3(Wh)) = \langle b, a \mid a^2b^{-1}a^{-1}b^3 = b^3a^{-1}b^{-1}a^2, (\mu\lambda)^{q-p}(\mu^2\lambda)^p = 1 \rangle,$$

where

$$(\mu\lambda)^{q-p}(\mu^2\lambda)^p = (ba^{-2}ba)^{q-p}(ba^{-2}b^3)^p.$$

Suppose $\circ$ is a left-ordering of $\pi_1(S_{*, 1+\frac{p}{q}}^3(Wh))$ that weakly order-detects some $[\beta] \in (3, 4) \subset \mathcal{S}(S_{*, 1+\frac{p}{q}}^3(Wh))$. Then m^3l and m^4l are of opposite signs under $\circ$. Changing to the opposite of $\circ$ if necessary, we may further assume that $m^3l = ab^{-3}a^2 >_{\circ} 1$.

If $m >_{\circ} 1$, then it follows immediately that $m^4l >_{\circ} 1$, which is a contradiction. So we assume $m = b^{-1}a <_{\circ} 1$. From $m = b^{-1}a <_{\circ} 1 <_{\circ} m^3l = ab^{-3}a^2$, we see that b and a are of the same sign. We consider cases based upon the signs of a and b .

Case 1: Both a, b are positive.

Rewrite the relator $(\mu\lambda)^{q-p}(\mu^2\lambda)^p = 1$ as

$$((ba^{-2}b)a)^{q-p}((ba^{-2}b)b^2)^p = 1.$$

Since b, a are positive, we have $ba^{-2}b <_{\circ} 1$ and so $b^{-1}a^2b^{-1} >_{\circ} 1$.

Now if $a^{-2}b^2 >_{\circ} 1$ and $p < q$, then we can again rewrite $(\mu\lambda)^{q-p}(\mu^2\lambda)^p = 1$ as

$$b((a^{-2}b^2)(b^{-1}ab))^{q-p}(a^{-2}b^2)b(b(a^{-2}b^2)b)^{p-1} = 1.$$

Since b and $a^{-2}b^2$ are positive, we must have $b^{-1}ab <_o 1$ and then $b^{-1}a^{-1}b >_o 1$ and $b^{-1}a^{-2}b >_o 1$. But now the relator $(\mu\lambda)^{q-p}(\mu^2\lambda)^p = 1$ can be rewritten again as

$$(b^2(b^{-1}a^{-2}b)a)^{q-p}(b^2(b^{-1}a^{-2}b)b^2)^p = 1,$$

where all the terms on the left-hand side are positive and the right-hand side is the identity. We arrive at a contradiction. Therefore, we must have $a^{-2}b^2 <_o 1$, or equivalently, $b^{-2}a^2 >_o 1$. Note that if $p = q = 1$ then the relator $b(a^{-2}b^2)b = 1$ implies $b^{-2}a^2 >_o 1$ in this case as well.

It follows immediately that $m^4l = (b^{-1}a^2b^{-1})(b^{-2}a^2) >_o 1$, which is a contradiction.

Case 2: Both a, b are negative.

We shall show that this case is not possible as well. The argument begins as in the previous case: Rewrite the relator $(\mu\lambda)^{q-p}(\mu^2\lambda)^p = 1$ as

$$((ba^{-2}b)a)^{q-p}((ba^{-2}b)b^2)^p = 1.$$

Since b, a are negative, we have $ba^{-2}b >_o 1$ and so $b^{-1}a^2b^{-1} <_o 1$.

Now if $a^{-2}b^2 <_o 1$ and $p < q$, then we can again rewrite $(\mu\lambda)^{q-p}(\mu^2\lambda)^p = 1$ as

$$b((a^{-2}b^2)(b^{-1}ab))^{q-p}(a^{-2}b^2)b(b(a^{-2}b^2)b)^{p-1} = 1.$$

Now since $b, (a^{-2}b^2)$ are negative, it follows that $b^{-1}ab >_o 1$ and then $b^{-1}a^{-1}b <_o 1$ and $b^{-1}a^{-2}b <_o 1$. But now the relator $(\mu\lambda)^{q-p}(\mu^2\lambda)^p = 1$ can be rewritten again as

$$(b^2(b^{-1}a^{-2}b)a)^{q-p}(b^2(b^{-1}a^{-2}b)b^2)^p = 1,$$

where all the terms on the left-hand side are positive and the right-hand side is the identity. We arrive at a contradiction. Hence, we must have $a^{-2}b^2 >_o 1$, that is, $b^{-2}a^2 <_o 1$. As in the previous case, when $p = q = 1$ then the relator $b(a^{-2}b^2)b = 1$ forces $b^{-2}a^2 <_o 1$.

