Integrability of magnetic geodesic flows

by

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Abstract

This thesis investigates some aspects of the integrability problem of a Hamiltonian system. The Hamiltonian system with Hamiltonian function

$$H = \sum_{i,j=1}^{n} \frac{1}{2} g^{ij}(x_1, \ldots, x_n) p_i p_j,$$

describes the geodesic flow of a Riemannian metric $ds^2 = \sum_{i,j=1}^{n} g_{ij}(x_1, \ldots, x_n) dx^i dx^j$ on an n-dimensional manifold. Some results from the research article, *Polynomials integrals of magnetic geodesic flows on the 2-torus on several energy levels* [3], are studied. In particular, a complex structure on the 2-torus is constructed to prove that if the geodesic flow with non-zero magnetic field on the 2-torus admits an additional cubic-in-momenta first integral on two different energy levels, then the magnetic field and the metric are functions of one variable.

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Contents

Introduction **1**

Motivation 1

Literature review 3

Outline of the thesis 5

1 Mathematical Background **7**

1.1 Integrable hamiltonian systems 7

1.2 The Liouville-Arnold theorem 16

1.3 Some definitions from complex analysis 25

2 Integrable Magnetic Geodesic Flows on the 2-torus **26**

2.1 Polynomial integrals of magnetic geodesic flows 26

2.1.1 Introduction 27

2.1.2 Integrability on several energy levels 29

2.1.3 First integral of degree three 32

3 Integrability on Several Energy Levels by the Complex Variables Approach **38**

3.1 Polynomial in momenta first integral of degree three 38

4 Conclusion **51**

Bibliography **53**

Introduction

Motivation

From the seventeen to the nineteen century, one of the main mathematical tasks was to solve mechanical and astronomical problems. Such problems are mostly modelled by *Hamilton's equations* (see Definition (1.2)). Therefore, solvable hamiltonian systems have significant importance.

Consider a point moving by inertia along a two-dimensional surface lying in three dimensional-space. The trajectory of this point is called a geodesic on the surface. Geodesics are described by a system of two second-order differential equations. These equations can be written as Hamilton's equations. Let M be a smooth manifold and $g = (g_{ij})$ be a Riemannian metric defined on M. In local coordinate system $q_1, \ldots, q_n$, pass from velocities $\dot{q}_i$ to momenta p_j by using the formula $p_j = g_{ij}\dot{q}_i$. Then the equations of geodesics in the new coordinates q_i, p_i $(= 1, \ldots, n)$, can be written as:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{1}$$

where H(the hamiltonian) is interpreted as the kinetic energy and is given by:

$$H = \frac{1}{2}\sum_{i,j=1}^{n} g^{ij} p_i p_j = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}\dot{q}_i\dot{q}_j.$$

Here g^{ij} are the coefficients of the tensor inverse to the metric tensor. This system of equations is called a *hamiltonian system* and is globally defined on the cotangent bundle $T^\star M$ (with the standard symplectic form $\omega = \sum dp_i \wedge dq_i$ (see Definitions (1.10) and (1.8)). The *geodesic flow* of the Riemannian manifold (M, g) is the one-parameter group of diffeomorphisms defined by the above differential equations.

In particular, we consider the global behaviour of geodesic flows on closed Riemannian manifolds. The question then arises whether there exist integrable geodesic flows on a given manifold or not? We can answer this question either by finding a topological property of the manifold that obstructs integrability or explicitly construct a metric on the manifold with an integrable geodesic flow. Examples show that the character of first integrals is significant to the solution of this problem. Most restrictive assumption is that the integrals are polynomial in momenta from past forty years. [10, 17]

The previous paragraph introduced the notion of integrability but let us introduce this notion in more depth. Historically, the work of *Liouville* on *hamiltonian mechanics* introduced the concept of an integrable system. The definition of a *Liouville integrable system* can be understood by developing *hamiltonian mechanics* on *symplectic* and *Poisson* manifolds. Consider a particle with the position coordinates $(q_1, \ldots, q_n)$ and the momentum coordinates $(p_1, \ldots, p_n)$, and a smooth real-valued *hamiltonian function* $H : \mathbb{R}^{2n} \to \mathbb{R}$ in the *phase space* $\mathbb{R}^{2n}$. This *hamiltonian dynamical system* is defined by the *Hamilton's equations* (1). Take an example of such system by considering the harmonic oscillator in one-dimension. For this system the coordinates of phase space $\mathbb{R}^2$ are (q, p) with hamiltonian $H = \frac{1}{2}(p^2 + q^2)$. The Hamilton's equations are:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = p, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -q,$$

which is equivalent to equation of motion of harmonic oscillator:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + q = 0.$$

Broadly speaking, for a Liouville integrable system on a phase space $M = \mathbb{R}^{2n}$ there exists n functionally independent conserved quantities F_i. With the hypothesis that F_i are independent on M, the Liouville-Arnold theorem states that every connected component of the phase space is diffeomorphic to the n-dimensional torus. Additionally, the system's time evolution is linear on this torus, so that the angular coordinates φ_i satisfy $\frac{d\varphi_i}{dt} = \omega_i$ (ω_i are frequencies of motion on a torus, see Theorem (1.16)). To see that the example of the one-dimensional harmonic oscillator is a Liouville integrable hamiltonian system, transformed this system into polar coordinates (r, θ) defined on the phase space $\mathbb{R}^2$. Let $q = r\sin\theta$ and $p = r\cos\theta$. The Hamilton's equations become

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1.$$

Note that the hamiltonian itself gives the only necessary conserved quantity. For a fixed level set $H(p, q) = E$, we see that the phase space is fibred into the 1-dimensional tori (circles), $r = \sqrt{2E}$. Also the system's time evolution is linear in the angular coordinate θ on these tori. [27]

Literature review

The examples of completely integrable hamiltonian systems showed that first integrals are polynomial in momenta. In 1979, V. I. Arnold in his book [4] presented that hamiltonian dynamical system corresponds to one-parameter groups of symmetries. He also presented in this book the famous Noether's theorem that states that, If the system (M, L) admits the one-parameter group of diffeomorphisms $h^s : M \to M$. $s \in \mathbb{R}$, then the Lagrangian system of equations corresponding to L has a first integral $I : TM \to \mathbb{R}$. In [5], M. L. Bialy considered a hamiltonian system on a two-dimensional torus with hamiltonian $H = \frac{p_1^2+p_2^2}{2} + V(x, y,)$ where V is a $2\pi-$periodic functions in both variables. For this flow the author found a potential V such that there exist a polynomial in momenta first integral F_n of degree n with $2\pi-$periodic coefficients. In [6], M. L. Bialy proved that for any non-zero magnetic field there always exist orbits of the motion of a charge on a conformally flat Riemannian torus which have conjugate points. In 1995, V. V. Kozlov and N. V. Denisova discussed the polynomial integrals of geodesic flows on a two-dimensional torus [21] . The authors proved the conjecture that the degree of an additional irreducible polynomial for such a flow on torus (genus $p{=}1$) can not exceed $4-2p$ (two). A. V. Bolsinov and B. Jovanović in [11], described a class of completely integrable G-invariant magnetic geodesic flows on coadjoint orbits. Fourth-degree polynomial integrals of a natural mechanical system on a two-dimensional torus is studied by S. V. Agapov and D. N. Aleksandrov in [1].

Different problems related to integrable geodesic flows on the 2-torus have been studied in [7–9] by the authors A. E. Mironov and M. L. Bialy. Some other problems related to integrals of geodesic flows on a 2-torus are studied by I. A. Taimanov in [28, 29]. In 2017, S. V. Agapov, M. Bialy and A. E. Mironov, proved in [2] that some integrability problems can be reduced to a question of finding smooth periodic solutions of a semi-Hamiltonian system of a quasi-linear PDEs. One of the two main results of this paper was that on the 2-torus there exist real analytic Riemannian metrics which are arbitrary close to Liouville metrics and non-zero analytic magnetic fields such that magnetic geodesic flows on the energy level $\{\mathrm{H}{=}\frac{1}{2}\}$ have a quadratic in momenta first integral. The other result stated that if magnetic geodesic flow with Riemannian metric $ds^2 = \Lambda(dx^2 + dy^2)$ admits quadratic in momenta first integral F_2 on all energy levels then metric and magnetic field are functions of one variable and there exists another integral F_1 linear in momenta such that F_2 can be written in terms of F_1 and H.

In [13], D. I. Efimov considered the magnetic geodesic flow on a simply connected homogeneous symplectic manifold, that is on an orbit of the coadjoint representation of a semi-simple compact Lie group. This flow is defined by the Riemannian metric which is induced by the Killing form and the magnetic field which is the Kirillov symplectic form. The author proved that this flow is integrable in the noncommutative sense. In [22], A. E. Mironov studied polynomial integrals of a mechanical system on a two-dimensional torus. Together with some other results it was shown that if a natural mechanical system defined on a two-dimensional torus and having a real analytic potential possesses an integral of momenta of degree five then there also exists an integral of the first degree in momenta for this system. In 2019, S. V. Agapov and A. A. Valyuzhenich discussed polynomial integrals of magnetic geodesic flows on the 2-torus on several energy levels [3]. Some results of this article is studied in detail in this thesis. The results of this paper are correct but there is a technical flaw in the proof. We use a different approach to arrive at the same results but avoiding such flaws in the proof.

Outline of the thesis

This thesis consists of following four chapters.

Chapter 1 contains the concepts that are important in understanding this thesis. Terms like, symplectic manifold, Poisson bracket and integrable system together with some other definitions are introduced here. The proof of Liouville-Arnold theorem is given in this chapter.

Chapter 2 summarizes the research article about the polynomial integrals of magnetic geodesic flows on the 2-torus. The proof for the case of an additional polynomial in momenta first integral of degree 3 is analyzed in detail.

Chapter 3 presents a different approach to proof the main theorem of Chapter 2. The technical mistake of the proof in Chapter 2 is removed using this method.

Chapter 4 compares the two approaches used for solving the integrability problem of the article cited in the Chapter 2. Suggestions for the further research work are presented in this chapter.

All the references used in this dissertation are listed in the bibliography.

1

Mathematical Background

In this chapter, we present some basic definitions and results related to integrable geodesic flows. These definitions and examples are mostly taken from the book [4], unless otherwise stated.

This chapter consists of three sections. The first section includes some important definitions with examples that are helpful in understanding hamiltonian vector fields and integrable systems. The second section presents the statement and proof of the Liouville-Arnold theorem. The third section contains some results from complex analysis.

1.1 Integrable hamiltonian systems

Definition 1.1 (Vector field [18]). A *vector field* is a vector-valued function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ from $\mathbb{R}^n$ to itself.

Consider a path $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ whose velocity vectors $\mathbf{x}'(t)$ are vectors in this vector field. Such a path is called a *flow line* of the vector field. The velocity vectors $\mathbf{x}'(t)$ are the vectors of vector field $\mathbf{F}$ if the following equation is satisfied

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)).$$

Another way of looking at this is that a flow line is a solution to a system of differential equation. When $n = 2$, the two differential equations are

$$x' = F_1(x, y),$$
$$y' = F_2(x, y).$$

Example 1.1.1. [18] For the constant vector field $\mathbf{F}(\mathbf{x}) = (\omega_1, \omega_2)$, where ω_1, ω_2 are some constants, the flow lines are the parallel straight paths $(x(t), y(t)) = (\omega_1 t + c_1, \omega_2 t + c_2)$ where different values of the constant c give different lines.

Example 1.1.2. [16] [24] Another important example of a vector field is a constant vector field on a torus. This example is used in our proof of the theorem presented in Chapter 3.

