THE UNIVERSITY OF MANITOBA

ASYMPTOTIC SOLUTIONS OF SECOND ORDER DIFFERENCE EQUATIONS

BY

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Abstract

In this thesis we study the asymptotic solutions of the second-order linear difference equation

$$y(n+2) + n^{M_a} a(n) y(n+1) + n^{M_b} b(n) y(n) = 0,$$

where both M_a and M_b are integers and the coefficient functions are

$$a(n) = \sum_{s=0}^{\infty} a_s n^{-s} \qquad \textit{and} \qquad b(n) = \sum_{s=0}^{\infty} b_s n^{-s}$$

for large values of n, and $a_0 \neq 0$, $b_0 \neq 0$. Our discussion is divided into two parts. The first part, in which $M_a \leq 0$ and $M_b = 0$, consists of three cases according to whether the roots of the characteristic equation $\rho^2 + a_0\rho + b_0 = 0$ are distinct, or equal and do not satisfy the auxiliary equation $a_1\rho + b_1 = 0$, or equal and do satisfy the auxiliary equation. The last case is further divided into three subcases, according to whether the zero of the indicial polynomial $\alpha(\alpha-1)\rho^2+(a_1\alpha+a_2)\rho+b_2$ do not differ by an integer, or differ by a nonzero integer, or are equal. In these cases, the formal series solution will be shown to be asymptotic. The approach is based on the method of successive approximation. In the second part, we shall deal with the general situation where M_a and M_b are arbitrary integers. Our discussion will also be divided into three cases according to whether the constant $K = 2M_a - M_b$ is zero, or negative, or positive. When K is positive, there are two subcases according to whether K is equal to one or not. When K is negative, there are three subcases according to whether K is equal to -1, or odd but not -1, or even. In all these cases, formal solutions are established.

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Contents

Chapter 1

Introduction

1.1 Introduction

The second-order linear difference equation

$$y(n+2) + n^{M_a}a(n)y(n+1) + n^{M_b}b(n)y(n) = 0, \tag{1.1}$$

where both M_a and M_b are integers and the coefficient functions are

$$a(n) = \sum_{s=0}^{\infty} a_s n^{-s} \qquad \textit{and} \qquad b(n) = \sum_{s=0}^{\infty} b_s n^{-s},$$

for large values of n and $a_0 \neq 0$, $b_0 \neq 0$, arises frequently in many branches of mathematics. For example, special functions of mathematical physics satisfy a three-term recurrence relation; a second-order linear differential equation when discretized, leads to a corresponding difference equation. Topics similar to this thesis have been developed by *E. W. Barnes* [2], *P. M. Batchelder* [3], *W. J. Culmer* and *W. A. Harris Jr.* [8], etc., but *Barnes* and *Batchelder* discussed only hypergeometric equations which are a part of our cases while *Culmer* and *Harris* dealt with a difference equation system with 2 by 2 matrix. Still, the solutions constructed by *Culmer* and *Harris Jr.* were not directly from the original system but from the eight canonical forms which are reduced from it.

We also know that there is a well-known theorem concerning nth-order linear difference equations represented by *G. D. Birkhoff* and *W. J. Trjitzinsky* ([7] p.3-4). The theorem indicates that the equation (1.1) admits two linearly independent formal series solutions with elements of general type

$$e^{Q(n)}S(n) \tag{1.2}$$

where

$$Q(n) = \mu n \log n + \gamma n + \delta n^{\frac{p-1}{p}} + \ldots + \nu n^{\frac{1}{p}},$$

$$S(n) = n^{r}\left\{\left(a^{(0)} + b^{(0)}n^{-\frac{1}{p}} + \ldots\right) + \left(a^{(m)} + b^{(m)}n^{-\frac{1}{p}} + \ldots\right)\log^{m} n\right\},$$

and where p is a positive integer, μp is an integer and m is 1 or 0. Here, p does not need to be the same as the integer, denoted by the same letter, occuring in connection with the coefficients of the equation (see *G.D.Birkhoff* and *W.J. Trjitzinsky* [7] p.4). They further proved that these formal solutions are indeed asymptotic. According to their definition, series (1.2) is called a formal series solution or a formal solution if, when is substituted in (1.1), the sum of the coefficients of the terms with the same powers of n are zero. We, in the thesis, use capital letters, for example, $Y(n)$, $Z(n)$ and $X(n)$ as formal solutions while small letter, $y(n)$, $z(n)$ and $x(n)$ as exact solutions. Also, when (1.2) is called an asymptotic solutions, denoted by

$$y(n) \sim e^{Q(n)}S(n), \qquad n \to \infty,$$

where $y(n)$ is the exact solution of equation (1.1), it means that for every $N \geq 1$, we can determine functions $E_N^{(j)}(n)$, $j = 0, m$, such that

$$
\begin{aligned}
e^{-Q(n)} n^{-r} y(n) = & \left(a^{(0)} + b^{(0)} n^{-\frac{1}{p}} + \ldots + c^{(0)} n^{-\frac{N-1}{p}}\right) \\
& + \left(a^{(m)} + b^{(m)} n^{-\frac{1}{p}} + \ldots + c^{(m)} n^{-\frac{N-1}{p}}\right) \log^m n \\
& + E_N^{(0)}(n) + E_N^{(m)}(n) \log^m n,
\end{aligned} \tag{1.3}
$$

and $E_N^{(j)}(n) = O(n^{-\frac{N}{p}})$, $j = 0, m$, as $n \to \infty$.

However, we found that their proof was essentially existential, and no direct method was given to compute successive coefficients in (1.2). From the viewpoint of the computation, the approach to the problem is always clumsy and quite unsatisfactory. Furthermore, the method of verification seems to be somewhat complicated. *G. K. Immik* ([9] p.128) has rightly pointed out that " Some of the proofs are incomplete and not all the conclusions seem to be justified." A short summary and comments on this massive work can also be found in *J. Wimp* ([12] Appendix).

This thesis will offer some types of formal series solutions in the light of the papers by *C. R. Adams* [1], and the earlier works of *Birkhoff* [5] and [6]. The reason for this is that in these papers, they classified the general equation into several cases (although incomplete as they seem to be), and gave the type of asymptotic solutions for each of them. These are the foundation of the paper of *Birkhoff* and *Trjitzinsky* [7].

We shall discuss the problem in two parts. First, we study equation(1.1) with $M_a \leq 0$ and $M_b = 0$. We give the explicit expression of solutions in Chapter 2. These expressions may be obtained by substitution and direct comparison. The proof that these formal solutions are indeed asymptotic, i. e. , $y(n) \sim Y(n)$

will be given in Chapter 3. We provide a new and elementary proof based on the method of successive approximation which is customarily used in differential equation theory ([11] p.229-235). Second, we shall deal with equation (1.1) where M_a and M_b have no restrictions. We shall give formal solutions in Chapter 4.

In Chapter 2, equation (1.1) may be rewritten as

$$y(n+2) + a(n)y(n+1) + b(n)y(n) = 0, \tag{1.4}$$

where $a(n)$ and $b(n)$ have power series expansions of the form

$$a(n) = \sum_{s=0}^{\infty} a_s n^{-s} \quad \text{and} \quad b(n) = \sum_{s=0}^{\infty} b_s n^{-s} \tag{1.5}$$

for large values of n, and $b_0 \neq 0$. The classification of cases depends on the roots of the characteristic equation

$$\rho^2 + a_0\rho + b_0 = 0. \tag{1.6}$$

If the two values of ρ are not equal, i. e. , $a_0^2 \neq 4b_0$, then *Birkhoff* [5] showed that equation (1.4) has two linearly independent solutions both of asymptotic expressions of the form

$$y(n) \sim \rho^n n^\alpha \sum_{s=0}^{\infty} c_s n^{-s}, \qquad n \to \infty. \tag{1.7}$$

Motivated by the terminologies in differential equation theory ([11] p.230), we call series of the form (1.7) normal series solutions or normal solutions. We shall give explicit recurrences relation for the α, c_s as (2.4) and (2.5).

If ρ is a multiple root of equation (1.6), but not a zero of the auxiliary equation

$$a_1\rho + b_1 = 0, \tag{1.8}$$

i. e. , $2b_1 \neq a_0a_1$, then *Adams* [1] gave two linearly independent solutions

$$y(n) \sim \rho^n e^{\gamma\sqrt{n}} \sum_{s=0}^{\infty} c_s n^{\alpha - \frac{1}{2}s}, \qquad n \to \infty, \tag{1.9}$$

which we shall call subnormal solutions ([11] p.231). In §2.2, formulas for the constants γ and α are given by (2.17) and (2.18), and recurrence relations for the coefficients c_s are derived in (2.19).

Furthermore, in §2.3 and §2.4, we shall discuss in detail three exceptional cases in which the double roots of characteristic equation (1.6) satisfy the auxiliary equation (1.8), and the zero α_1 and α_2 (Re $\alpha_2 \geq$ Re α_1, where Re α is the real part of the complex number α) of the indicial polynomial

$$q(\alpha) = \alpha(\alpha - 1)\rho^2 + (a_1\alpha + a_2)\rho + b_2 \tag{1.10}$$

satisfy

$$\begin{aligned} &\text{(i)} && \alpha_2 - \alpha_1 \neq 0, 1, 2, \ldots, \\ &\text{(ii)} && \alpha_2 - \alpha_1 = 1, 2, \ldots, \\ &\text{(iii)} && \alpha_2 - \alpha_1 = 0. \end{aligned} \tag{1.11}$$

In case (i), it will be shown that equation (1.4) has two independent asymptotic solutions of the form (1.7), where $\alpha = \alpha_i$, $i = 1, 2$, are zeros of (1.10), and that the coefficients c_s can be determined by the recurrence formula (2.23). The possibility that the subnormal solutions may be reduced to the normal solutions in this case was also pointed out by *Adams* ([1] p.510).

In the other two cases, *Birkhoff* [6] suggested that one of the two independent solutions, say $y_1(n)$, is the same solution as in case (i). The second solution involves a logarithmic term

$$y_2(n) = z(n) + Cy_1(n)\log n, \tag{1.12}$$

where C is a constant and $z(n)$ has an expansion

$$z(n) \sim \rho^n n^\alpha \sum_{s=0}^{\infty} d_s n^{-s}, \qquad n \to \infty. \tag{1.13}$$

In case (ii), C is specified by (2.38), $\alpha = \alpha_2$ and d_s can be determined iteratively by formulas (2.35), (2.37) and (2.39). In case (iii), $\alpha = \alpha_1 = \alpha_2$ in (1.13). C is a non-zero constant specified by (2.44), and d_s may be determined iteratively by (2.43), (2.45) and (2.46).

Chapter 4 is devoted to the general situation where M_a and M_b have no restriction. In §4.1 three cases will be obtained by transformation. The classification is according to whether $K = 2M_a - M_b$ is equal to zero, positive or negative. The cases in which $K = 0$ or negative and even shall be shown to be included in Chapter 2. In §4.2 the formal solution

$$Y_i(n) = [(n-2)!]^{-\lambda_i} \rho_i^n n^{\alpha_i} \sum_{s=0}^{\infty} c_s^{(i)} n^{-s}, \quad i = 1, 2, \tag{1.14}$$

will be obtained for $K > 0$. λ_i and ρ_i may be determined from (4.9), (4.12) and (4.13). The determination of α_i, $c_s^{(i)}$ is dependent of K. If $K = 1$, α_i and $c_s^{(i)}$ are from recurrence formulas (4.29) and (4.30) for $i = 1$, and from (4.40) and

(4.41) for $i = 2$. If $K > 1$, α_i and $c_s^{(i)}$ are determined from (4.31), (4.32) and (4.33) for $i = 1$, and from (4.42),(4.43) and (4.44) for $i = 2$. In §4.3, the formal solutions

$$Y_i(n) = [(n-2)!]^{-\lambda_i} \rho_i^n e^{\gamma_i \sqrt{n}} n^{\alpha_i} \sum_{s=0}^{\infty} c_s^{(i)} n^{-\frac{s}{2}}, \tag{1.15}$$

$i = 1, 2$, are formed for $K < 0$ and odd. λ_i and ρ_i can be obtained from (4.15) and (4.16). The determination of γ_i, α_i and $c_s^{(i)}$ is dependent on whether $K = -1$ or not. If $K = -1$, γ_i, α_i and $c_s^{(i)}$ may be obtained from (4.52), (4.53) and (4.54) for $i = 1, 2$. If $K < -1$, $\gamma = 0$. α_i and $c_s^{(i)}$ may be odtained from (4.53), (4.57) and (4.58) for $i = 1, 2$. Throughout this thesis, without loss of generality, we shall always assume that $c_0 = 1$ in (1.7), (1.9), (1.14) and (1.15).

Chapter 2

Formal Solutions (1)

2.1 Case 1, $\rho_1 \neq \rho_2$

We begin by showing that the infinite series in (1.7) is indeed a formal solution of equation (1.4) with $a_0^2 \neq 4b_0$ or $\rho \neq -a_0/2$. Let $\mathcal{L}$ denote the linear difference operator

$$\mathcal{L}\{Y(n)\} = Y(n+2) + a(n)Y(n+1) + b(n)Y(n). \tag{2.1}$$

Inserting (1.7) into (2.1), and making use of the identity

$$(n+\mu)^{\alpha-s} = n^{\alpha} \sum_{k=0}^{\infty} \binom{\alpha - s}{k} \mu^k n^{-(s+k)}, \qquad \mu = 1, 2, \tag{2.2}$$

we have

$$\mathcal{L}\{Y(n)\} = \rho^n n^{\alpha} \sum_{s=0}^{\infty} \left\{ \rho^2 \sum_{j=0}^{s} 2^{s-j} \binom{\alpha - j}{s-j} c_j \right.$$
$$\left. + \rho \sum_{k=0}^{s} \left[\sum_{j=0}^{k} \binom{\alpha - j}{k-j} c_j \right] a_{s-k} + \sum_{j=0}^{s} b_{s-j} c_j \right\} n^{-s}.$$

By exchanging summations in the middle term of the right-hand side, the above equality may be rewritten as

$$\mathcal{L}\{Y(n)\} = \rho^n n^\alpha$$
$$\cdot \sum_{s=0}^{\infty} \left\{ \sum_{j=0}^{s} \left[\rho^2 2^{s-j} \binom{\alpha - j}{s - j} + \rho \sum_{k=j}^{s} \binom{\alpha - j}{k - j} a_{s-k} + b_{s-j} \right] c_j \right\} n^{-s}.$$

Equating the coefficients of terms in n^{-s} to zero gives the relations

$$\sum_{j=0}^{s} \left\{ \rho^2 2^{s-j} \binom{\alpha - j}{s - j} + \rho \sum_{k=j}^{s} \binom{\alpha - j}{k - j} a_{s-k} + b_{s-j} \right\} c_j = 0. \tag{2.3}$$

It is readily found that when $s = 0$, (2.3) reduces to the characteristic equation (1.6). When $s = 1$ in (2.3), it can be shown by using (1.6) that

$$\alpha = -\frac{a_1 \rho + b_1}{2\rho^2 + \rho a_0} = \frac{a_1 \rho + b_1}{a_0 \rho + 2b_0}. \tag{2.4}$$

Since we are dealing with the case of distinct characteristic roots, i. e. , $\rho \neq -(2b_0)/a_0$, the denominator in (2.4) is not zero. Now notice that the coefficients of c_s and c_{s-1} in (2.3) are $\rho^2 + \rho a_0 + b_0$ and

$$\left[\left(2\rho^2 + \rho a_0 \right) \alpha + \rho a_1 + b_1 \right] + (1 - s)(2\rho + a_0)\rho,$$

respectively. In view of (1.6) and (2.4), these are, in turn, zero and $(s-1)(a_0\rho + 2b_0)$. Thus (2.3) becomes

$$c_{s-1} = \frac{1}{(1-s)(a_0\rho + 2b_0)}$$

$$\cdot \sum_{j=0}^{s-2} \left\{ \rho^2 2^{s-j} \binom{\alpha - j}{s - j} + \rho \sum_{k=j}^{s} \binom{\alpha - j}{k - j} a_{s-k} + b_{s-j} \right\} c_j,$$

$$s = 2, 3, \ldots. \tag{2.5}$$

We also find that the denominator in (2.5) is not zero.

