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A CLASS OF COMPLEMENTARY
SEQUENCES OF INTEGERS

BY
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A THESIS SUBMITTED TO THE FACULTY OF GRADUATE
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Abstract

A pair of sequences $\{u_n\}$, $\{v_n\}$ for $n = 1, 2, 3, \dots$ are called complementary sequences if together they contain the positive integers without repetition or omission. This paper deals exclusively with Beatty sequences which are complementary sequences of the form $u_n = [n(1 + \frac{1}{\alpha})]$, $v_n = [n(1 + \alpha)]$ where $\alpha > 0$ is irrational.

The pairs (u_n, v_n) for particular values of α are winning pairs in a generalization of Wythoff's game. The generalized game is defined and the complete solution given. The particular values of α give rise to a generalization of the Fibonacci numbers; arithmetical properties of these numbers are discussed.

The subscript rules give explicit values for u_{p_n} and v_{q_n} where p_n/q_n is a continued fraction convergent to α .

The property $u_n = u_n + v_n$, $n = 1, 2, 3, \dots$, called reduplication, is enjoyed by only a discrete set of values of α . These special values are determined and the allied problem: 'for what values of $\alpha > 0$ can

$$\left[\frac{[n(1+\alpha)]}{\alpha} \right] = \left[\frac{n(1+\alpha)}{\alpha} \right]$$

be true for every positive integer n ' is completely solved.

Reduplication leads to the idea of continued subscripts which is developed in detail.

Beatty sequences can be split into r -fold complementary sequences. For example the sequence $\{u_n\}$ can be split into the two sequences $\{u_{u_n}\}$ and $\{u_{v_n}\}$. The number of r -fold sequences is determined.

The Eames representation leads to an algorithm for determining the Beatty sequences from the continued fraction for α and conversely.

A class of Beatty sequences is discussed which is found to be related to Fan sequences.

Van der Waerden's theorem for Beatty sequences leads to an interesting conjecture.

Finally, a brief description of the work of several other writers in the subject is given.

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INTRODUCTION

0.1 The Wythoff Game.

This investigation was initiated by the discovery of certain interesting properties of the numbers which occur in connection with Wythoff's game¹. The game is defined as follows :

- (i) there are two players who play alternately
- (ii) there are two piles of matches, an arbitrary number in each pile
- (iii) a player may take an arbitrary number of matches from one pile or an equal number from both piles but he must take at least one match
- (iv) the player who takes the last match wins the game.

If, after his move, a player leaves one match in one pile and two in the other he cannot fail to win; for if the other player takes one match from the pile containing two he can take both remaining matches, and so on for the other possibilities. Thus (1,2) is called a winning pair and in Table I each pair (u_n, v_n) is a winning pair in the following sense:

- (a) in a single move a player can change a non-winning pair into a winning pair
- (b) any move will change a winning pair into a non-winning pair.

¹see W.W.Rouse Ball and H.S.M. Coxeter, Mathematical Recreations and Essays, New York, 1947, p.p.38-39.

Table I

n	u_n	v_n
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26
11	17	28
12	19	31
13	21	34

For the purposes of (a) we call the pair (0,0) a winning pair. Assuming the validity of (a) and (b) for the moment, we see that if a player's move creates a winning pair and if each of his subsequent moves creates a winning pair he cannot fail to win for ultimately the winning pair must be (0,0) which means he took the last match.

The table was constructed by the rules:

$$(i) \quad u_1 = 1, \quad (0.1.1)$$

$$(ii) \quad v_n = u_n + n, \quad (0.1.2)$$

(iii) u_{n+1} is the least integer¹ which is distinct from the $2n$ integers $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$. It is seen that the sequences $\{u_n\}$ and $\{v_n\}$ exhaust the integers without repetition and for this reason are called

¹ Throughout this paper 'integer' will mean 'positive integer' when it is clear from the context that it must be so.

complementary sequences, a term introduced by J. Lambek and L. Moser.¹

Proof of statement (a): Let the number of matches in the two piles be x and y with $x \leq y$. If $x = y$ or $x = 0$ the player can move to the winning pair $(0, 0)$. Thus let $0 < x < y$. ^{Then} x is either a u_n or a v_n . If $x = v_n$ the player can move to (u_n, v_n) by reducing y since $u_n < v_n = x < y$. If $x = u_n$ then either $y > v_n$ or $y < v_n$. In the former case he can reduce y to v_n . In the latter case $y - x = m < n$ and he can reduce the smaller pile to u_m taking the same amount from the larger pile. Then the larger pile will contain $u_m + m = v_m$ matches which is a winning position.

Proof of statement (b): Since an integer occurs only once in Table I taking matches from one pile could not result in a winning pair. Taking an equal number of matches from both piles could not result in a winning pair by condition (0.1.2).

0.2 Some Conjectures Relating to the Sequences $\{u_n\}$ and $\{v_n\}$.

The Fibonacci sequences are defined by

$$w_{n+2} = w_{n+1} + w_n, \quad (0.2.1)$$

$$p_{n+2} = p_{n+1} + p_n, \quad (0.2.2)$$

with the starting conditions

$$w_0 = 0, \quad w_1 = 1, \quad (0.2.3)$$

$$p_0 = 2, \quad p_1 = 1. \quad (0.2.4)$$

¹ Inverse and Complementary Sequences of Natural Numbers, American Mathematical Monthly, Sept. 1954, p.454.

Table II

n	ω_n	p_n
0	0	2
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76

Inspecting tables I and II we were led to make the following conjectures:

$$U\omega_{2n-1} = \omega_{2n}, \quad (0.2.5)$$

$$V\omega_{2n-1} = \omega_{2n+1}, \quad (0.2.6)$$

$$U\omega_{2n} = \omega_{2n+1} - 1, \quad (0.2.7)$$

$$V\omega_{2n} = \omega_{2n+2} - 1, \quad (0.2.8)$$

all for $n = 1, 2, 3, \dots$.

(0.2.6) and (0.2.8) follow from (0.2.5) and (0.2.7) by means of (0.1.2) and (0.2.1).

Similarly it was conjectured that

$$Up_{2n} = p_{2n+1}, \quad (0.2.9)$$

$$Vp_{2n} = p_{2n+2} \quad (0.2.10)$$

$$u_{p_{2n+1}} = p_{2n+2} - 1, \quad (0.2.11)$$

$$v_{p_{2n+1}} = p_{2n+3} - 1, \quad (0.2.12)$$

for $n = 1, 2, 3, \dots$.

(0.2.10) and (0.2.12) follow from (0.2.9) and (0.2.11).

Finally it was conjectured that

$$u_{v_n} = u_n + v_n \quad (0.2.13)$$

for $n = 1, 2, 3, \dots$. This proved to be a deeper conjecture.

All these conjectures are proved below where the formulas¹

$$u_n = \left[n \left(\frac{1+\sqrt{5}}{2} \right) \right] \quad (0.2.14)$$

$$v_n = \left[n \left(\frac{3+\sqrt{5}}{2} \right) \right] \quad (0.2.15)$$

play an essential role. These formulas are proved in sec. 1.2.

Later, generalizations of the sequences $\{u_n\}$ and $\{v_n\}$ lead to a variety of results. For example the question is raised: For what values of $\alpha > 0$ can

$$\left[\frac{[n(1+\alpha)]}{\alpha} \right] = \left[\frac{n(1+\alpha)}{\alpha} \right] \quad (0.2.16)$$

be true for every integer n ?

¹ In this paper square brackets always denote the integral part function. That is, $[x]$ is the algebraically greatest integer (positive or negative) not exceeding x .

CHAPTER I

1.1 Beatty Sequences.

S.Beatty¹ proposed the following problem: If α is any irrational greater than zero then the sequences $\{u_n\}$ and $\{v_n\}$ defined by

$$u_n = \left[n \left(1 + \frac{1}{\alpha} \right) \right] , \quad (1.1.1)$$

$$v_n = \left[n(1 + \alpha) \right] \quad (1.1.2)$$

for $n = 1, 2, 3, \dots$ are the integers without repetitions or omissions; that is $\{u_n\}$ and $\{v_n\}$ are complementary sequences. Proofs were provided by A.Ostrowski and J.Hyslop² and by A.C.Aitken³. We call the sequences $\{u_n\}$, $\{v_n\}$ Beatty sequences of argument α . (More properly a Beatty pair of sequences).

¹ American Mathematical Monthly, March 1926, p.159.

² ibid. March 1927, p.159.

³ ibid. March 1927, p.159. He proves the following result for rational α : If $\alpha = p/q$, $(p, q) = 1$ then $\{u_n\}$, $\{v_m\}$ for $n = 1, 2, \dots, p-1$; $m = 1, 2, \dots, q-1$ are the integers

$$1, 2, \dots, p+q-2$$

without repetitions or omissions.

The problem was generalized by S.H.Gould¹.

We shall prove Beatty's statement by means of the 'Schubfachprinzip' or the 'pigeon-hole principle'² :

If there are n boxes containing n objects and no box is empty then each box contains precisely one object.

Theorem 1: Beatty sequences are complementary.

proof: Let the first n integers contain r members of $\{u_i\}$ and s members of $\{v_i\}$. Then

$$u_r \leq n < u_{r+1},$$

and

$$v_s \leq n < v_{s+1}$$

or³

$$r(1 + \frac{1}{\alpha}) < n+1 < (r+1)(1 + \frac{1}{\alpha}),$$

$$s(1 + \alpha) < n+1 < (s+1)(1 + \alpha).$$

¹ ibid. Nov.1947,p.549. Let b be a fixed integer and α an irrational such that $0 < \alpha < 1$. Call the interval $(m, m+1)$ a gap if it does not contain a multiple of $b + \alpha$. ($m = 1, 2, \dots$). Then every set of b successive gaps contains exactly one multiple of $1 + b/\alpha$.

² The principle is usually stated in the form: If n boxes contain $n + 1$ objects then some box contains more than one object.

³ Strict inequalities because α is irrational.

Multiplying the former by $\alpha > 0$, we get

$$r(1+\alpha) < (n+1)\alpha < (r+1)(1+\alpha).$$

Adding this to the previous line and dividing the sum by $1+\alpha > 0$,

$$r+s < n+1 < r+s+2.$$

Since all quantities are integers,

$$n = r+s.$$

Thus the first n integers contain n members of $\{u_i\}$ and $\{v_i\}$; in particular the integer 1 occurs once. Hence the first $n+1$ integers contain $n+1$ members of $\{u_i\}$ and $\{v_i\}$ and this $(n+1)^{\text{st}}$ member must be the integer $n+1$. The first n integers occur at least once among the first n members and therefore precisely once.

In some sense Beatty sequences form a relatively small subclass of the class of complementary sequences; for interchanging any pair u_m and v_n in a pair of Beatty sequences yields upon rearrangement a pair of complementary sequences which are not Beatty sequences. Further, arbitrarily dividing the integers into two sets yields a pair of complementary sequences which are 'seldom' Beatty sequences.

However from another point of view the number of Beatty sequences is about the same as the number of complementary sequences. Roughly, if a is the cardinal number of the set of integers and c the cardinal number of the continuum then the cardinal number of the set of Beatty sequences is c since

α is any irrational. On the other hand in a complementary sequence an integer may be either a u_n or a v_n so that the cardinal number of the set of complementary sequences is¹

$$2^a = C.$$

1.2 Wythoff Sequences and the k-Wythoff Game.

We define Wythoff sequences to be Beatty sequences of argument

$$\alpha = \frac{k + \sqrt{k^2 + 4}}{2} \quad (1.2.1)$$

for $k = 1, 2, 3, \dots$. That is for each integer k there is a pair of Wythoff sequences²

Theorem 2: If $\{u_n\}$ and $\{v_n\}$ are Wythoff sequences then

$$(i) \quad u_1 = 1,$$

$$(ii) \quad v_n = u_n + k n, \quad (1.2.2)$$

(iii) u_{n+1} is the smallest integer distinct from the $2n$ integers $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$. Hence (u_n, v_n) are winning pairs for the Wythoff game.

proof: Clearly $\alpha > 0$. α is irrational for otherwise

$$k^2 + 4 = a^2$$

¹ See E. Kamke, Theory of Sets, p.47.

² We write $k u_n$ and $k v_n$ when it is necessary to draw attention to the particular α .

for some integer a . If k is odd then

$$k^2 \equiv 1 \pmod{8}$$

and

$$a^2 \equiv 5 \pmod{8}$$

which is impossible. If $k = 2k_1$, then $a = 2a_1$, and

$$k_1^2 + 1 = a_1^2,$$

which is also impossible. Thus (1.2.1) is an admissible $\times$ and the series are complementary. Now

$$\sqrt{k^2 + 4} > k,$$

so that

$$\alpha > k,$$

and

$$1 + \frac{1}{\alpha} < 1 + \frac{1}{k}.$$

Hence $u_1 = 1$ for $k = 1, 2, 3, \dots$.

$$\frac{1}{\alpha} = \frac{\sqrt{k^2 + 4} - k}{2}$$

Thus¹

$$u_n + kn = \left[n \left(1 + \frac{\sqrt{k^2 + 4} - k}{2} \right) \right] + kn$$

$$= \left[n \left(1 + \frac{\sqrt{k^2 + 4} + k}{2} \right) \right] = v_n$$

¹ Using the theorem $[m + x] = m + [x]$ for any integer m , positive or negative, and any x .

Since $u_n < u_{n+1}$ and $u_n < v_n$, (iii) follows from the complementary property.

Setting $k=1$ we verify equations (0.2.14) and (0.2.15).

In general,

$$u_n = \left[n \left\{ \frac{-k+2+\sqrt{k^2+4}}{2} \right\} \right] \quad (1.2.3)$$

$$v_n = \left[n \left\{ \frac{k+2+\sqrt{k^2+4}}{2} \right\} \right]. \quad (1.2.4)$$

Since (1.2.3.) and (1.2.4) are generalizations of (0.2.14) and (0.2.15) it is natural to enquire what the corresponding generalization is of the Wythoff game. Define the k -Wythoff game by the following rules:

(i) there are two players who play alternately,

(ii) initially there are two piles of matches, an arbitrary number in each pile,

(iii) a player may take an arbitrary multiple of k matches from one pile or an equal number from both piles, but he must take at least one match,

(iv) the player who cannot make a legitimate move loses the game.

Let x and y be the number of matches in the piles after a player has moved. Since 'stalemate' cannot occur, for this player the pair (x, y) is either:

(a) a winning pair, i.e., whatever moves his opponent makes he can win by proper strategy, or,

(b) a losing pair, i.e., a winning pair for his opponent,

after his opponent's (correct) move.

To show that the set (a_n, b_n) is the set of all winning pairs we must show that

(α) an arbitrary pair (x, y) is either a pair (a_n, b_n) or can be reduced to such a pair in one move,

(β) a move on the pair (a_n, b_n) produces a pair which is not in the set, and

(γ) all the pairs on which no legitimate move can be made are in the set.

Thus if a player's move results in a winning pair (a_n, b_n) his opponent's move will result in a losing pair which he in turn can reduce to a winning pair $(a_{n'}, b_{n'})$. By repeated application of this strategy he will ultimately win.

Define

$$u_{i,n} = \left[\left(n + \frac{i}{k} \right) \left(\frac{-k+2+\sqrt{k^2+4}}{2} \right) \right] \quad (1.2.5)$$

and

$$v_{i,n} = \left[\left(n + \frac{i}{k} \right) \left(\frac{k+2+\sqrt{k^2+4}}{2} \right) \right], \quad (1.2.6)$$

for

$$i = 0, 1, 2, \dots, k$$

and

$$n = 0, 1, 2, \dots$$

Thus

$$u_{0,n} = u_{k,n-1} = u_n, \quad (1.2.7)$$

and

$$v_{0,n} = v_{k,n-1} = v_n, \quad (1.2.8)$$

for

$$n = 1, 2, 3, \dots$$

We introduce the notation¹

$$A = \frac{k + \sqrt{k^2 + 4}}{2}, \quad (1.2.9)$$

$$\bar{A} = \frac{k - \sqrt{k^2 + 4}}{2}, \quad (1.2.10)$$

reserving the symbol α for an arbitrary real number.

Since

$$1 + A = 1 + \frac{1}{A} + k,$$

$$\begin{aligned} v_{i,n} &= \left[\left(n + \frac{i}{k} \right) \left(1 + \frac{1}{A} + k \right) \right] \\ &= u_{i,n} + nk + i. \end{aligned} \quad (1.2.11)$$

$$\begin{aligned} u_{i,0} &= \left[\frac{i}{k} \left(1 + \frac{1}{A} \right) \right] \\ &\leq \left[\frac{k-1}{k} \left(1 + \frac{1}{A} \right) \right] \end{aligned}$$

for $i = 0, 1, 2, \dots, k-1$.

Now $\frac{k-1}{k} \left(1 + \frac{1}{A} \right) < 1$

for $(k-1)(1+A) < kA$,

i.e. $k-1 < A$,

or $k^2 - 2k + 1 < k^2 + 4$.

¹ When necessary we write A_k and $\bar{A}_k$.

Thus $u_{i,0} = 0$ and by (1.2.11) $v_{i,0} = i$
for $0 \leq i \leq k-1$.

Theorem 3: The sequences $\{u_{i,n}\}$ and $\{v_{k-i,n-1}\}$
for $0 \leq i \leq k-1$ and for $n = 1, 2, 3, \dots$
are complementary¹.

proof:

$$v_{k-i,n-1} = \left[\left(n - \frac{i}{k} \right) (1+A) \right] \quad (1.2.12)$$

$u_{i,n} \geq 1$ for $n \geq 1$ and since $i < k$, $v_{k-i,n-1} \geq 1$
for $n \geq 1$. Let the first N integers contain r members of
 $\{u_{i,n}\}$ and s members of $\{v_{k-i,n-1}\}$.

¹ Th. Skolem, *Mathematica Scandinavica*, Vol. 5, 1957, p. 61, has
found explicit expressions for complementary sequences which
obey (1.2.11) (for negative 1) :

If $u'_{i,n} = \left[\left(n - \frac{i}{k} \right) \left(1 + \frac{1}{A} \right) + \frac{2i}{k} \right]$

and $v'_{i,n} = \left[\left(n - \frac{i}{k} \right) (1+A) + \frac{2i}{k} \right]$

for $0 \leq i \leq k-1$

then

$$v'_{i,n} = u'_{i,n} + kn - i$$

and the sequences $\{u'_{i,n}\}$ and $\{v'_{i,n}\}$ are complementary.

This result can be extended to $-1 \leq i \leq k-1$,

but no further (not even for particular values of k). The

proof follows easily by the 'pigeon-hole' method. The

restriction on i arises from demanding $u'_{i,1} = 1$.

Then

$$(r + \frac{i}{k})(1 + \frac{1}{A}) < N+1 < (r+1 + \frac{i}{k})(1 + \frac{1}{A}),$$

or

$$(r + \frac{i}{k})(1+A) < (N+1)A < (r+1 + \frac{i}{k})(1+A),$$

and

$$(s - \frac{i}{k})(1+A) < N+1 < (s+1 - \frac{i}{k})(1+A).$$

Adding the last two lines and dividing the result by $1+A$,

$$r+s < N+1 < r+s+2.$$

Since all these quantities are integers $N = r + s$ and the theorem follows by the pigeon-hole argument.

Theorem 4: The set of winning pairs for the k -Wythoff game comprises the k sets

$$\{(u_{i,n}, v_{i,n})\}, \quad (1.2.13)$$

for $i = 0, 1, 2, \dots, k-1,$

and $n = 0, 1, 2, \dots$

proof: (α) If (x, y) is not a pair of type (1.2.13) we must show that it can be reduced to such a pair by a single move.

Let

$$x \leq y, \quad y - x \equiv i \pmod{k}, \quad (1.2.14)$$

so that $y = x + i + rk,$ (1.2.15)

and $r \geq 0, \quad 0 \leq i \leq k-1.$

Then by theorem 3 either

$$(i) \quad X = 0,$$

$$(ii) \quad X = u_{i,n}, \quad n \geq 1, \quad \text{or}$$

$$(iii) \quad X = v_{k-i, n-1}, \quad n \geq 1.$$

(1)

$$X = u_{i,0} = 0, \quad y = i + rk, \quad r \geq 0.$$

If $r=0$ the pair (X, y) is the pair $(u_{i,0}, v_{i,0})$ which is of type (1.2.13). If $r > 0$ the pile y can be reduced by rk to give this pair.

(ii) Either

$$(a) \quad y > v_{i,n},$$

$$(b) \quad y = v_{i,n},$$

$$(c) \quad y < v_{i,n}.$$

(a)

$$y = x + i + rk = v_{i,n} + (r-n)k$$

Thus the pile y can be reduced by $r-n > 0$ multiples of k to yield the pair $(u_{i,n}, v_{i,n})$.

(b) (X, y) is already of type (1.2.13).

(c) Here $r < n$. Take s matches from each pile so that

$$X - s = u_{i,n} - s = u_{i,r}.$$

Since $u_{i,n} > u_{i,n-1}$ by (1.2.5), $s > 0$. Then

$$\begin{aligned} y - s &= x + i + rk - s \\ &= u_{i,r} + i + rk = v_{i,r}, \end{aligned}$$

and (X, y) has been reduced to $(u_{i,r}, v_{i,r})$ where $r \geq 0$.

$$(iii) \quad y \geq x = v_{k-i, n-1} > u_{k-i, n-1},$$

so that

$$\begin{aligned} y &= x + i + rk = v_{k-i, n-1} + i + rk \\ &= u_{k-i, n-1} + k - i + (n-1)k + i + rk \\ &= u_{k-i, n-1} + (n+r)k, \end{aligned}$$

and the move is obvious.

