THE UNIVERSITY OF MANITOBA

CONSTRUCTIONS OF EXTENDED TRIPLE SYSTEMS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS AND ASTRONOMY

WINNIPEG, MANITOBA R3T 2N2

April 1976

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ABSTRACT

An extended triple system is a pair (S, W) where S is a finite set and W is a collection of unordered triples from S, where each triple may have repeated elements, such that every pair of elements of S, not necessarily distinct, belongs to exactly one triple of W. Algebraically, an extended triple system on an n-set S is equivalent to a quasigroup on S satisfying the laws $x(xy) = y$ and $(yx)x = y$. If all elements of the quasigroup are idempotent, then the system is equivalent to a Steiner triple system. We define $\{n; b\}$ to be the class of all extended triple systems on n elements with b idempotent elements. If $\{n; b\}$ is non-empty, we shall say $\{n; b\}$ exists. It is known that necessary conditions for the existence of $\{n; a\}$, $0 \le a \le n$, are:

(1) if $n \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$;

(2) if $n \not\equiv 0 \pmod 3$, then $a \equiv 1 \pmod 3$;

(3) if n is even, then $a \le \frac{n}{2}$;

(4) if $a = n-1$, then $n = 2$.

This thesis is concerned mainly with the sufficiency of the known necessary conditions for the existence of $\{n; a\}$. A direct method of construction is used to show that if the necessary conditions are satisfied, then $\{n; a\}$ exists. In addition, we shall give recursive methods of construction including the use of the direct and singular direct product of quasigroups. It is shown that, with a few obvious exceptions, there exist at least two non-isomorphic systems in each class $\{n; a\}$.

ACKNOWLEDGEMENTS

I am greatly indebted to Professor N.S. Mendelsohn, who granted me the privilege of pursuing this research under his supervision. His patience, encouragement and many helpful suggestions throughout the various stages of this work have been greatly appreciated.

Thanks are also due to Mr. Barry Wolk for calling my attention to the work of Th. Skolem, and to Professor C.C. Lindner for making other useful references readily available.

The author is particularly grateful to Professors D.M. Johnson and M. Doob for their valuable comments and help in checking the original version of this manuscript, and to Mrs. M. McTavish, who patiently and skilfully performed the formidable task of typing this thesis.

Finally, I wish to express my great indebtedness to the University of Manitoba for financial assistance in the form of a Graduate Fellowship during this study.

CHAPTER 1

Introduction

A Steiner triple system, briefly STS, is a pair (S, T) where S is a finite set and T a collection of 3-subsets of S, called triples, such that every pair of distinct elements of S is contained in exactly one triple of T . The number $|S|$ is called the order of (S, T) . It is known (see, e.g., [5]) that there is an STS of order n, briefly STS(n), if and only if $n \equiv 1$ or 3(mod 6).

Algebraically the set of all Steiner triple systems may be looked upon as a variety of quasigroups satisfying the three identities

$$\text{(i)} \quad x^2 = x \quad \text{(idempotent)}$$

$$\left.\begin{array}{ll} \text{(ii)} & x(xy) = y \\ \text{(iii)} & (yx)x = y \,. \end{array}\right\} \text{(totally symmetric)}$$

Quasigroups satisfying (i), (ii), (iii) are called Steiner quasigroups. Quasigroups satisfying (ii) and (iii) are commutative, and are called totally symmetric quasigroups.

An extended triple system is a pair (S, W) where S is a finite set and W is a collection of non-ordered triples from S, where each triple may have repeated elements, such that every pair of elements of S, not necessarily distinct, belongs to exactly one triple of W .

The class of all extended triple systems is co-extensive with the variety of quasigroups satisfying only the totally symmetric identities $x(xy) = y$ and $(yx)x = y$.

Quasigroups satisfying the totally symmetric identities are not necessarily idempotent. We shall denote by $\{n; b\}$ the class of all extended triple systems on n elements which have b idempotents. Clearly, $\{n; n\}$ is the class of all STS(n) and this class is non-empty if and only if $n \equiv 1$ or $3 \pmod 6$.

We say $\{n; a\}$ exists if there exist systems with parameters n and a. If (S, W) belongs to $\{n; a\}$, we simply write $W \in \{n; a\}$.

Johnson and Mendelsohn [8] gave the following conditions which are necessary for the existence of $\{n; a\}$, $0 \le a \le n$:

(1) if $n \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$;

(2) if $n \not\equiv 0 \pmod 3$, then $a \equiv 1 \pmod 3$;

(3) if n is even, then $a \le \frac{n}{2}$;

(4) if $a = n-1$, then $n = 2$.

It was also conjectured in [8] that the necessary conditions for the existence of $\{n; a\}$ given above were sufficient.

Let (S, W) and (S^*, W^*) be two extended triple systems. If $S \subseteq S^*$ and $W \subseteq W^*$, we shall say that (S, W) is a <u>subsystem</u> of (S^*, W^*) and that (S^*, W^*) contains (S, W). If $W \cap W^* = \emptyset$, then we say that (S, W) and (S^*, W^*) are <u>disjoint</u>. If there is a bijection $\alpha : S \longrightarrow S^*$ such that $(W)\alpha = W^*$, then we say that (S, W) and (S^*, W^*) are <u>isomorphic</u> (or <u>equivalent</u>).

There are essentially two inequivalent systems based on the set $S = \{1, 2, 3\}$. We shall denote these by

$$\mathfrak{T}_0 = \{(1, 1, 2), (2, 2, 3), (3, 3, 1)\} \in \{3; 0\},$$

$$\mathfrak{T}_3 = \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3)\} \in \{3; 3\}.$$

There are two inequivalent systems on $S = \{1, 2, 3, 4\}$. We denote these by

$\mathcal{K}_4 = \{(1, 1, 1), (2, 2, 1), (3, 3, 1), (4, 4, 1), (2, 3, 4)\} \in \{4; 1\}$,

$\mathcal{K}_4^* = \{(1, 1, 1), (2, 2, 1), (3, 3, 2), (4, 4, 2), (1, 3, 4)\} \in \{4; 1\}$.

Algebraically $\mathcal{K}_4$ is associated with the "four-group". Up to isomorphism, there is only one system on $S = \{1, 2, 3, 4, 5\}$. We denote this system by

$$\mathcal{V}_5 = \{(1,1,1), (2,2,3), (3,3,4), (4,4,5), (5,5,2), (1,2,4), (1,3,5)\} \in \{5; 1\}.$$

A system $W \in \{n; a\}$ which contains the maximum possible number $\left[\frac{a}{3}\right]$ mutually disjoint copies of $\mathcal{T}_3$ is called a consistent system. ($[x]$ denotes the largest integer not exceeding x.)

For convenience and completeness we state

Theorem 1.1 (Johnson and Mendelsohn [8]). Suppose $\{n; a\}$ exists with $0 \le a \le n$. Then

(1) if $n \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$;

(2) if $n \not\equiv 0 \pmod 3$, then $a \equiv 1 \pmod 3$;

(3) if n is even, then $a \le \frac{n}{2}$;

(4) if $a = n-1$, then $n = 2$.

Chapter 2 of this thesis deals mainly with some of the essential properties of $\{n; a\}$. Some basic constructions are given, and examples of consistent systems, for small values of n, are included at the end. These examples provide enough initial cases for recursive constructions which follow. In Chapter 3 we use direct construction methods to show that the necessary conditions for the existence of $\{n; a\}$ given in Theorem 1.1 are also sufficient. Small embeddings of extended triple systems are considered in Chapter 4, and a recursive method of construction is developed. An algebraic approach is taken in

Chapter 5. In this chapter we construct consistent systems using Direct Products and Singular Direct Products. In Chapter 6 we comment on the existence of non-isomorphic extended triple systems, and some examples of non-isomorphic systems are given. Apart from a few exceptions, it is shown that there exist at least two non-isomorphic systems in each class $\{n; a\}$.

CHAPTER 2

Properties of $\{n; a\}$ and some basic constructions

1. Preliminaries.

A partial Steiner triple system is a pair (P, T) where P is a finite set and T a collection of 3-subsets of P, called triples, such that every pair of distinct elements of P is contained in at most one triple of T. Unlike Steiner triple systems, there is no cardinality restriction on P. The number $|P|$ is called the order of the partial Steiner triple system (P, T). A partial Steiner triple system (P, T) is called maximal if $|T'| \leq |T|$ for every partial Steiner triple system (P, T'). It is known [7, 19] that a maximal partial Steiner triple system of order n exists for every positive integer n. In particular, if $n \equiv 1$ or $3 \pmod 6$, it is clear that a maximal partial Steiner triple system of order n is indeed an STS(n). In the following section we shall be concerned mainly with maximal partial Steiner triple systems.

2. Some useful connections.

It is already clear that $\{n; n\}$ exists if and only if there exists an STS(n). The purpose of this section is to establish further connections between a certain class of extended triple systems and maximal partial Steiner triple systems. We shall utilize some properties of $\{n; a\}$ determined in [8].

If there is a system $W \in \{n; a\}$, the triples of W are essentially of three types: (1) (i, i, i), (2) (j, j, k), (3) (i, j, k), where i, j, k are pairwise distinct. Johnson and Mendelsohn [8] proved the following

Lemma 2.1. Suppose there is a system $W \in \{n; a\}$.

(i) If n is even, then amongst the type (2) triples (j, j, k) each idempotent appears an odd number of times as the single element k, and each non-idempotent appears an even number of times.

(ii) If n is odd, then amongst the type (2) triples (j, j, k) each non-idempotent must appear at least once as the element k. In particular, if the non-idempotents are $1, 2, 3, \cdots, b$, then in the triples $(1, 1, \alpha_1), (2, 2, \alpha_2), (3, 3, \alpha_3), \cdots, (b, b, \alpha_b)$ the elements $\alpha_1, \alpha_2, \cdots, \alpha_b$ are a permutation of $1, 2, \cdots, b$ with $\alpha_i \neq i$.

(iii) The number of type (3) triples (i, j, k) is

$$\frac{n(n-1)}{6} - \frac{(n-a)}{3}.$$

J. Schönheim [19] proved in essence

Lemma 2.2. Let (P, T) be a maximal partial Steiner triple system of order n. Then

$$|T| = \begin{cases} 6k^2 - 2k & \text{if} \quad n = 6k, \\ 6k^2 + k & \text{if} \quad n = 6k+1, \\ 6k^2 + 2k & \text{if} \quad n = 6k+2, \\ 6k^2 + 5k + 1 & \text{if} \quad n = 6k+3, \\ 6k^2 + 6k + 1 & \text{if} \quad n = 6k+4, \\ 6k^2 + 9k + 2 & \text{if} \quad n = 6k+5\ . \end{cases}$$

We are now in a position to prove the following theorems.

Theorem 2.3. There is a system $W \in \{n; \frac{n}{2}\}$ if and only if there is an $STS(n+1)$.

Proof. Let (S, T) be an $STS(n+1)$ where $S = \{1, 2, \cdots, n+1\}$. We shall eliminate from T the collection of all triples containing a particular element, say $n+1$. Let $T^{(n+1)}$ be the collection of all triples of T containing the element $n+1$ and assume, without loss of generality, $T^{(n+1)} = \{(n+1, 1, 2), (n+1, 3, 4), \cdots, (n+1, n-1, n)\}$. Let $T^* = \{(1, 1, 2), (2, 2, 2), (3, 3, 4), (4, 4, 4), \cdots, (n-1, n-1, n),$
$(n, n, n)\}$.

Put $W = (T - T^{(n+1)}) \cup T^*$. Then $W \in \{n; \frac{n}{2}\}$, based on $S - \{n+1\}$.

Assume there is a system $W \in \{n; \frac{n}{2}\}$ based on the set $S^* = \{1, 2, \cdots, n\}$. Introduce a new element ∞ and put $S = S^* \cup \{\infty\}$. We now derive from W an $STS(n+1)$.

By Theorem 1.1, $n \equiv 0$ or $2 (\text{mod } 6)$. So, by Lemma 2.1, we find that amongst the type (2) triples (j, j, k) in W, the element k must be idempotent. So, up to permuting the elements of S^*, W must contain the triples $J = \{(1, 1, 2), (2, 2, 2), (3, 3, 4), (4, 4, 4),$
$\cdots, (n-1, n-1, n), (n, n, n)\}$.

Let $T_\infty = \{(1, 2, \infty), (3, 4, \infty), \cdots, (n-1, n, \infty)\}$. Put $T = (W - J) \cup T_\infty$. Then it is easily seen that (S, T) is an $STS(n+1)$, which completes the proof.

Remark 1. In our construction of $W \in \{n; \frac{n}{2}\}$ there are several possibilities for the collection T^*, because in each of the $\frac{n}{2}$ disjoint pairs $(1, 2), (3, 4), \cdots, (n-1, n)$ any one of the two elements may be idempotent in W. This is a crucial point, since our choice of T^* may very well determine the internal structure of the system W.

For example, let (S, T) be the $STS(7)$ where
$S = \{1, 2, 3, 4, 5, 6, 7\}$, and
$T = \{(1, 2, 4), (1, 3, 7), (1, 5, 6), (2, 3, 5), (2, 6, 7), (3, 4, 6), (4, 5, 7)\}$.
$T^{(7)} = \{(1, 3, 7), (2, 6, 7), (4, 5, 7)\}$.
Let W_1 and W_2 be the following two systems derived from (S, T) by deleting $T^{(7)}$ from the collection T:

$$W_1 = \{(1, 1, 1), (3, 3, 1), (2, 2, 2), (6, 6, 2), (5, 5, 5), (4, 4, 5), (1, 2, 4), (1, 5, 6), (2, 3, 5), (3, 4, 6)\} \in \{6; 3\}.$$

$$W_2 = \{(1, 1, 1), (3, 3, 1), (2, 2, 2), (6, 6, 2), (4, 4, 4), (5, 5, 4), (1, 2, 4), (1, 5, 6), (2, 3, 5), (3, 4, 6)\} \in \{6; 3\}.$$

It is easy to check that W_1 contains no copy of $\mathfrak{J}_3$, while W_2 contains a copy of $\mathfrak{J}_3$ on the set $\{1, 2, 4\}$. This remark is not restricted to the construction given in Theorem 2.3. It is worth noting that the collection $\{(i, i, i), (j, j, i)\}$ may always be replaced with the collection $\{(i, i, j), (j, j, j)\}$ within any system. This procedure is treated more generally in the next section.

Theorem 2.4. There is a system $W \in \{6n+4;\ 3n+1\}$ if and only if there is a maximal partial Steiner triple system of order $6n+4$.

Proof. Let (P, T) be a maximal partial Steiner triple system where $P = \{1, 2, \cdots, 6n+4\}$. It follows from Lemma 2.2 that $|T| = 6n^2+6n+1$ and so the triples of T give rise to a total of $18n^2+18n+3$ unordered pairs. As a result, there are $\binom{6n+4}{2} - (18n^2+n+3) = 3n+3$ pairs of elements from P not contained in any triple of T . The number of pairs containing a particular element in (P, T) is an even number, and each element is contained in an odd number $6n+3$ pairs, counting all pairs from P . Consequently, one particular element must appear in 3 of these $3n+3$ excluded pairs, while all other elements appear in one pair. Up to permuting the elements of P , we may assume that the pairs $(1, 2)$, $(1, 3)$, $(1, 4)$, $(5, 6)$, $(7, 8)$, $\cdots$, $(6n+3, 6n+4)$ do not appear in any triple of T , but every other pair appears in exactly one triple of T .

Let
$$T^* = \{(1, 1, 1), (2, 2, 1), (3, 3, 1), (4, 4, 1), (5, 5, 5), (6, 6, 5), (7, 7, 7), (8, 8, 7), \cdots, (6n+3, 6n+3, 6n+3), (6n+4, 6n+4, 6n+3)\} .$$

Put $W = T \cup T^*$. It is easily verified that $W \in \{6n+4;\ 3n+1\}$ based on P . Conversely, let us assume that there is a system $W \in \{6n+4;\ 3n+1\}$ based on some set $P = \{1, 2, \cdots, 6n+4\}$. Let T be the collection of all type (3) triples (i, j, k) in W . By Lemma 2.1, $|T| = \frac{(6n+4)(6n+3)}{6} - (n+1) = 6n^2+6n+1$. It is clear that (P, T) is indeed a maximal partial Steiner triple system of order $6n+4$. This completes the proof of the theorem.

<u>Theorem 2.5.</u> There is a system $W \in \{6n+5; 6n+1\}$ if and only if there is a maximal partial Steiner triple system of order $6n+5$.

<u>Proof.</u> Let (P, T) be a maximal partial Steiner triple system, where $P = \{1, 2, \cdots, 6n+5\}$. By Lemma 2.2, $|T| = 6n^2 + 9n+2$. Thus the triples of T give rise to $18n^2+27n+6$ unordered pairs. There must be $\binom{6n+5}{2} - (18n^2+27n+6) = 4$ pairs of elements of P not contained in any triple of T. Each element is contained in an even number $6n+4$ pairs, counting all pairs from P. In (P, T) the number of pairs containing a particular element is always even. So we may assume, without loss of generality, that the pairs $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 1)$ do not appear in any triple of T, but that every other pair appears in exactly one triple of T.

Let $T^* = \{(1, 1, 2), (2, 2, 3), (3, 3, 4), (4, 4, 1), (5, 5, 5), (6, 6, 6), \cdots, (6n+5, 6n+5, 6n+5)\}$.

Put $W = T \cup T^*$. Clearly $W \in \{6n+5; 6n+1\}$ based on the set P.

Conversely, let us assume that $W \in \{6n+5; 6n+1\}$ based on some set $P = \{1, 2, \cdots, 6n+5\}$. Let T be the collection of all type (3) triples (i, j, k) of W. Then it is easily verified that $|T| = 6n^2+9n+2$ and (P, T) is a maximal partial Steiner triple system of order $6n+5$. This completes the proof.

<u>Theorem 2.6.</u> There is a system $W \in \{6n+4; 3n+1\}$ if and only if there is a system $W^* \in \{6n+5; 6n+1\}$.

<u>Proof.</u> Suppose there is a system $W \in \{6n+4; 3n+1\}$ based on the set $S = \{1, 2, \cdots, 6n+4\}$. Then by Lemma 2.1, each of the $3n+1$

idempotents must appear an odd number of times as the element k in type (2) triples (j, j, k). Up to a permutation on S, it is clear that W must contain one of the two collections of triples:

$A = \{(1, 1, 1), (2, 2, 1), (3, 3, 1), (4, 4, 1), (5, 5, 6), (6, 6, 6),$
$(7, 7, 8), (8, 8, 8), \cdots, (6n+3, 6n+3, 6n+4), (6n+4, 6n+4, 6n+4)\}$,

or

$B = \{(1, 1, 1), (2, 2, 1), (3, 3, 2), (4, 4, 2), (5, 5, 6), (6, 6, 6),$
$(7, 7, 8), (8, 8, 8), \cdots, (6n+3, 6n+3, 6n+4), (6n+4, 6n+4, 6n+4)\}$.

Let us introduce a new element ∞, and put $S^* = S \cup \{\infty\}$. We shall construct $W^* \in \{6n+5; 6n+1\}$ based on S^* as follows.

If W contains the collection A, we let

$A^* = \{(1, 1, 2), (2, 2, \infty), (\infty, \infty, 3), (3, 3, 1), (1, 4, \infty), (4, 4, 4),$
$(5, 5, 5), \cdots, (6n+4, 6n+4, 6n+4), (5, 6, \infty), (7, 8, \infty), \cdots,$
$(6n+3, 6n+4, \infty)\}$.

Then put $W^* = (W - A) \cup A^*$. It is easily verified that $W^* \in \{6n+5; 6n+1\}$, based on S^*.

If W contains the collection B, we let

$B^* = \{(1, 1, \infty), (\infty, \infty, 3), (3, 3, 2), (2, 2, 1), (2, 4, \infty), (4, 4, 4),$
$(5, 5, 5), \cdots, (6n+4, 6n+4, 6n+4), (5, 6, \infty), (7, 8, \infty), \cdots,$
$(6n+3, 6n+4, \infty)\}$.

Put $W^* = (W - B) \cup B^*$. Then $W^* \in \{6n+5; 6n+1\}$. Conversely, we assume there is a system $W^* \in \{6n+5; 6n+1\}$ based on the set $S^* = \{0, 1, 2, \cdots, 6n+4\}$. Again by Lemma 2.1, we find that, up to a permutation on S, W^* must contain the collection of triples

$T^* = \{(0, 0, 1), (1, 1, 2), (2, 2, 3), (3, 3, 0), (4, 4, 4), (5, 5, 5), \cdots,$
$(6n+4, 6n+4, 6n+4)\}$.

We shall construct $W \in \{6n+4;\ 3n+1\}$ by eliminating some triples from W^*, but we must be careful which triples we delete. If we delete all triples containing one of the elements 0, 1, 2, 3, this procedure will always guarantee us the desired result. Otherwise, we may be forced into a situation where the end result is not always a system in $\{6n+4;\ 3n+1\}$. To this end we let T_0^* be the collection of all triples in W^* containing the element 0 and assume, without any loss of generality,

$$T_0^* = \{(0,\ 0,\ 1),\ (3,\ 3,\ 0),\ (0,\ 2,\ 4),\ (0,\ 5,\ 6),\ (0,\ 7,\ 8), \cdots, (0,\ 6n+3,\ 6n+4)\}.$$

Let

$$A^* = \{(2,\ 2,\ 2),\ (1,\ 1,\ 2),\ (3,\ 3,\ 2),\ (4,\ 4,\ 2),\ (5,\ 5,\ 6),\ (6,\ 6,\ 6), (7,7,8), (8,8,8), \cdots, (6n+3, 6n+3, 6n+4), (6n+4, 6n+4, 6n+4)\}.$$

Put $W = (W^* - (T^* \cup T_0^*)) \cup A^*$. It can be checked that $W \in \{6n+4;\ 3n+1\}$ based on $S^* - \{0\}$. This completes the proof of the theorem.

<u>Theorem 2.7.</u> There is a system $W \in \{6n+4;\ 3n+1\}$ containing a copy of $\mathcal{K}_4$ if and only if there is a system $W^* \in \{6n+5;\ 6n+1\}$ containing a copy of $\mathcal{V}_5$.

<u>Proof.</u> Assume there is a system $W \in \{6n+4;\ 3n+1\}$ based on the set $S = \{1,\ 2, \cdots, 6n+4\}$ and containing a copy of $\mathcal{K}_4$ based on the set $\{1,\ 2,\ 3,\ 4\}$. We may assume without loss of generality that W contains the collection

$$A = \{(1,\ 1,\ 1),\ (2,\ 2,\ 1),\ (3,\ 3,\ 1),\ (4,4,1), (2,3,4), (5,5,6), (6,6,6), (7,7,8), (8,8,8), \cdots, (6n+3, 6n+3, 6n+4), (6n+4, 6n+4, 6n+4)\}.$$

Introduce a new element ∞ and put $S^* = S \cup \{\infty\}$.

Let

$A^* = \{(1, 1, 2), (2, 2, \infty), (\infty, \infty, 3), (3, 3, 1), (2, 3, 4), (1, 4, \infty),$

$(4,4,4), (5,5,5), \cdots, (6n+4, 6n+4, 6n+4), (5, 6, \infty), (7, 8, \infty), \cdots,$

$(6n+3, 6n+4, \infty)\}$.

Put $W^* = (W - A) \cup A^*$. Then $W^* \in \{6n+5; 6n+1\}$ and W^* contains a copy of $\mathcal{V}_5$ based on the set $\{1, 2, 3, 4, \infty\}$. Conversely, assume there is a system $W^* \in \{6n+5; 6n+1\}$ based on the set $S^* = \{0, 1, 2, \cdots, 6n+4\}$ and containing a copy of $\mathcal{V}_5$ based on the set $\{0, 1, 2, 3, 4\}$. We may assume that W^* contains the following two collections of triples:

$T^* = \{(0, 0, 1), (1, 1, 2), (2, 2, 3), (3, 3, 0), (0, 2, 4), (1, 3, 4),$

$(4,4,4), (5,5,5), \cdots, (6n+4, 6n+4, 6n+4)\}$, and

$T_0^* = \{(0, 0, 1), (3, 3, 0), (0, 2, 4), (0, 5, 6), (0, 7, 8), \cdots,$

$(0, 6n+3, 6n+4)\}$.

Let

$A^* = \{(2, 2, 2), (1, 1, 2), (3, 3, 2), (4, 4, 2), (1, 3, 4), (5, 5, 6),$

$(6,6,6), (7,7,8), (8,8,8), \cdots, (6n+3, 6n+3, 6n+4),$

$(6n+4, 6n+4, 6n+4)\}$.

Put $W = (W^* - (T^* \cup T_0^*)) \cup A^*$. Then $W \in \{6n+4; 3n+1\}$ and W contains a copy of $\mathcal{K}_4$ based on the set $\{1, 2, 3, 4\}$.

3. The replacement property.

So far we have seen how extended triple systems may be derived from Steiner and similar triple systems. Perhaps more important is the fact that extended triple systems may be derived from each other. We outline here a technique that will prove very effective in most of our constructions that appear in subsequent chapters.

Lemma 2.8. Suppose there is a system $W \in \{n; a\}$ containing a subsystem $Q \in \{q; r\}$. Suppose there is another system $Q^* \in \{q; s\}$. Then there is a system $W^* \in \{n; a-r+s\}$ containing a copy of Q^* .

Proof. We may assume, without loss of generality, that the system Q^* is based on the same set of elements as Q ; for otherwise we can relabel the elements to achieve this. We put $W^* = (W - Q) \cup Q^*$. It is readily verified that $W^* \in \{n; a-r+s\}$.

Remark 2. The fact that we can remove subsystems, and appropriately replace them, will prove quite crucial in most of our constructions. It is worth noting that even if Q and Q^* are isomorphic in Lemma 2.8, then W and W^* need not be isomorphic. For example, in $\mathcal{K}_4$ we may replace the sybsystem $Q = \{(1, 1, 1), (2, 2, 1)\}$ with $Q^* = \{(1, 1, 2), (2, 2, 2)\}$ thus obtaining a copy of $\mathcal{K}_4^*$.

Theorem 2.9. Suppose there is a system $W \in \{n; a\}$ containing t mutually disjoint copies of $\mathfrak{J}_3$. Then there is a system $W_k \in \{n; a-3k\}$ containing at least k mutually disjoint copies of $\mathfrak{J}_0$ and $t-k$ mutually disjoint copies of $\mathfrak{J}_3$, where $k = 1, 2, \cdots, t$.

Proof. The proof follows directly by applying Lemma 2.8. We obtain W_1 from W by removing one of the t disjoint copies of $\mathfrak{J}_3$ and replacing it with the appropriate copy of $\mathfrak{J}_0$. For $k = 1, 2, \cdots, t-1$, we obtain W_{k+1} from W_k by the same procedure.

Corollary 2.10. If there exists a consistent system $W \in \{n;\ a\}$, then there exists a consistent system $W_k \in \{n;\ a-3k\}$ where $k = 1, 2, \cdots, \left[\frac{a}{3}\right]$.

Example 2.11. Let $W \in \{9;\ 9\}$ be given by

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),
(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),
(1, 4, 7), (1, 5, 8), (1, 6, 9), (2, 4, 8), (2, 5, 9), (2, 6, 7),
(3, 4, 9), (3, 5, 7), (3, 6, 8).

We obtain $W_1 \in \{9;\ 6\}$:

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),
(7, 7, 8), (8, 8, 9), (9, 9, 7), (1, 4, 7), (1, 5, 8), (1, 6, 9), (2, 4, 8),
(2, 5, 9), (2, 6, 7), (3, 4, 9), (3, 5, 7), (3, 6, 8).

$W_2 \in \{9;\ 3\}$:

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 5), (5, 5, 6), (6, 6, 4), (7, 7, 8), (8, 8, 9), (9, 9, 7), (1, 4, 7),
(1, 5, 8), (1, 6, 9), (2, 4, 8), (2, 5, 9), (2, 6, 7), (3, 4, 9), (3, 5, 7),
(3, 6, 8).

$W_3 \in \{9;\ 0\}$:

(1, 1, 2), (2, 2, 3), (3, 3, 1), (4, 4, 5), (5, 5, 6), (6, 6, 4), (7, 7, 8),
(8, 8, 9), (9, 9, 7), (1, 4, 7), (1, 5, 8), (1, 6, 9), (2, 4, 8), (2, 5, 9),
(2, 6, 7), (3, 4, 9), (3, 5, 7), (3, 6, 8).

For an application of Lemma 2.8, let us consider the following theorem and example.

Theorem 2.12. Let $n \equiv 2$ or $4 \pmod 6$, $n \geq 4$. Then there exists a system $W \in \{n; 1\}$ which contains at least one copy of $\mathcal{K}_4$.

Proof. Let (S, T) be any $STS(n-1)$. We add to S a new element ∞, and put $S^* = S \cup \{\infty\}$. Let $T^* = \{(x, x, \infty) \mid x \in S^*\}$ and $W = T \cup T^*$. Then $W \in \{n; 1\}$, based on S^*. Since $n \geq 4$, T contains at least one triple (a, b, c). Hence, W contains the collection $K = \{(\infty, \infty, \infty), (a, a, \infty), (b, b, \infty), (c, c, \infty), (a, b, c)\}$, which is a copy of $\mathcal{K}_4$.

Remark 3. The system W constructed in the proof of Theorem 2.12 is associated algebraically with a totally symmetric loop (see for example [2]). It is clear that W does not contain a copy of $\mathcal{K}_4^*$, but contains as many copies of $\mathcal{K}_4$ as there are triples in T. The copies of $\mathcal{K}_4$ pairwise intersect in (∞, ∞, ∞).

Corollary 2.13. Let $n \equiv 2$ or $4 \pmod 6$, $n \geq 4$. Then there exists a system $W^* \in \{n; 1\}$ containing a unique copy of $\mathcal{K}_4^*$.

Proof. Let $W \in \{n; 1\}$ be as constructed in the proof of Theorem 2.12. Let Q be a copy of $\mathcal{K}_4$ in W. Let Q^* be a copy of $\mathcal{K}_4^*$, based on the same set as Q. Put $W^* = (W - Q) \cup Q^*$. A straightforward verification shows $W^* \in \{n; 1\}$ contains Q^* as its only copy of $\mathcal{K}_4^*$.

Example 2.14. This example illustrates Theorem 2.12 and Corollary 2.13 for the cases $n = 8, 10$.

(i)

$$W \in \{8;\ 1\}$$

$(\underline{\infty,\ \infty,\ \infty})$, $(\underline{1,\ 1,\ \infty})$, $(\underline{2,\ 2,\ \infty})$, $(\underline{3,\ 3,\ \infty})$, $(\underline{1,\ 2,\ 3})$,
$(4,\ 4,\ \infty)$, $(5,\ 5,\ \infty)$, $(6,\ 6,\ \infty)$, $(7,\ 7,\ \infty)$, $(1,4,7)$, $(1,5,6)$, $(2,4,5)$,
$(2,\ 6,\ 7)$, $(3,\ 4,\ 6)$, $(3,\ 5,\ 7)$.

$$W^* \in \{8;\ 1\}$$

$(\infty,\ \infty,\ \infty)^*$, $(1,\ 1,\ \infty)^*$, $(2,\ 2,\ 1)^*$, $(3,\ 3,\ 1)^*$, $(2,\ 3,\ \infty)^*$,
$(4,\ 4,\ \infty)$, $(5,\ 5,\ \infty)$, $(6,\ 6,\ \infty)$, $(7,\ 7,\ \infty)$, $(1,4,7)$, $(1,5,6)$, $(2,4,5)$,
$(2,\ 6,\ 7)$, $(3,\ 4,\ 6)$, $(3,\ 5,\ 7)$.

(ii)

$$W \in \{10;\ 1\}$$

$(\underline{\infty,\ \infty,\ \infty})$, $(\underline{1,\ 1,\ \infty})$, $(\underline{2,\ 2,\ \infty})$, $(\underline{3,\ 3,\ \infty})$, $(\underline{1,\ 2,\ 3})$,
$(4,\ 4,\ \infty)$, $(5,\ 5,\ \infty)$, $(6,\ 6,\ \infty)$, $(7,\ 7,\ \infty)$, $(8,8,\infty)$, $(9,9,\infty)$, $(1,4,7)$,
$(1,\ 5,\ 9)$, $(1,\ 6,\ 8)$, $(2,\ 4,\ 9)$, $(2,\ 5,\ 8)$, $(2,6,7)$, $(3,4,8)$, $(3,5,7)$,
$(3,\ 6,\ 9)$, $(4,\ 5,\ 6)$, $(7,\ 8,\ 9)$.

$$W^* \in \{10;\ 1\}$$

$(\infty,\ \infty,\ \infty)^*$, $(1,\ 1,\ \infty)^*$, $(2,\ 2,\ 1)^*$, $(3,\ 3,\ 1)^*$, $(2,\ 3,\ \infty)^*$,
$(4,\ 4,\ \infty)$, $(5,\ 5,\ \infty)$, $(6,\ 6,\ \infty)$, $(7,\ 7,\ \infty)$, $(8,8,\infty)$, $(9,9,\infty)$, $(1,4,7)$,
$(1,\ 5,\ 9)$, $(1,\ 6,\ 8)$, $(2,\ 4,\ 9)$, $(2,\ 5,\ 8)$, $(2,6,7)$, $(3,4,8)$, $(3,5,7)$
$(3,\ 6,\ 9)$, $(4,\ 5,\ 6)$, $(7,\ 8,\ 9)$.

Remark 4. In (i) and (ii) of Example 2.14 the systems W and W^* are clearly non-isomorphic. Algebraically the system $W \in \{8;\ 1\}$ is a group, but the system $W \in \{10;\ 1\}$ cannot be associated with any group.

Before we proceed to list some examples of systems which are consistent, we give some simple examples to illustrate that there are systems which are not consistent.

