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HIGH ENERGY DESCRIPTION OF FORWARD AND BACKWARD SCATTERING

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ABSTRACT

Analytic expressions are obtained for two-body forward and backward scattering amplitudes by three different methods. The formalism is extended to scattering from a bound state assuming the particles to be distinguishable. We give criteria for splitting the scattering amplitudes into forward and backward parts. The two-body impact parameter amplitudes and scattering amplitudes are calculated at intermediate energies by a phase-shift analysis and the results extended to higher energies. We conclude that the backward amplitude for proton-deuteron scattering is somewhat larger than predicted by other authors. The applications of the theory are limited by our present lack of detailed information of the high energy scattering amplitudes.

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CHAPTER I

INTRODUCTION

In this work we consider the description of high energy scattering of particles. We are interested in high energy scattering because at these energies the wave lengths of incident particles are very small and the particles therefore act as sensitive probes of the structure of the target particles and of the mechanism of short-range strong interactions. However, by high energy we mean that the potential is still a valid representation of the interaction between two systems. The validity of Schrödinger equation is assumed at such high energies.

At low energies experimental observations are reasonably well understood in terms of the partial wave expansion of the scattering amplitude. At high energies however this description is not very useful since, as the energy increases, large numbers of partial waves must be included, making a reliable analysis difficult. Also, large-angle cross-sections are very small compared with those in the forward direction, which implies that strong cancellation between various partial waves occurs, and hence the partial wave analysis becomes very sensitive to approximations.

In recent years two descriptions of high energy scattering have evolved. The first is a high energy small angle approximation of the scattering amplitude called the eikonal approximation. It was first proposed by Molière[1)] and subsequently developed by many authors[2,3)]. It describes a high energy particle passing through a scattering region in a semi-classical sense at an impact parameter b on a straight line trajectory (see Fig. 1). The wave-function of the particle undergoes a phase-change which can be calculated from the complex potential, and from this phase-change the scattering amplitude can be obtained. The second high energy description is due to Blankenbecler and Goldberger[4)] and contains only a high energy approximation and has no angular restriction. However, to date, most attention has been focussed on scattering in the forward direction. The angular distributions at high energies are characterised by a large forward peak and in some instances also by a prominent backward peak. The multiple scattering theory of Glauber, which is developed in the eikonal approximation, is very successful in explaining the observed data in the forward direction. The backward peak has not, however, been explained as successfully,[5,6,7] yet backward scattering is important since the Coulomb interaction is negligible and the inherent large momentum transfer means that backward scattering contains far more short range information about the target than does forward scattering at the same energy. The purpose of this work is, therefore, to formulate a Glauber-type eikonal model for backward as well as forward scattering from a composite particle. It is a relatively simple task to obtain forward and backward scattering amplitudes for a two-body system since the criteria for splitting the scattering amplitude are well defined. The problem of splitting the scattering amplitude in the three-body system is not as simple[8], and the effect is further compounded in multi-bodied systems. The problem of splitting is connected with the problem of definite signature. Signatures for scattering of a two-body system are well defined, but we obtain mixed signatures for scattering in a general three-body system. We will therefore restrict our discussion of the three-body system to the problem of scattering from a bound state. This allows us to split the scattering amplitude in a physically reasonable way, and to construct three-body scattering amplitudes from two-body amplitudes. We choose to apply our results specifically to proton-deuteron scattering since this has aroused much interest in recent years. Previous analyses of backward scattering[5,6,7] indicate that a simple nucleon exchange diagram gives an insufficient differential cross-section in the backward direction in the Born approximation. Several calculations[7] have been performed using esoteric components of the wave function of the deuteron which supposedly help to boost backward scattering, at least qualitatively. However, to our knowledge no unitarized calculations for backward scattering have been performed at high energies.

* * * * *

Chapter II is devoted to two-body scattering. In Section II.1 we present the basic formalism used in the derivation of the scattering amplitudes, and in Section II.2 we present criteria for splitting the scattering amplitude into its forward and backward parts. In Section II.3 we derive general off-energy-shell forward and backward scattering amplitudes corresponding to those obtained in the forward direction by Glauber and by Blankenbecler and Goldberger.

In Section II.4 we give an alternative method of deriving these scattering amplitudes, and discuss why we may not extend the method directly to three-body scattering. In Section II.5 we derive the scattering amplitudes using an operator method which we may extend to the three-body system. Finally in Section II.6 we deduce the form of the unitarity relations which these amplitudes satisfy in the high-energy limit.

In Chapter III we consider scattering from a bound state. We derive the forward and backward amplitudes and give criteria for splitting the amplitudes

in Section III.1. We then show how we may approximate the amplitudes obtained in order to make their application to physical problems easier in Section III.2.

Chapter IV is devoted to applying the two-body scattering analysis to proton-proton and proton-neutron scattering and using these two-body amplitudes to construct proton-deuteron differential cross-sections. We have to make some assumptions about the ratio of real to imaginary parts of the two-body scattering amplitudes, as these are not completely determined by the differential cross-sections. We calculate the scattering amplitudes and profile functions at intermediate energies using a partial wave analysis, and use the conclusions we draw from these results as a basis for our assumptions at high energy. We then evaluate the proton-deuteron forward and backward differential cross-sections.

We end this work with a discussion of the results.

CHAPTER II

TWO-BODY SCATTERING

II.1 Basic Approach

Our approach is based on the fact that the on-shell scattering amplitude for elastic scattering of two spinless particles may be written in terms of a Fourier-Bessel transform[4,9]

$$t(\underline{k}_f, \underline{k}_i) = i4\pi k \int_0^\infty b\, db\, J_0(b\sqrt{-t})\,\Gamma(s;b)$$

where k ($=\sqrt{s}$) and θ are momentum and scattering angle in the centre of mass system and we have assumed $m=\hbar=1$. The variable t is the negative of the momentum transfer and is given by

$$t = -4k^2 \sin^2 \frac{\theta}{2}$$

and J_0 is the zeroth order Bessel function. The parameter b is identified with the classical impact parameter in the forward direction, but is otherwise quite general. This equation is exact and valid for all physical values of energy and angle. Also, since the invariant momentum transfer occurs naturally with the impact parameter as a variable in the argument of the Bessel function, this equation may be readily extended to include relativistic effects. The impact parameter amplitude, or profile function $\Gamma(s,b)$, is related to the partial wave amplitude by

$$\Gamma(s,b) = 2\sum_{l=0}^{\infty}(2l+1)\, f_l(s)\, \frac{J_{2l+1}(kb)}{b}$$

where

$$f_l(s) = e^{i\delta_l(s)} - 1$$

This suggests that the impact parameter amplitude may be represented by the form

$$\Gamma(s,b) = e^{i\chi(s,b)} - 1$$

since using the relation

$$2\sum_{l=0}^{\infty}(2l+1)\,\frac{J_{2l+1}(\beta)}{\beta} = 1$$

one obtains

$$e^{i\chi(s,b)} = 2\sum_{l=0}^{\infty}(2l+1)\, e^{2i\delta_l(s)}\, \frac{J_{2l+1}(kb)}{b}$$

In the eikonal approximation, the phase function $\chi(s,b)$ is given by the Glauber expression

$$\chi(s,b) = -\frac{1}{v}\int_{-\infty}^{\infty} dz\, V(\underline{b} + \underline{n}\, z)$$

where v is the velocity of the projectile in the direction $\underline{n} = \underline{k}/k$ and V is the interaction potential.

This suggests that one can analagously write down the Glauber-type representation for the forward and backward directions in the presence of direct and space exchange potentials. Noble[10] was therefore able to derive exactly solvable coupled integral equations for the forward and backward scattering amplitudes in the eikonal approximation. However his analysis is far from complete. The amplitudes derived are only for on-energy-shell scattering and no clear-cut criteria for splitting the amplitude are given. There is no interpretation of the direction of linearization of the Green's function (see Appendix B), and he does not deal with the difficulties involved in extending his method to n-body systems. We shall deal systematically with these problems.

II.2 Splitting the Scattering Amplitude

Our starting point is the Lippmann-Schwinger[11)] equation. This is an inhomogenous integral equation describing scattering, which may be derived from the time-independent Schroedinger equation. It is satisfied by the stationary scattering matrix t and may be written in momentum space as

$$t(\underline{k}_f,\underline{k}_i;k^2) = V(\underline{k}_f,\underline{k}_i) + \frac{1}{(2\pi)^3}\int d^3k' \frac{V(\underline{k}_f,\underline{k}')\,t(\underline{k}',\underline{k}_i;k^2)}{\frac{1}{2}(k^2-k'^2+i\epsilon)} \qquad 2.1$$

where we have assumed $m = \hbar = 1$. Here, $\underline{k}_f$ and $\underline{k}_i$ are final and initial momenta in the centre of mass system. If the scattering is on the energy shell $|\underline{k}_f| = |\underline{k}_i| = k$, but otherwise these need not be equal. The particle interaction potential is represented by V. Normally, the potential is assumed to be a local, spherically symmetric, direct potential. It may however also contain a term which exchanges the coordinates of the particles (see Fig. 2). If we consider this exchange potential also to be local and spherically symmetric, we may express the potential operator as

$$V = V^d + V^{ex}P^r$$

where P^r is the space exchange operator. In momentum space this becomes

$$V(\underline{k}_f, \underline{k}_i) = V^d(|\underline{k}_f - \underline{k}_i|) + V^{ex}(|\underline{k}_f + \underline{k}_i|) \qquad 2.2$$

where the superscripts denote direct and exchange potentials respectively. If we now substitute this potential into the Lippmann-Schwinger equation (eq.2.1) the scattering amplitude can be split into two parts. This becomes clearer if we iterate the equation and write down the first few terms:

$$\begin{aligned} t(\underline{k}_f, \underline{k}_i, k^2) &= V^d(|\underline{k}_f - \underline{k}_i|) + V^{ex}(|\underline{k}_f + \underline{k}_i|) + \\ &+ \frac{1}{(2\pi)^3}\int d^3k' \left[\tfrac{1}{2}(k^2 - k'^2 + i\varepsilon)\right]^{-1} \times \\ &\times \Big\{ V^d(|\underline{k}_f - \underline{k}'|)\, V^d(|\underline{k}' - \underline{k}_i|) + V^{ex}(|\underline{k}_f + \underline{k}'|)\, V^{ex}(|\underline{k}' + \underline{k}_i|) \qquad 2.3 \\ &+ V^d(|\underline{k}_f - \underline{k}'|)\, V^{ex}(|\underline{k}' + \underline{k}_i|) + V^{ex}(|\underline{k}_f + \underline{k}'|)\, V^d(|\underline{k}' - \underline{k}_i|) \Big\} + \\ &+ \cdots \end{aligned}$$

Let us consider these terms in some detail. Two of the second order terms are

$$\frac{1}{(2\pi)^3}\int d^3k' \, V^d(|\underline{k}_f - \underline{k}'|)\, V^d(|\underline{k}' - \underline{k}_i|) \left[\tfrac{1}{2}(k^2 - k'^2 + i\varepsilon)\right]^{-1} \qquad 2.4a$$

and

$$\frac{1}{(2\pi)^3}\int d^3k' \, V^{ex}(|\underline{k}_f + \underline{k}'|) V^{ex}(|\underline{k}' + \underline{k}_i|)\left[\tfrac{1}{2}(k^2 - k'^2 + i\varepsilon)\right]^{-1} \qquad 2.4b$$

Since the integration is performed over all space, we may change the sign of the variable without altering the value of the integral. Hence, if we change the integration variable in the second of these terms we obtain

$$\frac{1}{(2\pi)^3}\int d^3k' \, V^{ex}(|\underline{k}_f - \underline{k}'|) V^{ex}(|\underline{k}' - \underline{k}_i|)\left[\tfrac{1}{2}(k^2 - k'^2 + i\varepsilon)\right]^{-1} \qquad 2.4c$$

which has the same form as expression 2.4a.

The remaining second order terms are

$$\frac{1}{(2\pi)^3}\int d^3k' \, V^{d}(|\underline{k}_f - \underline{k}'|) V^{ex}(|\underline{k}' + \underline{k}_i|)\left[\tfrac{1}{2}(k^2 - k'^2 + i\varepsilon)\right]^{-1} \qquad 2.5a$$

and

$$\frac{1}{(2\pi)^3}\int d^3k' \, V^{ex}(|\underline{k}_f + \underline{k}'|) V^{d}(|\underline{k}' - \underline{k}_i|)\left[\tfrac{1}{2}(k^2 - k'^2 + i\varepsilon)\right]^{-1} \qquad 2.5b$$

If we now change both the sign of the integration variable and the sign of $\underline{k}_f$ in these expressions we obtain

$$\frac{1}{(2\pi)^3}\int d^3k' \, V^{ex}(|\underline{k}_f \pm \underline{k}'|) V^d(|\underline{k}' \pm \underline{k}_i|) [\tfrac{1}{2}(k^2 - k'^2 + i\varepsilon)]^{-1} \qquad 2.5c$$

which has the same form as expressions 2.4a and 2.4b. This similarity in terms may be observed to all orders in the potentials. We may use the fact that we have to change the sign of $\underline{k}_f$ in some of the terms to obtain similarity to split the series. It splits into two subseries, one containing an even number of factors V^{ex} and one containing an odd number of such factors. Consequently, following Blankenbecler et al[12)] we may split the amplitude into even and odd parts

$$t(\underline{k}_f, \underline{k}_i) = t_e(\underline{k}_f, \underline{k}_i) + t_o(\underline{k}_f, \underline{k}_i) \qquad 2.6$$

where the subscripts denote even and odd amplitudes. These amplitudes then satisfy the equations

$$t_e(\underline{k}_f, \underline{k}_i) = V^d(\underline{k}_f, \underline{k}_i) + \frac{1}{(2\pi)^3}\int d^3k' [\tfrac{1}{2}(k^2 - k'^2 + i\varepsilon)]^{-1} \times$$

$$\times \left\{ V^d(\underline{k}_f, \underline{k}') t_e(\underline{k}', \underline{k}_i) + V^{ex}(\underline{k}_f, \underline{k}') t_o(\underline{k}', \underline{k}_i) \right\} \qquad 2.7a.$$

and

$$t_o(\underline{k}_f,\underline{k}_i) = V^{ex}(\underline{k}_f,\underline{k}_i) + \frac{1}{(2\pi)^3}\int d^3k' \left[\tfrac{1}{2}(k^2 - k'^2 + i\varepsilon)\right]^{-1} \times \tag{2.7b}$$

$$\times \left\{ V^d(\underline{k}_f,\underline{k}')\, t_o(\underline{k}',\underline{k}_i) + V^{ex}(\underline{k}_f,\underline{k}')\, t_e(\underline{k}',\underline{k}_i) \right\}$$

We may then represent ~~the~~ even and odd amplitudes by Fredholm series[13] which are uniformly convergent in the upper half k-plane. The splitting of the scattering amplitude in this way is closely connected with the idea of definite signatured amplitudes which are required for an analytic continuation into the complex angular momentum plane. Thus, since we must make the angular momentum continuous to obtain the impact parameter amplitude, it is natural to divide the equation in this way.

II.3 Derivation of Two-Body Scattering Amplitudes

We now develop integral equations for the scattering amplitudes with a fixed impact parameter in terms of the matrix elements of the potentials. We are essentially performing a Fourier-Bessel transform of eq. 2.7. We prefer to work with these transforms as the equations are automatically exact at all physical values of energy and angle as was discussed in Section II.1. To this end we introduce the Fourier-Bessel transforms

$$t_e(\mathbf{k}_f,\mathbf{k}_i) = -4\pi\int_0^\infty b\,db\,J_0(|\mathbf{k}_f-\mathbf{k}_i|_\perp b)\,H_e(\mathbf{k}_f,b,\mathbf{k}_i) \qquad 2.8a$$

$$t_o(\mathbf{k}_f,\mathbf{k}_i) = -4\pi\int_0^\infty b\,db\,J_0(|\mathbf{k}_f+\mathbf{k}_i|_\perp b)\,H_o(\mathbf{k}_f,b,\mathbf{k}_i) \qquad 2.8b$$

$$V^d(\mathbf{k}_f,\mathbf{k}_i) = -4\pi\int_0^\infty b\,db\,J_0(|\mathbf{k}_f-\mathbf{k}_i|_\perp b)\,B_e(\mathbf{k}_f,b,\mathbf{k}_i) \qquad 2.8c$$

$$V^{ex}(\mathbf{k}_f,\mathbf{k}_i) = -4\pi\int_0^\infty b\,db\,J_0(|\mathbf{k}_f+\mathbf{k}_i|_\perp b)\,B_o(\mathbf{k}_f,b,\mathbf{k}_i) \qquad 2.8d$$

where J_0 is the zeroth order Bessel function. The choice of arguments of the Bessel functions is guided by our previous discussion. Then by substituting eq. 2.8 into eq. 2.7 we find

$$\int_0^\infty b\,db\,J_0(|\underline{k}_f-\underline{k}_i|_\perp b)\{H_e(\underline{k}_f,b,\underline{k}_i)-B_e(\underline{k}_f,b,\underline{k}_i)\}=$$

$$=\frac{1}{2\pi^2}\int_0^\infty k'^2dk'[k'^2-k_i^2-i\varepsilon]^{-1}\int_0^\infty b'db'\int_0^\infty b''db''\times \tag{2.9a}$$

$$\times\{B_e(\underline{k}_f,b',\underline{k}')H_e(\underline{k}',b'',\underline{k}_i)\int d\Omega_{k'}J_0(|\underline{k}_f-\underline{k}'|_\perp b')J_0(|\underline{k}'-\underline{k}_i|_\perp b'')+$$

$$+B_o(\underline{k}_f,b',\underline{k}')H_o(\underline{k}',b'',\underline{k}_i)\int d\Omega_{k'}J_0(|\underline{k}_f+\underline{k}'|_\perp b')J_0(|\underline{k}'+\underline{k}_i|_\perp b'')\}$$

and

$$\int_0^\infty b\,db\,J_0(|\underline{k}_f+\underline{k}_i|_\perp b)\{H_o(\underline{k}_f,b,\underline{k}_i)-B_o(\underline{k}_f,b,\underline{k}_i)\}=$$

$$=\frac{1}{2\pi^2}\int_0^\infty k'^2dk'[k'^2-k_i^2-i\varepsilon]^{-1}\int_0^\infty b'db'\int_0^\infty b''db''\times \tag{2.9b}$$

$$\times\{B_e(\underline{k}_f,b',\underline{k}')H_o(\underline{k}',b'',\underline{k}_i)\int d\Omega_{k'}J_0(|\underline{k}_f-\underline{k}'|_\perp b')J_0(|\underline{k}'+\underline{k}_i|_\perp b'')+$$

$$+B_o(\underline{k}_f,b',\underline{k}')H_e(\underline{k}',b'',\underline{k}_i)\int d\Omega_{k'}J_0(|\underline{k}_f+\underline{k}'|_\perp b')J_0(|\underline{k}'-\underline{k}_i|_\perp b'')\}$$

If we now define our coordinate system by

$$\hat{k}_i = (0, 0, 1)$$

$$\hat{k}_f = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\hat{k}' = (\sin\beta\cos\alpha,\ \sin\beta\sin\alpha,\ \cos\beta)$$

we find

$$|\underline{k}_f - \underline{k}'|_\perp = 2k_f \sin\frac{\Theta}{2}$$

$$|\underline{k}_f + \underline{k}'|_\perp = 2k_f \cos\frac{\Theta}{2}$$

$$|\underline{k}' - \underline{k}_i|_\perp = 2k' \sin\beta/2$$

$$|\underline{k}' + \underline{k}_i|_\perp = 2k' \cos\beta/2$$

where

$$\cos\Theta = \hat{k}'\cdot\hat{k}_f = \sin\theta\sin\beta\cos(\phi-\alpha) + \cos\theta\cos\beta$$

We now wish to perform the angular integrations of eq. 2.9. There are four integrals involved

$$\frac{1}{4\pi}\int d\Omega_{k'}\, J_0\!\left(2k_f b' \sin\frac{\Theta}{2}\right) J_0\!\left(2k' b' \sin\beta/2\right) \qquad 2.10a$$

$$\frac{1}{4\pi}\int d\Omega_{k'} J_0(2k_f b' \cos\tfrac{\Theta}{2}) J_0(2k' b'' \cos\beta/_2) \qquad 2.10b$$

$$\frac{1}{4\pi}\int d\Omega_{k'} J_0(2k_f b' \sin\tfrac{\Theta}{2}) J_0(2k' b'' \cos\beta/_2) \qquad 2.10c$$

$$\frac{1}{4\pi}\int d\Omega_{k'} J_0(2k_f b' \cos\tfrac{\Theta}{2}) J_0(2k' b'' \sin\beta/_2) \qquad 2.10d$$

but these fall into only two categories. Since

$$\cos\left(\frac{\pi-\alpha}{2}\right) = \sin\frac{\alpha}{2} \qquad \sin\left(\frac{\pi-\alpha}{2}\right) = \cos\frac{\alpha}{2}$$

eq. 2.10b may be reduced to eq. 2.10a and eq. 2.10d to eq. 2.10c. If we denote 2.10a by I and 2.10c by I' we may evaluate these as (see Appendix A)

$$I(k_f, k') = \int_0^\infty b\,db\, G_0(k_f, k'; b', b''; b) J_0(2k_f b \sin\theta/_2) \qquad 2.11a$$

$$I'(k_f, k') = \int_0^\infty b\,db\, G_0'(k_f, k'; b', b''; b) J_0(2k_f b \cos\theta/_2) \qquad 2.11b$$

where

$$G(\underline{k}_f,\underline{k}';b',b'';b)=\frac{2k_f^2}{\pi}\int_0^{\pi}d\phi\int_0^{\pi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,J_0(2k_f b\sin\theta/2)\times$$ 2.12a

$$\times\frac{J_1(2\beta(\theta,\phi))}{\beta(\theta,\phi)}$$

and

$$G'(\underline{k}_f,\underline{k}';b',b'';b)=\frac{2k_f^2}{\pi}\int_0^{\pi}d\phi\int_0^{\pi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,J_0(2k_f b\cos\theta/2)\times$$ 2.12b

$$\times\frac{J_1(2\beta'(\theta,\phi))}{\beta'(\theta,\phi)}$$

Here

$$\beta^2(\theta,\phi)=k_f^2b'^2+k'^2b''^2-2k_fk'\,b'\,b''\cos\theta/_2\cos\phi$$

and

$$\beta'^2(\theta,\phi)=k_f^2b'^2+k'^2b''^2-2k_fk'\,b'\,b''\sin\theta/_2\cos\phi$$

Thus, by substituting eq. 2.11 in eq. 2.9 we find

$$H_e(\underline{k}_f,b,\underline{k}_i)=B_e(\underline{k}_f,b,\underline{k}_i)+\frac{2}{\pi}\int k'^2dk'\left[k'^2-k_i^2-i\varepsilon\right]^{-1}\times$$
$$\times\int_0^{\infty}b'db'\int_0^{\infty}b''db''\Big\{B_e(\underline{k}_f,b',\underline{k}')H_e(\underline{k}',b'',\underline{k}_i)+$$ 2.13a
$$+B_o(\underline{k}_f,b',\underline{k}')H_o(\underline{k}',b'',\underline{k}_i)\Big\}G(\underline{k}_f,\underline{k}';b',b'';b)$$

and

$$H_o(\underline{k}_f, b, \underline{k}_i) = B_o(\underline{k}_f, b, \underline{k}_i) + \frac{2}{\pi}\int_0^\infty k'^2 dk' \left[k'^2 - k_i^2 - i\varepsilon\right]^{-1} \times$$

$$\times \int b' db' \int b'' db'' \Big\{ B_e(\underline{k}_f, b', \underline{k}') H_o(\underline{k}', b'', \underline{k}_i) + \qquad 2.13b$$

$$+ B_o(\underline{k}_f, b', \underline{k}') H_e(\underline{k}', b'', \underline{k}_i) \Big\} G'(\underline{k}_f, \underline{k}'; b', b''; b)$$