Since $\mu^3\lambda = ab^{-3}a^2 = (ab^{-1})(b^{-2}a^2) >_o 1$, we must have $ab^{-1} >_o 1$. Rewrite the relator $(\mu\lambda)^{q-p}(\mu^2\lambda)^p = 1$ one more time as

$$((ba^{-1})(a^{-1}b^2)(b^{-1}a))^{q-p}((ba^{-1})(a^{-1}b^2)b)^p = 1.$$

Since $ba^{-1}, b^{-1}a$ and b are all negative, we must have $a^{-1}b^2 >_o 1$ and so $b^{-2}a <_o 1$. But then

$$m^2l = ab^{-3}ab = (ab^{-1})(b^{-2}a)b <_o 1,$$

contradicting $m^3l >_o 1$ and $m <_o 1$.

□

Corollary 6.2.4. *Suppose p, q are coprime positive integers with $p/q \in (1, \infty)$. If $[\beta] \in (1, \infty) \subset \mathcal{S}(S_{*,p/q}^3(\text{Wh}))$, then $[\beta]$ is not weakly order-detected.*

Proof. Using $\bar{m}, \bar{l} \in \pi_1(S_{*,p/q}^3(\text{Wh}))$ to denote the image of the peripheral elements $m, l \in \pi_1(M)$, Lemma 6.2.2 implies that $N(\bar{m}\bar{l}) \cap N(\bar{m}) \neq \emptyset$.

If $p/q \in (1, 2]$, combining this with Proposition 6.2.3 we may apply Theorem 6.2.1 to conclude that no $[\beta] \in (1, \infty)$ is weakly order-detected.

On the other hand suppose $p/q \in (2, \infty)$ and that $[\alpha] \in (2, \infty) \subset \mathcal{S}(S_{*,p/q}^3(\text{Wh}))$ is weakly order-detected. Then we can use the short exact sequence

$$\{\text{id}\} \longrightarrow \langle\langle \mu^p \lambda^q \rangle\rangle \longrightarrow \pi_1(M) \longrightarrow \pi_1(S_{*,p/q}^3(\text{Wh})) \longrightarrow \{\text{id}\}$$

to argue that $(\emptyset, \emptyset; [\alpha], [\mu^p \lambda^q])$ is order-detected, contradicting Theorem 6.1.1 in the case $n = 0$. We can now use Theorem 6.2.1 to conclude that no $[\beta] \in (1, \infty) \subset \mathcal{S}(S_{*,p/q}^3(\text{Wh}))$ is weakly order-detected. □

Proposition 6.2.5. *Suppose $(p_1/q_1, p_2/q_2) \in (1, \infty) \times (1, \infty)$, where p_i, q_i are coprime positive integers. Then $(\{2\}, \{2\}; [m^{p_1} l^{q_2}], [\mu^{p_2} \lambda^{q_2}])$ is not order-detected by any left-ordering σ of $\pi_1(M)$.*

Proof. For contradiction, suppose that $(\{2\}, \{2\}; [m^{p_1} l^{q_2}], [\mu^{p_2} \lambda^{q_2}])$ is order-detected by a left-ordering σ of $\pi_1(M)$. Then there exists an σ -convex normal subgroup $C \subset \pi_1(M)$ such that $C \cap \pi_1(T_2) = \langle \mu^{p_2} \lambda^{q_2} \rangle$ and $C \cap \pi_1(T) \subset \langle m^{p_1} l^{q_1} \rangle$. In particular, $\langle \langle \mu^{p_2} \lambda^{q_2} \rangle \rangle \subset C$, so there exists a homomorphism

$$\phi : \pi_1(S_{*,p_2/q_2}^3(\text{Wh})) \cong \pi_1(M) / \langle \langle \mu^{p_2} \lambda^{q_2} \rangle \rangle \rightarrow \pi_1(M) / C.$$