Example 2.15. In each case the system W given below is easily checked to be inconsistent:

(i) $W \in \{7;\ 7\}$

(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5),
(6, 6, 6), (7, 7, 7), (1, 2, 4), (1, 3, 7), (1, 5, 6),
(2, 3, 5), (2, 6, 7), (3, 4, 6), (4, 5, 7).

(ii) $W \in \{11;\ 7\}$

(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6),
(7, 7, 7), (8, 8, 9), (9, 9, 10), (10, 10, 11), (11, 11, 8),
(1, 2, 3), (1, 4, 8), (1, 5, 11), (1, 6, 10), (1, 7, 9), (2, 4, 10),
(2, 5, 9), (2, 6, 11), (2, 7, 8), (3, 4, 5), (3, 6, 7), (3, 8, 10),
(3, 9, 11), (4, 6, 9), (4, 7, 11), (5, 6, 8), (5, 7, 10).

Observe $W \in \{11;\ 7\}$ contains a copy of $\mathcal{V}_5$ on $\{3, 8, 9, 10, 11\}$.

(iii) $W \in \{12;\ 6\}$

(1, 1, 1), (2, 2, 2), (5, 5, 5), (6, 6, 6), (9, 9, 9),
(10, 10, 10), (7, 7, 1), (4, 4, 2), (12, 12, 5), (8, 8, 6),
(11, 11, 9), (3, 3, 10), (1, 2, 3), (1, 4, 11), (1, 5, 10),
(1, 6, 9), (1, 8, 12), (2, 5, 9), (2, 6, 12), (2, 7, 11),
(2, 8, 10), (3, 4, 8), (3, 5, 7), (3, 6, 11), (3, 9, 12),
(4, 7, 12), (4, 9, 10), (5, 8, 11), (6, 7, 10), (7, 8, 9), (10, 11, 12).

<u>Remark 5.</u> There are examples of consistent systems in $\{11; 7\}$ and $\{12; 6\}$, but there can be no consistent $W \in \{7; 7\}$ since the copies of $\mathfrak{T}_3$ in such a W pairwise intersect.

4. <u>Illustrative Examples</u>: <u>Consistent systems.</u>

Theorem 2.9 will be one of our most effective tools used in future constructions. We list here some examples, for small values of n, of systems W to which the theorem applies. The techniques outlined in the previous sections have been utilized in constructing these examples. The parameters n, a, t are as in the statement of Theorem 2.9. The examples provide a source for future reference.

$W \in \{8; 4\}$ $\qquad$ $n = 8,\ a = 4,\ t = 1$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(7, 7, 7), (6, 6, 1), (5, 5, 2), (4, 4, 3), (8, 8, 7),
(1, 4, 7), (1, 5, 8), (2, 4, 8), (2, 6, 7), (3, 5, 7),
(3, 6, 8), (4, 5, 6).

$W \in \{9; 9\}$ $\qquad$ $n = 9,\ a = 9,\ t = 3$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),
(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),
(1, 4, 7), (1, 5, 8), (1, 6, 9), (2, 4, 8), (2, 5, 9),
(2, 6, 7), (3, 4, 9), (3, 5, 7), (3, 6, 8).

Observe $W \in \{9; 9\}$ and $W \in \{8; 4\}$ are derived from the same STS(9).

$W \in \{10; 4\}$ $n = 10$, $a = 4$, $t = 1$.

(a) (1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(6, 6, 6), (7, 7, 6), (5, 5, 1), (4, 4, 2), (10, 10, 3),
(8, 8, 10), (9, 9, 10), (1, 4, 9), (1, 6, 10), (1, 7, 8),
(2, 5, 10), (2, 6, 8), (2, 7, 9), (3, 4, 8), (3, 5, 7),
(3, 6, 9), (4, 5, 6), (4, 7, 10), (5, 8, 9).

(b) (1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (8, 8, 4), (5, 5, 1), (6, 6, 2), (7, 7, 3),
(9, 9, 1), (10, 10, 1), (1, 4, 6), (1, 7, 8), (2, 4, 5),
(2, 7, 9), (2, 8, 10), (3, 4, 9), (3, 5, 8), (3, 6, 10),
(4, 7, 10), (5, 6, 7), (5, 9, 10), (6, 8, 9) .

(c) (1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),
(4, 4, 4), (8, 8, 4), (5, 5, 1), (6, 6, 2), (7, 7, 3),
(9, 9, 5), (10, 10, 5), (1, 4, 6), (1, 7, 8), (1, 9, 10),
(2, 4, 5), (2, 7, 9), (2, 8, 10), (3, 4, 9), (3, 5, 8),
(3, 6, 10), (4, 7, 10), (5, 6, 7), (6, 8, 9).

We observe the following properties concerning the three systems (a), (b), (c).

(i) (a) contains no copy of $\mathcal{K}_4$ or $\mathcal{K}_4^*$.

(ii) (b) contains a copy of $\mathcal{K}_4$ on $\{1, 5, 9, 10\}$.

(iii) (c) contains a copy of $\mathcal{K}_4^*$ on $\{1, 5, 9, 10\}$, and (c) is derived from (b) by an application of Lemma 2.8.

(iv) (a) is derived from $W \in \{11; 7\}$ which follows.

$W \in \{11; 7\}$ $\quad n = 11,\ a = 7,\ t = 2$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),

(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),

(7, 7, 7), (8, 8, 10), (9, 9, 11), (10, 10, 9), (11, 11, 8),

(1, 4, 9), (1, 5, 11), (1, 6, 10), (1, 7, 8), (2, 4, 11),

(2, 5, 10), (2, 6, 8), (2, 7, 9), (3, 4, 8), (3, 5, 7),

(3, 6, 9), (3, 10, 11), (4, 7, 10), (5, 8, 9), (6, 7, 11) .

$W \in \{12; 6\}$ $\quad n = 12,\ a = 6,\ t = 2$.

(3, 3, 3), (6, 6, 6), (11, 11, 11), (3, 6, 11),

(4, 4, 4), (7, 7, 7), (12, 12, 12), (4, 7, 12),

(10, 10, 3), (8, 8, 6), (9, 9, 11), (2, 2, 4), (1, 1, 7),

(5, 5, 12), (1, 2, 3), (4, 5, 6), (7, 8, 9), (10, 11, 12),

(1, 4, 11), (1, 5, 10), (1, 6, 9), (1, 8, 12), (2, 5, 9),

(2, 6, 12), (2, 7, 11), (2, 8, 10), (3, 4, 8), (3, 5, 7),

(3, 9, 12), (4, 9, 10), (5, 8, 11), (6, 7, 10).

$W \in \{13; 13\}$ $\quad n = 13,\ a = 13,\ t = 4$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),

(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),

(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),

(10, 10, 10), (11, 11, 11), (12, 12, 12), (10, 11, 12),

(13, 13, 13), (1, 4, 11), (1, 5, 10), (1, 6, 9), (1, 7, 13),

(1, 8, 12), (2, 4, 13), (2, 5, 9), (2, 6, 12), (2, 7, 11),

(2, 8, 10), (3, 4, 8), (3, 5, 7), (3, 6, 11), (3, 9, 12),

(3, 10, 13), (4, 7, 12), (4, 9, 10), (5, 8, 11), (5, 12, 13),

(6, 7, 10), (6, 8, 13), (9, 11, 13) .

Note $W \in \{13;\ 13\}$ and $W \in \{12;\ 6\}$ come from the same STS(13) .

$W \in \{14;\ 7\}$ $n = 14,\ a = 7,\ t = 2$.

(1, 1, 1), (5, 5, 5), (9, 9, 9), (1, 5, 9),

(2, 2, 2), (7, 7, 7), (14, 14, 14), (2, 7, 14),

(8, 8, 1), (11, 11, 5), (12, 12, 9), (10, 10, 2), (4, 4, 7), (13, 13, 14),

(3, 3, 3), (6, 6, 3), (1, 2, 3), (1, 4, 14), (1, 6, 12), (1, 7, 10),

(1, 11, 13), (2, 4, 12), (2, 5, 13), (2, 6, 8), (2, 9, 11), (3, 4, 11),

(3, 5, 7), (3, 8, 13), (3, 9, 10), (3, 12, 14), (4, 5, 6), (4, 8, 10),

(4, 9, 13), (5, 8, 12), (5, 10, 14), (6, 7, 11), (6, 9, 14), (6, 10, 13),

(7, 8, 9), (7, 12, 13), (8, 11, 14), (10, 11, 12).

$W \in \{15;\ 15\}$ $n = 15,\ a = 15,\ t = 5$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),

(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),

(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),

(10, 10, 10), (11, 11, 11), (12, 12, 12), (10, 11, 12),

(13, 13, 13), (14, 14, 14), (15, 15, 15), (13, 14, 15),

(1, 4, 14), (1, 5, 9), (1, 6, 12), (1, 7, 10), (1, 8, 15), (1, 11, 13),

(2, 4, 12), (2, 5, 13), (2, 6, 8), (2, 7, 14), (2, 9, 11), (2, 10, 15),

(3, 4, 11), (3, 5, 7), (3, 6, 15), (3, 8, 13), (3, 9, 10), (3, 12, 14),

(4, 8, 10), (4, 9, 13), (4, 7, 15), (5, 11, 15), (5, 8, 12), (5, 10, 14),

(6, 7, 11), (6, 9, 14), (6, 10, 13), (7, 12, 13), (8, 11, 14), (9, 12, 15).

$W \in \{15;\ 15\}$ and $W \in \{14;\ 7\}$ are derived from the same STS(15) .

$W \in \{16; 7\}$ $n = 16,\ a = 7,\ t = 2$.

(1, 1, 1), (6, 6, 6), (8, 8, 8), (1, 6, 8),

(2, 2, 2), (4, 4, 4), (12, 12, 12), (2, 4, 12),

(15, 15, 15), (13, 13, 15), (14, 14, 15), (16, 16, 15), (7, 7, 1), (3, 3, 6),

(5, 5, 8), (11, 11, 2), (10, 10, 4), (9, 9, 12), (1, 2, 3), (1, 4, 13),

(1, 5, 9), (1, 10, 14), (1, 11, 15), (1, 12, 16), (2, 5, 13), (2, 6, 14),

(2, 7, 15), (2, 8, 16), (2, 9, 10), (3, 4, 11), (3, 5, 14), (3, 7, 16),

(3, 8, 10), (3, 9, 15), (3, 12, 13), (4, 5, 6), (4, 7, 14), (4, 8, 15),

(4, 9, 16), (5, 7, 12), (5, 10, 15), (5, 11, 16), (6, 7, 11), (6, 9, 13),

(6, 10, 16), (6, 12, 15), (7, 8, 9), (7, 10, 13), (8, 12, 14), (8, 13, 16),

(9, 11, 14), (10, 11, 12), (13, 14, 16).

$W \in \{17; 13\}$ $n = 17,\ a = 13,\ t = 4$.

(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3),

(4, 4, 4), (5, 5, 5), (6, 6, 6), (4, 5, 6),

(7, 7, 7), (8, 8, 8), (9, 9, 9), (7, 8, 9),

(10, 10, 10), (11, 11, 11,), (12, 12, 12), (10, 11, 12),

(13, 13, 13), (14, 14, 15), (15, 15, 16), (16, 16, 17), (17, 17, 14),

(1, 4, 13), (1, 5, 9), (1, 6, 8), (1, 7, 17), (1, 10, 14), (1, 11, 15),

(1, 12, 16), (2, 4, 12), (2, 5, 13), (2, 6, 14), (2, 7, 15), (2, 8, 16),

(2, 9, 10), (2, 11, 17), (3, 4, 11), (3, 5, 14), (3, 6, 17), (3, 7, 16),

(3, 8, 10), (3, 9, 15), (3, 12, 13), (4, 7, 14), (4, 8, 15), (4, 9, 16),

(4, 10, 17), (5, 7, 12), (5, 8, 17), (5, 10, 15), (5, 11, 16), (6, 7, 11),

(6, 9, 13), (6, 10, 16), (6, 12, 15), (7, 10, 13), (8, 12, 14), (8, 13, 16),

(9, 11, 14), (9, 12, 17), (13, 14, 16), (13, 15, 17).

Concerning the systems $W \in \{16;\ 7\}$, $W \in \{17;\ 13\}$ note the following:

(i) $W \in \{16;\ 7\}$ is derived from $W \in \{17;\ 13\}$ by deleting all triples containing the number 17 from the system $W \in \{17;\ 13\}$.

(ii) $W \in \{16;\ 7\}$ contains a copy of $\mathcal{K}_4$ on $\{13, 14, 15, 16\}$.

(iii) $W \in \{17;\ 13\}$ contains a copy of $\mathcal{V}_5$ on $\{13, 14, 15, 16, 17\}$.

CHAPTER 3

Direct construction of extended triple systems

1. Introduction.

The main purpose of this chapter is to show that the necessary conditions for the existence of $\{n;\ a\}$ given in Theorem 1.1 are also sufficient. This result is proved in Section 2, using direct construction methods and applying the techniques of the previous chapter. In Section 3 a graphical method is used to construct a system in $\{6n+5;\ 6n+1\}$ which contains a copy of $\mathcal{U}_5$ for every integer $n \geq 0$. Similarly, a system in $\{6n+4;\ 3n+1\}$ containing a copy of $\mathcal{K}_4$ is obtained. These systems are inequivalent to those constructed in Section 2. Finally, it is shown that there exists a consistent system in $\{n;\ a\}$ for every $n \neq 7$ and every appropriate value of $a, 0 \leq a \leq n$.

2. Existence Theorems.

The result of our first theorem is contained in [8]. For completeness we shall give the construction and establish the existence of $\{n;\ 0\}$, $n \equiv 0 \pmod 3$, using Lemma 2.8.

Theorem 3.1. If $n \equiv 0 \pmod 3$, then $\{n;\ 3\}$ and $\{n;\ 0\}$ exist. If $n \not\equiv 0 \pmod 3$, then $\{n;\ 1\}$ exists.

Proof. Let $S = \{1, 2, \cdots, n\}$.

Let $W = \{(x, y, z) \mid x, y, z \in S\ ;\ x+y+z \equiv 0 \pmod n\}$.

If $n \equiv 0 \pmod 3$, then it is easily checked that $W \in \{n;\ 3\}$, where the

idempotents of W are $\frac{n}{3}$, $\frac{2n}{3}$, n. Clearly, W contains a copy of $\mathfrak{T}_3$, based on $\{\frac{n}{3}, \frac{2n}{3}, n\}$. By Lemma 2.8, there also exists a system $W_1 \in \{n; 0\}$ containing a copy of $\mathfrak{T}_0$.
If $n \not\equiv 0 (\mathrm{mod}\ 3)$, then $W \in \{n; 1\}$, where the idempotent of W is the number n.

<u>Theorem 3.2.</u> For every non-negative integer n, there exists a consistent system $W \in \{6n+3; 6n+3\}$.

<u>Proof.</u> It is clear that the theorem is true for $n = 0, 1, 2,$ in view of examples already given. More generally, we have the following construction.

Let $S = \{0, 1, 2, \cdots, 6n+2\}$. Divide S into three mutually disjoint sets A_0, A_1, A_2 where

$$A_0 = \{0, 1, 2, \cdots, 2n\},$$

$$A_1 = \{2n+1, 2n+2, \cdots, 4n+1\},$$

$$A_2 = \{4n+2, 4n+3, \cdots, 6n+2\}.$$

In what follows subscripts are taken modulo 3.

Let B and C be the following collection of triples:

$$B = \{(x, x+2n+1, x+4n+2) \mid x \in A_0\},$$

$$C = \{(x, y, z) \mid x, y \in A_i, z \in A_{i+1}, i = 0, 1, 2; \ x \neq y, \ x+y \equiv 2z (\mathrm{mod}\ 2n+1)\}.$$

Let $T = B \cup C$. Then (S, T) is an $STS(6n+3)$.

This construction is essentially due to R.C. Bose [1], and was reproduced by Th. Skolem [21].

Now let $J = \{(x, x, x) \mid x \in S\}$.

Finally, put $W = T \cup J$. Then $W \in \{6n+3;\ 6n+3\}$ and W contains $2n+1$ pairwise disjoint copies of $\mathfrak{J}_3$ based on the sets $B_k = \{k, k+2n+1, k+4n+2\}$, $k = 0, 1, 2, \cdots, 2n$. Consequently, W is consistent.

<u>Corollary 3.3.</u> Let $a \equiv 0 \pmod 3$, $0 \le a \le 6n+3$, where n is a non-negative integer. Then there exists a consistent system $W \in \{6n+3;\ a\}$.

<u>Proof.</u> The result follows from Theorems 2.9 and 3.2.

<u>Theorem 3.4.</u> There exists a consistent system $W \in \{6n+1;\ 6n+1\}$ if and only if $n \neq 1$.

<u>Proof.</u> Let $S = \{0, 1, 2, \cdots, 6n\}$.

For $i = 0, 1, 2$, put $L_i = \{x + 2n\,i \mid x = 0, 1, 2, \cdots, n-1\}$,

$R_i = \{x + n + 2n\,i \mid x = 0, 1, 2, \cdots, n-1\}$.

We have the following:

L_0	R_0
$0, 1, \cdots\cdots, n-1$	$n, n+1, \cdots, 2n-1$
L_1	R_1
$2n, 2n+1, \cdots, 3n-1$	$3n, 3n+1, \cdots, 4n-1$
L_2	R_2
$4n, 4n+1, \cdots, 5n-1$	$5n, 5n+1, \cdots, 6n-1$

In what follows all subscripts are reduced modulo 3.

Let B, C, D be the following collection of triples:

$$B = \{(x,\ x+2n,\ x+4n) \mid x = 0, 1, \cdots, n-1\},$$

$$C = \begin{cases} (x+n,\ x+2n,\ 6n) \\ (x+3n,\ x+4n,\ 6n) \\ (x+5n,\ x,\ 6n) \end{cases} : \quad x = 0, 1, \cdots, n-1,$$

$$D = \{(x, y, z) \mid x, y \in L_i \cup R_i \text{ ; if } x+y \text{ is even, then } z \in L_{i+1} \text{ and satisfies } x+y \equiv 2z \pmod{2n}\text{; if } x+y \text{ is odd, then } z \in R_{i+1} \text{ and satisfies } x+y \equiv 2z+1 \pmod{2n}\text{; } x \neq y\}.$$

Put $T = B \cup C \cup D$. Then (S, T) is an $STS(6n+1)$. This construction is due to Skolem [21].

Now let $J = \{(x, x, x) \mid x \in S\}$ and put $W = T \cup J$. Clearly, $W \in \{6n+1;\ 6n+1\}$. We now prove that W is consistent, if $n \neq 1$, by considering four cases. It is sufficient to show that the collection T contains $2n$ pairwise disjoint triples in each case.

<u>Case 1.</u> $n \equiv 0 \pmod 4$. It can be checked that T contains the following collection of triples.

$$A_1 = \begin{cases} (n+4k,\ n+2+4k,\ 2n+1+4k) \\ (3n+4k,\ 3n+2+4k,\ 4n+1+4k) \\ (5n+4k,\ 5n+2+4k,\ 1+4k) \end{cases} : \quad k = 0, 1, \cdots, \frac{n-4}{4},$$

$$A_2 = \begin{cases} (n+1+4k,\ n+3+4k,\ 2n+2+4k) \\ (3n+1+4k,\ 3n+3+4k,\ 4n+2+4k) \\ (5n+1+4k,\ 5n+3+4k,\ 2+4k) \end{cases} : \quad k = 0, 1, \cdots, \frac{n-4}{4},$$

$$A_3 = \begin{cases} (3+4k,\ 2n+3+4k,\ 4n+3+4k) \\ (4+4k,\ 2n+4+4k,\ 4n+4+4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-8}{4}$$

$$A_4 = \{(0,\ 2n,\ 4n),\ (n-1,\ 3n-1,\ 5n-1)\} \text{ if } n \geq 4 .$$

$$A_4 = A_1 = A_2 = A_3 = \phi \text{ if } n = 0 .$$

Set $A = A_1 \cup A_2 \cup A_3 \cup A_4$. Then it is easily verified that A contains $2n$ mutually disjoint triples. As an example, we may consider the case $n = 4$.

S :									
0	1	2	3	:	4	5	6	7	
8	9	10	11	:	12	13	14	15	
16	17	18	19	:	20	21	22	23	24

Disjoint triples in A :

(4, 6, 9)	(5, 7, 10)	(0, 8, 16)
(12, 14, 17)	(13, 15, 18)	(3, 11, 19)
(20, 22, 1)	(21, 23, 2)	

<u>Case 2.</u> $n \equiv 1 (\mathrm{mod}\ 4),\ n \neq 1$. T contains

$$A_1 = \begin{cases} (n+1+4k,\ n+3+4k,\ 2n+2+4k) \\ (3n+1+4k,\ 3n+3+4k,\ 4n+2+4k) \\ (5n+1+4k,\ 5n+3+4k,\ 2+4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-5}{4} ,$$

$$A_2 = \begin{cases} (n+2+4k,\ n+4+4k,\ 2n+3+4k) \\ (3n+2+4k,\ 3n+4+4k,\ 4n+3+4k) \\ (5n+2+4k,\ 5n+4+4k,\ 3+4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-5}{4} ,$$

$$A_3 = \begin{cases} (4+4k,\ 2n+4+4k,\ 4n+4+4k) \\ (5+4k,\ 2n+5+4k,\ 4n+5+4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-9}{4},$$

$$A_4 = \{(0, 1, 3n),\ (2n, 2n+1, 5n),\ (4n, 4n+1, n),\ (n-1, 3n-1, 5n-1)\}.$$

Set $A = A_1 \cup A_2 \cup A_3 \cup A_4$. Then it is easily checked that A contains $2n$ mutually disjoint triples. As an example, let $n = 5$.

S :											
0	1	2	3	4	:	5	6	7	8	9	
10	11	12	13	14	:	15	16	17	18	19	
20	21	22	23	24	:	25	26	27	28	29	30

Disjoint triples in A : (6, 8,12) (7, 9,13) (0, 1,15) (4,14,24)

(16,18,22) (17,19,23) (10,11,25)

(26,28, 2) (27,29, 3) (20,21, 5)

Case 3. $n \equiv 2 \pmod 4$. T contains

$$A_1 = \begin{cases} (n+1+4k,\ n+3+4k,\ 2n+2+4k) \\ (3n+1+4k,\ 3n+3+4k,\ 4n+2+4k) \\ (5n+1+4k,\ 5n+3+4k,\ 2+4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-6}{4},$$

$$A_2 = \begin{cases} (n+2+4k,\ n+4+4k,\ 2n+3+4k) \\ (3n+2+4k,\ 3n+4+4k,\ 4n+3+4k) \\ (5n+2+4k,\ 5n+4+4k,\ 3+4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-6}{4},$$

$$A_3 = \begin{cases} (4k,\ 2n+4k,\ 4n+4k) \\ (1+4k,\ 2n+1+4k,\ 4n+1+4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-6}{4},$$

$$A_4 = \{(n-1, n, 4n-1), (3n-1, 3n, 6n-1), (5n-1, 5n, 2n-1), (n-2, 3n-2, 5n-2)\}.$$

Set $A = A_1 \cup A_2 \cup A_3 \cup A_4$. Then A contains $2n$ mutually disjoint triples. As an example, let $n = 6$.

$$S : \begin{array}{cccccc|ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & \\ 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 \end{array}$$

Disjoint triples in A : (7, 9, 14) (8, 10, 15) (0, 12, 24) (5, 6, 23)

(19, 21, 26) (20, 22, 27) (1, 13, 25) (17, 18, 35)

(31, 33, 2) (32, 34, 3) (4, 16, 28) (29, 30, 11)

<u>Case 4.</u> $n \equiv 3 \pmod 4$. T contains

$$A_1 = \begin{cases} (n-1+4k,\ n+1+4k,\ 2n+4k) \\ (3n-1+4k,\ 3n+1+4k,\ 4n+4k) \\ (5n-1+4k,\ 5n+1+4k,\ 4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-3}{4},$$

$$A_2 = \begin{cases} (n+4k,\ n+2+4k,\ 2n+1+4k) \\ (3n+4k,\ 3n+2+4k,\ 4n+1+4k) \\ (5n+4k,\ 5n+2+4k,\ 1+4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-3}{4},$$

$$A_3 = \begin{cases} (2+4k,\ 2n+2+4k,\ 4n+2+4k) \\ (3+4k,\ 2n+3+4k,\ 4n+3+4k) \end{cases} \quad : \quad k = 0, 1, \cdots, \frac{n-7}{4}.$$

Set $A = A_1 \cup A_2 \cup A_3$. Then A contains $2n$ mutually disjoint triples. As an example, let $n = 7$.

$$S : \begin{array}{ccccccc|cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \\ 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & \\ 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 \end{array}$$

Disjoint triples in A :

(6, 8, 14) (7, 9, 15) (10, 12, 18) (11, 13, 19) (2, 16, 30)

(20, 22, 28) (21, 23, 29) (24, 26, 32) (25, 27, 33) (3, 7, 31)

(34, 36, 0) (35, 37, 1) (38, 40, 4) (39, 41, 5)

Combining these results, we have shown that the system $W \in \{6n+1;\ 6n+1\}$ is consistent, provided $n \neq 1$. By Remark 5, the proof of the theorem is complete.

Corollary 3.5. Let $a \equiv 1 (\mathrm{mod}\ 3)$, $1 \leq a \leq 6n+1$, where $n \neq 1$. Then there exists a consistent system $W \in \{6n+1;\ a\}$.

Proof. The corollary follows from the proofs of Theorems 2.9 and 3.4.

Example 3.6. There can be no consistent system $W \in \{7;\ 7\}$, since $W \in \{7;\ 7\}$ is equivalent to the projective plane of order 2. However, if there exists a $W \in \{7;\ 7\}$, then W necessarily contains a copy of $\mathfrak{J}_3$. So both $\{7;\ 7\}$ and $\{7;\ 4\}$ exist. By Theorem 3.1, $\{7;\ 1\}$ exists. We list three systems on $\{1, 2, \cdots, 7\}$.

(i) $\{7;\ 7\}$: (1,1,1),(2,2,2),(3,3,3),(4,4,4),(5,5,5),(6,6,6),
(7,7,7),(1,2,4),(1,3,7),(1,5,6),(2,3,5),(2,6,7),
(3,4,6),(4,5,7) .

(ii) $\{7;\ 4\}$: (1,1,1),(2,2,2),(3,3,3),(6,6,6),(4,4,5),(5,5,7),(7,7,4),
(1,2,4),(1,3,7),(1,5,6),(2,3,5),(2,6,7),(3,4,6) .

(iii) $\{7;\ 1\}$: (1,1,1),(2,2,3),(3,3,4),(4,4,5),(5,5,6),(6,6,7),
(7,7,2),(1,2,5),(1,3,6),(1,4,7),(2,4,6),(3,5,7) .

Theorem 3.7. For every positive integer n, there exists a consistent system $W \in \{6n;\ 3n\}$.

Proof. Let (S, T) be the STS$(6n+1)$ constructed in the proof of Theorem 3.4. Then T contains the two collections:

$$B = \{(x, x+2n, x+4n) \mid x = 0, 1, 2, \cdots, n-1\}, \text{ and}$$

$$C = \begin{cases} (x+n, x+2n, 6n) \\ (x+3n, x+4n, 6n) \\ (x+5n, x, 6n) \end{cases} : \quad x = 0, 1, 2, \cdots, n-1 .$$

To construct $W \in \{6n; 3n\}$ on the elements $\{0, 1, 2, \cdots, 6n-1\}$, we shall eliminate the collection C from T.

Let

$$J_1 = \begin{cases} (x, x, x) \\ (x+2n, x+2n, x+2n) \\ (x+4n, x+4n, x+4n) \end{cases} : \quad x = 0, 1, \cdots, n-1,$$

$$J_2 = \begin{cases} (x+5n, x+5n, x) \\ (x+n, x+n, x+2n) \\ (x+3n, x+3n, x+4n) \end{cases} : \quad x = 0, 1, \cdots, n-1 .$$

Put $W = (T - C) \cup J_1 \cup J_2$.

Then $W \in \{6n; 3n\}$ and W contains n pairwise disjoint copies of $\mathfrak{J}_3$, based on the sets $B_k = \{k, k+2n, k+4n\}$, $k = 0, 1, \cdots, n-1$, and consequently W is consistent.

<u>Corollary 3.8.</u> Let $a \equiv 0 \pmod 3$, $0 \le a \le 3n$, where n is any positive integer. Then there exists a consistent system $W \in \{6n; a\}$.

<u>Proof.</u> The proof follows directly from Theorems 2.9 and 3.7.

<u>Theorem 3.9.</u> For every non-negative integer n, there exists a consistent system $W \in \{6n+2; 3n+1\}$.

__Proof.__ Let (S, T) be the $STS(6n+3)$ constructed in the proof of Theorem 3.2. Let $T^{(0)}$ be the collection of all triples of T containing the element 0. We shall eliminate $T^{(0)}$ from T in order to construct $W \in \{6n+2; 3n+1\}$. We shall consider two cases, and in each case a representation of $T^{(0)}$ is given in a form that will facilitate verification of the results.

__Case 1.__ $n \equiv 0 \pmod 2$. $T^{(0)}$ can be represented as follows:

$$T^{(0)} = \begin{cases} (0,\ 2n+1,\ 4n+2), & \\ (0,\ 2x,\ x+2n+1) & : \ x = 1, 2, \cdots, 2n, \\ (0,\ 2x+4n+1,\ 6n+4-2x) & : \ x = 1, 2, \cdots, n\ ; \end{cases}$$

where $2x$ is taken modulo $2n+1$.

Let

$$J_1 = \begin{cases} (2n+1,\ 2n+1,\ 2n+1), & \\ (x+2n+1,\ x+2n+1,\ x+2n+1) & : \ x = 1, 2, \cdots, 2n, \\ (2x+4n+1,\ 2x+4n+1,\ 2x+4n+1) & : \ x = 1, 2, \cdots, n\ ; \end{cases}$$

$$J_2 = \begin{cases} (4n+2,\ 4n+2,\ 2n+1), & \\ (2x,\ 2x,\ x+2n+1) & : \ x = 1, 2, \cdots, 2n, \\ (6n+4-2x,\ 6n+4-2x,\ 2x+4n+1) & : \ x = 1, 2, \cdots, n\ ; \end{cases}$$

where $2x$ is taken modulo $2n+1$.

Put $W = (T - T^{(0)}) \cup J_1 \cup J_2$. It is easily seen that $W \in \{6n+2; 3n+1\}$ on the set $\{1, 2, \cdots, 6n+2\}$. We now show that W is consistent. First we observe that $T - T^{(0)}$ contains the following collection D of n mutually disjoint triples:

$$D = \begin{cases} (2n+1+2x,\ 2n+2+2x,\ 5n+3+2x) & : \quad x = 0, 1, \cdots, \frac{n-2}{2}\ , \\ (3n+2+2x,\ 3n+3+2x,\ 4n+3+2x) & : \quad x = 0, 1, \cdots, \frac{n-2}{2}\ . \end{cases}$$

$D = \phi$ if $n = 0$.

It follows that the collection $D \cup J_1$ provides n pairwise disjoint copies of $\mathfrak{T}_3$ in W . So W is consistent.

Case 2. $n \equiv 1 (\mathrm{mod}\ 2)$. $T^{(0)}$ may be represented as follows:

$$T^{(0)} = \begin{cases} (0,\ 2n+1,\ 4n+2), & \\ (0,\ 2x,\ x+2n+1) & : \ x = 1, 2, \cdots, 2n\ , \\ (0,\ 2x+4n+2,\ 6n+3-2x) & : \ x = 1, 2, \cdots, n\ ; \end{cases}$$

where $2x$ is taken modulo $2n+1$.

Let

$$J_1 = \begin{cases} (2n+1,\ 2n+1,\ 2n+1), & \\ (x+2n+1,\ x+2n+1,\ x+2n+1) & : \ x = 1, 2, \cdots, 2n\ , \\ (2x+4n+2,\ 2x+4n+2,\ 2x+4n+2) & : \ x = 1, 2, \cdots, n\ ; \end{cases}$$

$$J_2 = \begin{cases} (4n+2,\ 4n+2,\ 2n+1), & \\ (2x,\ 2x,\ x+2n+1) & : \ x = 1, 2, \cdots, 2n\ , \\ (6n+3-2x,\ 6n+3-2x,\ 2x+4n+2) & : \ x = 1, 2, \cdots, n\ ; \end{cases}$$

where $2x$ is taken modulo $2n+1$.

Set $W = (T - T^{(0)}) \cup J_1 \cup J_2$. It is easy to check that $W \in \{6n+2;\ 3n+1\}$ on the set $\{1, 2, \cdots, 6n+2\}$. We shall now show that $T - T^{(0)}$ contains a collection D of n pairwise disjoint triples, each element of a triple being idempotent in W .

Let $D = \begin{cases} (2n+1+2x,\ 2n+2+2x,\ 5n+3+2x) & : \quad x = 0, 1, \cdots, \frac{n-1}{2}, \\ (3n+3+2x,\ 3n+4+2x,\ 4n+4+2x) & : \quad x = 0, 1, \cdots, \frac{n-3}{2}. \end{cases}$

It is a routine matter to check that D contains n pairwise disjoint triples, and that $D \cup J_1$ provides n mutually disjoint copies of $\mathfrak{J}_3$ in W. This completes the proof of the theorem.

<u>Corollary 3.10.</u> Let $a \equiv 1 \pmod 3$, $1 \le a \le 3n+1$, where n is a non-negative integer. Then there exists a consistent system $W \in \{6n+2;\ a\}$.

<u>Proof.</u> The proof is a consequence of Theorems 2.9 and 3.9.

<u>Remark 6.</u> In the proofs of Theorems 3.7 and 3.9 our construction of J_1 and J_2 determined the consistency of the system W. There are many other possibilities for J_1 and J_2 in general, but some may give rise to inconsistent systems.