Given real numbers r, R with $0 < r < R$, take a circle of radius r in the xz plane, with center at $(R, 0)$, and rotate about the z-axis. The resulting surface is an example of a manifold called a 2-torus, $\mathbb{T}^2$ and is given by the equation:

$$\mathbb{T}^2 = \{(x, y, z) \mid (\sqrt{x^2 + y^2} - R)^2 + z^2 = 1\}.$$

That is, the 2-torus $\mathbb{T}^2$ is the product $S^1 \times S^1$ of two circles, where $S^1 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$. Consider the quotient group $\mathbb{R}/\mathbb{Z}$. There is a bijection from $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ onto $\mathbb{T}^2$. This defines a topology on $\mathbb{T}^2$. There is another natural bijection between $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ and $\mathbb{R}^2/\mathbb{Z}^2$. This means we can think of $\mathbb{T}^2$ as the quotient set $\mathbb{R}^2/\mathbb{Z}^2$. We use this representation of $\mathbb{T}^2$ to define the differential equations on it. Let $u_1(x, y)$, $u_2(x, y)$ be two real-valued functions of two real variables x and y. Assume these functions are periodic of period 1 in both variables. The planar vector field $Y(x, y) = (u_1(x, y), u_2(x, y))$ is a map from $\mathbb{R}^2 \to \mathbb{R}^2$. With the periodicity conditions, it induces a well-defined map from $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2$. This is known as a vector field on the torus $\mathbb{T}^2$ and the associated system of differential equations is:

$$\begin{aligned} x' &= u_1(x,y), \\ y' &= u_2(x,y). \end{aligned} \tag{1.1}$$

The solutions of this system of differential equations on $\mathbb{T}^2$ are obtained by taking the orbits in $\mathbb{R}^2$ and projecting them down to $\mathbb{T}^2$. For example consider the constant vector field in $\mathbb{R}^2$, $Y(x,y) = \omega_1\partial_x + \omega_2\partial_y$. Here ω_1, ω_2 are positive real numbers. The associated system of differential equations is $x' = w_1$, $y' = w_2$. This system satisfied the periodicity conditions and we get a differential equation on $\mathbb{T}^2$. Now consider the orbits of this constant or linear vector field on $\mathbb{T}^2$. The solutions to 1.1 in $\mathbb{R}^2$ are the lines $x(t) = \omega_1 t + c_1$, $y(t) = \omega_2 t + c_2$ where each such line has slope $\omega = \frac{\omega_2}{\omega_1}$ and passes through the point (c_1, c_2) (see figure 1.1). On the torus $\mathbb{T}^2$ we take $(x(t), y(t))$ mod 1. Then the first case is when $\frac{\omega_2}{\omega_1}$ is rational. In this case all the solutions in $\mathbb{T}^2$ are periodic of the same period T. And the corresponding periodic orbit eventually closes up as its winds around the torus. When $\frac{\omega_2}{\omega_1}$ is irrational then the orbits are not periodic and do not close up as they wind around the torus. As the orbit winds around the torus, it passes arbitrarily close to each point of the torus. That is, every orbit in $\mathbb{T}^2$ is dense in $\mathbb{T}^2$.

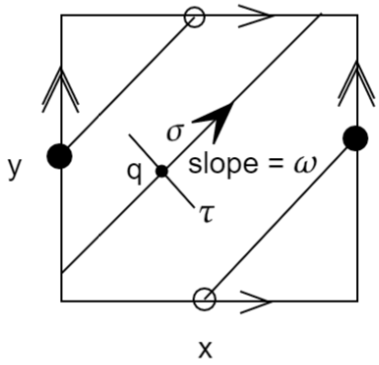

Figure 1.1: Flow lines on a 2-torus viewed as a rectangle

Definition 1.2 (Hamilton's equations)**.** The system of Lagrange's equations $\dot{\boldsymbol{p}} = \partial L/\partial \boldsymbol{q}$, where $\boldsymbol{p} = \partial L/\partial \dot{\boldsymbol{q}}$, with a given lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is equivalent to the system of 2n first-order equations (*Hamilton's equations*)

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}} \qquad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \tag{1.2}$$

where $H(\boldsymbol{p}, \boldsymbol{q}, t) = \boldsymbol{p}\dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ is the Legendre transform of the Lagrangian function viewed as a function of $\dot{\boldsymbol{q}}$.

Example 1.1.3. Consider the one-dimensional motion $\ddot{q} = -\partial U/\partial q$ with the kinetic energy $T = \frac{1}{2}\dot{q}^2$, the potential energy $U = U(q)$ and $p = \dot{q}$, then the hamiltonian $H = \frac{1}{2}p^2 + U(q)$ and Hamilton's equations take the form

$$\dot{q} = p \qquad \dot{p} = -\frac{\partial U}{\partial q}.$$

Definition 1.3 (Differentiable manifold)**.** A set M is given the structure of a differentiable manifold if M is provided with a finite or countable collections of *charts*, so that every point is represented in at least one chart.

A chart is an open set U in the euclidean coordinates $\mathbf{q} = (q_1, \ldots, q_n)$, together with a one-to-one mapping φ of U onto some subset of M, $\varphi : U \to \varphi U \subset M$.

We assume that if points $\mathbf{p}$ and $\mathbf{p}'$ in two charts U and U' have the same image in M, then $\mathbf{p}$ and $\mathbf{p}'$ have neighborhoods $V \subset U$ and $V' \subset U'$ with the same image in M. In this way we get a mapping ${\varphi'}^{-1}\varphi : V \to V'$.

This is a mapping of the region V of the euclidean space $\mathbf{q}$ onto the region V' of the euclidean space $\mathbf{q}'$, and it is given by n functions of n variables, $\mathbf{q}' = \mathbf{q}'(\mathbf{q})$, $(\mathbf{q} = \mathbf{q}(\mathbf{q}'))$. The charts U and U' are *compatible* if these functions are differentiable.

An *atlas* is a union of compatible charts. Two atlases are *equivalent* if there union is also an atlas.

A differentiable manifold is a class of equivalent atlases.

Example 1.1.4. Euclidean space $\mathbb{R}^n$ is a manifold, with an atlas consisting of one chart.

Example 1.1.5. [16] 2-dimensional sphere S^2 that can be conveniently described as a level surface inside $\mathbb{R}^3$ as follows:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Definition 1.4 (Tangent space)**.** If M is a k-dimensional manifold embedded in E^n, then at every point $\mathbf{x}$ we have a k-dimensional tangent space $TM_{\mathbf{x}}$. Namely, $TM_{\mathbf{x}}$ is the orthogonal complement to $\{\text{grad } f_1, ..., \text{grad } f_{n-k}\}$. The vectors of the tangent space $TM_{\mathbf{x}}$ based at $\mathbf{x}$ are called tangent vectors to M at $\mathbf{x}$. We can also define these vectors directly as velocity vectors of curves in M:

$$\dot{\mathbf{x}} = \lim_{t \to 0} \frac{\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}(0)}{t} \qquad \text{where } \boldsymbol{\varphi}(0) = \mathbf{x}, \boldsymbol{\varphi}(t) \in M.$$

Example 1.1.6. [16] Consider the sphere $S^n \subseteq \mathbb{R}^{n+1}$, given as the set of x such that $\|x\|^2 = 1$. A curve $\gamma(t)$ lies in S^n if and only if $\|\gamma(t)\| = 1$. Taking the derivative of the equation $\gamma(t) \cdot \gamma(t) = 1$ at $t = 0$, we obtain $p \cdot \frac{d\gamma}{dt} = p \cdot \gamma(t)^{\cdot} = 0$ (after dividing by 2, and using $\gamma(0) = p$). That is, T_pM consists of vectors $v \in \mathbb{R}^{n+1}$ that are orthogonal to $p \in \mathbb{R}^n \setminus \{0\}$. It is not hard to see that every such vector v is of the form $\gamma(0)^{\cdot}$, where $\gamma(t) = (p + tv)/\|p + tv\|$. Hence the tangent space T_pS^n is the hyperplane orthogonal to the line through p, mathematically

$$T_pS^n = (\mathbb{R}p)^{\perp}.$$

Definition 1.5 (Tangent bundle)**.** The union of the tangent spaces to M at the various points, $\bigcup_{\mathbf{x} \in M} TM_{\mathbf{x}}$, has a natural differentiable manifold structure, the dimensions of which is twice the dimension of M.

This manifold is called the *tangent bundle* of M and is denoted by TM. A point of TM is a vector $\boldsymbol{\xi}$, tangent to M at some point $\mathbf{x}$.

Example 1.1.7. [23] The tangent bundle TM of a manifold M is a vector bundle of rank$(TM) = \dim(M)$. The definition of the vector bundle is given in [23].

Definition 1.6 (1-form)**.** A form of degree 1 (or a 1-form) is a linear function $\omega : \mathbb{R}^n \to \mathbb{R}$, i.e.,

$$\omega(\lambda_1\boldsymbol{\xi_1} + \lambda_2\boldsymbol{\xi_2}) = \lambda_1\omega(\boldsymbol{\xi_1}) + \lambda_2\omega(\boldsymbol{\xi_2}), \qquad \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \boldsymbol{\xi_1}, \boldsymbol{\xi_2} \in \mathbb{R}^n.$$

Example 1.1.8. If a uniform force field $\mathbf{F}$ is given on euclidean $\mathbb{R}^3$, its work A on the displacement vector $\boldsymbol{\xi}$ is a 1-form acting on $\boldsymbol{\xi}$.

Definition 1.7 (The exterior product of two 1-forms)**.** If ω^k is a k form and ω^l is an l form on $\mathbb{R}^n$, then their exterior product $\omega^k \wedge \omega^l$ will be a $k + l$ form. The exterior product of 1-forms, associates to every pair of 1-forms ω_1, ω_2 on $\mathbb{R}^n$ a 2-form $\omega^1 \wedge \omega^2$ on $\mathbb{R}^n$.

Let $\boldsymbol{\xi}$ be a vector in $\mathbb{R}^n$. Give two 1-forms ω_1 and ω_2, we can define a mapping of $\mathbb{R}^n$ to the plane $\mathbb{R} \times \mathbb{R}$ by associating to $\boldsymbol{\xi} \in \mathbb{R}^n$ the vectors $\omega(\boldsymbol{\xi})$ with components $\omega_1(\boldsymbol{\xi})$ and $\omega_2(\boldsymbol{\xi})$ in the plane with coordinates ω_1, ω_2.

The value of the exterior product $\omega_1 \wedge \omega_2$ on the pair of vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^n$ is the oriented area of the image of the parallelogram with sides $\omega(\boldsymbol{\xi}_1)$ and $\omega(\boldsymbol{\xi}_2)$ on the ω_1, ω_2-plane:

$$(\omega_1 \wedge \omega_2)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \begin{vmatrix} \omega_1(\boldsymbol{\xi}_1) & \omega_2(\boldsymbol{\xi}_1) \\ \omega_1(\boldsymbol{\xi}_2) & \omega_2(\boldsymbol{\xi}_1) \end{vmatrix}$$

Definition 1.8 (Symplectic manifold)**.** Let M^{2n} be an even-dimensional differentiable manifold. A *symplectic structure* on M^{2n} is a closed nondegenerate differential

2-form ω^2 on M^{2n}:

$$d\omega^2 = 0 \quad \text{and} \quad \forall \boldsymbol{\xi} \neq 0 \quad \exists \boldsymbol{\eta} : \omega^2\left(\boldsymbol{\xi}, \boldsymbol{\eta}\right) \neq 0 \quad \left(\boldsymbol{\xi}, \boldsymbol{\eta} \in TM_{\mathbf{x}}\right).$$

The pair (M^{2n}, ω^2) is called a *symplectic manifold* and ω is called a *symplectic form.*

Example 1.1.9. Consider the vector space $\mathbb{R}^{2n}$ with coordinates p_i, q_i and let $\omega^2 = \sum dp_i \wedge dq_i$.

Definition 1.9 (Cotangent space)**.** Let V be an n-dimensional differentiable manifold. A 1-form on the tangent space to V at a point $\mathbf{x}$ is called a cotangent vector to V at $\mathbf{x}$. The set of all cotangent vectors to V at $\mathbf{x}$ forms an n-dimensional vector space, dual to the tangent space $TV_{\mathbf{x}}$. This vector space of cotangent vectors is denoted by $T^*V_{\mathbf{x}}$ and is called the *cotangent space* to V at $\mathbf{x}$.

Definition 1.10 (Cotangent bundle)**.** The union of the cotangent spaces to the manifold at all of its points is called the *cotangent bundle* of V and is denoted by T^*V.

Definition 1.11 (Hamiltonian vector field)**.** A Riemannian structure or a symplectic structure on a manifold establishes an isomorphism between the spaces of tangent vectors and 1-forms. Mathematically, to each vector $\boldsymbol{\xi}$, tangent to a symplectic manifold (M^{2n}, ω^2) at the point $\mathbf{x}$, we associate a 1-form ω^1_ξ on $TM_{\mathbf{x}}$ by the formula

$$\omega^1_\xi(\boldsymbol{\eta}) = \omega^2(\boldsymbol{\eta}, \boldsymbol{\xi}) \qquad \forall \boldsymbol{\eta} \in TM_{\mathbf{x}}.$$

Let I be the isomorphism $I : T^*M_{\mathbf{x}} \to TM_{\mathbf{x}}$ as constructed above and H be a smooth function on M^{2n}. Then dH is a differential 1-form on M, and at every point there is a tangent vector to M associated to it. In this way we obtain a vector field $I\ dH$ on M.

The vector field $I\ dH$ is called a *hamiltonian vector field*; H is called the *hamiltonian function.*

Example 1.1.10. If $M^{2n} = \mathbb{R}^{2n} = \{(\mathbf{p}, \mathbf{q})\}$, then we obtain the phase velocity vector field of Hamilton's canonical equations:

$$\dot{\mathbf{x}} = I\, dH(\mathbf{x}) \Leftrightarrow \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \quad \text{and} \quad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad (\mathbf{x} = (\mathbf{q}, \mathbf{p})).$$

Definition 1.12 (Poisson bracket)**.** Let (M^{2n}, ω^2) be a symplectic manifold. To a given function $H : M^{2n} \to \mathbb{R}$ on the symplectic manifold there corresponds a one-parameter local group $g_H^t : M^{2n} \to M^{2n}$ of canonical transformations of M^{2n}—the phase flow of the hamiltonian function equal to H. g_H^t be a group if integral curves exists for all t. Let $F : M^{2n} \to \mathbb{R}$ be another function on M^{2n}.