Set $K = \ker(\phi)$ and consider the short exact sequence

$$\{\text{id}\} \longrightarrow K \longrightarrow \pi_1(S_{*,p_2/q_2}^3(\text{Wh})) \xrightarrow{\phi} \pi_1(M) / C \longrightarrow \{\text{id}\}.$$

Equip $\pi_1(M)/C$ with the quotient left-ordering σ' defined by $gC <_{\sigma'} hC$ if and only if $g <_{\sigma} h$ whenever $gC \neq hC$, and equip K with any left-ordering whatsoever. Using these left-orderings, construct a lexicographic left-ordering σ'' of $\pi_1(S_{*,p_2/q_2}^3(\text{Wh}))$.

Note that for all $m^r l^s \in \pi_1(T_1) \setminus \langle m^{p_1} l^{q_1} \rangle$, we have $m^r l^s >_{\sigma} 1$ if and only if $m^r l^s C >_{\sigma'} C$. Therefore, if we denote by $\bar{m}, \bar{l}$ the images of m, l in $\pi_1(S_{*,p_2/q_2}^3(\text{Wh}))$, then $\bar{m}^r \bar{l}^s >_{\sigma''} 1$ if and only if $m^r l^s >_{\sigma} 1$. Thus σ'' weakly order-detects $[m^{p_1} l^{q_1}] \in \mathcal{S}(S_{*,p_2/q_2}^3(\text{Wh}))$. This contradicts Corollary 6.2.4. □

We require the next remark for our final proof of this section.

Remark 6.2.6. Note that there is an automorphism $f : \pi_1(M) \rightarrow \pi_1(M)$ given by $f(a) = a^{-1}b^3$ and $f(b) = a^{-1}ba$, and this automorphism satisfies

$$f(m) = \mu, f(l) = \lambda, f(\mu) = m \text{ and } f(\lambda) = l.$$

Therefore, if we let $\sigma : \{1, 2\} \rightarrow \{1, 2\}$ denote the transposition $\sigma(1) = 2$ and $\sigma(2) = 1$, then $(J, K; [\alpha_1], [\alpha_2])$ is order-detected if and only if $(\sigma(J), \sigma(K); f_*([\alpha_1]), f_*([\alpha_2]))$ is order-detected. Here we have used f_* to denote the induced map $\mathcal{S}(T_i) \rightarrow \mathcal{S}(T_{\sigma(i)})$ for $i = 1, 2$.

Theorem 6.2.7. *If $(r_1, r_2) \in [1, \infty) \times [1, \infty)$ is a pair of rational numbers, then (r_1, r_2) -surgery on the mirror image of the Whitehead link produces a manifold whose fundamental group is not left-orderable.*

Proof. Note that $S_{1,*}^3(\text{Wh})$ is the trefoil knot complement, so the conclusion holds true if $r_1 = 1$ or $r_2 = 1$, since Dehn fillings of the trefoil yield non-left-orderable fundamental groups when the filling slope is greater than or equal to one.

Now suppose that $(r_1, r_2) = (p_1/q_1, p_2/q_2) \in (1, \infty) \times (1, \infty)$, where p_i/q_i is written in lowest terms, and that $\pi_1(S_{r_1, r_2}^3(\text{Wh}))$ is left-orderable. Then we apply Remark 4.2.2 and Theorem 4.5.6 to conclude that one of $(\{1\}, \{1\}; [m^{p_1}l^{q_1}], [\mu^{p_2}\lambda^{q_2}])$ or $(\{2\}, \{2\}; [m^{p_1}l^{q_1}], [\mu^{p_2}\lambda^{q_2}])$ is order-detected. However, $(\{2\}, \{2\}; [m^{p_1}l^{q_1}], [\mu^{p_2}\lambda^{q_2}])$ is not order-detected as this would contradict Proposition 6.2.3.

