AN INVESTIGATION OF FREQUENCY RESPONSE STABILITY CRITERIA FOR NONLINEAR SYSTEMS

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ABSTRACT

Some of the more recent stability criteria for single-loop, continuous time, feedback systems which contain a single nonlinearity are considered in this thesis. Emphasis is placed on Popov's criterion. It is shown that the criterion can be applied to systems containing a certain class of hysteretic nonlinearities. It is also shown that the criterion may be used to predict zero steady-state error operation of certain systems.

Graphical and analytical applications of Popov's criterion are considered. Several systems for which Popov's criterion verifies Aizerman's conjecture are given. To complete the investigation, criteria which are more restrictive on the nonlinearity than Popov's criterion but less restrictive on the linear subsystem are presented and discussed.

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CHAPTER I

INTRODUCTION

General Considerations

The mathematical model of many physical systems can be represented by the block diagram of Figure 1.

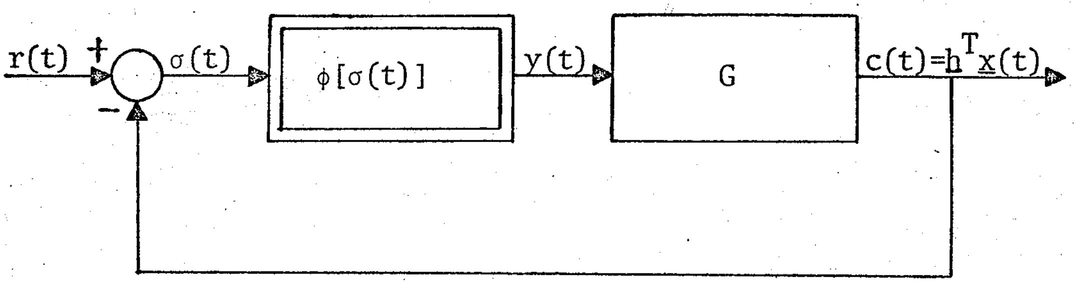

Figure 1 - System S

The block $\phi\,[\sigma\,(t)]$ represents a predominant nonlinearity in the system. G is a linear, time-invariant, causal operator whose Laplace transform is given by G(s).

A lumped parameter system can be described by the following set of state-space equations.

$$\dot{\underline{x}}(t)^{1} = A\underline{x}(t) + \underline{b}y(t). \tag{1a}$$

$$y(t) = \phi\,[\sigma\,(t)]\,. \tag{1b}$$

$$\sigma(t) = r(t) - \underline{h}^{T}\,\underline{x}(t), \qquad \text{for all } t \geq 0. \tag{1c}$$

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[1] Here and throughout a (.) denotes differentiation with respect to time.

In the above equations; $\underline{x}$ the state of the system is a real n-dimensional vector; A is a real constant n x n stable matrix; y(t) the input to the linear system, c(t) the output and r(t) the forcing function are real scalar time functions. The input transformation $\underline{b}$ and the output transformation $\underline{h}$ are real constant n-dimensional vectors.

An alternate and more general description which includes distributed parameter systems is the functional equation,

$$\sigma(t) = r(t) - z(t) - \int_0^t h(t - \tau)\, y(\tau)\, d\tau. \tag{1d}$$

In (1d), z(t) is the zero-input response of G and h(t) is the impulse response of G.

One of the first considerations in the analysis or design of a system is its stability. An unstable system will not only have unsuitable operating characteristics but may present a hazard to the operator. Stability theory of systems is therefore of prime importance. Also as stated in [7][2]; "The current conventional design techniques in control theory are all directly or indirectly derived from stability criteria".

The problem of stability for linear, time-invariant systems is resolved by the necessary and sufficient conditions given by the familiar Routh-Hurwitz or Nyquist criteria. For nonlinear systems such as shown in Figure 1, the stability problem has been more intractable. The first general explicit stability criterion was due to Lyapunov [14] . This criterion known as the direct method of Lyapunov states that if for the autonomous, finite system of Figure 1 there exists a real scalar function, $V(\underline{x})$, of the state vector $\underline{x}$ such that:

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[2] [ ] denotes bibliographical references wherever it appears throughout this study.

(1) $V(\underline{x})$ is positive definite over the whole n-dimensional space and $V(\underline{0}) = 0$,

(2) $V(\underline{x})$ is continuous with continuous first partial derivatives,

(3) $V(\underline{x})$ tends to infinity as $||\underline{x}||$ tends to infinity,

(4) The set of points for which $V(\underline{x}) < d$ is bounded. Here d is an arbitrarily large number,

(5) $dV(\underline{x})/d\underline{x}$ is a negative semidefinite function over the whole n-dimensional space,

(6) The solution of $dV(\underline{x})/d\underline{x} = 0$ does not correspond to any trajectory of the system S except the trivial one, $\underline{x} = \underline{0}$.

Then the origin of system S is globally, asymptotically stable[3].

The criterion is only sufficient and successful application of it depends on the choice of the $V(\underline{x})$ or as it is commonly called, Lyapunov function. Numerous Lyapunov functions have been proposed. One of these is the "quadratic plus integral" form of A.I. Lure [14]. This function is

$$V(\underline{x}) = \underline{x}^T A\underline{x} + \beta \int_0^{\sigma} \phi(\sigma)\, d\sigma. \tag{1e}$$

A is a real, symmetric, positive definite matrix and β is a fixed, real number.

Engineering applications of Lyapunov's direct method have been limited. The proper choice of a Lyapunov function usually requires a great deal of ingenuity and considerable insight into the problem. For engineering purposes criteria which are simple to apply

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[3] For all initial states of S, $\underline{x}$ tends to zero as t tends to infinity.

and to interpret are required. Recently techniques based on the frequency response of G have been developed. These techniques are readily acceptable for engineering purposes since:

(1) Experimental methods to determine the frequency response of physical systems are well-known. A system may thus be specified experimentally rather than analytically.

(2) Many of the techniques of analyzing linear systems are based on frequency response methods. Therefore the well-developed, powerful tools of linear analysis can be applied to the nonlinear system.

As a prelude to the stability criteria investigated in this thesis the following criterion is presented. Consider system S and assume that it satisfies the following assumptions.

(1) The input r(t) is bounded, in $L_2(0, \infty)$[4] and tends to zero as t tends to infinity.

(2) $\phi(\sigma, t)$ is an arbitrary, single-valued nonlinearity which satisfies

$$-\infty < \alpha \leq \frac{\phi(\sigma, t)}{\sigma} \leq \beta < \infty \text{ for all } \sigma \neq 0, \text{ for all } t \geq 0.$$

$\phi(0, t)$ is zero for all $t \geq 0$.

That is $\phi(\sigma, t)$ is a time-varying nonlinearity which always lies in sector $[\alpha, \beta]$[5] and passes through the origin.

(3) The impulse response of G can be expressed as $\gamma + h(t)$ for $t \geq 0$, the zero input response as $z_\infty + z(t)$ where γ, z_∞ are

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[4] A function, f(t), is said to be in $L_p(0, \infty)$ if $\int_0^\infty |f(t)|^p \, dt < \infty$.

[5] Here and throughout square brackets indicate the closed interval while open brackets indicate the open interval.

constants ≥ 0 and h(t), z(t) are bounded and members of $L_1(0, \infty)$, and z(t) tends to zero as t tends to infinity.

When G is a lumped parameter system its transfer function G(s) is a ratio of two polynomials. Assumption (3) can then be stated as: G is completely controllable and observable, the degree of the numerator polynomial is less than that of the denominator and all the poles of G(s) lie in the open left-hand s-plane[6]. If G(s) has poles on the jw-axis then G(s) is required to be stable - in - the - limit[7].

If S satisfies the above three conditions then for S to be L_2 - stable it is sufficient that its frequency response locus G(jw) satisfy the following conditions for all $w \geq 0$:

Case (a) $\alpha > 0$: The locus of G(jw) lies strictly outside the circle C of radius $1/2\ [1/\alpha - 1/\beta]$ centered on the real axis of the s-plane at $[-(1/2)(1/\alpha + 1/\beta), 0]$ and does not encircle the circle.

Case (b) $\alpha = 0$: Real part $[G(jw)] + 1/\beta > 0$.

Case (c) $\alpha < 0$: The locus of G(jw) lies strictly inside the circle C of radius $1/2\ [1/\beta - 1/\alpha]$ centered on the real axis of the complex plane at $[-(1/2)(1/\alpha + 1/\beta), 0]$.

The above criterion is known as the circle criterion and was originally presented in this form by Sandberg [20] . The criterion is illustrated graphically in Figure 2. Cases (b) and (c) apply only to the principal case while Case (a) may be applied to particular cases

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[6] G(s) is called a principal case if all its poles lie in the open left-hand s-plane. When some of the poles of G(s) lie on the jw-axis it is called a particular case.

[7] The concept of stability - in - the - limit is explained on Pages 8 and 9.

as well. A system is called L_2 - stable if for any finite state; (i) $\sigma(t)$ is bounded, in $L_2(0, \infty)$ and tends to zero as t tends to infinity, (ii) for the autonomous case the null solution is globally, asymptotically stable.

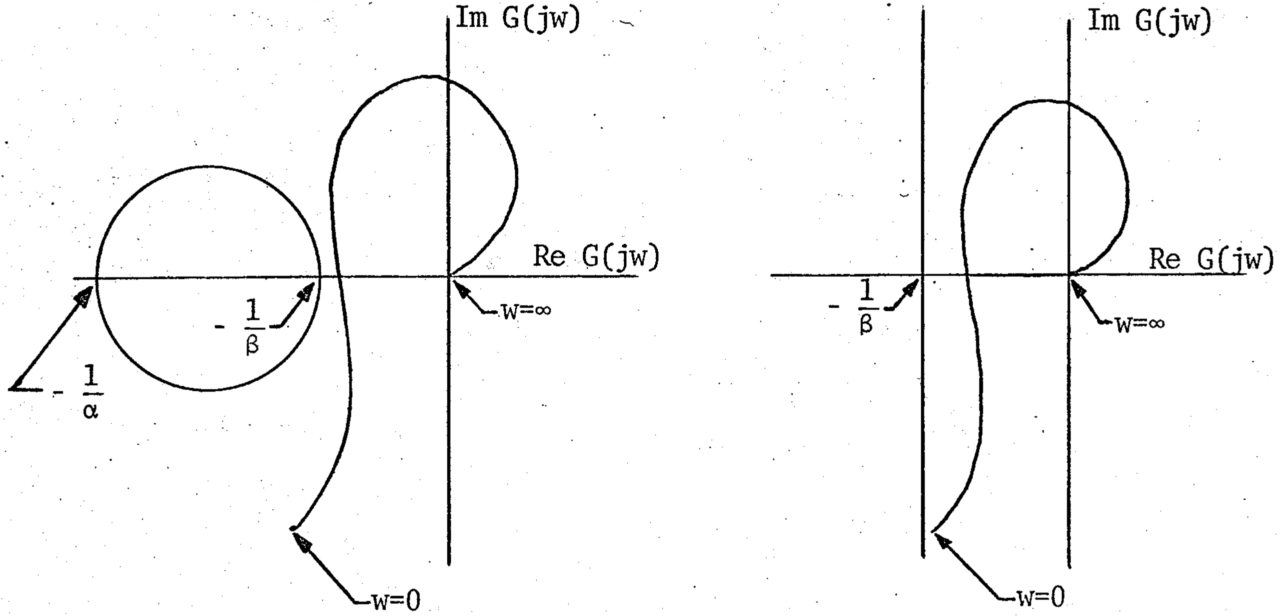
Satisfaction of the circle criterion by system S also implies that S is stable in the bounded input-bounded output sense. James [15] shows that even if system S has a multi-valued nonlinearity the circle criterion can be applied to it to prove bounded input-bounded output stability. In [4], Bergen and Iwens show that the circle criterion may be used to predict zero steady-state operation of the system S.

The circle criterion imposes a minimum of restrictions on the nonlinearity. Greater restriction on the nonlinearity should provide greater freedom in the linear operator G. Restriction of the nonlinearity to be time-invariant results in the V.M. Popov theorem [2].

This thesis is concerned primarily with the V.M. Popov stability criterion. The theorem is stated in the next section. Chapter II extends the theorem in two directions. The application of the theorem to several systems is considered in Chapter III. Stability criteria which arise when the nonlinearity is restricted further are presented in Chapter IV which is followed by the concluding remarks of Chapter V.

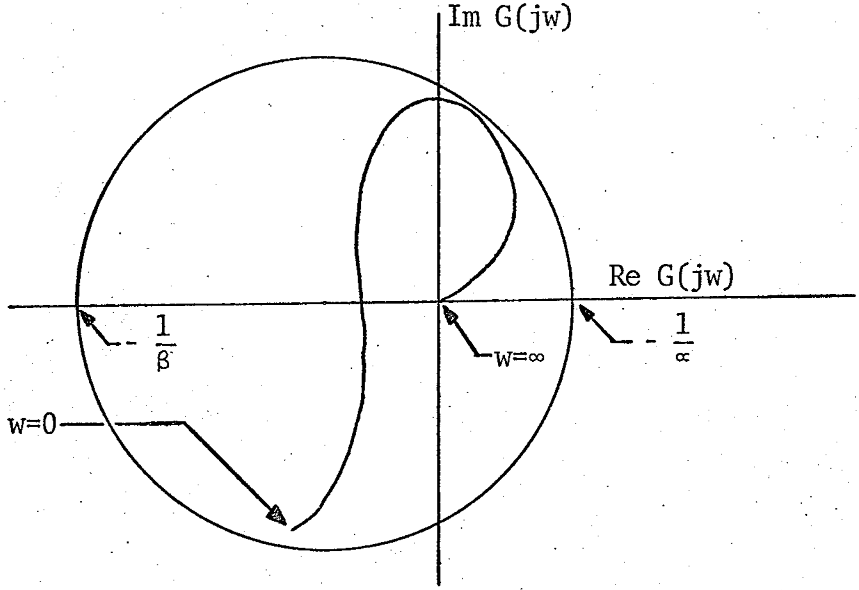
The V.M. Popov Theorem

The theorem as originally proven by V.M. Popov applied to autonomous, lumped parameter systems. Since then the theorem has been extended to nonautonomous and distributed systems [9, 21, 6]. The following presentation takes these extensions into account and follows



Case (a): $\alpha > 0$

Case (b): $\alpha = 0$



Case (c): $\alpha < 0$

Figure 2 - Graphical Interpretation of the Circle Criterion

closely that of reference [6] .