These are exact equations for the scattering amplitudes H_e and H_o. Unfortunately it does not seem possible to analytically integrate eq. 2.12 and we are forced to make a high energy approximation[15]. This simplifies eq. 2.11 as

$$I = \tfrac{1}{2}\int_{-1}^{1} d(\cos\theta)\, J_o(2k b'' \sin\beta/2)\, J_o(2k_f b' \sin\theta/2 \cos\beta/2) \times \qquad 2.14a$$

$$\times J_o(2k_f b' \sin\beta/2 \cos\theta/2) \xrightarrow[k_f \text{ or } k' \to \infty]{} J_o(2k_f b' \sin\theta/2)\, \frac{\delta(b'' - k_f/k'\, b')}{2 k_f k' b'}$$

$$I' = \tfrac{1}{2}\int_{-1}^{1} d(\cos\theta)\, J_0(2k'b''\sin\beta/_2)\, J_0(2k_f b'\cos\theta/_2\cos\beta/_2)\times$$

$$\times J_0(2k_f b'\sin\beta/_2\sin\theta/_2) \xrightarrow[k_f \text{ or } k' \to \infty]{} \qquad 2.14b$$

$$J_0(2k_f b'\sin\theta/_2)\,\frac{\delta(b'' - \frac{k_f}{k'} b')}{2k_f k' b'}$$

So in the high energy limit the propagator functions become

$$G(\underline{k}_f, \underline{k}'; b', b''; b) \approx (2k_f k' b' b)^{-1}\delta(b-b')\,\delta(b'' - \frac{k_f}{k'} b') \qquad 2.15a$$

$$G'(\underline{k}_f, \underline{k}'; b', b''; b) \approx (2k_f k' b' b)^{-1}\delta(b-b')\,\delta(b'' - \frac{k_f}{k'} b') \qquad 2.15b$$

In consequence, eq. 2.13 simplifies to

$$H_e(\underline{k}_f, b, \underline{k}_i) = B_e(\underline{k}_f, b, \underline{k}_i) + \frac{1}{\pi}\int_0^\infty dk' [k'^2 - k_i^2 - i\varepsilon]^{-1} \times \qquad 2.16a$$

$$\times \Big\{ B_e(\underline{k}_f, b, \underline{k}')\, H_e(\underline{k}', \frac{k_f}{k'} b, \underline{k}_i) + B_o(\underline{k}_f, b, \underline{k}')\, H_o(\underline{k}', \frac{k_f}{k'} b, \underline{k}_i) \Big\}$$

and
$$H_c(k_f, b, k_i) = B_c(k_f, b, k_i) + \frac{1}{\pi}\int_C^\infty dk' \left[k'^2 - k_i^2 - i\epsilon\right]^{-1}$$
$$\times \left\{ B_e(k_f, b, k') H_c(k', k_f b/k_i, k_i) + B_c(k_f, b, k') \times H_e(k', k_f b/k_i, k_i) \right\} \qquad 2\cdot 16b$$

The eq. (2.16) could be solved by an iteration. For the purpose of iteration, we re-write the eq. (2.16) in a matrix notation
$$H(k_f, b, k_i) = \beta(k_f, b, k_i) + \frac{1}{\pi}\int_C^\infty dq \left[q^2 - k_i^2 - i\epsilon\right]^{-1} B(k_f, b, q) H(q, \tfrac{k_f b}{q}, k_i) \qquad (2\cdot 17)$$
where matrices $H = \begin{pmatrix} H_e \\ H_c \end{pmatrix}$, $\beta = \begin{pmatrix} B_e \\ B_c \end{pmatrix}$ and $B = \begin{pmatrix} B_e & B_c \\ B_c & B_e \end{pmatrix}$ are defined with appropriate dependence of the variables as given in (2.17). The solution of (2.17) is given by
$$H(k_f, b, k_i) = \beta(k_f, b, k_i) + \sum_{n=1}^{\infty} H^{n+1}(k_f, b, k_i) \qquad 2\cdot 18$$

where $n+1$st term is
$$H^{n+1}(k_f, b, k_i) = \left(\frac{1}{\pi}\right)^n \int \prod_{i=1}^{n} dq_i \quad \frac{B(k_f, b, q_1) B(q_1, k_f b/q_1, q_2) \cdots B(q_{n-1}, k_f b/q_{n-1}, q_n) \beta(q_n, k_f b/q_n, k_i)}{(q_1^2 - k_i^2 - i\epsilon) \cdots\cdots (q_n^2 - k_i^2 - i\epsilon)} \qquad 2\cdot 19$$

There are $n+1$ equivalent ways of writing the expression (2.19) i.e.
$$H^{n+1}(k_f, b, k_i) = \left(\frac{1}{\pi}\right)^n \int \prod_{i=1}^{n} dq_i \left[\frac{B(k_f, b, q_1) \cdots B(q_{l-1}, k_f b/q_{l-1}, q_l)}{(q_1^2 - k_i^2 - i\epsilon) \cdots (q_{l-1}^2 - k_i^2 - i\epsilon)}\right] B(q_l, k_f b/q_l, q_{l+1})$$
$$\times \left[\frac{B(q_{l+1}, k_f b/q_{l+1}, q_{l+2}) \cdots\cdots B(q_{n-1}, k_f b/q_{n-1}, q_n)}{(q_{l+1}^2 - k_i^2 - i\epsilon) \cdots\cdots (q_{n-1}^2 - k_i^2 - i\epsilon)}\right] \frac{\beta(q_n, k_f b/q_n, k_i)}{q_n^2 - k_i^2 - i\epsilon}$$

i.e. there are ℓ terms in the first bracket and $n-\ell-1$ in the second bracket. If no approximations are made, then, of course, the value of this expression will not depend upon how this decomposition is done. (ℓ can vary from 0 to n). Defining

$$\vec{q}_i = \vec{k}_f - \vec{p}_1 - \vec{p}_2 \cdots - \vec{p}_i = \vec{k}_f - \vec{P}_i \qquad i = 1, 2, \cdots \ell$$

and
$$\vec{q}_j = \vec{k}_i + \vec{p}_j + \vec{p}_{j+1} \cdots + \vec{p}_n = \vec{k}_i + \vec{P}'_j \qquad j = \ell+1, \ell+2, \cdots n \quad ,$$

we make 'eikonal approximation' (see Appendix B) to the propagators by writing

$$q_i^2 - k_i^2 - i\epsilon = -2\vec{k}_f \vec{P}_{iz} - \Delta k^2 - i\epsilon \qquad i = 1, 2, \cdots \ell$$
$$q_j^2 - k_i^2 - i\epsilon = +2k_i P'_{j\xi} - i\epsilon \qquad j = \ell+1, \ell+2, \cdots n$$

where $\Delta k^2 = k_i^2 - k_f^2$ and $P'_{j\xi}$ is the projection of $\vec{P}'_j$ in the direction $\vec{k}_i$. Then next step is to assume that in the first ℓ rungs of the ladder representation (Note that n+1th order term can be represented by $n+1$ ladders in the Feynmann diagram) are essentially on the energy shell. Then an off-energy-shell 'kick' occurs and again for the remaining $n-\ell$ rungs, the scattering is on the energy shell; but the energy is now different from the initial energy. Thus at high energies $\{k_f, k_i \to \infty$ but $k_f \neq k_i\}$ we have $q_i^2 \approx k_f^2$ and $q_j^2 \approx k_i^2$ and $dq_i = -dp_{iz}$ and $dq_j = dp_{j\xi}$. Also for first ℓ rungs, $B(q_i, k_f b/q_i, q_{i+1}) \to B(k_f, b, k_f)$ and for the last $n-\ell$ rungs $B(q_j, k_f b/q_j, q_{j+1}) \to B(k_i, k_f b/k_i, k_i)$

We get

$$H^{n+1}(k_f, b, k_i) \approx \int_{+\infty}^{-\infty} \left(\frac{1}{\pi}\right)^n \prod_{i=1}^{\ell} \frac{dp_{iz}}{2k_f P_{iz} + \Delta k^2 + i\epsilon} \int_{+\infty}^{-\infty} \prod_{j=\ell+1}^{n} \frac{dp_{j\xi}}{-2k_i P'_{j\xi} + i\epsilon} \left[B(k_f, b, k_f)\right]^{\ell}$$
$$\times B(k_f\ b\ k_i) \left[B(k_i, k_f b/k_i, k_i)\right]^{n-\ell-1} \beta(k_i, k_f b/k_i, k_i)$$

Once we made the eikonal approximation, the value of H^{n+1} th term depends on the value of ℓ. (In the language of Feynmann diagrams, this choice corresponds to the choice of vertex at which over all momentum conservation is imposed. One can be more sophisticated by assuming not only one kick but a series of kicks with off-energy-shell occurs. But, the final result for on-energy-shell will be independent of the procedure). To overcome this, Levy and Sucher[16] replaced it by average over ℓ i.e.

$$\overline{H}^{n+1} = \frac{1}{n+1} \sum_{\ell=0}^{n} H^{n+1}$$

Since the product of B's is invariant under separate permutations of p's we can sum over all such permutations in the integrand and divide by their total no. $\ell!(n-\ell)!$. Thus we get

$$\overline{H}^{n+1} = \frac{1}{n+1} \frac{1}{(\pi)^n} \sum_{\ell=0}^{\infty} \frac{1}{(n-\ell)!(\ell)!} \sum_{\pi_i} \int_{+\infty}^{-\infty} \prod_{i=1}^{\ell} \frac{dp_{iz}}{\Delta k^2 + k_f p_{iz} + i\epsilon} \sum_{\pi_j} \int_{+\infty}^{-\infty} \prod_{j=\ell+1}^{n} \frac{dp_{jz}}{-2k_i p_{jz} + i\epsilon} [B(k_f, b, k_f)]^{\ell}$$

$$\times B(k_f, b, k_i) [B(k_i, k_f b/k_i, k_i)]^{n-\ell-1} \beta(k_i, k_f b/k_i, k_i)$$

where π_i and π_j are permutations of i's and j's . Using result proven by Levy and Sucher[16]

$$\sum_{\pi} \prod_{\xi=1}^{n} \frac{1}{A_\xi + x + i\epsilon} = -ix \int_0^{\infty} d\beta \, e^{i\beta x} \prod_{\xi=1}^{m} \frac{1 - e^{i\beta\xi}}{a_\xi}$$

And $$\int_{+\infty}^{-\infty} \frac{dp_{jz}}{2k_i p_{jz} - i\epsilon} = \frac{-i\pi}{2k_i} \quad \text{and} \quad \int_{+\infty}^{-\infty} \frac{dp_{iz} \, (1 - e^{-2i\beta k_f p_{iz}})}{-2k_f p_{iz} + i\epsilon} = \frac{i\pi\,\theta(\beta)}{2k_f}$$

where $$A_\xi = a_1 + a_2 + \cdots + a_\xi$$ we get

$$\overline{H}^{n+1}(k_f, b, k_i) = \frac{i^n}{n+1} B(k_f, b, k_i) \sum_{\ell=0}^{\infty} \frac{1}{\ell!(n-\ell)!} \left[\frac{B(k_f, b, k_f)}{2k_f} \right]^{\ell} \left[\frac{B(k_i, k_f b/k_i, k_i)}{2k_i} \right]^{n-\ell-1} \beta(k_i, k_f b/k_i, k_i)$$

Hence we get $$H(k_f, b, k_i) = \beta(k_f, b, k_i) + \sum_{n=1}^{\infty} \frac{i^n}{(n+1)!} B(k_f, b, k_i) \left[\frac{B(k_f, b, k_f)}{2k_f} + \frac{B(k_i, k_f b/k_i, k_i)}{2k_i} \right]^{r}$$

$$\times B^{-1}(k_i, k_f b/k_i, k_i) \, \beta(k_i, k_f b/k_i, k_i) \qquad (2.20)$$

The eq. (2.20), then, is most general form of off-energy-shell impact parameter amplitude. We shall, now, restrict ourselves to on-energy-shell amplitude. In that situation eq. (2.20) takes a much simpler form, i.e.

$$H(k_f, b, k_i) = \left\{ 1 + \sum_{n=1}^{\infty} \frac{i^n}{(n+1)!} \left[\frac{B(k_f, b, k_i)}{k}\right]^n \right\} \beta(k_f, b, k_i)$$
$$= k \left[e^{iB(k_f,b,k_i)/k} - 1 \right] B^{-1}(k_f, b, k_i)\, \beta(k_f, b, k_i) \qquad (2.21)$$

Using the identity

$$e^{iB/k} - 1 = \frac{i}{k}\begin{pmatrix} B_e & B_o \\ B_o & B_e \end{pmatrix} + \frac{i^2}{2!\,k^2}\begin{pmatrix} B_e & B_o \\ B_o & B_e \end{pmatrix}^2 + \cdots$$
$$= \begin{pmatrix} e^{iB_e/k}\cos(B_o/k) - 1 & i\,e^{iB_e/k}\sin(B_o/k) \\ i\,e^{iB_e/k}\sin(B_o/k) & e^{iB_e/k}\cos(B_o/k) - 1 \end{pmatrix}$$

We get

$$\begin{pmatrix} H_e(k_f, b, k_i) \\ H_o(k_f, b, k_i) \end{pmatrix} = k \begin{pmatrix} e^{iB_e/k}\cos(B_o/k) - 1 & i\,e^{iB_e/k}\sin(B_o/k) \\ i\,e^{iB_e/k}\sin(B_o/k) & e^{iB_e/k}\cos(B_o/k) - 1 \end{pmatrix} B^{-1}(k_f, b, k_i)\, \beta(k_f, b, k_i) \qquad 2.22$$

where dependence of B_e and B_o are dropped for convenience.

Defining $\chi(b) = \frac{1}{k} B_e(k_f, b, k_i)$ and $\lambda(b) = \frac{1}{k} B_o(k_f, b, k_i)$ we get substituting eq. (2.22) into eq. (2.8) the final expression

$$t_e(k_f, k_i) = 4\pi k \int_0^\infty b\,db\,J_0(|k_f - k_i|_\perp b)\left\{e^{i\chi(b)} \cos\lambda(b) - 1\right\} \qquad 2.23a$$

and

$$t_o(k_f, k_i) = -4\pi k \int_0^\infty b\,db\,J_0(|k_f + k_i|_\perp b)\, e^{i\chi(b)} \sin\lambda(b) \qquad 2.23b$$

Clearly, since $J_0(|k_f - k_i|_\perp b)$ is maximum in the forward direction and minimum in the backward direction, and $J_0(|k_f + k_i|_\perp b)$ is minimum in the forward direction and maximum in the backward direction, t_e and t_o dominate in the forward and backward directions respectively. Eq. 2.23a replaces the standard Glauber result for forward scattering. These expressions are similar to those derived by Noble[10] for two-body elastic scattering.

We now derive an alternative form of eq. 2.23 which is an extension of that obtained by Blankenbecler and Goldberger. We proceed by integrating eq. 2. 16. In the high energy limit $P\left(\frac{1}{k_f^2 - k'^2}\right)$ is negligible compared with $i\pi\delta(k'^2 - k_f^2)$ so for diagonal elements we obtain

$$H_e(\underline{k},b,\underline{k}) = B_e(\underline{k},b,\underline{k}) + \frac{i}{2k}\left\{H_e(\underline{k},b,\underline{k})B_e(\underline{k},b,\underline{k}) + H_o(\underline{k},b,\underline{k})B_o(\underline{k},b,\underline{k})\right\} \tag{2.24a}$$

and

$$H_o(\underline{k},b,\underline{k}) = B_o(\underline{k},b,\underline{k}) + \frac{i}{2k}\left\{H_e(\underline{k},b,\underline{k})B_o(\underline{k},b,\underline{k}) + H_o(\underline{k},b,\underline{k})B_e(\underline{k},b,\underline{k})\right\} \tag{2.24b}$$

These expressions give us

$$H_e = \left[B_e\left[1-\frac{i}{2k}B_e\right] + \frac{i}{2k}B_o^2\right]\left[\left[1-\frac{i}{2k}B_e\right]^2 + \frac{1}{4k^2}B_o^2\right]^{-1} \tag{2.25a}$$

and

$$H_o = \left[B_o\left[1-\frac{i}{2k}B_e\right] + \frac{i}{2k}B_oB_e\right]\left[\left[1-\frac{i}{2k}B_e\right]^2 + \frac{1}{4k^2}B_o^2\right]^{-1} \tag{2.25b}$$

Equation 2.25a reduces to the familiar form obtained by Blankenbecler and Goldberger if we neglect terms containing B_o. By substituting 2.25a and 2.25b into 2.8a and 2.8b we obtain the even and odd scattering amplitudes:

$$t_e(\underline{k}_f,\underline{k}_i) = -4\pi\int_0^\infty b\,db\,J_0(|\underline{k}_f-\underline{k}_i|_\perp b) \times \frac{\left[B_e(1-{}^{i}\!/\!{}_{2k}B_e) + {}^{i}\!/\!{}_{2k}B_o^2\right]}{\left[(1-{}^{i}\!/\!{}_{2k}B_e)^2 + {}^{1}\!/\!{}_{4k^2}B_o^2\right]} \tag{2.26a}$$

$$t_0(k_f,k_i) = -4\pi \int_0^\infty b\,db\, J_0(|k_f + k_i|_\perp b) \times$$

$$\times \frac{\left[B_0(1 - i/2k\,B_e) + i/2k\, B_0 B_e\right]}{\left[(1 - i/2k\,B_e)^2 + 1/4k^2 B_0^2\right]} \qquad 2.26b$$

Clearly, the forward and backward profile functions in eq. 2.23 are not the same as those in eq. 2.26, since the Blankenbecler-Goldberger-type amplitudes are derived in a high-energy approximation, while the Glauber-type amplitudes are derived in a small angle high-energy limit. However a simple expansion of these amplitudes in powers of 1/k reveals that to second order they are the same. It is interesting to note that although the Glauber amplitude is derived for small angles it appears to be valid at all angles in the high-energy limit. This is due to the fact that the first order term is equivalent to the Born approximation which has no such angular restriction.

II.4 Alternative Derivation 1

We now derive eq. 2.23 by a different method in which one can more readily see the physical reasons for splitting the amplitude into its forward and backward parts. This derivation is done in configuration space rather than momentum space. Therefore, we take the potential as

$$\langle \underline{r}' | V | \underline{r} \rangle = V^d(r)\,\delta(\underline{r}-\underline{r}') + V^{ex}(r)\,\delta(\underline{r}+\underline{r}') \qquad 2.27$$

and we choose to work with the integral form of Schroedinger's equation, given by

$$\psi_K(\underline{r}) = e^{i\underline{k}\cdot\underline{r}} - \frac{2\mu}{4\pi\hbar^2}\int d\underline{r}'\, \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \times \qquad 2.28$$

$$\times \left\{ V^d(r')\,\psi_K(\underline{r}') + V^{ex}(r')\,\psi_K(-\underline{r}') \right\}$$

If we now suppose that $\psi_k(r)$ can be split into forward and backward waves

$$\psi_K(\underline{r}) = e^{i\underline{k}\cdot\underline{r}}\phi_+(\underline{r}) + e^{-i\underline{k}\cdot\underline{r}}\phi_-(\underline{r}) \qquad 2.29$$

where $\phi_+(r)$ and $\phi_-(r)$ are slowly varying functions with the boundary conditions

$$\left.\begin{aligned} \phi_+(-\infty) &= 1 \\ \phi_-(\infty) &= 0 \end{aligned}\right\}$$

eq 2.28 becomes

$$e^{i\underline{k}\cdot\underline{r}}\phi_+(\underline{r}) + e^{-i\underline{k}\cdot\underline{r}}\phi_-(\underline{r}) = e^{i\underline{k}\cdot\underline{r}} -$$

$$-\frac{2\mu}{4\pi\hbar^2}\int d^3r' \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} V^d(\underline{r}')\left\{e^{i\underline{k}\cdot\underline{r}'}\phi_+(\underline{r}') + e^{-i\underline{k}\cdot\underline{r}'}\phi_-(\underline{r}')\right\} - \quad 2.30$$

$$-\frac{2\mu}{4\pi\hbar^2}\int d^3r' \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} V^{ex}(\underline{r}')\left\{e^{-i\underline{k}\cdot\underline{r}'}\phi_+(-\underline{r}') + e^{i\underline{k}\cdot\underline{r}'}\phi_-(-\underline{r}')\right\}$$

Hence, changing variables, eq. 2.30 can be split into two equations: one containing forward travelling waves, and the other containing backward travelling waves as

$$\phi_+(\underline{r}) = 1 - \frac{2\mu}{4\pi\hbar^2}\int d^3r'' \frac{e^{ikr''}}{r''} e^{-i\underline{k}\cdot\underline{r}''} \times$$

$$\times\left\{V^d(\underline{r}-\underline{r}'')\phi_+(\underline{r}-\underline{r}'') + V^{ex}(\underline{r}-\underline{r}'')\phi_-(\underline{r}''-\underline{r})\right\} \quad 2.31a$$

and

$$\phi_-(\underline{r}) = -\frac{2\mu}{4\pi\hbar^2}\int d^3r'' \frac{e^{ikr''}}{r''}\cdot e^{i\underline{k}\cdot\underline{r}''} \times$$

$$\times\left\{V^d(\underline{r}-\underline{r}'')\,\phi_-(\underline{r}-\underline{r}'') + V^{ex}(\underline{r}-\underline{r}'')\,\phi_+(\underline{r}''-\underline{r})\right\} \qquad 2.31b$$

If we now integrate these expressions by parts, we obtain

$$\phi_+(\underline{r}) = 1 + \frac{2\mu}{4\pi\hbar^2}\int r''^2 dr'' d\phi \left[\frac{e^{ikr''(1-\mu)}}{ikr''}\left\{V^d(r-r'')\,\phi_+(r-r'') + \right.\right.$$

$$\left.\left. + V^{ex}(\underline{r}-\underline{r}'')\,\phi_-(\underline{r}''-\underline{r})\right\}\right]_{\mu=-1}^{\mu=1} \qquad 2.32a$$

$$- \frac{2\mu}{4\pi\hbar^2}\int r''^2 dr'' d\phi \frac{e^{ikr''(1-\mu)}}{ikr''}\frac{\partial}{\partial\mu}\left\{V^d(\underline{r}-\underline{r}'')\,\phi_+(\underline{r}-\underline{r}'') + \right.$$

$$\left. + V^{ex}(\underline{r}-\underline{r}'')\,\phi_-(\underline{r}''-\underline{r})\right\}$$

and

$$\phi_-(\underline{r}) = -\frac{2\mu}{4\pi\hbar^2}\int r''^2 dr'' d\phi \left[\frac{e^{ikr''(1+\mu)}}{ikr''}\left\{V^{ex}(\underline{r}''-\underline{r})\,\phi_+(\underline{r}''-\underline{r}) + \right.\right.$$

$$\left.\left. + V^d(\underline{r}-\underline{r}'')\,\phi_-(\underline{r}-\underline{r}'')\right\}\right]_{\mu=-1}^{\mu=1} \qquad 2.32b$$

$$+ \frac{2\mu}{4\pi\hbar^2}\int r''^2 dr'' d\phi \frac{e^{ikr''(1+\mu)}}{ikr''}\frac{\partial}{\partial\mu}\left\{V^{ex}(\underline{r}''-\underline{r})\,\phi_+(\underline{r}''-\underline{r}) + \right.$$

$$\left. + V^d(\underline{r}-\underline{r}'')\,\phi_-(\underline{r}-\underline{r}'')\right\}$$

Since, by assumption, $\phi_+(r)$ and $\phi_-(r)$ are slowly varying functions of r within the wavelength of a particle 1/k, we assume that $V\phi$ also varies slowly. Hence, there will be a negligably small contribution to the integrals in eq. 2.32 from the region in which the exponential term oscillates rapidly. The largest contributions to the integrals come from values of $\underline{r}''$ lying close to the direction of $\underline{k}$ since there the exponential is nearly stationary. In fact, the term containing the derivative of $V\phi$ can be shown explicitly to be of higher order in 1/k than the rest of the expression.[17] Hence, in the equations for the wave modulations we drop both the last terms and also one of the terms obtained when taking the limit of the μ integration. This gives us the expressions

$$\phi_+(\underline{r}) = 1 - \frac{i}{\hbar v}\int_0^\infty d\underline{r}''\left\{V^d(\underline{r}-\underline{r}'')\,\phi_+(\underline{r}-\underline{r}'') + V^{ex}(\underline{r}-\underline{r}'')\,\phi_-(\underline{r}''-\underline{r})\right\} \qquad 2.33a$$

r" parallel to k

and

$$\phi_-(\underline{r}) = -\frac{i}{\hbar v}\int_0^\infty d\underline{r}''\left\{V^d(\underline{r}-\underline{r}'')\,\phi_-(\underline{r}-\underline{r}'') + V^{ex}(\underline{r}-\underline{r}'')\,\phi_+(\underline{r}''-\underline{r})\right\} \qquad 2.33b$$

r" anti-parallel to k

If we now take

$$\Gamma_{\parallel}^{\beta} = v\tau$$

and assume that v is in the z-direction, we obtain by differentiation

$$\frac{\partial \phi_+}{\partial z} = -\frac{i}{v}\left(V^d \phi_+ + V^{ex} \phi_-\right) \qquad 2.34a$$

$$\frac{\partial \phi_-}{\partial z} = -\frac{i}{v}\left(V^{ex} \phi_+ + V^d \phi_-\right) \qquad 2.34b$$

since we have assumed that the potentials are spherically symmetric.