<u>Example 3.11.</u> This example illustrates the construction in Theorem 3.9 for the case $n = 2$. (S, T), $T^{(0)}$, J_1, J_2 and D are as in the proof of the theorem. The system $W \in \{14;\ 7\}$ is easily seen to be consistent.

J_1^* and J_2^* are modifications of J_1 and J_2 respectively. It can be checked without much difficulty that the system $W^* \in \{14;\ 7\}$ is <u>not</u> consistent.

(S, T)

(0, 1, 8)	(0, 2, 6)	(0, 3, 9)	(0, 4, 7)	(0, 5,10)	(0,11,14)	(0,12,13)
(1, 2, 9)	(1, 3, 7)	(1, 4, 5)	(1, 6,11)	(1,10,12)	(1,13,14)	(2, 3, 5)
(2, 4, 8)	(2, 7,12)	(2,10,14)	(2,11,13)	(3, 4, 6)	(3, 8,13)	(3,10,11)
(3,12,14)	(4, 9,14)	(4,10,13)	(4,11,12)	(5, 6,13)	(5, 7,11)	(5, 8,14)
(5, 9,12)	(6, 7,14)	(6, 8,12)	(6, 9,10)	(7, 8,10)	(7, 9,13)	(8, 9,11)

$W \in \{14;\ 7\}$

$T - T^{(0)}$				J_1	J_2
(1, 2, 9)	(2, 4, 8)	(3,12,14)	(5, 9,12)	(5, 5, 5)	(10,10, 5)
(1, 3, 7)	(2, 7,12)	(4, 9,14)	(6, 7,14)	(6, 6, 6)	(2, 2, 6)
(1, 4, 5)	(2,10,14)	(4,10,13)	(6, 8,12)	(7, 7, 7)	(4, 4, 7)
(1, 6,11)	(2,11,13)	(4,11,12)	(6, 9,10)	(8, 8, 8)	(1, 1, 8)
(1,10,12)	(3, 4, 6)	(5, 6,13)	(7, 8,10)	(9, 9, 9)	(3, 3, 9)
(1,13,14)	(3, 8,13)	(5, 7,11)	(7, 9,13)	(11,11,11)	(14,14,11)
(2, 3, 5)	(3,10,11)	(5, 8,14)	(8, 9,11)	(13,13,13)	(12,12,13)

$D = \{(5, 6, 13), (8, 9, 11)\}$.

$W^* \in \{14;\ 7\}$

$T - T^{(0)}$				J_1^*	J_2^*
(1, 2, 9)	(3, 4, 8)	(3,12,14)	(5,9,12)	(1, 1, 1)	(8, 8, 1)
(1, 3, 7)	(2, 7,12)	(4, 9,14)	(6,7,14)	(2, 2, 2)	(6, 6, 2)
(1, 4, 5)	(2,10,14)	(4,10,13)	(6,8,12)	(3, 3, 3)	(9, 9, 3)
(1, 6,11)	(2,11,13)	(4,11,12)	(6,9,10)	(4, 4, 4)	(7, 7, 4)
(1,10,12)	(3, 4, 6)	(5, 6,13)	(7,8,10)	(10,10,10)	(5, 5,10)
(1,13,14)	(3, 8,13)	(5, 7,11)	(7,9,13)	(11,11,11)	(14,14,11)
(2, 3, 5)	(3,10,11)	(5, 8,14)	(8,9,11)	(12,12,12)	(13,13,12)

In [7], A.J.W. Hilton gave a direct method for constructing maximal partial Steiner triple systems. Hilton's final note is essentially the following basic lemma.

<u>Lemma 3.12.</u> Let $S = \{1, 2, 3, \cdots, 6n+3\}$ and let $\{x, y\}$ be two numbers of S such that $x - y \not\equiv 3n+1, 3n+2, 0 \pmod{6n+3}$.

Let $A = \{(k, k+3i+1, k+6i+3) \mid k \in S \; ; \; i = 0, 1, \cdots, n-1 \; ;$ each number taken modulo $6n+3\}$.

Then $\{x, y\}$ is contained in exactly one of the triples of A .

<u>Proof.</u> Let B be the following set of numbers:

$$B = \{3i+1, 3i+2, 6i+3 \mid i = 0, 1, \cdots, n-1\} .$$

Then for every pair $\{x, y\}$ satisfying the lemma one of the differences of $\{x, y\}$ is equal to an element of B . So every pair $\{x, y\}$ satisfying the lemma is contained in at least one triple of A . There are at most $3n(6n+3)$ distinct unordered pairs arising from the triples in A , and there are $\binom{6n+3}{2} - (6n+3) = 3n(6n+3)$ pairs satisfying the lemma. So every pair satisfying the lemma is contained in exactly one triple of A , as required.

One immediate consequence of Lemma 3.12 is the following theorem.

<u>Theorem 3.13.</u> Let $n \equiv 3 \pmod 6$. There exists a system $W \in \{n; 0\}$ which does not contain any proper subsystem. If W is based on the set $S = \{1, 2, \cdots, n\}$, then up to a permutation on S , W contains the collection

$$\{(1, 1, 2), (2, 2, 3), (3, 3, 4), \cdots, (n-1, n-1, n), (n, n, 1)\} .$$

Proof. Let $n = 6m+3$, where m is a non-negative integer. Let $S = \{1, 2, \cdots, 6m+3\}$.

Let $A = \{(k, k+3i+1, k+6i+3) \mid k \in S ; i = 0, 1, \cdots, m-1 ;$ each number taken modulo $6m+3\}$.

Let $B = \{(k, k, k+3m+1) \mid k \in S ;$ each number taken modulo $6m+3\}$.

Set $W = A \cup B$. It follows from Lemma 3.12 that $W \in \{n; 0\}$ and W satisfies the conditions of the theorem.

Example 3.14. (i) Let $n = 9$ in Theorem 3.13, then we have $W \in \{9; 0\}$:

(1,1,5), (5,5,9), (9,9,4), (4,4,8), (8,8,3), (3,3,7),
(7,7,2), (2,2,6), (6,6,1), (1,2,4), (2,3,5), (3,4,6),
(4,5,7), (5,6,8), (6,7,9), (7,8,1), (8,9,2), (9,1,3).

(ii) Let $n = 9$ in Theorem 3.1, then we have $W_1 \in \{9; 0\}$:

(3,3,6), (6,6,9), (9,9,3), (1,1,7), (7,7,4), (4,4,1), (2,2,5),
(5,5,8), (8,8,2), (1,2,6), (1,3,5), (1,8,9), (2,3,4), (2,7,9),
(3,7,8), (4,5,9), (4,6,8), (5,6,7).

This example shows that there are at least two inequivalent systems in $\{9; 0\}$. Observe that the system in (ii) contains three mutually disjoint copies of $\mathfrak{T}_0$ on the sets $\{1,4,7\}$, $\{2,5,8\}$ and $\{3,6,9\}$.

Theorem 3.15. For every non-negative integer n , there exists a consistent system $W \in \{6n+4; 3n+1\}$.

Proof. Let $P = \{1, 2, \cdots, 6n+4\}$.

Let $A = \{(k, k+3i+1, k+6i+3) \mid k = 1, 2, \cdots, 6n+3\ ;\ i = 0, 1, \cdots, n-1\ ;$ each number taken modulo $6n+3\}$.

Let $B = \{(6n+4, k, k+3n+2) \mid k = 1, 2, \cdots, 3n+1\}$.

Set $T = A \cup B$. Applying Lemmas 2.2 and 3.12, it can be verified that (P, T) is a maximal partial Steiner triple system of order $6n+4$ (cf. [7]).

Now the $3n+3$ pairs $\{(3n+2, 6n+4)\} \cup \{(k, k+3n+1) : k = 1, 2, \cdots, 3n+2\}$ do not appear in any triple of T , while every other pair from P is contained in exactly one triple of T . To obtain $W \in \{6n+4;\ 3n+1\}$ based on the set P , we shall consider three cases as follows:

Case 1. $n \equiv 0 \pmod 3$.

If $n \geq 3$, it is easily verified, putting $i = \frac{2n}{3}$ in A , T contains the following collection D of n pairwise disjoint triples;

$D = \{(6n+4, n+1, 4n+3)\} \cup \{(k, 2n+1+k, 4n+3+k) \mid k = 2, 3, \cdots, n\}$.

Now let

$$J_1 = \begin{cases} \{(1,1,1)\} & \text{if } n = 0\ , \\ \{(6n+4,\ 6n+4,\ 6n+4),\ (n+1,\ n+1,\ n+1)\ , \end{cases}$$

$(4n+3,\ 4n+3,\ 4n+3),\ (n+3,\ n+3,\ n+3)\} \cup \{(k,\ k,\ k)\ ,$

$(2n+1+k,\ 2n+1+k,\ 2n+1+k),\ (4n+3+k,\ 4n+3+k,\ 4n+3+k) \mid$

$k = 2,\ 3,\ \cdots,\ n\}$ if $n \geq 3$.

Let

$$J_2 = \begin{cases} \{(2,2,1),\ (3,3,2),\ (4,4,2)\} & \text{if } n = 0\ , \\ \{(1,1,3n+2),\ (3n+2,\ 3n+2,\ 6n+4),\ (6n+3,\ 6n+3,\ 3n+2)\ , \end{cases}$$

$(n+2,\ n+2,\ 4n+3),\ (4n+2,\ 4n+2,\ n+1),$

$(4n+4,\ 4n+4,\ n+3)\} \cup \{(3n+1+k,\ 3n+1+k,\ k),$

$(5n+2+k,\ 5n+2+k,\ 2n+1+k),\ (n+2+k,\ n+2+k,\ 4n+3+k) \mid$

$k = 2,\ 3,\ \cdots,\ n\}$ if $n \geq 3$.

Set $W = T \cup J_1 \cup J_2$. A straightforward verification shows $W \in \{6n+4; 3n+1\}$, and W contains n pairwise disjoint copies of $\mathfrak{T}_3$ arising from $D \cup J_1$ if $n \geq 3$.

Case 2. $n \equiv 1 (\mathrm{mod}\ 3)$.

Let D be the following n pairwise disjoint triples obtained by putting $i = \left[\frac{n}{3}\right]$ in A :

$$D = \{(k, n+k, 2n+1+k) \mid k = 1, 2, \cdots, n\} .$$

Let $J_1 = \{(2n+1, 2n+1, 2n+1)\} \cup \{(k, k, k), (n+k, n+k, n+k), (2n+1+k, 2n+1+k, 2n+1+k) \mid k = 1, 2, \cdots, n\}$, and

$J_2 = \{(5n+2, 5n+2, 2n+1), (6n+3, 6n+3, 3n+2), (6n+4, 6n+4, 3n+2)\} \cup \{(3n+1+k, 3n+1+k, k), (4n+1+k, 4n+1+k, n+k), (5n+2+k, 5n+2+k, 2n+1+k) \mid k = 1, 2, \cdots, n\}$.

Put $W = T \cup J_1 \cup J_2$. It can be checked that $W \in \{6n+4; 3n+1\}$ and W contains n mutually disjoint copies of $\mathfrak{T}_3$ provided by $D \cup J_1$.

Case 3. $n \equiv 2 (\mathrm{mod}\ 3)$.

T contains n pairwise disjoint triples in the collection $D = \{(k, 2n+k, 4n+1+k) \mid k = 1, 2, \cdots, n\}$, putting $i = \left[\frac{2n}{3}\right]$ in A .

Let $J_1 = \{(3n+1, 3n+1, 3n+1)\} \cup \{(k, k, k), (2n+k, 2n+k, 2n+k), (4n+1+k, 4n+1+k, 4n+1+k) \mid k = 1, 2, \cdots, n\}$, and

$J_2 = \{(6n+2, 6n+2, 3n+1), (6n+3, 6n+3, 3n+2), (6n+4, 6n+4, 3n+2)\} \cup \{(3n+1+k, 3n+1+k, k), (5n+1+k, 5n+1+k, 2n+k), (n+k, n+k, 4n+1+k) \mid k = 1, 2, \cdots, n\}$.

Set $W = T \cup J_1 \cup J_2$. Then $W \in \{6n+4; 3n+1\}$ and $D \cup J_1$ provides n pairwise disjoint copies of $\mathfrak{T}_3$ in W .

This completes the proof of Theorem 3.15.

Corollary 3.16. Let $a \equiv 1 (\mathrm{mod}\ 3)$, $1 \leq a \leq 3n+1$, where n is a non-negative integer. Then there exists a consistent system $W \in \{6n+4; a\}$.

Proof. The result follows from Theorems 2.9 and 3.15.

Example 3.17. We give an example illustrating the construction of W in Theorem 3.15 for the case $n = 2$. By modifying J_1 and J_2 we also exhibit a system W^* which is not consistent. A, B, J_1, J_2 and D are as in the proof of Theorem 3.15.

$W \in \{16;\ 7\}$

A		B	J_1	J_2
(1, 2, 4)	(1, 5,10)	(1, 9,16)	(1, 1, 1)	(14,14, 7)
(2, 3, 5)	(2, 6,11)	(2,10,16)	(5, 5, 5)	(15,15, 8)
(3, 4, 6)	(3, 7,12)	(3,11,16)	(10,10,10)	(16,16, 8)
(4, 5, 7)	(4, 8,13)	(4,12,16)	(2, 2, 2)	(8, 8, 1)
(5, 6, 8)	(5, 9,14)	(5,13,16)	(6, 6, 6)	(9, 9, 2)
(6, 7, 9)	(6,10,15)	(6,14,16)	(11,11,11)	(3, 3,10)
(7, 8,10)	(7,11, 1)	(7,15,16)	(7, 7, 7)	(4, 4,11)
(8, 9,11)	(8,12, 2)			(12,12, 5)
(9,10,12)	(9,13, 3)			(13,13, 6)
(10,11,13)	(10,14, 4)			
(11,12,14)	(11,15, 5)			
(12,13,15)	(12, 1, 6)			
(13,14, 1)	(13, 2, 7)			
(14,15, 2)	(14, 3, 8)			
(15, 1, 3)	(15, 4, 9)			

$D = \{(1, 5, 10),\ (2, 6, 11)\}$.

$W^* \in \{16;\ 7\}$

A		B	J_1^*	J_2^*
(1, 2, 4)	(1, 5,10)	(1, 9,16)	(1,1,1)	(8, 8,1)
(2, 3, 5)	(2, 6,11)	(2,10,16)	(2,2,2)	(9, 9,2)
(3, 4, 6)	(3, 7,12)	(3,11,16)	(3,3,3)	(10,10,3)
(4, 5, 7)	(4, 8,13)	(4,12,16)	(4,4,4)	(11,11,4)
(5, 6, 8)	(5, 9,14)	(5,13,16)	(5,5,5)	(12,12,5)
(6, 7, 9)	(6,10,15)	(6,14,16)	(6,6,6)	(13,13,6)
(7, 8,10)	(7,11, 1)	(7,15,16)	(7,7,7)	(14,14,7)
(8, 9,11)	(8,12, 2)			(15,15,8)
(9,10,12)	(9,13, 3)			(16,16,8)
(10,11,13)	(10,14, 4)			
(11,12,14)	(11,15, 5)			
(12,13,15)	(12, 1, 6)			
(13,14, 1)	(13, 2, 7)			
(14,15, 2)	(14, 3, 8)			
(15, 1, 3)	(15, 4, 9)			

<u>Theorem 3.18.</u> Let $a \equiv 1 \pmod 3$, $1 \le a \le 6n+1$, where n is a non-negative integer. Then there exists a system $W \in \{6n+5;\ a\}$.

<u>Proof.</u> By Theorem 3.1, $\{n;\ 1\}$ exists for all $n \not\equiv 0 \pmod 3$. In particular, $\{6n+5;\ 1\}$ exists for all $n \ge 0$. To complete the proof we shall show there is a system $W^* \in \{6n+5;\ 6n+1\}$ containing a minimum of $2n-1$ mutually disjoint copies of $\mathfrak{J}_3$ for all $n \ge 1$.

Let $P = \{1, 2, \cdots, 6n+5\}$.

Let $A = \{(k, k+3i+1, k+6i+3) \mid k = 1, 2, \ldots, 6n+3;\ i = 0, 1, \cdots, n-1;$ each number taken modulo $6n+3\}$.

Let $B = \begin{cases} (6n+4, k, k+3n+1) \\ (6n+5, k, k+3n+2) \end{cases} \quad : \quad k = 1, 2, \cdots, 3n+1$.

Put $T = A \cup B$. Then (P, T) is easily verified to be a maximal partial Steiner triple system, by applying Lemmas 2.2 and 3.12 (cf. [7]). Now the four pairs $(3n+2, 6n+3)$, $(6n+3, 6n+4)$, $(6n+4, 6n+5)$, $(3n+2, 6n+5)$ are not contained in any triple of T, while every other pair from P is contained in exactly one triple of T. We construct $W^* \in \{6n+5;\ 6n+1\}$ on P as follows:

Let $F = \{3n+2, 6n+3, 6n+4, 6n+5\}$, and $P^* = P - F$.

Let $J_1 = \{(k, k, k) \mid k \in P^*\}$, and

$J_2 = \{(3n+2, 3n+2, 6n+3), (6n+3, 6n+3, 6n+4),$
$(6n+4, 6n+4, 6n+5), (6n+5, 6n+5, 3n+2)\}$.

Set $W^* = T \cup J_1 \cup J_2$. Then it is easy to see that $W^* \in \{6n+5;\ 6n+1\}$. We show W^* contains at least $2n-1$ mutually disjoint copies of $\mathfrak{J}_3$. Consider three cases:

<u>Case 1.</u> $n \equiv 0 \pmod 3$, $n \geq 3$.

Putting $i = \frac{2n}{3}$ in A, we find that T contains the following collection D of $2n-1$ disjoint triples:

$D = \{(2n+k,\ 4n+1+k,\ k) \mid 1 \leq k \leq 2n,\ k \neq n+2\}$.

It is easily checked that $D \cup J_1$ provides $2n-1$ pairwise disjoint copies of $\mathfrak{T}_3$.

<u>Case 2.</u> $n \equiv 1 \pmod 3$, $n \geq 1$.

If $n = 1$, W^* contains two disjoint copies of $\mathfrak{T}_3$ based on the sets $\{1,7,8\}$, $\{3,4,6\}$. If $n \geq 4$, then T contains the following collection D of $2n-1$ pairwise disjoint triples ($i = 0, \left[\frac{2n}{3}\right]$ in A):

$D = \{(6n-1,\ 6n,\ 6n+2)\} \cup \{(k,\ 2n-1+k,\ 4n-1+k) \mid 1 \leq k \leq 2n-1,\ k \neq n+3\}$.

It can be verified that $D \cup J_1$ contains $2n-1$ pairwise disjoint copies of $\mathfrak{T}_3$.

<u>Case 3.</u> $n \equiv 2 \pmod 3$, $n \geq 2$.

If $n = 2$, then W^* contains four mutually disjoint copies of $\mathfrak{T}_3$ based on the sets $\{4,5,7\}$, $\{9,10,12\}$, $\{13,14,1\}$, $\{2,6,11\}$ (see Example 3.19).

If $n \geq 5$, then we observe, with $i = \left[\frac{2n}{3}\right]$ in A, that T contains the collection D of $2n-1$ disjoint triples:

$D = \{(k,\ 2n+k,\ 4n+1+k) \mid 1 \leq k \leq 2n,\ k \neq n+2\}$.

A straightforward verification shows that $D \cup J_1$ contains $2n-1$ pairwise disjoint copies of $\mathfrak{T}_3$.

Combining these cases together, we have proved that the system $W^* \in \{6n+5;\ 6n+1\}$ contains a minimum of $2n-1$ mutually disjoint copies of $\mathfrak{T}_3$ for all $n \geq 1$. By Theorem 2.9, the proof of Theorem 3.18

is complete.

<u>Example 3.19.</u> Let $n = 2$ in the construction of W^* in the preceding theorem. We show the system $W^* \in \{17;\ 13\}$ is consistent. Let A, B, J_1, J_2 be as in the proof of Theorem 3.18. Explicitly, we give the following construction of $W^* \in \{17;\ 13\}$.

$$W^* \in \{17;\ 13\}$$

A		B	J_1	J_2
(1, 2, 4)	(1, 5,10)	(1, 8,16)	(1, 1, 1)	(8, 8,15)
(2, 3, 5)	(2, 6,11)*	(2, 9,16)	(2, 2, 2)	(15,15,16)
(3, 4, 6)	(3, 7,12)	(3,10,16)	(3, 3, 3)	(16,16,17)
(4, 5, 7)*	(4, 8,13)	(4,11,16)	(4, 4, 4)	(17,17, 8)
(5, 6, 8)	(5, 9,14)	(5,12,16)	(5, 5, 5)	
(6, 7, 9)	(6,10,15)	(6,13,16)	(6, 6, 6)	
(7, 8,10)	(7,11, 1)	(7,14,16)	(7, 7, 7)	
(8, 9,11)	(8,12, 2)	(8,15,16)	(9, 9, 9)	
(9,10,12)*	(9,13, 3)	(9, 1,17)	(10,10,10)	
(10,11,13)	(10,14, 4)	(10, 2,17)	(11,11,11)	
(11,12,14)	(11,15, 5)	(11, 3,17)	(12,12,12)	
(12,13,15)	(12, 1, 6)	(12, 4,17)	(13,13,13)	
(13,14, 1)*	(13, 2, 7)	(13, 5,17)	(14,14,14)	
(14,15, 2)	(14, 3, 8)	(14, 6,17)		
(15, 1, 3)	(15, 4, 9)	(15, 7,17)		

The triples marked with an asterisk determine four disjoint copies of $\mathfrak{T}_3$ in $W^* \in \{17;\ 13\}$. So W^* is consistent.

For future reference we add the following theorem:

Theorem 3.20. Let n be a positive integer.

(i) There exists a consistent system $W \in \{6n+4; 3n+1\}$ such that W does not contain a copy of $\mathcal{K}_4$ or $\mathcal{K}_4^*$.

(ii) There exists a system $W \in \{6n+5; 6n+1\}$ such that W contains at least $2n-1$ mutually disjoint copies of $\mathcal{J}_3$, and no copy of $\mathcal{V}_5$.

Proof. (i) Let $W \in \{6n+4; 3n+1\}$ be as constructed in the proof of Theorem 3.15. Then each idempotent of W appears exactly once as the element k in triples of the type (j, j, k). So the possibility of W containing a copy of $\mathcal{K}_4$ is ruled out. On the other hand, a straightforward verification shows that any copy of $\mathcal{K}_4^*$ in W must essentially be based on the set $\{1, 3n+2, 6n+3, 6n+4\}$. However, the pair $(1, 6n+4)$ is contained in the triple $(6n+4, 1, 3n+3)$. So W cannot contain a copy of $\mathcal{K}_4^*$. W was shown to be consistent in Theorem 3.15.

(ii) Let $W^* \in \{6n+5; 6n+1\}$ be as constructed in the proof of Theorem 3.18. W^* contains at least $2n-1$ mutually disjoint copies of $\mathcal{J}_3$. By construction, any copy of $\mathcal{V}_5$ in W^* must contain the collection of triples $\{(3n+2, 3n+2, 6n+3), (6n+3, 6n+3, 6n+4), (6n+4, 6n+4, 6n+5), (6n+5, 6n+5, 3n+2)\}$. So any copy of $\mathcal{V}_5$ in W^* must be based on some set of five elements which contains $3n+2$, $6n+3$, $6n+4$, $6n+5$. Now the pair $(3n+2, 6n+4)$ is contained in the triple $(1, 3n+2, 6n+4)$, while the pair $(6n+3, 6n+5)$ is contained in the triple $(3n+1, 6n+3, 6n+5)$. Obviously W^* cannot contain a copy of $\mathcal{V}_5$.

This completes the proof of the theorem.

We are now in a position to state the main theorem of this section.

<u>Theorem 3.21.</u> Let n be a positive integer. Let a be an integer such that $0 \leq a \leq n$. Then necessary and sufficient conditions for the existence of $\{n; a\}$ are :

(1) if $n \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$;

(2) if $n \not\equiv 0 \pmod 3$, then $a \equiv 1 \pmod 3$;

(3) if n is even, then $a \leq \frac{n}{2}$;

(4) if $a = n-1$, then $n = 2$.

<u>Proof.</u> The necessity follows from Theorem 1.1. The sufficiency follows from Theorem 3.18, Corollaries 3.3, 3.5, 3.8, 3.10, 3.16, and Example 3.6 .

3. <u>Construction of a system in $\{6n+5; 6n+1\}$ containing $\mathcal{V}_5$.</u>

In this section we construct, for all integers $n \geq 0$, a system in $\{6n+5; 6n+1\}$ which contains a copy of $\mathcal{V}_5$. Equivalently, we obtain a system in $\{6n+4; 3n+1\}$ which contains a copy of $\mathcal{K}_4$. A consistent system in $\{6n+5; 6n+1\}$ for all $n \geq 0$ is also constructed. We shall use standard graph notation and definitions, as found in [6].

Let G be a graph. We denote by $V(G)$ the set of vertices of G , and by $E(G)$ the set of edges of G . A <u>factor</u> of G is a spanning subgraph of G. A factorization of G is a set of edge-disjoint factors whose union is G. A one-factor of G is a factor which is regular of degree 1. A one-factorization of G is a factorization whose factors are all one-factors. G is said to be one-factorable if it admits a one-factorization. For undefined graph-theoretic notions which appear in this thesis, see [6].

Construction 3.22.

Let (S, T) be the $STS(6n+3)$ constructed in the proof of Theorem 3.2.

$$S = A_0 \cup A_1 \cup A_2 \text{, where}$$

$$A_0 = \{0, 1, \cdots, 2n\},$$

$$A_1 = \{2n+1, 2n+2, \cdots, 4n+1\},$$

$$A_2 = \{4n+2, 4n+3, \cdots, 6n+2\}.$$

And $T = B \cup C$, where

$$B = \{(x, x+2n+1, x+4n+2) \mid x = 0, 1, \cdots, 2n\} \text{, and}$$

$$C = \{(x, y, z) \mid x+y \equiv 2z \pmod{2n+1}\ ;\ x, y \in A_i,\ z \in A_{i+1},\ i = 0, 1, 2\ ;\ x \neq y \text{, all subscripts reduced modulo } 3\}.$$

Let $T^{(0)}$ be the collection of all triples in T containing the element 0. Let $T^* = T - (B \cup T^{(0)})$ and $P = S - \{0\}$. Then (P, T^*) is clearly a partial Steiner triple system of order $6n+2$.

Now let G be the graph with $V(G) = P$ and $E(G) = \{[x, y] \mid (x, y)$ is not contained in any triple of $T^*\}$. Explicitly, $E(G)$ consists of the following $9n+1$ edges.

There are $6n$ edges $\{[x, x+2n+1], [x, x+4n+2],$ $[x+2n+1, x+4n+2] \mid x = 1, 2, \cdots, 2n\}$ arising from triples in B, and the following $3n+1$ edges arising from triples in $T^{(0)}$:

$$\begin{cases} [2n+1, 4n+2] , \\ [2x, x+2n+1], \; x = 1, 2, \cdots, 2n, \\ [x+4n+2, (2n+1-x)+4n+2], \; x = 1, 2, \cdots, n ; \end{cases}$$

where $2x$ is reduced modulo $2n+1$ when necessary.
Let $A = \{1, 2, \cdots, 2n\}$ and $G^* = G - (2n+1) - (4n+2)$, the graph obtained from G by removing the vertices $2n+1$ and $4n+2$. G^* is easily seen to be a regular graph of degree 3.

If we denote by C_x the component of G^* containing the vertex x, then it follows that $C_x = C_{2n+1-x}$ for each $x \in A$. Clearly, $G^* = \bigcup_{x \in A} C_x$. We illustrate the graph G^* for the cases $n = 2$ (Figure 1) and $n = 3$ (Figure 2). In general, the component C_x is essentially one of two types:

1) as illustrated in Figure 1, when $2^k x \equiv -x \pmod{2n+1}$ for some k; or, noting that $2^m x \equiv x \pmod{2n+1}$ for some m,

2) as illustrated in Figure 2, otherwise.

Type (1) and type (2) graphs are one-factorable, since it is always possible to find a hamiltonian cycle in each case. In Figures 1 and 2 the edges of such a cycle are marked alternately with $|$ and $||$, and the remaining edges are marked with $|||$ so that the set of all edges marked with any one of $|$, $||$ or $|||$ is a one-factor. Consequently, the graph G^* is one-factorable.

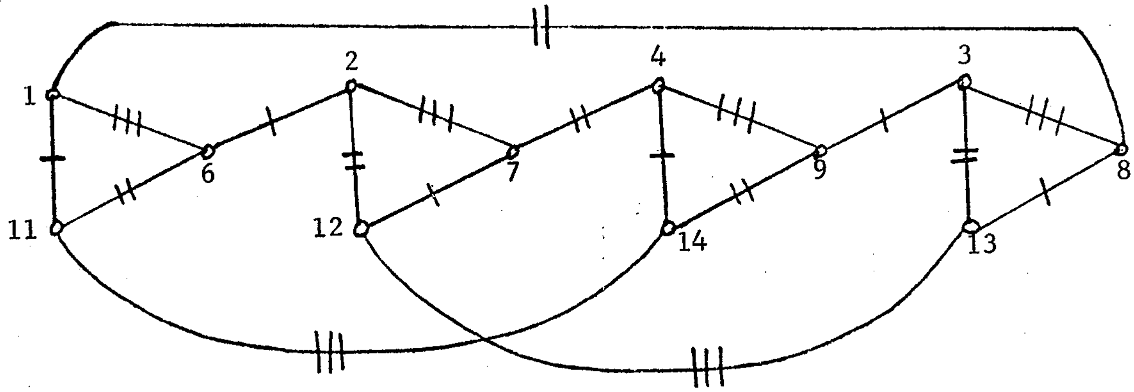

Figure 1.

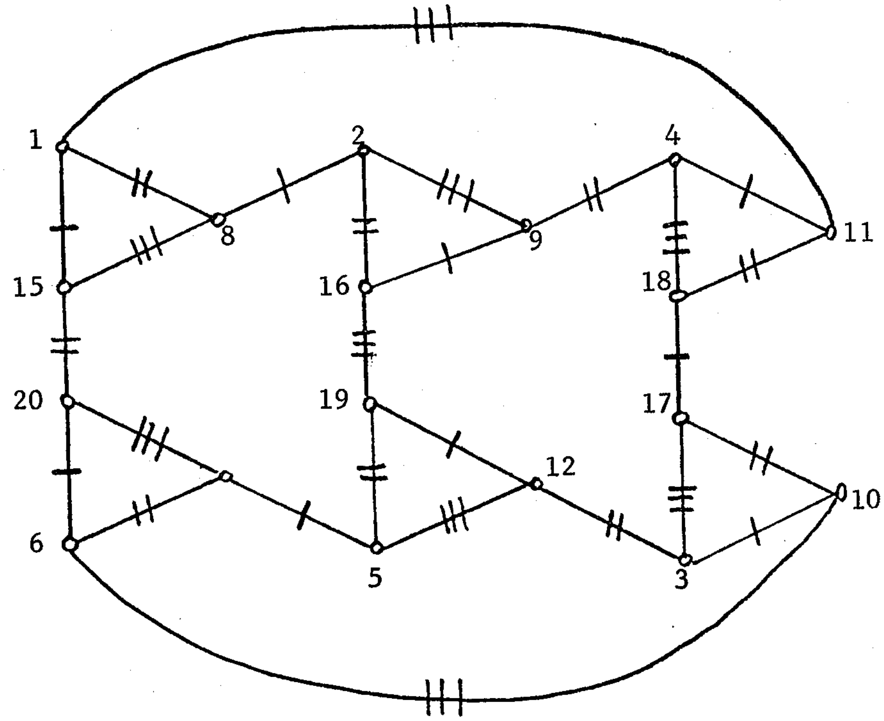

Figure 2.

Let (P, T^*) be as defined previously. Introduce three new elements $\infty_1, \infty_2, \infty_3$ and put $S^* = P \cup \{\infty_1, \infty_2, \infty_3\}$. We shall construct on S^* a system $W \in \{6n+5; 6n+1\}$ containing a copy of $\mathcal{V}_5$ as follows.

Let $P^* = P - \{2n+1, 4n+2\}$.

Let $\mathfrak{F}^* = \{F_1^*, F_2^*, F_3^*\}$ be a one-factorization of G^*.

Let $T^{**} = \{(\infty_i, x, y) \mid [x, y] \in F_i^*, \ i = 1, 2, 3\}$, and

$J = \{(x, x, x) \mid x \in P^*\}$.

Let V be any copy of $\mathcal{V}_5$ based on the elements $\infty_1, \infty_2, \infty_3, 2n+1, 4n+2$.

Put $W = T^* \cup T^{**} \cup J \cup V$.

It is easily checked that $W \in \{6n+5; 6n+1\}$.

Hence, we have proved the following theorem.

<u>Theorem 3.23.</u> For every integer $n \geq 0$, there exists a system $W \in \{6n+5; 6n+1\}$ such that W contains a copy of $\mathcal{V}_5$.

<u>Corollary 3.24.</u> For every integer $n \geq 0$, there exists a system $W \in \{6n+4; 3n+1\}$ such that W contains a copy of $\mathcal{K}_4$.

<u>Proof.</u> This follows from Theorems 2.7 and 3.23.

<u>Theorem 3.25.</u> For every non-negative integer n, there exists a consistent system $W^* \in \{6n+5; 6n+1\}$.

<u>Proof.</u> We shall consider three cases in what follows.

<u>Case 1.</u> $n \equiv 0 \pmod 3$.

Let W^* be as $W \in \{6n+5;\ 6n+1\}$ in Construction 3.22. If $n \geq 3$, T^* contains the following collection D of $2n$ disjoint triples:

$$D = \{(k,\ k+2,\ k+2n+2),\ (k+2n+1,\ k+2n+3,\ k+4n+3),$$
$$(k+4n+2,\ k+4n+4,\ k+1)\ |\ k = 1,\ 4,\ 7,\ \cdots,\ 2n-2\}\ .$$

So W^* contains $2n$ mutually disjoint copies of $\mathfrak{T}_3$ in $D \cup J$.

