Topological centers of semigroups and semigroup algebras

by

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Abstract

This thesis investigates the topological centres and Arens regularity of semigroups and their algebras. Following the work of Richard Arens, who introduced two distinct products (the Arens products) on the second dual of Banach algebras, the thesis' main objective is to study the properties of the topological centres of semigroup algebras, particularly focusing on specific types of semigroups such as the bicyclic semigroup and partially bicyclic semigroup.

The thesis begins by introducing concepts such as Arens products, topological centres, and semigroups, including their Stone–Čech compactification. Then the thesis explore the Arens irregularity of the bicyclic semigroup S_1 and the partially bicyclic semigroups S_2 and $S_{1,1}$. We shall prove that these semigroups are strongly Arens irregular, and that their left topological centres are determined by sets consisting of two points. The study also extends to the measure algebras $M(\beta S_1)$, $M(\beta S_2)$, and $M(\beta S_{1,1})$, proving that their left topological centres are similarly determined by sets that consist of two-point.

Moreover, This thesis investigates Arens regularity of inverse semigroup algebras. We shall prove that for an inverse semigroup S if $E(S)$ is uniformly locally finite then $\ell^1(S)$ is Arens regular if and only if the maximal subgroups of S are finite, and each $\mathcal{D}$-class has finitely many idempotents.

Contents

1

Introduction

Richard Arens introduced two distinct multiplications $\Box$ and $\Diamond$ on the second dual of a Banach algebra. The left topological centre $\xi_t^\ell(\mathcal{A}^{**})$ of the second dual $\mathcal{A}^{**}$ of a Banach algebra $\mathcal{A}$ is a Banach algebra consists of elements F in $\mathcal{A}^{**}$ satisfying $F \,\Box\, G = F \diamond G$ for every $G \in \mathcal{A}^{**}$. A Banach algebra is left Arens regular if $\xi_t^\ell(\mathcal{A}^{**}) = \mathcal{A}^{**}$, and is left Arens irregular otherwise. In the extreme case, when $\xi_t^\ell(\mathcal{A}^{**}) = \mathcal{A}$ we say $\mathcal{A}$ is strongly left Arens irregular. Since then, the study of topological centres and Arens regularity has been a focal point in the field of Banach algebras and functional analysis. For instance, Arens regularity of group algebras, C^*-algebras, and Fourier algebras have been investigated by many authors. However, Little is known about topological centres and Arens regularity of semigroup algebras.

Semigroups, which generalize groups by relaxing the requirement for inverses, provide a rich framework for exploring various mathematical structures and their applications. Inspired by Arens products, one may define two distinct product on the Stone–Čech compactification βS of S. Moreover, notions such as topological centres and Arens regularity can be taken to the context of semigroups. The second dual of $\ell^1(S)$ is the Banach algebra $M(\beta S)$ of measures on the Stone–Čech compactification βS of S.

The first author to study Arens regularity of semigroup algebras was Butcher [8]. For a countable, cancellative abelian semigroup S such that $S^* = \beta S \setminus S$ is the union of two disjoint left ideals of βS, he showed that $\ell^1(S)$ is strongly Arens irregular. Butcher's result was extend to all cancellative semigroups by Lau [20].

Dales, Lau, and Strauss [12] showed that, for each weakly cancellative and nearly right cancellative semigroup S, S and $\ell^1(S)$ are strongly Arens irregular. Moreover, they showed that there are sets determining for the left topological center (abbreviated DLTC) of βS and $M(\beta S)$ that consist of two points.

Finally, Dales and Strauss [13] proved that for a totally ordered, abelian semigroup S, both S and $\ell^1(S)$ are strongly Arens irregular. Moreover, they showed that cardinality of a DLTC set can be arbitrary large.

In this thesis, we delve into the Arens irregularity of specific semigroups, called the bicyclic semigroup S_1, and the partially bicyclic semigroups S_2 and $S_{1,1}$. Bicyclic and partially bicyclic semigroups have played very important roles in amenability theory of semigroups and semigroup algebras. Early in 1970's, J. Duncan and I. Namioka showed that S_1 is an amenable semigroup, but the semigroup algebra $\ell^1(S_1)$ is not amenable [15, 16]. This contrasts with the B. E. Johnson's celebrated result that asserts that a group G is amenable if and only if the group algebra $\ell^1(G)$ is amenable [28]. In fact, $\ell(S_1)$ is not even weakly amenable [7, Theorem 2.8], and it is not approximately amenable as well although S_1 is an amenable inverse semigroup [23]. A semigroup S is left reversible if every two right ideals intersect. It has been known due to [27, 19] that S is left amenable $\Rightarrow$ S is left reversible $\Rightarrow$ $\mathrm{WAP}(S)$ has a left invariant mean $\Rightarrow$ $\mathrm{AP}(S)$ has a left invariant mean, where $\mathrm{AP}(S)$ and $\mathrm{WAP}(S)$ are, respectively, the space of almost periodic functions and the space of weakly almost periodic functions on S. The converse of the first implication above is clearly untrue. Whether the converse of any of the other implications is true was a long-standing open question. Examining the partially bicyclic semigroups $S_{1,1}$ and S_2, Lau and Zhang answered the question negatively in [22]. Precisely, they showed that $\mathrm{AP}(S_{1,1})$ has an invariant mean but $\mathrm{WAP}(S_{1,1})$ has no left invariant means; While $\mathrm{WAP}(S_2)$ has a left invariant but S_2 is not left reversible.

Our primary objective is to prove the strong Arens irregularity of the semigroups S_1, S_2, and $S_{1,1}$. Moreover, we explore the Stone–Čech compactification of these semigroups and prove that their left topological centres are determined by sets consisting of precisely two points. These results provides new insights into the compactification structures of these semigroups and their topological centres.

In addition to these findings, we shall study the Banach algebras $\ell^1(S_1)$, $\ell^1(S_2)$, and $\ell^1(S_{1,1})$. We shall show that these algebras are also strongly Arens irregular. Furthermore, we examine the measure algebras $M(\beta S_1)$, $M(\beta S_2)$, and $M(\beta S_{1,1})$, and confirm that their left topological centres are also determined by sets consisting of two points.

In Chapters 2 and 3 we review notions and results about topological centres and Arens regularity. In particular, we focus on results concerning discrete semigroups and their convolution algebras. Chapters 4 and 5 are our original contribution. This thesis contributes to the deeper understanding of the structural and topological properties of bicyclic and partial bicyclic semigroups.

2

Preliminaries

2.1 Arens Products

In the present section, we shall recall some standard properties of Banach algebras and their second duals. In particular, we shall recall the notations and definitions of Arens products, Arens regularity, and the topological centers. For a comprehensive overview of Banach algebra theory, we refer to [11].

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^{**}$ be its second dual. We regard $\mathcal{A}$ as a subalgebra of $\mathcal{A}^{**}$, the second dual of $\mathcal{A}$. There are two products on $\mathcal{A}^{**}$ which extend the product of $\mathcal{A}$; these products are called the first and the second Arens products on $\mathcal{A}^{**}$; the original definitions of the two products were given in [2], [3]. Denote the dual space of $\mathcal{A}$ by $\mathcal{A}^*$. For $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, we write the action of f at a as $\langle a, f\rangle$, i.e. $\langle a, f\rangle = f(a)$. In the thesis, the *first Arens product* $\Box$ and the *second Arens product* $\Diamond$ on $\mathcal{A}^{**}$ are defined in the following way.

Definition 2.1.1. Let $\mathcal{A}$ be a Banach algebra and, $\mathcal{A}^*$ and $\mathcal{A}^{**}$ be its first and second duals respectively. Suppose that $a, b \in \mathcal{A}$, $\lambda \in \mathcal{A}^*$, and $M, N \in \mathcal{A}^{**}$. Then the first Arens product and the second Arens product on $\mathcal{A}^{**}$ are $M \,\Box\, N$ and $M \diamond N$ defined as follows.

First Arens Product $\Box$

1. $\langle b, \lambda \cdot a\rangle = \langle ab, \lambda\rangle$,
2. $\langle a, M \cdot \lambda\rangle = \langle M, \lambda \cdot a\rangle$,
3. $\langle M \,\Box\, N, \lambda\rangle = \langle M, N \cdot \lambda\rangle$.

Second Arens Product $\Diamond$

1. $\langle b, a \cdot \lambda\rangle = \langle ba, \lambda\rangle$,
2. $\langle a, \lambda \cdot M\rangle = \langle M, a \cdot \lambda\rangle$,
3. $\langle M \diamond N, \lambda\rangle = \langle N, \lambda \cdot M\rangle$.

In the process, $\lambda \cdot a$ and $a \cdot \lambda$ denote the natural A-module actions on $\mathcal{A}^*$, $M \cdot \lambda$ and $\lambda \cdot M$ are elements of A^* given by the formula in step 2 of the process.

These products can be implemented through the weak* limits in the following way.

$$M \,\square\, N = w^* \lim_\alpha \lim_\beta x_\alpha y_\beta \quad \textit{and} \quad M \diamond N = w^* \lim_\beta \lim_\alpha x_\alpha y_\beta \quad (M, N \in \mathcal{A}^{**}),$$

where (x_α) and (y_β) are nets in $\mathcal{A}$ such that $w^* \lim_\alpha x_\alpha = M$ and $w^* \lim_\beta y_\beta = N$.

Theorem 2.1.2 ([2]). *Let $\mathcal{A}$ be a Banach algebra. Then both $(\mathcal{A}^{**}, \square)$ and $(\mathcal{A}^{**}, \Diamond)$ are Banach algebras containing $\mathcal{A}$ as a closed subalgebra.*

In general $\square$ and $\Diamond$ are different products on $\mathcal{A}^{**}$. The algebra $\mathcal{A}$ is called *Arens regular* if the Arens products $\square$ and $\Diamond$ coincide on $\mathcal{A}^{**}$.

Definition 2.1.3. Let $\mathcal{A}$ be a Banach algebra. The *left and right topological centres*, $\xi_t^{(\ell)}(\mathcal{A}^{**})$ and $\xi_t^{(r)}(\mathcal{A}^{**})$, of $\mathcal{A}^{**}$ are

$$\xi_t^{(\ell)}(\mathcal{A}^{**}) = \{M \in \mathcal{A}^{**} : M \,\square\, N = M \diamond N \quad (\text{for all } N \in \mathcal{A}^{**})\} \tag{2.1.1}$$

$$\xi_t^{(r)}(\mathcal{A}^{**}) = \{M \in \mathcal{A}^{**} : N \,\square\, M = N \diamond M \quad (\text{for all } N \in \mathcal{A}^{**})\} \tag{2.1.2}$$

respectively. It is readily seen that

$$\mathcal{A} \subset \xi_t^{(\ell)}(\mathcal{A}^{**}) \cap \xi_t^{(r)}(\mathcal{A}^{**}).$$

It is also easy to see that $\mathcal{A}$ is Arens regular if and only if

$$\xi_t^{(\ell)}(\mathcal{A}^{**}) = \xi_t^{(r)}(\mathcal{A}^{**}) = \mathcal{A}^{**}.$$

In the case where $\mathcal{A}$ is a commutative Banach algebra, we have

$$\xi_t^{(\ell)}(\mathcal{A}^{**}) = \xi_t^{(r)}(\mathcal{A}^{**}),$$

and $\mathcal{A}$ is Arens regular if and only if $(\mathcal{A}^{**}, \square)$ is commutative.
We call $\mathcal{A}$ *left strongly Arens irregular* if $\xi_t^{(\ell)}(\mathcal{A}^{**}) = \mathcal{A}$, and *right strongly Arens irregular* if $\xi_t^{(r)}(\mathcal{A}^{**}) = \mathcal{A}$. We call $\mathcal{A}$ *strongly Arens irregular* if it is both left and right strongly Arens irregular.

The proof of the following proposition is elementary.

Proposition 2.1.4 ([12, page 24]). *Let $\mathcal{A}$ be a Banach algebra, and $\Phi \in \mathcal{A}^{**}$. Then the following are equivalent.*

- $\Phi \in \xi_t^{(\ell)}(\mathcal{A}^{**})$;
- $L_\Phi : \Psi \mapsto \Phi \,\square\, \Psi$ *is continuous with respect to the weak* topology of* $\mathcal{A}^{**}$.
- $\Phi \cdot a_\alpha \xrightarrow{wk^*} \Phi \,\square\, \Psi$ *whenever* (a_α) *is a net in* $\mathcal{A}$ *such that* $w^* \lim_\alpha a_\alpha = \Psi$.

Definition 2.1.5. Let $\mathcal{A}$ be a Banach algebra. A linear functional $\lambda \in \mathcal{A}^*$ is (weakly) *almost periodic* if the map

$$a \mapsto a \cdot \lambda, \qquad \mathcal{A} \to \mathcal{A}^*,$$

is (respectively weakly) compact, meaning that it maps every bounded set to a norm (respectively weakly) precompact set. The spaces of almost periodic and weakly almost periodic functionals on $\mathcal{A}$ are denoted by $AP(\mathcal{A})$ and $WAP(\mathcal{A})$, respectively.

Arens regularity of a Banach algebra can be determined (usually) by using the following well-known characterization.

Theorem 2.1.6 ([11, Theorem 2.6.17]). *Let $\mathcal{A}$ be a Banach algebra. Then the following conditions on $\mathcal{A}$ are equivalent.*

(a) $\mathcal{A}$ *is Arens regular;*

(b) for each $\Phi \in \mathcal{A}^{**}$, *the map* $\Psi \mapsto \Psi \diamond \Phi$ *is continuous with respect to the weak* topology of* $\mathcal{A}^{**}$;

(c) for each $\Phi \in \mathcal{A}^{**}$, *the map* $\Psi \mapsto \Phi \,\square\, \Psi$ *is continuous with respect to the weak* topology of* $\mathcal{A}^{**}$;

(d) for each $\lambda \in \mathcal{A}^*$ *and bounded sequences* (a_m) *and* (b_n) *in* $\mathcal{A}$,

$$\lim_m \lim_n \langle a_m b_n, \lambda \rangle = \lim_n \lim_m \langle a_m b_n, \lambda \rangle$$

whenever both the iterated limits exist;

(f) $WAP(\mathcal{A}) = \mathcal{A}^*$.

Consider a family of Banach algebras denoted as $\{A_\lambda : \lambda \in \Lambda\}$. Then the ℓ^1-direct sum denoted as

$$\ell^1\text{-}\bigoplus\{A_\lambda : \lambda \in \Lambda\},$$

is a Banach algebra with respect to component wise operations. In particular, multiplication in the λ^{th} component is given by the multiplication of A_λ.

Theorem 2.1.7 ([4, Theorem 6])**.** *Let $\{A_i : i \in I\}$ be a family of Banach algebras. Then $\ell^1\text{-}\bigoplus\{A_i : i \in I\}$ is Arens regular if and only if each A_i is Arens regular.*

Proof. If $A = \ell^1\text{-}\bigoplus_{i\in I} A_i$, then $A^* = \ell^\infty\text{-}\bigoplus_{i\in I} A_i^*$. Let $B = c_0\text{-}\bigoplus_{i\in I} A_i^*$. Then $A^{**} = B^* \bigoplus_1 B^\perp$, and $B^* = \ell^1\text{-}\bigoplus_{i\in I} A_i^{**}$.

It is easy to see that $A \cdot A^* \subseteq B$ and $A^* \cdot A \subseteq B$.

So $B^\perp \cdot A^* = A^* \cdot B^\perp = 0$. As a consequence,

$$A^{**} \,\square\, B^\perp = B^\perp \diamond A^{**} = 0. \tag{2.1.3}$$

This implies further, using $A^{**} = B^* \bigoplus_1 B^\perp$,

$$B^\perp \,\square\, A^{**} = B^\perp \,\square\, B^* \quad \text{and} \quad A^{**} \diamond B^\perp = B^* \diamond B^\perp. \tag{2.1.4}$$

Note that $B^* \cdot A^* \subseteq B$ and $A^* \cdot B^* \subseteq B$. We then have $B^\perp \,\square\, B^* = 0$ and $B^* \diamond B^\perp = 0$.

Comparing with (2.1.3) and (2.1.4), we have

$$A^{**} \,\square\, A^{**} = B^* \,\square\, B^* \quad \text{and} \quad A^{**} \diamond A^{**} = B^* \diamond B^*.$$

In the sense that, if $U_1, U_2 \in B^*$ and $M_1, M_2 \in B^\perp$, then

$$(U_1 \oplus M_1) \,\square\, (U_2 \oplus M_2) = U_1 \,\square\, U_2 \quad \text{and} \quad (U_1 \oplus M_1) \diamond (U_2 \oplus M_2) = U_1 \diamond U_2.$$

Computation through coordinates ensures $U_1 \,\square\, U_2 = U_1 \diamond U_2$, since all A_i are Arens regular. ☐

In [17], Esslamzadeh introduced the ℓ^1-Munn algebras which is a generalization of Munn-algebras. He studied Arens regularity of ℓ^1-Munn algebras in [18] and applied his results to study various properties of semigroup algebras. Ramsden [31] characterized some homological properties of semigroup algebras. He showed that, when a semigroup is uniformly locally finite, then its algebra is a direct sum of some

ℓ^1-Munn algebras (see Theorem 5.1.1), which is significant for our Chapter 5. We start by stating the definitions of Munn-algebras and ℓ^1-Munn algebras.

Definition 2.1.8. Let $\mathcal{A}$ be a unital algebra, let $m, n \in \mathbb{N}$, and let P be a matrix in $\mathbb{M}_{m,n}(\mathcal{A})$. The *Munn algebra* over $\mathcal{A}$ with sandwich matrix P (denoted by $\mathcal{M}(\mathcal{A}, P, n, m)$) is the matrix algebra $\mathbb{M}_{n,m}(\mathcal{A})$ with the product defined by

$$a \circ b = aPb \qquad (a, b \in \mathbb{M}_{n,m}(\mathcal{A})),$$

where the product on the right side is the normal matrix multiplication. The Munn algebra is also a Banach algebra for the norm

$$\|(a_{ij})\| = \sum\{\|a_{ij}\| : i \in \mathbb{N}_n, j \in \mathbb{N}_m\} \qquad ((a_{ij}) \in \mathbb{M}_{n,m}(\mathcal{A})),$$

where $\mathbb{N}_n = \{1, 2, 3, \cdots, n\}$.

Definition 2.1.9. Consider a Banach algebra $\mathcal{A}$ with identity element, and let I and J be arbitrary index sets. Let $P = (P_{ij})$ be a non-zero $J \times I$ matrix over $\mathcal{A}$, such that

$$\|P\|_\infty = \sup\{\|P_{ji}\| : j \in J, i \in I\} \leq 1.$$

Define $\mathscr{LM}(\mathcal{A}, P)$ as the vector space of all $I \times J$ matrices $M = (M_{ij})$ over $\mathcal{A}$ such that

$$\|M\|_1 = \sum_{i \in I, j \in J} \|M_{ij}\| < \infty.$$

Then, $\mathscr{LM}(\mathcal{A}, P)$, equipped with the product $A \cdot B = APB$ for $A, B \in \mathscr{LM}(\mathcal{A}, P)$ and the above ℓ^1-norm, forms a Banach algebra. We refer to this algebra as the ℓ^1-Munn $I \times J$ matrix algebra over $\mathcal{A}$ with the sandwich matrix P, or simply the ℓ^1-Munn algebra. When $I = J$ and P is the identity matrix of size $J \times J$ over $\mathcal{A}$, we denote $\mathscr{LM}(\mathcal{A}, P)$ by $\mathbb{M}_J(\mathcal{A})$. In particular, when $|J| = m < \infty$, $\mathbb{M}_J(\mathcal{A})$ represents the algebra of $m \times m$ complex matrices over $\mathcal{A}$.

For Arens regularity of ℓ^1-Munn algebras we have the following theorem. We state it without proof since the proof needs to much preparation.

Theorem 2.1.10 ([18, Theorem 4.2])**.** *Suppose $\mathcal{A}$ admits a nonzero character. Then $\mathscr{LM}(\mathcal{A}, P)$ is Arens regular if and only if one of the following conditions holds:*

(i) $\mathcal{A}$ is Arens regular and both of the index sets are finite.

(ii) $\mathcal{A}$ is finite dimensional and one of the index sets is finite.

Lemma 2.1.11 ([17, Lemma 3.3])**.** *Let $\mathcal{A}$ be a Banach algebra and I be an arbitrary index set. Then $\mathbb{M}_I(\mathcal{A})$ is Banach algebra isometrically isomorphic to $\mathbb{M}_I(\mathbb{C})\hat{\otimes}\mathcal{A}$, where $\hat{\otimes}$ denotes the projective tensor product.*

We recall that a Banach algebra $\mathcal{A}$ is *weakly sequentially complete* if every Cauchy sequence in $\mathcal{A}$ is convergent in the weak topology. Ülger [35] studied Arens regularity of weakly sequentially complete Banach algebras. The following is one of his results.