(β) We must show that a move on a pair of type (1.2.13) results in a pair not of that type. Since

$$\begin{aligned} v_{i,n} &\equiv u_{i,n} + i \pmod{k}, \\ \therefore v_{i,n} - s &\equiv u_{i,n} + i - s \pmod{k}, \\ v_{i,n} - rk &\equiv u_{i,n} + i \pmod{k}, \quad \text{and} \\ v_{i,n} &\equiv u_{i,n} + i - rk \pmod{k}. \end{aligned}$$

Hence if the result of a move on the pair $(u_{t,n}, v_{t,n})$ is to be a pair of type (1.2.13) it must be in fact of type $(u_{t,n'}, v_{t,n'})$. Now $n' < n$, $u_{t,n'} < u_{t,n}$, $v_{t,n'} < v_{t,n}$, so that the move must have removed s matches from each pile.

Then

$$v_{t,n} - s = u_{t,n} - s + nk + t,$$

i.e.

$$v_{t,n'} = u_{t,n'} + nk + t,$$

which is impossible by (1.2.11).

(γ) The pairs on which no legitimate move can be made are clearly

$$(u_{0,0}, v_{0,0}), (u_{1,0}, v_{1,0}), \dots, (u_{k-1,0}, v_{k-1,0}),$$

which are included in (1.2.13). This completes the proof.

Notice that if a player hits upon the winning pair (u_0, v_0) i.e. (u_n, v_n) he need only know the pairs of this sequence for pairs $(u_{i,n}, v_{i,n})$ with $i > 0$ do not arise. If k is even the sequence $(u_{\frac{1}{2}k,n}, v_{\frac{1}{2}k,n})$ is similarly self-contained. However in the other cases the player will in general have to know two sequences, corresponding to i and $k-1$.

Incidentally if k is even, $\{u_{\frac{1}{2}k,n}\}$ and $\{v_{\frac{1}{2}k,n}\}$ are complementary sequences, i.e.,

$$\left[\left(n + \frac{1}{2} \right) \left(-k + 1 + \sqrt{k^2 + 1} \right) \right],$$

and

$$\left[\left(n - \frac{1}{2} \right) \left(k + 1 + \sqrt{k^2 + 1} \right) \right],$$

for $n = 1, 2, 3, \dots$ and k a positive integer are complementary. (We have written k for $k/2$).

Following are short tables of winning pairs for the first few Wythoff games.

Table III

$k = 1:$			$k = 2:$				
n	$u_{0,n}$	$v_{0,n}$	n	$u_{0,n}$	$v_{0,n}$	$u_{1,n}$	$v_{1,n}$
0	0	0	0	0	0	0	1
1	1	2	1	1	3	2	5
2	3	5	2	2	6	3	8
3	4	7	3	4	10	4	11
4	6	10	4	5	13	6	15
5	8	13	5	7	17	7	18
6	9	15	6	8	20	9	22
7	11	18	7	9	23	10	25
8	12	20	8	11	27	12	29
9	14	23	9	12	30	13	32
10	16	26	10	14	34	14	35
11	17	28	11	15	37	16	39
12	19	31	12	16	40	17	42

$k = 3:$

n	$u_{0,n}$	$v_{0,n}$	$u_{1,n}$	$v_{1,n}$	$u_{2,n}$	$v_{2,n}$
0	0	0	0	1	0	2
1	1	4	1	5	2	7
2	2	8	3	10	3	11
3	3	12	4	14	4	15
4	5	17	5	18	6	20
5	6	21	6	22	7	24
6	7	25	8	27	8	28
7	9	30	9	31	9	32
8	10	34	10	35	11	37
9	11	38	12	40	12	41
10	13	43	13	44	13	45
11	14	47	14	48	15	50
12	15	51	16	53	16	54

 $k = 4:$

n	$u_{0,n}$	$v_{0,n}$	$u_{1,n}$	$v_{1,n}$	$u_{2,n}$	$v_{2,n}$	$u_{3,n}$	$v_{3,n}$
0	0	0	0	1	0	2	0	3
1	1	5	1	6	1	7	2	9
2	2	10	2	11	3	13	3	14
3	3	15	4	17	4	18	4	19
4	4	20	5	22	5	23	5	24
5	6	26	6	27	6	28	7	30
6	7	31	7	32	8	34	8	35
7	8	36	8	37	9	39	9	40
8	9	41	10	43	10	44	10	45
9	11	47	11	48	11	49	12	51
10	12	52	12	53	12	54	13	56
11	13	57	13	58	14	60	14	61
12	14	62	15	63	15	65	15	66

These tables were constructed using theorem 3, thus obviating the direct calculation of (1.2.5) and (1.2.6).

1.3 Continued Fractions.

It is convenient now to list some elementary results from the theory of continued fractions. The quantity

$$\xi = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \dots + \frac{b_N}{a_N}}},$$

is called a finite continued fraction; if it does not terminate it is called an infinite continued fraction. It is 'simple' if $b_i = 1$ for every i and a_i is a positive integer for every i (except that a_0 may be zero). We assume henceforth that ξ is simple and infinite.¹

A convenient notation is

$$\xi = [a_0, a_1, a_2, \dots],$$

or

$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}.$$

The n^{th} convergent is defined to be the ordinary fraction

$$\frac{P_n}{Q_n} = [a_0, a_1, \dots, a_n],$$

for $n = 0, 1, 2, \dots$.

A special type of ξ is a periodic continued fraction:

¹ For a discussion of the general case see Chrystal, Algebra, or Perron, Kettenbrüche.

$$\xi = [a_0, a_1, \dots, a_r, b_1, b_2, \dots, b_s, b_1, b_2, \dots],$$

The sequence $b_1, b_2, \dots, b_s$ is repeated, the situation being analogous to the decimal expansion of a rational. We denote such a ξ by¹

$$\xi = [a_0, a_1, \dots, a_r, \overline{b_1, b_2, \dots, b_s}].$$

Theorem 5: For

$$\xi = [a_0, a_1, \dots],$$

(1) ξ represents an irrational number α in the following sense:

$$(2) \quad \lim_{n \rightarrow \infty} \frac{P_n}{Q_n} = \alpha,$$

(3) Every irrational α determines a unique ξ , and conversely.

$$(4) \quad P_0 = a_0, \quad P_1 = a_1 a_0 + 1, \quad (1.3.1)$$

$$Q_0 = 1, \quad Q_1 = a_1, \quad (1.3.2)$$

$$P_{n+2} = a_{n+2} P_{n+1} + P_n, \quad (1.3.3)$$

$$Q_{n+2} = a_{n+2} Q_{n+1} + Q_n, \quad (1.3.4)$$

$$(5) \quad P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}, \quad (1.3.5)$$

Hence

¹ As in Perron, Kettenbrüche.

(6)¹ $(P_n, Q_n) = 1$ and the fraction P_n/Q_n is already in its lowest terms.

(7) The even convergents, viz: P_{2n}/Q_{2n} increase strictly with n and are all less than α . They are called inferior convergents. The odd convergents decrease strictly with n , are all greater than α and are called superior convergents.

$$(8)^2 \quad \left| \alpha - \frac{P_n}{Q_n} \right| < \frac{1}{Q_n^2}, \quad (1.3.6)$$

(9) If ξ is periodic then α is a quadratic surd, i.e., an irrational root of a quadratic equation with integral coefficients; and conversely.

(10) Of any two consecutive convergents to ξ one at least satisfies

$$\left| \alpha - \frac{P}{Q} \right| < \frac{1}{2Q^2}.$$

(11) If

$$\left| \alpha - \frac{P}{Q} \right| < \frac{1}{2Q^2},$$

then P/Q is a convergent.

(12)³ If P_n/Q_n is a convergent and $Q \leq Q_n$ and

¹ (a, b) means the greatest common divisor of the integers a and b .

² $|x|$ means the absolute value of x .

³ Thus the convergents are the best possible rational approximations to α .

$$\frac{P}{Q} \neq \frac{P_n}{Q_n},$$

then

$$|\alpha - \frac{P_n}{Q_n}| < |\alpha - \frac{P}{Q}|,$$

(13)

$$|\alpha - \frac{P}{Q}| < \frac{1}{Q^2 \sqrt{5}},$$

has infinitely many solutions P/Q . (These are necessarily convergents). The statement becomes false if $\sqrt{5}$ is replaced by any larger number.

All of these statements are proved in G.H.Hardy and E.M.Wright, Introduction to the Theory of Numbers.

We now write $\alpha = [a_0, a_1, \dots]$ in the sense of (2).

Theorem 6:

$$(i) \quad A_k = k + \frac{1}{k+} \frac{1}{k+} \dots = [\overline{k}], \quad (1.3.7)$$

$$(ii) \quad P_{n+2} = k P_{n+1} + P_n, \quad (1.3.8)$$

$$Q_{n+2} = k Q_{n+1} + Q_n, \quad (1.3.9)$$

$$(iii) \quad P_n = \frac{1}{\sqrt{k^2+4}} \{A^{n+2} - \bar{A}^{n+2}\}, \quad (1.3.10)$$

$$Q_n = P_{n-1}. \quad (1.3.11)$$

In particular, $Q_0 = \frac{1}{\sqrt{k^2+4}} \{A - \bar{A}\} = 1.$

proof: (i) Since

$$A = \frac{k + \sqrt{k^2+4}}{2},$$

$$A^2 - kA - 1 = 0 \quad (1.3.12)$$

$$\begin{aligned} \therefore A &= k + \frac{1}{A} = k + \frac{1}{k + \frac{1}{A}}, \\ &= \dots = k + \frac{1}{k + \frac{1}{k + \dots}} \end{aligned}$$

Since this continued fraction is infinite we have an alternative proof that A is irrational.¹

(ii) $a_n = k$ for every n and (1.3.8), (1.3.9) follow from (1.3.3) and (1.3.4).

(iii) We first show that P_n defined by (1.3.10) satisfies the recurrence relation (1.3.8).

$$\begin{aligned} kP_{n+1} + P_n &= \frac{1}{\sqrt{k^2+4}} \{kAA^{n+2} - k\bar{A}\bar{A}^{n+2} + A^{n+2} - \bar{A}^{n+2}\} \\ &= \frac{1}{\sqrt{k^2+4}} \{(kA+1)A^{n+2} - (k\bar{A}+1)\bar{A}^{n+2}\} \\ &= \frac{1}{\sqrt{k^2+4}} \{A^{n+4} - \bar{A}^{n+4}\} = P_{n+2}, \end{aligned}$$

since A and $\bar{A}$ are the two roots of equation (1.3.12).

It follows that Q_n defined by (1.3.11) satisfies (1.3.9).

Now

$$P_0 = \frac{1}{\sqrt{k^2+4}} \{A^2 - \bar{A}^2\} = \frac{1}{\sqrt{k^2+4}} \{(kA+1) - (k\bar{A}+1)\} = k,$$

¹ That the above procedure gives the correct continued fraction expansion is guaranteed by $0 < \frac{1}{A} < 1$, which is what the continued fraction algorithm requires.

C.f. Hardy and Wright, loc.cit.

$$\begin{aligned}
 P_1 &= \frac{1}{\sqrt{k^2+4}} \{ (kA+1)A - (k\bar{A}+1)\bar{A} \} \\
 &= \frac{1}{\sqrt{k^2+4}} \{ k(kA+1) + A - k(k\bar{A}+1) - \bar{A} \} \\
 &= \frac{1}{\sqrt{k^2+4}} \{ k^2\sqrt{k^2+4} + \sqrt{k^2+4} \} = k^2+1.
 \end{aligned}$$

Thus (1.3.1) and (1.3.2) are also satisfied so that (1.3.10) and (1.3.11) are correct.

Notice that (1.3.10) and (1.3.11) serve to define P_n and Q_n for negative n which still satisfy (1.3.8) and (1.3.9). In fact, noting that

$$A\bar{A} = \frac{k + \sqrt{k^2+4}}{2} \cdot \frac{k - \sqrt{k^2+4}}{2} = -1, \quad (1.3.13)$$

$$\begin{aligned}
 P_{-n} &= \frac{1}{\sqrt{k^2+4}} \{ A^{-n+2} - \bar{A}^{-n+2} \} \\
 &= \frac{1}{\sqrt{k^2+4}} \left\{ \frac{\bar{A}^{n-2} - A^{n-2}}{A^{n-2} \bar{A}^{n-2}} \right\} \\
 &= (-1)^{n-1} P_{n-4}, \quad (1.3.14)
 \end{aligned}$$

and

$$\begin{aligned}
 Q_n &= P_{-n-1} \\
 &= (-1)^n P_{n-3} \\
 &= (-1)^n Q_{n-2}. \quad (1.3.15)
 \end{aligned}$$

We shall make use of this extended definition.

We conclude this chapter by proving a more complicated theorem which like theorems 5 and 6 will be required later.

We first need the lemma

Theorem 7:

$$\omega_{m-1}\omega_{i+1} - \omega_m\omega_i = (-1)^i \omega_{m-i-1}, \quad (1.3.16)$$

proof: The convergents to

$$A_1 = 1 + \frac{1}{1+} \frac{1}{1+} \dots = \frac{1+\sqrt{5}}{2}, \quad (1.3.17)$$

are $P_0 = 1, \quad P_1 = 2,$

$$Q_0 = 1, \quad Q_1 = 1,$$

with $P_{n+2} = P_{n+1} + P_n,$

$$Q_{n+2} = Q_{n+1} + Q_n.$$

Comparing these with (0.2.1) and (0.2.3) we see that

$$P_n = \omega_{n+2},$$

and $Q_n = \omega_{n+1}.$

Hence, by (1.3.10),

$$\omega_n = \frac{1}{\sqrt{5}} \{A_1^n - \bar{A}_1^n\} = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}, \quad (1.3.18)$$

Recalling

$$A_1 \bar{A}_1 = -1,$$

$$\begin{aligned} \omega_{m-1}\omega_{i+1} - \omega_m\omega_i &= \frac{1}{5} \{ (A_1^{m-1} - \bar{A}_1^{m-1})(A_1^{i+1} - \bar{A}_1^{i+1}) - (A_1^m - \bar{A}_1^m)(A_1^i - \bar{A}_1^i) \} \\ &= \frac{1}{5} \{ -A_1^{m-1}\bar{A}_1^{i+1} - \bar{A}_1^{m-1}A_1^{i+1} + A_1^m\bar{A}_1^i + \bar{A}_1^m A_1^i \} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5} (-1)^i \{ A_1^{m-i-2} + \bar{A}_1^{m-i-2} + A_1^{m-i} + \bar{A}_1^{m-i} \} \\
&= \frac{1}{5} (-1)^i \{ A_1^{m-i-1} \left(\frac{1}{A_1} + A_1 \right) + \bar{A}_1^{m-i-1} \left(\frac{1}{\bar{A}_1} + \bar{A}_1 \right) \} \\
&= \frac{1}{\sqrt{5}} (-1)^i \{ A_1^{m-i-1} - \bar{A}_1^{m-i-1} \} = (-1)^i \omega_{m-i-1}.
\end{aligned}$$

where we have used

$$\frac{1}{A_1} + A_1 = \frac{\sqrt{5}-1}{2} + \frac{\sqrt{5}+1}{2} = \sqrt{5},$$

and

$$\frac{1}{\bar{A}_1} + \bar{A}_1 = -\frac{1-\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2} = -\sqrt{5}.$$

Notice that (1.3.18) defines ω_n for positive and negative n so that the theorem is true for positive and negative m and i , and

$$\omega_{-n} = (-1)^{n-1} \omega_n.$$

Define the numbers

$$A_{m,k} = [k, \overbrace{1, 1, \dots, 1}^{m-1}], \quad (1.3.19)$$

where there are $m-1$ ones after the k . Thus

$$A_k = A_{1,k}, \quad (1.3.20)$$

and

$$A_1 = A_{m,1}, \quad (1.3.21)$$

for any m .

Theorem 8:

$$(1) \quad A_{m,k} = \frac{k\omega_m + \sqrt{k^2\omega_m^2 + 4k\omega_m\omega_{m-1} + 4\omega_m\omega_{m-2}}}{2\omega_m}, \quad (1.3.22)$$

(ii) The convergents to $A_{m,k}$ satisfy the recurrence relations

$$P_{(r+2)m+i} = (k\omega_m + 2\omega_{m-1})P_{(r+1)m+i} - (-1)^m P_{rm+i}, \quad (1.3.23)$$

$$Q_{(r+2)m+i} = (k\omega_m + 2\omega_{m-1})Q_{(r+1)m+i} - (-1)^m Q_{rm+i}, \quad (1.3.24)$$

where $r = 0, 1, 2, \dots$ and $0 \leq i < m$.

proof: We can verify (1.3.22) immediately for $m = 1$.

$$\omega_{-1} = 1, \quad \omega_0 = 0, \quad \omega_1 = 1,$$

$$\therefore A_{1,k} = A_k = \frac{k + \sqrt{k^2 + 4}}{2},$$

as we had before.

$$A_{2,k} = k + \frac{1}{1 + \frac{1}{A_{2,k}}} = k + \frac{A_{2,k}}{A_{2,k} + 1},$$

$$\begin{aligned} A_{3,k} &= k + \frac{1}{1 + \frac{1}{1 + \frac{1}{A_{3,k}}}} = k + \frac{1}{1 + \frac{A_{3,k}}{A_{3,k} + 1}} \\ &= k + \frac{A_{3,k} + 1}{2A_{3,k} + 1}. \end{aligned}$$

In general if

$$A_{m,k} = k + \frac{\omega_{m-1} A_{m,k} + \omega_{m-2}}{\omega_m A_{m,k} + \omega_{m-1}}, \quad (1.3.25)$$

then

$$A_{m+1,k} = k + \frac{1}{1 + \frac{\omega_{m-1} A_{m+1,k} + \omega_{m-2}}{\omega_m A_{m+1,k} + \omega_{m-1}}}, \quad (1.3.26)$$

since the fraction in the denominator in (1.3.26) is arrived at in exactly the same way as the fraction in (1.3.25).

$$\begin{aligned} \therefore A_{m+1,k} &= k + \frac{\omega_m A_{m+1,k} + \omega_{m-1}}{(\omega_m + \omega_{m-1}) A_{m+1,k} + (\omega_{m-1} + \omega_{m-2})} \\ &= k + \frac{\omega_m A_{m+1,k} + \omega_{m-1}}{\omega_{m+1} A_{m+1,k} + \omega_m}. \end{aligned}$$

Hence (1.3.25) is true for all m by induction. That is

$$\omega_m A_{m,k}^2 - k \omega_m A_{m,k} - (k \omega_{m-1} + \omega_{m-2}) = 0.$$

Solving we get (1.3.22). (we take the positive root since $A_{m,k} > 0$).

$$(ii) \quad P_{n+2} = a_{n+2} P_{n+1} + P_n,$$

where $a_{n+2} = 1$ if $n+2 \neq rm$,

and $a_{n+2} = k$ if $n+2 = rm$ for some r .

Thus

$$P_{rm} = k P_{rm-1} + P_{rm-2}, \quad (1.3.27)$$

$$P_{rm+1} = P_{rm} + P_{rm-1},$$

$$P_{rm+2} = 2 P_{rm} + P_{rm-1},$$

$$P_{rm+3} = 3 P_{rm} + 2 P_{rm-1},$$

$$\vdots$$

$$P_{rm+i} = \omega_{i+1} P_{rm} + \omega_i P_{rm-1}, \quad (1.3.28)$$

$$\vdots$$

$$P_{(r+1)m-1} = \omega_m P_{rm} + \omega_{m-1} P_{rm-1}, \quad (1.3.29)$$

Using (1.3.27) - (1.3.29),

$$\begin{aligned}
P_{(r+2)m+i} &= \omega_{i+1} P_{(r+2)m} + \omega_i P_{(r+2)m-1} \\
&= \omega_{i+1} \{k P_{(r+2)m-1} + P_{(r+2)m-2}\} + \omega_i P_{(r+2)m-1} \\
&= \omega_{i+1} \{k (\omega_m P_{(r+1)m} + \omega_{m-1} P_{(r+1)m-1}) \\
&\quad + (\omega_{m-1} P_{(r+1)m} + \omega_{m-2} P_{(r+1)m-1})\} \\
&\quad + \omega_i \{\omega_m P_{(r+1)m} + \omega_{m-1} P_{(r+1)m-1}\} \\
&= k \omega_m P_{(r+1)m+i} + k \omega_{i+1} \omega_{m-1} P_{(r+1)m-1} \\
&\quad - k \omega_i \omega_m P_{(r+1)m-1} \\
&\quad + 2 \omega_{m-1} P_{(r+1)m+i} - \omega_{m-1} \omega_{i+1} P_{(r+1)m} \\
&\quad - \omega_{m-1} \omega_i P_{(r+1)m-1} \\
&\quad + \omega_{m-2} \omega_{i+1} P_{(r+1)m-1} + \omega_i \omega_m P_{(r+1)m} \\
&= (k \omega_m + 2 \omega_{m-1}) P_{(r+1)m+i} \\
&\quad + (\omega_{m-2} \omega_{i+1} - \omega_{m-1} \omega_i) P_{(r+1)m-1} \quad (1.3.30) \\
&\quad + (\omega_m \omega_i - \omega_{m-1} \omega_{i+1}) P_{(r+1)m-2}.
\end{aligned}$$

Now

$$\begin{aligned}
P_{rm+i} &= P_{rm+i+2} - P_{rm+i+1} \\
&= 2P_{rm+i+2} - P_{rm+i+3} \\
&= 2P_{rm+i+4} - 3P_{rm+i+3}.
\end{aligned}$$

If

$$P_{rm+i} = (-1)^j \{ \omega_{j+1} P_{rm+i+j} - \omega_j P_{rm+i+j+1} \},$$

Then

$$\begin{aligned} P_{r+m+i} &= (-1)^j \{ \omega_{j+1} P_{r+m+i+j+2} - (\omega_{j+1} + \omega_j) P_{r+m+i+j+1} \} \\ &= (-1)^{j+1} \{ \omega_{j+2} P_{r+m+i+j+1} - \omega_{j+1} P_{r+m+i+j+2} \}, \end{aligned}$$

so that the formula is valid by induction for $j < m-i-1$.

