Regular Orderings of Left Orderable Groups

by

Dana Kapoostinsky

A thesis submitted to the Faculty of Graduate Studies of
The University of Manitoba
in partial fulfillment of the requirements of the degree of

Master of Science

Department of Mathematics
University of Manitoba
Winnipeg

Abstract

This thesis explores the left orderings of finitely generated torsion-free abelian groups and finitely generated torsion-free nilpotent groups through the lens of computational complexity theory. We demonstrate that both types of groups admit regular left orderings. In chapters 2 and 3, the preliminaries of orderable groups and formal languages are introduced. In Chapter 4, we establish that the set of regular left orderings of $(\mathbb{Z}^n, +)$ is dense within the space of left orderings, $\mathrm{LO}(\mathbb{Z}^n)$. This result is achieved by approaching the problem from a geometric perspective and applying techniques from lattice theory. In Chapter 5, we extend this finding to finitely generated torsion-free nilpotent groups to obtain an analogous result, that is, we show that the set of regular left orderings of a finitely generated torsion-free nilpotent group is also dense within its space of orderings.

Acknowledgements

I would like to express my sincerest gratitude to Dr. Adam Clay for being an exceptional supervisor and an exemplary educator. I am grateful to Dr. Susan Cooper and Dr. Thomas Kucera for taking the time to carefully read my thesis and provide me with valuable feedback. Thank you to my family for making this possible. Thank you to Colin for the countless conversations about math.

Dedicated to Dvora

Contents

Abstract **i**

Acknowledgements **i**

List of Figures **vi**

List of Tables **vi**

1 Introduction **1**

2 Orderable Groups **5**

2.1 Left Orderable Groups 5

2.1.1 Bi-orderings and Archimedean Orderings 8

2.1.2 Conradian Orderings 12

2.2 Positive Cones 15

2.3 Spaces of Orderings 16

3 Formal Languages and Automata **21**

3.1 Finite State Automata 22

3.1.1 Transition Diagrams 25

3.2 Closure Properties of Regular Languages 28

3.3 Positive Cone Languages 30

4 **Regular Orderings on $\mathbb{Z}^n$** **36**

4.1 Regularity in the 2−Dimensional Case 37

4.2 Rational Orderings in Higher Dimensions 41

4.3 Rational Flag Orderings are Regular 44

4.4 Density of Rational Flag Orderings 52

5 **Regular Left Orderings of Nilpotent Groups** **57**

5.1 Preliminaries 57

5.2 Nilpotent Groups and Conradian Orderings 60

5.3 Hirsch Rank 64

5.4 Density of Regular Orderings 69

5.5 Directions for Future Work 74

Appendix **75**

References **80**

List of Figures

2.1 Geometric representation of a positive cone of $(\mathbb{Z}^2, +)$. 17

3.1 The Chomsky hierarchy. 21

3.2 A machine $\mathcal{M}$ reading a tape containing the word 101011. 23

3.3 An FSA accepting the language $\mathcal{L}_1 = \{0^n 1^m \mid n \in \mathbb{N}, m \geqslant 0\}$. 25

3.4 A generalised FSA constructed to accept the union of two regular languages. 29

3.5 FSA accepting the positive cone language $\mathcal{L}$. 31

3.6 An FSA accepting the language $P = \langle x, y \rangle^+$. 34

4.1 Vectors $\vec{v}_1, \vec{v}_2$ and points x, y in the plane. 38

4.2 A generalised FSA accepting the language X_{n+1}. 45

1

Introduction

A group G is left orderable if there exists a strict total ordering on its elements that is invariant under left multiplication. Each left ordering corresponds to a unique positive cone, that is, if $<$ is a left ordering of G, then $P = \{g \in G \mid g > 1\}$ is the corresponding positive cone. A standard tool used when studying the orderings of a group G is the *space of left orderings* of G, denoted by $\mathrm{LO}(G)$. The space of left orderings is constructed by taking all the positive cones of G and endowing the set with a topology. This naturally allows us to take what was initially a purely algebraic structure and view it as a topological structure.

A recent approach in the field of orderable groups has been to describe the positive cones of a group using tools from computational complexity theory. The central goal of our study will be to categorise the complexity of certain left orderings by associating their corresponding positive cones with formal languages and seeing where those languages fit within the *Chomsky hierarchy*. Regular languages are the "simplest" and sit at the bottom of the hierarchy. Therefore, it is of particular desire to find a left ordering of a group G that can be described by a positive cone language which is regular. In this case, we will say that G admits *regular left orderings*.

More precisely, the focus of this thesis will be to examine the regular left orderings of finitely generated torsion-free abelian groups and finitely generated torsion-free nilpotent groups. We not only show that regular orderings of such groups exist, but we also show that these orderings are dense in their respective spaces of orderings. This is summarised in the following two theorems, both of which are original results.

Theorem 1.0.1. *(Corollary 4.4.6). For $n \geqslant 1$, the set of regular left orderings is dense in* $\mathrm{LO}(\mathbb{Z}^n)$.

Theorem 1.0.2. *(Corollary 5.4.4) If G is a finitely generated left orderable group, then the set of regular left orderings of G is dense in* $\mathrm{LO}(G)$.

When studying the space of left orderings of G, one of the first questions that is typically asked is *what is its homeomorphism type?* The space $\mathrm{LO}(G)$ is either

1. finite;
2. uncountable and contains *no* isolated points;
3. uncountable and contains isolated points.

It is a basic fact that if $\mathrm{LO}(G)$ is finite, then all the positive cones in $\mathrm{LO}(G)$ are isolated points. Furthermore, if a positive cone P is finitely generated as a semigroup, then it is an isolated point in $\mathrm{LO}(G)$ (Proposition 2.3.8). In fact, most known examples of isolated points in the literature are finitely generated semigroups. For an example of a group that admits isolated orderings, see Example 3.3.5.

There is an intimate connection between the space of orderings of a group G and the complexity of the orderings that G admits. In some special cases, a classification of the complexity of the positive cones admitted by G can lead directly to a conclusion about the homeomorphism type of $\mathrm{LO}(G)$. We give some concrete examples below. To avoid giving the reader false impressions, we note that the purpose of our investigation is not to determine the homeomorphism types of the spaces of orderings of the aforementioned groups of interest, as these results are already known in the literature[1]. However, we would still like to emphasize the relevance of these new tools, *i.e.,* formal language and automata theory, to the longstanding questions and approaches of the field.

In the case when *all* positive cones of a left orderable group G are regular, then we can conclude that $\mathrm{LO}(G)$ is finite. This is due to the following result by Antolín, Rivas, and Su:

Theorem 1.0.3. *[1, Theorem 4.9] A group only admits regular left ordering if and only if it is Tararin poly-$\mathbb{Z}$.*

It is a well known result that if G is a Tararin group, then $\mathrm{LO}(G)$ is finite and consists entirely of isolated points. An example of such a group is the fundamental group of the Klein bottle, which is discussed extensively throughout this thesis.

Alternatively, if we can show that a group G does *not* admit any regular orderings, then this would imply that G does not admit any positive cones which are finitely generated as semigroups. This follows from the fact that a positive cone which is finitely generated as a semigroup is regular. While this would not definitively tell us that $\mathrm{LO}(G)$ does *not* contain isolated points, it is an incremental step towards determining the homeomorphism type of $\mathrm{LO}(G)$, since this would allow us to eliminate the existence of isolated points of the most common kind. From this perspective, an investigation into the existence of regular orderings can offer a *refinement* for distinguishing between different spaces of left orderings when the homeomorphism type fails to do so.

Consider the following result, due to Hermiller and Šunić, on the complexity of the orderings of free products:

Theorem 1.0.4. *[12, p. 3] Let A and B be two nontrivial, finitely generated, left orderable groups. There exists no left ordering on $G = A * B$ such that its positive cone is represented*

[1] If G is a finitely generated torsion-free abelian group, then $\mathrm{LO}(G)$ is either finite or uncountable without isolated points (see Sikora's Theorem (4.4.1). The same is true if G is a finitely generated torsion-free nilpotent group (see [20, Theorem 1.1]).

by a regular language.

By the reasoning above, this result alone should lead us to believe that the space of orderings of $A * B$ does not contain isolated points. This turns out to be true, due to the following theorem by Rivas:

Theorem 1.0.5. *[19, Theorem A] Let A and B be two left orderable groups. Then the free product $A * B$ has no isolated left orderings.*

In contrast, the space of orderings of $(\mathbb{Z}^n, +)$ is of the same homeomorphism type as $\mathrm{LO}(A*B)$ (Theorem 4.4.1), however $\mathbb{Z}^n$ not only admits regular orderings but does so in abundance. In this case, the existence of regular positive cones (or lack thereof) allows us to discern between two homeomorphic spaces.

2

Orderable Groups

2.1 Left Orderable Groups

Definition 2.1.1. A group $G = (G; \cdot, ^{-1}, 1)$ is *left orderable* if there exists a strict total ordering $<$ on the elements of G that is invariant under left multiplication, that is, for all $f, g, h \in G$, if $f < h$ then $gf < gh$. The ordering $<$ is called a left ordering of G.

One could equivalently define a right ordering. Fix a group G and a left ordering $<$. Define a new ordering $\prec$ on G by declaring $h^{-1} \prec f^{-1}$ whenever $f < h$. Since $<$ is left invariant, $gf < gh \implies h^{-1}g^{-1} \prec f^{-1}g^{-1}$, which means $\prec$ is invariant under right multiplication. Thus, any left ordering induces a right ordering and vice versa.

Remark 2.1.2. In general, and especially when the group operation is unspecified, we will use multiplicative notation in the context of groups. When working with abelian groups, we will occasionally switch to additive notation.

Remark 2.1.3. A group equipped with an ordering $<$ will be called an *ordered group*, and any element g such that $1 < g$ will be called a *positive element* of G with respect to $<$. Similarly, an element $g < 1$ will be called a *negative element*. The product of two positive (negative) elements is positive (negative).

Example 2.1.4. The groups $(\mathbb{R}, +), (\mathbb{Q}, +)$, and $(\mathbb{Z}, +)$ are left ordered groups with their usual orderings.

A necessary condition for left orderability of a group G is that G must be torsion-free. As a result, one can immediately disregard all finite groups in the study of left orderable groups.

Proposition 2.1.5. *[7, Proposition 1.3] Let $(G, <)$ be a left ordered group, then G is torsion-free.*

Proof. Suppose towards a contradiction that G is not torsion-free, that is, there exists an element $g \in G$ such that $g^n = 1$ for some $n \in \mathbb{Z}$. Without loss of generality, suppose that $1 < g$, then

$$1 < g < g^2 < \cdots < g^n = 1,$$

which is a contradiction. □

"Torsion-free" is not a sufficient condition for left orderability.

Example 2.1.6. [10, p. 42] The fundamental group of the Weeks manifold

$$\pi_1(W) = \langle a, b \,|\, babab = ab^{-2}a, ababa = ba^{-2}b \rangle$$

is torsion-free and not left orderable.

The group $\pi_1(W)$ is torsion-free because W is the fundamental group of a hyperbolic manifold [2]. To see that it is not left orderable, suppose towards a contradiction that $<$ is a left ordering on $\pi_1(W)$. Observe that from the two relators we obtain:

$$b^{-1}ab^{-2}a = (ab)^2 = ba^{-2}ba^{-1} \tag{2.1.1}$$

$$a^{-1}ba^{-2}b = (ba)^2 = ab^{-2}ab^{-1}. \tag{2.1.2}$$

Suppose that $1 < a$. There are three subcases that need to be considered:

- Case 1: Suppose $b < 1$, then $1 < b^{-1}$ and $1 < b^{-1}ab^{-2}a$, since the product of positive elements remains positive. Similarly, $a^{-1} < 1$, so $ba^{-2}ba^{-1} < 1$, but from equation (1.1.1), $1 < b^{-1}ab^{-2}a = ba^{-2}ba^{-1} < 1$, which is a contradiction.

- Case 2: Suppose $1 < a < b$, then $b^{-1}a < 1$ and $b^{-1}ab^{-2}a = (b^{-1}a)(b^{-1})(b^{-1}a) < 1$, since the product of negative elements is negative. Similarly, $1 < (ab)^2$. However, from equation (1.1.1), it follows that $1 < (ab)^2 = b^{-1}ab^{-2}a < 1$, which is a contradiction.

- Case 3: Suppose $1 < b < a$, then $a^{-1}b < 1$ and $a^{-1}ba^{-2}b = (a^{-1}b)(a^{-1})(a^{-1}b) < 1$. Also, $1 < (ba)^2$. From equation (1.1.2), it follows that $1 < (ba)^2 = a^{-1}ba^{-2}b < 1$, again a contradiction.

There are three remaining cases to check, which are symmetric to the first three cases. Their arguments follow similarly. Therefore, the group $\pi_1(W)$ is not left orderable.

Proposition 2.1.7. *[7, Problem 1.8] Let G be a group with normal subgroup N. If N and G/N admit left orderings $<_K$ and $<_Q$, respectively, then a left ordering on G can be constructed from $<_K$ and $<_Q$.*

Proof. Consider the short exact sequence

$$1 \to N \xhookrightarrow{i} G \xrightarrow{q} G/N \to 1,$$

where i is the inclusion map and q is the quotient map. Suppose that $<_K$ and $<_Q$ are left orderings on the kernel and quotient, respectively. Define an ordering $<$ on G as follows: if $q(g_1) <_Q q(g_2)$, then declare $g_1 < g_2$. Similarly, if $q(g_1) >_Q q(g_2)$, then set $g_1 > g_2$. Otherwise $q(g_1) =_Q q(g_2)$, so $g_1, g_2 \in N$. In this case, set $g_1 < g_2$ if $g_1 <_K g_2$. The new ordering $<$ on G is a strict total ordering, as a result of the fact that the cosets of G/N partition G and since $<_K$ and $<_Q$ are strict total orderings. Finally, left invariance of $<$ is inherited from $<_K$ and $<_Q$. □

Remark 2.1.8. The ordering defined in Proposition 2.1.7 is a type of *lexicographic ordering.*

Orderings obtained via short exact sequences will be of central importance in later chapters. For more on such orderings, see Examples 2.1.10 and 5.1.3.

2.1.1 Bi-orderings and Archimedean Orderings

Definition 2.1.9. A group G is *bi-orderable* if there exists a strict total ordering $<$ on the elements of G that is simultaneously invariant under left and right multiplication.

It is easy to see that any left orderable abelian group is also bi-orderable. Not all left orderable groups are bi-orderable. The fundamental group of the Klein bottle is a standard example of such a group.

Example 2.1.10. The fundamental group of the Klein bottle is given by the presentation

$$\pi_1(K) = \langle x, y \mid xyx^{-1} = y^{-1} \rangle.$$

The subgroup $N = \langle y \rangle$ is isomorphic to $\mathbb{Z}$ because y has infinite order. The fact that x and y both have infinite order becomes apparent by recognising $\pi_1(K)$ as an amalgamated product via the Seifert-Van Kampen Theorem. From this, it also follows that $\pi_1(K)/\langle y \rangle = \langle xN \rangle \cong \mathbb{Z}$, so $\pi_1(K)$ fits into the short exact sequence:

$$1 \to \langle y \rangle \hookrightarrow \pi_1(K) \to \pi_1(K)/\langle y \rangle \to 1.$$

Thus, by Proposition 2.1.7, $\pi_1(K)$ is left orderable. In fact, all left orderings of $\pi_1(K)$ are obtained from this short exact sequence [7, Example 9.2]. Since each of the kernel and quotient in this short exact sequence admits only two left orderings, it follows that $\pi_1(K)$ admits exactly four left orderings (for an explicit description of these orderings see Example 3.3.5).

Now, suppose that it is also bi-orderable, so there exists a bi-ordering $<$ on $\pi_1(K)$. Suppose further that $1 < y$, then

$$
\begin{aligned}
&x < xy && \text{by left multiplication}\\
&1 < xyx^{-1} && \text{by right multiplication}\\
&1 < y^{-1} && \text{substituting by the relator,}
\end{aligned}
$$

which is a contradiction since y and y^{-1} cannot both be positive. The argument follows similarly if we begin with the assumption that $1 < y^{-1}$. Therefore, $\pi_1(K)$ is not bi-orderable.

Lemma 2.1.11. *[7, Lemma 2.5] If G is a group admitting a bi-ordering $<$, and G does not have a least positive element with respect to $<$, then for any positive p there exists a positive q such that $1 < q^2 < p$.*

Proof. Suppose $1 < p$. Since G does not have a least positive element with respect to $<$, there exist elements $s, r \in G$ such that

$$1 < s < r < p.$$

From this inequality, it follows that $1 < rs^{-1}$. If $(rs^{-1})^2 < r$, then take $q = rs^{-1}$. Alternatively, if $r \leqslant (rs^{-1})^2$, then

$$r \leqslant rs^{-1}rs^{-1} \implies 1 \leqslant s^{-1}rs^{-1} \implies s^2 \leqslant r < p.$$

□

Definition 2.1.12. Let $(G, <)$ be a left ordered group. The ordering $<$ is *Archimedean* if for all positive $g, h \in G$ there exists an integer $n > 0$ such that $h < g^n$.

One of the characteristic features of Archimedean left orderings is that they are all bi-orderings. Our aim is to establish this fact in the next few lemmas. We then use this property to show that if G is a group admitting an Archimedean ordering, then G must be abelian (see Lemma 2.1.16).

Lemma 2.1.13. *[7, Problem 1.26] Let $(G,<)$ be a left ordered group. The ordering $<$ is also a bi-ordering if and only if for any $p > 1$ and any $g \in G$, $1 < gpg^{-1}$.*

Proof. For the forward direction, suppose that G admits a bi-ordering $<$. Let $p > 1$ and $g \in G$. Then

$$\begin{aligned} 1 < p \implies g &< gp && \text{by left invariance} \\ gg^{-1} &< gpg^{-1} && \text{by right invariance} \\ 1 &< gpg^{-1}. \end{aligned}$$

For the other direction, suppose that for all $p > 1$ and $g \in G$, $1 < gpg^{-1}$. Let $f, h \in G$ and without loss of generality, suppose that $f < h$. We want to show that $<$ is right invariant, *i.e.*, $f < h \implies fg < hg$. Consider

$$\begin{aligned} f < h \implies & 1 < f^{-1}h \\ & 1 < g^{-1}f^{-1}hg && \text{by assumption} \\ & fg < fgg^{-1}f^{-1}hg && \text{by left invariance} \\ & fg < hg. \end{aligned}$$

□

Lemma 2.1.14. *[7, Lemma 2.2] Every Archimedean left ordering is a bi-ordering.*

Proof. Let G be a group equipped with an Archimedean left ordering $<$. By Lemma 2.1.13, to see that $<$ is also a bi-ordering, it suffices to show that for any $p > 1$ and any $g \in G$, $1 < g^{-1}pg$.

Case 1. Suppose $1 < g$. Since $<$ is Archimedean, there exists $n > 0$ such that $g < p^n$, which implies $1 < g^{-1}p^n$. Since the product of two positive elements is positive, it follows that $1 < g^{-1}p^n g = (g^{-1}pg)^n$. So, $g^{-1}pg$ must also be positive.

Case 2. Suppose $g < 1$ and suppose, in hopes of a contradiction, that $g^{-1}pg < 1$. Then $1 < (g^{-1}pg)^{-1} = g^{-1}p^{-1}g$. Since g^{-1} is positive, by case 1, it follows that $1 < g(g^{-1}p^{-1}g)g^{-1} = p^{-1}$, which is a contradiction since we assumed that p is positive. $\square$

Lemma 2.1.15. *[7, Problem 2.3] Let G be a group admitting an Archimedean ordering $<$. For every positive element g and every element $h \in G$, there exists $k \in \mathbb{Z}$ such that $g^k \leqslant h < g^{k+1}$.*

Proof. Case 1. Suppose h is positive. Then, by definition of Archimedean, there exists an $n > 0$ such that $h < g^n$. By assumption, g is positive, so either $1 < h < g$ or $1 < g < h$. If $1 < h < g$, then $g^0 < h < g^{0+1}$, so take $k = 0$. On the other hand, if $1 < g < h < g^n$, then clearly there exists some k such that $1 \leqslant k < n$ and $g^k \leqslant h < g^{k+1}$.

Case 2. If $h < 1$, then h^{-1} is positive, so by case 1 there exists some $k \in \mathbb{Z}$ such that $g^k \leqslant h^{-1} < g^{k+1}$. By Lemma 2.1.14, since $<$ is also a bi-ordering, $g^k \leqslant h^{-1} < g^{k+1} \implies g^{-(k+1)} \leqslant h < g^{-k}$. $\square$

Lemma 2.1.16. *[7, Lemma 2.4] If G admits an Archimedean left ordering, then G is abelian.*

Proof. Suppose $<$ is an Archimedean ordering on G.

Case 1. Suppose there exists a least positive element p with respect to $<$. We claim that $G = \langle p \rangle$. If $G \neq \langle p \rangle$, then there exists a positive element $g \in G \backslash \langle p \rangle$. By Lemma 2.1.15, there exists some $k \in \mathbb{Z}$ such that

$$p^k < g < p^{k+1},$$

which implies that $1 < p^{-k}g < p$. This contradicts the assumption that p is the least positive element. Thus, G must be cyclic and therefore abelian.

Case 2. Suppose that G does not a contain a least positive element with respect to $<$. Suppose that G is not abelian, then there exist elements $g, h \in G$ such that g and h do not commute. This means that their commutator $ghg^{-1}h^{-1} \neq 1$. Without loss of generality, suppose that $1 < ghg^{-1}h^{-1}$. By Lemma 2.1.11, there exists an element x such that $1 < x^2 < ghg^{-1}h^{-1}$, and by Lemma 2.1.15, there exist $n, m \in \mathbb{Z}$ such that $x^m \leqslant g < x^{m+1}$ and $x^n \leqslant h < x^{n+1}$. This implies that $g^{-1} \leqslant x^{-m}$ and $h^{-1} \leqslant x^{-n}$. Therefore,

$$ghg^{-1}h^{-1} < x^{m+1}x^{n+1}x^{-m}x^{-n} = x^2,$$

which is a contradiction. Therefore, G must be abelian. □

2.1.2 Conradian Orderings

A Conradian ordering is a special kind of left ordering, which will be of interest in Chapter 5. Before giving the definition of a Conradian ordering, it is necessary to establish a notion of convex subgroups.