Theorem 2.1.12 ([35, Theorem 3.4])**.** *Let $\mathcal{A}$ be a weakly sequentially complete commutative Banach algebra with a bounded approximate identity. Then $\mathcal{A}$ is Arens regular if and only if it is semisimple and finite-dimensional.*

Ülger in [34] investigated Arens regularity of tensor product of two Banach algebras. The following is his main result.

Theorem 2.1.13 ([34, Corollary 3.5])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. If $\mathcal{A}\hat{\otimes}\mathcal{B}$ is Arens regular then so are $\mathcal{A}$ and $\mathcal{B}$.*

Civin and Yood [9] investigate Arens regularity of group algebras. They showed that an abelian group G, Arens regularity of $L^1(G)$ is equal to finiteness of G. Their result were generalized by Young [36] to all locally compact groups.

Theorem 2.1.14 ([36])**.** *For a locally compact group G, $L^1(G)$ is Arens regular if and only if G is finite.*

Proof. When G is finite, the group algebra $L^1(G)$ is a finite-dimensional Banach algebra, so $L^1(G)$ is Arens regular.

Now we consider the case where G is an infinite locally compact group. To show that $L^1(G)$ is Arens irregular, we will construct two sequences of compact sets C_n and D_m in G, with positive Haar measure, which are pairwise disjoint.

For the non-compact case, we proceed inductively to construct the sets.

- Let C_1 and D_1 be any arbitrary compact sets of positive Haar measure such that $C_1 \cap D_1 = \emptyset$.
- Inductively, assume we have constructed pairwise disjoint compact sets $C_1, \ldots, C_n$ and $D_1, \ldots, D_n$ such that their products C_rD_s for $1 \leq r, s \leq n$ are also pairwise disjoint.

Now, define the following union:

$$H = C_1 \cup \cdots \cup C_n \cup \bigcup_{r,s,t \leq n} C_r D_s D_t^{-1}.$$

Then H is compact. There exists an element $x \in G$ that is not in H, because G is not compact.

Since the sets D_r are pairwise disjoint,

$$G \setminus \bigcup_{r \neq s \leq n} D_r D_s^{-1}$$

is a neighborhood of the identity element e in G, and hence

$$\{(y, z) : y^{-1}z \notin D_r D_s^{-1}; 1 \leq r \neq s \leq n\},$$

defines a neighborhood around (x, x) in $G \times G$. Therefore, this neighborhood includes a set of the form $U \times U$, where U is an open neighborhood of x. The set $U \setminus H$ is also an open neighborhood of x and has positive measure. Since open sets are regular with respect to the Haar measure, we can find a compact subset $C_{n+1} \subseteq U \setminus H$ with positive measure. The set C_{n+1} is disjoint from $C_1, \ldots, C_n$, and from the construction, it is clear that $C_{n+1} D_t$ and $C_r D_s$ (for $s, t \leq n$ and $r \leq n + 1$) are disjoint unless $r = n + 1$ and $s = t$. The set D_{n+1} can be constructed similarly, ensuring that the sets $\{C_n D_m : n, m \geq 1\}$ are pairwise disjoint.

Suppose now that G is compact. Since G is infinite, it is not discrete. We will construct compact sets C_n and D_m, each having positive measure, such that for all positive integer indices:

1. $e \notin \bigcup_{r \geq 1} C_r \cup D_r$,
2. The families $\{C_r : r \geq 1\}$ and $\{D_r : r \geq 1\}$ are pairwise disjoint, and $C_r \cap (D_s \cup D_s^{-1}) = \emptyset$,
3. $C_r D_s \cap C_p D_q = \emptyset$ whenever $r < s$ and $p > q$,
4. $e \notin C_r D_s D_t^{-1}$ when $r < s$, and $e \notin C_r^{-1} D_s D_t$ when $s > t$.

Let C_1 be an arbitrary compact set of positive measure that does not contain the identity element e, and let D_1 be a compact set of positive measure that is disjoint from $\{e\} \cup C_1 \cup C_1^{-1}$. The sets C_1 and D_1 satisfy the conditions 1 through 4.

Next, assume that $C_1, \ldots, C_n$ and $D_1, \ldots, D_n$ already selected in a way satisfy the required hypotheses. Let H be defined as the following union

$$H = \bigcup_{r \le n} \left(C_r \cup D_r \cup C_r^{-1} \cup D_r^{-1} \right) \bigcup \bigcup_{r,s \le n} \left(C_r D_s \cup C_r D_s^{-1} \right) \bigcup \bigcup_{t \le n, r < s \le n} \left(C_r D_s D_t^{-1} \right)$$

Then H is a compact set, and $e \notin H$. By 1, $G \setminus \bigcup_{r \le n}(D_r)$ is a neighbourhood of e, so that, arguing as in the non-compact case, one can find a neighbourhood U of e disjoint from H and such that $U^{-1}U \cap D_r = \emptyset$, $r \le n$. Since G is not discrete, there exists a compact set C_{n+1} of positive measure, contained in U and not containing e such that:

$$C_{n+1} \cap H = \emptyset$$

$$C_{n+1}^{-1} C_{n+1} \cap \left(\bigcup_{r \le n} D_r \right) = \emptyset$$

Now, it is easy to verify that C_{n+1}, satisfies conditions 1 to 4.

We now construct D_{n+1}. Let X be the union of the sets:

$$X = \bigcup_{r \le n} (C_r \cup D_r \cup C_r^{-1} \cup D_r^{-1}) \bigcup \bigcup_{r,s \le n} (C_r D_s \cup C_r^{-1} D_s) \bigcup \bigcup_{r,s,t \le n} (C_r^{-1} C_s D_t)$$

X is compact and does not contain e. As before, we can select a compact set D_{n+1} of positive measure, not containing e, and such that:

$$D_{n+1} \cap X = \emptyset, \quad D_{n+1} D_{n+1}^{-1} \cap \bigcup_{r \le n} C_r = \emptyset.$$

One can verify, conditions 1-4 are preserved.

Now, we use the sets C_n and D_n to proof the claim of the theorem. Define characteristic functions $f_n = \frac{1}{\mu(C_n)} \chi_{C_n}$ and $g_m = \frac{1}{\mu(D_m)} \chi_{D_m}$, where μ is the Haar measure on G and χ_A is the characteristic function of the set A. These functions f_n and g_m belong to $L^1(G)$ and have unit norm.

Next, define a function $h \in L^\infty(G)$ such that

$$h(x) = \begin{cases} 1, & \text{if } x \in \cup_{m > n \ge 1} C_n D_m \\ 0, & \text{otherwise.} \end{cases}$$

Now we have

$$\langle f_n * g_m, h\rangle = \begin{cases} 1, & \text{if } m > n, \\ 0, & \text{if } m < n. \end{cases}$$

The two repeated limits $\lim_{n\to\infty}\lim_{m\to\infty}\langle f_n * g_m, h\rangle$ and $\lim_{m\to\infty}\lim_{n\to\infty}\langle f_n * g_m, h\rangle$ both exist but are unequal. Therefore, the two Arens products on $(L^1(G))^{**}$ do not coincide by Theorem 2.1.6(d). $\square$

Example 2.1.15. (i) Let $\mathcal{A} = c_0$, the Banach algebra of complex sequences vanishing at infinity with coordinate wise operations. Then as Banach spaces $\mathcal{A}^* = \ell^1$, the Banach space of all absolutely summable complex sequences. Moreover, $\mathcal{A}^{**} = \ell^\infty$, the Banach algebra of all bounded complex sequences with coordinate wise operations. For $\Phi = (\phi_n) \in \ell^\infty$, and $\Psi = (\psi_n) \in \ell^\infty$ we define $\Phi \cdot \Psi = (\phi_n \psi_n)$. Let $P_n(\Phi) = (\phi_1, \phi_2, \cdots, \phi_n, 0, \cdots)$, we have $P_n(\Phi) \in c_0$ and $P_n(\Phi) \longrightarrow \Phi$ in $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$, the weak* topology on $\mathcal{A}^{**}$. Now, suppose that $\Phi, \Psi \in \ell^\infty$. We have $\Phi \cdot \Psi = \lim\limits_m \lim\limits_n P_m(\Phi) P_n(\Psi)$ in $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$. Therefore, both $\Phi \,\square\, \Psi$ and $\Phi \diamond \Psi$ coincide with the coordinate product $\cdot$ of ℓ^∞. In particular, c_0 is Arens regular.

(ii) Set $\mathcal{A} = (\ell^1, *)$, where $*$ denotes convolution. Let δ_x be the mass point measure at x. Consider the bounded sequences (a_n) and (b_n) in $\mathcal{A}$ by setting $a_n = \delta_{2^{2n}}$ and $b_n = \delta_{2^{2n+1}}$ for $n \in \mathbb{N}$. Set $S = \{2^{2m} + 2^{2n+1} : m < n\}$. Then each element of S is a number with a unique representation. Now, for $\lambda = \chi_S \in \mathcal{A}^*$ we have

$$\lim_m \lim_n \langle a_m * b_n, \lambda\rangle = 1 \qquad \lim_n \lim_m \langle a_m * b_n, \lambda\rangle = 0.$$

Therefore, $\mathcal{A}$ is not Arens regular.

For C^*-algebras we have the following theorem, originally proved by Civin and Yood [9]. We state the theorem without proof as it requires a lot of preparation.

Theorem 2.1.16 ([11, Corollary 3.2.37]). *Every C^*-algebra is Arens regular.*

2.2 Semigroups and their algebras

In this section, we will review definitions and fundamental characteristics of semigroups and semigroup algebras. Our standard reference for the algebraic theory of semigroups is [26].

A *semigroup* S is a non-empty set with an associative product, usually denoted by juxtaposition

$$(s,t) \mapsto st, \quad S \times S \to S.$$

The semigroup S is *abelian* (or commutative) if $st = ts$ $(s,t \in S)$. A non-empty subset T of S is a *subsemigroup* if T is a semigroup with the product of S. Let S and T be two semigroups. A *morphism* from S to T is a mapping $\theta : S \to T$ such that $\theta(s_1s_2) = \theta(s_1)\theta(s_2)$ for $s_1, s_2 \in S$. An *epimorphism* is a surjective morphism, and an *isomorphism* is a bijective morphism. The semigroups S and T are isomorphic, denoted $S \cong T$, if there exists an isomorphism from S onto T. Let S be a semigroup, and let $0 \in S$ be such that $s0 = 0s = 0$ for all $s \in S$. Then 0 is a zero for the semigroup S. If S does not have a zero, we may associate a zero to it: Let $T = S \cup \{0\}$, and define $s0 = 0s = 0$ for all $s \in S$ and $0^2 = 0$. Then T is a semigroup containing S as a subsemigroup; we say that T is formed by adjoining a zero to S, and write $T = S^0$. Suppose that S has an identity e_S. Then e_S is also the identity of T. An element $p \in S$ is *idempotent* if $p^2 = p$. The set of idempotents is denoted by $E(S)$. A semigroup S is a *semilattice* if S is commutative and $E(S) = S$. The *canonical partial order* on $E(S)$ is given by

$$p \leq q \iff p = pq = qp \quad \text{for } p, q \in E(S).$$

An idempotent p is *maximal* if $p = q$ whenever $p \leq q$. Let S be a semigroup with a zero 0. Then an idempotent p is *primitive* if $p \neq 0$ and $q = p$ or $q = 0$ whenever $q \in E(S)$ with $q \leq p$. Let $s \in S$, an element $s^* \in S$ is an *inverse* of s if $ss^*s = s$ and $s^*ss^* = s^*$. In general, the inverse may not be unique. An element $s \in S$ is regular if there exists $t \in S$ with $sts = s$. Clearly, if s has an inverse, then s is regular; less obviously, the converse is also true. Indeed, suppose that there is $t \in S$ with $sts = s$. Set $s^* = tst$. Then we have

$$ss^*s = (sts)ts = sts = s \quad \text{and} \quad s^*ss^* = t(sts)tst = t(sts)t = s^*.$$

The set of regular elements of S is denoted by $R(S)$. The semigroup S is *regular* if $S = R(S)$. An element that is not regular is called singular, and the set of singular elements is denoted by $N(S)$.

Definition 2.2.1. Let S be a semigroup. Then S is an *inverse semigroup* if S is regular and every element has a unique inverse.

We have the following characterization theorem.

Proposition 2.2.2 ([26, Proposition 5.1.1]). *A semigroup S is an inverse semigroup if and only if S is regular and the idempotents commute.*

A *left (respectively, right) ideal* in a semigroup S is a non-empty subset T of S such that $ST \subset T$ (respectively, $TS \subset T$); a subset which is both a left and right ideal is an ideal. An ideal I in S is *prime ideal* if $I \neq S$ and $S \setminus I$ is a subsemigroup of S. Let T be a non-empty subsemigroup of S such that $s, t \in T$ whenever $s, t \in S$ and $st \in T$. Then $S \setminus T$ is a prime ideal in S. A *minimal left ideal* in S is a left ideal which is minimal in the family of left ideals of S when this family is ordered by inclusion. Similarly, we define *minimum left ideals*, etc. Let I be an ideal in a semigroup S, and let L be a minimum left ideal in S. Then $L \subset I$. The minimum ideal of S (if it exists) is denoted by $K(S)$.

Definition 2.2.3. Let S be a semigroup. A *principal sequence of ideals* for S is a chain

$$S = I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_m = K(S)$$

where $I_1, I_2, \cdots, I_m$ are ideals in S and there is no ideal of S strictly between I_j and I_{j+1} for each $j \in \mathbb{N}_{m-1}$.

If a partially ordered set is not finite, we may consider local finiteness of it defined as follows.

Definition 2.2.4. Let P be a partially ordered set. For $p \in P$, we define $(p] = \{x : x \leq p\}$ and $[p) = \{x : p \leq x\}$. Then P is called *locally finite* if $|(p]|$ is finite for every $p \in P$, and P is locally C-finite for some constant $C > 1$ if $|(p]| < C$ for each $p \in P$. A partially ordered set that is locally C-finite for some C is *uniformly locally finite*.

Suppose that S is an inverse semigroup. There is a natural way to extend the partial order on $E(S)$ to S. Let $s, t \in S$ we define the partial order on S by

$$s \leq t \Leftrightarrow s = ss^*t.$$

In fact, there are various equivalent definitions of the partial order on S. For instance,

$$s \leq t \Leftrightarrow s = ts^*s \Leftrightarrow s = tp \text{ for some } p \in E(S).$$

Definition 2.2.5. Let S be an inverse semigroup. Then S is said to be [locally finite/C-locally finite/uniformly locally finite] respectively if the partially ordered set $(E(S), \leq)$ has the same property.

There are various significant equivalence relations on a semigroup, commonly referred to as Green's equivalences. The specific equivalence relation of interest to us is as follows.

Definition 2.2.6. Suppose S is a semigroup, and let s and t belong to S. We establish a relation $\mathcal{D}$ on S by defining $s\mathcal{D}t$ if and only if there exists an element x in S such that $Ss \cup \{s\} = Sx \cup \{x\}$ and $xS \cup \{x\} = tS \cup \{t\}$.

In an inverse semigroup the $\mathcal{D}$-classes can be characterized as follows.

Proposition 2.2.7 ([26, Proposition 5.1.2(4)])**.** *Let S be an inverse semigroup, and let $s, t \in S$. Then $s\mathcal{D}t$ if and only if there exists $x \in S$ with $s^*s = xx^*$ and $t^*t = x^*x$.*

Let S denote an inverse semigroup, and let $p \in E(S)$. We define

$$G_p = \{s \in S : ss^* = s^*s = p\}.$$

Consequently, G_p forms a group with identity p, and it contains all subgroups of S with identity p. Hence, G_p is called the *maximal subgroup* of S at p. Suppose that idempotents p and q lie in the same $\mathcal{D}$-class. Consider $x \in S$ such that $p = xx^*$ and $q = x^*x$. Then the map

$$s \mapsto x^*sx, \qquad G_p \to G_q,$$

is an isomorphism.

To advance further, we need to introduce some notations. Let S be a semigroup. For any $s \in S$, we define

$$L_s(t) = st \quad \text{and} \quad R_s(t) = ts \quad \text{for } t \in S.$$

Consider a non-empty subset U of S. Then, we express

$$s^{-1}U = L_s^{-1}(U) = \{t \in S : st \in U\},$$

and

$$Us^{-1} = R_s^{-1}(U) = \{t \in S : ts \in U\}.$$

Additionally, we denote $s^{-1}x$ for the set $s^{-1}\{x\}$, and so forth. Let V be another non-empty subset. Then we denote

$$V^{-1}U = \left\{s^{-1}U : s \in V\right\}$$
$$UV^{-1} = \left\{Us^{-1} : s \in V\right\}.$$

In the context of a semigroup S, an element $s \in S$ is called *left (or right) cancellable* if L_s (or R_s) is injective on S. If s is both left and right cancellable, it is called *cancellable.* The semigroup S is referred to as *left (or right) cancellative* if each of its elements is left (or right) cancellable, and simply *cancellative* if every element is cancellable. Furthermore, S is *weakly left (or right) cancellative* if $s^{-1}F$ (or Fs^{-1}) is finite for every $s \in S$ and each finite subset F of S, and *weakly cancellative* if it is both weakly left and weakly right cancellative.

Moreover, semigroup S is called *nearly right cancellative* if there exists a subset X of S such that it has the same cardinality as S and for every pair of distinct elements s and t in S, the set $\{x \in X \mid sx = tx\}$ is finite. This subset X is termed a "witness" in S. Similarly, a semigroup S is *nearly left cancellative* if its opposite semigroup, denoted as S^{op}, is nearly right cancellative. Moreover, a semigroup is called *nearly cancellative* if it is both nearly right and nearly left cancellative.

Additionally, semigroups can have a non-discrete topology. A semigroup S which is a topological space is called a *left (respectively right) topological semigroup* if the left (respectively right) translation mapping L_t (respectively R_t), is continuous for every element t in S. S is called a *semi-topological semigroup* if the product map $m_S : (s,t) \mapsto st$ is separately continuous. Finally, S is called a *topological semigroup* if the product map m_S is jointly continuous.

Definition 2.2.8. Suppose that the unit circle and open unit disc in the complex plane $\mathbb{C}$ are denoted by $\mathbb{T}$ and $\mathbb{D}$. Let S be a semigroup. A *semi-character* (respectively, *character*) on S is a mapping $\varphi : S \to \overline{\mathbb{D}}$ (respectively, $\varphi : S \to \mathbb{T}$) such that $\varphi(st) = \varphi(s)\varphi(t)$ for all $s, t \in S$ and $\varphi \neq 0$. The sets of semi-characters and characters on S are denoted by Φ_S and Ψ_S, respectively. There is always one character on S, given by

$$1 : s \mapsto 1, \qquad S \to \mathbb{T},$$

which is referred to as the *augmentation character*. The augmentation character may be the only semi-character on S.

Example 2.2.9. Consider an infinite set S with a product operation given by $st = t$, where $s, t \in S$. With this operation, S become a semigroup known as the right zero semigroup. S is a left cancellative, but not weakly right cancellative. Similarly, a left zero semigroup is a semigroup with the product $st = s$ for $s, t \in S$, resulting in right cancellative semigroup.

Definition 2.2.10. Let S be a semigroup. Throughout, δ_s denotes the point mass function at $s \in S$. The *convolution* product $*$ is uniquely defined by requiring that $\delta_s * \delta_t = \delta_{st}$ $(s, t \in S)$. Let $f = \sum_{r\in S} \alpha_r \delta_r$ and $g = \sum_{s\in S} \beta_s \delta_s$, where $\alpha_r, \beta_s \in \mathbb{C}$. Then convolution product of f and g is

$$f * g = (\sum_{r\in S} \alpha_r \delta_r) * (\sum_{s\in S} \beta_s \delta_s) = \sum\{(\sum_{rs=t} \alpha_r \beta_s)\delta_t : t \in S\},$$

where $\sum_{rs=t} \alpha_r \beta_s = 0$ when there are no elements r and s in S with $rs = t$.

Now, we are ready to define semigroup algebra.

Definition 2.2.11. Let S be a semigroup. The Banach algebra $\ell^1(S)$, called the *semigroup algebra* of S, is consisting of all linear combinations $a = \sum_{s\in S} c_s \delta_s$ $(c_s \in \mathbb{C})$ such that $\sum_{s\in S} |c_s| < \infty$, with the convolution product and the ℓ^1-norm $\|a\|_1 = \sum_{s\in S} |c_s|$.

A character on an algebra $\mathcal{A}$ is a non-zero homomorphism from $\mathcal{A}$ onto $\mathbb{C}$. For every $\varphi \in \Phi_S$, the mapping

$$\sum \alpha_s \delta_s \mapsto \sum \alpha_s \varphi(s)$$

is a character on $\ell^1(S)$. In fact, every character on $\ell^1(S)$ arises in this manner. Thus, we establish an identification between the character space of the Banach algebra $\ell^1(S)$ and the semi-character space Φ_S. The character on $\ell^1(S)$ corresponding to the augmentation character on S is denoted as φ_S, and defined by

$$\varphi_S : \sum \alpha_s \delta_s \mapsto \sum \alpha_s.$$

Semigroup algebras can posses an identity. The following discuss the conditions needed for a semigroup algebra to admit an identity.