If we now define

$$\chi = -1/k\int_{-\infty}^{z} V^d\left(\sqrt{b^2+z^2}\right) dz$$

and

$$\Phi_\lambda = \phi_\lambda e^{-i\chi(b)} \qquad \lambda = +, -$$

we obtain

$$\frac{d\Phi_+}{dz} = -\frac{i}{v} V^{ex} \Phi_- \qquad 2.35a$$

$$\frac{d\Phi_-}{dz} = \frac{-i}{v} V^{ex} \Phi_+ \qquad 2.35b$$

If we eliminate Φ we have

$$\frac{k^2}{V^{ex2}} \frac{d^2\Phi_+}{dz^2} - \frac{k^2}{V^{ex3}} \frac{dV^{ex}}{dz} \frac{d\Phi_+}{dz} + \Phi_+ = 0 \qquad 2.36$$

which may be written as

$$\frac{d}{dz}\left(\frac{k^2}{V^{ex2}} \left(\frac{d\Phi_+}{dz}\right)^2 + \Phi_+^2 \right) = 0 \qquad 2.37$$

Hence if we define

$$\lambda(b) = -1/k \int_{-\infty}^{z} V^{ex}(\sqrt{b^2 + z^2})\, dz$$

and use the boundary condition that $\chi(-\infty) = 0$, we obtain the solutions

$$\phi_+(b) = e^{i\chi(b)} \cos \lambda(b) \qquad 2.38a$$

$$\phi_-(b) = i\, e^{i\chi(b)} \sin \lambda(b) \qquad 2.38b$$

Hence we may express the forward and backward scattering amplitudes as

$$t_f(\underline{k}_f, \underline{k}_i) = i4\pi k \int_0^\infty b\,db\,J_0(|\underline{k}_f - \underline{k}_i|b)\left\{e^{i\chi(b)}\cos\lambda(b) - 1\right\} \quad 2.39a$$

and

$$t_b(\underline{k}_f, \underline{k}_i) = -4\pi k \int_0^\infty b\,db\,J_0(|\underline{k}_f + \underline{k}_i|b)\, e^{i\chi(b)}\sin\lambda(b) \quad 2.39b$$

These expressions are the same as eq. 2.23. This result may not be easily extended to the three-body system using the methods outlined above, as we encounter several difficulties. The splitting of the scattering amplitude into its forward and backward parts is non-trivial[8)] for the three-body system and therefore the direction of linearization of the Green's function in each term is not clear, (as in eq. 2.20). Also the manipulation of the exchange potentials is cumbersome as there are different exchange operators arising from several different pairs of particles. Hence, although we may derive three-body differential equations corresponding to eq. 2.34 their solution is not easily obtained. We will therefore give a third method of deriving eq. 2.39 which can be readily extended to the three-body system.

II.5 Alternative Derivation 2

To avoid some of the difficulties mentioned in the previous section we exploit the fact that Glauber theory may be represented in operator form. For this reason we prefer to treat the impact parameter amplitude $\Gamma(s,b)$ as an operator containing various permutation operators. The expectation value of the impact parameter amplitude then gives us the required scattering amplitude. The space exchange operator for the nucleon-nucleon system is related to the product of spin and isospin exchange operators by

$$\mathcal{P}^{r}_{ij} = P^{\sigma}_{ij} P^{\tau}_{ij} \qquad 2.40$$

where the superscripts denote the space in which the coordinates are exchanged, and the subscripts denote the particles which are permuted. Hence, although in our derivation we ignore explicit spin and isospin dependence, the method may easily be extended to include this dependence.

In this case the potential operator is given by

$$V = V^{d} + V^{ex}P^{r} \qquad 2.41$$

and the impact parameter amplitude is given by

$$\Gamma(s,b) = e^{-\frac{i}{v}\int_{-\infty}^{\infty} dz\, V(\underline{b} + \underline{n}z)} - 1 \qquad 2.42$$

Although the exponential in eq. 2.42 contains the space exchange operator P^{r}, the profile function can be reduced to a function linear in the exchange operator by expanding the exponential in a power series. Since

$$P^{r}\, P^{r} = I \qquad 2.43$$

where I is the identity operator we may recombine the terms into two series involving even and odd numbers of the exchange operator. This gives

$$\Gamma(s,b) = \Gamma^{f}(s,b) + i\Gamma^{b}(s,b)P^{r} \qquad 2.44$$

where

$$\Gamma^{f}(s,b) = e^{i\chi(s,b)} \cos\lambda(s,b) - 1 \qquad 2.45a$$

and

$$\mathcal{T}^{b}(s,b) = e^{i\chi(s,b)} \sin \lambda(s,b) \qquad 2.45b$$

and the phase functions are defined by

$$\chi(s,b) = -1/k \int_{-\infty}^{\infty} dz\, V^{d}(\sqrt{b^2+z^2}) \qquad 2.46a$$

and

$$\lambda(s,b) = -1/k \int_{-\infty}^{\infty} dz\, V^{ex}(\sqrt{b^2+z^2}) \qquad 2.46b$$

We assume that the initial and final states may be taken as plane waves, and by taking the expectation value of the operator $\mathcal{T}(s,b)$ with respect to these states we get the scattering amplitude:

$$t(\underline{k}_f,\underline{k}_i) = i2k\int d^2b \left\{ e^{-i(\underline{k}_f-\underline{k}_i)\cdot\underline{b}}\, \mathcal{T}^{f}(s,b) + \right.$$
$$\left. + i e^{-i(\underline{k}_f+\underline{k}_i)\cdot\underline{b}}\, \mathcal{T}^{b}(s,b) \right\} \qquad 2.47$$

Defining the direct momentum transfer by $\underline{\Delta} = \underline{k}_f - \underline{k}_i$ and the exchange momentum transfer by

$\underline{\Lambda} = \underline{k}_f + \underline{k}_i$ we may rewrite eq. 2.47 as

$$t(\underline{k}_f,\underline{k}_i) = t^f(\underline{k}_f,\underline{k}_i) + t^b(\underline{k}_f,\underline{k}_i) \qquad 2.48$$

where

$$t^f(\underline{k}_f,\underline{k}_i) = i4\pi k\int_0^\infty bdb\,J_0(b\sqrt{-t})\,\Gamma^f(s,b) \qquad 2.49a$$

and

$$t^b(\underline{k}_f,\underline{k}_i) = -4\pi k\int_0^\infty bdb\,J_0(b\sqrt{-u})\,\Gamma^b(s,b) \qquad 2.49b$$

where we have defined $s = k^2$, $t = -(\underline{k}_f - \underline{k}_i)^2$ $= -2s(1-\cos\theta)$, $u = -(\underline{k}_f+\underline{k}_i)^2 = -2s(1+\cos\theta)$ and $4s + t + u = 0$. This is the same result as we obtained before.

Before we progress onto scattering from a bound state it is worthwhile investigating the form of the unitarity relations for the forward and backward scattering amplitudes, as any approximations in these amplitudes are restricted by unitarity.

II.6 Unitarity

We may obtain the profile functions in terms of the forward and backward scattering amplitudes by taking the Fourier-Bessel transforms of eq. 2.49.

$$\Gamma^f(s,b) = \frac{-i}{8\pi^2 k} \int d^2\Delta \; e^{i\underset{\sim}{\Delta}\cdot\underset{\sim}{b}} \; t^f(s,t) \qquad 2.50a$$

$$\Gamma^b(s,b) = -\frac{1}{8\pi^2 k} \int d^2\pi \; e^{i\underline{\pi}\cdot\underline{b}} \; t^b(s,u) \qquad 2.50b$$

These can be shown to satisfy a very simple form of the unitarity relation in the high-energy limit. It has been considered that the principal disadvantage of the impact parameter formalism as compared to the partial wave expansion is its complicated unitarity relation[18)]. However there is no essential difference between the theoretical calculations of partial wave and impact parameter amplitudes if inelastic channels are considered[19)]. The only difference is that a non-zero imaginary part of the phase function does not imply the opening of a reaction channel, whereas a non-zero imaginary part of the partial wave phase shift does. This is so because even in the case of elastic scattering, the impact parameter of the outgoing particle is not the same as that of the incoming particle.

The unitarity condition allows us to complete our set of dynamical equations for the scattering amplitude. This requirement states that

$$\mathrm{Im}\, t(\mathbf{k}_f,\mathbf{k}_i) = \frac{-k}{(4\pi)^2}\int d\Omega'\, t^*(\mathbf{k}',\mathbf{k}_f)\, t(\mathbf{k}',\mathbf{k}_i) \qquad 2.51$$

If we split the scattering matrix into its forward and backward parts the relation becomes

$$\mathrm{Im}\, t(s,t_{13}) + \mathrm{Im}\, t(s,u_{13}) =$$

$$= \frac{-k}{(4\pi)^2}\int d\Omega' \left\{ t^*(s,t_{12}) + t^*(s,u_{12}) \right\} \cdot \left\{ t(s,t_{23}) + t(s,u_{23}) \right\} \qquad 2.52$$

where

$$t_{ij} = -2s(1-\hat{k}_i\cdot\hat{k}_j) \qquad i\neq j \;\; i,j=1,2,3$$
$$u_{ij} = -2s(1+\hat{k}_i\cdot\hat{k}_j)$$

The unitarity relation may be equivalently expressed in terms of the two-body profile functions as

$$\mathrm{Im}\left\{ -i\int_0^\infty b\,db\, J_0(b\sqrt{-t_{13}})\,\Gamma^f(s,b) + \int_0^\infty b\,db\, J_0(b\sqrt{-u_{13}})\,\Gamma^b(s,b) \right\} =$$

$$= -\int_0^\infty b'db' \int_0^\infty b''db'' \left\{ \Gamma^{f*}(s,b')\,\Gamma^f(s,b'')\, I_{ff}(b'\sqrt{-t_{12}},\, b''\sqrt{-t_{23}}) + \right.$$

$$+ i T^{f*}(s,b')T^{b}(s,b'')\, I_{fb}(b'\sqrt{-t_{12}}, b''\sqrt{-u_{23}}) -$$
$$- i T^{b*}(s,b')T^{f}(s,b'')\, I_{bf}(b'\sqrt{-u_{12}}, b''\sqrt{-t_{23}}) + \qquad 2.53$$
$$+ T^{b*}(s,b')T^{b}(s,b'')\, I_{bb}(b'\sqrt{-u_{12}}, b''\sqrt{-u_{23}}) \Big\}$$

where

$$I_{aa'}(b'A, b''A') = \frac{k^2}{4(2\pi)^4}\int d\Omega'\, J_0(b'A)\, J_0(b''A')$$

$$a, a' = f, b$$
$$A = \sqrt{-t_{12}}\ ,\ \sqrt{-u_{12}}$$
$$A' = \sqrt{-t_{23}}\ ,\ \sqrt{-u_{23}}$$

These integrals may be evaluated as before using Appendix A.

The unitarity relation then becomes

$$\mathrm{Im}\{-iT^{f}(s,b)\} = \int_0^\infty b'db' \int_0^\infty b''db'' \times$$
$$\times \Big\{T^{f*}(s,b')T^{f}(s,b'') + T^{b*}(s,b')T^{b}(s,b'')\Big\}\, G(b,b',b'',s) \qquad 2.54$$

and

$$\mathrm{Im}\{T'^{b}(s,b)\} = i\int_0^\infty b'db'\int_0^\infty b''db'' \times$$

$$\times\{T'^{f*}(s,b')T'^{b}(s,b'') - T'^{b*}(s,b')T'^{f}(s,b'')\}\,G'(b,b',b'',s) \qquad 2.54b$$

where G and G' are defined by eq. 2.12. Taking the high energy limit for G and G' as in eq. 2.15 the unitarity relation finally becomes

$$\mathrm{Im}\{-iT'_{f}(s,b)\} = \frac{1}{2k}\{T'^{f*}(s,b)T'^{f}(s,b) + T'^{b*}(s,b)T'^{b}(s,b)\} \qquad 2.55a$$

$$\mathrm{Im}\{T'^{b}(s,b)\} = \frac{i}{2k}\{T'^{f*}(s,b)T'^{b}(s,b) - T'^{*}_{b}(s,b)T'^{f}(s,b)\} \qquad 2.55b$$

These unitarity relations are consistent with those derived by Blankenbecler and Goldberger.[4] If the scattering amplitudes are analytic in energy then the determination of the profile functions is a one-cut mapping problem involving only physical values of energy. Although we may not integrate analytically G and G', the fact that there exists an exact unitarity statement (eq.2.54) shows that the Fourier-Bessel representation of the forward and backward scattering amplitudes can form a complete dynamics, at least in principle[20).

CHAPTER III

Scattering from a Bound State

III.1 Derivation of Amplitudes

In this section the scattering of an incident particle from a two-body bound state is studied. We assume that particle 1 is the projectile, and that particles 2 and 3 form a composite system. The particles are all assumed to interact via local potentials and we assume that the bound-state particles are instantaneously "frozen" at the moment of impact[21]. Without this assumption, the methods used in the previous section may not be extended to this system. However, by finally taking the expectation values of the coordinates of the particles using the bound state wave function, we are effectively taking their motion into account.

The initial and final states of the system are then given by

$$\psi_i(\underline{r},\underline{\rho}) = \phi_B(\underline{r})\, e^{i\underline{k}_i\cdot\underline{\rho}} \tag{3.1a}$$

and

$$\psi_f(\underline{r},\underline{\rho}) = \phi_B(\underline{r})\, e^{i\underline{k}_f\cdot\underline{\rho}} \tag{3.1b}$$

where $\phi_B(\underline{r})$ is the bound state wave function. The centre of mass variables $\underline{r}$ and $\underline{\rho}$ represent the separation between particles 2 and 3, and between particle 1 and the centre of mass of 2 and 3 respectively (see Fig. 3). We assume that all the particles are distinguishable, but of equal mass. Hence particle 1 undergoes scattering from two fixed scattering centres located at $\pm\frac{1}{2}\underline{r}$ with the interaction given in operator form by

$$V^d = V_{12}^d(|\rho + \tfrac{1}{2}\underline{r}|) + V_{31}^d(|\rho - \tfrac{1}{2}\underline{r}|) \qquad 3.2a$$

and

$$V^{ex} = V_{12}^{ex}(|\rho + \tfrac{1}{2}\underline{r}|)P_{12}^r + V_{31}^{ex}(|\rho - \tfrac{1}{2}\underline{r}|)P_{31}^r \qquad 3.2b$$

where the direct and exchange potentials are denoted by superscripts, and the subscript indicates the interacting particles. Then we may consider the three-body profile function to be an operator:

$$\Gamma = \exp\{ i\chi_{12}(s, \rho_\perp + \tfrac{1}{2}\underline{r}_\perp) + i\lambda_{12}(s, \rho_\perp + \tfrac{1}{2}\underline{r}_\perp)P_{12}^r + i\chi_{31}(s, \rho_\perp - \tfrac{1}{2}\underline{r}_\perp) + i\lambda_{31}(s, \rho_\perp - \tfrac{1}{2}\underline{r}_\perp)P_{31}^r \} \qquad 3.3$$

where the phase functions are given by

$$\chi_{ij}(s, \rho_\perp \pm \tfrac{1}{2}\underline{r}_\perp) = -\frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} d\rho_{\parallel} V_{ij}^d(\rho_\perp \pm \tfrac{1}{2}\underline{r}_\perp, \rho_{\parallel} \pm \tfrac{1}{2}r_{\parallel})$$

$$\lambda_{ij}(s, \rho_\perp \pm \tfrac{1}{2}\underline{r}_\perp) = -\frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} d\rho_{\parallel} V_{ij}^{ex}(\rho_\perp \pm \tfrac{1}{2}\underline{r}_\perp, \rho_{\parallel} \pm \tfrac{1}{2}r_{\parallel})$$

This may be written in terms of the two-body profile functions as

$$\Gamma = (\Gamma^f_{12}+1)(\Gamma^f_{31}+1) + i P^r_{12}\Gamma^b_{12}(\Gamma^f_{31}+1) + + i(\Gamma^f_{12}+1)\Gamma^b_{31}P^r_{31} - P^r_{12}\Gamma^b_{12}\Gamma^b_{31}P^r_{31} \qquad 3.4$$

where

$$\Gamma^f_{ij} = e^{i\chi(s,b)}\cos\lambda(s,b) - 1 \qquad 3.5a$$

$$\Gamma^b_{ij} = e^{i\chi(s,b)}\sin\lambda(s,b) \qquad 3.5b$$

The terms appearing in eq. 3.4 have a simple diagrammatic representation which may be used to split the equation into its forward and backward parts. The single scattering terms involve only one factor Γ_{ij} and are represented by Fig. 4a. The composite particle is denoted by a double line, and the square vertex represents the on-energy-shell two particle scattering amplitude in the forward and backward direction. The double scattering terms which contain two factors Γ_{ij} are analogously represented by Fig. 4b. Clearly we obtain contributions to scattering in the forward direction not only from particles which scatter forwards off both particles, but also from those which scatter backwards from both particles. Similarly, double scattering contributions in the backward direction occur when the incident particle scatters backwards off one particle in the bound state system and forwards off the other.

Since the impact parameter amplitudes represent scattering in a small cone parallel or anti-parallel to the incident direction, we may use these diagrams to split eq. 3.4 into its forward and backward components. Then

$$\Gamma = \Gamma^f + \Gamma^b$$

where

$$\Gamma^f = \Gamma^f_{12} + \Gamma^f_{31} + \Gamma^f_{12}\Gamma^f_{31} - P^r_{12}\Gamma^b_{12}\Gamma^b_{31}P^r_{31} \tag{3.6a}$$

and

$$\Gamma^b = iP^r_{12}\Gamma^b_{12} + i\Gamma^b_{31}P^r_{31} + iP^r_{12}\Gamma^b_{12}\Gamma^f_{31} + i\Gamma^f_{12}\Gamma^b_{31}P^r_{31} \tag{3.6b}$$

The physical criterion for splitting the amplitude is equivalent to splitting it according to whether each term contains an even or odd number of factors P^r_{ij}.

We may now calculate the forward and backward scattering amplitudes by taking the expectation values of the profile functions with respect to the final and initial states given by eq. 3.1. We will work in terms of the more familiar $f(\underline{k}_f,\underline{k}_i)$ where $\left|f(\underline{k}_f,\underline{k}_i)\right|^2 = \frac{d\sigma}{d\Omega}$ and in the two-body system

$$f(\underline{k}_f,\underline{k}_i) = -1/4\pi \; t(\underline{k}_f,\underline{k}_i)$$

The three-body scattering amplitude is therefore given by

$$f(\underline{k}_f,\underline{k}_i) = \frac{k}{2\pi i}\int d\tau \, \psi_f^*(\underline{r},\underline{\rho})\, \tau \, \psi_i(\underline{r},\underline{\rho})$$

$$= \frac{k}{2\pi i}\int d\tau \, \phi^*(\underline{r})\, e^{-i\underline{k}_f\cdot\underline{\rho}}\, \tau \, \phi(\underline{r})\, e^{i\underline{k}_i\cdot\underline{\rho}} \qquad 3.7$$

where the integration is over all space, and hence the forward and backward amplitudes are given by

$$f^{for}(\underline{k}_f,\underline{k}_i) = \frac{k}{2\pi i}\int d^3r\, d^3\rho \left\{ |\phi_B(r)|^2 e^{-i\underline{\rho}\cdot(\underline{k}_f-\underline{k}_i)} \times \right.$$
$$\times\left[\tau_{12}^f + \tau_{31}^f + \tau_{12}^f\tau_{31}^f\right] + \phi_B^*(-\underline{\rho}+\tfrac{1}{2}\underline{r})\, e^{i\underline{k}_f\cdot(\frac{1}{2}\underline{\rho}+\frac{3}{4}\underline{r})} \times \qquad 3.8a$$
$$\left. \times\, \tau_{12}^b\tau_{31}^b\, \phi_B(\underline{\rho}+\tfrac{1}{2}\underline{r})\, e^{i\underline{k}_i\cdot(-\frac{1}{2}\underline{\rho}+\frac{3}{4}\underline{r})} \right\}$$

and

$$f^{bac}(\underline{k}_f,\underline{k}_i) = \frac{k}{2\pi}\int d^3r\, d^3\rho \left\{ \phi_B^*(-\underline{\rho}+\tfrac{1}{2}\underline{r})\, e^{i\underline{k}_f\cdot(\frac{1}{2}\underline{\rho}+\frac{3}{4}\underline{r})} \times \right.$$
$$\times\, \tau_{12}^b(1+\tau_{31}^f)\, \phi_B(\underline{r})\, e^{i\underline{k}_i\cdot\underline{\rho}} + \phi_B^*(\underline{r})\, e^{i\underline{k}_f\cdot\underline{\rho}} \times \qquad 3.8b$$
$$\left. \times\, (\tau_{12}^f+1)\,\tau_{31}^b\, \phi_B(\underline{\rho}+\tfrac{1}{2}\underline{r})\, e^{i\underline{k}_i\cdot(-\frac{1}{2}\underline{\rho}+\frac{3}{4}\underline{r})} \right\}$$

We choose to express the profile functions in terms of the two-body scattering amplitudes using eq. 2.50 since these are more readily obtained experimentally. If we now define the bound state form factor by

$$S(q) = \int d^3r\, e^{iq\cdot r} \, |\phi_B(r)|^2 \tag{3.9}$$

we may evaluate expressions 3.8a and 3.8b.