Definition 2.1.17. Let $(G, <)$ be a left ordered group and let C be a subgroup of G. Suppose $g, h \in C$ and $f \in G$. The subgroup C is a *convex subgroup* of G relative to $<$ if whenever $g < f < h$, then $f \in C$.

Example 2.1.18. Recall the Klein bottle group $\pi_1(K) = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$. The subgroup $\langle y \rangle$ is convex relative to every left ordering of the group. To see this, begin by fixing a left ordering $<$ of $\pi_1(K)$. Note that the elements of $\pi_1(K)$ have normal form $y^n x^m$, $m, n \in \mathbb{Z}$, due to the defining relator of the group. Suppose that $y^n x^m$ is an element that satisfies

$$y^a < y^n x^m < y^b.$$

To see that $\langle y \rangle$ is convex, we need to show that $y^n x^m \in \langle y \rangle$. Let $k \in \mathbb{Z}, l > 0$.

Case 1. Suppose $x > 1$. If $1 < x^l < y^k$, then $1 < x < y^k$, which implies that $1 < x^{-1}y^k$, *i.e.,* $x^{-1}y^k$ is a positive element. Therefore, the product $x^{-1}y^k \cdot x$ must be positive, but

$x^{-1}y^kx = y^{-k} < 1$, which is a contradiction. So, it must be that $1 < y^k < x^l$ ($\star$), or similarly $x^{-l} < y^{-k} < 1$ ($\star\star$). Now, suppose that $y^a < y^nx^m < y^b$, then $y^{a-n} < x^m < y^{b-n}$. If $m > 0$, then $x^m < y^{b-n}$ contradicts ($\star$). Alternatively, if $m < 0$, then $y^{a-n} < x^m$ contradicts ($\star\star$).

Case 2. If $x < 1$, then by a similar argument we obtain $x^l < y^{-k} < 1 < y^k < x^{-l}$, which contradictions $y^{a-n} < x^m < y^{b-n}$.

Hence, the inequality $y^a < y^nx^m < y^b$ holds only if $m = 0$, in which case $y^nx^m = y^n \in \langle y \rangle$. Therefore, $\langle y \rangle$ is convex with respect to every left ordering $<$.

Convex subgroups are ordered by set inclusion.

Proposition 2.1.19. *[7, Problem 2.9] Let $(G, <)$ be a left ordered group. Suppose C and D are distinct convex subgroups relative to $<$, then either $C \subset D$ or $D \subset C$.*

Proof. Suppose C and D are nontrivial convex subgroups relative to $<$. In hopes of a contradiction, suppose that $C \not\subset D$ and $D \not\subset C$. Since $C \not\subset D$, there exists an element $c \in C$ such that $c \notin D$. Assume $1 < c$. Choose an element $d \in D$ such that $1 < d$. If $c < d$, then $c \in D$ by convexity, which is a contradiction. So, it must be that $1 < d < c$, for all positive elements $d \in D$, from which it follows that $D \subset C$—also a contradiction. A similar argument follows if we suppose $c < 1$. $\square$

The next piece needed to define a Conradian ordering is the definition of a *convex jump*.

Definition 2.1.20. If C and D are convex subgroups of G relative to the ordering $<$, then (C, D) is called a *convex jump* if for any other convex subgroup C' such that $C \subset C' \subset D$, it follows that either $C' = C$ or $C' = D$.

Definition 2.1.21. Let $(G, <)$ be a left ordered group. Then (C, D) is a *Conradian jump* with respect to $<$ if

1. (C, D) is a convex jump,

2. C is normal in D, and

3. the natural ordering on D/C is Archimedean.

To clarify the last point of Definition 2.1.21, the natural ordering on a quotient G/N of a left ordered group $(G,<)$ is the ordering inherited via the quotient map. More precisely, for $g^{-1}h \notin N$, $gN < hN$ in G/N if and only if $g < h$ in G.

Definition 2.1.22. A left ordering $<$ on G is a *Conradian left ordering* if all the convex jumps in G with respect to $<$ are Conradian jumps.

Example 2.1.23. The Klein bottle group $\pi_1(K)$ admits Conradian left orderings. From Example 2.1.18, we saw that the subgroup $\langle y \rangle \leqslant \pi_1(K)$ is convex with respect to any left ordering $<$ of $\pi_1(K)$. Next, we show that $\{1\}, \langle y \rangle, \pi_1(K)$ are the only convex subgroups (with respect to $<$) in $\pi_1(K)$.

Suppose there exists another convex subgroup $H \leqslant \pi_1(K)$. By Proposition 2.1.19, since convex subgroups are ordered by inclusion, either $H \subset \langle y \rangle$ or $\langle y \rangle \subset H$.

If $H \subset \langle y \rangle$, then $H = \langle y^k \rangle$ for some k. We can assume $k > 0$, and if $k > 1$, then $1 < y < y^k$ and $y \in H$ by convexity. Therefore, $\langle y \rangle \subset H$ and $H = \langle y \rangle$.

Alternatively, suppose $\langle y \rangle \subset H$. Then, by [7, Problem 2.14], $H/\langle y \rangle$ is convex in $\pi_1(K)/\langle y \rangle$. From Example 2.1.10, $\pi_1(K)/\langle y \rangle \simeq \mathbb{Z}$. By the previous argument, the only convex subgroups contained in $\mathbb{Z}$ are the trivial subgroup and $\mathbb{Z}$. If $H/\langle y \rangle = \{1\}$, then $H = \langle y \rangle$. If $H/\langle y \rangle \simeq \mathbb{Z}$, then $H = \pi_1(K)$. Therefore, $\{1\}, \langle y \rangle, \pi_1(K)$ are the only convex subgroups.

Finally, the only convex jumps in $\pi_1(K)$ (with respect to $<$) are therefore $(\{1\}, \langle y \rangle)$ and $(\langle y \rangle, \pi_1(K))$. By definition, both of these jumps are Conradian. So, all left orderings on $\pi_1(K)$ are Conradian orderings.

The following characterization of Conradian orderings will be paramount to proving Proposition 5.2.2.

Theorem 2.1.24. *[7, Theorem 9.5] A group G admits a Conradian left ordering $<$ if and only if for every pair of positive elements $g, h \in G$ there exists an integer $n > 0$ such that*

$g < hg^n$.

2.2 Positive Cones

Any left ordering of a group G can be encoded by a positive cone. Given a group and a left ordering $(G, <)$, the corresponding positive cone P is the set consisting of all strictly positive elements in G under the ordering $<$, *i.e.,* $P = \{g \in G \mid g > 1\}$. It is desirable to represent a left ordering in terms of its positive cone as this will simplify the presentation of the space of left orderings of a given group, which will be introduced in the next section.

Definition 2.2.1. A *positive cone* P of a group G is a subset $P \subset G$ such that

1. P is closed under the group operation, *i.e.,* $P \cdot P \subset P$, and
2. P partitions G, that is, $G = P^{-1} \sqcup \{1\} \sqcup P$.

Consider a group G with the left ordering $<$ and let P be its corresponding positive cone. Suppose that for some $g, h \in G$, $g < h$, then

$$g < h \iff 1 < g^{-1}h \iff g^{-1}h \in P,$$

which shows that no information is lost by representing an ordering via its positive cone.

In the next example, we give a geometric method for constructing positive cones of the group $(\mathbb{Z}^2, +)$. In this example, we use the word "lattice", which comes up again in Chapter 4. When we refer to a *lattice* in this thesis, we always mean a discrete additive subgroup of $\mathbb{R}^n$ and *not* an ordered structure.

Definition 2.2.2. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a set of linearly independent elements in $\mathbb{Z}^n$. A *lattice* $\mathcal{L}$ is the set of integer linear combinations of elements in $\mathcal{B}$. We say that $\mathcal{B}$ is a basis of $\mathcal{L}$.

Example 2.2.3. Consider the group $G = (\mathbb{Z}^2, +)$. Each element of the group can be viewed as an integer lattice point in $\mathbb{R}^2$. Let ℓ_1 be a line passing through the origin having normal vector $\vec{v}_1$. If ℓ_1 has irrational slope, then the line ℓ_1 does not intersect any integer lattice points, and so a positive cone P of G is determined by taking the dot product with $\vec{v}_1$, namely

$$P = \{\vec{z} \in \mathbb{Z}^2 \mid \vec{z} \cdot \vec{v}_1 > 0\}.$$

On the other hand, if ℓ_1 has rational slope, then taking the dot product with $\vec{v}_1$ is not sufficient, since for any integer lattice point $\vec{z}$ that lies on ℓ_1, $\vec{z} \cdot \vec{v}_1 = 0$. In the case that ℓ_1 has rational slope, a second line ℓ_2 is introduced. In particular, choose ℓ_2 with normal vector $\vec{v}_2$ such that $\vec{v}_1 \cdot \vec{v}_2 = 0$, *i.e.,* ℓ_1 and ℓ_2 are perpendicular. In this case, the positive cone is given by

$$P = \{\vec{z} \in \mathbb{Z}^2 \mid [\vec{z} \cdot \vec{v}_1 > 0] \text{ or } [\vec{z} \cdot \vec{v}_1 = 0 \text{ and } \vec{z} \cdot \vec{v}_2 > 0]\}.$$

One can check Definition 2.2.1 to confirm that in either case, whether ℓ_1 has rational slope or not, P is indeed a positive cone of $\mathbb{Z}^2$. A positive cone for the case when ℓ_1 has rational slope is pictured in Figure 2.1. First, all of the black integer lattice points are ordered by $\vec{v}_1$, then the blue lattice points are ordered by $\vec{v}_2$.

Proposition 2.2.4. *If P_1 and P_2 are positive cones of a left orderable group G and $P_1 \subset P_2$, then $P_1 = P_2$.*

Proof. Suppose $P_1 \subset P_2$, then $P_1^{-1} \subset P_2^{-1}$. For the sake of contradiction, suppose that $P_2 \not\subset P_1$. Then there exists an element $g \in P_2$ such that $g \notin P_1$. Since P_1 partitions G, it must be that $g \in P_1^{-1}$. Therefore, $g \in P_1^{-1} \subset P_2^{-1}$, however this contradicts the assumption that $g \in P_2$. $\square$

2.3 Spaces of Orderings

A standard approach to studying the left orderings admitted by a group G is to take the collection of all positive cones of G and endow it with a topology.

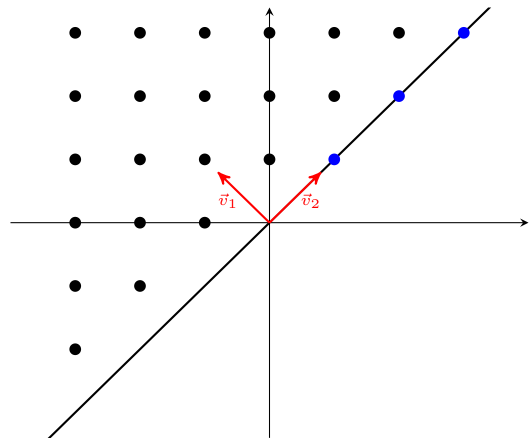


Figure 2.1: Geometric representation of a positive cone of $(\mathbb{Z}^2, +)$.

Definition 2.3.1. Let G be a left orderable group. We represent the left orderings of G by their positive cones, and denote the *set of left orderings of* G by

$$\mathrm{LO}(G) = \{P \mid P \text{ is a positive cone of } G\}.$$

Definition 2.3.2. Let X be a set and $\mathcal{P}(X)$ be the power set of X. The *Tychonoff topology* τ on $\mathcal{P}(X)$ is defined by declaring the sets

$$U_x = \{A \subset X \mid x \in A\} \text{ and } U_x^{\complement} = \{A \subset X \mid x \notin A\}$$

to be open, *i.e.,* they form a subbasis for τ.

We topologize $\mathrm{LO}(G)$ by imposing the Tychonoff topology on $\mathcal{P}(G\backslash\{1\})$ and endow $\mathrm{LO}(G)$ with the subspace topology. When $\mathrm{LO}(G)$ is endowed with a topology, we call it the *space of left orderings of* G.

Remark 2.3.3. A basic open set of $\mathrm{LO}(G)$ is a finite intersection of the form

$$\left(\bigcap_{i=1}^{n} U_{x_i} \cap \bigcap_{j=1}^{m} U_{x_j}^{\complement}\right) \cap \mathrm{LO}(G),$$

but since $U_{x_j}^{\complement} \cap \mathrm{LO}(G) = U_{x_j^{-1}} \cap \mathrm{LO}(G)$ we can throw away the complements. That is, sets of the form $\bigcap_{i=1}^{n} U_{x_1} \cap \mathrm{LO}(G) = \{P \in \mathrm{LO}(G) \,|\, x_1, \ldots, x_n \in P\}$ are sufficient to describe any basic open set in the subspace topology.

If G is a countable group, then G can be topologized by defining the following metric.

Definition 2.3.4. Let G be a countable left orderable group and fix an enumeration of $G \backslash \{1\} = \{g_1, g_2, g_3, \ldots\}$. Define the metric d on $\mathrm{LO}(G)$ as follows: if $P, P' \in \mathrm{LO}(G)$, then $d(P, P') = 2^{-n}$ if the first element that P and P' differ on is g_n, and $d(P, P') = 0$ if $P = P'$.

Proposition 2.3.5. *[7, Problem 1.41] Let G be a countable group. Then the subspace topology on* $\mathrm{LO}(G)$ *is equivalent to the topology induced by the metric defined in Definition 2.3.4.*

Proof. Let $\mathcal{B}_M$ and $\mathcal{B}_S$ denote bases for the topology induced by the metric and the subspace topology, respectively. To see that the two topologies are equivalent, we show that any basic open set $B \in \mathcal{B}_M$ is open in the subspace topology and that any basic open set $B \in \mathcal{B}_S$ is open in the topology induced by the metric.

A basic open set in the topology induced by the metric is an open ball of the form

$$B(P, 2^{-n})$$

where P is an element of $\mathrm{LO}(G)$ and $n \in \mathbb{N}$.

Similarly, by Remark 2.3.3, a basic open set in the subspace topology consists of all orderings in which some finite collection of elements $x_1, \ldots, x_k$ are declared to be positive.

Fix an enumeration $\{g_1, g_2, g_3, \ldots\}$ of $G \backslash \{1\}$. Suppose $B \in \mathcal{B}_M$, then $B = B(P, 2^{-n})$ for some positive cone P and $n \in \mathbb{N}$. For any positive cone $Q \in B$, P and Q agree on the first $n-1$ elements, *i.e.*, $P \cap \{g_1, \ldots, g_{n-1}\} = Q \cap \{g_1, \ldots, g_{n-1}\}$. Without loss of generality, suppose that $g_{k_1}, \ldots, g_{k_i} \in P \cap \{g_1, \ldots, g_{n-1}\}$ and $g_{k_{i+1}}, \ldots, g_{k_{n-1}} \notin P \cap \{g_1, \ldots, g_{n-1}\}$, which means that $g_{k_{i+1}}^{-1}, \ldots, g_{k_{n-1}}^{-1} \in P$. Therefore, $B(P, 2^{-n})$ is precisely the set containing all orderings in which the finite collection of elements $\{g_{k_1}, \ldots, g_{k_i}, g_{k_{i+1}}^{-1}, \ldots, g_{k_{n-1}}^{-1}\}$ are positive, so $B(P, 2^{-n})$ is open in the subspace topology.

The other direction follows similarly. ☐

Corollary 2.3.6. *Let G be a countable group and D be a subset of* $\mathrm{LO}(G)$*, then the following are equivalent:*

1. *the set D is dense in* $\mathrm{LO}(G)$*;*
2. *for any positive cone $P \in \mathrm{LO}(G)$ and any finite subset $F \subset P$, there exists a distinct positive cone $Q \in D$ such that $F \subset Q$;*
3. *for any positive cone $P \in \mathrm{LO}(G)$, there exists a positive cone $Q \in D$ arbitrarily close to P.*

Definition 2.3.7. An ordering P of G is *isolated* in $\mathrm{LO}(G)$ if the singleton set $\{P\}$ is open in $\mathrm{LO}(G)$. That is, P is an isolated ordering if it is the only ordering of G in which some finite collection of elements $g_1, \ldots, g_n$ is positive.

Proposition 2.3.8. *[9, Proposition 2.2.41] Let G be a left orderable group with positive cone P. If P is finitely generated as a semigroup, then P is an isolated ordering in* $\mathrm{LO}(G)$*.*

Proof. Suppose $P = \langle x_1, \ldots, x_k \rangle^+$ and it is not an isolated point in $\mathrm{LO}(G)$. Then for any finite collection of elements C in P, there exists a positive cone $P' \in \mathrm{LO}(G)$ such that $C \subset P'$ and $P \neq P'$. Take C to be the generating set $\{x_1, \ldots, x_k\}$ and suppose $C \subset P'$. By the closure property of positive cones, any word over $\{x_1, \ldots, x_k\}$ is contained in P', or equivalently $P \subset P'$. By Proposition 2.2.4, this means that $P = P'$, which is a contradiction. Therefore, P is an isolated point. ☐

If $\mathrm{LO}(G)$ is finite, then all positive cones $P \in \mathrm{LO}(G)$ are isolated points. The Klein bottle group is an example of a group whose positive cones are all isolated points in the space of left orderings (see Example 3.3.5). In general, $\mathrm{LO}(G)$ is either finite or uncountable.

Theorem 2.3.9. *[16] If G is a left orderable group, then* $\mathrm{LO}(G)$ *is either finite or uncountable.*

We will later see that if all left orderings of a group are regular, then the space of orderings must consist entirely of isolated points.

3

Formal Languages and Automata

Formal languages and automata are central tools in the study of computational complexity theory. A decision problem can be recognised as a formal language, which can then be categorised into one of four classes: *regular*, *context-free*, *context-sensitive*, or *recursively enumerable*. These classes form the *Chomsky hierarchy* (see Figure 3.1). Regular languages are at the bottom of the hierarchy making them the "simplest" languages, and as we move up the hierarchy, we encounter languages that are more "complex". In our context, the decision problem we are working with is left ordering a given group.

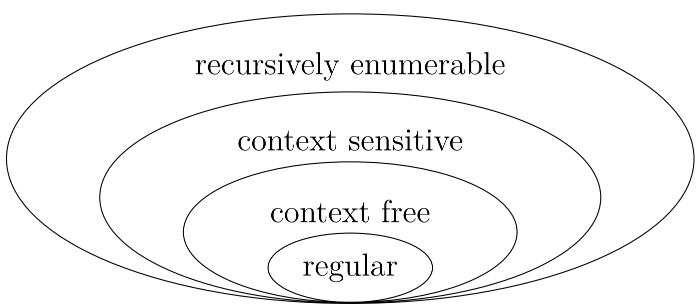


Figure 3.1: The Chomsky hierarchy.

The complexity of a language is measured by the sophistication of the *automaton*, or machine, required to recognise that language. The word "automaton" is of Latin origin and refers to a "self operating mechanism" [22, Automaton]. Simply put, an automaton is a machine that takes in as input words over some alphabet, which it can either *accept* or *reject*. Certain automata are equipped with additional capabilities that make them more powerful. Given a formal language, we are interested in finding the least complex automaton required to recognise that language.

A classic example of an automaton is a Turing machine, although we will not work with such machines in our study. *Finite state automata* (FSA), the machines we are primarily concerned with, are far simpler than Turing machines. This means that there exist languages (or equivalently decision problems) that can be parsed by a Turing machine but not by an FSA. Much of this chapter will be devoted to defining and proving certain properties of finite state automata. This is done with the purpose of eventually defining and working with regular languages.

Definition 3.0.1. An *alphabet* Σ is a set of symbols. A *word* w over Σ is a finite sequence of symbols $x_1 x_2 \cdots x_n$, where each $x_i \in \Sigma$. A *subword* of w is a word u over Σ such that $w = w'uw''$, where w' and w'' are also words over Σ that may be empty.

Let Σ^* denote the set of all finite words over Σ. The *empty word*, which is denoted by ϵ, is also an element of Σ^*. The *length* of a word w will be denoted by $|w|$.

Definition 3.0.2. A *language* $\mathcal{L}$ over Σ is a subset of Σ^*.

Example 3.0.3. If $\Sigma = \{0, 1\}$ is an alphabet, then Σ^* is the set of all finite binary strings. The language $\mathcal{L}_1 = \{0^n 1^m \mid n \in \mathbb{N}, m \geqslant 0\}$ is a subset of Σ^*.

3.1 Finite State Automata

As a visual aid, one should imagine a finite state automata (FSA) to be a simple machine that contains a reader head and an infinite tape. The tape is divided into squares, and each square contains a single symbol from some finite alphabet Σ. The tape moves from right to left, and as it moves, the reader head reads one square at a time. With each symbol that it reads, the machine can transition from one state to another or stay in the same state. Unlike a Turing machine, an FSA does not have the ability to write on the tape. In other words, it cannot produce output in the form of words over Σ, rather it can only produce "output" in the form of an *accept* or *reject* state.

In Figure 3.2, the machine $\mathcal{M}$ began reading the tape at the far left on the first square that reads a "1" and stopped reading the tape on the last square that reads a "1". As the tape moves one square to the left, the machine $\mathcal{M}$ changes states. If $\mathcal{M}$ enters an accept state when reading the last letter on the tape, then $\mathcal{M}$ accepts the word $w = 101011$. Otherwise, $\mathcal{M}$ rejects w.

