A DEVELOPMENT OF ANALYTICAL GEOMETRY BY THE USE OF LINE CO-ORDINATES,

BY

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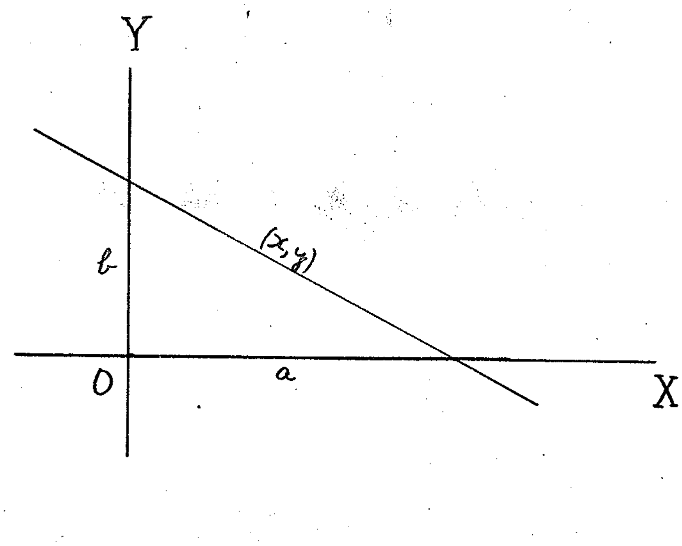
Chapter 1.

The Point Fundamental Relations and Transformation.

1. Introduction.

In the present paper a system of line co-ordinates are employed to develop the usual relations of Analytical Geometry. Rectangular axes referred to as the X and Y axes respectively are used throughout and the customary convention regarding signs are observed. Thus distances perpendicular to the Y axis are positive, when measured to the right and negative to the left. Distances perpendicular to the X axis are positive, when measured upward and negative, when measured downward. As is usual in Geometry, angles are considered positive, when measured in a counter clock-wise direction and unless otherwise stated, positive angles will be used throughout.

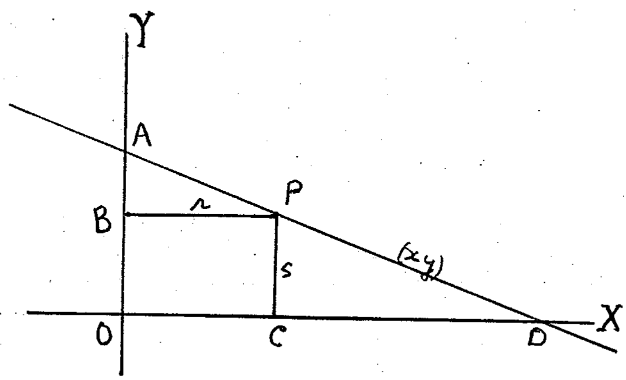
The co-ordinates of a line are defined as the reciprocals of the intercepts of the line on the axes. Thus if the intercepts are <u>a</u> and <u>b</u> the co-ordinates of the line are $\frac{1}{a}$ and $\frac{1}{b}$ which we denote by (x, y).



The co-ordinates of a line parallel to the Y axis are (x,o) and of a line parallel to the X axis are (o,y). The co-ordinates of a line at infinity are (o,o) whereas the co-ordinates of a line passing through the origin are (∞,∞).

It will be seen as we proceed that points are represented by linear equations and conics by equations of the second degree.

2. <u>The equation of a point defined by its perpendicular distances from the axes.</u>



Let P be the given point whose distances from the X and Y axes are <u>r</u> and <u>s</u> respectively. Through P draw any line (x,y). From similar triangles, (see figure), we have: $\frac{BP}{OD} = \frac{BA}{OA}$

Therefore:

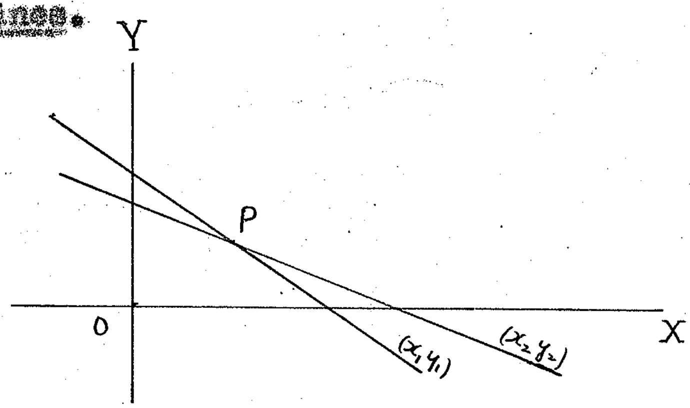
$$\frac{r}{\frac{1}{x} - r} = \frac{\frac{1}{y} - s}{s}$$

Hence:
$$r\,x + s\,y - 1 = 0 \qquad (1)$$

This relation is satisfied by the co-ordinates of any line passing through P. We shall refer to it as the standard form of the equation of the point P.

Any equation of the form $Ax+By+C = o$ with $C \neq o$ may be reduced to this form and so represents a point. If $C = o$, the equation is satisfied by the co-ordinates (o,o) and thus represents an infinitely distant point.

3. The equation of the point of intersection of two given lines.



Let the co-ordinates of the two lines be (x_1, y_1) and (x_2, y_2) and their point of intersection be P. By section 2 the equation of P is of the form

$$rx + sy - 1 = o.$$

But the co-ordinates of the two lines satisfy the above equation, therefore we have:

$$rx_1 + sy_1 - 1 = o,$$

and $$rx_2 + sy_2 - 1 = o.$$

The condition for consistency of these three equations is:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = o \qquad (2)$$

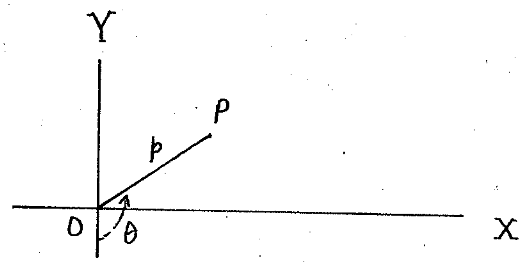
which is accordingly the equation of the point P.

This equation may also be written in the form:

$$\frac{y-y_1}{x-x_1} = \frac{y_1-y_2}{x_1-x_2}. \tag{3}$$

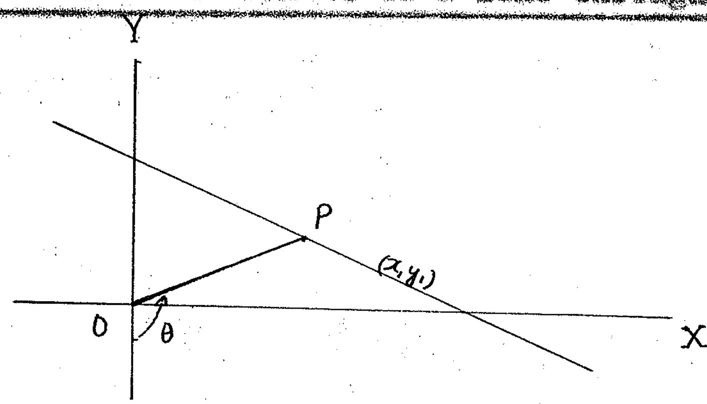
4. <u>Radius vector and vectorial angle.</u>

The distance OP from the origin O to any point P is called the radius vector of P and is represented by p, whereas the angle between the negative extension of the Y axis and OP is called the vectorial angle and is denoted by θ.



If the point P has the equation: $rx + sy - 1 = 0$, then evidently $\tan\theta = -\frac{r}{s} = \frac{dy}{dx}$.

5. <u>The equation of a point in terms of its vectorial angle and the co-ordinates of a line through it.</u>



Let $rx + sy - 1 = 0$ represent the given point and (x_1, y_1) be the given line. Then $rx + sy - 1 = 0$.

But $\tan\theta = -\frac{r}{s}$ therefore $s\tan\theta + r = 0$.

Whence for consistency we have:

$$\begin{vmatrix} x & y & -1 \\ x_1 & y_1 & -1 \\ 1 & \tan\theta & 0 \end{vmatrix} = 0,$$

reducing we get $y - y_1 = m(x - x_1)$ (4)

as the required equation where $m = \tan\theta$.

If (x_2, y_2) is a second line passing through the point P, then $y_2 - y_1 = m(x_2 - x_1)$.