Since all the terms are evaluated in a similar manner we will show explicitly the evaluation of one of them and then give the result for the rest. Consider

$$\int d\tau\, \phi_B^*(r)\, e^{-ik_f\cdot\rho} P_{12}^r \left(\Gamma_{12}^b + \Gamma_{12}^b \Gamma_{31}^f\right) \phi_B(r)\, e^{ik_i\cdot\rho} \tag{3.10}$$

Operating through the expression with the permutation operator P_{12}^r gives

$$\int d\tau\, \phi_B^*(-\rho+\tfrac{1}{2}r)\, e^{ik_f\cdot(\frac{1}{2}\rho+\frac{3}{4}r)} \left\{\Gamma_{12}^b(\rho+\tfrac{1}{2}r) + \Gamma_{12}^b(\rho+\tfrac{1}{2}r)\,\Gamma_{31}^f(\rho-\tfrac{1}{2}r)\right\} \phi_B(r)\, e^{ik_i\cdot\rho} \tag{3.11}$$

If we now substitute eq. 2.50 in eq. 3.11 and use

$$\phi_B(r) = \int d^3p\, e^{ip\cdot r}\, \phi_B(p)$$

the expression becomes

$$\int d^3q\, d^3p\, d^3p'\, \phi_B^*(p)\, e^{-ip\cdot(-\rho+\frac{1}{2}r)}\, \phi_B(p')\, e^{i(\frac{1}{2}k_f+k_i)\cdot\rho} \times$$

$$\times\, e^{i\frac{3}{4}k_f\cdot r}\, e^{iq\cdot(\rho+\frac{1}{2}r)+ip'\cdot r}\, d^3r\, d^3\rho\, \times$$

$$\times\left\{ f_{12}^{bac}(q) + \int d^3q'\, e^{iq'\cdot(\rho-\frac{1}{2}r)}\, f_{12}^{bac}(q)\, f_{31}^{for}(q') \right\} \qquad 3.12$$

Then, performing some of the integrations we obtain

$$\int d^3q'\, \phi_B^*\left(-q' + \frac{\Pi - k_i}{2}\right) \phi_B\left(-q' + \frac{\Pi - k_f}{2}\right) f_{12}^{bac}(q'-\Pi) +$$

$$+\int d^3q\, d^3q'\, \phi_B^*\left(-q-q'+\frac{k_f}{2}\right) \phi_B\left(-q+\tfrac{1}{2}k_i\right) f_{12}^{bac}(q-\Pi)\, f_{31}^{for}(q') \qquad 3.13$$

We may evaluate the other terms similarly giving the final result

$$f^{for}(k_f k_i) = S\left(\tfrac{\Delta}{2}\right) f_{12}^{for}(\Delta) + S\left(-\tfrac{\Delta}{2}\right) f_{31}^{for}(\Delta) +$$

$$+\frac{i}{2\pi k}\int d^2q\, S(q)\, f_{12}^{for}\left(q+\tfrac{\Delta}{2}\right) f_{31}^{for}\left(q-\tfrac{\Delta}{2}\right) +$$

$$+\frac{i}{2\pi k}\int \frac{d^2p_\perp\, d^2p'_\perp}{(2\pi)^2}\, \phi_B^*\left(p_\perp+\tfrac{1}{2}p'_\perp\right) \phi_B\left(-p_\perp+\tfrac{1}{2}p'_\perp\right) \times \qquad 3.14a$$

$$\times\, f_{12}^{bac}\left(-k_f - \frac{k_i}{2} + p_\perp + \tfrac{1}{2}p'_\perp\right) f_{31}^{bac}\left(k_i + \tfrac{1}{2}k_f - p_\perp - \tfrac{1}{2}p'_\perp\right)$$

$$f^{bac}(k_f, k_i) = \frac{1}{(2\pi)^2}\int d^2q \Big\{ \phi_B^*(q+\tfrac{1}{2}k_f)\, \phi_B(q+\tfrac{1}{2}k_i) \times$$
$$\times f_{12}^{bac}(-\Pi-q) + \phi_B^*(q-\tfrac{1}{2}k_f)\phi_B(q-\tfrac{1}{2}k_i) f_{31}^{bac}(\Pi+q)\Big\}$$
$$+ \frac{i}{2\pi k}\cdot\frac{1}{(2\pi)^2}\int d^2q\, d^2q' \Big\{ \phi_B^*(q+\tfrac{1}{2}k_f)\phi_B(q'+\tfrac{1}{2}k_i) \times \qquad 3.14b$$
$$\times f_{12}^{bac}(-\Pi-q') f_{31}^{for}(q'-q) +$$
$$+ \phi_B^*(q-\tfrac{1}{2}k_f)\phi_B(q'-\tfrac{1}{2}k_i) f_{31}^{bac}(\Pi-q') f_{12}^{for}(q-q')\Big\}$$

III.2 Approximation of Three-Body Amplitudes

These expressions are exact within the eikonal approximation. However, we now choose to approximate them so that we may obtain them completely in terms of the bound-state form factor rather than the wave-functions, since this is more readily obtainable experimentally. It also allows us to express the equations in a more elegant form. The approximation we make, is consistent with those already made and agrees in substance with that of Bertocchi and Capella[5]. Since most of the scattering at the energies we are considering takes place within a small cone parallel or antiparallel to the incident direction, we assume that if the component of any momentum parallel to the incident direction is small, then its transverse component is also small compared with the momentum of the incident particle (see Appendix C). This approximation gives us

$$f^{for}(\underline{k}_f,\underline{k}_i) \approx S(\underline{\Delta}/2) f_{12}^{for}(\underline{\Delta}) + S(-\underline{\Delta}/2) f_{31}^{for}(\underline{\Delta}) + \frac{i}{2\pi k}\int d^2q\, S(\underline{q}) \Big\{ f_{12}^{for}(\underline{q}+\tfrac{1}{2}\underline{\Delta})\, f_{31}^{for}(\underline{q}-\tfrac{1}{2}\underline{\Delta}) + f_{12}^{bac}(\underline{k}_f+\tfrac{1}{2}\underline{k}_i+\tfrac{1}{2}\underline{q})\, f_{31}^{bac}(-\underline{k}_i-\tfrac{1}{2}\underline{k}_f-\tfrac{1}{2}\underline{q}) \Big\} \qquad 3.15a$$

and

$$f^{bac}(\underline{k}_f,\underline{k}_i) \approx S(\underline{\Delta}/2) f_{12}^{bac}(-\underline{\pi}) + S(\underline{\Delta}/2) f_{31}^{bac}(\underline{\pi}) + \frac{i}{2\pi k} f_{12}^{bac}(-\underline{\pi}) \int d^2q\, S(\underline{q}-\tfrac{1}{2}\underline{\Delta})\, f_{31}^{for}(\underline{q}) + \frac{i}{2\pi k} f_{31}^{bac}(\underline{\pi}) \int d^2q\, S(\underline{q}+\tfrac{1}{2}\underline{\Delta})\, f_{12}^{for}(\underline{q}) \qquad 3.15b$$

The approximation has the effect of slightly increasing the scattering amplitude, since the functions under the integral sign fall off more slowly than before.

CHAPTER IV

APPLICATIONS

IV.1 Nucleon-nucleon Scattering

As an application of the two-body formalism we have developed in previous chapters we consider nucleon-nucleon scattering; in particular proton-proton and proton-neutron* forward and backward scattering. The analysis does not contain explicit spin-dependence but this dependence is taken into account indirectly in the input parameters.

We shall first calculate the scattering amplitudes at intermediate energies from partial waves and then use the results as a basis for the assumptions we must make about the high-energy scattering amplitudes which are to a large extent unknown. Once we decide the form of these amplitudes we may evaluate the profile

*We refer to proton-neutron differential cross-sections although the experimental data used is from neutron-proton experiments.

Clearly we cannot use data extracted from proton-deuteron scattering data, which is the usual method of obtaining proton-neutron data functions at high energies and compare them with those we obtain at intermediate energies from a phase-shift analysis.

Before continuing however, we give a short discussion of the nucleon-nucleon experimental situation which will provide some of the input parameters in our three-body calculations. Proton-proton differential cross-sections are reasonably well known over a wide range of energies[22]. The information about proton-neutron differential cross-sections is somewhat sparce. We restrict our discussion of the behavior of nucleon-nucleon differential cross-sections to narrow cones about the forward and backward directions. Since the p-p elastic differential cross-section is symmetric about 90 deg. (because of identity of particles), comments about the forward p-p peak apply equally well to the backward p-p peak. Also the forward p-n diffraction peak is similar to that found in p-p elastic scattering, and hence we may discuss these simultaneously. In the forward region the differential cross-section decreases approximately exponentially in t $(= -2k^2(1 - \cos\theta))$, and this region extends to higher values of momentum transfer as the energy increases. The diffraction peak shrinks rapidly at first and then more slowly as the momentum of the incident particle increases. The data is well fitted by the expression

$$\frac{d\sigma}{dt} = \left.\frac{d\sigma}{dt}\right)_{t=0} e^{at} \qquad 4.1$$

where $\left.\frac{d\sigma}{dt}\right)_{t=0}$ remains approximately constant, and a increases with energy.

Backward proton-neutron scattering is, however, complicated by the presence of charge-exchange scattering. The backward peak falls off very rapidly as the exchange momentum transfer increases and is fitted by a curve of the form:

$$\frac{d\sigma}{du} = \left.\frac{d\sigma}{du}\right)_{u=0} e^{au} \qquad 4.2$$

The parameter $\left.\frac{d\sigma}{du}\right)_{u=0}$ is energy dependent and is proportional to the inverse square of the incident beam momentum, whereas a passes through a maximum and then decreases[23]. This maximum occurs near the one-pion threshold, but the mechanism of such a reaction is not yet clear.

These analytic expressions for the two-body differential cross-sections are used in our calculations with the parameters given in Table 1[22].

IV.2 Intermediate Energy Scattering Amplitudes

We may calculate both the scattering amplitudes and the impact parameter amplitudes at intermediate energies from partial wave analysis since the necessary information is available between 400 and 750 MeV[24]. Then, as our formalism for the two-body system holds at intermediate energies as well as high energies we use our calculations as a basis for assumptions we make about high energy scattering amplitudes. Unfortunately we cannot directly calculate the scattering amplitudes in the 1 GeV energy region since the information is not available.

We may calculate the scattering amplitude from the partial waves[24] using Appendix D, and for simplicity we restrict our calculations to the spin independent part $a(E,\theta)$ of the general scattering matrix. The phase-shifts used in the calculations are taken from Macgregor et al[25]. The results of the calculation are shown in Fig. 5. The values of the real and imaginary parts of nucleon-nucleon scattering do not show very marked dependence on the energy at 0° and 180°. The forward and backward real and imaginary parts of the proton-proton scattering amplitudes are not equal, as might be expected of an identical particle system, since

$$a(\theta) + m(\theta) = -\{ a(\pi-\theta) + m(\pi-\theta) \}$$

and we plot only a(Θ), and not the spin-dependent coefficient m(Θ). Hence it is reasonable to assume at these energies and at these angles that the real and imaginary parts of the scattering amplitudes vary only slowly with energy.

It is possible to perform these calculations at any energy up to 750 MeV, but we choose only a few in the intermediate energy region to illustrate the results.

We may also calculate the real and imaginary parts of the isovector (I=1) and isoscalar (I=0) parts of proton-neutron spin independent amplitudes. This gives us a clear indication of the charge-exchange behavior since this is calculated from the difference between the isovector and isoscalar parts of the scattering amplitudes. The results are shown in Fig.6. Clearly, the difference between the two curves shows that the real part of the (charge exchange) scattering amplitude is dominant over much of the energy region. We use this behavior later to argue that at high energies the charge-exchange amplitude may be taken to be real.

Since we have calculated the real and imaginary parts of the scattering amplitudes it is relatively easy to calculate the ratio

$$\alpha(\theta, E) = \frac{|\mathrm{Re}\, t(\theta, E)|}{\mathrm{Im}\, t(\theta, E)} \qquad 4.3$$

This may also be obtained from the interference of the Coulomb and nuclear interactions in the forward direction, and the energy dependence of this ratio at $\theta = 0^{\circ}$ is given in Fig. 7. Clearly, as the energy increases the ratios for p-p and p-n (no charge exchange) scattering oscillate to zero, suggesting that at high energies the imaginary part of the forward scattering amplitude is dominant. We may also calculate from phase-shift analyses this ratio at angles other than 0° and the theoretical angular dependence of the ratio is shown in Fig. 8 at two energies, 570 MeV and 750 MeV. This shows that the ratio α is virtually independent of angle in this energy region in the forward and backward directions, and hence we may use the same value of α in our calculations at different angles. This suggests that it is quite plausible to assume that the same angular behavior holds at high energies.

The results of the phase-shift analysis[25] are not to be taken absolutely, although the analysis is the best to date. There is still much work to be done in this direction since, for example, the inelasticity parameter is incorrectly included, and the authors themselves warn about the accuracy of the results above 450 MeV.

IV.3. High Energy Scattering Amplitudes

Due to the lack of experimental information about high-energy scattering amplitudes, we make assumptions about them, based in part on the results of the previous section, but also on general theoretical arguments. We will consider first the forward p-p and p-n scattering amplitudes. If the nucleon-nucleon total cross-section remains constant with increasing energy, or increases, as recent experiments suggest for the proton-proton total cross-section[26], then, by the optical theorem, the imaginary part of the forward scattering amplitude increases at least linearly in k. So we may expect that at sufficiently high energies Im $t(0)$ greatly exceeds $|\mathrm{Re}\ t(0)|$ in the elastic diffraction peak for both p-p and p-n scattering. It seems that $|\mathrm{Re}\ t(0)|$ is negligible at the energies with which we are concerned. Since the proton-proton system is symmetric about 90 deg., we may make the same assumptions about the backward p-p scattering amplitude as we do about the forward amplitude.

We base our assumptions about the backward p-n scattering amplitude on the form of the forward p-n charge-exchange amplitude, since these systems are equivalent. The forward exchange amplitude is related by general isotopic-spin arguments[27)] to the difference between the total cross-sections for p-p and p-n scattering.

$$\mathrm{Im}\; t^{pn}_{\text{exchange}}(0) = -\frac{k}{(4\pi)^2}\left(\sigma_T(pp) - \sigma_T(pn)\right) \qquad 4.4$$

where σ_T is total cross-section. This total cross-section difference is very small at intermediate energies and vanishes at high energies. Hence, at high energies the imaginary part of the scattering amplitude is negligible compared with the real part. This conclusion is general and does not depend on any particular charge exchange model. Consequently, we take the backward p-n scattering amplitude to be real.

These assumptions about the scattering amplitude are fairly reasonable at high energies. Serber[28)] uses a completely imaginary optical potential to fit high energy elastic p-p data with good results. The assumption that the backward p-n amplitude is real is consistent with Arushanov[29)] who assumes this and finds good qualitative agreement with the measured differential cross-sections in the full angular range.

We also extend our findings from the intermediate energy results to assume that at high energy the ratio of real to imaginary parts is independent of angle.

IV.4 Evaluation of Impact Parameter Amplitudes

We are now in a position to be able to find a functional form for the profile functions. Under the assumptions made in the last three sections we may write the p-p and p-n scattering amplitudes as

$$t^f_{pp}(\underset{\sim}{\Delta}) = iAe^{-a\Delta^2} \tag{4.5a}$$

$$t^f_{pn}(\underset{\sim}{\Delta}) = iCe^{-c\Delta^2} \tag{4.5b}$$

$$t^b_{pp}(\pi) = iAe^{-a\pi^2} \tag{4.5c}$$

$$t^b_{pn}(\pi) = De^{-d\pi^2} \tag{4.5d}$$

where A,C,D,a,c,d are assumed to be real constants.

Eq. 2.50 gives the profile functions in terms of the scattering amplitudes and hence taking a Fourier-Bessel transform of the scattering amplitudes we obtain

$$\Gamma^{f}_{pp} = e^{i\chi_{pp}(b)}\cos\lambda_{pp}(b) - 1 = \frac{A}{8\pi ka}\, e^{-b^2/4a} \qquad 4.6a$$

$$\Gamma^{b}_{pp} = \sin\lambda_{pp}(b)\, e^{i\chi_{pp}(b)} = -\frac{iAe^{-b^2/4a}}{8\pi ka} \qquad 4.6b$$

and

$$\Gamma^{f}_{pn} = e^{i\chi_{pn}(b)}\cos\lambda_{pn}(b) - 1 = \frac{Ce^{-b^2/4c}}{8\pi kc} \qquad 4.6c$$

$$\Gamma^{b}_{pn} = e^{i\chi_{pn}(b)}\sin\lambda_{pn}(b) = -\frac{De^{-b^2/4d}}{8\pi kd} \qquad 4.6d$$

From our assumptions about the scattering amplitudes we know that in the Born approximation both $\chi_{pp}(b)$ and $\lambda_{pp}(b)$ are imaginary and negative, and therefore eqs. 4.6a and 4.6b give the constraint that

$$\chi_{pp}(b) = \lambda_{pp}(b) \qquad 4.7$$

This is eminently reasonable as we expect the interactions which give rise to forward and backward scattering to be equivalent. Hence the profile functions for elastic p-p scattering are given by

$$\lambda_{pp}(b) = \chi_{pp}(b) = -\frac{i}{2}\ln\left\{1 + \frac{A}{8\pi ka}e^{-b^2/4a}\right\} \qquad 4.8$$

For p-n scattering we obtain the two profile functions from eqs. 4.6c and 4.6d as

$$\lambda_{pn}(b) = -\tan^{-1}\left\{\frac{De^{-b^2/4d}}{8\pi kd\left(1 + \frac{Ce^{-b^2/4e}}{8\pi kc}\right)}\right\} \qquad 4.9a$$

and

$$\chi_{pn}(b) = -\frac{i}{2}\ln\left\{\left(\frac{Ce^{-b^2/4c}}{8\pi kc} + 1\right)^2 + \frac{D^2e^{-b^2/2d}}{64\pi^2k^2d^2}\right\} \qquad 4.9b$$

Obviously $\chi_{pn}(b) \neq \lambda_{pn}(b)$ as the differential cross-sections are not symmetric about 90 deg. and since one potential is real and one imaginary. The Gaussian form of the differential cross-section (eq. 4.1) is obtained in the Born approximation from a Gaussian potential. Eqs. 4.8 and 4.9 therefore suggest that the nucleon-nucleon interaction is similar to a Gaussian interaction. Clearly both $\chi(b)$ and $\lambda(b)$ decrease as b increases, which is to be expected, since nuclear potentials are in general short range potentials.

The impact parameter amplitudes between 400 and 750 MeV may also be calculated from phase-shift analyses (see Appendix D). The dependence of the real and imaginary parts of the nucleon-nucleon profile functions on the impact parameter is shown in Fig. 9a. Clearly, the magnitudes of the real and imaginary parts are almost equal for given b, and their magnitudes increase with increasing energy. As the impact parameter increases the real and imaginary parts tend to zero. This is reasonable, since the potential from which they are originally constructed, in practice, acts only over a small distance.

Our analysis is not completely spin-independent, as we mentioned earlier, as the nucleon-nucleon amplitudes are constructed from contributions from singlet and triplet states (see Appendix D). Again we may calculate the contribution of each state using a standard partial wave analysis, and the results are shown in Fig. 9b. The real and imaginary parts are almost equal in magnitude and tend to zero as the impact parameter increases since the interaction region is finite in size. Their dependence on energy is indicated in Fig. 10, and again the magnitude of the real and imaginary parts increases with increasing energy.

These results are consistent with the behavior of our proposed impact parameter amplitudes given in eqs. 4.8 and 4.9.

IV.5 Proton-deuteron Scattering

We are now in a position to be able to construct high energy proton-deuteron differential cross-sections from the two-body scattering amplitudes. However we are constrained in this by the lack of detailed information about the two-body scattering amplitudes and our results can be only as good as the assumptions we make and models we choose.

IV.5.1 The Deuteron Form Factor

In order to evaluate the proton-deuteron differential cross-sections using eq. 3.15 we need to know the form factor of the deuteron. Since, however, neither the deuteron ground state wave function nor the form factor are known very accurately, particularly for small neutron-proton separations, we prefer to choose a reasonable theoretical model. The non-relativistic deuteron wave functions that are used in the prediction of form factors, originate from solutions of the Schroedinger equation for the two-nucleon problem using various nucleon interaction potentials. The success of the wave functions may be measured by their ability to predict the elastic electron-deuteron scattering cross-section at large momentum transfers.

Several reasonably realistic deuteron wave functions exist for low and intermediate energies. They contain various D-state admixtures and hard and soft cores. The more successful examples are those due to Hamada and Johnson[30], Bressel[31], Feshbach and Lomon[32], and Hulthèn[33]. These models give reasonable values for small momentum transfers, but the situation at large momentum transfers, ie scattering in the backward direction, is far from clear. A recent comparison[34] of the Breit and Hulthèn wave functions at low energy suggests that the Hulthèn wave function gives better agreement with experiment at a scattering angle of 180 deg., although at all other angles the data seem to support a hard core hypothesis. However at the energies with which we are concerned the forms of the deuteron wave function and form factor are unresolved.

An analysis of several simply plausible high-energy deuteron wave functions with a manageable analytic form has been performed by Franco and Glauber[35]. They obtain a reasonable fit to experimental data in the high-energy region using the Gaussian wave function

$$\phi_B = \left(\frac{0.1922}{\pi}\right)^{3/4} e^{-0.0961 r^2} \tag{4.10}$$

The values of the constants are those found by Verde[36] using a variational technique.

We may integrate eq. 4.10 immediately to give the deuteron form factor

$$S(\delta) = e^{-\delta^2/.7688} \tag{4.11}$$

Since the deuteron wave function is derived from the nucleon-nucleon interaction and we know that, in the Born approximation, we may obtain the form of the two-body amplitudes given by eqs. 4.1 and 4.2 from a Gaussian potential, it is consistent to use such a Gaussian wave function, and it also facilitates the evaluation of several integrals. Also it is not worthwhile to use a sophisticated deuteron wave function containing a hard core or D-state admixture since we have already had to make many assumptions about the nucleon-nucleon scattering amplitudes, and such a wave function would not significantly increase the accuracy of our results.

IV.5.2 Proton-Deuteron Differential Cross-sections

We now apply the three-body formalism, which was developed in Section III, to the calculation of proton-deuteron differential cross-sections at an incident proton energy of 1 GeV, 1.3 GeV, and 1.5 GeV. We compare the theoretical results with the experimental data of Bennett et al[37] and Coleman et al[38]. The two-body parameters used as input information are given in Table 1. We attempt only to fit the first forward p-d peak since our analysis is good only for small values of momentum transfer. The results are shown in Figs. 11 and 12.

In the forward direction the calculated curves at 1 GeV (Fig. 11a) vary slightly from the experimental results, although they agree at the optical point. This shifting appears to be due to the assumption that the real part of the scattering amplitude in the forward direction is negligible. This may be seen if we adopt the parameters for nucleon-nucleon scattering by Bassel and Wilkin[39] who include a real part of the scattering amplitude ($\alpha_{pp} = \alpha_{pn} = .325$). These parameters give a good fit to the data. Their analysis is however performed at only one energy.