Hence

$$P_{r+m+i} = (-1)^{m-i} \{ \omega_{m-i-1} P_{(r+1)m-2} - \omega_{m-i-2} P_{(r+1)m-1} \}.$$

Thus

$$\begin{aligned} -(-1)^m P_{r+m+i} &= (-1)^i \{ \omega_{m-i-2} P_{(r+1)m-1} - \omega_{m-i-1} P_{(r+1)m-2} \} \\ &= (\omega_{m-2} \omega_{i+1} - \omega_{m-1} \omega_i) P_{(r+1)m-1} \\ &\quad + (\omega_m \omega_i - \omega_{m-1} \omega_{i+1}) P_{(r+1)m-2}, \end{aligned}$$

by theorem 7. Comparing this with (1.3.30) yields (1.3.23).

No where in the proof of (1.3.23) did we use the starting values of $\{P_n\}$, P_0 and P_1 . Thus (1.3.23) is true for any sequence $\{X_n\}$ which satisfies the recurrence

$$X_{n+2} = a_{n+2} X_{n+1} + X_n,$$

$$a_{rm} = k, \quad a_n = 1, \quad \text{otherwise.}$$

In particular it is true for $X_n = Q_n$ and this proves (1.3.24).

Chapter II

2.1 k-Fibonacci Sequences.

Define these sequences by¹

$$q_n = \frac{1}{\sqrt{k^2+4}} \{A^n - \bar{A}^n\}, \quad (2.1.1)$$

$$r_n = A^n + \bar{A}^n. \quad (2.1.2)$$

Theorem 9:

$$q_{n+2} = k q_{n+1} + q_n, \quad (2.1.3)$$

$$r_{n+2} = k r_{n+1} + r_n, \quad (2.1.4)$$

where $q_0 = 0, \quad q_1 = 1, \quad (2.1.5)$

$$r_0 = 2, \quad r_1 = k. \quad (2.1.6)$$

In particular

$$q_n = \omega_n, \quad r_n = \rho_n. \quad (2.1.7)$$

proof: As in the proof of theorem 6 A and $\bar{A}$ are both roots of

$$x^2 - kx - 1 = 0.$$

Thus

$$\begin{aligned} k q_{n+1} + q_n &= \frac{1}{\sqrt{k^2+4}} \{(kA+1)A^n - (k\bar{A}+1)\bar{A}^n\} \\ &= \frac{1}{\sqrt{k^2+4}} \{A^{n+2} - \bar{A}^{n+2}\} = q_{n+2}. \end{aligned}$$

Similarly

$$\begin{aligned} k r_{n+1} + r_n &= (kA+1)A^n + (k\bar{A}+1)\bar{A}^n \\ &= A^{n+2} + \bar{A}^{n+2} = r_{n+2}. \end{aligned}$$

¹ When necessary we write $k q_n, k r_n$.

$$q_0 = \frac{1}{\sqrt{k^2+4}}(1-1) = 0,$$

$$q_1 = \frac{1}{\sqrt{k^2+4}} \left\{ \frac{k+\sqrt{k^2+4}}{2} - \frac{k-\sqrt{k^2+4}}{2} \right\} = 1,$$

$$r_0 = 1+1 = 2,$$

$$r_1 = \frac{k+\sqrt{k^2+4}}{2} + \frac{k-\sqrt{k^2+4}}{2} = k,$$

$$A\bar{A} = -1,$$

$$\begin{aligned} q_{-n} &= \frac{1}{\sqrt{k^2+4}} \{A^{-n} - \bar{A}^{-n}\} \\ &= \frac{1}{\sqrt{k^2+4}} \left\{ \frac{\bar{A}^n - A^n}{(A\bar{A})^n} \right\} = (-1)^{n-1} q_n, \end{aligned} \quad (2.1.8)$$

$$r_{-n} = \frac{\bar{A}^n + A^n}{(A\bar{A})^n} = (-1)^n r_n. \quad (2.1.9)$$

It follows that q_n and r_n are integral for all n .

Comparing (0.2.1)-(0.2.4) with (2.1.3)-(2.1.6) yields (2.1.7). Thus¹

$$\omega_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}, \quad (2.1.10)$$

$$p_n = \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n, \quad (2.1.11)$$

Referring to theorem 6 we see that the convergents

¹ (2.1.10) occurred before: (1.3.18).

to A are given by

$$\begin{aligned} P_n &= q_{n+2}, \\ Q_n &= q_{n+1}, \end{aligned} \quad (2.1.12)$$

Theorem 10: Any solution of the recurrence

$$X_{n+2} = kX_{n+1} + X_n, \quad (2.1.13)$$

is a linear combination of q_n and r_n .

proof: Set

$$X_0 = 2b, \quad X_1 = a + kb,$$

so that

$$a = X_1 - \frac{kX_0}{2}, \quad b = \frac{X_0}{2}.$$

Then

$$X_0 = aq_0 + br_0,$$

If

$$X_1 = aq_1 + br_1,$$

$$X_n = aq_n + br_n, \quad (2.1.14)$$

and

$$X_{n+1} = aq_{n+1} + br_{n+1},$$

then

$$\begin{aligned} kX_{n+1} + X_n &= a(kq_{n+1} + q_n) + b(kr_{n+1} + r_n) \\ &= aq_{n+2} + br_{n+2}. \end{aligned}$$

so that (2.1.14) is generally true.

If Δ is the forward differencing operator:

$$\Delta X_n = X_{n+1} - X_n,$$

then (2.1.13) can be rewritten as

$$\Delta^2 X_n - (k-2)\Delta X_n - kX_n = 0,$$

and this theorem becomes an example of the theorem that a second order difference equation has only two linearly independent solutions (apart from arbitrary multiplicative functions which are periodic with period 1).

We may take

$$X_0 = q_m, X_1 = q_{m+1},$$

or

$$X'_0 = r_m, X'_1 = r_{m+1}.$$

Then

$$X_n = (q_{m+1} - \frac{1}{2}k q_m) q_n + \frac{1}{2} q_m r_n,$$

or

$$2X_n = (q_{m+1} + q_{m-1}) q_n + q_m r_n.$$

Similarly

$$2X'_n = (r_{m+1} + r_{m-1}) q_n + r_m r_n.$$

But clearly

$$X_n = q_{m+n}, X'_n = r_{m+n}.$$

Hence we have

Theorem 11:

$$2q_{m+n} = (q_{m+1} + q_{m-1}) q_n + q_m r_n, \quad (2.1.15)$$

$$2r_{m+n} = (r_{m+1} + r_{m-1}) q_n + r_m r_n. \quad (2.1.16)$$

We now follow Hardy and Wright¹ extending their results to $k = 2, 3, \dots$.

Theorem 12:

$$(a) \quad 2q_{m+n} = q_m r_n + q_n r_m, \quad (2.1.17)$$

$$(b) \quad r_n^2 - (k^2 + 4) q_n^2 = (-1)^n 4, \quad (2.1.18)$$

$$(c) \quad q_n^2 - q_{n+1} q_{n-1} = (-1)^{n+1}, \quad (2.1.19)$$

$$(d) \quad r_n^2 - r_{n+1} r_{n-1} = (-1)^n (k^2 + 4), \quad (2.1.20)$$

proof: (a) (2.1.17) will follow from (2.1.15) if we can show

¹ loc.cit., p.p.148-150.

$$q_{m+1} + q_{m-1} = r_m. \quad (2.1.21)$$

If it is true for m it is true for $-m$ by (2.1.8) and (2.1.9).

Since $q_{-1} = q_1 = 1$ and $r_0 = 2$, it is true for $m = 0$.

Since $q_2 = k$, $q_0 = 0$, $r_1 = k$, it is true for $m = 1$.

If it is true for m and $m+1$ then

$$\begin{aligned} q_{m+3} + q_{m+1} &= k(q_{m+2} + q_m) + q_{m+1} + q_{m-1} \\ &= k r_{m+1} + r_m = r_{m+2}. \end{aligned}$$

Thus it is true for all m .

$$\begin{aligned} (b) \quad r_n^2 - (k^2 + 4) q_n^2 &= A^{2n} + 2(A\bar{A})^n + \bar{A}^{2n} - (A^{2n} - 2(A\bar{A})^n + \bar{A}^{2n}) \\ &= 4(A\bar{A})^n = (-1)^n 4. \end{aligned}$$

(c) follows from (2.1.12) and theorem 5 part (5).

$$\begin{aligned} (d) \quad r_n^2 - r_{n+1} r_{n-1} &= A^{2n} + 2(A\bar{A})^n + \bar{A}^{2n} - A^{2n} - (A\bar{A})^n \left(\frac{A}{\bar{A}} + \frac{\bar{A}}{A} \right) - \bar{A}^{2n} \\ &= (-1)^n (k^2 + 4). \end{aligned}$$

Theorem 13:

$$(i) \quad (q_n, q_{n+1}) = 1. \quad (2.1.22)$$

$$(ii) \quad (r_n, r_{n+1}) = \begin{cases} 1 & k \text{ odd} \\ 2 & k \text{ even} \end{cases}. \quad (2.1.23)$$

$$(iii)^1 \quad q_n \mid q_{sn} \quad \text{for } s = 0, \pm 1, \pm 2, \dots \quad (2.1.24)$$

¹ $r \mid s$ or 'r divides s' means that the integer r is a factor of the integer s.

(iv)

$$\begin{aligned}
 (q_n, r_n) &= 1 \begin{cases} k \text{ odd}, n \not\equiv 0 \pmod{3} \\ k \text{ even}, n \text{ odd} \end{cases} \\
 &= 2 \begin{cases} k \text{ odd}, n \equiv 0 \pmod{3} \\ k \text{ even}, n \text{ even} \end{cases}
 \end{aligned}
 \tag{2.1.25}$$

(v) If

$$\begin{aligned}
 (m, n) &= d \\
 (q_m, q_n) &= q_d.
 \end{aligned}
 \tag{2.1.26}$$

(vi) If

$$\begin{aligned}
 (m, n) &= 1 \\
 q_m \cdot q_n &\mid q_{mn}
 \end{aligned}
 \tag{2.1.27}$$

Proof: (i) follows from theorem 12 (a).

$$\begin{aligned}
 \text{(ii)} \quad (r_{n+1}, r_n) &= (kr_n + r_{n-1}, r_n) \\
 &= (r_{n-1}, r_n) = \dots \\
 &= (r_1, r_0) = (k, 2).
 \end{aligned}$$

(iii) If it is true for s it is true for $-s$. It is true for $s = 0$. Let it be true for $s = 1, 2, \dots, R-1$.

We have by (a) theorem 12

$$2q_{Rn} = q_n r_{(R-1)n} + q_{(R-1)n} r_n.$$

If q_n is odd, $q_n \mid q_{(R-1)n}$ ex hypoth., $q_n \mid 2q_{Rn}$ so that $q_n \mid q_{Rn}$. If q_n is even, r_n is even by (b), $q_{(R-1)n}$ is even ex hypoth., and $r_{(R-1)n}$ is even by (b). Thus

$$q_{Rn} = q_n \cdot \frac{1}{2} r_{(R-1)n} + q_{(R-1)n} \cdot \frac{1}{2} r_n,$$

so that

$$q_n \mid q_{Rn}.$$

(iv) By (2.1.21),

$$r_n = k q_n + 2 q_{n-1},$$

so that

$$\begin{aligned} (q_n, r_n) &= (q_n, 2 q_{n-1}) \\ &= 1 \quad \text{if } q_n \text{ is odd} \\ &= 2 \quad \text{if } q_n \text{ is even.} \end{aligned}$$

If k is odd it follows from

$$q_{n+2} = k q_{n+1} + q_n,$$

that q_n is even only if $n \equiv 0 \pmod{3}$.

If k is even q_n is even only if n is even.

(v) If $(m, n) = d$ there exist (not necessarily positive) integers a and b such that

$$am + bn = d.$$

Thus

$$2q_d = q_{am} r_{bn} + q_{bn} r_{am}.$$

Let

$$(q_m, q_n) = \Delta.$$

Then

$$\Delta \mid q_m, \quad \Delta \mid q_n,$$

so that

$$\Delta \mid q_{am}, \quad \Delta \mid q_{bn}. \quad \therefore \Delta \mid 2q_d.$$

If Δ is odd $\Delta \mid q_d$. If Δ is even so are q_{am}, q_{bn}, r_{am} , and r_{bn} .

$$\therefore q_d = q_{am} \cdot \frac{1}{2} r_{bn} + q_{bn} \cdot \frac{1}{2} r_{am}.$$

$$\therefore \Delta \mid q_d.$$

Also

$$q_d \mid q_m \quad \text{and} \quad q_d \mid q_n.$$

$$\therefore q_d \mid \Delta. \quad \therefore q_d = \Delta.$$

(vi) is a corollary of (iii) and (v).

Theorem 14:¹ For a fixed k ,

(a) every prime divides one q_n

(b) and therefore an infinity of q_n .

proof: By (2.1.1)³

$$2^{n-1} q_n = \binom{n}{1} k^{n-1} + \binom{n}{3} k^{n-3} (k^2+4) + \binom{n}{5} k^{n-5} (k^2+4)^2 + \dots \quad (2.1.28)$$

where the last term is

$$(k^2+4)^{\frac{1}{2}(n-1)},$$

if n is odd and

$$n k (k^2+4)^{\frac{1}{2}n-1},$$

if n is even.

Let p be an odd prime.

$$p \mid \binom{p}{m} \quad \text{for} \quad 1 \leq m \leq p-1,$$

for the numerator of $\binom{p}{m}$ contains p while the denominator does not.

$$2^{p-1} \equiv 1 \pmod{p},$$

by Euler's theorem² so that

¹ For the case $k=1$ earlier writers called (a) the law of apparition and (b) the law of repetition.

² Hardy and Wright, op.cit., p.63.

³ $\binom{n}{m}$ denotes the binomial coefficient with the convention $\binom{0}{0} = 1$.

$$\frac{n(n-1)\dots(n-m+1)}{1 \cdot 2 \cdot 3 \dots m}$$

$$g_p \equiv (k^2 + 4)^{\frac{1}{2}(p-1)} \pmod{p}.$$

If $k^2 + 4$ is a multiple of p then

$$g_p \equiv 0 \pmod{p}, \text{ i.e. } p \mid g_p.$$

If $k^2 + 4$ is not a multiple of p then

$$g_p \equiv \pm 1 \pmod{p},$$

by the formula for quadratic residues¹, and

$$g_p^2 \equiv 1 \pmod{p}.$$

Thus

$$g_{p-1} g_{p+1} \equiv 0 \pmod{p},$$

by (2.1.19) so that p divides at least one of g_{p-1} and g_{p+1} .

We have noted already that

$$2 \mid g_{2n} \quad \text{if } k \text{ is even,}$$

$$\text{and } 2 \mid g_{3n} \quad \text{if } k \text{ is odd.}$$

Thus every prime p divides some g_n : $p \mid g_{np}$.

Hence $p \mid g_{sn_p}$, $s = 0, \pm 1, \pm 2, \dots$,

by (2.1.24) and statement (b) of the theorem follows.

Returning to the case p an odd prime and $k^2 + 4$ not a multiple of p , since $(g_{p-1}, g_{p+1}) = g_2 = k$, p divides both g_{p-1} and g_{p+1} only when k is a multiple of p . In the case k not a multiple of p it is easy to establish a criterion to decide whether

¹ Hardy and Wright, op.cit., p.69.

$$p \mid q_{p-1} \quad \text{or} \quad p \mid q_{p+1}.$$

$$\begin{aligned} 2^p q_{p+1} &= \binom{p+1}{1} k^p + \binom{p+1}{3} k^{p-2} (k^2+4) \\ &\quad + \dots + (p+1) k (k^2+4)^{\frac{1}{2}(p-1)}. \end{aligned}$$

Only the first and last terms are not divisible by p and

$$\binom{p+1}{1} = p+1 \equiv 1 \pmod{p}.$$

Thus

$$2^p q_{p+1} \equiv k + k (k^2+4)^{\frac{1}{2}(p-1)} \pmod{p}.$$

We now define Legendre's symbol for any $m \not\equiv 0 \pmod{p}$ by

$$\left(\frac{m}{p}\right) = \begin{cases} +1 & \text{if there exists an integer } x \text{ such that} \\ & x^2 \equiv m \pmod{p} \\ -1 & \text{otherwise.} \end{cases}$$

By the theorem¹

$$\left(\frac{m}{p}\right) \equiv m^{\frac{1}{2}(p-1)} \pmod{p}, \quad (2.1.29)$$

we have

$$2^p q_{p+1} \equiv k \left\{ 1 + \left(\frac{k^2+4}{p}\right) \right\} \pmod{p}.$$

Since p divides one of q_{p+1} , q_{p-1} ,

$$p \mid q_{p-1} \quad \text{if} \quad \left(\frac{k^2+4}{p}\right) = +1, \quad (2.1.30)$$

$$p \mid q_{p+1} \quad \text{if} \quad \left(\frac{k^2+4}{p}\right) = -1. \quad (2.1.31)$$

¹ Hardy and Wright, op.cit., p.69.



Recapitulating we have,

$$\begin{array}{ll}
 k = 2r & 2 \mid g_{25}, \\
 k \neq 2r & 2 \mid g_{35}, \\
 (k^2+4) = pt & p \mid g_{ps}, \\
 (k^2+4) \neq pt \quad k = pr & p \mid g_{(p-1)s}, \quad p \mid g_{(p+1)s} \\
 (k^2+4) \neq pt \quad k \neq pr & \begin{cases} p \mid g_{(p-1)s} \text{ if } k^2+4 \text{ is a} \\ \text{quadratic residue mod } p \\ p \mid g_{(p+1)s} \text{ if } k^2+4 \text{ is a} \\ \text{quadratic non-residue mod } p. \end{cases}
 \end{array}$$

Some examples are;¹

$$\underline{k=1} \quad \left(\frac{5}{3}\right) = \left(\frac{2}{3}\right) = -1 \quad \therefore 3 \mid w_4,$$

$$k^2+4 = 5 \quad \therefore 5 \mid w_5,$$

$$\left(\frac{5}{7}\right) = -1 \quad \therefore 7 \mid w_8,$$

$$\left(\frac{5}{11}\right) = +1 \quad \therefore 11 \mid w_{10},$$

$$\underline{k=6} \quad k = 3r \quad \therefore 3 \mid g_2, \quad 3 \mid g_4$$

$$k^2+4 = 5t \quad \therefore 5 \mid g_5,$$

$$\left(\frac{40}{7}\right) = \left(\frac{5}{7}\right) = -1 \quad \therefore 7 \mid g_8,$$

$$\left(\frac{40}{11}\right) = \left(\frac{7}{11}\right) = -1 \quad \therefore 11 \mid g_{12}.$$

¹ We use $\left(\frac{m}{p}\right) = \left(\frac{m'}{p}\right)$ if $m \equiv m' \pmod{p}$.

Theorem 14 is not true for p_n . We have in regard to part (a)

$$2|p_3, 3|p_2, 7|p_4, 11|p_5,$$

$$19|p_9, 23|p_{12}, 29|p_7, \dots$$

But the primes 5, 13, 17, ... do not divide any p_n ; the residues of p_n are periodic with period r , the residue 0 never occurring, according to the following scheme:

$$p=5, r=4; p=13, r=26; p=17, r=36; \dots$$

No general rule was observed.

However the 'law of repetition' is still true; in fact

$$r_n | r_{3n} | r_{9n} | r_{27n} | \dots,$$

since

$$A^{3n} + \bar{A}^{3n} = (A^n + \bar{A}^n)(A^{2n} - (-1)^n + \bar{A}^{2n}),$$

or

$$r_{3n} = r_n \{ r_{2n} - (-1)^n \}. \quad (2.1.32)$$

Theorem 15: The generating function of the numbers X_n

defined by (2.1.13) is

$$\frac{X_0 + (X_1 - kX_0)t}{1 - kt - t^2} = \sum_{n=0}^{\infty} X_n t^n. \quad (2.1.33)$$

In particular,

$$\frac{t}{1 - kt - t^2} = \sum_{n=0}^{\infty} q_n t^n, \quad (2.1.34)$$

$$\frac{2 - kt}{1 - kt - t^2} = \sum_{n=0}^{\infty} r_n t^n. \quad (2.1.35)$$

Thus

$$\frac{t}{1-t-t^2} = \sum_{n=0}^{\infty} \omega_n t^n, \quad (2.1.36)$$

$$\frac{2-t}{1-t-t^2} = \sum_{n=0}^{\infty} \rho_n t^n. \quad (2.1.37)$$

proof: Here we are interested only in formal identities: the right hand side of each equation of the theorem is the result of the division indicated on the left hand side. However it is not hard to see that each sum has a finite radius of convergence (and therefore its sum must be the left hand side of the equation); for

$$|\bar{A}| < 1 < A, \quad \sqrt{k^2+4} > 2$$

so that

$$0 < q_n < A^n, \quad r_n < 2A^n$$

Similarly

$$|x_n| < K A^n$$

for some constant K.

Multiplying (2.1.33) through by $(1-kt-t^2)$

and comparing coefficients we get

$$\begin{aligned} X_0 &= X_0, \\ X_1 - kX_0 &= X_1 - kX_0, \\ 0 &= -X_0 - kX_1 + X_2, \\ &\vdots \\ 0 &= -X_n - kX_{n+1} + X_{n+2}, \end{aligned}$$

which proves the theorem.

Similarly we have

$$\frac{t}{1-kt-t^2} = - \sum_{n=0}^{\infty} q_{-n} t^{-n}, \quad (2.1.38)$$

(convergent for large $|t|$).

2.2 The Subscript Rules.

Theorem 16: Let $\alpha > 1$. If

$$\alpha = [a_0, a_1, \dots]$$

has convergents P_n/Q_n and $1/\alpha$ has convergents P'_n/Q'_n then

$$P'_0 = 0, \quad Q'_0 = 1, \quad (2.2.1)$$

$$P'_n = Q_{n-1}, \quad Q'_n = P_{n-1}. \quad (2.2.2)$$

proof: Since $\alpha > 1$, $a_0 \geq 1$. Thus

$$\frac{1}{\alpha} = \frac{1}{a_0 + \frac{1}{a_1 + \dots}} = [0, a_0, a_1, \dots],$$

and (2.2.1) and (2.2.2) follow directly.