2.2 Case 2, $\rho_1 = \rho_2$ but $a_1\rho + b_1 \neq 0$

As $\rho_1 = \rho_2$ implies $a_0\rho + 2b_0 = 0$, the denominators of both (2.4) and (2.5) vanish. However, two independent subnormal solutions (1.9) were suggested by *Adams* ([1] p.509). Before inserting (1.9) into equation (2.1), we observe the formal identity

$$e^{\gamma\sqrt{n+\mu}} = e^{\gamma\sqrt{n}} \sum_{s=0}^{\infty} G_s^{(\mu)}(\gamma) n^{-\frac{1}{2}s}, \qquad \mu = 1, 2, \tag{2.6}$$

where

$$G_s^{(\mu)}(\gamma) = \sum_{\sigma(\boldsymbol{l}_p)=s} \prod_{j=1}^{p} \frac{1}{l_j!} \left[\binom{\frac{1}{2}}{j} \mu^j \gamma \right]^{l_j}, \tag{2.7}$$

the summation being taken over all multi-indices $\boldsymbol{l}_p = (l_1, \ldots, l_p)$ for non-negative integers l_s, $s = 1, 2, \ldots, p$, such that

$$\sigma(\boldsymbol{l}_p) = l_1 + 3l_2 + \ldots + (2p-1)l_p = s, \tag{2.8}$$

and the last value $l_p \neq 0$ if $p > 1$. Elementary calculatons give

$$
\begin{array}{lll}
 & G_0^{(1)}(\gamma) = 1, & G_0^{(2)}(\gamma) = 1, \\
 & G_1^{(1)}(\gamma) = \frac{1}{2}\gamma, & G_1^{(2)}(\gamma) = \gamma, \\
(2.9) & G_2^{(1)}(\gamma) = \frac{1}{8}\gamma^2, & G_2^{(2)}(\gamma) = \frac{1}{2}\gamma^2, \\
 & G_3^{(1)}(\gamma) = \frac{1}{48}\gamma^3 - \frac{1}{8}\gamma, & G_3^{(2)}(\gamma) = \frac{1}{6}\gamma^3 - \frac{1}{2}\gamma, \\
 & G_4^{(1)}(\gamma) = \frac{1}{384}\gamma^4 - \frac{1}{16}\gamma^2, & G_4^{(2)}(\gamma) = \frac{1}{24}\gamma^4 - \frac{1}{2}\gamma^2.
\end{array}
$$

We also have another identity

$$
(2.10) \qquad \sum_{s=0}^{\infty} c_s (n+\mu)^{\alpha - \frac{1}{2}s} = n^{\alpha} \sum_{s=0}^{\infty} F_s^{(\mu)}(\alpha) n^{-\frac{1}{2}s}, \qquad \mu = 1, 2,
$$

where

$$
(2.11) \qquad F_s^{(\mu)}(\alpha) = \sum_{j=0}^{s} \frac{1+(-1)^{s-j}}{2} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}(s-j)} \mu^{\frac{1}{2}(s-j)} c_j.
$$

In view of (2.6) and (2.10), substitution of (1.9) in (2.1) gives

$$
\begin{aligned}
\mathcal{L}\{Y(n)\} &= e^{\gamma\sqrt{n}} \rho^n n^{\alpha} \\
&\cdot \sum_{s=0}^{\infty} \left\{ \rho^2 \sum_{k=0}^{s} G_{s-k}^{(2)}(\gamma) F_k^{(2)}(\alpha) + \rho \sum_{k=0}^{s} \left[\frac{1+(-1)^k}{2} a_{\frac{k}{2}} \sum_{l=0}^{s-k} G_{s-l-k}^{(1)}(\gamma) F_l^{(1)}(\alpha) \right] \right. \\
&\qquad \left. + \sum_{j=0}^{s} \frac{1+(-1)^{s-j}}{2} b_{\frac{1}{2}(s-j)} c_j \right\} n^{-\frac{1}{2}s}.
\end{aligned}
$$

Making use of (2.11) and exchanging summations, we have

$$
\begin{aligned}
\mathcal{L}\{Y(n)\} &= e^{\gamma\sqrt{n}} \rho^n n^{\alpha} \\
&\cdot \sum_{s=0}^{\infty} \left\{ \sum_{j=0}^{s} \left[\rho^2 \sum_{k=0}^{s-j} \frac{1+(-1)^k}{2} 2^{\frac{1}{2}k} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}k} G_{s-k-j}^{(2)}(\gamma) \right. \right. \\
&\qquad + \rho \sum_{k=0}^{s-j} \frac{1+(-1)^k}{2} a_{\frac{1}{2}k} \sum_{l=0}^{s-k-j} \frac{1+(-1)^l}{2} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}l} G_{s-l-k-j}^{(1)}(\gamma)
\end{aligned}
$$

$$+\frac{1+(-1)^{s-j}}{2} b_{\frac{1}{2}(s-j)} \Bigg] c_j \Bigg\} n^{-\frac{1}{2}s}. \tag{2.12}$$

Equating coefficients of $n^{-\frac{1}{2}}$ to zero gives

$$\begin{aligned}\sum_{j=0}^{s} \Bigg\{ \sum_{k=0}^{s-j} \frac{1+(-1)^k}{2} \Bigg[\rho^2 2^{\frac{1}{2}k} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}k} G^{(2)}_{s-k-j}(\gamma) \\ + \rho a_{\frac{1}{2}k} \sum_{l=0}^{s-k-j} \frac{1+(-1)^l}{2} \binom{\alpha-\frac{1}{2}j}{\frac{1}{2}l} G^{(1)}_{s-l-k-j}(\gamma) \Bigg] \\ + \frac{1+(-1)^{s-j}}{2} b_{\frac{1}{2}(s-j)} \Bigg\} c_j = 0, \qquad s = 0,1,2,\ldots.\end{aligned} \tag{2.13}$$

Since $G_0^{(1)}(\gamma) = G_0^{(2)}(\gamma) = 1$ and $c_0 = 1$, (2.13) reduces to (1.6) if $s = 0$. We observe the coefficients of c_s and c_{s-1} in (2.13). In view of (1.6), they are zero and $\rho\gamma(\rho + \frac{1}{2}a_0)$, respectively. Since ρ is a double root of (1.6), or $\rho = -\frac{1}{2}a_0$ in this case, the second coefficient vanishes as well. Consequently, (2.13) becomes

$$\begin{aligned}\sum_{j=0}^{s-2} \Bigg\{ \sum_{k=0}^{s-j} \frac{1+(-1)^k}{2} \Bigg[\rho^2 2^{\frac{1}{2}k} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}k} G^{(2)}_{s-k-j}(\gamma) \\ + \rho a_{\frac{1}{2}k} \sum_{l=0}^{s-k-j} \frac{1+(-1)^l}{2} \binom{\alpha-\frac{1}{2}j}{\frac{1}{2}l} G^{(1)}_{s-l-k-j}(\gamma) \Bigg] \\ + \frac{1+(-1)^{s-j}}{2} b_{\frac{1}{2}(s-j)} \Bigg\} c_j = 0, \qquad s = 2,3,\ldots.\end{aligned} \tag{2.14}$$

The parameters γ and α in (1.9) should necessarily satisfy (2.14) for $s = 2$ and $s = 3$. Now we set $s = 2$ and $s = 3$. In view of (2.9) and the fact that $\rho = -\frac{1}{2}a_0$, we have

$$\left(\frac{1}{4}\rho^2\gamma^2 + \rho a_1 + b_1\right) c_0 = 0 \tag{2.15}$$

and

$$\rho\gamma\left[\rho\left(\frac{1}{8}\gamma^2+\alpha-\frac{1}{4}\right)+\frac{1}{2}a_1\right]c_0+\left(\frac{1}{4}\rho^2\gamma^2+\rho a_1+b_1\right)c_1=0. \tag{2.16}$$

Solving equation (2.15) gives, in view of the fact that $\rho=-\frac{1}{2}a_0$ and $\rho^2=b_0$,

$$\gamma=\pm 2\sqrt{\frac{a_0a_1-2b_1}{2b_0}}. \tag{2.17}$$

By taking into account that $a_1\rho+b_1\neq 0$ in our case, we find that γ does not vanish. Therefore substituting (2.17) into (2.16) yields

$$\alpha=\frac{1}{4}+\frac{b_1}{2b_0}. \tag{2.18}$$

Finally, we note that the coefficients of c_{s-2} and c_{s-3} in (2.14) are, respectively,

$$\frac{1}{4}\rho^2\gamma^2+\rho a_1+b_1$$

and

$$\rho\gamma\left[\rho\left(\frac{1}{8}\gamma^2-\frac{1}{4}+\alpha\right)+\frac{1}{2}a_1\right]-\frac{1}{2}\rho^2\gamma(s-3).$$

Looking back to (2.17) and (2.18), these are in turn equal to 0 and $-\frac{1}{2}\rho^2\gamma(s-3)$. Therefore, (2.14) can be written as

$$\begin{aligned}c_{s-3}=\frac{2}{\rho^2\gamma(s-3)}\sum_{j=0}^{s-4}\Bigg\{\sum_{k=0}^{s-j}\frac{1+(-1)^k}{2}\Bigg[&\rho^2 2^{\frac{1}{2}k}\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}k}G^{(2)}_{s-k-j}(\gamma)\\&+\rho a_{\frac{1}{2}k}\sum_{l=0}^{s-k-j}\frac{1+(-1)^l}{2}\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}l}G^{(1)}_{s-l-k-j}(\gamma)\Bigg]\\&+\frac{1+(-1)^{s-j}}{2}b_{\frac{1}{2}(s-j)}\Bigg\}c_j,\qquad s=4,5,\ldots.\end{aligned} \tag{2.19}$$

The coefficients $c_1, c_2, \cdots$ in (1.9) are now determined recursively by (2.19). In particular, we have

$$c_1 = \frac{1}{24b_0^2\gamma}(a_0^2a_1^2 - 24a_0a_1b_0 + 8a_0a_1b_1 - 24a_0a_2b_0$$
$$-9b_0^2 - 32b_1^2 + 24b_0b_1 + 48b_0b_2).$$

2.3 Exceptional Case (i)

In this section we shall discuss the exceptional case (i) in (1.11) where ρ is a double root of the characteristic equation (1.6), and the difference of the roots of the indicial polynomial $q(\alpha)$ (1.10) , $\alpha_2 - \alpha_1$, is not an integer.

We shall show that the subnormal solution may "reduce" to the normal solution if we insert (1.9) into (2.1) as in the last section. Since $\rho a_1 + b_1 = 0$, γ must be zero to satisfy the relation (2.15). Consequently, from (2.7) and (2.8), we find all $G_s^{(\mu)}(\gamma) = 0$ except $G_0^{(\mu)}(\gamma) = 1$, $\mu = 1, 2$. Thus (2.14) becomes

$$\sum_{j=0}^{s-2}\left\{\frac{1+(-1)^{s-j}}{2}\left[\rho^2 2^{\frac{1}{2}(s-j)}\binom{\alpha - \frac{1}{2}j}{\frac{1}{2}(s-j)}\right.\right.$$
$$\left.\left. + \rho\sum_{k=0}^{s-j}\frac{1+(-1)^k}{2}a_{\frac{1}{2}k}\binom{\alpha - \frac{1}{2}j}{\frac{1}{2}(s-j-k)} + b_{\frac{1}{2}(s-j)}\right]\right\}c_j = 0, \tag{2.20}$$
$$s = 2, 3, \ldots.$$

By the same argument as in the last section, the coefficients of c_{s-2}, c_{s-3} are

$$(2\rho + a_0)\rho(\alpha - \frac{1}{2}s + 1) + (\rho a_1 + b_1)$$

and zero, respectivily. In view of the fact that $\rho = -\frac{1}{2}a_0$ and $\rho a_1 + b_1 = 0$ in this case, these both vanish. Thus (2.20) becomes

$$\sum_{j=0}^{s-4}\left\{\frac{1+(-1)^{s-j}}{2}\left[\rho^2 2^{\frac{1}{2}(s-j)}\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}(s-j)}\right.\right.$$

$$\left.\left.+\rho\sum_{k=0}^{s-j}\frac{1+(-1)^k}{2}a_{\frac{1}{2}k}\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}(s-j-k)}+b_{\frac{1}{2}(s-j)}\right]\right\}c_j=0,$$

$$s=4,5,\ldots. \tag{2.21}$$

Now, when we set $s=4$ in (2.21), we find that α should satisfy the indicial polynomial (1.10).

Next, we find that the coefficient of c_{s-4} in (2.21) is

$$q\left(\alpha-\frac{s}{2}+2\right)$$

$$=\rho^2\left(\alpha-\frac{s}{2}+2\right)\left(\alpha-\frac{s}{2}+1\right)+\rho a_1\left(\alpha-\frac{s}{2}+2\right)+\rho a_2+b_2.$$

As $\alpha_2-\alpha_1$ (assume Re $\alpha_2>$ Re α_1) is not an integer, both $q\left(\alpha_i-\frac{1}{2}s+2\right)$, $i=1,2$, do not vanish when $s>4$. Hence, from (2.21) we have the recurrence relation,

$$c_{s-4}=\frac{-1}{q(\alpha-\frac{1}{2}s+2)}$$

$$\sum_{j=0}^{s-6}\left\{\frac{1+(-1)^{s-j}}{2}\left[\rho^2 2^{\frac{1}{2}(s-j)}\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}(s-j)}\right.\right.$$

$$\left.\left.+\rho\sum_{k=0}^{s-j}\frac{1+(-1)^k}{2}a_{\frac{1}{2}k}\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}(s-j-k)}+b_{\frac{1}{2}(s-j)}\right]\right\}c_j,$$

$$s=5,6,\ldots, \tag{2.22}$$

where the coefficient of c_{s-5} vanishes. Furthermore we find that if s is odd, c_{s-4}

is the linear combination of all c_j with j odd. As $c_1 = 0$ when $s = 5$ in (2.22), we find that all c_s vanish when s is odd. This indicates that the subnormal solution (1.9) reduces to the normal solution (1.7). Thus we must change (2.22) to reflect the coefficients of (1.7). We first replace s, j and k by $2s'$, $2j'$ and $2k'$, and then replace $c_{2(s'-2)}$ and $c_{2j'}$ by $c_{(s-2)}$ and c_j, and finally change s', j' and k' back to s, j and k. We obtain the recurrence formula for the coefficients in (1.7),

$$c_{s-2} = \frac{-1}{q(\alpha - s + 2)} \sum_{j=0}^{s-3} \left\{ \rho^2 2^{s-j} \begin{pmatrix} \alpha - j \\ s - j \end{pmatrix} + \rho \sum_{k=0}^{s-j} a_k \begin{pmatrix} \alpha - j \\ s - j - k \end{pmatrix} + b_{s-j} \right\} c_j, \tag{2.23}$$
$$s = 3, 4, \ldots,$$

where $q(\alpha)$ is the indicial polynomial (1.10). Thus we obtain two linearly independent formal series solutions to the difference equation (1.4).