On the other hand, if $(\{1\}, \{1\}; [m^{p_1}l^{q_1}], [\mu^{p_2}\lambda^{q_2}])$ is order-detected then, then an application of Remark 6.2.6 shows that this case also contradicts Proposition 6.2.3. $\square$

Conclusion

The aim of this thesis has been to investigate (non-)left-orderability of the fundamental groups of 3-manifolds via the techniques of order detection. By introducing, refining, and applying the notion of order detection, this article extends existing work on knot exteriors to manifolds with multiple boundary components, providing new examples and applications to the study of Dehn surgeries on cable knots, the Whitehead link, and a class of two-component links. The results in this article also emphasize the versatility of order detection as a tool, capable of accommodating both theoretical generalizations and concrete applications.

Appendix A

Tools for computing JN-realisable tuples

We follow the notation introduced at Sections 4.7 and 5.4 that matches that of [4]. These results are from [4, Appendix], the proofs of which follow from an analysis of [24, 42, 43, 58].

Set $\bar{\tau}_i = \tau_i - \lfloor \tau_i \rfloor \in [0, 1)$ for $i = 1, \dots, r$ and $b = -(\lfloor \tau_1 \rfloor + \dots + \lfloor \tau_r \rfloor)$. Then it is straightforward to see that $(J; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_r)$ is JN-realisable if and only if $(J; b; \gamma_1, \dots, \gamma_n; \bar{\tau}_1, \dots, \bar{\tau}_r)$ is JN-realisable.

Firstly, consider the case where there exists j such that τ_j is an integer, and use J^0 to denote $J \setminus \{j : \tau_j \in \mathbb{Z}\}$. Since JN realizability is invariant under permutation of $(\tau_1, \dots, \tau_r)$, we may assume that the τ_j 's are indexed in such a way that $\tau_1, \dots, \tau_{r_1}$ are not integers, and $\tau_{r_1+1}, \dots, \tau_r$ are integers and $J \cap \{r_1 + 1, \dots, r\} = \{r_2 + 1, \dots, r\}$ for some $r_2 \geq r_1$. Then $(J; b; \gamma_1, \dots, \gamma_n; \bar{\tau}_1, \dots, \bar{\tau}_r)$ is JN-realisable if and only if $(J^0; b; \gamma_1, \dots, \gamma_n; \bar{\tau}_1, \dots, \bar{\tau}_{r_2})$ is JN-realisable since $\tau_j \in \mathbb{Z}$ with $j \in \mathbb{Z}$ forces the function g_j corresponding to $\bar{\tau}_j$ to be the identity. Therefore, in the case where there exists j such that τ_j is an integer, we can reduce to considering the case where $j \in J$ implies $\tau_j \notin \mathbb{Z}$. To handle this case, we have the following theorem.

Theorem A.0.1 ([4, Theorem A.1] or [43, Theorem 1]). *Suppose that if $j \in J$ then $\tau_j \notin \mathbb{Z}$, and let s be the number of τ_j which are integers. If $s > 0$, then $(J; b; \gamma_1, \dots, \gamma_n; \bar{\tau}_1, \dots, \bar{\tau}_r)$ is JN-realisable if and only if $2 - s \leq b \leq n + r - 2$.*

On the other hand, if no τ_j is an integer, then we have:

Theorem A.0.2 ([24, 43, 58] and [4, Theorem A.2]). *Suppose that $n + r \geq 3$, $J \subset \{1, 2, \dots, r\}$, $b \in \mathbb{Z}$ and $0 < \gamma_1, \dots, \gamma_n, \bar{\tau}_1, \dots, \bar{\tau}_r < 1$. Then we have the following.*