Case 2. $n \equiv 2 \pmod 3$.

Let W^* be as $W \in \{6n+5;\ 6n+1\}$ in Construction 3.22, where the idempotent of V is ∞_1.

Since one of $[2n,\ 4n+1]$, $[2n,\ 6n+2]$, $[4n+1,\ 6n+2]$ must belong to the one-factor F_1^*, T^{**} must contain one of the triples $(\infty_1,\ 2n,\ 4n+1)$, $(\infty_1,\ 2n,\ 6n+2)$, $(\infty_1,\ 4n+1,\ 6n+2)$. In addition, T^* contains the following $2n-1$ disjoint triples $D = \{(k, k+2, k+2n+2)$, $(k+2n+1, k+2n+3, k+4n+3), (k+4n+2, k+4n+4, k+1)\ |\ k = 1, 4, 7, \cdots, 2n-3\}$. So in $T^* \cup T^{**}$ we can find $2n$ pairwise disjoint triples such that no triple contains any of the elements $\infty_2, \infty_3, 2n+1, 4n+2$. Consequently, W^* contains $2n$ mutually disjoint copies of $\mathfrak{T}_3$.

Case 3. $n \equiv 1 \pmod 3$.

We first of all make the observation that in Construction 3.22 the system $W \in \{6n+5;\ 6n+1\}$ contains a subsystem $Q \in \{11;\ 7\}$ based on the set

$$E = \{\infty_1, \infty_2, \infty_3, 2n+1,\ 4n+2,\ \frac{2n+1}{3},\ \frac{8n+4}{3},\ \frac{14n+7}{3},$$
$$\frac{4n+2}{3},\ \frac{10n+5}{3},\ \frac{16n+8}{3}\}\ .$$

This arises from the fact that in this case (S, T) contains an STS(9) based on the elements $0, 2n+1, 4n+2, \frac{2n+1}{3}, \frac{8n+4}{3}, \frac{14n+7}{3}, \frac{4n+2}{3}, \frac{10n+5}{3}, \frac{16n+8}{3}$. This is easy to verify and details are omitted.

Now let $W^* = (W - Q) \cup Q^*$, where $Q^* \in \{11; 7\}$ is a consistent system based on E.

Clearly $W^* \in \{6n+5; 6n+1\}$. We now show that W^* is consistent.

Let

$$D = \left\{ \begin{array}{l} \left(k,\ k+\frac{2n+1}{3},\ k+\frac{10n+5}{3}\right) \\ \left(k+2n+1,\ k+\frac{8n+4}{3},\ k+\frac{16n+8}{3}\right) \\ \left(k+4n+2,\ k+\frac{14n+7}{3},\ k+\frac{4n+2}{3}\right) \end{array} \right. : k = 1, 2, \cdots, \frac{2n-2}{3}.$$

Then D consists of $2n-2$ pairwise disjoint triples from T^*, and each element of a triple in D is idempotent in W^*. Moreover, no triple of D contains an element from E. So W^* contains a total of $(2n-2)+2 = 2n$ mutually disjoint copies of $\mathfrak{T}_3$. This completes the proof of Theorem 3.25.

<u>Corollary 3.26.</u> Let $a \equiv 1 \pmod 3$, $1 \le a \le 6n+1$, where n is a non-negative integer. Then there exists a consistent system $W \in \{6n+5; a\}$.

<u>Proof.</u> The proof follows from Theorems 3.25 and 2.9.

Combining the result of Corollary 3.26 with results from the previous section, we can now state the following theorem:

<u>Theorem 3.27.</u> Let $n \neq 7$ be a given positive integer. Then there exists a consistent system $W \in \{n; a\}$ for all values of a satisfying the

necessary conditions for the existence of $\{n; a\}$.

Proof. The statement follows from Corollaries 3.3, 3.5, 3.8, 3.10, 3.16, 3.26.

Remark 7. One can directly construct a system $W \in \{6n+4; 3n+1\}$ via the one-factorization $\mathfrak{F}^* = \{F_1^*, F_2^*, F_3^*\}$ in Construction 3.22 as follows: Let P, P^*, T^* be as in Construction 3.22. Suppose $F_3^* = \{[x, y] \mid x \in X, y \in Y\}$, where $P^* = X \cup Y$. Then introduce two new elements ∞_1, ∞_2 , and put $S^* = P \cup \{\infty_1, \infty_2\}$.
Let $T^{**} = \{(\infty_i, x, y) \mid [x, y] \in F_i^*, \; i = 1, 2\}$,

$$J = \{(x, x, x), (y, y, x) \mid x \in X, y \in Y, [x, y] \in F_3^*\} .$$

Let K be a copy of $\mathcal{K}_4$ on $\{\infty_1, \infty_2, 2n+1, 4n+2\}$. Put $W = T^* \cup T^{**} \cup J \cup K$. Then it is readily verified that $W \in \{6n+4; 3n+1\}$, based on S^* . Apart from the fact that W contains a copy of $\mathcal{K}_4$, it is quite difficult to assess the internal structure of the system in general. This results from not knowing precisely what the idempotents and non-idempotents of the system are (see Remark 1).

Example 3.28. This example illustrates Construction 3.22 for the case $n = 2$. We construct a system $W \in \{16; 7\}$ which contains $\mathcal{K}_4$ and is consistent. The consistent system $W^* \in \{17; 13\}$ containing $\mathcal{U}_5$ is obtained as outlined in the proof of Theorem 3.25.

(S, T)

(0, 5,10)	(1, 6,11)	(2, 7,12)	(3, 8,13)	(4, 9,14)
(0, 1, 8)	(0, 2, 6)	(0, 3, 9)	(0, 4, 7)	(0,11,14)
(0,12,13)	(1, 2, 9)	(1, 3, 7)	(1, 4, 5)	(1,10,12)
(1,13,14)	(2, 3, 5)	(2, 4, 8)	(2,10,14)	(2,11,13)
(3, 4, 6)	(3,10,11)	(3,12,14)	(4,10,13)	(4,11,12)
(5, 6,13)	(5, 7,11)	(5, 8,14)	(5, 9,12)	(6, 7,14)
(6, 8,12)	(6, 9,10)	(7, 8,10)	(7, 9,13)	(8, 9,11)

(P, T^*)

(1, 2, 9)	(1, 3, 7)	(1, 4, 5)	(1,10,12)	(1,13,14)
(2, 3, 5)	(2, 4, 8)	(2,10,14)	(2,11,13)	(3, 4, 6)
(3,10,11)	(3,12,14)	(4,10,13)	(4,11,12)	
(5, 7,11)	(5, 8,14)	(5, 9,12)	(6, 7,14)	
(6, 9,10)	(7, 8,10)	(7, 9,13)	(8, 9,11)	

Let $F_1^* = \{[1,11], [2, 6], [3, 9], [4,14], [7,12], [8,13]\}$,

$F_2^* = \{[1, 8], [2,12], [3,13], [4, 7], [6,11], [9,14]\}$, and

$F_3^* = \{[1, 6], [2, 7], [3, 8], [4, 9], [11,14], [12,13]\}$.

Let $X = \{1, 2, 3, 4, 11, 12\}$, and

$Y = \{6, 7, 8, 9, 13, 14\}$.

$W \in \{16;\ 7\}$

$(P,\ T^{*})$		T^{**}	J		K
(1, 2, 9)	(4,10,13)	$(\infty_1,1,11)$	(1, 1, 1)	(6, 6, 1)	(5, 5, 5)
(1, 3, 7)	(4,11,12)	$(\infty_1,2,\ 6)$	(2, 2, 2)	(7, 7, 2)	$(\infty_1,\infty_1,\ 5)$
(1, 4, 5)	(5, 6,13)	$(\infty_1,3,\ 9)$	(3, 3, 3)	(8, 8, 3)	$(\infty_2,\infty_2,\ 5)$
(1,10,12)	(5, 7,11)	$(\infty_1,4,14)$	(4, 4, 4)	(9, 9, 4)	(10,10, 5)
(1,13,14)	(5, 8,14)	$(\infty_1,7,12)$	(11,11,11)	(14,14,11)	$(\infty_1,\infty_2,10)$
(2, 3, 5)	(5, 9,12)	$(\infty_1,8,13)$	(12,12,12)	(13,13,12)	
(2, 4, 8)	(6, 7,14)	$(\infty_2,1,\ 8)$			
(2,10,14)	(6, 8,12)	$(\infty_2,2,12)$			
(2,11,13)	(6, 9,10)	$(\infty_2,3,13)$			
(3, 4, 6)	(7, 8,10)	$(\infty_2,4,\ 7)$			
(3,10,11)	(7, 9,13)	$(\infty_2,6,11)$			
(3,12,14)	(8, 9,11)	$(\infty_2,9,14)$			

$W \in \{16;\ 17\}$ contains two disjoint copies of $\mathfrak{T}_3$ on $\{2,3,5\}$, $\{4,11,12\}$.

$W^* \in \{17;\ 13\}$

$(P,\ T^*)$		T^{**}		J	V
(1, 2, 9)	(4,10,13)	$(\infty_1,1,11)$	$(\infty_3,\ 1,\ 6)$	(1, 1, 1)	$(\infty_1,\infty_1,\infty_1)$
(1, 3, 7)	(4,11,12)	$(\infty_1,2,\ 6)$	$(\infty_3,\ 2,\ 7)$	(2, 2, 2)	$(\infty_2,\infty_2,\infty_3)$
(1, 4, 5)	(5, 6,13)	$(\infty_1,3,\ 9)$	$(\infty_3,\ 3,\ 8)$	(3, 3, 3)	$(\infty_3,\infty_3,\ 5)$
(1,10,12)	(5, 7,11)	$(\infty_1,4,14)$	$(\infty_3,\ 4,\ 9)$	(4, 4, 4)	(5, 5,10)
(1,13,14)	(5, 8,14)	$(\infty_1,7,12)$	$(\infty_3,11,14)$	(6, 6, 6)	$(10,10,\infty_2)$
(2, 3, 5)	(5, 9,12)	$(\infty_1,8,13)$	$(\infty_3,12,13)$	(7, 7, 7)	$(\infty_1,\ 5,\infty_2)$
(2, 4, 8)	(6, 7,14)	$(\infty_2,1,\ 8)$		(8, 8, 8)	$(\infty_1,10,\infty_3)$
(2,10,14)	(6, 8,12)	$(\infty_2,2,12)$		(9, 9, 9)	
(2,11,13)	(6, 9,10)	$(\infty_2,3,13)$		(11,11,11)	
(3, 4, 6)	(7, 8,10)	$(\infty_2,4,\ 7)$		(12,12,12)	
(3,10,11)	(7, 9,13)	$(\infty_2,6,11)$		(13,13,13)	
(3,12,14)	(8, 9,11)	$(\infty_2,9,14)$		(14,14,14)	

By Theorem 3.25, $W^* \in \{17;\ 13\}$ contains 4 pairwise disjoint copies of $\mathfrak{J}_3$ based on the sets $\{\infty_1,4,14\}$, $\{1,3,7\}$, $\{2,11,13\}$ and $\{6,8,12\}$.

CHAPTER 4

Embeddings of extended triple systems

1. Introduction.

If there exists a system $W \in \{n; a\}$ based on S and a system $W^* \in \{m; b\}$ based on S^* such that $S \subsetneq S^*$ and $W \subseteq W^*$, we shall say that W is embedded in W^*. If W is embedded in W^*, then elementary considerations show that $m \geq 2n$ and $b \geq a$. In particular, if n is even, then according to Lemma 2.1 m must be even.

Lemma 2.8 suggests that in order to embed a system W in some system W^* we need not pay any attention to the triples of the system W. However, it may be necessary to impose conditions on W if the embedding system W^* is to possess certain properties. We shall show that under certain conditions the embedding system is either consistent or contains a specified minimum number of mutually disjoint copies of $\mathfrak{T}_3$. The embeddings also provide examples of systems which are not consistent. Finally, we give a recursive method for constructing extended triple systems.

2. Preliminaries.

We shall denote by K_{2n} the complete graph on $2n$ vertices. If $V(K_{2n}) = \{1, 2, \cdots, 2n\}$, then it is well-known [6] that $\mathcal{G}_{2n} = \{G_1, G_2, \cdots, G_{2n-1}\}$ is a one-factorization of K_{2n} where $G_i = \{\{[2n, i]\} \cup \{[i-j, i+j] \mid j = 1, 2, \cdots, n-1\}\}$, i, $i-j$ and $i+j$ being taken modulo $2n-1$ in the range $\{1, 2, \cdots, 2n-1\}$. Wallis [23] has shown that there are at least two non-isomorphic one-factorizations of K_{2n} for every integer $n \geq 4$.

The following two definitions appear in [14,15]:

Definition 4.1. An (A, k)-system is a set of k disjoint pairs (p_r, q_r) covering the elements $\{1, 2, \cdots, 2k\}$ exactly once and such that $q_r - p_r = r$ for $r = 1, 2, \cdots, k$.

Definition 4.2. A (B, k)-system is a set of k disjoint pairs (p_r, q_r) covering the elements $\{1, 2, \cdots, 2k-1, 2k+1\}$ exactly once and such that $q_r - p_r = r$ for $r = 1, 2, \cdots, k$.

It is known [7,16,20,21] that an (A, k)-system exists if and only if $k \equiv 0$ or $1 \pmod 4$, and a (B, k)-system exists if and only if $k \equiv 2$ or $3 \pmod 4$.

3. Embedding Theorems.

In what follows we shall consider the embedding of a system $W \in \{n; a\}$ in a system $W^* \in \{m; b\}$, where $2n \leq m \leq 2n+8$ and $a \leq b \leq m$.

Theorem 4.3. If there exists a system $W \in \{n; a\}$, then W can be embedded in a system $W^* \in \{2n; a\}$. The system W^* is consistent if and only if W is consistent.

Proof. Let W be based on the set $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_n\}$ such that $S_1 \cap S_2 = \emptyset$. Set $S^* = S_1 \cup S_2$. On S^* we construct $W^* \in \{2n; a\}$ as follows:

Case 1. n odd.

Let $W_1 = \{(\beta_i, \beta_j, \alpha_k) \mid 1 \le i, j \le n, i \ne j; i+j \equiv 2k (\text{mod } n), 1 \le k \le n\}$,

$W_2 = \{(\beta_i, \beta_i, \alpha_i) \mid i = 1, 2, \cdots, n\}$.

Put $W^* = W \cup W_1 \cup W_2$.

Case 2. n even, $n = 2m$.

Let $W_1 = \{(\beta_i, \beta_j, \alpha_k) \mid 1 \le i, j \le 2m, i \ne j$; if $i+j$ is even, then $i+j \equiv 2k (\text{mod } 2m)$ and $1 \le k \le m$; if $i+j$ is odd, then $i+j \equiv 2k+1 (\text{mod } 2m)$ and $m+1 \le k \le 2m\}$,

$W_2 = \{(\beta_i, \beta_i, \alpha_i), (\beta_{m+i}, \beta_{m+i}, \alpha_i) \mid i = 1, 2, \cdots, m\}$.

Put $W^* = W \cup W_1 \cup W_2$.

In each case it is easily verified that $W^* \in \{2n; a\}$, and also W^* is consistent if and only if W is consistent.

Theorem 4.4. Let n be an odd integer. Suppose there exists a system $W \in \{n; a\}$. Then W can be embedded in a system $W^* \in \{2n+1; a+n+1\}$.

Proof. Let W be based on $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_{n+1}\}$ such that $S_1 \cap S_2 = \phi$. Set $S^* = S_1 \cup S_2$. We shall construct $W^* \in \{2n+1; a+n+1\}$ on S^*.

Let $\mathfrak{F} = \{F_1, F_2, \cdots, F_n\}$ be any one-factorization of K_{n+1} where $V(K_{n+1}) = S_2$.

Let $W_1 = \{(\alpha_i, x, y) \mid [x, y] \in F_i, i = 1, 2, \cdots, n\}$,

$W_2 = \{(\beta_i, \beta_i, \beta_i) \mid i = 1, 2, \cdots, n+1\}$.

Put $W^* = W \cup W_1 \cup W_2$. A straightforward verification shows $W^* \in \{2n+1; a+n+1\}$.

Theorem 4.5. Let n be an odd integer. Suppose there exists a system $W \in \{n; a\}$ with $0 \le a < \frac{n+1}{2}$. Then W cannot be embedded in any consistent system $W^* \in \{2n+1; a+n+1\}$.

Proof. Let $W \in \{n; a\}$ be based on $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_{n+1}\}$ where $S_1 \cap S_2 = \emptyset$. Let $S^* = S_1 \cup S_2$ and assume there is a system $W^* \in \{2n+1; a+n+1\}$ on S^* such that $W \subseteq W^*$. Since W^* contains $a+n+1$ idempotents, it follows that each element of S_2 must be idempotent in W^*. We shall show that W^* is not consistent. First of all, W^* cannot contain a copy of $\mathfrak{I}_3$ based on three elements from the set S_2. For suppose there is a copy of $\mathfrak{I}_3$ on $\{\beta_1, \beta_2, \beta_3\}$. Since β_1 is idempotent, each of the n pairs (α_k, β_1), $k = 1, 2, \cdots, n$, is contained in some triple of the form $(\alpha_k, \beta_1, \gamma)$ where $\gamma \in \{\beta_4, \beta_5, \cdots, \beta_{n+1}\}$. Consequently, some pair (β_1, γ) appears in more than one triple of W^* which is impossible.

There are only two possibilities for a copy of $\mathfrak{I}_3$ in W^*: A copy of $\mathfrak{I}_3$ in W^* is based on (1) a three-element set of the type $\{\alpha_i, \alpha_j, \alpha_k\}$, or (2) a three-element set of the type $\{\alpha_i, \beta_j, \beta_k\}$. There can be at most $\frac{n+1}{2}$ disjoint copies of $\mathfrak{I}_3$ of the type (2). It is clear that the number of disjoint copies of $\mathfrak{I}_3$ in W^* cannot exceed a, when all the possibilities are considered. On the other hand, the number of disjoint copies of $\mathfrak{I}_3$ required for W^* to be consistent is $\omega = [(a+n+1)/3]$. We consider three cases and show $\omega > a$. Observe $0 \le a < \frac{n+1}{2}$ and Theorem 1.1 is applied in each case.

Case 1. If $n = 6k+1$, then $a = 3m+1$, m an integer such that $0 \le m < k$. So $\omega = m+2k+1 > 3m+1 = a$.

Case 2. If $n = 6k+3$, then $a = 3m$, m an integer such that $0 \le m \le k$. So $\omega = m+2k+1 \ge 3m+1 > a$.

Case 3. If $n = 6k+5$, then $a = 3m+1$, m an integer such that $0 \le m \le k$. So $\omega = m+2k+2 \ge 3m+2 > a$.

This completes the proof of the theorem.

Corollary 4.6. Let n be an odd integer. If there exists a system $W \in \{n;\ a\}$ with $0 \le a < \frac{n+1}{2}$, then there exists a system $W^* \in \{2n+1;\ a+n+1\}$ which cannot be consistent.

Proof. The result is an immediate consequence of Theorems 4.4 and 4.5.

Example 4.7. (i) By Corollary 3.3, there is a system $W \in \{6n+3;\ a\}$ for $a = 0, 3, 6, \cdots, 3n$. By Corollary 4.6, there is an inconsistent system $W^* \in \{12n+7;\ a+6n+4\}$ for $a = 0, 3, 6, \cdots, 3n$.

(ii) According to Corollary 3.5, there is a $W \in \{6n+1;\ a\}$ for $a = 1, 4, 7, \cdots, 3n-2$. By Corollary 4.6, there is an inconsistent $W^* \in \{12n+3;\ a+6n+2\}$ for $a = 1, 4, 7, \cdots, 3n+2$.

(iii) Theorem 3.18 guarantees the existence of a $W \in \{6n+5;\ a\}$ for $a = 1, 4, 7, \cdots, 3n+1$. Consequently, by Corollary 4.6, there is an inconsistent $W^* \in \{12n+11;\ a+6n+6\}$ for $a = 1, 4, 7, \cdots, 3n+1$.

Theorem 4.8. Let $n \geq 5$ be an odd integer. A system $W \in \{n; a\}$ can be embedded in a consistent system $W^* \in \{2n+1; a+n+1\}$ if and only if $\frac{n+1}{2} \leq a \leq n$ and W contains at least $[(2a-n-1)/6]$ mutually disjoint copies of $\mathfrak{J}_3$.

Proof. Following the argument in the proof of Theorem 4.6, the condition $\frac{n+1}{2} \leq a \leq n$ is obviously necessary. Furthermore, there are at most $\frac{n+1}{2}$ disjoint copies of $\mathfrak{J}_3$ not contained in W , and since $[(a+n+1)/3]$ disjoint copies of $\mathfrak{J}_3$ are required for W^* to be consistent, W must contain at least $[(a+n+1)/3] - \left(\frac{n+1}{2}\right) = [(2a-n-1)/6]$ disjoint copies of $\mathfrak{J}_3$.

Conversely, let us assume there is a system $W \in \{n; a\}$ with $\frac{n+1}{2} \leq a \leq n$ and W contains at least $[(2a-n-1)/6]$ disjoint copies of $\mathfrak{J}_3$. Let W be based on the set $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, and let I be the set of idempotents of W . Without any loss of generality, we shall make the following assumptions:

(i) if $n = 4m+1$, then I contains the set

$$J = \{\alpha_{2i} \mid i = 1, 2, \cdots, 2m\} \cup \{\alpha_{2m+1}\} ,$$

(ii) if $n = 4m+3$, then I contains the set

$$J = \{\alpha_i \mid i = 2, 3, \cdots, m, m+2, 2m+1, 3m+3, 3m+4, \cdots, 4m+3\} ,$$

(iii) W contains $[(2a-n-1)/6]$ disjoint copies of $\mathfrak{J}_3$ on elements from $I - J$ in cases (i) and (ii) above.

Observe in (i) and (ii) $|J| = \frac{n+1}{2}$. Let $J^* = \{(x, x, x) \mid x \in J\}$. Now let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_{n+1}\}$ such that $S_1 \cap S_2 = \phi$. Set $S^* = S_1 \cup S_2$. We construct W^* on S^* as follows.

Let $W_1 = \{(\beta_i, \beta_j, \alpha_k) \mid 1 \le i, j \le n, i \ne j; i+j \equiv 2k (\mathrm{mod}\ n), 1 \le k \le n\}$,

$W_2 = \{(\beta_i, \beta_{n+1}, \alpha_i) \mid 1 \le i \le n\}$;

$W_3 = \{(\beta_i, \beta_i, \beta_i) \mid 1 \le i \le n+1\}$.

Put $W^* = W \cup W_1 \cup W_2 \cup W_3$.

It is easily checked that $W^* \in \{2n+1; a+n+1\}$. We shall show W^* is consistent. Consider two cases:

<u>Case 1.</u> $n = 4m+1$. It is readily verified that $W_1 \cup W_2$ contains the following collection D of $2m+1 = \frac{n+1}{2}$ disjoint triples:

$$D = \begin{cases} (\beta_{2m+1}, \beta_{4m+2}, \alpha_{2m+1}), \\ (\beta_{2i-1}, \beta_{2i}, \alpha_{2m+2i}) : i = 1, 2, \cdots, m, \\ (\beta_{2m+2i}, \beta_{2m+2i+1}, \alpha_{2i}) : i = 1, 2, \cdots, m. \end{cases}$$

So $\frac{n+1}{2}$ mutually disjoint copies of $\mathfrak{T}_3$ can be found in $W_1 \cup W_2 \cup W_3 \cup J^*$. Consequently, W^* contains a total of $[(2a-n-1)/6] + \left(\frac{n+1}{2}\right) = [(a+n+1)/3]$ mutually disjoint copies of $\mathfrak{T}_3$.

<u>Case 2.</u> $n = 4m+3$. It is readily checked that $W_1 \cup W_2$ contains the following collection D of $2m+2 = \frac{n+1}{2}$ disjoint triples:

$$D = \begin{cases} (\beta_{3m+3}, \beta_{3m+4}, \alpha_{m+2}), (\beta_{2m+1}, \alpha_{2m+1}, \beta_{4m+4}), \\ (\beta_i, \beta_{2m+i+1}, \alpha_{3m+2+i}) : i = 1, 2, \cdots, m+1, \\ (\beta_{m+1+i}, \beta_{3m+4+i}, \alpha_{1+i}) : i = 1, 2, \cdots, m-1. \end{cases}$$

So $W_1 \cup W_2 \cup W_3 \cup J^*$ contains $\frac{n+1}{2}$ mutually disjoint copies of $\mathfrak{T}_3$, and W^* contains a total of $[(a+n+1)/3]$ mutually disjoint copies of $\mathfrak{T}_3$ as in the previous case.

In each case W^* is consistent, and the proof of Theorem 4.8 is complete.

Remark 8. A system W satisfying the conditions imposed in Theorem 4.8 need not be consistent. For example, a system $W \in \{7; 7\}$ will satisfy the necessary conditions in the theorem. Hence W can be embedded in a consistent $W^* \in \{15; 15\}$. Consequently, a consistent system may contain an inconsistent subsystem.

Theorem 4.9. Let n be an odd integer, $n \geq 5$. Suppose there exists a system $W \in \{n; r\}$ with $0 \leq r < \frac{n+1}{2}$. Then for each $k \in \{0, 1, \cdots, r\}$, there exists a system $W^* \in \{2n+1; r+n+1-3k\}$ which cannot be consistent.

Proof. By Theorems 4.5 and 2.9, it is sufficient to show that the system $W \in \{n; r\}$ can be embedded in some $W^* \in \{2n+1; r+n+1\}$ containing r disjoint copies of $\mathfrak{J}_3$. This is easily achieved by extending W as we did in the proof of Theorem 4.8, assuming the idempotents of W are all contained in the set J. Clearly, the embedding system W^* so obtained contains r pairwise disjoint copies of $\mathfrak{J}_3$ in $W_1 \cup W_2 \cup W_3 \cup J^*$.

Example 4.10. Theorem 4.9 can be used to improve the results we have in Example 4.7 as follows:

(i) $W \in \{6n+1;\ 3n-2\} \Rightarrow$ inconsistent $W^* \in \{12n+3;\ 9n-3k\}$,

$k = 0, 1, 2, \cdots, 3n-2$.

(ii) $W \in \{6n+3;\ 3n\} \Rightarrow$ inconsistent $W^* \in \{12n+7;\ 9n+4-3k\}$,

$k = 0, 1, 2, \cdots, 3n$.

(iii) $W \in \{6n+5;\ 3n+1\} \Rightarrow$ inconsistent $W^* \in \{12n+11;\ 9n+7-3k\}$,

$k = 0, 1, 2, \cdots, 3n+1$.

Theorem 4.11. Let n be an even integer. If there exists a system $W \in \{n;\ a\}$, then W can be embedded in a system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$.

Proof. Let $W \in \{n;\ a\}$ be based on $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, and let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_{n+2}\}$ such that $S_1 \cap S_2 = \phi$. Let $\mathfrak{F} = \{F_1, F_2, \cdots, F_{n+1}\}$ be any one-factorization of K_{n+2} where $V(K_{n+2}) = S_2$.

Assume $F_{n+1} = \{[\beta_{i_1}, \beta_{i_2}], [\beta_{i_3}, \beta_{i_4}], \cdots, [\beta_{i_{n+1}}, \beta_{i_{n+2}}]\}$ where $i_1, i_2, \cdots, i_{n+2}$ are a permutation of $1, 2, \cdots, n+2$.

Let $W_1 = \{(\alpha_i, x, y) \mid [x, y] \in F_i,\ i = 1, 2, \cdots, n\}$,

$W_2 = \{(\beta_{i_k}, \beta_{i_k}, \beta_{i_k}) \mid k = 1, 3, 5, \cdots, n+1\}$,

$W_3 = \{(\beta_{i_k}, \beta_{i_k}, \beta_{i_{k-1}}) \mid k = 2, 4, 6, \cdots, n+2\}$.

Put $W^* = W \cup W_1 \cup W_2 \cup W_3$. Then it is easily seen that $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$, based on $S_1 \cup S_2$.

Theorem 4.12. Let n be an even integer. Suppose there exists a system $W \in \{n;\ a\}$ with $0 \le a < \left[\frac{n+2}{4}\right]$. Then W cannot be embedded in any consistent system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$.

Proof. An argument similar to that given in the proof of Theorem 4.5 shows that if the system $W \in \{n;\ a\}$, $a < \left[\frac{n+2}{4}\right]$, is embedded in some $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$, then the number of disjoint copies of $\mathfrak{T}_3$ in W^* cannot exceed a. On the other hand, the number required for W^* to be consistent is $\omega = [(2a+n+2)/6]$. We consider the residues of n modulo 12 in the cases which follow and show that $\omega > a$. Note we are assuming $0 \le a < \left[\frac{n+2}{4}\right]$, and Theorem 1.1 is applied in each case.

Case 1. If $n = 12k$, then $a = 3m$, m an integer such that $0 \le m < k$, and $\omega = m+2k > 3m = a$.

Case 2. If $n = 12k+2$, then $a = 3m+1$, m an integer such that $0 \le m < k$, and $\omega = m+2k+1 > 3m+1 = a$.

Case 3. If $n = 12k+4$, then $a = 3m+1$, m an integer such that $0 \le m < k$, and $\omega = m+2k+1 > 3m+1 = a$.

Case 4. If $n = 12k+6$, then $a = 3m$, m an integer such that $0 \le m \le k$, and $\omega = m+2k+1 \ge 3m+1 > a$.

Case 5. If $n = 12k+8$, then $a = 3m+1$, m an integer such that $0 \le m \le k$, and $\omega = m+2k+2 \ge 3m+2 > a$.

Case 6. If $n = 12k+10$, then $a = 3m+1$, m an integer such that $0 \le m \le k$, and $\omega = m+2k+2 \ge 3m+2 > a$.

This proves that W cannot be embedded in any consistent system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$.

Theorem 4.13. Let $n \equiv 8 \pmod{12}$. Suppose there exists a system $W \in \{n;\ a\}$. Then W cannot be embedded in any consistent system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$.

Proof. Suppose there exists a $W \in \{n;\ a\}$ and a system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$ such that $W \subseteq W^*$. If $0 \le a < \left[\frac{n+2}{4}\right]$, we have already shown in Theorem 4.12 W^* cannot be consistent. In what follows we assume $\left[\frac{n+2}{4}\right] \le a \le \frac{n}{2}$. When all possibilities are considered, the maximum number of mutually disjoint copies of $\mathfrak{J}_3$ which W^* can contain is $\lambda = \left[\frac{n+2}{4}\right] + \left[\left(a - \left[\frac{n+2}{4}\right]\right)/3\right]$. However, the number of disjoint copies of $\mathfrak{J}_3$ required for W^* to be consistent is $\omega = [(2a+n+2)/6]$. By Theorem 1.1, if $n = 12k+8$, then $a = 3m+1$, $k < m \le 2k+1$. So $\lambda = 3k+2+(m-k-1) = m+2k+1 < \omega = m+2k+2$. Consequently, W^* cannot be consistent.

Corollary 4.14. Let n be an even integer. Suppose there exists a system $W \in \{n;\ a\}$ with $0 \le a < \left[\frac{n+2}{4}\right]$. Then there exists a system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$ which cannot be consistent.

Proof. The proof follows directly from Theorems 4.11 and 4.12.

Corollary 4.15. Let $n \equiv 8 \pmod{12}$. Suppose there exists a system $W \in \{n;\ a\}$. Then there exists a system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$ which cannot be consistent.

Proof. The result follows from Theorems 4.11 and 4.13.

<u>Theorem 4.16.</u> Let n be an even integer, $n \not\equiv 8 \pmod{12}$. A system $W \in \{n;\ a\}$ can be embedded in a consistent system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$ if and only if $\left[\frac{n+2}{4}\right] \le a \le \frac{n}{2}$ and W contains at least $\left[\left(a - \left[\frac{n+2}{4}\right]\right)/3\right]$ disjoint copies of $\mathfrak{T}_3$.

<u>Proof.</u> Suppose $W \in \{n;\ a\}$ is embedded in some consistent $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$. By Theorem 4.12, the condition $\left[\frac{n+2}{4}\right] \le a \le \frac{n}{2}$ is clearly necessary. Also there can be at most $\left[\frac{n+2}{4}\right]$ disjoint copies of $\mathfrak{T}_3$ not contained in W , each such copy involving an idempotent of W . For W^* to be consistent, it is easy to check that W must contain at least $\left[\left(a - \left[\frac{n+2}{4}\right]\right)/3\right]$ disjoint copies of $\mathfrak{T}_3$.

Conversely, we assume there is a system $W \in \{n;\ a\}$ with $\left[\frac{n+2}{4}\right] \le a \le \frac{n}{2}$ and W contains at least $\left[\left(a - \left[\frac{n+2}{4}\right]\right)/3\right]$ disjoint copies of $\mathfrak{T}_3$. Let W be based on $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, and let I be the set of idempotents of W .

We shall assume, without loss of generality,

(i) if $n = 4m$, I contains the set

$$J = \{\alpha_{2i-1} \mid i = 1, 2, \cdots, m\} ;$$

(ii) if $n = 4m+2$, I contains the set

$$J = \{\alpha_{2i-1} \mid i = 1, 2, \cdots, m+1\} ;$$

(iii) W contains $\left[\left(a - \left[\frac{n+2}{4}\right]\right)/3\right]$ disjoint copies of $\mathfrak{T}_3$ on elements from $I - J$ in the cases (i) and (ii) above.

Note that $|J| = \left[\frac{n+2}{4}\right]$. Let $J^* = \{(x, x, x) \mid x \in J\}$.
Let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_{n+2}\}$ such that $S_1 \cap S_2 = \phi$.
Set $S^* = S_1 \cup S_2$. We shall construct W^* on S^* as follows.