The *Poisson bracket* $\{F, H\}$ of functions F and H given on a symplectic manifold (M^{2n}, ω^2) is the derivative of the function F in the direction of the phase flow with hamiltonian function H:

$$\{F, H\}(x) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} F(g_H^t(x)) = \langle IdH, dF \rangle.$$

Thus, the Poisson bracket of two functions on M is again a function on M. If the phase space of the system is parametrized by canonical coordinates $\mathbf{q} = (q_1, \ldots, q_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$, then the general definition of the Poisson Bracket for any two functions $F(\mathbf{q}, \mathbf{p}, t)$ and $G(\mathbf{q}, \mathbf{p}, t)$ in an n degrees of freedom problem is [25]:

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right). \tag{1.3}$$

Example 1.1.11. [25] Given $H = T + U = \frac{p^2}{2m} + mgz$, we can find $\frac{\mathrm{d}T}{\mathrm{d}t}$ by using the

Poisson bracket as follows:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \{H, T\} + \frac{\partial T}{\partial t} = \frac{\partial H}{\partial p}\frac{\partial T}{\partial z} - \frac{\partial H}{\partial z}\frac{\partial T}{\partial p} = -gp.$$

Definition 1.13 (First integral)**.** A function F is a first integral of the phase flow with hamiltonian function H if and only if its Poisson bracket with H is identically zero:

$$\{F, H\} \equiv 0.$$

Definition 1.14 (Integrable system [10])**.** Two functions F_1 and F_2 on a symplectic manifold are in *involution* if their Poisson bracket is equal to zero.

In a system with n degrees of freedom (i.e., with a $2n$-dimensional phase space), if n independent first integrals in involution are known, then the system is integrable by quadratures.

The geodesic flow described by the Hamilton's equations is called *completely Liouville integrable system,* if it admits n smooth functions $f_1(\mathbf{q}, \mathbf{p}), \ldots, f_n(\mathbf{q}, \mathbf{p})$ satisfying three conditions:

1. $f_i(\mathbf{q}, \mathbf{p})$ is an integral of the geodesic flow, i.e., is constant along each geodesic line $(q(t), p(t))$;
2. $f_1, \ldots, f_n$ pairwise commute with respect to the standard Poisson bracket on T^*M, i.e., $\{f_i, f_j\} = \sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial q_i}\frac{\partial f_j}{\partial p_i} - \frac{\partial f_j}{\partial p_i}\frac{\partial f_i}{\partial q_i}\right) = 0$;
3. $f_1, \ldots, f_n$ are functionally independent on T^*M.

Example 1.1.12. [12] Consider a system with one degree of freedom with $M = \mathbb{R}^2$ and the hamiltonian $H(p, q) = \frac{1}{2}p^2 + V(q)$. The Hamilton's equations of this system are, $\dot{q} = p$ and $\dot{p} = -\frac{\mathrm{d}V}{\mathrm{d}q}$. The hamiltonian itself is a first integral as $\{H, H\} = 0$. Then for a constant energy E, we can write $\frac{1}{2}p^2 + V(q) = E$. Now we can integrate the equation $\dot{q} = p$, where $p = \pm\sqrt{2(E - V(q))}$. Thus the solution of this system in the implicit form is:

$$t = \pm \int \frac{dq}{\sqrt{2(E - V(q))}}.$$

Example 1.1.13. [12] All time-independent hamiltonian system with two-dimensional phase spaces (like the harmonic oscillator) are integrable.

Definition 1.15 (Geodesic flow [15])**.** A geodesic flow of the Riemannian manifold M with metric $ds^2 = \sum g_{ij} dq_i dq_j$ is a Lagrangian system in the tangent bundle TM with Lagrangian function $L = \sum g_{ij} \dot{q}_i \dot{q}_j$.

The geodesic flow acts in the cotangent bundle T^*M, and the corresponding system is Hamiltonian, and the hamiltonian H is the Legendre transform of the Lagrangian function L, that is,

$$H = \sum a_{ij} p_i p_j,$$

where the matrix $A = (a_{ij})$ that defines the hamiltonian H is connected with the matrix $G = (g_{ij})$ that defines L by the relation

$$A = \frac{1}{4} G^{-1}.$$

1.2 The Liouville-Arnold theorem

Theorem 1.16 (Liouville-Arnold theorem [4])**.** Suppose that we are given n functions in involution on a symplectic $2n$-dimensional manifold

$$F_1, \dots, F_n \quad \{F_i, F_j\} \equiv 0, \quad i, j = 1, 2, \dots, n.$$

Consider the level set of the functions F_i

$$M_{\mathbf{f}} = \{x : F_i(x) = f_i, i = 1, \dots, n\}.$$

Assume that the n functions F_i are independent on $M_{\mathbf{f}}$ (i.e., the n 1-forms dF_i are linearly independent at each point of $M_{\mathbf{f}}$). Then

1. $M_{\mathbf{f}}$ is a smooth manifold, invariant under the phase flow with hamiltonian function $H = F_1$.

2. If the manifold $M_{\mathbf{f}}$ is compact and connected, then it is diffeomorphic to the n-dimensional torus
$$T^n = \{(\varphi_1, \ldots, \varphi_n) \text{mod } 2\pi\}.$$
Or, we can rephrase this statement as:
Every connected component of the manifold $M_{\mathbf{f}}$ is diffeomorphic to the n-dimensional torus
$$T^n = \{(\varphi_1, \ldots, \varphi_n) \text{mod } 2\pi\}.$$

3. The phase flow with hamiltonian function H determines a conditionally periodic motion on $M_{\mathbf{f}}$, i.e., in angular coordinates $\varphi = (\varphi_1, \ldots, \varphi_n)$ we have
$$\frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} = \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{f}).$$

4. The canonical equations with hamiltonian function H can be integrated by quadratures.

Proof. This proof is illustrated by V. I. Arnold in the book [4]. We follow exactly the same lines for proof of this theorem, with small portion included from the source [27]. For a simplicity of exposition, we will assume that $M^{2n} = \mathbb{R}^{2n}$ and the Poisson bracket is standard. But that the general proof does not require these two extra assumptions.

Now we first prove the first point of this theorem. Let us consider the level set of the integrals:
$$M_{\mathbf{f}} = \{x : F_i(x) = f_i, i = 1, \ldots, n\}.$$

By the assumption that the n 1-forms dF_i are linearly independent at each point of $M_{\mathbf{f}}$, that is, $dF_1 \wedge \ldots \wedge dF_n \neq 0$ on $M_{\mathbf{f}}$, it follows immediately from the implicit function theorem that $M_{\mathbf{f}}$ is an n-dimensional submanifold of the $2n$-dimensional phase space M. Now let $I : T^*M_{\mathbf{x}} \to TM_{\mathbf{x}}$ be the isomorphism and F_i be a hamiltonian function on the phase space M^{2n}. Then through the symplectic structure of this phase space, the operator I carries the 1-forms dF_i to the field $I\ dF_i$ of phase velocities of this system.

Since the n 1-forms dF_i are linearly independent at each point on $M_{\mathbf{f}}$, and the isomorphism I is nonsingular, therefore $I\ dF_i$ is independent at every point of $M_{\mathbf{f}}$. The fields $I\ dF_i$ commute with one another because the Poisson bracket of their Hamiltonian functions $\{F_i, F_j\} \equiv 0$. For any $i, j = 1, \ldots, n$, $\{F_i, F_j\} = \langle dF_i, I\ \ dF_j\rangle = 0$, which means that the derivative of the function F_i in the direction of the field $I\ \ dF_j$ is equal to zero. Thus the fields $I\ \ dF_i$ are tangent to $M_{\mathbf{f}}$. Under the Hamiltonian flow of $H = F_1$ we have $\frac{\mathrm{d}F_i}{\mathrm{d}t} = \{F_i, F_1\} = 0$, this gives that $M_{\mathbf{f}}$ is invariant under this flow. This proves the first point of the theorem and also the following lemma:

Lemma 1.17. On the n-dimensional manifold $M_{\mathbf{f}}$ there exist n tangent vector fields which commute with one another and which are linearly independent at every point.

To prove the second point of this theorem let us consider the one-parameter groups of diffeomorphisms of M corresponding to the n given vector fields defined by g_i^t, $i = 1, \ldots, n$. Since $M_{\mathbf{f}}$ is compact, the flows are complete so these g_i^t are groups. We are assuming that these fields and hence the groups g_i^t and g_i^t commute. Now define an action g of the commutative group $\mathbb{R}^n = \{\mathbf{t}\}$ on the manifold M by setting

$$g^{\mathbf{t}} : M \to M \qquad g^{\mathbf{t}} = g_1^{t_1} \ldots g_n^{t_n}, \qquad (\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n).$$

Observe that, $g^{\mathbf{t}+\mathbf{s}} = g^{\mathbf{t}} \circ g^{\mathbf{s}}$, for all $\mathbf{t}, \mathbf{s} \in \mathbb{R}^n$. For a fixed point $x_0 \in M_{\mathbf{f}}$ we define a map

$$g : \mathbb{R}^n \to M \qquad g(\mathbf{t}) = g^{\mathbf{t}}(x_0).$$

Since $M^n_{\mathbf{f}}$ is compact and $\mathbb{R}^n$ is not, therefore the map $g : \mathbb{R}^n \to M^n_{\mathbf{f}}$ is not one-to-one. We know that the *stationary group* of point x_0 is the set Γ of points $\mathbf{t} \in \mathbb{R}^n$ satisfying $g^{\mathbf{t}}x_0 = x_0$. The point $\mathbf{t} = 0$ belongs to Γ which is a well-defined subgroup of $\mathbb{R}^n$ independent of the point x_0. Γ is a *discrete subgroup* as its points lie discretely in $\mathbb{R}^n$.

Lemma 1.18. Let Γ be a discrete subgroup of $\mathbb{R}^n$. Then there exist k $(0 \leq k \leq n)$ linearly independent vectors $\mathbf{e_1}, \ldots, \mathbf{e_k} \in \Gamma$ such that Γ is exactly the set of all their integral linear combinations. This can be rephrased as:

Any discrete subgroup $\Gamma \in \mathbb{R}^n$ is isomorphic to integral lattice $\mathbb{Z}^k$ generated by some k-tuple of vectors, i.e. $\Gamma = \{a_1\mathbf{e_1}, \ldots, a_k\mathbf{e_k} \mid a_i \in \mathbb{Z}\} \cong \mathbb{Z}^k$.

We use induction to prove this lemma. For $n = 1$, it is clear that a discrete subgroup $\Gamma \subset \mathbb{R}$ must consist of integer multiples of a vector $\mathbf{e_1} \in \Gamma$ with smallest Euclidean length. Otherwise, if there were a vector $\mathbf{v} \in \Gamma$ and some $k \in \mathbb{Z}$ such that $\mathbf{v} = r\mathbf{e_1}$ and $k < r < k + 1$, then the length of the vector $\mathbf{v} - k\mathbf{e_1}$ would be strictly less than that of $\mathbf{e_1}$ itself, which is a contradiction. If $n = 2$, again take $\mathbf{e_1} \in \Gamma$- to be a vector with smallest Euclidean length and let $V_1 = span(e_1)$. If $\Gamma \subset V_1$, then by the results of the case $n = 1$ the proof is complete. If not, we choose a vector $\mathbf{e_2} \in \Gamma$ at (non-zero) minimal distance from the line V_1. We claim that Γ consists of integer linear combinations of $\mathbf{e_1}, \mathbf{e_2}$. To get a contradiction, suppose that there exists a vector $\mathbf{v}$ which is not contained in the integer lattice spanned by $\mathbf{e_1}, \mathbf{e_2}$. Consider this lattice as partitioning the plane into parallelograms spanned by the vectors $\mathbf{e_1}, \mathbf{e_2}$, then the vector $\mathbf{v}$ cannot coincide with any of the vertices of these parallelograms. Using the case $n = 1$, we can translate $\mathbf{v}$ by a suitable integer linear combination of $\mathbf{e_1}$ and $\mathbf{e_2}$. We obtain a vector $\mathbf{v}'$ which is closer to V_1 than $\mathbf{e_2}$, which is again a contradiction. Continuing this way, we can obtain a set of linearly independent vectors $\mathbf{e_1}, \ldots, \mathbf{e_k}$ such that $\Gamma = \{a_1\mathbf{e_1}, \ldots, a_k\mathbf{e_k} \mid a_i \in \mathbb{Z}\}$. Lemma (1.18) is proved.