Proposition 2.2.12 ([1, Proposition 2.1])**.** *Let S be an inverse semigroup and let M denote the set of maximal idempotents of S. Then $\ell^1(S)$ has an identity e if and only if M is finite and $E(S) = \bigcup_{p\in\mathsf{M}}(p]$. Furthermore, $\|e\|_1 = 2^{|\mathsf{M}|} - 1$.*

Proof. If $|\mathsf{M}| = 1$, the result is evident. Now assume $|\mathsf{M}| = 2$, and let p' and p'' be the maximal elements of $E(S)$. Define $e = \delta_{p'} + \delta_{p''} - \delta_{p'p''}$. Based on the assumption, for any $x \in S$, we have $x^*x \leq p'$ or $x^*x \leq p''$, or possibly both. Let's assume $x^*x \leq p'$; then, we get:

$$\delta_x e = \delta_{xx^*x}\delta_{p'} + \delta_{xx^*x}\delta_{p''} - \delta_{xx^*x}\delta_{p'p''} = \delta_x(\delta_{x^*x}\delta_{p'} + \delta_{x^*x}\delta_{p''} - \delta_{x^*x}\delta_{p'p''}).$$

Simplifying this expression:

$$= \delta_{xx^*x} + \delta_{xx^*x}\delta_{p''} - \delta_{xx^*x}\delta_{p''} = \delta_{xx^*x} = \delta_x.$$

The same reasoning applies for the other cases. Now, let $a = \sum_{x \in S} \lambda_x \delta_x \in \ell^1(S)$, then by the earlier calculation:

$$ae = \sum_{x \in S} \lambda_x(\delta_x e) = \sum_{x \in S} \lambda_x \delta_x = a.$$

For the case where $|\mathsf{M}| = k$, let $p_1, p_2, \ldots, p_k$ be the maximal elements of $E(S)$. Using induction, we can show that:

$$e = \sum_{i=1}^{k} \delta_{p_i} - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \delta_{p_i p_j} + \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-1} \sum_{l=j+1}^{k} \delta_{p_i p_j p_l} - \cdots + (-1)^{k+1} \prod_{i=1}^{k} \delta_{p_i}$$

is an identity for $\ell^1(S)$. Furthermore, the norm $\|e\|_1$ is given by:

$$\|e\|_1 = \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k} = 2^k - 1.$$

Conversely, suppose that $\ell^1(S)$ has an identity element $e = \sum_{s \in S} \lambda_s \delta_s$. For each $t \in S$, we have:

$$\delta_t = \delta_t e = \delta_t \sum_{s \in S} \lambda_s \delta_s = \sum_{u \in S} \left(\sum_{ts=u} \lambda_s \right) \delta_u.$$

It follows that:

$$\sum_{ts=u} \lambda_s = \begin{cases} 1, & \text{if } u = t \\ 0, & \text{if } u \neq t \end{cases}.$$

Choose a finite subset $F \subset S$ such that $\sum_{s \notin F} |\lambda_s| < \frac{1}{2}$. We will show that $E(S) \subseteq \langle F \rangle$, where $\langle F \rangle = \bigcup_{x \in F} (x]$. Assume, for contradiction, that $E(S) \nsubseteq \langle F \rangle$. Let $p \in E(S) \backslash \langle F \rangle$, then $\{s \in S : ps = p\} \cap \langle F \rangle = \emptyset$, leading to the contradiction:

$$1 = \sum_{ps=p} \lambda_s \leq \sum_{ps=p} |\lambda_s| \leq \sum_{s \notin \langle F \rangle} |\lambda_s| \leq \sum_{s \notin F} |\lambda_s| < \frac{1}{2}.$$

Thus, for every $q \in E(S)$, there exists $x \in F$ such that $q \leq x$, implying $q \leq p = xx^*$, which shows that $E(S) = \bigcup_{p \in \mathsf{M}} (p]$. Additionally, for each $p \in \mathsf{M}$, we have $\{s \in S : ps = p\} = \{p\}$, so:

$$|\mathsf{M}| = \sum_{p \in \mathsf{M}} \sum_{ps=p} \lambda_s = \sum_{p \in \mathsf{M}} \lambda_p \leq \sum_{s \in S} |\lambda_s| = \|e\|_1.$$

Thus, $|\mathsf{M}| = k$ for some positive integer k. Similar to the "if" part, we obtain an identity for $\ell^1(S)$ of the form:

$$f = \sum_{i=1}^{k} \delta_{p_i} - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \delta_{p_i p_j} + \sum_{i=1}^{k-2} \sum_{j=i+1}^{k-1} \sum_{l=j+1}^{k} \delta_{p_i p_j p_l} - \cdots + (-1)^{k+1} \prod_{i=1}^{k} \delta_{p_i}.$$

Uniqueness of the identity implies that $e = f$ and $\|e\|_1 = 2^k - 1$. □

We end this chapter by recalling that semigroup algebras are weakly sequentially complete.

Theorem 2.2.13 ([11, Theorem A.4.4]). *Let S be a semigroup. Then $\ell^1(S)$ is weakly sequentially complete.*

3

Topological centres for semigroups and semigroup algebras

In this chapter, we review the conventional theory of Stone–Čech compactifications for locally compact spaces, focusing particularly on discrete spaces. We shall show that there are two products $\Box$ and $\Diamond$ on the Stone-Čech compactification βS of a semigroup S such that $(\beta S, \Box)$ and $(\beta S, \Diamond)$ are semigroups. Finally, we shall consider the second duals of semigroup algebras and some of their basic properties.

3.1 The Stone–Čech compactification of a semigroup

The *Stone–Čech compactification* βS of a locally compact Hausdorff space S is a compact space containing S as a dense subset such that every continuous map from S into a compact space K has a continuous extension from βS into K. A full account on Stone–Čech compactification is [24]. Another characterization of βS is that it is the character space (or maximal ideal space) of the commutative C^*-algebra $C_b(S)$ of all bounded continuous functions on S with the pointwise product. This correspondence is discussed in [11, §4.2]. The closure in βS of a subset U of S is denoted by $\overline{U}$, and we set

$$U^* = \overline{U} \setminus S.$$

It is called the growth of U, As a subspace of βS, U^* is compact.

Let X be a non-empty, locally compact Hausdorff space. The space of all complex-valued, regular Borel measures on X is denoted by $M(X)$.

Let S be a non-empty set with discrete topology. The space $\ell^\infty(S)$ of all bounded functions on S is a Banach space with respect to the uniform norm $|f|_S = \sup\{|f(s)| : s \in S\}$ $(f \in \ell^\infty(S))$. It is a commutative C^*-algebra with the pointwise product; $c_0(S)$ is a C^*-subalgebra of $\ell^\infty(S)$ consisting of all functions on S vanishing at infinity. The space βS is homeomorphic to the character space of $(\ell^\infty(S), |\cdot|_S)$, and $(\ell^\infty(S), |\cdot|_S)$ is isometrically isomorphic to $(C(\beta S), |\cdot|_{\beta S})$; we shall identify these two Banach algebras. The space βS is extremely disconnected (i.e. the closure of every open subset of βS is open), and $\overline{F \cap G} = \overline{F} \cap \overline{G}$ whenever F and G are subsets of S [11, Proposition 4.2.8]. Let T be a subset of S. Then we identify βT with $\overline{T}$, the closure of T in βS. By Riesz–Markov theorem, the dual Banach space of $\ell^\infty(S) = C(\beta S)$ is $M(\beta S)$, the Banach space of complex regular Borel measures on βS. We have

$$\langle \mu, \chi_F \rangle = \mu(\overline{F}) = \int_{\overline{F}} d\mu \qquad (\mu \in M(\beta S))$$

for each subset F of S.

Regarding βS and S^*, we note the following result.

Proposition 3.1.1 ([25, Theorem 3.36]). *Let S be an infinite set. Then each non-empty G_δ-set in S^* has a non-empty interior in S^*.*

Proposition 3.1.2 ([12, Proposition 5.2]). *Let S be a non-empty set, let T be a subset of S, and let $f : \beta S \longrightarrow \beta S$ be a continuous function with $f(S) \subset S$. Then $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$.*

When S is a discrete semigroup with multiplication $*$, we can define two products $\Box$ and $\Diamond$ on βS such that $(\beta S, \Box)$ and $(\beta S, \Diamond)$ are semigroups. For each $s \in S$, there is a continuous extension of the map

$$L_s : t \mapsto s * t, \qquad S \to S \subset \beta S,$$

to a map $L_s : \beta S \to \beta S$. Now, for each $u \in \beta S$ we define $s \,\Box\, u = L_s(u)$. Also, the map

$$R_u : s \mapsto s \,\Box\, u, \qquad S \to \beta S,$$

has a continuous extension to a map $R_u : \beta S \to \beta S$. Finally, for $u, v \in \beta S$, we set

$$u \,\Box\, v = R_v(u).$$

Moreover, since for each $v \in \beta S$, the map R_v is continuous, and, for each $s \in S$, the map L_s is continuous, this product can also be implemented in the following way

$$u \,\square\, v = \lim_{\alpha} \lim_{\beta} s_\alpha t_\beta, \quad (u, v \in \beta S)$$

where s_α and t_β are nets in S such that $\lim_{\alpha} s_\alpha = u$ and $\lim_{\beta} t_\beta = v$. Similarly, one can define another product $\Diamond$ on βS such that

$$u \diamond v = \lim_{\beta} \lim_{\alpha} s_\alpha t_\beta, \quad (u, v \in \beta S)$$

where s_α and t_β are nets in S such that $\lim_{\alpha} s_\alpha = u$ and $\lim_{\beta} t_\beta = v$.

The following is elementary and is easy to verify.

Theorem 3.1.3 ([25, Theorem 4.4], [12, Theorem 6.1])**.** *$(\beta S, \Box)$ and $(\beta S, \Diamond)$ are semigroups containing S as a subsemigroup. Further, the semigroup $(\beta S, \Box)$ is a compact, right topological semigroup.*

Example 3.1.4. Let S be an infinite set with the product given by $st = t \quad (s, t \in S)$. In this case, $u \,\square\, v = v \quad (u, v \in \beta S)$, and so $(\beta S, \Box)$ is also a right zero semigroup. Furthermore, L_u is continuous for each $u \in \beta S$.

Example 3.1.5. Set $S = \mathbb{N}_\vee$, the semigroup of natural numbers with product $n \vee m = \max\{n, m\}$. Then $u \,\square\, v = v$ and $u \diamond v = u$ for $u, v \in S^*$. Thus $(S^*, \Box)$ is a right zero semigroup, and $(S^*, \Diamond)$ is a left zero semigroup. We note, in this case, $u \,\square\, s = u$ if $u \in S^*$ and $s \in S$.

Remark 3.1.6. Let S and T be semigroups, and let $\varphi : S \longrightarrow T$ be a homomorphism. Then φ extends to a continuous map $\varphi : \beta S \longrightarrow \beta T$ such that $\varphi : (\beta S, \Box) \longrightarrow (\beta T, \Box)$ and $\varphi : (\beta S, \Diamond) \longrightarrow (\beta T, \Diamond)$ are homomorphisms. In fact $(\beta S, \Box)$ is the largest right topological semigroup compactification of S, in the sense that any other such compactification is a continuous homomorphic image of $(\beta S, \Box)$. Since $V = (\beta S, \Box)$ is a compact, right topological semigroup, we have definitions of $L_v \mu$ and $R_v \mu$ for each $v \in V$ and $\mu \in M(V)$. It is evident that $L_v \mu = \delta_v \,\square\, \mu$ and $R_v \mu = \mu \,\square\, \delta_v$.

The following definition is analogues of the topological centres for second duals of Banach algebras.

Definition 3.1.7. Let S be a semigroup. The left and right topological centres, $\xi_t^{(\ell)}(\beta S)$ and $\xi_t^{(r)}(\beta S)$, of βS are

$$\xi_t^{(\ell)}(\beta S) = \{u \in \beta S : u \,\square\, v = u \diamond v\} \qquad (v \in \beta S) \tag{3.1.1}$$

$$\xi_t^{(r)}(\beta S) = \{u \in \beta S : v \,\Box\, u = v \diamond u\} \qquad (v \in \beta S) \tag{3.1.2}$$

respectively. Clearly, $S \subseteq \xi_t^{(\ell)}(\beta S), \xi_t^{(r)}(\beta S) \subseteq \beta S$. The semigroup S is called Arens regular if

$$\xi_t^{(\ell)}(\beta S) = \xi_t^{(r)}(\beta S) = \beta S;$$

It is called left strongly Arens irregular if $\xi_t^{(\ell)}(\beta S) = S$, right strongly Arens irregular if $\xi_t^{(r)}(\beta S) = S$, and strongly Arens irregular if it is both left and right strongly Arens irregular.

In many sources, the left topological center of βS is simply called the topological center of βS and is denoted by $\Lambda(\beta S)$ or $\xi(\beta S)$.

In general, right and left Arens irregularity are different. A simple example to verify this is as follows.

Example 3.1.8 ([12, Example 6.12]). Let $S = G \times G$, where G is infinite group. Then S is a semigroup with the product specified by

$$(a, x) \cdot (b, y) = (ab, ay) \qquad (a, b, x, y \in G).$$

S is a semigroup which is right, but not left, strongly Arens irregular.

Theorem 3.1.9 ([12, Theorem 6.16])**.** *Let S be an infinite semigroup. Then:*

- *S^* is a left ideal in $(\beta S, \Box)$ if and only if S is weakly left cancellative;*
- *S^* is an ideal in $(\beta S, \Box)$ if and only if S is weakly cancellative.*

Let S be a semigroup and $\ell^1(S)$ be the semigroup algebra with convolution. We saw earlier that the dual of $\ell^\infty(S) = \ell^1(S)^*$ is $M(\beta S)$, and therefore, we may identify $M(\beta S)$ with $\ell^1(S)^{**}$. This identification gives us two product $\Box$ and $\Diamond$ on $M(\beta S)$. The Banach algebras $(M(\beta S), \Box)$ and $(M(\beta S), \Diamond)$ are discussed in details in [14]. In fact, Parsons [29] proved the following.

Theorem 3.1.10. *Let S be an abelian discrete semigroup. Then $(\ell^1(S)^{**}, \Box)$ and $(M(\beta S), *)$ are isomorphic, where $*$ denotes the convolotion product on $M(\beta S)$.*

If $\mathcal{A}$ is a Banach algebra, then its dual space $\mathcal{A}^*$ is a Banach $\mathcal{A}$-bimodule with the module actions defined by $\langle b, a \cdot f\rangle = \langle ba, f\rangle$ and $\langle b, f \cdot a\rangle = \langle ab, f\rangle$ for $a, b \in \mathcal{A}$ and $f \in \mathcal{A}^*$.

Definition 3.1.11. Let $\mathcal{A}$ be a Banach algebra. We say that $\mathcal{A}$ is a dual Banach algebra if there is a closed submodule E of $\mathcal{A}^*$ such that $E^* = \mathcal{A}$, i.e. E is a predual of $\mathcal{A}$.

In addition to Theorem 3.1.9, we have the following equivalent result. Here we omit the proof to avoid a long preparation.

Theorem 3.1.12 ([12, Theorem 7.11])**.** *The following conditions are equivalent, when S is an infinite semigroup:*

(i) $\ell^1(S)$ *is a dual Banach algebra with the predual* $c_0(S)$*;*

(ii) S *is weakly cancellative;*

(iii) S^* *is an ideal in* $(\beta S, \Box)$*;*

(iv) $\ell^1(S^*)$ *is an ideal in* $(\ell^1(\beta(S)), \Box)$*;*

(v) $M(S^*)$ *is an ideal in* $(M(\beta S), \Box)$*.*

We close this section by stating the following theorem.

Theorem 3.1.13 ([12, Proposition 4.4])**.** *Suppose V is a compact, right topological semigroup. Let $\mu \in M(V)$ and $v \in V$ be right cancellable in V. Then* $\|R_v(\mu)\| = \|\mu\|$.

3.2 Rees matrix semigroups

Definition 3.2.1. Let $(S, \cdot)$ be a semigroup and X be a set. An action of the semigroup S on the set X is a function $* : S \times X \to X$ that satisfies the following conditions,

1. **Identity Preservation**: For all $x \in X$, $e * x = x$, if S has identity e.

2. **Compatibility with the Semigroup Operation**: For all $s, t \in S$ and $x \in X$, $(s \cdot t) * x = s \cdot (t * x)$.

Moreover, we say that S act bijectively on X if $* : x \mapsto s * x$ is a bijection.

Example 3.2.2. In this example we shall introduce a famous semigroup S called Rees matrix semigroup. We will describe $\ell^1(S)$, βS, and $M(\beta S)$. Later we will discus Arens regularity of S and $\ell^1(S)$. Let H be a semigroup, and let G be a group that acts bijectively on H such that e_G is the identity on H. Suppose that $m, n \in \mathbb{N}$, and

$P_G = (a_{ij}) \in \mathbb{M}_{m,n}(G)$, where $\mathbb{M}_{m,n}(G)$ is the set of all m,n-matrices with entries in G. For $x \in H$, $i \in \mathbb{N}_m$, and $j \in \mathbb{N}_n$, let $(x)_{ij}$ be the element of $\mathbb{M}_{n,m}(H^0)$ with x at the $(i,j)^{\text{th.}}$ place and 0 elsewhere. Let T be the set that consists of the collection of all these matrices $(x)_{ij}$. We define a multiplication for T by the formula

$$(x)_{ij}(y)_{kl} = (xa_{jk}y)_{il} \qquad (x, y \in G, i, k \in \mathbb{N}_n, j, l \in \mathbb{N}_m);$$

with this multiplication T becomes a semigroup which we denote by $\mathcal{M}(H, P_G, m, n)$. Similarly, we have the semigroup $\mathcal{M}^0(H, P_G, m, n)$ where the elements of this semigroup are those of $\mathcal{M}(H, P_G, m, n)$, together with the element 0 that is identified with the matrix with 0 in each place (so that 0 is the zero of $\mathcal{M}^0(H, P_G, m, n)$). The entries of P_G are now allowed to belong to G^0, the semigroup formed by adjoining a zero to G. The matrix P_G is called the sandwich matrix in each case. The semigroup $\mathcal{M}^0(H, P_G, m, n)$ is called Rees matrix semigroup with a zero over H. In the particular case that $H = G$ is a group, we obtain a Rees matrix semigroup with a zero over G, which we denote by $S = \mathcal{M}^0(G, P_G, m, n)$

If every row and column of the sandwich matrix P_G contains at least one (non-zero) entry in G, then the sandwich matrix is called regular. It is well-known that the semigroup $\mathcal{M}^0(G, P_G, m, n)$ is a regular semigroup if and only if P_G is regular [10, Lemma 3.1].

We now describe the semigroup algebra $\ell^1(S)$. For $x \in G$, $(x)_{ij}$ stands for the element of $\mathbb{M}_{n,m}(\ell^1(G))$ with δ_x at the $(i,j)^{\text{th.}}$. Then one can identify $\ell^1(S)$ with $\mathbb{M}_{n,m}(\ell^1(G)) \bigcup \mathbb{C}\delta_0$. The formula for the multiplication of f and g in $\ell^1(S)$, where $f = (f_{ij})$ and $g = (g_{ij})$ belong to $\mathbb{M}_{n,m}(\ell^1(G))$ is given by

$$(f *_P g)_{il} = \sum_{(j,k)\in N(P)} f_{ij} * \delta_{a_{jk}} * g_{kl} + \sum_{(j,k)\in Z(P)} \varphi_G(f_{ij})\varphi_G(g_{kl})\delta_0 \tag{3.2.1}$$

in which $N(P) = \{(j,k) \in \mathbb{N}_n \times \mathbb{N}_m : a_{jk} \neq 0\}$, $Z(P) = \{(j,k) \in \mathbb{N}_n \times \mathbb{N}_m : a_{jk} = 0\}$, and φ_G is the augmentation character on G. In particular, we have

$$f *_P \delta_0 = \delta_0 *_P f = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \varphi_G(f_{ij}) \right) \delta_0 = \varphi_S(f)\delta_0 \tag{3.2.2}$$

Therefore, the semigroup algebra $\ell^1(S)$ corresponding to the Rees semigroup $S = \mathcal{M}^0(G, P, m, n)$ is the Munn algebra $\mathcal{M}^0(\ell^1(G), P, m, n)$, for more details see [12, page 50].