2.4 Exceptional Case (ii), (iii)

When $\alpha_2 - \alpha_1 = p$ is a non-negative integer, the recursive formula (2.23) for the coefficients breaks down when α is replaced by α_2 and $s = p + 2$. Hence the preceding argument for case (i) yields only one formal series solution, namely

$$Y_1(n) = \rho^n n^{\alpha_1} \sum_{s=0}^{\infty} c_s n^{-s}, \tag{2.24}$$

the coefficients c_s being determined by (2.23) with α replaced by α_1. *Birkhoff* in ([6] p.213) presented the second solution in the form

$$Y_2(n) = \rho^n n^r \left[S_2(n) + S_1(n) \log n \right],$$

where r is an unspecified constant, and

$$S_i(n) = \sum_{s=0}^{\infty} d_s^{(i)} n^{-s}, \qquad\qquad i = 1, 2.$$

Here we shall show that the second formal series solution has, in fact, the simpler form given by (1.12) and (1.13), i. e. ,

$$Y_2(n) = Z(n) + CY_1(n) \log n \tag{2.25}$$

and

$$Z(n) = \rho^n n^{\alpha_2} \sum_{s=0}^{\infty} d_s n^{-s}, \tag{2.26}$$

C being a constant, and $Y_1(n)$ the first formal solution. To show that $Y_2(n)$ formally satisfies equation (1.4), we make use of the expansion

$$\log(n+\mu) = \log n + \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{s} \left(\frac{\mu}{n}\right)^s, \qquad\qquad \mu = 1, 2. \tag{2.27}$$

Substituting (2.27) and (2.26) into (2.1) gives

$$\begin{aligned}
\mathcal{L}\{Y_2(n)\} &= \mathcal{L}\{Z(n)\} + C\mathcal{L}\{Y_1(n)\} \log n \\
&+ C\rho^n \sum_{s=1}^{\infty} \left\{ \sum_{j=0}^{s-1} \left[\sum_{l=j}^{s-1} \frac{(-1)^{s+1-l}}{s-l} \left\{ \rho^2 2^{s-j} \binom{\alpha_1 - j}{l-j} \right.\right.\right. \\
&\left.\left.\left. + \rho \sum_{k=j}^{l} \binom{\alpha_1 - j}{k-j} a_{l-k} \right\} \right] c_j \right\} n^{\alpha_1 - s},
\end{aligned} \tag{2.28}$$

where c_j are the coefficients of $Y_1(n)$ in (2.24). In view of the fact that $2\rho + a_0 = 0$, the coefficients of $n^{\alpha_1 - 1}$ and c_{s-1} are zero. Thus (2.28) becomes

$$(2.29) \quad \mathcal{L}\{Y_2(n)\} = \mathcal{L}\{Z(n)\} + C\mathcal{L}\{Y_1(n)\}\log n + C\rho^n \sum_{s=2}^{\infty} Y_c(s) n^{\alpha_1 - s},$$

where, for convenience, we use the notation

$$Y_c(s) =$$

$$(2.30) \quad \sum_{j=0}^{s-2}\left\{\sum_{l=j}^{s-1}\frac{(-1)^{s+1-l}}{s-l}\left[\rho^2 2^{s-j}\begin{pmatrix}\alpha_1 - j\\ l-j\end{pmatrix} + \rho\sum_{k=j}^{l}\begin{pmatrix}\alpha_1 - j\\ k-j\end{pmatrix}a_{l-k}\right]\right\}c_j.$$

By the argument used for case (i), we have

$$(2.31) \qquad \mathcal{L}\{Z(n)\} = \rho^n \sum_{s=2}^{\infty} Z_d(s) n^{\alpha_2 - s},$$

where we also use the notation

$$Z_d(s) = q(\alpha_2 - s + 2)d_{s-2} + \sum_{j=0}^{s-3}\left[\rho^2 2^{s-j}\begin{pmatrix}\alpha_2 - j\\ s-j\end{pmatrix}\right.$$

$$(2.32) \qquad \left. + \rho\sum_{l=0}^{s-j}\begin{pmatrix}\alpha_2 - j\\ s-l-j\end{pmatrix}a_l + b_{s-j}\right]d_j,$$

which can be obtained directly from (2.23) by replacing α_1 and c_j by α_2 and d_s. It is easy to see that $Z_d(2) = 0$. Thus (2.31) may be rewritten as

$$(2.33) \qquad \mathcal{L}\{Z(n)\} = \rho^n \sum_{s=3}^{\infty} Z_d(s) n^{\alpha_2 - s}.$$

First we consider case (ii) where $\alpha_2 - \alpha_1 = p$ is a positive integer. Inserting

(2.33) into (2.29), and rearranging the order produces

$$\mathcal{L}\{Y_2(n)\} = C\mathcal{L}\{Y_1(n)\}\log n + \rho^n \sum_{s=3}^{p+1} Z_d(s) n^{\alpha_2 - s}$$
$$+\rho^n \left[Z_d(p+2) + CY_c(2)\right] n^{\alpha_2 - p - 2}$$
$$+\rho^n \sum_{s=p+3}^{\infty} \left[Z_d(s) + CY_c(s-p)\right] n^{\alpha_2 - s}. \tag{2.34}$$

Note that the coefficients of n^{-s} in $\mathcal{L}\{Y_1(n)\}$ are equal to zero. Thus, upon equating coefficients of n^{α_2-s}, s from three to $p+1$, in (2.34) to zero, we obtain

$$d_{s-2} = \frac{-1}{q(\alpha_2 - s + 2)} \sum_{j=0}^{s-3} \left[\rho^2 2^{s-j} \binom{\alpha_2 - j}{s-j} + \rho \sum_{l=0}^{s-j} \binom{\alpha_2 - j}{s-l-j} a_l + b_{s-j}\right] d_j \qquad s = 3, 4, \ldots, p+1. \tag{2.35}$$

We find that the denominator is not zero. Now, upon equating the coefficients of $n^{\alpha_2 - p - 2}$ to zero, we have

$$Z_d(p+2) + CY_c(2) = 0. \tag{2.36}$$

We find the coefficient of d_p in (2.36) is $q(\alpha_1) = 0$. This implies d_p can be an arbitrary constant. For convenience we may choose

$$d_p = 1. \tag{2.37}$$

Thus, from (2.36) we obtain the constant

$$C = \frac{-1}{\rho(2\rho\alpha_1 + a_1 - \rho)}$$

$$(2.38) \cdot \sum_{j=0}^{p-1} \left[\rho^2 2^{p+2-j} \binom{\alpha_2 - j}{p+2-j} + \rho \sum_{l=0}^{p+2-j} \binom{\alpha_2 - j}{p+2-l-j} a_l + b_{p+2-j} \right] d_j.$$

The denominator is not zero since α_1 is not a double root of the indicial polynomial (1.10), or $\alpha_1 \neq (\rho - a_1)/2\rho$.

Finally equating the coefficients of $n^{\alpha_2 - s}$, s from $p+3$ to ∞, yields

$$\begin{aligned} d_{s-2} = & \frac{-1}{q(\alpha_2 - s + 2)} \\ & \cdot \left\{ \sum_{j=0}^{s-3} \left[\rho^2 2^{s-j} \binom{\alpha_2 - j}{s-j} + \rho \sum_{l=0}^{s-j} \binom{\alpha_2 - j}{s-l-j} a_l + b_{s-j} \right] d_j \right. \\ & + C \sum_{j=0}^{s-p-2} \sum_{l=j}^{s-p-1} \frac{(-1)^{s-p+1-l}}{s-p+1-l} \left[\rho^2 2^{s-p-j} \binom{\alpha_1 - j}{l-j} \right. \\ & \left. \left. + \rho \sum_{k=j}^{l} \binom{\alpha_1 - j}{k-j} a_{l-k} \right] c_j \right\}, \end{aligned}$$

$$(2.39) \qquad s = p+3, p+4, \ldots.$$

The denominator $q(\alpha_2 - s + 2)$ is not zero.

Next, we consider $p = 0$ or $\alpha_1 = \alpha_2 = \alpha$. (2.34) becomes

$$\mathcal{L}\{Y_2(n)\} = C\mathcal{L}\{Y_1(n)\} \log n + \rho^n \sum_{s=2}^{\infty} [Z_d(s) + CY_c(s)] n^{\alpha - s},$$

where α_1 and α_2 in $Y_c(s)$ and $Z_d(s)$ are replaced by α. Equating coefficients of

n^{-s} to zero, we obtain

$$Z_d(s) + CY_c(s) = 0, \qquad s = 2, 3, \ldots. \tag{2.40}$$

Since $q(\alpha) = 0$ and $(2\rho\alpha_1 + a_1 - \rho) = 0$, (2.40) for $s = 2$ is the trivial identity $0 = 0$. If

$$Y_c(3) = Y_c(4) = \ldots = Y_c(Q-1) \qquad \text{and} \qquad Y_c(Q) \neq 0, \tag{2.41}$$

for some integer $Q \geq 3$, in order to have (2.40) satisfied, $Z_d(s)$ also has to be zero for $s = 3, 4, \ldots, Q-1$. Thus, we obtain

$$\begin{aligned} d_{s-2} = & \frac{-1}{q(\alpha - s + 2)} \\ & \cdot \sum_{j=0}^{s-3} \left\{ \rho^2 2^{s-j} \binom{\alpha - j}{s - j} + \rho \sum_{l=0}^{s-j} \binom{\alpha - j}{s - l - j} a_l + b_{s-j} \right\} d_j, \\ & \qquad s = 3, 4, \ldots, Q-1. \end{aligned} \tag{2.42}$$

If we assume $d_0 = 0$, this implies

$$d_0 = d_1 = \ldots = d_{Q-3} = 0. \tag{2.43}$$

Consequently, from (2.30), (2.32) and (2.40) for $s = Q$, we obtain

$$\begin{aligned} C = -\frac{Z_d(Q)}{Y_c(Q)} = & -q(\alpha - Q + 2) \\ & \cdot \left\{ \sum_{j=0}^{Q-2} \left[\sum_{l=j}^{Q-1} \frac{(-1)^{Q+1-l}}{Q-l} \left\{ \rho^2 2^{Q-j} \binom{\alpha - j}{l - j} \right. \right. \right. \\ & \left. \left. \left. + \rho \sum_{k=j}^{l} \binom{\alpha - j}{k - j} a_{l-k} \right\} \right] c_j \right\}^{-1}, \end{aligned} \tag{2.44}$$

where we assume

$$d_{Q-2} = 1. \tag{2.45}$$

Then by making use of (2.30) and (2.32), d_s for $s > Q-2$ can be determined recursively from (2.40),

$$\begin{aligned} d_{s-2} = {} & \frac{-1}{q(\alpha - s + 2)} \\ & \cdot \left\{ \sum_{j=Q-2}^{s-3} \left[\rho^2 2^{s-j} \binom{\alpha - j}{s - j} + \rho \sum_{l=0}^{s-j} \binom{\alpha - j}{s - l - j} a_l + b_{s-j} \right] d_j \right. \\ & \left. + C \sum_{j=0}^{s-2} \sum_{l=j}^{s-1} \frac{(-1)^{s+1-l}}{s-l} \left[\rho^2 2^{s-j} \binom{\alpha - j}{l - j} + \rho \sum_{k=j}^{l} \binom{\alpha - j}{k - j} a_{l-k} \right] c_j \right\}, \\ & \qquad s = Q+1, Q+2, \ldots . \end{aligned} \tag{2.46}$$

Thus substitution (2.43), (2.44), (2.45) and (2.46) in (2.25) yields the second independent formal solution.

It should be pointed out that if we assume $d_0 = 1$ and $C = 1$, we obtain an alternate second independent formal solution, say,

$$\bar{Y}_2(n) = \bar{Z}(n) + Y_1(n) \log n, \tag{2.47}$$

where

$$\bar{Z}(n) = \rho^n n^\alpha \sum_{s=0}^{\infty} \bar{d}_s n^{-s}.$$

Looking at (2.23) and (2.42), from (2.40) we find

$$\bar{d}_s = c_s, \qquad s = 0, 1, \ldots, Q-3. \tag{2.48}$$

Then from (2.40) again, making use of (2.30) and (2.32), we obtain

$$\bar{d}_{s-2} = \frac{-1}{q(\alpha - s + 2)}$$

$$\cdot \left\{ \sum_{j=0}^{s-3} \left[\rho^2 2^{s-j} \binom{\alpha - j}{s - j} + \rho \sum_{l=0}^{s-j} \binom{\alpha - j}{s - l - j} a_l + b_{s-j} \right] \bar{d}_j \right.$$

$$\left. + \sum_{j=0}^{s-2} \sum_{l=j}^{s-1} \frac{(-1)^{s+1-l}}{s-l} \left[\rho^2 2^{s-j} \binom{\alpha - j}{l - j} + \rho \sum_{k=j}^{l} \binom{\alpha - j}{k - j} a_{l-k} \right] c_j \right\},$$

$$s = Q, Q+1, \ldots. \tag{2.49}$$

These two second formal solutions $Y_2(n)$ and $\bar{Y}_2(n)$ do not yield a contradiction, since $Y_1(n)$ is a linear combination of $Y_2(n)$ and $\bar{Y}_2(n)$. In fact, we can show that

$$Y_1(n) = \bar{Y}_2(n) - \frac{1}{C} Y_2(n). \tag{2.50}$$

From (2.25) and (2.47), we have

$$\bar{Y}_2(n) - \frac{1}{C} Y_2(n) = \bar{Z}(n) - \frac{1}{C} Z(n) = \rho^n n^\alpha \sum_{s=0}^{\infty} \left(\bar{d}_s - \frac{1}{C} d_s \right) n^{-s}. \tag{2.51}$$

We now investigate the coefficients of terms in n^{-s} in (2.51). In view of (2.43) and (2.48), we find

$$\bar{d}_s - \frac{1}{C} d_s = \bar{d}_s = c_s, \qquad s = 0, 1, \ldots, Q - 3.$$

By making use of (2.44) and the fact that $d_{Q-2} = 1$,

$$\bar{d}_{Q-2} - \frac{1}{C} d_{Q-2} = \frac{-1}{q(\alpha - s + 2)}$$

$$\cdot\left\{\sum_{j=0}^{Q-3}\left[\rho^2 2^{s-j}\begin{pmatrix}\alpha-j\\ s-j\end{pmatrix}+\rho\sum_{l=0}^{s-j}\begin{pmatrix}\alpha-j\\ s-l-j\end{pmatrix}a_l+b_{s-j}\right]c_j+Y_c(Q)\right\}$$

$$+\frac{Y_c(Q)}{q(\alpha-s+2)}=c_{Q-2}$$

Finally, in view of (2.48), (2.49) and (2.46), we obtain

$$\bar{d}_{s-2}-\frac{1}{C}d_{s-2}=\frac{-1}{q(\alpha-s+2)}$$

$$\cdot\left\{\sum_{j=0}^{Q-3}\left[\rho^2 2^{s-j}\begin{pmatrix}\alpha-j\\ s-j\end{pmatrix}+\rho\sum_{l=0}^{s-j}\begin{pmatrix}\alpha-j\\ s-l-j\end{pmatrix}a_l+b_{s-j}\right]c_j\right.$$

$$\left.+\sum_{j=Q-2}^{s-3}\left[\rho^2 2^{s-j}\begin{pmatrix}\alpha-j\\ s-j\end{pmatrix}+\rho\sum_{l=0}^{s-j}\begin{pmatrix}\alpha-j\\ s-l-j\end{pmatrix}a_l+b_{s-j}\right]\left[\bar{d}_j-\frac{1}{C}d_j\right]\right\},$$

for $s=Q+1,Q+2,\ldots$. Applying the method of induction, we immediately obtain

$$\bar{d}_{s-2}-\frac{1}{C}d_{s-2}=c_{s-2},\qquad s=Q+1,Q+2,\ldots. \tag{2.52}$$

Hence, the assertion (2.50) follows.