Let $W_1 = \{(\beta_i, \beta_j, \alpha_k) \mid 1 \le i, j \le n+1, i \ne j, i+j \equiv 2k (\mathrm{mod}\ n+1),$

$1 \le k \le n\}$,

$W_2 = \{(\alpha_i, \beta_i, \beta_{n+2}) \mid 1 \le i \le n\}$,

$W_3 = \{(\beta_i, \beta_i, \beta_i) \mid i = \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n+1\}$,

$W_4 = \{(\beta_i, \beta_i, \beta_{n+1-i}) \mid i = 1, 2, \cdots, \frac{n}{2}\} \cup \{(\beta_{n+2}, \beta_{n+2}, \beta_{n+1})\}$.

Put $W^* = W \cup W_1 \cup W_2 \cup W_3 \cup W_4$. It is a routine matter to check $W^* \in \{2n+2; a+\frac{n+2}{2}\}$. We show W^* is consistent. Let

$\lambda = \left[\frac{n+2}{4}\right] + \left[\left(a - \left[\frac{n+2}{4}\right]\right)/3\right]$.

Case 1. $n = 4m$. Then W_1 contains the following collection D of $m = \left[\frac{n+2}{4}\right]$ disjoint triples:

$D = \{(\beta_{2m+2i-1}, \beta_{2m+2i}, \alpha_{2i-1}) \mid i = 1, 2, \cdots, m\}$.

So $J^* \cup W_1 \cup W_3$ contains $\left[\frac{n+2}{4}\right]$ mutually disjoint copies of $\mathfrak{T}_3$.

Thus W^* contains at least λ disjoint copies of $\mathfrak{T}_3$.

Case 2. $n = 4m+2$. Then W_1 contains the following collection D of $m+1 = \left[\frac{n+2}{4}\right]$ disjoint triples:

$D = \{(\beta_{2m+2i}, \beta_{2m+2i+1}, \alpha_{2i-1}) \mid i = 1, 2, \cdots, m+1\}$.

Hence $J^* \cup W_1 \cup W_3$ contains $\left[\frac{n+2}{4}\right]$ mutually disjoint copies of $\mathfrak{T}_3$, and W^* contains at least λ disjoint copies of $\mathfrak{T}_3$ as in the previous case.

Now the number of disjoint copies of $\mathfrak{T}_3$ required for W^* to be consistent is $\omega = [(2a+n+2)/6]$.

In what follows we consider five cases and show $\lambda = \omega$. Noting that $\left[\frac{n+2}{4}\right] \le a \le \frac{n}{2}$ and applying Theorem 1.1, we have:

(1) if $n = 12k$, then $a = 3m'$, $k \le m' \le 2k$, m' an integer, and

$$\lambda = 3k + \left[\frac{3m' - 3k}{3}\right] = m' + 2k = \left[\frac{3m' + 6k + 1}{3}\right] = \omega ;$$

(2) if $n = 12k + 2$, then $a = 3m' + 1$, $k \le m' \le 2k$, m' an integer,

and $\lambda = 3k + 1 + \left[\frac{3m' - 3k}{3}\right] = m' + 2k + 1 = \left[\frac{3m' + 6k + 3}{3}\right] = \omega$;

(3) if $n = 12k + 4$, then $a = 3m' + 1$, $k \le m' \le 2k$, m' an integer,

and $\lambda = 3k + 1 + \left[\frac{3m' - 3k}{3}\right] = m' + 2k + 1 = \left[\frac{3m' + 6k + 4}{3}\right] = \omega$;

(4) if $n = 12k + 6$, then $a = 3m'$, $k + 1 \le m' \le 2k + 1$, m' an integer,

and $\lambda = 3k + 2 + \left[\frac{3m' - 3k - 2}{3}\right] = m' + 2k + 1 = \left[\frac{3m' + 6k + 4}{3}\right] = \omega$;

(5) if $n = 12k + 10$, then $a = 3m' + 1$, $k + 1 \le m' \le 2k + 1$, m' an integer,

and $\lambda = 3k + 3 + \left[\frac{3m' - 3k - 2}{3}\right] = m' + 2k + 2 = \left[\frac{3m' + 6k + 7}{3}\right] = \omega$.

This completes the proof of our theorem.

<u>Illustrative example</u>: Any system $W \in \{4; 1\}$ trivially satisfies the conditions in Theorem 4.16. Suppose W is based on $S_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and assume α_1 is idempotent in W . Let $S_2 = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$. Following the construction in Theorem 4.16, we obtain

$$W_1 = \{(\alpha_1, \beta_2, \beta_5), (\alpha_1, \beta_3, \beta_4), (\alpha_2, \beta_1, \beta_3), (\alpha_2, \beta_4, \beta_5),$$
$$(\alpha_3, \beta_1, \beta_5), (\alpha_3, \beta_2, \beta_4), (\alpha_4, \beta_1, \beta_2), (\alpha_4, \beta_3, \beta_5)\};$$

$$W_2 = \{(\alpha_1, \beta_1, \beta_6), (\alpha_2, \beta_2, \beta_6), (\alpha_3, \beta_3, \beta_6), (\alpha_4, \beta_4, \beta_6)\};$$

$$W_3 = \{(\beta_3, \beta_3, \beta_3), (\beta_4, \beta_4, \beta_4), (\beta_5, \beta_5, \beta_5)\};$$

$$W_4 = \{(\beta_1, \beta_1, \beta_4), (\beta_2, \beta_2, \beta_3), (\beta_6, \beta_6, \beta_5)\}.$$

Then $W^* = W \cup W_1 \cup W_2 \cup W_3 \cup W_4 \in \{10; 4\}$ contains $\mathfrak{T}_3$ on $\{\alpha_1, \beta_3, \beta_4\}$.

<u>Theorem 4.17.</u> Let n be an even integer. Suppose there exists a system $W \in \{n; r\}$ with $0 \le r < \left[\frac{n+2}{4}\right]$. Then for each $k \in \{0, 1, \cdots, r\}$, there exists a system $W^* \in \{2n + 2;\ r + \frac{n+2}{2} - 3k\}$ which cannot be consistent.

Proof. Following the construction in Theorem 4.16, we may embed W in $W^* \in \{2n+2;\ r+\frac{n+2}{2}\}$ in such a way that W^* contains r mutually disjoint copies of $\mathfrak{T}_3$. We simply assume that all the idempotents of W are contained in the set J so that eventually $J^* \cup W_1 \cup W_3$ contains r disjoint copies of $\mathfrak{T}_3$. The result then follows from Theorems 4.12 and 2.9.

Theorem 4.18. Let $n \equiv 8 \pmod{12}$. Suppose there exists a system $W \in \{n;\ a\}$ with $\left[\frac{n+2}{4}\right] \le a \le \frac{n}{2}$. Then there exists a system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$ containing λ disjoint copies of $\mathfrak{T}_3$, where $\lambda = \left[\frac{n+2}{4}\right] + \left[\left(a - \left[\frac{n+2}{4}\right]\right)/3\right]$, provided W contains at least $\left[\left(a - \left[\frac{n+2}{4}\right]\right)/3\right]$ disjoint copies of $\mathfrak{T}_3$. Furthermore, W^* cannot be consistent.

Proof. We embed the system W in a system $W^* \in \{2n+2;\ a+\frac{n+2}{2}\}$ exactly as we did in the proof of Theorem 4.16. The system W^* contains λ disjoint copies of $\mathfrak{T}_3$ and $\lambda = \omega - 1$, where $\omega = [(2a+n+2)/6]$.

Example 4.19. Corollary 3.10 guarantees the existence of a consistent system $W \in \{12n+8;\ 6n+4\}$. By Theorem 4.18, there exists an inconsistent system $W^* \in \{24n+18;\ 12n+9\}$ which contains $\lambda = 4n+2$ mutually disjoint copies of $\mathfrak{T}_3$. Consequently, there exists an inconsistent system in $\{24n+18;\ 12n+9-3k\}$ for each $k = 0, 1, \cdots, 4n+2$.

Theorem 4.20. Let n be an odd integer. Suppose there exists a system $W \in \{n;\ a\}$. Then W can be embedded in a system $W^* \in \{2n+3;\ a\}$. W^* is consistent if and only if W is consistent.

Proof. Let W be based on the set $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.
Let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_{n+3}\}$ such that $S_1 \cap S_2 = \phi$.
Let $\mathfrak{F} = \{F_1, F_2, \cdots, F_{n+2}\}$ be any one-factorization of K_{n+3} where $V(K_{n+3}) = S_2$.
Let $G = F_{n+1} \cup F_{n+2}$. Then G consists of disjoint cycles of even length $k \geq 4$.
Corresponding to each cycle C of G given by $\beta_{i_0}, \beta_{i_1}, \cdots, \beta_{i_k}$, where $i_k = i_0$, let $J_C = \{(\beta_{i_t}, \beta_{i_t}, \beta_{i_{t+1}}) \mid t = 0, 1, \cdots, k-1\}$.
Set $W_C = \bigcup_{C \subseteq G} J_C$.

Let $W_1 = \{(\alpha_i, x, y) \mid [x, y] \in F_i, i = 1, 2, \cdots, n\}$.
Put $W^* = W \cup W_1 \cup W_C$.
It is readily verified that $W^* \in \{2n+3; a\}$, and that W^* is consistent if and only if W is consistent.

Remark 9. W.D. Wallis [23] proved that if G_i and G_k are any two one-factors in $\mathcal{G}_{2n}$, then $G_i \cup G_k$ consists of a cycle of length $v_{ik}+1$ and $\frac{1}{2}(d_{ik}-1)$ cycles of length $2v_{ik}$, where d_{ik} is the greatest common divisor of $i-k$ and $2n-1$, and $v_{ik} = (2n-1)/d_{ik}$.

Clearly if $\mathfrak{F}$, in the proof of Theorem 4.20, is isomorphic to $\mathcal{G}_{n+3}$, we can always choose F_{n+1} and F_{n+2} so that $F_{n+1} \cup F_{n+2}$ is a cycle of length $n+3$. On the other hand, if $n+2$ is not a prime, we can also choose F_{n+1} and F_{n+2} so that $F_{n+1} \cup F_{n+2}$ is not a cycle of length $n+3$. Consequently, the embedding system $W^* \in \{2n+3; a\}$ need not be the same for every choice of F_{n+1} and F_{n+2} .

Analogous to a special case of Rosa's Theorem 2 [17] on Steiner triple systems, we have the following:

<u>Theorem 4.21.</u> Let $n \equiv 3 \pmod 6$. Suppose there exists a system $W \in \{n;\ a\}$. Then W can be embedded in a system $W^* \in \{2n+3;\ a+n+3\}$. Furthermore, W^* contains $\frac{n}{3}+1$ disjoint copies of $\mathfrak{T}_3$, each disjoint from W. Consequently, W^* is consistent if W is consistent.

<u>Proof.</u> The method of construction used is the same as that for the STS$(2n+3)$ (see [17]).

Let W be based on the set $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_n\}$, $S_3 = \{\gamma_1, \gamma_2, \gamma_3\}$ so that S_1, S_2, S_3 are pairwise disjoint. Let W_1 be a copy of $\mathfrak{T}_3$ on S_3. Let $m = \frac{n-1}{2} = 3t+1$ and let $\Omega = \{(p_r, q_r) \mid q_r - p_r = r,\ r = 1, 2, \cdots, m\}$ be an (A, m)-system or (B, m)-system according to whether $m \equiv 0$ or $1 \pmod 4$ or $m \equiv 2$ or $3 \pmod 4$.

Let $\Omega^* = \Omega - \{(p_{2t+1}, q_{2t+1})\}$.

Let $S_2^* = S_2 - V$, where $V = \{\beta_i \mid i = p_r \text{ or } q_r;\ (p_r, q_r) \in \Omega^*\}$.

Clearly $|S_2^*| = 3$. Let $S_2^* = \{\beta_{j_i} \mid i = 1, 2, 3\}$.

Set $W_2 = \{(\gamma_i, \alpha_k, \beta_{j_i+k-1}) \mid i = 1, 2, 3;\ k = 1, 2, \cdots, n\}$,

$W_3 = \{(\beta_k, \beta_{k+2t+1}, \beta_{k+4t+2}) \mid k = 1, 2, \cdots, 2t+1\}$,

$W_4 = \{(\alpha_k, \beta_{p_r+k-1}, \beta_{q_r+k-1}) \mid k = 1, 2, \cdots, n;\ (p_r, q_r) \in \Omega^*\}$,

$W_5 = \{(\beta_k, \beta_k, \beta_k) \mid k = 1, 2, \cdots, n\}$,

where subscripts are reduced modulo n as necessary.

Finally, put $W^* = W \cup W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5$. It can be readily verified that $W^* \in \{2n+3;\ a+n+3\}$. Since W_3 contains $\frac{n}{3}$ pairwise disjoint triples it is clear that $W_1 \cup W_3 \cup W_5$ contains $\frac{n}{3}+1$ disjoint copies of $\mathfrak{T}_3$, and our theorem has been proved.

Corollary 4.22. If $n \equiv 3 \pmod 6$ and there exists a system $W \in \{n; a\}$, then W can be embedded in a system $W^* \in \{2n+3; a+3t\}$, where $t = 0, 1, 2, \cdots, \frac{n}{3}+1$.

Proof. The result follows directly from Theorems 4.21 and 2.9.

Example 4.23. Let $n \equiv 3 \pmod 6$. Then Theorem 3.1 guarantees the existence of $\{n; 0\}$. Also $\{n; n\}$ exists. By Corollary 4.22, the existence of $\{n; 0\}$ and $\{n; n\}$ together will imply the existence of $\{2n+3; t\}$ for all $t = 0, 3, 6, \cdots, 2n+3$.

In [17], A. Rosa showed how an STS(n) can be embedded in an STS(2n+7), provided $n \geq 7$. In our next theorem we give the analogue for extended triple systems. We follow the construction for Steiner triple systems which can be found in [14, 15].

Theorem 4.24. Let $n \geq 7$ be an odd integer. Suppose there exists a system $W \in \{n; a\}$. Then W can be embedded in a system $W^* \in \{2n+7; a+n+7\}$ which contains a subsystem $W_1 \in \{7; 7\}$ disjoint from W.

Proof. Let W be based on the set $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_n\}$, $S_3 = \{\gamma_1, \gamma_2, \cdots, \gamma_7\}$; S_1, S_2, S_3 pairwise disjoint. Let $W_1 \in \{7; 7\}$ be based on S_3.
Let $m = \frac{n-1}{2}$ and $\Omega = \{(p_r, q_r) \mid q_r - p_r = r, r = 1, 2, \cdots, m\}$ be an (A, m)-system or (B, m)-system according to whether $m \equiv 0$ or $1 \pmod 4$ or $m \equiv 2$ or $3 \pmod 4$.

Let $S_2^* = S_2 - V$ where $V = \{\beta_i \mid i = p_r$ or q_r, $r = 4, 5, \cdots, m$; $(p_r, q_r) \in \Omega\}$.

Then $|S_2^*| = 7$. Let $S_2^* = \{\beta_{j_i} \mid i = 1, 2, \cdots, 7\}$.

Let $W_2 = \{(\gamma_i, \alpha_k, \beta_{j_i+k-1}) \mid i = 1, 2, \cdots, 7;\ k = 1, 2, \cdots, n\}$,

$W_3 = \{(\alpha_k, \beta_{p_r+k-1}, \beta_{q_r+k-1}) \mid k = 1, 2, \cdots, n;$
$r = 4, 5, \cdots, m;\ (p_r, q_r) \in \Omega\}$,

$W_4 = \{(\beta_i, \beta_{i+1}, \beta_{i+3}) \mid i = 1, 2, \cdots, n\}$, and

$W_5 = \{(\beta_i, \beta_i, \beta_i) \mid i = 1, 2, \cdots, n\}$, where all subscripts are reduced modulo n in the range $\{1, 2, \cdots, n\}$.

Finally, put $W^* = W \cup W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5$.

A straightforward verification shows that $W^* \in \{2n+7;\ a+n+7\}$.

It is clear that W_1 is disjoint from W.

Theorem 4.25. Let $n \geq 7$ be an odd integer. Suppose there exists a system $W \in \{n; a\}$. Then W can be embedded in a system $W^* \in \{2n+7;\ a+n+7-3t\}$, where $t = 0, 1, \cdots, \left[\frac{n}{4}\right]+1$.

Proof. We extend the system W to the system $W^* \in \{2n+7;\ a+n+7\}$ constructed in the proof of Theorem 4.24. Then we observe that W^* contains a copy of $\mathfrak{T}_3$ in W_1 and $\left[\frac{n}{4}\right]$ disjoint copies of $\mathfrak{T}_3$ based on $\{\beta_{1+4k}, \beta_{2+4k}, \beta_{4+4k}\}$, $k = 0, 1, \cdots, \left[\frac{n}{4}\right]-1$. So W^* contains $\left[\frac{n}{4}\right]+1$ mutually disjoint copies of $\mathfrak{T}_3$, which are disjoint from W. The result follows immediately.

Example 4.26. (i) By Theorem 4.25, the existence of any $W \in \{6n-3;\ 0\}$ implies the existence of a system $W^* \in \{12n+1;\ 6n+4-3t\}$, where $t = 0, 1, \cdots, \left[\frac{6n-3}{4}\right]+1$, provided $n \geq 2$. Similarly we have for

$n \geq 2$, $W \in \{6n-3;\ 6n-3\} \Rightarrow W^* \in \{12n+1;\ 12n+1-3t\}$, where $t = 0, 1, \cdots, \left[\frac{6n-3}{4}\right]+1$.

Similar results are obtained when $n \equiv 1$ or $5 \pmod 6$ in Theorem 4.25. It is worth noting that no assumption regarding the system W is made so far.

(ii) If $n = 7$ in the construction of W^* in Theorem 4.24, then $W_3 = \phi$. If we choose $W \in \{7;\ 7\}$, it is easily checked that the embedding system $W^* \in \{21;\ 21\}$ contains 7 pairwise disjoint copies of $\mathfrak{J}_3$, and is thus consistent.

<u>Remark 10.</u> In what follows we shall not restrict our attention to any particular (A, m)-system or (B, m)-system. For convenience, in our next theorem we shall impose conditions on the system $W \in \{n;\ a\}$ similar to those required in Theorem 4.8. These conditions are not absolutely necessary to obtain the end result in every case.

<u>Theorem 4.27.</u> Let n be an odd integer, $n \geq 9$. Suppose there exists a system $W \in \{n;\ a\}$ with $a \geq \frac{n+1}{2}$, and W contains at least $[(2a-n-1)/6]$ disjoint copies of $\mathfrak{J}_3$. Then W can be embedded in a system $W^* \in \{2n+7;\ a+n+7\}$ such that W^* contains at least $[(a+n+1)/3]$ disjoint copies of $\mathfrak{J}_3$.

<u>Proof.</u> Let W be based on the set $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Let I be the set of idempotents of W . Let us assume, without loss of generality, I contains the set $J = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ where $m = \frac{n-1}{2}$, and W contains $[(2a-n-1)/6]$ disjoint copies of $\mathfrak{J}_3$ on elements from $I - J$. Let $J^* = \{(x, x, x) \mid x \in J\}$.

Let W_i , $i = 1, 2, 3, 4, 5$, be as in the proof of Theorem 4.24, and put $W^* = W \cup W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5$.

Then $W^* \in \{2n+7;\ a+n+7\}$.

Now $W_1 \in \{7;\ 7\}$ contains at least one copy of $\mathfrak{J}_3$.

For any (A, m)-system or (B, m)-system, W_3 contains the following collection D of $m = \frac{n-1}{2}$ disjoint triples:

$$D = \{(\alpha_k,\ \beta_{p_m+k-1},\ \beta_{q_m+k-1}) \mid k = 1, 2, \cdots, m\} .$$

So $J^* \cup W_3 \cup W_5$ contains $m = \frac{n-1}{2}$ mutually disjoint copies of $\mathfrak{J}_3$.

Consequently, W^* contains at least $[(2a-n-1)/6] + \frac{n-1}{2} + 1 = [(a+n+1)/3]$ mutually disjoint copies of $\mathfrak{J}_3$, as required.

__Theorem 4.28.__ Let n be an odd integer, $n \geq 9$. Suppose there exists a system $W \in \{n;\ a\}$ with $a \geq \frac{n+1}{2}$, and W contains at least $[(2a-n-1)/6]$ disjoint copies of $\mathfrak{J}_3$. Then there exists a system $W^* \in \{2n+7;\ a+n+1\}$ such that W^* contains at least $[(a+n-2)/3]$ disjoint copies of $\mathfrak{J}_3$.

__Proof.__ If we choose $W_1 \in \{7;\ 1\}$ in the proof of Theorem 4.27, the result follows immediately.

__Corollary 4.29.__ Let n be an odd integer, $n \geq 9$. Suppose there exists a system $W \in \{n;\ a\}$ with $a \geq \frac{n+1}{2}$, and W contains at least $[(2a-n-1)/6]$ disjoint copies of $\mathfrak{J}_3$. Then there exists a system $W^* \in \{2n+7;\ a+n+7-3t\}$, where $t = 0, 1, \cdots,\ [(a+n+4)/3]$

__Proof.__ The result follows directly from Theorems 4.25, 4.28 and 2.9 .

Theorem 4.30. Let n be an odd integer, $n \geq 7$. Suppose there exists a system $W \in \{n; a\}$. Then W can be embedded in a system $W^* \in \{2n+6; a+3\}$ such that W^* contains a subsystem $W_1 \in \{6; 3\}$ disjoint from W.

Proof. Let W be based on the set $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.
Let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_n\}$, $S_3 = \{\gamma_1, \gamma_2, \cdots, \gamma_6\}$, where S_1, S_2, S_3 are pairwise disjoint.
Let $W_1 \in \{6; 3\}$ be based on S_3.
Let $m = \frac{n-1}{2}$ and $\Omega = \{(p_r, q_r) \mid q_r - p_r = r, \ r = 1, 2, \cdots, m\}$ be an (A, m)-system or (B, m)-system according to whether $m \equiv 0$ or $1 \pmod 4$ or $m \equiv 2$ or $3 \pmod 4$.
Set $S_2^* = S_2 - V$ where $V = \{\beta_i \mid i = p_r \text{ or } q_r, \ r = 4, 5, \cdots, m; \ (p_r, q_r \in \Omega\}$.
Clearly $|V| = n-7$ and $|S_2^*| = 7$.
Let $S_2^* = \{\beta_{j_i} \mid i = 1, 2, \cdots, 7\}$.
Set $W_2 = \{(\gamma_i, \alpha_k, \beta_{j_i+k-1}) \mid i = 1, 2, \cdots, 6; \ k = 1, 2, \cdots, n\}$,

$W_3 = \{(\alpha_k, \beta_{p_r+k-1}, \beta_{q_r+k-1}) \mid k = 1, 2, \cdots, n; \ r = 4, 5, \cdots, m; \ (p_r, q_r) \in \Omega\}$,

$W_4 = \{(\beta_i, \beta_{i+1}, \beta_{i+3}) \mid i = 1, 2, \cdots, n\}$, and

$W_5 = \{(\beta_{j_7+k-1}, \beta_{j_7+k-1}, \alpha_k) \mid k = 1, 2, \cdots, n\}$, where the subscripts are reduced modulo n in the range $\{1, 2, \cdots, n\}$.

Put $W^* = W \cup W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5$. It is readily verified that $W^* \in \{2n+6; a+3\}$, and our theorem is proved.

Corollary 4.31. Let n be an odd integer, $n \geq 7$. If there exists a system $W \in \{n; a\}$, then W can be embedded in a system $W^* \in \{2n+6; a\}$ which contains a subsystem $W_1 \in \{6; 0\}$ disjoint from W.

Proof. The corollary follows from the proof of Theorem 4.30 and Lemma 2.8.

Remark 11. The subsystem $W_1 \in \{6; 3\}$ in Theorem 4.30 is obviously unique. We exhibited two systems in $\{6; 3\}$, one containing a copy of $\mathfrak{J}_3$ and the other not containing a copy of $\mathfrak{J}_3$ (see Remark 1 following Theorem 2.3). As a result, the embedding system $W^* \in \{2n+6; a+3\}$ is not unique. Further, W^* is consistent only if W_1 contains a copy of $\mathfrak{J}_3$.

Theorem 4.32. Let n be an even integer, $n \geq 6$. Suppose there exists a system $W \in \{n; a\}$. Then W can be embedded in a system $W^* \in \{2n+8; a+\frac{n}{2}+4\}$ such that W^* contains a subsystem $W_1 \in \{7; 7\}$ disjoint from W.

Proof. Let W be based on $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.
Let $S_2 = \{\beta_1, \beta_2, \cdots, \beta_{n+1}\}$, $S_3 = \{\gamma_1, \gamma_2, \cdots, \gamma_7\}$; S_1, S_2, S_3 pairwise disjoint. Let $W_1 \in \{7; 7\}$ be based on S_3.
Let $m = \frac{n}{2}$ and $\Omega = \{(p_r, q_r) \mid q_r - p_r = r, r = 1, 2, \cdots, m\}$ be an (A, m)-system or (B, m)-system according to whether $m \equiv 0$ or $1 \pmod 4$ or $m \equiv 2$ or $3 \pmod 4$.
Set $S_2^* = S_2 - V$ where $V = \{\beta_i \mid i = p_r$ or q_r, $r = 4, 5, \cdots, m$; $(p_r, q_r) \in \Omega\}$.

Then $|S_2^*| = 7$. Let $S_2^* = \{\beta_{j_i} \mid i = 1, 2, \cdots, 7\}$.

Set $W_2 = \{(\gamma_i, \alpha_k, \beta_{j_i+k-1}) \mid i = 1, 2, \ldots, 7 ; k = 1, 2, \cdots, n\}$,

$$W_3 = \{(\alpha_k, \beta_{p_r+k-1}, \beta_{q_r+k-1}) \mid k = 1, 2, \cdots, n ; r = 4, 5, \cdots, m ; (p_r, q_r) \in \Omega\} ,$$

$$W_4 = \{(\beta_i, \beta_{i+1}, \beta_{i+3}) \mid i = 1, 2, \cdots, n+1\} ,$$

$$W_5 = \{(\beta_{j_i+n}, \beta_{j_i+n}, \gamma_i) \mid i = 1, 2, \cdots, 7\} \text{, and}$$

$$W_6 = \{(\beta_{p_r+n}, \beta_{p_r+n}, \beta_{p_r+n}) , (\beta_{q_r+n}, \beta_{q_r+n}, \beta_{p_r+n}) \mid r = 4, 5, \cdots, m ; (p_r, q_r) \in \Omega\} ,$$

where subscripts are reduced modulo $n+1$ in the range $\{1, 2, \cdots, n+1\}$.

Finally, put $W^* = W \cup W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5 \cup W_6$.

Then $W^* \in \{2n+8; a+\frac{n}{2}+4\}$. The verification is direct.

4. Recursive construction of extended triple systems.

For a given positive integer n the integers a which satisfy the necessary conditions for the existence of $\{n; a\}$ will be called admissible.

We shall apply the results of the previous section to show that, for every positive integer n , $\{n; a\}$ exists for all admissible values of a . Our method of proof will be inductive, and we refer to Section 4 of Chapter 2 for all the initial cases required for induction. We also assume the existence of $\{n; 0\}$ for $n \equiv 0 (\mathrm{mod}\ 3)$, and $\{n; 1\}$ for $n \not\equiv 0 (\mathrm{mod}\ 3)$, established in Theorem 3.1.

Lemma 4.33. Let $n \equiv 1$ or $3 (\mathrm{mod}\ 6)$, $n \geq 7$. There exists a system $W \in \{n; n\}$ containing at least $\left[\frac{n-6}{3}\right]$ disjoint copies of $\mathfrak{T}_3$.

<u>Proof.</u> The statement is obviously true for $n = 7, 9, 13, 15$, and hence, by Example 4.26 and Theorems 4.8 and 4.27, for $n = 21, 25, 27$.

Let $n \geq 31$ and assume the statement is true for all $m < n$; $m \geq 9$, $m \equiv 1$ or $3 \pmod 6$.

If $n \equiv 1$ or $9 \pmod{12}$, then $m = \frac{1}{2}(n-7) \equiv 1$ or $3 \pmod 6$. Since $12 \leq m < n$, there exists a system in $\{m;\ m\}$ containing at least $\left[\frac{m-6}{3}\right]$ disjoint copies of $\mathfrak{T}_3$. So by Theorem 4.27, there exists a system $W \in \{2m+7;\ 2m+7\} = \{n;\ n\}$ containing at least $\left[\frac{2m+1}{3}\right] = \left[\frac{n-6}{3}\right]$ disjoint copies of $\mathfrak{T}_3$.

If $n \equiv 3$ or $7 \pmod{12}$, then $m = \frac{1}{2}(n-1) \equiv 1$ or $3 \pmod 6$. Since $15 \leq m < n$, there exists a system in $\{m;\ m\}$ containing at least $\left[\frac{m-6}{3}\right]$ disjoint copies of $\mathfrak{T}_3$. By Theorem 4.8, there exists a system $W \in \{2m+1;\ 2m+1\} = \{n;\ n\}$ containing $\left[\frac{2m+1}{3}\right] = \left[\frac{n}{3}\right] > \left[\frac{n-6}{3}\right]$ disjoint copies of $\mathfrak{T}_3$. This proves our lemma.

<u>Lemma 4.34.</u> Let $n \equiv 5 \pmod 6$, $n \geq 11$. There exists a system $W \in \{n;\ n-4\}$ containing at least $\left[\frac{n-10}{3}\right]$ disjoint copies of $\mathfrak{T}_3$ and a copy of $\mathfrak{V}_5$.

<u>Proof.</u> The statement is true for $n = 11, 17$ (see illustrative examples of Chapter 2), and hence for $n = 23$, by Theorem 4.8.

Let $n \geq 29$ and assume the statement is true for all $m < n$; $m \geq 11$, $m \equiv 5 \pmod 6$.

Let $n \equiv 5 \pmod{12}$. Then $m = \frac{1}{2}(n-7) \equiv 5 \pmod 6$. Since $11 \leq m < n$, there exists a system in $\{m;\ m-4\}$ containing at least $\left[\frac{m-10}{3}\right]$ disjoint copies of $\mathfrak{T}_3$ and a copy of $\mathfrak{V}_5$. By Theorem 4.27, there exists a system $W \in \{2m+7;\ 2m+3\} = \{n;\ n-4\}$ such that W

contains at least $\left[\frac{2m-3}{3}\right] = \left[\frac{n-10}{3}\right]$ disjoint copies of $\mathcal{T}_3$ and a copy of $\mathcal{V}_5$.

Let $n \equiv 11 \pmod{12}$. Then $m = \frac{1}{2}(n-1) \equiv 5 \pmod 6$. Since $14 \le m < n$, there exists a system in $\{m;\ m-4\}$ containing at least $\left[\frac{m-10}{3}\right]$ disjoint copies of $\mathcal{T}_3$ and a copy of $\mathcal{V}_5$. By Theorem 4.8, there exists a system $W \in \{2m+1;\ 2m-3\} = \{n;\ n-4\}$ such that W contains $\left[\frac{2m-3}{3}\right] = \left[\frac{n-4}{3}\right] > \left[\frac{n-10}{3}\right]$ disjoint copies of $\mathcal{T}_3$ and a copy of $\mathcal{V}_5$. This completes the proof of the lemma.

<u>Corollary 4.35.</u> Let $n \equiv 4 \pmod 6$, $n \ge 10$. There exists a system $W \in \{n;\ \frac{n}{2} - 1\}$ containing a copy of $\mathcal{K}_4$.

<u>Proof.</u> The result follows from the proof of Theorem 2.7 and Lemma 4.34.

<u>Remark 12.</u> Corollary 4.35 can be proved by recursively applying Theorems 4.11 and 4.32.

<u>Theorem 4.36.</u> Let $n \ge 1$ be an odd integer. Then $\{n;\ a\}$ exists for all admissible values of a.

<u>Proof.</u> The statement is certainly true for all odd n, $1 \le n \le 17$, and $n = 21$ (see Chapter 2 and Example 4.26). By Theorem 3.1, $\{n;\ 0\}$ and $\{n;\ 3\}$ exist for $n \equiv 0 \pmod 3$, and $\{n;\ 1\}$ exists for $n \not\equiv 0 \pmod 3$. For the remainder of the proof we consider the following cases.

<u>Case 1.</u> $n \equiv 3$ or $7 \pmod{12}$, $n \ge 15$.

Then $n = 2m+1$, where $m \equiv 1$ or $3 \pmod 6$, $m \ge 7$. By Lemma 4.33, there exists a system $W \in \{m;\ m\}$ containing at least

$\left[\frac{m-6}{3}\right]$ disjoint copies of $\mathfrak{T}_3$. By Theorem 4.8, W can be extended to a consistent system $W^* \in \{n; n\}$. So $\{n; a\}$ exists for all admissible values of a , by Theorem 2.9.

<u>Case 2.</u> $n \equiv 1$ or $9 \pmod{12}$, $n \geq 25$.

Then $n = 2m+7$, where $m \equiv 1$ or $3 \pmod 6$, $m \geq 9$. By Lemma 4.33, there exists a system $W \in \{m; m\}$ containing at least $\left[\frac{m-6}{3}\right]$ disjoint copies of $\mathfrak{T}_3$. By Corollary 4.29, there exists a system $W^* \in \{n; n-3t\}$, where $t = 0, 1, \cdots, \left[\frac{n-3}{3}\right]$.

<u>Case 3.</u> $n \equiv 11 \pmod{12}$, $n \geq 23$.

Then $n = 2m+1$, where $m \equiv 5 \pmod 6$, $m \geq 11$. By Lemma 4.34, there exists a $W \in \{m; m-4\}$ containing at least $\left[\frac{m-10}{3}\right]$ disjoint copies of $\mathfrak{T}_3$. So by Theorem 4.8, there exists a consistent system $W^* \in \{n; n-4\}$. Hence $\{n; a\}$ exists for all admissible values of a .