Consider system S which is characterized as follows:

(1) The input $r(t)$ and its derivative $\dot{r}(t)$ are bounded for all $t \geq 0$[8].

(2) The nonlinearity is single-valued, piecewise continuous and time-invariant with $\phi(0) = 0$.

(3) The linear operator G satisfies the following conditions;

- (i) The impulse response satisfies $\int_0^\infty e^{2\alpha t} h^2(t)\, dt < A < \infty$, where A is some positive number and $\alpha > 0$ is a sufficiently small number.
- (ii) The zero-input response $z(t)$ and its derivative $\dot{z}(t)$ are bounded for all $t \geq 0$ by decaying exponentials.
- (iii) $G(s) = \int_0^\infty h(t)\, e^{-st}\, dt$ is analytic in the domain $\mathrm{Re}\, s \geq -\alpha$.

Assumption (3) describes the linear operator by its impulse and zero-input responses. Thus distributed systems are included. As in the case of the circle criterion assumption (3) for lumped parameter systems, provided that G is completely controllable and observable, may be stated as: G(s) is a ratio of two polynomials with all of its poles in the closed left-hand s-plane and with the degree of the numerator polynomial less than the degree of the denominator polynomial.

Particular cases of G are required to be stable - in - the - limit. A particular case is stable - in - the - limit if system S is asymptotically stable for a linear gain, $\phi(\sigma) = \varepsilon\sigma$, where $\varepsilon > 0$ is

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[8] When G is a lumped parameter system, this assumption may be replaced by the requirement that $r(t)$ and $d\phi/d\sigma$ be bounded [22]. Baker and Bergen [3] show that these assumptions are both necessary and sufficient to prove stability in the bounded input-bounded output sense.

arbitrarily small. When G is a lumped, parameter system stability - in - the - limit requires that all the jw-axis poles move into the left-hand s-plane when the linear gain is increased from zero. Necessary and sufficient conditions for lumped parameter systems to be stable - in - the - limit are stated in Appendix A. When G is a distributed system it will be stable - in - the - limit if the linear system of Figure 3 satisfies assumption (3).

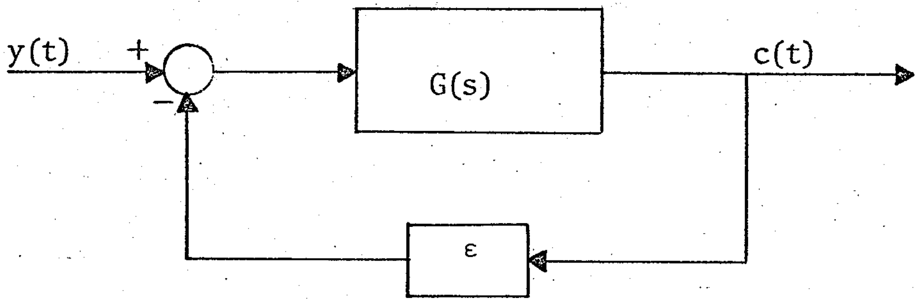


Figure 3 - Linear System Used to Prove Stability - in - the - Limit

When the system satisfies the above three assumptions then the V.M. Popov theorem states that if there exists a real q and a positive δ such that for all $w \geq 0$ the following inequality is satisfied

$$\mathrm{Re}\ [(1 + jwq)\ G(jw)] + 1/k \geq \delta > 0 \tag{1f}$$

then for nonlinearities contained in the sector $[0, k]$ for the principal case and in the sector $[\varepsilon, k]$ for the particular cases the system S is:

(1) Absolutely stable in the sense that for any finite initial state the output will be bounded for all time ≥ 0.

(2) If r(t) is identically equal to zero (autonomous case) then S is globally asymptotically stable.

(3) If r(t) and $\dot{r}(t)$ are in $L_2(0, \infty)$ and tend to zero as t tends to infinity, as well as being bounded, then S is L_2- stable.

Particular cases require that the nonlinearity lie in the sector [ε, k]. The sector [ε, k] is smaller than sector (0, k]. For example, nonlinearities which have a zero slope at σ = 0 will always lie outside the sector [ε, k] but could lie inside the sector (0, k]. However, since the nonlinearity is never specified that exactly at the origin the difference between the two sectors has little practical significance.

For the case where the blocks N and G are interchanged then as shown by Sandberg [21] the assumptions on the derivative of the input are not required. This is due to the smoothing properties of G. Thus, in contrast with linear systems, the placement of the nonlinear block plays an important role in the stability of the system.

Inequality (1f) is basically Popov's criterion. When S is autonomous the inequality may be relaxed slightly to obtain more general results for global asymptotic stability of S. These results are summarized in Appendix B.

Though the criterion is only sufficient it is quite powerful. In particular, Yakubovitch [23] has shown that for the autonomous, lumped parameter system the Popov criterion subsumes all results that can be proven by the Lyapunov method using a "quadratic plus integral" Lyapunov function. Since the criterion has a simple geometrical interpretation it is much easier to apply than Lyapunov methods. The geometrical interpretation of the criterion is accomplished by plotting the modified frequency response of G.

The modified frequency response, $G^*(jw)$, is related to $G(jw)$ by;

$$\text{Re } G^*(jw) \equiv \text{Re } G(jw) \equiv X$$

$$\text{Im } G^*(jw) \equiv w \text{ Im } G(jw) \equiv Y$$

Inequality (1f) may therefore be written as

$$X - q\,Y + 1/k \geq \delta > 0. \tag{1g}$$

The equation $X - q\,Y + 1/k = 0$ represents the locus of a straight line with slope $1/q$ and intercept $- 1/k$. For inequality (1g) to be satisfied the locus of $G^*(jw)$ must lie strictly to the right of this line. This is illustrated in Figure 4.

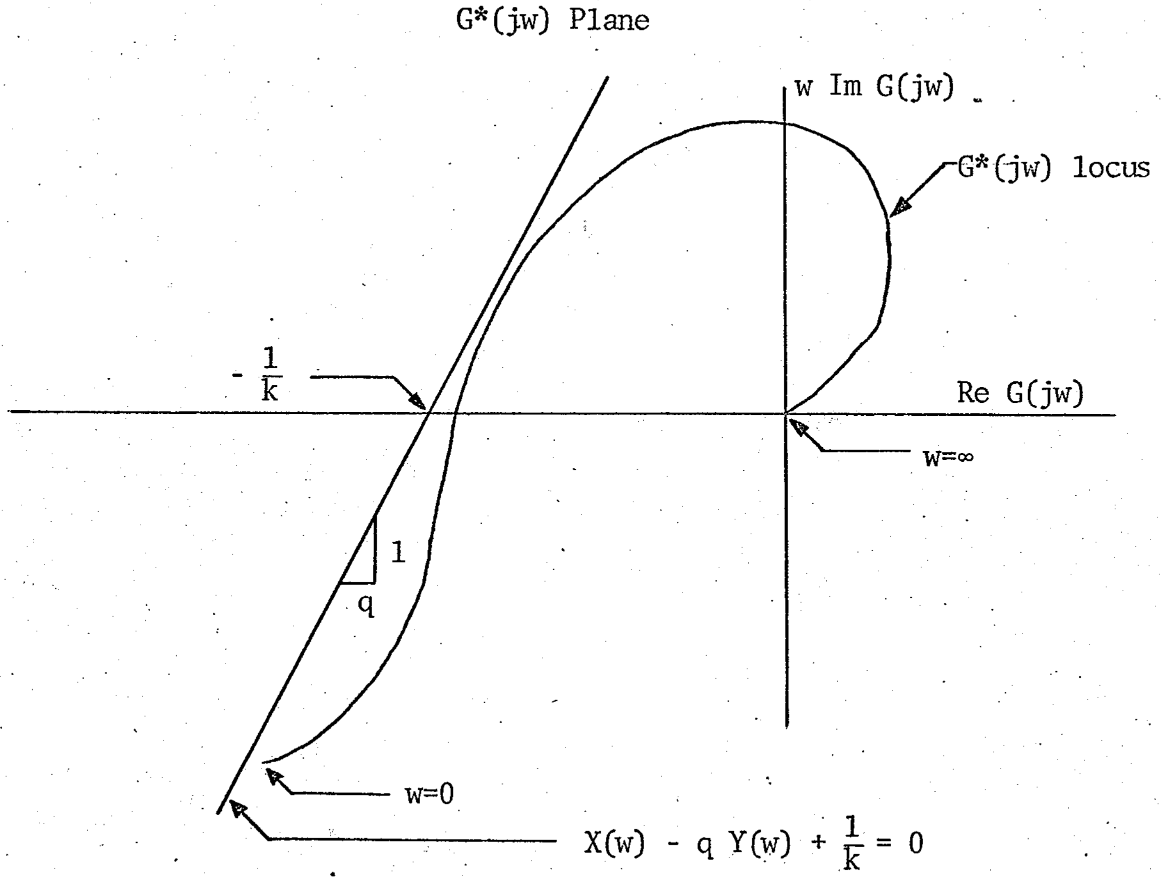


Figure 4 - Graphical Interpretation of the Popov Criterion

CHAPTER II

TWO EXTENSIONS OF THE V.M. POPOV CRITERION

I. EXTENSION TO HYSTERETIC NONLINEARITIES

General Considerations

Let system S satisfy assumptions (1) and (3) of the V.M. Popov criterion. Assumption (2) is now relaxed so that the nonlinearity may be hysteretic. The nonlinearity is assumed to have the following qualitative features.

(1) $\phi\,[\sigma(t),\,\phi_0]$ is defined for all σ and ϕ_0 in $(-\infty,\,\infty)$ and lies in the sector $[0,\,k]$ for principal cases and $[\varepsilon,\,k]$ for particular cases. ϕ_0 indicates the memory of the nonlinearity.

(2) $\phi\,[0,\,\phi_0]$ is zero.

(3) The integral $\int_0^{\sigma(T)} \phi\,[\sigma,\,\phi_0]\,d\sigma$ is > 0 where $T > 0$ is arbitrary. In graphical terms this requirement restricts the hysteretic loops to be traversed in a clockwise direction. A suitable $\phi\,[\sigma,\,\phi_0]$ is shown in Figure 5.

For a nonlinearity satisfying these three assumptions Theorem 1 shows that the V.M. Popov criterion is still applicable to system S.

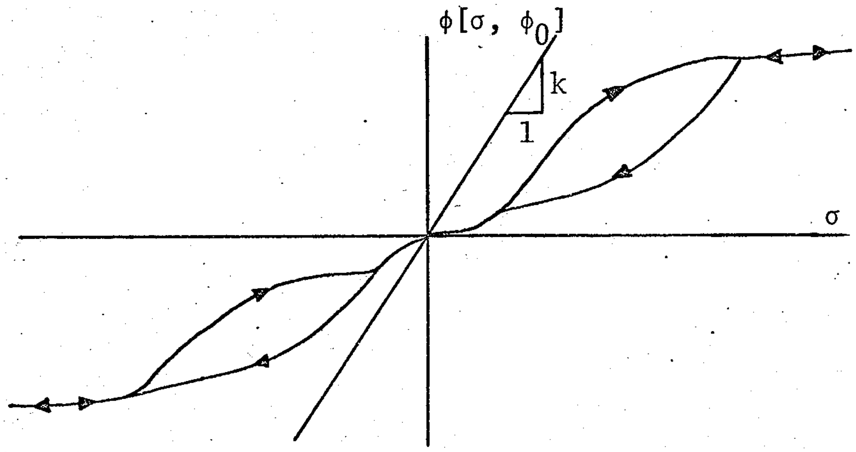


Figure 5 - Allowable Hysteretic Nonlinearity

Theorem 1

For the preceding hysteretic nonlinearity system S will be;

(a) Absolutely stable in the bounded input-bounded output sense,

(b) L_2 - stable if $r(t)$ and $\dot{r}(t)$ are in $L_2(0, \infty)$,

if there exists a finite number $q \geq 0$ and a positive number δ such that for all $w \geq 0$ the V.M. Popov criterion $\mathrm{Re}\,[(1 + jwq)\,G(jw)] + 1/k \geq \delta > 0$ is satisfied.

The proof of Theorem 1 is based on the following lemma and follows that given in [6] for the single-valued nonlinearity.

Lemma

When system S is a principal case and satisfies all the conditions of Theorem 1 then the following inequality holds for a sufficiently small $\alpha > 0$.

$$\left[\int_0^t e^{2\alpha\tau}\, y^2(\tau)\,d\tau\right]^{\frac{1}{2}} \leq \left[1/\delta^2 \int_0^t e^{2\alpha\tau}\,[r(\tau) - z(\tau) + q\,\{\dot{r}(\tau) - \dot{z}(\tau)\}]^2\,d\tau + 2q/\delta \int_0^{\sigma(0)} \phi\,[\sigma, \phi_0]\,d\sigma\right]^{\frac{1}{2}} \quad \text{for all } t \geq 0. \tag{2a}$$

This lemma is identical to the Main Lemma in [6] but is extended to include the hysteretic nonlinearities of Theorem 1. Its proof is given in Appendix C.