Now, we can describe βS and S^*. From [12, Proposition 6.5(ii)] we know that G

acts bijectively on βG and G^*, and therefore, it makes sense to consider $\mathcal{M}^0(\beta G, P_G, m, n)$ and $\mathcal{M}^0(G^*, P_G, m, n)$. Now, the definition of $\Box$ implies that

$$(\beta S, \Box) = \mathcal{M}^0((\beta G, \Box), P_G, m, n)$$

and

$$(S^*, \Box) = \mathcal{M}^0((G^*, \Box), P_G, m, n).$$

Now, we can identify $M(\beta S)$ with $\mathbb{M}_{n,m}(M(\beta G)) \bigcup \mathbb{C}\delta_0$. To describe the product in $(M(\beta S), \Box)$, we use the equations (3.2.1) and (3.2.2) and take iterated limit. Let $\mu = (\mu)_{ij}$ and $\nu = (\nu)_{ij}$ belong to $\mathbb{M}_{n,m}(M(\beta G))$. Then

$$(\mu \,\Box\, \nu)_{il} = \sum_{(j,k)\in N(P)} \mu_{ij} \,\Box\, \delta_{a_{jk}} \,\Box\, \nu_{kl} + \sum_{(j,k)\in Z(P)} \varphi_G''(\mu_{ij})\varphi_G''(\nu_{kl})\delta_0 \tag{3.2.3}$$

in which $N(P) = \{(j,k) \in \mathbb{N}_n \times \mathbb{N}_m : a_{jk} \neq 0\}$, $Z(P) = \{(j,k) \in \mathbb{N}_n \times \mathbb{N}_m : a_{jk} = 0\}$, and φ_G'' is the second adjoint of the augmentation character on G. Moreover, we have

$$\mu \,\Box\, \delta_0 = \delta_0 \,\Box\, \mu = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} \varphi_G''(\mu_{ij})\right)\delta_0 \tag{3.2.4}$$

Therefore, $M(\beta S)/\mathbb{C}\delta_0 = \mathcal{M}(M(\beta G), P_G, m, n)$, and $(M(\beta S), \Diamond)$ can be described with a similar formula. Besides, we see that

$$M(S^*) = \mathcal{M}(M(G^*), P_G, m, n),$$

so $M(S^* \cup \{0\})$ is a closed, complemented ideal in $(M(\beta S), \Box)$.

Let us consider a special case. Let G be an infinite group, and

$$P = \begin{pmatrix} e_G & e_G \\ e_G & e_G \end{pmatrix}.$$

Take $S = \mathcal{M}^0(G, P, 2, 2)$. Then P is a regular sandwich matrix, and therefore, S is a regular semigroup. Take $\mu \in M(G^*)$ with $\mu \neq 0$, and consider the element

$$\Phi = \begin{pmatrix} \mu & -\mu \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(M(G)).$$

Then the equation (3.2.3) implies that $\Phi \,\square\, \nu = 0$ for every $\nu \in \mathbb{M}_2(M(G))$, and the equation (3.2.4) implies that $\Phi \,\square\, \delta_0 = 0$. Hence, $\Phi \,\square\, \Psi = 0$ for every $\Psi \in \ell^1(S)^{**} = M(\beta S)$. Similarly, $\Phi \diamond \Psi = 0$ for every $\Psi \in \ell^1(S)^{**} = M(\beta S)$. Therefore, $\Phi \in \xi_t^{(\ell)}(M(\beta S))$, whereas $\Phi \notin \ell^1(S)$. It follows that $\ell^1(S)$ is not strongly Arens irregular. We shall see in Corollary 3.4.12 that if $S = \mathcal{M}^0(G, P, n, n)$, and P is an invertible sandwich matrix in $\mathbb{M}_n(M(G))$, then $\ell^1(S)$ is strongly Arens irregular.

3.3 Topological centres for semigroups

We start with discussing the Arens regularity of some special semigroups and their algebras. These examples illustrate the diverse possibilities for Arens regularity in semigroups and semigroup algebras.

Example 3.3.1 ([12, Example 7.30]). Let S be a right zero semigroup with $|S| \geq 2$. We have already seen that $u \,\square\, v = u \diamond v = v$ for every $u, v \in \beta S$. Thus S is Arens regular. Moreover, we have

$$\mu \,\square\, \nu = \mu \diamond \nu = \langle \mu, 1 \rangle \nu \qquad (\mu, \nu \in M(\beta S))\,,$$

and therefore, $\ell^1(S)$ is Arens regular.

Example 3.3.2 ([12, Example 7.31]). Let $S = \mathbb{Z}^2$, and define

$$(m_1, n_1) \cdot (m_2, n_2) = (m_1 + m_2, n_2) \qquad (m_1, n_1, m_2, n_2 \in \mathbb{Z}).$$

Then S is a non-abelian semigroup which is left cancellative, but not weakly right cancellative. Take $U = 0 \times \mathbb{Z}$, so that U is a subsemigroup of S , and take $u \in U^*$. Then one can see that that $u \in \xi_t^{(\ell)}(\beta S)$ and $u \in \xi_t^{(r)}(\beta S)$, and so S is neither left nor right strongly Arens irregular. Since $\mathbb{Z} \times \{0\}$ is a subgroup of S and $\ell^1(\mathbb{Z})$ is not Arens regular, $\ell^1(S)$ is not Arens regular.

Example 3.3.3 ([12, Example 7.32]). Let $S = \mathbb{N}_\vee$. We have already seen that $u \,\square\, v = v$ and $u \diamond v = u$ for $u, v \in S^*$, which means $(S^*, \square)$ is a right zero semigroup, and $(S^*, \diamondsuit)$ is a left zero semigroup. It is clear that $\xi_t^{(\ell)}(\beta S) = \xi_t^{(r)}(\beta S) = S$, and so S is strongly Arens irregular. Moreover, we have

$$\mu \,\square\, \nu = \langle \mu, 1 \rangle \nu \qquad (\mu \in M(\beta S), \nu \in M(S^*))\,,$$

$$\mu \diamond \nu = \langle \nu, 1 \rangle \mu \qquad (\mu \in M(S^*), \nu \in M(\beta S))\,.$$

Therefore, if $\mu \in M(S^*)$ satisfies $\mu \,\square\, \delta_a = \mu \diamond \delta_a$ and $\mu \,\square\, \delta_b = \mu \diamond \delta_b$ for two distinct points $a, b \in S^*$, then $\mu = 0$. It follows that $\ell^1(S)$ is strongly Arens irregular.

Example 3.3.4 ([12, Example 7.33]). Let $S = \mathbb{N}_\wedge$, the semigroup of natural numbers with the product $m \wedge m = \min\{n, m\}$. Then similar to $\mathbb{N}_\vee$, $u \,\square\, v = u$ and $u \diamond v = v$ for $u, v \in S^*$, which means $(S^*, \Box)$ is a left zero semigroup. It is clear that $\xi_t^{(\ell)}(\beta S) = \xi_t^{(r)}(\beta S) = S$, and so S is strongly Arens irregular. Moreover, we have

$$\mu \,\square\, \nu = \langle \nu, 1\rangle \mu \qquad (\mu \in M(\beta S), \nu \in M(S^*))\,,$$

$$\mu \diamond \nu = \langle \mu, 1\rangle \nu \qquad (\mu \in M(S^*), \nu \in M(\beta S))\,.$$

Therefore, if $\mu \in M(S^*)$ satisfies $\mu \,\square\, \delta_a = \mu \diamond \delta_a$ and $\mu \,\square\, \delta_b = \mu \diamond \delta_b$ for two distinct points $a, b \in S^*$, then $\mu = 0$. It follows that $\ell^1(S)$ is strongly Arens irregular.

Example 3.3.5 ([12, Example 7.34]). Let $S = \mathbb{N} \times \{0, 1\}$, with the operation

$$(m, i) \cdot (n, j) = (m + n, 0) \qquad (m, n \in \mathbb{N}, i, j \in \{0, 1\}).$$

Then S is an abelian, countable, weakly cancellative semigroup. However S is not nearly cancellative because we have $(1, 0) \neq (1, 1)$, but $(1, 0) \cdot x = (1, 1) \cdot x$ for each $x \in S$. βS can be identified with $\beta\mathbb{N} \times \{0, 1\}$, and we have

$$(u, i) \,\square\, (v, j) = (u \,\square\, v, 0), \qquad (u, i) \diamond (v, j) = (u \diamond v, 0), \qquad (u, v \in \beta\mathbb{N}, i, j \in \{0, 1\}).$$

Also, $M(\beta S)$ can be identified with $M(\beta\mathbb{N}) \times \{0, 1\}$, and we have

$$(\mu, i) \,\square\, (\nu, j) = (\mu \,\square\, \nu, 0), \quad (\mu, i)\diamond(\nu, j) = (\mu\diamond\nu, 0), \qquad (\mu, \nu \in M(\beta\mathbb{N}), i, j \in \{0, 1\}).$$

Let $u \in \mathbb{N}^*$. Then for every $(v, j) \in \beta S$, we have

$$((u, 0) - (u, 1)) \,\square\, (v, j) = (u \,\square\, v, 0) - (u \,\square\, v, 0) = (0, 0),$$

and

$$((u, 0) - (u, 1)) \diamond (v, j) = (u \diamond v, 0) - (u \diamond v, 0) = (0, 0).$$

Thus, $(u, 0) - (u, 1) \in \xi_t^{(\ell)}(M(\beta S))$ while $(u, 0) - (u, 1) \notin \ell^1(S)$, so $\ell^1(S)$ is not strongly Arens irregular. However, we shall see later that S itself is strongly Arens irregular.

Definition 3.3.6. Let S be a semigroup, and let $\lambda \in \ell^\infty(S)$. Set $LO(\lambda) = \{\lambda \cdot t :$

$t \in S\}$ where $\lambda \cdot t \in \ell^\infty(S)$ is defined by $\lambda \cdot t(s) = \lambda(ts)$. The function λ is weakly almost periodic (respectively, almost periodic) if the set $LO(\lambda)$ is relatively compact in the weak (respectively, $\|\cdot\|$) topology on $\ell^\infty(S)$. The spaces of these functions are denoted by $WAP(S)$ and $AP(S)$, respectively.

The spaces $WAP(S)$ and $AP(S)$ are easily identified with $WAP(\ell^1(S))$ and $AP(\ell^1(S))$, respectively.

Theorem 3.3.7 ([5, Theorem 12.1.]). *The following are equivalent for a semigroup S;*

(a) The convolution algebra $\ell^1(S)$ is Arens regular;

(b) $WAP(S) = \ell^\infty(S)$;

(c) there do not exist sequences (s_m) and (t_n) in S such that the two sets $\{s_m t_n : m < n\}$ and $\{s_m t_n : m > n\}$ are disjoint.

Esslamzadeh [18] has investigated the Arens regularity of semigroup algebras. He showed that under some conditions, arens regularity of $\ell^1(S)$ implies finiteness of S.

Theorem 3.3.8 ([18, Theorem 5.1]). *Let S be a semigroup such that $\ell^1(S)$ is Arens regular. If either of the following conditions holds, then S is finite.*

(i) S is a regular semigroup with a finite number of idempotents.

(ii) S is an inverse semigroup which admits a principal series.

Remark 3.3.9. The conditions of the previous theorem cannot be eliminated. For instance, let G be a finite group and let $n \in \mathbb{N}$. Consider the Rees semigroup $S = \mathcal{M}^0(G, P, n, n)$, where $P \in \mathbb{M}_n(G^0)$ is a regular sandwich matrix. Then S and $E(S)$ are infinite, and since P is regular, S is regular as well. However, Esslamzadeh showed that $\ell^1(S)$ is Arens regular (see [18, Example 5.3]).

Butcher [8] investigates Arens regularity of semigroup algebras for countable, cancellative abelian semigroup. He proved the following.

Theorem 3.3.10 ([8, Theorem 4.4.2]). *$\ell^1(S)$ is strongly Arens irregular whenever S is a countable, cancellative abelian semigroup such that S^* is the union of two disjoint left ideals of βS.*

In [20], Lau investigate the topological center of βS, with different terminology he showed the following theorem.

Theorem 3.3.11 ([20, Theorem 1]). *Let S be a cancellative semigroup, and suppose that $t \in \beta S$ is such that $t \,\square\, \mu = t \diamond \mu$ for each $\mu \in M(\beta S)$. Then $t \in S$.*

Remark 3.3.12. In [20], the result is claimed to hold only under the condition that S is weakly cancellative. However, Bami in [6] pointed out that the proof did not work under this level of generality. Later, Lau, Dales, and Strauss in [12] recovered a stronger form of this theorem for weakly cancellative semigroups. We will state this theorem later.

3.4 Sets determining for topological centres

In [12] Dales, Lau, and Strauss studied the topological center of βS and $M(\beta S)$ for various semigoups, which generalize most of the aforementioned results on semigroups and groups. They introduced the notion of sets determining for the topological centres.

Definition 3.4.1. Let S be an infinite semigroup. A subset V of S^* is determining for the left topological centre (abbreviated as DLTC) of $M(\beta S)$ if the only element μ of $M(S^*)$ such that $\mu \,\square\, \nu = \mu \diamond \nu$ for each $\nu \in V$ is $\mu = 0$. A subset V of S^* is determining for the left topological centre of βS if there are no elements u of S^* such that $u \,\square\, v = u \diamond v$ for each $v \in V$.
A DRTC set is defined similarly. In the case where the semigroup S is abelian, we use the phrase determining for the topological centre (DTC).

The research on DLTC and DRTC sets focuses usually on whether there exist such sets which are finite. In particular when we can have a two-point set that is DLTC.

Lemma 3.4.2 ([12, Theorem 12.5]). *Let S be an infinite semigroup, and let K be a non-empty, closed subset of S^*. Suppose there exist subsets U and V of S and right cancellable elements $a \in U^*$ and $b \in V^*$ such that the following conditions hold:*

(i) $U \cap V = \emptyset$;

(ii) $S \,\square\, a \subseteq \overline{U}$ *and* $S \,\square\, b \subseteq \overline{V}$;

(iii) *For each* $x \in K$, *either* $(x \diamond S) \cap \overline{U} = \emptyset$ *or* $(x \diamond S) \cap \overline{V} = \emptyset$.

Then, given $\mu \in M(\beta S)$, $\nu \in M(K)$ and $\varepsilon > 0$,there exist two functions $\lambda_a, \lambda_b \in C(\beta S)_1$, the unit ball of $C(\beta S)$, with

$$\langle \mu \,\square\, \delta_a, \lambda_a \rangle > \|\mu\| - \varepsilon \quad \text{and} \quad \langle \mu \,\square\, \delta_b, \lambda_b \rangle > \|\mu\| - \varepsilon \tag{3.4.1}$$

such that either

$$\langle \nu \diamond \delta_x, \lambda_a \rangle \leq \frac{1}{2}\|\nu\| \quad (x \in \beta S) \tag{3.4.2}$$

or

$$\langle \nu \diamond \delta_x, \lambda_b \rangle \leq \frac{1}{2}\|\nu\| \quad (x \in \beta S). \tag{3.4.3}$$

Moreover, if $\mu \in M(K)$ satisfies $\mu \,\square\, \delta_a = \mu \diamond \delta_a$ and $\mu \,\square\, \delta_b = \mu \diamond \delta_b$, then $\mu = 0$.

Proof. Since a and b are right cancellable in βS, it follows from Theorem 3.1.13 that $\|\mu \,\square\, \delta_a\| = \|\mu \,\square\, \delta_b\| = \|\mu\|$. Therefore, there exist $\lambda_a, \lambda_b \in C(\beta S)_1$ such that equation (3.4.1) holds.

Using condition (ii), we have $\beta S \,\square\, a \subseteq \overline{U}$ and $\beta S \,\square\, b \subseteq \overline{V}$, so:

$$\operatorname{supp}(\mu \,\square\, \delta_a) \subseteq \overline{U} \quad \text{and} \quad \operatorname{supp}(\mu \,\square\, \delta_b) \subseteq \overline{V}.$$

By condition (i) and the fact that βS is extremely disconnected, $\overline{U} \cap \overline{V} = \emptyset$. Hence, by replacing λ_a with $\lambda_a \cdot \chi_U$ and λ_b with $\lambda_b \cdot \chi_V$, we can assume λ_a and λ_b vanish outside of $\overline{U}$ and $\overline{V}$, respectively.

Let $f = \sum_{i=1}^m \alpha_i x_i$, where $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ such that $\sum_{i=1}^m \alpha_i = 1$, and $x_1, \ldots, x_m \in K$. For some $i \in \mathbb{N}_m$, by condition (iii),

$$\text{either} \quad \langle x_i \diamond s, \lambda_a \rangle = 0 \quad (s \in S) \quad \text{or} \quad \langle x_i \diamond s, \lambda_b \rangle = 0 \quad (s \in S).$$

Thus,

$$\text{either} \quad |\langle f \diamond s, \lambda_a \rangle| \leq \frac{1}{2} \quad (s \in S) \quad \text{or} \quad |\langle f \diamond s, \lambda_b \rangle| \leq \frac{1}{2} \quad (s \in S),$$

which implies that:

$$|\langle \nu \diamond \delta_s, \lambda_a \rangle| \leq \frac{1}{2}\|\nu\| \quad (s \in S) \quad \text{or} \quad |\langle \nu \diamond \delta_s, \lambda_b \rangle| \leq \frac{1}{2}\|\nu\| \quad (s \in S).$$

Since the map $x \mapsto \nu \diamond \delta_x$, from βS to $M(S^*)$, is continuous, either equation (3.4.2) or (3.4.3) must hold.

Finally, consider $\mu \in M(K)$ such that $\mu \,\square\, \delta_a = \mu \diamond \delta_a$ and $\mu \,\square\, \delta_b = \mu \diamond \delta_b$. Assuming

$\mu \neq 0$ for contradiction, we may set $\|\mu\| = 1$. Applying the earlier argument with $\nu = \mu$ and $\varepsilon = 1/2$, we are led to a contradiction, and hence, $\mu = 0$. $\square$

With the above lemma, one may obtain the following result that enhance some early result in [8].

Corollary 3.4.3 ([12, Corollary 12.6]). *Suppose that S is an infinite semigroup, and assume that U and V are disjoint, infinite subsemigroups of S such that U^* and V^* are left ideals in S^*. Then, for each pair $\{a, b\}$ of right cancellable elements with $a \in U^*$ and $b \in V^*$, the set $\{\delta_a, \delta_b\}$ is determining for the left topological centre of $M(\beta S)$.*

Moreover, if S is cancellative, then such a pair $\{a, b\}$ of right cancellable elements exists, and $\ell^1(S)$ is left strongly Arens irregular.

Note that Example 3.3.5 provides an infinite, abelian, weakly cancellative semigroup S such that $\ell^1(S)$ is not strongly Arens irregular. Thus the term 'cancellative' cannot be replace by 'weakly cancellative' in the previous theorem.

Dales, Lau, and Strauss proved the following for a countable weakly cancellative semigroup S. The proof is complicated and we will present the ideal of the proof later in our results in section .

Theorem 3.4.4 ([12, Theorem 12.7]). *Let S be an infinite, countable semigroup such that S is weakly cancellative and nearly right cancellative. Then $\ell^1(S)$ is strongly left Arens irregular, and there exist a and b in S^* that are right cancellable in $(\beta S, \Box)$ such that the two-element set $\{\delta_a, \delta_b\}$ is determining for the left topological centre of $M(\beta S)$.*

Furthermore, if S is also nearly left cancellative, then $\ell^1(S)$ is strongly Arens irregular.

In the previous theorem, if 'weakly cancellative' is replaced by 'cancellative', then for most pairs $\{a, b\}$ in S^* the set $\{\delta_a, \delta_b\}$ is determining for the left topological centre of $M(\beta S)$.

Theorem 3.4.5 ([12, Theorem 12.8]). *Let S be an infinite, countable, cancellative semigroup. Let P and Q be infinite subsets of S. Then there exist elements $a \in P^*$ and $b \in Q^*$ that are right cancellable in $(\beta S, \Box)$, and such that the set $\{\delta_a, \delta_b\}$ is determining for the left topological centre of $M(\beta S)$.*

For the topological center of βS, we have the following.

Theorem 3.4.6 ([12, Theorem 12.9])**.** *Let S be an infinite, countable, weakly cancellative semigroup. Then there exist elements a and b in S^* such that the set $\{a, b\}$ is determining for the left topological centre of βS. Moreover, S is strongly Arens irregular.*

While the previous theorems ensure the existence of a two-point set that is determining for the topological centre of βS and $M(\beta S)$, finding these points is not simple. In the case where the semigroup is abelian the following theorem might be helpful. Still, we state the theorem without a proof.

Theorem 3.4.7 ([12, Theorem 12.10])**.** *Let S be an infinite, weakly cancellative, abelian semigroup, and let Ω be a non-empty, compact space. Let $\varphi : \beta S \longrightarrow \Omega$ be a continuous function such that*

$$\varphi(u \,\Box\, v) = \varphi(v) \qquad (u \in \beta S, v \in S^*).$$

Take $a, b \in S^*$ with $\varphi(a) \neq \varphi(b)$. Then $\{a, b\}$ is determining for the topological centre of βS. Suppose, further, that a and b are right cancellable. Then $\{\delta_a, \delta_b\}$ is determining for the topological centre of $M(\beta S)$.

