Cutting Rules For Feynman Diagrams at Finite Temperature

by

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Abstract

The imaginary part of the retarded self energy is of particular interest as it contains a lot of physical information about particle interactions. In higher order loop diagrams the calculation become extremely tedious and if we have to do the same at finite temperature, it includes an extra dimension to the difficulty. In such a condition we require to switch between bases and select the best basis for a particular diagram. We have shown in our calculation that in higher order loop diagrams, at finite temperature, the R/A basis is most convenient on summing over the internal vertices and very efficient on calculating some particular diagrams while the result is most easily interpretable in the Keldysh basis for most other complex diagrams.

Dedication

This work is dedicated to my parents, who have set a new definition of ultimate sacrifice and love for me and my siblings.

Acknowledgments

Any work at the graduate level is challenging and demands a lot of effort. Being an overseas student is another challenge, but I was lucky enough to have extremely high-caliber friendly people as my supervisors, teachers, colleagues, and fellow students. Without their help this work would remain undone. It's impossible to name all of them. I am really grateful to Dr. Randy Kobes and Dr. Margaret Carrington for supervising my work; they are well known in their fields and their reputations precede them. Nonetheless, I would like to acknowledge their patience and encouraging words. Anytime I was frustrated or disappointed, they motivated me and gave valuable advice. I am indebted to all my colleagues (especially those in room 501 Allen) as they were always helping me with their valuable comments about my research editing. During my course of study, I worked in three universities: University of Manitoba, University of Winnipeg, and Brandon University. I am appreciative for the warm hospitality shown by the people of these institutions.

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Publications

U. Chowdhury, M. E. Carrington, T. Fugleberg, R. Kobes, "Calculating imaginary parts of Feynman Diagrams," in preparation.

Chapter 1

Introduction

Quantum Field Theory (QFT) is a proven successful framework for explaining particle interactions and predicting the possible outcome where we have to consider the many-body-systems as well as the creation and annihilation of the particles [1]. QFT evolved as a result of the demand to fulfill the requirement of a method that can explain different particle phenomena. Quantum Mechanics (QM) was developed and was successful on interpreting single particle quantum level occurrences, but for some problems QM was literally helpless [2, 3]. Many scientists worked together to give QFT the present day's form. Among them Feynman, Dyson, Bogoliubov, Schwinger are prominent. Feynman not only contributed significantly explaining different phenomenons, but also gave an easy to use form to QFT. He did it by introducing some diagrams, that look very simple, but contain extremely valuable information about the physics behind the diagrams, and thus make the calculation significantly simpler. A lot of work has been done since the Feynman diagram was first introduced and in present days Feynman diagrams are almost indispensable for particle physicists. The diagrams are much easier to calculate in zero temperature approximation and

for some problems that is enough. The importance of finite temperature calculation is realized when we want to explain the interaction of the elementary particles at the very beginning, just after the big bang, when there was an enormously high temperature and particles had interactions with the heat bath [4]. At this stage in early universe, everything was just in quark gluon plasma (QGP) phase. This QGP stage is created artificially in some laboratories as well [5]. QFT is so far found successful on predicting the chain of events that proves that indeed the QGP was created.

Finite temperature means the existence of a heat-bath and the particles in the system interact with the heat-bath that we have to take into consideration. In this thesis we have shown that Optical theorem [3] holds even at finite temperature. We have shown that, if we change the basis, some particular diagrams vanish that are not cuttable although the sum over indices of vertices becomes enormous. The diagrams that are not cuttable we refer them as "problamatic" from the point to view of "cutting rules", although they can be written in terms of an amplitude in the *primary* basis. These problamatic terms vanish in the Retarded/Advanced and Keldysh basis and this fact was generalized in this thesis. We will see that the problamatic terms/diagrams are nothing but the diagrams in which the circled and uncircled vertices do not form a connected region. We were also able to develop a method to simplify the enormous sums when Feynman diagrams are calculated. We found that it is convenient to use Retarded/Advanced basis for some diagrams and Keldysh for most others as the physics is easily extractable in the Keldysh basis.

For convenience we have performed all our calculations for Bosons. A little effort is needed to derive the rules for Fermions. For this reason, although in general the wavy lines are used to represent the bosons, we have used the plane lines for bosons from Chapter-3 onward. This thesis is organized as follows.

In Chapter-2 we discuss briefly about the scattering amplitude and the relation between scattering amplitude and Feynman diagram. In Chapter-3 we discuss about different basis and advantages of using a particular basis as discussed by Francois Gelis [6]. Chapter-4 is written to explain what is the importance of imaginary parts and how we can calculate them from the forward scattering expressions. We start with the methods developed by Kobes and Semenoff [7], we also discuss the method of P. F. Bedaque, A. Das and S. Naik [8]. At the end of Chapter-4 we will be able to see that the calculation of the imaginary part cannot be continued from the forward scattering expression and we need to develop a smarter way to proceed if we want get *cut* diagrams [6]. We review how this is done with circling rules. When we want to calculate the higher order loop diagrams, from Chapter-3 we know that we need a way to handle the enormously large calculation and that way is changing the basis. We also see how we can transform the so called *circling rules* to different basis. In Chapter-4 we also discuss the motivation behind calculating the imaginary part of the retarded self energy as it is discussed by H. A. Weldon [9] and in Chapter-5 we give some examples where we use all the knowledge we acquire in preceding chapters that will help us find usefulness of the R/A and Keldysh basis. In Chapter-6 we explain how the diagrams are always split into two pieces in R/A and Keldysh basis and any diagram that does not split into two pieces vanish. In Chapter-7 we present our conclusion.

In this thesis, we have used R/A and Keldysh basis to give a cutting description to the higher order loop diagrams. We have shown that the R/A and Keldysh basis [6, 10] are successful on explaining the particle interactions at finite temperature and easily produces the cutting rules that are very complex to derive in the other basis (+/- or sometimes named as 1/2 basis).

Chapter 2

Feynman diagram and Scattering amplitude

2.1 Introduction

Any physical theory must pass the acid test of experiment. If any theory fails to reproduce or predict the result of the experiment then that theory does not sustain. In particle physics most of the experiments involve scattering and thus we are always interested to calculate the scattering rate of some measurable process. In the language of mathematics, the scattering rate is nothing but a measure of probability that tells us what is the probability of a particular reaction to occur [11]. This reaction may result in the annihilation or creation of a particle that was not present prior to the incident of scattering. This is a phenomena that can only be treated using Quantum Field Theory. In classical mechanics or even if we consider quantum mechanics, there is no such thing happening, the particles are always there, no particles escape the system or emerge into the system. This fact made the calculation of Rutherford Scattering

significantly simpler [12]. When we proceed further and bring Quantum Mechanics (QM) to explain different properties of nucleus we still do not need to consider the annihilation or creation of any particle, for example shell model or optical model etc. But when we want to explain the pair production or electron positron annihilation, those models do not work any more and we rush towards Quantum Field Theory. Quantum Field Theory is so far found to be extremely successful on explaining the different elementary particle interactions. Each scattering amplitude can be presented as a Feynman diagram.

We discuss this below. We take the example of electron-positron scattering. We will see what are the different outcomes of this scattering and how we can predict the results. We will see how we can present this incident through Feynman diagrams, at the end how we can extract valuable information regarding the incident and specifically, why are Feynman diagrams so important in QFT. As Feynman diagrams have undergone lot of modification since it was developed and lot of smart ways were found by different scientists depending of specific problems under consideration, we discuss about Feynman diagram as generally as we can. Feynman diagram for electron positron annihilation is a simple but "good" example to start learning about Feynman diagrams and also about QFT. In the center-of-mass co-ordinate the process is presented diagrammatically[1] as shown in Fig. (2.1). Say the electron and positrons carry momentum p and p' respectively. They collide head on and the particles coming out (muon here) come out with momenta k and k' for muon and anti-muon respectively. The scattering angle is "ϕ". The reaction is now defined, we

[1]To draw the Feynman diagrams we have used the software Jaxodraw-4 [13].

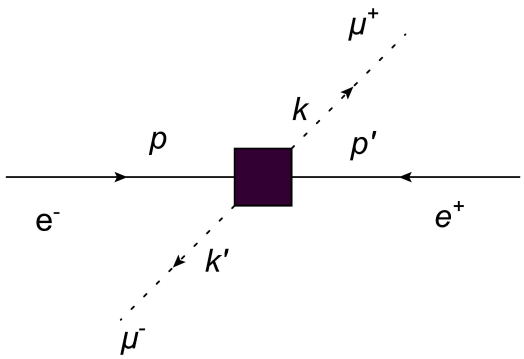


Figure 2.1: Electron positron annihilation in the center of mass co-ordinate system.

write it as

$$e^+e^- \longrightarrow \mu^+\mu^-. \tag{2.1.1}$$

We choose the center-of-mass (CM) co-ordinate system to make the calculation simpler. In CM co-ordinate system

$$\mathbf{p} = -\mathbf{p}', \quad \mathbf{k} = -\mathbf{k}'. \tag{2.1.2}$$

and we consider the beam energy E to be high enough so that we can write:

$$|\mathbf{p}| = |\mathbf{p}'| = |\mathbf{k}| = |\mathbf{k}'| = E \equiv E_{cm}/2. \tag{2.1.3}$$

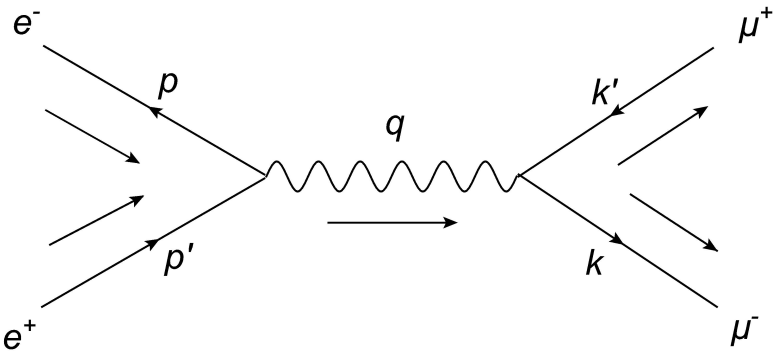
For a particular spin orientation (for details see [3]), it is convenient that we find the differential cross section ($d\sigma$) for this process [14]. We are interested to know the rate of *muon* production within a solid angle $d\Omega$,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{cm}^2}|\mathcal{M}|^2. \tag{2.1.4}$$

This quantity $\mathcal{M}$ is called the "quantum-mechanical" scattering amplitude and $|\mathcal{M}|^2$ presents the statistical probability for the certain reaction to take place. The quantities in the both sides of Eq. (2.1.4) are dimensionless as the dimension of E_{cm}^2 cancels out the dimension of $|\mathcal{M}|^2$. Now if we have an expression for $\mathcal{M}$ the whole calculation is very easy, but unfortunately the exact form of $\mathcal{M}$ is not known. The only solution is that we go for a perturbative series [15] to obtain an expression for $\mathcal{M}$, and then as usual, try to evaluate as many terms as we want depending on how precise a result we want. Next obstacle is that, getting the perturbative terms is also not easy. The basic physical principle is to expand around the classical solution which minimizes the action. The procedure is mathematically well defined, but it is difficult to identify which terms need to be included at each order in perturbation theory. At this state Richard Feynman [16] came with his revolutionary idea of presenting the perturbative terms diagrammatically which are now called Feynman diagrams [17]. He also gave a formula to "calculate" the diagrams that are known as Feynman rules [18]. So, the *black box* at the center of the Fig. (2.1) is opened by Feynman, his diagrams show the possible *pictures* (processes) inside and his rules tell us how to calculate them. The procedure is described in many standard texts [2, 3, 19]. To start, we draw the diagram that correspond to the process we are interested in to the desired order. For example, for electron-positron scattering, the first perturbative term, which is the most important in any perturbative series can be drawn as it is shown in Eq. (2.2a).

Once a diagram is drawn, the next task is to write the mathematical expression for that diagram and do some calculation to get our *scattering amplitude*. A diagram is a combination of three components:

1. The incoming and outgoing lines (straight lines in our example, particles that are



(a) Feynman diagram for the first order term in electron-positron scattering.

(b) Fermions.

(c) Bosons.

Figure 2.2: Feynman diagram with Fermion and Boson lines.

detectable), we call them on-shell particle, and the lines themselves as external lines.

2. The line between them (i.e. wavy one in our current example), we call them *internal particles* or *off-shell particles*, and the lines themselves as internal lines.

3. The vertices where internal and external lines meet (and evidently, means an interaction).

The Feynman rules can be summarized as follows [2]:

1. We write down the propagators corresponding to a line carrying momentum k. For example, for bosons the propagator is

$$\Delta_F(k) = \frac{i}{k^2 - m^2 + i\epsilon}. \tag{2.1.5}$$

2. For each interaction vertex we associate a coupling factor of $-ig$ and $(2\pi)^4\delta^4(\sum_i k_i - \sum_j k_j)$, this makes sure that the momentum at each vertex is conserved. Here $\sum_i k_i$ is the sum of all momenta coming in to the vertex and $\sum_j k_j$ the sum of all momenta going out.

3. For each independent four momenta (for multi loop diagrams) there is a 4-dimentional integration of the form $\int \frac{d^4k}{2\pi^4}$.

4. We multiply the diagrams with their corresponding symmetry factors.

5. We do not write the delta function that confirms the momentum conservation at each vertex, but its presence is understood.

6. We also do not write the propagators for the external lines as the propagators $\prod_{a=1}^{4} \Delta_F(k_a)$ is common in all the possible diagrams we can draw for the scattering under consideration. Removing these propagators gives us the so called *amputated diagram* [20].

7. Finally, we perform the integration to get the required *scattering amplitudes*.

Using these rules and performing the calculation we can show that the diagram in Fig. (2.2a) gives us a total scattering cross-section of

$$\sigma_{total} = \frac{4\pi\alpha^2}{3E_{cm}^2}, \tag{2.1.6}$$

where $\alpha = \frac{e^2}{4\pi}$. If we want to have a more accurate result, we must calculate corrections to Fig. (2.2a) such as those shown in Fig. (2.3).

But calculating the diagrams that have a closed loop in them becomes extremely sophisticated and tricky that requires some smart *tricks* to be applied. The major

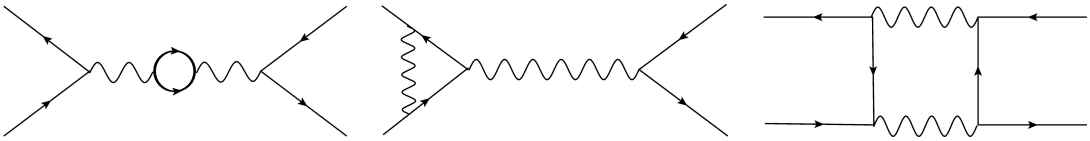


Figure 2.3: Some contributions to the scattering amplitude in the electron-positron scattering.

problem arises because of the fact that the particles in the loop are in fact not the real ones, they are called virtual particles and by the law of perturbation their momentum have to be integrated from zero to infinity (means they can take any value) which sometimes results in an infinity. The smart *trick* is re-normalization [21], that takes us out of this trouble. In this section we have tried to give a brief idea of how the calculation is performed from a Feynman diagram, but details of getting the expression for propagators, renormalization and the discussion of path integral [2, 21, 22] is skipped to make this thesis more compact.

2.2 Zero temperature Feynman diagram

Calculating the zero temperature diagrams are in most ways very easy when compared to the finite temperature ones, as we have mentioned earlier. It is because at zero temperature the particles are considered to obey the law of conservation of energy at every vertex which is not true in the case of finite temperature where the particles can exchange energy with the heat bath. In the language of mathematics, at zero temperature the propagators are proportional to the step function $\theta(\pm k^0)$ while at finite temperature they are proportional to $\theta(\pm k^0) \pm n_x(|k^0|)$ [23], where, $x \equiv B$ for bosons and $x \equiv F$ for fermions. and n_B, n_F are the bosons and fermion distribution functions respectively (see Eq. 2.3.6). It also means that the direction of energy flow

is constrained at zero temperature by the step function θ.

2.3 Finite temperature Feynman diagram

A particle in a heat bath that is described by fields $\phi(x) = \phi(t, \vec{x})$ are either periodic (bosons) or anti periodic(fermions) in complex time. The period is $-i\beta$ and the temperature is β^{-1}, then the integrals over time follow a path C in the complex time plane connecting a point $-\mathcal{T}$ to another point $-\mathcal{T} - i\beta$ ($\mathcal{T}$ presents time coordinate, not the temperature). In the imaginary time formalism (ITF) (one chooses $\mathcal{T} = 0$) the straight path connecting 0 and $-i\beta$, resulting in time arguments $t = i\tau$ that are purely imaginary. The advantage of the ITF is that there is no doubling of degrees of freedom (see below), and the rules have the same forms as at zero temperature with frequency integrals replaced by discrete sums while the perturbation theory remains valid [9]. One disadvantage of the ITF is that, for physical quantities which are defined for real times, an analytic continuation is involved. Also, the ITF can not be used for out of equilibrium situations [24]. The time path in the ITF is shown in Fig. (2.4).

In real-time formalisms we choose the contour in Fig. (2.5) [25]. Note that the path has a component along the real axis. The thermal Green functions are time-ordered along this path:

$$\Delta_C(x_1, \ldots, x_n) = \langle \mathcal{T}_C \phi(x_1) \ldots \phi(x_n) \rangle. \tag{2.3.1}$$

In Fig. (2.5) σ is an arbitrary parameter with the condition $0 \leq \sigma \leq \beta$ [26, 27]. For closed time path formalism $\sigma = 0$, whereas $\sigma = \frac{1}{2}$ for thermo field dynamics [27]. I

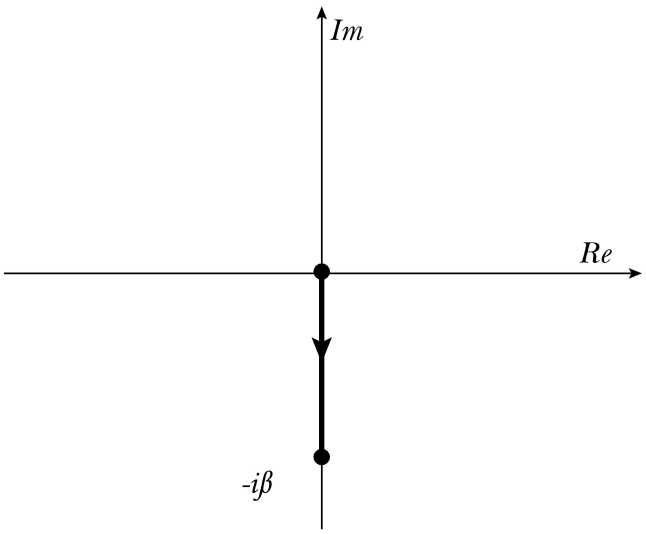


Figure 2.4: Time path in the *imaginary* time formalism.