Chapter 3

Existence Theorem

3.1 Case 1 (i)

In this case, equation (1.6) has two roots which are distinct and different from zero. Then as we have shown in §2.1, equation (2.1) has two formal series solutions of the form

$$Y_i(n) = \rho_i^n n^{\alpha_i} \sum_{s=0}^{\infty} c_s^{(i)} n^{-s}, \qquad i = 1, 2,$$

where the exponent α_i is determined by (2.4) with ρ replaced by ρ_i and the coefficients $c_s^{(i)}$, $i = 1, 2$, are determined recursively by (2.5). In this and the following sections, it will be proven that these formal solutions are indeed asymptotic.

Throughout this Chapter, we shall assume that $|\rho_2| \geq |\rho_1|$. First, we consider the solution of $y_1(n)$. In the sense of (1.3), we set

$$y_1(n) = L_N^{(1)}(n) + \varepsilon_N^{(1)}(n), \tag{3.1}$$

where

$$L_N^{(1)}(n) = \rho_1^n n^{\alpha_1} \sum_{s=0}^{N-1} c_s^{(1)} n^{-s},$$

and $\varepsilon_N^{(1)}(n)$ is the corresponding error term. In view of (1.6), (2.4) and (2.5), it is readily found that the coefficients of $\rho_1^n n^{\alpha_1 - s}$ for $s = 0, 1, \ldots, N$ in $\mathcal{L}\left\{L_N^{(1)}(n)\right\}$ are zero. Accordingly,

$$\mathcal{L}\left\{L_N^{(1)}(n)\right\} = \rho_1^n n^{\alpha_1} R_N^{(1)}(n), \tag{3.2}$$

where $R_N^{(1)}(n) = O(n^{-N-1})$ as $n \to \infty$. Therefore, from

$$0 = \mathcal{L}\left\{y_1(n)\right\} = \mathcal{L}\left\{L_N^{(1)}(n)\right\} + \mathcal{L}\left\{\varepsilon_N^{(1)}(n)\right\},$$

we have the equation for $\varepsilon_N^{(1)}(n)$,

$$\varepsilon_N^{(1)}(n+2) + a(n)\varepsilon_N^{(1)}(n+1) + b(n)\varepsilon_N^{(1)}(n) = -\rho_1^n n^{\alpha_1} R_N^{(1)}(n). \tag{3.3}$$

To solve the last equation, we retain the leading terms in the expansions of $a(n)$ and $b(n)$, and transfer the rest to the right-hand side. Thus, we have

$$\begin{aligned} &\varepsilon_N^{(1)}(n+2) + a_0\varepsilon_N^{(1)}(n+1) + b_0\varepsilon_N^{(1)}(n) \\ &\quad = -\rho_1^n n^{\alpha_1} R_N^{(1)}(n) - \left[a(n) - a_0\right]\varepsilon_N^{(1)}(n+1) - \left[b(n) - b_0\right]\varepsilon_N^{(1)}(n). \end{aligned} \tag{3.4}$$

Using the method of variation of parameters ([4] p.49), we obtain the equation

$$\begin{aligned} \varepsilon_N^{(1)}(n) = &\sum_{k=n}^{\infty} K(n,k) \\ &\cdot \left\{\rho_1^k k^{\alpha_1} R_N^{(1)}(k) + \left[a(k) - a_0\right]\varepsilon_N^{(1)}(k+1) + \left[b(k) - b_0\right]\varepsilon_N^{(1)}(k)\right\}, \end{aligned} \tag{3.5}$$

where

$$K(n,k) = \frac{\rho_2^{n-k-1} - \rho_1^{n-k-1}}{\rho_2 - \rho_1}.$$

It is easily verified that every solution of (3.5) is a solution of (3.4).

The equation (3.5) will be solved by the method of successive approximation. We express

$$\varepsilon_N^{(1)}(n) = \sum_{s=0}^{\infty} \{h_{s+1}(n) - h_s(n)\}, \tag{3.6}$$

where the sequence $\{h_s(n)\}$ is defined by $h_0(n) = 0$ and

$$\begin{aligned} h_{s+1}(n) &= \sum_{k=n}^{\infty} K(n,k) \\ &\cdot \left\{\rho_1^k k^{\alpha_1} R_N^{(1)}(k) + [b(k) - b_0]\, h_s(k) + [a(k) - a_0]\, h_s(k+1)\right\}. \end{aligned} \tag{3.7}$$

We shall show that the series (3.6) is absolutely and uniformly convergent in n, and that its sum $\varepsilon_N^{(1)}(n)$ is a solution of (3.5).

Let $n_0(N)$ be sufficiently large so that

$$|R_N^{(1)}(n)| \le B_N n^{-N-1}$$

for some positive constant B_N and for all $n \ge n_0(N)$. Since $|\rho_2| \ge |\rho_1|$, a simple estimation gives

$$|K(n,k)| \le \frac{2}{|\rho_2 - \rho_1|} |\rho_1|^{n-k-1}. \tag{3.8}$$

From (3.7) it follows that

$$|h_1(n)| \le \frac{2B_N}{|\rho_2 - \rho_1|} |\rho_1|^{n-1} \sum_{k=n}^{\infty} k^{\sigma-N-1} \tag{3.9}$$

for $n \geq n_0(N)$, where $\sigma = \mathrm{Re}\ \alpha_1$. If $p > 1$, we have

$$\sum_{k=n}^{\infty} \frac{1}{k^p} \leq \int_{n-1}^{\infty} \frac{1}{x^p} dx = \frac{1}{p-1}(n-1)^{-p+1}. \tag{3.10}$$

For sufficiently large $n \geq n_0(p)$, it follows that

$$\sum_{k=n}^{\infty} \frac{1}{k^p} \leq \frac{2}{p-1} n^{-p+1}. \tag{3.11}$$

Coupling (3.9) and (3.11), we obtain

$$|h_1(n)| \leq \frac{2^2 B_N}{|\rho_2 - \rho_1|(N-\sigma)} |\rho_1|^{n-1} n^{\sigma - N}, \tag{3.12}$$

if $N > \sigma$, and without loss of generality, we may assume that this holds for all $n \geq n_0(N)$.

The same argument gives

$$|h_1(n+1)| \leq \frac{2^2 B_N}{|\rho_2 - \rho_1|(N-\sigma)} |\rho_1|^{n} n^{\sigma - N}. \tag{3.13}$$

By induction, we shall prove that

$$|h_{s+1}(n) - h_s(n)| \leq \frac{2^{2(s+1)} B_N \beta^s}{|\rho_2 - \rho_1|^{s+1}(N-\sigma)^{s+1}} |\rho_1|^{n-s-1} n^{-(N-\sigma)}, \tag{3.14}$$

and

$$\begin{aligned} &|h_{s+1}(n+1) - h_s(n+1)| \\ &\qquad \leq \frac{2^{2(s+1)} B_N \beta^s}{|\rho_2 - \rho_1|^{s+1}(N-\sigma)^{s+1}} |\rho_1|^{n-s} n^{-(N-\sigma)}, \end{aligned} \tag{3.15}$$

for $s = 0, 1, \ldots$, and $n \geq n_0(N)$, where

$$\beta = \sup \{k [|b(k) - b_0| + |a(k) - a_0||\rho_1|] : \quad k \geq 1\}.$$

From (3.12) and (3.13), we find that when $s = 0$ the last two statements are true. We now assume that when $s = t$, (3.14) and (3.15) are true, then

$$\begin{aligned} h_{t+2}(n) - h_{t+1}(n) &= \sum_{k=n}^{\infty} K(n,k) \\ &\cdot \{[b(k) - b_0][h_{t+1}(n) - h_t(n)] + [a(k) - a_0][h_{t+1}(n+1) - h_t(n+1)]\}. \end{aligned}$$

Applying (3.8), (3.7), (3.14) and (3.15) to the last equation gives

$$|h_{t+2}(n) - h_{t+1}(n)| \leq \frac{2^{2t+3} B_N \beta^{t+1}}{|\rho_2 - \rho_1|^{t+2} (N - \sigma)^{t+1}} |\rho_1|^{n-t-2} \sum_{k=n}^{\infty} k^{-(N-\sigma+1)}.$$

From (3.11), it then follows that (3.14) is also true when $s = t + 1$. By the same argument, (3.15) can be proven as well.

From (3.14), it is now evident that the series in (3.6) is uniformly convergent in n, if we choose N sufficiently large so that

$$(3.16) \qquad |\rho_1(\rho_2 - \rho_1)(N - \sigma)| > 4\beta.$$

Summation of (3.6) gives

$$(3.17) \qquad \varepsilon_N^{(1)}(n) = \rho_1^n n^{\alpha_1} O(n^{-N}).$$

(We first fix N from (3.16) and then choose $n_0(N)$ in (3.12).) Since (3.6) can be also written as

$$\varepsilon_N^{(1)}(n) = \lim_{s\to\infty} h_s(n),$$

by taking $s \to \infty$ in (3.7), we have established that $\varepsilon_N^{(1)}(n)$ is a solution to (3.5), and therefore to (3.3) satisfying (3.17).

3.2 Case 1 (ii)

We next show that the formal series $y_2(n)$ in (1.7) is also asymptotic. We set

$$y_2(n) = L_N^{(2)}(n) + \varepsilon_N^{(2)}(n),$$

where

$$L_N^{(2)}(n) = \rho_2^n n^{\alpha_2} \sum_{s=0}^{N-1} c_s^{(2)} n^{-s},$$

and $\varepsilon_N^{(2)}(n)$ is the corresponding error term. $\varepsilon_N^{(2)}(n)$ will be shown in this section to be

$$\varepsilon_N^{(2)} = \rho_2^n n^{\alpha_2} O(n^{-N}), \qquad \text{as } n \to \infty. \tag{3.18}$$

If $|\rho_2| = |\rho_1|$ then the analysis in §2.1 can be repeated with only the roles played by ρ_1 and ρ_2 being interchanged. However, if $|\rho_2| > |\rho_1|$ then this argument fails at (3.8), and an alternative method must be sought. A natural attempt is to use the method of varition of parameters (also known as the method of reduction of order ([4], p.43)), which is to set $y_2(n) = v(n)y_1(n)$ and show that the difference $w(n) \equiv v(n+1) - v(n)$ satisfies the first-order equation

$$y_1(n+2)w(n+1) - b(n)y_1(n)w(n) = 0.$$

From this, one readily obtains (see [4], p. 38–39)

$$v(n) = v(0) + [v(1) - v(0)] \sum_{k=0}^{n-1} \prod_{l=0}^{k-1} \frac{b(l) y_1(l)}{y_1(l+2)},$$

but the behavior of $y_2(n)$ is difficult to derive from the above equation. Therefore, we make the following alteration. If we set

$$y_2(n) = [K_N(n) + \delta_N(n)]\, y_1(n), \tag{3.19}$$

where $y_1(n)$ is the first solution found in §2.1,

$$K_N(n) = \left(\frac{\rho_2}{\rho_1}\right)^n n^{\alpha_2 - \alpha_1} \sum_{s=0}^{N-1} d_s n^{-s}$$

and the coefficients d_s, $s = 0, 1, \ldots, N-1$, are determined by

$$c_s^{(2)} = \sum_{t=0}^{s} d_t c_{s-t}^{(1)}.$$

Note that this can always be done. It is easily shown that

$$K_N(n) y_1(n) = L_N^{(2)}(n) + \rho_2^n n^{\alpha_2} O(n^{-N-1}).$$

In view of (3.18), this implies that it is equivalent to prove

$$\delta_N(n) = \left(\frac{\rho_2}{\rho_1}\right)^n n^{\alpha_2 - \alpha_1} O(n^{-N}). \tag{3.20}$$

From the fact that ρ_2 satisfies the characteristic equation (1.6), it readily follows that

$$\mathcal{L}\left\{K_N(n)y_1(n)\right\} = \mathcal{L}\left\{L_N^{(2)}(n)\right\} + \rho_2^n n^{\alpha_2} O(n^{-N-1}). \tag{3.21}$$

By making use of (3.21) and $\mathcal{L}\left\{L_N^{(2)}(n)\right\} = \rho_2^n n^{\alpha_2} O(n^{-N-1})$, cf. (3.2), and substituting (3.19) into (1.4), we have

$$\begin{aligned} y_1(n+2)\delta_N(n+2) + a(n)y_1(n+1)\delta_N(n+1) + b(n)y_1(n)\delta_N(n) \\ = \rho_2^n n^{\alpha_2} O(n^{-N-1}). \end{aligned} \tag{3.22}$$

Furthermore, since $y_1(n)$ is a solution of (2.1), i. e.

$$a(n)y_1(n+1) = -\left[y_1(n+2) + b(n)y_1(n)\right],$$

it can be verified from (3.22) that the difference

$$\Delta_N(n) \equiv \delta_N(n+1) - \delta_N(n) \tag{3.23}$$

satisfies the first order equation

$$y_1(n+2)\Delta_N(n+1) - b(n)y_1(n)\Delta_N(n) = \rho_2^n n^{\alpha_2} O(n^{-N-1}). \tag{3.24}$$

From (1.5), (1.7) and in view of (2.2) we have

$$y_1(n+2) = \rho_1^{n+2} n^{\alpha_1}[1 + \sigma_1(n)]$$

and

$$b(n)y_1(n) = \rho_1^n n^{\alpha_1}[b_0 + \sigma_2(n)],$$

where

$$\sigma_1(n) = O(n^{-1}), \qquad \text{and} \qquad \sigma_2(n) = O(n^{-1}).$$

Now define $\xi_N(n)$ by

$$\Delta_N(n) = \left(\frac{\rho_2}{\rho_1}\right)^n \xi_N(n). \tag{3.25}$$

In terms of $\xi_N(n)$, equation (3.24) becomes

$$\rho_1\rho_2\left[1+\sigma_1(n)\right]\xi_N(n+1) - \left[b_0+\sigma_2(n)\right]\xi_N(n) = n^{\alpha_2-\alpha_1}O(n^{-N-1}).$$

Since $\rho_1\rho_2 = b_0$, the last equation can be rewritten as

$$\xi_N(n+1) - \xi_N(n) = n^{\alpha_2-\alpha_1}E_N(n) - \sigma_1(n)\xi_N(n+1) + \sigma_2^*(n)\xi_N(n), \tag{3.26}$$

where $\sigma_2^*(n) = \sigma_2(n)/b_0$ and

$$|E_N(n)| \le K_N n^{-N-1},$$

K_N being some constant independent of n. Treating (3.26) as a first-order linear equation with the right-hand side being the nonhomogeneous term, one can formally derive the equation

$$\xi_N(n) = -\sum_{k=n}^{\infty}\left[k^{\alpha_2-\alpha_1}E_N(k) - \sigma_1(k)\xi_N(k+1) + \sigma_2^*(k)\xi_N(k)\right]. \tag{3.27}$$

Obviously, every solution of (3.27) is a solution of (3.26). We shall show that equation (3.27) has a solution satisfying

$$\xi_N(n) = n^{\alpha_2-\alpha_1}O(n^{-N}), \qquad n\to\infty, \tag{3.28}$$

which can be obtained by method of successive approximation.