Theorem 3.4.8 ([12, Theorem 12.20])**.** *Let S be an infinite, weakly cancellative semigroup. Then there is subset V of S^* of cardinality 2 such that V is determining for the left topological centre of βS , S is strongly Arens irregular, and the map*

$$L_v : u \mapsto v \,\Box\, u, \qquad \beta S \longrightarrow \beta S,$$

is continuous if and only if $v \in S$.

Example 3.4.9 ([12, Example 12.21])*.* Let $S = \mathbb{N} \times \{0, 1\}$, with the operation

$$(m, i) \cdot (n, j) = (m + n, 0) \qquad (m, n \in \mathbb{N}, i, j \in \{0, 1\}).$$

In Example 3.3.5 we examined that S is an abelian, countable, weakly cancellative semigroup for which $\ell^1(S)$ is not strongly Arens irregular. However, by the previous theorem, S is strongly Arens irregular.

For non-countable weakly cancellative semigroups, we collect the following result without a proof.

Theorem 3.4.10 ([12, Theorem 12.15]). *Let S be an infinite semigroup such that S is weakly cancellative and nearly right cancellative. Then there exist a and b in S^* that are right cancellable in $(\beta S, \Box)$ and such that the two-element set $\{\delta_a, \delta_b\}$ is determining for the left topological centre of $M(\beta S)$. Suppose that the semigroup S is cancellative and that P and Q are subsets of S such that $|P| = |Q| = |S|$. Then we can choose $a \in P^*$ and $b \in Q^*$.*

As a consequence we have the following.

Corollary 3.4.11 ([12, Corollary 12.16]). *Let S be a weakly cancellative and nearly cancellative semigroup. Then $\ell^1(S)$ is strongly Arens irregular.*

In particular, from Theorem 3.4.10, for a Rees matrix semigroups we have the following.

Corollary 3.4.12 ([12, Corollary 12.17]). *Let $S = \mathcal{M}_0(G, P, n, n)$, where G is a group and P is an invertible sandwich matrix in $\mathbb{M}_n(\ell^1(G))$. Then there is a subset V of S^* of cardinality $2n$ such that V is determining for the left topological centre of $M(\beta S)$; furthermore, in the case where G is abelian, $2n$ is the minimum cardinality of such a set V.*

A Banach algebra $\mathcal{A}$ is said to be amenable if it has a virtual diagonal, that is, there exists an element $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $a \cdot M = M \cdot a$ and $\pi_{\mathcal{A}}^{**}(M)a = a$ for every $a \in \mathcal{A}$, where $\pi_{\mathcal{A}} : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ is the multiplication map specified by $\pi_{\mathcal{A}}(a \otimes b) = ab$ $(a, b \in \mathcal{A})$ and $\pi_{\mathcal{A}}^{**} : (\mathcal{A}\hat{\otimes}\mathcal{A})^{**} \to \mathcal{A}^{**}$ is the bidual operator of $\pi_{\mathcal{A}}$. The relation between amenability and Arens regularity is not studied in general, however, Dales, Lau, and Strauss proved the following result for semigroup algebras.

Theorem 3.4.13 ([12, Corollary 12.18]). *Let S be a semigroup such that the Banach algebra $\ell^1(S)$ is amenable. Then there is a finite subset of S^* which is determining for the left topological centre of $M(\beta S)$. Furthermore, $\ell^1(S)$ is strongly Arens irregular.*

One cannot expect that there always exist a finite set that is DLTC.

Example 3.4.14. [12, Example 12.19] Let $\{S_j : j \in \mathbb{N}\}$ be a countable family of pairwise disjoint infinite abelian groups, and define $S = (\bigcup_{j\in\mathbb{N}} S_j) \cup \{0\}$ such that $S_iS_j = 0$ when $i \neq j$. Then S is an abelian semigroup. We claim that every set that is determining for the topological centre of βS is infinite. Suppose that V is determining for the topological centre of βS. First we show that $V \cap \overline{S_i} \neq 0$ for every $i \in \mathbb{N}$. Set $T = S \setminus S_i$, and assume that $V \subset \overline{T}$. Then $S_iT = \{0\}$, and so $u \,\Box\, v = u \diamond v = 0$ for each $u \in S_i$ and $v \in V$, a contradiction. It follows that V is an infinite set.

3.5 Totally ordered semigroups

We have already seen that, when the semigroup S is cancellative, $\ell^1(S)$ is strongly Arens irregular and that there is a DLTC set consisting of two points of S^*. However, there is little known about the Arens regularity of $\ell^1(S)$ when S is not cancellative. In [13], Dales and Strauss investigated totally ordered, abelian semigroups where the semigroup operation is defined as $(s,t) \longrightarrow s \wedge t$. This kind of semigroups are usually non-cancellative. They characterized the topological centers of both βS and $M(\beta S)$ for such semigroups. We review their results below.

First, we shall introduce some notations. Let T be an infinite, totally ordered set. For any s, t belonging to T, we define $s \wedge t = \min\{s,t\}$, making T a semi lattice. Additionally, we assume T possesses a minimum element, denoted as 0, and a maximum element, denoted as ∞. To say T being complete, we mean every non-empty subset of T has both a supremum and an infimum. We equip T with its interval topology, where closed intervals serve as a subbase for closed sets, and intervals in the form (a,b), $[0,a)$, $(a,\infty]$, and $[0,\infty]$ form a subbase for open sets in T, referred to as open intervals. Consequently, T becomes a compact topological semigroup. Moreover, every increasing or decreasing net in T converges to its supremum or infimum, respectively. Let S be an arbitrary infinite subset of T. Then S is a sub-semigroup of $(T, \wedge)$ that is also an abelian, idempotent semigroup. For subsets A and B of S, we write $A \leq B$ if $s \leq t$ for every $s \in A$ and $t \in B$. The set S with the discrete topology is denoted by S_d. Let $X = \beta S_d$ and $X^* = X \setminus S$. The closures of a subset A of S in T and in X are denoted as $\mathrm{cl}_T A$ and $\mathrm{cl}_X A$, respectively. Let $\pi : X \to T$ denote the continuous extension of the inclusion map of S into T. $\pi(X) = \mathrm{cl}_T S$. We use F_t to denote the fiber $\{x \in X : \pi(x) = t\}$ for $t \in \mathrm{cl}_T S$. We define $F_t^* = F_t \cap X^*$, so that $F_t^* = F_t$ for $t \in T \setminus S$ and $F_t^* = F_t \setminus \{t\}$ for $t \in S$. Let E denote the set of accumulation points of S in T. Then $E \neq \emptyset$. Take $t \in T$. Then F_t^* is a closed and hence compact subspace of X, and it is evident that $F_t^* \neq \emptyset$ if and only if $t \in E$. For $\mu \in M(X)$, we define $\mu_\pi \in M(T)$ by $\mu_\pi(B) = \mu(\pi^{-1}(B))$ where B is a Borel subset of T.

Take $t \in E$. Throughout we shall write

$$A_t = S \cap [0,t) \quad \text{and} \quad B_t = S \cap (t, \infty];$$

at least one of these sets is non-empty. It follows that

$$F_t^* \subseteq \mathrm{cl}_X\, A_t \cup \mathrm{cl}_X\, B_t,$$

and hence the sets $F_t^* \cap \mathrm{cl}_X\, A_t$ and $F_t^* \cap \mathrm{cl}_X\, B_t$ are disjoint, compact subspaces of F_t^* whose union is F_t^*.

To proceed, we require several lemmas, which we will present without proof as their need extensive preparations.

Lemma 3.5.1 ([13, Lemma 2.6]). *Let $\mu \in M(X)_1$, and take $t \in E$. Suppose that*

$$\mu \mid (F_t^* \cap U) = 0,$$

where $U = \mathrm{cl}_X A_t$ or $U = \mathrm{cl}_X B_t$. Then

$$\mu \,\Box\, p = p \,\Box\, \mu \quad (p \in F_t^* \cap U).$$

Lemma 3.5.2 ([13, Proposition 2.7]). *Let $\mu \in M(X)$. Take $t \in E$, and set*

$$U = F_t^* \cap \mathrm{cl}_X A_t \quad or \quad U = F_t^* \cap \mathrm{cl}_X B_t.$$

Suppose that $p, q \in U$ such that $p \neq q$, $\mu \,\Box\, p = p \,\Box\, \mu$ and $\mu \,\Box\, q = q \,\Box\, \mu$. Then $\mu \mid U = 0$.

Theorem 3.5.3 ([13, Theorem 2.11]). *Let $\mu \in M(X)$. Then $\mu \in \xi_t(M(X))$ if and only if $\mu|F_t^* = 0$ $(t \in E)$.*

A topological space Y is scattered if each non-empty subset of Y contains a point that is isolated in the subset.

Theorem 3.5.4 ([13, Theorem 2.14]). *The semigroup algebra $(\ell^1(S), *)$ is strongly Arens irregular if and only if $\mathrm{cl}_T(S)$ is scattered.*

Proof. First, assume that $\mathrm{cl}_T S$ is scattered. Consider a measure $\mu \in M(X)$. In this case, $|\mu|_\pi$ belongs to $\ell^1(\mathrm{cl}_T S)$. By Theorem 3.5.3, $\mu|F_t^* = 0$ for $t \in E$, which implies that $|\mu|_\pi(\{t\}) = |\mu|(\{t\})$ for $t \in \mathrm{cl}_T S$. Moreover, we have $|\mu|_\pi(\{t\}) = 0$ for $t \in \mathrm{cl}_T S \setminus S$, since $\pi^{-1}(\{t\}) \subseteq X^*$. Consequently, $|\mu|_\pi \in \ell^1(S)$. As $\||\mu|_\pi\|_1 = \||\mu|\|_1$, it follows that $|\mu| \in \ell^1(S)$, which means $\mu \in \ell^1(S)$. Thus, we can conclude that $M(X) = \ell^1(S)$, demonstrating that $\ell^1(S)$ is strongly Arens irregular.

Conversely, suppose that $\mathrm{cl}_T S$ is not scattered. We can select a continuous probability measure μ on $\mathrm{cl}_T S$ and define the set

$$L = \{f \circ \pi : f \in C(\mathrm{cl}_T S)\},$$

Then L is a closed linear subspace of $C(X)$. Next, we define a continuous linear functional ν on L by setting

$$\nu(f \circ \pi) = \mu(f) \quad \text{for} \quad f \in C(\mathrm{cl}_T S).$$

we have $\|\nu\| = \mu(1_{\mathrm{cl}_T S}) = 1$. By the Hahn-Banach theorem, ν can be extended to a continuous linear functional on $C(X)$, still satisfying $\|\nu\| = \nu(1_{\mathrm{cl}_T S}) = 1$. This allows us to treat ν as a probability measure on X. Given that $\nu(F_t^*) = 0$ for $t \in E$, and following from Theorem 3.5.3, we conclude that $\nu \in \xi_t(M(X))$. However, since $\nu \notin \ell^1(S)$, this implies that the Banach algebra $\ell^1(S)$ is not strongly Arens irregular. $\square$

Corollary 3.5.5 ([13, Corollary 2.15])**.** *Suppose that $\mathrm{cl}_T(S)$ is not scattered. Then*

$$\ell^1(S) \subsetneq \xi_t(M(X)) \subsetneq M(X),$$

and so the algebra $\ell^1(S)$ is neither Arens regular nor strongly Arens irregular.

Theorem 3.5.6 ([13, Proposition 3.1])**.** *(i) Suppose that the set E is finite. Then there is a finite DTC set for the semigroup S.*

(ii) Suppose that the set E is infinite (respectively uncountable). Then there is no finite (respectively countable) DTC set for the semigroup S.

Theorem 3.5.7 ([13, Theorem 3.2])**.** *Suppose that the set E is countable. Then the semigroup algebra $\ell^1(S)$ is strongly Arens irregular and has a DTC set consisting of at most four elements in $M(X^*)^+$.*

Proof. The set E is scattered because it is a countable, compact space, which implies that $cl_T S$ is also scattered. By Theorem 3.5.4, $\ell^1(S)$ is strongly Arens irregular.

Define

$$A = \{t \in E : F_t^* \cap \mathrm{cl}_X A_t \neq \emptyset\}$$

and

$$B = \{t \in E : F_t^* \cap \mathrm{cl}_X B_t \neq \emptyset\}.$$

(At least one of these sets is non-empty, though one may be empty.) Let $\{s_n : n \in I\}$ and $\{t_n : n \in J\}$ represent the enumerations of A and B, respectively, where I and J are subsets of $\mathbb{N}$. For each $n \in I$, select two distinct points $p_{1,n}$ and $p_{2,n}$ from $F_{s_n}^* \cap \mathrm{cl}_X A_{s_n}$, and for each $n \in J$, select two distinct points $q_{1,n}$ and $q_{2,n}$ from $F_{t_n}^* \cap \mathrm{cl}_X B_{t_n}$.

Next, consider the two measures

$$\mu_j = \sum_{n \in I} \frac{1}{2^n} \delta_{p_j,n} \quad (j = 1, 2),$$

defined when $A \neq \emptyset$, and similarly define μ_3, μ_4 when $B \neq \emptyset$. This gives us at most four measures in $M(X^*)^+$, forming a set V. We claim that V is a DTC set for $M(X)$. Suppose $\mu \in M(X)$ and $\mu \,\square\, \omega = \omega \,\square\, \mu$ for some $\omega \in V$. Take $n \in I$, and set $\nu_n = \mu\Big|(F_{s_n}^* \cap A_{s_n})$. Using Lemma 3.5.1, we have

$$(\mu - \nu) \,\square\, p_{j,n} = p_{j,n} \,\square\, (\mu - \nu_n) \quad (j = 1, 2).$$

Now, set $\nu = \sum_{n \in I} \nu_n$. For each $n \in I$, it follows that

$$(\mu - \nu)\Big|(F_{s_n}^* \cap A_{s_n}) = (\mu - \nu_n)\Big|(F_{s_n}^* \cap A_{s_n})$$

because $\nu_m\Big|(F_{s_n}^* \cap A_{s_n}) = 0$ for $m \in I \setminus \{n\}$. Now, take $m_n \in I$ where $m \neq n$, and by Lemma 3.5.1, we have

$$\nu_m \,\square\, p_{j,n} = p_{j,n} \,\square\, \nu_m \quad (j = 1, 2). \tag{3.5.1}$$

This leads to

$$(\mu - \nu_m) \,\square\, p_{j,n} = p_{j,n} \,\square\, (\mu - \nu_m) \quad (j = 1, 2),$$

which shows that

$$(\mu - \nu) \,\square\, p_{j,n} = p_{j,n} \,\square\, (\mu - \nu) \quad (n \in I, j = 1, 2).$$

Thus,

$$(\mu - \nu) \,\square\, \mu_j = \mu_j \,\square\, (\mu - \nu) \quad (j = 1, 2).$$

By the assumption, we have

$$\nu \,\square\, \mu_j = \mu_j \,\square\, \nu \quad (j = 1, 2).$$

Now, from equation (3.5.1), we conclude that

$$\nu_n \,\square\, p_{j,n} = p_{j,n} \,\square\, \nu_n \quad (j = 1, 2).$$

By Proposition 3.5.2, $\nu_n = 0$ for $n \in I$, meaning $\mu\Big|(F^*_{s_n} \cap A_{s_n}) = 0$ for $n \in I$.

Similarly, $\mu\Big|(F^*_{t_n} \cap \mathrm{cl}_X B_{t_n}) = 0$ for $n \in J$. It follows that $\mu\Big|F^*_t = 0$ for $t \in E$, and by Theorem 3.5.3, $\mu \in \xi_t(M(X)) = \ell^1(S)$, which completes the proof. $\square$

Corollary 3.5.8 ([13, Corollary 3.3])**.** *Suppose that the semigroup S is countable and the semigroup algebra $\ell^1(S)$ is strongly Arens irregular. Then $\ell^1(S)$ has a DTC set consisting of at most four elements in $M(X^*)^+$.*

Theorem 3.5.9 ([13, Proposition 3.4])**.** *Suppose that $|E| = \kappa$, where $\kappa \geq \aleph_1$. Then there is no DTC set for $M(X)$ with cardinality less than κ.*

Proof. Assume V is a DTC set for $M(X)$ such that $|V| < \kappa$. For each $\nu \in V$, the set $\{t \in E : \nu|F^*_t \neq 0\}$ is countable because the family $\{F^*_t : t \in E\}$ consists of pairwise-disjoint, non-empty, compact sets. Therefore, there exists some $t \in E$ such that $\nu|F^*_t = 0$ for $\nu \in V$. Select $p \in F^*_t$. According to Lemma 3.5.1, it follows that $p \,\square\, \nu = \nu \,\square\, p$ for all $\nu \in V$. Based on the assumption, $p \in \xi_t(M(X))$, leading to a contradiction with Theorem 3.5.3.

Consequently, it follows that no DTC set V exists for $M(X)$ with $|V| < \kappa$. $\square$

Corollary 3.5.10 ([13, Corollary 3.5])**.** *Suppose that the set E is uncountable. Then there is no countable DTC set for $M(X)$.*

The following example provide multiple instances demonstrating that the minimum cardinality of DTC sets can be arbitrarily large. Specifically, we have an example of a countable, totally ordered, abelian semigroup S equipped with the operation $\wedge$, where there exists no countable DTC set for either βS or $M(\beta S)$.

Example 3.5.11 ([13, Example 3.6]).

(i) Let's consider the semigroup $(\mathbb{Q}, \wedge)$. Theorem 3.5.6 establishes that there is no countable DTC set for this semigroup. Additionally, corollary 3.5.5 implies that $\ell^1(\mathbb{Q}, \wedge)$ is neither Arens regular nor strongly Arens irregular. Furthermore, according to Corollary 3.5.10, there is no countable DTC set for $M(X)$.

(ii) Now, let's examine the subset S of $T := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ defined as numbers of the form $n - x$, where $n \in \mathbb{Z}$ and $x \in \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$. This results in the set $E = \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, which is countable. Therefore, by Theorem 3.5.7, the semigroup algebra $\ell^1(S)$ is strongly Arens irregular. Theorem 3.5.6 indicates that there is no finite DTC set for the semigroup S. But according to Theorem 3.5.7, $M(X)$ possesses a finite DTC set in $M(X^+)$.

(iii) Lastly, let's consider the semigroup $S = T = ([0, \kappa], \wedge)$, where κ is a cardinal with $\kappa \geq \aleph_1$. As T is scattered, by Theorem 3.5.4, the algebra $\ell^1(S)$ is strongly Arens irregular. Theorem 3.5.9 asserts that there is no DTC set for $M(X)$ with cardinality strictly less than κ. Thus, this illustrates that the cardinality of a DTC set can be arbitrarily large, even when $\ell^1(S)$ is strongly Arens irregular.

4

Arens irregularity of bicyclic semigroups

In the previous chapters we have reviewed important works in the field of Arens regularity for semigroups and semigroup algebras. In the current chapter we focus on three special semigroups. The results in this chapter are our original contribution to the field.

4.1 The DLTC and DRTC sets for βS_1, βS_2, and $\beta S_{1,1}$

The bicyclic semigroup is the semigroup generated by a unit e and two more elements p and q subject to the relation $pq = e$. We denote it by $S_1 = \langle e, p, q \mid pq = e \rangle$. The semigroup generated by a unit e and three more elements a, b and c subject to the relations $ab = ac = e$ is denoted by $S_2 = \langle e, a, b, c \mid ab = e, ac = e \rangle$.The semigroup generated by a unit e and four more elements a, b , c and d subject to the relations $ac = bd = e$ is denoted by $S_{1,1} = \langle e, a, b, c, d \mid ac = e, bd = e \rangle$. S_2 and $S_{1,1}$ are called partially bicyclic semigroups. Properties of these simigroups, in particular amenability properties of them, have been discussed in [22] and [15]. We note that non of these semigroups are cancellative. Here we show that they are weakly cancellative.

Proposition 4.1.1. *S_1 is weakly cancellative.*

Proof. First, we show it is left weakly cancellative. Let $x_1 = q^{n_1}p^{m_1}$ and $x_2 = q^{n_2}p^{m_2}$ be two element in S_1. We find the solution for the equation $x_1 y = x_2$ ($y \in S_1$). We can write $y = q^n p^m$. We have

$$x_1 y = q^{n_1} p^{m_1} q^n p^m = q^{n_1+n-\min(m_1,n)} p^{m_1+m-\min(m_1,n)}.$$

Hence y is a solution for the equation if an only if

$$n_1 + n - \min(m_1, n) = n_2 \quad \text{and} \quad m_1 + m - \min(m_1, n) = m_2.$$

We consider two cases:

- If $n \geq m_1$, then $n_1 + n - m_1 = n_2$ and $m = m_2$. Therefore, $n = n_2 + m_1 - n_1$ and $m = m_2$. Hence, $q^{n_2+m_1-n_1} p^{m_2}$ is the unique solution for the equation if $n_2 + m_1 - n_1 \geq 0$, otherwise, the equation has no solution.