- (1) If $(J; b; \gamma_1, \dots, \gamma_n; \overline{\tau}_1, \dots, \overline{\tau}_r)$ is JN-realisable, then $1 \leq b \leq n + r - 1$.
- (2) If $2 \leq b \leq n + r - 2$, then $(J; b; \gamma_1, \dots, \gamma_n; \overline{\tau}_1, \dots, \overline{\tau}_r)$ is JN-realisable in $\widetilde{\text{PSL}}(2, \mathbb{R})$.
- (3) $(J; n + r - 1; \gamma_1, \dots, \gamma_n; \overline{\tau}_1, \dots, \overline{\tau}_r)$ is JN-realisable if and only if $(J; 1; 1 - \gamma_1, \dots, 1 - \gamma_n; 1 - \overline{\tau}_1, \dots, 1 - \overline{\tau}_r)$ is JN-realisable.
- (4) $(J; 1; \gamma_1, \dots, \gamma_n; \overline{\tau}_1, \dots, \overline{\tau}_r)$ is JN-realisable if and only if there are coprime integers $0 < A < N$ and some permutation $(\frac{A_1}{N}, \frac{A_2}{N}, \dots, \frac{A_{n+r}}{N})$ of $(\frac{A}{N}, 1 - \frac{A}{N}, \frac{1}{N}, \dots, \frac{1}{N})$ such that
 - $\gamma_i < \frac{A_i}{N}$ for all $1 \leq i \leq n$;
 - $\overline{\tau}_j < \frac{A_{n+j}}{N}$ for all $j \in J$;
 - $\overline{\tau}_j \leq \frac{A_{n+j}}{N}$ for all $j \in \{1, 2, \dots, r\} \setminus J$.

We can use these theorems to develop a notion of “relative JN-realisability” as follows. Following [4, Appendix], for a fixed tuple $\tau_* = (\tau_1, \dots, \tau_{r-1})$, we set

- $r_1 = |\{j : \tau_j \notin \mathbb{Z}, 1 \leq j \leq r - 1\}|$, the number of non-integral τ_j ;
- $s_0 = |\{j : \tau_j \in \mathbb{Z} \text{ and } j \in \{1, 2, \dots, r - 1\} \setminus J\}|$, the number of integral τ_j whose indices not in J ;
- $b_0 = -(\lfloor \tau_1 \rfloor + \dots + \lfloor \tau_{r-1} \rfloor)$;
- $m_0 = b_0 - (n + r_1 + s_0 - 1)$;
- $m_1 = b_0 + s_0 - 1$.

For a Seifert fibered manifold as in Section [TBA], fix $J \subset \{1, \dots, r - 1\}$ and $\tau_* = (\tau_1, \dots, \tau_{r-1}) \in \mathbb{R}^{r-1}$. Set

$$\mathcal{T}(M; J; \tau_*) = \{\tau' \in \mathbb{R} : (J; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_{r-1}, \tau') \text{ is JN-realisable}\}$$

and

$$\mathcal{T}_{\text{str}}(M; J; \tau_*) = \{\tau' \in \mathbb{R} : (J \cup \{r\}; 0; \gamma_1, \dots, \gamma_n; \tau_1, \dots, \tau_{r-1}, \tau') \text{ is JN-realisable}\}.$$

Theorem A.0.3 ([4, Proposition A.4¹]). *Fix $J \subset \{1, \dots, r - 1\}$ and $\tau_* = (\tau_1, \dots, \tau_{r-1}) \in \mathbb{R}^{r-1}$. Suppose that $n + r_1 + s_0 \geq 2$. Then we have the following.*

- (1) (a) $(m_0, m_1) \subset \mathcal{T}_{\text{str}}(M; J; \tau_*) \subset \mathcal{T}(M; J; \tau_*) \subset (m_0 - 1, m_1 + 1)$.

¹The statements of 2(a)(iii) and 3(a)(iii) are slightly changed from that of [4] Proposition A.4, where there is a small error in the case where $n + r_1 = 2$.