Whence for the intersection of the two lines (x_1, y_1) and (x_2, y_2) the vectorial angle is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{5}$$

Special Cases.

(a) If the given line is parallel to the X axis its co-ordinates are (o, y_1) and equation (4) reduces to

$$y = mx + y_1. \tag{6}$$

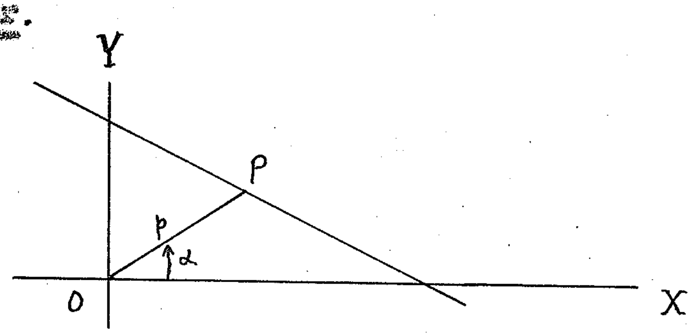
Any point $Ax + By + C = 0$ with $B \neq 0$ can be written in the form of (6). The value of m may be found in such a case by dividing the coefficient of x by that of y and changing the sign of the quotient, i.e. $m = -\frac{A}{B} = \frac{dy}{dx}$.

The direction of an infinitely distant point may be readily found. Thus for the point $3x - 4y = 0$, $m = \frac{3}{4}$.

(b) If the given line is parallel to the Y axis, its co-ordinates are (x_1, o) and equation (4) reduces to

$$y = m(x - x_1). \tag{7}$$

6. <u>The equation of a point in terms of its vector and the angle between the positive extension of the X axis and the vector.</u>



Let α denote the angle between the vector and the X axis. By section 3 the equation of P is of the form:

$$rx + sy - 1 = 0$$

where $s = p \sin\alpha$ and $r = p \cos\alpha$

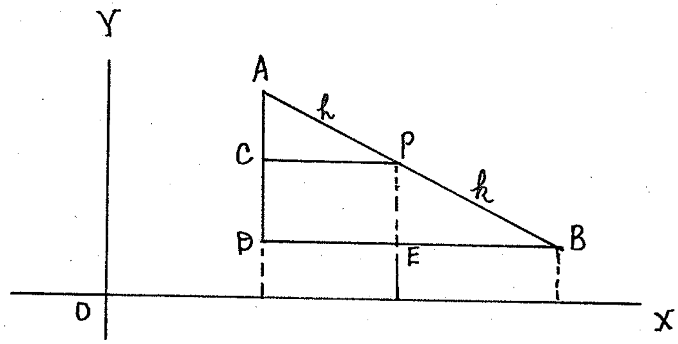
Substituting we get:

$$p \cos\alpha \,.\, x + p \sin\alpha \,.\, y - 1 = 0,$$

or $$x \cos\alpha + y \sin\alpha - \frac{1}{p} = 0 \qquad (8)$$

is the required equation of P in terms of α and p.

7. **The equation of a point P which divides a line joining two given points in the ratio $h:k$.**



Let the equations of A, P and B, be

$$r_1 x + s_1 y - 1 = 0,$$

$$r x + s y - 1 = 0$$

and $\quad r_2 x + s_2 y - 1 = 0 \quad$ respectively.

From similar triangles we have:

$$\frac{CA}{EP} = \frac{CP}{EB} = \frac{AP}{PB}$$

i.e. $\quad \dfrac{s_1 - s}{s - s_2} = \dfrac{r - r_1}{r_2 - r} = \dfrac{h}{k},$

whence

$$\left.\begin{aligned} r &= \frac{kr_1 + hr_2}{h + k} \\ s &= \frac{ks_1 + hs_2}{h + k} \end{aligned}\right\} \qquad (9)$$

Substituting for r and s we get:

$$h(r_2 x + s_2 y - 1) + k(r_1 x + s_1 y - 1) = 0 \qquad (10)$$

as the required equation.

Also the equation of any point on the line AB is

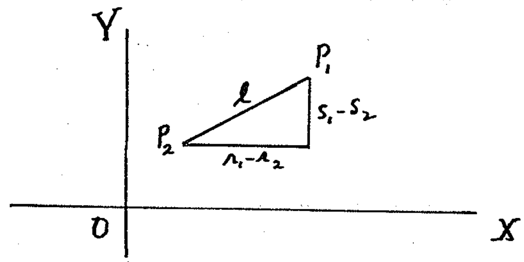
$$r_1 x + s_1 y - 1 + K(r_2 x + s_2 y - 1) = 0.$$

Corollary.

If P bisects the line AB, then the equation P reduces to

$$\left.\begin{aligned} &\left(\frac{r_1 + r_2}{2}\right) x + \left(\frac{s_1 + s_2}{2}\right) y - 1 = 0, \\ \text{or} \quad &(r_1 x + s_1 y - 1) + (r_2 x + s_2 y - 1) = 0. \end{aligned}\right\} \qquad (11)$$

8. <u>The distance between two given points.</u>



Let the equations of the given points be

$r_1 x + s_1 y - 1 = 0$

$r_2 x + s_2 y - 1 = 0$ respectively.

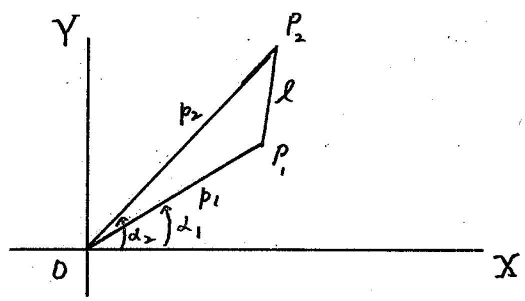
Obviously $l = \sqrt{(r_1 - r_2)^2 + (s_1 - s_2)^2}$. (12)

9. <u>Distance between two given points.</u>

Let the equations of the given points be:

$x \cos \alpha_1 + y \sin \alpha_1 - \frac{1}{p_1} = 0$

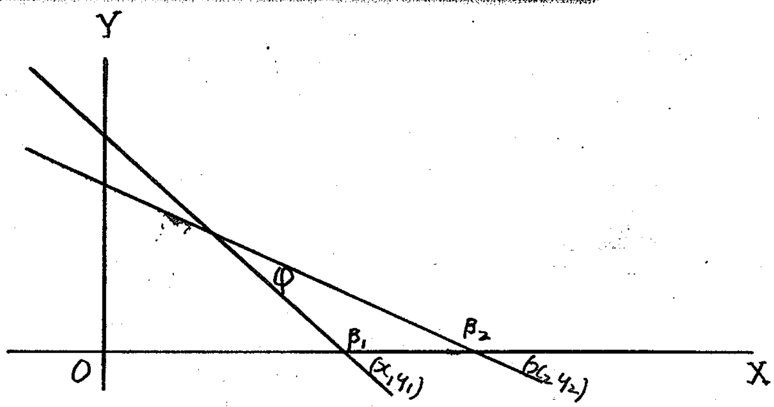
$x \cos \alpha_2 + y \sin \alpha_2 - \frac{1}{p_2} = 0$.



Obviously: $l^2 = p_1^2 + p_2^2 - 2 p_1 p_2 \cos (\alpha_2 - \alpha_1)$

Therefore $l = \sqrt{p_1^2 + p_2^2 - 2 p_1 p_2 \cos (\alpha_2 - \alpha_1)}$. (13)

10. The angle between two given lines.



Let φ be the angle between the two lines (x_1, y_1) and (x_2, y_2).
Let the lines (x_1, y_1) and (x_2, y_2) make the angles β_1 and β_2 respectively, with positive extension of the X axis.

Then $\varphi = \beta_2 - \beta_1$; $\tan \beta_1 = -\dfrac{x_1}{y_1}$; $\tan \beta_2 = -\dfrac{x_2}{y_2}$

$$\tan\varphi = \tan(\beta_2 - \beta_1) = \frac{\tan\beta_2 - \tan\beta_1}{1 + \tan\beta_2 \tan\beta_1} = \frac{x_1 y_2 - x_2 y_1}{x_1 x_2 + y_1 y_2}$$

Therefore
$$\varphi = \tan^{-1} \frac{x_1 y_2 - x_2 y_1}{x_1 x_2 + y_1 y_2} \qquad (14)$$

If lines are parallel, then $\tan \varphi = 0$

i.e. $x_1 y_2 - x_2 y_1 = 0$

or
$$\frac{x_1}{x_2} = \frac{y_1}{y_2} \qquad (15)$$

is the condition for parallelism.