- If $m_1 > n$, then $n_1 = n_2$ and $m_1 + m - n = m_2$. Hence, $n_1 = n_2$ and $m - n = m_2 - m_1$. Since $m_1 > n$ we have finitely many choices for n and since $m - n = m_2 - m_1$ we have finitely many choices for m. Therefore, the equation has finitely many solutions. This shows that S_1 is left weakly cancellative.

Similarly, by symmetry, one may show that S_1 is right weakly cancellative. $\square$

Now similar to S_1, S_2 is also weakly cancellative. Before we state the proposition we remark that each element $x \in S_2$ can be uniquely written as fa^k, where $f \in \langle e, b, c \rangle$ and $k = 0, 1, 2, \cdots$. We denote the length of f by $|f|$.

Proposition 4.1.2. *S_2 is weakly cancellative.*

Proof. First, we show it is left weakly cancellative. Let $x_1 = f_1 a^{k_1}$ and $x_2 = f_2 a^{k_2}$ be two elements in S_2. We find the solution for the equation $x_1 y = x_2$ $(y \in S_2)$. We can write $y = fa^k$. We consider two cases:

- If $|f| \leq k_1$, then we have

  $$x_1 y = f_1 a^{k_1} f a^k = f_1 a^{k_1 - |f| + k},$$

  so the equation $x_1 y = x_2$ has a solution if and only if $f_1 = f_2$ and $k_1 - |f| + k = k_2$. Since $|f| \leq k_1$ and $k = k_2 + |f| - k_1$, we have at most finitely many choices for f and k.

- If $|f| > k_1$, then we have

  $$x_1 y = f_1 a^{k_1} f a^k = f_1 f' a^k,$$

for some f' with $|f'| = |f| - k_1$. In this case the equation $x_1y = x_2$ has a solution if and only if $f_1f' = f_2$ and $k = k_2$. Hence, $|f_1| + |f'| = |f_2|$, so $|f_1| + |f| - k_1 = |f_2|$. Then $f_1f' = f_2$ has at most finite number of solutions for f'. Thus there are only finite number of solutions for f to the equation $f_1a^{k_1}f = f_2$.

Now, we show S_2 is right weakly cancellative. Let x_1, x_2, and y be as previous. We have to find the number of the solution of the equation $yx_1 = x_2$. We consider two cases:

- If $|f_1| \leq k$, then we have

$$yx_1 = fa^kf_1a^{k_1} = fa^{k-|f_1|+k_1},$$

  so the equation $yx_1 = x_2$ has a solution if and only if $f = f_2$ and $k - |f_1| + k_1 = k_2$. Therefore, if $|f_1| - k_1 + k_2 \geq 0$, the equation has a unique solution. Otherwise, it has no solution.

- If $|f_1| > k$, then we have

$$yx_1 = fa^kf_1a^{k_1} = ff'a^{k_1},$$

  for some f' with $|f'| = |f_1| - k$. In this case the equation $yx_1 = x_2$ has a solution if and only if $ff' = f_2$ and $k_1 = k_2$. Hence, $|f| + |f'| = |f_2|$, so $|f_1| + |f| - k = |f_2|$. Therefore, we have $|f| = |f_2| + k - |f_1|$. Since $|f_1| > k$ and $|f| = |f_2| + k - |f_1|$ we have at most finitely many choices for f and k.

Therefore, S_2 is right weakly cancellative. □

Now we consider $S_{1,1}$.It is worthy mentioning that any element x in $S_{1,1}$ can be written uniquely in the form $x = f_1g_1f_2g_2 \cdots f_ng_n$ where $g_i \in \langle e, a, c | ac = e\rangle$ and $f_i \in \langle e, b, d | bd = e\rangle$ for $i = 1, 2, \cdots n$.

Proposition 4.1.3. *$S_{1,1}$ is weakly cancellative.*

Proof. Let $x_1, x_2 \in S_{1,1}$ be fixed and consider the equation $x_1y = x_2$. Let $x_1 = f_1g_1f_2g_2 \cdots f_ng_n$ and $|x_1| \neq 0$. Suppose $|g_n| \neq 0$. There are two cases.

1. If $g_n = c^k$ and $k > 0$. In this case the equation has at most one solution, since no cancellation can be made in x_1y.

2. If $g_n = c^{m_1}a^{n_1}$, where $n_1 = 1, 2, \cdots$, and $m_1 = 0, 1, 2, \cdots$. In this case the solutions may have the form $y = g'_1 f'_2 g'_2 \cdots f'_{k'} g'_{k'}$, where $g'_1 = c^{m'}a^{n'}$, $n' = 0, 1, 2, \cdots$, $m' = 0, 1, 2, \cdots$, and $n' + m' > 0$ (If $n' + m' = 0$, then $x_1 y = x_2$ clearly has no more than one solution). In this case we have $x_1 y = f_1 g_1 f_2 g_2 \cdots f_n c^{m_1+m'-\min(n_1,m')} a^{n_1+n'-\min(n_1,m')} f'_2 g'_2 \cdots f'_{k'} g'_{k'}$.

   - If $n_1 \geq m'$, then $x_1 y = f_1 g_1 f_2 g_2 \cdots f_n c^{m_1} a^{n_1+n'-m'} f'_2 g'_2 \cdots f'_{k'} g'_{k'}$. So $x_1 y = x_2$ has a solution only if $x_2 = f_1 g_1 f_2 g_2 \cdots f_n c^{m_1} a^{n_2} f'_2 g'_2 \cdots f'_{k'} g'_{k'}$, where $n_2 = n_1 + n' - m'$. Since $m' \leq n_1$ and $n' = n_2 + m' - n_1$, we have at most finitely many choices for m' and n'.
   - If $m' > n_1$, then $x_1 y = f_1 g_1 f_2 g_2 \cdots f_n c^{m_1+m'-n_1} a^{n'} f'_2 g'_2 \cdots f'_{k'} g'_{k'}$. So the equation has no solution, unless $x_2 = f_1 g_1 f_2 g_2 \cdots f_n c^{m_2} a^{n_2} f'_2 g'_2 \cdots f'_{k'} g'_{k'}$. In this case, $y = c^{m_2+n_1-m_1} a^{n_2} f'_2 g'_2 \cdots f'_{k'} g'_{k'}$ is a unique solution for the equation.

If $|g_n| = 0$, we may switch f and g in the above argument. The proof is similar. Therefore, $S_{1,1}$ is left weakly cancellative. Similarly, by symmetry, one may show that $S_{1,1}$ is right weakly cancellative. $\square$

We now consider sets determining for the left and right topological centres of βS_1.

Theorem 4.1.4. *S_1 is strongly Arens irregular. Moreover, there is a DLTC set (DRTC set) consists of two points for βS_1.*

Proof. Let $A_1 = \{q^n : n = 1, 2, \cdots\}$ and $B_1 = \{q^n p^m : n = 0, 1, 2, \cdots, \text{and } m = 1, 2, \cdots\}$. Then $A_1 \cap B_1 = \emptyset$, and since βS_1 is extremely disconnected, we have $\overline{A_1} \cap \overline{B_1} = \emptyset$. Similarly, if we define $A_2 = \{q^n p^m : n < m\}$ and $B_2 = \{q^n p^m : n > m\}$, then $A_2 \cap B_2 = \emptyset$, and therefore, $\overline{A_2} \cap \overline{B_2} = \emptyset$.

Suppose that $u_p \in P^*$ and $u_q \in Q^*$, where $P = \{p^n : n = 1, 2, \cdots\}$ and $Q = \{q^n : n = 1, 2, \cdots\}$. We claim that $\{u_p, u_q\}$ is a DLTC set. Suppose that $u_p = \lim_j p^{k_j}$ and $u_q = \lim_j q^{k_j}$. Let $v \in S_1^*$. Then

1. If $v = \lim_i q^{n_i} p^{m_i}$ and $m_i \longrightarrow \infty$, then

$$v \,\square\, u_q = \lim_i \lim_j q^{n_i} p^{m_i} q^{k_j} \in \overline{A_1}, \quad \text{and}$$
$$v \diamond u_q = \lim_j \lim_i q^{n_i} p^{m_i} q^{k_j} \in \overline{B_1}.$$

   Therefore, $v \,\square\, u_q \neq v \diamond u_q$.

2. If $v = \lim_i q^{n_i}p^{m_i}$ and there is an M such that $m_i \leq M$ for every i, then we must have $n_i \longrightarrow \infty$. Now we have

$$v \,\Box\, u_p = \lim_i \lim_j q^{n_i}p^{m_i}p^{k_j} \in \overline{A_2}, \quad \textit{and}$$
$$v \diamond u_p = \lim_j \lim_i q^{n_i}p^{m_i}p^{k_j} \in \overline{B_2}.$$

Therefore, $v \,\Box\, u_p \neq v \diamond u_p$.

This shows that S_1 is strongly left Arens irregular and $\{u_p, u_q\}$ is a DLTC set.

Similarly, one can show that S_1 is strongly right Arens irregular and $\{u_p, u_q\}$ is a DRTC set. □

Now we discuss Arens regularity of S_2 and the determining sets for the topological centres of its Stone–Čech compactification.

Theorem 4.1.5. *S_2 is strongly Arens irregular. Moreover,*

1. *there is a DLTC set consists of two points for βS_2.*

2. *there is a DRTC set consists of two points for βS_2.*

Proof. First, we show that S_2 is strongly right Arens irregular and has a DRTC set of two points.

Let $A_1 = \{fa^k : f \in \langle b, c\rangle, |f| \geq 1, k = 0, 1, \cdots\}$ and $B_1 = \{a^k : k = 1, 2, \cdots\}$. Then $A_1 \cap B_1 = \emptyset$, and therefore, $\overline{A_1} \cap \overline{B_1} = \emptyset$, since βS_2 is extremely disconnected. Similarly, if we define $A_2 = \{fa^k : f \in \langle b, c\rangle,\ |f| < k\}$ and $B_2 = \{fa^k : f \in \langle b, c\rangle,\ |f| > k\}$, then $A_2 \cap B_2 = \emptyset$, and therefore, $\overline{A_2} \cap \overline{B_2} = \emptyset$.

Let $U_1 = \{a^n : n = 1, 2, \cdots\}$, and $U_2 = \{c^n : n = 1, 2, \cdots\}$. Take $u_1 \in U_1^*$ and $u_2 \in U_2^*$. We claim that $\{u_1, u_2\}$ is a DRTC set. For every $v \in S_2^*$ we can write $v = \lim_i f_i a^{k_i}$ where $k_i \geq 0$ and $f_i \in \langle b, c\rangle$ for every i. Now going through subnets if necessary, we have two cases.

1. If $|f_i| \longrightarrow \infty$, then

$$u_1 \,\Box\, v = \lim_j \lim_i a^{n_j} f_i a^{k_i} \in \overline{A_1}, \quad \text{and}$$
$$u_1 \diamond v = \lim_i \lim_j a^{n_j} f_i a^{k_i} \in \overline{B_1}.$$

Therefore, $u_1 \,\Box\, v \neq u_1 \diamond v$.

2. If there is an M such that $|f_i| \leq M$ for every i, then we must have $k_i \longrightarrow \infty$. Now we have

$$u_2 \,\square\, v = \lim_j \lim_i c^{n_j} f_i a^{k_i} \in \overline{A_2}, \quad \text{and}$$
$$u_2 \diamond v = \lim_i \lim_j c^{n_j} f_i a^{k_i} \in \overline{B_2}.$$

Therefore, $u_2 \,\square\, v \neq u_2 \diamond v$.

Hence, S_2 is strongly right Arens irregular and $\{u_1, u_2\}$ is a DRTC set.

Now we show that S_2 is also strongly left Arens irregular and has a DLTC set consisting of two points. Let $A_1 = \{f : f \in \langle b, c\rangle, |f| \geq 1\}$ and $B_1 = \{fa^k : f \in \langle b, c\rangle, k \geq 0\}$. Then $A_1 \cap B_1 = \emptyset$, and therefore, $\overline{A_1} \cap \overline{B_1} = \emptyset$. Let A_2, B_2, U_1, and U_2 be the same sets as in the previous part. Take $u_1 \in U_1^*$ and $u_2 \in U_2^*$. Like before, for every $v \in S_2^*$ we can write $v = \lim_i f_i a^{k_i}$ where $k_i = 0, 1, \cdots$ and $f_i \in \langle b, c\rangle$ for every i. Now, easily we can see that

1. If there is an M such that $k_i \leq M$ for every i, then we must have $|f_i| \longrightarrow \infty$. Now we have

$$v \,\square\, u_1 = \lim_i \lim_j f_i a^{k_i} a^{n_j} \in \overline{A_2}, \quad \text{and}$$
$$v \diamond u_1 = \lim_j \lim_i f_i a^{k_i} a^{n_j} \in \overline{B_2}.$$

Therefore, $v \,\square\, u_1 \neq v \diamond u_1$.

2. If $k_i \longrightarrow \infty$, then

$$v \,\square\, u_2 = \lim_i \lim_j f_i a^{k_i} c^{n_j} \in \overline{A_1}, \quad \text{and}$$
$$v \diamond u_2 = \lim_j \lim_i f_i a^{k_i} c^{n_j} \in \overline{B_1}.$$

Therefore, $v \,\square\, u_2 \neq v \diamond u_2$. This shows that $\{u_1, u_2\}$ is a DLTC set for βS_2 and S_2 is strongly Arens irregular.

□

We end this section by examining $S_{1,1}$.

Theorem 4.1.6. *$S_{1,1}$ is strongly Arens irregular. Moreover, there is a DLTC set (DRTC set) consists of two points for $\beta S_{1,1}$.*

Proof. Suppose that $v \in S_{1,1}^*$. Then we can write $v = \lim_i f_i$ where $f_i \in S_{1,1}$ for every i.

Let $U_1 = \{a^n : n = 1, 2, \cdots\}$, and $U_2 = \{b^n : n = 1, 2, \cdots\}$. Take $u_1 \in U_1^*$ and $u_2 \in U_2^*$. We claim that $\{u_1, u_2\}$ is a DLTC set. We Consider two cases.

1. If the net (f_i) has a subnet that all of its element end with a $\Big($i.e. $f_{i_j} = g_{i_j} a^{k_{i_j}}$, where $g_{i_j} \in S_{1,1}$ and $k_{i_j} = 1, 2, \cdots\Big)$. Then $f_{i_j} \longrightarrow v$ in $\beta S_{1,1}$. In this case, by passing to a subnet if necessary, we may assume without lose of generality that $f_i = g_i a^{k_i}$ for every i.

   Now, let $A_1 = \{fb^k : k \in \mathbb{N},\ f \in S_{1,1} \text{ ends with } a, \text{and} \, |f| < k\}$ and $B_1 = \{fb^k : k \in \mathbb{N},\ f \in S_{1,1} \text{ ends with } a, \text{and} \, |f| > k\}$. Then $A_1 \cap B_1 = \emptyset$, and therefore, $\overline{A_1} \cap \overline{B_1} = \emptyset$, since $\beta S_{1,1}$ is extremely disconnected. We have

   $$\begin{aligned} v \,\square\, u_2 &= \lim_i \lim_j f_i b^{k_j} \in \overline{A_1}, \quad \text{and} \\ v \diamond u_2 &= \lim_j \lim_i f_i b^{k_j} \in \overline{B_1}. \end{aligned}$$

   Therefore, $v \,\square\, u_2 \neq v \diamond u_2$.

2. If the net (f_i) has no subnet with all element, ending with a. Then the net (f_i) has a subnet that non of its elements end with a. In this case, by passing to a subnet if necessary, we may assume without lose of generality that all elements of (f_i) does not end with a. Now, we define $A_2 = \{fa^k : k \in \mathbb{N},\ f \in S_{1,1} \text{ does not end with } a, \text{and} \, |f| < k\}$ and $B_2 = \{fa^k : k \in \mathbb{N},\ f \in S_{1,1} \text{ does not end with } a, \text{and} \, |f| > k\}$. Then $A_2 \cap B_2 = \emptyset$, and therefore, $\overline{A_2} \cap \overline{B_2} = \emptyset$. Hence, we have

   $$\begin{aligned} v \,\square\, u_1 &= \lim_i \lim_j f_i a^{k_j} \in \overline{A_2}, \quad \text{and} \\ v \diamond u_1 &= \lim_j \lim_i f_i a^{k_j} \in \overline{B_2}. \end{aligned}$$

   Therefore, $v \,\square\, u_1 \neq v \diamond u_1$.

Therefore, $S_{1,1}$ is strongly left Arens irregular and $\{u_1, u_2\}$ is a DLTC set.

By its symmetric structure, one sees easily that $S_{1,1}$ is also strongly right Arens irregular with a DRTC set of two points. $\square$

4.2 The DLTC and DRTC sets for $M(\beta S_1)$, $M(\beta S_2)$, and $M(\beta S_{1,1})$

Now we discuss Arens irregularity of $\ell^1(S)$ for $S = S_1$, $S = S_2$, and $S = S_{1,1}$. This task turns out to be more difficult. First, we need to introduce some notations. Let S be a countable semigroup with an identity e. We may enumerate S as a sequence $(s_n)_{n\in\mathbb{Z}^+}$, where $s_0 = e$. Fix an enumeration, we define a relation $\preccurlyeq$ on S by

$$s_n \preceq s_m \Longleftrightarrow n \le m, \text{ and } s_n \prec s_m \Longleftrightarrow n < m.$$

This makes S a totally ordered set.

Moreover, for every $t \in S$, we define $[t] = \{s \in S : s \preceq t\}$ and for every subset $T \subset S$, we define $[T] = \cup\{[t] : t \in T\}$.

Theorem 4.2.1. *$\ell^1(S_1)$ is strongly Arens irregular. Moreover, there is a DLTC set (DRTC set) consists of two points for $M(\beta S_1)$.*

Proof. First, as above we enumerate S_1 as a sequence $(s_n)_{n\in\mathbb{Z}^+}$, where $s_0 = e$. Let $P = \{e, p, p^2, \cdots\}$. Then every element in P is right cancellable. We construct a sequence (t_n) in P by induction. Set $t_0 = s_0 = e$. Take $n \in \mathbb{Z}^+$ and suppose that $t_0, \cdots, t_n$ have been chosen in P. We define $T_n = \{s_0, \cdots, s_n, t_0, \cdots, t_n\}$. Since S_1 is weakly cancellative, the sets $T_n^{-1}[T_n^2]$, $[T_n^2]T_n^{-1}$, $T_n^{-1}[T_n]^2$, and $[T_n]^2T_n^{-1}$ are finite. We choose t_{n+1} in P such that

$$\max\{t : t \in (T_n^{-1}[T_n^2]) \cup ([T_n^2]T_n^{-1}) \cup (T_n^{-1}[T_n]^2) \cup ([T_n]^2T_n^{-1})\} \prec t_{n+1}.$$

The inductive construction continues. Note that the sequence (t_n) satisfies the following conditions:

(i) $sT_n \cap [T_n] = \emptyset$ whenever $s \in S_1$ with $t_{n+1} \preceq s$;

(ii) $st_{n+1} \neq tt_{n+1}$ whenever $s, t \in [T_n]$ with $s \neq t$, since $t_{n+1} \in P$ and P is right cancellable;

(iii) $rs \prec tt_{n+1}$ whenever $r, s, t \in [T_n]$.

Note that $s_n \preceq t_{n+1}$ $(n \in \mathbb{Z}^+)$, and so $\cup\{[t_n] : n \in \mathbb{Z}^+\} = S_1$. Also $S_1^* = (S_1 \setminus [t_k])^*$ for each $k \in \mathbb{N}$.

Define $\varphi : S_1 \to \mathbb{Z}^+$ by setting

$$\varphi(s) = \min\{n \in \mathbb{Z}^+ : s \in [t_n]\} \quad (s \in S_1).$$

Assume that $\varphi(s) = m \in \mathbb{N}$. According to condition (iii), if $n > m$, then $t_{n-1} \prec st_n \prec t_{n+1}$, therefore,

$$\varphi(st_n) \in \{n, n+1\} \quad (n > m). \tag{4.2.1}$$

Now suppose $m \geq 2$ and $k \leq m-2$. By condition (i), $t_{m-2} \prec ss_k$ because $sT_{m-2} \cap [T_{m-2}] = \emptyset$. Additionally, by condition (iii), $ss_k \prec t_{m+1}$ since $s \in [T_m]$. Thus, $\varphi(ss_k) \in \{m-1, m, m+1\}$, leading to

$$\varphi(ss_k) \in \{m-1, m, m+1\} \quad (m \geq k+2). \tag{4.2.2}$$

For $s \in S_1$, define $\gamma(s) \equiv \varphi(s) \pmod 8$. Then (4.2.2) and the definition of γ imply that

$$\gamma(ss_k) \in \{\gamma(s)-1, \gamma(s), \gamma(s)+1\} \subset \mathbb{Z}_8 \quad (m \geq k+2). \tag{4.2.3}$$

Now, $\gamma : S_1 \to \mathbb{Z}_8$ can be extended continuously to a map $\gamma : \beta S_1 \to \mathbb{Z}_8$. Let $x \in S_1^*$ and $s \in S_1$. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a net in S_1 that converges to x. Without lose of generality we may assume that $\varphi(x_n) \geq \varphi(s) + 2$ for all $n \in \mathbb{N}$. From (4.2.3), for every $n \in \mathbb{N}$ we get

$$\gamma(x_n s) \in \{\gamma(x_n)-1, \gamma(x_n), \gamma(x_n)+1\}. \tag{4.2.4}$$

It follows that

$$\gamma(x \,\Box\, s) = \gamma(x \diamond s) = \lim_n \gamma(x_n s) \in \{\gamma(x)-1, \gamma(x), \gamma(x)+1\} \subset \mathbb{Z}_8. \tag{4.2.5}$$

Define

$$A = \{t_n : \gamma(t_n) = 1\}, \quad B = \{t_n : \gamma(t_n) = 5\},$$

so A and B are infinite subsets of S_1. Then define

$$U = \{s \in S_1 : \gamma(s) \in \{1,2\}\}, \quad V = \{s \in S_1 : \gamma(s) \in \{5,6\}\},$$

so that $A \subset U$ and $B \subset V$.