Proof of Theorem

The system is described by the equation,

$$\sigma(t) = r(t) - z(t) - \int_0^t h(t - \tau)\,y(\tau)\,d\tau, \tag{2b}$$

or $\sigma(t) = r(t) - z(t) - \int_0^t e^{\alpha(t-\tau)}\, h(t - \tau)\, e^{-\alpha(t-\tau)}\, y(\tau)\, d\tau.$ (2c)

By the triangle inequality and Schwarz's inequality (2c) becomes

$$|\sigma(t)| \le |r(t) - z(t)| + \left[\int_0^\infty e^{2\alpha x}\, h^2(x)\, dx\right]^{\frac{1}{2}} e^{-\alpha t} \cdot \left[\int_0^t e^{2\alpha\tau}\, y^2(\tau)\, d\tau\right]^{\frac{1}{2}}. \qquad (2d)$$

Use of the Lemma in (2d) yields

$$|\sigma(t)| \le |r(t) - z(t)| + \left[\int_0^\infty e^{2\alpha x}\, h^2(x)\, dx\right]^{\frac{1}{2}} \cdot \left[1/\delta^2 \int_0^t e^{-2\alpha(t-\tau)} \cdot [r(\tau) - z(\tau) + q\,\{\dot{r}(\tau) - \dot{z}(\tau)\}]^2\, d\tau + 2q/\delta \int_0^{\sigma(0)} \phi[\sigma, \phi_0]\, d\sigma\right]^{\frac{1}{2}}. \qquad (2e)$$

The linear system G is a principal case and therefore its impulse response satisfies the inequality:

$|h(t)| < k_1\, e^{-k_2 t}$ where k_1, k_2 are positive constants.

Thus α can be found, $0 < \alpha < k_2$, so that the following holds,

$$\int_0^\infty e^{2\alpha x}\, h^2(x)\, dx \le A < \infty.$$

The second integral in (2e) is bounded since it is a convolution of the impulse response of a strictly stable linear system with a bounded input. Therefore the right-hand side of inequality (2e) is bounded for all $t \ge 0$, i.e.,

$|\sigma(t)| \le B < \infty$ for all $t \ge 0$. Q.E.D.

Example

Let $G(s)$ be given by $1/s(s + a)$ with $a > 0$. The V.M. Popov criterion is $(-1 + qa) / (w^2 + a^2) + 1/k \geq \delta > 0$ for all $w \geq 0$. The criterion is satisfied for $q > 1/a$. Thus for this $G(s)$ system S is L_2 - stable and bounded input-bounded output stable for all hysteretic nonlinearities lying in the sector $[\varepsilon, \infty]$ provided that the hysteresis loops are traversed clockwise.

Remarks

Theorem 1 was proven only for the principal case. However, as shown in [2] for the autonomous case, particular cases can be transformed into principal cases. Exactly the same arguments apply to the nonautonomous case.

Similar results for the autonomous system have been established by Ghelig [13]. The qualitative description of the nonlinearity was in fact suggested by [13]. The originality of Theorem 1 is the proof that the theorem established by Bergen, Iwens and Rault, [6], is still valid for this nonlinearity.

The class of hysteretic nonlinearities covered by Theorem 1 is rather limited. Practical hysteretic nonlinearities are usually such that the hysteretic loops are traversed counterclockwise. Theorem 1 therefore has limited practical value.

II. PREDICTION OF ZERO STEADY-STATE ERROR OPERATION BY MEANS OF POPOV'S CRITERION

In the Introduction it was stated that when system S satisfies Popov's criterion then with suitable restrictions on the input and its derivative L_2 - stability of the system is implied. It was also pointed out that when the nonlinear and linear blocks are interchanged the restrictions on the derivative may be removed. Theorem 2 shows that for this system, S_1, (Figure 6) it is possible to predict zero steady-state error operation by means of Popov's criterion.

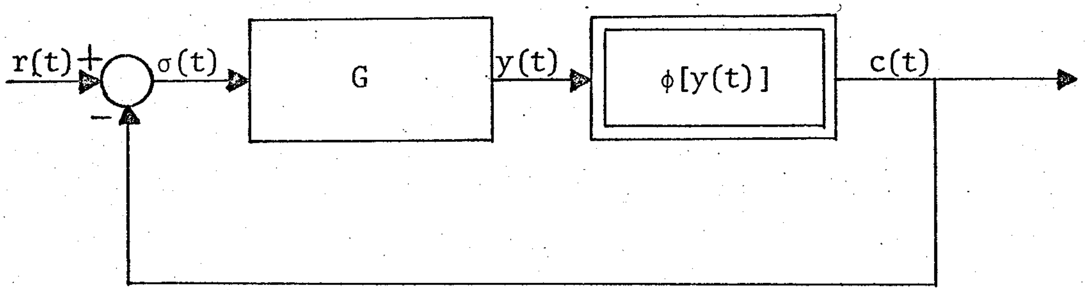


Figure 6 - System S_1

Theorem 2

If the linear companion feedback system, S_L, (Figure 7) follows an input with zero steady-state error then system S_1 will follow the same input with zero steady-state error provided that;

(1) The nonlinearity lies in the sector [ε, k] for both the principal and particular cases of G where ε is a small positive number,

(2) Popov's criterion is satisfied by system S_1,

(3) The linear operator G satisfies assumption (3) of the V.M. Popov theorem.

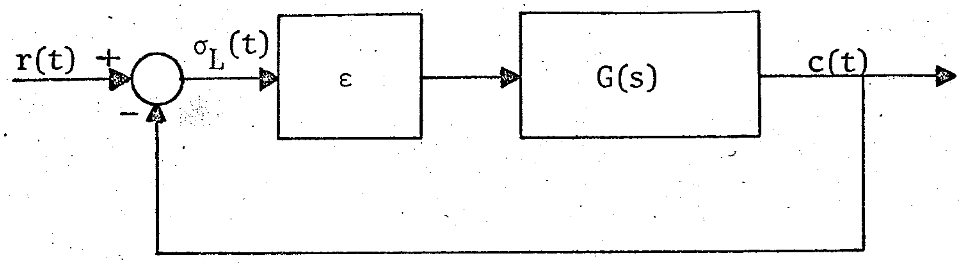


Figure 7 - Linear Companion Feedback System, S_L

Proof of Theorem 2

Transform system S_1 by means of the transformation

$$\bar{\phi}(\ ) \equiv \phi(\ ) - \varepsilon y. \tag{2e}$$

Transformation (2e) results in the equivalent system, S_{eq}, of Figure 8. The nonlinearity $\bar{\phi}(\ )$ lies in the sector $[0, k - \varepsilon]$.

Popov's criterion for system S_{eq} is,

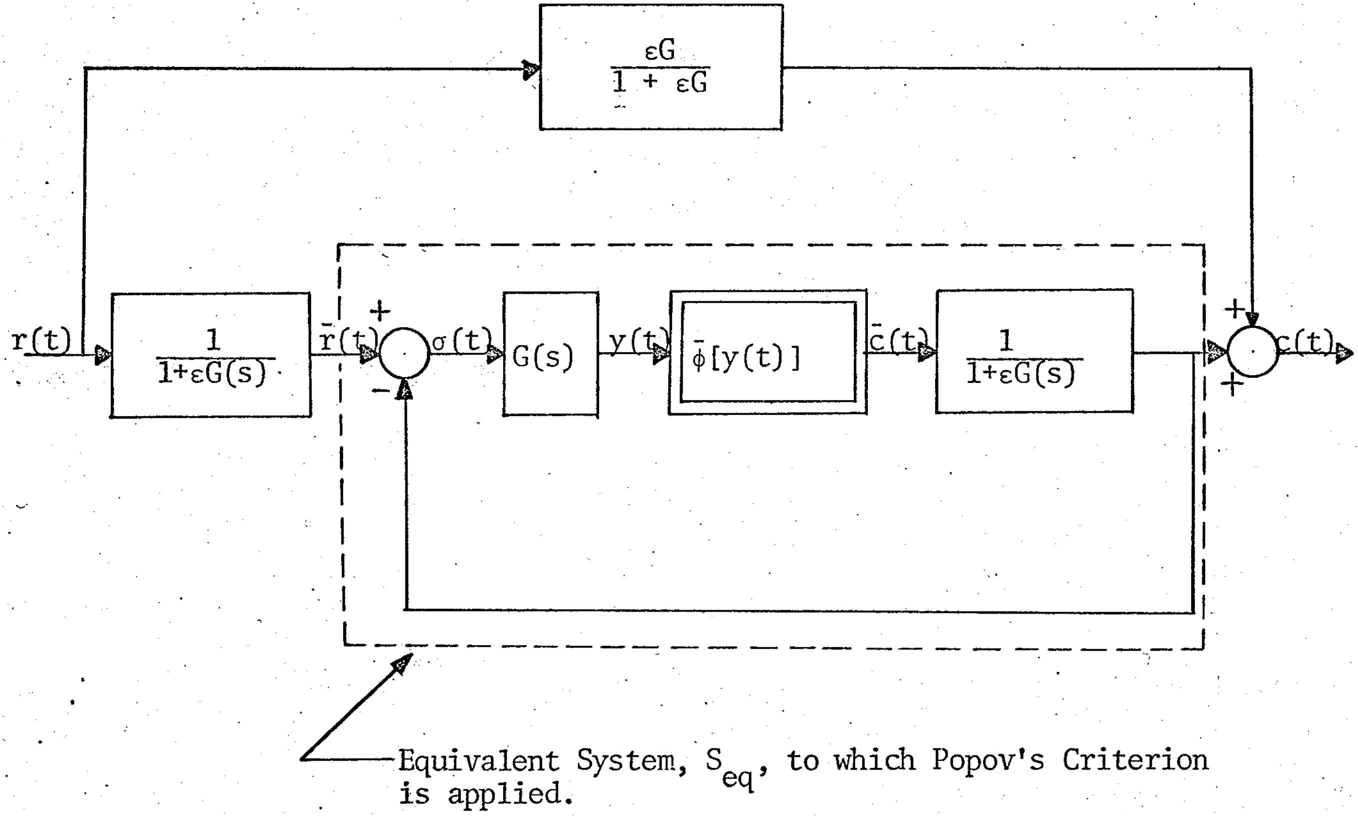
$$\mathrm{Re}\left[(1 + jwq)\frac{G(jw)}{1 + \varepsilon G(jw)}\right] + \frac{1}{k - \varepsilon} \geq \delta > 0 \quad \text{for all} \quad w \geq 0. \tag{2f}$$

Inequality (2f) may be rewritten as

$$\frac{1}{|1 + \varepsilon G(jw)|^2}\left[\mathrm{Re}\{(1 + jwq)\ G(jw)\} + \frac{1}{k}\right] + \frac{\varepsilon k}{k-\varepsilon}\ \frac{|G(jw) + 1/k|^2}{|1 + \varepsilon G(jw)|^2}$$

$$\geq \delta > 0 \quad \text{for all} \quad w \geq 0. \tag{2g}$$

But it is assumed that $\mathrm{Re}[(1 + jqw)\ G(jw)] + 1/k \geq \delta_1 > 0$ for all $w \geq 0$. Since δ, δ_1 are arbitrarily small numbers then inequality (2g) holds for both principal and particular cases of G. For particular cases of G, the limit of the left-hand side of (2g) as w approaches a pole of G is $k/[\varepsilon(k-\varepsilon)]$. Thus an ε can always be chosen such that $k/[\varepsilon(k-\varepsilon)] \geq \delta$.



Equivalent System, S_{eq}, to which Popov's Criterion is applied.

Figure 8 - Equivalent System Derived by Transforming System S_1

The input to system S_{eq} is related to the input of system S_1 by $\bar{R}(s) = \{1/[1 + \varepsilon\ G(s)]\}\ R(s)$. (2h)

But (2h) is the same as the expression for the error of the linear companion feedback system. Since the hypothesis of Theorem 2 states that $\lim_{t\to\infty} \sigma_L(t)$ is zero then the $\lim_{t\to\infty} \bar{r}(t)$ is zero and system S_{eq} is therefore L_2-stable by hypothesis.

Thus $\lim_{t\to\infty} \sigma(t)$ is zero for both systems S_{eq} and S_1 and therefore for system S_1:

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} r(t). \qquad \text{Q.E.D.}$$

Example

Let G(s) be given by $1/[s(s + a)]$. If the input to system S_1 is a step function the error of the linear companion system is $(s + a)/\{s(s + a) + \varepsilon\}$. The final value of the error is given by $\lim_{s\to 0} s(s + a)/\{s(s + a) + \varepsilon\}$ and is equal to zero. System S_1 satisfies Popov's criterion for nonlinearities lying in the sector $[\varepsilon, \infty)$. Therefore Theorem 2 implies that for this particular G(s), system S_1 will follow the step function with zero steady-state error for nonlinearities in the sector $[\varepsilon, \infty)$.

Remarks

The preceding example points out that zero steady-state error operation of system S_1 is possible even when the input is not in $L_2(0, \infty)$. Since the input $\bar{R}(s)$ is related to R(s) by $1/\{1 + \varepsilon\ G(s)\}$ any input whose jw-axis poles are cancelled by the poles of G(s) will result in zero steady-state error.

Theorem 2 is similar to the Theorem 2 proven by Bergen and Iwens, [4], in both its statement and proof. Bergen and Iwens however considered the circle criterion whereas the Popov criterion is considered here.