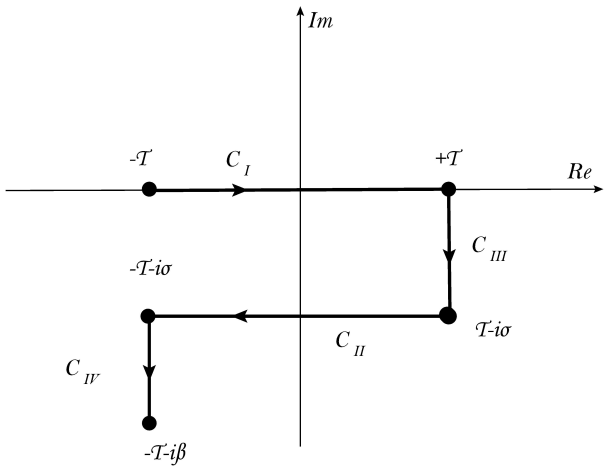


Figure 2.5: Time path in the *real* time formalism.

have used $\sigma = 0$ throughout this thesis except in few places where σ is kept for the sake of generality. Evidently C_{III} and C_{IV} can be neglected for $\mathcal{T} \to \infty$. At this point it is convenient to introduce the thermal doublet

$$\phi_a = \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}, \tag{2.3.2}$$

where the subscript $a = +, -$. The field $\phi_+(x) = \phi(x)$ is defined on the forward segment C_I, and $\phi_-(x) = \phi(t - i\sigma, x)$ is defined on the backward segment C_{II} of the time contour. This facilitates a matrix form of the thermal Green function:

$$\Delta_{a_1 \ldots a_n}(x_1, \ldots, x_n) = \langle \mathcal{T}_C \phi_{a_1}(x_1) \ldots \phi_{a_n}(x_n) \rangle; \qquad a_i \in \{+, -\}. \tag{2.3.3}$$

We discuss in the next few paragraphs how to write propagators and vertices in the primary basis. The word *primary* does not have any other importance and there is no reason to think that this basis is something fundamental. In other works this basis is called the $+/-$ basis or the $1/2$ basis [6, 8]. Now we discuss shortly on the matrix form of propagators. The form of the propagator of a free particle in momentum space is

$$\Delta_{ab}(k) = \begin{bmatrix} \Delta_{++}(k) & \Delta_{+-}(k) \\ \Delta_{-+}(k) & \Delta_{--}(k) \end{bmatrix}. \tag{2.3.4}$$

We define $\eta = 1$ for bosons and $\eta = -1$ for fermions and obtain:

$$\begin{aligned}
&\Delta_{++}(k) = \Delta^*_{--}(k) = \theta(k^0)\Delta_F(k) - \theta(-k^0)\Delta^*_F(k) + \eta\epsilon(k^0)n(x)[\Delta_F(k) + \Delta^*_F(k)], \\
&\Delta_{+-}(k) = e^{-x}\Delta_{-+}(k) = \eta\epsilon(k^0)n(x)[\Delta_F(k) + \Delta^*_F(k)],
\end{aligned} \tag{2.3.5}$$

where $\Delta_F(k)$ is the time ordered Feynman propagator of Eq. (2.1.5) for scalar bosons and

$$n(x) = \frac{1}{e^x - \eta}; \tag{2.3.6}$$

is the distribution function, where $x = \beta(k^0 - \mu)$ with μ being the chemical potential. For $\eta = 1$ we have the Bose-Einstein distribution and for $\eta = -1$ we have the Fermi-Dirac distribution. These distribution functions obey the following rules:

$$e^x n(x) = -\eta n(-x); \quad n(x) + n(-x) = -\eta. \tag{2.3.7}$$

From here on we consider zero chemical potential and set $\mu = 0$. We also consider bosons and set $\eta = 1$. In many applications, Green functions other than the time-ordered one arise. A common example is the linear response of the field to a weak external current [28, 29], which involves the retarded propagator Δ_R. The retarded propagator for bosons can be written as

$$\Delta_{ret}(k) = \Delta_{++}(k) - \Delta_{+-}(k) = \theta(k^0)\Delta_F(k) - \theta(-k^0)\Delta^*_F(k), \tag{2.3.8}$$

where

$$\Delta_{ret}(k) = \frac{i}{k^2 - m^2 + i\epsilon k^0}. \tag{2.3.9}$$

The propagators in Eq. (2.3.5) satisfy the relations

$$\Delta_{++}(k) + \Delta_{--}(k) - \Delta_{+-}(k) - \Delta_{-+}(k) = 0, \tag{2.3.10a}$$

$$\Delta_{++}(k) + \Delta_{--}(k) - e^{-\beta k^0}\Delta_{-+}(k) - e^{\beta k^0}\Delta_{+-}(k) = 0, \tag{2.3.10b}$$

which means that, all the propagator components are not independent [26]. If we re-write the propagator matrix in Eq. (2.3.4) by performing a change of basis, the new propagator $\widehat{\Delta}$ will also not contain four independent components. This will be our key for choosing a convenient basis. This is discussed in detail in Chapter-3. Full vertices are written $\Gamma_{a_1,a_2\ldots,a_n}(k_1, k_2, \ldots, k_n)$ with all momenta are taken to be incoming. We shall consider the particular combinations

$$\Gamma_{ret}(k_1; k_2, \ldots, k_n) \equiv \sum_{a_i=\pm} [\Gamma_{+a_2\ldots a_n}(k_1, \ldots, k_n)], \tag{2.3.11a}$$

$$\Gamma_{adv}(k_1; k_2, \ldots, k_n) \equiv \sum_{a_i=\pm} [\Gamma_{+a_2\ldots a_n}(k_1, \ldots, k_n) \prod_{a_i=-} e^{-i\beta k_i^0}], \tag{2.3.11b}$$

which are real-time representations of the one-particle-irreducible contributions of the generalized $n-$point retarded and advanced products [29, 30, 31]. They are related by complex conjugations;

$$\Gamma_{adv}(k_1; k_2, \ldots, k_n) = \Gamma^*_{ret}(k_1; k_2, \ldots, k_n). \tag{2.3.12}$$

One can show that these definitions (2.3.11) correspond to the usual definitions in co-ordinate space:

$$\Gamma_{ret}(X_1; X_2, \ldots X_n) = \theta(X_1 - X_2), \theta(X_2 - X_3) \ldots \langle [[[\ldots \phi(X_1), \phi(X_2)], \phi(X_3)] \ldots \phi(X_n)] \rangle. \tag{2.3.13}$$

The bare vertices can be written as $g^{\{a_i=+,-\}}$ [6], i.e. we take all the combinations of $+$ and $-$. For ϕ^3 theory, they will look like $g^{+++}, g^{++-}, \ldots g^{---}$. The only non-zero components are:

$$g^{+++} = -g^{---} = -ig. \tag{2.3.14}$$

The same concept is applicable to 4-particle interaction or ϕ^4-theory so that we may write:

$$g^{++++} = -g^{----} = -ig. \tag{2.3.15}$$

We use: g^+ or g^- which are short hand versions of g^{+++} or g^{++++} and g^{---} or g^{----} respectively and so on for higher number of particle interactions. In any given diagram it is clear from the figure (by looking at the vertices) if g^+ refers to g^{+++} or g^{++++}.

Chapter 3

Different bases

3.1 Introduction

The concept of different bases is similar to the concept of different co-ordinates systems: we choose the co-ordinate that is best for a certain problem under consideration. For example, if we are trying to find the moment of inertia of a cylindrical object, it is convenient if we use cylindrical co-ordinate system. Similarly in QFT (from now on we shall use only QFT for finite temperature QFT) we notice that some bases that we use are convenient over some other for a certain problem. The physics remains unchanged, irrespective to our choice of basis, as it does not matter whether I am writing the expression in spherical co-ordinate or Cartesian co-ordinate, the trajectory of a projectile is alway a parabola, just the mathematical expression has different appearance in different co-ordinate systems. First of all we define a basis in general and then we discuss the variety of the basis that are of common use in QFT. We will use the indices $(a, b, c, \ldots) \in \{+, -\}$ for the primary basis, $(X, Y, Z, \ldots) \in \{R, A\}$ for R/A basis and $X, Y, Z, \ldots . \in \{1, 2\}$ for Keldysh basis in the general discussion. When

we specify a basis, to make the expressions more *compact* we replace the X, Y, Z etc. with α.

The fundamental formula that is used to transform propagator in the primary basis Δ_{ab} to the propagator in the *new* basis $\widehat{\Delta}_{XY}$ is:

$$\widehat{\Delta}(k) = U(k)\Delta(k)U^T(-k), \tag{3.1.1}$$

here T stands for transpose and U is a 2×2 matrix that rotates the propagator to the new basis. Generalizing this we can (as done by Aurenche and Becherrawy [30, 32]) introduce transformed amputated $n-$point functions $\Gamma_{X_1 \ldots X_n}(k_1, \ldots, k_n)$ and we write

$$\widehat{\Gamma}_{X_1 \ldots X_n}(k_1, \ldots, k_n) = V_{X_1 a_1 \ldots X_n a_n}(k_1 \ldots k_n)\Gamma_{a_1 \ldots a_n}(k_1, \ldots, k_n), \tag{3.1.2}$$

with all momentum incoming. Here, by definition:

$$V(k) \equiv \left[U^T(-k)\right]^{-1}. \tag{3.1.3}$$

The linear relations (Eqs. 2.3.10) are satisfied by the new basis $\widehat{\Delta}_{XY}$. For example, the amputated transformed self energy $\widehat{\Pi}(k)$ is given by for example:

$$\widehat{\Pi}(k) = V(k)\Pi(k)V^T(-k), \tag{3.1.4}$$

and the bare 3-point vertex transforms as:

$$\widehat{g}^{XYZ}(k_1, k_2, k_3) = V_{Xa}(k_1)V_{Yb}(k_2)V_{Zc}(k_3)g^{abc}(k_1, k_2, k_3), \tag{3.1.5}$$

Using these results we find that a diagram corresponding to a given transformed $n-$point functions can be written solely in terms of transformed vertices connected by transformed internal propagators. A simple illustration of this transformation of Feynman rules in a one-loop self-energy graph involving cubic interactions is:

$$-i\Pi_{aa'}(p) = (-i)^2 \int \frac{d^4k}{(2\pi)^4} g^{abc}(p,-k,q)\Delta_{bb'}(k) g^{a'b'c'}(-p,k,-q)\Delta_{c'c}(q), \qquad (3.1.6)$$

where $q = k - p$ and only thermal indices are retained (see Fig. (3.1)). Using Eqs.

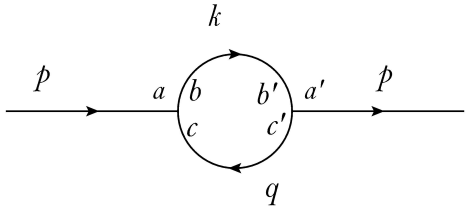


Figure 3.1: A simple loop diagram.

(3.1.1), (3.1.4), and (3.1.5) one can show:

$$-i\widehat{\Pi}_{XX'}(p) = V_{Xa}(p)[-i\Pi_{aa'}(p)]V^T_{X'a'}(-p) \qquad (3.1.7)$$

3.2 Retarded/Advanced (R/A) Basis

In equilibrium situation, one can consider transformation in which both of the constraints of Eqs. (2.3.10) translate into the trivial vanishing of two of the transformed functions $\widehat{\Delta}_{XY}(k)$ of Eq. (3.1.1). This might be viewed as an attempt to economize the information contained in the original propagators $\Delta_{ab}(k)$. We first look at the one that results by defining $\widehat{\Delta}_{11}(k)$, so that it vanishes by the first expression of Eq. (2.3.10) and $\widehat{\Delta}_{22}(k)$ so that it vanishes by the second expression of Eq. (2.3.10). We will find in this case that the transformation automatically leads to an off-diagonalization in terms of the retarded and advanced propagators [30].

$$\begin{aligned} U(k) &= \frac{1}{a(-k^0)} \begin{bmatrix} a(k^0)a(-k^0) & -e^{\sigma k^0} a(k^0)a(-k^0) \\ -\eta n(-x) & -\eta e^{\sigma k^0} n(x) \end{bmatrix} \\ &= \frac{1}{a(-k^0)} \begin{bmatrix} a(k^0)a(-k^0) & -a(k^0)a(-k^0) \\ -n(-x) & -n(x) \end{bmatrix}; \quad \because \sigma = 0 \text{ and } \eta = 1, \end{aligned} \tag{3.2.1}$$

where $a(k^0)$ is an arbitrary function of k^0, non vanishing at $k^0 = 0$. The matrix $V(k)$ is, from Eq. (3.2.1) and (3.1.3),

$$\begin{aligned} V(k) &= \frac{1}{a(-k^0)} \begin{bmatrix} -\eta n(-x) & \eta e^{\sigma k^0} n(x) \\ a(k^0)a(-k^0) & e^{\sigma k^0} a(k^0)a(-k^0) \end{bmatrix} \\ &= \begin{bmatrix} -n(-x) & n(x) \\ a(k^0)a(-k^0) & a(k^0)a(-k^0) \end{bmatrix} \quad \because \quad \sigma = 0, \quad \eta = 1. \end{aligned} \tag{3.2.2}$$

The propagator is

$$\widehat{\Delta}(k) = \begin{bmatrix} \widehat{\Delta}_{RR}(k) & \widehat{\Delta}_{RA}(k) \\ \widehat{\Delta}_{AR}(k) & \widehat{\Delta}_{AA}(k) \end{bmatrix} = \begin{bmatrix} 0 & \Delta_A(k) \\ \Delta_R(k) & 0 \end{bmatrix}. \tag{3.2.3}$$

By Eq. (3.1.2) and (3.2.2) we then define R/A $n-$point functions $\mathcal{R}_{\alpha_1\ldots\alpha_n}(k_1,\ldots,k_n)$. Using the largest/smallest time equation relations of Eq. (2.3.11), one finds in general [30, 32]:

$$\mathcal{R}_{RR\ldots R}(k_1,\ldots,k_n) = \mathcal{R}_{AA\ldots A}(k_1,\ldots,k_n) = 0, \tag{3.2.4a}$$

$$\mathcal{R}_{RA\ldots A}(k_1,\ldots,k_n) = \frac{a(k_2^0)\ldots a(k_n^0)}{a(-k_1^0)}\Gamma_{ret}(k_1;k_2,\ldots,k_n), \tag{3.2.4b}$$

$$\mathcal{R}_{AR\ldots R}(k_1,\ldots,k_n) = (-1)^n\frac{a(k_1^0)n(-k_2^0)\ldots n(-k_n^0)}{n(k_1^0)a(-k_2^0)\ldots a(-k_n^0)}\Gamma_{adv}(k_1;k_2,\ldots,k_n), \tag{3.2.4c}$$

where $\Gamma_{ret}(k_1;k_2,\ldots,k_n)$ and $\Gamma_{adv}(k_1;k_2,\ldots,k_n)$ appear in Eq. (2.3.11). The R/A transformation matrix satisfies a complex conjugation relation. Using Eq. (2.3.11), one finds

$$\mathcal{R}^*_{\alpha_1\ldots\alpha_n}(k_1,k_2,\ldots,k_n) = -\mathcal{R}_{\bar{\alpha_1}\ldots\alpha_n}(k_1,k_2,\ldots,k_n)\prod_{\alpha_i=R}\frac{n(k_i^0)}{a(k_i^0)}a(-k_i^0)\prod_{\alpha_i=A}\frac{a(k_i^0)}{a(-k_i^0)}n(k_i^0). \tag{3.2.5}$$

Where $\bar{\alpha} = A, R$ for $\alpha = R, A$. We see by Eq. (3.2.5) that it is not possible to choose $a(k^0)$ so as to cancel the factors of the distribution function $n(k^0)$ arising in this complex conjugation relation.

We have two more constraints

$$\Gamma^{\{R\ldots R\}} = 0, \quad \Gamma^{\{A\ldots A\}} = 0. \tag{3.2.6}$$

Bare vertices in the R/A basis are obtained from Eqns. (2.3.14), (3.1.2), (3.2.2)

$$\frac{\widehat{g}^{\alpha_1\ldots\alpha_n}(k_1,\ldots k_n)}{g(k_1,\ldots k_n)} = \prod_{\alpha_i=R}\frac{n(k_i^0)}{a(-k_i^0)}\prod_{\alpha_i=A}a(k_i^0)\left[\prod_{\alpha_i=A}e^{\beta k^0}-1\right]. \tag{3.2.7}$$

Thus, as for the full $n-$point functions in Eq. (3.2.4), bare vertices with all "R" or all "A" indices vanish:

$$\widehat{g}^{RR\ldots R} = \widehat{g}^{AA\ldots A} = 0. \tag{3.2.8}$$

Two convenient choices of the free parameter $a(k^0)$ can be made based on these considerations. One is $a(k^0) = 1$. In this case the full $n-$point function $\mathcal{R}_{RAA\ldots A}$ of Eq. (3.2.4) is normalized to Γ_{ret}. The other choice, as made in Refs. [32, 30], is $a(k^0) = -n(k^0)$. For this choice the full $n-$point function $\mathcal{R}_{ARR\ldots R}$ of Eq. (3.2.4) is normalized to Γ_{adv}. We choose $a(k^0) = -n(k^0)$ and using the Eq. (3.2.7) we can write the possible values of the bare 3-point vertices as:

$$\begin{aligned}
&g^{AAA}(k_1,k_2,k_3) = g^{RRR}(k_1,k_2,k_3) = 0\\
&g^{ARR}(k_1,k_2,k_3) = g^{RAR}(k_1,k_2,k_3) = g^{RRA}(k_1,k_2,k_3) = -ig\\
&g^{RAA}(k_1,k_2,k_3) = -ig(1+n_B(k_2^0)+n_B(k_3^0))\\
&g^{ARA}(k_1,k_2,k_3) = -ig(1+n_B(k_1^0)+n_B(k_3^0))\\
&g^{AAR}(k_1,k_2,k_3) = -ig(1+n_B(k_1^0)+n_B(k_2^0)).
\end{aligned} \tag{3.2.9}$$

Similarly, for ϕ^4-theory the bare 4-point vertices can be written as:

$$
\begin{aligned}
g^{AAAA}(k_1,k_2,k_3,k_4) &= g^{RRRR}(k_1,k_2,k_3,k_4) = 0 \\
g^{ARRR}(k_1,k_2,k_3,k_4) &= g^{RARR}(k_1,k_2,k_3,k_4) = g^{RRAR}(k_1,k_2,k_3,k_4) \\
&= g^{RRRA}(k_1,k_2,k_3,k_4) = -ig \\
g^{RRAA}(k_1,k_2,k_3,k_4) &= -ig(1+n_B(k_3^0)+n_B(k_4^0)) \\
g^{RARA}(k_1,k_2,k_3,k_4) &= -ig(1+n_B(k_2^0)+n_B(k_4^0)) \\
g^{AARR}(k_1,k_2,k_3,k_4) &= -ig(1+n_B(k_1^0)+n_B(k_2^0)) \\
g^{AAAR}(k_1,k_2,k_3,k_4) &= -ig(1+n_B(k_1^0)+n_B(k_2^0)+n_B(k_3^0)+ \\
&\qquad n_B(k_1^0)n_B(k_2^0)+n_B(k_2^0)n_B(k_3^0)+n_B(k_1^0)n_B(k_3^0)) \\
g^{AARA}(k_1,k_2,k_3,k_4) &= -ig(1+n_B(k_1^0)+n_B(k_2^0)+n_B(k_4^0)+ \\
&\qquad n_B(k_1^0)n_B(k_2^0)+n_B(k_2^0)n_B(k_4^0)+n_B(k_1^0)n_B(k_4^0)) \\
g^{ARAA}(k_1,k_2,k_3,k_4) &= -ig(1+n_B(k_1^0)+n_B(k_3^0)+n_B(k_4^0)+ \\
&\qquad n_B(k_1^0)n_B(k_3^0)+n_B(k_3^0)n_B(k_4^0)+n_B(k_1^0)n_B(k_4^0)) \\
g^{RAAA}(k_1,k_2,k_3,k_4) &= -ig(1+n_B(k_2^0)+n_B(k_3^0)+n_B(k_4^0)+ \\
&\qquad n_B(k_2^0)n_B(k_3^0)+n_B(k_3^0)n_B(k_4^0)+n_B(k_2^0)n_B(k_4^0)).
\end{aligned}
\tag{3.2.10}
$$

In the R/A basis, as we have seen earlier, we have only two off-diagonal matrix elements in the propagator matrix. This property of having less matrix element present make the summing over internal vertices extremely simple. A basic difference with other two bases is that, the propagator in this basis are zero temperature propagators. All the thermal weights are carried by the vertices.

3.3 Keldysh(1/2) Basis

The Keldysh basis is constructed to be as simple as possible, and still valid at out of equilibrium. The Keldysh transformation uses only the first of two relations of Eq. (2.3.10) to eliminate one of the transformed functions $\widehat{\Delta}_{XY}(k)$ [29, 33, 34, 35]. The second relation of Eq. (2.3.10), which follows from the KMS condition, does not hold at out of equilibrium. We will find that this class of transformations automatically involves the retarded and advanced propagators $\Delta_R^*(k) = -\Delta_A(k)$ of Eq. (2.3.8), but there is some freedom in the form of the third non-zero element. The transformation matrix is:

$$\begin{aligned} U(k) &= \frac{1}{\sqrt{2}b(-k^0)} \begin{bmatrix} b(k^0)b(-k^0) & e^{\sigma k^0}b(k^0)b(-k^0) \\ 1 & -e^{\sigma k^0} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}b(-k^0)} \begin{bmatrix} b(k^0)b(-k^0) & b(k^0)b(-k^0) \\ 1 & -1 \end{bmatrix}; \quad \because \ \sigma = 0, \end{aligned} \tag{3.3.1}$$

where $b(k^0)$ is an arbitrary function of k^0 non-vanishing at $k^0 = 0$. Using Eq. (3.1.1) and (3.3.1), with $\alpha_i = 1, 2$, the transformed propagator is:

$$\widehat{\Delta}_{\alpha_1\alpha_2}(k) = \begin{bmatrix} \widehat{\Delta}_{11}(k) & \widehat{\Delta}_{12}(k) \\ \widehat{\Delta}_{21}(k) & \widehat{\Delta}_{22}(k) \end{bmatrix} = \begin{bmatrix} \Delta_S^T(k) & \Delta_R(k) \\ \Delta_A(k) & 0 \end{bmatrix}, \tag{3.3.2}$$

where the symmetric product $\Delta_S^T(p)$ is

$$\begin{aligned} \Delta_S^T(k) &= b(k^0)b(-k^0)\left[\Delta_{++}(k) + \Delta_{--}(k)\right] \\ &= b(k^0)b(-k^0)\coth^\eta(x/2)\Delta_S(k); \quad \Delta S(k) = \Delta_R(k) - \Delta_A(k). \end{aligned} \tag{3.3.3}$$

We then define *Keldysh* $n-$point function $\mathcal{K}_{\alpha_1\ldots\alpha_n}(k_1,\ldots,k_n)$ in terms of the $+/-$ $n-$ point functions $\Gamma_{a_1\ldots a_n}(k_1,\ldots,k_n)$ by Eq. (3.1.2) and (3.1.3):

$$V=\frac{1}{\sqrt{2}b(-k^0)}\begin{bmatrix}1 & 1\\ b(k^0)b(-k^0) & -b(k^0)b(-k^0)\end{bmatrix}. \tag{3.3.4}$$

The amputated 2-point functions are:

$$\mathcal{K}_{11}(k_1,k_2)=0, \tag{3.3.5a}$$

$$\mathcal{K}_{12}(k_1,k_2)=\Pi_{adv}(k_1,k_2), \tag{3.3.5b}$$

$$\mathcal{K}_{21}(k_1,k_2)=\Pi_{ret}(k_1,k_2), \tag{3.3.5c}$$

$$\mathcal{K}_{22}(k_1,k_2)=b(k_1^0)b(k_2^0)\coth(x/2)[\Pi_{ret}(k_1,k_2)-\Pi_{adv}(k_1,k_2)]. \tag{3.3.5d}$$

In general it is easy to show, using the largest/smallest time equation and Eq. (2.3.11), that

$$\mathcal{K}_{11\ldots1}(k_1,\ldots,k_n)=\mathcal{K}_{21\ldots1}(k_1,\ldots,k_n)\ \ =2^{(1-\frac{n}{2})}\frac{b(k_1^0)}{b(-k_2^0)\ldots b(-k_n^0)}\Gamma^R(k_1;k_2,\ldots,k_n), \tag{3.3.6}$$

The bare vertices are obtained from Eqs.(3.1.2), (3.3.4):

$$\frac{\widehat{g}^{\alpha_1\ldots\alpha_n}(k_1,\ldots k_n)}{g(k_1,\ldots k_n)}=\frac{1-(-1)^{\#2}}{2^{\frac{n}{2}}}\prod_{k_i=1}\frac{1}{b(-k_i^0)}\prod_{k_i=2}b(k_i^0), \tag{3.3.7}$$

where "#2" denotes the number of "2" indices present. Just to avoid any confusion we clarify that the n in $2^{\frac{n}{2}}$ is the number of particle interacting at a vertex while the n 's that appear as subscripts are the *labels* for the momenta. Thus, bare vertices with

an even number of "2" indices vanish. The non-vanishing ones satisfy the following relation

$$\frac{\widehat{g}^{\alpha_1\ldots\alpha_n}(-k_1,\ldots,-k_n)}{\widehat{g}^{\alpha_1\ldots\alpha_n}(k_1,\ldots,k_n)} = \frac{g(-k_1,\ldots,-k_n)}{g(k_1,\ldots,k_n)} \prod_{\alpha_i=1,2} \frac{b(-k_i^0)}{b(k_i^0)}. \tag{3.3.8}$$

In these consideration there is no obvious advantage to choosing the free parameter $b(k^0)$ to be momentum dependent, and so choice such as $b(k^0) = 1$ or $1/\sqrt{2}$ are usually most convenient. In my calculation I have taken $b(k^0) = 1$.