Choose $a \in A^*$ and $b \in B^*$. We claim that a is right cancellable in βS_1. To demonstrate this, let u_1 and u_2 be distinct points in βS_1, and select disjoint sets N_1

and N_2 in S_1 with $u_j \in \overline{N_j}$ for $j = 1, 2$. For $j = 1, 2$, define

$$Y_j = \{s_m t_n : s_m \in N_j, t_n \in A, m < n\},$$

so that $u_j \,\Box\, a \in \overline{Y_j}$. Choose $m_1, m_2, n_1, n_2 \in \mathbb{N}$ with $m_1 < n_1$, $m_2 < n_2$, and $m_1 \neq m_2$. Then $s_{m_1} t_{n_1} \neq s_{m_2} t_{n_2}$; this is true for $n_1 < n_2$ by condition (iii) and for $n_1 = n_2$ by condition (ii). It follows that $Y_1 \cap Y_2 = \emptyset$, and so $u_1 \,\Box\, a \neq u_2 \,\Box\, a$, as required. Similarly, b is right cancellable in βS_1.

From (4.2.1), we have $S_1 \,\Box\, a \subset \overline{U}$ and $S_1 \,\Box\, b \subset \overline{V}$.

Let $x \in S_1^*$, set $k = \gamma(x)$, and take $s \in S_1$. Suppose $x \diamond s \in \overline{U}$. Then, by (4.2.5), $k \in \{0, 1, 2, 3\}$. Suppose $x \diamond s \in \overline{V}$. Then, similarly, $k \in \{4, 5, 6, 7\}$. Thus, either

$$(x \diamond S_1) \cap \overline{U} = \emptyset \quad \text{ or } \quad (x \diamond S_1) \cap \overline{V} = \emptyset. \tag{4.2.6}$$

We claim that $\{\delta_a, \delta_b\}$ is a DLTC set for $M(\beta S_1)$. Suppose to the contrary that there is a non-zero measure $\mu \in M(S^*)$ such that $\mu \,\Box\, \delta_a = \mu \diamond \delta_a$ and $\mu \,\Box\, \delta_b = \mu \diamond \delta_b$. Without lose of generality we may suppose that $\|\mu\| = 1$. Since a, b are right cancellable in βS_1, it follows from Theorem 3.1.13 that $\|\mu \,\Box\, \delta_a\| = \|\mu \,\Box\, \delta_b\| = \|\mu\| = 1$. Therefore, by supremum properties, there are $\lambda_a, \lambda_b \in C(\beta S_1)_{[1]}$, the unit ball of $C(\beta S_1)$, such that both

$$\langle \mu \,\Box\, \delta_a, \lambda_a \rangle > \|\mu\| - \frac{1}{2} = \frac{1}{2} \quad \text{ and } \quad \langle \mu \,\Box\, \delta_b, \lambda_b \rangle > \|\mu\| - \frac{1}{2} = \frac{1}{2}. \tag{4.2.7}$$

Since $S_1 \,\Box\, a \subset \overline{U}$ and $S_1 \,\Box\, b \subset \overline{V}$ we have $\beta S_1 \,\Box\, a \subset \overline{U}$ and $\beta S_1 \,\Box\, b \subset \overline{V}$, and so, $\operatorname{supp}(\delta_x \,\Box\, \delta_a) = \operatorname{supp}(\delta_{x \,\Box\, a}) \subset \overline{U}$ for every $x \in S_1^*$. Thus for every $f \in \ell^1(S_1^*)$ we have $\operatorname{supp}(f \,\Box\, a) \subset \overline{U}$. It follows that $\operatorname{supp}(\mu \,\Box\, \delta_a) \subset \overline{U}$ and $\operatorname{supp}(\mu \,\Box\, \delta_b) \subset \overline{V}$. Since $\overline{U} \cap \overline{V} = \emptyset$, by replacing λ_a by $\lambda_a \cdot \chi_{\overline{U}}$ and λ_b by $\lambda_b \cdot \chi_{\overline{V}}$, we may suppose that λ_a and λ_b vanish outside $\overline{U}$ and $\overline{V}$, respectively. Let $f = \sum_{i=1}^m \alpha_i \delta_{x_i}$, where $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ such that $\sum_{i=1}^m |\alpha_i| = 1$, and $x_1, \ldots, x_m \in S_1^*$. Consider $i \in \mathbb{N}_m$. According to (4.2.6), either

$$\langle x_i \diamond s, \lambda_a \rangle = 0 \quad (s \in S_1) \quad \text{ or } \quad \langle x_i \diamond s, \lambda_b \rangle = 0 \quad (s \in S_1).$$

Thus, either

$$|\langle f \diamond \delta_s, \lambda_a \rangle| \leq \frac{1}{2} \quad (s \in S_1) \quad \text{ or } \quad |\langle f \diamond \delta_s, \lambda_b \rangle| \leq \frac{1}{2} \quad (s \in S_1),$$

and hence either

$$|\langle \mu \diamond \delta_s, \lambda_a \rangle| \leq \frac{1}{2} \quad (s \in S_1) \quad \text{or} \quad |\langle \mu \diamond \delta_s, \lambda_b \rangle| \leq \frac{1}{2} \quad (s \in S_1),$$

Since the mapping $x \mapsto \nu \diamond x$, $\beta S_1 \to M(S_1^*)$, is continuous, it follows that either

$$|\langle \mu \diamond \delta_x, \lambda_a \rangle| \leq \frac{1}{2} \quad (x \in \beta S_1) \quad \text{or} \quad |\langle \mu \diamond \delta_x, \lambda_b \rangle| \leq \frac{1}{2} \quad (x \in \beta S_1), \tag{4.2.8}$$

From (4.2.7) and (4.2.8) it follows that either

$$\mu \,\square\, \delta_a \neq \mu \diamond \delta_a \quad \text{or} \quad \mu \,\square\, \delta_b \neq \mu \diamond \delta_b.$$

This is a contradiction. So $\mu = 0$, and therefore, $\{\delta_a, \delta_b\}$ is a DLTC set for $M(\beta S_1)$. As a consequence $\ell^1(S_1)$ is strongly left Arens irregular.

We can repeat the argument by replacing P by $Q = \{e, q, q^2, \cdots\}$ to show that $\ell^1(S_1)$ is strongly right Arens irregular and $M(\beta S_1)$ has a DRTC set consists of two points. $\square$

By the same argument we can see that $\ell^1(S_2)$ is also strongly Arens irregular.

Theorem 4.2.2. *$\ell^1(S_2)$ is strongly Arens irregular. Moreover, there is a DLTC set (DRTC set) consists of two points for $M(\beta S_2)$.*

Proof. First, as before, we denote S_2 as a sequence $(s_n)_{n \in \mathbb{Z}^+}$, where $s_0 = e$. Let $P = \{e, a, a^2, \cdots\}$. Each element in P is right cancellable. We construct a sequence (t_n) in P by induction. Set $t_0 = s_0 = e$. Take $n \in \mathbb{Z}^+$ and assume $t_0, \cdots, t_n$ have been specified in P. Define $T_n = \{s_0, \cdots, s_n, t_1, \cdots, t_n\}$. Because S_2 is weakly cancellative, the sets $T_n^{-1}[T_n^2]$, $[T_n^2]T_n^{-1}$, $T_n^{-1}[T_n]^2$, and $[T_n]^2 T_n^{-1}$ are finite. We choose t_{n+1} in P such that

$$\max\{t : t \in (T_n^{-1}[T_n^2]) \cup ([T_n^2]T_n^{-1}) \cup (T_n^{-1}[T_n]^2) \cup ([T_n]^2 T_n^{-1})\} \prec t_{n+1}.$$

The inductive construction proceeds. Note that the sequence (t_n) satisfies the following conditions:

(i) $sT_n \cap [T_n] = \emptyset$ whenever $s \in S_2$ with $t_{n+1} \preceq s$;

(ii) $st_{n+1} \neq tt_{n+1}$ whenever $s, t \in T_n$ with $s \neq t$;

(iii) $rs \prec tt_{n+1}$ whenever $r, s, t \in [T_n]$.

Note that $s_n \preceq t_n$ ($n \in \mathbb{Z}^+$), so $\cup\{[t_n] : n \in \mathbb{Z}^+\} = S_2$. Also, $S_2^* = (S_2 \setminus [t_k])^*$ for each $k \in \mathbb{N}$.

Define $\varphi : S_2 \to \mathbb{Z}^+$ by setting

$$\varphi(s) = \min\{n \in \mathbb{Z}^+ : s \in [t_n]\} \quad (s \in S_2).$$

Assume $\varphi(s) = m \in \mathbb{N}$. According to condition (iii), if $n > m$, then $t_{n-1} \prec st_n \prec t_{n+1}$, therefore,

$$\varphi(st_n) \in \{n, n+1\} \quad (n > m). \tag{4.2.9}$$

Now suppose $m \geq 2$ and $k \leq m-2$. By condition (i), $t_{m-2} \prec ss_k$ because $sT_{m-2} \cap [T_{m-2}] = \emptyset$. Additionally, by condition (iii), $ss_k \prec t_{m+1}$ since $s \in T_m$. Thus, $\varphi(ss_k) \in \{m-1, m, m+1\}$, leading to

$$\varphi(ss_k) \in \{m-1, m, m+1\} \quad (m \geq k+2). \tag{4.2.10}$$

For $s \in S_2$, define $\gamma(s) \equiv \varphi(s) \pmod 8$. Then $\gamma : S_2 \to \mathbb{Z}_8$ can be extended continuously to a map $\gamma : \beta S_2 \to \mathbb{Z}_8$. From (4.2.10), for each $x \in S_2^*$ and $s \in S_2$, we get

$$\gamma(x \,\square\, s) = \gamma(x \diamond s) \in \{\gamma(x)-1, \gamma(x), \gamma(x)+1\} \subset \mathbb{Z}_8. \tag{4.2.11}$$

Define

$$A = \{t_n : \gamma(t_n) = 1\}, \quad B = \{t_n : \gamma(t_n) = 5\},$$

so A and B are infinite subsets of S_2. Then define

$$U = \{s \in S_2 : \gamma(s) \in \{1,2\}\}, \quad V = \{s \in S_2 : \gamma(s) \in \{5,6\}\},$$

so that $A \subset U$ and $B \subset V$.

Choose $a \in A^*$ and $b \in B^*$. We claim that a is right cancellable in βS_2. To demonstrate this, let u_1 and u_2 be distinct points in βS_2, and select disjoint sets N_1 and N_2 in S_2 with $u_j \in N_j$ for $j = 1, 2$. For $j = 1, 2$, define

$$Y_j = \{s_m t_n : s_m \in N_j, t_n \in A, m < n\},$$

so that $u_j \,\square\, a \in \overline{Y_j}$. Choose $m_1, m_2, n_1, n_2 \in \mathbb{N}$ with $m_1 < n_1$, $m_2 < n_2$, and $m_1 \neq m_2$. Then $s_{m_1}t_{n_1} \neq s_{m_2}t_{n_2}$; this is true for $n_1 < n_2$ by condition (iii) and for $n_1 = n_2$ by condition (ii). It follows that $Y_1 \cap Y_2 = \emptyset$, and so $u_1 \,\square\, a \neq u_2 \,\square\, a$, as required.

Similarly, b is right cancellable in βS_2.

From (4.2.9), we have $S_2 \,\Box\, a \subset \overline{U}$ and $S_2 \,\Box\, b \subset \overline{V}$.

Let $x \in S_2^*$, set $k = \gamma(x)$, and take $s \in S_2$. Suppose $x \diamond s \in U$. Then, by (4.2.11), $k \in \{0, 1, 2, 3\}$. Suppose $x \diamond s \in V$. Similarly, $k \in \{4, 5, 6, 7\}$. Thus, either

$$(x \diamond S_2) \cap U = \emptyset \quad \text{or} \quad (x \diamond S_2) \cap V = \emptyset. \tag{4.2.12}$$

We claim that $\{\delta_a, \delta_b\}$ is a DLTC for $M(\beta S_2)$. Suppose, on the contrary, that there is a non-zero measure $\mu \in M(S^*)$ such that $\mu \,\Box\, \delta_a = \mu \diamond \delta_a$ and $\mu \,\Box\, \delta_b = \mu \diamond \delta_b$. Without loss of generality, suppose $\|\mu\| = 1$. Since a, b are right cancellable in βS_2, it follows from Theorem 3.1.13 that $\|\mu \,\Box\, \delta_a\| = \|\mu \,\Box\, \delta_b\| = \|\mu\| = 1$. By definition of norm,

$$\langle \mu \,\Box\, \delta_a, \lambda_a \rangle > \|\mu\| - \frac{1}{2} = \frac{1}{2} \quad \text{and} \quad \langle \mu \,\Box\, \delta_b, \lambda_b \rangle > \|\mu\| - \frac{1}{2} = \frac{1}{2}. \tag{4.2.13}$$

Since $S_2 \,\Box\, a \subset \overline{U}$ and $S_2 \,\Box\, b \subset \overline{V}$, we have $\beta S_2 \,\Box\, a \subset \overline{U}$ and $\beta S_2 \,\Box\, b \subset \overline{V}$, and so, $\mathrm{supp}(\mu \,\Box\, \delta_a) \subset \overline{U}$ and $\mathrm{supp}(\mu \,\Box\, \delta_b) \subset \overline{V}$. Since $\overline{U} \cap \overline{V} = \emptyset$, by replacing λ_a with $\lambda_a \cdot \chi_{\overline{U}}$ and λ_b with $\lambda_b \cdot \chi_{\overline{V}}$, we may suppose that λ_a and λ_b vanish outside $\overline{U}$ and $\overline{V}$, respectively. Let $f = \sum_{i=1}^m \alpha_i \delta_{x_i}$, where $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ such that $\sum_{i=1}^m |\alpha_i| = 1$, and $x_1, \ldots, x_m \in S_2^*$. Consider $i \in \mathbb{N}_m$. According to (4.2.12), either

$$\langle x_i \diamond s, \lambda_a \rangle = 0 \quad (s \in S_2) \quad \text{or} \quad \langle x_i \diamond s, \lambda_b \rangle = 0 \quad (s \in S_2).$$

Thus, either

$$|\langle f \diamond \delta_s, \lambda_a \rangle| \le \frac{1}{2} \quad (s \in S_2) \quad \text{or} \quad |\langle f \diamond \delta_s, \lambda_b \rangle| \le \frac{1}{2} \quad (s \in S_2),$$

and hence either

$$|\langle \mu \diamond \delta_s, \lambda_a \rangle| \le \frac{1}{2} \quad (s \in S_2) \quad \text{or} \quad |\langle \mu \diamond \delta_s, \lambda_b \rangle| \le \frac{1}{2} \quad (s \in S_2),$$

Since the mapping $x \mapsto \nu \diamond x$, $\beta S_2 \to M(S_2^*)$, is continuous, it follows that either

$$|\langle \mu \diamond \delta_x, \lambda_a \rangle| \le \frac{1}{2} \quad (x \in \beta S_2) \quad \text{or} \quad |\langle \mu \diamond \delta_x, \lambda_b \rangle| \le \frac{1}{2} \quad (x \in \beta S_2), \tag{4.2.14}$$

From (4.2.13) and (4.2.14), it follows that either

$$\mu \,\square\, \delta_a \neq \mu \diamond \delta_a \quad \text{or} \quad \mu \,\square\, \delta_b \neq \mu \diamond \delta_b.$$

This is a contradiction. This implies that $\mu = 0$, and therefore, $\{\delta_a, \delta_b\}$ is a DLTC for $M(\beta S_2)$. Moreover, it follows that $\ell^1(S_2)$ is strongly left Arens irregular.

We can repeat the argument by replacing P with $Q = \{e, b, b^2, \cdots\}$ to show that $\ell^1(S_2)$ is strongly right Arens irregular and $M(\beta S_2)$ has a DRTC consisting of two points. $\square$

Finally, we finish this chapter by discussing Arens regularity of $\ell^1(S_{1,1})$.

Theorem 4.2.3. *$\ell^1(S_{1,1})$ is strongly Arens irregular. Moreover, there is a DLTC set (DRTC set) consists of two points for $M(\beta S_{1,1})$.*

Proof. First, we represent $S_{1,1}$ as a sequence $(s_n)_{n\in\mathbb{Z}^+}$, where $s_0 = e$. Let $P = \{e, a, a^2, \ldots\}$. Each element in P is right cancellable. We construct a sequence (t_n) within P by induction. Set $t_0 = s_0 = e$. For any $n \in \mathbb{Z}^+$, assume $t_0, \ldots, t_n$ have been specified in P. Define $T_n = \{s_0, \ldots, s_n, t_1, \ldots, t_n\}$. Since $S_{1,1}$ is weakly cancellative, the sets $T_n^{-1}[T_n^2]$, $[T_n^2]T_n^{-1}$, $T_n^{-1}[T_n]^2$, and $[T_n]^2T_n^{-1}$ are finite. We choose t_{n+1} in P such that

$$\max\{t : t \in (T_n^{-1}[T_n^2]) \cup ([T_n^2]T_n^{-1}) \cup (T_n^{-1}[T_n]^2) \cup ([T_n]^2T_n^{-1})\} \prec t_{n+1}.$$

The inductive construction proceeds. It should be noted that the sequence (t_n) has to the following conditions:

(i) $sT_n \cap [T_n] = \emptyset$ whenever $s \in S_{1,1}$ with $t_{n+1} \preceq s$;

(ii) $st_{n+1} \neq tt_{n+1}$ whenever $s, t \in T_n$ with $s \neq t$;

(iii) $rs \prec tt_{n+1}$ whenever $r, s, t \in [T_n]$.

Observe that $s_n \preceq t_n$ $(n \in \mathbb{Z}^+)$, thus $\cup\{[t_n] : n \in \mathbb{Z}^+\} = S_{1,1}$. Also, $S_{1,1}^* = (S_{1,1} \setminus [t_k])^*$ for each $k \in \mathbb{N}$.

Define $\varphi : S_{1,1} \to \mathbb{Z}^+$ by setting

$$\varphi(s) = \min\{n \in \mathbb{Z}^+ : s \in [t_n]\} \quad (s \in S_{1,1}).$$

Assume $\varphi(s) = m \in \mathbb{N}$. According to condition (iii), if $n > m$, then $t_{n-1} \prec st_n \prec t_{n+1}$, therefore,

$$\varphi(st_n) \in \{n, n+1\} \quad (n > m). \tag{4.2.15}$$

Now suppose $m \geq 2$ and $k \leq m-2$. By condition (i), $t_{m-2} \prec ss_k$ because $sT_{m-2} \cap [T_{m-2}] = \emptyset$. Additionally, by condition (iii), $ss_k \prec t_{m+1}$ since $s \in T_m$. Thus, $\varphi(ss_k) \in \{m-1, m, m+1\}$, leading to

$$\varphi(ss_k) \in \{m-1, m, m+1\} \quad (m \geq k+2). \tag{4.2.16}$$

For $s \in S_{1,1}$, define $\gamma(s) \equiv \varphi(s) \pmod 8$. Then $\gamma : S_{1,1} \to \mathbb{Z}_8$ can be extended continuously to a map $\gamma : \beta S_{1,1} \to \mathbb{Z}_8$. From (4.2.16), for each $x \in S_{1,1}^*$ and $s \in S_{1,1}$, we get

$$\gamma(x \mathbin{\square} s) = \gamma(x \diamond s) \in \{\gamma(x) - 1, \gamma(x), \gamma(x) + 1\} \subset \mathbb{Z}_8. \tag{4.2.17}$$

Define

$$A = \{t_n : \gamma(t_n) = 1\}, \quad B = \{t_n : \gamma(t_n) = 5\},$$

so A and B are infinite subsets of $S_{1,1}$. Then define

$$U = \{s \in S_{1,1} : \gamma(s) \in \{1, 2\}\}, \quad V = \{s \in S_{1,1} : \gamma(s) \in \{5, 6\}\},$$

so that $A \subset U$ and $B \subset V$.