If lines are perpendicular, then $\tan \varphi = \infty$

Whence
$$x_1 x_2 + y_1 y_2 = 0 \qquad (16)$$

is the condition for perpendicularity of two lines.

11. To find a line, parallel to a given line and passing through a given point.

Let $r x + s y - 1 = 0$ be the given point and (x_1, y_1) the given line. Let (x_2, y_2) be the required line.

Then $\frac{x_2}{x_1} = \frac{y_2}{y_1} = k$

Therefore: $x_2 = k x_1$, $y_2 = k y_1$

These co-ordinates satisfy the equation of the point.

Therefore: $k(r x_1 + s y_1) - 1 = 0$.

Hence $k = \frac{1}{r x_1 + s y_1}$.

Therefore: the required line is

$$\frac{x_1}{r x_1 + s y_1}, \frac{y_1}{r x_1 + s y_1} \qquad (17)$$

12. <u>To find a line perpendicular to a given line, passing through a given point.</u>

Let $r x + s y - 1 = 0$ be the given point and (x_1, y_1) the given line. Let (x_2, y_2) be the required line.

For perpendicularity $x_1 x_2 + y_1 y_2 = 0$

i.e. $\frac{x_2}{y_1} = -\frac{y_2}{x_1} = k$.

Then $x_2 = k y_1$

$y_2 = -k x_1$

These co-ordinates satisfy the equation of the point.

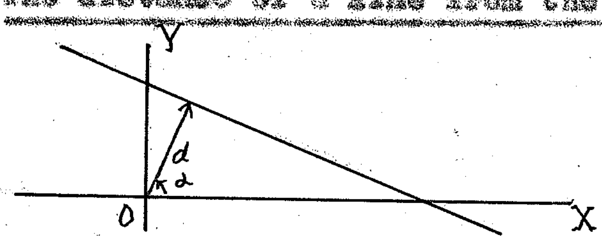
Therefore $k(r y_1 - s x_1) - 1 = 0$

$k = -\frac{1}{s x_1 - r y_1}$.

Hence the required line is

$$\left(-\frac{y_1}{s x_1 - r y_1}, \frac{x_1}{s x_1 - r y_1}\right). \qquad (18)$$

13. <u>The distance of a line from the origin.</u>

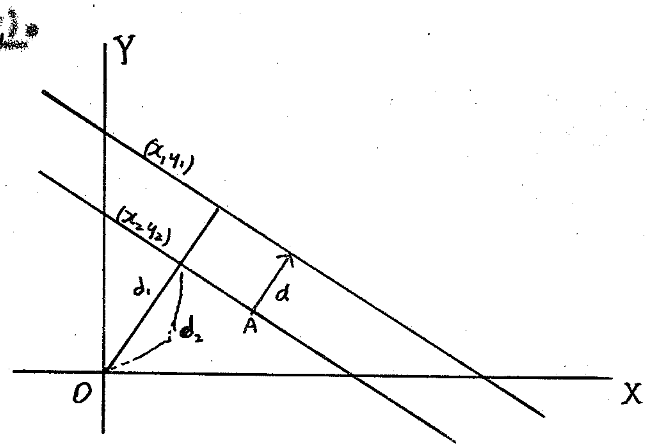


Let (x_1, y_1) be the given line, and α the angle that the perpendicular from the origin upon the line makes with the X axis. Let d denote the length of this perpendicular. Then we have: $\sin\alpha = d\,y_1$, $\cos\alpha = d\,x_1$,

but $\sin^2\alpha + \cos^2\alpha = d^2(x_1^2+y_1^2) = 1$

Therefore $d = \dfrac{1}{\sqrt{x^2+y^2}}$. (19)

14. <u>The distance from the point $r\,x + s\,y - 1 = 0$ to the line $(x_1\,y_1)$.</u>



Let A be the given point $r\,x + s\,y - 1 = 0$ and let $(x_2\,y_2)$ represent a line passing through A parallel to the given line $(x_1\,y_1)$. Let d be the required distance. Since the lines $(x_1\,y_1)$ and $(x_2\,y_2)$ are parallel we have:

$$x_2 = k\,x_1, \qquad y_2 = k\,y_1 .$$

But the co-ordinates $(x_2\,y_2)$ satisfy the equation of A.

Therefore $k\,r\,x_1 + k\,s\,y_1 - 1 = 0$

Hence $k = \dfrac{1}{r\,x_1 + s\,y_1}$; $x_2 = \dfrac{x_1}{r\,x_1 + s\,y_1}$, and

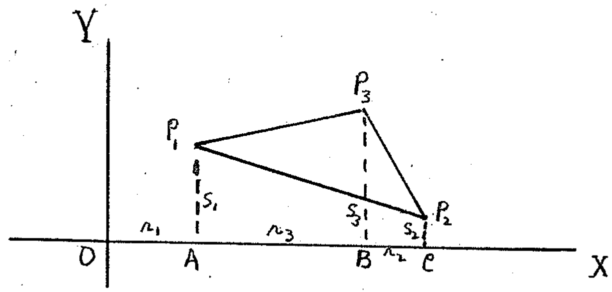
$$y_2 = \frac{y_1}{r\,x_1 + s\,y_1}$$

If d_1 and d_2 are the distances from O to lines $(x_1\,y_1)$ and $(x_2\,y_2)$, then $d = d_1 - d_2 = \dfrac{1}{\sqrt{x_1^2+y_1^2}} - \dfrac{1}{\sqrt{\dfrac{x_1^2+y_1^2}{(r\,x_1 + s\,y_1)^2}}}$

Therefore $d = \dfrac{1 - r\,x_1 - s\,y_1}{\sqrt{x_1^2+y_1^2}}$ (20)

is the required distance.

15. The area of a triangle whose vertices are known.



Let $P_1\,P_2\,P_3$ be the given triangle and let their equations be

$$r_1 x + s_1 y - 1 = 0$$

$$r_2 x + s_2 y - 1 = 0$$

$$r_3 x + s_3 y - 1 = 0 \quad \text{respectively.}$$

The area of the triangle $P_1\,P_2\,P_3$ = area of $A\,B\,P_3\,P_1$ + area of $B\,C\,P_2\,P_3$ − area of $A\,C\,P_2\,P_1$

$$= (r_3 - r_1)\frac{(s_3 + s_1)}{2} + (r_2 - r_3)\frac{(s_2 + s_3)}{2} - (r_2 - r_1)\frac{(s_2 + s_1)}{2}.$$

Therefore area of the triangle $= \frac{1}{2}\left[r_1(s_2 - s_3) + r_2(s_3 - s_1) + r_3(s_1 - s_2)\right]$

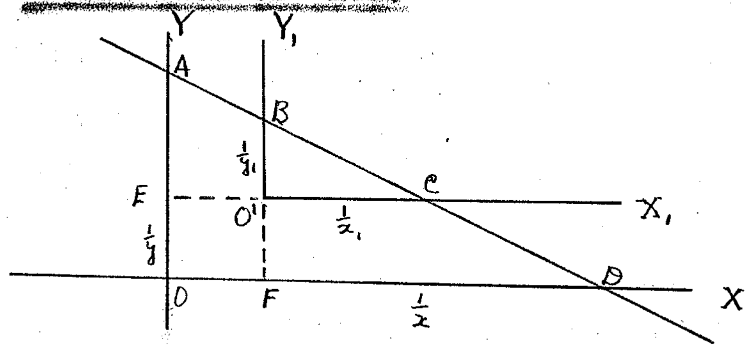
$$\text{or} \quad = \frac{1}{2}\begin{vmatrix} r_1 & s_1 & 1 \\ r_2 & s_2 & 1 \\ r_3 & s_3 & 1 \end{vmatrix} \qquad (21)$$

Corollary.

If the sides of a triangle have co-ordinates $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$ respectively, then the equations of the vertices may be found by (2), and (21) reduces to

$$A = \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2}{2\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}} \qquad (22)$$

16. Translation of Axes.



Translate the axes to the point $r\,x + s\,y - 1 = 0$ as a new origin. Referring to the figure the triangles $O'\,B\,C$ and $E\,A\,C$ are similar.