Theorem 17: If $\{U_n\}$ and $\{V_n\}$ are Beatty sequences of argument α and if α has convergents P_n/Q_n then

$$U_{P_{2n}} = P_{2n} + Q_{2n} - 1, \quad (2.2.3)$$

$$U_{P_{2n+1}} = P_{2n+1} + Q_{2n+1}, \quad (2.2.4)$$

$$V_{Q_{2n}} = P_{2n} + Q_{2n}, \quad (2.2.5)$$

$$V_{Q_{2n+1}} = P_{2n+1} + Q_{2n+1} - 1, \quad (2.2.6)$$

all for $n = 0, 1, 2, \dots$.

proof: We can take $\alpha > 1$; for taking $0 < \alpha < 1$ is equivalent to interchanging α and $1/\alpha$ which interchanges

$\{u_n\}$ and $\{v_n\}$ which gives the same Beatty pair. In these circumstances the P's and Q's are interchanged and so is the parity of the subscripts, and it is easy to see that (2.2.3)-(2.2.6) are changed into an equivalent set.

By theorems 5 and 16,

$$\alpha = \frac{P_{2n}}{Q_{2n}} + \gamma_1, \quad 0 < \gamma_1 < \frac{1}{Q_{2n}^2},$$

$$\alpha = \frac{P_{2n+1}}{Q_{2n+1}} - \gamma_2, \quad 0 < \gamma_2 < \frac{1}{Q_{2n+1}^2},$$

$$\frac{1}{\alpha} = \frac{Q_{2n}}{P_{2n}} - \gamma_3, \quad 0 < \gamma_3 < \frac{1}{P_{2n}^2},$$

$$\frac{1}{\alpha} = \frac{Q_{2n+1}}{P_{2n+1}} + \gamma_4, \quad 0 < \gamma_4 < \frac{1}{P_{2n+1}^2}.$$

$$u_{P_{2n}} = \left[P_{2n} \left\{ 1 + \frac{Q_{2n}}{P_{2n}} - \gamma_3 \right\} \right]$$

$$= P_{2n} + Q_{2n} - 1,$$

since

$$-1 < -P_{2n} \gamma_3 < 0.$$

The rest follow just as easily.

By (2.1.12) we have the corollary

Theorem 18: If $\{u_n\}$ and $\{v_n\}$ are Wythoff sequences

$$u_{q_{2n-1}} = q_{2n-1} + q_{2n-2}, \quad (2.2.7)$$

$$u_{q_{2n}} = q_{2n} + q_{2n-1} - 1, \quad (2.2.8)$$

$$v_{q_{2n-1}} = q_{2n} + q_{2n-1} , \quad (2.2.9)$$

$$v_{q_{2n}} = q_{2n+1} + q_{2n} - 1 . \quad (2.2.10)$$

for $n = 1, 2, 3, \dots$

Setting $k=1$ we have proved conjectures (0.2.5)-(0.2.8). The conjectures (0.2.9)-(0.2.12) are corollaries of the next theorem.

Theorem 19: If $\{u_n\}$ and $\{v_n\}$ are Wythoff sequences

$$u_{r_{2n}} = r_{2n} + r_{2n-1} , \quad (2.2.11)$$

$$u_{r_{2n+1}} = r_{2n+1} + r_{2n} - 1 , \quad (2.2.12)$$

$$v_{r_{2n}} = r_{2n+1} + r_{2n} , \quad (2.2.13)$$

$$v_{r_{2n+1}} = r_{2n+2} + r_{2n+1} - 1 , \quad (2.2.14)$$

for $n = 1, 2, 3, \dots$

proof:

$$\begin{aligned} u_{r_n} &= \left[(A^n + \bar{A}^n) \left(1 + \frac{1}{A} \right) \right] \\ &= r_n + r_{n-1} + \left[\bar{A}^n \left(\frac{1}{A} - \frac{1}{\bar{A}} \right) \right] \\ &= r_n + r_{n-1} + \left[\sqrt{k^2 + 4} \bar{A}^n \right] . \end{aligned}$$

Now

$$-1 < \bar{A} < 0$$

and the least subscript which occurs on r_n is 2 so that (2.2.11) and (2.2.12) will follow if we can show

$$\sqrt{k^2+4} \left(\frac{\sqrt{k^2+4} - k}{2} \right)^2 < 1,$$

$$\text{i.e. } \sqrt{k^2+4} (k^2+2 - k\sqrt{k^2+4}) < 2,$$

$$\text{or } (k^2+2)\sqrt{k^2+4} < 2 + k(k^2+4).$$

Squaring:

$$k^6 + 8k^4 + 20k^2 + 16 < k^6 + 8k^4 + 4k^3 + 16k^2 + 16k + 4,$$

$$\text{or } 4k^2 + 16 < 4k^3 + 16k + 4,$$

which is true for $k \geq 1$.

$$v_{2n} = u_{2n} + k r_{2n} \quad \text{by (1.2.2)}$$

$$= r_{2n} + (r_{2n-1} + k r_{2n})$$

$$= r_{2n} + r_{2n+1} \quad \text{by (2.1.4)}$$

and similarly for (2.2.14).

2.3 A Restatement of Theorem 1.

Theorem 20: The sequences

$$u_n = [n\xi], \quad v_n = [n\eta], \quad (2.3.1)$$

for $n = 1, 2, 3, \dots$ are complementary if and only if $\xi > 1$, ξ is irrational and

$$\frac{1}{\xi} + \frac{1}{\eta} = 1. \quad (2.3.2)$$

proof: If the conditions of the theorem are satisfied and if we write $\xi = 1 + \frac{1}{\alpha}$,then $\alpha > 0$, α is irrational and

$$\eta = \frac{\xi}{\xi - 1} = \frac{1 + \frac{1}{\alpha}}{1/\alpha} = 1 + \alpha.$$

By theorem 1 the sequences are complementary.

Conversely, let the sequences (2.3.1) be complementary.

If there are r members of $\{u_n\}$ and s members of $\{v_n\}$ less than N

$$r\xi < N < (r+1)\xi \quad \text{or} \quad r < \frac{N}{\xi} < r+1,$$

$$s\eta < N < (s+1)\eta \quad \text{or} \quad s < \frac{N}{\eta} < s+1.$$

Since $r+s = N-1$,

$$N-1 < N\left(\frac{1}{\xi} + \frac{1}{\eta}\right) < N+1,$$

for all N . Therefore

$$\frac{1}{\xi} + \frac{1}{\eta} = 1.$$

Since ξ and η are positive this demands they both exceed 1.

Writing $\xi = 1 + \frac{1}{\alpha}$ we get $\eta = 1 + \alpha$. If α were rational we would have

$\xi = 1 + \frac{a}{b}$, $\eta = 1 + \frac{b}{a}$ then $u_b = a + b$ and $v_a = a + b$ which is impossible. Hence α is irrational.

Thus we can say that the sequences (1.1.1) and (1.1.2) are complementary if and only if $\alpha > 0$, and α is irrational.

Taking $\eta = \psi(\xi)$ we have

$$\frac{1}{\xi} + \frac{1}{\psi(\xi)} = 1, \quad (2.3.3)$$

an equation in ξ . For example if

$$\eta = \frac{a\xi + b}{c}, \quad (2.3.4)$$

where a, b and c are integers then

$$\xi = \frac{(a-b+c) + \sqrt{(a-b+c)^2 + 4ab}}{2a}. \quad (2.3.5)$$

In terms of the Beatty Sequences (2.3.4) is

$$v_{cn} = u_{an} + bn. \quad (2.3.6)$$

For this to be meaningful we demand that a and c be positive integers and b any integer such that $\alpha > 1$. The latter condition amounts to

$$b > 2(c-a),$$

where we have used the expression for α given in the following theorem.

Theorem 21: If $c\eta = a\xi + b$, a and c positive integers, b an integer and $b > 2(c-a)$

then
$$\alpha = \frac{(a+b-c) + \sqrt{(a+b-c)^2 + 4ac}}{2c}, \quad (2.3.7)$$

and
$$v_{cn} = u_{an} + bn.$$

(2.3.7) is obtained by straightforward calculation.

Setting

$$a = k\omega_{m-1} + \omega_{m-2}, \quad (2.3.8)$$

$$b = k\omega_{m-2} + \omega_{m-1}, \quad (2.3.9)$$

$$c = \omega_m, \quad (2.3.10)$$

we obtain, after reduction and use of theorem 8,

$$\alpha = A_{m,k}.$$

Theorem 22: If $\{u_n\}, \{v_n\}$ are Beatty sequences of argument $A_{m,k}$ then

$$v_{\omega_m n} = u_{(k\omega_{m-1} + \omega_{m-2})n} + (k\omega_{m-2} + \omega_{m-1})n. \quad (2.3.11)$$

Examples of this theorem are:

$$m=1, \quad \alpha = \frac{k + \sqrt{k^2 + 4}}{2}, \quad v_n = u_n + kn, \quad (1.2.2)$$

$$m=2, \quad \alpha = \frac{k + \sqrt{k^2 + 4k}}{2}, \quad v_n = u_{kn} + n. \quad (2.3.12)$$

This theorem can be improved for special values of k :

Theorem 23: If $\alpha = A_m, k\omega_m + 1$, $k = 0, 1, 2, \dots$,

$$\text{then } v_n = u_{(k\omega_{m-1}+1)n} + (k\omega_{m-2}+1)n. \quad (2.3.13)$$

We omit the proof, which is straightforward.

If

$$\eta = b - a\xi, \quad (2.3.14)$$

$$\begin{aligned} \text{then } [n\eta] &= [n(b - a\xi)] = nb + [-an\xi] \\ &= nb - [an\xi] - 1, \end{aligned}$$

i.e.

$$v_n = bn - 1 - u_{an}. \quad (2.3.15)$$

If $a=1$, $b=5, 6, 7, \dots$ the sequences $\{u_n\}$ and $\{v_n\}$ are known as Fan sequences; these will be considered in more detail later.

2.4 Reduplication.

The title of this section refers to the following striking subscript property exhibited by a certain class of Beatty sequences:

$$u_{v_n} = u_n + v_n. \quad (2.4.1)$$

If (2.4.1) is true for all n we say that reduplication holds for that value of α . Using the notation introduced in the previous section (2.4.1) is

$$[[n\eta]\xi] = [n\xi] + [n\eta] . \quad (2.4.2)$$

We assume, as before, that $1 < \xi < 2$. Let

$$[n\xi] = n\xi - \epsilon, \quad 0 < \epsilon < 1, \quad (2.4.3)$$

and

$$[n\eta] = n\eta - \delta, \quad 0 < \delta < 1. \quad (2.4.4)$$

Then (2.4.2) implies

$$[n\xi] + [n\eta] < [n\eta]\xi < [n\xi] + [n\eta] + 1,$$

or

$$n(\xi + \eta) - \epsilon - \delta < n\eta\xi - \delta\xi < n(\xi + \eta) - \epsilon - \delta + 1,$$

or, since $\xi + \eta = \eta\xi$,

$$-\epsilon - \delta < -\delta\xi < -\epsilon - \delta + 1.$$

Now $-\delta\xi < -\epsilon - \delta + 1$ is invariably true since $\xi > 1$

$\delta > 0$ and $\epsilon < 1$. Thus (2.4.2) (hence (2.4.1)) is equivalent to

$$-\epsilon - \delta < -\delta\xi,$$

i.e.

$$\delta(\xi - 1) < \epsilon. \quad (2.4.5)$$

Since $\xi = 1 + \frac{1}{\alpha}$ this may be written as

$$\delta < \alpha\epsilon. \quad (2.4.6)$$

Define

$$\lambda(n) = 1 \quad \text{if (2.4.6) is true,}$$

$$\lambda(n) = 0 \quad \text{if (2.4.6) is false,}$$

and

$$\Lambda(N) = \sum_{n=1}^N \lambda(n) \quad (2.4.7)$$

Thus $\Lambda(N)$ is the number of $n \leq N$ for which (2.4.1) is valid. Hence $\Lambda(N) = N$ for all N if and only if reduplication holds.

We shall determine all α for which reduplication holds and evaluate $\Lambda(N)$ (asymptotically) in the other cases. However first we prove two easy theorems which will be improved upon later.

Theorem 24: Let $\{u_n\}$ and $\{v_n\}$ be Beatty sequences. If there exists a formula

$$v_n = u_{an} + bn, \quad (2.4.8)$$

valid for all n , with fixed positive integers a and b then reduplication holds.

proof: (2.4.8) implies

$$\eta = a\xi + b, \quad (2.4.9)$$

If $r(x)$ denotes the fractional part of x , $x - [x]$, then

$$r(n\eta) = r(na\xi).$$

Thus¹

$$\delta = r(n\eta) \leq ar(a\xi) = a\epsilon. \quad (2.4.10)$$

Now

$$\alpha = \eta - 1 > a,$$

since b is a positive integer and $\xi > 1$. (2.4.10) thus guarantees (2.4.6) and the theorem is proved.

¹ It is here we need a positive integer: $r(a\epsilon) \leq a\epsilon$.

We shall see later that (2.4.8) is also a necessary condition for reduplication.

The notation $f(n) \sim g(n)$ means

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1,$$

and $f(n) = o(g(n))$ means

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0.$$

Theorem 25:

$$\Lambda(N) > \frac{N}{2} \left(\frac{\alpha-1}{\alpha} \right) + o(1). \quad (2.4.11)$$

proof: If (2.4.2) is true

$$[n\eta]\xi < [n\xi] + [n\eta] + 1,$$

and if it is false

$$[n\eta]\xi < [n\xi] + [n\eta].$$

In either case

$$[n\eta]\xi < [n\xi] + [n\eta] + \lambda(n). \quad (2.4.12)$$

A theorem of W. Sierpinski¹ states that for x irrational

$$\sum_{n=1}^N [nx] = \frac{N(N+1)}{2} x - \frac{N}{2} + o(1). \quad (2.4.13)$$

Adding inequalities (2.4.12) and applying (2.4.13)

$$\frac{N(N+1)}{2} \eta \xi - \frac{N}{2} \xi + o(1) < \frac{N(N+1)}{2} (\xi + \eta) - N + \Lambda(N),$$

$$\text{i.e. } \Lambda(N) > \frac{N}{2} (2 - \xi) + o(1) = \frac{N}{2} \left(1 - \frac{1}{\alpha} \right) + o(1),$$

which is the theorem.

¹ Bull. Inter. Acad. Sc., Cracovie, 1909, II, p.p. 725-7.

Let S and T be sets of real numbers. We say that S is enumerable if its elements can be written in succession:

$$S = \{x_1, x_2, x_3, \dots\}.$$

S is dense in T if for every element z of T there is an element x of S such that

$$|z - x| < \epsilon,$$

for arbitrary $\epsilon > 0$.

For example it is well known that the set of all rational numbers is dense in the set of all real numbers.

Let T be the set of all real numbers in the unit interval:

$0 \leq z \leq 1$ and let S be an enumerable set $\{x_n\}$. Denote by I any subinterval of the unit interval and by N_I the number of elements x_n with $n \leq N$ in I . Letting I denote the length of the interval I , if

$$\lim_{N \rightarrow \infty} \frac{N_I}{N} = I,$$

for arbitrary I we say the set S is uniformly distributed in T . This property certainly implies that S is dense in T .

These definitions readily extend to two (and more) dimensions. If P is any point of T and if there exists a point Q of S arbitrarily near P then S is dense in T .

If x and y are cartesian coordinates in the euclidean plane let T be the unit square:

$$0 \leq x \leq 1,$$

$$0 \leq y \leq 1,$$

Let S be an enumerable set of points $\{P_n\}$ in T . Let Γ be a contiguous closed curve bounding a simply connected area γ in T and denote by N_γ the number of P_n with $n \leq N$ in γ . If

$$\lim_{N \rightarrow \infty} \frac{N_\gamma}{N} = \gamma,$$

for arbitrary Γ S is uniformly distributed in T . Again it is clear that uniform distribution implies density.

Finally we need to define linear dependence. The numbers $\theta_1, \theta_2, \dots, \theta_n$ are linearly dependent if there exist integers a_i not all zero such that

$$a_1 \theta_1 + a_2 \theta_2 + \dots + a_n \theta_n = 0.$$

If no such integers exist the numbers are linearly independent.

The following theorem (stated for the special case of two dimensions) was given first in terms of density by Kronecker.¹ Weyl² and others showed that the distributions are in fact uniform. We omit the proof.

Theorem 26: If ξ, η and 1 are linearly independent³ then the set of points with coordinates

$$(r(n\xi), r(n\eta)) \quad , \quad n = 1, 2, 3, \dots \quad (2.4.14)$$

is uniformly distributed in the unit square. If one and only one relation

¹ See Hardy and Wright, *op.cit.*, ch.23.

² *Math. Ann.*, 77 (1916), 313-52. Also Koksma, *Diophantische*

Approximationen. ³ Hence ξ and η are irrational.

$$c\eta = a\xi + b, \quad (2.4.15)$$

for integers a, b, c obtains then the set (2.4.14) is uniformly distributed on the segments within the unit square of the lines

$$cy = ax + v, \quad (2.4.16)$$

where v is any integer. The values of v which give rise to segments in the unit square are

$$\text{a positive} \quad -a < v < c,$$

$$\text{a negative} \quad 0 < v < c - a.$$

Our condition for reduplication is

$$\delta < \alpha \epsilon.$$

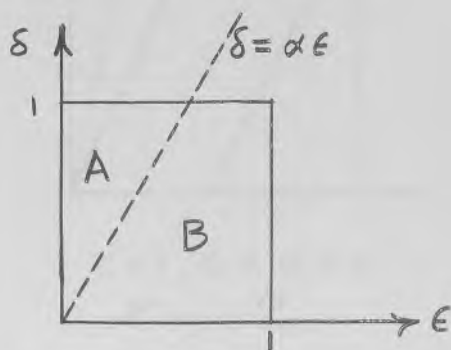


Fig. 1.

Thus plotting the points (ϵ, δ) we see that if a point lies in region A (fig.1) then (2.4.1) fails; if the point lies in region B (2.4.1) is true. Thus reduplication cannot hold if

ξ, η and 1 are linearly independent for A contains infinitely many points. In this case by the uniform distribution

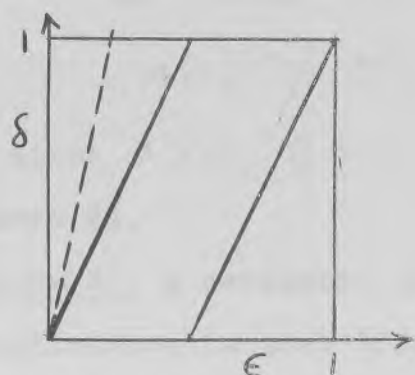
$$\lim_{N \rightarrow \infty} \frac{\Lambda(N)}{N} = \text{area of region B.}$$

That is

$$\Lambda(N) \sim N\left(\frac{2\alpha-1}{2\alpha}\right), \quad (2.4.17)$$

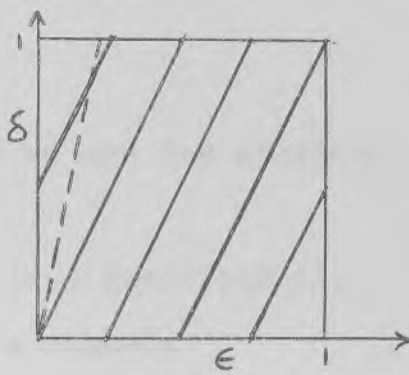
(c.f. theorem 25).

If there exists a relation (2.4.15) the possible situations are shown in the following diagrams. We disregard $a=0$ or $c=0$ which correspond to η or ξ rational, for by the additional condition (2.3.2) they would then both be rational and two relations of type (2.4.15) would exist. We take $c > 0$.



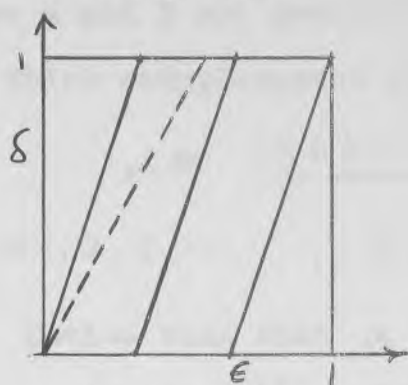
$$c=1, 0 < a < \alpha$$

Fig. 2.



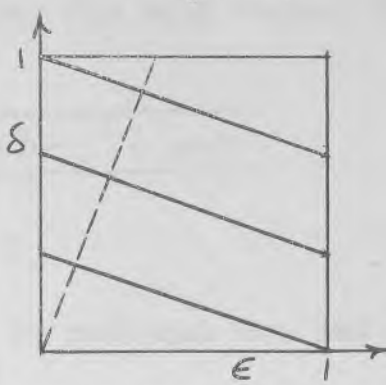
$$c > 1, 0 < a/c < \alpha$$

Fig. 3.



$$c > 0, a/c > \alpha$$

Fig. 4.



$$c > 0, a < 0$$

Fig. 5.

We see that reduplication can occur only in the case

$c=1, 0 < a < \alpha$. From

$$\frac{1}{\xi} + \frac{1}{\eta} = 1, \quad \eta = a\xi + b = 1 + \alpha,$$

we calculate

$$\begin{aligned} \alpha &= \frac{(a+b-1) + \sqrt{(a-b+1)^2 + 4ab}}{2} \\ &= \frac{(a+b-1) + \sqrt{(a+b-1)^2 + 4a}}{2}. \end{aligned} \quad (2.4.18)$$

$\alpha > a$ gives

$$\sqrt{(a-b+1)^2 + 4ab} > a-b+1,$$

or

$$4ab > 0,$$

and since $a > 0, b > 0$. Thus we can now complete theorem 24.