Define $g_0(n) = 0$ and

$$(3.29) \qquad g_{s+1}(n) = \sum_{k=n}^{\infty} \left[-k^{\alpha_2-\alpha_1} E_N(k) + \sigma_1(k) g_s(k+1) - \sigma_2^*(k) g_s(k) \right]$$

for $s = 0, 1, 2, \cdots$. Put $m = \text{Re}\,(\alpha_2 - \alpha_1)$ and choose $N > m$. We have

$$|g_1(n)| \le K_N \sum_{k=n}^{\infty} k^{m-N-1} \le \frac{K_N}{N-m}(n-1)^{m-N}.$$

Let $n_0(N)$ be sufficiently large so that $(n-1)^{m-N} \le 2n^{m-N}$ for all $n \ge n_0(N)$. Consequently,

$$|g_1(n)| \le \frac{2K_N}{N-m} n^{m-N}, \qquad n \ge n_0(N).$$

By the same argument, we have

$$|g_1(n+1)| \le K_N \sum_{k=n+1}^{\infty} k^{m-N+1} \le K_N \sum_{k=n}^{\infty} k^{m-N+1} \le \frac{2K_N}{N-m} n^{m-N}.$$

In terms of the method of induction, it can be established that

$$\begin{aligned} |g_{s+1}(n) - g_s(n)|, \quad & |g_{s+1}(n+1) - g_s(n+1)| \\ & \le \frac{K_N}{\beta} \left(\frac{2\beta}{N-m} \right)^{s+1} n^{m-N} \end{aligned}$$

for $n \ge n_0(N), s = 0, 1, 2, \cdots$, where

$$\beta = \sup \left\{ k \left[|\sigma_1(k)| + |\sigma_2(k)| \right] : \quad k \ge 1 \right\}.$$

Let N be larger than $m + 2\beta$ so that the series

$$\lim_{s\to\infty} g_s(n) = \sum_{s=0}^{\infty} [g_{s+1}(n) - g_s(n)] \tag{3.30}$$

is absolutely and uniformly convergent in n. (We first fix N and then choose the integer $n_0(N)$). Since $\xi_N(n)$ can also be written in the form

$$\xi_N(n) = \lim_{s\to\infty} g_s(n), \tag{3.31}$$

by taking the limit as $s \to \infty$ in (3.29), we conclude that $\xi_N(n)$ is a solution to (3.27). Substituting (3.30) into (3.31), it is evident that $\xi_N(n)$ satisfies (3.28).

By definition (3.23), we can rewrite (3.25) as

$$\delta_N(n+1) - \delta_N(n) = \left(\frac{\rho_2}{\rho_1}\right)^n \xi_N(n).$$

Solving this first-order equation, we obtain

$$\delta_N(n) = \delta_N(1) + \sum_{k=1}^{n-1} \left(\frac{\rho_2}{\rho_1}\right)^k \xi_N(k). \tag{3.32}$$

From (3.28), if we set

$$|\xi_N(k)| \le M k^{m-N}, \qquad k \ge 1,$$

for some $M > 0$, we have

$$\left|\sum_{k=1}^{n-1} \left(\frac{\rho_2}{\rho_1}\right)^k \xi_N(k)\right| \le M \left|\frac{\rho_2}{\rho_1}\right|^n n^{m-N} \sum_{k=1}^{n-1} \left|\frac{\rho_2}{\rho_1}\right|^{k-n} \left(\frac{k}{n}\right)^{m-N}$$

$$= M \left|\frac{\rho_2}{\rho_1}\right|^n n^{m-N} \sum_{t=1}^{n-1} \left|\frac{\rho_1}{\rho_2}\right|^t \left(\frac{n}{n-t}\right)^{m-N}.$$

Since

$$\sum_{j=1}^{n-1} \rho^j \left(\frac{n}{n-j}\right)^p = O(1),$$

for any positive integer p and any real number $\rho \in (0,1)$, we have

$$\sum_{k=1}^{n-1} \left(\frac{\rho_2}{\rho_1}\right)^k \xi_N(k) = \left(\frac{\rho_2}{\rho_1}\right)^n n^{\alpha_2-\alpha_1-N} O(1). \tag{3.33}$$

Thus the required result (3.20) now follows from (3.32) and (3.33).

3.3 Case 2 (i)

In §2.3, we established two formal solutions of equation (1.4) where the roots of the characteristic equation (1.6) are equal, but different from the zero of the auxiliary equation (1.8). Both solutions are of the form

$$y(n) = \rho^n e^{\gamma\sqrt{n}} \sum_{s=0}^{\infty} c_s n^{\alpha - \frac{s}{2}}, \tag{3.34}$$

where γ an α are determined by (2.17) and (2.18), and the coefficients c_s are determined recursively by (2.19). It is easy to see that we may assume without loss of generality that $\rho = 1$ or, equivalently, $a_0 = -2$ and $b_0 = 1$.

To show that these formal series solutions are asymptotic, as in (3.1), we now set

$$y(n) = L_N(n) + E_N(n) \tag{3.35}$$

with

$$L_N(n) = e^{\gamma\sqrt{n}} n^\alpha \sum_{s=0}^{N-1} c_s n^{-\frac{s}{2}}.$$

Since the coefficients of c_s, c_{s-1}, and c_{s-2} in (2.12) are all zero, in view of (2.19), it is easily verified that

$$L_N(n+2) + a(n)L_N(n+1) + b(n)L_N(n) = e^{\gamma\sqrt{n}} n^\alpha R_N(n) \tag{3.36}$$

where

$$R_N(n) = O\left(n^{-\frac{1}{2}(N+3)}\right). \tag{3.37}$$

Substituting (3.35) in (1.4), we obtain from (3.36)

$$E_N(n+2) + a(n)E_N(n+1) + b(n)E_N(n) = -e^{\gamma\sqrt{n}} n^\alpha R_N(n).$$

Write

$$E_N(n) = e^{\gamma\sqrt{n}} n^\alpha \varepsilon_N(n).$$

Then $\varepsilon_N(n)$ satisfies the equation

$$\varepsilon_N(n+2) + a^\star(n)\varepsilon_N(n+1) + b^\star(n)\varepsilon_N(n) = R_N^\star(n), \tag{3.38}$$

where

$$a^\star(n) = e^{\gamma\sqrt{n+1}-\gamma\sqrt{n+2}} \left(\frac{n+1}{n+2}\right)^\alpha a(n),$$

$$b^\star(n) = e^{\gamma\sqrt{n}-\gamma\sqrt{n+2}} \left(\frac{n}{n+2}\right)^\alpha b(n), \tag{3.39}$$

and

$$R_N^*(n) = -e^{\gamma\sqrt{n}-\gamma\sqrt{n+2}}\left(\frac{n}{n+2}\right)^{\alpha} R_N(n). \tag{3.40}$$

Recalling that we have assumed $\rho = 1$, or equivalently $a_0 = -2$ and $b_0 = 1$, a simple calculation shows that

$$\begin{aligned} a^*(n) = &-2 + \gamma n^{-\frac{1}{2}} + \left(a_1 - \frac{1}{4}\gamma^2 + 2\alpha\right) n^{-1} \\ &+ \left(-\frac{1}{2}a_1 - \frac{3}{4} + \frac{1}{24}\gamma^2 - \alpha\right)\gamma n^{-\frac{3}{2}} + R_a(n), \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} b^*(n) = &1 - \gamma n^{-1/2} + \left(b_1 + \frac{1}{2}\gamma^2 - 2\alpha\right) n^{-1} \\ &+ \left(\frac{1}{2} - \frac{1}{6}\gamma^2 + 2\alpha - b_1\right)\gamma n^{-3/2} + R_b(n), \end{aligned} \tag{3.42}$$

where

$$R_a(n), R_b(n) = O(n^{-2}), \qquad \text{as } n \to \infty, \tag{3.43}$$

Furthermore,

$$R_N^*(n) = O\left(n^{-\frac{1}{2}(N+3)}\right), \qquad \text{as } n \to \infty.$$

Since $a_0 = -2$ and $b_0 = 1$, we have from (2.17) and (2.18)

$$\gamma^2 = -4(a_1 + b_1), \qquad \alpha = \frac{1}{4} + \frac{1}{2}b_1.$$

Inserting (3.41) and (3.42) in (3.38) then gives

$$
\begin{aligned}
\varepsilon_N(n+2) + \Big[-2 + \gamma n^{-\frac{1}{2}} &+ \left(\frac{1}{2} + 2a_1 + 2b_1\right) n^{-1} \\
&- \left(1 + \frac{2}{3}a_1 + \frac{2}{3}b_1\right) \gamma n^{-\frac{3}{2}}\Big] \varepsilon_N(n+1) \\
+ \Big[1 - \gamma n^{-\frac{1}{2}} - \left(\frac{1}{2} + 2a_1 + 2b_1\right) n^{-1} &+ \left(1 + \frac{2}{3}a_1 + \frac{2}{3}b_1\right) \gamma n^{-\frac{3}{2}}\Big] \varepsilon_N(n) \\
&= R_N^\star(n) - R_a(n)\varepsilon_N(n+1) - R_b(n)\varepsilon_N(n), \qquad (3.44)
\end{aligned}
$$

which can also be written as

$$
\begin{aligned}
\Delta\varepsilon_N(n+1) - \Big[1 - \gamma n^{-\frac{1}{2}} &- \left(\frac{1}{2} + 2a_1 + 2b_1\right) n^{-1} \\
&+ \left(1 + \frac{2}{3}a_1 + \frac{2}{3}b_1\right) \gamma n^{\frac{3}{2}}\Big] \Delta\varepsilon_N(n) \\
&= R_N^\star(n) - R_a(n)\varepsilon_N(n+1) - R_b(n)\varepsilon_N(n), \qquad (3.45)
\end{aligned}
$$

with $\Delta\varepsilon_N(n) = \varepsilon_N(n+1) - \varepsilon_N(n)$. For convenience, we introduce the notation

$$
(3.46)\quad \theta(n) = 1 - \gamma n^{-\frac{1}{2}} - \left(\frac{1}{2} + 2a_1 + 2b_1\right) n^{-1} + \left(1 + \frac{2}{3}a_1 + \frac{2}{3}b_1\right) \gamma n^{-\frac{3}{2}},
$$

and

$$
(3.47)\quad q\left(\varepsilon_N(n+1), \varepsilon_N(n), n\right) = R_N^\star(n) - R_a(n)\varepsilon_N(n+1) - R_b(n)\varepsilon_N(n).
$$

Equation (3.45) then becomes

$$
(3.48)\qquad \Delta\varepsilon_N(n+1) - \theta(n)\Delta\varepsilon_N(n) = q\left(\varepsilon_N(n+1), \varepsilon_N(n), n\right).
$$

To show that the formal series solution (3.34) is asymptotic, it sufficies to prove that equation (3.44), or equivalently equation (3.48), has a solution $\varepsilon_N(n)$ satisfying

$$\varepsilon_N(n) = O\left(n^{-\frac{N}{2}}\right), \qquad \text{as } n \to \infty. \tag{3.49}$$

Considering (3.48) as a first-order linear nonhomogeneous equation, one can formally derive the equations

$$\varepsilon_N(n) = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} \theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j)q\left(\varepsilon_N(j+1), \varepsilon_N(j), j\right), \tag{3.50}$$

where $\theta^{-1}(j) \equiv 1/\theta(j)$, and

$$\varepsilon_N(n) = -\sum_{i=n}^{\infty}\sum_{j=1}^{i-1} \theta(i-1)\theta(i-2)\cdots\theta(j+1)q\left(\varepsilon_N(j+1), \varepsilon_N(j), j\right), \tag{3.51}$$

where it is understood that $\theta(i-1)\theta(i-2)\cdots\theta(j+1) = 1$ when $j = i-1$. It is easily verified that every solution of (3.50) and every solution of (3.51) is a solution of (3.48). Now recall that the constant γ in (3.34) has two possible values given in (2.17). We shall show that when $\operatorname{Re}\gamma \le 0$ then equation (3.50) has a solution satisfying (3.49), and that when $\operatorname{Re}\gamma > 0$ then equation (3.51) has such a solution. This will complete our investigation of the subnormal solutions.

In this section, we are concerned only with the case $\operatorname{Re}\gamma \le 0$. Before proceeding, we first record some preliminary results.

LEMMA 1 *For positive integers $j \ge i \ge 1$, the function $\theta(n)$ in (3.46) satisfies*

$$\theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j) = e^{2\gamma\left(\sqrt{j}-\sqrt{i}\right)}\sqrt{\frac{j}{i}}\left[1 + O\left(i^{-1/2}\right)\right], \tag{3.52}$$

as $i \to \infty$, where the O-term is uniform with respect to j.

Proof: First we recall the well-known asymptotic formulas ([11] p.292)

$$(3.53)\qquad \sum_{k=1}^{n-1} k^z - \zeta(-z) = \frac{n^{z+1}}{z+1} + O(n^z), \qquad n \to \infty,$$

and

$$(3.54)\qquad \sum_{k=1}^{n-1} \frac{1}{k} = \log n + c + O\left(n^{-1}\right), \qquad n \to \infty,$$

where $\zeta(z)$ is the *Riemann* Zeta function and c denotes the *Euler* constant. Since

$$\log(1-x) = -x - \frac{1}{2}x^2 + O(x^3), \qquad x \to 0,$$

and the fact that $\gamma^2 = -4\,(a^2+b^2)$, we have from (3.46)

$$\log\theta(k) = -\gamma k^{-\frac{1}{2}} - \frac{1}{2}k^{-1} + O\left(k^{-\frac{3}{2}}\right), \qquad k \to \infty.$$

Upon summation, we obtain

$$(3.55)\quad -\sum_{k=i}^{j} \log\theta(k) = 2\gamma\left(\sqrt{j}-\sqrt{i}\right) + \frac{1}{2}\log\left(\frac{j}{i}\right) + O\left(i^{-\frac{1}{2}}\right), \quad i \to \infty,$$

uniformly for $j \geq i \geq 1$. The result (3.52) now follows from (3.55) by exponentiation. □

LEMMA 2 *For $Re\,\gamma \leq 0$, $\gamma \neq 0$, and $N \geq 1$, we have*

$$(3.56)\qquad \sum_{j=i}^{\infty} e^{2\gamma\left(\sqrt{j}-\sqrt{i}\right)} j^{-\frac{N}{2}-1} = -\frac{1}{\gamma} i^{-\frac{N}{2}-\frac{1}{2}} + O\left(i^{-\frac{N}{2}-1}\right), \qquad i \to \infty.$$