The theoretical backward p-d differential cross-section at 1 GeV (Fig. 12a) appears to be in good agreement with experiment. The curve however falls below the experimental points if we use the parameters of Bassel and Wilkin, and also if we add a small real part to the forward scattering amplitude ($\alpha = .3, .5$) and a small imaginary part to the backward p-n scattering amplitude ($1/\alpha = 1/.3, 1/.5$). At all three energies the inclusion of a small real and imaginary part in the respective amplitudes lowers the theoretical predictions of the backward differential cross-sections. This arbitrary shifting of the predicted differential cross-sections is due to our lack of precise knowledge of high-energy scattering amplitudes.

It is interesting that, at 1.5 GeV (see Fig. 12c) even with the inclusion of small real and imaginary parts, the theoretical curve still rises above the experimental data, since previous analyses[5,6,7] have always produced differential cross-sections which are too low. Since the Gaussian form factor falls off quite fast, the single scattering terms are negligible in the backward direction and the differential cross-section is caused entirely by the interference terms. However our approximation of the deuteron wave function in the interference terms (see Appendix C) has the effect of increasing the magnitude of the scattering amplitude, so the backwards differential cross-sections depend sensitively on:

i) the bound-state wave function approximation (see Appendix C)

ii) the assumptions about the two-body scattering amplitudes

iii) the choice of bound-state form factor

Hence, the theory of backward scattering developed in this work can not be critically tested until we know more about the proton-proton and proton-neutronscattering amplitudes.

CHAPTER V

DISCUSSION

We have rigorously derived expressions for two-body off-energy-shell forward and backward scattering amplitudes and used three different methods to derive the corresponding on-energy-shell amplitudes. We have extended our formalism to obtain forward and backward scattering amplitudes for scattering from a composite system, in particular from a two-body state, but by a method which is easily extended to n-body composite systems. Our expressions contain only single and double scattering terms similar to Glauber theory[2]. Our amplitudes may also be derived from the multiple scattering series of Faddeev equations[40] by linearizing the Green's function as in Appendix B. Our results, however, are significantly different from those obtained by Bhasin[6]. Since we assume the particles to be distinguishable we have no term corresponding to mass exchange scattering, in which the incident particle replaces one of the target particles as

$$1 + (23) \rightarrow (12) + 3 \qquad 5.1$$

In our formalism, the input is clearly the forward and backward two particle scattering amplitudes, giving a clear indication as to which of the terms contribute to the forward direction and which to the backward direction scattering. Bhasin's analysis is, however, based on a second order approximation obtained from the Faddeev equations[40].

$$U_{\beta\alpha} = (s - H_0)(1 - \delta_{\alpha\beta}) + \sum_{\gamma \neq \beta} t_\gamma G_0 U_{\gamma\alpha} \qquad 5.2$$

The amplitude for the pick-up process is given by U_{31}. The multiple scattering series for this process is obtained by iterating eq. 5.2.

$$U_{31} = (s - H_0) + t_{31} + \cdots \qquad 5.3$$

The zeroth order term $(s - H_0)$ in such a rearrangement series corresponds to the Born approximation for the pick-up process (eq.5.1) and it is well known[40] that this yields a peak in the backward direction. Bhasin however assumes that the first order term t_{31} contributes only to backward scattering, and hence the backward amplitude is given entirely by $U_{\beta\alpha}$ $(\beta \neq \alpha)$ when it is approximated to second order. This implicitly ignores the contribution of the elastic scattering amplitude $U_{\alpha\alpha}$ which also contains some of the same first order terms as $U_{\beta\alpha}$, and implies that the backward scattering peak is negligible for distinguishable particles. Such an assumption, however is in clear disagreement with experiment, since such a backward scattering peak has already been observed in $\bar{\pi}$-d scattering[41].

The shadow effect has, in our formalism, contributions from both forward and backward elementary particle scattering, as is clear from eq. 3.15. Hence we expect our predicted cross-sections to be higher than those predicted by Bertocchi and Capella[5], and this is borne out by our results. There is clearly a need for more nucleon-nucleon and proton-deuteron scattering in the 200-800 MeV energy region. For example, although there is a reasonable amount of backward p-d scattering in the range 200-600 MeV, there is only one forward scattering experiment in this energy range, at 580 MeV. Proton-neutron charge exchange information is also needed at many more energies than is presently available.

In conclusion, we have constructed a complete high energy dynamics for forward and backward scattering of elementary particles, but we are limited in its application by our present lack of detailed experimental information.

Table 1

Nucleon-nucleon parameters used to calculate proton-deuteron differential cross-sections. E_p is the energy of the incident proton in the proton-deuteron system. The parameters given are obtained in a least squares fit[22] to the data as:

$$\left.\frac{d\sigma}{dt}\right|_{pp} = Ae^{at}$$

$$\left.\frac{d\sigma}{dt}\right|_{pn} = Ce^{ct}$$

$$\left.\frac{d\sigma}{du}\right|_{pn} = De^{dt}$$

Table 1

Ep(Gev)	A	a	C	c	D	d
1.0	56.38	.87	53.88	7.53	127.4	57.63
1.3	147.51	6.27	73.61	10.54	54.65	38.64
1.5	151.40	6.31	80.10	10.95	41.41	33.20

Figure 1

Eikonal description of high energy scattering. The incident particle passes through a scattering region at impact parameter b, and is scattered through an angle of θ.

Eikonal description of high energy scattering

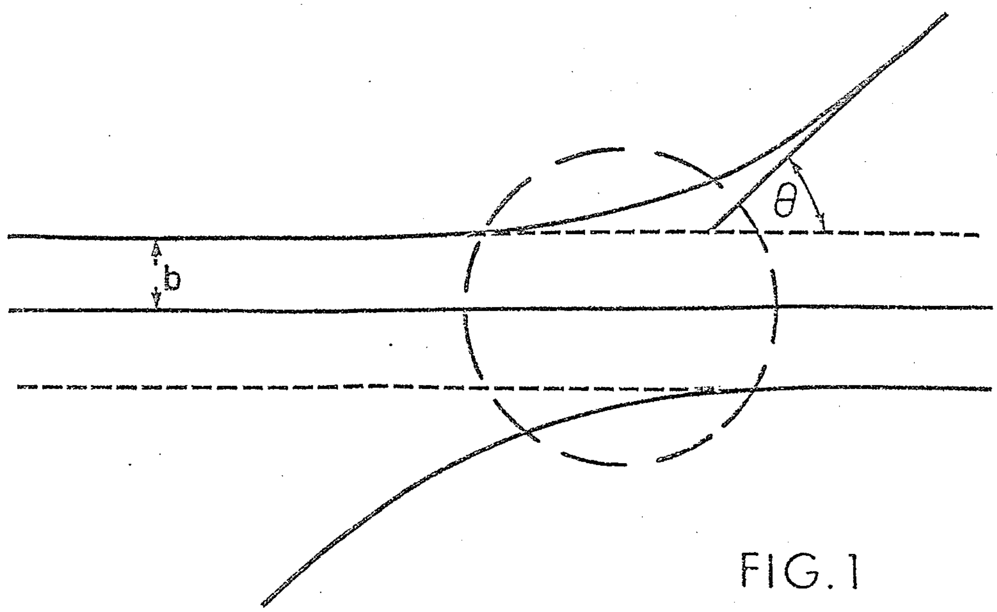


FIG. 1

Figure 2

Feynmann diagram showing a direct interaction (a) and a space exchange interaction (b) between two particles.

Feynmann diagram showing a direct interaction (a) and a space exchange interaction (b)

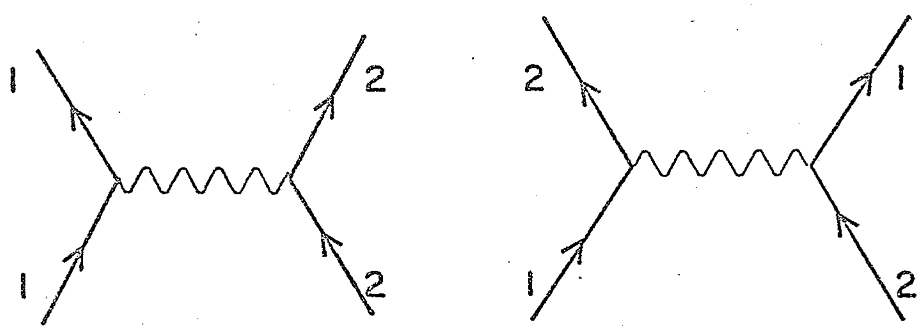


FIG. 2a FIG. 2b

Figure 3

Geometry for fixed scattering centre problem. The incoming particle 1 is scattered from fixed particles 2 and 3 at $\frac{1}{2}\underline{r}$ and $-\frac{1}{2}\underline{r}$ respectively. The origin is at the centre of mass of the system of 2 and 3.

Geometry for fixed scattering centre problem

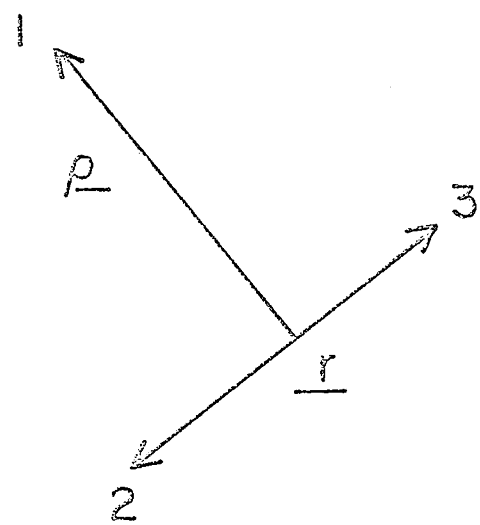


FIG. 3

Figure 4

Single (a) and double (b) scattering terms in the forward and backward scattering amplitudes for scattering from a bound state. The incident particle is labelled 1, and particles 2 and 3 form a bound state denoted by a double line. The square vertex represents two-body scattering amplitudes.

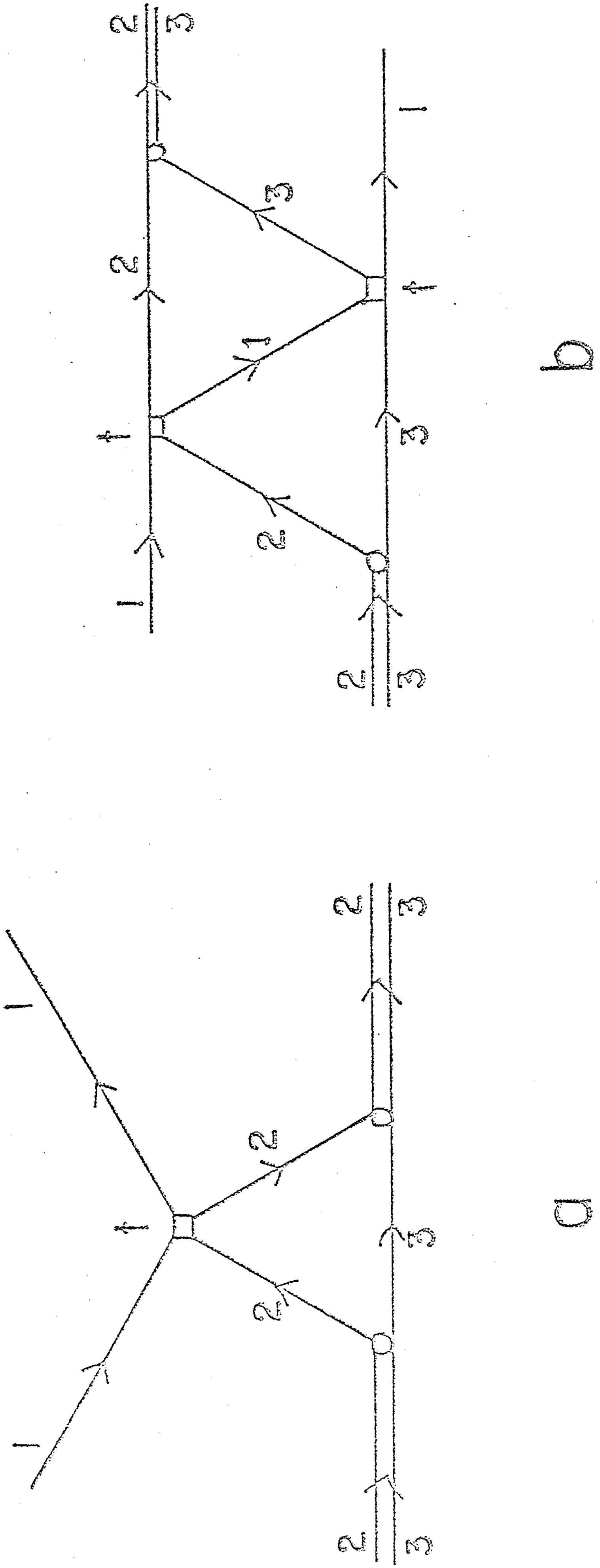
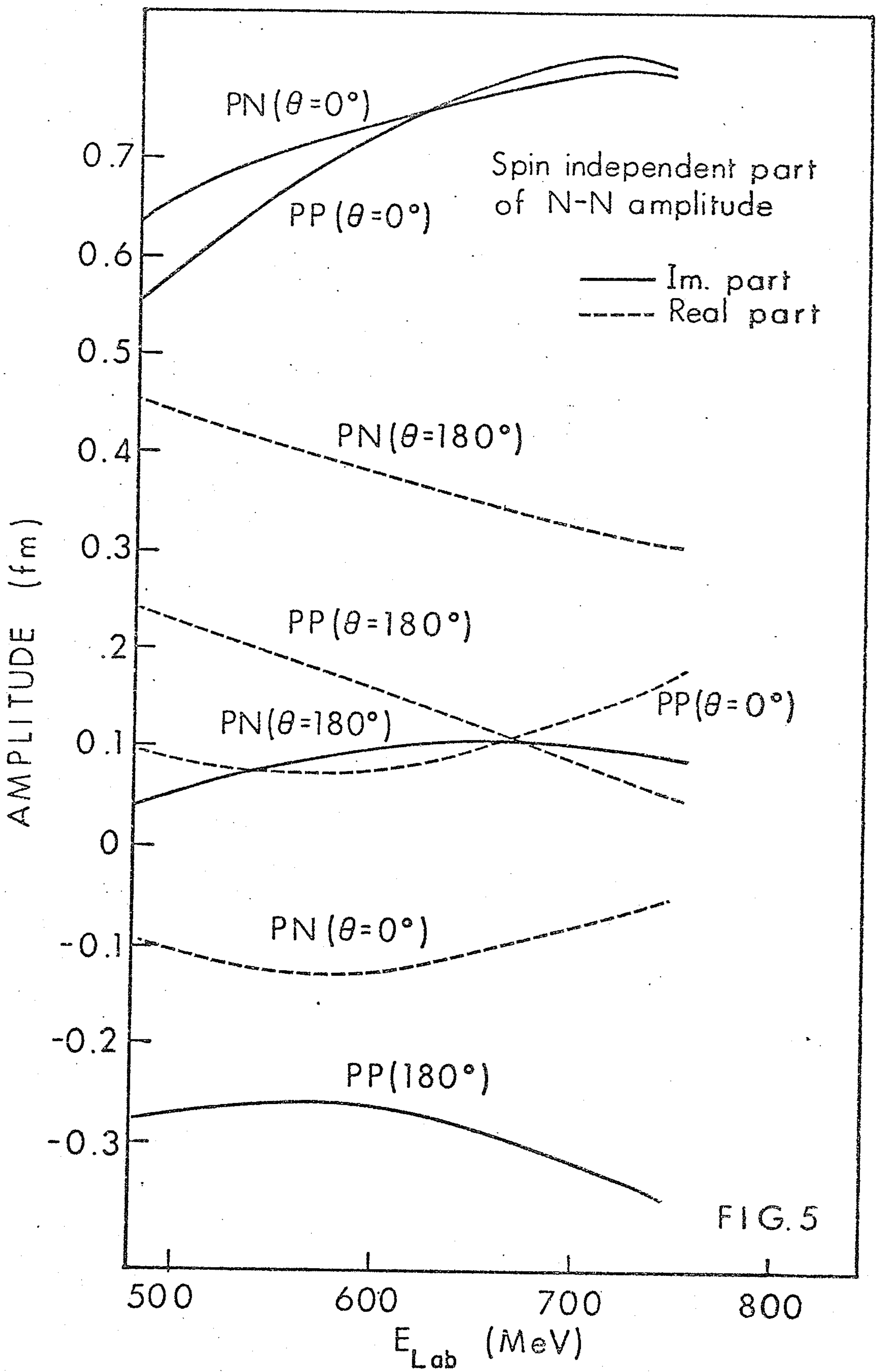


FIG. 4

Figure 5

Real and imaginary parts of the spin independent part of proton-proton and proton-neutron amplitudes. They are calculated in the forward and backward directions ie at a scattering angle of 0° and 180°.



PN (θ=0°)
PP (θ=0°)
Spin independent part of N-N amplitude
Im. part
Real part
PN (θ=180°)
PP (θ=180°)
PP(θ=0°)
PN(θ=180°)
PN (θ=0°)
PP(180°)
AMPLITUDE (fm)
0.7
0.6
0.5
0.4
0.3
.2
0.1
0
-0.1
-0.2
-0.3
500
600
700
800
E Lab (MeV)

FIG. 5

Figure 6

Real and imaginary parts of the isovector and iso-scalar parts of the Nucleon-Nucleon spin independent amplitudes.

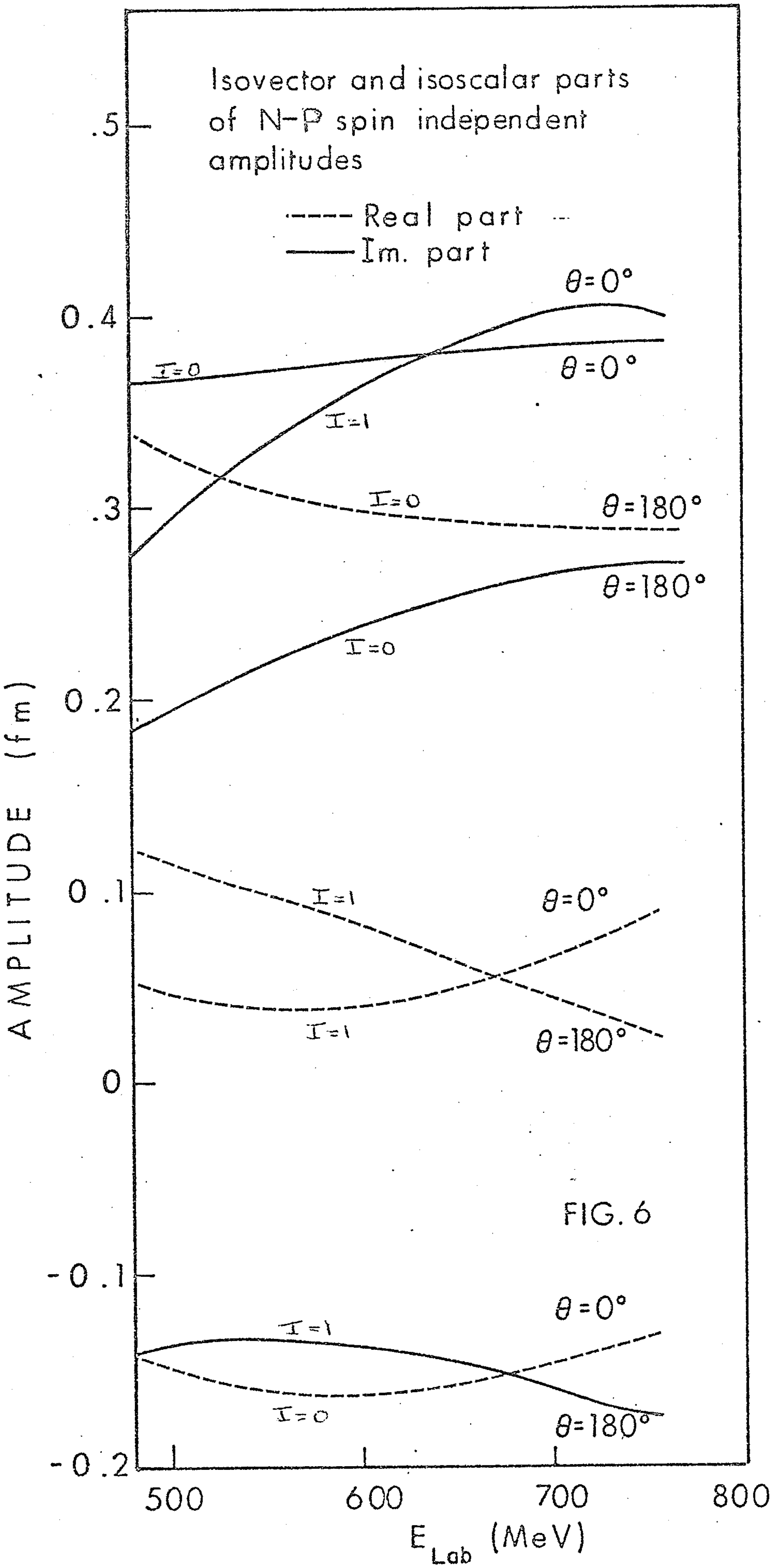

Isovector and isoscalar parts of N-P spin independent amplitudes
----- Real part
——— Im. part
θ=0°
θ=0°
θ=180°
θ=180°
θ=0°
θ=180°
θ=0°
θ=180°
I=0
I=1
I=0
I=0
I=1
I=1
I=1
I=0
AMPLITUDE (fm)
.5
0.4
.3
0.2
0 .1
0
-0 .1
-0 .2
500
600
700
800
E Lab (MeV)

FIG. 6

Figure 7

Energy dependence of the ratio of real to imaginary parts of the nucleon-nucleon scattering amplitudes in the forward direction ($\theta = 0^{\circ}$).