Theorem 27: A necessary and sufficient condition for reduplication is that there exist a formula

$$v_n = u_{an} + bn,$$

where a and b are positive integers. The only values of α for which reduplication holds are

$$\alpha = \frac{(a+b-1) + \sqrt{(a+b-1)^2 + 4a}}{2},$$

$$a = 1, 2, 3, \dots, \quad b = 1, 2, 3, \dots.$$

Notice that such α cannot be rational; for let

$$(a+b-1)^2 + 4a = k^2,$$

and let $k = a + b - 1 + h$. Then

$$4a = 2h(a + b - 1) + h^2,$$

Therefore h is even: $h = 2h_1$ and

$$a = h_1(a + b - 1) + h_1^2.$$

Thus h_1 is a positive integer; but $b \geq 1$. Hence

$$a \geq h_1(a + h_1),$$

which is impossible for a a positive integer.

Let us write (2.4.2) in terms of α .

$$\left[\left[n(1 + \alpha) \right] \left(1 + \frac{1}{\alpha} \right) \right] = \left[n \left(1 + \frac{1}{\alpha} \right) \right] + \left[n(1 + \alpha) \right],$$

or
$$\left[\frac{\left[n(1 + \alpha) \right]}{\alpha} \right] = \left[\frac{n(1 + \alpha)}{\alpha} \right].$$

Theorem 28: The statement

$$\left[\frac{\left[n(1 + \alpha) \right]}{\alpha} \right] = \left[\frac{n(1 + \alpha)}{\alpha} \right], \quad (2.4.19)$$

for $n = 1, 2, 3, \dots$,

is true for

(i) positive integers, $\alpha = k$.

(ii) the irrational numbers

$$\alpha = \frac{(a + b - 1) + \sqrt{(a + b - 1)^2 + 4a}}{2},$$

for $a = 1, 2, 3, \dots$, $b = 1, 2, 3, \dots$,

and is false for all other values of $\alpha > 0$. Written in

order of magnitude the possible values of α are

$$1, \frac{1 + \sqrt{5}}{2}, 2, 1 + \sqrt{2}, 1 + \sqrt{3}, 3, \frac{3 + \sqrt{13}}{2}, \frac{3 + \sqrt{17}}{2}, \frac{3 + \sqrt{21}}{2}, 4, \dots$$

proof: We first dispose of $0 < \alpha < 1$. Set $n = 1$. Then

$$\left[\frac{[1+\alpha]}{\alpha} \right] = \left[\frac{1}{\alpha} \right].$$

On the other hand

$$\left[\frac{1+\alpha}{\alpha} \right] = 1 + \left[\frac{1}{\alpha} \right],$$

and (2.4.19) is violated for this value of n .

(2.4.19) is clearly true if α is a positive integer.

But it is false for any non-integral rational for if

$$\alpha = \frac{p}{q}, \quad (p, q) = 1, \quad 1 < q < p,$$

set $n = p$ and let

$$p^2 = rq + s, \quad 0 < s < q.$$

Then

$$\begin{aligned} \left[\frac{[n(1+\alpha)]}{\alpha} \right] &= \left[(p+r) \frac{q}{p} \right] = q + \left[\frac{p^2-s}{p} \right] \\ &= p+q + \left[-\frac{s}{p} \right] = p+q-1, \end{aligned}$$

since

$$0 < s < p.$$

But

$$\begin{aligned} \left[\frac{n(1+\alpha)}{\alpha} \right] &= \left[(p+r + \frac{s}{q}) \frac{q}{p} \right] \\ &= p+q. \end{aligned}$$

Finally $\alpha > 1$, α irrational, gives rise to the values (ii) and no others by the last theorem.

Notice there are k values of α of type (ii) between k and $k+1$, for $a+b-1 < \alpha < a+b$.

We list without proof specific values of n for which (2.4.19) is violated for some values of α not in the 'spectrum'. The ranges of α for a given n nest about the points $\alpha = 1, (1+\sqrt{5})/2$, etc. (2.4.19) is violated for every value of α in the range $R(\alpha)$ for the given value of n .

(i) The range $1 < \alpha < 2$

(a) $1 < \alpha < 3/2$.

$$\begin{array}{l} n = m \\ m = 2, 3, 4, \dots \end{array} \quad R(\alpha) = \left(\frac{2m}{2m-1}, \frac{m+1}{m} \right),$$

(b) $3/2 < \alpha < \frac{1+\sqrt{5}}{2}$

$$\begin{array}{l} n = \omega_{2^m+1} \\ m = 2, 3, \dots \end{array} \quad R(\alpha) = \left(\frac{\omega_{2^{m-1}+2}}{\omega_{2^{m-1}+1}}, \frac{\omega_{2^m+2}}{\omega_{2^m+1}} \right),$$

(c) $\frac{1+\sqrt{5}}{2} < \alpha < 2$,

$$\begin{array}{l} n = \omega_{2m+1} \\ m = 1, 2, 3, \dots \end{array} \quad R(\alpha) = \left(\frac{\omega_{2m+1}}{\omega_{2m+2}}, \frac{\omega_{2m-1}}{\omega_{2m}} \right),$$

As the parameter m varies in each case the various $R(\alpha)$ abut, and the points of abutment are rational (which points have been considered above). Thus for every α not in the spectrum in the range $(1, 2)$ a specific n has been exhibited for which (2.4.19) is false.

In the remainder we combine the individual $R(\alpha)$ to get a continuous range. For example the combined range of (i)(a) would be $R(\alpha) = (1, 3/2)$.

$$(ii) \quad k < \alpha < k + \frac{1}{k}, \quad k = 2, 3, \dots$$

$$(a) \quad n = k+1, k+2, \dots,$$

$$R(\alpha) = \left(k, k + \frac{1}{k^2+1} \right).$$

$$(b) \quad n = k^2 + mk + 1, \quad m = 0, 1, 2, \dots, k-1,$$

$$R(\alpha) = \left(k + \frac{1}{k^2+1}, k + \frac{k}{k^2+k+1} \right),$$

$$(c) \quad n = g_{2m+1}, \quad m = 1, 2, 3, \dots,$$

$$R(\alpha) = \left(k + \frac{k}{k^2+k+1}, \frac{k + \sqrt{k^2+4}}{2} \right),$$

$$(d) \quad n = r g_{2m+2} + g_{2m+1}, \quad m = 1, 2, 3, \dots,$$

$$r = 0, 1, 2, \dots, k-1,$$

$$R(\alpha) = \left(\frac{k + \sqrt{k^2+4}}{2}, k + \frac{1}{k} \right),$$

$$(iii) \quad k + \frac{k}{k+2} < \alpha < k+1,$$

$$n = t_m, \quad m = 1, 2, 3, \dots \quad (\text{see p. 101}),$$

$$R(\alpha) = \left(k + \frac{k}{k+2}, \frac{k + \sqrt{k^2+4k}}{2} \right) \text{ and}$$

$$R(\alpha) = \left(\frac{k + \sqrt{k^2 + 4k}}{2}, k+1 \right).$$

It happens that every α in the range $(2,3)$ is accounted for. However for larger values of α , except those listed, I was unable to find appropriate n .

Noticing that the numbers

$$\frac{k + \sqrt{k^2 + 4k}}{2},$$

arise by setting $a=1$, $b=k$ in theorem 27 we have

Theorem 29: Reduplication holds for Wythoff sequences.

(2.4.17) gives the asymptotic value of $\Lambda(N)$ in the case that ξ, η and 1 are linearly independent. If they are dependent let s be the total length of line segments in region B (figure 1), and d the perpendicular distance between successive lines. Then $\Lambda(N) \sim Nsd$

by the uniform distribution and the fact that the total length of segments in the square is $\frac{1}{2}$. However the expression for s in the various cases is very complicated and we content ourselves with

Theorem 30: If $\alpha > 1$ is irrational but not a quadratic surd then

$$\Lambda(N) \sim N \left(\frac{2\alpha - 1}{2\alpha} \right).$$

Chapter III

3.1 Continued Subscripts.

By theorem 29, if $\{u_n\}$ and $\{v_n\}$ are Wythoff sequences

$$u_{v_n} = u_n + v_n. \quad (3.1.1)$$

Hence

$$v_{v_n} = u_n + (k+1)v_n, \quad (3.1.2)$$

by (1.2.2). Solving,

$$u_n = \frac{1}{k} \{ (k+1)u_{v_n} - v_{v_n} \}, \quad (3.1.3)$$

$$v_n = \frac{1}{k} \{ v_{v_n} - u_{v_n} \}. \quad (3.1.4)$$

Applying the reduplication formula once more,

$$u_n = \frac{1}{k^2} \{ (k^2 + 2k + 2)u_{v_{v_n}} - (k+2)v_{v_{v_n}} \}, \quad (3.1.5)$$

$$v_n = \frac{1}{k^2} \{ 2v_{v_{v_n}} - (k+2)u_{v_{v_n}} \}, \quad (3.1.6)$$

and so on.

Let us write

$$u_n^{(1)} = u_n, \quad u_n^{(2)} = u_{v_n}, \quad u_n^{(3)} = u_{v_{v_n}}, \dots,$$

and similarly

$$v_n^{(1)} = v_n, \quad v_n^{(2)} = v_{v_n}, \quad v_n^{(3)} = v_{v_{v_n}}, \dots$$

Suppose that

$$v_n = \frac{1}{k^{r-1}} \{ \alpha_r v_n^{(r)} - \beta_r u_n^{(r)} \}, \quad (3.1.7)$$

$$u_n = \frac{1}{k^{r-1}} \{ \gamma_r u_n^{(r)} - \delta_r v_n^{(r)} \} . \quad (3.1.8)$$

Then

$$v_n = \frac{1}{k^r} \left\{ \alpha_r v_n^{(r+1)} - \alpha_r u_n^{(r+1)} - \beta_r (k+1) u_n^{(r+1)} + \beta_r v_n^{(r+1)} \right\} , \quad (3.1.9)$$

$$u_n = \frac{1}{k^r} \left\{ \gamma_r (k+1) u_n^{(r+1)} - \gamma_r v_n^{(r+1)} - \delta_r v_n^{(r+1)} + \delta_r u_n^{(r+1)} \right\} , \quad (3.1.10)$$

Thus

$$\alpha_{r+1} = \alpha_r + \beta_r , \quad (3.1.11)$$

$$\beta_{r+1} = (k+1)\beta_r + \alpha_r = k\beta_r + \alpha_{r+1} , \quad (3.1.12)$$

$$\gamma_{r+1} = (k+1)\gamma_r + \delta_r = k\gamma_r + \delta_{r+1} , \quad (3.1.13)$$

$$\delta_{r+1} = \delta_r + \gamma_r , \quad (3.1.14)$$

The starting conditions are

$$\alpha_1 = \gamma_1 = 1 , \quad \beta_1 = \delta_1 = 0 ,$$

corresponding to

$$v_n = v_n^{(1)} \quad \text{and} \quad u_n = u_n^{(1)} .$$

Theorem 31:

$$\beta_r = \delta_r .$$

(3.1.15)

proof: $\beta_1 = \delta_1 = 0$ and $\beta_2 = \delta_2 = 1$.

Suppose

$$\beta_{r-1} = \delta_{r-1} \quad \text{and} \quad \beta_r = \delta_r .$$

$$\text{i.e. } (k+1)\beta_{r-1} + \alpha_{r-1} = \gamma_{r-1} + \delta_{r-1},$$

$$\text{or } k\beta_{r-1} + \alpha_{r-1} = \gamma_{r-1} \quad (3.1.16)$$

Thus we must show

$$k\beta_r + \alpha_r = \gamma_r,$$

$$\begin{aligned} \text{i.e. } k(k+1)\beta_{r-1} + k\alpha_{r-1} + \alpha_{r-1} + \beta_{r-1}, \\ = (k+1)\gamma_{r-1} + \delta_{r-1}, \end{aligned}$$

which reduces to (3.1.16).

We have proved in the process

Theorem 31:

$$\gamma_r = \alpha_r + k\beta_r \quad (3.1.17)$$

Theorem 32: (i) If $k=1$,

$$\alpha_r = \omega_{2r-3}, \quad (3.1.18)$$

$$\beta_r = \omega_{2r-2} = \delta_r, \quad (3.1.19)$$

$$\gamma_r = \omega_{2r-1}. \quad (3.1.20)$$

(ii) If $k=2$,¹

$$\alpha_{2r+1} = 2^r q_{2r-1}, \quad (3.1.21)$$

$$\alpha_{2r+2} = 2^r (q_{2r-1} + q_{2r}), \quad (3.1.22)$$

¹ Here q_n stands for ${}_2q_n$.

$$\beta_{2r+1} = 2^r q_{2r}, \quad (3.1.23)$$

$$\beta_{2r+2} = 2^r (q_{2r} + q_{2r+1}), \quad (3.1.24)$$

$$\gamma_r = \frac{1}{2} \alpha_{r+2}. \quad (3.1.25)$$

proof: (i) $\alpha_1 = \omega_{-1} = 1,$
 $\beta_1 = \omega_0 = 0 = \delta_1,$
 $\gamma_1 = \omega_1 = 1,$

$$\begin{aligned} \alpha_{r+1} &= \alpha_r + \beta_r = \omega_{2r-3} + \omega_{2r-2} = \omega_{2r-1}, \\ \beta_{r+1} &= \beta_r + \alpha_{r+1} = \omega_{2r-2} + \omega_{2r-1} = \omega_{2r}, \\ \gamma_{r+1} &= \gamma_r + \delta_{r+1} = \omega_{2r-1} + \omega_{2r} = \omega_{2r+1}. \end{aligned}$$

(ii) $\alpha_1 = q_{-1} = 1,$
 $\alpha_2 = q_{-1} + q_0 = 1,$
 $\beta_1 = q_0 = 0,$
 $\beta_2 = q_0 + q_1 = 1,$

$$\begin{aligned} \alpha_{2r+3} &= \alpha_{2r+2} + \beta_{2r+2} = 2^r (q_{2r-1} + 2q_{2r} + q_{2r+1}), \\ &= 2^{r+1} q_{2r+1}, \end{aligned}$$

$$\begin{aligned} \beta_{2r+3} &= 2\beta_{2r+2} + \alpha_{2r+3} = 2^{r+1} (q_{2r} + 2q_{2r+1}), \\ &= 2^{r+1} q_{2r+2}, \end{aligned}$$

$$\alpha_{2r+4} = \alpha_{2r+3} + \beta_{2r+3} = 2^{r+1} (q_{2r+1} + q_{2r+2}),$$

$$\begin{aligned}\beta_{2r+4} &= 2\beta_{2r+3} + \alpha_{2r+4} = 2^{r+1}(q_{2r+1} + 3q_{2r+2}), \\ &= 2^{r+1}(q_{2r+2} + q_{2r+3}).\end{aligned}$$

Finally (3.1.25) follows from (3.1.17):

$$\begin{aligned}\gamma_{2r+1} &= 2^r(q_{2r-1} + 2q_{2r}) = 2^r q_{2r+1} \\ &= \frac{1}{2}\alpha_{2r+3},\end{aligned}$$

$$\begin{aligned}\gamma_{2r+2} &= 2^r(q_{2r-1} + 3q_{2r} + 2q_{2r+1}) \\ &= 2^r(q_{2r+1} + q_{2r+2}) = \frac{1}{2}\alpha_{2r+4}.\end{aligned}$$

Returning to the general case we have

Theorem 33:

$$\alpha_r = \sum_{s=0}^{r-1} \binom{r-1}{s} q_{s-1}, \quad (3.1.26)$$

$$\beta_r = \sum_{s=0}^{r-1} \binom{r-1}{s} q_s, \quad (3.1.27)$$

$$\gamma_r = \sum_{s=0}^{r-1} \binom{r-1}{s} q_{s+1}, \quad (3.1.28)$$

proof:

$$\alpha_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} q_{-1} = 1,$$

$$\beta_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} q_0 = 0,$$

$$\alpha_{r+1} = \alpha_r + \beta_r = \sum_{s=0}^{r-1} \binom{r-1}{s} (q_{s-1} + q_s) =$$

$$\sum_{s=0}^r \left\{ \binom{r-1}{s} + \binom{r-1}{s-1} \right\} q_{s-1} = \sum_{s=0}^r \binom{r}{s} q_{s-1}.$$

$$\begin{aligned} \beta_{r+1} &= (k+1)\beta_r + \alpha_r \\ &= \sum_{s=0}^{r-1} \binom{r-1}{s} \{ (k+1)q_s + q_{s-1} \} \\ &= \sum_{s=0}^{r-1} \binom{r-1}{s} \{ q_{s+1} + q_s \} \\ &= \sum_{s=0}^r \left\{ \binom{r-1}{s} + \binom{r-1}{s-1} \right\} q_s = \sum_{s=0}^r \binom{r}{s} q_s, \end{aligned}$$

(3.1.28) follows from (3.1.17).

Comparing this theorem with the previous,

Theorem 34:

$$(i) \quad \omega_{2r-1} = \sum_{s=0}^{r-1} \binom{r-1}{s} \omega_{s+1} \quad (3.1.29)$$

and two similar relations.

$$(ii) \quad {}_2q_{2r-1} = \frac{1}{2^{r-1}} \sum_{s=0}^{2r-1} \binom{2r}{s}_2 q_{s-1}, \quad (3.1.30)$$

$${}_2q_{2r} = \frac{1}{2^{r-1}} \sum_{s=0}^{2r-1} \binom{2r}{s}_2 q_s, \quad (3.1.31)$$

Examples of case (ii):

$${}_2q_3 = \frac{1}{3} \left\{ \binom{4}{0}_2 q_{-1} + \binom{4}{1}_2 q_0 + \binom{4}{2}_2 q_1 + \binom{4}{3}_2 q_2 \right\} =$$

$$= \frac{1}{3} \{1 + 0 + 6 + 8\} = 5,$$

$$\begin{aligned} {}_2q_4 &= \frac{1}{3} \left\{ \binom{4}{0}_2 q_0 + \binom{4}{1}_2 q_1 + \binom{4}{2}_2 q_2 + \binom{4}{3}_2 q_3 \right\} \\ &= \frac{1}{3} \{0 + 4 + 12 + 20\} = 12. \end{aligned}$$

Case (i) is an example of the more general formula

Theorem 35:¹

$$\omega_{2r+c} = \sum_{s=0}^r \binom{r}{s} \omega_{s+c} \quad (3.1.32)$$

proof: Let E be the shift operator:

$$E\omega_c = \omega_{c+1},$$

Then $(E+1)\omega_c = \omega_{c+2},$

and $(E+1)^r \omega_c = \omega_{c+2r},$

Expanding the left side binomially yields the theorem.

An alternative expression for α_r (which is easily verified) is

$$\alpha_r = \frac{1}{k} \sum_{s=0}^r \binom{r}{s} (q_{s-1} - q_{s-2}), \quad (3.1.33)$$

There are analogous expressions for β_r and γ_r .

Comparison with theorem 32 for the case $k=2$ gives

$${}_2q_{2r-1} = \frac{1}{2^{r+1}} \sum_{s=0}^{2r+1} \binom{2r+1}{s} ({}_2q_{s-2} + {}_2q_{s-3}), \quad (3.1.34)$$

¹ This appeared as a problem in The American Mathematical Monthly, Jan. 1959, p.61.

$${}_2q_{2r} = \sum_{s=0}^{2r+1} \left\{ \frac{1}{2^{r+1}-1} \binom{2r+2}{s} - \frac{1}{2^{r+1}} \binom{2r+1}{s} \right\} ({}_2q_{s-2} + {}_2q_{s-3}) \quad (3.1.35)$$

The latter formula is rather curious as the following example shows:

$$\begin{aligned} {}_2q_2 &= 3 \left(\frac{1}{3} \binom{4}{0} - \frac{1}{4} \binom{3}{0} \right) - \left(\frac{1}{3} \binom{4}{1} - \frac{1}{4} \binom{3}{1} \right) \\ &\quad + \left(\frac{1}{3} \binom{4}{2} - \frac{1}{4} \binom{3}{2} \right) + \left(\frac{1}{3} \binom{4}{3} - \frac{1}{4} \binom{3}{3} \right) \\ &= \frac{3}{12} - \frac{7}{12} + \frac{15}{12} + \frac{13}{12} = 2. \end{aligned}$$

The formula for

$$\beta_r = \frac{1}{k} \sum_{s=0}^r \binom{r}{s} (q_s - q_{s-1}), \quad (3.1.36)$$

gives

$${}_2q_{2r} = \frac{1}{2^{r+1}} \sum_{s=0}^{2r+1} \binom{2r+1}{s} ({}_2q_{s-1} + {}_2q_{s-2}), \quad (3.1.37)$$

$${}_2q_{2r+1} = \sum_{s=0}^{2r+1} \left\{ \frac{1}{2^{r+1}-1} \binom{2r+2}{s} - \frac{1}{2^{r+1}} \binom{2r+1}{s} \right\} ({}_2q_{s-1} + {}_2q_{s-2}). \quad (3.1.38)$$

Combination of (3.1.37) and (3.1.35) gives

$$\sum_{s=0}^{2r+1} \binom{2r+1}{s} {}_2q_{s-1} = \frac{2^r}{2^{r+1}-1} \sum_{s=0}^{2r+1} \binom{2r+2}{s} ({}_2q_{s-2} + {}_2q_{s-3}).$$

As an example of (3.1.38),

$$\begin{aligned} {}_2q_5 &= -\left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{7} \cdot 6 - \frac{1}{8} \cdot 5\right) + \left(\frac{1}{7} \cdot 15 - \frac{1}{8} \cdot 10\right) + 3\left(\frac{1}{7} \cdot 20 - \frac{1}{8} \cdot 10\right) \\ &\quad + 7\left(\frac{1}{7} \cdot 15 - \frac{1}{8} \cdot 5\right) + 17\left(\frac{1}{7} \cdot 6 - \frac{1}{8}\right) = \frac{287}{7} - \frac{96}{8} = 29. \end{aligned}$$

There are numerous other relations of the same sort.