Transformation (2e) may also be used to predict zero steady-state error operation of system S when the nonlinearity lies in the sector $[\varepsilon, k]$. This is shown by Corollary 1.

Corollary 1

If the input to system S is continuous and bounded and if the linear companion feedback system follows this input with zero steady-state error then system S will follow the same input with zero steady-state error.

Proof of Corollary 1

The transformation (2e) results in the system of Figure 9. From the final and initial value theorems of Laplace transform theory the following values for $\bar{r}(t)$ and $\dot{\bar{r}}(t)$ are established;

$$\lim_{t\to0^+} \bar{r}(t) = \lim_{s\to\infty} \frac{D(s)}{D(s) + \varepsilon\, N(s)} \cdot \lim_{s\to\infty} sR(s) = 0, \tag{2i}$$

$$\lim_{t\to\infty} \bar{r}(t) = \lim_{s\to0} \frac{D(s)}{D(s) + \varepsilon\, N(s)} \cdot \lim_{s\to0} sR(s) = \lim_{t\to\infty} \sigma_L(t) = 0, \tag{2j}$$

$$\lim_{t\to0^+} \dot{\bar{r}}(t) = \lim_{s\to\infty} \frac{D(s)}{D(s) + \varepsilon\, N(s)} \cdot \lim_{s\to\infty} s^2\, R(s) = \text{Finite quantity}, \tag{2k}$$

$$\lim_{t\to\infty} \dot{\bar{r}}(t) = \lim_{s\to0} \frac{sD(s)}{D(s) + \varepsilon\, N(s)} \cdot \lim_{s\to0} sR(s) = 0. \tag{2\ell}$$

Limits (2i) and (2k) follow from the restriction on r(t) to be continuous while (2j) and (2ℓ) follow from the hypothesis that system S_L follows r(t) with zero steady-state error. Thus $\bar{r}(t)$ and $\dot{\bar{r}}(t)$ are in $L_2(0,\infty)$ and therefore $\lim_{t\to\infty} \sigma(t)$ is zero.

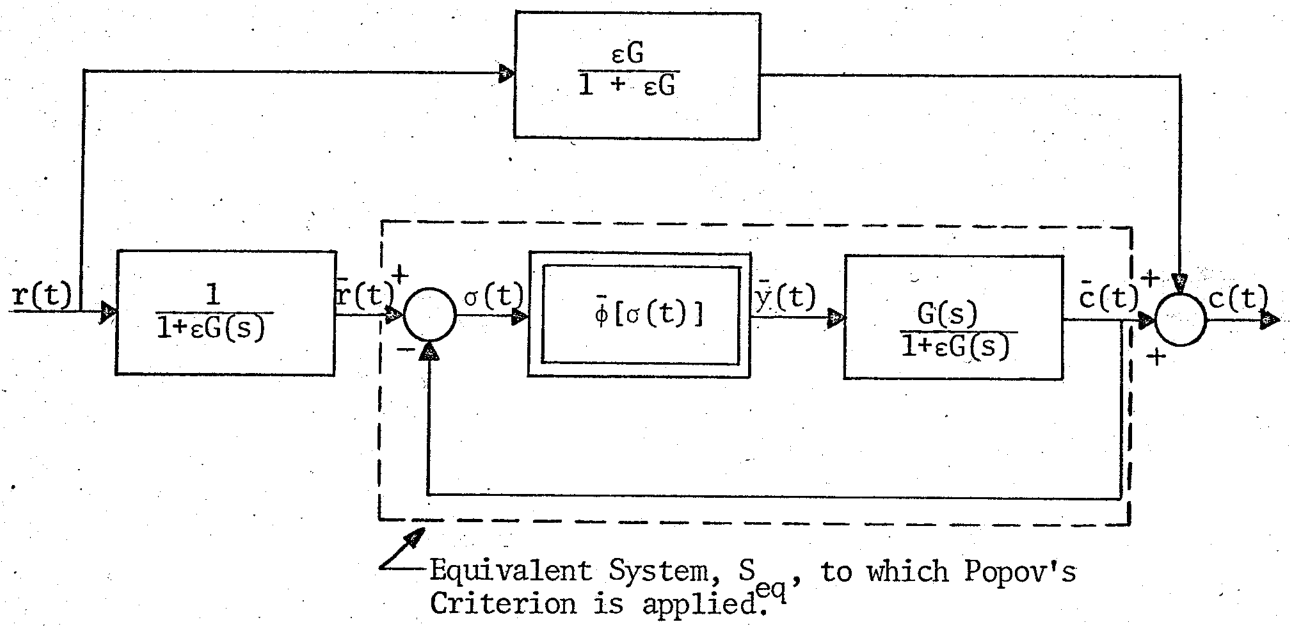


Figure 9 - Equivalent System Derived by Transforming System S

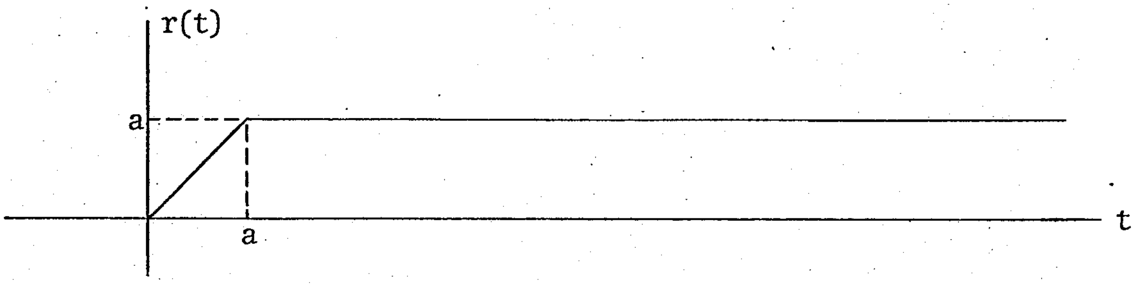


Figure 10 - Allowable Input for Corollary 1

Example

Consider the input of Figure 10 applied to system S with $G(s) = 1/[s(s + a)]$. The input r(t) is continuous with the Laplace transform $R(s) = (1 - e^{-as})/s^2$. The transform of the error of system S_L for this input is $\{(s + a)(1 - e^{-as})\}/\{s^2(s + a) + \varepsilon s\}$ and its final value is zero. Therefore by Corollary 1 system S follows this input with zero steady-state error for nonlinearities in the sector $[\varepsilon, \infty)$.

CHAPTER III

APPLICATION OF POPOV'S CRITERION

I. GRAPHICAL APPLICATION OF POPOV'S CRITERION

The application of Popov's criterion assumes that the nonlinearity is constrained to lie in the sector $[0, k]$ for principal cases and $[\varepsilon, k]$ for particular cases. For these cases the criterion is easily tested by the graphical technique presented in the Introduction. However, when the nonlinearity lies in the general sector $[k_1, k_2]$, $-\infty < k_1 < k_2 < \infty$, the technique can only be used after the transformation,

$$\bar{\phi}(\sigma) \equiv \phi(\sigma) - k_1\sigma \,, \tag{3a}$$

is applied to system S.

The system resulting from transformation (3a) is shown in Figure 11, where the nonlinearity $\bar{\phi}(\sigma)$ lies in the sector $[0, k_2 - k_1]$ and the linear subsystem $\bar{G}$ is strictly stable. To apply the graphical technique of the Introduction to system S_{k1} the modified frequency response locus of $\bar{G}(jw)$ has to be plotted. To avoid this the following develops a graphical technique for the application of Popov's criterion directly to the linear subsystem G.

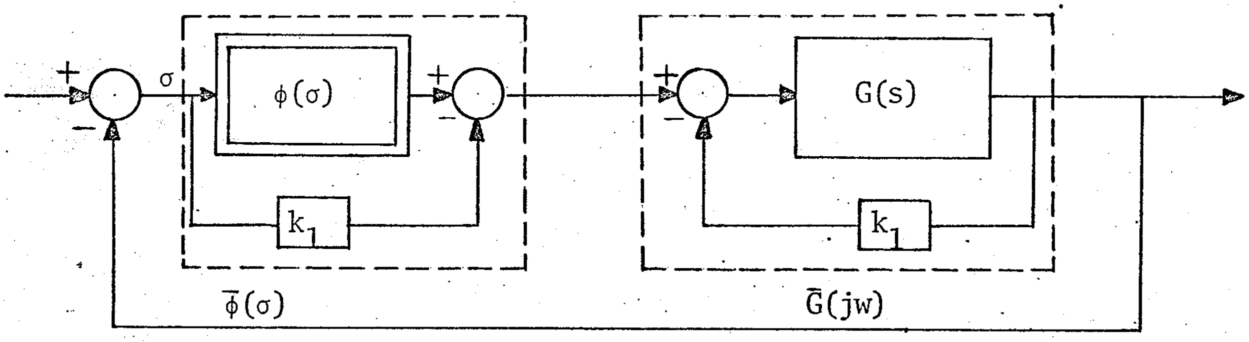


Figure 11 - System S_{k1}

Graphical Technique for the Direct Application of Popov's Criterion to G(jw) when the Nonlinearity Lies in Sector $[k_1, k_2]$

Popov's criterion for system S_{k1} is,

$$\mathrm{Re}\left[(1 + jqw)\, G(jw)/\, [1 + k_1\, G(jw)]\right] + 1/k \geq \delta > 0$$

for $\leq w \leq \infty$, (3a)

where k is identical to $(k_2 - k_1)$.

In terms of the real and imaginary parts of G(jw), X(w) and Y(w) respectively, criterion (3a) may be rewritten as

$$\frac{X(w) + k_1\, [X^2(w) + Y^2(w)]}{1 + 2k_1\, X(w) + k_1^2\, [X^2(w) + Y^2(w)]}$$

$$- \frac{qw\, Y(w)}{1 + 2k_1\, X(w) + k_1^2\, [X^2(w) + Y^2(w)]} + 1/k \geq \delta > 0$$

for $0 \leq w \leq \infty$. (3b)

If G(s) is a particular case then when $w \to w_0$ where jw_0 is a pole of G(s) inequality (3a) becomes $1/k_1 + 1/k \geq \delta > 0$ and a δ can always be found such that this is true.

When δ is set to zero in (3b), then (3b) may be rewritten as

$$\left[X(w) + \frac{1 + 2k_1/k}{2(k_1 + k_1^2/k}\right]^2 + \left[Y(w) - \frac{qw}{2(k_1 + k_1^2/k)}\right]^2$$

$$> \frac{(qw)^2 + 1}{4k_1^2(1 + k_1/k)^2} \quad \text{for } 0 \leq w \leq \infty. \qquad (3c)$$

If equality is allowed in (3c) then the resulting equation will be a circle in the X(w), Y(w) plane with centre at

$$X(w) = -\frac{1 + 2k_1/k}{2(k_1 + k_1^2/k)}, \quad Y(w) = \frac{qw}{2(k_1 + k_1^2/k)}$$

and radius $r^2 = \dfrac{(qw)^2 + 1}{4k_1^2\,[1 + k_1/k]^2}$.

The intersections of this circle with the $X(w)$ axis occur at $-1/(k_1 + k), - 1/k_1$ and are independent of w.

Define qw to be $\bar{w}$ and to a convenient scale draw circles for the various $\bar{w}$ as in Figure 12. The frequency response of G(jw) is plotted on a separate sheet of transparent paper to the same scale as Figure 12. The locus of G(jw) is graduated in terms of $\bar{w}$. The transparent sheet of paper is then laid over Figure 12 and inequality (3c) is tested by seeing if each $\bar{w}$ on the G(jw) locus falls strictly outside the corresponding $\bar{w}$ circle. The parameter q is used to adjust the $\bar{w}$ positions on the G(jw) locus.

Example

Consider system S with $G(s) = 100/s(s + 1)(s^2 + 0.8\,s + 16)$. The frequency response locus of G(jw) as well as several $\bar{w}$ circles are shown in Figure 13. The linear system is stable in the sector (0, 0.7) while the Popov criterion shows that the nonlinear sector is $[\varepsilon, 0.26]$[1]. For q = 0.1 the preceding graphical technique shows that Popov's criterion is satisfied in the sector (0.139, 0.555).

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[1] Dewey & Jury, [11] , p. 60.