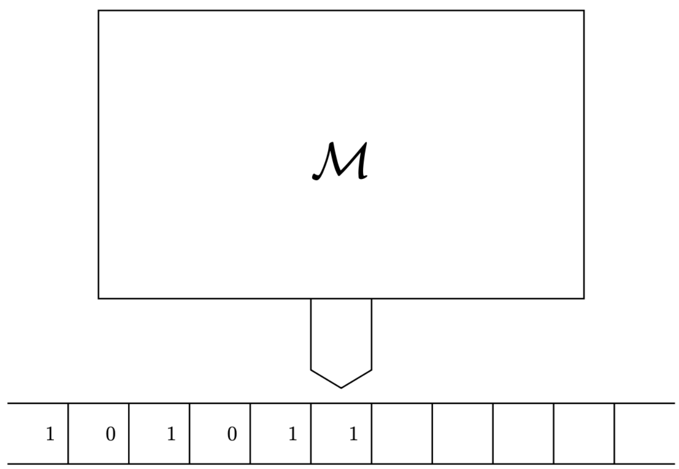


Figure 3.2: A machine $\mathcal{M}$ reading a tape containing the word 101011.

Definition 3.1.1. A *finite state automaton* (FSA) is a quintuple $\mathcal{M} = (Q, F, \Sigma, \tau, q_0)$, where

1. Q is a finite set of states,

2. F is a subset of Q (called the *accept* or *final states*),

3. Σ is a finite alphabet,

4. $\tau \subset Q \times \Sigma \times Q$ (called the *set of transitions*), and

5. $q_0 \in Q$ is the *initial state.*

A *transition*, or a *triple* $(q_i, x, q_j) \in Q \times \Sigma \times Q$, corresponds to the machine $\mathcal{M}$ being in state

q_i, reading the symbol x on the tape, and then transitioning to the state q_j.

Definition 3.1.2. A *generalised FSA* is an FSA $\mathcal{M}$ where $\tau \subset Q \times (\Sigma \cup \{\epsilon\}) \times Q$. In other words, in addition to the conditions given in Definition 3.1.1, we allow for transitions of the type (q_i, ϵ, q_j). These are called $\epsilon-$*transitions*.

An $\epsilon-$transition (q_i, ϵ, q_j) corresponds to the machine $\mathcal{M}$ being in state q_i, reading a blank square, and then transitioning to the state q_j.

Definition 3.1.3. An FSA $\mathcal{M} = (Q, F, \Sigma, \tau, q_0)$ is *deterministic* if for all $q \in Q$ and $x \in \Sigma$ there is exactly one $q' \in Q$ such that the triple $(q, x, q') \in \tau$ is a transition of $\mathcal{M}$.

The machine $\mathcal{M}$ is deterministic if whenever $\mathcal{M}$ is in state q reading the symbol x, there is exactly one and only one state q' that $\mathcal{M}$ can enter.

It turns out that FSAs, generalised FSAs, and deterministic FSAs are equivalent (see appendix, Theorem A.0.5). For the most part, we will work with deterministic FSAs, however generalised FSAs will be useful for proving certain closure properties of regular languages in Section 3.2.

Definition 3.1.4. A *computation* C of $\mathcal{M}$ is a sequence $q_0, x_1, q_1, x_2, q_2, \ldots, x_n, q_n$, where $n \geqslant 0$ and $(q_{i-1}, x_i, q_i) \in Q \times \Sigma \times Q$ for all $1 \leqslant i \leqslant n$. The word or label corresponding to C is $x_1 x_2 \cdots x_n$.

It is always assumed that a computation begins at the same initial state q_0.

Definition 3.1.5. A computation $C = q_0, x_1, q_1, x_2, q_2, \ldots, x_n, q_n$ is *successful* if $q_n \in F$.

Definition 3.1.6. A word $w = x_1 \cdots x_n$ is *accepted* by $\mathcal{M}$ if there is a successful computation with label $x_1 \cdots x_n$. Additionally, the language $\mathcal{L}(\mathcal{M})$ denotes the set of all words w that are accepted by $\mathcal{M}$.

Definition 3.1.7. The language $\mathcal{L}$ over Σ is said to be *accepted* or *recognised* by $\mathcal{M}$ if $\mathcal{L} = \mathcal{L}(\mathcal{M})$.

Definition 3.1.8. A language $\mathcal{L}$ over Σ is *regular* if there exists a finite state automaton $M = (Q, F, \Sigma, \tau, q_0)$ that accepts $\mathcal{L}$.

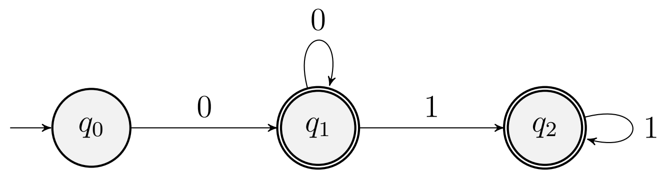


Figure 3.3: An FSA accepting the language $\mathcal{L}_1 = \{0^n 1^m \mid n \in \mathbb{N}, m \geqslant 0\}$.

3.1.1 Transition Diagrams

Finite state automata are perhaps easier to understand and work with when they are represented by their corresponding transition diagrams. These are finite digraphs that model the computations a particular FSA can preform. As such, examples of FSAs and regular languages are best deferred until a notion of transition diagrams has been established.

Definition 3.1.9. Let $\mathcal{M} = (Q, F, \Sigma, \tau, q_0)$ be a FSA. A *transition diagram* T representing $\mathcal{M}$ is a finite digraph whose vertices are labelled by elements of Q and edges are labeled by elements Σ such that there is a directed labeled edge x leaving q_i and entering q_j if and only if $(q_i, x, q_j) \in \tau$.

Additionally, we adopt the following conventions when working with transition diagrams:

1. the initial state q_0 is indicated by an arrow starting from nowhere, and
2. accept states are indicated by a double circle.

If T is a transition diagram as given in Definition 3.1.9 and $x_1 \cdots x_n$, $x_i \in \Sigma$, is a path in T beginning at the initial state vertex q_0, then $\mathcal{M}$ accepts the word $x_1 \cdots x_n$ if and only if the path terminates at an accept state vertex.

Example 3.1.10. Recall the language $\mathcal{L}_1 = \{0^n 1^m \mid n \in \mathbb{N}, m \geqslant 0\}$ introduced earlier. This language is regular since there exists a FSA that recognises it. To see this, consider the transition diagram T given in Figure 3.3.

Let $\mathcal{L}(\mathcal{M})$ denote the language accepted by $\mathcal{M}$. To see that $\mathcal{M}$ accepts $\mathcal{L}_1$, we need to show that $\mathcal{L}_1 = \mathcal{L}(\mathcal{M})$.

First, to see that $\mathcal{L}_1 \subset \mathcal{L}(\mathcal{M})$ we begin with an arbitrary word $w \in \mathcal{L}_1$ and show that there exists a path in T, beginning at q_0 and ending in an accept state, whose edge labels spell out the word w.

- Case 1: $w = 0^n$. The path in T whose edge labels correspond to w is obtained by starting at the initial vertex, following the 0 edge, and walking around the 0 loop $n-1$ times. Such a path always ends in state q_1, which is an accept state, so all words of this form are accepted by $\mathcal{M}$.

- Case 2: $w = 0^n1^m$, for $m > 0$. To obtain the corresponding path in T begin with the path given in case 1 and continue by following the 1 edge leaving q_1 and entering q_2. Walk around the 1 loop $m-1$ many times. Since at the end of this path we arrive at vertex q_2, an accept state, w is accepted by $\mathcal{M}$.

To see that $\mathcal{L}(\mathcal{M}) \subset \mathcal{L}_1$, we show that the word w corresponding to the edge labels of any path that begins at q_0 and ends in an accept state is contained in $\mathcal{L}_1$.

- Case 1: Suppose the path p ends at vertex q_1. The only way to arrive at q_1 from q_0 is to follow the 0 edge and walk around the 0 loop. A word w corresponding to such a path p is of the form $p = 0^n$, $n \in \mathbb{N}$.

- Case 2: Suppose the path p ends at vertex q_2. The only way to arrive at q_2, starting from q_0, is to walk through q_1 (this is known is a bottleneck in the graph). It is easy to see that any word w corresponding to p is of the form $w = 0^n1^m$, $n, m \in \mathbb{N}$.

There are no other accept state vertices in T, so there are no other paths or cases to consider. Thus, $\mathcal{L}_1$ is precisely the language accepted by $\mathcal{M}$.

The proof given in Example 3.1.10 is the standard approach to showing that a language $\mathcal{L}$ is accepted by an FSA $\mathcal{M}$, or that $\mathcal{L}$ is regular. As such proofs are just a matter of exhaustively checking cases, they are typically omitted, however it is worth seeing at least once to better understand how to read transition diagrams. In the future, we will consider it sufficient to construct a transition diagram for a language $\mathcal{L}$ to show that it is regular.

Example 3.1.11. Any finite language $\mathcal{L}$ is regular. This is true because it is always possible to construct a transition diagram T that accepts $\mathcal{L}$ by creating a distinct path in T for each word in $\mathcal{L}$. While this may not be an optimal solution, in the sense that it will not produce a graph with the minimal number of vertices and edges, it will produce a finite digraph that accepts the language $\mathcal{L}$.

There exist languages that are not regular. In fact, regular languages are only a *small* class of languages contained in the class of computable languages, as suggested by the Chomsky hierarchy (3.1). Consider, as a non-example, the language $\mathcal{L}_1$ given in Example 3.1.10 with a slight modification.

Example 3.1.12. Let $\mathcal{L}_2 = \{0^n 1^n \mid n \in \mathbb{N}\}$. This language is very similar to $\mathcal{L}_1$, and in fact $\mathcal{L}_2 \subset \mathcal{L}_1$. However, the same transition diagram T as given for $\mathcal{L}_1$ will not work for $\mathcal{L}_2$, since T accepts words that are not contained in $\mathcal{L}_2$. To see that $\mathcal{L}_2$ is *not* regular, we will need some stronger machinery.

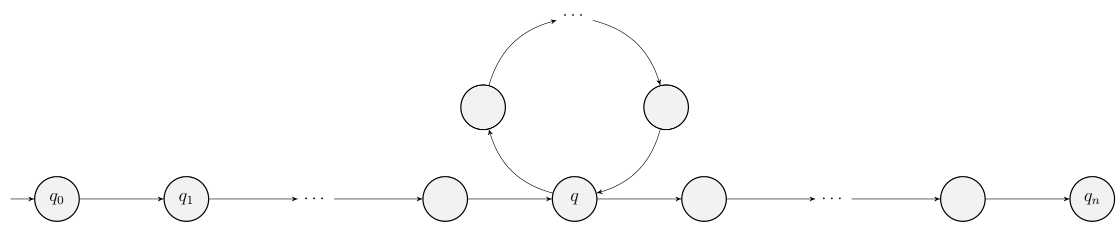
Lemma 3.1.13. *[6, Lemma 1.8] (Pumping Lemma). Let $\mathcal{L}$ be a regular language. There exists some $k \in \mathbb{N}$ such that any word $w \in \mathcal{L}$ with $|w| \geqslant k$ can be expressed as $w = uvz$, where $|v| > 0$, $|uv| \leqslant k$, and $uv^iz \in \mathcal{L}$ for all $i \geqslant 0$.*

Proof. Let $\mathcal{L}$ be a regular language and suppose the FSA $\mathcal{M} = (Q, F, \Sigma, \tau, q_0)$ accepts $\mathcal{L}$. Take k to be $|Q|$. Consider a word $w = x_1 x_2 \cdots x_n$, with $n \geqslant k$. Since $\mathcal{M}$ accepts w, there is a successful computation C represented by the sequence

$$q_0, x_1, q_1, \ldots, x_{n-1}, q_{n-1}, x_n, q_n,$$

i.e., the computation C has label w. Since this sequence contains at least $k+1$ states and $|Q| = k$, the path in the transition diagram corresponding to C passes through some vertex q at least twice.

Next, take uv to be the shortest prefix of w whose corresponding path in the graph passes through q twice. (Let z represent the suffix of w such that $w = uvz$.) Suppose that u is the prefix corresponding to the subpath from q_0 to q and v is the loop going from q to q.



By construction, $|v| > 0$ and $|uv| \leqslant k$ (at most, uv covers every vertex in the graph). Therefore, any word of the form uv^iz corresponds to a path going from q_0 to q, then looping i many times around q, and finally walking the path corresponding to z. Since, by assumption, such a path will always end at the vertex q_n (an accept state), the word uv^iz will be accepted by $\mathcal{M}$. $\square$

Example 3.1.14. Recall the language $\mathcal{L}_2 = \{0^n1^n \mid n \in \mathbb{N}\}$. Suppose for the sake of contradiction that it is regular. By the Pumping Lemma, there exists some $k \in \mathbb{N}$ satisfying the hypothesis of the lemma statement. Consider the word $w = 0^k1^k$. There exists a decomposition $w = uvz$, where $|v| > 0$ and $|uv| \leqslant k$. Thus, the subword v must be composed of only 0's, in fact, it must contain at least one 0. Finally, according to the Pumping Lemma, the word uv^2z will also be contained in $\mathcal{L}_2$, however this is a contradiction since uv^2z will contain more 0's than 1's. Therefore, $\mathcal{L}_2$ cannot be regular.

3.2 Closure Properties of Regular Languages

The class of regular languages is closed under certain operations such as taking unions, intersections, complements, and word concatenation (see [6] for complete proofs). These properties will be integral to proving later results.

Proposition 3.2.1. *[6, Lemma 1.5] Regular languages are closed under taking unions and word concatenation.*

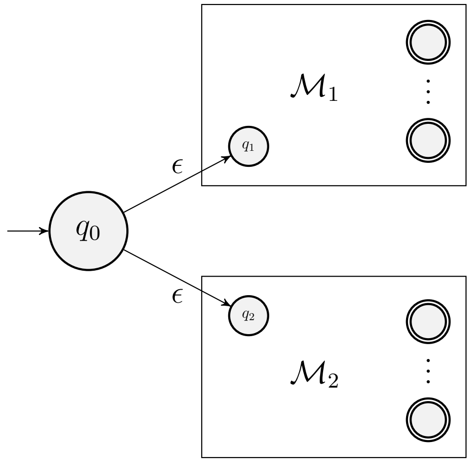


Figure 3.4: A generalised FSA constructed to accept the union of two regular languages.

Proof. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are regular languages over an alphabet Σ. Additionally, suppose that $\mathcal{M}_1 = (Q_1, F_1, \Sigma, \tau_1, q_1)$ and $\mathcal{M}_2 = (Q_2, F_2, \Sigma, \tau_2, q_2)$ are FSAs accepting $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

1. *Closure under unions.* Construct a generalised FSA $\mathcal{M} = (Q, F, \Sigma, \tau, q_0)$, where $Q = Q_1 \cup Q_2$, $F = F_1 \cup F_2$, and

$$\tau = \tau_1 \cup \tau_2 \cup \{(q_0, \epsilon, q_1), (q_0, \epsilon, q_2)\}.$$

   By construction, the machine $\mathcal{M}$ accepts the language $\mathcal{L}_1 \cup \mathcal{L}_2$ (see Figure 3.4 for a visual of this construction).

2. *Closure under concatenation.* Construct a generalised FSA $\mathcal{M} = (Q, F, \Sigma, \tau, q_1)$, where $Q = Q_1 \cup Q_2$, $F = F_2$, and

$$\tau = \tau_1 \cup \tau_2 \cup \{(q_i, \epsilon, q_2) \,|\, q_i \in F_1\}.$$

   By construction, $\mathcal{M}$ accepts the language $\mathcal{L}_1 \cdot \mathcal{L}_2$.

Therefore, by Theorem A.0.5, $\mathcal{L}_1 \cup \mathcal{L}_2$ and $\mathcal{L}_1 \cdot \mathcal{L}_2$ are regular languages. $\square$

As indicated by Examples 3.1.10 and 3.1.12, we can already see that regular languages are not closed under taking subsets since $\mathcal{L}_2 \subset \mathcal{L}_1$, but $\mathcal{L}_2$ is not regular.

Proposition 3.2.2. *[13, Proposition 2.5.14] The class of regular languages is closed under inverse homomorphisms.*

Proof. Let A and B be finite alphabets and let $\mathcal{L}$ be a regular language over B. Suppose we are given a monoid homomorphism $\phi : A^* \to B^*$. Since $\mathcal{L}$ is regular, by Theorem A.0.5, there exists a deterministic FSA $\mathcal{M} = (Q, F, B, \tau, q_0)$ that accepts $\mathcal{L}$. Define a new FSA $\mathcal{M}' = (Q, F, A, \tau', q_0)$, where $\tau' = \{(q, a, q') \mid (q, \phi(a), q') \in \tau\}$. By construction, $\mathcal{M}'$ recognises the language $\phi^{-1}(\mathcal{L}) \subseteq A^*$. Therefore, $\phi^{-1}(\mathcal{L})$ is regular. $\square$

3.3 Positive Cone Languages

In this section, we formalize the notion of associating a positive cone to a formal language and offer some preliminary examples of regular positive cones.

The set Σ^* along with the binary operation $\cdot$ of word concatenation forms the free monoid over Σ. The identity element of $(\Sigma^*, \cdot)$ is the empty word ϵ.

Definition 3.3.1. [12, p. 2] Let G be a left ordered group with positive cone P. Let Σ be a finite alphabet and $\phi : \Sigma^* \to G$ be a surjective homomorphism from the free monoid Σ^* onto the group G. A *positive cone language* is a language $\mathcal{L} \subseteq \Sigma^*$ such that $\phi(\mathcal{L}) = P$.

A positive cone P is *regular* if there exists a regular positive cone language $\mathcal{L}$ such that $\phi(\mathcal{L}) = P$. In this case, we say that P can be expressed by $\mathcal{L}$ or that $\mathcal{L}$ evaluates to P under ϕ.

Example 3.3.2. [12] As a first example, consider the group $(\mathbb{Z}^2, +)$ ordered lexicographically with respect to the standard generating elements $x = (1, 0)$ and $y = (0, 1)$. More precisely, the corresponding positive cone is given by

$$P = \{x^n y^m \,|\, n \geqslant 1 \text{ and } m \in \mathbb{Z}\} \cup \{y^m \,|\, m \geqslant 1\}.$$

The underlying alphabet is $\Sigma = \{x, y, X, Y\}$ and the homomorphism $\phi : \Sigma^* \to \mathbb{Z}^2$ is defined by

$$\begin{aligned} \phi &: x \mapsto x \\ \phi &: y \mapsto y \\ \phi &: X \mapsto x^{-1} \\ \phi &: Y \mapsto y^{-1}. \end{aligned}$$

The language

$$\mathcal{L} = \{x^n y^m \,|\, n \geqslant 1 \text{ and } m \geqslant 0\} \cup \{x^n Y^m \,|\, n \geqslant 1 \text{ and } m \geqslant 0\} \cup \{y^m \,|\, m \geqslant 1\}$$

over Σ evaluates to P, *i.e.,* $\phi(\mathcal{L}) = P$. Therefore, the positive cone P is regular since there exist a FSA that recognises its positive cone language $\mathcal{L}$ (see Figure 3.5).

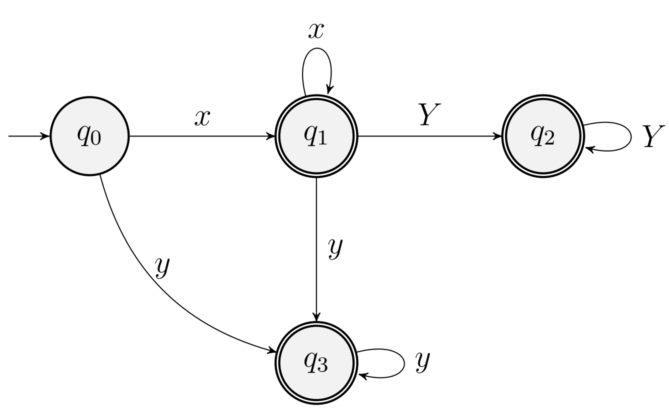


Figure 3.5: FSA accepting the positive cone language $\mathcal{L}$.

In this example, we used capital letters to identify inverse elements. This is done to emphasize the fact that a monoid need not have inverse elements. We could have instead identified the group element x^{-1} with a monoid element x^{-1}, as long as we treat x^{-1} as just another symbol and not allow for cancellations to happen. In this case, the homomorphism ϕ could just be defined by the canonical map. In the future, when showing that a positive cone P is regular, we will avoid explicitly defining the corresponding positive cone language with the understanding that there is an underlying canonical homomorphism and that cancellations are not permitted.

Remark 3.3.3. The lexicographic ordering on $\mathbb{Z}^2$ corresponds to the geometric ordering defined in Example 2.2.3. In particular, the lexicographic ordering with the standard generating set corresponds to taking normal vectors $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = (0, 1)$ in the geometric ordering. Example 3.3.2 will be of central importance in Chapter 4.

If the group G fits into a short exact sequence where both the kernel and quotient admit regular positive cones, then G also admits a regular positive cone. The following result will be useful in Chapter 5.

Lemma 3.3.4. *[1, Proposition 3.3] Let G, Q, and K be finitely generated groups such that G is an extension of Q by K, that is, the groups K, G, and Q fit into the short exact sequence*

$$1 \to K \hookrightarrow G \to Q \to 1.$$

Let $f : G \to Q$ be a surjective homomorphism. Suppose that P_K and P_Q are regular positive cones of K and Q, respectively, then $P = f^{-1}(P_Q) \cup P_K$ is a regular positive cone of G.

Proof. First, we need to check that $P = f^{-1}(P_Q) \cup P_K$ is indeed a positive cone of G, which has already been done in Proposition 2.1.7.

Next, we need to show that for some finite alphabet Σ there exists a regular positive cone language $\mathcal{L} \subseteq \Sigma^*$ and a surjective monoid homomorphism $\phi : \Sigma^* \to G$ such that $\phi(\mathcal{L}) = P$. By assumption, P_K and P_Q are regular, so there exist regular positive cone languages $\mathcal{L}_K$ and $\mathcal{L}_Q$ of P_K and P_Q, respectively. Suppose the underlying alphabet of $\mathcal{L}_K$ and $\mathcal{L}_Q$ are X

and Y, respectively. Finally, there exist surjective monoid homomorphisms $\phi_K : X^* \to K$ and $\phi_Q : Y^* \to Q$ such that $\phi_K(\mathcal{L}_K) = P_K$ and $\phi_Q(\mathcal{L}_Q) = P_Q$.