In Keldysh basis we do not write the arguments (k_1, k_2, k_3) etc anymore as the bare vertices are not the functions of the momenta in this basis. So, for the 3-point and 4-point functions we write the bare vertices explicitly:

$$\begin{aligned} g^{111} &= g^{122} = g^{221} = g^{212} = 0 \\ g^{112} &= g^{121} = g^{211} = -ig\frac{1}{\sqrt{2}} \\ g^{222} &= -ig\frac{1}{\sqrt{2}}. \end{aligned} \tag{3.3.9}$$

For the 4-point function we have:

$$\begin{aligned} &g^{1111} = g^{1122} = g^{1221} = g^{2211} = g^{2121} = g^{1212} = g^{2222} = 0 \\ &g^{1112} = g^{1121} = g^{1211} = g^{2111} = \frac{-ig}{2} \\ &g^{2221} = g^{2212} = g^{2122} = g^{1222} = \frac{-ig}{2}. \end{aligned} \tag{3.3.10}$$

In Keldysh basis we do not have a simpler matrix form for the propagator matrices as we had in the R/A basis, but in this basis the thermal factors appear in the propagators which looks more *natural*, simpler to work with, and could be confirmed

after doing some examples. Another advantage of Keldysh basis is that we can use it even in situations where we are out of equilibrium, without any modification [6].

Chapter 4

Imaginary parts and circling rules

4.1 The Optical Theorem

The Optical Theorem [3] relates the forward scattering amplitude to total cross section. If we consider multi particle intermediate states, we can show that for two point functions we end up getting branch cut singularities. This is because the scattering amplitude as a function of energy has a branch cut on the positive real axis [3]. The imaginary part of the scattering amplitude could be shown as a discontinuity across this branch cut. Optical theorem tells us that the imaginary part of the forward scattering amplitude is proportional to the total cross section. This fact is the foundation of this thesis and also the motivation of calculating the retarded self energy. It is interesting that at zero temperature, the time ordered product is the standard for calculational convenience, but at finite temperature, the retarded product is more widespread. Part of the reason is historical. Initially the imaginary time formalism was the only formalism used to any extent, and the analytic continuation from imaginary time formalism is simple. The other reason is more fundamental, the imaginary

part of the retarded self energy at finite temperature is related to the expected form in terms of certain decay rates weighted by thermal factors. If one uses the imaginary part of the time ordered product instead, this is no longer true. There is a crucial sign difference. In other words, the imaginary part of the retarded and time-ordered products at finite temperature differ, and physically, the retarded self energy is preferred for a physical interpretation. In section-4.4 we demonstrate this explicitly for a one loop example. The imaginary part of the retarded self energy gives us the rate of particle creation and annihilation at finite temperature [9] which can be written as follows:

$$\Gamma_{ret}(k) = \Gamma_d(k) - \Gamma_c(k), \tag{4.1.1}$$

where $\Gamma_d(k)$ is the decay rate while $\Gamma_c(k)$ is the rate of creation. This property is clear from the examples that are done in this thesis. One can also see that the imaginary part of the retarded self energy appears in the full time ordered propagator. We discuss this next. It is easiest to see in the symmetric representation with $\sigma = 1/2$. One can show that the bare propagator can be diagonalized as

$$D^F_{ab}(k) = U_{ac}(k) \begin{bmatrix} D^F_0(k) & 0 \\ 0 & D^{F*}_0(k) \end{bmatrix}_{cd} U_{db}(k), \tag{4.1.2}$$

where $D^F_0(k)$ is the $T = 0$ Feynman propagator of Eq. (2.1.5). and $U(k)$ is given by

$$U(k) = \begin{bmatrix} \sqrt{1 + n(k^0)} & \sqrt{n(k^0)} \\ \sqrt{n(k^0)} & \sqrt{1 + n(k^0)} \end{bmatrix}, \tag{4.1.3}$$

where $n(k^0) = \frac{1}{e^{\beta|k^0|}-1}$.

It follows from the spectral representation that the full propagator has the same form [7].

We write

$$\mathcal{D}(k) = U(k) \begin{bmatrix} \bar{D}(k) & 0 \\ 0 & \bar{D}^*(k) \end{bmatrix} U(k). \tag{4.1.4}$$

The Dyson equation relates the full propagator to the bare one

$$\mathcal{D}_{ab}(k) = D_{ac}(k) + \mathcal{D}_{ac}(k)(-i\Pi_{cd}(k))\mathcal{D}_{db}(k). \tag{4.1.5}$$

So we can write

$$-i\Pi(k) = U^{-1}(k) \begin{bmatrix} -i\bar{\Pi}(k) & 0 \\ 0 & (-i\bar{\Pi}(k))^* \end{bmatrix} U^{-1}(k), \tag{4.1.6}$$

and

$$\bar{D}(k) = \frac{i}{|k^2 - m^2 - \bar{\Pi}(k)|}. \tag{4.1.7}$$

From the Eq. (4.1.6) we obtain directly

$$\text{Re}\ \Pi_{++}(k) = \text{Re}\ \bar{\Pi}(k) \tag{4.1.8a}$$

$$\text{Im}\ \Pi_{++}(k) = \text{Sign}\ (k^0) N(k^0) \text{Im}\ \bar{\Pi}(k). \tag{4.1.8b}$$

We will show in section-4.4 that

$$\text{Im}\,\Pi_{++}(k) = N(k^0)\text{Im}\,\Pi_{ret}(k). \tag{4.1.9}$$

Combining Eq. (4.1.8) and Eq. (4.1.9) we have

$$\begin{aligned} N(k^0)\text{Im}\,\Pi_{ret}(k) &= \text{Sign}(k^0)N(k^0)\text{Im}\,\bar{\Pi}(k) \\ \Rightarrow \text{Im}\,\bar{\Pi}(k) &= \text{Sign}(k^0)\text{Im}\,\Pi_{ret}(k). \end{aligned} \tag{4.1.10}$$

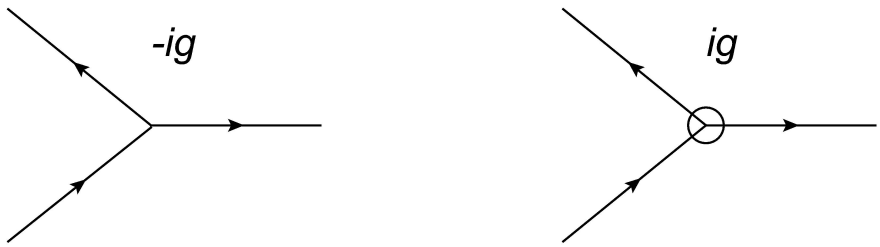
We see that the poles of $\bar{D}(k)$ (Eq. (4.1.8a)) occur at

$$k^2 - m^2 - \bar{\Pi} = k^2 - m^2 - (\text{Re}\,\Pi_{++}(k) + \text{Sign}(k^0)\text{Im}\,\Pi_{ret}(k)) = 0, \tag{4.1.11}$$

so we see that the imaginary part of the retarded propagator appears in the pole of the time ordered propagator but with a crucial "Sign" difference.

In this section we present a trick to calculate the imaginary part of a diagram called the *circling* rules.

In any given diagram each vertex can be either circled or uncircled. We define a circled vertex is simply the complex conjugate of the corresponding uncircled vertex.



(a) An uncircled vertex (b) A circled vertex

Figure 4.1: Circled and un-circled vertex.

Propagators connect pairs of vertices. The propagators are most easily defined in co-ordinate space. In the primary basis we decompose $\Delta_{++}(X,Y)$ into two pieces:

$$\Delta_{++}(X,Y) = \theta(X_0 - Y_0)\Delta_{-+}(X-Y) + \theta(Y_0 - X_0)\Delta_{+-}(X-Y). \tag{4.1.12}$$

Similarly for $\Delta_{--}(X,Y)$:

$$\Delta_{--}(X,Y) = \theta(X_0 - Y_0)\Delta_{+-}(X-Y) + \theta(Y_0 - X_0)\Delta_{-+}(X-Y). \tag{4.1.13}$$

The effect of circling (appears as underscores in propagators) on individual propagator is such that the diagonal elements are related as:

$$\Delta_{\pm\pm}(X,Y) = -\Delta_{--}(X,Y), \tag{4.1.14}$$

and off-diagonal elements remain unchanged by circling

$$\Delta_{\pm-}(X,Y) = \Delta_{+-}(X,Y); \quad \Delta_{-+}(X,Y) = \Delta_{-+}(X,Y). \tag{4.1.15}$$

In matrix form in the primary basis the propagators for different circling are given by

$$\Delta_{ab}(k) = \begin{bmatrix} \Delta_{++} & \Delta_{+-} \\ \Delta_{-+} & \Delta_{--} \end{bmatrix}; \quad \text{when } no \text{ vertex is circled,} \tag{4.1.16a}$$

$$\Delta_{\underline{a}b}(k) = \begin{bmatrix} \Delta_{+-} & \Delta_{+-} \\ \Delta_{-+} & \Delta_{-+} \end{bmatrix}; \quad \text{when } only \text{ vertex } a \text{ is circled,} \tag{4.1.16b}$$

$$\Delta_{a\underline{b}}(k) = \begin{bmatrix} \Delta_{-+} & \Delta_{+-} \\ \Delta_{-+} & \Delta_{+-} \end{bmatrix}; \quad \text{when } only \text{ vertex } b \text{ is circled,} \tag{4.1.16c}$$

$$\Delta_{\underline{ab}}(k) = \begin{bmatrix} \Delta_{--} & \Delta_{+-} \\ \Delta_{-+} & \Delta_{++} \end{bmatrix}; \quad \text{when both the vertices } a \text{ and } b \text{ are circled.} \tag{4.1.16d}$$

There are two important points to the circling rules:

(1) when we add up the diagrams with all possible circlings, including the diagrams where all the vertices are circled and the one where none of the vertices are circled, we get zero [36]. This fact is deduced from the so called *largest time equation* and also possible to deduce from the *smallest time equation*. We know that by Fourier transformation we can always switch to momentum space and then the equations can be written as:

$$\sum_{circlings} \Gamma_{a_1 \ldots a_n}(k_1, \ldots k_n) = 0, \tag{4.1.17}$$

where a_i's represent the label of the vertices. We can write the Eq. (4.1.17) as:

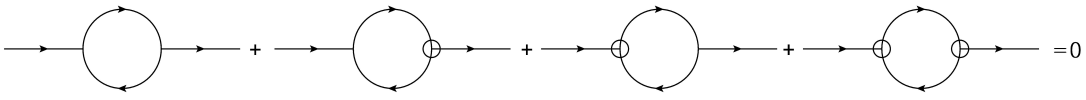


Figure 4.2: Sum of all circling is zero.

$$\Gamma_{a_1\ldots a_n}(k_1,\ldots k_n)+\Gamma_{\underline{a}_1\ldots\underline{a}_n}(k_1,\ldots k_n)=-\sum_{circling}{}'\,\Gamma_{a_1\ldots a_n}(k_1,\ldots k_n), \tag{4.1.18}$$

here the prime ($'$) means that we add all the diagrams except the two that have been extracted on the left hand side in which all the vertices are circled, and none of the vertices are circled. For one loop, this is shown diagrammatically in Fig. (4.2).

(2) We can show that the left hand side of Eq. (4.1.18) gives the imaginary part of Γ and Eq. (4.1.18) becomes:

$$\mathrm{Im}\Gamma_{a_1\ldots a_n}(k_1,\ldots k_n)=-\frac{1}{2i}\sum_{circling}{}'\,\Gamma_{a_1\ldots a_n}(k_1,\ldots k_n). \tag{4.1.19}$$

We follow the Kobes-Semenoff approach [7], where the propagators with external vertices with label $a_i=+/-$ are written as the sum over the internal vertices v_i of type $+/-$

$$\Gamma^{\{a_i\}}(\{k_i\})=\sum_{\{v_i\}}\Gamma^{\{a_i\}\{v_i\}}(\{k_i\}). \tag{4.1.20}$$

Eq. (4.1.19) can now be written:

$$\mathrm{Im}\,(\Gamma^{\{a_i\}}(\{k_i\}))=-\frac{1}{2i}\sum_{circling}{}'\sum_{\{v_i\}}\Gamma^{\{v_i\}}(\{k_i\}). \tag{4.1.21}$$

This equation can be considered as a generalization of zero temperature Cutkosky's

rule [37].

4.2 Circling rules in the R/A and Keldysh bases

The transformation matrix that transforms the propagators in the primary basis to the propagators in R/A basis is given by matrix of Eq. (3.2.1) and the matrix that transforms the vertices is given by Eq. (3.2.2). Now it is simple to transform the circling rules from the primary basis to the R/A basis. We just use the Eq. (3.1.1), (3.2.1), (4.1.16) and for different circlings we get:

$$\Delta_{XY}(k) = \begin{bmatrix} 0 & \Delta_A(k) \\ \Delta_R(k) & 0 \end{bmatrix}; \quad \Delta_{\underline{X}Y}(k) = \begin{bmatrix} 0 & -\Delta_S(k) \\ 0 & 0 \end{bmatrix} \tag{4.2.1}$$

$$\Delta_{X\underline{Y}}(k) = \begin{bmatrix} 0 & 0 \\ \Delta_S(k) & 0 \end{bmatrix}; \qquad \Delta_{\underline{XY}}(k) = \begin{bmatrix} 0 & -\Delta_R(k) \\ -\Delta_A(k) & 0 \end{bmatrix}$$

Similarly in Keldysh basis the transformation matrix is given by Eq. (3.3.1). Using Eq. (3.1.1), (3.3.1) and (4.1.16) [30, 32, 38, 39]:

$$\Delta_{XY}(k) = \begin{bmatrix} \Delta_S^T(k) & \Delta_R(k) \\ \Delta_A(k) & 0 \end{bmatrix}; \qquad \Delta_{\underline{X}Y}(k) = \begin{bmatrix} \Delta_S^T(k) & 0 \\ \Delta_S(k) & 0 \end{bmatrix} \tag{4.2.2}$$

$$\Delta_{X\underline{Y}}(k) = \begin{bmatrix} \Delta_S^T(k) & -\Delta_S(k) \\ 0 & 0 \end{bmatrix}; \qquad \Delta_{\underline{XY}}(k) = \begin{bmatrix} \Delta_S^T(k) & -\Delta_A(k) \\ -\Delta_R(k) & 0 \end{bmatrix}.$$

Using Eq. (3.1.2) the vertices are given by:

$$g^{\underline{XYZ}}(k_1,k_2,k_3) = \sum_{\{a,b,c=+,-\}} V(k_1)V(k_2)V(k_3)g^{\underline{abc}}(k_1,k_2,k_3) = -g^{XYZ}(k_1,k_2,k_3). \tag{4.2.3}$$

In any basis the circled vertex is the negative of the uncircled vertex. In all examples this factor is written in the form $(-1)^{\#c}$ where $\#c$ is the number of circled vertices.

4.3 A one-loop example

We go back to the one loop self energy diagram shown in Fig. (3.1). From Eq. (2.3.8) with $\sigma = 0$ (CTP) and Dyson's equation, the retarded self energy is given by:

$$\Pi_{ret} = \Pi_{++} + \Pi_{+-}, \tag{4.3.1}$$

as shown in Fig.(4.3).

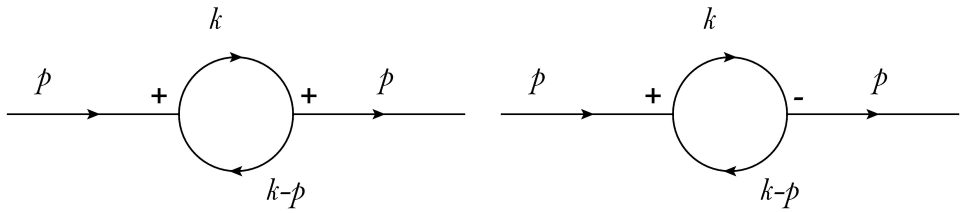


Figure 4.3: Retarded self-energy in the *primary* basis.

We have:

$$-i\Pi_{ret}(p) = -g^2 \int [\Delta_{++}(k)\Delta_{++}(p-k) - \Delta_{+-}(k)\Delta_{-+}(p-k)]\, d^4k. \tag{4.3.2}$$

First we discuss how one would calculate the imaginary part of this diagram without the benefit of the circling rules. The concept of calculating the imaginary part is same as finding the imaginary part of a complex number z by subtracting the complex conjugate z^* and dividing the result by $2i$. i.e.

$$\mathrm{Im}\, z = \frac{z - z^*}{2i}.$$

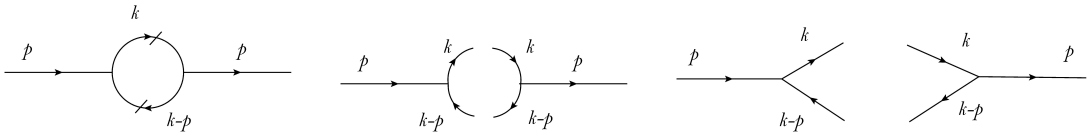
If we take the complex conjugate of Eq. (4.3.2) we have:

$$i\Pi^*_{ret}(p) = ig^2 \int [\Delta_{--}(k)\Delta_{--}(p-k) - \Delta_{+-}(k)\Delta_{-+}(p-k)]\, d^4k, \tag{4.3.3}$$

The expression for the imaginary part of the self-energy is then:

$$\mathrm{Im}\Pi_{ret}(p) = -ig^2 \int \frac{d^4k}{(2\pi)^4} [\Delta_{+-}(k)\Delta_{-+}(k-p) - \Delta_{-+}(k)\Delta_{+-}(k-p)]. \tag{4.3.4}$$

Using Eq. (2.3.5) (with $\sigma = 0$) and Eq. (2.1.5) it is easy to see that $\Delta_{+-}(k)$ and $\Delta_{-+}(k)$ are both proportional to $\delta(k^2 - m^2)$. This means the internal lines have been put on shell by taking the imaginary part. Now they correspond to physical particles. We say that the internal lines have been cut, and we write the imaginary part of the 2-point function as the product of two amplitudes. This process is shown in Fig. (4.4).



(a) Delta functions cut the propagators

(b) Loop is opened

(c) Loop diagram is reduced to tree diagrams

Figure 4.4: Imaginary part of a single loop diagram, step by step.

When there is more than one loop however, the procedure is very complicated. We want to show that the imaginary part of Π (at any loop level) can be written as the product of two amplitudes (as in Fig. 4.4). This result is guaranteed by the optical theorem but it is hard to show explicitly. This fact is not possible to realize in the one loop diagram but in the higher loop diagrams it becomes evident. To illustrate the circling rules, we redo the one loop example above. For the purpose of illustration, we present the calculation of the one loop diagram explicitly in all three bases. It is easy to show that when calculating the imaginary part of the retarded self energy, we must have the highest time vertex always circled (the vertex *from* which the external momentum flows out) and the smallest time vertex (the vertex *to* which the external momentum flows in) always uncircled. The diagrams that do not fulfill this requirement vanish trivially. Fig. (4.5) shows how this is done with appropriate labeling.

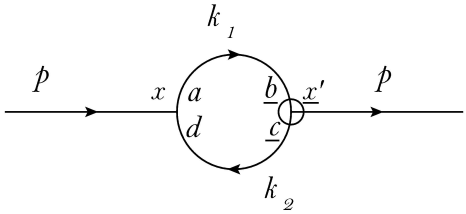


Figure 4.5: Imaginary part of the retarded self-energy of the one loop diagram in R/A basis.

In general

$$\text{Im } \Pi_{ret}(p) = i \int \frac{d^4k}{(2\pi)^4} \Delta_{a\underline{b}}(k_1) g^{xad} \Delta_{\underline{c}d}(k_2) g^{\underline{x'bc}}. \tag{4.3.5}$$

4.3.1 One loop diagram for ϕ^3 theory in the primary basis

Summing over indices and using $g^{+++} = -ig$ and $g^{---} = ig$ we can re-write the Eq. (4.3.5) in the primary basis as:

$$\text{Im } \Pi_{ret}(p) = ig^2 \int \frac{d^4k}{(2\pi)^4} [\Delta_{+-}(k_1)\Delta_{-+}(k_2) - \Delta_{+-}(k_2)\Delta_{-+}(k_1)], \tag{4.3.6}$$

which agrees with Eq. (4.3.4). Here we see that this diagram (Fig. 4.5) can be split into two parts as in Fig. (4.4). One part (rhs) will have the circled vertex when the other (lhs) will have the vertex that is not circled.