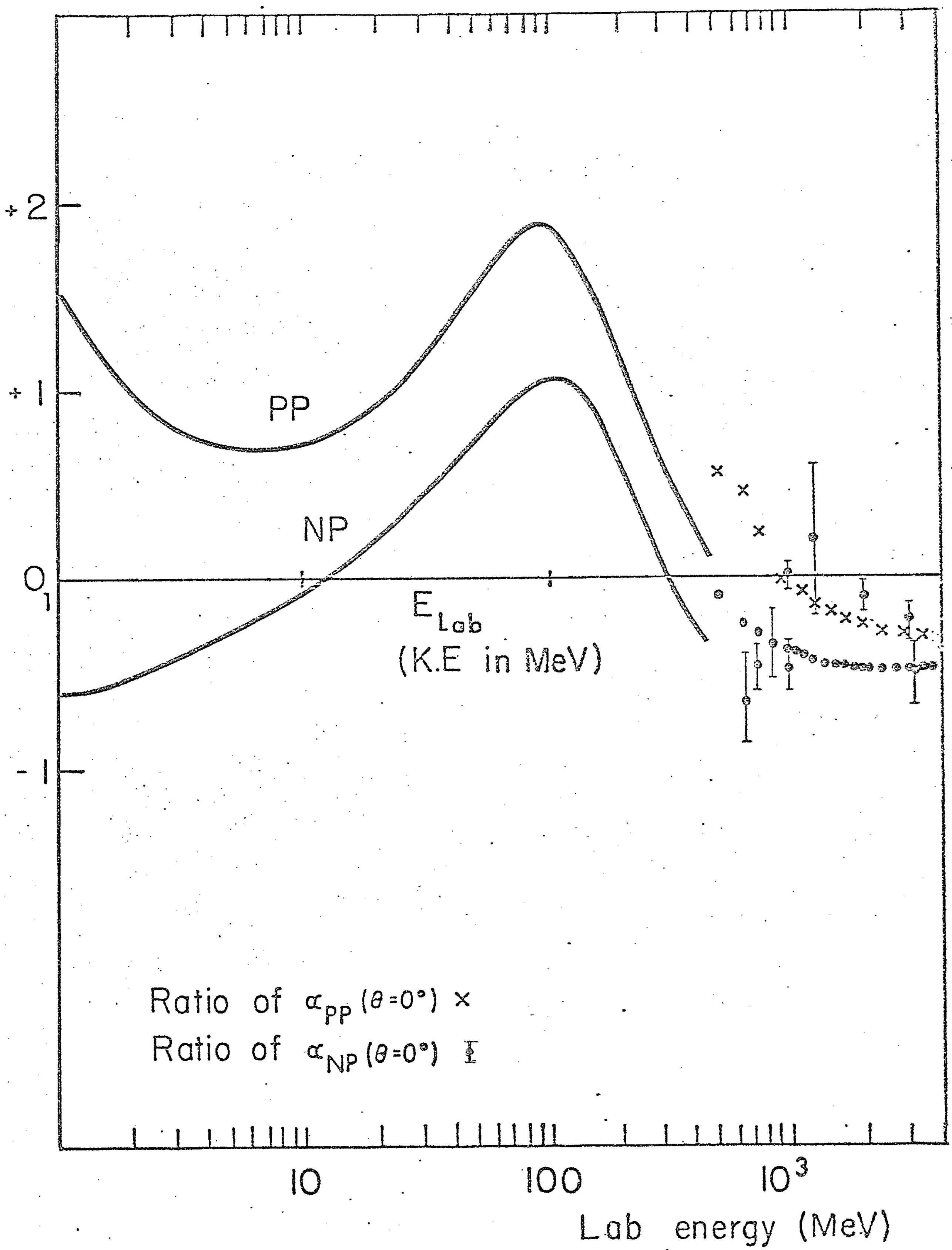

+2
+1
0
-1
PP
NP
E_{Lab}
(K.E in MeV)
Ratio of α_{PP} (θ=0°) ×
Ratio of α_{NP} (θ=0°)
10
100
10^3
Lab energy (MeV)

Figure 8

Angular dependence of the ratio of real and imaginary parts of the nucleon-nucleon scattering amplitudes at 570 MeV and 750MeV in the forward and backward directions.

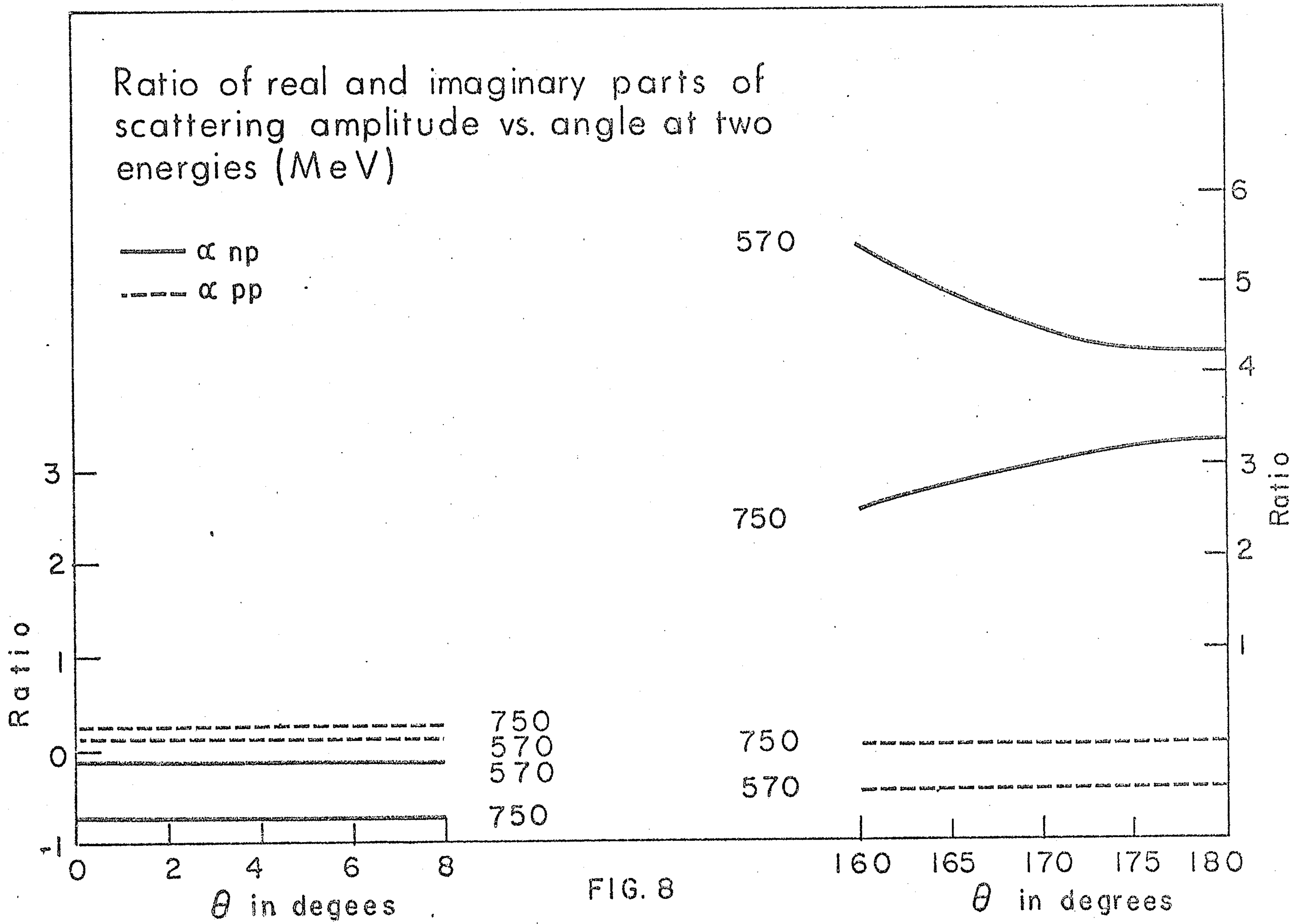

Ratio of real and imaginary parts of scattering amplitude vs. angle at two energies (MeV)
α np
α pp
570
750
750
570
570
750
750
570
Ratio
Ratio
θ in degees
θ in degrees
0
2
4
6
8
160
165
170
175
180
-1
0
1
2
3
1
2
3
4
5
6

FIG. 8

Figure 9

Real and imaginary parts of the impact parameter amplitudes for the nucleon-nucleon system at 750 MeV (a) in the forward and backward directions, and (b) for singlet and triplet states.

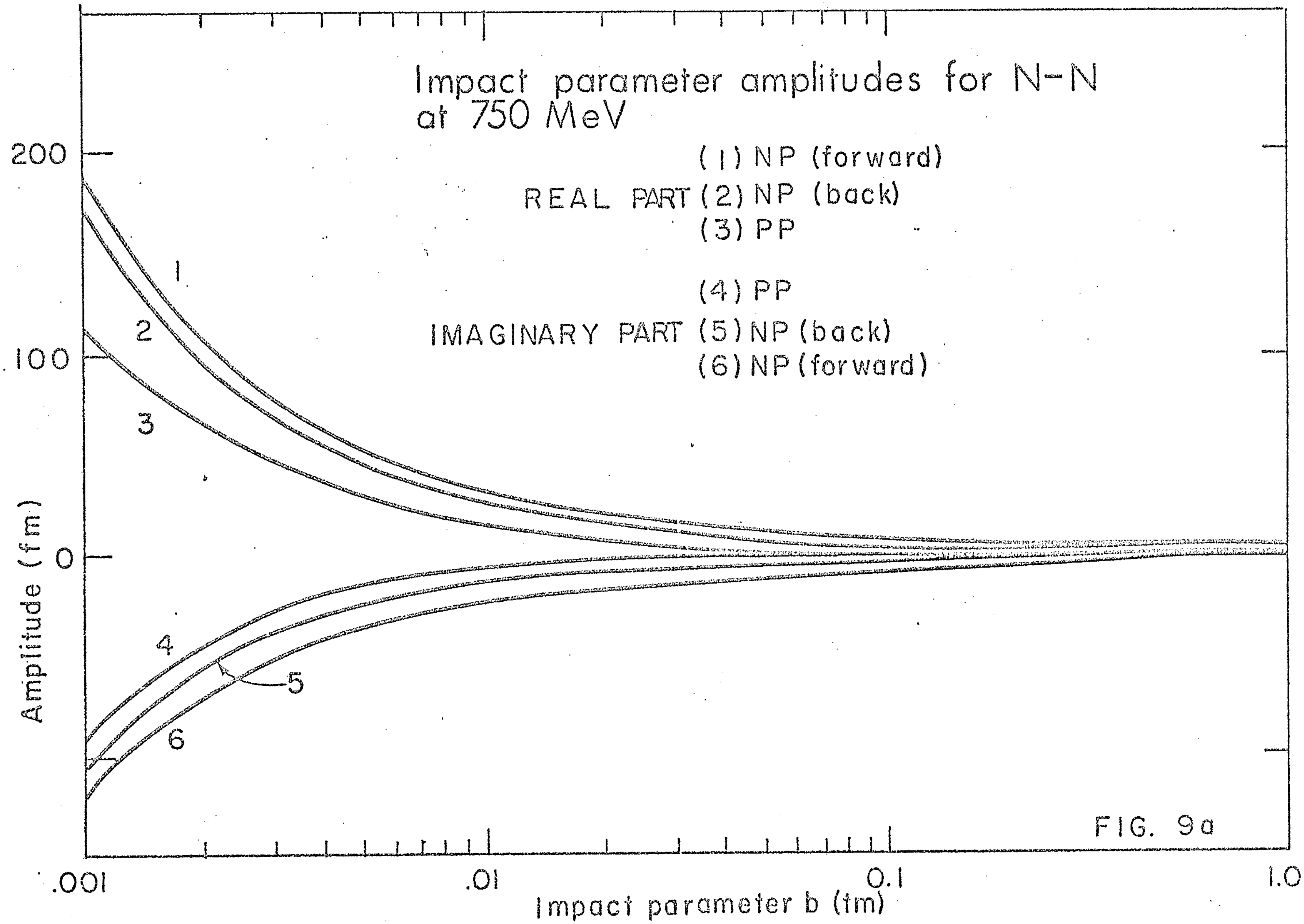

Impact parameter amplitudes for N-N at 750 MeV
REAL PART (1) NP (forward)
(2) NP (back)
(3) PP
IMAGINARY PART (4) PP
(5) NP (back)
(6) NP (forward)
200
100
0
Amplitude (fm)
1
2
3
4
5
6
.001
.01
0.1
1.0
Impact parameter b (fm)

FIG. 9a

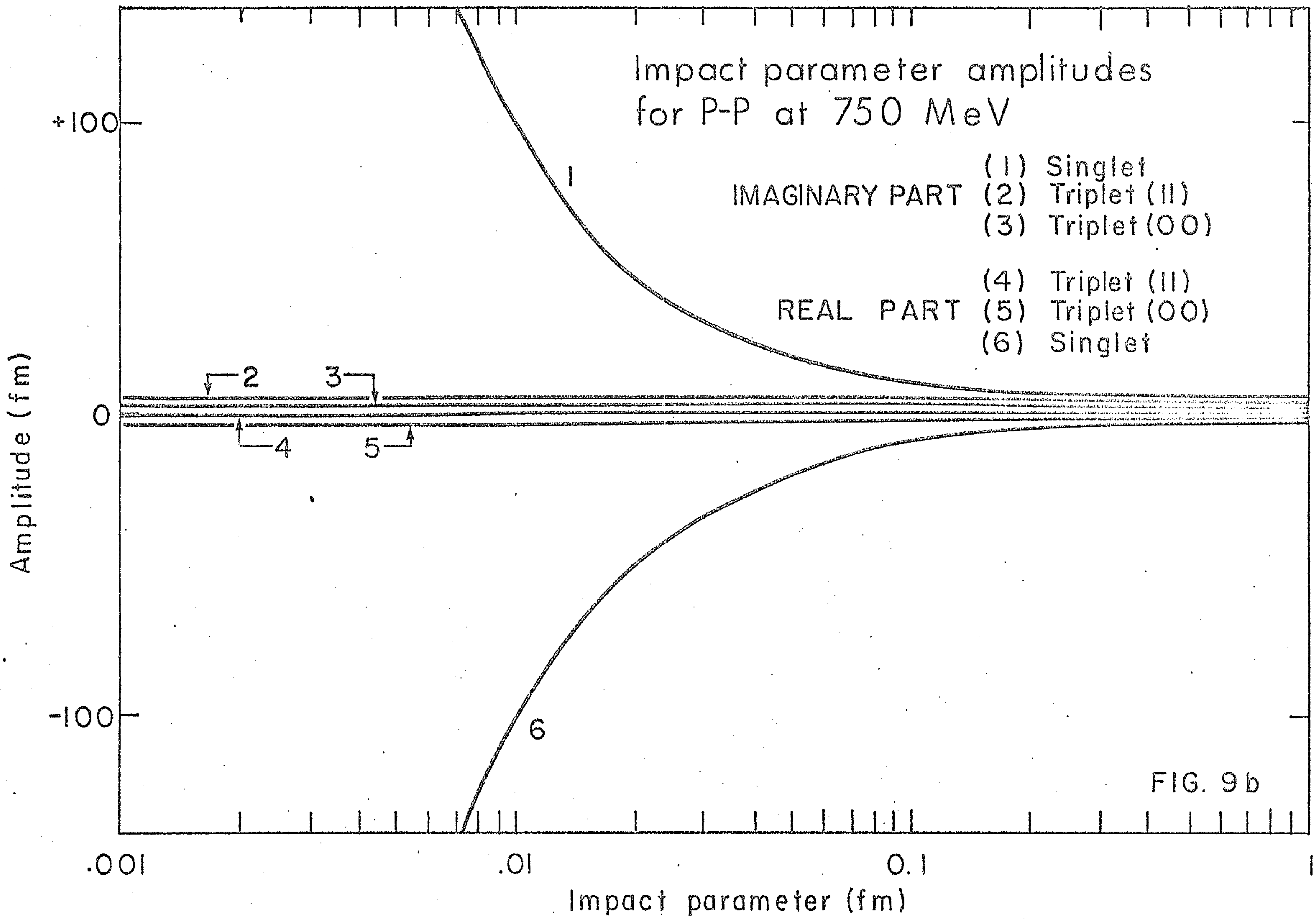

Impact parameter amplitudes
for P-P at 750 MeV
IMAGINARY PART (1) Singlet
(2) Triplet (11)
(3) Triplet (00)
REAL PART (4) Triplet (11)
(5) Triplet (00)
(6) Singlet
+100
0
-100
Amplitude (fm)
.001
.01
0.1
1
Impact parameter (fm)
1
2
3
4
5
6

FIG. 9b

Figure 10

Real and imaginary parts of the impact parameter amplitudes for a) proton-proton scattering b) proton-neutron forward scattering and c) proton-neutron backward scattering at 480 MeV, 570 MeV, and 750 MeV.

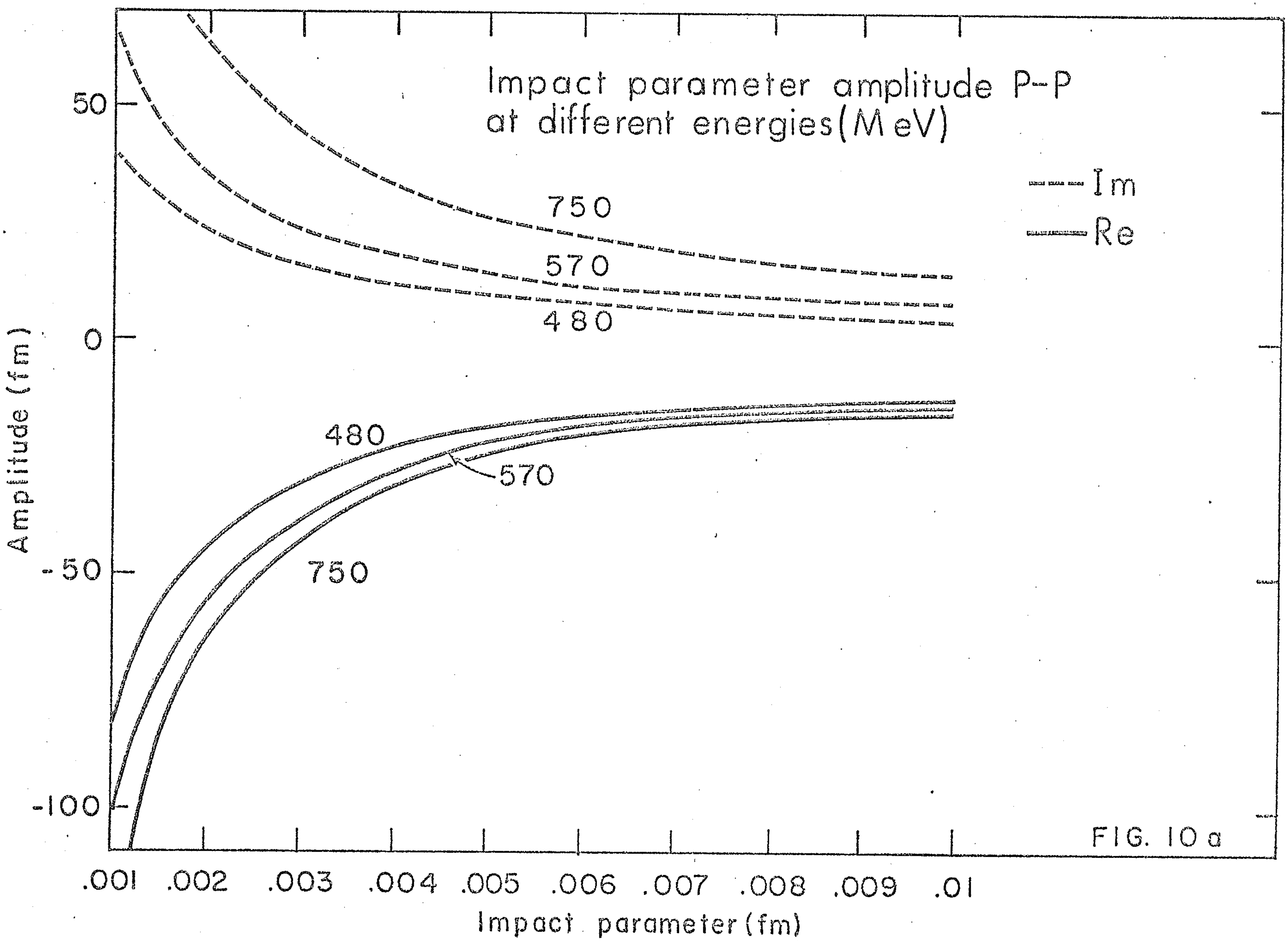

Impact parameter amplitude P-P
at different energies (MeV)
Im
Re
750
570
480
480
570
750
Amplitude (fm)
50
0
-50
-100
.001
.002
.003
.004
.005
.006
.007
.008
.009
.01
Impact parameter (fm)

FIG. 10 a

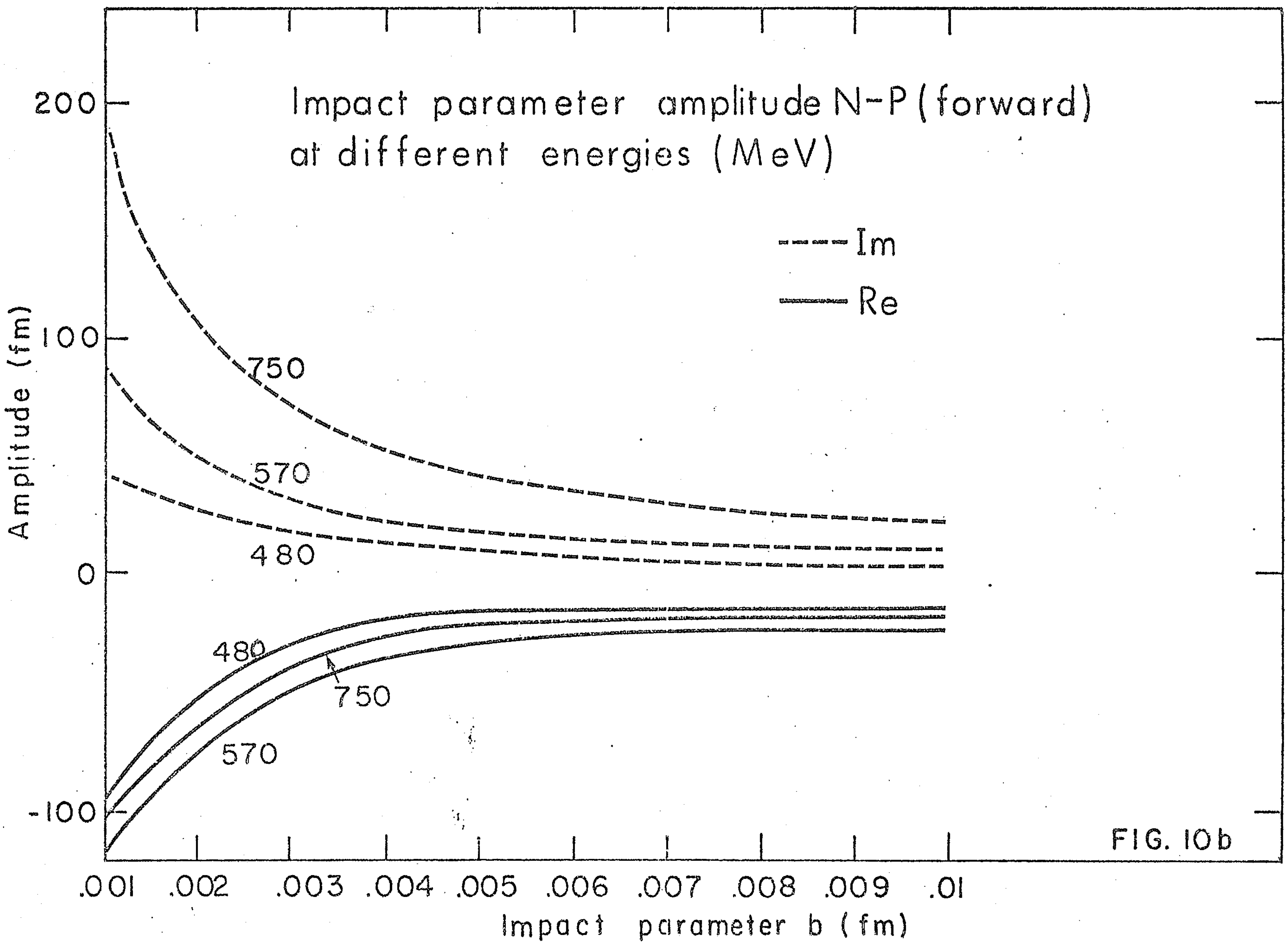

Impact parameter amplitude N-P (forward)
at different energies (MeV)
Im
Re
750
570
480
480
750
570
200
100
0
-100
Amplitude (fm)
.001
.002
.003
.004
.005
.006
.007
.008
.009
.01
Impact parameter b (fm)