Now consider the direct product of k circles and $n-k$ straight lines, $\mathbb{T}^k \times \mathbb{R}^{n-k} = \{(\varphi_1, \ldots, \varphi_k; y_1, \ldots, y_{n-k})\}$, $\boldsymbol{\varphi}$ mod 2π, together with the natural map $p : \mathbb{R}^{2n} \to T^k \times \mathbb{R}^{n-k}$, $p(\boldsymbol{\varphi}, \mathbf{y}) = (\boldsymbol{\varphi} \bmod 2\pi, \mathbf{y})$. With this map the points $\mathbf{f_1}, \ldots, \mathbf{f_k} \in \mathbb{R}^n$ are mapped to 0. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an isomorphism that maps the vector space $\mathbb{R}^n = \{(\boldsymbol{\varphi}, \mathbf{y})\}$ onto the space $\mathbb{R}^n = \{\mathbf{t}\}$ such that $\mathbf{f_i} \mapsto \mathbf{e_i}$. Where $\mathbf{e_i}, \ldots, \mathbf{e_k} \in \Gamma \subset \mathbb{R}^n$ are the generators of the group Γ. Note that $\mathbb{R}^n = \{(\boldsymbol{\varphi}, \mathbf{y})\}$ gives charts for $\mathbb{T}^k \times \mathbb{R}^{n-k}$ and $\mathbb{R}^n = \{\mathbf{t}\}$ gives charts for manifold $M_{\mathbf{f}}$. Or we can observe that the quotient space $\mathbb{R}^n/\mathbb{Z}^k$ is diffeomorphic to the cylinder $\mathbb{T}^k \times \mathbb{R}^{n-k}$. But since the manifold $M_{\mathbf{f}}$ is compact by hypothesis, we must have $k = n$, and $M_{\mathbf{f}}$ is diffeomorphic to an n-dimensional torus. This proves the second point of the theorem and the following lemma:

Lemma 1.19. Let M^n be a compact and connected differentiable n-dimensional manifold, on which we are given n pairwise commuting and linearly independent at each point vector fields. Then M^n is diffeomorphic to an n-dimensional torus.

Note that we have constructed the angular coordinates $\varphi_1, \ldots, \varphi_n$ mod 2π on $M_{\mathbf{f}}$. By this construction, under the flow of $I\ dF_1$, these coordinates vary uniformly with time, that is

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \omega_i, \qquad \omega_i = \omega_i(\mathbf{f}), \qquad \boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(0) + \boldsymbol{\omega} t,$$

or we can write it as

$$\frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} = \boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{f}).$$

Hence we prove the third statement of the theorem.

To prove that the canonical equations with hamiltonian function H can be integrated by quadratures, we will find the symplectic coordinates $(\mathbf{I}, \boldsymbol{\varphi})$ such that the first integral $\mathbf{F}$ depends only on $\mathbf{I}$ and $\boldsymbol{\varphi}$ are angular coordinates on the torus $M_{\mathbf{f}}$.

We know that $M_{\mathbf{f}} = \{x : \mathbf{F}(x) = \mathbf{f}\}$ is an n-dimensional torus, invariant with respect to the phase flow. With angular coordinates φ_i on M, the phase flow with hamiltonian function $H = F_1$ can be written as:

$$\frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} = \boldsymbol{\omega}(\mathbf{f}), \qquad \boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(0) + \boldsymbol{\omega} t.$$

Now in the coordinates $(\mathbf{F}, \boldsymbol{\varphi})$, the phase flow with hamiltonian function $H = F_1$ takes the form of $2n$ ordinary differential equations:

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} = \boldsymbol{\omega}(\mathbf{F}), \tag{1.4}$$

which can be integrated and gives, $\mathbf{F}(t) = \mathbf{F}(0)$, $\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(0) + \boldsymbol{\omega}(\mathbf{F}(0))t$. Therefore, if we can find the explicit form of $\boldsymbol{\varphi}$, we can integrate the original system of differential equations explicitly. These variables $(\mathbf{F}, \boldsymbol{\varphi})$ are not symplectic coordinates in general. But, there are functions of $\mathbf{F}$, which can be denoted by $\mathbf{I} = \mathbf{I}(\mathbf{F})$, $\mathbf{I} = (I_1, \ldots, I_n)$, such that the variables $(\mathbf{I}, \boldsymbol{\varphi})$ are symplectic coordinates. The original symplectic structure ω^2 in these symplectic coordinates takes the form

$$\omega^2 = \sum dI_i \wedge d\varphi_i.$$

Hence we obtain the action-angle system of canonical coordinates in a neighbourhood of $M_{\mathbf{f}}$ by these action variables $\mathbf{I}$ and the angle variables $\boldsymbol{\varphi}$. I_i are functions of first integrals F_j and thus are first integrals of the system satisfying $H = F_1$. Hence, $H = F_1 = H(\mathbf{I})$. Now we can rewrite the flow equation (1.4) using action-angle variables as

$$\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} = \boldsymbol{\omega}(\mathbf{I}).$$

Let us construct the action-angle variables for one-dimensional case. For this, we construct a canonical transformation $p, q \to I, \varphi$ using its generating functions $S(I, q)$, where

$$p = \frac{\partial S(I,q)}{\partial q} \qquad \varphi = \frac{\partial S(I,q)}{\partial I} \qquad H\left(\frac{\partial S(I,q)}{\partial q}, q\right) = h(I), \tag{1.5}$$

and $(p, q) \to (I, \varphi)$ satisfies the following two conditions:

$$I = I(h), \qquad \text{and} \qquad \oint_{M_h} d\varphi = 2\pi. \tag{1.6}$$

Assume that the function $h(I)$ is known and invertible, that is every curve M_h is determined by I ($M_h = M_{h(I)}$) and for a fixed value of I, equation (1.5) implies

$$dS|_{I=const} = pdq. \tag{1.7}$$

This determines a differential 1-form dS on the curve $H = h$. Integrating this 1-form on the curve $M_{h(I)}$ we obtain a function

$$S(I, q) = \int_{q_0}^{q} p\, dq,$$

in the neighbourhood of a point q_0.

This function will be the generating function in the transformations (1.5) in a neighbourhood of point (I, q_0). The first condition of equation (1.6) is satisfied automatically. To prove the second condition, observe that

$$\Delta S(I) = \oint_{M_{h(I)}} p\, dq,$$

which is equal to the area Π enclosed by the curve $M_{h(I)}$. Thus, the function S is a multiple-valued function on $M_{h(I)}$ and is determined up to addition of integral multiple of Π. The derivative $\partial S(I,q)/\partial q$ is unchanged and the multi-valuedness of $\varphi = \partial s/\partial I$ is given by this term. The formulas (1.5) define a 1-form $d\varphi$ on the curve $M_{h(I)}$, and the integral of this 1-form is equal to $d\Delta S(I)/dI$. Now to meet the second condition, we need that

$$\frac{d\Delta S(I)}{dI} = 2\pi \qquad I = \frac{\Delta S}{2\pi} = \frac{\Pi}{2\pi},$$

where $\Pi = \oint_{M_{h(I)}} p\, dq$ is the area bounded by the phase curve $H = h$. Finally, with the conclusion that if $d\Pi/dh \neq 0$. Then the inverse $I(h)$ of the function $h(I)$ is defined, the action-angle variables in the one-dimension are constructed.

We are now considering a system given by a hamiltonian function $H(\mathbf{p},\mathbf{q})$ in $\mathbb{R}^{2n} = \mathbf{p},\mathbf{q}$ with n degrees of freedom. This system has n first integrals $F_1 = H$, $F_2, \ldots, F_n$, that are in involution. Using the reasoning of one-dimensional case, we set

$$I_i(\mathbf{f}) = \frac{1}{2}\oint_{\gamma_i} \mathbf{p}\, d\mathbf{q},$$

we define n action variables $I_i(\mathbf{f})$ where $\gamma_1, \ldots, \gamma_n$ are the basis of one-dimensional cycles on the torus $M_{\mathbf{f}}$. Assume that $\det(\partial \mathbf{I}/\partial \mathbf{f})|_{\mathbf{f}} \neq 0$, that is for n integrals $F_i = f_i$, the n quantities I_i are independent. Now we can take the variables $\mathbf{I}, \boldsymbol{\varphi}$ as coordinates in the neighbourhood of the torus $M_{\mathbf{f}}$.

Since the manifold $M_{\mathbf{f}}$ is null (i.e., the 2-form ω^2 is zero on $TM_{\mathbf{f}}|_x$), the differential 1-form $\mathbf{p}\, d\mathbf{q}$ on $M_{\mathbf{f}}$ is closed and its exterior derivative $\omega^2 = d\mathbf{p} \wedge d\mathbf{q}$ is identically equal to zero. Then by Stokes' formula the following relation does not change with the change of path of integration,

$$S(x) = \int_{x_0}^{x} \mathbf{p}\, d\mathbf{q}|M_{\mathbf{f}}.$$

Hence, $S(x)$ is a multiple-valued function on $M_{\mathbf{f}}$ with periods given by

$$\Delta_i S = \int_{\gamma_i} dS = 2\pi I_i.$$

Now consider a point x_0 on $M_{\mathbf{f}}$ in a neighbourhood of $M_{\mathbf{f}}$ where the n variables $\mathbf{q}$ are coordinates of $M_{\mathbf{f}}$ such that $M_{\mathbf{f}} \subset \mathbb{R}^{2n}$ is given by n equations of the form $\mathbf{p} = \mathbf{p}(\mathbf{I}, \mathbf{q}), \mathbf{q}(x_0) = \mathbf{q}_0$. Define a single-valued function in the simply connected neighbourhood of the point $\mathbf{q}_0$ as

$$S(\mathbf{I}, \mathbf{q}) = \int_{q_0}^{q} \mathbf{p}(\mathbf{I}, \mathbf{q})\, d\mathbf{q}.$$

Using this as the generating function of a canonical transformation $\mathbf{p}, \mathbf{q} \to \mathbf{I}, \boldsymbol{\varphi}$, we have

$$\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}}, \qquad \boldsymbol{\varphi} = \frac{\partial S}{\partial \mathbf{I}}.$$

These formulas give a canonical transformation in a neighbourhood of $M_{\mathbf{f}}$. The coordinates $\boldsymbol{\varphi}$ are multiple-valued with periods

$$\Delta_i \varphi_j = \Delta_i \frac{\partial S}{\partial I_j} = \frac{\partial \Delta_i S}{\partial I_j} = \frac{2\pi I_i}{\partial I_j} = 2\pi \delta_{ij}.$$

This proves the following theorem,

Theorem 1.20. The transformation $\mathbf{p}, \mathbf{q} \to \mathbf{I}, \boldsymbol{\varphi}$ is canonical, i.e.,

$$\sum dp_i \wedge dq_i = \sum dI_i \wedge d\varphi_i.$$

Now note that the problem of integrating a canonical system with $2n$ equations, of which n first integrals in involution are known, is solved by only using algebraic operations and quadratures (calculating the integrals of known functions). Hence the last assertion of Liuoville-Arnold theorem is also proved. $\square$

1.3 Some definitions from complex analysis

Definition 1.21 (Holomorphic [26])**.** Suppose z is a complex number and f is a complex function define in the domain D. If $z_0 \in D$ and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, we denote this limit by $f'(z_0)$ and call it the derivative of f at z_0. If $f'(z_0)$ exists for every $z_0 \in D$, we say that f is *holomorphic (or analytic)* in D.

Example 1.3.1. [26] z^n is holomorphic in the whole plane for $n = 0, 1, 2, \ldots$.

Definition 1.22 (Entire [26])**.** An *entire* function f is a complex-valued function which is holomorphic in the whole complex plane.

Theorem 1.23 (Liouville's theorem [26])**.** Every bounded entire function is constant.

Theorem 1.24 (Cauchy Riemann equations [14])**.** Suppose that $f(z) = u(x, y) + iv(x, y)$ is complex differentiable, that is, $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy–Riemann equations

$$u_x(x_0 + iy_0) = v_y(x_0 + iy_0), \qquad u_y(x_0 + iy_0) = -v_x(x_0 + iy_0),$$

and

$$f'(x_0 + iy_0) = u_x(x_0 + iy_0) + iv_x(x_0 + iy_0) = v_y(x_0 + iy_0) - iu_y(x_0 + iy_0).$$

2

Integrable Magnetic Geodesic Flows on the 2-torus

This chapter includes one section. This section presents the brief review of the article, titled as *Polynomial integrals of magnetic geodesic flows on the 2-torus on several energy levels* [3].

2.1 Polynomial integrals of magnetic geodesic flows

This section is about finding the polynomial integrals of magnetic geodesic flows on the 2-torus on several energy levels. It includes some details from the article [3]. In this paper, the geodesic flow on a 2-torus in a non-zero magnetic field is considered with the assumption that this flow admits an additional first integral F on $N+2$ different energy levels. This first integral F is a polynomial in momenta of arbitrary degree N and its coefficients are analytic periodic. With these assumptions it is proven that the magnetic field and metrics are functions of one variable and there exists a linear in momenta first integral on all energy levels.

The article is divided into six sections. The first section contains some discussion about the research articles related to this topic. It also presents an example of integrable magnetic geodesic flow. In the section 2, the authors prove that for the 2-torus, if first integral of arbitrary degree N exists on $N+2$ different energy levels then the integrability of this flow on $N+2$ different energy levels is equivalent to the integrability on all energy levels. There are three theorems of this article that are the part of this chapter. The Theorem 1 and 2 are proven in the section 3 and 4. While the sections 5 and 6 prove the Theorem 3. We only discuss some results from sections 1 to 3 in the following subsections.