4.3.2 One loop diagram for ϕ^3 theory in the R/A basis

In the R/A basis the calculation of the diagram in Fig. (4.5) is particularly easy (the summations are especially easy to perform) since both the propagators in the diagrams have one vertex circled and the other uncircled and these propagators have only one non-zero component (see Eq. 4.2.2). In this case we will find that only one term is present in the final expression. We can use the same diagram that we used for primary and R/A basis (Fig. 4.5), but this time the labeling of the vertices, i.e. a, b, c and d will take values either "R" or "A" while $x = R, x' = A$ remains fixed for

retarded part. From Eq. (4.2.2) and (4.3.5) we can write the retarded self energy as:

$$\begin{aligned}
\mathrm{Im}\Pi_{RA}(p) &= -i \int \frac{d^4k_1}{(2\pi)^4} g_1^{\mathrm{RAA}} g_2^{\mathrm{ARR}} [\Delta_{AR}(k_1)\Delta_{RA}(k_2)] \\
&= ig^2 \int \frac{d^4k_1}{(2\pi)^4} (1 + n_B(-k_1^0) + n_B(k_2^0))[\Delta_S(k_1)(-\Delta_S(k_2))] \\
&= -ig^2 \int \frac{d^4k_1}{(2\pi)^4} (n_B(k_2^0) - n_B(k_1^0))[\Delta_S(k_1)\Delta_S(k_2)],
\end{aligned} \tag{4.3.7}$$

here we have used the relations of Eq. (3.2.9) and the third relation of Eq. (2.3.7). It is straight forward to show that Eq. (4.3.7) is identical to Eq. (4.3.6).

What we saw in this calculation is that, we don't have to do any sum over the internal vertices as the superscripts of the vertices are constrained by the matrix structure of the propagators. This aspect of R/A basis makes the higher order loop diagrams significantly simpler. We will see this when we calculate the two-loop diagrams also.

4.3.3 One loop diagram for ϕ^3 theory in the Keldysh basis

In Keldysh basis the thermal weight is carried by the "11" element of each of the propagator matrices irrespective to circling. The vertices remain temperature independent as in the primary basis. For the diagram under consideration we have the propagator matrices (see Eq. (4.2.3)):

$$\Delta_{a\underline{b}}(k_1) = \begin{bmatrix} \Delta_S^T(k_1) & -\Delta_S(k_1) \\ 0 & 0 \end{bmatrix}; \quad \Delta_{\underline{c}d}(k_2) = \begin{bmatrix} \Delta_S^T(k_2) & 0 \\ \Delta_S(k_2) & 0 \end{bmatrix}. \tag{4.3.8}$$

The vertices are again labeled as g_1^{xad} and $g_2^{\underline{x'}bc}$ and from the propagator matrices Eq. (4.3.8) we find the values the superscripts can take. Since we are looking at the

retarded self energy, the external indices take the values $x = 2$ and $x' = 1$. From the first part of Eq. (4.3.8) we have $a = 1$, and the second part gives $d = 1$. Combining we have:

$$g_1^{xad} = g_1^{211} \tag{4.3.9a}$$

$$g_1^{x'bc} = g_2^{112}, \text{ or } g_2^{121} \tag{4.3.9b}$$

which means that, when we carry out the sum over the vertices, we have two terms. In the Keldysh basis, we can write the expression for retarded self-energy as:

$$\begin{aligned}
&\mathrm{Im}\Pi_{21}(p) = -i\int \frac{d^4k_1}{(2\pi)^4}[g_1^{211}g_2^{112}(\Delta_{11}(k_1)\Delta_{21}(k_2)) + g_1^{211}g_2^{121}(\Delta_{12}(k_1)\Delta_{11}(k_2))] \\
&= -i\frac{g^2}{2}\int \frac{d^4k_1}{(2\pi)^4}[\Delta_S^T(k_1)\Delta_S(k_2) + (-\Delta_S(k_1))\Delta_S^T(k_2)] \\
&= -i\frac{g^2}{2}\int \frac{d^4k_1}{(2\pi)^4}[(n_B(k_1^0) - n_B(-k_1^0))\Delta_S(k_1)\Delta_S(k_2) - (n_B(k_2^0) - n_B(-k_2^0))\Delta_S(k_2)\Delta_S(k_1)] \\
&= -i\frac{g^2}{2}\int \frac{d^4k_1}{(2\pi)^4}[((1 + n_B(k_1^0))n_B(k_2^0) - n_B(k_1^0)(1 + n_B(k_2^0)))\Delta_S(k_1)\Delta_S(k_2)]
\end{aligned} \tag{4.3.10}$$

Doing some simple algebra we can show that this expression for retarded self-energy is identical to the one we got for R/A basis (Eq. 4.3.7), and thus same as the one we got in the primary basis (Eq. 4.3.6).

4.4 One loop example interpreted

In this section we explain the result we got for one loop diagram in different basis. We can see that the Eqs. (4.3.6), (4.3.7), and (4.3.10) are all the same. Now I discuss

the interpretation of the result for the one loop diagram. We can re-write the Eq. (4.3.10) in form:

$$\text{Im } \Pi_{ret}(p) = -i\frac{g^2}{2}\int \frac{d^4k_1}{(2\pi)^4}[(n_B(k_1^0)n_B(-k_2^0) - (1+n_B(k_1^0))(1+n_B(-k_2^0)))\Delta_S(k_1)\Delta_S(k_2)]. \tag{4.4.1}$$

The factor $(1+n_B(k^0))$ has the form of a statistical emission factor and $n_B(k^0)$ has the form of a statistical absorption factor. So, Eq. (4.4.1) contains the rate to absorb two particles of energy $E_1 \equiv k_1^0$ and $E_2 \equiv -k_2^0$ minus the rate to emit the same. This is written $\Gamma_{ret}(k) = \Gamma_d(k) - \Gamma_c(k)$ as in Eq. (4.1.1). To compare, if we calculate the time ordered product, we get the following expression:

$$\begin{aligned}\text{Im } \Sigma_{++}(p) &= -i\frac{g^2}{2}\iint dk_1(n_B(k_1^o) + n_B(k_2^o) + 2n_B(k_1^o)n_B(k_2^o))\Delta_{+-}(k_1)\Delta_{-+}(k_2) \\ &= -i\frac{g^2}{2}\iint dk_1((1+n_B(k_1^o))n_B(k_2^o) + n_B(k_1^o)(1+n_B(k_2^o)))\Delta_{+-}(k_1)\Delta_{-+}(k_2)\end{aligned} \tag{4.4.2}$$

which agrees with Eq. (4.3.10) except for a sign. This can not be written as $(\Gamma_d(k) - \Gamma_c(k))$. Which is why, at finite temperature, the imaginary part of the retarded self energy draws more attention than any other elements.

4.5 Some important rules

Before we go for the 2-loop and 3-loop examples, we summarize some basic rules related to circling, some already seen, some of which will be illustrated in Chapter-5:

1. The on-shell (or external) particles are physical particles and they do not prop-

agate *through* the medium while the off-shell (or internal) particles do.

2. On-shell particles are always represented by a delta function of the form $\delta(k^2 - m^2)$, where k is the vector momentum of the corresponding particle.

3. When we take the imaginary part of a diagram, some of the internal propagators are put on shell and these become external propagators. We call these cut propagators.

4. Cut propagators pass between pairs of vertices where one is circled and one not. Cut line is the line that crosses only off shell propagators that have only one vertex circled and the other uncircled. Each diagram contain one and only one cut line.

5. In two loop diagrams, for some circlings the cut line does not open all loops. All loops which are not opened by cut line contain one tic-ed propagator which contains an on-shell delta (δ) function. Thus all loops are opened by the *cut-lines* alone or with a combination of cut lines and *tics* [11].

6. After we have taken the imaginary part, the diagram can be written as the product of two amplitudes and in this sense it is divided into two parts. All the vertices in one half (lhs for Π_{ret}) are uncircled and all the vertices on the other half (rhs for Π_{ret}) are circled. The diagram has form (amplitude-A) $\times$ (amplitude-B)* form.

7. The results we get by using the circling rules are physically interpretable and consistent with the optical theorem. Any external line that represents an emitted particle has a thermal factor of $(1 + n_B(k))$ while an external line that represents an absorbed particle has a thermal factor of $n_B(k)$.

Chapter 5

Examples in primary, R/A and Keldysh bases

5.1 Introduction

In this chapter we see some examples of two and three loop diagrams, compare the results and try to find out why one basis is convenient over the others for a particular example. We will always consider the retarded 2-point function which is written Π_{ret}. We will see that the diagrams in which the circled and uncircled vertices do not form a connected region has zero contribution in the self energy. Below we list some useful tricks that are used in this section.

- For ϕ^3-theory a very important relation between thermal factors corresponding

to the momentum meeting at a particular vertex is

$$1 + N(k_1^0) + N(k_2^0) + N(k_3^0) + N(k_1^0)N(k_2^0) + N(k_2^0)N(k_3^0) + N(k_1^0)N(k_3^0) = 0;$$
$$\text{for} \quad k_1 + k_2 + k_3 = 0, \tag{5.1.1}$$

where $N(k_i^0) = (1 + 2n_B(k_i^0))$.

- We also define

$$\mathcal{N}(k_1^0, k_2^0, k_3^0, \ldots) = \left[(1 + n(k_1^0))(1 + n_B(k_2^0))(1 + n_B(k_3^0)) \ldots - n(k_1^0)n_B(k_2^0)n_B(k_3^0) \ldots\right] \tag{5.1.2}$$

which is used with the identity $n_B(x) + n_B(-x) = -1$, the second relation of Eq. (2.3.7) for bosons.

- We can write Eq. (5.1.1) in a more useful form for the two cases where (a) one momentum leaving the vertex and two momenta entering the vertex and; (b) one momentum entering the vertex and two momenta leaving the vertex. The two relations corresponding to the two possibilities are given in Eqs. (5.1.3).

$$n_B(k_{out}^0) + n_B(k_{in_1}^0)n_B(k_{out}^0) + n_B(k_{in_2}^0)n_B(k_{out}^0) - n_B(k_{in_1}^0)n_B(k_{in_2}^0) = 0, \tag{5.1.3a}$$

$$n_B(k_{in}^0) + n_B(k_{out_1}^0)n_B(k_{in}^0) + n_B(k_{out_2}^0)n_B(k_{in}^0) - n_B(k_{out_1}^0)n_B(k_{out_2}^0) = 0. \tag{5.1.3b}$$

- The relation between thermal factors for ϕ^4 theory is which is the ϕ^4-version of

Eq. (5.1.1):

$$N(k_1^0) + N(k_2^0) + N(k_3^0) + N(k_4^0) + N(k_1^0)N(k_2^0)N(k_3^0) + N(k_1^0)N(k_3^0)N(k_4^0) + N(k_2^0)N(k_3^0)(k_4^0) = 0, \tag{5.1.4}$$

provided $k_1 + k_2 + k_3 + k_4 = 0$.

- We define a *principle* propagator as the principle part:

$$\Delta_P(k) = \frac{1}{2}[\Delta_R(k) + \Delta_A(k)]. \tag{5.1.5}$$

This propagator will help us keep the equations in examples in the R/A basis in a compact form.

- Consider the loop formed by all retarded or all advanced propagators in which the loop momenta is flowing either clockwise or anti-clockwise and there are no thermal weights which depends on the loop momentum. These terms vanish when the loop integration is performed. For example

$$\int \left[\prod_1^n \Delta_R(k_i)\right] dk_l = 0; \quad \text{when } k_1 \pm k_2 \ldots \pm k_n = 0, \tag{5.1.6}$$

where k_l is the momentum that circulates along the loop and n is the total number of propagators forming the loop. This integration also vanish by the *time ordering* in mixed space [40] and by the causality law [32] in the momentum space.

- Use

$$\Delta_S(k_i) = \Delta_R(k_i) - \Delta_A(k_i), \tag{5.1.7}$$

  as defined in Eq. (3.3.3).

- Consider the terms where in a particular vertex all the momentum (entering or leaving) are delta functions (δ). Mathematically

$$\prod_1^n \Delta_S(k_i) = 0, \tag{5.1.8}$$

  where n is the number of fields interacting at the vertex.

- To take the advantage of the two relations in Eq. (5.1.6) and Eq. (5.1.8) use the substitutions:

$$\Delta_A(k_i) = \Delta_R(k_i) - \Delta_S(k_i), \quad \text{and} \quad \Delta_R(k_i) = \Delta_A(k_i) + \Delta_S(k_i) \tag{5.1.9}$$

The trick to avoid the lengthy sum is to identify the nonzero vertices to get only the valid/non-zero terms. A major challenge is to simplify this sum. For example in a two-loop self energy diagram we have ten internal indices and each of them can take two distinct values (R,A in the R/A basis or 1,2 in the Keldysh basis) which makes a total $2^{10} = 1024$ terms. However, only few of them are non zero. We are able to find these few valid terms without performing this huge sum. This procedure is much more difficult in the primary basis [36, 40]. The two-loop diagram in the Fig. (5.1) is also a good example of another problem with the primary basis: we can have circling in a way where the circled vertices do not form a connected region.

As a result, it requires a lot of manipulation to show that the imaginary part of the diagram can be written as the product of two connected amplitude as required by the optical theorem. It requires grouping the terms properly which is extremely laborious and we will get an idea of the level of difficulty of grouping the terms when we will see the two-loop example in the R/A basis. We use the knowledge we acquired in the previous chapters and the similar approach of last section (for one loop diagram). As mentioned earlier, since we calculate the imaginary part of the retarded self-energy, we always calculate the $\Pi_{RA}(p)$ in the R/A basis and $\Pi_{21}(p)$ in the Keldysh basis. We always have the right hand external vertex circled and the left hand external vertex uncircled.

5.2 Two-loop diagram for ϕ^3 theory

There are two diagrams for ϕ^3-theory at two-loop level which we call the *self energy* and *vertex diagrams*. We show the details of how the sum over internal vertices is simplified in R/A and Keldysh basis in the appendices (appendix A for the self-energy diagram and appendix B for the vertex diagram).

5.2.1 Two-loop self-energy diagram

To start with we take the diagram shown in Fig. (5.1). We will call this a *self-energy* diagram because the loop forms a self-energy type correction on the upper line. The mathematical expression for the imaginary part of the diagram in Fig. (5.1) is given

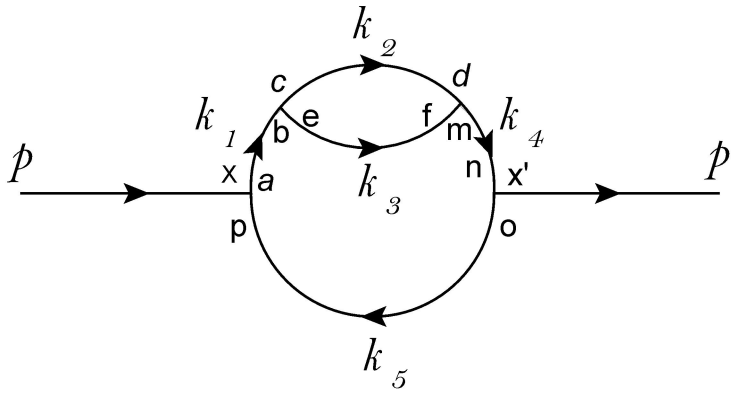


Figure 5.1: Two-loop self energy diagram.

by:

$$\mathrm{Im}\Pi_{xx'}(p) = -i(-1)^{\#c} \iint dk_1 dk_2 [g_1^{xap} g_2^{bce} g_3^{mdf} g_4^{x'no} \Delta_{ab}(k_1)\Delta_{cd}(k_2)\Delta_{ef}(k_3)\Delta_{mn}(k_4)\Delta_{op}(k_5)]. \tag{5.2.1}$$

where $\#c$ is the number of circled vertices. Since we calculate Π_{ret}, the vertex g_4 is always circled and the vertex g_1 always uncircled. The diagrams that result from the sum over circlings are shown in Fig. (5.2). The propagator and vertices in the R/A and Keldysh basis are given in Eq. (3.2.1), Eq. (3.2.2) and Eq. (3.3.1), Eq. (3.3.4) respectively. The trick to reduce the number of terms in the sum is to identify the non-zero components in the vertex. This is explained in detail in Appendix A.

As explained in rule # 4 of section-4.5, it is easy to see that the first three diagrams can be written as the product of two amplitudes. This is shown in Fig. (5.3). Fig. (5.2d) *cannot* be written as a product of two amplitudes. However, in R/A and Keldysh basis it is easy to show that its contribution is zero. We can see that the circled vertices in this diagram do not form a connected region and from the vertex configuration we can see that in R/A basis this diagram vanishes. It is easy to show

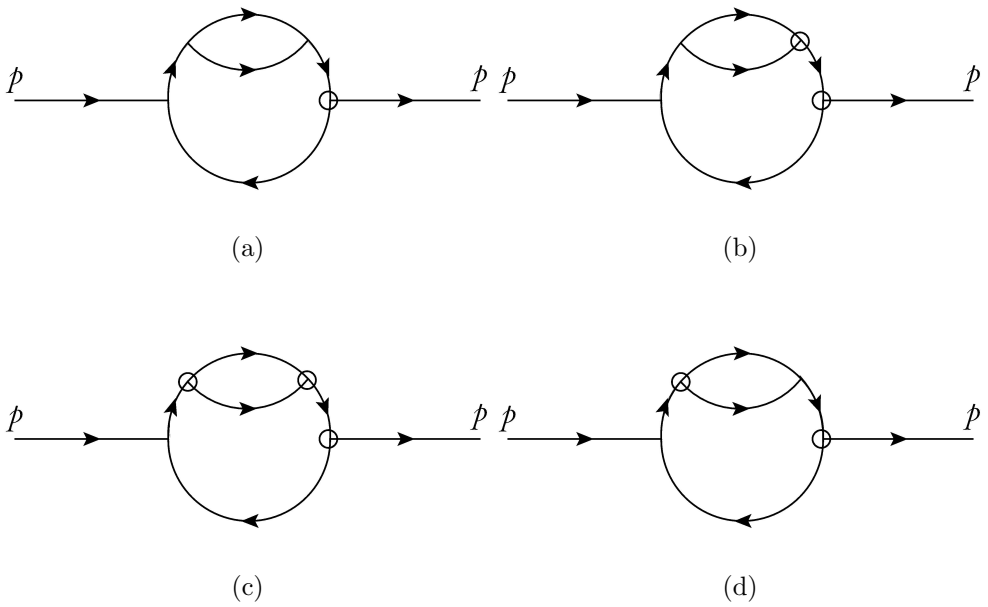


Figure 5.2: Imaginary part of the retarded self-energy for the *self-energy* diagram.

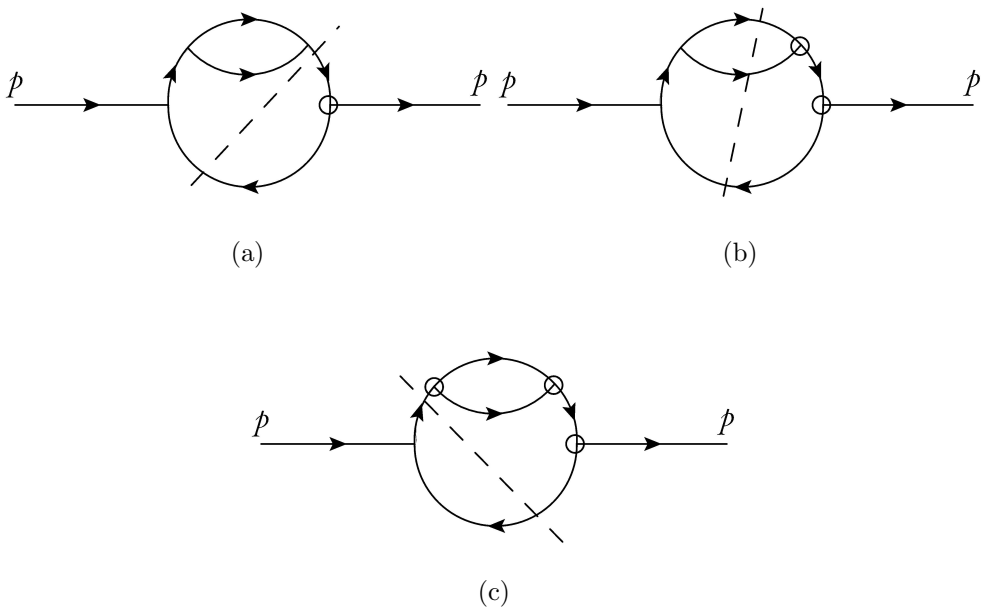


Figure 5.3: Imaginary part of the retarded self-energy for the *self-energy* diagram with cut lines drawn as dashed lines.

that for the second vertex of the diagram in Fig. (5.2d):

$$g_2^{bce} = g_2^{RRR} = 0. \tag{5.2.2}$$

The result is straight forward:

$$\mathrm{Im}\,\Pi_{RA}^{(d)}(p) = 0. \tag{5.2.3}$$

The reason is discussed in detail in the Appendix.