Many of the formulas which appear in this section can be inverted according to the following theorem:

Theorem 36:¹ If

$$g(n) = \sum_{v=0}^n \binom{n}{v} f(v), \quad (3.1.39)$$

then

$$f(n) = \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} g(v). \quad (3.1.40)$$

proof: $f(0) = g(0)$ so that the theorem is true for $n=0$.

$$\begin{aligned} \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} g(v) &= \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} \sum_{\mu=0}^v \binom{v}{\mu} f(\mu) \\ &= \sum_{\mu=0}^n f(\mu) \sum_{v=\mu}^n (-1)^{n-v} \binom{n}{v} \binom{v}{\mu} = f(n) + \sum_{\mu=0}^{n-1} f(\mu) h(\mu), \text{ say.} \end{aligned}$$

But

$$\begin{aligned} h(\mu) &= \sum_{v=\mu}^n (-1)^{n-v} \binom{n}{v} \binom{v}{\mu} = \binom{n}{\mu} \sum_{v=\mu}^n (-1)^{n-v} \binom{n-\mu}{v-\mu} \\ &= \binom{n}{\mu} (-1)^{n-\mu} (1-1)^{n-\mu} = 0. \end{aligned}$$

and the theorem is proved.

Some applications of this result are

¹ For a symbolic method for inverting this and more general 'triangular' matrices see N.S. Mendelsohn, Applications of Combinatorial Formulae to Generalizations of Wilson's Theorem, Canadian J. of Math., 1949, p. 329.

Theorem 37:

$$q_r = \sum_{s=0}^{r+1} (-1)^{r+1-s} \binom{r+1}{s} \alpha_{s+1} \quad (3.1.41)$$

$$= \sum_{s=0}^r (-1)^{r-s} \binom{r}{s} \beta_{s+1} \quad (3.1.42)$$

$$= \sum_{s=0}^{r-1} (-1)^{r-1-s} \binom{r-1}{s} \gamma_{s+1}, \quad (3.1.43)$$

$$\omega_{r+c} = \sum_{s=0}^r (-1)^{r-s} \binom{r}{s} \omega_{2s+c}, \quad (3.1.44)$$

(3.1.41)-(3.1.43) arise from (3.1.26)-(3.1.28) and (3.1.44) from (3.1.32).

Substituting (3.1.26)-(3.1.28) in (3.1.7) and (3.1.8)

we obtain

Theorem 38:

$$u_n = \frac{1}{k^{r-1}} \sum_{s=0}^{r-1} \binom{r-1}{s} (u_n^{(r)} q_{s+1} - v_n^{(r)} q_s), \quad (3.1.45)$$

$$v_n = \frac{1}{k^{r-1}} \sum_{s=0}^{r-1} \binom{r-1}{s} (v_n^{(r)} q_{s+1} - u_n^{(r)} q_s), \quad (3.1.46)$$

In solving these equations for $u_n^{(r)}$ and $v_n^{(r)}$ we need the lemma

Theorem 39:

$$\alpha_r \gamma_r - \beta_r^2 = k^{r-1}. \quad (3.1.47)$$

proof: It is true for $r=1$ and we suppose it true for r .

Then $\alpha_{r+1} \gamma_{r+1} - \beta_{r+1}^2 =$

$$\begin{aligned}
 & (\alpha_r + \beta_r) \{ (k+1) \gamma_r + \beta_r \} - \{ (k+1) \beta_r + \alpha_r \}^2 = \\
 & (k+1) \alpha_r \gamma_r + \alpha_r \beta_r + (k+1) \beta_r \gamma_r + \beta_r^2 \\
 & \quad - (k+1)^2 \beta_r^2 - 2(k+1) \beta_r \alpha_r - \alpha_r^2 = \\
 & \alpha_r \beta_r + (k+1) \beta_r \gamma_r + \beta_r^2 - k(k+1) \beta_r^2 \\
 & \quad - 2(k+1) \beta_r \alpha_r - \alpha_r^2 + (k+1) k^{r-1} \quad (\text{ex hypoth.}) \\
 & = \alpha_r \beta_r + \beta_r^2 - (k+1) \beta_r \alpha_r - \alpha_r^2 + (k+1) k^{r-1} \\
 & \hspace{25em} \text{by (3.1.17)} \\
 & = \beta_r^2 - \alpha_r \gamma_r + (k+1) k^{r-1} \hspace{10em} \text{by (3.1.17)} \\
 & = k^r \quad \text{ex hypoth. and the induction follows.}
 \end{aligned}$$

Theorem 40:

$$U_n^{(r)} = \alpha_r U_n + \beta_r V_n, \quad (3.1.48)$$

$$V_n^{(r)} = \beta_r U_n + \gamma_r V_n. \quad (3.1.49)$$

proof: By (3.1.7) and (3.1.8)

$$k^{r-1} U_n = \gamma_r U_n^{(r)} - \beta_r V_n^{(r)},$$

$$k^{r-1} V_n = \alpha_r V_n^{(r)} - \beta_r U_n^{(r)},$$

Solving,

$$U_n^{(r)} = \begin{vmatrix} k^{r-1} U_n & -\beta_r \\ k^{r-1} V_n & \alpha_r \end{vmatrix} \div \begin{vmatrix} \gamma_r & -\beta_r \\ -\beta_r & \alpha_r \end{vmatrix}$$

$$= \alpha_r u_n + \beta_r v_n \quad \text{by theorem 39. Similarly for (3.1.49).}$$

Writing (3.1.48) and (3.1.49) explicitly,

Theorem 41:

$$u_n^{(r)} = \sum_{s=0}^{r-1} \binom{r-1}{s} (u_n q_{s-1} + v_n q_s), \quad (3.1.50)$$

$$v_n^{(r)} = \sum_{s=0}^{r-1} \binom{r-1}{s} (u_n q_s + v_n q_{s+1}). \quad (3.1.51)$$

Theorem 42:

$$q_{s-1} q_{t+1} - q_s q_t = (-1)^t q_{s-t-1}, \quad (3.1.52)$$

$$q_{s-1} q_{t-1} + q_s q_t = q_{s+t-1}. \quad (3.1.53)$$

proof: Theorem 7 is a special case of this theorem and the proof follows identical lines. By (2.1.1) and recalling that

$$A \bar{A} = -1,$$

$$\begin{aligned} & q_{s-1} q_{t+1} - q_s q_t = \\ & \frac{1}{k^2+4} \left\{ A^{s+t} - \bar{A}^{s-1} A^{t+1} - A^{s-1} \bar{A}^{t+1} + \bar{A}^{s+t} \right. \\ & \quad \left. - A^{s+t} + \bar{A}^s A^t + A^s \bar{A}^t - \bar{A}^{s+t} \right\} \\ & = \frac{(-1)^t}{k^2+4} \left\{ A^{s-t-1} (A - \bar{A}) - \bar{A}^{s-t-1} (A - \bar{A}) \right\} \\ & = (-1)^t q_{s-t-1}. \end{aligned}$$

(3.1.53) follows from (3.1.52) by writing $-t$ for t and

$$\text{noting } q_{-n} = (-1)^{n+1} q_n.$$

Since $\alpha_{r+1}\gamma_{r+1} - \beta_{r+1}^2 = k^r$,

$$\left\{ \sum_{s=0}^r \binom{r}{s} q_{s-1} \right\} \left\{ \sum_{s=0}^r \binom{r}{s} q_{s+1} \right\} - \left\{ \sum_{s=0}^r \binom{r}{s} q_s \right\}^2 = k^r,$$

i.e.

$$\sum_{s=0}^r \sum_{t=0}^r \binom{r}{s} \binom{r}{t} \{ q_{s-1} q_{t+1} - q_s q_t \} = k^r$$

which by (3.1.52) gives

Theorem 43:

$$\sum_{s=0}^r \sum_{t=0}^r (-1)^t \binom{r}{s} \binom{r}{t} q_{s-t-1} = k^r. \quad (3.1.54)$$

For example

$$\sum_{s=0}^r \sum_{t=0}^r (-1)^t \binom{r}{s} \binom{r}{t} \omega_{s-t-1} = 1. \quad (3.1.55)$$

Noticing that

$$u_n^{(r)} = u v_n^{(r-1)}, \quad (3.1.56)$$

we have

$$v_n^{(r)} = v v_n^{(r-1)} = u v_n^{(r-1)} + k v_n^{(r-1)} = u_n^{(r)} + k v_n^{(r-1)},$$

Hence

Theorem 44:

$$\begin{aligned} v_n^{(r)} &= u_n^{(r)} + k u_n^{(r-1)} + k^2 u_n^{(r-2)} + \dots \\ &\quad \dots + k^{r-1} u_n, \end{aligned} \quad (3.1.57)$$

Theorem 45:

$$u_{q_{2n+1}}^{(r)} = \sum_{s=0}^r \binom{r}{s} q_{2n+s}, \quad (3.1.58)$$

$$v_{q_{2n+1}}^{(r)} = \sum_{s=0}^r \binom{r}{s} q_{2n+s+1}. \quad (3.1.59)$$

proof: By theorem 41 and theorem 18

$$\begin{aligned} u_{q_{2n+1}}^{(r)} &= \sum_{s=0}^{r-1} \binom{r-1}{s} \{ (q_{2n} + q_{2n+1}) q_{s-1} + (q_{2n+1} + q_{2n+2}) q_s \} \\ &= \sum_{s=0}^{r-1} \binom{r-1}{s} \{ q_{2n+s} + q_{2n+s+1} \} \end{aligned}$$

by theorem 42

$$\begin{aligned} &= \sum_{s=0}^r q_{2n+s} \left\{ \binom{r-1}{s-1} + \binom{r-1}{s} \right\} \\ &= \sum_{s=0}^r q_{2n+s} \binom{r}{s} \end{aligned}$$

(3.1.59) follows in the same way.

(3.1.58) and (3.1.59) can be inverted by theorem 36:

$$q_{2n+r} = \sum_{s=0}^r (-1)^{r-s} \binom{r}{s} u_{q_{2n+1}}^{(s)}, \quad u_{q_{2n+1}}^{(0)} = q_{2n} \quad (3.1.60)$$

$$= \sum_{s=0}^{r-1} (-1)^{r-1-s} \binom{r-1}{s} v_{q_{2n+1}}^{(s)}, \quad v_{q_{2n+1}}^{(0)} = q_{2n+1}. \quad (3.1.61)$$

Theorem 46: (i) If $k=1$,

$$u_{\omega_{2n+1}}^{(r)} = \omega_{2n+2r}, \quad (3.1.62)$$

$$v_{w_{2n+1}}^{(r)} = w_{2n+2r+1}. \quad (3.1.63)$$

(11) If $k=2$,

$$u_{2q_{2n+1}}^{(2r+1)} = 2^r (2q_{2n+2r} + 2q_{2n+2r+1}), \quad (3.1.64)$$

$$u_{2q_{2n+1}}^{(2r)} = 2^r 2q_{2n+2r}, \quad (3.1.65)$$

$$v_{2q_{2n+1}}^{(2r+1)} = 2^r (2q_{2n+2r+1} + 2q_{2n+2r+2}), \quad (3.1.66)$$

$$v_{2q_{2n+1}}^{(2r)} = 2^r 2q_{2n+2r+1}, \quad (3.1.67)$$

proof: Part (i) is an application of theorem 35 to theorem 45. (ii) If E is the shift operator

$$(E+1)2q_m = 2q_m + 2q_{m+1},$$

$$\begin{aligned} (E+1)^2 2q_m &= 2q_m + 22q_{m+1} + 2q_{m+2} \\ &= 22q_{m+2}. \end{aligned}$$

Hence

$$(E+1)^{2r} 2q_m = 2^r 2q_{m+2r},$$

$$(E+1)^{2r+1} 2q_m = 2^r (2q_{m+2r} + 2q_{m+2r+1}),$$

That is to say

$$\sum_{s=0}^{2r} \binom{2r}{s} {}_2q_{m+s} = 2^r {}_2q_{m+2r},$$

$$\sum_{s=0}^{2r+1} \binom{2r+1}{s} {}_2q_{m+s} = 2^r ({}_2q_{m+2r} + {}_2q_{m+2r+1}),$$

which by theorem 45 give (3.1.64)-(3.1.67).

Examples of (i):

$$u_{\omega_3}^{(4)} = u_2^{(4)} = u_{v v_2} = u_{v v_5} = u_{v_{13}} = u_{34} = 55 = \omega_{10},$$

$$v_{\omega_5}^{(3)} = v_5^{(3)} = v_{v v_5} = v_{v_{13}} = v_{34} = 89 = \omega_{11}.$$

It is in a sense fortuitous that formulas such as (3.1.62)-(3.1.67) exist. There do not seem to be comparable formulas for $k > 2$; that is expressions for $u_{q_n}^{(r)}$ and $v_{q_n}^{(r)}$ in terms of a bounded number (as $r \rightarrow \infty$) of the q_m . Inspecting Pascal's triangle,

$$\begin{array}{cccccc} & & & 1 & & & \\ & & & & 1 & & \\ & & 1 & & 1 & & \\ & & & 1 & 2 & 1 & \\ & 1 & & 3 & 3 & 1 & \\ & & 1 & 4 & 6 & 4 & 1 \\ & 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

we see that the second row corresponds to

$$1 \cdot \omega_m + 1 \cdot \omega_{m+1} = \omega_{m+2},$$

and the third row to

$$1 \cdot {}_2q_m + 2 \cdot {}_2q_{m+1} + {}_2q_{m+2} = 2 {}_2q_{m+2}.$$

But, e.g., the fourth row applied to ${}_3q_m$ gives

$$\begin{aligned} {}_3q_m + 3 {}_3q_{m+1} + 3 {}_3q_{m+2} + {}_3q_{m+3} \\ = 4 {}_3q_{m+2} + {}_3q_{m+3}, \end{aligned}$$

which cannot be reduced further.

We conclude this section with the minor observation

Theorem 47:

$$q_{2n} = q_{2r+1} q_{2n+2r+2} - q_{2r+2} q_{2n+2r+1}, \quad (3.1.68)$$

$$q_{2n+1} = q_{2r-1} q_{2n+2r+1} - q_{2r} q_{2n+2r}, \quad (3.1.69)$$

proof: These follow from (3.1.52) by setting

$$(i) \quad s = 2r+2, \quad t = 2n+2r+1,$$

$$(ii) \quad s = 2r, \quad t = 2n+2r.$$

For example

$$\begin{aligned} 1 &= q_{2r-1} q_{2r+1} - q_{2r}^2 \\ &= q_{2r-1}^2 - q_{2r} q_{2r-2}, \end{aligned}$$

as we saw in theorem 12.

3.2 Similar Beatty Sequences.

Two Beatty sequences of argument α and α' are said to be similar if

(i) α and α' are quadratic surds:

$$\alpha = \frac{a+b\sqrt{m}}{c}, \quad \alpha' = \frac{a'+b'\sqrt{m'}}{c'}, \quad (3.2.1)$$

where a, a' are integers, b, b', c, c' are non-zero integers and m, m' are positive quadratfrei¹ integers larger than 1.

(ii) $m = m'$. (3.2.2)

We may always take $c = c'$.

Theorem 48: If $\{u_n\}, \{v_n\}$ and $\{u'_n\}, \{v'_n\}$ are similar Beatty sequences with argument

$$\alpha = \frac{a+b\sqrt{m}}{c},$$

and $\alpha' = \frac{a'+b'\sqrt{m}}{c}$ respectively, then

$$v_{b'cn} + (a'+c)b'n = v'_{bcn} + (a+c)b'n. \quad (3.2.3)$$

proof:

$$\begin{aligned} v_{b'cn} &= \left[b'cn \left\{ 1 + \frac{a+b\sqrt{m}}{c} \right\} \right] \\ &= b'cn + ab'n + [bb'n\sqrt{m}], \end{aligned}$$

$$\begin{aligned} v'_{bcn} &= \left[bcn \left\{ 1 + \frac{a'+b'\sqrt{m}}{c} \right\} \right] \\ &= bcn + a'b'n + [bb'n\sqrt{m}], \end{aligned}$$

¹ i.e. no squared factor.

and (3.2.3) follows directly.

Relations of the sort (3.2.3) are the motivation for considering similar sequences,

The remainder of this section will be concerned with Wythoff sequences $\{k u_n\}, \{k v_n\}$; k is called the order.

Theorem 49: Wythoff sequences of order k and $k r_{2n+1}$ are similar.

proof: Clearly (1) is satisfied.

(11) For Wythoff sequences of order k , m is the quadratfrei part of $k^2 + 4$,

and for order $k r_{2n+1}$, m' is the quadratfrei part of

$$k r_{2n+1}^2 + 4 = (k^2 + 4) q_{2n+1}^2, \quad (3.2.4)$$

(by theorem 12), i.e. the quadratfrei part of

$$k^2 + 4.$$

Theorem 50:

$$k r_{2n+1} u_t = k u_{t q_{2n+1}} - t (k q_{2n+1} + k q_{2n} - 1), \quad (3.2.5)$$

$$k r_{2n+1} v_t = k v_{t q_{2n+1}} - t (k q_{2n+1} - k q_{2n} - 1), \quad (3.2.6)$$

proof: Omitting the left subscript k ,

$$A_{r_{2n+1}} = \frac{r_{2n+1} + q_{2n+1} \sqrt{k^2 + 4}}{2},$$

so that

$$\begin{aligned}
 r_{2n+1} v_t &= \left[t \left\{ 1 + \frac{r_{2n+1} + q_{2n+1} \sqrt{k^2+4}}{2} \right\} \right] \\
 &= \left[t \left\{ 1 + \frac{q_{2n} + q_{2n+2} + q_{2n+1} \sqrt{k^2+4}}{2} \right\} \right]
 \end{aligned}$$

by (2.1.21)

$$= \left[t \left\{ 1 + q_{2n} - q_{2n+1} + q_{2n+1} \left(1 + \frac{k + \sqrt{k^2+4}}{2} \right) \right\} \right],$$

which is (3.2.6). Thus

$$\begin{aligned}
 r_{2n+1} u_t &= r_{2n+1} v_t - t r_{2n+1} \\
 &= k v_t q_{2n+1} - t(q_{2n+1} - q_{2n} - 1) - t(q_{2n} + q_{2n+2}) \\
 &= k u_t q_{2n+1} - t(q_{2n+1} + q_{2n+2} - 1) + k t q_{2n+1} \\
 &= k u_t q_{2n+1} - t(q_{2n+1} + q_{2n} - 1).
 \end{aligned}$$

Examples are:

$$r_3 = k(k^2+3), \quad q_2 = k, \quad q_3 = k^2+1,$$

$$k(k^2+3) u_t = k u_{t(k^2+1)} - t k(k+1),$$

$$k(k^2+3) v_t = k v_{t(k^2+1)} - t k(k-1),$$

$$\text{e.g. } {}_{36}u_t = {}_{310}u_t - 12t,$$

$${}_{36}v_t = {}_{310}v_t - 6t,$$

Various identities can be developed, one example being:

Theorem 51:

$$k_{r_{2n+1}} U_{k q_{2m+1}} - k_{r_{2m+1}} U_{k q_{2n+1}} \\ = k_{q_{-2(m-n)}} + k_{q_{2m+1}} - k_{q_{2n+1}}.$$

proof: Omitting the left subscript k , by the previous theorem

$$r_{2n+1} U_{q_{2m+1}} = k U_{q_{2m+1} q_{2n+1}} - q_{2m+1} (q_{2n+1} + q_{2n} - 1),$$

$$r_{2m+1} U_{q_{2n+1}} = k U_{q_{2m+1} q_{2n+1}} - q_{2n+1} (q_{2m+1} + q_{2m} - 1),$$

and the result follows by a simple application of theorem 42.

3.3 R-fold Complementary Sequences.

Let $\{u_n\}$ and $\{v_n\}$ be Beatty sequences. They form a 2-fold complementary sequence in the sense that the integers have been split into two disjoint subsets. Since n is either a u or a v , u_n is either a u_u or a u_v . Thus (omitting subscripts) u_u, u_v, v are three mutually disjoint subsets of the integers whose union is the set of integers.

We say

$$u_u, u_v, v,$$

is a 3-fold complementary sequence. Similarly

$$u, v_u, v_v,$$

is a 3-fold complementary sequence.

Starting with u, v we enquire how many r -fold complementary sequences we can generate. Denote this number by $v(r)$. Thus $v(2) = 1$. Define $v(1) = 1$.

Theorem 52:

$$v(r) = \frac{1}{r} \binom{2r-2}{r-1}, \quad (3.3.1)$$

proof:¹ Suppose in an r -fold complementary sequence u is split into s components; that is there are s terms u with various subscripts. The number of ways this can be done is $v(s)$. v has been split into $r-s$ components and the number of ways this can be done is $v(r-s)$. Thus summing over all possible values of s ,

$$v(r) = \sum_{s=1}^{r-1} v(s) v(r-s). \quad (3.3.2)$$

Let

$$\phi(x) = v(1)x + v(2)x^2 + \dots + v(r)x^r + \dots, \quad (3.3.3)$$

Then $\{\phi(x)\}^2 = (v(1))^2 x^2 + (v(1)v(2) + v(2)v(1))x^3$

$$+ \dots + \sum_{s=1}^{r-1} v(s)v(r-s)x^r + \dots$$

$$= v(2)x^2 + v(3)x^3 + \dots + v(r)x^r + \dots$$

$$= \phi(x) - v(1)x = \phi(x) - x,$$

$$\therefore \{\phi(x)\}^2 - \phi(x) + x = 0.$$

¹ Due to Dr. N. S. Mendelsohn.