Therefore $\quad \dfrac{O'B}{O'C} = \dfrac{E\,A}{E\,C} \quad$ i.e.

$$\frac{x_1}{y_1} = \frac{\frac{1}{y} - s}{\frac{1}{x_1} + r} = \frac{1 - s\,y}{1 + r\,x_1} \cdot \frac{x_1}{y}$$

Therefore $\quad y = \dfrac{y_1}{1 + r\,x_1 + s\,y_1}$

From similar triangles $O'\,B\,C$ and $O\,A\,D$ $\quad \dfrac{x}{y} = \dfrac{x_1}{y_1}$.

Therefore $\quad x = y\,\dfrac{x_1}{y_1} = \dfrac{x_1}{1 + r\,x_1 + s\,y_1}$.

The equations of transformation are

$$\left.\begin{aligned} x &= \frac{x_1}{1 + r\,x_1 + s\,y_1} \\ y &= \frac{y_1}{1 + r\,x_1 + s\,y_1} \end{aligned}\right\} \qquad (23)$$

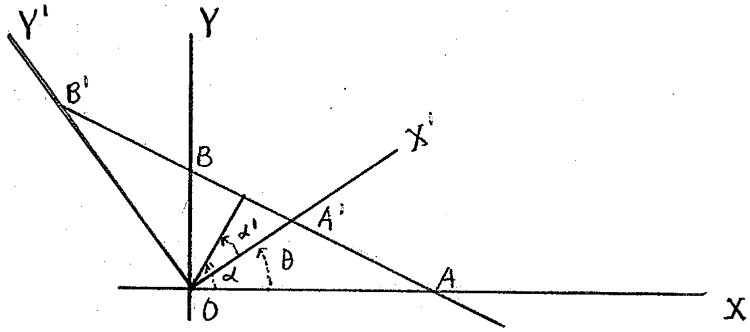
The inverse transformations are evidently,

$$\left.\begin{aligned} x_1 &= \frac{x}{1 - r\,x + s\,y} \\ y_1 &= \frac{y}{1 - r\,x + s\,y} \end{aligned}\right\} \qquad (24)$$

17. Rotation of Axes.

Let the axes O X and O Y be rotated about O through the angle θ to the new position O X' and O Y'. Let A B be any line whose old and new co-ordinates are $(x\ y)$ and $(x'\,y')$ respectively. Let p be the perpendicular from O to

A B, making angles α and α' with the axes O X and O X'



Evidently $$x = \frac{\cos\alpha}{p} = \frac{\cos(\alpha' + \theta)}{p} = \frac{\cos\alpha' \cos\theta - \sin\alpha' \sin\theta}{p}$$

$$\left.\begin{array}{ll} \text{Whence} & x = x' \cos\theta - y' \sin\theta \\ \text{Similarly} & y = x' \sin\theta + y' \cos\theta \end{array}\right\} \qquad (25)$$

Solving for x' and y' we get the inverse transformations

$$\left.\begin{array}{l} x' = \dfrac{\begin{vmatrix} x, & -\sin\theta \\ y & \cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta, & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}} = \dfrac{x\cos\theta + y\sin\theta}{\cos^2\theta + \sin^2\theta} = x\cos\theta + y\sin\theta \\ \\ y' = \dfrac{\begin{vmatrix} \cos\theta & x \\ \sin\theta & y \end{vmatrix}}{\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}} = \dfrac{-x\sin\theta + y\cos\theta}{\cos^2\theta + \sin^2\theta} = -x\sin\theta + y\cos\theta \end{array}\right\} (26)$$

16. General transformation of co-ordinates.

Translation followed by rotation gives:

$$\left.\begin{array}{l} x'' = x' \cos\theta + y'\sin\theta = \dfrac{x\cos\theta + y\sin\theta}{1 - r\,x - s\,y} \\ y'' = -x' \sin\theta + y' \cos\theta = \dfrac{-x\sin\theta + y\cos\theta}{1 - r\,x - s\,y} \end{array}\right\} \qquad (27)$$

The denominator equated to zero gives the equation of the new origin referred to the old axes. Solving these equations we get the direct equations of transformation,

$$\left.\begin{array}{l} x = \dfrac{x''\cos\theta - y''\sin\theta}{1 + (r\cos\theta + s\sin\theta)x'' + (-r\sin\theta + s\cos\theta)y''} \\ y = \dfrac{x''\sin\theta + y''\cos\theta}{1 + (r\cos\theta + s\sin\theta)x'' + (-r\sin\theta + s\cos\theta)y''} \end{array}\right\} \qquad (28)$$

Formulae (28) may be written as:

$$x = \frac{x''\cos\theta - y''\sin\theta}{1 + r''x'' + s''y''}$$
$$y = \frac{x''\sin\theta + y''\cos\theta}{1 + r''x'' + s''y''} \qquad (29)$$

Where $r''x'' + s''y'' + 1 = 0$ is the equation of the old origin with respect to the new axes. Equations (29) give the inverse transformations to (27) and so might have been obtained from (27) by changing the sign of θ, r and s.

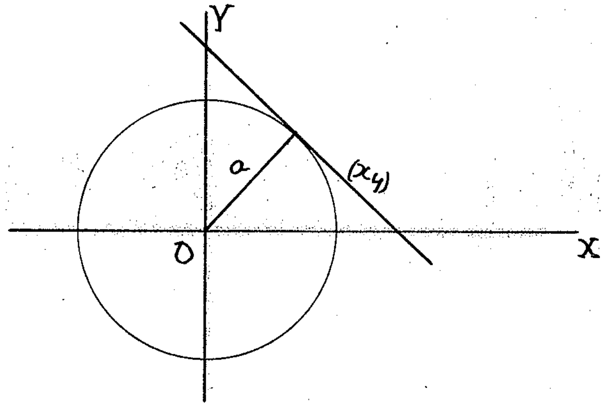
Chapter 2.

CONICS.

19. Circle.

Definition:

The Circle is the envelope of a line which is at a constant distance from a fixed point called the centre.



Let the line (x,y) envelope the circle whose radius is a. Then by formula (20) $a = \frac{1}{\sqrt{x^2 + y^2}}$

Therefore the equations of the circle is:

$$x^2 + y^2 = \frac{1}{a^2} \tag{1}$$

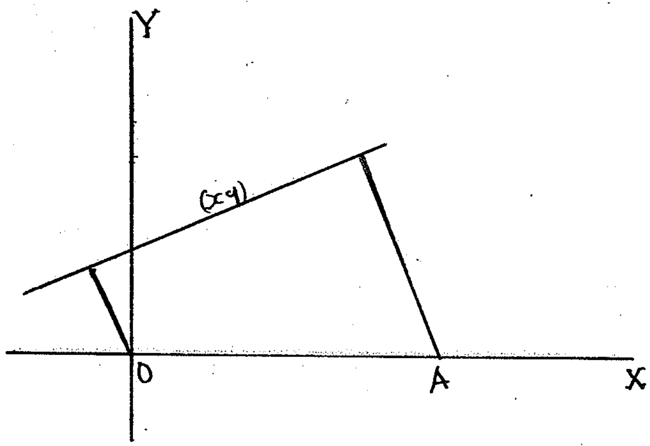
If the centre of the circle is at the point $r x + s y - 1 = 0$, then equation (1) reduces on transforming co-ordinates to the form:

$$x^2 + y^2 = \frac{1}{a^2} (r x + s y - 1)^2 \tag{2}$$

20. The general conic.

Definition.

A conic is the envelope of a line, which moves so, that the product of its distances from two fixed points called the foci is a constant. The line joining the foci is called the principal axis.



Let the principal axes be chosen as the X axis and let one focus be taken as the origin and the other as $2cx - 1 = 0$.

Then $\dfrac{1}{\sqrt{x^2+y^2}} \cdot \dfrac{1-2cx}{\sqrt{x^2+y^2}} = k$

where k is a constant.

Therefore $\qquad k(x^2+y^2) + 2cx - 1 = 0$

whence $\qquad x^2+y^2 = \dfrac{1}{k} - \dfrac{2cx}{k} \qquad (3)$

is the general equation of any conic.

From the definition, it appears, that a conic, situated as above, is symmetrical to the X axis.

21. Ellipse.

If the fixed points be on the same side of the moving line, then k is positive and may be represented by b^2. If, in addition, c is finite, the locus is called an ellipse. Its equation, when a focus is taken as origin and the principal axis as X axis, has the form

$$x^2+y^2 = \frac{1}{b^2} - \frac{2cx}{b^2} \qquad (4)$$

when $c = 0$, this equation reduces to

$x^2+y^2 = \dfrac{1}{b^2}$. Which represents a circle of radius b.