5.2.1.1 In R/A basis

In R/A basis we obtain the results given in Eqs. (5.2.4)-(5.2.6) for the first three diagrams in Fig. (5.2). In each equation the first line results in from direct substitution of vertex and propagators in Eq. (5.2.1). Then we want to rearrange the terms to get the required form with appropriate thermal factors. The process of rearranging for the the self energy diagram in the R/A basis is much easier compared to the vertex diagram (Fig. (5.5). Still, we require to use Eqs. (5.1.1), (5.1.6), (5.1.8), (5.1.3) to simplify and get the expression in a form that could be explained in terms of decay and creation rate. Now we write down the expressions for the imaginary parts of the

retarded self energy for each diagram.

$$\begin{aligned}
\mathrm{Im}\Pi^{(a)}_{RA}(p) = -ig^4 \iint dk_1 dk_2 [&(n_B(k_1^0) - n_B(k_2^0))(n_B(k_3^0) - n_B(k_4^0))\Delta_A(k_1)\Delta_R(k_2) \\
&\Delta_A(k_3)\Delta_S(k_4)(-\Delta_S(k_5))+ \\
&(n_B(k_1^0) - n_B(k_3^0))(n_B(k_2^0) - n_B(k_4^0))\Delta_A(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_S(k_4)(-\Delta_S(k_5))+ \\
&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_3^0) - n_B(k_4^0))\Delta_R(k_1)\Delta_R(k_2)\Delta_R(k_3)\Delta_S(k_4)(-\Delta_S(k_5))+ \\
&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_2^0) - n_B(k_4^0))\Delta_R(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_S(k_4)(-\Delta_S(k_5))+ \\
&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_3^0) - n_B(k_4^0))\Delta_R(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_S(k_4)(-\Delta_S(k_5))]
\end{aligned}$$

$$\begin{aligned}
= -ig^4 \iint dk_1 dk_2 [&(n_B(k_1^0)n_B(k_3^0) - n_B(k_1^0)n_B(k_4^0) - n_B(k_2^0)n_B(k_3^0) + n_B(k_2^0)n_B(k_4^0)) \\
&\Delta_A(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5)+ \\
&(n_B(k_1^0)n_B(k_2^0) - n_B(k_1^0)n_B(k_4^0) - n_B(k_3^0)n_B(k_2^0) + n_B(k_3^0)n_B(k_4^0)) \\
&\Delta_A(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S(k_5)+ \\
&(n_B(k_5^0) + n_B(k_5^0)n_B(k_3^0) + n_B(k_5^0)n_B(k_2^0) - n_B(k_1^0) - n_B(k_1^0)n_B(k_3^0) + n_B(k_1^0)n_B(k_2^0)) \\
&\Delta_R(k_1)\Delta_R(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S(k_5)+ \\
&(n_B(k_5^0)n_B(k_2^0) - n_B(k_5^0)n_B(k_4^0) - n_B(k_1^0)n_B(k_2^0) + n_B(k_1^0)n_B(k_4^0)) \\
&\Delta_R(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S(k_5)+ \\
&(n_B(k_5^0)n_B(k_3^0) - n_B(k_5^0)n_B(k_4^0) - n_B(k_1^0)n_B(k_3^0) + n_B(k_1^0)n_B(k_4^0)) \\
&\Delta_R(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5)]
\end{aligned} \tag{5.2.4}$$

$$= ig^4 \iint dk_1 dk_2 [(n_B(k_5^0) n_B(k_2^0) - n_B(k_4^0) n_B(k_2^0) + \frac{1}{2}(n_B(k_5^0) - n_B(k_4^0)))\Delta_A(k_1)\Delta_S(k_2)$$
$$\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5) + (n_B(k_5^0) n_B(k_3^0) - n_B(k_4^0) n_B(k_3^0) + \frac{1}{2}(n_B(k_5^0) - n_B(k_4^0)))$$
$$\Delta_A(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)]$$
$$= ig^4 \iint dk_1 dk_2 [\frac{1}{2}((1 + n_B(k_4^0)) n_B(k_5^0) - (n_B(k_4^0)(1 + n_B(k_5^0)))(1 + 2n_B(k_2^0)))\Delta_R(k_1)\Delta_S(k_2)$$
$$\Delta_R(k_3)\Delta_S(k_4)\Delta_S(k_5) + \frac{1}{2}(n_B(k_4^0) n_B(k_3^0) - n_B(k_4^0)(1 + n_B(k_5^0)))(1 + 2n_B(k_3^0))$$
$$\Delta_R(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)]$$
$$= ig^4 \iint dk_1 dk_2 [\frac{N(k_2^0)}{2}\mathcal{N}(k_4^0, -k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S(k_5)$$
$$+ \frac{N(k_3^0)}{2}\mathcal{N}(k_4^0, -k_5^0)\Delta_R(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)]$$

$$\mathrm{Im}\Pi_{RA}^{(b)}(p) = -ig^4 \iint dk_1 dk_2 [(n_B(k_5^0) - n_B(k_1^0))(n_B(-k_3^0) - n_B(k_2^0))\Delta_R(k_1)\Delta_S(k_2)$$
$$\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)]$$
$$= -ig^4 \iint dk_1 dk_2 [(n_B(k_5^0) + n_B(k_5^0) n_B(k_3^0) + n_B(k_5^0) n_B(k_2^0) + n_B(k_2^0) n_B(k_3^0))$$
$$\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)]$$
$$= -ig^4 \iint dk_1 dk_2 [(1 + n_B(k_2^0))(1 + n_B(k_3^0)) n_B(k_5^0) - (n_B(k_2^0) n_B(k_3^0)(1 + n_B(k_5^0)))$$
$$\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)]$$
$$= -ig^4 \iint dk_1 dk_2 [\mathcal{N}(k_2^0, k_3^0, -k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)] \tag{5.2.5}$$

$$
\begin{aligned}
\mathrm{Im}\Pi^{(c)}_{RA}(p) = -ig^4 \iint dk_1 dk_2 [&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_3^0) - n_B(k_4^0))\Delta_S(k_1)\Delta_A(k_2) \\
&\Delta_R(k_3)\Delta_A(k_4)(-\Delta_S(k_5))+ \\
&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_4^0) - n_B(p_0^0))\Delta_S(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_R(k_4)(-\Delta_S(k_5))- \\
&(n_B(k_5^0) - n_B(k_1^0))(1 + n_B(k_2^0) + n_B(k_3^0))\Delta_S(k_1)\Delta_A(k_2)\Delta_A(k_3)\Delta_A(k_4)(-\Delta_S(k_5))+ \\
&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_2^0) - n_B(k_4^0))\Delta_S(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_A(k_4)(-\Delta_S(k_5))+ \\
&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_4^0) - n_B(p_0^0))\Delta_S(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_R(k_4)(-\Delta_S(k_5))] \\
= ig^4 \iint dk_1 dk_2 [&(n_B(k_5^0)n_B(k_3^0) - n_B(k_1^0)n_B(k_3^0) + \frac{1}{2}(n_B(k_1^0) - n_B(k_5^0))) \\
&\Delta_S(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)+ \\
&(n_B(k_5^0)n_B(k_2^0) - n_B(k_1^0)n_B(k_2^0) + \frac{1}{2}(n_B(k_1^0) - n_B(k_5^0))) \\
&\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_A(k_4)\Delta_S(k_5)] \\
= ig^4 \iint dk_1 dk_2 [&\frac{1}{2}(1 + 2n_B(k_3^0))((1 + n_B(k_1^0))n_B(k_5^0) - n_B(k_1^0)(1 + n_B(k_5^0))) \\
&\Delta_S(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)+ \\
&\frac{1}{2}(1 + 2n_B(k_2^0))(1 + n_B(k_1^0)n_B(k_5^0) - n_B(k_1^0)(1 + n_B(k_5^0))) \\
&\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_A(k_4)\Delta_S(k_5)] \\
= ig^4 \iint dk_1 dk_2 [&\frac{N(k_2^0)}{2}\mathcal{N}(k_1^0, -k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_A(k_4)\Delta_S(k_5)+ \\
&\frac{N(k_3^0)}{2}\mathcal{N}(k_1^0, -k_5^0)\Delta_S(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)]
\end{aligned}
\tag{5.2.6}
$$

5.2.1.2 In Keldysh basis

In Keldysh basis, as we have discussed earlier, the thermal weights are carried by the propagator matrices. As a result, the propagator that are on-shell, can easily be associated with the appropriate thermal factor to give a easily explainable physical interpretation. Summing over the indices is a not as easy as it is in the R/A basis, but this disadvantage is fully compensated by the advantage of having the thermal factors in proper line. The results for the first three diagrams in Fig. (5.2) are given in Eqs. (5.2.7)-(5.2.9). In each equation the first line results in from the direct substitution of the propagators and vertices. Then, as in R/A basis, we rearrange the terms to

find the appropriate propagators in the right places with right thermal factors.

$$
\begin{aligned}
\mathrm{Im}\Pi_{21}^{(a)}(p) = -i\frac{g^4}{2^2}\iint dk_1 dk_2[&\Delta_R(k_1)\Delta_R(k_2)\Delta_S^T(k_3)\Delta_S(k_4)\Delta_S^T(k_5)+\\
&\Delta_R(k_1)\Delta_R(k_2)\Delta_S^T(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5)) + \Delta_R(k_1)\Delta_S^T(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S^T(k_5)+\\
&\Delta_R(k_1)\Delta_S^T(k_2)\Delta_R(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5)) + \Delta_S^T(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S^T(k_5)+\\
&\Delta_S^T(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5)) + \Delta_S^T(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S^T(k_5)+\\
&\Delta_S^T(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5))]\\
= -i\frac{g^4}{4}\iint dk_1 dk_2[&(1+2n_B(k_3^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)-\\
&(1+2n_B(k_3^0))(1+2n_B(k_4^0))\Delta_R(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)+\\
&(1+2n_B(k_2^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_R(k_3) - \Delta_S(k_4)\Delta_S(k_5)-\\
&(1+2n_B(k_2^0))(1+2n_B(k_4^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S(k_5)]\\
= -ig^4\iint dk_1 dk_2[&\frac{1}{2}(1+2n_B(k_3^0))((1+n_B(k_4^0))n_B(k_5) - n_B(k_4^0)(1+n_B(k_5^0)))\\
&\Delta_R(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)+\\
&\frac{1}{2}(1+2n_B(k_2^0))((1+n_B(k_4^0))n_B(k_5) - n_B(k_4^0)(1+n_B(k_5^0)))\\
&\Delta_R(k_1)\Delta_S(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S(k_5)]\\
= -ig^4\iint dk_1 dk_2[&\frac{N(k_3^0)}{2}\mathcal{N}(k_4^0,-k_5^0)\Delta_R(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)+\\
&\frac{N(k_2^0)}{2}\mathcal{N}(k_4^0,-k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S(k_5)]
\end{aligned}
\tag{5.2.7}
$$

$$\begin{aligned}
\mathrm{Im}\Pi_{21}^{(b)}(p) &= -i\frac{g^4}{2^2}\iint dk_1 dk_2[-\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)\Delta_R(k_4)(-\Delta_S(k_5))- \\
&\Delta_R(k_1)\Delta_S(k_2)\Delta_S^T(k_3)\Delta_R(k_4)\Delta_S^T(k_5) - \Delta_R(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S^T(k_5)+ \\
&\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5)) + \Delta_R(k_1)\Delta_S(k_2)\Delta_S^T(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5))+ \\
&\Delta_R(k_1)(-\Delta_S(k_2))(-\Delta_S(k_3))\Delta_R(k_4)(-\Delta_S(k_5))] \\
&= -i\frac{g^4}{4}\iint dk_1 dk_2[(1+2n_B(k_2^0))(1+2n_B(k_3^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)- \\
&(1+2n_B(k_3^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \\
&(1+2n_B(k_2^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)- \\
&\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)] \\
&= -ig^4\iint dk_1 dk_2[(1+n_B(k_2^0))(1+n_B(k_3^0)n_B(k_5^0) - n_B(k_2^0)n_B(k_3^0)(1+n_B(k_5^0))) \\
&\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)] \\
&= -ig^4\iint dk_1 dk_2[\mathcal{N}(k_2^0,k_3^0,-k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)]
\end{aligned} \tag{5.2.8}$$

$$\begin{aligned}
&\mathrm{Im}\Pi_{21}^{(c)}(p) = -i\frac{g^4}{2^2}\iint dk_1 dk_2[\Delta_S^T(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_S(k_4)(-\Delta_S(k_5))+\\
&\Delta_S^T(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_S(k_4)\Delta_S^T(k_5) + \Delta_S^T(k_1)\Delta_S^T(k_2)\Delta_R(k_3)\Delta_R(k_4)(-\Delta_S(k_5))+\\
&\Delta_S^T(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_S(k_4)(-\Delta_S(k_5)) + \Delta_S^T(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_R(k_4)\Delta_S^T(k_5)+\\
&\Delta_S^T(k_1)\Delta_R(k_2)\Delta_S^T(k_3)\Delta_R(k_4)(-\Delta_S(k_5)) + \Delta_S(k_1)\Delta_S^T(k_2)\Delta_A(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5))+\\
&\Delta_S(k_1)\Delta_S^T(k_2)\Delta_A(k_3)\Delta_A(k_4)\Delta_S^T(k_5) + \Delta_S(k_1)\Delta_A(k_2)\Delta_S^T(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5))+\\
&\Delta_S(k_1)\Delta_A(k_2)\Delta_S^T(k_3)\Delta_A(k_4)\Delta_S^T(k_5)\Delta_S(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)\Delta_R(k_4)(-\Delta_S(k_5))]\\
&= -i\frac{g^4}{4}\iint dk_1 dk_2[-(1+2n_B(k_1^0))(1+2n_B(k_2^0))\Delta_S(k_1)\Delta_S(k_2)\Delta_R(k_3)\Delta_R(k_4)\Delta_S(k_5)-\\
&(1+2n_B(k_1^0))(1+2n_B(k_3^0))\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)+\\
&(1+2n_B(k_2^0))(1+2n_B(k_5^0))\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_A(k_4)\Delta_S(k_5)-\\
&(1+2n_B(k_3^0))(1+2n_B(k_5^0))\Delta_S(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)]\\
&= -i\frac{g^4}{4}\iint dk_1 dk_2[((1+2n_B(k_2^0))(1+2n_B(k_5^0)) - (1+2n_B(k_1^0))(1+2n_B(k_2^0)))\\
&\Delta_S(k_1)\Delta_S(k_2)\Delta_R(k_3)\Delta_R(k_4)\Delta_S(k_5)-\\
&((1+2n_B(k_1^0))(1+2n_B(k_3^0)) - (1+2n_B(k_3^0))(1+2n_B(k_5^0)))\\
&\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)]\\
&= -ig^4\iint dk_1 dk_2[\frac{1}{2}(1+2n_B(k_2^0))((1+n_B(k_1^0))n_B(k_5^0) - n_B(k_1^0)(1+n_B(k_5^0)))\\
&\Delta_S(k_1)\Delta_S(k_2)\Delta_R(k_3)\Delta_R(k_4)\Delta_S(k_5)-\\
&\frac{1}{2}(1+2n_B(k_3^0))((1+n_B(k_1^0))n_B(k_5^0) - n_B(k_1^0)(1+n_B(k_5^0)))\\
&\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)]
\end{aligned} \tag{5.2.9}$$

$$= -ig^4 \iint dk_1 dk_2 [\frac{N(k_2^0)}{2} \mathcal{N}(k_1^0, -k_5^0) \Delta_S(k_1) \Delta_S(k_2) \Delta_A(k_3) \Delta_A(k_4) \Delta_S(k_5) -$$
$$\frac{N(k_3^0)}{2} \mathcal{N}(k_1^0, -k_5^0) \Delta_S(k_1) \Delta_A(k_2) \Delta_S(k_3) \Delta_A(k_4) \Delta_S(k_5)] \tag{5.2.10}$$

5.2.2 Results for two-loop self energy diagram

The result in Eq. (5.2.7)-(5.2.9) agree with the corresponding expressions in Eq. (5.2.4)-(5.2.6) in section-5.2.1.1 from the R/A basis. Collecting results we have for the self energy diagram:

$$\text{Im } [\Pi^a_{ret}(p) + \Pi^b_{ret}(p) + \Pi^c_{ret}(p) + \Pi^d_{ret}(p)] =$$
$$- ig^4 \iint dk_1 dk_2 [\frac{N(k_3^0)}{2} \mathcal{N}(k_4^0, -k_5^0) \Delta_R(k_1) \Delta_R(k_2) \Delta_S(k_3) \Delta_S(k_4) \Delta_S(k_5) +$$
$$\frac{N(k_2^0)}{2} \mathcal{N}(k_4^0, -k_5^0) \Delta_R(k_1) \Delta_S(k_2) \Delta_R(k_3) \Delta_S(k_4) \Delta_S(k_5) +$$
$$\mathcal{N}(k_2^0, k_3^0, -k_5^0) \Delta_R(k_1) \Delta_S(k_2) \Delta_S(k_3) \Delta_R(k_4) \Delta_S(k_5) -$$
$$\frac{N(k_3^0)}{2} \mathcal{N}(k_1^0, -k_5^0) \Delta_S(k_1) \Delta_R(k_2) \Delta_S(k_3) \Delta_R(k_4) \Delta_S(k_5) +$$
$$\frac{N(k_2^0)}{2} \mathcal{N}(k_1^0, -k_5^0) \Delta_S(k_1) \Delta_S(k_2) \Delta_R(k_3) \Delta_R(k_4) \Delta_S(k_5)] \tag{5.2.11}$$

These five terms corresponding to the five possible ways to open all the loops by cutting and ticing lines, as explained in section-4.5. They are shown diagrammatically in Fig. (5.4).

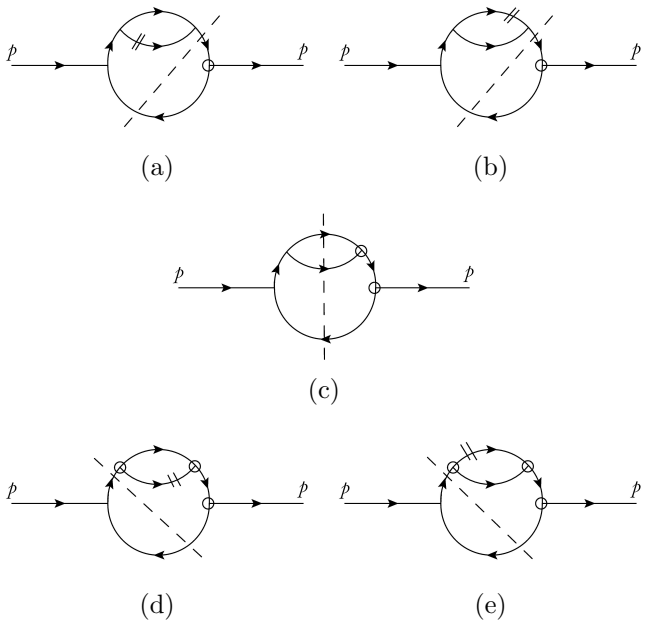


Figure 5.4: Imaginary part of the retarded self-energy for *self-energy* diagram with cut lines and tic-ed lines.

5.2.3 Two-loop vertex diagram

In this section we calculate the retarded self energy of the *vertex diagram* which is shown in Fig. (5.5).

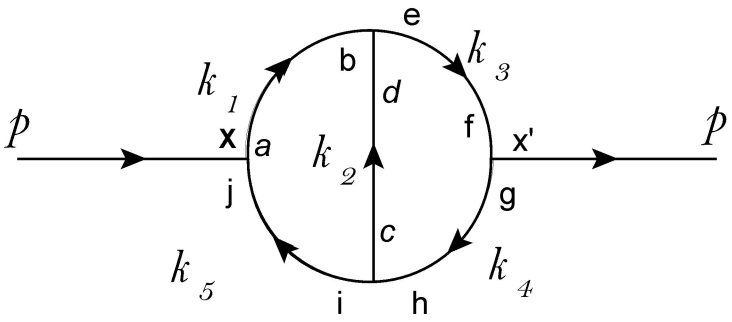


Figure 5.5: Two-loop vertex diagram.

The mathematical expression for the imaginary part of the diagram in Fig.(5.5) is

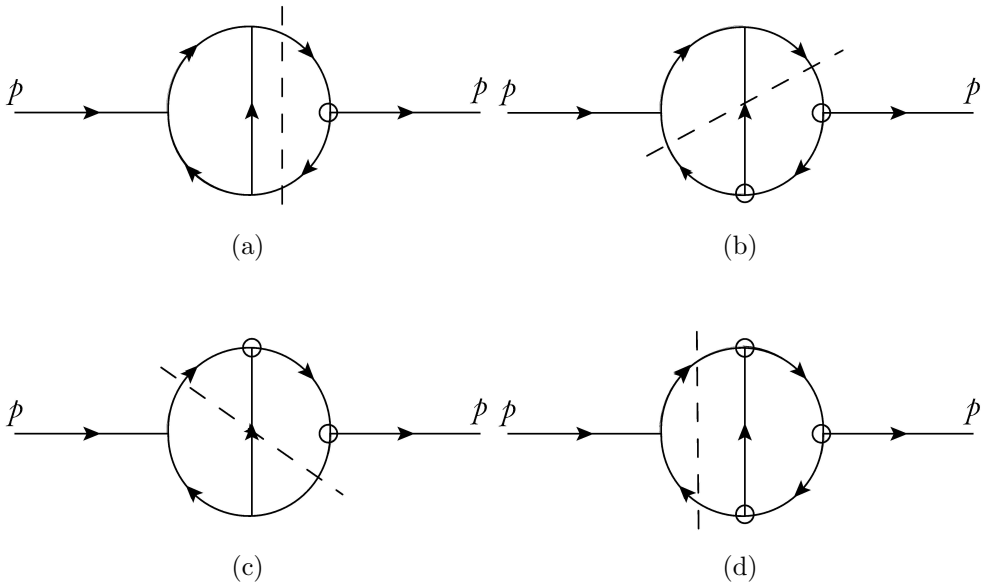


Figure 5.6: Imaginary part of the retarded self-energy for the vertex diagram (the cut lines are drawn as dashed lines).

given by:

$$\mathrm{Im}\Pi_{xx'}(p) = -i(-1)^{\#c} \iint dk_1 dk_2 [g_1^{xaj} g_2^{bed} g_3^{ich} g_4^{x'fg} \Delta_{ab}(k_1)\Delta_{cd}(k_2)\Delta_{ef}(k_3)\Delta_{gh}(k_4)\Delta_{ij}(k_5)]. \tag{5.2.12}$$

with $xx' = RA$ and $xx' = 21$ and where $\#c$ is the number of circled vertices. The imaginary part of the retarded amplitude is the sum of the diagrams in Fig. (5.6).