Proof: Recall the *Euler-Maclaurin* formula ([11] p.281)

$$f(0)+\cdots+f(n)=\int_0^n f(x)dx+\frac{1}{2}[f(0)+f(n)]+\int_0^n \tilde{\omega}_1(x)f'(x)dx,$$

where the saw-tooth function $\tilde{\omega}_1(x)=x-[x]-\frac{1}{2}$, $[x]$ is the greatest integer less than or equal to x. Applying this formula to the function

$$f(x)=e^{2\gamma\left(\sqrt{x+i}-\sqrt{i}\right)}(x+i)^{-\frac{N}{2}-1},$$

and letting $n\to\infty$, in view of Re $\gamma\le 0$, we obtain

$$\begin{aligned}\sum_{j=i}^{\infty} e^{2\gamma\left(\sqrt{j}-\sqrt{i}\right)}j^{-\frac{N}{2}-1} &= \int_i^\infty e^{2\gamma\left(\sqrt{x}-\sqrt{i}\right)}x^{-\frac{N}{2}-1}dx+\frac{1}{2}i^{-\frac{N}{2}-1}\\ &+\int_i^\infty \tilde{\omega}_1(x-i)\frac{d}{dx}\left[e^{2\gamma\left(\sqrt{x}-\sqrt{i}\right)}x^{-\frac{N}{2}-1}\right]dx. \qquad (3.57)\end{aligned}$$

Making use of the fact that $|\tilde{\omega}_1(x-i)|<1$, the last term of (3.57) has the estimate

$$\begin{aligned}&\left|\int_i^\infty \tilde{\omega}_1(x-i)\frac{d}{dx}\left[e^{2\gamma\left(\sqrt{x}-\sqrt{i}\right)}x^{-\frac{N}{2}-1}\right]dx\right|\\ &\le e^{-2\sigma\sqrt{i}}\int_i^\infty\left|\frac{d}{dx}\left[e^{2\gamma\sqrt{x}}x^{-\frac{N}{2}-1}\right]\right|dx\le\left(1-\frac{2|\gamma|}{\sigma}\right)i^{-\frac{N}{2}-1},\end{aligned}$$

where $\sigma=$Re γ. Integration by parts twice gives

$$\int_i^\infty e^{2\gamma\left(\sqrt{x}-\sqrt{i}\right)}x^{-\frac{N}{2}-1}dx=-\frac{1}{\gamma}i^{-\frac{N}{2}-\frac{1}{2}}+O\left(i^{-\frac{N}{2}-1}\right).$$

Coupling the last two results, we obtain the approximation (3.56). □

We now return to equation (3.50), and define the successive approximants $h_0(n) \equiv 0$ and

$$(3.58) \quad h_{s+1}(n) = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} \theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j)q\left(h_s(j+1), h_s(j), j\right),$$

for $s = 0, 1, \cdots$. In particular, we have

$$(3.59) \qquad h_1(n) = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} \theta^{-1}(i)\cdots\theta^{-1}(j)R_N^*(j),$$

where $R_N^*(j)$ is given in (3.40). From (3.37), it is readily seen that $R_N(n)$ has an asymptotic approximation

$$R_N(n) = cn^{-\frac{N}{2}-\frac{3}{2}} + O\left(n^{-\frac{N}{2}-2}\right),$$

where c is some constant whose exact value is immaterial for our purpose. Hence (3.40) gives

$$(3.60) \qquad R_N^*(n) = -cn^{-\frac{N}{2}-\frac{3}{2}} + O\left(n^{-\frac{N}{2}-2}\right).$$

Inserting (3.60) in (3.59), and applying *Lemma 1*, we obtain

$$|h_1(n)| \le |c| \sum_{i=n}^{\infty} \left| \sum_{j=i}^{\infty} e^{2\gamma\left(\sqrt{j}-\sqrt{i}\right)} j^{-\frac{N}{2}-1} i^{-\frac{1}{2}} \right| + \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} \left| O\left(j^{-\frac{N}{2}-\frac{3}{2}}\right) i^{-\frac{1}{2}} \right|.$$

Applying *Lemma 2*, and making use of (3.10) and (3.11), we obtain

$$h_1(n) = O\left(n^{-\frac{N}{2}}\right).$$

Let M be a positive number such that

$$|h_1(n)|, \quad |h_1(n+1)| \leq Mn^{-\frac{N}{2}}.$$

In view of (3.43) and *Lemma 1*, it is possible to choose $\beta > 0$ so that

$$|R_a(j)| + |R_b(j)| \leq \beta j^{-2}$$

and

$$\left|\theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j)\right| \leq \beta\sqrt{\frac{j}{i}},$$

for $j \geq i \geq 1$. A combination of (3.58) and (3.47) yields, by induction,

$$\begin{aligned} |h_{s+1}(n+1) - h_s(n+1)|, &\quad |h_{s+1}(n) - h_s(n)| \\ &\leq M\left[\frac{16\beta^2}{N(N+1)}\right]^s n^{-\frac{N}{2}}, \end{aligned}$$

if $n \geq n_0(N)$ for $s = 0, 1, 2, \cdots$. Now we choose N so that

$$\frac{16\beta^2}{N(N+1)} < 1.$$

The series

$$\varepsilon_N(n) = \lim_{s\to\infty} h_{s+1}(n) = \sum_{s=0}^{\infty}[h_{s+1}(n) - h_s(n)]$$

is then uniformly convergent in n, and its sum $\varepsilon_N(n)$ satisfies (3.50) and (3.49).

3.4 Case 2 (ii)

When the constant γ in (3.34) has a positive real component, then the series in (3.56) is divergent and hence equation (3.50) cannot be used to establish the existence of a solution of (3.48) satisfying (3.49). In this case, we shall use, instead, equation (3.51). Define the sequence $\{h_s(n)\}$ of successive approximants by $h_0(n) = 0$ and

$$h_{s+1}(n) =$$

$$-\sum_{i=n}^{\infty}\sum_{j=1}^{i-1}\theta(i-1)\theta(i-2)\cdots\theta(j+1)q\left(h_s(j+1), h_s(j), j\right), \tag{3.61}$$

where it is understood that $\theta(i-1)\theta(i-2)\cdots\theta(j+1) = 1$ when $j = i-1$. From (3.52), we have

$$\theta(i-1)\theta(i-2)\cdots\theta(j+1) = e^{2\gamma(\sqrt{j+1}-\sqrt{i-1})}\sqrt{\frac{j+1}{i-1}}\left[1 + O(j^{-1/2})\right],$$

as $j \to \infty$, uniformly with respect to $i \geq j+2$. Noting that

$$\left|e^{2\gamma(\sqrt{j+1}-\sqrt{i-1})}\sqrt{\frac{j+1}{i-1}}\right| \leq \sqrt{2}e^{2\sigma}e^{2\sigma(\sqrt{j}-\sqrt{i-1})}\sqrt{\frac{j}{i-1}},$$

where $\sigma =$Re γ, thus it is possible to choose a constant $\beta > 1$ such that

$$|\theta(i-1)\theta(i-2)\cdots\theta(j+1)| \leq \beta e^{2\sigma(\sqrt{j}-\sqrt{i-1})}\sqrt{\frac{j}{i-1}},$$

for $i \geq j+1$,

$$|R_N^*(j)| \leq \beta j^{-\frac{N}{2}-\frac{3}{2}},$$

and

$$|R_a(j)| + |R_b(j)| \leq \beta j^{-2}$$

also holds, when $j \geq 1$. From (3.61) and (3.47), it then follows that

$$|h_1(n)| \leq \beta^2 \sum_{i=n}^{\infty} e^{-2\sigma\sqrt{i-1}}(i-1)^{-1/2} \sum_{j=1}^{i-1} e^{2\sigma\sqrt{j}} j^{-\frac{N}{2}-1}.$$

Using the *Euler-Maclaurin* formula, it can be shown that

$$\sum_{j=1}^{i-1} e^{2\sigma\sqrt{j}} j^{-\frac{N}{2}-1} \leq M_0 e^{2\sigma\sqrt{i-1}}(i-1)^{-\frac{N}{2}-\frac{1}{2}}$$

for some constant $M_0 > 0$, and $i \geq n \geq n_0(N)$. Therefore

$$|h_1(n)| \leq M_0\beta^2 \sum_{i=n}^{\infty}(i-1)^{-\frac{N}{2}-1} \leq \frac{4M_0\beta^2}{N} n^{-\frac{N}{2}}$$

for sufficiently large n, $n \geq n_0(N)$. By induction, the same argument gives

$$|h_{s+1}(n) - h_s(n)|, \quad |h_{s+1}(n+1) - h_s(n+1)| \leq \left(\frac{4M_0\beta^2}{N}\right)^{s+1} n^{-\frac{N}{2}}$$

for $n \geq n_0(N)$. As before, we now choose $N > 4M_0\beta^2$ so that the series

$$\varepsilon_N(n) = \lim_{s\to\infty} h_{s+1}(n) = \sum_{s=0}^{\infty} [h_{s+1}(n) - h_s(n)]$$

converges uniformly in n, and its sum $\varepsilon(n)$ satisfies (3.51) and (3.49).

3.5 Exceptional Case (i)

In §1.5, we established the formal series

$$Y(n) = \rho^n n^\alpha \sum_{s=0}^{\infty} c_s n^{-s}.$$

Without loss of generality we may assume that $\rho = 1$ or equivalently, $a_0 = -2$ and $b_0 = 1$, as in the last two sections. From (1.8), it follows that

$$a_1 + b_1 = 0. \tag{3.62}$$

As in the previous section, we set

$$y(n) = L_N(n) + E_N(n) \tag{3.63}$$

with

$$L_N(n) = \sum_{s=0}^{N-1} c_s n^{\alpha - s}.$$

Using (2.23), it is easily verified that

$$L_N(n+2) + a(n)L_N(n+1) + b(n)L_N(n) = n^\alpha R_N(n)$$

where

$$R_N(n) = O\left(n^{-N-2}\right), \qquad \text{as } n \to \infty.$$

Hence

$$E_N(n+2) + a(n)E_N(n+1) + b(n)E_N(n) = -n^\alpha R_N(n). \tag{3.64}$$

If we write

$$E_N(n) = n^\alpha \varepsilon_N(n),$$

then $\varepsilon_N(n)$ satisfies the equation

$$\varepsilon_N(n+2) + a^\star(n)\varepsilon_N(n+1) + b^\star(n)\varepsilon_N(n) = R_N^\star(n), \tag{3.65}$$

where

$$a^\star(n) = \left(\frac{n+1}{n+2}\right)^\alpha a(n),$$

$$b^\star(n) = \left(\frac{n}{n+2}\right)^\alpha b(n),$$

and

$$R_N^\star(n) = -\left(\frac{n}{n+2}\right)^\alpha R_N(n). \tag{3.66}$$

Recalling that we have assumed $\rho = 1$ or, equivalently $a_0 = -2$ and $b_0 = 1$, using (3.62), a simple calculation shows that

$$a^\star(n) = -2 + (2\alpha + a_1)n^{-1} + R_a(n), \tag{3.67}$$

$$b^\star(n) = \quad 1 - (2\alpha + a_1)n^{-1} + R_b(n), \tag{3.68}$$

where

$$R_a(n), \quad R_b(n) = O\left(n^{-2}\right), \text{as } n \to \infty. \tag{3.69}$$

Furthermore

$$R_N^\star(n) = O\left(n^{-2}\right), \text{as } n \to \infty. \tag{3.70}$$

Inserting (3.67), (3.68) in (3.65), we obtain

$$\begin{aligned}
&\varepsilon_N(n+2) + \left[-2 + (2\alpha + a_1)n^{-1}\right]\varepsilon_N(n+1) + \left[1 - (2\alpha + a_1)n^{-1}\right]\varepsilon_N(n) \\
&\qquad = R_N^*(n) - R_a(n)\varepsilon_N(n+1) - R_b(n)\varepsilon_N(n), \qquad (3.71)
\end{aligned}$$

which can be written as

$$\Delta\varepsilon_N(n+1) - \theta(n)\Delta\varepsilon_N(n) = q\left(\varepsilon_N(n+1), \varepsilon_N(n), n\right), \qquad (3.72)$$

where

$$\Delta\varepsilon_N(n) = \varepsilon_N(n+1) - \varepsilon_N(n),$$

$$\theta(n) = 1 - (2\alpha + a_1)n^{-1}, \qquad (3.73)$$

and

$$q\left(\varepsilon_N(n+1), \varepsilon_N(n), n\right) = R_N^*(n) - R_a(n)\varepsilon_N(n+1) - R_b(n)\varepsilon_N(n).$$

To prove that the solution $y(n) \sim Y(n)$, it suffices to show that

$$\varepsilon_N(n) = O\left(n^{-N}\right), \qquad \text{as } n \to \infty. \qquad (3.74)$$

By considering (3.72) as a first-order linear nonhomogeneous equation, one can formally derive the equation

$$\varepsilon_N(n) = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j)q\left(\varepsilon_N(j+1), \varepsilon_N(j), j\right). \qquad (3.75)$$

It is easily verified that every solution of (3.75) is a solution of (3.72). To solve the equation (3.75) which satisfies (3.74), we first prove the following analogue of *Lemma 1.*

LEMMA 3 *For positive integers $j \geq i \geq 1$, the function $\theta(n)$ in (3.73) satisfies*

$$\theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j) = \left(\frac{j}{i}\right)^{2\alpha+a_1}\left[1+O\left(i^{-1}\right)\right], \tag{3.76}$$

as $i \longrightarrow \infty$, where the O-term is uniform with respect to j.

Proof: We first recall the well-known asymptotic approximations

$$\sum_{k=1}^{n-1} k^{-2} = \frac{\pi^2}{6} - \frac{1}{n} + O(n^{-2}), \qquad n \longrightarrow \infty,$$

and

$$\sum_{k=1}^{n-1} k^{-1} = \log n + c + O\left(n^{-1}\right), \qquad n \longrightarrow \infty,$$

where c denotes the *Euler* constant; cf. (3.53) and (3.54). Since $\log(1-x) = -x + O\left(x^2\right)$, as $x \longrightarrow 0$, we have from (3.73)

$$\log\theta(k) = -(2\alpha+a_1)k^{-1} + O\left(k^{-2}\right), \qquad k \longrightarrow \infty,$$

where the remainder is uniformly bounded when $k \geq 1$.

Upon summation, we obtain

$$-\sum_{k=i}^{j}\log\theta(k) = (2\alpha+a_1)\log\left(\frac{j}{i}\right) + O\left(i^{-1}\right), \qquad i \longrightarrow \infty,$$

uniformly for $j \geq i \geq 1$. The result in (3.76) now follows by exponentiation. □

We now return to equation (3.75), and define the successive approximants $h_0(n) \equiv 0$ and

$$(3.77) \quad h_{s+1}(n) = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} \theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j)q\left(h_s(j+1), h_s(j), j\right),$$

for $s = 0, 1, \cdots$.. When $s = 0$, we have

$$(3.78) \qquad h_1(n) = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} \theta^{-1}(i)\cdots\theta^{-1}(j)R_N^*(j)$$

where $R_N^*(j)$ is given in (3.66). From (3.70) and (3.75) there is a positive constant M_1 and a positive integer n_1 such that

$$(3.79) \qquad \left|\theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j)\right| \leq M_1\left(\frac{j}{i}\right)^{\sigma},$$

and

$$(3.80) \qquad |R^*(j)| \leq M_1 j^{-N-2},$$

for $j \geq i \geq n_1$, where $\sigma = \text{Re }(2\alpha + a_1)$. Consequently,

$$(3.81) \qquad |h_1(n)| \leq M_1^2 \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} i^{-(2\alpha+a_1)} j^{-N-2+\sigma}$$

for all $n > n_1$. Let $N + 1 > \sigma$ and put $\beta = 4M_1^2$, choosing $n_0(N) \geq n_1$ so that two applications of (3.11) and (3.81) give

$$|h_1(n+1)|, \quad |h_1(n)| \leq \frac{\beta}{N(N+1-\sigma)} n^{-N},$$

for all $n \geq n_0(N)$. In view of (3.69), the constant M_1 and integer n_1 in (3.79) and (3.80) may be chosen so that

$$|R_a(j)| + |R_b(j)| \leq M_1 j^{-2}$$

for $j \geq n_1$. Induction then shows that

$$\begin{aligned} &|h_{s+1}(n+1) - h_s(n+1)|\,, \ |h_{s+1}(n) - h_s(n)| \\ &\qquad \leq \left[\frac{\beta}{N(N+1-\sigma)}\right]^{s+1} n^{-N}, \end{aligned}$$

if $n \geq n_0(N)$ for $s = 0, 1, 2, \cdots$. As long as $N(N+1-\sigma) > \beta$ the series

$$\varepsilon_N(n) = \lim_{s \to \infty} h_{s+1}(n) = \sum_{s=0}^{\infty} [h_{s+1}(n) - h_s(n)]$$

is then uniformly convergent in n for $n \geq n_0(N)$, where we first fix N and then choose $n_0(N) \geq n_1$. Thus the sum of $\varepsilon_N(n)$ satisfies (3.74) and (3.75).