2.1.1 Introduction

The problem of finding a Riemannian metric that gives an integrable geodesic flow on a 2-surface is a classical problem that has been studied by many researchers. There is no analytic Riemannian metrics with integrable geodesic flows on any surface of genus $g > 1$. This was proved in [19]. Interestingly, on a 2-sphere and 2-torus there exist metrics admitting additional first integrals that are polynomial in momenta. This paper considered the 2-torus on which the existence of two Riemannian metrics with integrable geodesic flow is already known. These metrics are of the form $ds^2 = \Lambda(\alpha x+\beta y)(dx^2+dy^2)$ or $ds^2 = (\Lambda_1(\alpha_1 x+\beta_1 y)+\Lambda_2(\alpha_2 x+\beta_2 y))(dx^2+dy^2)$. With any of these metrics there exists an additional polynomial in momenta first integral of degree 1 and 2 but the existence of metrics with irreducible polynomial in momenta first integral of degree greater than 2 is unknown. In [20] and [21] the conjecture that on the 2-torus there exists no metric with irreducible polynomial integrals of degree > 2 was proved with some assumptions either on the first integrals or on the metrics. But for general case this conjecture is still not proven.

Consider the geodesic flow in the magnetic field on a 2-surface given by the Hamiltonian system:

$$\dot{x}^j = \{x^j, H\}_{mg}, \quad \dot{p}_j = \{p_j, H\}_{mg}, \quad j = 1, 2 \tag{2.1}$$

with Hamiltonian $H = g^{ij}p_ip_j$ and the Poisson bracket of the following form:

$$\{F, H\}_{mg} = \sum_{i=1}^{2} \left(\frac{\partial F}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial x^i} \right) + \Omega \left(x^1, x^2 \right) \left(\frac{\partial F}{\partial p_1}\frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2}\frac{\partial H}{\partial p_1} \right). \qquad (2.2)$$

The function $F(x^1, x^2, p_1, p_2)$ is called the first integral of the magnetic geodesic flow given by equation (2.1), if $\{F, H\}_{mg} = 0$.

Here is an example with the only known Riemannian metric on the 2-torus with integrable magnetic geodesic flow on all energy levels:

Example 2.1.1. Let the metric and the magnetic field on the 2-torus be given by $ds^2 = \Omega(y)(dx^2 + dy^2)$ and $\omega = -u'(y)dx \wedge dy$, respectively. Then there exists a first integral linear in momenta and the magnetic geodesic flow is integrable on all energy levels.

Since $u'(y) = \frac{\mathrm{d}u}{\mathrm{d}y}$, the magnetic field can be written as $\omega = -u'(y)dx \wedge dy = -dx \wedge du$. Let Hamiltonian $H = H(y, p_x, p_y)$ and replace the variable y with u, then $H = H(u, p_x, p_u)$. Now we find some Poisson brackets using the following relation:

$$\{F, H\}_{mg} = \sum_{i=1}^{2} \left(\frac{\partial F}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial x^i} \right) + \Omega \left(x^1, x^2 \right) \left(\frac{\partial F}{\partial p_1}\frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2}\frac{\partial H}{\partial p_1} \right), \qquad (2.3)$$

That is,

$$\{F, H\}_{mg} = \sum_{i=1}^{2} \left(\frac{\partial F}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial x^i} \right) + (-dx \wedge du) \left(\frac{\partial F}{\partial p_x}\frac{\partial H}{\partial p_u} - \frac{\partial F}{\partial p_u}\frac{\partial H}{\partial p_x} \right).$$

This implies,

$$\{x, u\}_{mg} = 0, \qquad \{u, p_u\}_{mg} = 1, \qquad \{x, p_x\}_{mg} = 1, \qquad \{p_x, p_u\}_{mg} = -\,1.$$

Similarly,

$$\begin{aligned}
\dot{x} &= \{x, H\} = \{x, p_x\}H_{p_x} = H_{p_x},\\
\dot{u} &= \{u, H\} = \{u, p_u\}H_{p_u} = H_{p_u},\\
\dot{p}_u &= \{p_u, H\} = \{p_u, u\}H_{p_u} + \{p_u, x\}H_{p_x} = -H_u + H_{p_x},\\
\dot{p}_x &= \{p_x, H\} = \{p_x, p_u\}H_{p_u} = -H_{p_u} = -\dot{u}.
\end{aligned}$$

From the last equation,

$$\dot{p}_x + \dot{u} = (p_x + u)^{\dot{}} = 0,$$

which implies,

$$p_x + u = constant = F_1,$$

or

$$F_1 = p_1 + u(y).$$

This F_1 is the first integral linear in momenta for the given magnetic geodesic flow.

2.1.2 Integrability on several energy levels

In this part, we will discuss the polynomial integrability of a magnetic geodesic flow on the two-torus on a finite number of different energy levels means that the flow is integrable on all energy levels.

To prove the above claim the authors consider the integrability of magnetic geodesic flow on two different energy levels through first integral in momenta of degree three and show that this implies the integrability of this flow on all energy levels. Since we can define the conformal coordinates (x, y) on the 2-torus globally, with metric $ds^2 = \Lambda(x,y)(dx^2+dy^2)$ and Hamiltonian $H = \frac{p_1^2+p_2^2}{2\Lambda}$. Therefore we will be considering these relations throughout this chapter. Let us prove the following lemma:

Lemma 2.1. Suppose that the geodesic flow (2.1) on the 2-torus in a non-zero magnetic field admits an additional polynomial in momenta first integral F of degree 3 on two different levels $\{H = E_1\}$ and $\{H = E_2\}$ where $E_1 \neq E_2$. Then F is the first integral of the same flow on all energy levels.

Proof. The proof of this Lemma is taken from [3]. To prove this lemma, we consider the fixed energy level $H = \frac{C}{2}$ and parameterize the momenta in the following way:

$$H(\mathbf{x},\mathbf{p}) = \frac{p_1^2+p_2^2}{2\Lambda} = \frac{1}{2\Lambda}\left(C\Lambda\cos^2\varphi + C\Lambda\sin^2\varphi\right) = \frac{C}{2},$$

That is,

$$p_1 = \sqrt{C\Lambda}\cos\varphi, \qquad p_2 = \sqrt{C\Lambda}\sin\varphi.$$

The equations (2.1) take the form

$$\dot{x} = \frac{\partial H}{\partial p_1} = \frac{\partial\left(\frac{p_1^2+p_2^2}{2\Lambda}\right)}{\partial p_1} = \frac{p_1}{\Lambda} = \sqrt{\frac{C}{\Lambda}}\cos\varphi,$$

$$\dot{y} = \frac{\partial H}{\partial p_2} = \frac{\partial\left(\frac{p_1^2+p_2^2}{2\Lambda}\right)}{\partial p_1} = \frac{p_2}{\Lambda} = \sqrt{\frac{C}{\Lambda}}\sin\varphi,$$

$$\dot{\varphi} = \left(\left(\frac{\partial\varphi}{\partial x}\frac{\partial H}{\partial p_1} - \frac{\partial\varphi}{\partial p_1}\frac{\partial H}{\partial x}\right) + \left(\frac{\partial\varphi}{\partial y}\frac{\partial H}{\partial p_2} - \frac{\partial\varphi}{\partial p_2}\frac{\partial H}{\partial y}\right)\right) + \Omega\left(\frac{\partial\varphi}{\partial p_1}\frac{\partial H}{\partial p_2} - \frac{\partial\varphi}{\partial p_2}\frac{\partial H}{\partial p_1}\right),$$

where,

$$\begin{aligned}
\frac{\partial H}{\partial p_1} &= \sqrt{\frac{C}{\Lambda}}\cos\varphi, & \frac{\partial H}{\partial p_2} &= \sqrt{\frac{C}{\Lambda}}\sin\varphi,\\
\frac{\partial H}{\partial x} &= \frac{-C\Lambda_x}{2\Lambda}, & \frac{\partial H}{\partial y} &= \frac{-C\Lambda_y}{2\Lambda},\\
\frac{\partial \varphi}{\partial p_1} &= \frac{-1}{\sin\varphi\sqrt{C\Lambda}}, & \frac{\partial \varphi}{\partial p_2} &= \frac{1}{\cos\varphi\sqrt{C\Lambda}},\\
\frac{\partial \varphi}{\partial x} &= \frac{\Lambda_x\cos\varphi}{2\Lambda\sin\varphi}, & \frac{\partial \varphi}{\partial y} &= \frac{\Lambda_y\cos\varphi}{2\Lambda\sin\varphi},
\end{aligned}$$

Hence,

$$\dot{\varphi} = \left(\frac{\sqrt{C}}{2\Lambda\sqrt{\Lambda}}\right)(\Lambda_y\cos\varphi - \Lambda_x\sin\varphi) - \frac{2\Omega}{\Lambda}.$$

The proof of our claim requires us to prove both $\Omega(x, y)$ and $\Lambda(x, y)$ depend on only one variable. So for simplicity we absorb the factor 2 in the $\Omega(x, y)$.

Let F be the cubic in momenta first integral of the following form:

$$\begin{aligned}
F(x, y, \varphi) &= a_0(x, y)p_1^3 + a_1(x, y)p_1^2p_2 + a_2(x, y)p_1p_2^2 + a_3(x, y)p_2^3 + b_0(x, y)p_1^2\\
&\quad + b_1(x, y)p_1p_2 + b_2(x, y)p_2^2 + c_0(x, y)p_1 + c_1(x, y)p_2 + d_0(x, y). \qquad (2.4)
\end{aligned}$$

The fact that $\dot{F} = 0$, implies

$$F_x\dot{x}+F_y\dot{y}+F_\varphi\dot{\varphi} = F_x\cos\varphi+F_y\sin\varphi+F_\varphi\left(\frac{\Lambda_y}{2\Lambda}\cos\varphi - \frac{\Lambda_x}{2\Lambda}\sin\varphi - \frac{\Omega}{\sqrt{C\Lambda}}\right) = 0. \quad (2.5)$$

Now we substitute the equation (2.4) into the equation (2.5) and equate the coefficients for $k = 0, \ldots, 4$ at $e^{ik\varphi}$ to 0. By doing this we will get the set of five equations. These equations are listed in the next subsection. By analyzing these equations we see that these are the necessary and sufficient conditions for F to be the first integral of this flow on the energy level $H = \frac{C}{2}$. These equations are linear in C and for the different energy levels $C = 2E_1$ or $C = 2E_2$ or $H = E_3$, with $E_1 \neq E_2 \neq$

E_3, we obtain same system of equations. That is, our equations become independent of the value of the energy level C. This proves that the integrability on two different energy levels via cubic in momenta first integral F is same as integrability on all energy levels through F. Hence the Lemma (2.1) is proved. □

2.1.3 First integral of degree three

This subsection presents the proof of the following theorem:

Theorem 2.2 ([3])**.** Suppose that the geodesic flow (2.1) on the 2-torus in a non-zero magnetic field admits an additional polynomial in momenta first integral F of degree 3 with analytic periodic coefficients on 2 different energy levels $\{H = E_1\}$, $\{H = E_2\}$. Then the magnetic field and metrics are functions of one variable, there exists a linear in momenta first integral F_1 on all energy levels and F can be expressed in terms of F_1 and the Hamiltonian H.

Proof. This proof is given in detail in [3]. Here we discuss the main points of this proof by assuming that on all energy levels, there exists an additional cubic in momenta first integral of the same form as the equation (2.4). That is,

$$\begin{aligned} F_3(x, y, \varphi) = {} & a_0(x, y)p_1^3 + a_1(x, y)p_1^2p_2 + a_2(x, y)p_1p_2^2 + a_3(x, y)p_2^3 + b_0(x, y)p_1^2 \\ & + b_1(x, y)p_1p_2 + b_2(x, y)p_2^2 + c_0(x, y)p_1 + c_1(x, y)p_2 + d_0(x, y). \end{aligned} \tag{2.6}$$

where all the coefficients are analytic periodic functions of variables x and y. Same as the previous subsection, using the equation (2.3) gives the polynomial in momenta of degree four. By equating the coefficients of this polynomial at different degrees to zero we get the system of PDEs on the metric $\Lambda(x, y)$ and the coefficients of the polynomial F_3. The important results from these group of the equations for different degrees are given below. The reader can find these equations in the article [3].