Let Σ be the disjoint union of X and Y. Define a homomorphism $\phi : \Sigma^* \to G$ by $\phi(x) = \phi_K(x)$, for all $x \in X$, and for each $y \in Y$ choose an element $z \in f^{-1}(y)$ and define $\phi(y) = z$. Next, define $\pi : \Sigma^* \to Y^*$ by the rule $\pi(x) = \epsilon$ for all $x \in X$ and $\pi(y) = y$ for all $y \in Y$. Let $\overline{\mathcal{L}_Q}$ be the preimage of $\mathcal{L}_Q$ under π. Next, we show that $\phi(\overline{\mathcal{L}_Q}) = f^{-1}(P_Q)$:

- First, observe that $f(\phi(\overline{\mathcal{L}_Q})) = P_Q$, by construction. So, $f^{-1}(f(\phi(\overline{\mathcal{L}_Q}))) = f^{-1}(P_Q)$ implies that $\phi(\overline{\mathcal{L}_Q}) \subseteq f^{-1}(\overline{\mathcal{L}_Q})$.
- To see that $f^{-1}(P_Q) \subseteq \phi(\overline{\mathcal{L}_Q})$, consider an arbitrary element $g \in f^{-1}(P_Q)$. Then $f(g) \in P_Q$, which means that there exists a word $w \in Y^*$ such that $\phi_Q(w) = f(g)$ or $f(\phi(w)) = f(g)$, which implies $\phi(w) = g$. By construction, w is also a word in $\overline{\mathcal{L}_Q}$, so $g = \phi(w) \in \phi(\overline{\mathcal{L}_Q})$. So, the desired set equality holds.

From the above argument, it follows that

$$\begin{aligned}\phi(\overline{\mathcal{L}_Q} \cup \mathcal{L}_K) &= \phi(\overline{\mathcal{L}_Q}) \cup \phi(\mathcal{L}_K)\\ &= f^{-1}(P_Q) \cup P_K.\end{aligned}$$

The language $\overline{\mathcal{L}_Q}$ is regular, since regular languages are closed under inverse homomorphisms (by Proposition 3.2.2), and $\overline{\mathcal{L}_Q} \cup \mathcal{L}_K$ is regular since regular languages are closed under taking unions (by Proposition 3.2.1). Therefore, we have found a regular positive cone language that evaluates to $P = f^{-1}(P_Q) \cup P_K$, so the claim holds. □

We have already seen in Example 2.1.10 that the Klein bottle group fits into a short exact sequence with a copy of $\mathbb{Z}$ in both the kernel and quotient. Combining this fact with Lemma 3.3.4 yields that all left orderings of $\pi_1(K)$ are regular. In the following example, we show that all of these left ordering have finitely generated positive cones.

Example 3.3.5. The Klein bottle group $\pi_1(K) = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$ admits regular positive cones. We claim that $P = \langle x, y \rangle^+$ is a positive cone of $\pi_1(K)$. To see this, we check that P is closed and that it partitions the group.

1. Suppose $w_1, w_2 \in P$ and consider their product $w_1 w_2$. Since w_1 and w_2 are words over $\{x, y\}$ with strictly positive exponents, their product will also have strictly positive exponents, so $w_1 w_2 \in P$.

2. Consider an arbitrary element $w \in G$. We show that P partitions G by checking that w is contained in only one of: P, P^{-1}, or $\{1\}$. First, if $w = 1$, then $w \notin P$ (or P^{-1}) since semigroups do not contain identity elements. Suppose that w is nontrivial. We show that it is contained in either P or P^{-1} by examining its spelling.

   Clearly, if w is a word over $\{x, y\}$ with strictly positive (negative) exponents, then $w \in P$ ($w \in P^{-1}$). An arbitrary word w over $\{x, y\}$ can always be expressed by a spelling with strictly positive (negative) exponents by inducting on the length of w (see Appendix, Lemma A.0.6 for proof of this claim).

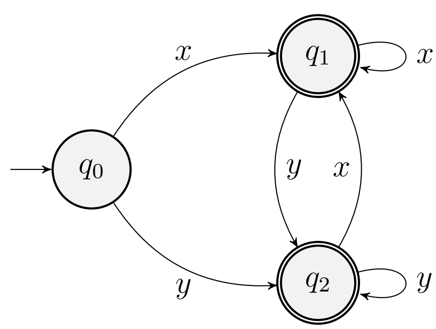


Figure 3.6: An FSA accepting the language $P = \langle x, y \rangle^+$.

In Figure 3.6 we give an FSA that recognizes the positive cone language of P. By a similar argument, we can show that $\langle x, y^{-1} \rangle^+, \langle x^{-1}, y \rangle^+$, and $\langle x^{-1}, y^{-1} \rangle^+$ are also regular positive cones of $\pi_1(K)$. From previous discussion, we know that $\pi_1(K)$ admits only four left orderings, so we have found all of them. Furthermore, since all four orderings are regular, we conclude that $\pi_1(K)$ admits only regular orderings.

Proposition 3.3.6. *[1, p. 23] Suppose G is a left orderable group such that all left orderings of G are regular, then* $\mathrm{LO}(G)$ *is finite.*

Proof. Suppose that all left orderings of G are regular. Since there exist only countably many FSAs, $\mathrm{LO}(G)$ is at most countable. By Theorem 2.3.9, $LO(G)$ must be finite. $\square$

Recall that if $\mathrm{LO}(G)$ is finite, then all left orderings of G are isolated points. While $\mathrm{LO}(G)$ being finite is a necessary condition for all positive cones of G to be regular, it is not sufficient. For more on regular positive cones, see [1].

4

Regular Orderings on $\mathbb{Z}^n$

In this chapter, we show that the set of regular left orderings on a finitely generated torsion-free abelian group G is dense in $\mathrm{LO}(G)$. We have already seen that left orderings of $(\mathbb{Z}^2, +)$ can be realised geometrically (see Example 2.2.3), and that at least one of these orderings is regular (see Example 3.3.2). The question we then asked was *which other orderings of $\mathbb{Z}^2$, if any, are regular?* In the 2−dimensional case, it turns out the left orderings determined by lines with rational slope are regular. Such orderings will be called *rational orderings*. Rational orderings form a dense set in $\mathrm{LO}(\mathbb{Z}^2)$, so we obtain that regular left orderings are dense in $\mathrm{LO}(\mathbb{Z}^2)$.

In generalising this result to higher dimensions, it will be necessary to first generalise the definition of a *rational ordering*. For $n > 2$, left orderings can no longer be realised by lines, instead they are determined by hyperplanes containing the origin. We are interested in a special subset of such orderings, which we will call *rational flag orderings*. We will first show that rational orderings are regular in the 2−dimensional case by providing a procedure for finding an explicit positive cone language using a tool from lattice theory. For the general case, we only show that regular positive cone languages exist for rational flag orderings without giving a procedure for explicitly computing them. In the last section of this chapter, we prove that regular left orderings are dense in $\mathrm{LO}(\mathbb{Z}^n)$ for all $n \geqslant 1$.

Notation 4.0.1. In the following section, the components of a vector $\vec{v}_i$ will be denoted by $(v_i^1, \dots, v_i^j, \dots, v_i^n)$, that is, the j^{th} component of the vector $\vec{v}_i$ is v_i^j.

4.1 Regularity in the 2−Dimensional Case

Strictly speaking, solving the 2−dimensional case separately is redundant in light of Theorem 4.3.13. However, the existence proof given in 4.3.13 falls short of providing the reader with the geometric intuition that is natural to this problem.

Definition 4.1.1. Let $\vec{v} = (v^1, v^2)$ be a vector such that $\frac{v^2}{v^1} \in \mathbb{R}\backslash\mathbb{Q}$. An *irrational left ordering* of $(\mathbb{Z}^2, +)$ is an ordering with positive cone

$$P = \{\vec{z} \in \mathbb{Z}^2 \mid \vec{z} \cdot \vec{v} > 0\}.$$

Definition 4.1.2. Let $\vec{v}_1$ and $\vec{v}_2$ be perpendicular vectors with rational slopes. A *rational left ordering* of $(\mathbb{Z}^2, +)$ is an ordering with positive cone

$$P = \{\vec{z} \in \mathbb{Z}^2 \mid [\vec{z} \cdot \vec{v}_1 > 0] \text{ or } [\vec{z} \cdot \vec{v}_1 = 0 \text{ and } \vec{z} \cdot \vec{v}_2 > 0]\}.$$

Rational and irrational orderings correspond precisely to the geometric constructions given in Example 2.2.3. It is easy to check that rational and irrational orderings as defined above satisfy the definition of a positive cone (see Definition 2.2.1). As already noted in Remark 3.3.3, the rational ordering determined by vectors $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = (0, 1)$ has positive cone

$$P = \{x^n y^m \mid n \geqslant 1 \text{ and } m \in \mathbb{Z}\} \cup \{y^m \mid m \geqslant 1\},$$

where $x = (1, 0)$ and $y = (0, 1)$. This positive cone has positive cone language $\mathcal{L}$, as given in Example 3.3.2. The language $\mathcal{L}$ (and therefore the positive cone P) is regular regardless of the choice of elements x and y. We will show that any rational positive cone of $\mathbb{Z}^2$ can be expressed by this language for some choice of x and y.

Definition 4.1.3. The elements $x, y \in \mathbb{Z}^2$ form a *regular basis* for the rational positive cone P if $P = \{x^n y^m \mid n \geqslant 1 \text{ and } m \in \mathbb{Z}\} \cup \{y^m \mid m \geqslant 1\}$.

Observe that for the standard lexicographic ordering, the regular basis elements are normal vectors $\vec{v}_1, \vec{v}_2$ that determine the ordering. Or, more precisely, since normal vectors are unique up to scalar multiplication, the regular basis elements correspond to the first positive integer lattice points lying on the lines ℓ_1 and ℓ_2 whose normal vectors are $\vec{v}_1$ and $\vec{v}_2$, respectively.

Question 4.1.4. Given a rational ordering determined by $\vec{v}_1$ and $\vec{v}_2$, can a regular basis always be obtained by taking the first positive integer lattice points lying on lines ℓ_1 and ℓ_2 whose respective normal vectors are $\vec{v}_1$ and $\vec{v}_2$?

Remark 4.1.5. In the work that follows, we alternate between multiplicative notation, *e.g.,* $x^n y^m$, and additive notation, *e.g.,* $nx + my$. In the context of $(\mathbb{Z}^2, +)$ the two are equivalent, however multiplicative notation is more natural when treating elements of $\mathbb{Z}^2$ as words in a language, hence we do not want to abandon it entirely.

Example 4.1.6. Consider the rational ordering P determined by normal vectors $\vec{v}_1 = (1, -2)$ and $\vec{v}_2 = (2, 1)$. Setting $x = (1, -2)$ and $y = (2, 1)$ does *not* yield a regular basis for P.

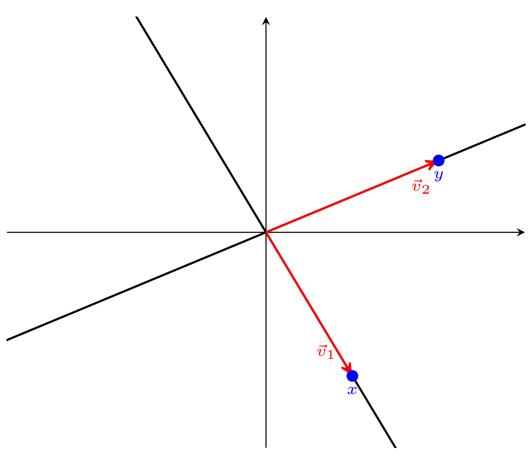


Figure 4.1: Vectors $\vec{v}_1, \vec{v}_2$ and points x, y in the plane.

Consider the element $g = (2, 0) \in \mathbb{Z}^2$. Since $g \cdot \vec{v}_1 = (2, 0) \cdot (1, -2) > 0$, $g \in P$. So, if x

and y are to form a regular basis of P, then g needs to be expressed as an integer linear combination of x and y:

$$g = x^n y^m = nx + my = n(1, -2) + m(2, 1) = (2, 0),$$

where $n \in \mathbb{Z}^+, m \in \mathbb{Z}$. From this, we obtain $m = 2n$ and $(5n, 0) = (2, 0)$, which implies that $n = \frac{2}{5} \notin \mathbb{Z}$. This example shows that the approach to finding a regular basis suggested by Question 4.1.4 will not work in general.

The reason the choice of $x = (1, -2)$ and $y = (2, 1)$ in the preceding example fail to form a regular basis is due to the following theorem.

Theorem 4.1.7. *[11, p. 4] Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a set of vectors in $\mathbb{Z}^n$. Let B be the $n \times n$ matrix formed by setting b_i to be the i^{th} row of B for all $1 \leqslant i \leqslant n$. The set $\mathcal{B}$ forms a basis for the integer lattice $\mathbb{Z}^n$ if and only if the matrix B is unimodular, i.e.,* $\det(B) = \pm 1$.

In other words, $x = (1, -2)$ and $y = (2, 1)$ fail to form a regular basis of P because they do not even form a basis for the integer lattice $\mathbb{Z}^2$, since

$$\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1 - (-4) = 5 \neq \pm 1.$$

Observation 4.1.8. The choice of x and y must meet the following two conditions to form a regular basis for a rational ordering determined by $\vec{v}_1$ and $\vec{v}_2$.

1. The elements x, y need to form a basis for the integer lattice $\mathbb{Z}^2$.

2. The element y must be the smallest positive integer lattice point lying on the line $\ell_1 : \vec{v}_1 \cdot t = 0$, $t \in \mathbb{R}$. The element x must lie in the positive half plane determined by ℓ_1, *i.e.*, $x \cdot \vec{v}_1 > 0$.

The second restriction is due to the form of the image of the positive cone language $\{x^n y^m \mid n \geqslant 1 \text{ and } m \in \mathbb{Z}\} \cup \{y^m \mid m \geqslant 1\}$. If x and y are vectors in $\mathbb{R}^2$, then the first set in the union,

$\{x^n y^m \mid n \geqslant 1 \text{ and } m \in \mathbb{Z}\}$, determines a collection of points lying in a single open half plane. Similarly, the second set in the union, $\{y^m \mid m \geqslant 1\}$, determines a collection of points lying on a half line. Therefore, the choice of y is fixed while the choice of x is not.

We give a procedure for finding an explicit regular basis:

1. Begin with any rational ordering of $\mathbb{Z}^2$ by fixing two perpendicular vectors $\vec{v}_1, \vec{v}_2$ having rational slopes. Let ℓ_1 and ℓ_2 denote the lines with normal vectors $\vec{v}_1$ and $\vec{v}_2$, respectively.

2. Set $y = (a, b)$, where (a, b) is the integer lattice point on ℓ_1 having minimal distance to the origin such that $(a, b) \cdot \vec{v}_2 > 0$.

3. Since (a, b) is a lattice point on ℓ_1 with minimal distance to the origin, it must be that $\gcd(a, b) = 1$. Therefore, by Bézout's identity, the equation $ax_1 + bx_2 = 1$ has a solution. In fact, if (c, d) is a solution, then $x_1 = c - kb$ and $x_2 = d + ka$, for $k \in \mathbb{Z}$, form an infinite family of solutions. This gives the freedom to choose a solution lying in the appropriate half plane. Without loss of generality, suppose that the initial solution (c, d) is such that $(c, d) \cdot \vec{v}_1 > 0$. Then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \pm 1.$$

   So, conditions 1 and 2 of Observation 4.1.8 are satisfied.

Proposition 4.1.9. *The elements x, y obtained from the procedure above form a regular basis of the positive cone*

$$P = \{\vec{z} \in \mathbb{Z}^2 \mid [\vec{z} \cdot \vec{v}_1 > 0] \text{ or } [\vec{z} \cdot \vec{v}_1 = 0 \text{ and } \vec{z} \cdot \vec{v}_2 > 0]\}.$$

Proof. Let $X = \{x^n y^m \mid n \geqslant 1 \text{ and } m \in \mathbb{Z}\} \cup \{y^m \mid m \geqslant 1\}$ be a language. We show that X is a positive cone language that evaluates to P. To see this, suppose ϕ is the underlying canonical homomorphism and $\phi(X) = \overline{X}$. We show that $\overline{X} = P$.

First, we show that $\overline{X} \subset P$. Let $x^n y^m$ with $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$ be an arbitrary element of $\overline{X}$, then

$$\begin{aligned} x^n y^m \cdot \vec{v}_1 &= [n(c,d) + m(a,b)] \cdot \vec{v}_1 \\ &= n(c,d) \cdot \vec{v}_1 + m(a,b) \cdot \vec{v}_1 \\ &= n(c,d) \cdot \vec{v}_1 && \text{since } (a,b) \cdot \vec{v}_1 = 0 \\ &> 0 && \text{since } n \geqslant 1 \text{ and } (c,d) \cdot \vec{v}_1 > 0 \text{ by choice.} \end{aligned}$$

Similarly, if we start with an arbitrary element of $\overline{X}$ of the form y^m, $m \geqslant 1$, then $y^m \cdot \vec{v}_1 = m(a,b) \cdot \vec{v}_1 = 0$ and $y^m \cdot \vec{v}_2 = m(a,b) \cdot \vec{v}_2 > 0$, by construction. So, $\overline{X} \subset P$.

For the other direction, start with an arbitrary element $g \in P$. Since x, y form a basis for the integer lattice $\mathbb{Z}^2$, it must be that $g = nx + my$ for some $n, m \in \mathbb{Z}$. If $n < 0$, then $g \cdot \vec{v}_1 < 0$, which contradicts the assumption that $g \in P$. Similarly, if $n = 0$ and $m < 1$, then $g \notin P$. So, by a process of elimination, it must be that ($n \geqslant 1$ and $m \in \mathbb{Z}$) or ($n = 0$ and $m \geqslant 1$). Therefore, $P \subset \overline{X}$ and the claim holds. □

It is possible to generalise this construction for rational flag orderings of $\mathbb{Z}^3$ using Hermite normal forms. For dimension $n > 3$, finding an explicit regular basis remains unsolved. Hence, we proceed with an existence proof.

4.2 Rational Orderings in Higher Dimensions

For $n > 2$, left orderings of $(\mathbb{Z}^n, +)$ can be obtained by extending the geometric construction given for the 2−dimensional case (see Example 2.2.3). In the 2−dimensional case, the integer lattice was ordered by cutting the plane using lines passing through the origin. In higher dimensions, lines will not suffice to cut the space $\mathbb{R}^n$. Thus, in generalising this construction, we replace lines with hyperplanes and slopes with normal vectors.

Definition 4.2.1. Let $\vec{v}$ be a nontrivial vector in $\mathbb{R}^n$ and $b \in \mathbb{R}$. A *hyperplane* π in $\mathbb{R}^n$ is the set of vectors $\pi = \{\vec{x} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{x} = b\}$.

In $\mathbb{R}^2$ and $\mathbb{R}^3$ hyperplanes are lines and planes, respectively. For the remainder of this section, we only consider hyperplanes containing the origin. Therefore, any hyperplane can be described by providing its normal vector.

Definition 4.2.2. A hyperplane π in $\mathbb{R}^n$ is said to be *irrational* if $\pi \cap \mathbb{Z}^n = \{0\}$ and *rational* if $\pi \cap \mathbb{Z}^n \simeq \mathbb{Z}^{n-1}$.

Observation 4.2.3. For $n > 2$, there exist hyperplanes in $\mathbb{R}^n$ that are *neither* rational nor irrational. Consider the hyperplane π in $\mathbb{R}^3$ with normal vector $\vec{v} = (1, \sqrt{2}, \sqrt{2})$. Then $\pi \cap \mathbb{Z}^3 = \{(0, k, -k) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}$.

Definition 4.2.4. Let π be an irrational hyperplane in $\mathbb{R}^n$ with normal vector $\vec{v}$. Then

$$P = \{\vec{z} \in \mathbb{Z}^n \mid \vec{v} \cdot \vec{z} > 0\}$$

is a *positive cone determined by* π. A left ordering of $\mathbb{Z}^n$ corresponding to such a positive cone P is called an *irrational ordering*.

Definition 4.2.5. Let π_1 and π_2 be hyperplanes in $\mathbb{R}^n$. We say that π_1 and π_2 are *orthogonal hyperplanes* if π_1 and π_2 have normal vectors $\vec{v}_1$ and $\vec{v}_2$, respectively, such that $\vec{v}_1 \cdot \vec{v}_2 = 0$.

Definition 4.2.6. Let $\mathcal{F} = \{\pi_n, \pi_{n-1}, \ldots, \pi_1\}$ be a family of mutually orthogonal rational hyperplanes in $\mathbb{R}^n$ with respective normal vectors $\vec{v}_n, \vec{v}_{n-1}, \ldots, \vec{v}_1$. Then

$$P = \{\vec{z} \in \mathbb{Z}^n \mid \vec{z} \cdot \vec{v}_n > 0 \text{ or } [\exists j \geqslant 1 \text{ s.t. } \vec{z} \cdot \vec{v}_i = 0 \text{ for } j < i \leqslant n \text{ and } \vec{z} \cdot \vec{v}_j > 0]\}$$

is a *positive cone determined by* $\mathcal{F}$. A left ordering of $\mathbb{Z}^n$ corresponding to such a positive cone P is called a *rational flag ordering*.

Remark 4.2.7. The definitions for irrational and rational orderings of $\mathbb{Z}^2$ coincide with Definitions 4.2.4 and 4.2.6, as expected.