If the axes are translated to the point $cx - 1 = 0$, as a new origin, the equation of the ellipse becomes

$$a^2x^2+b^2y^2 = 1. \qquad (5)$$

where $a^2 = b^2 + c^2$. Equation (5) shows symmetry with both axes. The X axis is cut at the points $\pm a\ x - 1 = 0$, which are called the vertices of the ellipse. The Y axis is cut at $\pm b\ y - 1 = 0$. The foci are at $\pm c\ x - 1 = 0$.

22. Parabola.

If, as above k is positive, but b and c each approach infinity, while $\frac{b^2}{c}$ remains finite and equal to p, then the general equation reduces to:

$$x^2 + y^2 = - \frac{2\ x}{p}. \tag{6}$$

This equation shows symmetry with the X axis which is cut at the point $- \frac{2\ x}{p} - 1 = 0$, which is called the vertex of the parabola. Translating the axes to the vertex as a new origin, the equation becomes:

$$y^2 = - \frac{2\ x}{p} \tag{7}$$

The distance between the vertex and the focus is $\frac{p}{2}$.

23. Hyperbola.

In case the fixed points lie on opposite sides of the moving line then k is negative and may be represented by $- b^2$. The general equation has the form:

$$x^2 + y^2 = - \frac{1}{b^2} + \frac{2\ c\ x}{b^2}. \tag{8}$$

If the axes are translated to the point $c\ x - 1 = 0$ as a new origin, the equation reduces to:

$$a^2 x^2 - b^2 y^2 = 1 \tag{9}$$

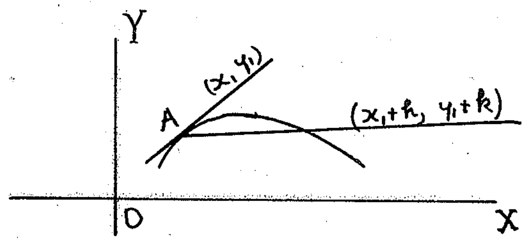
where $a^2 = c^2 - b^2$.

This equation shows symmetry with both axes. The X axis is cut at the points $\pm a\ x - 1 = 0$.

24. <u>The equation of the point of contact of a tangent to a curve whose equation is:</u>

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

Draw a tangent (x_1,y_1) and secant $(x_1+h,\ y_1+k)$ passing through point A of the curve.



The equation of the point A is $\frac{Y-Y_1}{X-X_1}=\frac{k}{h}$

The co-ordinates $(x_1+h,\ y_1+k)$ and (x_1,y_1) satisfy the equation of the curve.

Whence:

$$A(x_1+h)^2+2H(x_1+h)(y_1+k)+B(y_1+k)^2+2G(x_1+h)+2F(y_1+k)+C=0$$

and

$$Ax_1^2+2Hx_1y_1+By_1^2+2Gx_1+2Fy_1+C=0$$

Subtracting we get:

$$A(2hx_1+h^2)+2H(kx_1+hy_1+hk)+B(2ky_1+k^2)+2Gh+2Fk=0$$

Therefore $\frac{k}{h}=-\frac{2Ax_1+Ah+Hk+2Hy_1+2G}{2Hx_1+Hh+2By_1+Bk+2F}$

Therefore limit $\frac{k}{h}=-\frac{Ax_1+Hy_1+G}{Hx_1+By_1+F}$

$h \doteq 0$
$k \doteq 0$

Whence eliminating k and h the equation of point A becomes:

$$\frac{Y-Y_1}{X-X_1}=-\frac{Ax_1+Hy_1+G}{Hx_1+By_1+F}$$

This reduces to the form:

$$Ax_1x+H(y_1x+x_1y)+By_1y+G(x+x_1)+F(y+y_1)+C=0 \qquad (10)$$

Rule:- The equation of the point of contact of a tangent (x_1,y_1) to a curve whose equation is of the second degree may be obtained by the following rule:- In the equation of the curve replace x^2 by x_1x, y^2 by y_1y, xy by $\frac{y_1x+x_1y}{2}$, x by $\frac{x+x_1}{2}$, and y by $\frac{y+y_1}{2}$. The result is the equation of the point of contact.

Note:- In the above method, it may readily be shown that $h \doteq 0$, and $k \doteq 0$ as the secant approaches coincidence with the tangent.

For, let $x_1 = \frac{1}{a}$ and $x_1 + h = \frac{1}{a+\alpha}$

then $h = \frac{-\alpha}{a(a+\alpha)} = \frac{-1}{\frac{a^2}{\alpha} + a}$

Hence $\lim\limits_{\alpha \doteq 0} h = 0.$

Similarly $\lim\limits_{\alpha \doteq 0} k = 0.$

The co-ordinates of a normal at any point of contact can be written at once by virtue of (18) Chapter 1.

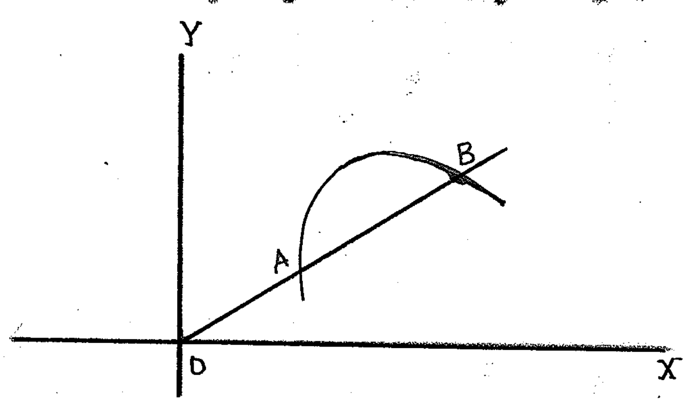
It will be shown later that the equation:

$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$ represents in general, the equation of a conic.

25. <u>The equation of the points in which a given radius vector cuts the conic.</u>

The equation of the conic is :

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0 \qquad (\alpha)$$



The equation of any point on a vector whose angle is $\theta = \tan^{-1} m$, is $y = mx + k$ $\qquad (\beta)$

This point lies on the conic if equations (α) and (β) give equal roots on solution.

Eliminating y we have:

$$(A + 2Hm + Bm^2)x^2 + 2(Hk + Bkm + G + Fm)x + Bk^2 + 2Fk + C = 0.$$

The condition for equal roots gives:

$$k^2(H^2 - AB) + 2k\left[(GH - AF) + m(BG - FH)\right] + (G^2 - AC) + 2m(FG - CH) + m^2(F^2 - BC) = 0.$$

Whence:

$$k = \frac{(AF - GH) + m(FH - BG) \pm \sqrt{-\Theta(A + 2Hm + Bm^2)}}{H^2 - AB}$$

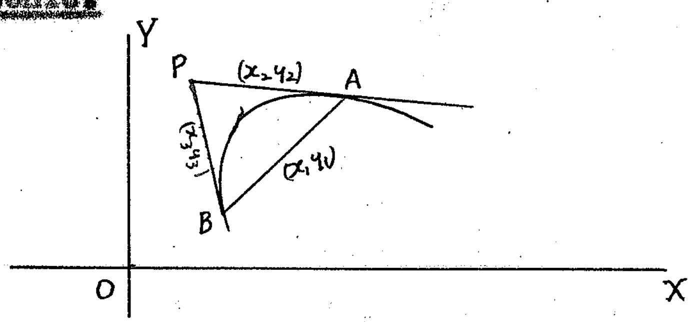
Where $\Theta = ABC + 2FGH - AF^2 - BG^2 - CH^2$

The equations of the required points are given by

$$y = mx + \frac{(AF - GH) + m(FH - BG) \pm \sqrt{\Theta(A + 2Hm + Bm^2)}}{H^2 - AB} \qquad (11)$$

An examination of these equations will give the co-ordinates of the tangent lines at the two points.

26. <u>The equation of the point of intersection of tangents to the conic.</u>



Let the tangents be $(x_2 y_2)$ and (x_3, y_3) and let (x_1, y_1) represent the chord of contact. The equations of A is

$$Ax_2x + H(y_2x + x_2y) + By_2y + G(x + x_2) + F(y + y_2) + C = 0.$$

Equation of B is:

$$Ax_3x + H(y_3x + x_3y) + By_3y + G(x + x_3) + F(y + y_3) + C = 0.$$

Both of these equations are satisfied by the co-ordinates $(x_1 y_1)$.