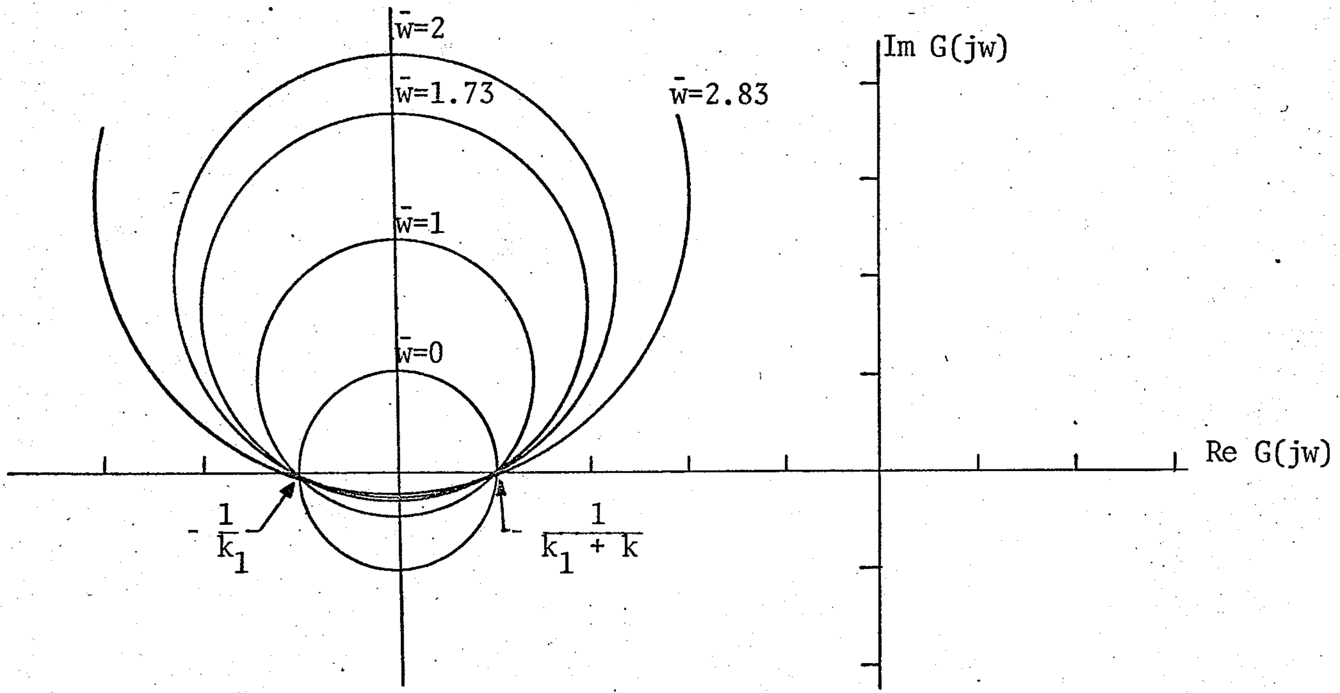

Figure 12 - $\bar{w}$ - Circles

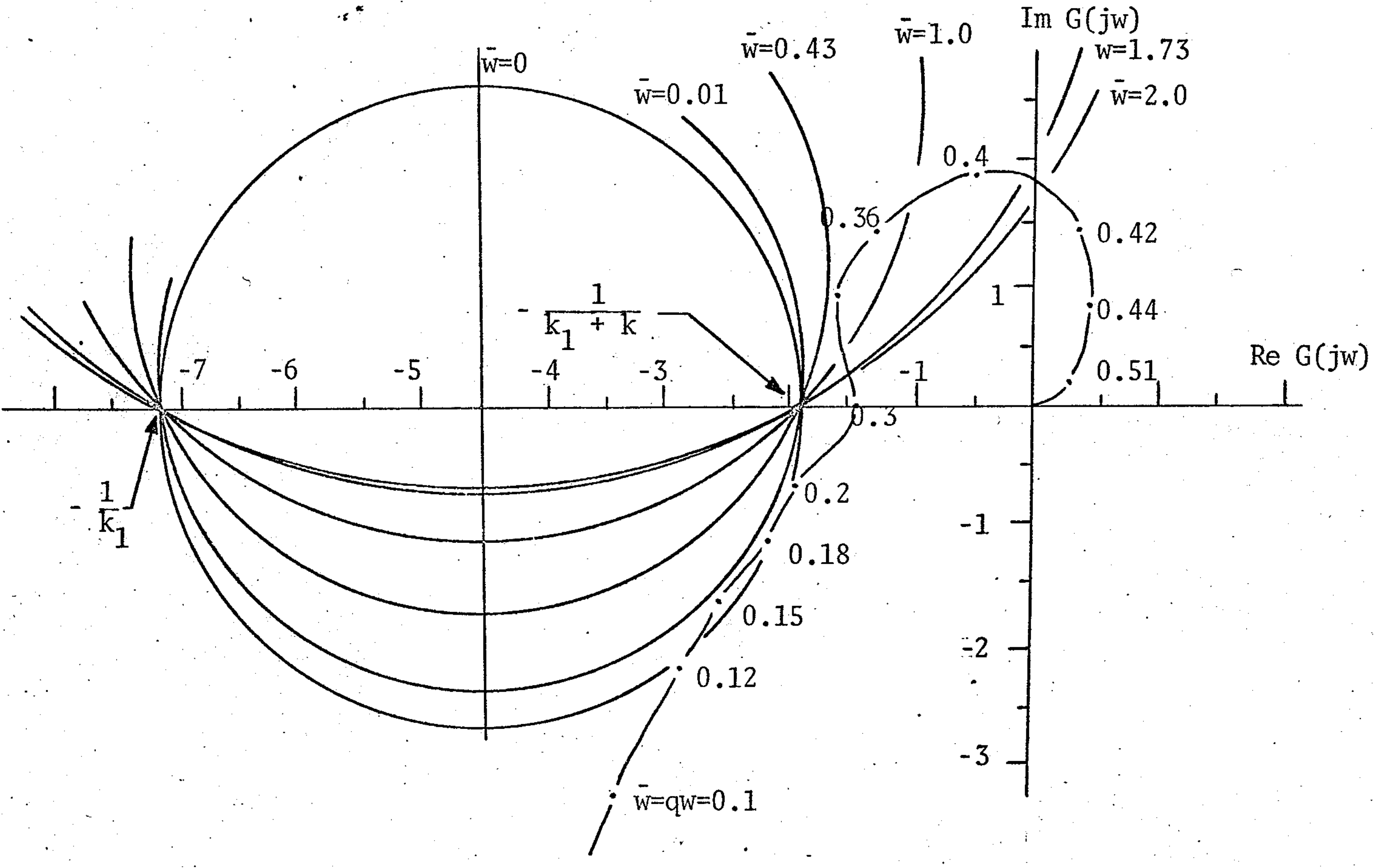

Figure 13 - Example of Graphical Technique

Remarks

The example illustrates that it is possible to obtain a better upper bound on the nonlinearity at the expense of the lower bound. The graphical technique is laborious and most suited to analysis problems where the sector $[k_1, k_2]$ is given. Its chief advantage would be that it deals directly with the frequency response locus of G(jw). This would simplify the problem of adjusting sector $[k_1, k_2]$ by compensation.

A graphical technique which is somewhat simpler to apply and interpret is presented in Reference [5]. This technique is called the Parabola test and deals with the modified frequency response locus G*(jw).

Parabola Test[2]

The test states that if the inequality

$$X^{*2}(w) + \frac{k + 2k_1}{k_1(k_1 + k)} \cdot X^*(w) - \frac{kq\, Y^*(w)}{k_1(k_1 + k)} + \frac{1}{k_1(k_1 + k)} > 0 \qquad (3d)$$

is satisfied for all $0 \leq w \leq \infty$, then system S satisfies Popov's criterion for nonlinearities in the sector $[k_1, k_2]$. $X^*(w)$ and $Y^*(w)$ are the real and imaginary parts of G*(jw).

When equality is allowed in (3d) the result is an equation whose locus is a parabola in the $X^*(w)$, $Y^*(w)$ plane. The intersections with the $X^*(w)$ axis occur at $-1/k$, $-1/(k_1 + k)$ and the slope of the tangents at these points is $1/q$ and $-1/q$ respectively. The construction and application of the parabola is shown in Figure 14.

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[2] Reference [5] proves the test for the autonomous case. However, the same considerations as in the graphical technique presented in this chapter show that it is applicable to the nonautonomous case.

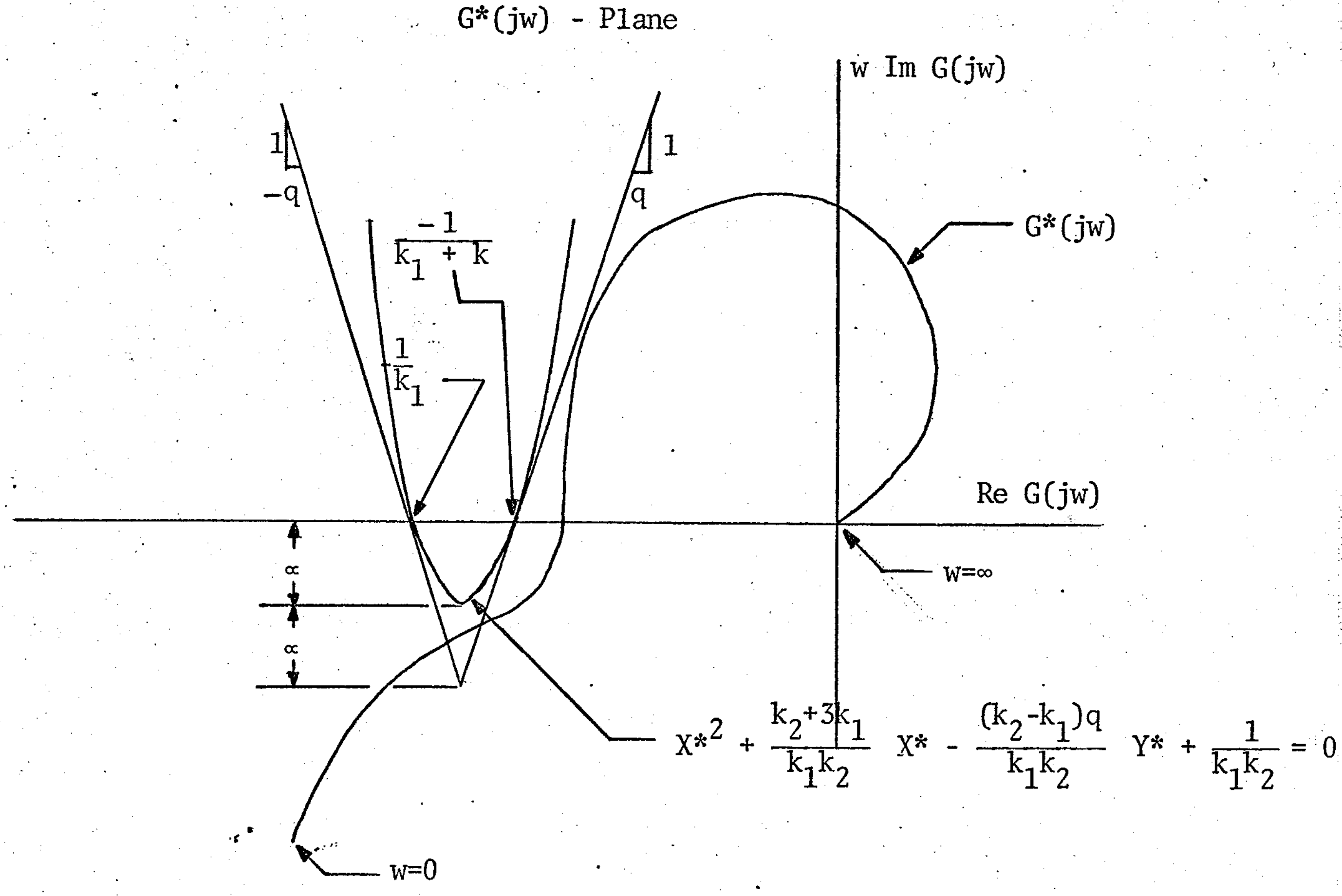

Figure 14 - Graphical Interpretation of the Parabola Test

II. ANALYTICAL APPLICATION OF POPOV'S CRITERION

When the linear subsystem G is specified numerically the graphical interpretation of Popov's criterion can be applied readily. However, if G is specified in general form then the Popov criterion must be applied analytically. The application involves the qualitative features of the modified frequency response locus. Generally the algebra involved becomes prohibitive but the criterion may be applied analytically with a fair amount of success to low order systems. The criterion is particularily useful in singling out systems which satisfy Aizerman's conjecture.

Aizerman's Conjecture

This conjecture first stated by M.A. Aizerman in 1947 postulates that the sector for global asymptotic stability of system S coincides with the Hurwitz sector. The conjecture was first disproved by Pliss [see 16]. Fitts [12] and Ackerman [1] recently presented experimental evidence that the conjecture is false. Thus it only remains now to identify the class of systems for which the conjecture is true. Application of Popov's criterion shows that the following systems satisfy the conjecture. The proofs are given in Appendix D.

(1) $G(s) = 1/ [s(s + a)] \qquad a > 0$

(2) $G(s) = 1/ [s(s^2 + as + b)] \qquad a, b > 0$

(3) $G(s) = [s + a] / [s(s^2 + bs + c)] \qquad a, b, c > 0$

(4) $G(s) = 1/ [s(s^3 + as^2 + bs + c)] \qquad a, b, c > 0,$
$ab - c > 0, a^3 - 2(ab - c) > 0$

(5) $G(s) = [s + as + b] / [(s^2 + 1) (s + c)] \qquad a, b, c > 0,$
$b - ac - 1 > 0, \quad bc + a - c > 0$

Systems (1) to (5) are all particular cases. When G(s) is a principal case the Hurwitz sector will be $[-k_1, k_2]$, $k_1, k_2 > 0$. Popov's criterion can be applied only after transformation (3a) has been applied to the system. The Hurwitz sector for the transformed system (Figure 11) will be $[0, k_1 + k_2]$. Popov's criterion is now applied to this system.

The effect of the transformation is to shift the poles[3] of

---

[3] The transformation is in effect the pole-shifting technique presented by Rekasius and Gibson, [19].

G(s) until some of its poles are on the jw-axis. The principal case is thus transformed into a particular case. Therefore only particular cases need to be considered which simplifies the algebra considerably. It can also be said then that Aizerman's conjecture is verified for the following more general systems.