Example 4.2.8. There are left orderings of $\mathbb{Z}^n$, for $n > 2$, that are neither irrational nor rational flag orderings. Consider the left ordering of $\mathbb{Z}^3$ determined by vectors $\vec{v}_2 = (1, \sqrt{2}, \sqrt{2})$ and $\vec{v}_1 = (\sqrt{2}, -1, 0)$, that is,

$$P = \{\vec{z} \in \mathbb{Z}^3 \mid \vec{v}_2 \cdot \vec{z} > 0 \text{ or } [\vec{v}_2 \cdot \vec{z} = 0 \text{ and } \vec{v}_1 \cdot \vec{z} > 0]\}.$$

One can check that P is indeed a positive cone of $\mathbb{Z}^3$ and that it does not satisfy Definition 4.2.4 nor 4.2.6 (see Observation 4.2.3). This type of ordering falls somewhere "in between" the irrational and rational flag orderings. We will not be working with these kinds of orderings.

Definition 4.2.9. Let π be a hyperplane in $\mathbb{R}^n$ having normal vector $\vec{v}$. A *closed half space* determined by π is the set of points $H = \{\vec{x} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{x} \geqslant 0\}$.

Definition 4.2.10. A *flag* in $\mathbb{R}^n$ is a sequence of closed half spaces $H_1 \subset H_2 \subset \cdots \subset H_n$, where the dimension of H_i is i for all $1 \leqslant i \leqslant n$.

Let $\mathcal{F} = \{\pi_n, \ldots, \pi_1\}$ be a family of mutually orthogonal rational hyperplanes with normal vectors $\vec{v}_n, \ldots, \vec{v}_1$. In addition to inducing a rational flag ordering of $\mathbb{Z}^n$, the family $\mathcal{F}$ also induces a flag in $\mathbb{R}^n$ as follows:

- set $H_n = \{\vec{x} \in \mathbb{R}^n \mid \vec{v}_n \cdot \vec{x} \geqslant 0\}$; and
- for $1 \leqslant k < n$, set $H_k = \{\vec{x} \in \mathbb{R}^n \mid \vec{v}_i \cdot \vec{x} = 0 \text{ for } k < i \leqslant n \text{ and } \vec{v}_k \cdot \vec{x} \geqslant 0\}$.

This flag construction will be useful in the proof of Theorem 4.3.13.

Lemma 4.2.11. *Let π be a hyperplane containing the origin in $\mathbb{R}^n$. If π has normal vector $\vec{v}$ such that all the components of $\vec{v}$ are rational, then π is a rational hyperplane.*

Proof. Suppose π is a hyperplane containing the origin with normal vector $\vec{v} = (v^1, \ldots, v^n)$ such that $v^i \in \mathbb{Q}$. Since normal vectors are unique up to scalar multiplication, we can assume that all of the components v^i are integers. Consider the set of vectors

$$(v^2, -v^1, 0, \ldots, 0)$$

$$(v^3, 0, -v^1, 0, \ldots, 0)$$
$$\vdots$$
$$(v^k, 0, \ldots, 0, -v^1, 0, \ldots, 0)$$
$$\vdots$$
$$(v^n, 0, \ldots, 0, -v^1).$$

This is a set of $n-1$ linearly independent vectors in π, which are all integer lattice points. Therefore, the dimension of $\mathbb{Z}^n \cap \pi$ is $n-1$, and π is a rational hyperplane. $\square$

4.3 Rational Flag Orderings are Regular

We claim that a rational flag ordering of $\mathbb{Z}^n$ can be expressed by a positive cone language of the form

$$\mathfrak{X}_n = \bigcup_{i=1}^{n} X_i,$$

where $X_1 = \{x_1^{k_1} \mid k_1 \in \mathbb{Z}^+\}$ and $X_i = \{x_i^{k_i} \cdots x_1^{k_1} \mid k_i \in \mathbb{Z}^+ \text{ and } k_{i-1}, \ldots, k_1 \in \mathbb{Z}\}$ for $1 < i \leqslant n$. We will call the symbols $x_1, \ldots, x_n$ the *generators* of $\mathfrak{X}_n$. Recall that any element of the form x_i^{-1} in this language is simply another element of the monoid and *not* an inverse element.

Remark 4.3.1. Suppose $\Sigma = \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ and $\phi : \Sigma^* \to \mathbb{Z}^n$ is the canonical homomorphism. Throughout this section, we use $\overline{\mathfrak{X}_n}$ to denote the image of $\mathfrak{X}_n$ under the homomorphism ϕ, *i.e.*, $\phi(\mathfrak{X}_n) = \overline{\mathfrak{X}_n}$. Similarly, the notation $\overline{x}_i$ will be used to indicate the image of an element x_i under ϕ, *i.e.*, $\phi(x_i) = \overline{x}_i$.

Definition 4.3.2. Let P be a positive cone of a rational flag ordering of $\mathbb{Z}^n$. The generators $x_1, \ldots, x_n$ of $\mathfrak{X}_n$ are called a *regular basis* of P if $\overline{\mathfrak{X}_n} = P$.

Remark 4.3.3. When $n = 2$, $\mathfrak{X}_2 = \{x_1^{k_1} \mid k_1 \in \mathbb{Z}^+\} \cup \{x_2^{k_2} x_1^{k_1} \mid k_2 \in \mathbb{Z}^+ \text{ and } k_1 \in \mathbb{Z}\}$, which corresponds to the positive cone language for a rational ordering of $\mathbb{Z}^2$ as seen in Example 3.3.2.

Lemma 4.3.4. *For all integers $n \geqslant 2$, the language $\mathfrak{X}_n = \bigcup_{i=1}^{n} X_i$ is regular.*

Proof. We prove the claim inductively.

Base case. For $n = 2$, $\mathfrak{X}_2 = X_1 \cup X_2 = \{x_1^{k_1} \mid k_1 \in \mathbb{Z}^+\} \cup \{x_2^{k_2} x_1^{k_1} \mid k_2 \in \mathbb{Z}^+ \text{ and } k_1 \in \mathbb{Z}\}$. The FSA given in Figure 3.5 accepts $\mathfrak{X}_2$, therefore $\mathfrak{X}_2$ is regular.

Inductive step. Fix an integer $n > 2$ and assume that $\mathfrak{X}_n$ is regular. We can construct a generalised FSA that that accepts the language X_{n+1} (see Figure 4.2). By Theorem A.0.5, X_{n+1} is a regular language. By Proposition 3.2.1, $\mathfrak{X}_{n+1} = X_{n+1} \cup \mathfrak{X}_n$ is regular. $\square$

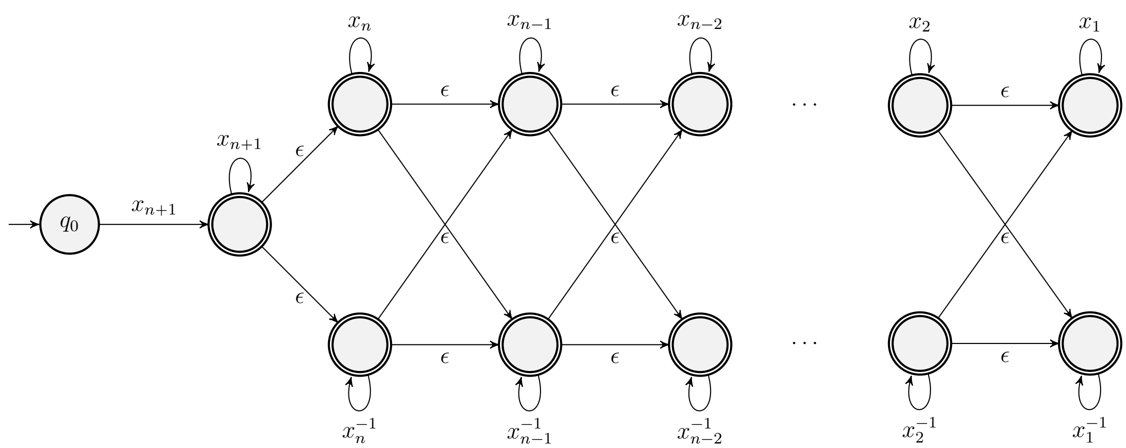


Figure 4.2: A generalised FSA accepting the language X_{n+1}.

Definition 4.3.5. Let $\mathcal{L}$ be a lattice. The lattice $\mathcal{L}'$ is a *sublattice* of $\mathcal{L}$ if $\mathcal{L}' \subseteq \mathcal{L}$.

Lemma 4.3.6. *Let $A \simeq \mathbb{Z}^m$ be a sublattice of $\mathbb{Z}^n$. If $\overline{x}_1, \ldots, \overline{x}_m \in A$ form a basis for A, then $\overline{\mathfrak{X}_m} = \bigcup_{i=1}^{m} \overline{X_i}$ generated by $x_1, \ldots, x_m$ is a positive cone of $\mathbb{Z}^m$.*

Proof. Suppose that $\overline{x}_1, \ldots, \overline{x}_m$ form a basis for $A \cong \mathbb{Z}^m$. To see that $\overline{\mathfrak{X}_m}$ is a positive cone, we show that

1. $\overline{\mathfrak{X}_m} \cdot \overline{\mathfrak{X}_m} \subset \overline{\mathfrak{X}_m}$, *i.e.*, the set is closed under the group operation,

2. for any $g \in A$, g is contained in exactly one of $\overline{\mathfrak{X}_m}$, $(\overline{\mathfrak{X}_m})^{-1}$, or $\{0\}$.

Suppose that $\overline{w_1} = \overline{x}_{i_1}^{k_{i_1}} \cdots \overline{x}_{i_l}^{k_{i_l}}$ and $\overline{w_2} = \overline{x}_{j_1}^{k_{j_1}} \cdots \overline{x}_{j_m}^{k_{j_m}}$ are arbitrary elements in $\overline{\mathfrak{X}_m}$. To see that the product $\overline{w}_1 \cdot \overline{w}_2$ is contained in $\overline{\mathfrak{X}_m}$, one only needs to show that there exists a word

$w \in \mathfrak{X}_m$ such that $\phi(w) = \overline{w}_1 \cdot \overline{w}_2$. The preimage of $\overline{w}_1$ is a word of the form $w_1 = x_{i_1}^{k_{i_1}} \cdots x_{i_l}^{k_{i_l}}$ up to a rearrangement of indices, since $\mathbb{Z}^m$ is abelian. Without loss of generality, suppose that the preimages of $\overline{w}_1$ and $\overline{w}_2$ under ϕ are $w_1 = x_{i_1}^{k_{i_1}} \cdots x_{i_l}^{k_{i_l}}$ and $w_2 = x_{j_1}^{k_{j_1}} \cdots x_{j_l}^{k_{j_l}}$, respectively, and that the indices $i_1, \dots, i_l, j_1, \dots, j_l$ appear in descending order. Then the word $w_1 \cdot w_2$ is precisely the w that we are looking for. Thus, (1) holds.

To show that $\overline{\mathfrak{X}_m}$ partitions A, first observe that $\overline{x}_n^0 \cdots \overline{x}_1^0 = 0$ is not contained in $\overline{\mathfrak{X}_m}$ nor $(\overline{\mathfrak{X}_m})^{-1}$. Next, consider an arbitrary element $g \in A$. Since $\overline{x}_1, \dots, \overline{x}_m$ forms a basis for A, $g = \overline{x}_m^{k_m} \cdots \overline{x}_1^{k_1}$, where $k_i \in \mathbb{Z}$. Without loss of generality, suppose that the indices of the letters in the spelling of g $(m, \dots, 1)$ appear in descending order. Suppose that k_i, for some $m \geqslant i \geqslant 1$, is the first non-zero exponent (reading from left to right). If $k_i < 0$, then $g \in (\overline{\mathfrak{X}_m})^{-1}$, otherwise $g \in \overline{\mathfrak{X}_m}$. $\square$

Given a rational flag ordering P, we prove that it is regular by showing that a regular basis for P exists. We do this by inducting on the dimension of the positive cone. First, we introduce some new terminology and notation.

Proposition 4.3.7. *[5, Corollary 3, p. 14] Let $M \subseteq \mathbb{Z}^n$ be a lattice and let $a_1, \dots, a_m$ $(m \leqslant n)$ be linearly independent vectors in M. The set $a_1, \dots, a_m$ can be extended to a basis of M if and only if any element $x \in M$ that can be expressed as a linear combination of $a_1, \dots, a_m$ must be an integer linear combination, i.e., if*

$$x = \lambda_1 a_1 + \cdots + \lambda_m a_m,$$

then $\lambda_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant m$.

Lemma 4.3.8. *Let $\mathcal{F} = \{\pi_n, \dots, \pi_1\}$ be a family of rational mutually orthogonal hyperplanes in $\mathbb{R}^n$ with respective normal vectors $\vec{v}_n, \dots, \vec{v}_1$. The intersection $A_k = \mathbb{Z}^n \cap \pi_n \cap \cdots \cap \pi_{k+1}$ forms a subgroup of $\mathbb{Z}^n$ of rank k for all $1 \leqslant k \leqslant n-1$.*

Proof. By backward induction on k.

Base Case. Consider $A_{n-1} = \mathbb{Z}^n \cap \pi_n$. Since π_n is a rational hyperplane, by definition, $A_{n-1} \simeq \mathbb{Z}^{n-1}$. So, the base case holds.

Inductive Step. Suppose $A_k = \mathbb{Z}^n \cap \pi_n \cap \cdots \cap \pi_{k+1} \simeq \mathbb{Z}^k$. Denote $B = A_k \cap \pi_k$, then $B \simeq \mathbb{Z}^h$ for some h. We want to show that $h = k-1$. Assume $h = k-2$ and $\{b_1, \ldots, b_{k-2}\}$ is a basis for B. The set $\{b_1, \ldots, b_{k-2}\}$ can be extended to a basis of A_k. Consider Proposition 4.3.7 and an element $\vec{z} \in A_k$ that can be expressed as a linear combination of the b_i's, *i.e.,* $\vec{z} = \lambda_1 b_1 + \cdots + \lambda_{k-2} b_{k-2}$. If $\vec{z} \in B \subset A_k$, then $\lambda_i \in \mathbb{Z}$, since by assumption $\{b_1, \ldots, b_{k-2}\}$ is a basis of B. Alternatively, suppose $\vec{z} \in A_k \backslash B$, then $\vec{v}_k \cdot \vec{z} \neq 0$ since $\vec{z} \notin \pi_k$ and

$$\begin{aligned} 0 \neq \vec{v}_k \cdot \vec{z} &= \vec{v}_k \cdot (\lambda_1 b_1 + \cdots + \lambda_{k-2} b_{k-2}) \\ &= \lambda_1 (\vec{v}_k \cdot b_1) + \cdots + \lambda_{k-2} (\vec{v}_k \cdot b_{k-2}) \\ &= 0, \end{aligned}$$

which is a contradiction. Therefore, any element in A_k that can be expressed as a linear combination of the b_i's can be expressed as an integral linear combination. So, there exists an extension of the basis $\{b_1, \ldots, b_{k-2}\}$ to a basis of A_k. Suppose this extension is obtained by adding the basis elements $a_1, a_2 \in A_k \backslash B$.

Similarly, there also exists an extension of $\{b_1, \ldots, b_{k-2}\}$ to a basis of $\mathbb{Z}^n \cap \pi_k$. Once again, consider an element $\vec{z} \in \mathbb{Z}^n \cap \pi_k$ that can be expressed as $\vec{z} = \lambda_1 b_1 + \cdots + \lambda_{k-2} b_{k-2}$. If $\vec{z} \in B$, then $\lambda_i \in \mathbb{Z}$ for $1 \leqslant i \leqslant k-2$. On the other hand, if $\vec{z} \in (\mathbb{Z}^n \cap \pi_k) \backslash B$, then $\vec{z} \cdot \vec{v}_j \neq 0$ for some $k+1 \leqslant j \leqslant n$. That is,

$$\begin{aligned} 0 \neq \vec{v}_j \cdot \vec{z} &= \vec{v}_j \cdot (\lambda_1 b_1 + \cdots + \lambda_{k-2} b_{k-2}) \\ &= \lambda_1 (\vec{v}_j \cdot b_1) + \cdots + \lambda_{k-2} (\vec{v}_j \cdot b_{k-2}) \\ &= 0, \end{aligned}$$

which is a contradiction. Therefore, there exists an extension of $\{b_1, \ldots, b_{k-2}\}$ to a basis of

$\mathbb{Z}^n \cap \pi_k$. Suppose the extension is obtained by adding the basis elements $\{c_1, \ldots, c_{n-k+1}\}$, so $\{b_1, \ldots, b_{k-2}, c_1, \ldots, c_{n-k+1}\}$ is a basis for $\mathbb{Z}^n \cap \pi_k$.

Observe that no a_i can be expressed as a linear combination of the elements $\{c_1, \ldots, c_{n-k+1}\}$. Similarly, no c_j can be expressed as a linear combination of the a_i's. To see this, suppose that $a_i = \sum_{j=1}^{n-k+1} \gamma_j c_j$, where $\gamma_j \in \mathbb{Z}$. Then $a_i \in \pi_k$, but this contradicts the fact that $a_i \in A_k \backslash B$. A similar contradiction follows if we suppose $c_j = \lambda_1 a_1 + \lambda_2 a_2$. Therefore, the set $\{b_1, \ldots, b_{k-2}, a_1, a_2, c_1, \ldots, c_{n-k+1}\}$ is linearly independent in $\mathbb{Z}^n$. In other words, we have found a set of $(k-2) + 2 + (n-k+1) = n+1$ linearly independent vectors in $\mathbb{Z}^n$. This is a contradiction.

We obtain a similar contradiction if we assume $h < k-2$. Additionally, if we suppose $h = k$, then this contradicts the assumption that the hyperplanes $\pi_i \in \mathcal{F}$ are mutually orthogonal. Therefore, $B \simeq \mathbb{Z}^{k-1}$. ☐

Lemma 4.3.9. *If A_k be a subgroup as defined in Lemma 4.3.8, then A_k is closed under taking roots. That is, if $g^h \in A_k$ for some $g \in \mathbb{Z}^n$ and $h \in \mathbb{Z}$, then $g \in A_k$.*

Proof. Suppose that $g^h \in A_k$, for some $g \in \mathbb{Z}^n$ and $h \in \mathbb{Z}\backslash\{0\}$. Then $g^h \cdot \vec{v}_i = 0$, for $k+1 \leqslant i \leqslant n$. More precisely,

$$g^h \cdot \vec{v}_i = h(g \cdot \vec{v}_i) = 0$$

implies that $g \cdot \vec{v}_i = 0$ for all $k+1 \leqslant i \leqslant n$. Therefore, $g \in A_k$ and A_k is closed under taking roots. ☐

Definition 4.3.10. An element $g \in \mathbb{Z}^n$ is *primitive* if it has no nontrivial roots, that is g cannot be expressed as h^k for some $h \in \mathbb{Z}^n$ and $k > 1$.

Observation 4.3.11. Let $\mathcal{F} = \{\pi_n, \ldots, \pi_1\}$ be a family of mutually orthogonal rational hyperplanes in $\mathbb{R}^n$ with respective normal vectors $\vec{v}_n, \ldots, \vec{v}_1$. The family $\mathcal{F}$ induces the rational flag ordering P of $\mathbb{Z}^n$. There is a "copy" of a rational flag ordering of $\mathbb{Z}^k$, for

$1 \leqslant k \leqslant n$, inside of P. Let A_k be as given in Lemma 4.3.8. It is easy to verify that

$$P_k = P \cap A_k = \{\vec{z} \in A_k \,|\, \exists j\, 1 \leqslant j < k+1 \text{ s.t. } \vec{v}_j \cdot \vec{z} > 0 \text{ and } \vec{v}_i \cdot \vec{z} = 0\, \forall j < i \leqslant n\}$$

is a positive cone of $A_k \simeq \mathbb{Z}^k$.

Definition 4.3.12. Let P and P_k be as defined in Observation 4.3.11. We call P_k the *restricted rational flag ordering of dimension* k in P.

Theorem 4.3.13. *Fix a family of mutually orthogonal rational hyperplanes* $\mathcal{F} = \{\pi_n, \ldots, \pi_1\}$ *in* $\mathbb{R}^n$. *Suppose* P *is the rational flag ordering and* $H_n, \ldots, H_1$ *is the flag induced by* $\mathcal{F}$. *There exists a regular basis* $x_1, \ldots, x_k$ *of the restricted rational flag ordering* P_k, *for all* $1 \leqslant k \leqslant n$, *such that* $x_1 \in H_1$ *and* $x_i \in H_i \backslash (H_{i-1} \cap \cdots \cap H_1)$ *for* $1 < i \leqslant k$.

Proof. We induct on k. Let $\mathcal{F} = \{\pi_n, \ldots, \pi_1\}$ be a given family of mutually orthogonal rational hyperplanes in $\mathbb{R}^n$ with respective normal vectors $\vec{v}_n, \ldots, \vec{v}_1$. Let $S(k)$ denote the statement that the restricted rational flag ordering P_k admits a regular basis $x_1, \ldots, x_k$ such that $x_1 \in H_1$ and $x_i \in H_i \backslash (H_{i-1} \cap \cdots \cap H_1)$ for $1 < i \leqslant k$.

Base case. Set $k = 1$. We are looking for a regular basis x_1 such that $\overline{\mathfrak{X}_1} = \{x_1^{k_1} \,|\, k_1 \in \mathbb{Z}^+\} = P_1$. The subset $A_1 = \mathbb{Z}^n \cap \pi_n \cap \cdots \cap \pi_2 \simeq \mathbb{Z}$ is a set of integer lattice points lying on a line ℓ. So, the restricted positive cone $P_1 = P \cap A_1 = \{\vec{z} \in A_1 \,|\, \vec{z} \cdot \vec{v}_1 > 0\}$ is precisely the set of non-zero integer lattice points lying on one half of the line ℓ. Take x_1 to be the integer lattice point on the positive half of ℓ ($x_1 \cdot \vec{v}_1 > 0$) having minimal distance to the origin. Then x_1 is a basis for the integer lattice $A_1 \simeq \mathbb{Z}$, and by Lemma 4.3.6, $\overline{\mathfrak{X}_1}$ is a positive cone of $\mathbb{Z}$. Clearly, $\overline{\mathfrak{X}_1} \subset P_1$, so by Lemma 2.2.4, it follows that $\overline{\mathfrak{X}_1} = P_1$. Additionally, $x_1 \in H_1$, so the base case holds.