Therefore $\quad Ax_2x_1 + H(y_2x_1 + x_2y_1) + By_2y_1 + G(x_1 + x_2) + F(y_1 + y_2) + C = 0$

and $\quad Ax_3x_1 + H(y_3x_1 + x_3y_1) + By_3y_1 + G(x_1 + x_3) + F(y_1 + y_3) + C = 0.$

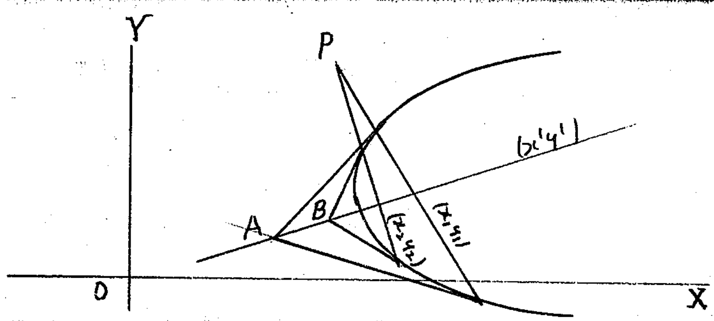
Whence the equation of the required point of intersection is:

$$\left.\begin{array}{l} Ax_1x + H(y_1x + x_1y) + By_1y + G(x + x_1) + F(y + y_1) + C = 0 \\ \text{or} \quad (Ax_1 + Hy_1 + G)x + (Hx_1 + By_1 + F)y + (Gx_1 + Fy_1 + C) = 0. \end{array}\right\} \quad (12)$$

27. **Definition.**

The pole of a line with respect to a conic is the envelope of the chords of contact of tangents to the conic from points on the given line. The line is called the polar of the point.

28. <u>The equation of the pole of a line $(x'y')$.</u>



Let A and B be any two points on the line $(x'y')$.

Let (x_1y_1) and (x_2y_2) be the chords of contact of tangents from A and B respectively.

The equation of A is:

$$(Ax_1 + Hy + G)x + (Hx_1 + By_1 + F)y + (Gx_1 + Fy_1 + C) = 0$$

The equation of B is:

$$(Ax_2 + Hy_2 + G)x + (Hx_2 + By_2 + F)y + (Gx_2 + Fy_2 + C) = 0.$$

These equations are satisfied by the co-ordinates $(x'y')$.

Therefore $(Ax_1 + Hy_1 + G)x' + (Hx_1 + By_1 + F)y' + (Gx_1 + Fy_1 + C) = 0$

and $(Ax_2 + Hy_2 + G)x' + (Hx_2 + By_2 + F)y' + (Gx_2 + Fy_2 + C) = 0$

whence $(Ax' + Hy' + G)x_1 + (Hx' + By' + F)y_1 + (Gx' + Fy' + C) = 0$

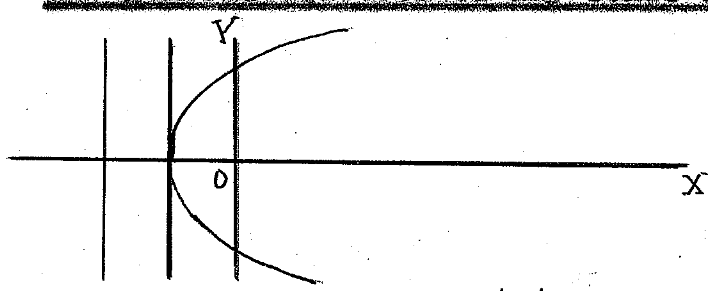
and $(Ax' + Hy' + G)x_2 + (Hx' + By' + F)y_2 + (Gx' + Fy' + C) = 0$

These equations show that (x_1y_1) and (x_2y_2) satisfy the equation $(Ax' + Hy' + G)x + (Hx' + By' + F)y + (Gx' + Fy' + C) = 0$. (13)

This equation is satisfied by the co-ordinates of any chord of contact. It must therefore represent the pole, which is therefore a point.

The form of the equation of the pole of (x', y') with respect to the conic is identical with that for the point of tangency of the tangent (x', y'), and also with that for the point of intersection of tangents at the extremity of a chord (x', y'). Hence if the polar is a tangent to the conic, its pole is the point of contact, and if the pole lies outside the conic, its polar is the chord of contact of tangents to the curve from the pole.

29. <u>The general equation of any conic is $kx^2 + ky^2 + 2cx - 1 = 0$</u>.



The pole of any line (x', y') with respect to this conic is: $\frac{(kx' + c)x}{1 - cx'} + \frac{ky'y}{1 - cx'} - 1 = 0$.

The equation of the focus is $ox \quad oy - 1 = 0$. Whence on equating coefficients the polar of the focus is seen to be the line $(-\frac{c}{k}, 0)$. This line is called the <u>directrix</u> of the conic.

A line through the focus perpendicular to the principal axis and terminated by the conic, is called the <u>latus rectum</u>. From the definition of a polar and from considerations of symmetry, it is seen that tangents at the extremity of the latus rectum meet at the point where the principal axis is cut by the directrix. The equation of this point is $kx + c = 0$.

Substituting $x = -\frac{c}{k}$ in the equation of the conic, we get

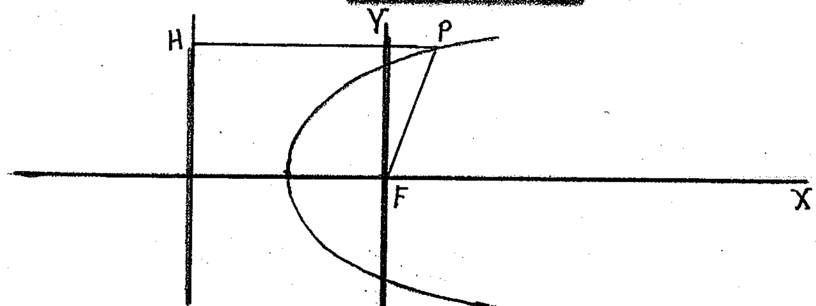
$$\frac{c^2}{k} + ky^2 - \frac{2c^2}{k} - 1 = 0.$$

Whence $y = \sqrt{\frac{k+c^2}{k}}$

The length of the latus rectum is then $\frac{2k}{\sqrt{k+c^2}}$

For the ellipse this becomes $\frac{2b^2}{a}$. The same value is obtained for the hyperbola, whereas after obvious reduction the value 4p is obtained in the case of the parabola.

The ratio of the distances from the focus and from the directrix to any point on a conic is a constant. This constant is called the <u>eccentricity</u> and is denoted by <u>e</u>.



Let $\frac{(kx'+c)x}{1-cx'} + \frac{ky'}{1-cx'}y - 1 = 0$ be any point of the conic $kx^2 + ky^2 + 2cx - 1 = 0$.

Then

$$e = \frac{FP}{HP} = \frac{\sqrt{\left(\frac{kx'+c}{1-cx'}\right)^2\left(\frac{ky'}{1-cx'}\right)^2}}{\frac{1-\left(\frac{kx'+c}{1-cx'}\right)\left(\frac{-c}{k}\right)}{\frac{c}{k}}} = \frac{c}{k+c^2}.$$

For the ellipse $e = \frac{c}{b^2+c^2} = \frac{c}{a} < 1$.

For the parabola we take the limit of $\frac{c}{b^2+c^2}$ as b and c become infinite but $\frac{b}{c}$ remaining finite. This gives $e=1$

Also for the hyperbola $e = \frac{c}{c^2-b^2} = \frac{c}{a} > 1$.

Chapter 3.

The General Equation of the Second Degree.

1. The condition under which the general second degree equation represents a pair of points.

The equation:

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0 \tag{1}$$

represents a pair of points whenever x can be expressed rationally in terms of y. Solving we get:

$$x = \frac{-2(Hy+G) \pm \sqrt{4(H^2-AB)y^2 + 8(GH-AF)y + 4(G^2-AC)}}{2A}$$

In order that x shall be rational, the expression under the radical must be a perfect square.