By equating the coefficients at degree four to zero, the first group of the equations gives the following results:

$$\Delta(a_2 - a_0) = \Delta(a_3 - a_1) = 0,$$

which implies

$$a_2 - a_0 = A_0, \qquad a_3 - a_1 = A_1,$$

where A_0 and A_1 are arbitrary constants and $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplace operator. Without loss of generality, assume that $A_0 = 1$ and $A_1 = 0$. Let $f(x, y) = b_0(x, y) - b_2(x, y)$ and $g(x, y) = b_1(x, y)$. Then from the second group of the equations we obtain the following relations:

$$\Omega(x, y) = \frac{1}{3}(f_y + g_x), \tag{2.7}$$

$$f_x = g_y. \tag{2.8}$$

The above mentioned group of equations along with the third and the last group of equations give necessary and the sufficient conditions for F_3 to be a first integral of the magnetic geodesic flow (2.1). These equations are given in the article [3]. By doing algebraic operations on some of these equations, and introducing the arbitrary constants K_1 and K_2, with

$$3c_0 + f^2 - g^2 = K_1, \qquad 3c_1 + 2fg = K_2, \tag{2.9}$$

and we obtain the relation

$$K_1 g + K_2 f + \frac{1}{3}g^3 - gf^2 = K_3, \tag{2.10}$$

where K_3 is also an arbitrary constant. Say $f = f(g)$, then the equation (2.8) can be written as:

$$g_y - f'(g)g_x = 0. \tag{2.11}$$

Using the equation (2.10) the value of $f'(g)$ can be written as:

$$f'(g) = \frac{\alpha(g)}{\beta(g)} = \frac{3K_2\left(\pm\sqrt{3}K_2 - Q\right) \mp 2\sqrt{3}g\left(2g^3 + 3K_3\right)}{6g^2 Q},$$

where $Q = \sqrt{3K_2^2 + 4g(g^3 + 3gK_1 - 3K_3)}$.

Thus we can write the equation (2.8) as:

$$\beta(g)g_y - \alpha(g)g_x = 0. \tag{2.12}$$

This equation does not admit non-constant global analytic periodic solutions. Characteristics of this equation are straight lines on $(x, y)-$plane with a velocity vector $(-\alpha(g), \beta(g))$. In the form of equation, these characteristics can be written as:

$$\beta(g)x - \alpha(g)y = c,$$

with c as an arbitrary real constant. Consider a smooth curve $\gamma(t) = (x(t), y(t))$ intersecting these characteristics transversally. To find the solution of the equation (2.12), consider a Cauchy problem for this partial differential equation with the initial data

$$g|_{\gamma(t)} = \phi(x, y), \tag{2.13}$$

where $\phi(x, y)$ is an arbitrary function. Assume that $\phi(x, y)$ is not a constant and take two points $\xi \neq \eta$ on γ, with $\phi(\xi) \neq \phi(\eta) \neq 0$. This mean that the solution blows up as the characteristics from these two points will intersect. Thus $g|_{\gamma(t)} = \phi(x, y)$ is a constant. Hence, $g(x, y) \equiv G$ is a constant function.

Then from the equation (2.8), we can write $f(x, y) = f_1(y)$. Thus the equation

(2.7) shows that the magnetic field Ω is the function of only one variable, that is

$$\Omega(y) = \frac{1}{3} f_1'(y). \tag{2.14}$$

Now we prove that metric $\Lambda(x, y)$ is also a function of one variable. For this, note that the equation (2.9) implies that $c_0(x, y)$ and $c_1(x, y)$ are functions of y only. Then using the value of $c_1(x, y)$ and the relation that $f_y + g_x = f_1'(y)$ in the equation involving $d_0(x, y)$ from the last group of the equations, we obtain

$$3d_0(x, y) - \frac{f_1'(y)}{3}(K_2 - 2Gf_1(y)) = 0.$$

All the functions involve in this equation are periodic and independent of x, therefore $d_0(x, y) = d_0(y)$. Thus, $f_1'(y)(K_2 - 2Gf_1(y)) = 0$. This relation can be written as

$$\frac{\partial}{\partial y}\Big(K_2 f_1(y) - G f_1^2(y)\Big) = 0,$$

which implies that $K_2 f_1(y) - G f_1^2(y) = C_0$, C_0 is an arbitrary constant. Now we arrive at two cases. First, if $K_2 \neq 0$ or $G \neq 0$, then this equation will have only constant solutions. But then, $\Omega(y) = \frac{1}{3} f_1'(y) \equiv 0$, and we end up having a standard(flat) geodesic flow. So we consider the case when $K_2 = G = 0$. In this case, $c_1(y) = 0$ and $c_0(y) = \frac{1}{3}\left(K_1 - f_1^2(y)\right)$. Then using an equation from the third group of equations implies that

$$\Big(K_1 - f_1^2(y)\Big)\Lambda_x = 0. \tag{2.15}$$

Assume that there is a point $q = (x_0, y_0)$ and $\Lambda_x \neq 0$ at q. This means that $\Lambda_x \neq 0$ in the neighbourhood U_q of q. Then from the equation (2.15) $f_1(y) \equiv \sqrt{K_1}$ or $f_1(y) \equiv -\sqrt{K_1}$. But we know that if an analytic function is constant on an open set, then it is identically constant on the whole domain. So if $f_1(y)$ is constant then Ω vanishes identically and we again have a standard geodesic flow. Therefore we consider $\Lambda_x \equiv 0$, which implies that

$$\Lambda(x, y) = \Lambda(y),$$

that is, the metric is the function of only one variable.

As from example (2.1.1), it is known that if metric and magnetic field depend only on one variable, then there exists a first integral linear in momenta F_1 on all energy levels. Here, $F_1 = p_1 - \frac{f_1(y)}{3}$. Also we know that hamiltonian H itself is a first integral. In this case, $H = \frac{p_1^2+p_2^2}{2\Lambda}$ is a quadratic in momenta first integral. Now it is obvious that we can write the cubic in momenta first integral F_3 as a combination of the linear first integral F_1 and the hamiltonian H. Here

$$F_3 = -F_1^3 + 2s_0 H F_1 + 2s_2 H + \frac{K_1 F_1}{3},$$

where s_0 and s_2 are some constants that satisfies the equations $s_0 = (1 + a_0)\Lambda$ and $s_2 = \Lambda_1(y)b_2(y) + \frac{1}{3}s_0 f_1(y)$.

This result together with the statement of Lemma (2.1) completes the proof of Theorem (2.2). $\square$

The statement of Theorem (2.2) is true for the polynomial in momenta first integral F of degree 3 and 4. The proof for the case of degree 4 is similar as discussed above for F of degree 3. These proofs are given in the article [3].

Remark 2.3. From the equation (2.10) it is clear that f and g are functionally dependent. But it is not clear that what is the relation between these two functions. The authors didn't mention the logic for considering $f = f(g)$. Because if we assume that $g = g(f)$, then using the equation (2.8) the role of variables will be interchanged from the equations (2.11) - (2.13). And we end up having,

$$\Omega(x) = \frac{1}{3}g_1'(x),$$

which will invalidate the results proved above.

To avoid this technical error in the proof we choose a different approach to prove the results of Theorem (2.2). In Theorem (2.2) the coefficients of the additional polynomial in momenta first integral F are analytic periodic while in our theorem in the next chapter the coefficients of F are differentiable periodic.

3

Integrability on Several Energy Levels by the Complex Variables Approach

In this chapter we present a new theorem similar to a theorem from the paper [3] and give a proof by using the theory of functions of a complex variable. By this approach we avoid the technical error that was presented in the proof as discussed in Chapter 2 and arrive at the same results with more general assumptions.

3.1 Polynomial in momenta first integral of degree three

Let M be a Riemannian manifold homeomorphic to a two-torus. The two-torus is homeomorphic to the quotient space $\mathbb{R}^2/\Gamma$ equipped with a conformally Euclidean metric,

$$ds^2 = \Lambda(x, y)(dx^2 + dy^2) = \Lambda(\theta, \bar{\theta})(dx^2 + dy^2) = \Lambda(\theta, \bar{\theta})d\theta d\bar{\theta},$$

where $\theta = x - iy$ is a complex-valued conformal coordinate on the two-plane $\mathbb{R}^2$ and $\Gamma \approx \mathbb{Z}^2$ is a lattice.

The hamiltonian system on the cotangent bundle T^*M with the standard Poisson structure defines the geodesic flow on the two-torus. The space of smooth functions on T^*M is extended to the space of smooth complex-valued functions on the same phase space. The Poisson bracket of two smooth functions F and G is defined in the same way as in the real-valued case. Under this Poisson structure, the Poisson bracket on the coordinate functions x, y and the coordinates momenta p_x, p_y on the cotangent bundle are as follows:

$$\begin{aligned}\{x, p_x\} = -\{p_x, x\} = \{y, p_y\} = -\{p_y, y\} = 1,\\ \{p_x, p_y\} = \{p_y, p_x\} = \{x, y\} = \{y, x\} = 0.\end{aligned}$$

We know that the complex line $\mathbb{C}$ and the fibers of cotangent bundle T^*M are $\mathbb{R}-$linearly isomorphic to each other, so we can naturally define the complex-valued functions on the fibers of T^*M in terms of coordinates momenta as, $z = p_x + ip_y$. The hamiltonian $H(u, p)$ of the geodesic flow of a metric $\Lambda(\theta, \bar{\theta})d\theta d\bar{\theta}$ takes the following form:

$$H(u, p) = \frac{1}{2\Lambda}(p_x^2 + p_y^2) = \frac{1}{2\Lambda}(z\bar{z}). \tag{3.1}$$

Theorem 3.1. Suppose that the hamiltonian $H = \frac{z\bar{z}}{2\Lambda}$ on the 2-torus is integrable by an additional cubic in momenta first integral F with differentiable periodic coefficients, then the magnetic field and metric are functions of one variable and F is reducible to linear in momenta integral.

Proof. We give the proof of this theorem. Note that in the article [3], the authors fixed the energy level by taking Hamiltonian $H = \frac{C}{2}$ and parametrizing the momenta as $p_x = \sqrt{C\Lambda}\cos\varphi$, and $p_y = \sqrt{C\Lambda}\sin\varphi$. Then $z = \sqrt{C\Lambda}e^{i\varphi}$, but we are not fixing the energy level and considering the case for all energy levels. As $\theta = x - iy$, then

$\bar{\theta} = x + iy$ and $\bar{z} = p_x - ip_y$. The standard relation for the Poisson bracket of the magnetic geodesic flow on a surface or 2-manifold is,

$$\begin{aligned} \{F, G\}_{mg} = &\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial p_x} - \frac{\partial F}{\partial p_x}\frac{\partial G}{\partial x}\right) + \left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial p_y} - \frac{\partial F}{\partial p_y}\frac{\partial G}{\partial y}\right) \\ &+ \Omega(x, y)\left(\frac{\partial F}{\partial p_x}\frac{\partial G}{\partial p_y} - \frac{\partial F}{\partial p_y}\frac{\partial G}{\partial p_x}\right). \end{aligned} \tag{3.2}$$

Using equation (3.2), the Poisson brackets of the complex-valued coordinates take the following values where we can see that θ, z and $\bar{\theta}, \bar{z}$ are canonically conjugate up to a factor of 2 if the magnetic field is zero,

$$\begin{aligned} &\{\theta, z\}_{mg} = -\{z, \theta\}_{mg} = \{\bar{\theta}, \bar{z}\}_{mg} = -\{\bar{z}, \bar{\theta}\}_{mg} = 2, \\ &\{\theta, \theta\}_{mg} = \{\bar{\theta}, \bar{\theta}\}_{mg} = \{z, z\}_{mg} = \{\bar{z}, \bar{z}\}_{mg} = 0, \\ &\{\theta, \bar{\theta}\}_{mg} = \{\bar{\theta}, \theta\}_{mg} = \{\theta, \bar{z}\}_{mg} = \{\bar{\theta}, z\}_{mg} = 0, \\ &\{z, \bar{z}\}_{mg} = -\{\bar{z}, z\}_{mg} = -2i\Omega. \end{aligned} \tag{3.3}$$

Note that the complex function $f(x, y) = u(x, y) + iv(x, y)$ can be written as a function of a complex variable $r = x + iy$ and its conjugate $\bar{r} = x - iy$ as follows:

$$f\left(\frac{r + \bar{r}}{2}, \frac{r - \bar{r}}{2i}\right) = u\left(\frac{r + \bar{r}}{2}, \frac{r - \bar{r}}{2i}\right) + iv\left(\frac{r + \bar{r}}{2}, \frac{r - \bar{r}}{2i}\right),$$

A differentiable function f is called a holomorphic function if f does not depend on $\bar{r}$, that is

$$\frac{\partial f}{\partial \bar{r}} = 0.$$

In order to find the definition of the complex partial derivatives $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \bar{r}}$, let us consider

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \bar{r}}d\bar{r}. \tag{3.4}$$

Since $x = \frac{r+\bar{r}}{2}$, $y = \frac{r-\bar{r}}{2i}$ this implies $dx = \frac{dr+d\bar{r}}{2}$, $dy = \frac{dr-d\bar{r}}{2i}$. Using these relations in equation (3.4), we get

$$\begin{aligned}
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\
&= \frac{\partial f}{\partial x}\left(\frac{dr + d\bar{r}}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{dr - d\bar{r}}{2i}\right) \\
&= \left(\frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}\right)dr + \left(\frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}\right)d\bar{r}.
\end{aligned}$$

We can see from above equations that, $\frac{\partial f}{\partial r} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial \bar{r}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}$. Or,

$$\begin{aligned}
2\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}, \\
2\frac{\partial f}{\partial \bar{r}} &= \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}. \qquad (3.5)
\end{aligned}$$