<u>Case 4.</u> $n \equiv 5 \pmod{12}$, $n \geq 29$.

Then $n = 2m+7$, where $m \equiv 5 \pmod 6$, $m \geq 11$. By Lemma 4.35, there exists a $W \in \{m; m-4\}$ containing at least $\left[\frac{m-10}{3}\right]$ disjoint copies of $\mathfrak{T}_3$. By Corollary 4.29, there exists a $W^* \in \{n; n-4-3t\}$, where $t = 0, 1, \cdots, \left[\frac{n-7}{3}\right]$. This completes the proof of Theorem 4.36.

<u>Corollary 4.37.</u> Let $n \geq 2$ be an even integer. Then $\{n; a\}$ exists for all admissible values of a .

<u>Proof.</u> The statement is true for all even n , $2 \leq n \leq 16$ (see examples in Chapter 2). In what follows we shall apply Theorems 4.3, 4.30 and Corollary 4.31, using the results of Theorem 4.36. It is clear that

$\{n; \frac{n}{2}\}$ exists for $n \equiv 0$ or $2 \pmod 6$, and $\{n; \frac{n}{2} - 1\}$ exists for $n \equiv 4 \pmod 6$.

By Theorems 4.36 and 4.3, we have

1) $W \in \{6k+1; a\}$, $k \geq 1$; $a = 1, 4, \cdots, 6k+1$,
$\Rightarrow W^* \in \{12k+2; a\}$, $k \geq 1$; $a = 1, 4, \cdots, 6k+1$;

2) $W \in \{6k+3; a\}$, $k \geq 0$; $a = 0, 3, \cdots, 6k+3$,
$\Rightarrow W^* \in \{12k+6; a\}$, $k \geq 0$; $a = 0, 3, \cdots, 6k+3$;

3) $W \in \{6k+5; a\}$, $k \geq 0$; $a = 1, 4, \cdots, 6k+1$,
$\Rightarrow W^* \in \{12k+10; a\}$, $k \geq 0$; $a = 1, 4, \cdots, 6k+1$.

By Theorems 4.36, 4.30 and Corollary 4.31, we have

4) $W \in \{6k-3; a\}$, $k \geq 2$; $a = 0, 3, \cdots, 6k-3$,
$\Rightarrow W^* \in \{12k; b\}$, $k \geq 2$; $b = 0, 3, \cdots, 6k$;

5) $W \in \{6k+1; a\}$, $k \geq 1$; $a = 1, 4, \cdots, 6k+1$,
$\Rightarrow W^* \in \{12k+8; b\}$, $k \geq 1$; $b = 1, 4, \cdots, 6k+4$;

6) $W \in \{6k-1; a\}$, $k \geq 2$; $a = 1, 4, \cdots, 6k-5$,
$\Rightarrow W^* \in \{12k+4; b\}$, $k \geq 2$; $b = 1, 4, \cdots, 6k-2$.

This completes the proof of Corollary 4.37.

CHAPTER 5

Recursive methods for constructing consistent systems

1. Introduction.

A totally symmetric quasigroup is a quasigroup satisfying the identities $x(xy) = y$ and $(yx)x = y$.

A Steiner quasigroup is an idempotent quasigroup satisfying the totally symmetric identities. It is known (see [8]; cf. also [2]) that an extended triple system is algebraically a totally symmetric quasigroup, and a Steiner triple system is algebraically a Steiner quasigroup. We shall say that an extended triple system (S, W) is associated with a totally symmetric quasigroup (S, o) provided that $(a, b, c) \in W$ if and only if $a \circ b = c$. In a similar manner, we associate a Steiner triple system (S, T) with a Steiner quasigroup (S, o) .

In this chapter we shall construct consistent systems, using the direct product and singular direct product of quasigroups. The constructions given are recursive and require the use of a smaller system on which we impose very little or no restrictions. We refer to the examples given in Chapter 2 for all initial cases of induction. We conclude with a remark on the use of tricovers. It will subsequently be shown that the constructions in this chapter provide examples of systems which are inequivalent to some previously given.

2. Direct Products.

Let $(Q, *)$ and (V, o) be totally symmetric quasigroups of orders n and m, respectively. Let $W = Q \times V$ and define on W a binary operation $\otimes$ as follows:

$$(p, v) \otimes (q, w) = (p * q, v \circ w),\ p, q \in Q;\ v, w \in V .$$

Then it is readily verified that $(W, \otimes)$ is a totally symmetric quasigroup of order mn, called the direct product of $(Q, *)$ and (V, o).

Let q be an idempotent of $(Q, *)$ and v be an idempotent of (V, o). Then it is easily checked that

(1) (q, v) is an idempotent of $(W, \otimes)$;

(2) $(Q \times \{v\}, \otimes)$ is a subsystem of $(W, \otimes)$ isomorphic to $(Q, *)$;

(3) $(\{q\} \times V, \otimes)$ is a subsystem of $(W, \otimes)$ isomorphic to (V, o).

Combinatorially we can state the following theorem (cf. [8, Theorem 2]).

Theorem 5.1. Suppose there exists a system $Q^* \in \{n; r\}$ based on Q. Suppose there exists a system $V^* \in \{m; s\}$ based on V. Then there exists a system $W^* \in \{mn; rs\}$ based on $W = Q \times V$ such that W^* contains r disjoint copies of V^* and s disjoint copies of Q^*.

3. Singular Direct Products.

In [18], A. Sade gave a construction for quasigroups which he calls the singular direct product. More recently, C.C. Lindner [9, 10, 11, 12] has given some generalizations of this construction. Most of the constructions in this chapter will be based on a special case of the following generalized singular direct product of quasigroups. All quasigroups considered will be finite, and we use the notation of Lindner. The results of this section may also be compared with [8, Section 4], where the singular direct product was used to construct examples of extended triple systems.

Let $(Q, \otimes)$ be any quasigroup and on the set Q define six binary operations $\otimes(1,2,3)$, $\otimes(1,3,2)$, $\otimes(2,1,3)$, $\otimes(2,3,1)$, $\otimes(3,1,2)$, $\otimes(3,2,1)$ as follows:

$a \otimes b = c$ if and only if

$$a \otimes(1,2,3)\, b = c\ ,$$
$$a \otimes(1,3,2)\, c = b\ ,$$
$$b \otimes(2,1,3)\, a = c\ ,$$
$$b \otimes(2,3,1)\, c = a\ ,$$
$$c \otimes(3,1,2)\, a = b\ ,$$
$$c \otimes(3,2,1)\, b = a\ .$$

The six (not necessarily distinct) quasigroups $(Q, \otimes(i,j,k))$ are called the <u>conjugates</u> of $(Q, \otimes)$ [22].

Denote by $(T, *)$ the Steiner quasigroup of order 3 where $T = \{1,2,3\}$. Let $(V, \odot)$ be any Steiner quasigroup and (V, t) the associated Steiner triple system. Let $t_1, t_2, \cdots, t_r$ be the triples of t. Then each $(t_i, \odot)$ is a subquasigroup of $(V, \odot)$ and is isomorphic to $(T, *)$. Let α_i be a fixed isomorphism of $(t_i, \odot)$ onto $(T, *)$. Let Q be a set and for each v in V let $o(v)$ be a binary operation on Q so that $(Q, o(v))$ is a totally symmetric quasigroup. Further suppose that $P \subseteq Q$ and $p\ o(v)\ q = p\ o(w)\ q$ for all $p, q \in P$ and all $v, w \in V$ and that $(P, o(v))$ is a subquasigroup of $(Q, o(v))$. Let $(P' = Q - P, \otimes)$ be any quasigroup. If $p, q \in P'$ and $v \neq w \in V$, by $p \otimes (v, w, v \odot w) q$ is meant the element $p \otimes (v\alpha_i, w\alpha_i, v\alpha_i * w\alpha_i) q$ where $(v, w, v \odot w) = t_i$. On the set $P \cup (P' \times V)$ define the binary operation $\oplus$ by means of the following <u>generalized singular direct product</u>:

(1) $p \oplus q = p\ o(v)\ q = p\ o(w)\ q$, if $p, q \in P$;

(2) $p \oplus (q, v) = (p\ o(v)\ q, v)$, if $p \in P$, $q \in P'$, $v \in V$;

(3) $(q, v) \oplus p = (q\ o(v)\ p, v)$, if $p \in P$, $q \in P'$, $v \in V$;

(4) $(p, v) \oplus (q, v) = p\ o(v)\ q$, if $p\ o(v)\ q \in P$, and

$$= (p\ o(v)\ q, v), \text{ if } p\ o(v)\ q \in P';$$

(5) $(p, v) \oplus (q, w) = p \otimes (v, w, v \odot w) q, v \odot w)$, $v \neq w$.

Lindner denotes the quasigroup constructed by $V(\odot) \times Q(o(v), P, P' \otimes (u, v, w))$.

<u>Remark 13.</u> (i) The operations $o(v)$ are not necessarily related other than agreeing on P, unlike in [18] where $o(v) = o(w)$ for all $v, w \in V$.

(ii) $(V, \odot)$ and $(Q, o(v))$ are both totally symmetric, but the quasigroup $(P', \otimes)$ is not necessarily totally symmetric.

(iii) The single operator $\otimes$ may be replaced by a set of operators $\otimes_i$, $i = 1, 2, \cdots, r$, associated with each triple $t_i \in t$. Further, $\otimes_1, \otimes_2, \cdots, \otimes_r$ are not necessarily related (see [12]).

The results of Lindner [9, 10, 11, 12] enable us to state the following lemma (see also [13]) :

Lemma 5.2. The singular direct product $(W = P \cup (P' \times V), \oplus) = V(\odot) \times Q(o(v), P, P' \otimes (u, v, w))$ defined previously is a totally symmetric quasigroup which contains at least one isomorphic copy of each of the quasigroups $(Q, o(v))$ and at least as many disjoint copies of $(V, \odot)$ as there are elements in P' which are idempotent under $\otimes$ and under every $o(v)$. If $(Q, o(v))$ is a Steiner quasigroup, then $(W, \oplus)$ is a Steiner quasigroup. In general, we have $|W| = v(q - p) + p$, where $|V| = v$, $|Q| = q$ and $|P| = p$.

Combinatorially we can state the following theorem (cf. [8, Theorem 7]).

Theorem 5.3. Suppose there exists a system $V^* \in \{v; v\}$ based on V. Suppose there exists a system $Q^* \in \{q; r\}$ based on Q and containing a subsystem $P^* \in \{p; t\}$ based on P. If $(P' = Q - P, \otimes)$ is any quasigroup having λ idempotents in common with Q^*, then there exists a system $W^* \in \{v(q-p)+p; (r-t)v+t\}$ on $W = P \cup (P' \times V)$ such that W^* contains at least v copies of Q^* and λ disjoint copies of V^*.

Remark 14. It is worth noting that the hypothesis of Theorem 5.3 is more general than that given in [8, Theorem 7], where the quasigroup $(P', \otimes)$ is assumed to be totally symmetric. Furthermore, Theorem 5.3 itself can be generalized (see also [8, Theorem 8]). If in place of $Q^* \in \{q; r\}$ we assume the existence of $Q_i^* \in \{q; r_i\}$, $i = 1, 2, \cdots, v$, such that $Q_1^*, Q_2^*, \cdots, Q_v^*$ have a common subsystem $P^* \in \{p; t\}$, then in this case we are guaranteed the existence of a system $W^* \in \{v(q-p)+p; t + \sum_{i=1}^{v}(r_i - t)\}$ such that W^* contains λ disjoint copies of V^* and a copy of each of $Q_1^*, Q_2^*, \cdots, Q_v^*$. However, in most of what follows we shall need only a simple application of Theorem 5.3, invariably choosing $V^* \in \{3; 3\}$.

4. Construction of consistent systems.

In Chapter 2, we have given some examples of consistent systems in $\{n; a\}$ for values of $n \le 17$. In what follows we shall apply Theorems 5.1 and 5.3 recursively to construct consistent systems in $\{n; a\}$ for $n \ge 18$.

Theorem 5.4. Let n be a positive integer. Suppose there exists a system $Q^* \in \{n; r\}$. Then there exists a consistent system $W^* \in \{3n; 3r\}$ such that W^* contains 3 disjoint copies of Q^*.

Proof. Let $Q^* \in \{n; r\}$ be based on Q, and choose $V^* \in \{3; 3\}$ based on V in Theorem 5.1. By Theorem 5.1, there exists a system $W^* \in \{3n; 3r\}$ based on $W = Q \times V$ such that W^* contains r disjoint copies of V^* and 3 disjoint copies of Q^*. This completes the proof of the theorem.

Example 5.5. Let n be a positive integer. Then, by Theorem 5.4,

(i) $Q^* \in \{6n; 3n\} \Rightarrow$ consistent $W^* \in \{18n; 9n\}$ containing 3 disjoint copies of Q^*;

(ii) $Q^* \in \{6n+1; 6n+1\} \Rightarrow$ consistent $W^* \in \{18n+3; 18n+3\}$ containing 3 disjoint copies of Q^*;

(iii) $Q^* \in \{6n+2; 3n+1\} \Rightarrow$ consistent $W^* \in \{18n+6; 9n+3\}$ containing 3 disjoint copies of Q^*;

(iv) $Q^* \in \{6n+3;\ 6n+3\} \Rightarrow$ consistent $W^* \in \{18n+9;\ 18n+9\}$ containing 3 disjoint copies of Q^* ;

(v) $Q^* \in \{6n+4;\ 3n+1\} \Rightarrow$ consistent $W^* \in \{18n+12;\ 9n+3\}$ containing 3 disjoint copies of Q^* ;

(vi) $Q^* \in \{6n+5;\ 6n+1\} \Rightarrow$ consistent $W^* \in \{18n+15;\ 18n+3\}$ containing 3 disjoint copies of Q^* .

<u>Theorem 5.6.</u> Let n be a positive integer. If there exists a system $Q^* \in \{6n+4;\ 3n+1\}$ containing a copy of $\mathcal{K}_4$, then there exists a consistent system $\mathfrak{W}^* \in \{18n+12;\ 9n+6\}$ containing a subsystem $\mathfrak{W}^{**} \in \{12;\ 6\}$.

<u>Proof.</u> Let $Q^* \in \{6n+4;\ 3n+1\}$ be based on the set $Q = \{1, 2, \cdots, 6n+4\}$ such that $1, 5, 6, \cdots, 3n+4$ are the idempotents of Q^* . Let $P^* \in \{4;\ 1\}$ be based on $P = \{1, 2, 3, 4\}$ so that $P^* \subseteq Q^*$. Let $V^* \in \{3;\ 3\}$ be based on $V = \{1, 2, 3\}$. Let $(Q, *)$ and $(V, \circ)$ be the quasigroups associated with Q^* and V^* , respectively, and consider the direct product $(W = Q \times V, \otimes)$. Let W^* be the triple system associated with $(W, \otimes)$. Then it is clear that $W^* \in \{18n+12;\ 9n+3\}$ and W^* contains $3n$ disjoint copies of $\mathfrak{T}_3$ on sets $S_i = \{(i+4, j) \mid j = 1, 2, 3\}$, $i = 1, 2, \cdots, 3n$. Further, W^* contains a subsystem $W^{**} \in \{12;\ 3\}$ based on the set $S = \{(i, j) \mid i = 1, 2, 3, 4;\ j = 1, 2, 3\}$. Let $\mathfrak{W}^{**} \in \{12;\ 6\}$ be a consistent system based on S . Finally, we put $\mathfrak{W}^* = (W^* - W^{**}) \cup \mathfrak{W}^{**}$. Then $\mathfrak{W}^* \in \{18n+12;\ 9n+6\}$ and $\mathfrak{W}^*$ contains $3n+2$ disjoint copies of $\mathfrak{T}_3$, which completes the proof of the theorem.

Theorem 5.7. Let n be a positive integer. If there exists a system $Q^* \in \{6n+5;\ 6n+1\}$ containing a copy of $\mathcal{V}_5$, then there exists a consistent system $\mathcal{W}^* \in \{18n+15;\ 18n+15\}$ containing a subsystem $\mathcal{W}^{**} \in \{15;\ 15\}$.

Proof. The proof is very similar to that of Theorem 5.6. We let $V^* \in \{3;\ 3\}$ be based on $V = \{1, 2, 3\}$. Let $Q^* \in \{6n+5;\ 6n+1\}$ be based on $Q = \{1, 2, \cdots, 6n+5\}$. Then via the direct product $(Q \times V, \otimes)$ we obtain a $W^* \in \{18n+15;\ 18n+3\}$ such that W^* contains $6n$ disjoint copies of $\mathcal{T}_3$ which are mutually disjoint from a subsystem $W^{**} \in \{15;\ 3\}$. By removing W^{**} and replacing it with a consistent $\mathcal{W}^{**} \in \{15;\ 15\}$, we obtain $\mathcal{W}^* = (W^* - W^{**}) \cup \mathcal{W}^{**}$ such that $\mathcal{W}^* \in \{18n+15;\ 18n+15\}$ contains $6n+5$ mutually disjoint copies of $\mathcal{T}_3$, which proves the theorem.

Theorem 5.8. Let n be a positive integer. Suppose there exists a system $Q^* \in \{n;\ a\}$ with $a \geq 1$. Then there exists a consistent system $W^* \in \{3n-2;\ 3a-2\}$ containing at least 3 copies of Q^*. If n is even, then W^* contains a copy of $\mathcal{K}_4$.

Proof. Let $V^* \in \{3;\ 3\}$. Let $Q^* \in \{n;\ a\}$ be based on Q. Since $a \geq 1$, Q^* contains a $P^* \in \{1;\ 1\}$ on $P = \{i\}$ for some $i \in Q$. Let $(P' = Q-P, \otimes)$ be any idempotent quasigroup. Clearly $(P', \otimes)$ contains $a-1$ idempotents in common with Q^*. By Theorem 5.3, there exists a system $W^* \in \{3n-2;\ 3a-2\}$ such that W^* contains $a-1$ disjoint copies of V^* and at least 3 copies of Q^*. So W^* is consistent. If n is even, we further observe that Q^* contains a subsystem $Q^{**} \in \{2; 1\}$ which contains P^*, and the result follows from the construction of W^*.

<u>Example 5.9.</u> Let n be a positive integer. Then, by Theorem 5.8,

(i) $Q^* \in \{6n; 3n\} \Rightarrow$ consistent $W^* \in \{18n-2; 9n-2\}$ containing 3 copies of Q^* and a copy of $\mathcal{K}_4$;

(ii) $Q^* \in \{6n+1; 6n+1\} \Rightarrow$ consistent $W^* \in \{18n+1; 18n+1\}$ containing 3 copies of Q^*;

(iii) $Q^* \in \{6n+2; 3n+1\} \Rightarrow$ consistent $W^* \in \{18n+4; 9n+1\}$ containing 3 copies of Q^* and a copy of $\mathcal{K}_4$;

(iv) $Q^* \in \{6n+3; 6n+3\} \Rightarrow$ consistent $W^* \in \{18n+7; 18n+7\}$ containing 3 copies of Q^*;

(v) $Q^* \in \{6n+4; 3n+1\} \Rightarrow$ consistent $W^* \in \{18n+10; 9n+1\}$ containing 3 copies of Q^* and a copy of $\mathcal{K}_4$;

(vi) $Q^* \in \{6n+5; 6n+1\} \Rightarrow$ consistent $W^* \in \{18n+13; 18n+1\}$ containing 3 copies of Q^*.

<u>Lemma 5.10.</u> If $n \equiv 4 \pmod 6$, there exists an idempotent quasigroup of order n containing a subquasigroup of order 4.

<u>Proof.</u> In what follows let $(Q_4, \cdot)$ and $(Q_4, *)$ be the quasigroups given by the accompanying tables.

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

$(Q_4, \cdot)$

$*$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

$(Q_4, *)$

Let $n = 6k+4$. Let S be the set $\{0, 1, 2, \cdots, 6k+3\}$. Let (S_0, T) be an $STS(6k+3)$ where $S_0 = S - \{0\}$ and T contains a collection Δ of $2k+1$ pairwise disjoint triples covering S_0 (see for example the proof of Theorem 3.2). Let $T^* = \{(x, x, 0) \mid x \in S\}$ and put $W = T \cup T^*$. Then $W \in \{6k+4; 1\}$ based on S. Since each triple of Δ gives rise to a copy of $\mathcal{K}_4$ in W, W contains $2k+1$ copies of $\mathcal{K}_4$ pairwise intersecting in $(0, 0, 0)$. Let $(S, \odot)$ be the quasigroup associated with W. Then $(S, \odot)$ contains $2k+1$ copies of $(Q_4, \cdot)$ pairwise intersecting in the idempotent 0. Let $(S, \otimes)$ be the result of removing the $2k+1$ copies of $(Q_4, \cdot)$ and replacing them with $2k+1$ copies of $(Q_4, *)$ pairwise intersecting in the idempotent 0. Then it is clear that $(S, \otimes)$ is an idempotent quasigroup of order $n = 6k+4$, and $(S, \otimes)$ contains a subquasigroup of order 4.

Remark 15. The quasigroup $(S, \otimes)$ constructed in the proof of Lemma 5.10 satisfies $x^2 = x$, $x(yx) = y$.

As an example, let $n=10$. $S=\{0,1,2,3,4,5,6,7,8,9\}$. Let (S_0, T) be an $STS(9)$ where $T = \{(1,2,3), (4,5,6), (7,8,9),$

(1, 4, 7), (1, 5, 9), (1, 6, 8), (2, 4, 9), (2, 5, 8), (2, 6, 7), (3, 4, 8), (3, 5, 7), (3, 6, 9)}, $\Delta = \{(1,2,3), (4,5,6), (7,8,9)\}$.

$(S, \otimes)$ is given by the table at right.

$\otimes$	0	1	2	3	4	5	6	7	8	9
0	0	3	1	2	6	4	5	9	7	8
1	2	1	3	0	7	9	8	4	6	5
2	3	0	2	1	9	8	7	6	5	4
3	1	2	0	3	8	7	9	5	4	6
4	5	7	9	8	4	6	0	1	3	2
5	6	9	8	7	0	5	4	3	2	1
6	4	8	7	9	5	0	6	2	1	3
7	8	4	6	5	1	3	2	7	9	0
8	9	6	5	4	3	2	1	0	8	7
9	7	5	4	6	2	1	3	8	0	9

Theorem 5.11. Let n be a positive integer. If there exists a system $Q^* \in \{6n+5;\ 6n+1\}$ containing a copy of $\mathcal{V}_5$, then there exists a consistent system $\mathcal{W}^* \in \{18n+13;\ 18n+13\}$ containing a subsystem $\mathcal{W}^{**} \in \{13;\ 13\}$.

Proof. Let $Q^* \in \{6n+5;\ 6n+1\}$ be based on the set $Q = \{1, 2, \cdots, 6n+5\}$ so that Q^* contains $P_1^* \in \{5;\ 1\}$ based on $P_1 = \{1, 2, 3, 4, 5\}$, where the element 1 is idempotent. Then $1, 6, 7, \cdots, 6n+5$ are idempotents of Q^* and so Q^* contains a subsystem $P^* \in \{1;\ 1\}$ based on $P = \{1\}$. Let $V^* \in \{3;\ 3\}$ be based on $V = \{1, 2, 3\}$. Let $(P' = Q - P, \otimes)$ be an idempotent quasigroup of order $6n+4$ containing a subquasigroup of order 4 based on the set $\{2, 3, 4, 5\}$.

Let $(V, \odot)$, $(Q, o(v))$, $(P, o(v))$ be the quasigroups associated with V^*, Q^*, P^*, respectively, and consider the singular direct product

$$(W, \oplus) = V(\odot) \times Q(o(v), P, P' \otimes(u,v,w)) .$$

Let W^* be the extended triple system associated with $(W, \oplus)$. Then $W^* \in \{18n+13;\ 18n+1\}$, and W^* contains a subsystem $W^{**} \in \{13;\ 1\}$, based on the set $S = \{(i, j) \mid i = 2, 3, 4, 5;\ j = 1, 2, 3\} \cup \{1\}$, and $6n$ disjoint copies of $\mathfrak{I}_3$, based on sets $S_i = \{(i+5, j) \mid j = 1, 2, 3\}$, $i = 1, 2, 3, \cdots, 6n$. Let $\mathfrak{W}^{**}$ be a consistent system in $\{13;\ 13\}$ based on S. Finally, put $\mathfrak{W}^* = (W^* - W^{**}) \cup \mathfrak{W}^{**}$. Then $\mathfrak{W}^* \in \{18n+13;\ 18n+13\}$ contains $6n+4$ mutually disjoint copies of $\mathfrak{I}_3$, and this completes the proof of the theorem.

<u>Theorem 5.12.</u> Let n be a positive integer. Suppose there exists a system $Q^* \in \{6n+4;\ 3n+1\}$ containing a copy of $\mathcal{K}_4$. Then

(i) there exists a consistent system $\mathfrak{W}_1^* \in \{18n+10;\ 9n+4\}$ containing a copy of $\mathcal{K}_4$ and a subsystem $\mathfrak{W}_1^{**} \in \{10;\ 4\}$;

(ii) there exists a consistent system $\mathfrak{W}_2^* \in \{18n+11;\ 18n+7\}$ containing a subsystem $\mathfrak{W}_2^{**} \in \{11;\ 7\}$.

<u>Proof.</u> (i) Let $Q^* \in \{6n+4;\ 3n+1\}$ be based on the set $Q = \{1, 2, \cdots, 6n+4\}$ so that Q^* contains a copy P_1^* of $\mathcal{K}_4$ on $P_1 = \{1, 2, 3, 4\}$. Assume the idempotents of Q^* are $1, 5, 6, \cdots, 3n+4$. It is clear that P_1^* contains a $P^* \in \{1;\ 1\}$ on $P = \{1\}$. Let $V^* \in \{3;\ 3\}$ on $V = \{1, 2, 3\}$. Let $(P' = Q-P, \otimes)$ be a Steiner quasigroup containing a subquasigroup of order 3 on $\{2, 3, 4\}$.

Let $(V, \odot)$, $(Q, o(v))$, $(P, o(v))$ be quasigroups associated with V^*, Q^*, P^*, respectively. Consider the singular direct product $(W, \oplus) = V(\odot) \times Q(o(v), P, P' \otimes (u, v, w))$.

Let W^* be the triple system associated with $(W, \oplus)$. Then $W^* \in \{18n+10; 9n+1\}$. W^* contains a subsystem $W^{**} \in \{10; 1\}$ based on $S = \{1\} \cup \{(i, j) \mid i = 2, 3, 4; j = 1, 2, 3\}$, and $3n$ mutually disjoint copies of $\mathfrak{T}_3$ on sets $S_i = \{(i+4, j) \mid j = 1, 2, 3\}$, $i = 1, 2, \cdots, 3n$. Let $\mathfrak{W}_1^{**} \in \{10; 4\}$ be based on S such that $\mathfrak{W}_1^{**}$ contains a copy of $\mathcal{K}_4$ and a copy of $\mathfrak{T}_3$. Put $\mathfrak{W}_1^* = (W^* - W^{**}) \cup \mathfrak{W}_1^{**}$. Then it is readily verified that $\mathfrak{W}_1^* \in \{18n+10; 9n+4\}$ contains a copy of $\mathcal{K}_4$ and $3n+1$ mutually disjoint copies of $\mathfrak{T}_3$, completing the proof of (i).

(ii) The system $\mathfrak{W}_1^* \in \{18n+10; 9n+4\}$ constructed in part (i) is based on the set $W = \{1\} \cup \{(i, j) \mid i = 2, 3, \cdots, 6n+4 ; j = 1, 2, 3\}$. Now let $W_2 = W \cup \{\infty\}$, where ∞ is a new element. Let $W_2^* \in \{18n+11; 18n+7\}$, based on W_2, be derived from $\mathfrak{W}_1^*$ using the construction given in the proof of Theorem 2.7. Then W_2^* contains a subsystem $W_2^{**} \in \{11; 7\}$ based on the set $S^* = \{1, \infty\} \cup \{(i, j) \mid i = 2, 3, 4; j = 1, 2, 3\}$. W_2^{**} contains $\mathfrak{V}_5$ and is thus not consistent. Let $\mathfrak{W}_2^{**}$ be a consistent system in $\{11; 7\}$ based on S^* and put $\mathfrak{W}_2^* = (W_2^* - W_2^{**}) \cup \mathfrak{W}_2^{**}$. then $\mathfrak{W}_2^* \in \{18n+11; 18n+7\}$, based on W_2, contains 2 disjoint copies of $\mathfrak{T}_3$ in $\mathfrak{W}_2^{**}$ in addition to $6n$ mutually disjoint copies of $\mathfrak{T}_3$ on the sets $S_i^* = \{(i+4, j) \mid j = 1, 2, 3\}$, $i = 1, 2, \cdots, 6n$. Thus $\mathfrak{W}_2^*$ is consistent, and the proof of the theorem is complete.

<u>Theorem 5.13.</u> Let n be a positive integer. Suppose there exists a system $Q^* \in \{6n; 3n\}$. Then there exists a consistent system $\mathfrak{W}^* \in \{18n-1; 18n-5\}$ containing a copy of $\mathcal{V}_5$.

<u>Proof.</u> We shall first construct a consistent system $W^* \in \{18n-2; 9n-2\}$ containing a copy of $\mathcal{K}_4$ as follows. Let $Q^* \in \{6n; 3n\}$ be based on $Q = \{1, 2, \cdots, 6n\}$. Assume the idempotents of Q^* are $1, 3, 4, \cdots, 3n+1$ and Q^* contains a subsystem $Q^{**} \in \{2; 1\}$ on the set $\{1, 2\}$. Clearly, Q^* contains a $P^* \in \{1; 1\}$ on $P = \{1\}$. Let $V^* \in \{3; 3\}$ be based on $V = \{1, 2, 3\}$. Let $(P' = Q-P, \otimes)$ be any idempotent quasigroup. Let $(V, \odot)$, $(Q, o(v))$, $(P, o(v))$ be the quasigroups associated with V^*, Q^*, P^*, respectively. Consider the singular direct product $(W, \oplus) = V(\odot) \times Q(o(v), P, P' \otimes (u, v, w))$. Let W^* be the triple system associated with $(W, \oplus)$. Then a straightforward verification shows $W^* \in \{18n-2; 9n-2\}$ contains a copy of $\mathcal{K}_4$ on the set $S = \{1\} \cup \{(i, j) \mid i = 2;\ j = 1, 2, 3\}$. In addition, W^* contains $3n-1$ mutually disjoint copies of $\mathcal{T}_3$ on sets $S_i = \{(i+2, j) \mid j = 1, 2, 3\}$, $i = 1, 2, \cdots, 3n-1$. So W^* is also consistent. Now let us add a new element ∞ to $W = \{1\} \cup \{(i, j) \mid i = 2, 3, \cdots, 6n;\ j = 1, 2, 3\}$. Put $W' = \{\infty\} \cup W$. Let $\mathfrak{W}^* \in \{18n-1; 18n-5\}$, based on W', be the system derived from W^* via the construction given in Theorem 2.7. Then $\mathfrak{W}^*$ contains a copy of $\mathcal{V}_5$ on the set $S^* = \{1, \infty\} \cup \{(i, j) \mid i = 2;\ j = 1, 2, 3\}$. Also $\mathfrak{W}^*$ contains $6n-2$ mutually disjoint copies of $\mathcal{T}_3$ on sets $S_i^* = \{(i+2, j) \mid j = 1, 2, 3\}$, $i = 1, 2, \cdots, 6n-2$. Hence $\mathfrak{W}^*$ is consistent and our theorem is proved.

<u>Theorem 5.14.</u> Let n be a positive integer. Suppose there exists a system $Q^* \in \{6n+2;\ 3n+1\}$. Then there exists a consistent system $\mathfrak{W}^* \in \{18n+5;\ 18n+1\}$ containing a copy of $\mathcal{V}_5$.

<u>Proof.</u> The proof of this theorem is identical to that given in Theorem 5.13 and the details are omitted.

<u>Theorem 5.15.</u> Let n be a positive integer. Suppose there exists a system $Q^* \in \{6n+5;\ 6n+1\}$ containing a copy of $\mathcal{V}_5$. Then there exists a consistent system $W^* \in \{18n+5;\ 18n+1\}$ containing 3 copies of Q^*.

<u>Proof.</u> Let $P^* \in \{5;\ 1\}$ and assume $P^* \subseteq Q^* \in \{6n+5;\ 6n+1\}$. Let $V^* \in \{3;\ 3\}$ be based on V, and assume Q^* and P^* are based on Q and P, respectively. Let $(P' = Q-P, \otimes)$ be any idempotent quasigroup. Then $(P', \otimes)$ has $6n$ idempotents in common with Q^*. By Theorem 5.3, there exists a system $W^* \in \{18n+5;\ 18n+1\}$ containing at least 3 copies of Q^* and $6n$ disjoint copies of V^*, which completes the proof of the theorem.

Similarly, one can prove

<u>Theorem 5.16.</u> Let n be a positive integer. Suppose there exists a system $Q^* \in \{6n+4;\ 3n+1\}$ containing a copy of $\mathcal{K}_4$. Then there exists a consistent system $W^* \in \{18n+4;\ 9n+1\}$ containing 3 copies of Q^*.

<u>Theorem 5.17.</u> Let $n \geq 2$ be an even integer. If there exists a system $Q^* \in \{n;\ a\}$ with $a \geq 1$, then there exists a consistent system $W^* \in \{3n-4;\ 3a-2\}$ containing 3 copies of Q^*.

Proof. Since n is even and $a \geq 1$, $Q^* \in \{n; a\}$ contains a subsystem $P^* \in \{2; 1\}$. Let Q^* and P^* be based on Q and P, respectively. Let $(P' = Q - P, \otimes)$ be any idempotent quasigroup. Let $V^* \in \{3; 3\}$ be based on V. By Theorem 5.3, there exists a system $W^* \in \{3n-4; 3a-2\}$ such that W^* contains at least 3 copies of Q^* and $a-1$ disjoint copies of V^*, which proves the theorem.