5.2.3.1 In the R/A basis

For the diagram of Fig. (5.6a) we obtain directly using the Eq. (5.2.12):

$$\begin{aligned}\mathrm{Im}\Pi^{(a)}_{RA}(p) = -ig^4 \iint dk_1 dk_2 [&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_2^0) - n_B(k_3^0))\Delta_R(k_1)\Delta_A(k_2) \\ &\Delta_S(k_3)(-\Delta_S(k_4))\Delta_A(k_5)+ \\ &(n_B(k_5^0) - n_B(k_1^0))(n_B(k_4^0) - n_B(k_2^0))\Delta_R(k_1)\Delta_R(k_2)\Delta_S(k_3)(-\Delta_S(k_4))\Delta_A(k_5)+ \\ &(n_B(k_1^0) - n_B(k_3^0))(n_B(k_4^0) - n_B(k_2^0))\Delta_A(k_1)\Delta_R(k_2)\Delta_S(k_3)(-\Delta_S(k_4))\Delta_A(k_5)+ \\ &(n_B(k_2^0) - n_B(k_3^0))(n_B(k_4^0) - n_B(k_5^0))\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)(-\Delta_S(k_4))\Delta_R(k_5)]\end{aligned} \tag{5.2.13}$$

As explained in rule # 8 of section-4.5, in order to have a physical interpretation we need the cut propagators to carry thermal factors corresponding to statistical weights for emission and absorption. The next step is to rearrange the results in Eq. (5.2.13). We expand the whole expression and group the terms so that they have factors $n_B(k_2^0) - n_B(k_3^0)$, $n_B(k_2^0) - n_B(k_4^0)$. We drop the terms that have three delta functions ($\Delta_S(k_i)$) such that the delta-function-lines meet at a single vertex, since not all three δ functions be simultaneously satisfied (Eq. (5.1.8)).

$$\begin{aligned}\mathrm{Im}\Pi^{(a)}_{RA}(p) = -ig^4 \iint dk_1 dk_2 [&n_B(k_1^0)(n_B(k_2^0) - n_B(k_3^0))\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)- \\ &n_B(k_2^0)n_B(k_3^0)\Delta_A(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)+ \\ &n_B(k_5^0)(n_B(k_2^0) - n_B(k_4^0))\Delta_R(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)- \\ &n_B(k_2^0)n_B(k_4^0)\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_R(k_5)].\end{aligned}$$

We expand this relation again and take advantage of the relation for ϕ^3 theory in Eq. (5.1.1). After expanding we use Eq. (5.1.6) the fact that the terms without thermal factors.After some algebra we are left with:

$$\begin{aligned}
&\text{Im}\,\Pi^{(a)}_{RA}(p) = -ig^4 \iint dk_1 dk_2 [n_B(k_2^0)(-n_B(k_3^0) + n_B(k_4^0)) \\
&\Delta_P(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_R(k_5) + \\
&n_B(k_1^0)(-n_B(k_3^0) + n_B(k_4^0))\Delta_S(k_1)\Delta_P(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5) + \\
&n_B(k_5^0)(-n_B(k_3^0) + n_B(k_4^0))\Delta_R(k_1)\Delta_P(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)] \\
&= -ig^4 \iint dk_1 dk_2 [\frac{N(k_1^0)}{2}\mathcal{N}(k_3^0, -k_4^0)\Delta_S(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_R(k_5) + \\
&\frac{N(k_3^0)}{2}\mathcal{N}(k_3^0, -k_4^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5) + \\
&\frac{N(k_5^0)}{2}\mathcal{N}(k_3^0, -k_4^0)\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)]
\end{aligned} \tag{5.2.14}$$

The result for diagrams in Fig. (5.6b), (5.6c) and (5.6d) are obtained the same way and are given in Eqs. (5.2.15), (5.2.3.1) and (5.2.15) respectively. For Fig. (5.6d) we found it convenient to calculate the complex conjugate of the diagram which means that we can calculate the diagram where only the vertex g_1^{xaj} is circled. This gives $\Pi_{AR}(p)$ instead of $\Pi_{RA}(p)$ and then if we take the complex conjugate we obtain result

for Fig. (5.6d). It was done in both ways and the result is the same.

$$\begin{aligned}
\mathrm{Im}\Pi^{(b)}_{RA}(p) &= -ig^4 \iint dk_1 dk_2 [(n_B(k_5^0) - n_B(k_1^0))(n_B(k_2^0) - n_B(k_3^0))\Delta_R(k_1)(-\Delta_S(k_2)) \\
&\quad \Delta_S(k_3)\Delta_R(k_4)(-\Delta_S(k_5))] \\
&= -ig^4 \iint dk_1 dk_2 [1 + n_B(k_2^0) + n_B(k_3^0) + n_B(k_5^0) + n_B(k_2^0)n_B(k_3^0) + n_B(k_3^0)n_B(k_5^0) + \\
&\quad n_B(k_2^0)n_B(k_5^0)]\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5) \\
&= -ig^4 \iint dk_1 dk_2 [\mathcal{N}(k_2^0, k_3^0, -k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)]
\end{aligned} \tag{5.2.15}$$

$$\begin{aligned}
\mathrm{Im}\Pi^{(c)}_{RA}(p) &= ig^4 \iint dk_1 dk_2 [(n_B(k_5^0) - n_B(k_1^0))(n_B(k_4^0) - n_B(k_2^0))\Delta_S(k_1)\Delta_S(k_2) \\
&\quad \Delta_A(k_3)(-\Delta_S(k_4))\Delta_A(k_5)] \\
&= ig^4 \iint dk_1 dk_2 [\mathcal{N}(k_1^0, -k_2^0, -k_4^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_A(k_5)].
\end{aligned} \tag{5.2.16}$$

$$\begin{aligned}
\mathrm{Im}\Pi^{(d)}_{RA}(p) &= ig^4 \iint dk_1 dk_2 [(n_B(k_5^0) - n_B(k_1^0))(n_B(k_2^0) - n_B(k_3^0))\Delta_S(k_1)\Delta_R(k_2) \\
&\Delta_R(k_3)\Delta_R(k_4)(-\Delta_S(k_5))+ \\
&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_3^0) - n_B(p_0^0))\Delta_S(k_1)\Delta_R(k_2)\Delta_A(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \\
&(n_B(k_5^0) - n_B(k_1^0))(n_B(k_4^0) - n_B(k_2^0))\Delta_S(k_1)\Delta_A(k_2)\Delta_A(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \\
&(n_B(k_5^0) - n_B(k_1^0))(1 + n_B(p_0^0) + n_B(k_4^0))\Delta_S(k_1)\Delta_A(k_2)\Delta_A(k_3)\Delta_A(k_4)\Delta_S(k_5)] \\
&= ig^4 \iint dk_1 dk_2 [\frac{N(k_2^0)}{2}\mathcal{N}(k_1^0, k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \\
&\frac{N(k_3^0)}{2}\mathcal{N}(k_1^0, -k_5^0)\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \\
&\frac{N(k_4)}{2}\mathcal{N}(k_1^0, -k_5^0)\Delta_S(k_1)\Delta_A(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5)]
\end{aligned} \tag{5.2.17}$$

5.2.3.2 In the Keldysh basis

Again in Keldysh basis the second part of the calculation is simpler for the same reason. We do not have to work hard for the thermal factors, we just re-arrange the terms to get the physically interpretable terms. The expressions we get for self energy for the diagrams in Fig. (5.6) from the general equation (Eq. 5.2.12) are given in Eq. (5.2.18) - (5.2.21). For each diagram, the first equation is obtained by directly substituting the appropriate propagators and vertices in Eq. (5.2.14). The second

and the third diagram's expressions are obtained by simple rearranging.

$$
\begin{aligned}
\mathrm{Im}\Pi_{21}^{(a)}(p) = -i \iint dk_1 dk_2 \frac{g^4}{2^2} [&\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_S^T(k_4)\Delta_A(k_5) + \\
&\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)(-\Delta_S(k_4))\Delta_A(k_5) + \Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S^T(k_4)\Delta_S^T(k_5) + \\
&\Delta_R(k_1)\Delta_A(k_2)\Delta_S^T(k_3)(-\Delta_S(k_4))\Delta_S^T(k_5) + \Delta_S^T(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S^T(k_4)\Delta_A(k_5) + \\
&\Delta_S^T(k_1)\Delta_R(k_2)\Delta_S^T(k_3)(-\Delta_S(k_4))\Delta_A(k_5)] \\
= -i\frac{g^4}{4} \iint dk_1 dk_2 [&(1+2n_B(k_2^0))(1+2n_B(k_4^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5) - \\
&(1+2n_B(k_2^0))(1+2n_B(k_3^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5) + \\
&(1+2n_B(k_4^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5) - \\
&(1+2n_B(k_3^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5) + \\
&(1+2n_B(k_1^0))(1+2n_B(k_4^0))\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5) - \\
&(1+2n_B(k_1^0))(1+2n_B(k_3^0))\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)] \\
= -i\frac{g^4}{2} \iint dk_1 dk_2 [&(n_B(k_4^0) - n_B(k_3^0) + 2n_B(k_2^0)n_B(k_4^0) - 2n_B(k_3^0)n_B(k_5^0)) \\
&\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5) + \\
&(n_B(k_4^0) - n_B(k_3^0 + 2n_B(k_4^0)n_B(k_5^0)) - 2n_B(k_3^0)n_B(k_5^0)) \\
&\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5) + \\
&(n_B(k_4^0) - n_B(k_3^0) + 2n_B(k_4^0)n_B(k_5^0) - 2n_B(k_3^0)n_B(k_5^0)) \\
&\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)]
\end{aligned}
\tag{5.2.18}
$$

$$= -i\frac{g^4}{2}\iint dk_1 dk_2[(n_B(k_4^0) - n_B(k_3^0) + 2n_B(k_2^0)n_B(k_4^0) - 2n_B(k_3^0)n_B(k_5^0))$$

$$\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)+$$

$$(n_B(k_4^0) - n_B(k_3^0 + 2n_B(k_4^0)n_B(k_5^0)) - 2n_B(k_3^0)n_B(k_5^0))$$

$$\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)+$$

$$(n_B(k_4^0) - n_B(k_3^0) + 2n_B(k_4^0)n_B(k_5^0) - 2n_B(k_3^0)n_B(k_5^0))$$

$$\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)]$$

$$= -ig^4\iint dk_1 dk_2[\frac{N(k_1^0)}{2}\mathcal{N}(k_3^0, -k_4^0)\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)+$$

$$\frac{N(k_2^0)}{2}\mathcal{N}(k_3^0, -k_4^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)+$$

$$\frac{N(k_5^0)}{2}\mathcal{N}(k_3^0, -k_4^0)\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)]$$

$$
\begin{aligned}
\mathrm{Im}\Pi_{21}^{(b)}(p) = -i \iint dk_1 dk_2 \frac{g^4}{2^2}[&-\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_A^T(k_5)+ \\
&\Delta_R(k_1)(-\Delta_S(k_2))\Delta_S(k_3)\Delta_S^T(k_4)\Delta_A^T(k_5) + \Delta_R(k_1)(-\Delta_S(k_2))\Delta_S^T(k_3)\Delta_R(k_4)\Delta_A^T(k_5)+ \\
&\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_S^T(k_4)(-\Delta_A(k_5)) + \Delta_R(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)\Delta_R(k_4)(-\Delta_A(k_5))- \\
&\Delta_R(k_1)(-\Delta_S(k_2))\Delta_S(k_3)\Delta_A(k_4)(-\Delta_A(k_5))] \\
= -i\frac{g^4}{4}[&-(1+2n_B(k_2^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)- \\
&(1+2n_B(k_3^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \\
&(1+2n_B(k_2^0))(1+2n_B(k_3^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)- \\
&\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)] \\
= -ig^4 \iint dk_1 dk_2[&(1+n_B(k_2^0)+n_B(k_3^0)+n_B(k_5^0)+n_B(k_2^0)n_B(k_3^0)+n_B(k_3^0)n_B(k_5^0)+ \\
&n_B(k_2^0)n_B(k_5^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)] \\
= -ig^4 \iint dk_1 dk_2[&\mathcal{N}(k_2^0,k_3^0,-k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)]
\end{aligned}
\tag{5.2.19}
$$

$$\begin{aligned}
\mathrm{Im}\Pi_{21}^{(c)}(p) &= -i\iint dk_1 dk_2 \frac{g^4}{2^2}[-\Delta_S^T(k_1)\Delta_S^T(k_2)\Delta_R(k_3)(-\Delta_S(k_4))\Delta_A(k_5)+ \\
&\Delta_S^T(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_S^T(k_4)\Delta_A(k_5) + \Delta_S^T(k_1)\Delta_S(k_2)\Delta_S^T(k_3)(-\Delta_S(k_4))\Delta_A(k_5)- \\
&(-\Delta_S(k_1))\Delta_S^T(k_2)\Delta_A(k_3)\Delta_S^T(k_4)\Delta_A(k_5) + \Delta_S(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)(-\Delta_S(k_4))\Delta_A(k_5)- \\
&\Delta_S(k_1)\Delta_S(k_2)\Delta_R(k_3)(-\Delta_S(k_4))\Delta_A(k_5)] \\
&= -i\frac{g^4}{4}\iint dk_1 dk_2[(1+2n_B(k_1^0))(1+2n_B(k_2^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)- \\
&(1+2n_B(k_1^0))(1+2n_B(k_3^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \\
&(1+2n_B(k_2^0))(1+2n_B(k_4^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)] \\
&= -ig^4\iint dk_1 dk_2[\mathcal{N}(k_1^0,-k_2^0,-k_4^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_A(k_5)]
\end{aligned} \tag{5.2.20}$$

$$\mathrm{Im}\Pi_{21}^{(d)}(p) = -i\iint dk_1 dk_2 \frac{g^4}{2^2}[\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_S^T(k_4)\Delta_A(k_5)+$$

$$\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)\Delta_S(k_4)\Delta_A(k_5) + \Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S^T(k_4)\Delta_S^T(k_5)+$$

$$\Delta_R(k_1)\Delta_A(k_2)\Delta_S^T(k_3)\Delta_S(k_4)\Delta_S^T(k_5) + \Delta_S^T(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S^T(k_4)\Delta_A(k_5)+$$

$$\Delta_S^T(k_1)\Delta_R(k_2)\Delta_S^T(k_3)\Delta_S(k_4)\Delta_A(k_5)]$$

$$= -i\frac{g^4}{4}\iint dk_1 dk_2[(1+2n_B(k_2^0))(1+2n_B(k_4^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)-$$

$$(1+2n_B(k_2^0))(1+2n_B(k_3^0))\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)+$$

$$(1+2n_B(k_4^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)-$$

$$(1+2n_B(k_3^0))(1+2n_B(k_5^0))\Delta_R(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)+$$

$$(1+2n_B(k_1^0))(1+2n_B(k_4^0))\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)-$$

$$(1+2n_B(k_1^0))(1+2n_B(k_3^0))\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)]$$

$$= -i\frac{g^4}{2}\iint dk_1 dk_2[(n_B(k_1^0) - n_B(k_5^0) + 2n_B(k_1^0)n_B(k_3^0) - 2n_B(k_3^0)n_B(k_5^0))$$

$$\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_A(k_5)+$$

$$(n_B(k_1^0) - n_B(k_5^0) + 2n_B(k_1^0)n_B(k_2^0) - 2n_B(k_2^0)n_B(k_5^0))$$

$$\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_R(k_4)\Delta_S(k_5)+$$

$$(n_B(k_1^0) - n_B(k_5^0) + 2n_B(k_1^0)n_B(k_4^0) - 2n_B(k_4^0)n_B(k_5^0))$$

$$\Delta_S(k_1)\Delta_A(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5)]$$

$$= -ig^4\iint dk_1 dk_2[\frac{N(k_2^0)}{2}\mathcal{N}(k_1^0, -k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \tag{5.2.21}$$

$$\frac{N(k_3^0)}{2}\mathcal{N}(k_1^0,-k_5^0)\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)+$$
$$\frac{N(k_4)}{2}\mathcal{N}(k_1^0,-k_5^0)\Delta_S(k_1)\Delta_A(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5)]$$

5.2.4 Result for two-loop vertex diagram

The results in Eq. (5.2.18)-(5.2.21) agree with the corresponding expressions (5.2.14)-(5.2.17) in section-5.2.3.1 from the R/A basis. Collecting results we have for the vertex diagram:

$$\begin{aligned}
&\text{Im } [\Pi^a_{ret}(p)+\Pi^b_{ret}(p)+\Pi^c_{ret}(p)+\Pi^d_{ret}(p)] = \\
&-ig^4\iint dk_1dk_2[\frac{N(k_1^0)}{2}\mathcal{N}(k_3^0,-k_4^0)\Delta_S(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_R(k_5)+ \\
&\frac{N(k_3^0)}{2}\mathcal{N}(k_3^0,-k_4^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_A(k_5)+ \\
&\frac{N(k_5^0)}{2}\mathcal{N}(k_3^0,-k_4^0)\Delta_R(k_1)\Delta_P(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)+ \\
&\mathcal{N}(k_2^0,k_3^0,-k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5)+ \\
&\mathcal{N}(k_1^0,-k_2^0,-k_4^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)(-\Delta_S(k_4))\Delta_A(k_5)+ \\
&\frac{N(k_2^0)}{2}\mathcal{N}(k_1^0,-k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \\
&\frac{N(k_3^0)}{2}\mathcal{N}(k_1^0,-k_5^0)\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)+ \\
&\frac{N(k_4)}{2}\mathcal{N}(k_1^0,-k_5^0)\Delta_S(k_1)\Delta_A(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5)].
\end{aligned} \tag{5.2.22}$$

These eight terms of Eq. (5.2.22) correspond to the five possible ways to open all the loops by cutting and ticing lines, as explained in section-4.5. These are shown diagrammatically in Fig. (5.7).

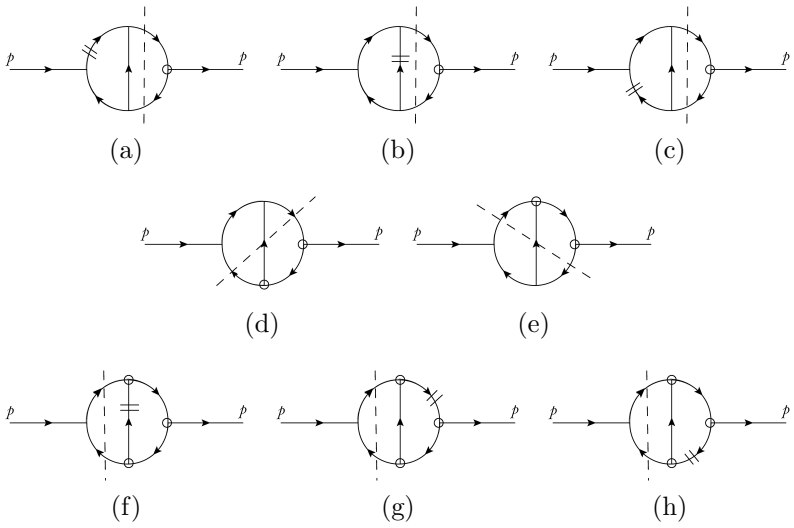


Figure 5.7: Imaginary part of the retarded self-energy for the *vertex diagram* with cut lines and tic-ed lines.

5.3 Calculation of production rate

Now we add up the diagrams in two-loop level for ϕ^3 interaction that has all the loops opened by the cut line (we call such a cut line the *central* cut). These diagrams are diagrams in Figs. (5.3b), (5.6b), (5.6c). We want to have k_2, k_3, and k_5 as on-shell, and we want to keep $p + k_5 = k_2 + k_3$. In Fig. (5.6b) we shift $k_2 \to -k_3$. In Fig. (5.6c) we shift $k_1 \to -k_5$, $k_3 \to -k_1$, $k_4 \to -k_3$, $k_5 \to -k_4$, $k_2 \to -k_2$. Next we re-label in a way so that we always have $p + p' = l + l'$, where p is the momenta of the external particle (the only on-shell particle before loops are opened) and p', l, l' are the new labels for the momentum of the cut propagators. In this way we can have three distinct choices:

1. $k_2 \to l$, $k_3 \to l'$, $k_5 \to p'$,

2. $k_2 \to l'$, $k_3 \to -p'$, $k_5 \to l$,

3. $k_2 \to -p'$, $k_3 \to l$, $k_5 \to -l'$.