For each zero α of the indicial polynomial (1.10), the above construction provides a solution to (3.75) and hence a solution to (3.71). Since, in the present case, (1.10) has two distinct zero α_1 and α_2 and $\alpha_2 - \alpha_1$ is not an integer, this establishes the existence of two asymptotic solutions to (1.4) both of which are of the form (2.24).

3.6 Exceptional Case (ii), (iii)

In these cases, one of the independent solutions $y_1(n)$ was proved in the previous section to be the asymptotic form (2.24). We set the second solution

$$y_2(n) = \sum_{s=0}^{N-1} d_s n^{\alpha_2 - s} + E_N(n) + c y_1(n) \log n, \tag{3.82}$$

cf. (2.25), where we assume as in the previous case, without loss of generality, that $\rho = 1$ or equivalently $a_0 = -2$ and $b_0 = 1$.

In case (ii), $p = \alpha_2 - \alpha_1$ is a positive integer, c and the coefficients d_s are given by (2.38), (2.35), (2.37) and (2.39). Substituting (3.82) into (2.1) gives

$$\mathcal{L}\{y_2(n)\} = \mathcal{L}\left\{\sum_{s=0}^{N-1} d_s n^{\alpha_2 - s}\right\} + \mathcal{L}\{E_N(n)\} + C\mathcal{L}\{y_1(n)\log n\}. \tag{3.83}$$

Refering to the formulas (2.31) and (2.32) we have

$$\mathcal{L}\left\{\sum_{s=0}^{N-1} d_s n^{\alpha_2 - s}\right\} = \sum_{s=2}^{N+1} Z_d(s) n^{\alpha_2 - s} + O\left(n^{\alpha_2 - N - 2}\right), \tag{3.84}$$

where the notation

$$Z_d(s) = \sum_{j=0}^{s-2}\left[2^{s-j}\binom{\alpha_2 - j}{s - j} + \sum_{l=0}^{s-j}\binom{\alpha_2 - j}{s - l - j} a_l + b_{s-j}\right] d_j.$$

Write

$$y_1(n) = \sum_{s=0}^{N-p+1} c_s n^{\alpha_1 - s} + O\left(n^{\alpha_2 - N - 2}\right)$$

where $\alpha_2 = \alpha_1 + p$ and

$$\log(n+\mu) = \log n + \sum_{s=1}^{N+1-p} \frac{(-1)^{s+1}}{s} \left(\frac{\mu}{n}\right)^s + O\left(n^{-(N+2-p)}\right)$$

Then we have, after many computations,

$$\begin{aligned} &\mathcal{L}\{y_1(n)\log n\} \\ &= \mathcal{L}\{y_1(n)\}\log n + \sum_{s=2}^{N+1-p} Y_c(s) n^{\alpha_1 - s} + O\left(n^{\alpha_2 - N - 2}\right) \end{aligned} \tag{3.85}$$

where

$$Y_c(s) = \sum_{j=0}^{s-2} \left\{ \sum_{l=j}^{s-1} \frac{(-1)^{s+1-l}}{s-l} \left[\rho^2 2^{s-j} \binom{\alpha_1 - j}{l-j} + \rho \sum_{k=j}^{l} \binom{\alpha_1 - j}{k-j} a_{l=k} \right] \right\} c_j,$$

for $N \geq \alpha_2 - \alpha_1 + 1$; cf. (2.32) and (2.30).

Upon substitution of (3.84) and (3.85) in (3.83), and noting that $y_1(n)$ is the solution of equation (1.4), in view of the recurrence equations for d_s and c, we obtain

$$\begin{aligned} &\mathcal{L}\{E_N(n)\} \\ &= E_N(n+2) + a(n)E_N(n+1) + b(n)E_N(n) = O\left(n^{\alpha_2 - N - 2}\right), \end{aligned} \tag{3.86}$$

which is exactly of the form (3.64) with $\alpha = \alpha_2$. Hence the existence of a solution to (3.86) satisfying

$$E_N(n) = O\left(n^{\alpha_2 - N}\right)$$

is guaranteed by the construction provided in previous section.

The analysis in the case where $\alpha_1 = \alpha_2$ is done in a manner similar to that given above. It needs to be noted that in this case $d_0 = d_1 = \cdots = d_{Q-3} = 0$, c, and d_s for $s \geq Q-2$ are given by (2.44)–(2.46), α_2 is replaced by α_1 and N should be $\geq Q-1$, where Q is defined in (2.41).

3.7 Examples

Example 1 *Behavior of the Legendre Polynomials of large degree.*

The *Legendre* polynomials $P_n(x)$ satisfy the recurrence relation

$$(n+2)y_{n+2}(x) - (2n+3)xy_{n+1}(x) + (n+1)y_n = 0, \qquad n = -1, 0, 1, \ldots, \tag{3.87}$$

for all x. The behavior of $P_n(x)$ as $n \to \infty$ for fixed x may be studied by considering the possible behavior of solutions of (3.87) as $n \to \infty$.

(3.87) can be changed to

$$y_{n+2}(x) + a(n)y_{n+1}(x) + b(n)y_n(x) = 0,$$

where

$$a(n) = -\frac{2n+3}{n+2}x = -2x + xn^{-1} - 2xn^{-2} + 4xn^{-3} + O\left(n^{-4}\right),$$

$$b(n) = \frac{n+1}{n+2} = 1 - n^{-1} + 2n^{-2} - 4n^{-3} + O\left(n^{-4}\right).$$

Thus the characteristic equation is

$$\rho^2 - 2x\rho + 1 = 0$$

which has two distinct roots if $x \neq 1$, namely

$$\rho_{1,2} = x \pm \sqrt{x^2 - 1}.$$

Note that if $|x| < 1$, then $\rho_{1,2}$ is complex. We know that two normal solutions of the form (1.7) can be obtained. α_i and $c_s^{(i)}$ can be determined from (2.4) and (2.5), i. e. ,

$$\alpha_i = \frac{a_1\rho + b_1}{a_1\rho + 2b_0} = -\frac{1}{2}$$

and

$$c_1^{(i)} = \left(\frac{1}{8(x\rho_i - 1)} - \frac{1}{8}\right).$$

Thus we obtain two independent normal solutions of (3.87), namely,

$$\begin{aligned} y_{n,1}(x) &= \left(x + \sqrt{x^2 - 1}\right)^n n^{-\frac{1}{2}} \\ &\cdot \left\{1 + \frac{1}{8}\left[\frac{1}{x^2 + x\sqrt{x^2 - 1} - 1} - 1\right] n^{-1} + O\left(n^{-2}\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} y_{n,2}(x) &= \left(x - \sqrt{x^2 - 1}\right)^n n^{-\frac{1}{2}} \\ &\cdot \left\{1 + \frac{1}{8}\left[\frac{1}{x^2 - x\sqrt{x^2 - 1} - 1} - 1\right] n^{-1} + O\left(n^{-2}\right)\right\}. \end{aligned}$$

By comparing the behavior of the *Legendre* polynomials [10]

$$P_n(x) \sim (2\pi n)^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{n+\frac{1}{2}}, \qquad n \to \infty; \quad x > 1,$$

$$P_n(x) \sim (-1)^n(2\pi n)^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{4}}\left(-x+\sqrt{x^2-1}\right)^{n+\frac{1}{2}}, \qquad n \to \infty; \quad x < -1,$$

while for $|x| < 1$

$$P_n(\cos\alpha) = \left(\frac{2}{\pi n \sin\alpha}\right)^{\frac{1}{2}} \sin\left(n\alpha + \frac{1}{2}\alpha + \frac{1}{4}\pi\right) + O\left(n^{-\frac{3}{2}}\right),$$

we find that

$$P_n(x) = (2\pi)^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{4}}\left(x+\sqrt{x^2-1}\right)^{\frac{1}{2}} y_1(x), \qquad x > 1,$$

$$P_n(x) = (2\pi)^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{4}}\left(-x+\sqrt{x^2-1}\right)^{\frac{1}{2}} y_2(x), \qquad x < 1,$$

and

$$P_n(\cos\alpha) = \frac{1}{2}\left(\frac{2}{\pi\sin\alpha}\right)^{\frac{1}{2}}\left[e^{\frac{1}{2}\alpha+\frac{1}{4}\pi}y_1(\cos\alpha) - e^{-\frac{1}{2}\alpha-\frac{1}{4}\pi}y_2(\cos\alpha)\right].$$

Example 2 *Behavior of the* Laguerre Polynomials $L_n^{(\beta)}(x)$ *of large degree. The recurrence relation for the* Laguerre Polynomials *, is*

$$(n+1)y_{n+1}^{(\beta)}(x) + (x-2n-\beta-1)y_n^{(\beta)}(x) + (n+\beta)y_{n-1}^{(\beta)}(x) = 0,$$

for $x > 0$.

Changing this into the form of (1.4) gives

$$y_{n+2}^{(\beta)}(x) + a(n)y_{n+1}^{(\beta)}(x) + b(n)y_n^{(\beta)}(x) = 0,$$

where

$$a(n) = \frac{x - 2n - \beta - 3}{n+2} = -2 + \frac{x - \beta + 1}{n} - \frac{2(x - \beta + 1)}{n^2} + O\left(n^{-3}\right),$$

$$b(n) = \frac{n + \beta + 1}{n+2} = 1 + \frac{\beta - 1}{n} - \frac{2(\beta - 1)}{n^2} + O\left(n^{-3}\right).$$

We find that the characteristic equation

$$\rho^2 - 2\rho + 1 = 0$$

has double roots $\rho_{1,2} = 1$ which do not satisfy the auxiliary equation for $x \neq 0$. Thus the subnormal solutions are given by (1.9), where γ_k, α_k and $c_s^{(k)}$ $(k = 1, 2)$ are from (2.17), (2.18) and (2.19), i. e. ,

$$\gamma_k = \pm 2\sqrt{\frac{a_0 a_1 - 2b_0}{2b_0}} = \pm 2\sqrt{x} i,$$

$$\alpha = \frac{\beta}{2} - \frac{1}{4},$$

and

$$c_1^{(k)} = \mp \frac{i}{48\sqrt{x}} \left(4x^2 - 12\beta^2 - 24x\beta - 24x + 3\right).$$

Thus the formal solutions are

$$y_{n,k}^{(\beta)}(x) = e^{\pm 2\sqrt{xn} i} n^{\frac{\beta}{2} - \frac{1}{4}}$$
$$\cdot \left\{1 \mp \frac{i}{48\sqrt{x}} \left(4x^2 - 12\beta^2 - 24x\beta - 24x + 3\right) n^{-\frac{1}{2}} + O\left(n^{-1}\right)\right\},$$

for $k = 1, 2$.

Refering to the behavior of the *Laguerre Polynomials* [10],

$$
\begin{aligned}
&L_n^{(\beta)}(x) \\
&\quad = \pi^{-\frac{1}{2}} e^{\frac{x}{2}} x^{-\frac{\beta}{2}-\frac{1}{4}} n^{\frac{\beta}{2}-\frac{1}{4}} \cos\left(2\sqrt{nx} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right)\left\{1 + O\left(n^{-\frac{1}{2}}\right)\right\} \\
&\quad + \pi^{-\frac{1}{2}} e^{\frac{x}{2}} x^{-\frac{\beta}{2}-\frac{1}{4}} n^{\frac{\beta}{2}-\frac{1}{4}} \sin\left(2\sqrt{nx} - \frac{\beta\pi}{2} - \frac{\pi}{4}\right) O\left(n^{-\frac{1}{2}}\right),
\end{aligned}
$$

we have

$$
L_n^{(\beta)}(x) = \frac{1}{2}\pi^{-\frac{1}{2}} e^{\frac{x}{2}} x^{-\frac{\beta}{2}-\frac{1}{4}} \left\{ e^{-\left(\frac{\beta\pi}{2}+\frac{\pi}{4}\right)i} y_{n,1}^{(\beta)}(x)(1-i) + e^{\left(\frac{\beta\pi}{2}+\frac{\pi}{4}\right)i} y_{n,2}^{(\beta)}(x)(1+i) \right\}.
$$

Chapter 4

Formal Solutions (2)

4.1 Introduction

So far we have dealt with the case in which $M_a \leq 0$ and $M_b = 0$ in the equation

$$y(n+2) + n^{M_a} a(n) y(n+1) + n^{M_b} b(n) y(n) = 0, \tag{4.1}$$

where $a(n)$ and $b(n)$ have power series expansions of the form

$$a(n) = \sum_{s=0}^{\infty} a_s n^{-s}, \qquad b(n) = \sum_{s=0}^{\infty} b_s n^{-s}$$

for large values of n, and $a_0 \neq 0$, $b_0 \neq 0$. For the general situation where M_a and M_b are arbitrary integers, the method of deriving formal solutions that we shall use was introduced by *C. R. Adams*[1] who applied it to equations of any order. The transformation

$$x(n) = n^{\lambda n} e^{-\lambda n} y(n) \tag{4.2}$$

will change equation (4.1) into

$$x(n+2) + n^{M_a + \lambda} \tilde{a}(n) x(n+1) + n^{M_b + 2\lambda} \tilde{b}(n) x(n) = 0, \tag{4.3}$$

where

$$(4.4) \qquad \tilde{a}(n) = \left(\frac{n+2}{n+1}\right)^{\lambda(n+1)} \left(\frac{n+2}{n}\right)^{\lambda} e^{-\lambda} a(n) = a(n)\left\{1 + O(n^{-1})\right\},$$

$$(4.5) \qquad \tilde{b}(n) = \left(\frac{n+2}{n}\right)^{\lambda n} \left(\frac{n+2}{n}\right)^{2\lambda} e^{-2\lambda} b(n) = b(n)\left\{1 + O(n^{-1})\right\}.$$

We find that the leading terms of the three coefficient functions in equation (4.3) are

$$(4.6) \qquad 1, \qquad a_0 n^{M_a+\lambda} \quad \text{and} \quad b_0 n^{M_b+\lambda}.$$

We can choose a suitable λ, such that two of these are of the same power while the other is lower. If we have two such λ's, the characteristic equation

$$(4.7) \qquad \rho^2 + a_0\rho + b_0 = 0$$

may be replaced by two characteristic equations each of which is associated with each λ. The characteristic equation can be given by taking the term in equation (4.7) whose coefficient is associated with the lower power in (4.6) out of the equation. The choice of λ depends on whether

$$(4.8) \qquad K = 2M_a - M_b$$

is zero, or positive or negative.

(1) When $K = 0$, we choose

$$\lambda = -M_a$$

so that equation (4.3) becomes

$$x(n+2) + \tilde{a}(n)x(n+1) + \tilde{b}(n)x(n) = 0.$$

It is clear that this case is included in Chapter 2.