Example 5.18. Let n be any positive integer. Then, by Theorem 5.17,

(i) $Q^* \in \{6n; 3n\}$ $\Rightarrow$ consistent $W^* \in \{18n-4; 9n-2\}$ containing 3 copies of Q^*;

(ii) $Q^* \in \{6n+2; 3n+1\}$ $\Rightarrow$ consistent $W^* \in \{18n+2; 9n+1\}$ containing 3 copies of Q^*;

(iii) $Q^* \in \{6n+4; 3n+1\}$ $\Rightarrow$ consistent $W^* \in \{18n+8; 9n+1\}$ containing 3 copies of Q^*.

Theorem 5.19. Let n be a positive integer. Suppose there exists a system $Q^* \in \{6n+4; 3n+1\}$ containing a copy of $\mathcal{K}_4$. Then there exists a consistent system $\mathfrak{W}^* \in \{18n+8; 9n+4\}$ containing a subsystem $\mathfrak{W}^{**} \in \{8; 4\}$.

Proof. Let $Q^* \in \{6n+4; 3n+1\}$ be based on the set $Q = \{1, 2, \cdots, 6n+4\}$ and assume $P_1^* \subseteq Q^*$ where P_1^* is a copy of $\mathcal{K}_4$ on $P_1 = \{1, 2, 3, 4\}$. Let the idempotents of Q^* be $1, 5, 6, \cdots, 3n+4$. Clearly, Q^* contains a subsystem $P^* \in \{2; 1\}$ on $P = \{1, 2\}$. Let $V^* \in \{3; 3\}$ be based on $V = \{1, 2, 3\}$. Let $(V, \odot)$, $(Q, o(v))$, $(P, o(v))$ be the quasigroups associated with V^*, Q^* and P^*, respectively. Let $(P' = Q - P, \otimes)$ be the quasigroup associated with an extended triple system $W' \in \{6n+2;\ 3n+1\}$ based on P' in which $4, 5, 6, \cdots, 3n+4$ are idempotents and the triple $(3,3,4)$ is contained in W'. Then $(P', \otimes)$ has $3n$ idempotents in common with $(Q, o(v))$. Consider the singular direct product $(W, \oplus) = V(\odot) \times Q(o(v), P, P' \otimes (u, v, w))$. Let W^* be the triple system associated with $(W, \oplus)$. Then $W^* \in \{18n+8;\ 9n+1\}$. Further, W^* contains a subsystem $W^{**} \in \{8;\ 1\}$ on $S = \{1, 2\} \cup \{(i, j) \mid i = 3, 4;\ j = 1, 2, 3\}$ and $3n$ mutually disjoint copies of $\mathfrak{T}_3$ on the sets $S_i = \{(i+4, j) \mid j = 1, 2, 3\}$, $i = 1, 2, \cdots, 3n$. Let $\mathfrak{W}^{**} \in \{8;\ 4\}$ be based on S so that $\mathfrak{W}^{**}$ contains a copy of $\mathfrak{T}_3$.

Put $\mathfrak{W}^* = (W^* - W^{**}) \cup \mathfrak{W}^{**}$. Then it is readily verified that $\mathfrak{W}^* \in \{18n+8;\ 9n+4\}$ contains $3n+1$ mutually disjoint copies of $\mathfrak{T}_3$, and so the proof of the theorem is complete.

Summarizing some of the results of this section we have proved inductively

<u>Theorem 5.20.</u> Let $n \geq 18$ be an integer. Then

(1) if $n \equiv 1$ or $3 \pmod 6$, then there exists a consistent system in $\{n;\ n\}$ containing a subsystem on m elements, where $3 < m < n$;

(2) if $n \equiv 0$ or $2 \pmod 6$, then there exists a consistent system in $\{n;\ \frac{n}{2}\}$ containing a subsystem on m elements, where $3 < m < n$;

(3) if $n \equiv 4 \pmod 6$, then there exists a consistent system in $\{n;\ \frac{n}{2} - 1\}$ containing a copy of $\mathcal{K}_4$;

(4) if $n \equiv 5$ or $17 \pmod{18}$, then there exists a consistent system in $\{n;\ n-4\}$ containing a copy of $\mathcal{V}_5$;

(5) if $n \equiv 11 \pmod{18}$, then there exists a consistent system in $\{n;\ n-4\}$ containing a subsystem in $\{11;\ 7\}$.

By Theorem 2.9, the result of Theorem 5.20 is sufficient to guarantee the existence of a consistent system in $\{n;\ a\}$ for every integer $n \geq 18$ and every admissible value of a .

5. <u>The use of Tricovers.</u>

The following notion of tricover may be found in [4] .

Let A_1, A_2, A_3 be three mutually disjoint nonempty sets, each containing the same number n of elements.

Let T be a set of subsets of $A_1 \cup A_2 \cup A_3$ such that

(i) each element of T is a triple of the form (a_1, a_2, a_3) where $a_i \in A_i$, $i = 1, 2, 3$;

(ii) all pairs of elements of $A_1 \cup A_2 \cup A_3$ not belonging to the same A_i belong to exactly one element of T .

Then T is called a <u>tricover</u> for the system (A_1, A_2, A_3) .
A tricover exists for every system (A_1, A_2, A_3) . For let $A_i = \{j_i \mid j = 1, 2, \cdots, n\}$, $i = 1, 2, 3$. Then it is readily checked that

$$T = \{(i_1, j_2, k_3) \mid 1 \leq i, j, k \leq n\ ;\ i+j+k \equiv 0 \pmod{n}\} \text{ is a}$$

tricover for (A_1, A_2, A_3) .

The object of this section is to illustrate how tricovers can be used in the construction of extended triple systems. The role which the tricover plays is similar to that of the quasigroup $(P', \otimes)$ in the singular direct product defined in the third section. We shall confine ourselves to only one illustrative example, but it is worth noting that most of the results in the previous section can be obtained using tricovers.

Example 5.21. In Theorem 5.13 we have shown that the existence of $\{6n; 3n\}$ implies the existence of a consistent system in $\{18n-1; 18n-5\}$ containing $\mathcal{V}_5$. In this example we shall assume the existence of a system $Q^* \in \{6n+1; 6n+1\}$ and establish the existence of a consistent system $W^* \in \{18n-1; 18n-5\}$ containing $\mathcal{V}_5$.

Let $Q^* \in \{6n+1; 6n+1\}$ be based on the set $Q = \{\infty_1, \infty_2, 1, 2, \cdots, 6n-1\}$ so that $(\infty_1, \infty_2, 1)$ belongs to Q^*.
Let $A_i = \{k_i \mid k = 1, 2, \cdots, 6n-1\}$; $i = 1, 2, 3$.
Let $A_i^* = A_i \cup \{\infty_1, \infty_2\}$ and $W_i \in \{6n+1; 6n+1\}$ be based on A_i^* so that apart from subscripts each W_i, $i = 1, 2, 3$, is the identical copy of Q^*. In particular, note that $(\infty_1, \infty_2, 1_i)$ belongs to W_i, $i = 1, 2, 3$.
Let $T = \{(i_1, j_2, k_3) \mid 1 \le i, j, k \le 6n-1;\ i+j \equiv 2k (\mathrm{mod}\ 6n-1)\}$ be a tricover for (A_1, A_2, A_3).
Let
$$V^* = \{(\infty_1, \infty_1, \infty_1), (\infty_2, \infty_2, \infty_2), (1_1, 1_1, 1_1), (1_2, 1_2, 1_2), (1_3, 1_3, 1_3), (\infty_1, \infty_2, 1_1), (\infty_1, \infty_2, 1_2), (\infty_1, \infty_2, 1_3), (1_1, 1_2, 1_3)\}.$$
Clearly V^* is contained in $W_1 \cup W_2 \cup W_3$.
Let
$$V = \{(\infty_1, \infty_1, \infty_1), (\infty_2, \infty_2, 1_1), (1_1, 1_1, 1_2), (1_2, 1_2, 1_3), (1_3, 1_3, \infty_2), (\infty_1, 1_1, 1_3), (\infty_1, \infty_2, 1_2)\}.$$
Then V is a copy of $\mathcal{V}_5$.
Put $W^* = ((W_1 \cup W_2 \cup W_3) - V^*) \cup V \cup T$. It is readily verified that

$W^* \in \{18n-1;\ 18n-5\}$ and W^* contains $6n-2$ mutually disjoint copies of $\mathfrak{T}_3$ on the sets $S_k = \{(k+1)_i \mid i = 1, 2, 3\}$, $k = 1, 2, \cdots, 6n-2$. Consequently, W^* is consistent and contains $\mathfrak{V}_5$.

CHAPTER 6

Non-isomorphic systems in $\{n; a\}$

1. Introduction.

Let (S, W) and (S^*, W^*) be two extended triple systems. An isomorphism from (S, W) onto (S^*, W^*) is a bijection $\alpha : S \to S^*$ such that $(W)\alpha = W^*$. If (S, W) and (S^*, W^*) are isomorphic, then they necessarily belong to the same class $\{n; a\}$ and we simply say the systems W and W^* are isomorphic in $\{n; a\}$.

The main purpose of this chapter is to show that, with a few exceptions to be found in Section 3, there exist at least two non-isomorphic systems in $\{n; a\}$ for a given n and a for which $\{n; a\}$ exists. Section 3 is devoted to examples of non-isomorphic systems in $\{n; a\}$ for values of $n \le 11$. The main result is obtained in Section 4 and is based almost entirely on the constructions given in the previous chapters. In Section 5 we give a simple application of the singular direct product. Finally, in Section 6 the thesis is concluded with some remarks and problems.

2. Preliminaries.

Suppose W and W^* are two non-isomorphic systems in $\{n; a\}$ each containing at least t mutually disjoint copies of $\mathfrak{J}_3$. Further suppose W and W^* do not contain a copy of $\mathfrak{J}_0$. Let W_k and W_k^* be, respectively, the systems derived from W and W^* by removing k of the t mutually disjoint copies of $\mathfrak{J}_3$ and appropriately replacing them with k mutually disjoint copies of $\mathfrak{J}_0$. Then W_k and W_k^* belong to $\{n; a - 3k\}$. We may also claim that W_k and W_k^* are non-isomorphic systems. For suppose there exists an isomorphism from W_k onto W_k^*. Then such an isomorphism must map the k mutually disjoint copies of $\mathfrak{T}_0$ in W_k onto the k mutually disjoint copies of $\mathfrak{T}_0$ in W_k^*. This immediately gives rise to an isomorphism from W onto W^*, contrary to what we have assumed. We now state the following basic lemma that will be applied in most of the constructions given in this chapter.

Lemma 6.1. If there exist two non-isomorphic systems in $\{n; a\}$ each containing at least t mutually disjoint copies of $\mathfrak{T}_3$ and no copy of $\mathfrak{T}_0$, then, for every $k \in \{1, 2, \cdots, t\}$, there exist at least two non-isomorphic systems in $\{n; a-3k\}$ each containing $t-k$ mutually disjoint copies of $\mathfrak{T}_3$ and k mutually disjoint copies of $\mathfrak{T}_0$.

3. Non-isomorphic systems in $\{n; a\}$, $1 \le n \le 11$.

For each class $\{n; a\}$, where $1 \le n \le 11$, we shall list a representative. In cases where there are non-isomorphic systems in $\{n; a\}$ we give at least two examples. It is instantly seen that $\{1; 1\}$, $\{2; 1\}$, $\{3; 0\}$, $\{3; 3\}$ have a unique representation. Apart from isomorphism, there is only one STS(n) for $n = 7$ or 9. So $\{7; 7\}$ and $\{9; 9\}$ are uniquely represented. Using Lemma 2.1 and a bit of manipulation, it is not unduly difficult to show that $\{5; 1\}$, $\{6; 0\}$, $\{7; 1\}$, $\{7; 4\}$, $\{8; 4\}$ and $\{9; 6\}$ are also uniquely represented. Apart from these exceptions, there exist at least two non-isomorphic systems in $\{n; a\}$ for $n \le 11$ and all values of a for which $\{n; a\}$ exists.

Class	Example
{1; 1}	(1, 1, 1).
{2; 1}	(1, 1, 1), (2, 2, 1).
{3; 0}	(1, 1, 2), (2, 2, 3), (3, 3, 1).
{3; 3}	(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3).
{4; 1}	W_1: (1, 1, 1), (2, 2, 1), (3, 3, 1), (4, 4, 1), (2, 3, 4). W_2: (1, 1, 1), (2, 2, 1), (3, 3, 2), (4, 4, 2), (1, 3, 4).
{5; 1}	(1, 1, 1), (2, 2, 3), (3, 3, 4), (4, 4, 5), (5, 5, 2), (1, 2, 4), (1, 3, 5).
{6; 0}	(1, 1, 2), (2, 2, 3), (3, 3, 1), (4, 4, 1), (5, 5, 2), (6, 6, 3), (1, 5, 6), (2, 4, 6), (3, 4, 5).
{6; 3}	W_1: (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 1), (5, 5, 2), (6, 6, 3), (1, 2, 3), (1, 5, 6), (2, 4, 6), (3, 4, 5). W_2: (1, 1, 1), (2, 2, 2), (6, 6, 6), (4, 4, 1), (5, 5, 2), (3, 3, 6), (1, 2, 3), (1, 5, 6), (2, 4, 6), (3, 4, 5). W_1 contains a copy of $\mathfrak{T}_3$, while W_2 does not contain a copy of $\mathfrak{T}_3$.
{7; 1}	(1, 1, 1), (2, 2, 3), (3, 3, 4), (4, 4, 5), (5, 5, 6), (6, 6, 7), (7, 7, 2), (1, 2, 5), (1, 3, 6), (1, 4, 7), (2, 4, 6), (3, 5, 7).
{7; 4}	(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 6), (6, 6, 7), (7, 7, 5), (1, 2, 5), (1, 3, 6), (1, 4, 7), (2, 3, 7), (2, 4, 6), (3, 4, 5).

Class	Example
{7; 7}	(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6), (7, 7, 7), (1, 2, 4), (1, 3, 7), (1, 5, 6), (2, 3, 5), (2, 6, 7), (3, 4, 6), (4, 5, 7).
{8; 1}	W_1: (8, 8, 8), (1, 1, 8), (2, 2, 8), (3, 3, 8), (1, 2, 3), (4, 4, 8), (5, 5, 8), (6, 6, 8), (7, 7, 8), (1, 4, 7), (1, 5, 6), (2, 4, 5), (2, 6, 7), (3, 4, 6), (3, 5, 7).
	W_2: (8, 8, 8), (1, 1, 8), (2, 2, 1), (3, 3, 1), (2, 3, 8), (4, 4, 8), (5, 5, 8), (6, 6, 8), (7, 7, 8), (1, 4, 7), (1, 5, 6), (2, 4, 5), (2, 6, 7), (3, 4, 6), (3, 5, 7).
	W_3: (8, 8, 8), (1, 1, 2), (2, 2, 3), (3, 3, 1), (7, 7, 8), (5, 5, 1), (4, 4, 2), (6, 6, 3), (1, 4, 7), (1, 6, 8), (2, 5, 8), (2, 6, 7), (3, 4, 8), (3, 5, 7), (4, 5, 6).
	W_1 contains a copy of $\mathcal{K}_4$, but does not contain a copy of $\mathcal{K}_4^*$. W_2 contains a copy of K_4^*. W_1 and W_2 do not contain $\mathfrak{J}_0$. W_3 contains a copy of $\mathfrak{J}_0$, but does not contain a copy of $\mathcal{K}_4$ or $\mathcal{K}_4^*$.
{8; 4}	(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 1), (6, 6, 2), (7, 7, 3), (8, 8, 4), (1, 2, 3), (1, 4, 7), (1, 6, 8), (2, 4, 5), (2, 7, 8), (3, 4, 6), (3, 5, 8), (5, 6, 7).
{9; 0}	W_1: (1, 1, 2), (2, 2, 3), (3, 3, 4), (4, 4, 5), (5, 5, 6), (6, 6, 7), (7, 7, 8), (8, 8, 9), (9, 9, 1), (1, 3, 7), (1, 4, 6), (1, 5, 8), (2, 4, 8), (2, 5, 7), (2, 6, 9), (3, 5, 9), (3, 6, 8), (4, 7, 9).

Class	Example
{9; 0}	W_2: (1,1,2), (2,2,3), (3,3,1), (4,4,5), (5,5,6), (6,6,7), (7,7,8), (8,8,9), (9,9,4), (1,4,7), (1,5,9), (1,6,8), (2,4,8), (2,5,7), (2,6,9), (3,4,6), (3,5,8), (3,7,9).
	W_3: (1,1,2), (2,2,3), (3,3,1), (4,4,5), (5,5,6), (6,6,4), (7,7,8), (8,8,9), (9,9,7), (1,4,7), (1,5,8), (1,6,9), (2,4,8), (2,5,9), (2,6,7), (3,4,9), (3,5,7), (3,6,8).
	W_1 does not contain a copy of $\mathfrak{T}_0$.
	W_2 contains precisely one copy of $\mathfrak{T}_0$.
	W_3 contains 3 mutually disjoint copies of $\mathfrak{T}_0$.
{9; 3}	W_1: (1,1,1), (2,2,2), (3,3,3), (4,4,5), (5,5,6), (6,6,7), (7,7,8), (8,8,9), (9,9,4), (1,2,8), (1,3,6), (1,4,7), (1,5,9), (2,3,4), (2,5,7), (2,6,9), (3,5,8), (3,7,9), (4,6,8).
	W_2: (1,1,1), (2,2,2), (3,3,3), (4,4,5), (5,5,6), (6,6,7), (7,7,8), (8,8,9), (9,9,4), (1,2,3), (1,4,7), (1,5,9), (1,6,8), (2,4,8), (2,5,7), (2,6,9), (3,4,6), (3,5,8).
	W_3: (1,1,1), (2,2,2), (3,3,3), (4,4,5), (5,5,6), (6,6,4), (7,7,8), (8,8,9), (9,9,7), (1,2,3), (1,4,7), (1,5,8), (1,6,9), (2,4,8), (2,5,9), (2,6,7), (3,4,9), (3,5,7), (3,6,8).

Class	Example
{9; 3}	W_1 does not contain a copy of $\mathfrak{J}_0$ or $\mathfrak{J}_3$. W_2 contains a copy of $\mathfrak{J}_3$, but does not contain a copy of $\mathfrak{J}_0$. W_3 contains a copy of $\mathfrak{J}_3$ and two disjoint copies of $\mathfrak{J}_0$.
{9; 6}	(1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6), (7,7,8), (8,8,9), (9,9,7), (1,2,3), (1,4,7), (1,5,8), (1,6,9), (2,4,8), (2,5,9), (2,6,7), (3,4,9), (3,5,7), (3,6,8), (4,5,6).
{9; 9}	(1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6), (7,7,7), (8,8,8), (9,9,9), (1,2,3), (1,4,7), (1,5,8), (1,6,9), (2,4,8), (2,5,9), (2,6,7), (3,4,9), (3,5,7), (3,6,8), (4,5,6), (7,8,9).
{10; 1}	W_1: (10,10,10), (1,1,10), (2,2,10), (3,3,10), (1,2,3), (4,4,10), (5,5,10), (6,6,10), (7,7,10), (8,8,10), (9,9,10), (1,4,7), (1,5,9), (1,6,8), (2,4,9), (2,5,8), (2,6,7). (3,4,8), (3,5,7), (3,6,9), (4,5,6), (7,8,9). W_2: (10,10,10), (1,1,10), (2,2,1), (3,3,1), (2,3,10), (4,4,10), (5,5,10), (6,6,10), (7,7,10), (8,8,10), (9,9,10), (1,4,7), (1,5,9), (1,6,8), (2,4,9), (2,5,8), (2,6,7), (3,4,8), (3,5,7), (3,6,9), (4,5,6), (7,8,9).

Class	Example
{10; 1}	W_3: (10,10,10), (1,1,2), (2,2,3), (3,3,1), (4,4,5), (5,5,1), (6,6,2), (7,7,10), (8,8,3), (9,9,5), (1,4,6), (1,7,8), (1,9,10), (2,4,7), (2,5,10), (2,8,9), (3,4,9), (3,5,7), (3,6,10), (4,8,10), (5,6,8), (6,7,9).
	W_4: (10,10,10), (1,1,8), (8,8,4), (4,4,2), (2,2,6), (6,6,8), (3,3,4), (5,5,10), (7,7,6), (9,9,2), (1,2,7), (1,3,6), (1,4,5), (1,9,10), (2,3,5), (2,8,10), (3,7,10), (3,8,9), (4,6,10), (4,7,9), (5,6,9), (5,7,8).

W_1 contains a copy of $\mathcal{K}_4$, but does not contain a copy of $\mathcal{K}_4^*$, $\mathcal{J}_0$ or $\mathcal{V}_5$.

W_2 contains a copy of $\mathcal{K}_4$ and $\mathcal{K}_4^*$, but does not contain a copy of $\mathcal{J}_0$ or $\mathcal{V}_5$.

W_3 contains a copy of $\mathcal{J}_0$ but does not contain a copy of $\mathcal{K}_4$, $\mathcal{K}_4^*$ or $\mathcal{V}_5$.

W_4 contains a copy of $\mathcal{V}_5$ on {2, 4, 6, 8, 10} , but does not contain a copy of $\mathcal{J}_0$, $\mathcal{K}_4$ or $\mathcal{K}_4^*$.

Observe that both W_1 and W_2 contain several copies of $\mathcal{K}_4$ and W_2 contains a unique copy of $\mathcal{K}_4^*$ on {1, 2, 3, 10} .

{10; 4}	W_1: (1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,1), (6,6,2), (7,7,3), (8,8,4), (9,9,1), (10,10,1), (1,2,3), (1,4,6), (1,7,8), (2,4,5), (2,7,9), (2,8,10), (3,4,9), (3,5,8), (3,6,10), (4,7,10), (5,6,7), (5,9,10), (6,8,9).

Class | Example

{10; 4}

W_2: (1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,1), (6,6,2), (7,7,3), (8,8,4), (9,9,5), (10,10,5), (1,2,3), (1,4,6), (1,7,8), (1,9,10), (2,4,5), (2,7,9), (2,8,10), (3,4,9), (3,5,8), (3,6,10), (4,7,10), (5,6,7), (6,8,9).

W_3: (1,1,1), (2,2,2), (3,3,3), (6,6,6), (4,4,2), (5,5,1), (7,7,6), (8,8,10), (9,9,10), (10,10,3), (1,2,3), (1,4,9), (1,6,10), (1,7,8), (2,5,10), (2,6,8), (2,7,9), (3,4,8), (3,5,7), (3,6,9), (4,5,6), (4,7,10), (5,8,9).

W_4: (1,1,1), (2,2,2), (7,7,7), (10,10,10), (3,3,10), (4,4,2), (5,5,1), (6,6,7), (8,8,10), (9,9,10), (1,2,3), (1,4,9), (1,6,10), (1,7,8), (2,5,10), (2,6,8), (2,7,9), (3,4,8), (3,5,7), (3,6,9), (4,5,6), (4,7,10), (5,8,9).

W_1 contains a copy of $\mathfrak{J}_3$ and a copy of $\mathcal{K}_4$.

W_2 contains a copy of $\mathfrak{J}_3$ and a copy of $\mathcal{K}_4^*$.

W_3 contains a copy of $\mathfrak{J}_3$ but does not contain a copy of $\mathcal{K}_4$ or $\mathcal{K}_4^*$.

W_4 does not contain $\mathfrak{J}_3$, $\mathcal{K}_4$ or $\mathcal{K}_4^*$.

{11; 1}

W_1: (11,11,11), (1,1,2), (2,2,3), (3,3,1), (4,4,5), (5,5,6), (6,6,4), (7,7,8), (8,8,9), (9,9,10), (10,10,7), (1,4,10), (1,5,7), (1,6,9), (1,8,11), (2,4,7), (2,5,9), (2,6,8), (2,10,11), (3,4,8), (3,5,11), (3,6,10), (3,7,9), (4,9,11), (5,8,10), (6,7,11).

Class	Example
{11; 1}	W_2: (11, 11, 11), (1, 1, 2), (2, 2, 3), (3, 3, 4), (4, 4, 5), (5, 5, 6), (6, 6, 7), (7, 7, 8), (8, 8, 9), (9, 9, 10), (10, 10, 1), (1, 3, 6), (1, 4, 7), (1, 5, 8), (1, 9, 11), (2, 4, 8), (2, 5, 11), (2, 6, 10), (2, 7, 9), (3, 5, 9), (3, 7, 11), (3, 8, 10), (4, 6, 9), (4, 10, 11), (5, 7, 10), (6, 8, 11).
	W_1 contains two disjoint copies of $\mathfrak{T}_0$.
	W_2 does not contain a copy of $\mathfrak{T}_0$.
{11; 4}	W_1: (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 6), (6, 6, 7), (7, 7, 8), (8, 8, 9), (9, 9, 10), (10, 10, 11), (11, 11, 5), (1, 2, 3), (1, 4, 8), (1, 5, 9), (1, 6, 10), (1, 7, 11), (2, 4, 6), (2, 5, 8), (2, 7, 10), (2, 9, 11), (3, 4, 11), (3, 5, 7), (3, 6, 9), (3, 8, 10), (4, 5, 10), (4, 7, 9), (6, 8, 11).
	W_2: (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 6), (6, 6, 7), (7, 7, 5), (8, 8, 10), (10, 10, 9), (9, 9, 11), (11, 11, 8), (1, 2, 3), (1, 4, 8), (1, 5, 11), (1, 6, 10), (1, 7, 9), (2, 4, 9), (2, 5, 10), (2, 6, 8), (2, 7, 11), (3, 4, 5), (3, 6, 9), (3, 7, 8), (3, 10, 11), (4, 6, 11), (4, 7, 10), (5, 8, 9).

Class	Example
$\{11;\ 4\}$	W_3: (1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,6), (6,6,7), (7,7,5), (8,8,10), (10,10,9), (9,9,11), (11,11,8), (1,2,5), (1,3,11), (1,4,8), (1,6,10), (1,7,9), (2,3,10), (2,4,9), (2,6,8), (2,7,11), (3,4,5), (3,6,9), (3,7,8), (4,6,11), (4,7,10), (5,8,9), (5,10,11).
	W_1 contains a copy of $\mathfrak{J}_3$, but does not contain a copy of $\mathfrak{J}_0$.
	W_2 contains a copy of $\mathfrak{J}_3$ and $\mathfrak{J}_0$, and
	W_3 contains a copy of $\mathfrak{J}_0$ but does not contain a copy of $\mathfrak{J}_3$.
$\{11;\ 7\}$	W_1: (1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6), (7,7,7), (8,8,10), (10,10,9), (9,9,11), (11,11,8), (1,2,3), (1,4,9), (1,5,11), (1,6,10), (1,7,8), (2,4,11), (2,5,10), (2,6,8), (2,7,9), (3,4,8), (3,5,7), (3,6,9), (3,10,11), (4,5,6), (4,7,10), (5,8,9), (6,7,11).
	W_2: (1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6), (7,7,7), (8,8,9), (9,9,10), (10,10,11), (11,11,8), (1,2,3), (1,4,8), (1,5,11), (1,6,10), (1,7,9), (2,4,10), (2,5,9), (2,6,11), (2,7,8), (3,4,5), (3,6,7), (3,8,10), (3,9,11), (4,6,9), (4,7,11), (5,6,8), (5,7,10).
	W_1 contains two disjoint copies of $\mathfrak{J}_3$ on $\{1, 2, 3\}, \{4, 5, 6\}$.

Class	Example
$\{11;\ 7\}$	W_2 contains a copy of $\mathcal{V}_5$ on $\{3, 8, 9, 10, 11\}$ and cannot contain two disjoint copies of $\mathcal{T}_3$. Observe that W_1 does not contain a copy of $\mathcal{V}_5$. It is easy to verify that a system in $\{11;\ 7\}$ which contains a copy of $\mathcal{V}_5$ cannot be consistent. Note that W_2 contains copies of $\mathcal{T}_3$ on $\{1, 2, 3\}$, $\{3, 4, 5\}$, $\{3, 6, 7\}$.

4. Main results.

I. Non-isomorphic consistent systems in $\{6n+1;\ a\}$, $n \geq 2$.

Example 6.2. There are essentially two non-isomorphic STS(13) and these are given in [5, p. 237]. We give here the two non-isomorphic extended triple systems which are associated with these STS(13) . The systems W and W^* in $\{13;\ 13\}$ differ only in the four triples marked with an asterisk, and each system contains 4 mutually disjoint copies of $\mathcal{T}_3$ on the sets $\{1,6,7\}$, $\{2,9,12\}$, $\{4,10,13\}$ and $\{5,8,11\}$. Hence W and W^* are consistent systems.

$W \in \{13;\ 13\}$:			$W^* \in \{13;\ 13\}$:		
(1, 1, 1)	(1, 2, 3)	$(3, 6, 13)^*$	(1, 1, 1)	(1, 2, 3)	$(3, 6, 10)^*$
(2, 2, 2)	(1, 4, 5)	(3, 7, 11)	(2, 2, 2)	(1, 4, 5)	(3, 7, 11)
(3, 3, 3)	(1, 6, 7)	$(3, 9, 10)^*$	(3, 3, 3)	(1, 6, 7)	$(3, 9, 13)^*$
(4, 4, 4)	(1, 8, 9)	(4, 7, 9)	(4, 4, 4)	(1, 8, 9)	(4, 7, 9)
(5, 5, 5)	(1, 10, 11)	(4, 10, 13)	(5, 5, 5)	(1, 10, 11)	(4, 10, 13)
(6, 6, 6)	(1, 12, 13)	(4, 11, 12)	(6, 6, 6)	(1, 12, 13)	(4, 11, 12)
(7, 7, 7)	(2, 4, 6)	$(5, 6, 10)^*$	(7, 7, 7)	(2, 4, 6)	$(5, 6, 13)^*$
(8, 8, 8)	(2, 5, 7)	(5, 8, 11)	(8, 8, 8)	(2, 5, 7)	(5, 8, 11)
(9, 9, 9)	(2, 8, 10)	$(5, 9, 13)^*$	(9, 9, 9)	(2, 8, 10)	$(5, 9, 10)^*$
(10, 10, 10)	(2, 9, 12)	(6, 8, 12)	(10, 10, 10)	(2, 9, 12)	(6, 8, 12)
(11, 11, 11)	(2, 11, 13)	(6, 9, 11)	(11, 11, 11)	(2, 11, 13)	(6, 9, 11)
(12, 12, 12)	(3, 4, 8)	(7, 8, 13)	(12, 12, 12)	(3, 4, 8)	(7, 8, 13)
(13, 13, 13)	(3, 5, 12)	(7, 10, 12)	(13, 13, 13)	(3, 5, 12)	(7, 10, 12)

Theorem 6.3. For every integer $n \geq 2$, there exist at least two non-isomorphic consistent systems in $\{6n+1;\ 6n+1\}$.

Proof. For $n = 2$, the result is established in Example 6.2. We now consider the case $n \geq 3$. J. Doyen [3] has shown that the $STS(6n+1)$ constructed in the proof of Theorem 3.4 contains no subsystems other than those of order 1, 3 and $6n+1$. Thus, by Theorem 3.4, the extended triple system $W \in \{6n+1;\ 6n+1\}$ associated with this $STS(6n+1)$ is consistent and contains no subsystems other than those on 1, 3 and $6n+1$ elements. On the other hand, Example 5.9 and Theorem 5.11 guarantee the existence of a consistent system

$W^* \in \{6n+1;\ 6n+1\}$ which contains a subsystem on m elements, where $3 < m < 6n+1$. It is clear that W and W^* are non-isomorphic systems and our theorem is proved.

<u>Corollary 6.3.</u> For every integer $n \geq 2$, there exist at least two non-isomorphic consistent systems in $\{6n+1;\ a\}$, where $1 \leq a \leq 6n+1$ and $a \equiv 1 (\mathrm{mod}\ 3)$.

<u>Proof.</u> The result is a consequence of Theorem 6.2. and Lemma 6.1.

II. <u>Non-isomorphic consistent systems in $\{6n+3;\ a\}$, $n \geq 2$.</u>

<u>Example 6.4.</u> In Remark 8 following Theorem 4.8 we pointed out that it is possible to construct a consistent system $W^* \in \{15;\ 15\}$ containing a subsystem in $\{7;\ 7\}$. Explicitly, we give the construction here. It is easily checked that the system $W^* \in \{15;\ 15\}$ contains a subsystem in $\{7;\ 7\}$ on $\{1, 2, \cdots, 7\}$ and 5 mutually disjoint copies of $\mathfrak{J}_3$ on the sets $\{1,2,4\}$, $\{3,9,12\}$, $\{5,8,13\}$, $\{6,10,11\}$ and $\{7,14,15\}$.

$W^* \in \{15;\ 15\}$:

(1, 1, 1)	(11, 11, 11)	(1, 10, 14)	(3, 9, 12)	(5, 11, 14)
(2, 2, 2)	(12, 12, 12)	(1, 12, 13)	(3, 10, 15)	(5, 12, 15)
(3, 3, 3)	(13, 13, 13)	(2, 3, 5)	(3, 11, 13)	(6, 8, 12)
(4, 4, 4)	(14, 14, 14)	(2, 6, 7)	(4, 5, 7)	(6, 9, 14)
(5, 5, 5)	(15, 15, 15)	(2, 8, 11)	(4, 8, 9)	(6, 10, 11)
(6, 6, 6)	(1, 2, 4)	(2, 9, 15)	(4, 10, 13)	(6, 13, 15)
(7, 7, 7)	(1, 3, 7)	(2, 10, 12)	(4, 11, 15)	(7, 8, 10)
(8, 8, 8)	(1, 5, 6)	(2, 13, 14)	(4, 12, 14)	(7, 9, 13)
(9, 9, 9)	(1, 8, 15)	(3, 4, 6)	(5, 8, 13)	(7, 11, 12)
(10, 10, 10)	(1, 9, 11)	(3, 8, 14)	(5, 9, 10)	(7, 14, 15)

Theorem 6.5. For every integer $n \geq 2$, there exist at least two non-isomorphic consistent systems in $\{6n+3;\ 6n+3\}$.