FIG. 10b

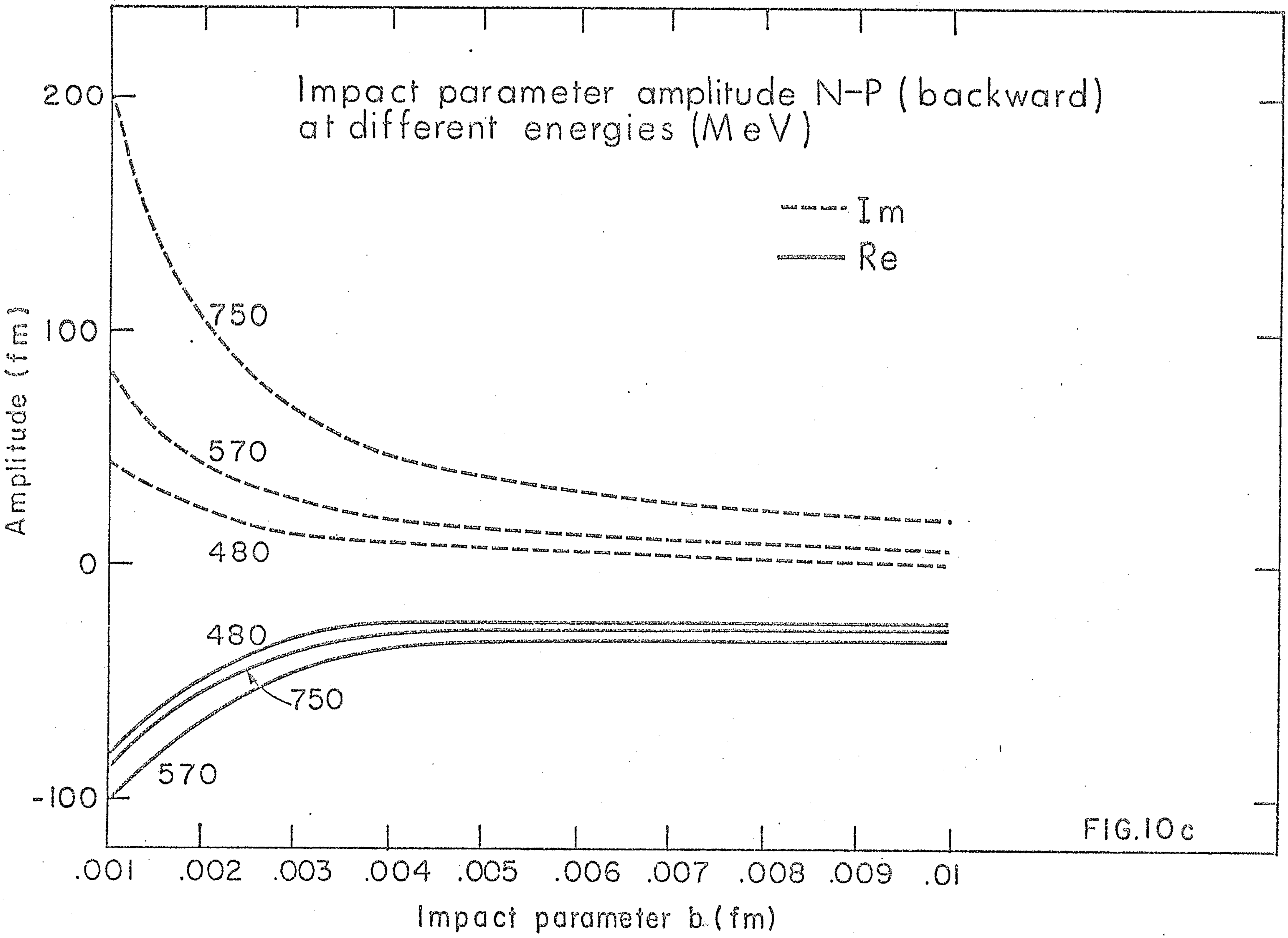
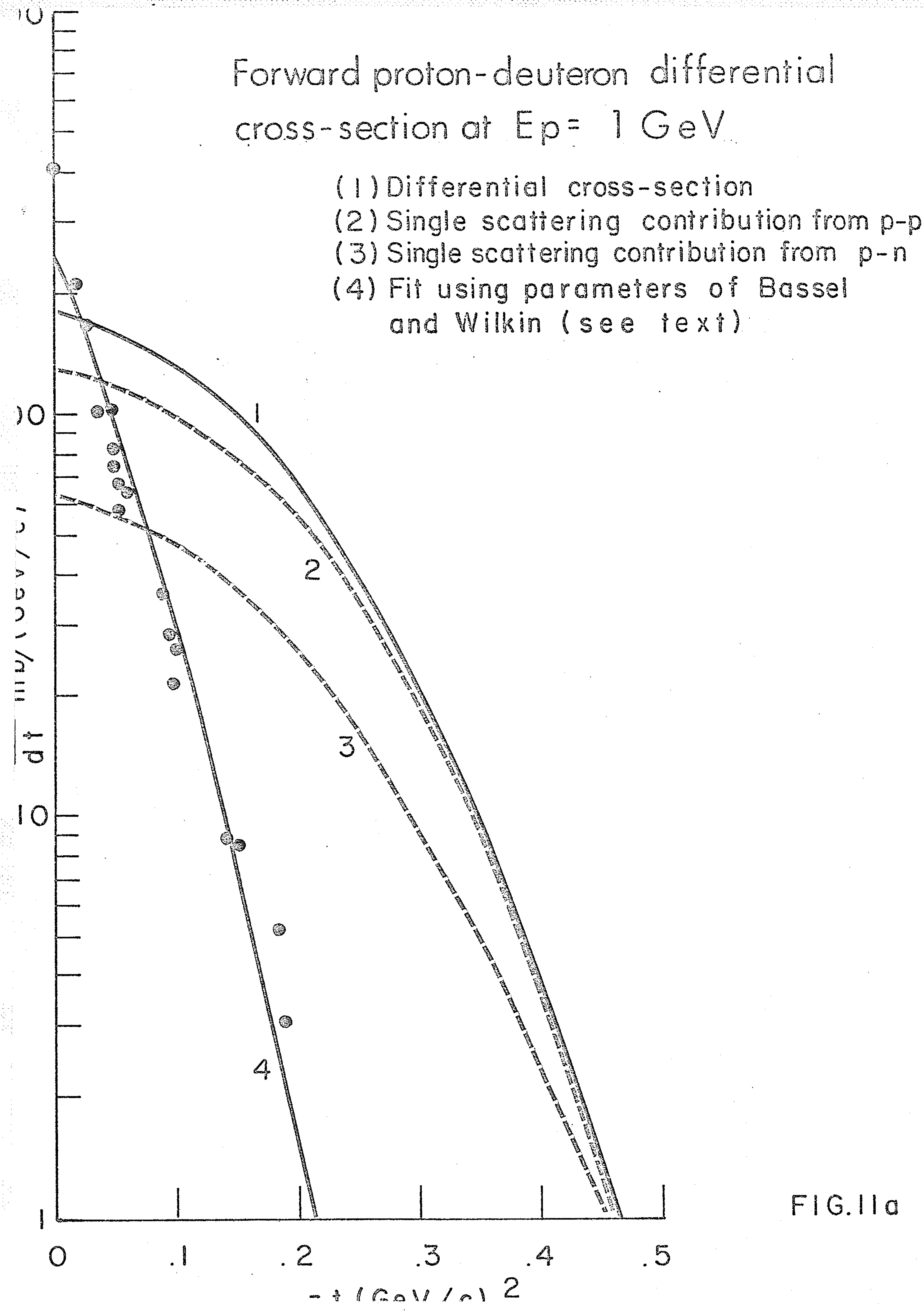

Impact parameter amplitude N-P (backward)
at different energies (MeV)
Im
Re
750
570
480
480
750
570
200
100
0
-100
Amplitude (fm)
.001
.002
.003
.004
.005
.006
.007
.008
.009
.01
Impact parameter b (fm)

FIG. 10c

Figure 11

Forward proton-deuteron differential cross-sections at (a) $E_p = 1$ GeV, (b) $E_p = 1.3$ GeV, and (c) $E_p = 1.5$ GeV. The solid line is the theoretical prediction for proton-deuteron forward scattering and the dashed lines show the contributions of the single scattering terms $S^2(\Delta/2)\left.\frac{d\sigma}{dt}\right)_{pp}$ and $S^2(\Delta/2)\left.\frac{d\sigma}{dt}\right)_{pn}$. The experimental points at 1 GeV are those of Bennett et al[37], and the parameters used to fit the experimental points are given by Bassel and Wilkin[39].



Forward proton-deuteron differential cross-section at Ep= 1 GeV
(1) Differential cross-section
(2) Single scattering contribution from p-p
(3) Single scattering contribution from p-n
(4) Fit using parameters of Bassel and Wilkin (see text)
1
2
3
4
10
1
0
.1
.2
.3
.4
.5

FIG. 11a

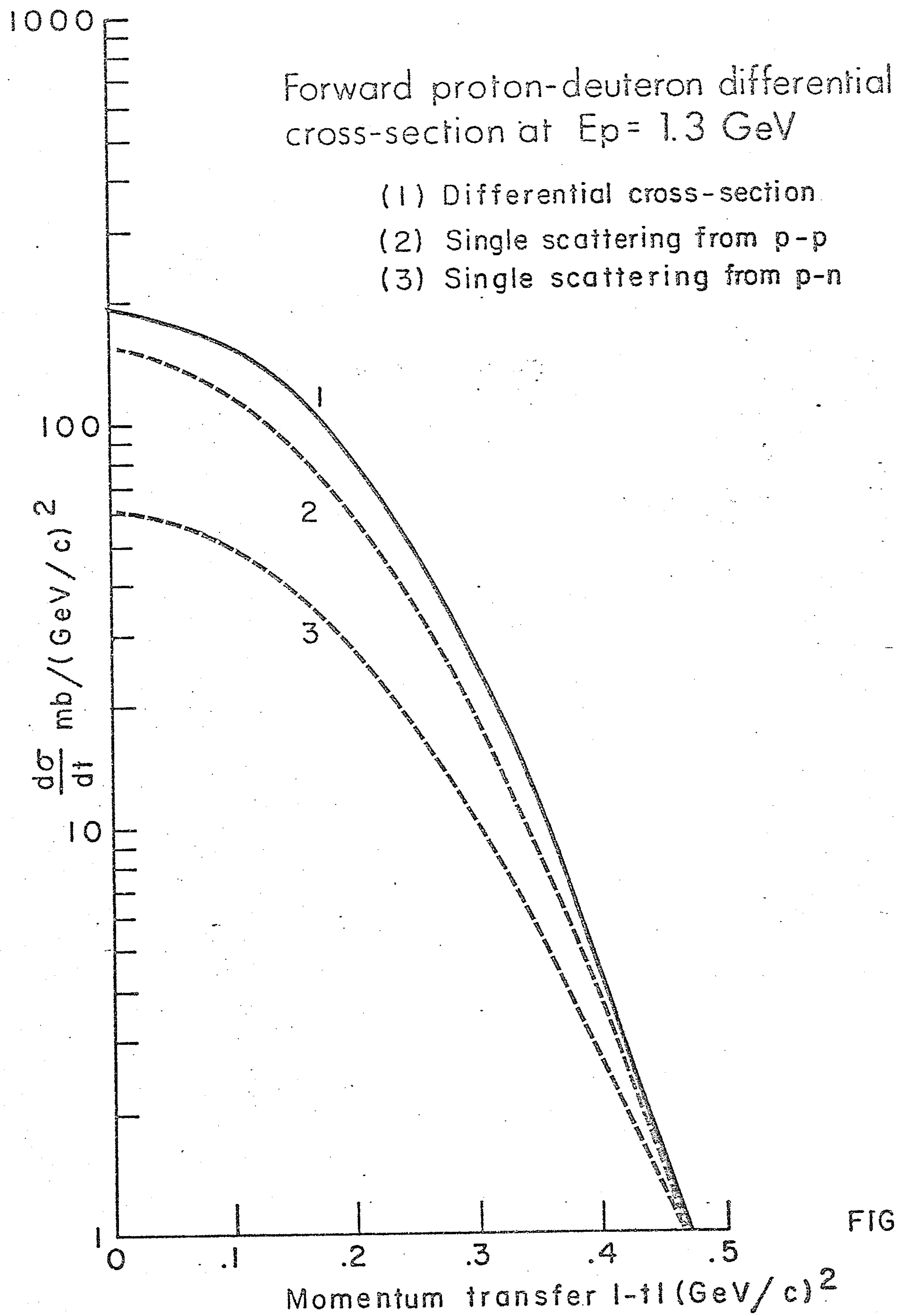

Forward proton-deuteron differential cross-section at Ep = 1.3 GeV
(1) Differential cross-section
(2) Single scattering from p-p
(3) Single scattering from p-n
1000
100
10
1
$\frac{d\sigma}{dt}$ mb/(GeV/c)^2
1
2
3
0
.1
.2
.3
.4
.5
Momentum transfer |-t| (GeV/c)^2

FIG. 11b

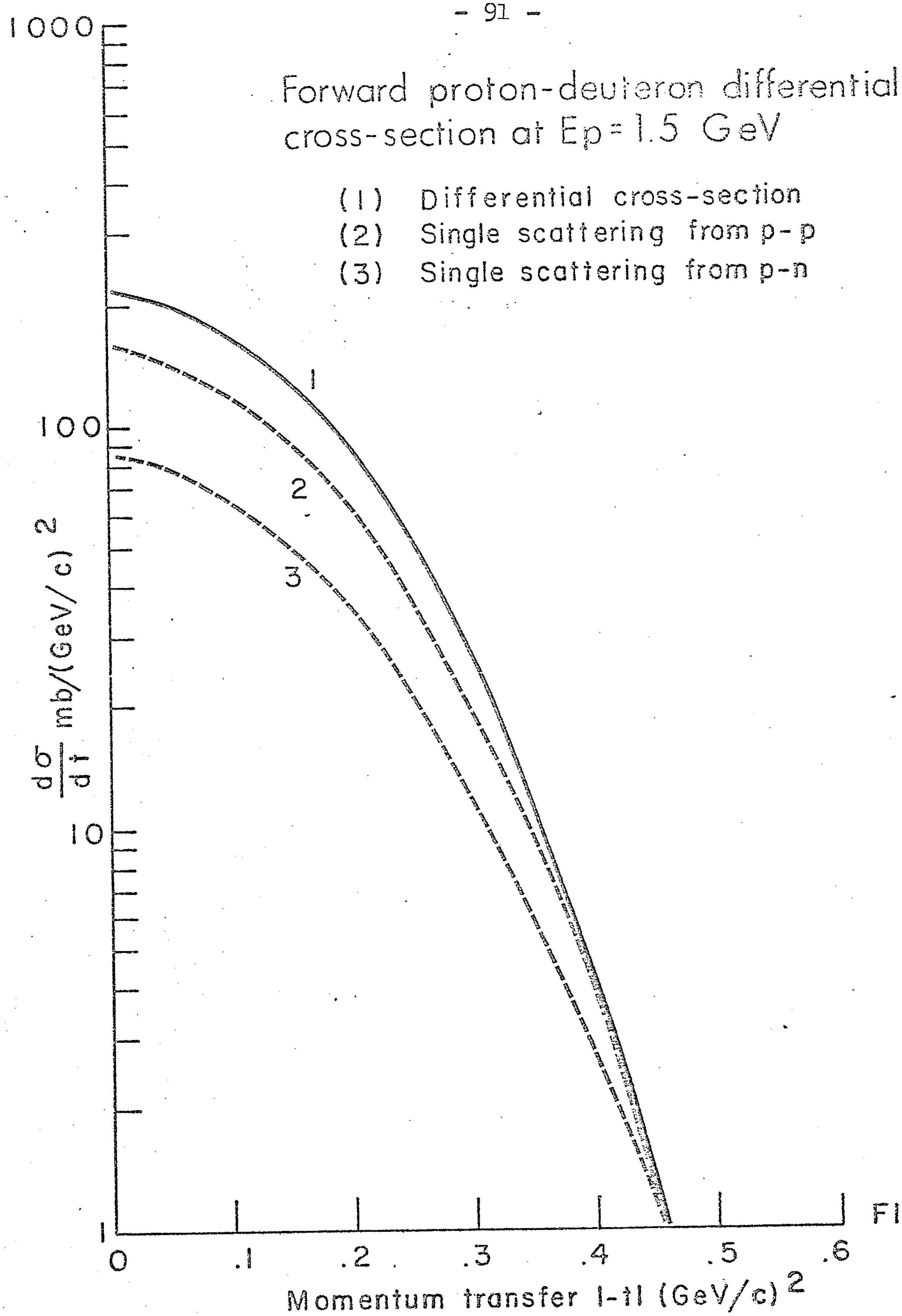
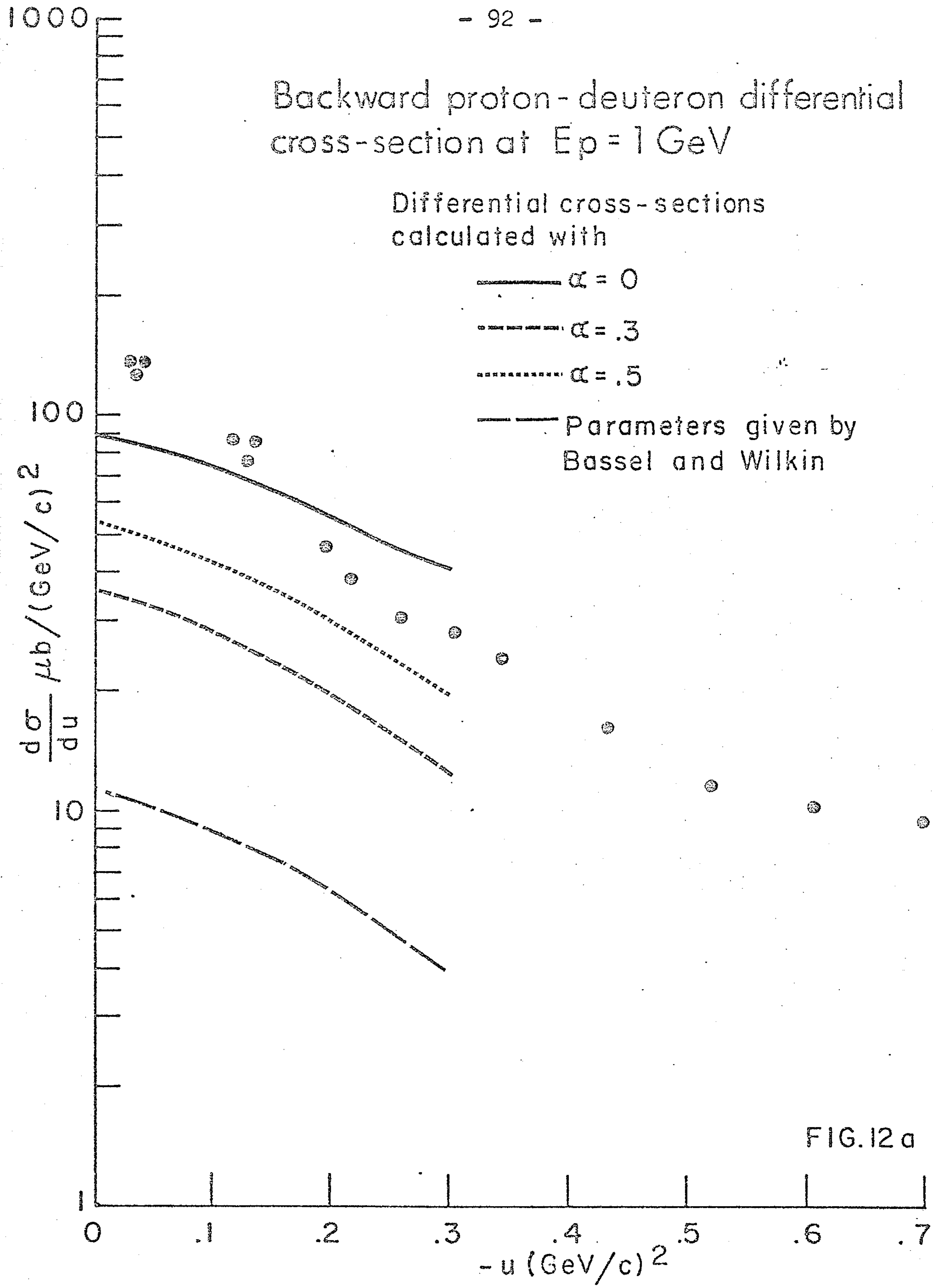

Forward proton-deuteron differential cross-section at Ep=1.5 GeV
(1) Differential cross-section
(2) Single scattering from p-p
(3) Single scattering from p-n
1000
100
10
1
1
2
3
dσ/dt mb/(GeV/c)^2
0
.1
.2
.3
.4
.5
.6
Momentum transfer l-tl (GeV/c)^2

FIG.11c



Backward proton-deuteron differential cross-section at Ep = 1 GeV
Differential cross-sections calculated with
α = 0
α = .3
α = .5
Parameters given by Bassel and Wilkin
1000
100
10
1
$\frac{d\sigma}{du}$ μb/(GeV/c)²
0
.1
.2
.3
.4
.5
.6
.7
-u (GeV/c)²