Inductive step. Fix some $m \leqslant n$ and suppose that $S(m-1)$ holds, that is, the restricted rational flag ordering P_{m-1} admits a regular basis $x_1, \ldots, x_{m-1}$ such that $x_1 \in H_1$ and $x_i \in H_i \backslash (H_{i-1} \cap \cdots \cap H_1)$ for $1 < i \leqslant m-1$. To show that $S(m)$ holds, we show that there exists an element $x_m \in H_m \backslash (H_{m-1} \cap \cdots \cap H_1)$ such that $x_1, \ldots, x_{m-1}, x_m$ forms a regular basis of P_m. That is, we show that $\overline{\mathfrak{X}_m}$ generated by $x_1, \ldots, x_{m-1}, x_m$ is precisely the positive cone P_m.

To see that there exists an element $x_m \in H_m \subset A_m$, consider the short exact sequence

$$0 \to A_{m-1} \overset{i}{\hookrightarrow} A_m \overset{q}{\to} A_m/A_{m-1} \to 0,$$

where i is the inclusion map and q is the quotient map.

First, we show that A_m/A_{m-1} is isomorphic to $\mathbb{Z}$:

- First, observe that A_m/A_{m-1} is abelian since quotients of abelian groups are abelian.
- Next, A_m/A_{m-1} is torsion-free. To see this, suppose not, *i.e.,* there exists an element $gA_{m-1} \in A_m/A_{m-1}$ such that $g \in A_m \backslash A_{m-1}$ and $(gA_{m-1})^k = A_{m-1}$, for some $k \in \mathbb{Z}$. Then, $(gA_{m-1})^k = (gA_{m-1})(gA_{m-1}) \cdots (gA_{m-1}) = g^k A_{m-1} = A_{m-1}$, which implies that $g^k \in A_{m-1}$. However, $g \notin A_{m-1}$ by assumption, which contradicts the fact that A_{m-1} is closed under taking roots (by Lemma 4.3.9).
- The quotient A_m/A_{m-1} is finitely generated because A_m is finitely generated. More explicitly, if $g_1, \ldots, g_m$ are generators of A_m, then some subset of $\{g_1A_{m-1}, \ldots, g_mA_{m-1}\}$ will generate A_m/A_{m-1}.

Thus, A_m/A_{m-1} is a torsion-free, finitely generated, abelian group. By the fundamental theorem of finitely generated abelian groups, this means that $A_m/A_{m-1} \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}$. It remains to show that $k = 1$.

Since $A_{m-1} \cong \mathbb{Z}^{m-1}$, there exists a choice of elements $b_1, \ldots, b_{m-1} \in A_{m-1}$ that are linearly independent over $\mathbb{Z}$. Similarly, since A_m/A_{m-1} has dimension k, there exists a choice of elements $c_1, \ldots, c_k \in A_m/A_{m-1}$ that are linearly independent over $\mathbb{Z}$. For each c_i, there is a choice of $a_i \in A_m$ such that $q(a_i) = c_i$.

Now, we want to show that $a_1, \ldots, a_k, b_1, \ldots, b_{m-1}$ are linearly independent over $\mathbb{Z}$ in A_m. So, suppose $\lambda_1 a_1 + \cdots + \lambda_k a_k + \gamma_1 b_1 + \cdots + \gamma_{m-1} b_{m-1} = 0$. Then

$$\begin{aligned} 0 &= q(\lambda_1 a_1 + \cdots + \lambda_k a_k + \gamma_1 b_1 + \cdots + \gamma_{m-1} b_{m-1}) \\ &= \lambda_1 q(a_1) + \cdots + \lambda_k q(a_k) + \gamma_1 q(b_1) + \cdots + \gamma_{m-1} q(b_m) && \text{since } q \text{ is a homomorphism} \\ &= \lambda_1 c_1 + \cdots + \lambda_k c_k && \text{since } q(b) = 0 \text{ for any } b \in A_{m-1}, \end{aligned}$$

which implies that $\lambda_1 = \cdots = \lambda_k = 0$, since the c_i's are linearly independent. From this, we obtain

$$\begin{aligned} 0 &= \lambda_1 a_1 + \cdots + \lambda_k a_k + \gamma_1 b_1 + \cdots + \gamma_{m-1} b_{m-1} \\ &= \gamma_1 b_1 + \cdots + \gamma_{m-1} b_{m-1}, \end{aligned}$$

but since the b_i's are also linearly independent, $\gamma_1 = \cdots = \gamma_{m-1} = 0$. Therefore, we have found $(m-1+k)-$many linearly independent vectors in A_m. If $k > 1$, then $m - 1 + k > m$, but the dimension of A_m is m, so we have a contradiction. Thus, it must be that $k = 1$ and $A_m/A_{m-1} \simeq \mathbb{Z}$.

Let gA_{m-1} be a generator of A_m/A_{m-1} such that g is a primitive element of A_m. Then $g \in A_m \backslash A_{m-1}$, therefore $g \cdot \vec{v}_m \neq 0$. If $g \cdot \vec{v}_m > 0$, then set $x_m = g$ (otherwise, set $x_m = -g$). This guarantees that $x_m \in H_m \backslash (H_{m-1} \cap \cdots \cap H_1)$. Taking $x_1, \ldots, x_{m-1}, x_m$ as the generators for $\overline{\mathfrak{X}_m}$ yields the containment $\overline{\mathfrak{X}_m} \subset P_m$. To see this, consider an arbitrary element $k_m x_m + k_{m-1} x_{m-1} + \cdots + k_1 x_1 \in \overline{X_m}$ and take its dot product with $\vec{v}_m$:

$$\begin{aligned} \left(\sum_{i=1}^{m} k_i x_i \right) \cdot \vec{v}_m &= (k_m x_m) \cdot \vec{v}_m + (k_{m-1} x_{m-1} + \cdots + k_1 x_1) \cdot \vec{v}_m \\ &= (k_m x_m) \cdot \vec{v}_m && \text{since } x_{m-1}, \ldots, x_1 \in A_{m-1} \\ &> 0 && \text{since } k_m \in \mathbb{Z}^+ \text{ and by the choice of } x_m. \end{aligned}$$

Since this dot product is positive, $\overline{X_m} \subset P_m$. Now, by the inductive hypothesis, $\overline{X_j}$ for all $1 \leqslant j < m$ is contained in P_j, and since $P_j \subset P_m$, it follows that $\overline{X_j} \subset P_m$. Thus, $\overline{\mathfrak{X}_m} \subset P_m$.

Finally, to see that $\overline{\mathfrak{X}_m} = P$, it remains to verify that $\overline{\mathfrak{X}_m}$ is a positive cone of $\mathbb{Z}^m$. By Lemma 4.3.6, it suffices to show that $x_1, \ldots, x_m$ is a basis for the lattice A_k. By the inductive hypothesis, $x_1, \ldots, x_{m-1}$ is a basis of A_{m-1}. Since x_m is a primitive element of A_m/A_{m-1}, $x_1, \ldots, x_{m-1}, x_m$ is a basis of A_m. Finally, by the inductive hypothesis, $x_i \in H_i \backslash (H_{i-1} \cap \cdots \cap H_1)$, for $1 \leqslant i \leqslant m-1$, so the theorem holds. □

4.4 Density of Rational Flag Orderings

Before proceeding to show that regular orderings are dense in $\mathrm{LO}(\mathbb{Z}^n)$, it is worth noting that, for $n > 2$, this result is not trivial. That is, the set of all regular orderings does *not* make up the entire space $\mathrm{LO}(\mathbb{Z}^n)$. This is due to Sikora's theorem.

Theorem 4.4.1 (Sikora)**.** *[21] For $n \geqslant 2$, $\mathrm{LO}(\mathbb{Z}^n)$ is uncountable.*

If all $P \in \mathrm{LO}(\mathbb{Z}^n)$ are regular, then by Proposition 3.3.6, this means that the space of orderings is finite, which contradicts Sikora's theorem. Therefore, asking if the set of regular orderings is dense in $\mathrm{LO}(\mathbb{Z}^n)$ is a nontrivial question. Note that in the case when $n = 1$, the result is trivial since $\mathbb{Z}$ admits only two orderings, both of which are regular.

Lemma 4.4.2. *[14, Lemma 2.2] Let $\Sigma = \{v_1, \ldots, v_k\} \subset \mathbb{Z}^n$, where k is finite and $k \geqslant n+1$. If Σ is not contained in a single closed half space, then the semigroup generated by Σ contains 0.*

Proof. If $n = 1$, then the claim is trivial. So, suppose $n \geqslant 2$. Suppose $v_1, \ldots, v_n, v_{n+1} \in \Sigma$ such that $v_1, \ldots, v_n, v_{n+1}$ are not all contained in any closed half space. The vectors $v_1, \ldots, v_{n+1}$ are necessarily linearly dependent, so there exist $a_1, \ldots, a_{n+1} \in \mathbb{Z}$ such that

$$a_1 v_1 + \cdots + a_{n+1} v_{n+1} = 0.$$

Let S denote the semigroup generated by Σ. Suppose, for the sake of contradiction, that $0 \notin S$. This means that there exists some a_i with $1 \leqslant i \leqslant n+1$, such that $a_i < 0$, otherwise $v_1^{a_1} \cdots v_{n+1}^{a_{n+1}} = 0 \in S$. Similarly, since

$$-(a_1 v_1 + \cdots + a_{n+1} v_{n+1}) = (-a_1) v_1 + \cdots + (-a_{n+1}) v_{n+1} = 0,$$

we conclude $(-a_j) < 0$ or $a_j > 0$ for some $1 \leqslant j \leqslant n+1$. Without loss of generality, suppose that $a_1 > 0$ and $a_2 < 0$. Now, consider the subspace π spanned by $\{v_3, \ldots, v_{n+1}\}$ and let $\vec{N}$ be any vector in $\pi^{\perp}$. Then

$$\begin{aligned}
(-a_2)(v_2 \cdot \vec{n}) &= -a_2 v_2 \cdot \vec{N} \\
&= (a_1 v_1 + a_3 v_3 + \cdots + a_{n+1} v_{n+1}) \cdot \vec{N} \\
&= a_1 v_1 \cdot \vec{N} + (a_3 v_3 + \cdots + a_{n+1} v_{n+1}) \cdot \vec{N} \\
&= a_1 v_1 \cdot \vec{N}.
\end{aligned}$$

Finally, $-a_2 v_2 \cdot \vec{N} = a_1 v_1 \cdot \vec{N}$ implies that $v_1 \cdot \vec{N}$ and $v_2 \cdot \vec{N}$ have the same sign, therefore v_1 and v_2 both lie in the same half space. In particular, all the vectors $v_1, \ldots, v_n, v_{n+1}$ are either contained on π or on one side of π, *i.e.*, $v_1, \ldots, v_n, v_{n+1}$ are all contained in a single closed half space $H = \{\vec{v} \in \mathbb{Z}^n \mid \vec{v} \cdot \vec{N} \geqslant 0\}$. This contradicts the original assumption, hence $0 \in S$. □

The following corollary is given as proposition 2.3 in [14]. We give a different proof below.

Corollary 4.4.3. *Let P be a positive cone of $\mathbb{Z}^n$ and $g_1, \ldots, g_k$ be a finite collection of elements in P. Then $g_1, \ldots, g_k$ are contained in a closed half space.*

Proof. It is a basic fact from linear algebra that n distinct points in $\mathbb{R}^n$ uniquely determine a hyperplane. Therefore, if $k \leqslant n$, there exists a hyperplane H in $\mathbb{R}^n$ containing $g_1, \ldots, g_k$. So, $g_1, \ldots, g_k$ are contained in a closed half space (in the case where $k < n$, the hyperplane

H is not unique). On the other hand, if $k > n$, then suppose for the sake of contradiction that $g_1, \ldots, g_k$ are not contained in a closed half space. Then, by Lemma 4.4.2, it follows that $0 \in P$, which is a contradiction. $\square$

Proposition 4.4.4. *[8, Lemma 3.3] All orderings of $\mathbb{Z}^n$ are determined geometrically, that is, for any positive cone P of $\mathbb{Z}^n$ there exists a collection of orthogonal vectors $\vec{v}_1, \ldots, \vec{v}_l$, where $l \leqslant n$, such that*

$$P = \{\vec{z} \in \mathbb{Z}^n \mid \vec{z} \cdot \vec{v}_l > 0 \textit{ or } [\exists j \geqslant 1 \textit{ s.t. } \vec{z} \cdot \vec{v}_i = 0 \textit{ for } j < i \leqslant l \textit{ and } \vec{z} \cdot \vec{v}_j > 0]\}.$$

Recall the metric on $\mathrm{LO}(G)$ introduced in Definition 2.3.4. Fix an enumeration $\{g_1, g_2, \ldots\}$ of $\mathbb{Z}^n$. If P and P' are two distinct positive cones in $\mathrm{LO}(\mathbb{Z}^n)$ and the first index on which P and P' differ is k, i.e., $g_k \in P$ and $g_k \notin P'$, then we define the distance between P and P' by

$$\mathrm{d}(P, P') = 2^{-k}.$$

We use this metric to show that rational flag orderings are dense in $\mathrm{LO}(\mathbb{Z}^n)$.

Theorem 4.4.5. *[14] Rational flag orderings of $\mathbb{Z}^n$ are dense in $\mathrm{LO}(\mathbb{Z}^n)$.*

Proof. Choose an arbitrary positive cone $P \in \mathrm{LO}(\mathbb{Z}^n)$. We show that for every $k > 0$ there exists a regular positive cone $P' \in \mathrm{LO}(\mathbb{Z}^n)$ such that $d(P, P') < 2^{-k}$. By Corollary 2.3.6, this is equivalent to choosing finitely many distinguished elements $g_1, \ldots, g_k \in P$ and showing that $g_1, \ldots, g_k \in P'$.

By Corollary 4.4.3, the elements $g_1, \ldots, g_k$ are contained in a single closed half space. Suppose this half space is determined by the hyperplane $H : \vec{N} \cdot \vec{x} = 0$, *i.e.,* $\vec{N} \cdot g_i \geqslant 0$. We claim that there exists a vector $\vec{w} \in \mathbb{R}^n$ such that $\vec{w} \cdot g_i > 0$ for all $1 \leqslant i \leqslant k$. If $g_1, \ldots, g_k$ are all contained in an open half space, then $\vec{N} \cdot g_i > 0$ for all $1 \leqslant i \leqslant k$, so take $\vec{w} = \vec{N}$. On the other hand, if any of the g_i's are contained in H, then we construct such a vector $\vec{w}$.

In the latter case, the positive cone P must be determined by some collection of orthogonal vectors $\vec{v}_1, \ldots, \vec{v}_l$, by Proposition 4.4.4. For each g_i, $1 \leqslant i \leqslant k$, there exists some $m \in \{1, \ldots, l\}$ such that $g_i \cdot \vec{v}_j = 0$ for $1 \leqslant j \leqslant m-1$ and $g_i \cdot \vec{v}_m > 0$. Let S_m denote the set of all g_i's for which $g_i \cdot \vec{v}_j = 0$ for $1 \leqslant j \leqslant m-1$ and $g_i \cdot \vec{v}_m > 0$.

We construct a vector $\vec{w}$ such that $g_i \cdot \vec{w} > 0$ for all $1 \leqslant i \leqslant k$. We do this using the following iterative process:

- Step 1. Set $\vec{w}_0 = v_l$, then $\vec{w}_0 \cdot g > 0$ for all $g \in S_l$.
- Step 2. For each $g \in S_{l-1}$, choose $r_g \in \mathbb{R}$ such that $r_g(g \cdot v_{l-1}) + (g \cdot v_l) > 0$. Set $r_{l-1} = \max_{g \in S_{l-1}}\{r_g\}$ and set $\vec{w}_1 = r_{l-1}\vec{v}_{l-1} + \vec{v}_l$. By construction, $\vec{w}_1 \cdot g > 0$ for all $g \in S_l \cup S_{l-1}$.
- Repeat step 2 with vectors $\vec{v}_{l-2}, \ldots, \vec{v}_1$. At each iteration, we obtain a vector $\vec{w}_i = r_{l-i}\vec{v}_{l-i} + \cdots + r_{l-1}\vec{v}_{l-1} + \vec{v}_l$. The last vector obtained by this process will be $\vec{w}_{l-1} = r_1\vec{v}_1 + \cdots + r_{l-1}\vec{v}_{l-1} + \vec{v}_l$. Set $\vec{w} = \vec{w}_{l-1}$. By construction, $\vec{w} \cdot g_i > 0$ for all $1 \leqslant i \leqslant k$.

For each g_i, define a function $f_i : \mathbb{R}^n \to \mathbb{R}$ by $f_i(\vec{x}) = \vec{x} \cdot g_i$. Since f_i is continuous, each set of the form $U_i = f_i^{-1}((0, \infty))$, for $1 \leqslant i \leqslant k$, is an open set in $\mathbb{R}^n$. Next, define the set $U = U_1 \cap U_2 \cap \cdots \cap U_k$. Since U is a finite intersection of open sets, U is an open set in $\mathbb{R}^n$. Furthermore, $\vec{w} \in U_i$, for all $1 \leqslant i \leqslant k$, so $\vec{w} \in U$. Therefore, U is nonempty.

Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, and U is a nonempty open subset of $\mathbb{R}^n$, it follows that $U \cap \mathbb{Q}^n$ is nonempty. So, there exists some vector $\vec{\nu} \in U$ with all rational components. Furthermore, $\vec{\nu} \cdot g_i > 0$ for all $1 \leqslant i \leqslant k$. By Lemma 4.2, since $\vec{\nu}$ has all rational components, it determines a rational hyperplane. Therefore, any sequence of normal vectors $\vec{\nu}, \vec{\nu}_{n-1}, \ldots, \vec{\nu}_1$ that determines a rational flag will also determine a rational flag ordering in which all $g_1, \ldots, g_k$ are positive. Thus, we have found the desired positive cone P'. □

Corollary 4.4.6. *For $n \geqslant 1$, the set of regular left orderings is dense in* $\mathrm{LO}(\mathbb{Z}^n)$.

Proof. By Theorem 4.4.5, the set of rational flag orderings is dense in $\mathrm{LO}(\mathbb{Z}^n)$. By Theorem

4.3.13, all rational flag orderings of $\mathbb{Z}^n$ are regular. Therefore, the set of regular orderings is dense in $\mathrm{LO}(\mathbb{Z}^n)$. $\square$

5

Regular Left Orderings of Nilpotent Groups

A left orderable finitely generated nilpotent group G admits regular orderings which form a dense set in $\mathrm{LO}(G)$. This fact is an extension of the central result from Chapter 4. All left orderings of a nilpotent group are Conradian (see Proposition 5.2.9), and Conradian orderings of a finitely generated group are derived from short exact sequences (see Proposition 5.2.1). As a result, any positive cone of a finitely generated nilpotent group can be decomposed into a union of preimages of positive cones of torsion-free abelian groups (see Theorem 5.4.2). This will be the key to proving that regular orderings of finitely generated nilpotent groups are dense in the space of orderings in the proof of Corollary 5.4.4.

5.1 Preliminaries

In this section, we state the definition of a nilpotent group and give a preliminary example to demonstrate how left orderings of a nilpotent group can be built from the group's central series.

Definition 5.1.1. The *upper central series* of a group G is the series of normal subgroups

$$1 = Z_0 \trianglelefteq Z_1 \trianglelefteq \cdots \trianglelefteq Z_n \trianglelefteq \cdots$$

such that Z_1 is the center of G and, for $n > 1$, Z_n/Z_{n-1} is the center of G/Z_{n-1}.

Definition 5.1.2. A group G is *nilpotent* if its upper central series terminates with $Z_n = G$ for finite n. In this case, G is said to be of *nilpotency class* n.

Example 5.1.3. The Heisenberg group over the integers

$$H = H(\mathbb{Z}) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\}$$

is nilpotent of class 2. This can be shown by calculating the group's upper central series. Taking the elements X, Y, Z to be generators of H, where

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

yields the presentation $H = \langle X, Y, Z \mid Z = XYX^{-1}Y^{-1}, XZ = ZX, YZ = ZY \rangle$.

First, to compute the center Z_1 of H, consider an arbitrary element $C \in Z_1$. This is a matrix of the form

$$C = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}$$

that commutes with all elements $A \in H$, *i.e.,* $AC = CA$. Taking

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

and the requirement that $AC = CA$ yields

$$\beta + a\gamma + b = b + \alpha c + \beta \implies a\gamma = \alpha c \implies a\gamma - \alpha c = 0.$$

Since a, c are arbitrary elements in $\mathbb{Z}$, it must be that $\gamma = \alpha = 0$. This means that the matrix C has the form

$$\begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, it is easy to see that the center of H is generated by the matrix z and $Z_1 = \langle Z \rangle$. The next subgroup in the series, Z_2, is a subgroup such that Z_2/Z_1 is the center of H/Z_1. If H/Z_1 is abelian, then the center of H/Z_1 is the entire group, *i.e.,* $Z_2/Z_1 = H/Z_1$, which implies that $Z_2 = H$. Therefore, it remains to show that H/Z_1 is abelian.

The quotient group H/Z_1 is abelian if and only if for all $h_1, h_2 \in H$ the commutator $[h_1, h_2] = h_1^{-1}h_2^{-1}h_1h_2 \in Z_1$. Taking two arbitrary matrices $h_1, h_2 \in H$ and computing the product $h_1^{-1}h_2^{-1}h_1h_2$ yields an element in Z_1 (see Computation A.0.7). Therefore, H/Z_1 is indeed abelian, and we obtain that H has the upper central series

$$1 = Z_0 \trianglelefteq Z_1 \trianglelefteq Z_2 = H.$$

Thus, H is of nilpotency class 2. From this calculation, we obtain that H fits into the following short exact sequence:

$$1 \to Z_1 \hookrightarrow H \to H/Z_1 \to 1.$$

It is easy to see, from the presentation of H and from the fact that H/Z_1 is abelian, that $H/Z_1 \simeq \mathbb{Z}^2$. Therefore, both $Z_1 \simeq \mathbb{Z}$ and $H/Z_1 \simeq \mathbb{Z}^2$ are left orderable, and by Proposition 2.1.7, this means that H is left orderable. More specifically, left orderings of H can be obtained lexicographically from the kernel and quotient of such a short exact sequence. Such a short exact sequence, with H in the middle and an abelian kernel and quotient, is not unique. Nonetheless, in the following sections, we will see that all left orderings of H are determined lexicographically via *some* short exact sequence (see Proposition 5.2.1 and Theorem 5.2.9).