Assume that f is holomorphic, then $\frac{\partial f}{\partial \bar{r}} = 0$ which can be written as

$$\begin{aligned}
\frac{\partial f}{\partial \bar{r}} = \frac{\partial u + \partial iv}{\bar{r}} &= \frac{\partial u}{\partial \bar{r}} + \frac{\partial iv}{\partial \bar{r}} \\
&= \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial iv}{\partial x} - \frac{1}{2i}\frac{\partial iv}{\partial y} \\
&= \left(\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}\right) + i\left(\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial y}\right) \\
&= 0,
\end{aligned}$$

that is, $u_x = v_y$ and $u_y = -v_x$. This shows that the holomorphic function f satisfies the Cauchy-Riemann equations (1.24). In our case we have assumed that the complex variable $\bar{r} = \theta = x - iy$ and its conjugate $r = \bar{\theta} = x + iy$. So if a function $g = g(\theta, \bar{\theta}) = g(x, y)$, is differentiable in the variable $\bar{\theta}$ then g is a holomorphic function and satisfies the Cauchy-Riemann equations. Thus we get

$$\frac{\partial g}{\partial \theta} = 0, \qquad \frac{\partial g}{\partial \bar{\theta}} = g'(\bar{\theta}).$$

Using the above definition of the complex partial derivatives and using the values of Poisson brackets from the equation (3.3), we transform the equation (3.2) in the complex coordinates as follows,

$$\begin{aligned}
\{F,G\}_{mg} &= F_\theta\{\theta,G\}_0 + F_{\bar{\theta}}\{\bar{\theta},G\}_0 + F_z\{z,G\}_0 + F_{\bar{z}}\{\bar{z},G\}_0 \\
&\quad + \Omega(\theta,\bar{\theta})\left(F_{p_x}G_{p_y} - F_{p_y}G_{p_x}\right) \\
&= F_\theta G_z\{\theta,z\} + F_{\bar{\theta}}G_{\bar{z}}\{\bar{\theta},\bar{z}\} + F_z G_\theta\{z,\theta\} + F_{\bar{z}}G_{\bar{\theta}}\{\bar{z},\bar{\theta}\} \\
&\quad + i\Omega(\theta,\bar{\theta})\left(F_{\bar{z}}G_z - F_z G_{\bar{z}} + G_z F_{\bar{z}} - F_z G_{\bar{z}}\right) \\
&= 2\left(F_\theta G_z + F_{\bar{\theta}}G_{\bar{z}} - F_z G_\theta - F_{\bar{z}}G_{\bar{\theta}}\right) + 2i\Omega(\theta,\bar{\theta})\left(F_{\bar{z}}G_z - F_z G_{\bar{z}}\right).
\end{aligned} \tag{3.6}$$

In order to find the equations of the first integral, we make use of the values of the following Poisson brackets,

$$\begin{aligned}
\{z,\Lambda\} &= \{z,\theta\}\Lambda_\theta + \{z,\bar{\theta}\}\Lambda_{\bar{\theta}} = -2\Lambda_\theta, \\
\{\bar{z},\Lambda\} &= \{\bar{z},\theta\}\Lambda_\theta + \{\bar{z},\bar{\theta}\}\Lambda_{\bar{\theta}} = -2\Lambda_{\bar{\theta}}, \\
\{\theta,\Lambda\} &= \{\bar{\theta},\Lambda\} = 0.
\end{aligned} \tag{3.7}$$

We take the Poisson bracket of conformal coordinates with the Hamiltonian and obtain the following Hamiltonian equations of this flow:

$$\begin{aligned}
\dot{\theta} &= \{\theta,H\} = \{\theta,z\}\frac{\bar{z}}{2\Lambda} + \{\theta,\bar{z}\}\frac{z}{2\Lambda} = \frac{\bar{z}}{\Lambda}, \\
\dot{\bar{\theta}} &= \{\bar{\theta},H\} = \{\bar{\theta},z\}\frac{\bar{z}}{2\Lambda} + \{\bar{\theta},\bar{z}\}\frac{z}{2\Lambda} = \frac{z}{\Lambda}, \\
\dot{z} &= \{z,H\} = \{z,\bar{z}\}\frac{z}{2\Lambda} - \{z,\Lambda\}\frac{\bar{z}z}{2\Lambda^2} = \frac{-i\Omega z}{\Lambda} + \frac{z\bar{z}\Lambda_\theta}{\Lambda^2}, \\
\dot{\bar{z}} &= \{\bar{z},H\} = \{\bar{z},z\}\frac{\bar{z}}{2\Lambda} - \{\bar{z},\Lambda\}\frac{\bar{z}z}{2\Lambda^2} = \frac{i\Omega\bar{z}}{\Lambda} + \frac{z\bar{z}\Lambda_{\bar{\theta}}}{\Lambda^2}.
\end{aligned} \tag{3.8}$$

Now we search for the real-valued function F on T^*M that is a homogeneous polynomial of degree three in momenta with analytic periodic coefficients. The

$F(\theta,\bar{\theta}) : T^*\mathbb{T}^2 \to \mathbb{R}$ is known as a first integral of the third degree and has the following form:

$$F = \sum_{k+l\leq 3} \underbrace{a_{kl}(\theta,\bar{\theta})z^k\bar{z}^l}_{F_{kl}}. \tag{3.9}$$

Note that the real-valued polynomial F consists of the monomials in variables z and $\bar{z}$ with the complex coefficients a_{kl}. And $a_{kl}(\theta,\bar{\theta}) = \overline{a_{kl}(\theta,\bar{\theta})} = \bar{a}_{lk}(\theta,\bar{\theta})$, $\forall\ k,l$. Take the derivative of the equation (3.9),

$$\dot{F} = F_\theta\dot{\theta} + F_{\bar{\theta}}\dot{\bar{\theta}} + F_z\dot{z} + F_{\dot{\bar{z}}}\dot{\bar{z}}. \tag{3.10}$$

By substituting the values of the Poisson bracket of conformal coordinates with the Hamiltonian from equation (3.8) in equation (3.10), we get

$$\begin{aligned}
\dot{F} = & \sum_{k+l\leq 3}\left(a_{kl,\theta}(z^k\bar{z}^l)\left(\frac{\bar{z}}{\Lambda}\right) + a_{kl,\bar{\theta}}(z^k\bar{z}^l)\left(\frac{z}{\Lambda}\right) + a_{kl}k(z^{k-1}\bar{z}^l)\left(\frac{-i\Omega z}{\Lambda} + \frac{\bar{z}z\Lambda_\theta}{\Lambda^2}\right)\right) + \\
& \sum_{k+l\leq 3}\left(a_{kl}l(z^k\bar{z}^{l-1})\left(\frac{i\Omega\bar{z}}{\Lambda} + \frac{\bar{z}z\Lambda_{\bar{\theta}}}{\Lambda^2}\right)\right) \\
= & \sum_{k+l\leq 3}\left(\frac{1}{\Lambda}\left(z^{k+1}\bar{z}^l\left(a_{kl,\bar{\theta}} + a_{kl}l\left(\frac{\Lambda_{\bar{\theta}}}{\Lambda}\right)\right) + z^k\bar{z}^{l+1}\left(a_{kl,\theta} + a_{kl}k\left(\frac{\Lambda_\theta}{\Lambda}\right)\right)\right)\right) + \\
& \sum_{k+l\leq 3}\left(\frac{1}{\Lambda}\left(z^k\bar{z}^l\left(i\Omega(l-k)\right)a_{kl}\right)\right).
\end{aligned} \tag{3.11}$$

The monomials $z^k\bar{z}^l$ acts as nodes along the z^k-axis and $\bar{z}^l$-axis. We can define the partial derivatives or vector fields in our coordinate system θ and $\bar{\theta}$ using equation (3.5) as follows:

$$\begin{aligned}
\partial &= 2\frac{\partial}{\partial\theta} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \\
\bar{\partial} &= 2\frac{\partial}{\partial\bar{\theta}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.
\end{aligned} \tag{3.12}$$

Now we expand the equation (3.11) consisting of monomials $z^k\bar{z}^l$. We only consider those monomials for which the sum of the powers $k+l \leq 3$ and collect the like terms together. That is, for the node $z^0\bar{z}^0$ the value of $k=0$ and $l=0$ and we see from equation (3.11) that there is no monomial that gives coefficients for $z^0\bar{z}^0$. But for the node $z^0\bar{z}^1$ the value of $k=0$, $l=1$, and the nodes $z^0\bar{z}^0$, $z^0\bar{z}^1$ contribute for the coefficients of monomial $z^0\bar{z}^1$ as follows:

$$\dot{F}_{01} = \frac{1}{\Lambda}\left[z^0\bar{z}^1 a_{00,\theta} + z^0\bar{z}^1\left(i\Omega\right)a_{01}\right].$$

From relations for ∂ and $\bar{\partial}$, we have $\partial a_{00} = 2a_{00,\theta}$ and $\bar{\partial} a_{00} = 2a_{00,\bar{\theta}}$. Also assume that $\Lambda = e^{\lambda}$ where $\lambda(\theta,\bar{\theta}) = \bar{\lambda}(\theta,\bar{\theta})$, then $\Lambda_\theta = \lambda_\theta e^{\lambda}$ and $\Lambda_{\bar{\theta}} = \lambda_{\bar{\theta}} e^{\lambda}$. That is, $\frac{\partial\Lambda}{\Lambda} = \partial\lambda$, $\frac{\bar{\partial}\Lambda}{\Lambda} = \bar{\partial}\lambda$. Since F is assumed to be a first integral, $\dot{F}_{01} = 0$. Under these assumptions the equation of coefficients for the node $z^0\bar{z}^1$ can be written as:

$$\frac{1}{2}\partial a_{00} + i\Omega a_{01} = 0. \tag{3.13}$$

Similarly we obtain the rest of equations for monomials of degree ≤ 3 as follows:

$$\frac{1}{2}\bar{\partial} a_{00} - i\Omega a_{10} = 0. \tag{3.14}$$

$$\frac{1}{2}\partial a_{01} + 2i\Omega a_{02} = 0. \tag{3.15}$$

$$\frac{1}{2}\bar{\partial} a_{10} - 2i\Omega a_{20} = 0. \tag{3.16}$$

$$\frac{1}{2}\partial a_{02} + 3i\Omega a_{03} = 0. \tag{3.17}$$

$$\frac{1}{2}\bar{\partial} a_{20} - 3i\Omega a_{30} = 0. \tag{3.18}$$

$$\frac{1}{2}\partial a_{03} = 0. \tag{3.19}$$

$$\frac{1}{2}\bar{\partial} a_{30} = 0. \tag{3.20}$$

$$\frac{1}{2}\left(\partial a_{11}+a_{11}\partial\lambda+\bar{\partial}a_{02}+2a_{02}\bar{\partial}\lambda\right)+i\Omega a_{12}=0. \tag{3.21}$$

$$\frac{1}{2}\left(\bar{\partial}a_{11}+a_{11}\bar{\partial}\lambda+\partial a_{20}+2a_{20}\partial\lambda\right)-i\Omega a_{21}=0. \tag{3.22}$$

$$\frac{1}{2}\left(\partial a_{12}+a_{12}\partial\lambda+\bar{\partial}a_{03}+3a_{03}\bar{\partial}\lambda\right)+2i\Omega a_{13}=0. \tag{3.23}$$

$$\frac{1}{2}\left(\bar{\partial}a_{21}+a_{21}\bar{\partial}\lambda+\partial a_{30}+3a_{30}\partial\lambda\right)-2i\Omega a_{31}=0. \tag{3.24}$$

$$\frac{1}{2}\left(\partial a_{10}+a_{10}\partial\lambda+\bar{\partial}a_{01}+a_{01}\bar{\partial}\lambda\right)=0. \tag{3.25}$$

$$\frac{1}{2}\left(\bar{\partial}a_{12}+2a_{12}\bar{\partial}\lambda+\partial a_{21}+2a_{21}\partial\lambda\right)=0. \tag{3.26}$$

From the equation (3.19), $\partial a_{03}=0$. That is, a_{03} is a differentiable function only of $\bar{\theta}$ and is therefore holomorphic in $\bar{\theta}$. Since a_{03} is also doubly periodic, Theorem 1.23 implies that a_{03} is constant. Without loss of generality we can take $a_{03}=A=1$.