Choose $a \in A^*$ and $b \in B^*$. We assert that a is right cancellable in $\beta S_{1,1}$. To demonstrate this, let u_1 and u_2 be distinct points in $\beta S_{1,1}$, and select disjoint sets N_1 and N_2 in $S_{1,1}$ with $u_j \in N_j$ for $j = 1, 2$. For $j = 1, 2$, define

$$Y_j = \{s_m t_n : s_m \in N_j, t_n \in A, m < n\},$$

so that $u_j \mathbin{\square} a \in \overline{Y_j}$. Choose $m_1, m_2, n_1, n_2 \in \mathbb{N}$ with $m_1 < n_1$, $m_2 < n_2$, and $m_1 \neq m_2$. Then $s_{m_1} t_{n_1} \neq s_{m_2} t_{n_2}$; this is true for $n_1 < n_2$ by condition (iii) and for $n_1 = n_2$ by condition (ii). It follows that $Y_1 \cap Y_2 = \emptyset$, and so $u_1 \mathbin{\square} a \neq u_2 \mathbin{\square} a$, as required. Similarly, b is right cancellable in $\beta S_{1,1}$.

From (4.2.15), we have $S_{1,1} \mathbin{\square} a \subset \overline{U}$ and $S_{1,1} \mathbin{\square} b \subset \overline{V}$.

Let $x \in S_{1,1}^*$, set $k = \gamma(x)$, and take $s \in S_{1,1}$. Suppose $x \diamond s \in U$. Then, by (4.2.17), $k \in \{0, 1, 2, 3\}$. Suppose $x \diamond s \in V$. Similarly, $k \in \{4, 5, 6, 7\}$. Thus, either

$$(x \diamond S_{1,1}) \cap U = \emptyset \quad \text{or} \quad (x \diamond S_{1,1}) \cap V = \emptyset. \tag{4.2.18}$$

We claim that $\{\delta_a, \delta_b\}$ is a DLTC for $M(\beta S_{1,1})$. Suppose, on the contrary, that there is a non-zero measure $\mu \in M(S^*)$ such that $\mu \,\Box\, \delta_a = \mu \diamond \delta_a$ and $\mu \,\Box\, \delta_b = \mu \diamond \delta_b$. Without loss of generality, suppose $\|\mu\| = 1$. Since a, b are right cancellable in $\beta S_{1,1}$, it follows from Theorem 3.1.13 that $\|\mu \,\Box\, \delta_a\| = \|\mu \,\Box\, \delta_b\| = \|\mu\| = 1$. By definition of norm,

$$\langle \mu \,\Box\, \delta_a, \lambda_a \rangle > \|\mu\| - \frac{1}{2} = \frac{1}{2} \quad \text{and} \quad \langle \mu \,\Box\, \delta_b, \lambda_b \rangle > \|\mu\| - \frac{1}{2} = \frac{1}{2}. \tag{4.2.19}$$

Since $S_{1,1} \,\Box\, a \subset \overline{U}$ and $S_{1,1} \,\Box\, b \subset \overline{V}$, we have $\beta S_{1,1} \,\Box\, a \subset \overline{U}$ and $\beta S_{1,1} \,\Box\, b \subset \overline{V}$, and so, $\operatorname{supp}(\mu \,\Box\, \delta_a) \subset \overline{U}$ and $\operatorname{supp}(\mu \,\Box\, \delta_b) \subset \overline{V}$. Since $\overline{U} \cap \overline{V} = \emptyset$, by replacing λ_a with $\lambda_a \cdot \chi_{\overline{U}}$ and λ_b with $\lambda_b \cdot \chi_{\overline{V}}$, we may suppose that λ_a and λ_b vanish outside $\overline{U}$ and $\overline{V}$, respectively. Let $f = \sum_{i=1}^m \alpha_i \delta_{x_i}$, where $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ such that $\sum_{i=1}^m |\alpha_i| = 1$, and $x_1, \ldots, x_m \in S_{1,1}^*$. Consider $i \in \mathbb{N}_m$. According to (4.2.18), either

$$\langle x_i \diamond s, \lambda_a \rangle = 0 \quad (s \in S_{1,1}) \quad \text{or} \quad \langle x_i \diamond s, \lambda_b \rangle = 0 \quad (s \in S_{1,1}).$$

Thus, either

$$|\langle f \diamond \delta_s, \lambda_a \rangle| \leq \frac{1}{2} \quad (s \in S_{1,1}) \quad \text{or} \quad |\langle f \diamond \delta_s, \lambda_b \rangle| \leq \frac{1}{2} \quad (s \in S_{1,1}),$$

and hence either

$$|\langle \mu \diamond \delta_s, \lambda_a \rangle| \leq \frac{1}{2} \quad (s \in S_{1,1}) \quad \text{or} \quad |\langle \mu \diamond \delta_s, \lambda_b \rangle| \leq \frac{1}{2} \quad (s \in S_{1,1}),$$

Since the mapping $x \mapsto \nu \diamond x$, $\beta S_{1,1} \to M(S_{1,1}^*)$, is continuous, it follows that either

$$|\langle \mu \diamond \delta_x, \lambda_a \rangle| \leq \frac{1}{2} \quad (x \in \beta S_{1,1}) \quad \text{or} \quad |\langle \mu \diamond \delta_x, \lambda_b \rangle| \leq \frac{1}{2} \quad (x \in \beta S_{1,1}), \tag{4.2.20}$$

From (4.2.19) and (4.2.20), it follows that either

$$\mu \,\Box\, \delta_a \neq \mu \diamond \delta_a \quad \text{or} \quad \mu \,\Box\, \delta_b \neq \mu \diamond \delta_b.$$

This is a contradiction. This implies that $\mu = 0$, and therefore, $\{\delta_a, \delta_b\}$ is a DLTC for $M(\beta S_{1,1})$. Moreover, it follows that $\ell^1(S_{1,1})$ is strongly left Arens irregular.

One may replicate the argument by replacing P with $Q = \{e, c, c^2, \cdots\}$ to demonstrate that $\ell^1(S_{1,1})$ is strongly right Arens irregular and $M(\beta S_{1,1})$ has a DRTC consisting of two points. $\square$

5

Arens regularity of inverse semigroup algebras

In this Chapter, we will give a characterization for Arens regularity of inverse semigroup algebras under the extra condition that $E(S)$ is uniformly locally finite. Using this, we will get a sufficient and necessary condition for the Brandt semigroup algebra to be Arens regular. The results of this chapter are also our original contribution.

5.1 Uniformly locally finite inverse semigroups

Ramsden in [31] showed that semigroup algebra of a uniformly locally finite inverse semigroup can be identified by a direct sum of some matrix algebras. We state his theorem here without proof.

Theorem 5.1.1 ([31, Theorem 2.18]). *Let S be a uniformly locally finite inverse semigroup with $\mathcal{D}$-classes $\{D_\lambda : \lambda \in \Lambda\}$. For each λ, take an idempotent $p_\lambda \in D_\lambda$. Then there is an isomorphism between Banach algebras so that*

$$\ell^1(S) \cong \ell^1\text{-}\bigoplus \left\{\mathbb{M}_{E(D_\lambda)}\left(\ell^1(G_{p_\lambda})\right) : \lambda \in \Lambda\right\}.$$

Theorem 5.1.2 ([11, Theorem A.4.4]). *Let S be a semigroup. Then $\ell^1(S)$ is weakly sequentially complete.*

Theorem 5.1.3. *Let S be an inverse semigroup such that $E(S)$ is uniformly locally finite. Then $\ell^1(S)$ is Arens regular if and only if the following conditions (i) and (ii) hold.*

(i) for every $p \in E(S)$, the maximal subgroup G_p is finite; and

(ii) each $\mathcal{D}$-classes has finitely many idempotents.

Proof. Let (i) and (ii) hold. Since G_p is finite, Theorem 2.1.14 imlies that $\ell^1(G_p)$ is Arens regular. Since each $\mathcal{D}$-class has finitely many idempotents, $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is Arens regular for every $\lambda \in \Lambda$ by Theorem 2.1.10. It follows from Theorem 2.1.7 that $\ell^1\text{-}\bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\}$ is Arens regular. Since $E(S)$ is uniformly locally finite, from Theorem 5.1.1 we have $\ell^1(S) \cong \ell^1\text{-}\bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\}$, and therefore, $\ell^1(S)$ is Arens regular.

Conversely, since $E(S)$ is uniformly locally finite, by Theorem 5.1.1 we have $\ell^1(S) \cong \ell^1\text{-}\bigoplus\{\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\}$. Therefore, by Theorem 2.1.7 $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is Arens regular for every $\lambda \in \Lambda$. From Lemma 2.1.11 we have $\mathbb{M}_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) \cong \mathbb{M}_{E(D_\lambda)}(\mathbb{C})\hat{\otimes}\ell^1(G_{p_\lambda})$. Theorem 2.1.13 implies that both $\mathbb{M}_{E(D_\lambda)}(\mathbb{C})$ and $\ell^1(G_{p_\lambda})$ are Arens regular. It follows from Theorem 2.1.10 that $E(D_\lambda)$ is finite, and therefore, (ii) holds. Moreover, Theorem 2.1.14 implies that G_{p_λ} is finite, so (i) holds as well. $\square$

Example 5.1.4. Let G be a group and let I be a non-empty set. Define the set

$$M^0(G, I) = \{(g)_{ij} : g \in G,\ i, j \in I\} \cup \{0\},$$

where $(g)_{ij}$ denotes the $I \times I$ matrix with entry $g \in G$ in the (i, j) position and zero elsewhere. Then $M^0(G, I)$, with the multiplication defined by

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases} \qquad (g, h \in G,\ i, j, k, l \in I),$$

forms an inverse semigroup with $(g)^*_{ij} = (g^{-1})_{ji}$, known as the Brandt semigroup over G with index set I. In the case that I is finite the Brandt semigroup become a particular case of Rees matrix semigroups. Let $S = M^0(G, I)$ It can be readily verified that the set of idempotents $E(S)$ is the union

$$\{(e_G)_{ii} : i \in I\} \cup \{0\}.$$

As $E(S)$ is a semilattice, for every $x \in E(S)$, the principal ideal generated by x, is given by

$$(x] = \{y \in E(S) : y = yx\} = E(S)x = \begin{cases} \{0, x\} & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}.$$

Therefore, $(E(S), \leq)$ is uniformly locally finite. On the other hand, one can easily verify that $G_p \cong G$ for every $p \in E(S)$. Besides, for non-zero elements $s = (g)_{ij}$ and $t = (f)_{kl}$ of S we have $s^*s = (e_G)_{jj}$ and $t^*t = (e_G)_{ll}$. Hence by choosing $x = (e_G)_{jl}$ we see that $s^*s = xx^*$ and $t^*t = x^*x$, so s and t belong to the same $\mathcal{D}$-class. Thus S has two $\mathcal{D}$-classes $\mathcal{D}_0 = \{0\}$ and $\mathcal{D}_1 = S \setminus \{0\}$. It follows that $|E(\mathcal{D}_1)| = |E(S) \setminus \{0\}| = |I|$. Now, Theorem 5.1.3 implies that $\ell^1(S)$ is Arens regular if and only if G and I are finite.

5.2 A problem for further investigation

Our discussion in Section 5.1 shows Arens regularity of inverse semigroup algebras is closely related to the behavior of the idempotent elements within the semigroup. Certain finiteness conditions on the set of idempotents $E(S)$ play a crucial role in determining whether $\ell^1(S)$ is Arens regular. However, the precise relationship between these finiteness conditions and Arens regularity remains a question. In this section, we try to reveal further the conditions under which $\ell^1(S)$ is Arens regular and propose a conjecture that characterizes this regularity in terms of specific finiteness properties of the inverse semigroup S.

First, we establish a key lemma, which demonstrates that if $\ell^1(S)$ is Arens regular, then the set of idempotents $E(S)$ is locally finite and can be expressed as a union of certain maximal ideals.

Lemma 5.2.1. *Let S be an inverse semigroup. If $\ell^1(S)$ is Arens regular, then $E(S)$ is locally finite, and $E(S) = \bigcup_{p \in M}(p]$, where M is the set of maximal elements of $E(S)$.*

Proof. We begin with defining a map for the inverse semigroup S. Let $\psi : S \longrightarrow E(S)$ be defined by $\psi(s) = ss^*$.

This map ψ can be extended to a bounded epimorphism $\Psi : \ell^1(S) \longrightarrow \ell^1(E(S))$ by defining $\Psi(\delta_s) = \delta_{\psi(s)}$ for each $s \in S$.

Since $\ell^1(S)$ is assumed to be Arens regular, it follows that $\ell^1(E(S))$ must also be Arens regular. Therefore, we may assume that S is a semilattice.

Let S be a semilattice, and assume that $\ell^1(S)$ is Arens regular. For any element $p \in S$, define $S_p = (p]$, the set of elements less than or equal to p. This set S_p forms an ideal in S so $\ell^1(S_p)$ is also Arens regular.

However, by Proposition 2.2.12, we know that $\ell^1(S_p)$ has an identity. From Theorem 2.1.12, we conclude that S_p must be finite. Therefore, we can conclude that S is locally finite, as every subset S_p is finite.

Next, we claim that every chain in S has a maximal element. Assume toward a contradiction, that there exists an infinite chain $P = \{p_1 < p_2 < \cdots\}$ in S, where each p_i is strictly less than p_{i+1}.

In this case, the chain P forms a subsemigroup of S, and we can define an isomorphism:

$$\psi : \ell^1(\mathbb{N}_\wedge) \longrightarrow \ell^1(P), \quad \delta_k \mapsto \delta_{p_k}.$$

However, it is known that $\ell^1(\mathbb{N}_\wedge)$ is not Arens regular. Therefore, $\ell^1(P)$ cannot be Arens regular either, which contradicts our original assumption that $\ell^1(S)$ is Arens regular. Hence, the chain P must be finite, and every chain in S has a maximal element.

Since every chain in S has a maximal element, we can conclude that $E(S) = \bigcup_{p \in M}(p]$, where M is the set of maximal elements of $E(S)$. $\square$

Question. For an inverse semigroup S , if $\ell^1(S)$ is Arens regular, is $E(S)$ uniformly locally finite?

If the question has a positive answer, then we can characterize Arens regularity of $\ell^1(S)$ as follows.

Conjecture. Let S be an inverse semigroup. Then $\ell^1(S)$ is Arens regular if and only if

(i) $E(S)$ is uniformly locally finite,

(ii) for every $p \in E(S)$, the maximal subgroup G_p is finite; and

(iii) each $\mathcal{D}$-class has finitely many idempotents.

Conclusion

In this thesis, we studied the topological centres and Arens regularity of semigroups and their algebras. Our main focus was on investigating Arens irregularity of specific semigroups such as the bicyclic semigroup S_1 and partially bicyclic semigroups S_2 and $S_{1,1}$.

We proved that the semigroups S_1, S_2, and $S_{1,1}$ and their algebras are strongly Arens irregular. Moreover, we showed that the left topological centers of these semigroups' Stone–Čech compactifications are determined by two-point sets. Additionally, the study shows that the topological centers of $M(\beta S_1)$, $M(\beta S_2)$, and $M(\beta S_{1,1})$, are determined by two-point sets as well.

Moreover, in this thesis, we investigated Arens regularity of inverse semigroup algebras and we proved that for an inverse semigroup S, if $E(S)$ is uniformly locally finite, then the algebra $\ell^1(S)$ is Arens regular if and only if the maximal subgroups of S are finite and each $\mathcal{D}$-class contains a finite number of idempotents.

The results presented in this thesis provide insight for further study of Arens regularity of other semigroup algebras.

References

[1] Askari-Sayah, M.; Pourabbas, A.; Sahami, A. *Johnson pseudo-contractibility of certain semigroup algebras. II.* Semigroup Forum **104** (2022), no. 1, 18-27.

[2] Arens. R, *Operations induced in function classes*, Monatsh Math., **55** (1951), 1–19.

[3] Arens. R, *The adjoint of a bilinear operation*, Proc.American Math. Soc., **2** (1951), 839–848.

[4] Arikan, N. *Arens regularity and reflexivity.* Quart. J. Math. Oxford Ser. (2) **32** (1981), no. 128, 383–388.

[5] J. W. Baker and A. Rejali, *On the Arens regularity of weighted convolution algebras*, J. London Math. Soc. (2), **40** (1989), 535–546.

[6] M. L. Bami, *The topological centers of* $LUC(S)^*$ *and* $M_a(S)^{**}$ *of certain foundation semigroups*, Glasgow Math. Journal, **42** (2000), 335–343.

[7] S. Bowling and J. Duncan, *First order cohomology of Banach semigroup algebras*, Semigroup Forum **56** (1998), 130-145.

[8] R. J. Butcher, Thesis, University of Sheffield, 1975.

[9] Civin, P; Yood, B. *The second conjugate space of a Banach algebra as an algebra.* Pacific J. Math. **11** (1961), 847–870.

[10] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Volumes I and II, American Math. Soc., Providence, Rhode Island, 1961 and 1967.

[11] Dales, H. G., *Banach algebras and automatic continuity*, London Math. Society Monographs, Volume **24**, Clarendon Press, Oxford, 2000.

[12] Dales, H. G.; Lau, A. T.-M.; Strauss, D. *Banach algebras on semigroups and on their compactifications.* Mem. Amer. Math. Soc.**205** (2010), no. 966, vi+165 pp.

[13] Dales, H. G., Strauss, D. *Arens regularity for totally ordered semigroups*, Semigroup Forum **105** (2022), no. 1, 172–190.

[14] Duncan, J.; Hosseiniun, S. A. R. *The second dual of a Banach algebra.* Proc. Roy. Soc. Edinburgh Sect. A **84** (1979), no. 3-4, 309–325.

[15] Duncan, J.; Namioka, I. *Amenability of inverse semigroups and their semigroup algebras.* Proc. Roy. Soc. Edinburgh Sect. A **80** (1978), no. 3-4, 309–321.

[16] J. Duncan and A. L. T. Paterson, *Amenability for discrete convolution semigroup algebras*, Math. Scand. **66** (1990), 141-146.

[17] *Banach algebra structure and amenability of a class of matrix algebras with applications.* J. Funct. Anal. **161** (1999), no. 2, 364–383.

[18] G. H. Esslamzadeh, *Duals and topological center of a class of matrix algebras with applications*, Proc. American Math. Soc., **128** (2000), 3493–3503.

[19] Lau, A. Amenability of semigroups, *The analytic and topological theory of semigroups* (ed. K. H. Hoffmann, J. D. Lawson, and J. S. Pym), de Gruyter, Berlin, 1990, 313-334

[20] Lau, A. *Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups*, Math. Proc. Cambridge Philosophical Soc.,**99** (1986), 273–283.

[21] Lau, A. Pym, J. *The topological centre of a compactification of a locally compact group*, Math. Z. **219** (1995), no. 4, 567–579.

[22] Lau, A. Zhang, Y. *Fixed point properties of semigroups of non-expansive mappings.* J. Funct. Anal. **254** (2008), no. 10, 2534–2554.

[23] F. Gheorghe and Y. Zhang, *A note on the approximate amenability of semigroup algebras*, Semigroup Forum, **79** (2009), 249-254.

[24] L. Gillman and N. Jerison, *Rings of continuous functions*, van Nostrand, Princeton, New Jersey, 1960.

[25] N. Hindman and D. Strauss, *Algebra in the Stone–Čechcompactification*, Expositions in Mathematics **27**, de Gruyter, Berlin, 1998.

[26] Howie, J.M. *Fundamentals of Semigroup Theory*. London Mathematical Society Monographs, vol. **12**. Clarendon Press, Oxford (1995).

[27] R. Hsu, Topics on weakly almost periodic functions, *Ph.D. Thesis*, SUNY at Buffalo, 1985.

[28] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., **127** (1972).

[29] D. J. Parsons, *The centre of the second dual of a commutative semigroup algebra*, Math. Proc. Cambridge Philosophical Soc.,**95** (1984), 71–92.

[30] Protasov, I. V.; Pym, J. S. *Continuity of multiplication in the largest compactification of a locally compact group*, Bull. London Math. Soc. **33** (2001), no. 3, 279–282.

[31] Ramsden, Paul. *Biflatness of semigroup algebras*. Semigroup Forum **79** (2009), no. 3, 515-530.

[32] Ruppert, W. *On semigroup compactifications of topological groups*, Proc. Roy. Irish Acad. A **79A**, 179-200 (1979)

[33] Strauss, D. *Semigroup structures on* $\beta\mathbb{N}$, Semigroup Forum **44** (1992), no. 2, 238–244.

[34] Ülger, A. *Arens regularity of the algebra* $A\hat{\otimes}B$. Trans. Amer. Math. Soc. **305** (1988), no. 2, 623–639.

[35] Ülger, A. *Arens regularity of weakly sequentially complete Banach algebras.* Proc. Amer. Math. Soc. **127** (1999), no. 11, 3221-3227.

[36] N. J. Young, *The irregularity of multiplication in group algebras*, Quarterly J. Math. Oxford (2), **24** (1973), 59–62.