Therefore: $G^2H^2 - 2AFGH + A^2F^2 - G^2H^2 + ABG^2 + ACH^2 - A^2BC = 0$

Whence, dividing by A, we have

$$\Theta = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0 \tag{2}$$

If $A = 0$, the same condition may be obtained by solving for y. If $A = 0$ and also $B = 0$, the same result may still be obtained by expressing (1) as the product of linear factors. Therefore in all cases (2) is the condition that (1) shall represent a pair of points.

2. Reduction of the general second degree equation to the standard form.

We shall show that equation (1) can be reduced to one of the five following types.

$$\left.\begin{array}{ll} & A_2x^2 + B_2y^2 + C = 0 \\ \text{or} & B_2y^2 + C = 0 \end{array}\right\} \text{ when } C \neq 0$$

$$\left.\begin{array}{ll} \text{and} & B_2y^2 + G_2x = 0 \\ & x(A_1x + 2H_1y + 2G_1) = 0 \text{ with } G_1 \neq 0 \\ \text{or} & Ax^2 + 2Hxy + By^2 = 0 \end{array}\right\} \text{ when } C = 0.$$

(a) $C \neq 0$. Translating the axes to the point

$rx + sy - 1 = 0$ as a new origin, then equation (1) becomes

$$(A + 2Gr + Cr^2)x^2 + 2(H + Gs + Fr + Crs)xy + (B + 2Fs + Cs^2)y^2$$

$$2(G + Cr)x + 2(F + Cs)y + C = 0 \qquad (3)$$

The x and y terms will vanish if:

$$r = -\frac{G}{C} \quad \text{and} \quad s = -\frac{F}{C}.$$

Equation (3) then reduces to:

$$\frac{(AC-G^2)}{C}x^2 + \frac{2(CH-FG)xy}{C} + \frac{(BC-F^2)y^2}{C} + C = 0$$

i.e. $$A_1 x^2 + 2H_1 xy + B_1 y^2 + C = 0 \qquad (4)$$

where $A_1 = \frac{AC-G^2}{C}$, $H_1 = \frac{CH-FG}{C}$, $B_1 = \frac{BC-F^2}{C}$.

On rotating the axes through an angle θ, given by

$$\tan 2\theta = \frac{2H_1}{A_1 - B_1} = \frac{2(CH-FG)}{AC-BC+F^2-G^2}$$

equation (4) reduces to the form:

$$A_2 x^2 + B_2 y^2 + C = 0. \qquad (5)$$

A_2 and B_2 evidently cannot both be zero.

If $A_2 = 0$, $B_2 \neq 0$, we have the form

$$B_2 y^2 + C = 0 \qquad (6)$$

If $B_2 = 0$, $A_2 \neq 0$, we get

$$A_2 x^2 + C = 0 \qquad (6)'$$

which is reducible to (6) on rotation.

Whence if $C \neq 0$, we get on reduction the two possible forms (5) and (6).

(b) $C = 0$, assume F and G are not both zero, as otherwise $\Theta = 0$ and the equation is $Ax^2 + 2Hxy + By^2 = 0$, which is factorable. On rotating the axes through an angle $\theta = \tan^{-1}\frac{F}{G}$, the transformation formulas become:

$$x = \frac{1}{\sqrt{F^2 + G^2}} (Gx_1 - Fy_1)$$

$$y = \frac{1}{\sqrt{F^2 + G^2}} (Fx_1 + Gy_1),$$

and equation (1) reduces to the form:

$$A_1 x^2 + 2H_1 xy + B_1 y^2 + 2C_1 x = 0 \qquad (7)$$

$$A_1 = AG^2 + 2FGH + BF^2$$

Where $H_1 = HG^2 + BFG - AFG - F^2H$

$$B_1 = AF^2 - 2FGH + BG^2 = -\Theta$$

$$C_1 = (F^2 + G^2)^{3/2} \neq 0 .$$

The co-ordinates of vertical tangents to (7) are

$\left(-\frac{2C_1}{A_1}, 0\right)$ and $(0,0)$, the latter being the line at infinity.

The equation of the point of contact corresponding to the first tangent is:

$$-C_1 x - \frac{2C_1 H_1}{A_1} y - \frac{2C_1^2}{A_1} = 0.$$

Since $C_1 \neq 0$, this is equivalent to:

$$\left(-\frac{A_1}{2C_1}\right) x + \left(-\frac{H_1}{C_1}\right) y - 1 = 0.$$

Translate axes to this point as a new origin.

Equation (7) reduces to the form:

$$B_1 y^2 + 2C_1 x = 0 . \qquad (8)$$

If $B_1 = -\Theta \neq 0$, this equation may be written as:

$$y = -\frac{2C_1}{B_1} x = \frac{2(-\xi)^{3/2}}{\Theta} x \qquad (8)'$$

where in general:

$$\xi = C(A + B) - (F^2 + G^2).$$

If $B_1 = -\Theta = 0$, then the equation (7) may be written as:

$x(A_1 x + 2H_1 y + 2C_1) = 0$, and so represents two points, one of which is an infinitely distant point on the X axis.

5. **Invariants under rotation and translation.**

(a) **Rotation.**

The transformationsfor rotation are identical with those in rectangular point co-ordinates. Hence under rotation there are the usual invariants,

viz: $C, A+B,\quad \Delta = H^2 - AB,\quad \xi = C(A+B)-(F^2+G^2)$

and

$$\Theta = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

(b) **Translation.**

It is obvious that under any translation C remains invariant.

Under the translation;

$$x = \frac{x_1}{1+rx_1+sy_1}, \quad y = \frac{y_1}{1+rx_1+sy_1}.$$

The expression $Ax^2+2Hxy+By^2$ reduces to:

$Ax_1^2+2H\,x_1y_1+By_1^2$ and so is invariant in form for any value of A H and B.

Hence when the above translation is applied to the general conic with $C \neq 0$,

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C = 0 \qquad (9)$$

reduces to the form:

$$A'x_1^2+2H'xy_1+B'y_1^2+2G'x+2F'y+C = 0 \qquad (10)$$

while $Ax^2+2Hxy+By^2+2Gx+2Fy+C+k(x^2+y^2) = 0 \qquad (11)$

reduces to:

$$A'x^2+2H'xy+B'y^2+2G'x+2F'y+C+k(x^2+y^2)=0. \qquad (12)$$

Equation (11) separates into linear factors

if $\begin{vmatrix} A+k & H & G \\ H & B+k & F \\ G & F & C \end{vmatrix} = 0$

i.e. if $Ck^2+\left[C(A+B)-(F^2+G^2)\right]k+\Theta = 0 \qquad (13)$

Similarly (12) degenerates if:

$$Ck^2 + \left[C(A' + B') - (F'^2 + G'^2)\right]k + \Theta' = 0 \qquad (14)$$

Equations (11) and (12) must degenerate for the same values of k. Hence equations (13) and (14) are equivalent.

Therefore: $\xi = C(A + B) - (F^2 + G^2)$ and

$$\Theta = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$ are invariant under translation.

The roots of the equations (13) and (14) remain equal as $C \doteq 0$. Hence our conclusions are also valid when $C = 0$.

It will be shown later that $\Delta = H^2 - AB$ has relation to the position of the origin and so is obviously not invariant under translation. $A + B$ is also not invariant under translation as this transformation applied to the general conic gives: $A' = A + 2Gr + Cr^2$

$B' = B + 2Fs + Cs^2$.

The invariants C, ξ and Θ are common to rotation and translation.

4. <u>Absolute invariants and reduction to standard forms.</u>

The expressions $\frac{\xi}{C^2}$, $\frac{\Theta}{C^3}$ and $\frac{\Theta^2}{\xi^3}$ are evidently absolute invariants under rotation and translation.

(a) It has been shown that the equation:

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

with $C \neq 0$ and $\Theta \neq 0$ may be reduced to the form:

$$ax^2 + by^2 - 1 = 0$$

Here $\xi' = -(a + b)$, $C' = -1$, $\Theta' = -ab$

Therefore: $a + b = -\frac{\xi'}{C'^2} = -\frac{\xi}{C^2}$

and $ab = \frac{\Theta'}{C'^3} = \frac{\Theta}{C^3}$.

Therefore: a and b are roots of the equation:

$$x^2 + \frac{\xi}{C^2}x + \frac{\Theta}{C^3} = 0$$

and so may be readily obtained.

(b) The equation: $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy = 0$ with $\Theta \neq 0$, has the absolute invariant $\dfrac{\xi^3}{\Theta^2}$.