Thus

$$\phi(x) = \frac{1 - \sqrt{1-4x}}{2},$$

(since $\phi(0) = 0$).

$$\begin{aligned}\phi(x) &= \frac{1}{2} \{1 - (1-4x)^{1/2}\} \\ &= \frac{1}{2} \left\{1 - 1 - \binom{1/2}{1}(-4x) - \binom{1/2}{2}(-4x)^2 - \dots\right\}.\end{aligned}$$

Now

$$\begin{aligned}\binom{1/2}{n} &= \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(\frac{3-2n}{2}\right) \cdot \frac{1}{n!} \\ &= \frac{(2n-3)(2n-5)\dots 3 \cdot 1}{2^n n!} (-1)^{n-1} \\ &= \frac{(2n-2)! (-1)^{n-1}}{2^n n! (n-1)! 2^{n-1}} = \frac{(-1)^{n-1}}{n 2^{2n-1}} \binom{2n-2}{n-1},\end{aligned}$$

$$\begin{aligned}\therefore \phi(x) &= x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 \\ &\quad + 132x^7 + \dots + \frac{1}{r} \binom{2r-2}{r-1} x^r + \dots,\end{aligned}$$

and

$$v(r) = \frac{1}{r} \binom{2r-2}{r-1}.$$

The number $v(r)$ is also the number of meaningful, distinct ways of bracketing r quantities¹ and there is a

¹ It was in this connection that Dr. Mendelsohn first used the proof of theorem 52.

natural one to one correspondence between r -fold complementary sequences and bracketings of r quantities. Write $(vuuu)$ for v_{uuu} , etc. If, for example, a component $(uvuvu)$ occurs in an r -fold complementary sequence then $(uvuvv)$ must occur in this sequence, and they both arose from the component $(uvuv)$ in some $(r-1)$ -fold sequence. Identify each component with a quantity x ; we need not distinguish between the r x 's.

Take the longest pair of components (there may be a choice) and bracket them together. This yields, on the one hand, an $(r-1)$ -fold sequence, as in the example above, and on the other a product of $(r-1)$ quantities where (xx) is regarded as one quantity. The process is now iterated. For example,

$$\begin{array}{ll}
 (uuu)(uv)(vu)(vv), & xxxxx, \\
 (uu) (uv)(vu)(vv), & (xx)xxx \\
 (u), (vu)(vv), & ((xx)x)xx, \\
 (u), (v), & ((xx)x)(xx),
 \end{array}$$

so that the 5-fold complementary sequence $(uuu)(uv)(vu)(vv)$ corresponds to the bracketing $((xx)x)(xx)$ of 5 quantities.

The possibilities for the first few values of r are displayed in the following list:

$r=1:$	$(1),$	$(x),$
$r=2:$	$(u)(v),$	$(xx),$
$r=3:$	$(uu)(uv)(v),$	$((xx)x),$
	$(u)(vu)(vv),$	$(x(xx)),$
$r=4:$	$(uuu)(uuv)(uv)(v),$	$((xx)x)x),$
	$(uu)(uvu)(uvv)(v),$	$((x(xx))x),$
	$(u)(vuu)(vuv)(vv),$	$(x(x(xx))),$
	$(u)(vuuv)(vuv)(vv),$	$(x((xx)x)),$
	$(uu)(uv)(vu)(vv),$	$((xx)(xx)),$

The case $r=3$ is, explicitly, :

$$[n(1+\frac{1}{2})](1+\frac{1}{2}), [n(1+\alpha)](1+\frac{1}{\alpha}), [n(1+\alpha)],$$

and

$$[n(1+\frac{1}{\alpha})], [n(1+\frac{1}{2})](1+\alpha), [n(1+\alpha)](1+\alpha),$$

are 3-fold complementary sequences.

If $\alpha = \frac{1+\sqrt{5}}{2}$ then,

$$u_{v_n} = u_n + v_n \quad \text{by theorem 29}$$

and

$$u_{u_n} = v_n - 1 \quad \text{by the general formula}^1$$

for Wythoff sequences

$$u_{u_n} = v_n - (k-1)u_n - R, \quad 1 \leq R \leq k.$$

¹ the proof is straightforward. The nature of R is not known in detail except in the case $k=1$.

Thus

$$\left[n \left(\frac{1+\sqrt{5}}{2} \right) \right] + \left[n \left(\frac{3+\sqrt{5}}{2} \right) \right], \left[n \left(\frac{3+\sqrt{5}}{2} \right) \right] - 1, \left[n \left(\frac{3+\sqrt{5}}{2} \right) \right],$$

are complementary.

3.4 The Eames' Representation.

W.Eames¹ discussed a representation of positive numbers based on the ideas of theorem 1. He defined

$$f(\alpha) = (a_1, a_2, a_3, \dots),$$

where $a_n = 1$ if n is a u

$a_n = 0$ if n is a v

for any $\alpha > 0$. (there were special provisions for rational α).

For our purposes it is more convenient to define the representation

$$\psi(\alpha) = (v_1, v_2, v_3, \dots), \quad v_n = [n(1+\alpha)],$$

for any irrational $\alpha > 1$. Clearly the representation uniquely determines α (if it is a possible representation) and conversely. Three obvious properties are

$$(i) \quad [\alpha] = v_1 - 1,$$

$$(ii) \quad \psi(\alpha - k) = (v_1 - k, v_2 - 2k, v_3 - 3k, \dots),$$

for any integer k , and

$$(iii) \quad \psi\left(\frac{1}{\alpha}\right) = (u_1, u_2, u_3, \dots).$$

¹ Thesis, University of Manitoba, 1953.

On the basis of these three properties we shall obtain

(a) an algorithm for the continued fraction of α given $\psi(\alpha)$.

(b) an algorithm for $\psi(\alpha)$, given the continued fraction for α , which is in effect an algorithm for the Beatty sequences of argument α .

Let the continued fraction for α be

$$\alpha = [a_0, a_1, a_2, \dots],$$

and define $\alpha_0, \alpha_1, \alpha_2, \dots$ by

$$\alpha_0 = [a_0, a_1, \dots] = \alpha,$$

$$\alpha_1 = [a_1, a_2, \dots],$$

⋮

$$\alpha_n = [a_n, a_{n+1}, \dots],$$

so that¹

$$\alpha = [a_0, \alpha_1],$$

$$= [a_0, a_1, \alpha_2], \text{ etc.}$$

and

$$\alpha_n = [a_n, \alpha_{n+1}],$$

$$\text{i.e.} \quad \alpha_n = a_n + \frac{1}{\alpha_{n+1}} \quad \text{or} \quad \alpha_{n+1} = \frac{1}{\alpha_n - a_n}.$$

The usual continued fraction algorithm is based on

¹ Relaxing for the moment the restriction that a continued fraction be simple and infinite.

the formula

$$a_n = [\alpha_n],$$

and the calculation proceeds as follows :

$$a_0 = [\alpha] \quad , \quad \alpha_1 = \frac{1}{\alpha - a_0} ,$$

$$a_1 = [\alpha_1] \quad , \quad \alpha_2 = \frac{1}{\alpha_1 - a_1} ,$$

$$a_2 = [\alpha_2] \quad , \quad \text{etc.}$$

When α is a quadratic surd the divisions on the right can be done formally and the amount of labour is relatively small. (In this case α is periodic; c.f. theorem 5). Otherwise the divisions must be performed with many significant figures to obtain an appreciable number of the a_n .

The algorithm given below is interesting in that only multiplications are involved, viz. the calculation of $v_1, v_2, v_3, \dots$ (and these calculations need usually be performed with only a few significant figures).

(a) Given $\psi(\alpha)$.

$$\psi(\alpha) \quad , \quad a_0 = [\alpha] \quad , \quad \psi(\alpha - a_0)$$

$$\psi(\alpha_1) \quad , \quad a_1 = [\alpha_1] \quad , \quad \psi(\alpha_1 - a_1)$$

$$\psi(\alpha_2) \quad , \quad \text{etc.}$$

For example suppose $\alpha = e = 2.718\dots$

Then $\nu_n = [3.718n]$ and

$$\psi(e) = (3, 7, 11, 14, 18, 22, 26, 29, 33, 37, 40, 44, 48, 52, 55, 59, \dots)$$

$$[e] = 2,$$

$$\psi(e-2) = (1, 3, 5, 6, 8, 10, 12, 13, 15, 17, 18, 20, 22, 24, 25, 27, \dots),$$

$$\psi(e_1) = (2, 4, 7, 9, 11, 14, 16, 19, 21, 23, 26, \dots),$$

$$[e_1] = 1,$$

$$\psi(e_1-1) = (1, 2, 4, 5, 6, 8, 9, 11, 12, 13, 15, \dots),$$

$$\psi(e_2) = (3, 7, 10, 14, \dots),$$

$$[e_2] = 2,$$

$$\psi(e_2-2) = (1, 3, 4, 6, \dots),$$

$$\psi(e_3) = (2, 5, \dots),$$

$$[e_3] = 1,$$

$$\psi(e_3-1) = (1, 3, \dots),$$

$$\psi(e_4) = (2, \dots),$$

$$[e_4] = 1.$$

Therefore¹

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{\dots}}}}$$

The disadvantage lies, of course, in having to calculate a large number of the ν_n ; but this becomes an advantage in the inverse algorithm.

¹ In fact

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots] = [2, \overline{1, 2n, 1}]_{n=1}^{\infty}$$

(b) Given $\alpha = [a_0, a_1, \dots]$.

Calculate in succession

$$\psi(\alpha_n), \psi\left(\frac{1}{\alpha_n}\right) = \psi(\alpha_{n-1} - a_{n-1}), \psi(\alpha_{n-1}),$$

$$\dots, \psi(\alpha_0 - a_0), \psi(\alpha_0) = \psi(\alpha).$$

For example, ¹

$$\pi = [3, 7, 15, 1, 292, \dots],$$

$$\psi(\pi_2) = (16, \dots),$$

$$\psi(\pi_1 - 7) = (1, 2, 3, \dots, 15, \dots),$$

$$\psi(\pi_1) = (8, 16, 24, \dots, 120, \dots),$$

$$\psi(\pi - 3) = (1, 2, 3, 4, 5, 6, 7, 9, 10, 11, \dots),$$

$$\psi(\pi) = (4, 8, 12, 16, 20, 24, 28, 33, 37, 41, \dots),$$

We have omitted most of the final answer in order to conserve space. The number of terms this method yields is often remarkable. Starting with

$$\psi(\pi_4) = (293, \dots),$$

one obtains

$$v_1 = 4, \dots, v_{32988} = 136,622,$$

whence

$$u_1 = 1, \dots, u_{103634} = 136,621.$$

¹ Perron, op.cit., p.42 : $\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 1, \dots]$.

'Das allgemeine Bildungsgesetz der Teilnenner von π ist nicht bekannt'.

If α is periodic with period n then $\alpha_n = \alpha$, and the algorithm can proceed without interruption; that is one need not go back to the beginning in order to obtain more terms of $\psi(\alpha)$. For example,

$$\alpha = [\overline{2, 3}] = \frac{1}{3}\{3 + \sqrt{15}\} = 2 + \frac{1}{\alpha_1},$$

$$\alpha_1 = [\overline{3, 2}] = \frac{1}{2}\{3 + \sqrt{15}\} = 3 + \frac{1}{\alpha},$$

$$\psi(\alpha) = (3, \dots),$$

$$\psi(\alpha, -3) = (1, 2, \dots),$$

$$\psi(\alpha_1) = (4, 8, \dots),$$

$$\psi(\alpha - 2) = (1, 2, 3, 5, 6, 7, \dots),$$

$$\psi(\alpha) = (3, 6, 9, 13, 16, 19, \dots),$$

$$\psi(\alpha_1 - 3) = (1, 2, 4, 5, 7, 8, 10, 11, 12, 14, 15, 17, 18, \dots),$$

$$\psi(\alpha_1) = (4, 8, 13, 17, 22, 26, 31, 35, 39, 44, 48, 53, 57, \dots),$$

$$\psi(\alpha - 2) = (1, 2, 3, 5, 6, 7, 9, 10, 11, 12, 14, \text{etc.} \dots),$$

$$\psi(\alpha) = (3, 6, 9, 13, 16, 19, 23, 26, 29, 32, 36, \text{etc.} \dots),$$

and so on.

This example was chosen because α is not in the class (2.4.18). If α is in the class (2.4.18) then the Beatty sequences can be calculated without a numerical value of α on the basis of the complementarity property and the relation $v_n = u_{an} + bn$. In this example we have only

$$v_{3n} = u_{2n} + 7n,$$

and the algorithm is a practical necessity.

The algorithm fails for the one number

$$\alpha = \frac{1+\sqrt{5}}{2} = A_1 = [1]$$

and numbers 'equivalent'¹ to A_1 , i.e. numbers of the form

$$\alpha = [a_0, a_1, a_2, \dots, a_r, 1, 1, 1, 1, \dots].$$

3.5 Rational Approximations to A .

We end this chapter with a theorem of a more miscellaneous character.

It was observed in theorem 5, part (13), that

$$\left| \alpha - \frac{P}{Q} \right| < \frac{1}{\sqrt{5} Q^2},$$

has infinitely many solutions where P and Q are integers. (P/Q is necessarily a convergent to α by part (11) of the same theorem). But for particular values of α the statement becomes false if $\sqrt{5}$ is replaced by any larger number. These particular values of α are actually

$$\alpha = A_1 = \frac{1+\sqrt{5}}{2},$$

and numbers equivalent to α . For certain classes of α more can be said.

Let α have the convergents

$$P_n/Q_n, \quad n=0, 1, 2, \dots$$

Define K' to be the largest number such that

¹ Hardy and Wright, op.cit., p.141.

$$\left| \alpha - \frac{P_n}{Q_n} \right| < \frac{1}{K' Q_n^2}, \quad (3.4.1)$$

for infinitely many n and define K' to be the largest number such that

$$\left| \alpha - \frac{P_n}{Q_n} \right| \leq \frac{1}{K'' Q_n^2}, \quad (3.4.2)$$

for all n . For

$$\alpha = A = \frac{k+R}{2}, \quad R = \sqrt{k^2+4}, \quad (3.4.3)$$

we have the result

Theorem 53:

$$K' = R, \quad K'' = A, \quad (3.4.4)$$

Further if

$$K'' < K < K',$$

$$\left| A - \frac{P_n}{Q_n} \right| < \frac{1}{K Q_n^2},$$

is true for all but a finite number of n . When $K = K'$ the inequality is true for odd n , false for even n .

proof:

$$P_n = q_{n+2}, \quad Q_n = q_{n+1},$$

$$\text{where } q_n = \frac{1}{R} (A^n - \bar{A}^n).$$

$$A - \frac{P_n}{Q_n} = A - \frac{A^{n+2} - \bar{A}^{n+2}}{A^{n+1} - \bar{A}^{n+1}}$$

$$= \frac{\bar{A}^{n+1}(\bar{A} - A)}{A^{n+1} - \bar{A}^{n+1}} = \frac{-R \bar{A}^{n+1}}{A^{n+1} - \bar{A}^{n+1}}$$

$$= \frac{R}{1 - \left(\frac{A}{\bar{A}}\right)^{n+1}} = \frac{R}{1 - (-1)^{n+1} \left(\frac{k^2 + 2 + kR}{2}\right)^{n+1}},$$

Thus

$$\zeta = \left| A - \frac{P_n}{Q_n} \right| = \frac{R}{\left(\frac{k^2 + 2 + kR}{2}\right)^{n+1} - (-1)^{n+1}},$$

$$\frac{1}{Q_n^2} = \frac{R^2}{\left(\frac{k^2 + 2 + kR}{2}\right)^{n+1} + \left(\frac{k^2 + 2 - kR}{2}\right)^{n+1} - 2(-1)^{n+1}}.$$

Setting

$$\zeta = \frac{1}{K_n Q_n^2},$$

$$K_n = \frac{R \left\{ \left(\frac{k^2 + 2 + kR}{2}\right)^{n+1} - (-1)^{n+1} \right\}}{\left(\frac{k^2 + 2 + kR}{2}\right)^{n+1} + \left(\frac{k^2 + 2 - kR}{2}\right)^{n+1} - 2(-1)^{n+1}}$$

$$= R + \frac{(-1)^{n+1} - \left(\frac{k^2 + 2 - kR}{2}\right)^{n+1}}{R Q_n^2}$$

$$= R + \frac{(-1)^{n+1} - \left(\frac{k-R}{2}\right)^{2n+2}}{R Q_n^2},$$

Since

$$0 < \frac{R-k}{2} < 1,$$

$$K_{2n} < R < K_{2n+1},$$

and

$$\lim_{n \rightarrow \infty} K_n = R.$$

Clearly $K' = R$ and the inequality is true for odd n , false for even n .

Setting $n = 2n_1$,

$$\frac{1 + (\bar{A})^{4n_1+2}}{RQ_{2n_1}^2},$$

is a strictly decreasing function so that

$$K'' = R - \frac{1 + \bar{A}^2}{RQ_0^2} = R - \frac{k\bar{A} + 2}{R}$$

$$= R - \frac{(k^2+4)R - k(k^2+4)}{2(k^2+4)} = R - \frac{R-k}{2}$$

$$= A.$$

4.1 A Class of Beatty Sequences.

Setting $a = k$, $b = 1$ in (2.4.18)

$$\alpha = \frac{k + \sqrt{k^2 + 4k}}{2}, \quad k = 1, 2, 3, \dots \quad (4.1.1)$$

Denote these numbers by B or B_k . In this section we study the Beatty sequences

$$u_n = \left[n \left(1 + \frac{1}{B} \right) \right] = \left[n \frac{B}{k} \right], \quad (4.1.2)$$

$$v_n = \left[n(1 + B) \right],$$

By theorem 27 these obey reduplication:

$$u_{v_n} = u_n + v_n. \quad (4.1.3)$$

Setting $m = 2$ in theorem 8,

$$B = \frac{k + \sqrt{k^2 + 4k}}{2} = [\overline{k, 1}], \quad (4.1.4)$$

Define

$$s_{n+1} = kt_n + s_n, \quad (4.1.5)$$

$$t_{n+1} = (k+1)t_n + s_n = t_n + s_{n+1}, \quad (4.1.6)$$

with the starting conditions,

$$s_0 = 0, \quad t_0 = 1. \quad (4.1.7)$$

Then

$$s_{n+2} - (k+2)s_{n+1} + s_n = 0, \quad (4.1.8)$$

and

$$t_{n+2} - (k+2)t_{n+1} + t_n = 0. \quad (4.1.9)$$

The characteristic equation is

$$E^2 - (k+2)E + 1 = 0, \quad (4.1.10)$$

which has the roots

$$\frac{k+2 \pm R}{2}, \quad R = \sqrt{k^2 + 4k}, \quad (4.1.11)$$

i.e. $1+B$ and $1+\bar{B}$.

Setting

$$S_n = K_1(1+B)^n + K_2(1+\bar{B})^n,$$

we have

$$S_0 = K_1 + K_2 = 0,$$

$$S_1 = K_1(1+B) + K_2(1+\bar{B}) = k,$$

Solving,

$$K_1 = -K_2 = k/R,$$

Hence

$$S_n = \frac{k}{R} \{ (1+B)^n - (1+\bar{B})^n \}, \quad (4.1.12)$$

It follows easily by induction that $k \mid S_n$.

Define

$$\sigma_n = S_n/k. \quad (4.1.13)$$

Thus

$$\sigma_{n+1} = \sigma_n + t_n. \quad (4.1.14)$$

Writing

$$t_n = H_1(1+B)^n + H_2(1+\bar{B})^n,$$

we have

$$t_0 = H_1 + H_2 = 1,$$

and

$$t_1 = H_1(1+B) + H_2(1+\bar{B}) = k+1.$$

Solving,

$$H_1 = \frac{R+k}{2R}, \quad H_2 = \frac{R-k}{2R}, \quad (4.1.15)$$

so that

$$\begin{aligned} t_n &= \left(\frac{R+k}{2R}\right)(1+B)^n + \left(\frac{R-k}{2R}\right)(1+\bar{B})^n, \\ &= \frac{B}{R}(1+B)^n - \frac{\bar{B}}{R}(1+\bar{B})^n. \end{aligned} \quad (4.1.16)$$

Theorem 54: The convergents to B are

$$\frac{P_{2n}}{Q_{2n}} = \frac{S_{n+1}}{t_n} \quad (\text{inferior}), \quad (4.1.17)$$

$$\frac{P_{2n-1}}{Q_{2n-1}} = \frac{t_n}{\sigma_n} \quad (\text{superior}), \quad (4.1.18)$$

proof: By theorem 8 the recurrence

$$X_{n+2} = (k+2)X_{n+1} - X_n, \quad (4.1.19)$$

is satisfied by the four sequences $P_{2n-1}, P_{2n}, Q_{2n-1}, Q_{2n}$;

e.g. $P_{2n+4} = (k+2)P_{2n+2} - P_{2n}.$

But (4.1.19) is also satisfied by S_n, t_n and σ_n

so that it remains to check the starting values:

$$\begin{cases} P_0 = S_1 = k, \\ P_2 = S_2 = k^2 + 2k, \\ P_1 = t_1 = k+1, \\ P_3 = t_2 = k^2 + 3k + 1, \\ Q_0 = t_0 = 1, \\ Q_2 = t_1 = k+1, \end{cases}$$

$$\begin{cases} Q_1 = \sigma_1 = 1, \\ Q_3 = \sigma_2 = k+2, \end{cases}$$

and these are found to be correct.

Theorem 55:

$$t_n^2 = s_n \sigma_{n+1} + 1 = \sigma_n s_{n+1} + 1, \quad (4.1.20)$$

proof: By theorem 5

$$P_{2n-1} Q_{2n} - P_{2n} Q_{2n-1} = 1,$$

which is (4.1.20).

Theorem 56:

$$v_{t_n} = t_{n+1}, \quad (4.1.21)$$

$$v_{\sigma_n} = \sigma_{n+1} - 1, \quad (4.1.22)$$

$$u_{t_n} = \sigma_{n+1}, \quad (4.1.23)$$

$$u_{s_n} = t_n - 1, \quad (4.1.24)$$

$$u_{\sigma_n} = \frac{t_n - 1}{k} = \sigma_n + \sigma_{n-1} + \dots + \sigma_1. \quad (4.1.25)$$

proof: All of these formulas are corollaries of theorem 17, except (4.1.25) which we now prove.