5.2 Nilpotent Groups and Conradian Orderings

Recall the definition of a Conradian left ordering from Section 2.1.2 (Definition 2.1.22). The goal of this section is to show that if G is a finitely generated left orderable nilpotent group, then all left orderings on G are Conradian.

Proposition 5.2.1. *[7, Proposition 9.19] Let G be a finitely generated group that admits a Conradian left order, and let P be a positive cone corresponding to such an ordering. There there exists a torsion-free abelian group A and a short exact sequence*

$$1 \to K \xrightarrow{i} G \xrightarrow{q} A \to 1$$

such that $P = i(Q) \cup q^{-1}(S)$, where Q and S are positive cones of K and A, respectively.

Proof. Suppose G is generated by $g_1, \ldots, g_n$. Fix a Conradian left ordering $<$ on G. Without loss of generality, suppose $1 < g_1 < \cdots < g_n$. Define

$$C = \bigcup_{\substack{H \text{ convex} \\ g_n \notin H}} H.$$

Since convex subgroups are closed under unions (see section 2.2 of [7] for proof of this fact), C is a convex subgroups of G. By definition, G is also a convex subgroup, so (C, G) is a convex jump in G (since any convex subgroup of G strictly greater than C contains all the generating elements $g_1, \ldots, g_n$). Furthermore, since $<$ is Conradian, (C, G) is a Conradian jump, so C is normal in G and G/C admits an Archimedean ordering. The quotient G/C is torsion-free abelian by Lemma 2.1.16. From this, we derive the short exact sequence

$$1 \to C \xrightarrow{i} G \xrightarrow{q} G/C \to 1.$$

Next, $P \cap C$ is a positive cone of C, so set $P \cap C = Q$. Similarly, $q(P)$ is a positive cone of A. Set $S = q(P)$. Then $P = i(Q) \cup q^{-1}(S)$. $\square$

Proposition 5.2.2. *[18] If G is a group that admits a non-Conradian ordering, then G contains a subsemigroup that is free on two letters.*

Proof. By Theorem 2.1.24, since $(G, <)$ does not admit a Conradian ordering, there exist positive elements $a, b \in G$ such that $ba^n < a$ for all $n \in \mathbb{N}$. Let S be the subsemigroup generated by a, b. We want to show that any arbitrary words $w_1, w_2 \in S$ with distinct spellings are not equal. Without loss of generality, suppose $w_1 = a\tilde{w}_1$ and $w_2 = b\tilde{w}_2$, where $\tilde{w}_1, \tilde{w}_2 \in S$ (since if w_1, w_2 start with the same letter, we can cancel on the left until $w_1 = a\tilde{w}_1$ and $w_2 = b\tilde{w}_2$).

Case 1. $\tilde{w}_2 = b^m a^n$, for some $m, n \geqslant 0$. Then

$$ba^n < a \qquad \text{by assumption}$$

$$
\begin{aligned}
b^2a^n &< ba && \text{by left multiplication} \\
b^2a^n &< ba < a && \text{by assumption.}
\end{aligned}
$$

Repeating the above argument yields $b^m a^n < a$. Since $\tilde{w}_1 \in S$, it is positive, and so $1 < \tilde{w}_1 \implies a < a\tilde{w}_1 = w_1$. Therefore, $w_2 < a < w_1$ and $w_1 \neq w_2$.

Case 2. $\tilde{w}_2 = w' b^m a^n$, where $m, n \geqslant 0$, $w' \in S$.

From case 1, $b^m a^n < a$ implies $\tilde{w}_2 = w' b^m a^n < w' a$. We want to show that $w'a < a$ by inducting on the total exponent sum of a in w'.

Base case. Suppose the exponent sum of a in w' is zero. Then $w' = b^l$ for some $l \in \mathbb{N}$, and $w'a = b^l a < a$, as shown in case 1.

Inductive step. Suppose that for any word w' with exponent sum of a equal to k the inequality $w'a < a$ holds. Next, suppose w'' is a word with exponent sum of a equal to $k+1$. Then

$$w'' = b^l a w',$$

where the exponent sum of a in w' is k and $l \geqslant 0$.

If $w'' = b^l a w'$, then

$$
\begin{aligned}
w'a &< a && \text{by the inductive hypothesis} \\
b^l a w' a &< b^l a^2 && \text{left multiplication.}
\end{aligned}
$$

However, by case 1, $b^l a^2 < a$, from which it follows that $w''a = b^l a w' a < a$. Since $\tilde{w}_2$ is positive, $1 < \tilde{w}_2 < w'a < a \implies b < b\tilde{w}_2 = w_2 < bw'a < ba$. Repeating the argument from the conclusion of case 1 yields $a < w_1$, and by assumption, $ba < a$. Therefore, by combining these inequalities, we obtain $w_2 < w_1$.

In either case, $w_1 \neq w_2$, and so the subsemigroup generated by a, b is free. □

Next, we use a tool from geometric group theory to conclude that all left orderings of finitely generated nilpotent groups are Conradian. In particular, we come to this conclusion by examining the *growth rate* of a left orderable finitely generated nilpotent group. To define the growth rate of a group, we first need to establish the notion of a *Cayley graph*. For a more detailed exposition on group growth, see [17, Chapter 6].

Definition 5.2.3. Let G be a group with generating set S. The *Cayley graph* of G with respect to S, denoted by $\mathrm{Cay}_S(G)$, is the directed graph $\Gamma = (V, E)$ where

1. the vertex set V is the set of elements of G, and
2. each edge in E is labelled by an element in S. There is a directed edge labelled by $g \in S$ leaving the vertex w_1 and entering the vertex w_2 if $w_1 \cdot g = w_2$.

Since the generating set of a group G is *not* unique, likewise a Cayley graph of G is not unique. However, distinct Cayley graphs for the same group G are quasi-isometric, *i.e.,* they are equivalent as metric spaces under the equivalence relation of quasi-isometry.

Definition 5.2.4. Let G be a finitely generated group with generating set S. The *growth function* of G (with respect to S) is defined by $f : \mathbb{N} \to \mathbb{N}$ such that $f : r \mapsto |\mathrm{B}(1, r)|$, where $\mathrm{B}(1, r)$ is the closed ball of radius r centered at the origin in $\mathrm{Cay}_S(G)$.

Definition 5.2.5. A group G is said to have *polynomial growth* (*exponential growth*) if the growth function exhibits polynomial growth (exponential growth).

Group growth rate is a quasi-isometric invariant, that is, if G and H are quasi-isometric then G and H must have the same growth rate. Therefore, the reader can rest assured knowing that the definition of group growth is well-defined.

Theorem 5.2.6. *[17, Theorem 6.3.1] Finitely generated nilpotent groups have polynomial growth.*

Theorem 5.2.7. *[17, Example 6.4.2] Let F be a free group on 2 or more generators, then F has exponential growth.*

Remark 5.2.8. If G contains a free subsemigroup on 2 or more generators, then it can be shown that G has exponential growth by mimicking the proof of Theorem 5.2.7.

Theorem 5.2.9. *[3] If G is a finitely generated left orderable nilpotent group, then any left ordering of G is Conradian.*

Proof. Suppose that G is a left orderable nilpotent group. Suppose that $<$ is a left order of G that is not Conradian. By Proposition 5.2.2, G contains a subsemigroup S that is free on two letters. Therefore, G has exponential growth, which contradicts theorem 5.2.6. Thus, there does not exist a left order $<$ on G that is *not* Conradian. $\square$

5.3 Hirsch Rank

In the proof of Theorem 5.4.2, we show that the positive cone of a finitely generated nilpotent group G can be decomposed into a union of preimages of positive cones of torsion-free abelian groups. This is done by inducting on the Hirsch rank of G, which is introduced in this section. The Hirsch rank of a group G comes from a subnormal series of G.

Definition 5.3.1. A *subnormal series* of a group G is a series of subgroups

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

where G_{i-1} is normal in G_i for all $0 \leqslant i < n$. Quotients of the form G_i/G_{i-1} are called the *factors* of the series.

Definition 5.3.2. Let G be a finitely generated nilpotent group and $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ be a subnormal series of G. The *Hirsch rank* of G, denoted $H(G)$, is defined by

$$H(G) = \sum_{i=1}^{n} r_i,$$

where $G_i/G_{i-1} \cong \mathbb{Z}^{r_i} \times \mathbb{Z}_{k_{i_1}} \times \cdots \times \mathbb{Z}_{k_{i_l}}$ for all $1 \leqslant i \leqslant n$.

Since a subnormal series of G need not be unique, we need to show that the Hirsch rank of G

is invariant under the choice of subnormal series. We establish the next few definitions and theorems with this goal in mind.

Definition 5.3.3. Let $m \leqslant n$ be natural numbers and $\{A_i\}_{i=1}^m$ and $\{B_j\}_{j=1}^n$ be two subnormal series of a group G. The series $\{B_j\}$ is called a *refinement* of $\{A_i\}$ if for all $i \in \{1, \ldots, m\}$, $A_i = B_j$ for some $j \in \{1, \ldots, n\}$.

Definition 5.3.4. Two subnormal series $\{H_i\}_{i=1}^m$ and $\{K_j\}_{i=1}^n$ of a group G are *isomorphic* if there is a bijection between their corresponding sets of factor groups.

Theorem 5.3.5 (Schreier Refinement Theorem)**.** *[4] Any two subnormal series of a group G have isomorphic refinements.*

Proposition 5.3.6. *[15, Lemma 1.2] Let G be a finitely generated nilpotent group. If H is a subgroup of G, then H is also finitely generated nilpotent.*

Observation 5.3.7. If G is a finitely generated nilpotent group then for any subnormal series of G there exists a refinement $G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n$ of the series such that the factors G_i/G_{i-1} are finitely generated abelian groups. Such a refinement exists due to the Schreier Refinement Theorem. The fact that the factors are abelian is due to the definition of a nilpotent group, and they are finitely generated as a result of Proposition 5.3.6. By the fundamental theorem of finitely generated abelian groups, this means that there exist $r_i, k_1, \ldots, k_l \in \mathbb{N}$ such that

$$G_i/G_{i-1} \cong \mathbb{Z}^{r_i} \times \mathbb{Z}_{k_1} \times \cdots \times \mathbb{Z}_{k_l}.$$

We use a *rank-nullity* argument to show that the Hirsch rank is invariant under the choice of subnormal series. The next definition and proposition is given with this purpose.

Definition 5.3.8. The *rank* of a torsion-free abelian group G is the cardinality of a maximal linearly independent set in G.

Proposition 5.3.9. *Let A, B, C be finitely generated abelian groups and*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence. Then $\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$.

Proof. Suppose A and C have rank n and m, respectively. Choose a maximal linearly independent set $a_1, \ldots, a_n \in A$ and a maximal linearly independent set $c_1, \ldots, c_m \in C$. For each c_i, there exists an element $b_i \in B$ such that $g(b_i) = c_i$.

Consider the following linear combination in B:

$$\sum_{i=1}^{n} \lambda_i f(a_i) + \sum_{i=1}^{m} \gamma_i b_i = 0,$$

where $\lambda_i, \gamma_i \in \mathbb{Z}$ for all i. We want to show that $\lambda_i = \gamma_i = 0$ for all i.

Observe that

$$\begin{aligned} 0 &= g\left(\sum_{i=1}^{n} \lambda_i f(a_i) + \sum_{i=1}^{m} \gamma_i b_i\right) \\ &= \sum_{i=1}^{n} \lambda_i g(f(a_i)) + \sum_{i=1}^{m} \gamma_i c_i \\ &= 0 + \sum_{i=1}^{m} \gamma_i c_i & \text{because } a_i \text{ is in the kernel.} \end{aligned}$$

In particular, since the c_i's are linearly independent, $\sum_{i=1}^{m} \gamma_i c_i = 0$ implies $\gamma_i = 0$ for all i. From this, we obtain

$$0 = \sum_{i=1}^{n} \lambda_i f(a_i) + \sum_{i=1}^{m} \gamma_i b_i = \sum_{i=1}^{n} \lambda_i f(a_i),$$

and it remains to show that $\lambda_i = 0$ for all i.

Consider:

$$0 = \sum_{i=1}^{n} \lambda_i f(a_i) = f\left(\sum_{i=1}^{n} \lambda_i a_i\right).$$

It follows that $\sum_{i=1}^{n} \lambda_i a_i = 0$, since f is injective, and since the a_i's are linearly independent, $\lambda_i = 0$ for all i. Therefore, $\{f(a_1), \ldots, f(a_n), b_1, \ldots, b_m\}$ is a linearly independent set in B

and

$$\operatorname{rank}(B) \geqslant n + m = \operatorname{rank}(A) + \operatorname{rank}(C).$$

Next, we show that $\operatorname{rank}(B) \leqslant m + n$. Let $X = \{f(a_1), \ldots, f(a_n), b_1, \ldots, b_m\}$. Suppose that X is not a maximal linearly independent set and thus can be extended to one – denote this maximal set by X'.

By assumption, there exists some element $x \in X' \backslash X$. Since the c_i's are a maximal linearly independent set, $g(x)$ can be expressed as

$$g(x) = \sum_{i=1}^{m} \lambda_i c_i,$$

for some choice of λ_i's.

Then

$$\begin{aligned} 0 &= \sum_{i=1}^{m} \lambda_i c_i - g(x) \\ &= g\left(\sum_{i=1}^{m} \lambda_i b_i - x\right) \end{aligned}$$

implies that $\sum_{i=1}^{m} \lambda_i b_i - x$ is in the kernel of g, which is equal to the image of f. So, there exists an elements $a' \in A$ such that

$$f(a') = \sum_{i=1}^{m} \lambda_i b_i - x.$$

However, $a' = \sum_{i=1}^{n} \gamma_i a_i$ for some choice of γ_i's. So,

$$f(a') = \sum_{i=1}^{m} \lambda_i b_i - x = \sum_{i=1}^{n} \gamma_i f(a_i)$$

implies $x = \sum_{i=1}^{m} \lambda_i b_i - \sum_{i=1}^{n} \gamma_i f(a_i)$. In other words, x can be expressed as a linear combination of the elements in X. This contradicts the assumption that X' is a non-trivial maximal extension of X. Thus, $\operatorname{rank}(B) \leqslant m + n$. $\square$

Proposition 5.3.10. *[23, p. 18] Let G be a finitely generated nilpotent group. The Hirsch rank of G is invariant under the choice of subnormal series.*

Proof. Suppose that $\{H_i\}$ and $\{K_j\}$ are two subnormal series of G, where $H_i/H_{i-1} \cong \mathbb{Z}^{r_i} \times \mathbb{Z}_{k_{i_1}} \times \cdots \times \mathbb{Z}_{k_{i_l}}$ and $K_j/K_{j-1} \cong \mathbb{Z}^{s_j} \times \mathbb{Z}_{k_{j_1}} \times \cdots \times \mathbb{Z}_{k_{j_l}}$. Also, let $a = \sum_{i=1}^{n} r_i$ and $b = \sum_{j=1}^{m} s_j$. We want to show that $a = b$. If $\{H_i\}$ and $\{K_i\}$ are equivalent, then there is nothing to show. So, suppose the two series are not isomorphic.

First, we show that the Hirsch rank obtained via $\{H_i\}$ and any refinement of $\{H_i\}$ are equivalent. Consider the subnormal series $\{H_i\}$ and the refinement

$$H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{j-1} \trianglelefteq \tilde{H} \trianglelefteq H_j \trianglelefteq \cdots \trianglelefteq H_n$$

of $\{H_i\}$, *i.e.,* the new series contains only one additional subgroup, namely $\tilde{H}$. By the third isomorphism theorem,

$$\frac{H_j/H_{j-1}}{\tilde{H}/H_{j-1}} \cong \frac{H_j}{\tilde{H}},$$

so we can construct the short exact sequence:

$$1 \to \tilde{H}/H_{j-1} \to H_j/H_{j-1} \to H_j/\tilde{H} \to 1.$$

Suppose $\operatorname{rank}(\tilde{H}/H_{j-1}) = n$ and $\operatorname{rank}(H_j/\tilde{H}) = m$. By Proposition 5.3.9, it follows that $\operatorname{rank}(H_j/H_{j-1}) = r_j = m + n$. This means that

$$a = \sum_{i=1}^{n} r_i = \sum_{i=1}^{j-1} r_i + r_j + \sum_{i=j+1}^{n} r_i = \sum_{i=1}^{j-1} r_i + (m+n) + \sum_{i=j+1}^{n} r_i,$$

(note that the RHS is the Hirsch rank of the refinement). Therefore, the Hirsch rank obtained from the original series $\{H_i\}$ and its refinement are the same. We can show that any other (finite) refinement of $\{H_i\}$ yields the same Hirsch rank through an iterative application of the previous argument.

Finally, by the Schreier refinement theorem, there exist refinements $\{H_i'\}$ and $\{K_j'\}$ of $\{H_i\}$ and $\{K_j\}$, respectively, which are isomorphic. Therefore the Hirsch rank associated to $\{H_i'\}$ is a and the Hirsch rank associated to $\{K_j'\}$ is b, but since the series $\{H_i'\}$ and $\{K_j'\}$ are equivalent, it must be that $a = b$. $\square$

We are now ready to prove the main result of this chapter.

5.4 Density of Regular Orderings

In this section, we show that finitely generated left orderable nilpotent groups admit regular left orderings and that this set of regular orderings is dense in the space of orderings.

Definition 5.4.1. Let G be a finitely generated left orderable nilpotent group with positive cone P. Let $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ be a subnormal series of G and let $q_i : G_i \to G_i/G_{i-1}$ be the quotient map. If $P = \bigcup_{i=1}^n q_i^{-1}(Q_i)$, where $Q_i \in \mathrm{LO}(G_i/G_{i-1})$, then we say that P is *induced* by the subnormal series $G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n$.

Theorem 5.4.2. *Suppose G is a finitely generated left orderable nilpotent group, and P is a positive cone in* $\mathrm{LO}(G)$*. Then there exists a subnormal series*

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

where each factor group G_i/G_{i-1} is torsion-free abelian such that $P = \bigcup_{i=1}^n q_i^{-1}(Q_i)$, where $q_i : G_i \to G_i/G_{i-1}$ is the quotient map and $Q_i \in$ $\mathrm{LO}(G_i/G_{i-1})$.

Proof. We induct on the Hirsch rank of G.

Base case. Suppose that the Hirsch rank of G is 1, then it is easy to see that $G \cong \mathbb{Z}$. Let P be a positive cone of G, then the subnormal series $\{1\} = G_0 \triangleleft G_1 = G$ induces P vacuously.

Inductive step. Let A be a finitely generated left orderable nilpotent group of Hirsch rank at most k and R a positive cone in $\mathrm{LO}(A)$. Suppose that for all such A and all such R, there exists a subnormal series of A that induces R.

Next, consider a finitely generated left orderable nilpotent group G of Hirsch rank $k+1$, and let P be a positive cone in $\mathrm{LO}(G)$.

By Proposition 5.2.9, the left ordering corresponding to P is Conradian. Therefore, there exists a short exact sequence

$$1 \to K \overset{i}{\hookrightarrow} G \xrightarrow{q} H \to 1$$

such that H is torsion-free abelian and $P = q^{-1}(Q) \cup (P \cap K)$, where Q is a positive cone of H.

First, observe that K must have Hirsch rank at most k. To see this, suppose $K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_l$ is a subnormal series of K. It is easy to see that such a series can be extended to a subnormal series of G:

$$K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_l \trianglelefteq G.$$

Since G has Hirsch rank $k+1$ and $H \cong G/K$ is a non-trivial finitely generated torsion-free abelian group, it must be that $G/K \cong \mathbb{Z}^n$, where $n \geqslant 1$. Therefore, if the Hirsch rank of K is $k+1$ or greater, this would contradict G having Hirsch rank $k+1$.

By the inductive hypothesis, there exists a subnormal series of K that induces $P \cap K$. That is

$$P \cap K = \bigcup_{i=1}^{l} q_i^{-1}(R_i),$$

where $q_i : K_i \to K_i/K_{i-1}$ are quotient maps and $R_i \in \mathrm{LO}(K_i/K_{i-1})$. Therefore,

$$P = q^{-1}(Q) \cup \bigcup_{i=1}^{l} q_i^{-1}(R_i),$$

which proves the theorem. $\square$

The following lemma is the converse of Theorem 5.4.2 with the added fact that a positive cone obtained for a subnormal series with abelian factors is also regular.

Proposition 5.4.3. *Suppose G is a finitely generated left orderable nilpotent group and that*

$$G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n$$

is a subnormal series of G, where each factor group G_i/G_{i-1} is torsion-free abelian. For each factor G_i/G_{i-1}, there exists a regular positive cone Q_i of G_i/G_{i-1}. If $P = \bigcup_{i=1}^n q_i^{-1}(Q_i)$, where $q_i : G_i \to G_i/G_{i-1}$ is the quotient map, then P is a regular positive cone of G.

Proof. First, we check that P satisfies the definition of a positive cone. We show that P *is closed under taking products.* Suppose $g_1, g_2 \in P$ such that $g_1 \in q_i^{-1}(P_i)$ and $g_2 \in q_j^{-1}(P_j)$ for some $i, j \in \{1, \ldots, n\}$. Clearly, if $i = j$, then $q_i(g_1) \cdot q_i(g_2) = q_i(g_1 g_2) \in Q_i$, and so $g_1 g_2 \in q_i^{-1}(Q_i) \subset P$.