(6) $1/\,[s^2 + as + b] \qquad a, b > 0$

(7) $1/\,[s^3 + as^2 + bs + c] \qquad a, b, c > 0$

(8) $1/\,[(s + d)\,(s^3 + as^2 + bs + c)] \qquad a, b, c, d > 0,$
$ab - c > 0, \qquad a^3c - 2c(ab - c) > 0$

Systems (4) and (5) satisfy Aizerman's conjecture only when restrictions on the pole locations in the s-plane are placed. For system (4) this restriction is, $a^3 - 2(ab - c) > 0$. This restriction may be interpreted in the s-plane by writing G(s) in the following equivalent form[4]

$$\bar{G}(s) = \frac{1}{s(s+1)(s^2 + \bar{a}s + \bar{b})} .$$

The restriction, $a^3 - 2(ab - c) > 0$ becomes

$$(\bar{a} + 1)^3 - 2\bar{a}(\bar{a} + 1) - 2\bar{a}\bar{b} > 0 \qquad \text{or} \qquad \bar{b} < \frac{(\bar{a} + 1)(\bar{a}^2 + 1)}{2\bar{a}} .$$

The poles of $\bar{G}(s)$ lie at

$$s_1 = 0 \qquad\qquad s_2 = -1$$

$$s_{3,4} = -\bar{a}/2 \pm 1/2\sqrt{\bar{a}^2 - 4\bar{b}} = -\bar{a}/2 \pm 1/2j\sqrt{4\bar{b} - \bar{a}^2} .$$

---

[4] To obtain $\bar{G}(s)$ from G(s) first use the time transformation $\tau = a_1 t$, where a_1 is the pole of G(s) on the negative real axis, and then normalize the resulting transfer function with respect to $1/a_1^2$.

The restriction on $\bar{b}$ limits the magnitude of Imag $s_{3,4}$ to

$$\frac{1}{2}\sqrt{\frac{\bar{a}^3 + \bar{a}^2 + \bar{a} + 1}{\bar{a}}}$$

A plot of the permissible pole location is shown in Figure 15.

When Aizerman's conjecture is not satisfied by system S then the nonlinear sector can be increased when greater information about the nonlinearity is available. Some of the frequency domain stability criteria that arise when the nonlinearity is restricted further are presented in the next chapter.

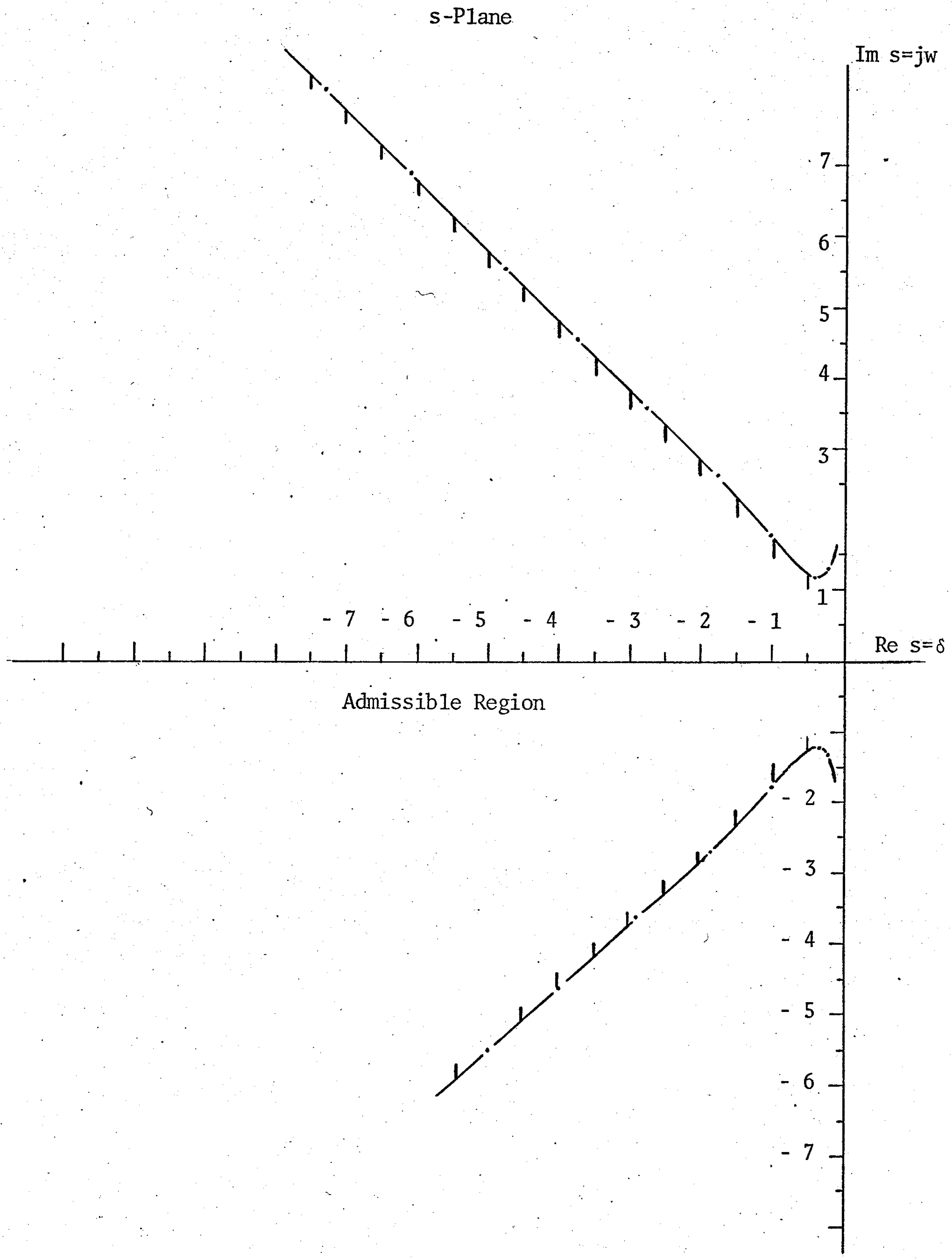

Figure 15 - Admissible Pole Locations of $G(s) = \frac{1}{s(s+1)(s^2+\bar{a}s+\bar{b})}$ for Aizerman's Conjecture to be Verified

CHAPTER IV

STABILITY CRITERIA WHICH ARISE WHEN THE NONLINEARITY IS RESTRICTED FURTHER

The previous chapter has shown that Popov's criterion fails to guarantee stability in the entire Hurwitz sector as postulated by Aizerman's conjecture. In an effort to increase the sector, other restrictions are placed on the nonlinearity.

The next logical restriction on the nonlinearity is its slope. Several papers [8, 10, 11, 15, 17, 18] have been published in which stability criteria are presented when the slope of $\phi(\sigma)$ is restricted. The stability criteria of only two papers [10, 11] will be presented here. These papers were chosen because;

(1) The criteria are applicable to distributed parameter systems.

(2) Certain classes of inputs are allowed.

(3) The criteria are expressed in terms of the frequency response of the linear subsystem G.

Stability Criteria Presented in [11]

Reference [11] states that if system S satisfies the following assumptions;

(1) The zero input response of G, $z(t)$, is bounded and uniformly continuous for $0 \le t < \infty$ and $z(t)$, $\dot{z}(t)$, $\ddot{z}(t)$, are elements of $L_2(0, \infty)$. The impulse response of G, $h(t)$, and its derivative are elements of $L_1(0, \infty)$.

(2) The input, $r(t)$, satisfies the same conditions as $z(t)$.

(3) $\phi(\sigma)$ is a single-valued, time-invariant nonlinearity which lies in the sector $[0, k]$ for $k < \infty$. $\phi(0)$ is zero and the slope of $\phi(0)$ is restricted so that

$$-k_1 \le \frac{d\phi(\sigma)}{d\sigma} \le k_2 \text{ for } k_1, k_2 \ge 0$$

then sufficient conditions for L_2-stability of S are that there exist a finite number q and a finite number $\mu \ge 0$ such that for all $w \ge 0$ the following inequality

$$\mathrm{Re}\,[(1 + jwq)\,G(jw)] + \frac{1}{k} + \mu w^2\,[1 + (k_2 - k_1)\,\mathrm{Re}\,G(jw) - k_1 k_2\,|G(jw)|^2] > 0 \tag{4a}$$

is satisfied and also

$$\lim_{|\sigma| \to \infty} q \left[\int_0^{\sigma} \phi(\sigma)\,d\sigma - \frac{\sigma\phi(\sigma)}{2} \right] = +\infty\,. \tag{4b}$$

When the slope restriction on $\phi(\sigma)$ is otherwise specified inequality (4a) becomes;

(1) For $-k_1 \le \frac{d\phi}{d\sigma}$

$$\mathrm{Re}\,[(1 + jwq)\,G(jw)] + \frac{1}{k} + \mu w^2\,[\mathrm{Re}\,G(jw) - k_1\,|G(jw)|^2] > 0 \tag{4c}$$

(2) For $k_2 \ge \frac{d\phi}{d\sigma}$

$$\mathrm{Re}\,[(1 + jwq)\,G(jw)] + \frac{1}{k} - \mu w^2\,[\mathrm{Re}\,G(jw) + k_2\,|G(jw)|^2] > 0 \tag{4d}$$

(3) For $0 \leq \frac{d\phi}{d\sigma} \leq k_2$

$$\text{Re}\ [(1 + jwq)\ G(jw)] + \frac{1}{k} + \mu w^2 [\text{Re}\ G(jw) + \frac{1}{k_2}] > 0 \qquad (4e)$$

(4) For $0 \leq \frac{d\phi}{d\sigma}$

$$\text{Re}\ [(1 + jwq + \mu w^2)\ G(jw)] + \frac{1}{k} > 0 \qquad (4f)$$

(5) For $0 \leq \frac{d\phi}{d\sigma} \leq k$

$$\text{Re} \left[1 + \frac{jwq}{1 + \mu w^2} \right] G(jw) + \frac{1}{k} > 0 \qquad (4g)$$

Due to the $|G(jw)|^2$ term, inequalities (4a), (4c) and (4d) apply to the principal case and simplest particular case[1] of G. Inequalities (4e), (4f), and (4g) apply to all particular cases. Particular cases must be stable - in - the - limit and the nonlinearity must lie in the sector $[\varepsilon, k]$.

The criteria have two adjustable parameters q and μ and are therefore difficult to interpret graphically. Similar criteria in which only one parameter is present are the following:

Stability Criteria Presented in [10]

If the system S satisfies the following conditions;

(1) Zero input response, $z(t)$, is bounded and in $L_1(0, \infty)$. Its derivative is bounded, uniformly continuous on $(0, \infty)$ and in $L_1(0, \infty)$.

---

[1] G is called the simplest particular case when the only jw-axis poles it has is a single pole at the origin.

(2) The input, $r(t)$, satisfies the same conditions that $z(t)$ does.

(3) $\phi(\sigma)$ is a single-valued, time-invariant nonlinearity which lies in the sector $[0, k]$ for principal cases and $[\varepsilon, k]$ for particular cases. $\phi(0)$ is zero and the slope of $\phi(\sigma)$ is restricted so that

$$- k_1 < \frac{d\phi(\sigma)}{d\sigma} < k_2 \qquad \text{for } k_1, k_2 > 0.$$

$\phi(\sigma)$ is bounded from above for principal cases and for particular cases, $\phi(\sigma) - \varepsilon\sigma$, is bounded from above. This bound always exists for a practical system. Then for S to be L_2-stable it is sufficient that there exists a finite number q such that for all $w \geq 0$ the inequality

$$\text{Re } [jwq\, G(jw)] + w^2 \,[1 + (k_2 - k_1) \text{ Re } G(jw) - k_1 k_2 \,|G(jw)|^2] \geq 0 \qquad (4h)$$

is satisfied and also

$$G(jw) \neq -\frac{1}{k}, \quad G(0) > -\frac{1}{k}. \qquad (4i)$$

When the slope is otherwise specified the resulting criteria are;

(1) For $- k_1 < \frac{d\phi}{d\sigma}$

$$\text{Re } [jwq\, G(jw)] + w^2 \,[\text{Re } G(jw) - k_1 \,|G(jw)|^2] \geq 0 \qquad (4j)$$

(2) For $k_2 > \frac{d\phi}{d\sigma}$

$$\text{Re } [jwq\, G(jw)] - w^2 \,[\text{Re } G(jw) + k_2 \,|G(jw)|^2] \geq 0 \qquad (4k)$$

(3) For $0 < \frac{d\phi(\sigma)}{d\sigma} < k_2$

$$\mathrm{Re}\,[jwq\,G(jw)] + w^2\,[\mathrm{Re}\,G(jw) + \frac{1}{K_2}] \geq 0 \tag{4ℓ}$$

(4) For $0 < \frac{d\phi}{d\sigma}$

$$\mathrm{Re}\,[(jwq + w^2)\,G(jw)] \geq 0 \tag{4m}$$

Again due to the $|G(jw)|^2$ term inequalities (4h), (4j) and (4k) can only be satisfied by the principal and simplest particular cases of G. As usual particular cases are required to be stable - in - the - limit.

Example

As an example of the above criteria consider the system S where

$$G(s) = \frac{s^2 + as + b}{(s^2 + 1)(s + c)} \qquad a, b, c > 0 \qquad b - ac - 1 < 0.$$

In Chapter III it was shown that this system satisfied Popov's criterion only if the additional restriction $(bc + a - c) > 0$ is placed on G(s). Appendix E shows that when the nonlinearity is monotone and satisfies restriction (4b) then criterion (4f) establishes L_2-stability with the restriction $(bc + a - c) > 0$ removed.

Discussion

Though the stability criteria (4a) - (4m) are more liberal with G(s) than Popov's criterion, they do not establish stability in the bounded input-bounded output sense. The criteria are also mainly useful for analysis purposes where k_1, k_2, and k are given.