Let us collect the information about the magnetic field $\Omega(\theta,\bar{\theta})$. The equation (3.17) implies that, $3i\Omega a_{03}=-\frac{1}{2}\partial a_{02}$. That is,

$$\Omega=\frac{i}{6}\partial a_{02}=\frac{-i}{6}\bar{\partial}a_{20}. \tag{3.27}$$

Plugging the equation (3.27) in the equations (3.14) and (3.13), give

$$\left(\partial a_{02}\right)a_{10}+3\bar{\partial}a_{00}=0, \tag{3.28}$$

$$\left(\bar{\partial}a_{20}\right)a_{01}+3\partial a_{00}=0. \tag{3.29}$$

From the equation (3.27), $\bar{\partial}a_{20}=-\partial a_{02}$, hence from the equation (3.29),

$$-a_{02}\partial a_{01}=\partial\left(-a_{02}a_{01}+3a_{00}\right). \tag{3.30}$$

Now multiplying the equation (3.15) with a_{02} and use the results from the equations (3.27) and (3.30), we obtain

$$\frac{1}{3}\left(\bar{\partial}a_{20}\right)a_{02}^{2}+\frac{1}{2}\partial\left(a_{02}a_{01}-3a_{00}\right)=0, \tag{3.31}$$

and again using the fact that $\bar{\partial}a_{20}=-\partial a_{02}$, we can write the above equation as:

$$\partial\left(-\frac{1}{9}a_{02}^{3}+\frac{1}{2}a_{02}a_{01}-\frac{3}{2}a_{00}\right)=0. \tag{3.32}$$

The conjugate of above equation implies,

$$\bar{\partial}\left(-\frac{1}{9}a_{20}^{3}+\frac{1}{2}a_{20}a_{10}-\frac{3}{2}a_{00}\right)=0. \tag{3.33}$$

Since $\partial\left(-\frac{1}{9}a_{02}^{3}+\frac{1}{2}a_{02}a_{01}-\frac{3}{2}a_{00}\right)=0$, this implies $-\frac{1}{9}a_{02}^{3}+\frac{1}{2}a_{02}a_{01}-\frac{3}{2}a_{00}=B$ is differential function only of $\bar{\theta}$, and is therefore holomorphic in $\bar{\theta}$. Since B is also doubly periodic, Theorem 1.23 implies that B is constant. That is,

$$-\frac{1}{9}a_{02}^{3}+\frac{1}{2}a_{02}a_{01}-\frac{3}{2}a_{00}=B, \tag{3.34}$$

where B is a constant. Use the value of Ω from the equation (3.27) into the equation (3.15), we obtain:

$$\begin{aligned}-\frac{1}{3}\left(\partial a_{02}\right)a_{02}+\frac{1}{2}\partial a_{01}&=0,\\ \partial\left(-\frac{1}{6}a_{02}^{2}+\frac{1}{2}a_{01}\right)&=0.\end{aligned} \tag{3.35}$$

By the same arguments used earlier for the constant B, we have

$$-\frac{1}{6}a_{02}^{2}+\frac{1}{2}a_{01}=C, \tag{3.36}$$

where C is a constant. We can use the same algebraic operations on the equation

(3.16) and can obtain the result as the conjugate of the equation (3.35). Write the equation for B in the terms of C, to obtain:

$$\begin{aligned} B &= -\frac{3}{2}a_{00} + \frac{1}{18}a_{02}^3 + Ca_{02}, \\ a_{00} &= \frac{1}{27}a_{02}^3 + \frac{2}{3}Ca_{02} - \frac{2}{3}B. \end{aligned}$$

From the equation (3.14), $i\Omega = \frac{\bar{\partial}a_{00}}{2a_{10}}$. Using the value of a_{00} and a_{10} in this relation gives the following result:

$$i\Omega = \frac{\bar{\partial}a_{02}\left(\frac{2}{3}C + \frac{1}{9}a_{02}^2\right)}{2\left(\overline{2C + \frac{1}{3}a_{02}^2}\right)}. \tag{3.37}$$

Let $a_{02} = s(\theta, \bar{\theta})$, $\psi_1(s) = 2C + \frac{1}{3}a_{02}^2$ and $\psi_2(s) = -\overline{\left(2C + \frac{1}{3}a_{02}^2\right)}$. Here $\psi_1(s) = -\overline{\psi_2(s)}$. Then from comparing the values of $i\Omega$ from the equations (3.27) and (3.37), we get

$$\left(\psi_1(s)\bar{\partial} - \psi_2(s)\partial\right)s = 0. \tag{3.38}$$

Thus we can define a vector field $Y(s)$ as follows:

$$Y = \psi_1(s)\bar{\partial} - \psi_2(s)\partial, \tag{3.39}$$

that is,

$$0 \equiv \frac{ds}{dt} = \langle Y, ds\rangle = \psi_1(s)\bar{\partial}s - \psi_2(s)\partial s = 2\psi_1(s)\frac{\partial s}{\partial\bar{\theta}} - 2\psi_2(s)\frac{\partial s}{\partial\theta}. \tag{3.40}$$

This shows that the directional derivative of the function s along the non-zero vector field Y is zero. Which means that s is constant along integral curves of Y.

Going back to the equation (3.39) we consider the new variables θ_1, θ_2 on the two-torus such that, $\theta = \theta_1 + i\theta_2$ and $\bar{\theta} = \theta_1 - i\theta_2$. Assume that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 2\psi_1(s)$,

$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}t} = -2\psi_1(s)$. Then $\frac{\mathrm{d}\theta_1}{\mathrm{d}t} - i\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = 2\psi_1(s)$ and $\frac{\mathrm{d}\theta_1}{\mathrm{d}t} + i\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = -2\psi_2(s)$. This gives us:

$$\begin{aligned} \frac{\mathrm{d}\theta_1}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t} = \psi_1(s) - \psi_2(s) = \omega_1(s), \\ \frac{\mathrm{d}\theta_2}{\mathrm{d}t} &= \frac{\mathrm{d}y}{\mathrm{d}t} = i(\psi_1(s) + \psi_2(s)) = \omega_2(s), \end{aligned}$$

where $\omega_1(s)$ and $\omega_1(s)$ are constant in time t as s is constant along solutions.

Since $\psi_1(s) = -\overline{\psi_2(s)}$, we can write

$$\begin{aligned} \omega_1(s) &= \psi_1(s) + \overline{\psi_1(s)} = 2\mathrm{Re}(\psi_1(s)), \\ \omega_2(s) &= i(\psi_1(s) - \overline{\psi_1(s)}) = -2\mathrm{Im}(\psi_1(s)). \end{aligned}$$

This shows that $\omega_1(s)$, $\omega_2(s) \in \mathbb{R}$. Denote $\omega(s) = \frac{\mathrm{d}\theta_2}{\mathrm{d}t} \Big/ \frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \frac{\omega_2(s)}{\omega_1(s)}$. Then by the example (1.1.2) presented in the section (1.1) and the equation (3.39), we arrive at two cases.

The first case is when $\omega(s) \in \mathbb{Q}$ then the solutions or orbits are periodic. Let $P \subset \mathbb{T}^2$ be the set of points where $\omega(s) \in \mathbb{Q}$. If $P \neq \emptyset$, then we claim $P = \mathbb{T}^2$ and ω is constant, hence s is a function of a single variable.

To prove $P = \mathbb{T}^2$, let $q = (x_0, y_0) \in P$. Then the solution curve σ to the vector Y through q is $t \to q + t\omega \mod 2\pi\mathbb{Z}^2$, $\omega = \omega(s(q))$. Let τ be a transversal to this solution curve through q (see figure 1.1). Note that τ can be extended so that if the solution curve σ takes only rational values then τ also takes rational values and thus becomes a close curve along the 2-torus. Now if there is a $q' \in \tau$ such that $\omega(s(q')) \notin \mathbb{Q}$, then the solution curve through q' is dense. In particular, this solution curve has times $t_n \to \infty$ as $n \to \infty$ such that $\sigma'(t_n) \to q$ as $n \to \infty$. Then $\omega(s(\sigma'(t_n))) \to \omega(s(q))$ by continuity and so $\omega(s(q)) = \omega(s(q'))$. But $\omega(s(q)) \in \mathbb{Q}$, $\omega(s(q')) \notin \mathbb{Q}$. Therefore, $\omega|\tau$ takes only rational vales and must be constant. Thus, there is an open neighbourhood Q of q such that ω is constant on this neighbourhood. If $\exists\, q' \in \mathbb{T}^2 \setminus Q$ and $\omega(q') \notin \mathbb{Q}$, then we can repeat the above argument and conclude

$\omega(q') \in \mathbb{Q}$. Hence a contradiction. This if $q' \in \mathbb{T}^2 \setminus Q$, then $\omega(q') \in \mathbb{Q}$. Therefore, $\omega : \mathbb{T}^2 \to \mathbb{Q}$, and ω is continuous, hence constant, that is, $Q = \mathbb{T}^2$. Thus, $P = Q = \mathbb{T}^2$. Hence our claim is proved. Finally, we can introduce coordinates ζ_1 (along solution curves) and ζ_2 (along the transversal τ) so that $s_{\zeta_1} \equiv 0$ and therefore $s = s(\zeta_2)$. Hence, s is a function of a single variable ζ_2.

The second case is when $\omega(s) \notin \mathbb{Q}$ then the solutions or orbits are dense. As s is constant on a dense set, hence s is constant. Thus s is a function of a single variable ζ_2 in a trivial sense.

Now analyze the metric function $\Lambda(\theta, \bar{\theta})$. Substitute the equation (3.36) and its conjugate in the equation (3.25), we obtain

$$\partial\overline{(2C + \frac{1}{3}a_{02}^2)} + \overline{(2C + \frac{1}{3}a_{02}^2)}\partial\lambda + \bar{\partial}(2C + \frac{1}{3}a_{02}^2) + (2C + \frac{1}{3}a_{02}^2)\bar{\partial}\lambda = 0,$$

which can be written as

$$\psi_1(s)\bar{\partial}\lambda - \psi_2(s)\partial\lambda = \partial\psi_2(s) - \bar{\partial}\psi_1(s). \tag{3.41}$$

This is a non-homogeneous partial differential equation which is linear in λ and so it suffices to find a particular solution and the general solution to the homogeneous equation. The related homogeneous equation is

$$0 \equiv X(s) = \psi_1(s)\bar{\partial}\lambda - \psi_2(s)\partial\lambda = 2\psi_1(s)\frac{\partial\lambda}{\partial\bar{\theta}} - 2\psi_2(s)\frac{\partial\lambda}{\partial\theta}, \tag{3.42}$$

that is,

$$X(\theta, \bar{\theta}) = Y(\theta, \bar{\theta}) = 2\psi_1(\theta, \bar{\theta})\frac{\partial}{\partial\bar{\theta}} - 2\psi_2(\theta, \bar{\theta})\frac{\partial}{\partial\theta}. \tag{3.43}$$

Since the equations (3.42) and (3.43) are the same as the equations (3.40) and (3.42), by similar arguments as used above, we conclude that $s = a_{02}$ is a function of one variable. Hence the general solution of the equation (3.41) depends only on one variable ζ_2.

To find a particular solution to the equation (3.41), we solve

$$\psi_1(s)\bar{\partial}\lambda_1 = -\bar{\partial}\psi_1(s),$$
$$\psi_2(s)\partial\lambda_2 = -\partial\psi_2(s),$$

which is equivalent to the following two equations:

$$\bar{\partial}\lambda_1 = -\bar{\partial}\ln\psi_1(s),$$
$$\partial\lambda_2 = -\partial\ln\psi_2(s).$$

Solutions to these two equations are $\lambda_1 = -\ln\psi_1$, and $\lambda_2 = -\ln\psi_2$, where $\lambda = \lambda_1 + \lambda_2$. Since ψ_1, and ψ_2 are functions of the constant C and s, therefore the particular solution of the equation (3.41) also depends only on C and s. Since s is a function of a single variable ζ_2, therefore, λ is a function of one variable ζ_2. That is, $\Lambda = e^{\lambda}$ is a function of one variable ζ_2.

To conclude, we have shown that if the hamiltonian H (see equation (3.1)) on the 2-torus is integrable by an additional cubic in momenta first integral F (see equation (3.9)), then the magnetic field Ω and the metric Λ are functions of one variable and F is reducible to a linear in momenta integral. This completes the proof. $\square$

4

Conclusion

In this thesis we considered integrable magnetic geodesic flows. In particular, we discussed the results of the research paper, *Polynomial integrals of magnetic geodesic flows on the 2-torus on several energy levels* [3], for the cubic case. Our main goal was to analyze the proof that if the geodesic flow on the 2-torus in a non-zero magnetic field admits an additional polynomial in momenta first integral F of degree 3 with analytic periodic coefficients on 2 different energy levels $\{H = E_1\}$, $\{H = E_2\}$, then the magnetic field and metrics are functions of one variable and F can be expressed in terms of the linear in momenta first integral F_1 and hamiltonian H.

We realize that these results are true but there is one technical error in the proof of this theorem in the paper [3]. To remove this error we used the theory of complex variables and reformulated the above theorem. For the proof of this new theorem, we considered the complex conformal coordinates on the 2-torus and transformed all the equations in these complex coordinates to prove the same results. These results were proved in the new coordinates ζ_1 (along solution curves) and ζ_2 (along the transversal τ) on the 2-torus $\mathbb{T}^2$. We found that ζ_1 was *cyclic* (as $\frac{\partial H}{\partial \zeta_1}$=0) and that the magnetic field and metrics are functions of coordinate ζ_2.

There are a lot of open problems related to the study of integrable magnetic geodesic flows. Some of them are listed below.

1. Apply topological methods to explain or prove the non-integrability in the presence of a magnetic field in order to avoid the detailed algebraic manipulations that are used in solving such problems.
2. Apply KAM theory to study the phenomenon of near-integrability for weak magnetic fields, this would be difficult especially for the case of non-exact magnetic fields.

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