We re-introduce the principle part as it is define in Eq. (5.1.5) to simplify our expressions. After summing the three individual choices and doing some algebra we are left with:

$$\text{Im}\ \left[\Pi_{ret}^{(b_{self-energy})}(p) + \Pi_{ret}^{(b_{vertex})}(p) + \Pi_{ret}^{(c_{vertex})}(p)\right] =$$
$$\frac{-ig^4}{3}\iint dk_1 dk_2 \mathcal{N}(p'^0, -l^0, l'^0)\Delta_S(l)\Delta_S(l')\Delta_S(p')[\Delta_P(-l+p) + \Delta_P(-l'+p) + \Delta_P(p+p')]^2. \tag{5.3.1}$$

Now, $[\Delta_P(p+p') + \Delta_P(-l+p) + \Delta_P(-l'+p)]^2 = |\mathcal{M}|^2$, where $\mathcal{M}$ is define in Eq. (2.1.4). This gives us the diagrams in Fig. (5.8). which represent the s, t, and u

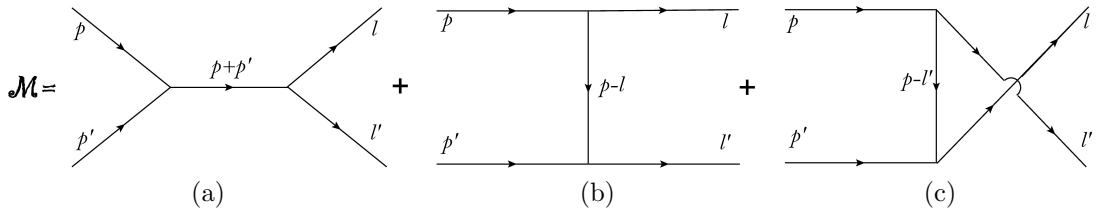


Figure 5.8: The final tree level diagrams presenting scattering amplitudes.

channels for 2-2 scattering and production process [3].

5.4 Two-loop diagram for ϕ^4-theory

Calculating any diagram for ϕ^4 theory, in any basis, is very difficult for a number of reasons. Firstly, the sum over internal vertices becomes enormous and secondly, the relation in Eq. (5.1.4) that we use to simplify and extract physical information from each diagram becomes larger and thus the replacement operation appears to be more and more complicated. But calculating the diagram in Fig. (5.9)is quite easy and very clearly shows the advantage of R/A basis over the Keldysh basis which for the previous two-loop examples for ϕ^3 interaction found to be the most convenient choice.

The general expression for imaginary part of the retarded self energy is given by:

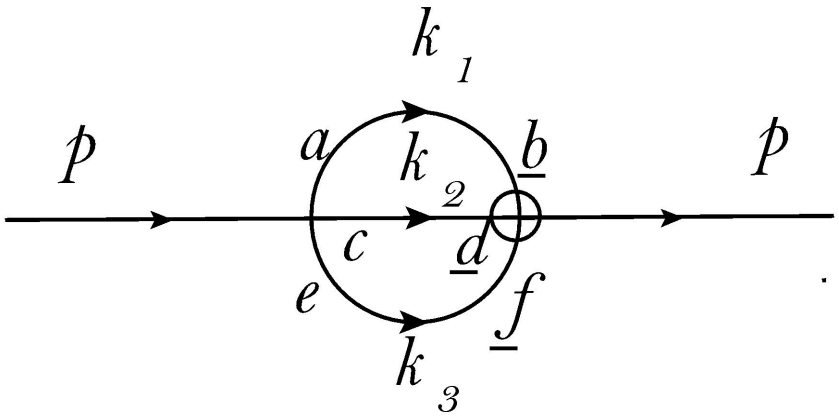


Figure 5.9: Two-loop diagram for ϕ^4 theory.

$$\text{Im } \Pi_{RA}(p) = \iint dk_1 dk_2 [g_1^{xace} g_2^{x'bdf} \Delta_{a\underline{b}}(k_1) \Delta_{c\underline{d}}(k_2) \Delta_{e\underline{f}}(k_3)]. \tag{5.4.1}$$

5.4.1 R/A basis

Following the same strategies of last section, from the general expression of Eq. (5.4.1) we can write the imaginary part of the retarded self energy in the R/A basis as:

$$
\begin{aligned}
&\mathrm{Im}\Pi_{RA}(p) = -i \iint dk_1 dk_2 [g_1^{RAAA} g_2^{ARRR} \Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3)] \\
&= -ig^2 \iint dk_1 dk_2 (1 + n_B(k_1^0) + n_B(k_2^0) + n_B(k_3^0) + n_B(k_1^0) n_B(k_2^0) + \\
&\quad n_B(k_2^0) n_B(k_3^0) + n_B(k_1^0) n_B(k_3^0)) \Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3) \\
&= -ig^2 \iint dk_1 dk_2 ((1 + n_B(k_1^0))(1 + n_B(k_2^0))(1 + n_B(k_3^0)) - n_B(k_1^0) n_B(k_2^0) n_B(k_3^0)) \\
&\quad \Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3) \\
&= -ig^2 \iint dk_1 dk_2 [\mathcal{N}(k_1^0, k_2^0, k_3^0) \Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3)].
\end{aligned}
\tag{5.4.2}
$$

5.4.2 Keldysh basis

Similarly in the Keldysh basis we have:

$$\begin{aligned}
\mathrm{Im}\Pi_{21}(p) &= -i \iint dk_1 dk_2 [g_1^{2111} g_2^{1112} \Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3) + g_1^{2111} g_2^{1121} \Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3) \\
&\quad + g_1^{2111} g_2^{1211} \Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3) + g_1^{2111} g_2^{1222} \Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3)] \\
&= -ig^2 \iint dk_1 dk_2 (1 + n_B(k_1^0) + n_B(k_2^0) + n_B(k_3^0) + n_B(k_1^0) n_B(k_2^0) + \\
&\quad n_B(k_2^0) n_B(k_3^0) + n_B(k_1^0) n_B(k_3^0))[\Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3)] \\
&= -ig^2 \iint dk_1 dk_2 ((1 + n_B(k_1^0))(1 + n_B(k_2^0))(1 + n_B(k_3^0)) - n_B(k_1^0) n_B(k_2^0) n_B(k_3^0)) \\
&\quad [\Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3)] \\
&= -ig^2 \iint dk_1 dk_2 [\mathcal{N}(k_1^0, k_2^0, k_3^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3)].
\end{aligned} \tag{5.4.3}$$

So we can see that Eq. (5.4.3) agrees with Eq. (5.4.2). The two-loop diagram in the ϕ^4 can be re-drawn as shown in Fig. (5.10). This term corresponding to the one and only possible way to open the loops by cut line as explained in section-5.1. It is

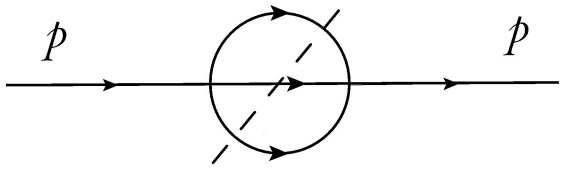


Figure 5.10: Two-loop diagram for ϕ^4 theory.

trivial to see that the results (in the R/A basis Eq. (5.4.2) which is the same as Eq. (5.4.3) in the Keldysh basis) has the form of Eq. (4.1.1).

5.5 Three-loop diagram for ϕ^4 theory

We were also able to show at three-loop level that the diagrams in which the circled or uncircled vertices do not form a connected region as well as the diagrams where we have circled and uncircled *island* of vertices vanish in the R/A and Keldysh basis. We picked a simple three-loop diagram as in Fig. (5.11) and calculated in R/A and Keldysh basis. The imaginary part of the three-loop diagram in Fig. (5.11) is given

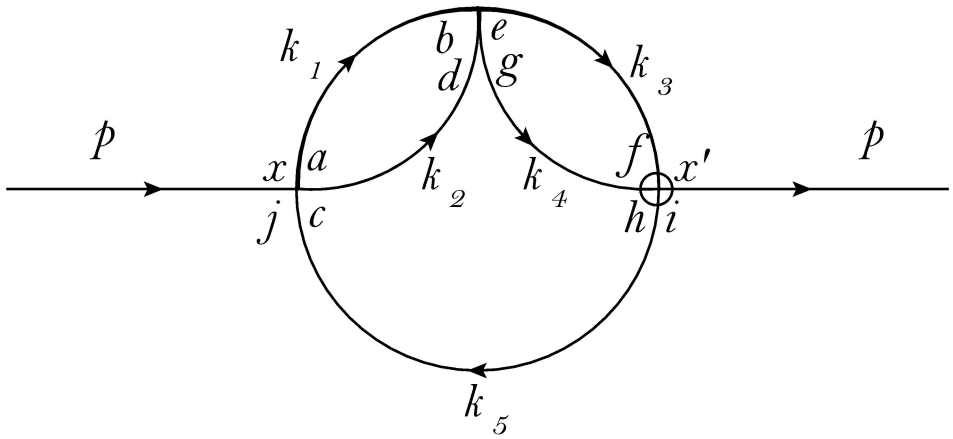


Figure 5.11: A three-loop self-energy diagram.

by the sum of the diagrams in Fig. (5.12) Following the same way of ϕ^3-theory we

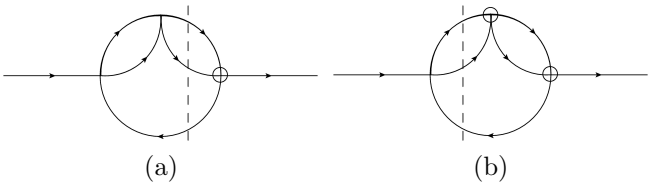


Figure 5.12: Imaginary part of the retarded self-energy for a three-loop diagram.

now write the expressions for the two diagrams in Fig. (5.12) in the next two sections.

5.5.1 In R/A basis

$$\begin{aligned}
&\mathrm{Im}\Pi^{(a)}_{RA}(p) = -g^4 \iint dk_1 dk_2 [(n_B(k_1^0) - n_B(k_2^0))(n_B(k_3^0) - n_B(k_4^0))\Delta_A(k_1)\Delta_R(k_2) \\
&\quad \Delta_S(k_3)\Delta_S(k_4)(-\Delta_S(k_5)) + \\
&\quad (n_B(k_1^0) - n_B(k_3^0))(n_B(k_2^0) - n_B(k_4^0))\Delta_A(k_1)\Delta_A(k_2)\Delta_S(k_3)\Delta_S(k_4)(-\Delta_S(k_5))] \\
&= g^4 \iint dk_1 dk_2 [\frac{N(k_1^0)}{2}\mathcal{N}(k_3^0, k_4^0, -k_5^0)\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5) + \\
&\quad \frac{N(k_2^0)}{2}\mathcal{N}(k_3^0, k_4^0, -k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)].
\end{aligned} \tag{5.5.1}$$

$$\begin{aligned}
&\mathrm{Im}\Pi^{(b)}_{RA}(p) = -g^4 \iint dk_1 dk_2 [(n_B(k_1^0) - n_B(k_2^0))(n_B(k_3^0) - n_B(k_4^0))\Delta_S(k_1)\Delta_S(k_2) \\
&\quad \Delta_A(k_3)\Delta(k_4)(-\Delta_S(k_5)) + \\
&\quad (n_B(k_1^0) - n_B(k_3^0))(n_B(k_2^0) - n_B(k_4^0))\Delta_A(k_1)\Delta_A(k_2)\Delta_R(k_3)\Delta_S(k_4)(-\Delta_S(k_5))] \\
&= g^4 \iint dk_1 dk_2 [\frac{1}{2}((n_B(k_4^0)(1 + n_B(k_5^0)) - (1 + n_B(k_4^0))n_B(k_5^0))(1 + 2n_B(k_2^0)))\Delta_S(k_1) \\
&\quad \Delta_S(k_2)\Delta_R(k_3)\Delta(k_4)\Delta_S(k_5) + \\
&\quad \frac{1}{2}(n_B(k_4^0)(1 + n_B(k_5^0)) - n_B(k_4^0)n_B(k_3^0))(1 + 2n_B(k_3^0))\Delta_S(k_1) \\
&\quad \Delta_S(k_2)\Delta_3(k_2)\Delta(k_4)\Delta_S(k_5)] \\
&= g^4 \iint dk_1 dk_2 [\frac{N(k_3^0)}{2}\mathcal{N}(k_1^0, k_2^0, -k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5) + \\
&\quad \frac{N(k_4)}{2}\mathcal{N}(k_1^0, k_2^0, -k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5)].
\end{aligned} \tag{5.5.2}$$

5.5.2 In Keldysh basis

$$
\begin{aligned}
\mathrm{Im}\Pi_{21}^{(a)}(p) = \frac{g^4}{2^3}\iint dk_1 dk_2 [&\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_S(k_4)(-\Delta_S(k_5))+ \\
&\Delta_R(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5)) + \Delta_R(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)\Delta_S(k_4)\Delta_S^T(k_5) \\
&+ \Delta_R(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_S^T(k_4)\Delta_S^T(k_5) + \Delta_S^T(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)(-\Delta_S(k_5))+ \\
&\Delta_S^T(k_1)\Delta_R(k_2)\Delta_S^T(k_3)\Delta_S^T(k_4)(-\Delta_S(k_5)) + \Delta_S^T(k_1)\Delta_R(k_2)\Delta_S^T(k_3)\Delta_S(k_4)\Delta_S^T(k_5)+ \\
&\Delta_S^T(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S^T(k_4)\Delta_S^T(k_5)] \\
= -g^4 \iint dk_1 dk_2 [&\frac{N(k_1^0)}{2}\mathcal{N}(k_3^0, k_4^0, -k_5^0)\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)+ \\
&\frac{N(k_2^0)}{2}\mathcal{N}(k_3^0, k_4^0, -k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5)].
\end{aligned}
\tag{5.5.3}
$$

The calculation of the second diagram is not as straight forwards as the first one. In the first step we get total 18 terms out of which only two terms can be dropped as they contain four delta (δ) functions meeting at the same point (see Eq. (5.1.8)), another four terms can be dropped because of forming loop of either retarded or advanced type propagators (see Eq. (5.1.6)). In the second step we replace some retarded (advanced) type propagators with a combination of delta (δ) functions and advanced (retarded) propagators using Eq. (5.1.9) and thus drop few more terms as they either contain four delta (δ) functions or form a loop as before. At the end we

are left with the terms on the rhs of Eq. (5.5.4).

$$
\begin{aligned}
\mathrm{Im}\Pi_{21}^{(b)}(p) = \frac{g^4}{2^3}\iint dk_1 dk_2 [&\Delta_S^T(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)\Delta_R(k_4)\Delta_S(k_5) + \\
&\Delta_S(k_1)\Delta_S(k_2)\Delta_S^T(k_3)\Delta_R(k_4)\Delta_S(k_5) - \\
&\Delta_S^T(k_1)\Delta_S(k_2)\Delta_S^T(k_3)\Delta_R(k_4)\Delta_S^T(k_5) - \Delta_S(k_1)\Delta_S^T(k_2)\Delta_S^T(k_3)\Delta_R(k_4)\Delta_S^T(k_5) + \\
&\Delta_S(k_1)^T\Delta_S^T(k_2)\Delta_R(k_3)\Delta_S^T(k_4)\Delta_S(k_5) + \Delta_S(k_1)\Delta_S(k_2)\Delta_R(k_3)\Delta_S^T(k_4)\Delta_S(k_5) - \\
&\Delta_S^T(k_1)\Delta_S(k_2)\Delta_R(k_3)\Delta_S^T(k_4)\Delta_S(k_5)^T - \Delta_S(k_1)\Delta_S^T(k_2)\Delta_R(k_3)\Delta_S^T(k_4)\Delta_S^T(k_5) \\
&\Delta_S^T(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5) + \Delta_S(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5) - \\
&\Delta_S(k_1)\Delta_S^T(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5) - \Delta_S^T(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_R(k_4)\Delta_S(k_5)] \\
= -g^4\iint dk_1 dk_2 [&\frac{N(k_3^0)}{2}\mathcal{N}(k_1^0, k_2^0, -k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5) + \\
&\frac{N(k_4)}{2}\mathcal{N}(k_1^0, k_2^0, -k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5)].
\end{aligned}
\tag{5.5.4}
$$

So we can see that this expression of Eq. (5.5.4) is the identical to its R/A counterpart Eq. (5.5.3). Adding Eqs. (5.5.4) and (5.5.3) we get Eq. (5.5.5) as:

$$
\begin{aligned}
\mathrm{Im}\ [\Pi_{ret}^a(p) + \Pi_{ret}^b(p)] = \\
-g^4\iint dk_1 dk_2 [&\frac{N(k_1^0)}{2}\mathcal{N}(k_3^0, k_4^0, -k_5^0)\Delta_S(k_1)\Delta_R(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5) + \\
&\frac{N(k_2^0)}{2}\mathcal{N}(k_3^0, k_4^0, -k_5^0)\Delta_R(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_S(k_4)\Delta_S(k_5) + \\
&\frac{N(k_3^0)}{2}\mathcal{N}(k_1^0, k_2^0, -k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_S(k_3)\Delta_A(k_4)\Delta_S(k_5) + \\
&\frac{N(k_4^0)}{2}\mathcal{N}(k_1^0, k_2^0, -k_5^0)\Delta_S(k_1)\Delta_S(k_2)\Delta_A(k_3)\Delta_S(k_4)\Delta_S(k_5)].
\end{aligned}
\tag{5.5.5}
$$

These four terms of Eq. (5.5.5) correspond to the four possible ways to open all loops by cutting and ticing lines as explained in section-4.5. Diagrammatically these four terms are shown in Fig. (5.13)

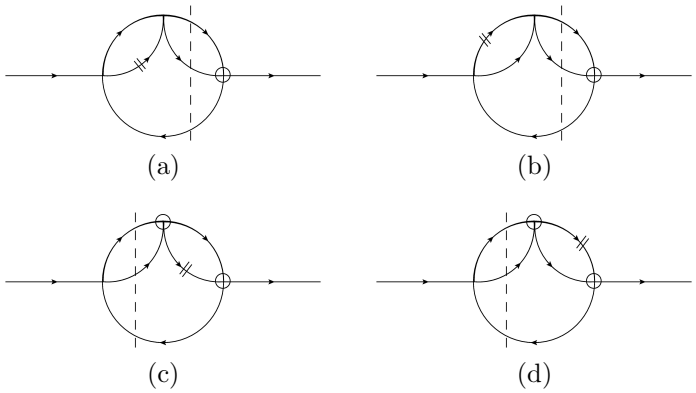


Figure 5.13: Final diagrams for Fig. (5.12a) (a-b) and (5.12b) (c-d).

5.6 Other three-loop diagrams

We were also able to check explicitly for the diagrams in Fig. (5.14) that the diagrams with isolated circled vertex or isolated circled island or isolated uncircled vertex or isolated uncircled island (see Chapter-6 for the explanation of these terms), vanish identically in R/A and Keldysh basis. Equivalently, each contribution to the imaginary part has one and only one cut line which divides the diagram into the product of two amplitudes. Using the techniques developed in this thesis, one can write a symbolic program to calculate any diagram. We are writing such a program that will be able to calculate all the diagrams in Fig. (5.14) and may be developed further for more complex diagrams. If we want to write the expressions for three-loop diagrams in the form of Eq. (4.1.1) we require to calculate all the three-loop diagram of ϕ^3. This work is done in Ref. [41], (which is our paper in preparation).

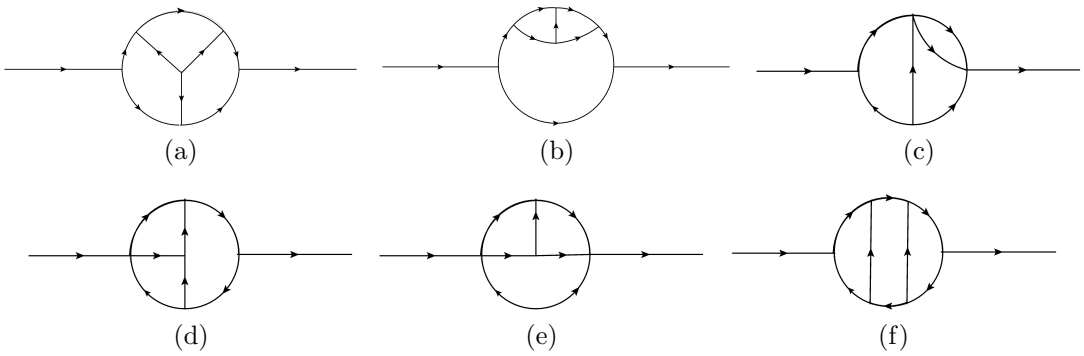
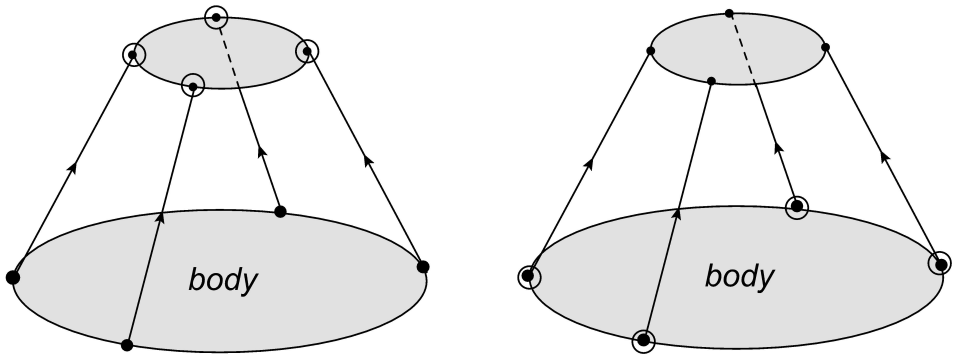


Figure 5.14: Other three-loop diagrams of ϕ^3 and ϕ^4-interaction.

Chapter 6

Cutting rules

In principle, the circling rules allow contributions to the imaginary part from diagrams that can not be divided into two pieces by one cut line. The diagram in Fig. (5.2d) is one example. As explained in section-4.5, we must be able to write the imaginary part as the product of two amplitudes. In fact, one can show that any diagram with this structure (uncuttable) is identically zero in the R/A and Keldysh basis. This is the subject of this chapter.