Restriction (4b) can always be satisfied in a practical situation since the nonlinearity is specified over a finite interval of σ only, say $[-\sigma_L, \sigma_L]$. Beyond this interval the nonlinearity can be extended arbitrarily. Restriction (4b) may therefore be written as

$$\lim_{|\sigma|\to\infty} q \left[\int_0^{\sigma_L} \phi(\sigma)d\sigma + \int_{\sigma_L}^{\sigma} \phi(\sigma)d\sigma - \frac{\sigma\phi(\sigma)}{2} \right] = +\infty .$$

The first integral is some finite, positive quantity, C, and if after σ_L the nonlinearity is defined by

$$\phi(\sigma) \equiv k_L(\sigma - \sigma_L) + \phi(\sigma_L),\ k_L < k,$$

restriction (4b) becomes

$$\lim_{|\sigma|\to\infty} q \left[C + \frac{k_L\sigma^2}{2} - \frac{k_L\sigma_L{}^2}{2} + (\phi(\sigma\ ) - k_L\sigma_L)\,\sigma - (\phi(\sigma_L) - k_L\sigma_L)\cdot \sigma_L - \frac{k_L\sigma^2}{2} - (\phi(\sigma_L) - k_L\sigma_L)\,\frac{\sigma}{2} \right] = +\infty \qquad (4n)$$

The limit of the left-hand side of (4n) is that of

$$q \lim_{|\sigma|\to\infty} [(\phi(\sigma_L) - k_L\sigma_L)\,\frac{\sigma}{2}\,] = +\infty .$$

This can always be satisfied by the proper choice of

$$\phi(\sigma_L) - k_L\sigma_L .$$

Inequality (4h) is similar to inequality (4a) when μ is set equal to 1. Since (4h) has one parameter while (4a) has two it would be suspected that satisfaction of (4h) would imply that (4a) is also satisfied. Here it is shown that if (4h) satisfies the stronger inequality

$$\mathrm{Re}\,[jwq\,G(jw)] + w^2\,[1 + (k_2 - k_1)\,\mathrm{Re}\,G(jw)\;k_1k_2\;|G(jw)|^2] \geq \delta > 0$$

where $\delta > 0$ is a sufficiently small number then (4a) is satisfied also.

The two inequalities can be written as

$$q_1 Z(w) + H(w) \geq \delta > 0 \quad \text{for all } w \geq 0 \tag{4o}$$[2]

and

$$qZ(w) + \mu H(w) + M(w) > 0 \quad \text{for all } w \geq 0 \tag{4p}$$

where

$$Z(w) \equiv \mathrm{Re}\,[jw\,G(jw)]$$

$$H(w) \equiv w^2\,[1 + (k_2 - k_1)\,\mathrm{Re}\,G(jw) - k_1k_2\,|G(jw)|^2]$$

$$M(w) \equiv \mathrm{Re}\,G(jw) + \frac{1}{k}\ .$$

For the principal case and the simplest particular case, $M(w)$ is always finite. Let the minimum value of $M(w)$ be C and let q/μ be identical to q_1. Inequality (4p) can therefore be written as

$$\mu[q_1 Z(w) + H(w) + C/\mu] > 0 \qquad \text{for all } w \geq 0 \tag{4q}$$

---

[2] The parameter q in inequality (4h) is re-defined to be q_1. This is permitted because the parameters in the two criteria are not related.

The minimum value of $q_1 Z(w) + H(w)$ is δ and when this is substituted into (4q) it becomes

$$\mu[\delta + C/\mu] > 0.$$

A $\mu>0$ can always be found such that

$$\delta + C/\mu > 0.$$

Therefore inequality (4o) implies inequality (4p).

When the nonlinearity lies in the sector $[\varepsilon, k]$ then as in Chapter II the conditions on $r(t)$ can be relaxed. Again transformation (2e) results in the equivalent system of Figure 9. If system S satisfies any of stability criteria presented in this chapter then so does the equivalent system[3].

The input $\bar{r}(t)$ is now required to satisfy all the conditions previously imposed on $r(t)$. Therefore the stability criteria may be applicable even if $r(t)$ does not satisfy all the conditions.

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[3] For stability criteria (4a)-(4g) see [11], p. 57. For stability criteria (4h)-(4m) see [10], p. 487.

CHAPTER V

CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER RESEARCH

This thesis has presented some of the latest results in the stability theory of nonlinear, feedback systems. The study was limited to single-loop, continuous time systems containing a single nonlinearity and to stability criteria which in the opinion of the investigator have direct engineering applications. For multi-loop and sampled data nonlinear systems counterparts of most of the stability criteria investigated can be found in the literature.

The emphasis of the thesis was on Popov's criterion. This criterion is very liberal in its restrictions on the nonlinearity, easy to apply and as illustrated by the examples quite powerful. The criterion was extended to a certain class of hysteretic nonlinearities and it was shown that the criterion could be used to predict zero steady-state error operation of certain systems.

Several systems which verify Aizerman's conjecture were singled out by means of Popov's criterion. Systems for which the criterion does not verify the conjecture were also presented. It was then pointed out that the nonlinear sector could be enlarged by placing greater restrictions on the nonlinearity. The stability criteria which arise out of these restrictions were stated and discussed.

The investigation has suggested several topics for further research. A fundamental problem is whether the stability criteria investigated are necessary as well as sufficient. This problem is one which would require considerable research. An attempt to disprove the necessity of Popov's criterion by means of a counter-example proved unsuccessful.

Other problems which should be less difficult are:

(1) The establishment of simple criteria for stability-in-the-limit when G is a distributed parameter system. The case where G contains a simple delay would merit special consideration.

(2) The extension of the stability criteria of Chapter IV to prove bounded input-bounded output stability.

(3) A comparison between the rigorous stability criteria of this investigation and the approximate stability criteria of the describing function method.

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APPENDICES

APPENDIX A

NECESSARY AND SUFFICIENT CONDITIONS FOR G TO BE STABLE - IN - THE - LIMIT

When G is a lumped parameter system then for G to be stable - in - the - limit it is necessary and sufficient that the multiplicity, v_o, of the jw-axis poles be not greater than two and that the following conditions hold[1], (w_o denotes a jw-axis pole):

(1) For $w_o = 0, \quad v_o = 1$:

$$\lim_{w \to 0^+} \operatorname{Im} G(jw) = -\infty.$$

(2) For $w_o = 0, \quad v_o = 2$:

$$\lim_{w \to 0^+} \operatorname{Re} G(jw) = -\infty \text{ and } \lim_{w \to 0^+} \operatorname{Im} G(jw) < 0.$$

(3) For $w_o \neq 0, \quad v_o = 1$:

when w traverses the point w_o going from $w < w_o$ to $w > w_o$, the frequency response G(jw) goes to a point at infinity in such a way that points on the negative real axis with abscissas of arbitrarily large magnitude remain to the left of G(jw).

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[1] Aizerman and Gantmacher, [2], pp. 67-79.

(4) For $w_o \neq 0$, $v_o = 2$:

$$\lim_{w \to w_o^-} \mathrm{Re}\, G(jw) = \lim_{w \to w_o^+} \mathrm{Re}\, G(jw) = -\infty$$

$$\text{and } \mathrm{Im}\, G(jw) \begin{cases} > 0 \text{ for } w_o - n < w < w_o \\ < 0 \text{ for } w_o < w < w_o + n \end{cases} \quad (n > 0 \text{ small}).$$

System G may also be proven to be stable - in - the - limit by means of its root locus. G will be stable - in - the - limit if its root locus stays entirely in the left-hand s-plane as the linear gain is increased from zero. This is shown in Figure 16.

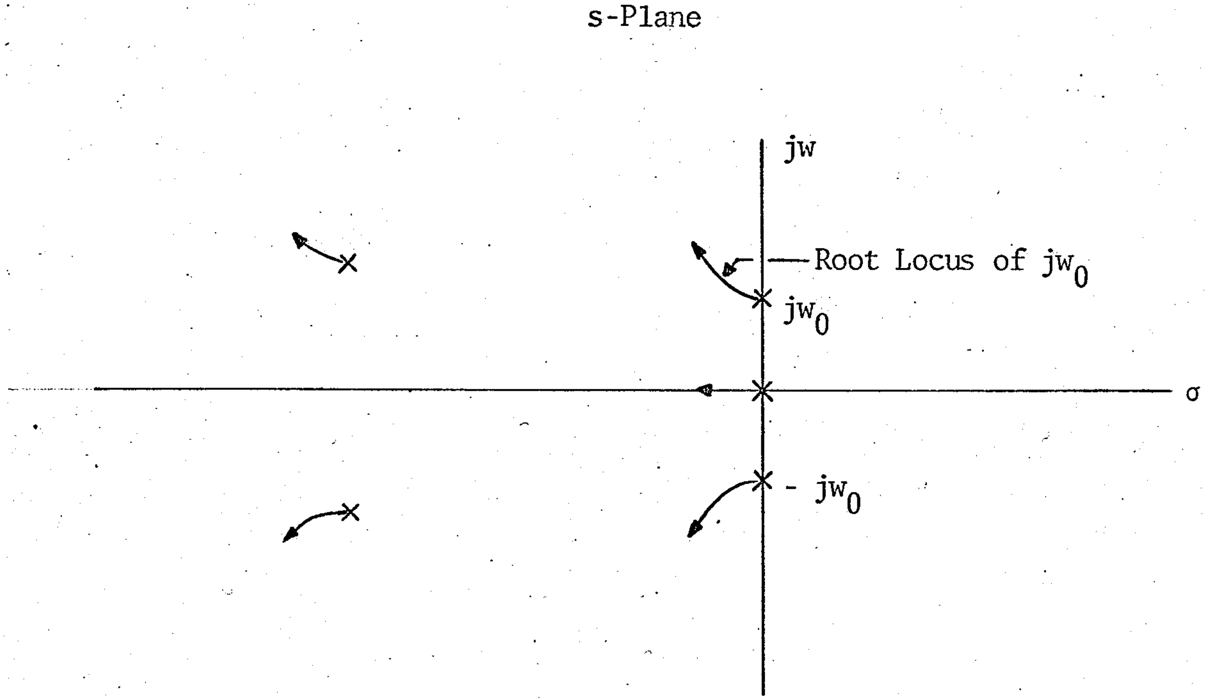

Figure 16 - Root Locus Interpretation of Stability - in - the - Limit

APPENDIX B

REFINEMENTS OF INEQUALITY (1f)

In the special case where system S is autonomous and G(s) is a lumped parameter system then inequality (1f) may be relaxed slightly. Sufficient conditions for the global asymptotic stability of S may now be stated as[1]:

(1) For the principal case of G(s) the satisfaction of the inequality

$$\mathrm{Re}\,[(1 + jwq)\,G(jw)] + 1/k > 0 \text{ for all } 0 \le w < \infty \qquad (P)$$

where the nonlinearity lies in the sector [0, k] and k may be finite or infinite. This is known as the basic Popov condition.

(2) When G(s) has a single pole at the origin and no other jw-axis poles the inequality (P) above must be satisfied. G(s) must be stable - in - the - limit and the nonlinearity must lie in the sector (0, k] where k may be finite or infinite.

(3) For all particular cases of G(s) the inequality

$$\mathrm{Re}\,[(1 + jwq)\,G(jw)] + 1/k \ge 0 \text{ for all } 0 \le w < \infty \qquad (P_-)$$

where the nonlinearity lies in the sector $[\varepsilon, k]$ with k a finite number and $G(jw) \ne - 1/k$. This is called the weakened Popov condition and allows the Popov line to have points in common with the $G^*(jw)$ locus.

(4) For all particular cases with a zero root the inequality

$$\mathrm{Re}\,[(1 + jwq)\,G(jw)] + 1/k > 0 \text{ for all } 0 \le w \le \infty\,. \qquad (P_+)$$

This is called the strengthened Popov condition. The nonlinearity must lie in the sector (0, k] where k may be infinite. When G(s) has

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[1] Aizerman and Gantmacher, [2], pp. 141-143.

a double pole at the origin the additional restriction,

$$\int_0^{\pm\infty} \phi(\sigma)d\sigma = \infty$$

is required.