Since $B \bar{B} = -k$,

$$\begin{aligned} \sigma_n + \sigma_{n-1} + \dots + \sigma_1 &= \sigma_n + \dots + \sigma_0 \\ &= \frac{1}{R} \left\{ (1+B)^n + \dots + (1+B)^0 - (1+\bar{B})^n - \dots - (1+\bar{B})^0 \right\} \\ &= \frac{1}{R} \left\{ \frac{(1+B)^{n+1} - 1}{B} - \frac{(1+\bar{B})^{n+1} - 1}{\bar{B}} \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{kR} \{ B(1+\bar{B})^{n+1} - B + \bar{B} - \bar{B}(1+B)^{n+1} \} \\
&= \frac{1}{kR} \{ (B-k)(1+\bar{B})^n + (k-\bar{B})(1+B)^n \} - \frac{1}{k} \\
&= \frac{1}{kR} \left\{ \left(\frac{k+R}{2} \right) (1+B)^n + \left(\frac{R-k}{2} \right) (1+\bar{B})^n \right\} - \frac{1}{k} \\
&= \frac{t_n - 1}{k},
\end{aligned}$$

which shows incidentally that $k \mid (t_n - 1)$.

$$\begin{aligned}
u_{\sigma_n} &= \left[\frac{1}{R} \{ (1+B)^n - (1+\bar{B})^n \} \frac{B}{k} \right] \\
&= \left[\frac{k+R}{2kR} \{ (1+B)^n - (1+\bar{B})^n \} \right] \\
&= \frac{t_n - 1}{k} + \left[\frac{1}{k} - (1+\bar{B})^n \left\{ \frac{R-k}{2kR} + \frac{k+R}{2kR} \right\} \right] \\
&= \frac{t_n - 1}{k} + \left[\frac{1}{k} \{ 1 - (1+\bar{B})^n \} \right] \\
&= \frac{t_n - 1}{k}, \quad \text{since } 0 < 1+\bar{B} < 1.
\end{aligned}$$

Let

$$g'_n = \frac{1}{R} \{ B^n - \bar{B}^n \}, \quad (4.1.26)$$

Theorem 57:

$$q'_{n+2} = k(q'_{n+1} + q'_n) \quad (4.1.27)$$

proof:

$$\begin{aligned} q'_{n+2} &= \frac{1}{R} \{ B^2 B^n - \bar{B}^2 \bar{B}^n \} \\ &= \frac{k}{R} \{ (1+B) B^n - (1+\bar{B}) \bar{B}^n \} \\ &= k(q'_{n+1} + q'_n). \end{aligned}$$

Theorem 58:

$$s_{-n} = -s_n, \quad (4.1.28)$$

$$\sigma_{-n} = -\sigma_n, \quad (4.1.29)$$

$$t_{-n} = t_{n-1}, \quad (4.1.30)$$

$$q'_{-n} = \frac{-1}{(-k)^n} q'_n. \quad (4.1.31)$$

proof:

Since $(1+B)(1+\bar{B}) = 1,$

$$s_{-n} = \frac{k}{R} \{ (1+B)^{-n} - (1+\bar{B})^{-n} \} = -s_n,$$

Hence by (4.1.13)

$$\sigma_{-n} = -\sigma_n.$$

$$\begin{aligned} t_{-n} &= \frac{B}{R} (1+B)^{-n} - \frac{\bar{B}}{R} (1+\bar{B})^{-n} \\ &= \frac{\bar{B}}{R} (1+\bar{B})^n - \frac{\bar{B}}{R} (1+B)^n \end{aligned}$$

$$= t_n - \frac{k}{R} \{ (1+B)^n - (1+\bar{B})^n \}$$

$$= t_n - s_n = t_{n-1}.$$

Finally, since $B\bar{B} = -k$,

$$q'_{-n} = \frac{1}{R} \{ B^{-n} - \bar{B}^{-n} \} = -\frac{1}{(-k)^n} q'_n.$$

Theorem 59:

$$q'_{2n} = k^n \sigma_n, \quad (4.1.32)$$

$$q'_{2n+1} = k^n t_n. \quad (4.1.33)$$

proof: By the previous theorem if (4.1.32) and (4.1.33) are true for n they are true for $-n$. Now $q'_0 = 0$, $q'_1 = 1$ so that they are true for $n=0$. Suppose them true for n .

Then

$$\begin{aligned} q'_{2n+2} &= k(q'_{2n+1} + q'_{2n}) \\ &= k(k^n t_n + k^n \sigma_n) = k^{n+1} \sigma_{n+1}, \end{aligned}$$

$$\begin{aligned} q'_{2n+3} &= k(q'_{2n+2} + q'_{2n+1}) \\ &= k(k^{n+1} \sigma_{n+1} + k^n t_n) \\ &= k^{n+1} (s_{n+1} + t_n) = k^{n+1} t_{n+1}, \end{aligned}$$

and the theorem is true by induction.

Theorem 60:

$$u_{q'_{2n-1}} = q'_{2n-1} + q'_{2n-2}, \quad (4.1.34)$$

$$u_{q'_{2n}} = q'_{2n} + q'_{2n-1} - 1, \quad (4.1.35)$$

$$v_{q'_{2n-1}} = q'_{2n} + q'_{2n-1}, \quad (4.1.36)$$

$$v_{q'_{2n}} = q'_{2n+1} + q'_{2n} - 1. \quad (4.1.37)$$

proof:

$$\begin{aligned} u_{q'_n} &= \left[\frac{1}{R} (B^n - \bar{B}^n) \left(1 + \frac{1}{B} \right) \right] \\ &= q'_n + q'_{n-1} + \left[\frac{1}{R} \bar{B}^n \left(\frac{1}{B} - \frac{1}{B} \right) \right] \\ &= q'_n + q'_{n-1} + \left[-\frac{1}{R} \bar{B}^n \right], \\ v_{q'_n} &= \left[\frac{1}{R} (B^n - \bar{B}^n) (1 + B) \right] \\ &= q'_n + q'_{n+1} + \left[-\bar{B}^n \right]. \end{aligned}$$

and the theorem follows from

$$-1 < \bar{B} < 0.$$

We notice from theorem 27 that

$$v_n = u_{kn} + n. \quad (4.1.38)$$

Most of the results of this section are illustrated by the following example:

$$k = 2, \quad B = [2, 1] = 1 + \sqrt{3}.$$

n	s_n	t_n	σ_n	q'_n
0	0	1	0	0
1	2	3	1	1
2	8	11	4	2
3	30	41	15	6
4	112	153	56	16
5	418	571	209	44
6	1560	2131	780	120
7	5822	7953	2911	328
8	21728	29681	10864	896
9	81090	110771	40545	2448
10	302632	413403	151316	6688

n	u_n	v_n	n	u_n	v_n	n	u_n	v_n
1	1	3	26	35	97	51	69	190
2	2	7	27	36	100	52	71	194
3	4	11	28	38	104	53	72	197
4	5	14	29	39	108	54	73	201
5	6	18	30	40	111	55	75	205
6	8	22	31	42	115	56	76	208
7	9	26	32	43	119	57	77	212
8	10	29	33	45	123	58	79	216
9	12	33	34	46	126	59	80	220
10	13	37	35	47	130	60	81	223
11	15	41	36	49	134	61	83	227
12	16	44	37	50	138	62	84	231
13	17	48	38	51	141	63	86	235
14	19	52	39	53	145	64	87	238
15	20	55	40	54	149	65	88	242
16	21	59	41	56	153	66	90	246
17	23	63	42	57	156	67	91	250
18	24	67	43	58	160	68	92	253
19	25	70	44	60	164	69	94	257
20	27	74	45	61	167	70	95	261
21	28	78	46	62	171	71	96	264
22	30	82	47	64	175	72	98	268
23	31	85	48	65	179	73	99	272
24	32	89	49	66	182	74	101	276
25	34	93	50	68	186	75	102	279

One can also define $r'_n = B^n + \bar{B}^n$ and get

$$r'_{2n} = k^{n+1}(\sigma_n + t_{n-1}), \quad r'_{2n+1} = k^{n+1}(\sigma_n + \sigma_{n+1}),$$

and a theorem analogous to theorem 60 (though the statement is more complicated).

4.2 Fan Sequences.

Ky Fan proposed¹ and solved² the following problem:
 Let $f(n)$ and $g(n)$ be two sequences of positive integers
 such that

$$(i) \quad f(1) = 1,$$

$$(ii) \quad g(n) = n(k+4) - 1 - f(n) \quad , k \text{ a positive integer}$$

(iii) $\{f(n)\}$ and $\{g(n)\}$ are complementary sequences:

$f(n+1)$ is the least integer distinct from the $2n$ integers
 $f(1), f(2), \dots, f(n), g(1), g(2), \dots, g(n)$.

Prove that there exist two numbers α and β such that

$$f(n) = [n\alpha] \quad , \quad g(n) = [n\beta], \quad (4.2.1)$$

Theorem 61:

$$\alpha = \overline{B} + 2 \quad , \quad \beta = B + 2, \quad (4.2.2)$$

so that

$$g(n) = v_n + n. \quad (4.2.3)$$

proof:

$$\alpha = 1 + \frac{1}{1+B},$$

$$\beta = 1 + (1+B),$$

and $1+B$ is irrational (the continued fraction is
 infinite) so that $[n\alpha]$ and $[n\beta]$ define Beatty sequences.

¹ American Mathematical Monthly, May 1950, p.343.

² ibid. Jan. 1952, p.48.

Further

$$\begin{aligned} n(k+4) - 1 - [n\alpha] &= n(k+4) - 1 + [-n\alpha] + 1 \\ &= [n\{k+4-\alpha\}] = [n\{k+4 - \frac{k+4-R}{2}\}] \\ &= [n\beta]. \end{aligned}$$

Finally

$$f(1) = [\alpha] = 1.$$

If P'_n/Q'_n are the convergents to $1+B$ then by theorem 54

$$\frac{P'_{2n}}{Q'_{2n}} = \frac{S_{n+1}}{t_n} + 1 = \frac{t_{n+1}}{t_n}, \quad (4.2.4)$$

$$\frac{P'_{2n-1}}{Q'_{2n-1}} = \frac{t_n}{\sigma_n} + 1 = \frac{\sigma_{n+1}}{\sigma_n}. \quad (4.2.3)$$

Hence by theorem 17,

Theorem 62:

$$f(t_n) = t_n + t_{n-1} - 1, \quad (4.2.4)$$

$$f(\sigma_n) = \sigma_n + \sigma_{n-1}, \quad (4.2.5)$$

$$g(t_n) = t_{n+1} + t_n, \quad (4.2.6)$$

$$g(\sigma_n) = \sigma_{n+1} + \sigma_n - 1. \quad (4.2.7)$$

It is easy to see that reduplication is false for $1+B$. That is,

$$f\{g(n)\} = f(n) + g(n), \quad (4.2.8)$$

is violated for some n . This follows from theorem 27; but we have the specific result

Theorem 63:

$$f(k+2) + g(k+2) = g(k+3). \quad (4.2.9)$$

Hence (4.2.8) is false for $n = k+2$.

proof: By (ii) page 110,

$$f(k+2) + g(k+2) = (k+2)(k+4) - 1$$

and by theorem 61,

$$g(k+3) = \left[(k+3) \left\{ 2 + \frac{k+R}{2} \right\} \right].$$

Thus we must show

$$(k+3) \left(2 + \frac{k+R}{2} \right) < (k+2)(k+4), \quad (4.2.10)$$

$$(k+3) \left(2 + \frac{k+R}{2} \right) > (k+2)(k+4) - 1. \quad (4.2.11)$$

(4.2.10) is

$$(k+3)R < k^2 + 5k + 4,$$

Squaring,

$$(k^2 + 6k + 9)(k^2 + 4k) < k^4 + 10k^3 + 25k^2 + 8k^2 + 40k + 16,$$

$$\begin{aligned} \text{i.e. } k^4 + 10k^3 + 33k^2 + 36k \\ < k^4 + 10k^3 + 33k^2 + 40k + 16, \end{aligned}$$

which is true. (4.2.11) is

$$\begin{aligned} (k+3)R &> k^2 + 5k + 2, \\ \text{i.e. } k^4 + 10k^3 + 33k^2 + 36k &> k^4 + 10k^3 + 29k^2 + 20k + 4, \end{aligned}$$

which is also true and (4.2.9) is proved.

$$f(k+2) + g(k+2) \neq f\{g(k+2)\},$$

follows from the fact that $\{f(n)\}$ and $\{g(n)\}$ are complementary.

4.3 Van der Waerden's Theorem.

Van der Waerden¹ proved the following theorem:

Let k and m be positive integers. Then there exists a positive integer $n = n(k, m)$ such that if n consecutive integers are divided arbitrarily into k sets then at least one of these sets contains an arithmetic progression of length m .

For our purposes we take a weakened statement of the theorem for the case $k = 2$: Of a pair of complementary sequences at least one contains an arithmetic progression of arbitrary length.

Theorem 64: If $\{u_n\}$ and $\{v_n\}$ are Beatty sequences they both contain arithmetic progressions of arbitrary length. But neither contains one of infinite length.
proof: Referring to the proof of theorem 17 we have

$$\begin{aligned} u_{mP_{2n+1}} &= m(P_{2n+1} + Q_{2n+1}) + [m\gamma_4 P_{2n+1}] \\ &= m(P_{2n+1} + Q_{2n+1}), \end{aligned}$$

$$\text{for } 1 \leq m \leq P_{2n+1},$$

¹ See A.Y. Khinchin, Three pearls of number theory.

Hence $\{u_n\}$ contains an arithmetic progression of length P_{2n+1} i.e. of arbitrary length since $P_{2n+3} > P_{2n+1}$.

Similarly

$$v_{mQ_{2n}} = m(P_{2n} + Q_{2n}), \quad (4.3.2)$$

for $1 \leq m \leq Q_{2n}$.

But on the other hand suppose $\{u_n\}$ contained the infinite a.p. $am + b$, $m = 1, 2, 3, \dots$.

If $u_n = [n\xi], \xi > 1$ then $u_n < n\xi < u_{n+1}$

$$\text{i.e.} \quad n - \frac{1}{\xi} < \frac{u_n}{\xi} < n,$$

$$\text{or} \quad 1 - \frac{1}{\xi} < r\left(\frac{u_n}{\xi}\right) < 1,$$

where $r(x)$ denotes the fractional part of x . This is the criterion that a number be in the sequence $\{u_n\}$. We have

$$1 - \frac{1}{\xi} < r\left(\frac{am+b}{\xi}\right) < 1, \quad m = 1, 2, 3, \dots$$

$$\text{or} \quad 1 - \frac{1}{\xi} < r\left(m \cdot \frac{a}{\xi} + \frac{b}{\xi}\right) < 1. \quad (4.3.3)$$

But a/ξ is irrational and therefore $r(m \frac{a}{\xi})$ is

dense in the interval $(0,1)$. Hence $r(m \frac{a}{\xi} + \frac{b}{\xi})$ is

dense in the interval $(0,1)$ which contradicts (4.3.3).

Thus $\{u_n\}$ cannot contain an infinite a.p. and this proof

is equally valid for $v_n = [n\eta]$.

A component of an r -fold complementary sequence cannot contain an infinite a.p. because it is a subset of either $\{u_n\}$ or $\{v_n\}$. We make the Conjecture: Each component of an arbitrary r -fold complementary sequence (generated by a Beatty sequence) contains an arithmetic progression of arbitrary length.

If $x_1, x_2, x_3, \dots$ are the members of such a component then clearly

$$x_n < Kn,$$

for some constant K . For example if $x_n = u_n v_n$ then K could be taken to be $\xi^2 \eta$. Thus the truth of this conjecture would follow from that of the following.

Conjecture: If $\{x_n\}$ is a strictly increasing sequence of positive integers such that $x_n < Kn$ for some fixed K then $\{x_n\}$ contains arithmetic progressions of arbitrary length.

This conjecture is weaker than Van der Waerden's theorem in that the sequence $\{x_n\}$ is infinite; but it gives the criterion $x_n < Kn$ and the proof (if it is true) would probably be more difficult.

4.4 Theorems on Inclusion.

T.Skolem¹ and T.Bang² have published the following theorems which we list without proof. Let ξ, η and ζ denote irrationals greater than 1 and let a and b denote positive integers.

Theorem 67: A necessary and sufficient condition that the sequences $[n\xi]$ and $[n\eta]$ be disjoint is that

$$\frac{a}{\xi} + \frac{b}{\eta} = 1.$$

Otherwise they have infinitely elements in common.

Theorem 68: $[n\xi]$ and $[n\eta]$ together contain all the integers if and only if

$$a\left(1 - \frac{1}{\xi}\right) + b\left(1 - \frac{1}{\eta}\right) = 1.$$

Theorem 69: $[n\xi]$ contains $[n\eta]$ if and only if

$$a\left(1 - \frac{1}{\xi}\right) + b/\eta = 1,$$

¹ On certain distributions of integers in pairs with given differences, *Mathematica Scandinavica*, 5 (1957), p.57 ; and *Über einige Eigenschaften der Zahlenmengen $[\alpha n + \beta]$ bei irrationalem α mit einleitenden Bemerkungen über einige kombinatorische Probleme*, *Det Kongelige Norske Videnskabers Selskabs For. b.* 30, 1957, Nr 19, p.118.

² On the sequence $[n\alpha]$, $n=1, 2, \dots$, supplementary note to the preceding paper by Th.Skolem, *Math.Scand.*, 5 (1957), p.69.

Using the a and b of this theorem,

Theorem 70: If $[n\xi]$ contains $[n\zeta]$ and $[n\zeta]$ contains $[n\eta]$ then either

$$(i) \quad \zeta = \eta/b, \quad \text{where } b_1 | b$$

(In this case $\zeta \geq \eta/b$), or

$$(ii) \quad \zeta = \frac{\xi}{\xi - a, (\xi - 1)} \quad \text{where } a_1 | a,$$

(In this case $\zeta \leq \eta/b$).

Theorem 71: $[n\xi + \theta]$ and $[n\eta + \phi]$ are disjoint if and only if

$$\frac{a}{\xi} + \frac{b}{\eta} = 1, \quad a \frac{\theta}{\xi} + b \frac{\phi}{\eta} \equiv 0 \pmod{1}.$$

~~(They are complementary if $a = b = 1$).~~ Otherwise they have infinitely elements in common.

Let $C(N)$ be the number of elements common to $[n\xi]$ and $[n\eta]$ for $n \leq N$. Then

Theorem 72:

$$C(N) \sim \frac{1}{\xi\eta},$$

if $\frac{1}{\xi}, \frac{1}{\eta}$ and 1 are linearly independent.

Theorem 73: $[n\xi], [n\eta]$ and $[n\zeta]$ cannot be disjoint; hence they cannot be complementary.

4.5 Inverse and Complementary Sequences.

J.Lambek and L.Moser¹ have shown the following theorems concerning general complementary sequences. We omit the proofs.

Let $f(n)$ be an increasing sequence of positive integers and define $f'(n)$ to be the number of m such that $f(m) < n$. Define $f(n)$ and $g(n)$ to be inverses if for every pair m, n

$$f(m) < n \quad \text{or} \quad g(n) < m,$$

but not both.

Theorem 74: The inverse of $f(n)$ is unique and coincides with $f'(n)$.

Theorem 75: $f(n)$ and $g(n)$ are inverses if and only if

$$F(n) = f(n) + n, \quad G(n) = g(n) + n.$$

are complementary.

Define $F^*(n) = F'(n+1)$ and $CF(n)$ to be the complementary sequence of $F(n)$. Further let

$$F_0(n) = n, \quad F_k(n) = F^*(F_{k-1}(n)) + n.$$

By the lemma

$$CF(n) = F^*(CF(n)) + n,$$

they prove

¹ Inverse and complementary sequences of natural numbers, American Mathematical Monthly, vol.61, 1954, p.454.

Theorem 76:

$$CF(n) = \lim_{k \rightarrow \infty} F_k(n) .$$

This limit is attained in a finite number of steps.

For example,

Theorem 77: If $F(m+1) - F(m) \geq m$,

then $CF(n) = F_2(n)$.

For Beatty sequences

$$f(n) = \left[\frac{n}{\alpha} \right] , \quad g(n) = [n\alpha] ,$$

$$F(n) = u_n , \quad G(n) = v_n .$$

By theorem 75, theorem 74 states that $[N\alpha]$ is the number of $\left[\frac{n}{\alpha} \right] < N$.

4.6 Some Concluding Remarks.

With regard to future study in the subject there are a number of conjectures which might be added to the list begun in section 4.3. For example, 'there are infinitely many primes in each of the sequences $\{u_n\}, \{v_n\}$ '. Indeed if $\pi(x, \xi)$ is the number of primes $\leq x$ in the sequence $u_n = [n\xi]$ we would expect

$$\pi(x, \xi) \sim \frac{x}{\xi \log x},$$

More generally if $(a, b) = 1$ and $\pi(x, \xi; a, b)$ is the number of primes $\leq x$ of the form $an + b$ then

$$\pi(x, \xi; a, b) \sim \frac{x}{\xi \phi(a) \log x},$$

where $\phi(a)$ is Euler's totient function.

These conjectures say of course that neither of the sequences $\{u_n\}, \{v_n\}$ is favoured and each receives its proper quota.

Similarly one would expect the number of squares to be apportioned fairly; and so on.

But perhaps the nature of theorem 28 should make one more cautious.

Finally we mention two lines of investigation which were not pursued at all in the thesis: the algebraic and the topological.

I. Niven¹ has defined the sequence $\{b_n\}$ to be the product

$$\{a_n\} \times \{b_n\}.$$

Under this operation the set of sequences of positive integers forms an associative, non-commutative groupoid. This clearly ties in with reduplication.

One can define a metric

$$\rho(\{a_n\}, \{b_n\}) = \frac{1}{N},$$

where N is the least n such that $a_n \neq b_n$, R.L. Duncan², using a slightly different metric, has discussed the topological properties of such metric spaces.

¹ The asymptotic density of sequences, Bulletin of the American Mathematical Society, 57, 1951, p. 431.

² A topology for sequences of integers I, American Mathematical Monthly, Jan., 1959, p. 34.