(a) Isolated circled island (b) Isolated uncircled island

Figure 6.1: Isolated circled and uncircled island.

Any diagram that cannot be divided into two pieces will contain an isolated island of circled or uncircled vertices. For example, in Fig. (5.2d), the circled vertex on the left hand side is isolated. In general this is shown in Fig. (6.1). A group of vertices circled or uncircled which can be seperated from the rest of the diagram is called an isolated island. The island could be just one vertex. Then we call it an isolated circled (or uncircled) vertex. We work in the R/A basis and it is straight forward to show that any isolated island of circled or uncircled vertices gives zero. Same calculation could be done for the Keldysh basis.

However, in the primary basis this is not true. In the primary basis we have different way for interpretation of those diagrams and we have less diagrams to consider but with the cost of *uncuttable* diagrams. The basic trick is that, the sum over internal indices is carried out by the circling itself. So, one does not have to perform the sum over internal indices of the vertices. This makes the whole calculation extremely easy. But we are interested in the R/A and Keldysh basis as in these bases the results are similar to that of zero temperature and matches with the simpler cutting rules given by Kobes, Carrington and Defu [11].

The difficulty in the R/A and Keldysh basis that the circlings do not absorb the sum over the internal vertices and this results in a very huge sums (an effective method of simplifying the sum is given in the appendices for two examples).

In R/A basis From Eq. (4.2.2) a propagator with one end circled and one uncircled has only one nonzero component: the circled end is always "R" and the uncircled end is always "A". This means that any isolated island of circled vertices can be written as an effective vertex $\Gamma^{RR\ldots R}$, and any isolated island of uncircled vertices can be written as an effective vertex $\Gamma^{AA\ldots A}$. Using Eq. (3.2.6) the contribution of these terms is always zero. Physically, these term vanish by the causality law as it is not possible

for all the particles to come out of nothing (without any field) or to vanish at a point leaving no trace (a field). This is clear from Eq. (3.2.6) and also from the examplse that we have done. We have generalized this fact in this thesis.

Chapter 7

Conclusion

In this thesis we have developed a set of cutting rules to calculate the imaginary part of the retarded self energy. We have carried out numerous examples. Doing the sum over internal indices is easiest in the R/A basis because of the zeros in the propagator. Arranging the thermal factors in the physical form is easier in the Keldysh basis. One important thing is that the imaginary part can always be written as the product of two amplitudes. We have to have this to write the rate as proportional to the imaginary part of Π^{ret} as in Eq. (2.1.4). In the primary basis this is very hard to see but in R/A and Keldysh basis it is simple. At higher loop orders, calculation like the ones described in this thesis become increasingly complicated.

So we can conclude that the calculation coule be performed in peaces; for the island argument we can use the R/A basis as it has more zeros in its propagator matrices and for servivors we can use the Keldysh basis so that extracting physics become simpler.

However, using the strategies developed in this thesis, it will be possible to develop a symbolic program which will calculate the imaginary part of the retarded self

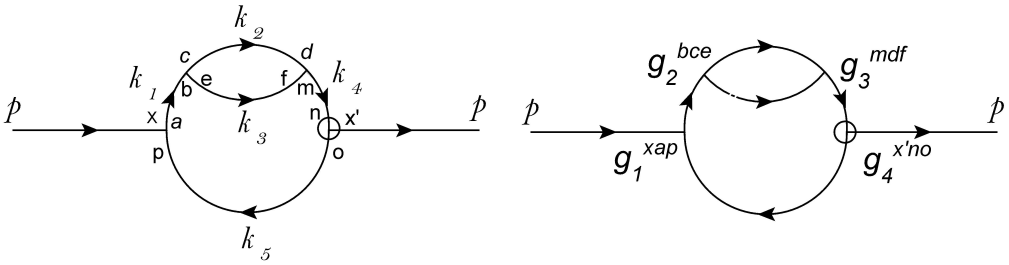
energy to much higher loop order. The first step involves the summation over R/A or Keldysh indices. This is essentially just matrix multiplication and can obviously be done by computer. The second step involves using the products of cut lines and the appropriate thermal factors. The development of a program is currently in progress.

Appendix A

Two-loop self-energy diagram

A.1 Allowed vertices for self-energy diagram in the R/A basis

For convenience and also to show the labeling of the vertices explicitly we re-draw the diagram of Fig. (5.1) in Fig. (A.1).



(a) Momentum and internal index labeled. (b) Vertex labeled.

Figure A.1: Two-loop self energy diagram for the single circling.

Step: 1

We note that, when calculating the retarded part, we always have $x = R$ and $x' = A$ in R/A basis. This gives (for the diagram in Fig. (A.1) with the help of the propagator matrices of Eq. (4.2.2)) $p = A, m = a, n = R, o = R$ *only* . Basing on these information we draw the table in Fig. (A.2). For all other indices we put "*"s which means that they may be replaced by either by an "A" or by an "R".

$$g_1^{xap} = g_1^{R*A}$$

$$g_2^{bce} = g_2^{***}$$

$$g_3^{mdf} = g_3^{A**}$$

$$g_4^{x'no} = g_4^{ARR}$$

Figure A.2: First step on simplifying the sum in the R/A basis for self energy diagram.

Step: 2

In the second step we replace all the "*"s by all the combinations of "R" and "A" except g^{AAA} and g^{RRR} which are identically zero. This is shown in Fig. (A.3). Each of the terms we get this way are what we will call *allowed* vertices.

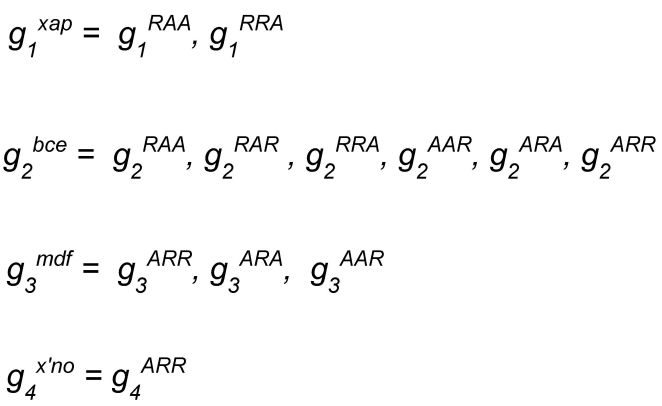


Figure A.3: Second step on simplifying the sum in the R/A basis for self energy diagram.

Step: 3

In the third step we draw the lines that connect the vertices which indicate the flow of momentum. Not all vertices can be connected as the momentum can flow only between the vertices that are connected by the matrix structure of the propagator matrix connecting the two vertices under consideration. The diagram in Fig. (A.4) shows how this is done. In ϕ^3-theory, each vertex could be connected to up to three partners. Any vertex that DOES NOT have appropriate index to be completely connected to all partners must be dropped from the table. We use the fact that the propagator matrix contains only off diagonal elements. From Fig. (A.4), the propagator $\Delta_{ab}(k_1)$ connects the middle index of g_1, to the first index of g_2 and the lines in the first level of the table show possible ways to connect the vertices. So that $\Delta_{ab}(k_1) = \Delta_{RA}(k_1), \Delta_{AR}(k_1)$. The vertices g_2 and g_3 are connected by the propagators $\Delta_{cd}(k_2)$ and $\Delta_{ef}(k_3)$. the arrows in the second level of the table show the possible ways to do this. Note that the vertex g_2^{ARR} drops out at this point since the propagator $\Delta_{cd}(k_2)$ connects only to g_3^{AAR} and the propagator $\Delta_{ef}(k_3)$ connects

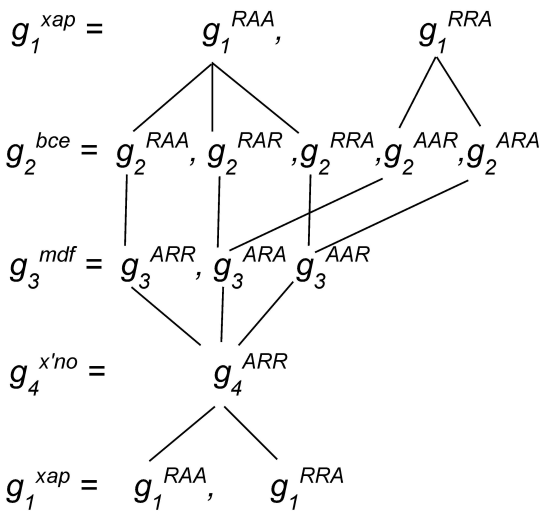


Figure A.4: Vertices in the R/A basis for self energy diagram from step: 3.

only to g_3^{ARR} and g_3^{ARA} (i.e. the other two). The vertices g_3 and g_4 are connected by means of the propagator $\Delta_{m\underline{n}}(k_4)$ and all three vertices of g_3 can be connected to g_4. The connection between g_1 and g_4 is shown by redrawing g_1 at the bottom of the table.

Step: 4

At the end we write the possible combination of all the vertices that are *valid* according to the steps 1 through 3. From the diagram in Fig. (A.4), we start from each vertex "1" and following any path to the same vertex at the bottom of the table. The expression for the self energy is the sum of all possible paths. For the diagram in Fig.

(A.1) we have:

$$
\begin{aligned}
&[\mathbf{a}] \quad g_1^{RAA} g_2^{RAA} g_3^{ARR} g_4^{ARR}, \\
&[\mathbf{b}] \quad g_1^{RAA} g_2^{RAR} g_3^{ARA} g_4^{ARR}, \\
&[\mathbf{c}] \quad g_1^{RAA} g_2^{RRA} g_3^{AAR} g_4^{ARR}, \\
&[\mathbf{d}] \quad g_1^{RRA} g_2^{AAR} g_3^{ARA} g_4^{ARR}, \\
&[\mathbf{e}] \quad g_1^{RRA} g_2^{ARA} g_3^{AAR} g_4^{ARR}.
\end{aligned}
$$

These five paths give the five terms in Eq. (5.2.4).

A.2 Allowed vertices for self-energy diagram in the Keldysh basis

On performing the sum the basic idea is the same in the R/A and Keldysh basis. Here we show without repeating the detailed discussion only the major steps how the vertices can be linked in Keldysh basis for a self energy diagram.

Step: 1

We put the values for the indices of all the vertices which are fixed to be either "1" or "2" with help of propagator matrix in Eqs. (4.2.3). For the diagram in Fig. (A.1) we have $x = 2, x' = 1$ and $p = 1, m = 1$ and all other being free to take values "1" or "2". This is shown in Fig. (A.5).

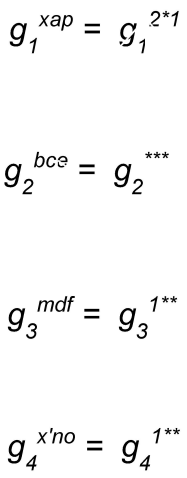


Figure A.5: Vertices in the Keldysh basis with single valued indices.

Step: 2

In the second step we replace the "*"s with all the possible choices of "1" and "2" except when the number of "2"s in any vertex is zero or any other even number since they those are zero. Fig. (A.6) shows the result of this step.

Step: 3

In the third step we connect (as shown in Fig. (A.7)) the vertices that are allowed by the propagator matrix in Eq. (4.2.3).

Step: 4

At the end we find the possible paths by traveling from the first vertex to the last one and back to the same first one again. For the Fig. (A.1) the allowed paths are as

$$g_1^{xap} = g_1^{211}$$

$$g_2^{bce} = g_2^{211}, g_2^{112}, g_2^{121}, g_2^{222}$$

$$g_3^{mdf} = g_3^{121}, g_3^{112}, g_3^{211}$$

$$g_4^{x'no} = g_4^{121}, g_4^{112}$$

Figure A.6: Vertices in the Keldysh basis with all possible choice of index.

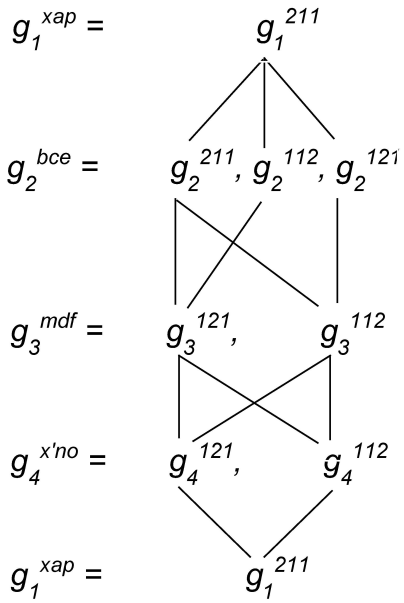


Figure A.7: Vertices in the Keldysh basis for self energy diagram.

follows: Traveling from the first vertex to the last one and back to the same first one again. For the Fig. (A.1) the allowed paths are as follows:

$$[\mathbf{a}] \quad g_1^{211} g_2^{211} g_3^{121} g_4^{121},$$

$$[\mathbf{b}] \quad g_1^{211} g_2^{211} g_3^{121} g_4^{112},$$

$$[\mathbf{c}] \quad g_1^{211} g_2^{211} g_3^{112} g_4^{121},$$

$$[\mathbf{d}] \quad g_1^{211} g_2^{211} g_3^{112} g_4^{112},$$

$$[\mathbf{e}] \quad g_1^{211} g_2^{112} g_3^{121} g_4^{121},$$

$$[\mathbf{f}] \quad g_1^{211} g_2^{112} g_3^{121} g_4^{112},$$

$$[\mathbf{g}] \quad g_1^{211} g_2^{121} g_3^{112} g_4^{121},$$

$$[\mathbf{h}] \quad g_1^{211} g_2^{121} g_3^{112} g_4^{112}.$$

Thus we are left with only eight possible terms after performing the sum over internal indices. These possible terms appear in Eq. (5.2.7).

Appendix B

Two-loop vertex diagram

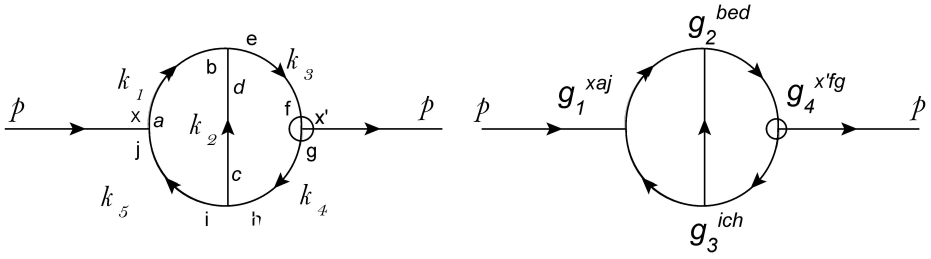
B.1 Allowed vertices for the vertex diagram in the R/A basis

For vertex diagram the strategy is the same as for self-energy diagram. But drawing the momentum flow diagram is a little complex as some vertices are connected to three vertices while in self-energy diagram it did not exceed two. Here we re-draw the diagram of Fig. (5.5) in Fig. (B.1) with a single circling and show the strategy to simplify the sum over the internal vertices.

Step: 1

In the first step we put the values of the indices that take one and only one value. The indices that can take more than one value and are not fixed from the propagator matrix which value they can take are presented with "*"s at their positions in the diagram of Fig. (B.2).

Step: 2



(a) Momentum and internal indices labeled.

(b) Vertex labeled.

Figure B.1: Two-loop vertex diagram for the single circling.

$$g_1^{xaj} = g_1^{R**}$$

$$g_2^{bed} = g_2^{*A*}$$

$$g_3^{ich} = g_3^{**A}$$

$$g_4^{x'fg} = g_4^{ARR}$$

Figure B.2: Vertices in the R/A basis for vertex diagram.

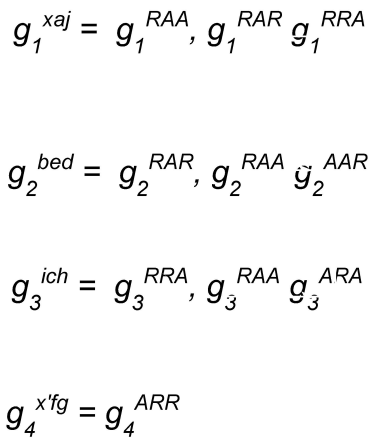


Figure B.3: Vertices in the R/A basis for vertex diagram from Fig. (B.2).

In the second step we replace all the "*" with either "R" or "A" except for the terms where we have three "A"s or three "R"s i.e g^{AAA} or g^{RRR} which are identically zero. This is shown in Fig. (B.3).

Step: 3

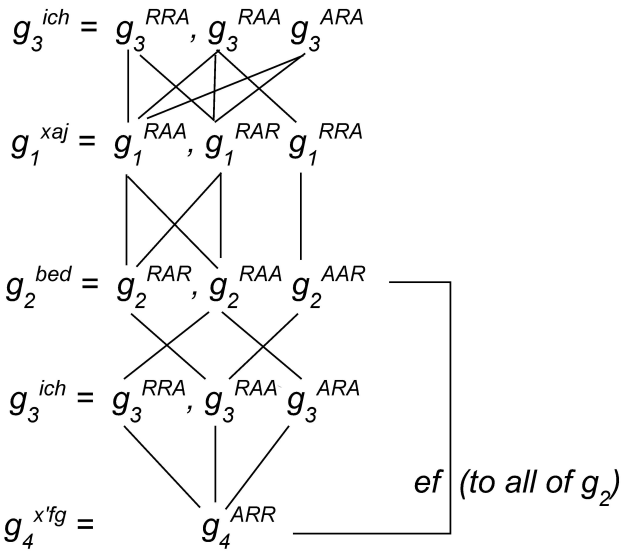


Figure B.4: Vertices in the R/A basis for vertex diagram.

Once we have all the possible vertices from step: 2, we now connect the vertices with lines that show the possible momentum flow along them. Any vertex that is not joined in a connected path must be dropped in step: 3.

Step: 4

At this step we rewrite the vertex combinations that correspond to connected paths. We note that the vertex g_3 on the first line and g_3 on the fourth line are identical and when crossing the one on the fourth line we must *go through* the same g_3 that

we started from on the first line.

$$[\mathbf{a}] \quad g_1^{RAA} g_2^{RAA} g_3^{RRA} g_4^{ARR},$$

$$[\mathbf{b}] \quad g_1^{RAA} g_2^{RAR} g_3^{RAA} g_4^{ARR},$$

$$[\mathbf{c}] \quad g_1^{RRA} g_2^{AAR} g_3^{RAA} g_4^{ARR},$$

$$[\mathbf{d}] \quad g_1^{RAR} g_2^{RAA} g_3^{ARA} g_4^{ARR}.$$

These four paths generate the four terms in Eq. (5.2.13).

B.2 Allowed vertices for the vertex diagram in the Keldysh basis

For this example, we do not repeat the same detailed descriptions again, but just write down the major steps. Carrying out the sum over internal vertices for the vertex diagram of Fig. (B.1) in Keldysh basis is done the same way. The only difference is that in Keldysh basis we naturally end up getting more terms compared to R/A basis.

Step: 1

$$g_1^{xaj} = g_1^{2**}$$

$$g_2^{bed} = g_2^{*1*}$$

$$g_3^{ich} = g_3^{2**}$$

$$g_4^{x'fg} = g_4^{121}$$

Figure B.5: Vertices in the R/A basis for vertex diagram.

Step: 2

$$g_1^{xaj} = g_1^{211}, g_1^{222}$$

$$g_2^{bed} = g_2^{211}, g_2^{112}$$

$$g_3^{ich} = g_3^{211}, g_3^{121}$$

$$g_4^{x'fg} = g_4^{121}, g_4^{112}$$

Figure B.6: Vertices in the Keldysh basis for vertex diagram.

Step: 3

Step-3 is shown in Fig. (B.7).

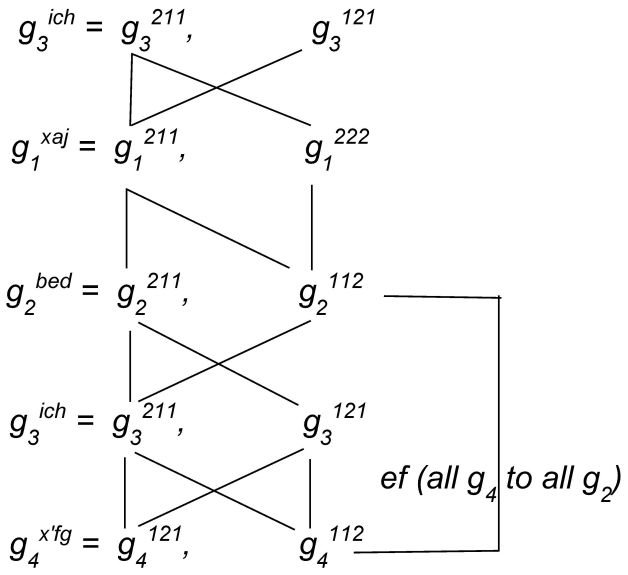


Figure B.7: Vertices in the Keldysh basis for vertex diagram.

Step: 4

In this step we write the paths as we did in the last example and again we remember that we have to go through the same g_3 that we start with in the first line.

$$[\mathbf{a}] \quad g_1^{211} g_2^{211} g_3^{211} g_4^{121},$$

$$[\mathbf{b}] \quad g_1^{211} g_2^{211} g_3^{211} g_4^{112},$$

$$[\mathbf{c}] \quad g_1^{211} g_2^{211} g_3^{121} g_4^{121},$$

$$[\mathbf{d}] \quad g_1^{211} g_2^{211} g_3^{121} g_4^{112},$$

$$[\mathbf{e}] \quad g_1^{211} g_2^{112} g_3^{211} g_4^{121},$$

$$[\mathbf{f}] \quad g_1^{211} g_2^{112} g_3^{211} g_4^{112}.$$

These six paths generate the six terms in Eq. (5.2.18).

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