APPENDIX C

PROOF OF LEMMA

The functional equation describing system S is

$$\sigma(t) = r(t) - z(t) - \int_0^t h(t - \tau)\ y(\tau)\ d\tau. \tag{C1}$$

When (C1) is differentiated the resulting equation is

$$\dot{\sigma}(t) = \dot{r}(t) - \dot{z}(t) - \int_0^t \dot{h}(t - \tau)\ y(\tau)\ d\tau - h(0)\ y(t). \tag{C2}$$

The time variables r(t), z(t) $\dot{r}(t)$, $\dot{z}(t)$ and y(t) are truncated at T and denoted by $r_T(t)$, $z_T(t)$, $\dot{r}_T(t)$, $\dot{z}_T(t)$, and $y_T(t)$. The functions $\sigma_T(t)$ and $\dot{\sigma}_T(t)$ are then defined by the following two equations,

$$\sigma_T(t) = r_T(t) - z_T(t) - \int_0^t h(t - \tau)\ y_T(\tau)\ d\tau \tag{C3}$$

$$\dot{\sigma}_T(t) = \dot{r}_T(t) - \dot{z}_T(t) - \int_0^t \dot{h}(t - \tau)\ y_T(\tau)\ d\tau - h(0)\ y_T(t). \tag{C4}$$

For t > T, $\sigma_T(t)$ and $\dot{\sigma}_T(t)$ satisfy the inequalities

$$|\sigma_T(t)| \leq k_3 e^{-k_2 t}, \qquad |\dot{\sigma}_T(t)| \leq k_4 e^{-k_2 t}$$

where k_4 and k_3 are positive constants and k_2 is defined in

$$|h(t)| < k_1 e^{-k_2 t}.$$

When equation (C4) is multiplied by q and added to (C3) the result after multiplying through by a negative sign is

$$- \sigma_T(t) - q\dot{\sigma}_T(t) = - [r_T(t) - z_T(t) + q(\dot{r}_T(t) - \dot{z}_T(t))] + \int_0^t [h(t - \tau)$$

$$+ q\dot{h}(t - \tau)] \; y_T(\tau) \, d\tau + qh(0) \, y_T(t) \tag{C5}$$

Add $[1/k - \gamma]\, y_T(t)$ to both sides of (C5) and multiply through by $e^{\alpha t}$ where $0 < \alpha < k_2$. The result is

$$[- \sigma_T(t) - q\dot{\sigma}_T(t) + (1/k - \gamma) \, y_T(t)] \, e^{\alpha t} = - [r_T(t) - z_T(t) +$$

$$q(\dot{r}(t) - \dot{z}_T(t))] \, e^{\alpha t} + \int_0^t e^{\alpha(t - \tau)} \, [h(t - \tau) + q\dot{h}(t - \tau)]$$

$$e^{\alpha\tau} \, y_T(\tau) \, d\tau + qh(0) \, e^{\alpha t} \, y_T(t) + [1/k - \gamma] \, e^{\alpha t} \, y_T(t). \tag{C6}$$

Define

$$f_1(t) \equiv [- \sigma_T(t) - q\dot{\sigma}_T(t) + (1/k - \gamma) \, y_T(t)] \, e^{\alpha t}$$

$$f_2(t) \equiv - [r_T(t) - z_T(t) + q(\dot{r}_T(t) - \dot{z}_T(t))] \, e^{\alpha t}$$

Equation (C6) is then rewritten as

$$f_1(t) = f_2(t) + \int_0^t e^{\alpha(t - \tau)} \, [h(t - \tau) + q\dot{h}(t - \tau)] \, e^{\alpha\tau} \, y_T(\tau)$$

$$d\tau + qh(0) \, e^{\alpha t} \, y_T(t) + (1/k - \gamma) \, e^{\alpha t} \, y_T(t). \tag{C7}$$

All the terms in (C7) belong to $L_2(0, \infty)$ and therefore the Fourier transform of (C7) is

$$F_1(jw) = F_2(jw) + \{ [1 + q(jw - \alpha)] \, G(jw - \alpha) + 1/k - \gamma \} \, Y_T(jw - \alpha). \tag{C8}$$

The lemma concerning the frequency domain analysis in the V.M. Popov theorem states that if three real functions $f_1(t)$, $f_2(t)$, $f_3(t)$ belong to $L_2(0, \infty)$ and if their Fourier transforms are related by

$$F_1(jw) = H(jw)\, F_3(jw) + F_2(jw)$$

where $\operatorname{Re} H(jw) \geq \beta > 0$ for all $w \geq 0$ then

$$-\int_0^\infty f_1(t)\, f_3(t)\, dt \leq \frac{1}{4\beta} \int_0^\infty [f_2(t)]^2\, dt.$$

(C8) satisfies the conditions of this lemma if $\beta = \delta - \gamma$ and

$$\operatorname{Re} \{[1 + q(jw - \alpha)]\, G(jw - \alpha)\} + 1/k - \gamma \geq \delta - \gamma > 0. \qquad \text{(C9)}$$

The inequality $\operatorname{Re} [(1 + jwq)\, G(jw)] + 1/k \geq \delta > 0$ implies that (C9) is true. Therefore the lemma yields

$$-\int_0^\infty f_1(t)\, y_T(t)\, e^{\alpha t}\, dt \leq \frac{1}{4(\delta - \gamma)} \int_0^\infty [f_2(t)]^2\, dt. \qquad \text{(C10)}$$

When the defining equations for $f_1(t)$ and $f_2(t)$ are substituted into (C10) the result is

$$\int_0^T (\sigma(t) - y(t)/k)\, y(t)\, e^{2\alpha t}\, dt + q \int_0^T \dot{\sigma}(t)\, y(t)\, e^{2\alpha t}\, dt$$
$$+ \gamma \int_0^T e^{2\alpha t}\, y^2(t)\, dt < \frac{1}{4(\delta - \gamma)} \int_0^T e^{2\alpha t}\, [r(t) - z(t)$$
$$+ q(\dot{r}(t) - \dot{z}(t))]^2\, dt. \qquad \text{(C11)}$$

Denote the right-hand side of (C11) by C(T), integrate ${}_0\int^T \dot{\sigma}(t)\, y(t)\, e^{2\alpha t}\, dt$ by parts and add $q \int_0^{\sigma(0)} \phi(\sigma, \phi_0)\, d\sigma$ to both sides. The result is

$$\int_0^T \left[\sigma - \frac{\phi(\sigma, \phi_0)}{k}\right] \phi(\sigma, \phi_0)\, e^{2\alpha t}\, dt + q\, e^{2\alpha T} \int_0^{\sigma(T)} \phi(\sigma, \phi_0)\, d\sigma$$

$$- 2q\alpha \int_0^T e^{2\alpha t} \left[\int_0^{\sigma(t)} \phi[\sigma, \phi_0]\, d\sigma\right] dt + \gamma \int_0^T e^{2\alpha t}\, y^2(t)\, dt$$

$$< C(T) + q \int_0^{\sigma(0)} \phi(\sigma, \phi_0)\, d\sigma. \qquad \text{(C12)}$$

Let $\phi(\sigma, \phi_0)$ be restricted further so that it lies in the sector $[\varepsilon, k - \varepsilon]$, $\varepsilon > 0$ is arbitrarily small. Then the following inequalities hold:

$$\text{(1)} \quad \int_0^{\sigma(t)} \phi(\sigma, \phi_0)\, d\sigma \leq k/2\, \sigma^2(t)$$

$$\text{(2)} \quad \varepsilon^2/k\, \alpha^2 \leq \left[\sigma - \frac{\phi(\sigma, \phi_0)}{k}\right] \phi(\sigma, \phi_0).$$

Inequality (C12) is then strengthened by using (1) and (2) and deleting the positive quantity

$$q e^{2\alpha T} \int_0^{\sigma(T)} \phi(\sigma, \phi_0)\, d\sigma$$

from the left-hand side. This is where the integral constraint on the hysteretic nonlinearity takes effect. If γ is set equal to $\delta/2$ inequality (C12) becomes

$$2/\delta \int_0^T e^{2\alpha t}\, [\varepsilon^2/k - kq\alpha]\, \sigma^2(t)\, dt + \int_0^T e^{2\alpha t}\, y^2(t)\, dt \leq 1/\delta^2$$

$$\int_0^T e^{2\alpha t}\,[r(t) - z(t) + q(\dot{r}(t) - \dot{z}(t))]^2\,dt + 2q/\delta \int_0^{\sigma(0)} \phi(\sigma, \phi_0)\,d\sigma\,. \tag{C13}$$

Define I_1 to be

$$\int_0^T e^{2\alpha t}\,[\varepsilon^2/k - kq\alpha]\,\sigma^2(t)\,dt$$

and note that for $\alpha > 0$, I_1 is > 0. Thus I_1 may be dropped from the left-hand side of (C13). The condition $\varepsilon^2/k - kq\alpha \geq 0$ is satisfied since for any $\varepsilon > 0$, $q < \infty$, $k < \infty$ an α such that $0 < \alpha \leq \varepsilon^2/(qk^2)$ can always be found.

Inequality (C13) becomes

$$\left[\int_0^T e^{2\alpha t}\,y^2(t)\,dt\right]^{\frac{1}{2}} \leq \left[1/\delta^2 \int_0^T e^{2\alpha t}\,[r(t) - z(t) + q(\dot{r}(t) - \dot{z}(t))]^2\,dt + 2q/\delta \int_0^{\sigma(0)} \phi(\sigma, \phi_0)\,d\sigma\right]^{\frac{1}{2}} \quad \text{for all } T > 0.$$

APPENDIX D

VERIFICATION OF AIZERMAN'S CONJECTURE FOR THE VARIOUS SYSTEMS

A - Systems $\frac{1}{s(s+a)}$, $\frac{1}{s(s^2+as+b)}$ and $\frac{1}{s(s^3+as^2+bs+c)}$ have been verified in the literature [8, pp. 256-257].

B - System $\frac{s+a}{s(s^2+bs+c)}$. $a, b, c > 0$.

The time transformation $\tau = at$ transforms the system into the equivalent one $(s+1)/[a^2 s(s^2+a_1 s+b_1)]$, $a_1 = b/a$, $b_1 = c/a^2$. Normalize the transfer function with respect to $1/a^2$. The stability sectors for the normalized system are:

(i) $0 < k < \infty \quad a_1 > 1$.

(ii) $0 < k < \frac{a_1 b_1}{1-a_1} \quad a_1 < 1$.

The modified frequency response locus is

$$G^*(jw) = \frac{b_1 - a_1 - w^2}{(b_1 - w^2)^2 + a_1^2 w^2} - j\,\frac{b_1 + w^2(a_1 - 1)}{(b_1 - w^2)^2 + a_1^2 w^2}\,.$$

If $a_1 > 1$ then Im $G^*(jw)$ is never zero and therefore the locus of $G^*(jw)$ always remains below the Re $G^*(jw)$ axis. Popov's criterion is then satisfied for the sector $[\varepsilon, \infty)$.

If $a_1 < 1$ then Im $G^*(jw)$ becomes zero at $w^2 = \frac{b_1}{1-a_1}$. The slope of $G^*(jw)$ at this frequency is

$$\frac{b_1(1 - a_1)(1 + b_1 - a_1)}{b_1(1 - a_1^2)^2 + b_1^2 + (1 - a_1)^2} .$$

Form the expression $X(w) - qY(w)$, differentiate with respect to w and set the derivative to zero. The result is

$$w^4[1 + q(1 - a_1)] + w^2[- 2(b_1 - a_1) - 2b_1 q]$$

$$+ [b_1^2 - b_1 a_1^2 + a_1^3 - 2b_1 a_1 + qb_1^2 + qb_1^2 a_1 - qb_1 a_1^2] = 0.$$

For real, positive w the function $X(w) - qY(w)$ has at most two critical points, a maximum and a minimum. If $1/q$ is set equal to the slope of $G^*(jw)$ at $w^2 = (b_1)/(1 - a_1)$ and these values of q and w are substituted into the derivative the resulting expression is zero. Therefore $X(w) - qY(w)$ has a relative minimum at this frequency of $-(1 - a_1)/(a_1 b_1)$.

If $X(0) - qY(0) + (1 - a_1)/(a_1 b_1)$ is greater than zero then this will be an absolute minimum as well. With the q chosen $X(0) - qY(0) + (1 - a_1)/(a_1 b_1)$ is

$$\frac{b_1 - a_1}{b_1^2} + \frac{b_1(1 - a_1^2) + b_1^2 + (1 - a_1)^2}{b_1^2(1 - a_1)(1 + b_1 - a_1)} + \frac{1 - a_1}{a_1 b_1} .$$ This can be rewritten as

$$\frac{1}{a_1 b_1^2(1 + b_1 - a_1)} \left[a_1(1 - a_1)^2 + b_1(1 + b_1) + \frac{a_1 b_1^2}{1 - a_1} \right]$$

which is always greater than zero. Therefore Popov's criterion is satisfied for the sector

$$\left[\varepsilon, \frac{a_1 b_1}{1 - a_1} - \varepsilon\right].$$

$\underline{C}$ - System $\dfrac{s + as + b}{(s^2 + 1)(s + c)}$ $\quad a, b, c > 0 \quad b - ac - 1 < 0.$

The linear system is stable in the sector $(0, \infty)$. $G^*(jw)$ is

$$\frac{c(b - w^2) + aw^2}{(1 - w^2)(c^2 + w^2)} + j\,\frac{w^2(ac - b - w^2)}{(1 - w^2)(c^2 + w^2)} .$$

Due to the $(1 - w^2)$ term in the denominator of $G^*(jw)$ q is constrained to be $(bc - c + a)/(ac - b + 1)$ in order that the numerator of $X(w) - qY(w)$ can be divisible by $(1 - w^2)$. $X(w) - qY(w)$ becomes

$$\frac{w^2\left(\dfrac{bc + a - c}{ac - b + 1}\right) + bc}{c^2 + w^2}$$

and is greater than zero if $(bc + a - c)$ is greater than zero. Therefor Popov's criterion is satisfied for the sector, $[\varepsilon, \infty)$.

APPENDIX E

APPLICATION OF CRITERION (4f) TO

$$G(s) = \frac{s^2 + as + b}{(s^2 + 1)(s + c)}$$

In terms of G(jw) criterion (4f) is

$$\frac{c(b - w^2) + aw^2}{(1 - w^2)(c^2 + w^2)} - \frac{qw^2(ac - b - w^2)}{(1 + \mu w^2)(1 - w^2)(c^2 + w^2)} > 0 \text{ for all}$$

$w \geq 0$. If the term $(1 - w^2)$ is to be divided out then q must be

$$\frac{(cb - c + a)(1 + \mu)}{ac - b + 1}.$$

For this value of q the criterion becomes

$$\frac{-w^2\left[\frac{(c - a - bc) + \mu(-ac^2 + a^2c - ab)}{ac - b + 1}\right] + bc}{(1 + \mu w^2)\ (c^2 + w^2)} > 0 \text{ for all}$$

$w \geq 0$. If $(c - a - bc) + (-ac^2 + a^2c - ab)$ is less than zero then the criterion is satisfied for the sector $[\varepsilon, \infty)$. Consider only the case $(a + bc - c) < 0$ since Popov's criterion applies when $(a + bc - c)$ is greater than zero. But $(a + bc - c) < 0$ implies that $(ac - c^2 - b)$ is less than zero. Therefore a $\mu > 0$ can be chosen that $(c - a - bc) + \mu a(ac - c^2 - b) < 0$.