FIG. 12a

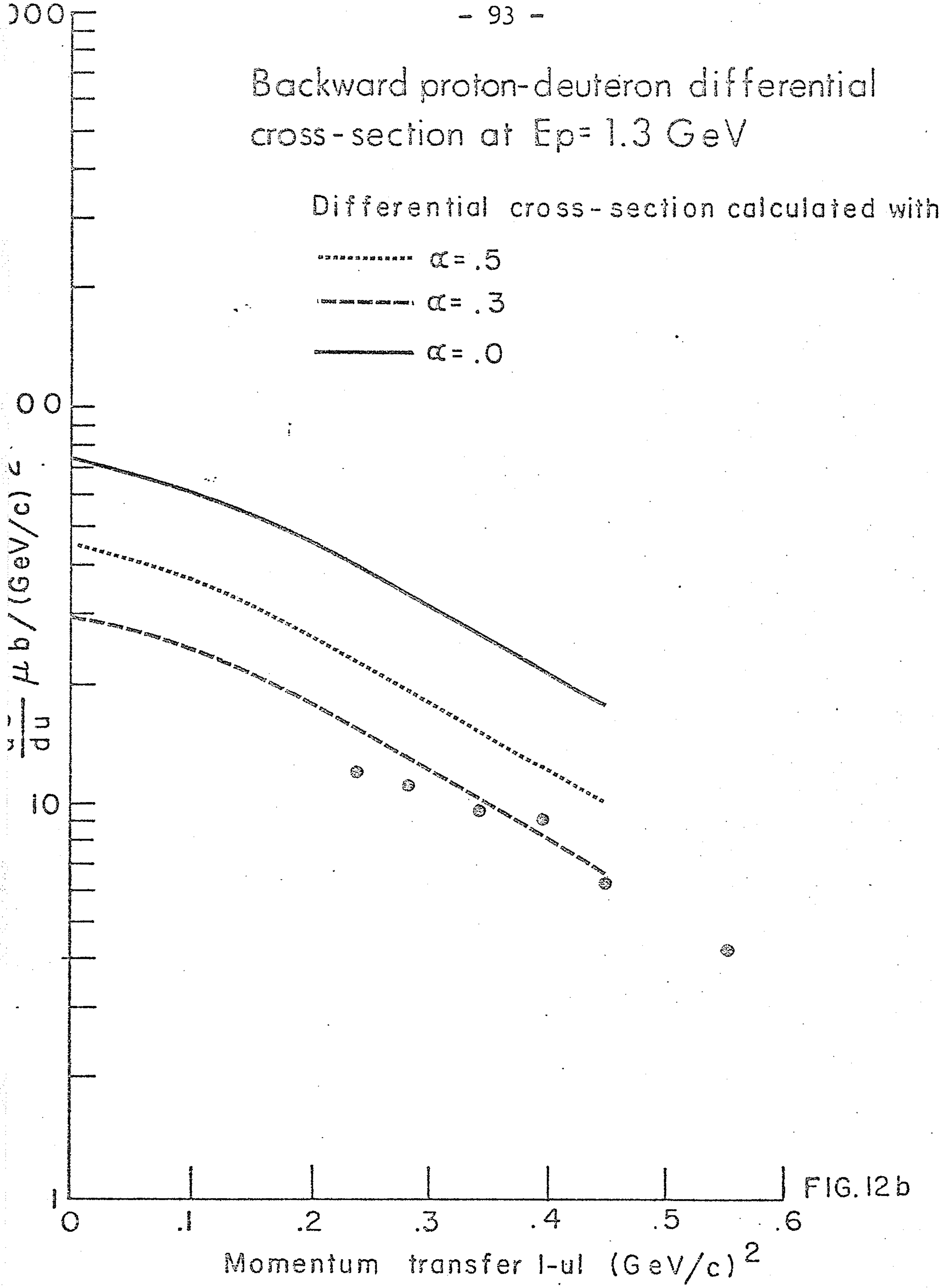
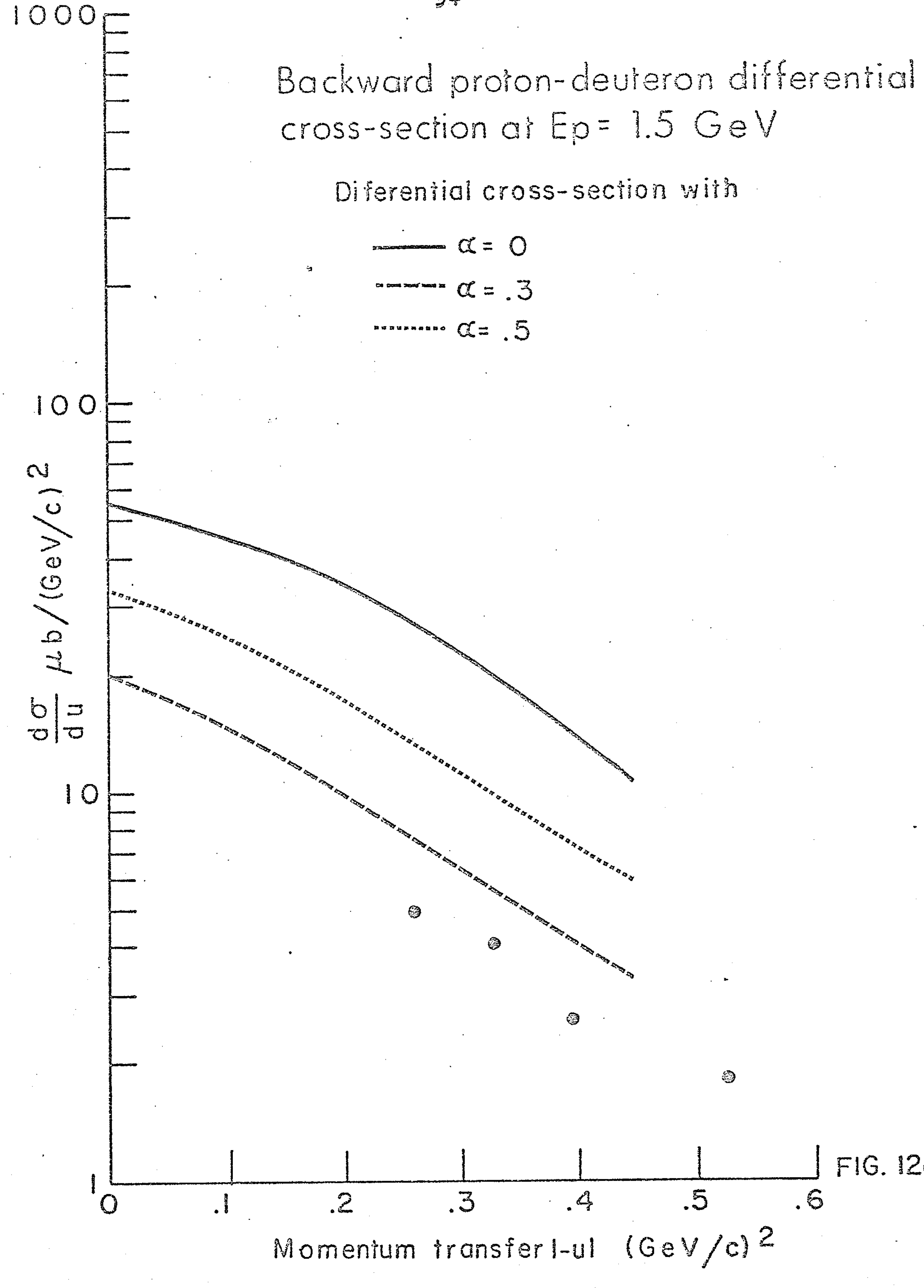

Backward proton-deuteron differential
cross-section at Ep= 1.3 GeV
Differential cross-section calculated with
α= .5
α= .3
α= .0
dσ/du μb/(GeV/c)²
00
10
1
0
.1
.2
.3
.4
.5
.6
Momentum transfer |-u| (GeV/c)²

FIG. 12b



Backward proton-deuteron differential cross-section at Ep= 1.5 GeV
Diferential cross-section with
α = 0
α = .3
α = .5
1000
100
10
1
$\frac{d\sigma}{du}$ μb/(GeV/c)^2
0
.1
.2
.3
.4
.5
.6
Momentum transfer I-uI (GeV/c)^2

FIG. 12c

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APPENDIX A

EVALUATION OF INTEGRALS I AND I'

We evaluate the integrals

$$I = \frac{1}{4\pi}\int d\Omega_{k'} J_0\left(2k_f b' \sin\frac{\Theta}{2}\right) J_0\left(2k' b'' \sin\beta/2\right) \qquad \text{A.1a}$$

and

$$I' = \frac{1}{4\pi}\int d\Omega_{k'} J_0\left(2k_f b' \cos\frac{\Theta}{2}\right) J_0\left(2k' b'' \sin\beta/2\right) \qquad \text{A.1b}$$

where $\cos\Theta = \sin(\theta)\sin\beta\cos(\phi - \alpha) + \cos\theta\cos\beta$

Using the relation

$$\int_0^{\pi} J_0\left\{\left(Z^2 + z^2 - 2zZ\cos\phi\right)^{1/2}\right\} d\phi = \pi J_0(z) J_0(Z) \qquad \text{A.2}$$

we can evaluate the integral

$$\int_0^{2\pi} J_0\left(2k_f b' \sin\frac{\Theta}{2}\right) d\alpha = 2\pi J_0\left(2k_f b' \sin\theta/2 \cos\beta/2\right) \times$$

$$\times J_0\left(2k_f b' \sin\beta/2 \cos\theta/2\right) \qquad \text{A.3}$$

Substituting A.3 into A.1a we obtain

$$I(k_f, k') = 1/2\int_{-1}^{1} d(\cos\beta) J_0\left(2k' b'' \sin\beta/2\right) J_0\left(2k_f b' \sin\theta/2 \cos\beta/2\right) \times$$

$$\times J_0\left(2k_f b' \sin\beta/2 \cos\theta/2\right) \qquad \text{A.4}$$

which may be written as

$$I(\underline{k}_f, \underline{k}') = \frac{1}{\pi}\int_0^{\pi} d\phi \, \frac{J_1(2\beta)}{\beta} \qquad \text{A.5}$$

where $\beta^2 = k_f^2 b'^2 + k'^2 b''^2 - 2k_f k' b' b'' \cos\theta/2 \cdot \cos\phi$

Now if we define

$$G(\underline{k}_f, \underline{k}', b', b'', b) = \frac{2k_f^2}{\pi}\int_0^{\pi} d\phi \int_0^{\pi} d\theta \sin\theta/_2 \cos\theta/_2 \times$$

$$\times J_0(2k_f b \sin\theta/_2) \, \frac{J_1(2\beta)}{\beta} \qquad \text{A.6}$$

then using

$$\int_0^{\infty} b\,db\,J_0(2k_f b \sin\theta/_2) J_0(2k_f b \sin\theta'/_2) = \frac{\delta\,(\sin\theta/2 - \sin\theta'/2)}{4k_f^2 \sin\theta/2} \qquad \text{A.7}$$

we obtain

$$I(\underline{k}_f, \underline{k}') = \int_0^{\infty} b\,db\,G(\underline{k}_f, \underline{k}', b', b'', b) \, J_0(2k_f b \sin\theta/_2) \qquad \text{A.8}$$

Similarly the integration of $I'(\underline{k}_f, \underline{k}')$ follows and we obtain

$$I'(\underline{k}_f, \underline{k}') = \int_0^{\infty} b\,db\,G'(\underline{k}_f, \underline{k}', b', b'', b) \, J_0(2k_f b \cos\theta/_2) \qquad \text{A.9}$$

where

$$G'(k_f, B', b', b'', b) = \frac{2k_f^2}{\pi} \int_0^{\pi} d\phi \int_0^{\pi} d\theta \sin\theta/_2 \cos\theta/_2 \, J_0(2k_f b \cos\theta/_2) \times \\ \times \frac{J_1(2\beta'(\theta,\phi))}{\beta'(\theta,\phi)} \qquad \text{A.10}$$

and $\beta'^2 = k_f^2 \, b'^2 + k'^2 b''^2 - 2k_f k' b' b'' \sin(\theta/2) \cos\phi$

APPENDIX B

LINEARIZATION OF THE GREEN'S FUNCTION

We consider the scattering of two high-energy particles. Let us denote by $\underline{k}$ the relative two-body centre of mass momentum of the system before the collision, and by μ the reduced mass. The non-interacting two-particle Hamiltonian is

$$h_0 = \frac{\underline{p}^2}{2\mu} \qquad \text{B.1}$$

where $\underline{p}$ is the momentum operator in the centre of mass frame. This may be rewritten as

$$h_0 = \frac{1}{2\mu} p^2 = \frac{1}{2\mu} (\underline{p} - \underline{k}) \cdot (\underline{p} + \underline{k}) + \frac{1}{2\mu} \underline{k}^2 \qquad \text{B.2}$$

If we consider scattering in the forward direction the incoming particle passes largely undeviated through the target so that all the important contributing intermediate states have momentum values near $\underline{k}$ so that $\underline{p} + \underline{k} \approx 2\underline{k}$. Hence we may approximate the Hamiltonian by one which is linear in $\underline{p}$:

$$h_0 = \frac{\underline{k}}{\mu} \cdot (\underline{p} - \underline{k}) + \frac{1}{2\mu} k^2 \qquad \text{B.3}$$

If we let V represent the operator which describes the interaction between the target and the incoming particle we may write the total eikonal Hamiltonian as

$$h = h_o + V \qquad \text{B.4}$$

and the resolvent for h is given by

$$g(z) = \frac{1}{h - z} \qquad \text{B.5}$$

where z is a complex number with Im $z \neq 0$, and in the limit $\mathrm{Im}(z) \rightarrow 0$, g(z) becomes the total Green's function with energy Re(z).

Hence, if we define $\underline{p} = \underline{k} + \underline{q}$ where $\underline{q}$ is small, then

$$g = \frac{1}{\underline{k}\cdot\underline{q} + i\varepsilon} \qquad \text{B.6}$$

Similarly the equivalent backward linearized Green's function is given by

$$g = \frac{1}{i\varepsilon - \underline{k}\cdot\underline{q}} \qquad \text{B.7}$$

APPENDIX C

APPROXIMATION OF THE BOUND-STATE WAVE FUNCTION

Let us consider first the single scattering terms in the forward and backward three-body amplitudes (eq. 3.14). Both of the single scattering terms are exact in the forward direction, so we shall deal first with the single scattering terms in the backward scattering amplitude. Consider the term from eq. 3.12

$$\int d^3r\, d^3\rho\, d^3q\, d^3p\, d^3p'\, \phi_B^*(\underline{p})\, \phi_B(\underline{p}')\, \exp\{-i\underline{p}\cdot(-\underline{\rho}+\tfrac{1}{2}\underline{r})+i\underline{p}'\cdot\underline{r}\}\times$$
$$\times\exp\{i\underline{\rho}\cdot(\tfrac{1}{2}\underline{k}_f+\underline{k}_i)+i\tfrac{3}{4}\underline{k}_f\cdot\underline{r}+i\underline{q}\cdot(\underline{\rho}+\tfrac{1}{2}\underline{r})\}\, f_{12}^{bac}(\underline{q}) \qquad \text{C.1}$$

If we replace $\underline{p}$ by $\underline{p}+\frac{1}{2}\underline{k}_f$ expression C.1 becomes

$$\int d^3r\, d^3\rho\, d^3p\, d^3p'\, d^2q\, \phi_B^*(\underline{p}+\tfrac{1}{2}\underline{k}_f)\, \phi_B(\underline{p}')\, f_{12}^{bac}(\underline{q})\times$$
$$\times\exp\{i\underline{\rho}\cdot(\underline{p}+\underline{k}_f+\underline{k}_i)+i\underline{r}\cdot(-\tfrac{1}{2}\underline{p}+\underline{p}'+\tfrac{1}{2}\underline{k}_f)+$$
$$+\,i\underline{q}\cdot(\underline{\rho}_\perp+\tfrac{1}{2}\underline{r}_\perp)\} \qquad \text{C.2}$$

Performing some of the integrations we obtain

$$\int d^3p\, d^3p'\, \phi_B^*(\underline{p}+\tfrac{1}{2}\underline{k}_f)\, \phi_B(\underline{p}')\, f_{12}^{bac}(-\underline{\pi}-\underline{p}_\perp)\times$$
$$\times\,\delta(p_\parallel+\pi_\parallel)\,\delta(p'_\parallel+k_{f\parallel}+\tfrac{1}{2}k_{i\parallel})\,\delta^2(\underline{p}'_\perp-\underline{p}_\perp-\tfrac{1}{2}\underline{k}_i) \qquad \text{C.3}$$

An approximation is now made of the arguments of the delta functions. We assume

$$p_{\parallel} = -\pi_{\parallel} \approx 0$$

$$p'_{\parallel} = -k_{f\parallel} - \tfrac{1}{2}k_{i\parallel} \approx \tfrac{1}{2}k \qquad \text{C.4}$$

Expression C.3 then becomes

$$\int d^2 p_\perp \phi_B^*(p_\perp + \tfrac{1}{2}k_f)\phi_B(p_\perp + \tfrac{1}{2}k_i) f_{12}^{bac}(-\pi) \qquad \text{C.5}$$

which becomes, using the Convolution theorem,

$$S(\Delta/2)\, f_{12}^{bac}(-\pi) \qquad \text{C.6}$$

Similarly the other single backward scattering term may be evaluated.

Now let us consider the double scattering terms. Consider the term taken from eq.3.12

$$\int d^3\rho\, d^3r\, d^3p\, d^3p'\, d^2q\, d^2q'\, \phi_B^*(p)\phi_B(p') f_{12}^{bac}(q) f_{31}^{for}(q') \times$$

$$\times \exp\{-ip\cdot(-\rho + \tfrac{1}{2}r) + ip'\cdot r - ik_f\cdot(-\tfrac{1}{2}\rho - \tfrac{3}{4}r)\} \times \qquad \text{C.7}$$

$$\times \exp\{ik_i\cdot\rho + iq\cdot(\rho_\perp + \tfrac{1}{2}r_\perp) + iq'\cdot(\rho_\perp - \tfrac{1}{2}r_\perp)\}$$

If we replace p by $p + \tfrac{1}{2}k_f$ and perform some of the integrations this becomes

$$\int d^3p\, d^3p'\, d^2q\, d^2q'\, \phi_B^*(\underline{p}+\tfrac{1}{2}\underline{k}_f)\, \phi_B(\underline{p}') \times$$

$$\times\, f_{12}^{bac}(q)\, f_{31}^{for}(q')\, \delta(p_{\parallel}+\pi_{\parallel})\, \delta(p'_{\parallel}+k_{f\parallel}+\tfrac{1}{2}k_{i\parallel}) \times$$

$$\times\, \delta^2(\underline{p}_\perp+\underline{\pi}_\perp+\underline{q}+\underline{q}')\, \delta^2\left(\underline{p}'_\perp-\frac{\underline{p}_\perp}{2}+\frac{\underline{k}_f}{2}+\frac{\underline{q}-\underline{q}'}{2}\right) \qquad \text{C.8}$$

Then making the approximation from the delta functions

$$p_{\parallel} = -\pi_{\parallel} \approx 0$$

$$p'_{\parallel} = -k_{f\parallel} - \tfrac{1}{2}k_{i\parallel} \approx \tfrac{1}{2}k \qquad \text{C.9}$$

and performing some of the integrations we obtain

$$f_{12}^{bac}(-\underline{\pi}) \int d^2\underline{p}_\perp\, d^2\underline{p}'_\perp\, \phi_B^*(\underline{p}_\perp+\tfrac{1}{2}\underline{k}_{f\perp})\, \phi_B(\underline{p}'_\perp+\tfrac{1}{2}\underline{k}_i)\, f_{31}^{for}(\underline{p}'_\perp-\underline{p}_\perp)$$

which finally becomes

$$f_{12}^{bac}(-\underline{\pi}) \int d^2q\, S(\underline{q}\cdot\underline{\Delta}/2)\, f_{31}^{for}(\underline{q})$$

The other second order terms may be similarly approximated.

APPENDIX D

PARTIAL WAVE ANALYSIS

The most general form of the nucleon-nucleon amplitude is given by

$$M = a + c(\underline{\sigma}^1 + \underline{\sigma}^2)\cdot\hat{n} + m(\underline{\sigma}^1\cdot\hat{n})(\underline{\sigma}^2\cdot\hat{n}) + \\ + g\{(\underline{\sigma}^1\cdot\hat{P})(\underline{\sigma}^2\cdot\hat{P}) + (\underline{\sigma}^1\cdot\hat{K})(\underline{\sigma}^2\cdot\hat{K})\} + \\ + h\{(\underline{\sigma}^1\cdot\hat{P})(\underline{\sigma}^2\cdot\hat{P}) - (\underline{\sigma}^1\cdot\hat{K})(\underline{\sigma}^2\cdot\hat{K})\} \qquad \text{D.1}$$

where $\underline{\sigma}^1$ and $\underline{\sigma}^2$ are Pauli spin matrices for particles 1 and 2. The unit vectors $\hat{n}$, $\hat{P}$, $\hat{K}$ are defined in the centre of mass system by initial and final momenta $\underline{k}_i$ and $\underline{k}_f$ as

$$\hat{P} = \frac{\underline{k}_i + \underline{k}_f}{|\underline{k}_i + \underline{k}_f|} \qquad \hat{n} = \frac{\underline{k}_i \times \underline{k}_f}{|\underline{k}_i \times \underline{k}_f|} \qquad \hat{K} = \frac{\underline{k}_f - \underline{k}_i}{|\underline{k}_f - \underline{k}_i|} \qquad \text{D.2}$$

The spin independent part of the amplitude $a(E,\theta)$ is given by

$$a = \frac{1}{4}(2M_{11} + M_{00} + M_{ss}) \qquad \text{D.3}$$

Where the subscripts denote the two particle spin projections in the initial and final states (s denotes singlet state).

For proton-proton scattering

$$M_{11} = \frac{2}{2k} \sum_{L\,odd}^{\infty} M_{11}^{L} P_L$$

where

$$M_{11}^{2} = (L+2)\alpha_L^{L+1} + (2L+1)\alpha_L^{L} + (L-1)\alpha_L^{L-1},$$

$$M_{00} = \frac{2}{k} \sum_{L\,odd}^{\infty} M_{00}^{L} P_L$$

where D.4

$$M_{00}^{2} = (L+1)\beta_L^{L+1} + L\beta_L^{L-1}$$

and

$$M_{SS} = \frac{2}{k} \sum_{L\,even} M_{SS}^{L} P_L$$

where

$$M_{SS}^{L} = (2L+1)\gamma_L$$

The expressions for proton-neutron scattering are similar, except that $2 \sum_{odd,\,even\,L}$ is replaced by $\sum_{L=0}^{\infty}$

Here

$$\alpha_J^J = \sin\delta_3^J e^{i\delta_3^J}$$

$$\alpha_{J-1}^J = \alpha^J\cos^2\varepsilon_J + \beta^J\sin^2\varepsilon_J - \left(\frac{J}{J+1}\right)^{1/2}\tfrac{1}{2}(\alpha^J-\beta^J)\sin 2\varepsilon_J$$

$$\alpha_{J+1}^J = \alpha^J\sin^2\varepsilon_J + \beta^J\cos^2\varepsilon_J - \left(\frac{J+1}{J}\right)^{1/2}\tfrac{1}{2}(\alpha^J-\beta^J)\sin 2\varepsilon_J$$

$$\beta^J_{J-1} = \alpha^J \cos^2\varepsilon_J + \beta^J \sin^2\varepsilon_J + \left(\frac{J+1}{J}\right)^{1/2} \tfrac{1}{2}(\alpha^J - \beta^J)\sin 2\varepsilon_J$$

$$\beta^J_{J+1} = \alpha^J \sin^2\varepsilon_J + \beta^J \cos^2\varepsilon_J + \left(\frac{J}{J+1}\right)^{1/2} \tfrac{1}{2}(\alpha^J - \beta^J)\sin 2\varepsilon_J$$

$$\alpha^J = \sin\delta_1^J \, e^{i\delta_1^J}$$

$$\beta^J = \sin\delta_2^J \, e^{2i\delta_2^J}$$

$$\gamma^J = \sin\delta_0^J \, e^{i\delta_0^J}$$

where δ denotes the phase-shift.

The impact parameter representation is obtained by replacing

$$P_L(\cos\theta) = P_L(1-2y^2) = \int_0^\infty d\beta \, J_0(\beta y) J_{2L+1}(\beta)$$

in D.4. For the proton-proton system we then get

$$M_{ss} = \int_0^\infty b\,db \left\{ J_0(b\sqrt{-t}) + J_0(b\sqrt{-u}) \right\} T_{ss}$$

$$M_{11} = \int_0^\infty b\,db \left\{ J_0(b\sqrt{-t}) - J_0(b\sqrt{-u}) \right\} T_{11} \qquad \text{D.5}$$

$$M_{00} = \int_0^\infty b\,db \left\{ J_0(b\sqrt{-t}) - J_0(b\sqrt{-u}) \right\} T_{00}$$

where

$$T^f_{ss} = \sum_{L\,\text{even}} M^L_{ss} \frac{J_{2L+1}(2kb)}{b} = T^b_{ss}$$

$$T^f_{11} = \tfrac{1}{2}\sum_{L\,\text{odd}} M^L_{11} \frac{J_{2L+1}(2kb)}{b} = T^b_{11}$$

$$T^f_{00} = \tfrac{1}{2}\sum_{L\,\text{odd}} M^L_{00} \frac{J_{2L+1}(2kb)}{b} = T^b_{00}$$

Substituting eq. D.5 into eq. D.3 we obtain the impact parameter form of a as

$$a(E,\theta) = \int_0^\infty b\,db\, J_0\left(b\sqrt{-t}\right) \Gamma^{a}(s,b)$$

$$\Gamma^{a}(s,b) = \frac{1}{2} \sum_{L} \left(M_{11}^{L} + M_{00}^{L} + M_{ss}^{L} \right) \frac{J_{2L+1}(2kb)}{b}$$

In a similar manner we may obtain the proton-neutron profile functions.