Proof. Let $G = \{1, a, a^2, \cdots, a^{2n}\}$ be a multiplicative cyclic group of order $2n+1$ and let us consider the Cartesian product $S = G \times \{0, 1, 2\}$. For every $i \in \{0, 1, 2\}$, the element (x, i) of the subset $G \times \{i\}$ will be denoted by $(x)_i$ or, where there is no danger of confusion, simply by x_i . Let φ be the permutation on $\{1, a, a^2, \cdots, a^{2n}\}$ defined as follows:

$$\varphi(a^j) = a^{j+1} \quad \text{if } 0 \leq j \leq n-2 ,$$
$$\varphi(a^{n-1}) = 1 ,$$
$$\varphi(a^j) = a^j \quad \text{if } n \leq j \leq 2n .$$

Let $A = \{(x_0, x_1, x_2) \mid x \in G\}$,

$B = \{(x_0, y_0, z_1) \mid x, y, z \in G ; x \neq y ; xy = z^2\}$,

$C = \{(x_1, y_1, z_2) \mid x, y, z \in G ; x \neq y ; xy = z^2\}$,

$D = \{(x_2, y_2, z_0) \mid x, y, z \in G ; x \neq y ; \varphi(x)\,\varphi(y) = (\varphi(z))^2\}$.

Put $T = A \cup B \cup C \cup D$. Then it can be verified without much difficulty that (S, T) is an $STS(6n+3)$. This construction is essentially due to Doyen [3], who has shown that the $STS(6n+3)$ described here contains no subsystems other than those of order 1, 3 and $6n+3$. Now let $J = \{(x, x, x) \mid x \in S\}$, and set $W = T \cup J$. Then $W \in \{6n+3; 6n+3\}$ and W contains $2n+1$ mutually disjoint copies of $\mathfrak{J}_3$ on the sets $\{x_0, x_1, x_2\}$, $x \in G$. Consequently, W is a consistent system which does not contain any subsystems other than those on 1, 3 and $6n+3$ elements. However, according to Examples 5.5 and 6.4 and Theorem 5.7, we can construct a consistent system $W^* \in \{6n+3; 6n+3\}$ such that W^* contains a subsystem on m elements, where $3 < m < 6n+3$. Clearly, the systems W and W^* are non-isomorphic and the proof is complete.

<u>Corollary 6.6.</u> For every integer $n \geq 2$, there exist at least two non-isomorphic consistent systems in $\{6n+3; a\}$, where $0 \leq a \leq 6n+3$ and $a \equiv 0 (\mathrm{mod}\ 3)$.

<u>Proof.</u> The proof follows from Theorem 6.5 and Lemma 6.1.

III. Non-isomorphic systems in $\{6n; a\}$, $n \geq 2$.

Theorem 6.7. Let n be a positive integer. Then there exists an inconsistent system $W^* \in \{6n; 3n\}$ such that W^* contains $n-1$ mutually disjoint copies of $\mathfrak{J}_3$.

Proof. Let (S, T) be the $STS(6n+1)$ described in the proof of Theorem 3.4. As in the proof of Theorem 3.7, we shall construct $W^* \in \{6n; 3n\}$ by deleting from T the collection C of all triples containing the number $6n$.

$$C = \begin{cases} (x+n, & x+2n, & 6n) \\ (x+3n, & x+4n, & 6n) \\ (x+5n, & x & , 6n) \end{cases} \quad : \quad x = 0, 1, 2, \cdots, n-1 .$$

We consider two cases as follows.

Case 1. n even.

Let

$$J_1^* = \begin{cases} (5n, 5n, 5n) , \\ (x, x, x) : x = 1, 2, \cdots, n-1 , \\ (x+n, x+n, x+n) : x = 0, 1, \cdots, n-1 , \\ (x+3n, x+3n, x+3n) : x = 0, 1, \cdots, n-1 ; \end{cases}$$

$$J_2^* = \begin{cases} (0, 0, 5n) , \\ (x+5n, x+5n, x) : x = 1, 2, \cdots, n-1 , \\ (x+2n, x+2n, x+n) : x = 0, 1, \cdots, n-1 , \\ (x+4n, x+4n, x+3n) : x = 0, 1, \cdots, n-1 . \end{cases}$$

Put $W^* = (T-C) \cup J_1^* \cup J_2^*$.

Case 2. n odd.

Let

$$J_1^* = \begin{cases} (6n-1,\ 6n-1,\ 6n-1)\ , \\ (x,\ x,\ x) : x = 0,\ 1,\ \cdots,\ n-2\ , \\ (x+n,\ x+n,\ x+n) : x = 0,\ 1,\ \cdots,\ n-1\ , \\ (x+3n,\ x+3n,\ x+3n) : x = 0,\ 1,\ \cdots,\ n-1\ ; \end{cases}$$

$$J_2^* = \begin{cases} (n-1,\ n-1,\ 6n-1)\ , \\ (x+5n,\ x+5n,\ x) : x = 0,\ 1,\ \cdots,\ n-2\ , \\ (x+2n,\ x+2n,\ x+n) : x = 0,\ 1,\ \cdots,\ n-1\ , \\ (x+4n,\ x+4n,\ x+3n) : x = 0,\ 1,\ \cdots,\ n-1\ . \end{cases}$$

Put $W^* = (T-C) \cup J_1^* \cup J_2^*$.

In each of the two cases it is clear that $W^* \in \{6n;\ 3n\}$. From the description of $(S,\ T)$ it is not difficult to show that W^* is inconsistent. If α and β are integers such that $0 \le \alpha \le 2n-1$ and $3n \le \beta \le 4n-1$, then neither $(\alpha,\ \beta,\ 5n)$ nor $(\alpha,\ \beta,\ 6n-1)$ is a triple of $T-C$, and when all the possibilities are considered W^* contains at most $n-1$ mutually disjoint copies of $\mathfrak{T}_3$. If n is even, then $T-C$ contains the following collection of $n-1$ disjoint triples:

$$D = \begin{cases} (x,\ x+1,\ x+3n) : x = 1,\ 3,\ 5,\ \cdots,\ n-1\ , \\ (x+n+1,\ x+n+2,\ x+3n+1) : x = 1,\ 3,\ 5,\ \cdots,\ n-3\ . \end{cases}$$

If $n \ge 3$ is odd, then $T-C$ contains $n-1$ disjoint triples:

$$D = \begin{cases} (x,\ x+1,\ x+3n) : x = 0,\ 2,\ 4,\ \cdots,\ n-3\ , \\ (x+n+1,\ x+n+2,\ x+3n+1) : x = 0,\ 2,\ 4,\ \cdots,\ n-3\ . \end{cases}$$

It is then readily verified that $W^* \in \{6n; 3n\}$ contains $n-1$ mutually disjoint copies of $\mathfrak{J}_3$, and our theorem is proved.

<u>Corollary 6.8.</u> For every integer $n \geq 2$, there exists an inconsistent system $W^* \in \{6n; a\}$, where $3 \leq a \leq 3n$ and $a \equiv 0 (\mathrm{mod}\ 3)$.

<u>Proof.</u> The result follows from Theorems 6.7 and 2.9.

<u>Theorem 6.9.</u> For every integer $n \geq 2$, there exist at least two non-isomorphic systems in $\{6n; a\}$, where $0 \leq a \leq 3n$ and $a \equiv 0 (\mathrm{mod}\ 3)$.

<u>Proof.</u> We shall consider two cases as follows.

<u>Case 1.</u> $n \geq 2$, $a = 0$.

Let $W = \{(x, y, z) \mid 1 \leq x, y, z \leq 6n; x+y+z \equiv 0 (\mathrm{mod}\ 6n)\}$. By Theorem 3.1, $W \in \{6n; 3\}$ and W contains a copy of $\mathfrak{J}_3$ on $\{2n, 4n, 6n\}$. We further claim that W does not contain a copy of $\mathfrak{J}_0$. Suppose W contains a collection $T = \{(i, i, j), (j, j, k), (k, k, i)\}$. Then we have $2i+j \equiv 2j+k \equiv 2k+i \equiv 0 (\mathrm{mod}\ 6n)$, and elementary considerations show $i = j = k$. So W does not contain a copy of $\mathfrak{J}_0$. Let $W^* \in \{6n; 0\}$ be the system derived from W by removing the copy of $\mathfrak{J}_3$ and replacing it with a copy of $\mathfrak{J}_0$. Then W^* contains a single copy of $\mathfrak{J}_0$. However, Theorems 2.9 and 3.7 guarantee the existence of a system $W \in \{6n; 0\}$ which contains n mutually disjoint copies of $\mathfrak{J}_0$. Since $n \geq 2$, the systems W and W^* are clearly non-isomorphic.

<u>Case 2.</u> $n \geq 2$, $3 \leq a \leq 3n$, $a \equiv 0 (\mathrm{mod}\ 3)$.

Let $W \in \{6n; a\}$ be as in Corollary 3.8 and $W^* \in \{6n; a\}$ be as in Corollary 6.8. Then W and W^* are non-isomorphic.

This completes the proof of Theorem 6.9.

Remark 16. It is known [3] that the $STS(6n+1)$ constructed in the proof of Theorem 3.4 contains no subsystems other than those of order 1, 3 and $6n+1$. It follows directly that the consistent system $W \in \{6n; 3n\}$ of Theorem 3.7, which is derived from this $STS(6n+1)$, contains no subsystems other than those on 1, 2, 3 and $6n$ elements. However, for $n \geq 3$ Theorem 5.20 guarantees the existence of a consistent system $W^* \in \{6n; 3n\}$ such that W^* contains a subsystem on m elements, where $3 < m < 6n$. Consequently, for $n \geq 3$ there exist at least two non-isomorphic consistent systems in $\{6n; a\}$, where $0 \leq a \leq 3n$ and $a \equiv 0 \pmod 3$.

IV. Non-isomorphic systems in $\{6n+2; a\}$, $n \geq 2$.

In what follows K_{2n} denotes the complete graph on $2n$ vertices. A one-factorization $\mathfrak{F} = \{F_1, F_2, \cdots, F_{2n-1}\}$ of K_{2n} is said to contain a one-factorization $\mathfrak{G} = \{G_1, G_2, \cdots, G_{2m-1}\}$ of K_{2m} if $V(K_{2m}) \subseteq V(K_{2n})$ and, for each $i \in \{1, 2, \cdots, 2m-1\}$, there exists a $j \in \{1, 2, \cdots, 2n-1\}$ such that $E(G_i) \subseteq E(F_j)$. Let (S, T) be any $STS(n)$, where $S = \{1, 2, \cdots, n\}$ and $(1, 2, 3) \in T$. Let $S^* = \{0, 1, 2, \cdots, n\}$. Let $\mathfrak{F} = \{F_i \mid i = 1, 2, \cdots, n\}$, where $F_i = \{[0, i]\} \cup \{[x, y] \mid (i, x, y) \in T\}$. Then it is readily verified that $\mathfrak{F}$ is a one-factorization of K_{n+1} with $V(K_{n+1}) = S^*$. Furthermore, $\mathfrak{F}$ contains a one-factorization of K_4 on $\{0, 1, 2, 3\}$, since $\{[0, 1], [2, 3]\} \subseteq F_1$, $\{[0, 2], [1, 3]\} \subseteq F_2$ and $\{[0, 3], [1, 2]\} \subseteq F_3$. Consequently, for $n \geq 1$ there is always a one-factorization of K_{6n+2} which contains a one-factorization of K_4.

We shall use this fact in our next theorem.

Theorem 6.10. Let n be a positive integer. Then there exists an inconsistent system $W^* \in \{12n+2;\ 6n+1\}$ such that W^* contains $2n-1$ mutually disjoint copies of $\mathfrak{I}_3$.

Proof. Let (S, T) be an $STS(6n+1)$, where $S = \{\alpha_1, \alpha_2, \cdots, \alpha_{6n+1}\}$ and T contains a set of $2n-1$ mutually disjoint triples $D = \{(\alpha_5, \alpha_6, \alpha_7), (\alpha_8, \alpha_9, \alpha_{10}), \cdots, (\alpha_{6n-1}, \alpha_{6n}, \alpha_{6n+1})\}$. The existence of such an $STS(6n+1)$ is guaranteed by the proof of Theorem 3.4. Let $S^* = \{1, 2, \cdots, 6n+2\}$. Let $\mathfrak{F} = \{F_i \mid i = 1, 2, \cdots, 6n+1\}$ be a one-factorization of K_{6n+2} with $V(K_{6n+2}) = S^*$ such that $\mathfrak{F}$ contains a one-factorization of K_4 on $\{1, 2, 3, 4\}$. Assume, without loss of generality, that $[i, 6n+2] \in F_i$, $i = 1, 2, \cdots, 6n+1$. Set $F_i^* = F_i - \{[i, 6n+2]\}$. Let

$$T^* = \{(\alpha_i, x, y) \mid [x, y] \in F_i^*;\ \ i = 1, 2, \cdots, 6n+1\},$$

$$J_1 = \{(\alpha_i, \alpha_i, \alpha_i) \mid i = 5, 6, \cdots, 6n+1\} \cup \{(i, i, i) \mid i = 1, 2, 3, 4\},$$

$$J_2 = \{(i, i, \alpha_i) \mid i = 5, 6, \cdots, 6n+1\} \cup \{(\alpha_i, \alpha_i, i) \mid i = 1, 2, 3, 4\}.$$

Put $W^* = T \cup T^* \cup J_1 \cup J_2$. Then it is readily checked that $W^* \in \{12n+2;\ 6n+1\}$ and W^* contains $2n-1$ mutually disjoint copies of $\mathfrak{I}_3$ on the sets $\{\alpha_5, \alpha_6, \alpha_7\}, \{\alpha_8, \alpha_9, \alpha_{10}\}, \cdots, \{\alpha_{6n-1}, \alpha_{6n}, \alpha_{6n+1}\}$. It is also easily verified that W^* contains no more than $2n-1$ mutually disjoint copies of $\mathfrak{I}_3$, since there cannot be two disjoint copies of $\mathfrak{I}_3$ on sets of the type $\{\alpha_r, s, t\}$, where $5 \le r \le 6n+1$ and $1 \le s < t \le 4$. Consequently, W^* is inconsistent and the proof of the theorem is complete.

<u>Theorem 6.11.</u> Let n be a positive integer. Then there exists an inconsistent system $W^* \in \{12n+8;\ 6n+4\}$ such that W^* contains $2n$ mutually disjoint copies of $\mathfrak{J}_3$.

<u>Proof.</u> We consider two cases.

<u>Case 1.</u> $n = 1$.

Let $W \in \{7;\ 7\}$. Then according to the construction given in the proof of Theorem 4.30, we can embed W in an inconsistent system $W^* \in \{20;\ 10\}$ which contains a pair of disjoint copies of $\mathfrak{J}_3$ (see Remark 11) .

<u>Case 2.</u> $n \geq 2$.

By Theorem 3.4, there exists a consistent system $W \in \{6n+1;\ 6n+1\}$. By Theorem 4.30 and Remark 11, W can be embedded in an inconsistent system $W^* \in \{12n+8;\ 6n+4\}$ such that W^* contains precisely the $2n$ mutually disjoint copies of $\mathfrak{J}_3$ in W . This completes the proof of the theorem.

<u>Corollary 6.12.</u> For every integer $n \geq 2$, there exists an inconsistent system $W^* \in \{6n+2;\ a\}$, where $4 \leq a \leq 3n+1$ and $a \equiv 1 (\mathrm{mod}\ 3)$.

<u>Proof.</u> The result is a consequence of Theorems 2.9, 6.10, and 6.11.

<u>Theorem 6.13.</u> For every integer $n \geq 2$, there exist at least two non-isomorphic systems in $\{6n+2;\ a\}$, where $1 \leq a \leq 3n+1$ and $a \equiv 1(\mathrm{mod}\ 3)$.

<u>Proof.</u> We shall consider two cases.

<u>Case 1.</u> $n \geq 1$, $a = 1$.

Let $W \in \{6n+2; 1\}$ be as constructed in the proof of Theorem 2.12 and $W^* \in \{6n+2; 1\}$ be as in Corollary 2.13. Then W and W^* are clearly non-isomorphic. In addition, Theorem 3.9 guarantees the existence of a system $W^{**} \in \{6n+2; 1\}$ which contains n mutually disjoint copies of $\mathfrak{I}_0$. It is easily checked that neither W nor W^* contains a copy of $\mathfrak{I}_0$, and so for $n \geq 1$ there are at least three non-isomorphic systems in $\{6n+2; 1\}$.

<u>Case 2.</u> $n \geq 2$, $4 \leq a \leq 3n+1$, $a \equiv 1 \pmod 3$.

Let $W \in \{6n+2; a\}$ be as in Corollary 3.10 and $W^* \in \{6n+2; a\}$ be as in Corollary 6.12. Obviously, W and W^* are non-isomorphic systems, and this completes the proof of the theorem.

<u>Remark 17.</u> If $n \geq 4$, then there are inconsistent systems in $\{6n; 3n\}$ and $\{6n+2; 3n+1\}$ which contain fewer than $n-1$ mutually disjoint copies of $\mathfrak{I}_3$. In fact, if n is any odd integer, there is a system in $\{6n; 3n\}$ which does not contain a copy of $\mathfrak{I}_3$ and, if n is even, there is a system in $\{6n+2; 3n+1\}$ which does not contain a copy of $\mathfrak{I}_3$. The following examples are designed to illustrate this point (cf. also Remark 6).

<u>Example 6.14.</u> Let (S, T) be the STS$(6n+1)$ constructed in the proof of Theorem 3.4. Then T contains

$$C = \begin{cases} (x+n, \; x+2n, \; 6n) \\ (x+3n, \; x+4n, \; 6n) : x = 0, 1, \cdots, n-1 \\ (x+5n, \; x, \; 6n) \end{cases} .$$

Let

$$J_1^* = \begin{cases} (x, x, x) \\ (x+n, x+n, x+n) & : x = 0, 1, \cdots, n-1 , \\ (x+4n, x+4n, x+4n) \end{cases}$$

$$J_2^* = \begin{cases} (x+5n, x+5n, x) \\ (x+2n, x+2n, x+n) & : x = 0, 1, \cdots, n-1 . \\ (x+3n, x+3n, x+4n) \end{cases}$$

Put $W^* = (T-C) \cup J_1^* \cup J_2^*$. Then $W^* \in \{6n; 3n\}$ and, by construction of (S, T), it is readily seen that W^* contains at most $\left[\frac{n}{2}\right]$ mutually disjoint copies of $\mathfrak{J}_3$.

<u>Example 6.15.</u> Let (S, T) be the STS$(6n+3)$ constructed in the proof of Theorem 3.2. Then T contains

$$T^{(0)} = \begin{cases} (0, 2n+1, 4n+2) , \\ (0, 2x, x+2n+1) & : x = 1, 2, \cdots, 2n , \\ (0, x+4n+2, 6n+3-x) & : x = 1, 2, \cdots, n ; \end{cases}$$

where $2x$ is reduced modulo $2n+1$.

Let

$$J_1^* = \begin{cases} (4n+2, 4n+2, 4n+2), \\ (2x, 2x, 2x) & : x = 1, 2, \cdots, 2n , \\ (x+4n+2, x+4n+2, x+4n+2) & : x = 1, 2, \cdots, n ; \end{cases}$$

$$J_2^* = \begin{cases} (2n+1, 2n+1, 4n+2), \\ (x+2n+1, x+2n+1, 2x) & : x = 1, 2, \cdots, 2n , \\ (6n+3-x, 6n+3-x, x+4n+2) & : x = 1, 2, \cdots, n ; \end{cases}$$

where $2x$ is reduced modulo $2n+1$.

Put $W^* = (T-T^{(0)}) \cup J_1^* \cup J_2^*$. Then $W^* \in \{6n+2; 3n+1\}$, and it is easily seen from the construction of (S, T) that W^* contains at most $\left[\frac{n+1}{2}\right]$ mutually disjoint copies of $\mathfrak{T}_3$.

Example 6.16. Let $n \equiv 1$ or $3 \pmod 6$. Let (S, T) be any STS(n), where $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.
Let $S^* = \{\beta_1, \beta_2, \cdots, \beta_n\}$ such that $S \cap S^* = \phi$.
Let $W_1 = \{(\beta_i, \beta_j, \alpha_k) \mid 1 \le i, j, k \le n;\ i \ne j;\ i+j \equiv 2k \pmod n\}$,

$W_2 = \{(\beta_i, \beta_i, \beta_i), (\alpha_i, \alpha_i, \beta_i) \mid i = 1, 2, \cdots, n\}$.

Put $W^* = T \cup W_1 \cup W_2$. A direct verification shows $W^* \in \{2n; n\}$, and it is clear that W^* does not contain a copy of $\mathfrak{T}_3$.

Remark 18. With regards to Remark 17 and Example 6.16, it should be pointed out that, for $n \equiv 0$ or $4 \pmod 6$, any system $W^* \in \{2n; n\}$ must contain a copy of $\mathfrak{T}_3$. For if $W^* \in \{2n; n\}$ is based on $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_n\}$ so that W^* contains the collection $\{(\alpha_i, \alpha_i, \alpha_i), (\beta_i, \beta_i, \alpha_i) \mid i = 1, 2, \cdots, n\}$, then a simple numerical argument shows that there can be at most $\frac{n(n-2)}{2}$ triples of the type $(\alpha_i, \alpha_j, \beta_k)$, where i, j, k are pairwise distinct. Consequently, there are at least $\frac{n(n-1)}{2} - \frac{n(n-2)}{2} = \frac{n}{2}$ pairs (α_i, α_j), with $i \ne j$, not contained in triples of the type $(\alpha_i, \alpha_j, \beta_k)$, where i, j, k are pairwise distinct, and W^* contains at least $\left\lceil \frac{n}{6} \right\rceil$ copies of $\mathfrak{T}_3$, where $\lceil x \rceil$ denotes the smallest integer which is not less than x.

V. Non-isomorphic consistent systems in $\{6n+4; a\}$, $n \ge 1$.

In Theorem 3.20 we established the existence of a consistent system $W \in \{6n+4; 3n+1\}$ which does not contain a copy of $\mathcal{K}_4$ or $\mathcal{K}_4^*$.

The illustrative examples given in Section 4 of Chapter 2 and the results of Chapter 5, summarized in Theorem 5.20, guarantee the existence of a consistent system $W^* \in \{6n+4;\ 3n+1\}$ which contains a copy of $\mathcal{K}_4$. A simple application of Lemma 2.1 shows that W^* cannot contain a copy of $\mathcal{K}_4^*$. However, the existence of W^* implies the existence of a consistent system $W^{**} \in \{6n+4;\ 3n+1\}$ which contains a copy of $\mathcal{K}_4^*$. It is immediately clear that the three systems W, W^*, W^{**} are non-isomorphic, and so we can state the following theorem:

Theorem 6.17. For every integer $n \geq 1$, there exist at least two non-isomorphic consistent systems in $\{6n+4;\ 3n+1\}$.

Corollary 6.18. For every integer $n \geq 1$, there exist at least two non-isomorphic consistent systems in $\{6n+4;\ a\}$, where $1 \leq a \leq 3n+1$ and $a \equiv 1 \pmod 3$.

Proof. The proof is an immediate consequence of Theorem 6.17 and Lemma 6.1.

VI. Non-isomorphic systems in $\{6n+5;\ a\}$, $n \geq 1$.

Theorem 6.19. For every integer $n \geq 1$, there exist at least two non-isomorphic systems in $\{6n+5;\ a\}$, where $1 \leq a \leq 6n+1$ and $a \equiv 1 \pmod 3$.

Proof. We consider two cases.

Case 1. $n \geq 1$, $a = 1$.

Let $W = \{(x, y, z) \mid 1 \leq x, y, z \leq 6n+5;\ x+y+z \equiv 0 \pmod{6n+5}\}$. By Theorem 3.1, $W \in \{6n+5;\ 1\}$. We claim W does not contain a copy of $\mathcal{J}_0$. If W contains a collection $T = \{(i, i, j), (j, j, k), (k, k, i)\}$, then we must have $2i+j \equiv 2j+k \equiv 2k+i \equiv 0 \pmod{6n+5}$, and it is readily verified that $i = j = k = 6n+5$ is our only solution. So our claim is justified. On the other hand, according to Theorem 3.25 there exists a consistent system in $\{6n+5;\ 6n+1\}$ for every positive integer n. This implies the existence of a system $W^* \in \{6n+5;\ 1\}$ which contains $2n$ mutually disjoint copies of $\mathfrak{T}_0$. Consequently, W and W^* are two non-isomorphic systems in $\{6n+5;\ 1\}$.

<u>Case 2.</u> $n \geq 1$, $4 \leq a \leq 6n+1$, $a \equiv 1 (\mathrm{mod}\ 3)$.

By Theorem 3.20, there exists a system $W \in \{6n+5;\ 6n+1\}$ which contains at least $2n-1$ mutually disjoint copies of $\mathfrak{T}_3$ and no copy of $\mathfrak{U}_5$. In the proof of Theorem 3.25 we considered three cases in constructing a consistent system in $\{6n+5;\ 6n+1\}$. For the cases $n \equiv 0 (\mathrm{mod}\ 3)$ and $n \equiv 2 (\mathrm{mod}\ 3)$, we constructed a consistent system $W^* \in \{6n+5;\ 6n+1\}$ containing a copy of $\mathfrak{U}_5$. For the case $n \equiv 1 (\mathrm{mod}\ 3)$, we constructed a consistent system $W^* \in \{6n+5;\ 6n+1\}$ such that W^* contains $2n-2$ mutually disjoint copies of $\mathfrak{T}_3$ which do not intersect a subsystem $Q^* \in \{11;\ 7\}$. Observing there is a system in $\{11;\ 7\}$ which contains a copy of $\mathfrak{U}_5$ and a copy of $\mathfrak{T}_3$, we have essentially shown, for every integer $n \geq 1$, there exists a system $W^* \in \{6n+5;\ 6n+1\}$ which contains a copy of $\mathfrak{U}_5$ and at least $2n-1$ mutually disjoint copies of $\mathfrak{T}_3$. Obviously, the systems W and W^* are non-isomorphic and by applying Lemma 6.1 we complete the proof of the theorem.

Gathering together the results of this section we have proved the main theorem:

<u>Theorem 6.20.</u> For every given integer $n \geq 10$ and every admissible value

of a, there exist at least two non-isomorphic systems in $\{n; a\}$.

5. An application of the singular direct product.

For large values of n it is expected that the number of non-isomorphic systems in $\{n; a\}$ will be very large indeed. C.C. Lindner [12] has shown that the singular direct product defined in Section 3 of Chapter 5 can be used to construct large numbers of non-isomorphic totally symmetric quasigroups, if the single operator $\otimes$ is replaced by a set of operators $\otimes_i$, $i = 1, 2, \ldots, r = (v^2 - v)/6$. As a special case of [12, Theorem 3], we can state

Theorem 6.21. Let (Q, o) be any totally symmetric quasigroup containing the subquasigroup (P, o). Then there are at least $((q-p)!\,(q-p-1)!\cdots 2!\,1!)^{(v^2-v)/6}/(v(q-p)+p)!$ non-isomorphic totally symmetric quasigroups of order $v(q-p)+p$, where $|Q| = q$, $|P| = p$, and $v \equiv 1$ or $3 \pmod 6$.

Using Lindner's argument in the proof of Theorem 6.21, we obtain the following combinatorial analogue:

Theorem 6.22. Suppose there is a system $V \in \{v; v\}$. Suppose there is a system $Q \in \{q; r\}$ containing a subsystem $P \in \{p; t\}$. Then there are at least $((q-p)!\,(q-p-1)!\cdots 2!\,1!)^{(v^2-v)/6}/(v(q-p)+p)!$ non-isomorphic systems in $\{v(q-p)+p\,;\,(r-t)v+t\}$.

The following example may be compared with that given in [12] for quasigroups satisfying $x(yx) = y$. The examples given in Section 3 of this chapter permit us to make the appropriate choices of V, Q and P in Theorem 6.22.

Example 6.23. (i) Take $V \in \{7; 7\}$, $Q \in \{10; 4\}$, and $P \in \{3; 3\}$. Then there are at least 10^9 non-isomorphic systems in $\{52; 10\}$.

(ii) Take $V \in \{7; 7\}$, $Q \in \{11; 7\}$, and $P \in \{3; 3\}$. Then there are at least 10^{29} non-isomorphic systems in $\{59; 31\}$.

(iii) Take $V \in \{7; 7\}$, $Q \in \{10; 4\}$, and $P \in \{1; 1\}$. Then there are at least 10^{59} non-isomorphic systems in $\{64; 22\}$.

6. Concluding remarks and problems.

In conclusion the author would like to make some remarks and mention a few problems.

In this thesis we have shown that, for every positive integer $n \neq 7$ and every admissible value of a, there exists a consistent system in $\{n; a\}$. In addition, examples of inconsistent systems in $\{n; a\}$ have been constructed for most values of n and a. Outstanding cases of interest to the author still to be considered are:

(i) $n \equiv 1$ or $3 \pmod 6$, $n \geq 15$; $a = n$.

(ii) $n \equiv 5 \pmod 6$, $n \geq 17$; $a = n-4$.

In this connection we may ask:

(a) For what values of n does there exist an inconsistent system $W \in \{n; a\}$ in cases (i) and (ii)?

(b) Is the system $W^* \in \{6n+5; 6n+1\}$ constructed in the proof of Theorem 3.18 consistent for every integer $n \geq 1$?

(c) Are the conditions on $W \in \{n; a\}$ in Theorem 4.27 sufficient for W to be embedded in a consistent system $W^* \in \{2n+7; a+n+7\}$ for every odd integer $n \geq 7$?

The answer to both (b) and (c) appears to be yes, but very little can be said about (a) .

Let n be an odd integer and suppose there exists a system $W \in \{n; a\}$ based on the set $S = \{1, 2, \cdots, n\}$, where $0 \le a < n$. Further suppose that $J = \{1, 2, \cdots, r\}$ is the set of non-idempotents of W . Then $|J| = r = n - a$. By Lemma 2.1, W necessarily contains a collection $J^* = \{(i, i, \alpha_i) \mid i = 1, 2, \cdots, r\}$, where $\alpha_1, \alpha_2, \cdots, \alpha_r$ are a permutation of $1, 2, \cdots, r$, with $\alpha_i \neq i$. Now corresponding to W , we let G be the graph where $V(G) = J$ and $E(G) = \{[i, \alpha_i] \mid i = 1, 2, \cdots, r ; (i, i, \alpha_i) \in J^*\}$. It follows that G consists of disjoint cycles of length k , $3 \le k \le n - a$. If G consists of a single cycle of length $n - a$, we shall say $W \in \{n; a\}$ is unicyclic. According to the results contained in this thesis, for every odd $n \neq 7$ and every admissible value of a , $0 \le a < n$, there exists a system $W \in \{n; a\}$ whose corresponding graph G consists of:

(i) $\frac{n-a}{3}$ disjoint cycles of length 3 , if $n \equiv 1$ or $3 \pmod 6$;

(ii) $\frac{n-a-4}{3}$ disjoint cycles of length 3 and one cycle of length 4, if $n \equiv 5 \pmod 6$.

For $n \ge a+6$ in (i) or $n \ge a+7$ in (ii) , it should be noted that such a system $W \in \{n; a\}$ is never unicyclic. Clearly, any system in $\{n; n-3\}$ or $\{n; n-4\}$ must be unicyclic for odd $n \ge 3$. For special values of n and a , the author has been able to construct unicyclic systems in $\{n; a\}$ (see for example Theorem 3.13, Theorem 4.20 combined with Remark 9, and the examples given in Section 3 of this chapter). However, the problem of constructing a unicyclic system in $\{n; a\}$ for every odd n and every admissible a , $0 \le a < n$, has not been solved. We may ask: For what values of n and a does there exist a unicyclic system $W \in \{n; a\}$?

Two Steiner triple systems (S, T_1) and (S, T_2) are said to be disjoint if $|T_1 \cap T_2| = 0$, and are said to be almost disjoint if $|T_1 \cap T_2| = 1$ (see [15]) . The existence of a pair of disjoint STS(n) has been shown for every $n \ge 7$, and the existence of a pair of almost disjoint STS(n) has been shown for every $n \ge 3$, where it is understood that $n \equiv 1$ or $3 (\mathrm{mod}\ 6)$.

Concerning extended triple systems, we give the following similar definitions: Two extended triple systems (S, W_1) and (S, W_2) in $\{n; a\}$ are said to be disjoint if $W_1 \cap W_2 = \phi$, and are said to be almost disjoint if $W_1 \cap W_2 = \{(x, x, x)\}$ for some $x \in S$. Note that if (S, W_1) and (S, W_2) are disjoint, then the latin squares corresponding to their associated quasigroups do not agree in any cell, while if (S, W_1) and (S, W_2) are almost disjoint, then the latin squares corresponding to their associated quasigroups agree in exactly one cell. We ask the following: For what values of n and a can a pair of disjoint (almost disjoint) systems (S, W_1), $(S, W_2) \in \{n; a\}$ be constructed? The author has obtained partial solutions to this problem, but several cases remain to be investigated.

For $n = 3, 7, 9$ there is one STS(n) , apart from isomorphism. There are essentially two non-isomorphic STS(13) and 80 non-isomorphic STS(15) . It is also known (see for example [14]) that the number of non-isomorphic STS(n) goes to infinity with n . In most cases, we have shown how extended triple systems can be derived from Steiner triple systems. It is also worth mentioning that the embedding theorems of Chapter 4 can be used to obtain information regarding the number of non-isomorphic systems in $\{n; a\}$, at least for some values of n and a (cf. [14]) . Concerning the results of this final chapter, it seems reasonable to expect that the number of non-isomorphic systems in $\{n; a\}$ will go to infinity with n , for every admissible value of a .

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