Alternatively, if $i \neq j$, suppose without loss of generality that $i < j$. Then $g_1 \in \ker(q_j)$, and so

$$q_j(g_1 g_2) = q_j(g_1) \cdot q_j(g_2) = q_j(g_2) \in Q_j,$$

therefore $g_1 g_2 \in q_j^{-1}(Q_j) \subset P$.

Next, we show that P *partitions* G. Choose an arbitrary element $g \in G$. Since $G = \bigcup_{i=1}^n q_i^{-1}(G_i/G_{i-1})$, $g \in q_i^{-1}(G_i/G_{i-1})$ for some i, so $g = q_i^{-1}(xG_{i-1})$ for some $xG_{i-1} \in G_i/G_{i-1}$. Since Q_i is a positive cone of G_i/G_{i-1}, then either xG_{i-1} is contained in Q_i, Q_i^{-1}, or $xG_{i-1} = G_{i-1}$.

If $xG_{i-1} \in Q_i$, then $g = q_i^{-1}(xG_{i-1}) \in q_i^{-1}(Q_i)$, so $g \in P$. Similarly, if $xG_{i-1} \in Q_i^{-1}$, then $g \in P^{-1}$. On the other hand, if $xG_{i-1} = G_{i-1}$, then $g \in G_{i-1}$ and so $g \in q_{i-1}^{-1}(G_{i-1}/G_{i-2})$. If $g \in q_{i-1}^{-1}(Q_{i-1})$ or $g \in q_{i-1}^{-1}(Q_{i-1}^{-1})$, then we are done. Otherwise, repeating the above argument, eventually we will find that either $g \in q_j^{-1}(Q_j)$ or $g \in q_j^{-1}(Q_j^{-1})$ for some $j < i$, or $g \in G_0 \implies g = 1$.

To see that P is regular, we induct on the subnormal series. In particular, let $S(k)$ denote the statement that $\bigcup_{i=1}^{k} Q_i$ is a regular positive cone of G_k.

Base case. There exists a regular positive cone of $G_1 \simeq G_1/G_0$, call it Q_1, since by assumption, G_1 is finitely generated torsion-free abelian.

Inductive step. Fix $k - 1 < n$. Suppose $\bigcup_{i=1}^{k-1} Q_i$ is a regular positive cone of G_{k-1}.

Consider the sequence

$$G_{k-1} \to G_k \xrightarrow{f} G_k/G_{k-1}.$$

By the inductive hypothesis, $\bigcup_{i=1}^{k-1} Q_i$ is a regular positive cone of G_{k-1}. By assumption, since G_k/G_{k-1} is torsion-free abelian, there exists a regular positive cone Q_k of G_k/G_{k-1}. Finally, by Lemma 3.3.4, $f^{-1}(Q_k) \cup (\bigcup_{i=1}^{k-1} Q_i)$ is a regular positive cone of G_k. □

Corollary 5.4.4. *If G is a finitely generated left orderable nilpotent group, then the set of regular left orderings of G is dense in* $\mathrm{LO}(G)$.

Proof. Let G be a finitely generated left orderable nilpotent group and let P be a positive cone in $\mathrm{LO}(G)$. We will show that there exists a regular positive cone $P' \in \mathrm{LO}(G)$ that is arbitrarily close to P. Since G is countable, by Corollary 2.3.6, this is equivalent to choosing finitely many distinguished elements $g_1, \ldots, g_m \in P$ and constructing a regular positive cone P' such that $g_1, \ldots, g_k \in P'$.

By Theorem 5.4.2, there exists a subnormal series

$$G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n$$

of G, where G_i/G_{i+1} is torsion-free abelian and

$$P = \bigcup_{i=1}^{n} q_i^{-1}(Q_i),$$

where Q_i is a positive cone of G_i/G_{i-1}. We construct a regular positive cone P' such that $P' = \bigcup_{i=1}^n q_i^{-1}(\tilde{Q}_i)$, where $\tilde{Q}_i \in \mathrm{LO}(G_i/G_{i-1})$.

Without loss of generality, suppose that $g_j, \ldots, g_k \in q_i^{-1}(Q_i)$. Since G_i/G_{i-1} is torsion-free abelian, there exists a regular positive cone $\tilde{Q}_i$ of G_i/G_{i-1} such that $q_i(g_j), \ldots q_i(g_k) \in \tilde{Q}_i$. For each $i \in \{1, \ldots, n\}$ such that $\{g_1, \ldots, g_m\} \cap q_i^{-1}(Q_i) \neq \varnothing$, set $\tilde{Q}_i$ to be such a regular positive cone of G_i/G_{i-1}. If $\{g_1, \ldots, g_m\} \cap q_i^{-1}(Q_i) = \varnothing$, then set $\tilde{Q}_i$ to be any regular positive cone of G_i/G_{i-1}. Then, by Lemma 5.4.3,

$$P' = \bigcup_{i=1}^{n} q_i^{-1}(\tilde{Q}_i)$$

is a regular positive cone of G and, by construction, P' contains the elements $g_1, \ldots, g_m$. Therefore, we have found the desired positive cone P'. $\square$

5.5 Directions for Future Work

In Chapter 4, we determined that the rational flag orderings of $\mathbb{Z}^n$ form dense sets in their respective space of orderings $\mathrm{LO}(\mathbb{Z}^n)$. In Section 4.4, we also saw that not all orderings of $\mathbb{Z}^n$ can be regular. Some questions of interest that remain unanswered include the following:

Question 5.5.1. For the case of $n = 2$, we found that the orderings in $\mathrm{LO}(\mathbb{Z}^2)$ could be partitioned into rational and irrational orderings. All rational orderings were shown to be regular, therefore at least some subset of the irrational orderings cannot be regular. Is it possible that *some* irrational orderings admit regular positive cones? What is the least level of language complexity required to recognise a positive cone language of an irrational ordering of $\mathbb{Z}^2$? Do all irrational positive cone languages of $\mathbb{Z}^2$ fall into the same complexity class?

Question 5.5.2. For $n > 2$, we saw that the orderings of $\mathbb{Z}^n$ could no longer be categorised into two classes as with the $n = 2$ case. As a result, we found that $\mathbb{Z}^n$ admits rational flag orderings, irrational orderings, and what we called the "in-between" orderings. What is the complexity upper bound for the irrational and the "in-between" orderings?

Finally, in this chapter, we saw how the results from Chapter 4 could be generalised to show that positive cones of finitely generated nilpotent groups can be built by taking preimages of positive cones of torsion-free abelian groups. As a result, finitely generated nilpotent groups also admit dense sets of regular orderings in their respective space of orderings.

Question 5.5.3. Using a similar argument, can we find an analogous result that generalises to *solvable groups*?

Appendix

Definition A.0.4. Let $\mathcal{M}$ be a generalised FSA and let T be a transition diagram representing $\mathcal{M}$. A path P in T is called an *$\epsilon-$path* if it is empty or if all the edges on P are labelled by ϵ.

Theorem A.0.5. *[6, Proposition 1.3] Let $\mathcal{L}$ be a language over the alphabet Σ. The following are equivalent:*

1. *$\mathcal{L}$ is recognised by a deterministic FSA;*
2. *$\mathcal{L}$ is recognised by a FSA;*
3. *$\mathcal{L}$ is recognised by a generalised FSA.*

Proof. Since we have the containment

$$\left\{ \begin{array}{c} \text{Deterministic} \\ \text{FSA} \end{array} \right\} \subset \left\{ \text{FSA} \right\} \subset \left\{ \begin{array}{c} \text{Generalised} \\ \text{FSA} \end{array} \right\}$$

it is clear that (1) $\implies$ (2) $\implies$ (3). We show that (3) $\implies$ (1).

Suppose $\mathcal{L}$ is recognised by the generalised FSA $\mathcal{M} = (Q, F, \Sigma, \tau, q_0)$. For any subset of states $X \subseteq Q$, define the closure

$$\overline{X} = \{S_n \in Q \mid S_1 \in X, S_2, \ldots, S_{n-1} \in Q \text{ such that } (s_i, \epsilon, s_{i+1}) \in \tau \text{ for } 1 \leqslant i < n\}.$$

In English, this means that $\overline{X}$ is the set of states reachable from X via $\epsilon-$paths.

Claim: $\overline{X} = \overline{\overline{X}}$.

Proof of claim: The fact that $\overline{X} \subseteq \overline{\overline{X}}$ is vacuously true. To see that $\overline{\overline{X}} \subseteq \overline{X}$, suppose $B \in \overline{\overline{X}}$. Then, there exists a state $A \in \overline{X}$ such that there exists an $\epsilon-$path P_2 starting at A and ending at B. By the definition of $\overline{X}$, this means that there exists some $x \in X$ and an $\epsilon-$path P_1 starting at x and ending at A. The concatenation of P_1 and P_2 is an $\epsilon-$path starting at x and ending at B. Therefore, $B \in \overline{X}$ and $\overline{\overline{X}} = \overline{X}$.

Next, we are going to construct a deterministic FSA that will recognise the language $\mathcal{L}$. Denote this machine by $\mathcal{M}' = (Q', F', \Sigma, \tau', q_0')$. Define Q' to be

$$Q' = \{X \subset Q \text{ such that } X = \overline{X}\}.$$

Define the set

$$\delta(X, a) = \{\overline{\text{states incident to edges labelled by } a \text{ which are incident to a vertex in } X}\},$$

and set $\tau' = \{(X, a, \delta(X, a)) \mid X \in Q', a \in \Sigma\}$. Finally, set $q_0' = \{\overline{q_0}\}$ and $F' = \{q \in Q'$ such that $q \cap F \neq \varnothing\}$. Then $\mathcal{M}'$ is deterministic due to the property $\overline{X} = \overline{\overline{X}}$.

It remains to show that $\mathcal{L} = \mathcal{L}(\mathcal{M}')$. Suppose $w \in \mathcal{L}(\mathcal{M})$, then there exists some

$$q_1, \ldots, q_n \in Q \text{ and } a_1, \ldots, a_n \in \Sigma \cup \{\epsilon\}$$

such that $w = a_1 a_2 \cdots a_n$ and $(q_{i-1}, a_i, q_i) \in \tau$ for all $1 \leqslant i \leqslant n$ and $q_n \in F$. Choose a sequence of b_i's from Σ such that $w = b_1 \cdots b_k$ and $b_i \neq \epsilon$ for $1 \leqslant i \leqslant k$, that is, we want to spell w without any instances of ϵ.

Since $\mathcal{M}'$ is deterministic, there is a unique sequence of states $q_1', q_2', \ldots, q_k' \in Q$ such that $\delta(q_{i-1}', b_i) = q_i \quad (\star)$.

Suppose that the first non-ϵ letter in $w = a_1 \cdots a_n$ is a_i, then $a_i = b_1$ and $a_1 = \cdots = a_{i-1} = \epsilon$.

This means that there is an $\epsilon-$path from q_0 to q_{i-1} in $\mathcal{M}$. So, by construction, $q_{i-1} \in \{\overline{q_0}\} = q'_0$, which implies that $q_i \in \delta(q_0, b_1) = q'_1$ (by ($\star$) and since $q_{i-1} \xrightarrow{a_i} q_i$ is a transition in $\mathcal{M}$). Continuing in this manner inductively on the subwords $a_i \cdots a_n$ and $b_2 \cdots b_k$, we eventually conclude that $q_n \in q'_k$. Since $q_n \in F$, this means that $q'_k \cap F \neq \varnothing$, hence $q'_k \in F'$. Therefore, $\mathcal{M}'$ accepts w and $\mathcal{L}(\mathcal{M}) \subset \mathcal{L}(\mathcal{M}')$.

To see that $\mathcal{L}(\mathcal{M}') \subset \mathcal{L}(\mathcal{M})$, begin by supposing that $w = a_1 \cdots a_n \in \mathcal{L}(\mathcal{M}')$. Then there exists a computation

$$q'_0, a_1, q'_1, \ldots, a_n, q'_n$$

of $\mathcal{M}'$ which is successful (*i.e.,* $q'_n \in F'$). Then, by construction, there exists a computation

$$q_{0_1}, \ldots, q_{0_{k_0}}, a_1, q_{1_1}, \ldots, q_{1_{k_1}}, a_2, \ldots, a_n, q_{n_1}, \ldots, q_{n_k},$$

where $q_{i_j} \in q'_i$ for all $0 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k_i$, and $q_{n_k} \in F$. Thus, $\mathcal{L}(\mathcal{M})$ accepts w and $\mathcal{L}(\mathcal{M}') \subset \mathcal{L}(\mathcal{M})$. □

Lemma A.0.6. *An arbitrary element w of the Klein bottle group*

$$\pi_1(K) = \langle x, y \,|\, xyx^{-1} = y^{-1} \rangle$$

can be spelled with strictly positive (negative) exponents.

Proof. We prove by inducting on the length of the word w.

Base case. Suppose w is a nontrivial word of length 2 which does not already have all positive (negative) exponents in its spelling. There are four cases to consider, each of which can be rewritten as follows due to the defining relator of the group:

- $xy^{-1} = yx$,
- $y^{-1}x = xy$,

- $x^{-1}y = y^{-1}x^{-1}$, and
- $yx^{-1} = x^{-1}y^{-1}$.

Inductive step. Fix $k \geqslant 2$ and suppose that any word of length at most k can be spelled with strictly positive (negative) exponents. Consider a word w of length $k+1$. Let $w = uv$, where u is a subword of length k. By the inductive hypothesis, there exists a spelling of u with strictly positive (negative) exponents.

- Case 1. Suppose u is spelled with strictly positive exponents. If $v = x$ or $v = y$, then we are done.

  Case 1, a. Suppose $v = y^{-1}$. Let $u = \tilde{u}u'$, where $\tilde{u}$ is a subword of length $k-1$. Then $w = \tilde{u}u'y^{-1}$.

  - If $u' = y$, then $w = \tilde{u}yy^{-1} = \tilde{u}$. Apply the inductive hypothesis to w.
  - If $u' = x$, then $w = \tilde{u}xy^{-1} = \tilde{u}yx$, *i.e.,* we have found a spelling of w with all positive exponents.

  Case 1, b. Now, suppose $v = x^{-1}$. Similarly, let $u = \tilde{u}u'$, then $w = \tilde{u}u'x^{-1}$.

  - If $u' = x$, then $w = \tilde{u}xx^{-1} = \tilde{u}$. Apply the inductive hypothesis.
  - If $u' = y$, then $w = \tilde{u}yx^{-1}$. Applying the last rewriting rule from the base case yields $w = \tilde{u}x^{-1}y^{-1}$. By the inductive hypothesis, $\tilde{u}x^{-1} = \omega$, where ω is a word with strictly positive (negative) exponents. If ω has all negative exponents, then we are done. If ω has all positive exponents, then apply case 1, a.

- Case 2. Suppose that u is spelled with strictly negative exponents. A similar case analysis follows as in case 1. □

Computation A.0.7. Recall the Heisenberg group

$$H = \langle X, Y, Z \mid Z = XYX^{-1}Y^{-1}, XZ = ZX, YZ = ZY \rangle.$$

We show that the quotient H/Z_1 is abelian, where $Z_1 = \langle Z \rangle$ is the center of H. The quotient H/Z_1 is abelian if and only if for all $h_1, h_2 \in H$ the commutator $[h_1, h_2] = h_1^{-1}h_2^{-1}h_1h_2 \in Z_1$. Let h_1 and h_2 be two arbitrary elements in H, then

$$h_1 = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad h_2 = \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix},$$

where $a, b, c, d, e, f \in \mathbb{Z}$. Consider the product:

$$\begin{aligned}
h_1^{-1}h_2^{-1}h_1h_2 &= \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -d & df-e \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -d & df-e \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a+d & e+af+b \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & af+b-dc \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & af-dc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in Z_1.
\end{aligned}$$

References

[1] Yago Antolín, Cristóbal Rivas, and Hang Lu Su. "Regular left-orders on groups". In: *J. Comb. Algebra* 6.3-4 (2022), pp. 265–314. ISSN: 2415-6302,2415-6310. DOI: 10.4171/jca/64. URL: https://doi.org/10.4171/jca/64.

[2] Matthias Aschenbrenner, Stefan Friedl, and Henry Wilton. *3-manifold groups*. 2013. arXiv: 1205.0202 [math.GT].

[3] J. C. Ault. "Right-Ordered Locally Nilpotent Groups". In: *Journal of the London Mathematical Society* s2-4.4 (1972), pp. 662–666. DOI: https://doi.org/10.1112/jlms/s2-4.4.662. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/jlms/s2-4.4.662. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/jlms/s2-4.4.662.

[4] Benjamin Baumslag. "A Simple Way of Proving the Jordan-Hölder-Schreier Theorem". In: *The American Mathematical Monthly* 113.10 (2006), pp. 933–935. DOI: 10.1080/00029890.2006.11920381. eprint: https://doi.org/10.1080/00029890.2006.11920381. URL: https://doi.org/10.1080/00029890.2006.11920381.

[5] J. W. S. Cassels. *An introduction to the geometry of numbers*. Classics in Mathematics. Corrected reprint of the 1971 edition. Springer-Verlag, Berlin, 1997, pp. viii+344. ISBN: 3-540-61788-4.

[6] Ian Chiswell. *A course in formal languages, automata and groups*. Universitext. Springer-Verlag London, Ltd., London, 2009, pp. x+157. ISBN: 978-1-84800-939-4. DOI: 10.1007/978-1-84800-940-0. URL: https://doi.org/10.1007/978-1-84800-940-0.

[7] Adam Clay and Dale Rolfsen. *Ordered groups and topology*. Vol. 176. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2016, pp. x+154. ISBN: 978-1-4704-3106-8. DOI: 10.1090/gsm/176. URL: https://doi.org/10.1090/gsm/176.

[8] Adam Clay and Dale Rolfsen. *Ordered groups, eigenvalues, knots, surgery and L-spaces*. 2010. arXiv: 1004.3615 [math.AT].

[9] B. Deroin, A. Navas, and C. Rivas. *Groups, Orders, and Dynamics*. 2016. arXiv: 1408.5805 [math.GR]. URL: https://arxiv.org/abs/1408.5805.

[10] Nathan M. Dunfield and Danny Calegari. "Laminations and groups of homeomorphisms of the circle". In: *Inventiones Mathematicae* 152.1 (Apr. 2003), pp. 149–204. ISSN: 1432-1297. DOI: 10.1007/s00222-002-0271-6. URL: http://dx.doi.org/10.1007/s00222-002-0271-6.

[11] Wolfgang Ebeling. *Lattices and codes*. Third. Advanced Lectures in Mathematics. A course partially based on lectures by Friedrich Hirzebruch. Springer Spektrum, Wiesbaden, 2013, pp. xvi+167. ISBN: 978-3-658-00359-3; 978-3-658-00360-9. DOI: 10.1007/978-3-658-00360-9. URL: https://doi.org/10.1007/978-3-658-00360-9.

[12] Susan Hermiller and Zoran Šunić. "No positive cone in a free product is regular". In: *Internat. J. Algebra Comput.* 27.8 (2017), pp. 1113–1120. ISSN: 0218-1967,1793-6500. DOI: 10.1142/S0218196717500527. URL: https://doi.org/10.1142/S0218196717500527.

[13] Derek F. Holt, Sarah Rees, and Claas E. Röver. *Groups, languages and automata*. Vol. 88. London Mathematical Society Student Texts. Cambridge University Press,

Cambridge, 2017, pp. xi+294. ISBN: 978-1-316-60652-0; 978-1-107-15235-9. DOI: 10.1017/9781316588246. URL: https://doi.org/10.1017/9781316588246.

[14] Thomas Koberda. *Faithful actions of automorphisms on the space of orderings of a group*. 2011. arXiv: 1011.2811 [math.GT].

[15] A. Leibman. "The structure of unitary actions of finitely generated nilpotent groups". In: *Ergodic Theory Dynam. Systems* 20.3 (2000), pp. 809–820. ISSN: 0143-3857,1469-4417. DOI: 10.1017/S0143385700000432. URL: https://doi.org/10.1017/S0143385700000432.

[16] Peter A. Linnell. "The space of left orders of a group is either finite or uncountable". In: *Bulletin of the London Mathematical Society* 43.1 (Nov. 2010), pp. 200–202. ISSN: 0024-6093. DOI: 10.1112/blms/bdq099. URL: http://dx.doi.org/10.1112/blms/bdq099.

[17] Clara Löh. *Geometric group theory*. Universitext. An introduction. Springer, Cham, 2017, pp. xi+389. ISBN: 978-3-319-72253-5; 978-3-319-72254-2. DOI: 10.1007/978-3-319-72254-2. URL: https://doi.org/10.1007/978-3-319-72254-2.

[18] Akbar Rhemtulla and Dale Rolfsen. "Local Indicability in Ordered Groups: Braids and Elementary Amenable Groups". In: *Proceedings of the American Mathematical Society* 130.9 (2002), pp. 2569–2577. ISSN: 00029939, 10886826. URL: http://www.jstor.org/stable/2699673 (visited on 05/01/2024).

[19] Cristóbal Rivas. *Left-orderings on free products of groups*. 2011. arXiv: 1106.2094 [math.GR]. URL: https://arxiv.org/abs/1106.2094.

[20] Cristóbal Rivas and Romain Tessera. *On the space of left-orderings of virtually solvable groups*. 2013. arXiv: 1209.3251 [math.GR]. URL: https://arxiv.org/abs/1209.3251.

[21] Adam S. Sikora. *Topology on the spaces of orderings of groups*. 2003. arXiv: math/0111002 [math.GR].

[22] J. A. Simpson and E. S. C. Weiner. *Oxford English dictionary*. Oxford, Eng, 2000.

[23] B. A. F. Wehrfritz. *Group and ring theoretic properties of polycyclic groups*. Vol. 10. Algebra and Applications. Springer-Verlag London, Ltd., London, 2009, pp. viii+128. ISBN: 978-1-84882-940-4. DOI: 10.1007/978-1-84882-941-1. URL: https://doi.org/10.1007/978-1-84882-941-1.