- (b) $[m_0, m_1] \subset \mathcal{T}(M; J; \tau_*)$.
- (c) If $s_0 > 0$, then $m_0 < m_1$ and $(m_0, m_1) = \mathcal{T}_{str}(M; J; \tau_*) \subset \mathcal{T}(M; J; \tau_*) = [m_0, m_1]$.
- (2) (a) If $\mathcal{T}(M; J; \tau_*) \cap (m_0 - 1, m_0) \neq \emptyset$, then
- i. $s_0 = 0$;
 - ii. $1 \geq |\{i : \gamma_i \leq \frac{1}{2}\}| + |\{j \in J : 0 < \overline{\tau}_j \leq \frac{1}{2}\}| + |\{j \notin J : 0 < \overline{\tau}_j \leq \frac{1}{2}\}|$;
 - iii. if $n + r_1 \geq 3$ then there is some $\eta \in (m_0 - 1, m_0) \cap \mathbb{Q}$ such that $\mathcal{T}_{str}(M; J; \tau_*) \cap (m_0 - 1, m_0] = (\eta, m_0]$ and $\mathcal{T}(M; J; \tau_*) \cap (m_0 - 1, m_0] = [\eta, m_0]$, or if $n + r_1 = 2$ then there is some $\eta \in (m_0 - 1, m_0) \cap \mathbb{Q}$ such that $\mathcal{T}_{str}(M; J; \tau_*) = (\eta, m_0)$ and $\mathcal{T}(M; J; \tau_*) = [\eta, m_0]$.
- (b) If $n + r_1 = 2$, then $\mathcal{T}(M; J; \tau_*) \cap (m_0 - 1, m_0) \neq \emptyset$ if and only if
- either $-(\gamma_1 + \dots + \gamma_n + \tau_1 + \dots + \tau_{r-1}) < m_0$,
 - or $n = 0$, $-\sum_{j=1}^r \tau_j = m_0$, $\tau_j \in \mathbb{Q}$ for all $j = 1, \dots, r$ and there is some $j \in J$ with τ_j not integral.
- (3) (a) If $\mathcal{T}(M; J; \tau_*) \cap (m_1, m_1 + 1) \neq \emptyset$, then
- i. $s_0 = 0$;
 - ii. $1 \geq |\{i : \gamma_i \geq \frac{1}{2}\}| + |\{j \in J : 0 < \overline{\tau}_j \geq \frac{1}{2}\}| + |\{j \notin J : 0 < \overline{\tau}_j \geq \frac{1}{2}\}|$;
 - iii. If $n + r_1 \geq 3$ then there is some $\xi \in (m_1, m_1 + 1) \cap \mathbb{Q}$ such that $\mathcal{T}_{str}(M; J; \tau_*) \cap [m_1, m_1 + 1) = [m_1, \xi)$ and $\mathcal{T}(M; J; \tau_*) \cap [m_1, m_1 + 1) = [m_1, \xi]$ or if $n + r_1 = 2$ then there is some $\xi \in (m_1, m_1 + 1) \cap \mathbb{Q}$ such that $\mathcal{T}_{str}(M; J; \tau_*) = (m_1, \xi)$ and $\mathcal{T}(M; J; \tau_*) = [m_1, \xi]$.
- (b) If $n + r_1 = 2$, then $\mathcal{T}(M; J; \tau_*) \cap (m_1, m_1 + 1) \neq \emptyset$ if and only if
- either $-(\gamma_1 + \dots + \gamma_n + \tau_1 + \dots + \tau_{r-1}) > m_0$,
 - or $n = 0$, $-\sum_{j=1}^r \tau_j = m_0$, $\tau_j \in \mathbb{Q}$ for all $j = 1, \dots, r$ and there is some $j \in J$ with τ_j not integral.
- (4) (a) $\mathcal{T}(M; J; \tau_*)$ is a closed subinterval of $(m_0 - 1, m_1 + 1)$ with rational endpoints.
- (b) Either $s_0 = 0$, $n + r_1 = 2$ and $\mathcal{T}_{str}(M; J; \tau_*) = \mathcal{T}(M; J; \tau_*) = \{m_0\}$ or $\mathcal{T}_{str}(M; J; \tau_*)$ is the interior of $\mathcal{T}(M; J; \tau_*)$.
- (c) $\mathcal{T}_{str}(M; J; \tau_*) = \{m_0\}$ if and only if $s_0 = 0$, $n + r_1 = 2$, $m_0 = -(\gamma_1 + \dots + \gamma_n + \tau_1 + \dots + \tau_{r-1})$ and either $n \neq 0$ or $\tau_j \notin \mathbb{Q}$ for some $j \in \{1, \dots, r\}$ or there is some $j \in J$ with τ_j not integral.

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