This equation may be reduced to the form: $y^2 + 2kx = 0$

whence $\xi' = -k^2$, $\Theta' = -k^2$

therefore: $\dfrac{\xi'^3}{\Theta'^2} = \dfrac{\xi^3}{\Theta^2} = \dfrac{-k^6}{k^4} = -k^2$

therefore:

$$k = \pm \frac{(-\xi)^{3/2}}{\Theta}$$ and the equation $y^2 + 2kx = 0$

takes the form:

$$y^2 = \pm \frac{2(-\xi)^{3/2}}{\Theta}x$$

The reduction may be effected so as to always obtain the form: $y^2 = \dfrac{2(-\xi)^{3/2}}{\Theta}x$.

6. <u>Nature of the locus of the general second degree equation.</u>

It has been shown that the equation :

$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$ may be reduced to one of five forms.

I. $A_2x^2 + B_2y^2 + C = 0$, $C \neq 0$, $A_2 \neq 0$, $B_2 \neq 0$.

Here $\Theta = A_2B_2C \neq 0$, and $\xi = C(A+B)$

This equation represents an ellipse if C differs in sign from A_2 and B_2. Hence if C and Θ agree in sign while $\xi < 0$ the general equation represents an ellipse. There is no real locus when A_2, B_2 and C agree in sign, i.e. the general equation has no real locus if C and Θ agree in sign and $\xi > 0$. We have a hyperbola when A_2 and B_2 differ in sign. In this case Θ is positive if C is negative and Θ is negative if C is positive. Therefore the general equation represents a hyperbola if C and Θ differ in sign while ξ has any value, positive, negative or zero.

II. $B_2 y^2 + C_2 x = 0$, $C = 0$, $B_2 \neq 0$, $C_2 \neq 0$.

Here $\Theta = -B_2 C_2^2 \neq 0$

$\xi = -C_2^2 < 0$

Hence the general equation represents a parabola if $C = 0$, $\Theta \neq 0$. This equation may be written in the form:

$$y^2 = \frac{2(-\xi)^{3/2}}{\Theta} x.$$

III. $B_2 y^2 + C = 0$, $C \neq 0$, $B_2 \neq 0$.

Here $\Theta = 0$, and $\xi = CB_2$. The equation factors if B_2 and C differ in sign and then represents two real points, otherwise it has no real locus. Hence the general equation represents two real and distinct points if $C \neq 0$, $\Theta = 0$ and $\xi < 0$. If $C \neq 0$, $\Theta = 0$ while $\xi > 0$, the general equation represents two imaginary points. The same conditions apply to the form:

$$A_2 x^2 + C = 0$$

IV. $x(A_1 x + 2H_1 y + 2G_1) = 0$, with $G_1 = (F^2 + G^2)^{3/2} \neq 0$

Here $C = 0$, $\Theta = 0$ and $\xi \neq 0$.

Hence if $C = 0$, $\Theta = 0$ and $\xi \neq 0$, the general equation represents two real points, one of which is infinitely distant on the X axis.

V $Ax^2 + 2Hxy + By^2 = 0$, $C = 0$, $F = 0$, $G = 0$.

Here $\Theta = 0$, and $\xi = 0$.

On rotation of axes, this equation reduces to the form:

$$A_2 x^2 + B_2 y^2 = 0.$$

For rotation $\Delta = H^2 - AB = -A_2 B_2$ is invariant. Accordingly under the above conditions, the general equation represents two real points when $\Delta > 0$ and two imaginary points if $\Delta < 0$. If $\Delta = 0$, either $A_2 = 0$, or $B_2 = 0$, and then the equation represents two infinitely distant coincident real points.

The above facts are summarised in the following table.

I	$\Theta \neq 0$ $C \neq 0$	Ellipse if $\xi < 0$ and C and Θ agree in sign. Imaginary locus if $\xi > 0$ and C and Θ agree in sign. Hyperbola if C and Θ differ in sign, while ξ has any value.
II	$\Theta \neq 0$ $C = 0$	Parabola. ξ is always negative.
III	$\Theta = 0$ $C \neq 0$	Two real points if $\xi < 0$. Two imaginary points if $\xi > 0$.
IV	$\Theta = 0$ $C = 0$ $\xi \neq 0$	Two distinct real points, one of which is infinitely distant, and the other is in a finite region of the plane.
V	$\Theta = 0$ $C = 0$ $\xi = 0$	Two infinitely distant real points if $\Delta > 0$. Two coincident and infinitely distant points if $\quad = 0$. Two imaginary points on the line at infinity if $\Delta < 0$.

6. <u>Centre of the conic represented by the general equation with $C \neq 0$ and $\Theta \neq 0$.</u>

Any central conic may have its equation reduced to the form: $ax^2 + by^2 - 1 = 0$. If (k_1x_1, k_1y_1) and (k_2x_1, k_2y_1) are any parallel tangents, then their points of contact have the equations $k_1(ax_1x + by_1y) - 1 = 0$

and $k_2(ax_1x + by_1y) - 1 = 0$

These points are co-linear with the origin, whose equation is

$$1 = 0,$$

since
$$\begin{vmatrix} k_1a & k_1b & 1 \\ k_2a & k_2b & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

Hence for central conics the chords joining points of contact of parallel tangents pass through the centre. Therefore the centre may be considered as the pole of the line at infinity, viz., the line (0,0). The pole of (0,0) with respect to the conic $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$

is $\quad Gx + Fy + C = 0.$ (15)

Hence the centre of the general conic is the point

$$Gx + Fy + C = 0.$$

7. <u>The polar of the origin with respect to the general conic.</u>

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0.$$

The general equation of the pole of line (x', y') is

$$(Ax' + Hy' + G)x + (Hx' + By' + F)y + Gx' + Fy' + C = 0 .$$

In order for this to be the point $1 = 0$, we have the equations

$$Ax' + Hy' + G = 0$$

$$Hx' + By' + F = 0$$

since $\quad Gx' + Fy' + C \neq 0$, otherwise the polar of the origin would pass through the centre and the origin would be at infinity. Solving we get:

$$\left.\begin{aligned} x' &= \frac{BG-FH}{H^2-AB} \\ y' &= \frac{AF-GH}{H^2-AB} \end{aligned}\right\} \qquad (16)$$

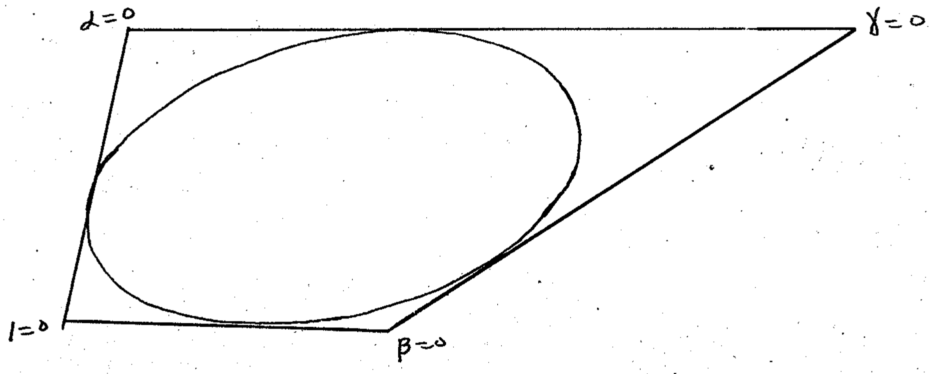
as the co-ordinates of the polar of the origin with respect to the general conic. If the origin lies on the conic, then its polar is the tangent at that point with co-ordinates (∞, ∞). Hence $\Delta \equiv H^2 - AB = 0$, is the condition that the general conic shall pass through the origin.

8. <u>The position of the origin with respect to the conic.</u>

In abridged notation.

$\alpha\beta + K\gamma\delta = 0$ represents any conic inscribed in the quadrilateral whose vertices are the points $\alpha = 0$, $\beta = 0$, $\gamma = 0$ and $\delta = 0$.

Likewise $\alpha\beta + K111\gamma = 0$ represents any conic inscribed in the quadrilateral whose vertices are the origin and the points $\alpha = 0$, $\beta = 0$, $\gamma = 0$.



In order that the general conic may be written in this latter form, the terms of second degree resolve into real factors. Hence $\Delta = (H^2 - AB) > 0$, is the condition that the origin shall lie outside the conic. Similarly if $\Delta = 0$ the origin lies on the conic, and if $\Delta < 0$ the origin lies within the conic.