(2) When $K > 0$, we choose

$$\lambda_1 = -M_a \quad \text{and} \quad \lambda_2 = M_a - M_b. \tag{4.9}$$

Therefore equation (4.3) becomes

$$x(n+2) + \tilde{a}(n)x(n+1) + n^{-K}\tilde{b}(n)x(n) = 0 \tag{4.10}$$

or

$$x(n+2) + n^{K}\tilde{a}(n)x(n+1) + n^{K}\tilde{b}(n)x(n) = 0 \tag{4.11}$$

respectively, corresponding to each λ. As the degrees of the coefficient functions in (4.10) and (4.11) are zero and K respectively, we shall show that the characteristic equations associated with each λ are

$$\rho_1 + a_0 = 0, \tag{4.12}$$

and

$$\rho_2 a_0 + b_0 = 0. \tag{4.13}$$

The normal solutions for both (4.10) and (4.11) were shown by *Adams* to be

$$x(n) \sim \rho_i^n n^{\alpha_i} \sum_{s=0}^{\infty} c_s^i n^{-s}, \qquad n \to \infty, \qquad i = 1, 2, \tag{4.14}$$

where ρ_i associated with each λ_i satisfies the characteristic equations (4.12) and (4.13) respectively.

(3) When $K < 0$, we choose

$$\lambda_1 = \lambda_2 = -M_b/2. \tag{4.15}$$

Equation of (4.3) becomes

$$x(n+2) + n^{\frac{1}{2}K}\tilde{a}(n)x(n+1) + \tilde{b}(n)x(n) = 0.$$

With the same reasoning, the characteristic equation is

$$\rho^2 + b_0 = 0. \tag{4.16}$$

It is easy to see that the case in which K is even is included in Chapter 2. When K is odd, the subnormal series solution was established by *Adams* (p.512) to be

$$x_i(n) \sim \rho_i^n e^{\gamma_i\sqrt{n}} n^{\alpha_i} \sum_{s=0}^{\infty} c_s^{(i)} n^{-\frac{s}{2}}, \qquad n \to \infty, \qquad i = 1, 2, \tag{4.17}$$

where ρ_i is the root of the characteristic equation (4.16). He also pointed out that the exponential factor will probably disappear. We shall find out in §4.3 that this occurs when $K < -1$ and odd.

It is noteworthy that *Adams* proved the existence theorems about the normal and subnormal solutions (4.17) and (4.14) except when $|\rho_1| = |\rho_2|$ and $\gamma_i \neq 0$ ([1] p.513), which will happen when $K = -1$ in our problem.

According to *C. R. Adams*[1], the choice of λ has a very simple geometric interpretation. Let $m = \max\{0, M_a, M_b\}$, then express equation (4.1) as

$$S_0(n)y(n+2) + S_1(n)y(n+1) + S_2(n)y(n) = 0,$$

where

$$S_0(n) = n^{-m} = \sum_{j=0}^{\infty} S_{0,j}n^{-j}, \qquad S_{0,m} = 1,$$

$$S_1(n) = n^{M_a-m}a(n) = \sum_{j=0}^{\infty} S_{1,j}n^{-j}, \qquad S_{1,m-M_a} = a_0,$$

$$S_2(n) = n^{M_b-m}b(n) = \sum_{j=0}^{\infty} S_{2,j}n^{-j}, \qquad S_{2,m-M_b} = b_0.$$

Thus $-m$, $M_a - m$ and $M_b - m$ are non-positive, and at least one of them is zero. If we denote by S_{i,j_i} the coefficient of the leading term of $S_i(n)$ ($i = 0, 1, 2$), we have

$$j_0 = m, \qquad j_1 = m - M_a, \qquad j_2 = m - M_b.$$

Choosing $i-$ and $j-$axes, and plotting the points $A_i = (i, j_i)$ on the plane, we may construct a broken line, convex upward, such that both ends of each segment of the line are points of the set (i, j_i) and all three points of the set lie upon or beneath the line. At least one of the points is situated on the $i-$axis (see Figures 1 to 11). Thus, the pamameter λ can be chosen as the slope [1] of any segment. Therefore the coefficients of the characteristic equation associated with λ or consequently, with each segment, are the S_{i,j_i} corresponding to points A_i actually on that segment, and the coefficient corresponding to a point A_i beneath

[1]In the paper of *C. R. Adams* [1], it may be a mistake to chose λ as the opposite sign of the slope. For example, in his paper, formula (8) will agree with (9) only if one exchanges the positions of funtions $f(x)$ and $g(x)$ in (8).

the segment is zero. (see *C.R.Adams* [1] p.511). If we denote $l_{A_iA_j}$ as the slope of segment A_iA_j, we have

$$l_{A_0A_1} = j_1 - j_0 = -M_a,$$

$$l_{A_0A_2} = \frac{j_2 - j_0}{2} = -\frac{M_b}{2},$$

$$l_{A_1A_2} = j_2 - j_1 = M_a - M_b.$$

Then in (1), in view of (4.8), $K = 0$ or $M_a = M_b/2 = M_b - M_a$. This implies that $l_{A_0A_1} = l_{A_0A_2} = l_{A_1A_2}$, i. e. A_1, A_2 and A_3 are on one line. (see Figures 1,2,3). λ should be chosen as

$$\lambda_1 = \lambda_2 = l_{A_0A_1} = -M_a.$$

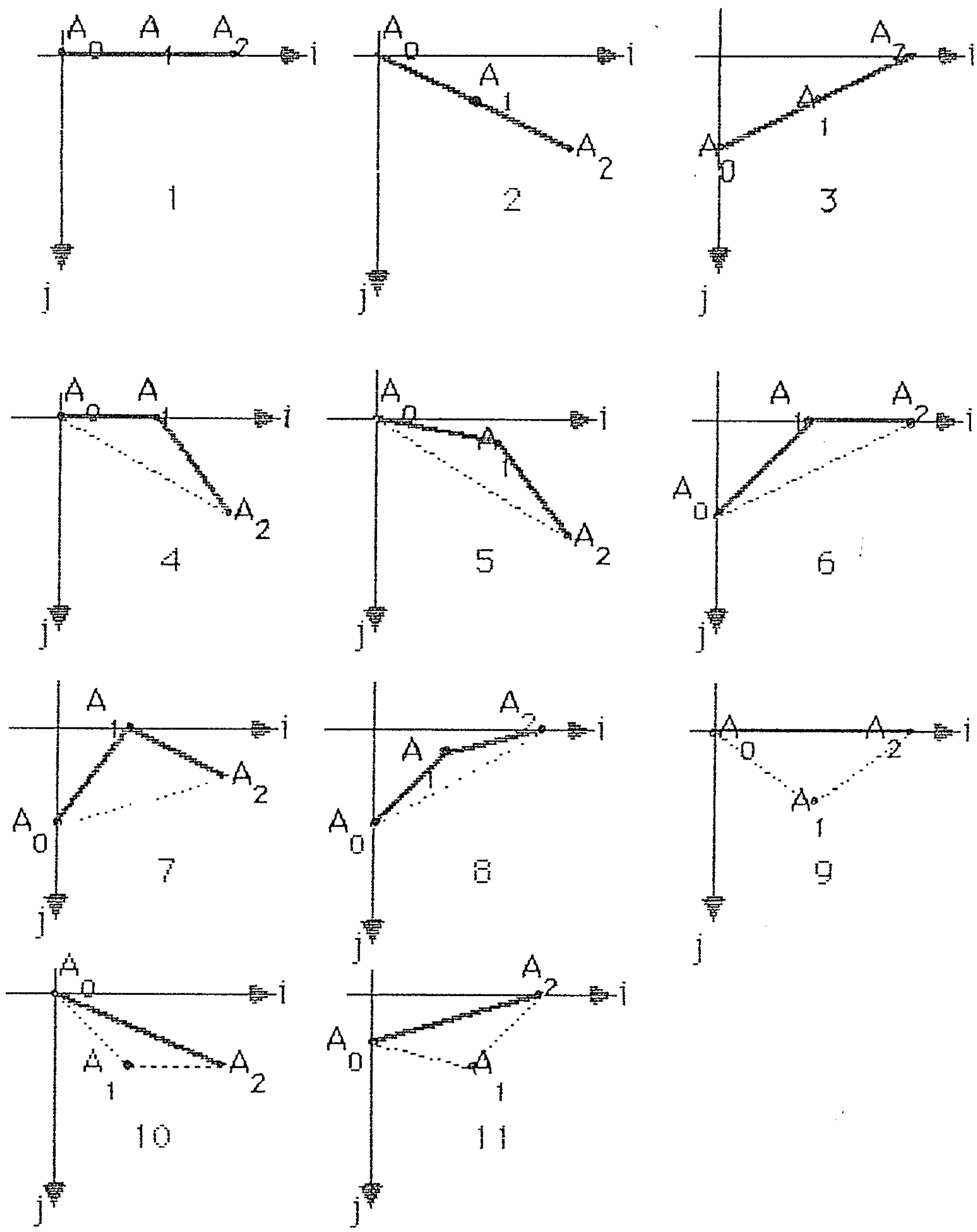
The characteristic equation is (4.7). In (2), $K > 0$ or $M_a > M_b/2$. This implies that A_1 is situated above the line $l_{A_0A_2}$. (see Figures 4,5,6,7,8). Thus we choose

$$\lambda_1 = l_{A_0A_1} = -M_a \quad \text{and} \quad \lambda_2 = l_{A_1A_2} = M_a - M_b.$$

For λ_1, A_0 and A_1 are on the line $l_{A_0A_1}$ and A_2 is beneath the line. The corresponding coefficients of A_0 and A_1 are 1 and a_0. Thus the characteristic equation associated with λ_1 is (4.12). Accordingly, the characteristic equation associated with λ_2 is (4.13). In (3), $K < 0$ or $M_a < M_b/2$. This implies that A_1 is situated beneath the line $l_{A_0A_2}$ (see Figures 9,10,11). Thus we choose

$$\lambda_1 = \lambda_2 = l_{A_0A_2} = -\frac{M_b}{2}.$$

The characteristic equation is therefore (4.16). When A_0 is one unit vertically below the segment $l_{A_0A_2}$, *Adams* ([1] p.512) pointed out that the exponential factor in the formal series (4.17) vanishes. This is just our case when $K < -1$.



FIGURES

Finally in this thesis, inspired by the famous *Stirling* formula

$$n! \sim n^n e^{-n}\sqrt{2\pi n},$$

we use the transformation

$$x(n) = [(n-2)!]^\lambda y(n) \tag{4.18}$$

instead of (4.2). Thus equation (4.1) becomes

$$x(n+2) + n^{M_a+\lambda}a^\star(n)x(n+1) + n^{M_b+2\lambda}b^\star(n)x(n) = 0, \tag{4.19}$$

where

$$a^\star(n) = \sum_{s=0}^{\infty} a_s^\star n^{-s} = a(n), \tag{4.20}$$

$$b^\star(n) = \sum_{s=0}^{\infty} b_s^\star n^{-s} = b(n)\left(\frac{n-1}{n}\right)^\lambda = \sum_{s=0}^{\infty}\left\{\sum_{l=0}^{s}(-1)^l \begin{pmatrix} \lambda \\ l \end{pmatrix} b_{s-l}\right\} n^{-s}, \tag{4.21}$$

for large values of n, and $a_0^\star = a_0 \neq 0$, $b_0^\star = b_0 \neq 0$. It is clear that (4.21) and (4.20) are simpler than (4.4) and (4.5).

4.2 Case 1, $K \geq 0$

In the case where $K = 0$, as described in condition (1) in the preceding section, λ may be chosen as $-M_a$. Then (4.19) becomes

$$x(n+2) + a^\star(n)x(n+1) + b^\star(n)x(n) = 0,$$

where

$$a^\star(n) = a(n), \tag{4.22}$$

$$b^\star(n) = \sum_{s=0}^{\infty}\left\{\sum_{l=0}^{s}(-1)^l \begin{pmatrix} -M_a \\ l \end{pmatrix} b_{s-l}\right\} n^{-s}, \qquad n \to \infty. \tag{4.23}$$

As $b_0^\star = b_0 \neq 0$, this case is included in Chapter 2. The normal or subnormal solutions may be obtained according to the properties of characteristic equation (1.6) , or auxiliary equation (1.8), or indicial polynomial (1.10) in Chapter 2.

In the case of $K > 0$, it has been pointed out that $\lambda_1 = -M_a$ and $\lambda_2 = M_a - M_b$ may be chosen for the transformation (4.18). When $\lambda = -M_a$, (4.19) becomes

$$x(n+2) + a^\star(n)x(n+1) + n^{-K}b^\star(n)x(n) = 0, \tag{4.24}$$

where $a^\star(n)$ and $b^\star(n)$ are the analogues of (4.22) and (4.23) in which $a_0^\star = a_0 \neq 0$ and $b_0^\star(n) = b_0 \neq 0$. To show that the infinite series (4.14) is indeed a formal series solution of equation (4.24), we substitute it into (4.24). Let $\mathcal{L}$ denote the linear difference operator

$$\begin{aligned} &\mathcal{L}\{X(n), \lambda\} \\ &\quad = X(n+2) + n^{M_a+\lambda}a^\star(n)X(n+1) + n^{M_b+2\lambda}b^\star(n)X(n), \end{aligned}$$

and make use of formula (2.2) and (4.23), we have

$$\begin{aligned} &\mathcal{L}\{X(n), -M_a\} \\ &\quad = \sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s}\left[2^{s-j}\rho \begin{pmatrix} \alpha - j \\ s - j \end{pmatrix} + \sum_{l=j}^{s}\begin{pmatrix} \alpha - j \\ l - j \end{pmatrix} a_{s-l}^\star\right] c_j\right\} \rho^{n+1} n^{\alpha - s} \end{aligned}$$

$$+\sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s} b_{s-j}^{\star} c_j\right\}\rho^n n^{\alpha-K-s}$$

$$=\sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s}\left[2^{s-j}\rho\binom{\alpha-j}{s-j}+\sum_{l=j}^{s}\binom{\alpha-j}{l-j}a_{s-l}\right]c_j\right\}\rho^{n+1}n^{\alpha-s}$$

$$+\sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s}\left[\sum_{l=0}^{s-j}(-1)^l\binom{-M_a}{l}b_{s-j-l}\right]c_j\right\}\rho^n n^{\alpha-K-s}. \tag{4.25}$$

In the last term above if we use the transform $l=\tilde{l}-j$ and then replace $\tilde{l}$ by l in the result, we obtain

$$\mathcal{L}\{X(n),-M_a\}$$

$$=\sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s}\left[2^{s-j}\rho\binom{\alpha-j}{s-j}+\sum_{l=j}^{s}\binom{\alpha-j}{l-j}a_{s-l}\right]c_j\right\}\rho^{n+1}n^{\alpha-s}$$

$$+\sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s}\left[\sum_{l=j}^{s}(-1)^{l-j}\binom{-M_a}{l-j}b_{s-l}\right]c_j\right\}\rho^n n^{\alpha-K-s}. \tag{4.26}$$

Equating the coefficients of terms with like powers of n to zero, we obtain

$$\sum_{j=0}^{s}\left[2^{s-j}\rho\binom{\alpha-j}{s-j}+\sum_{l=j}^{s}\binom{\alpha-j}{l-j}a_{s-l}\right]c_j=0, \qquad s=0,1,\dots.K-1, \tag{4.27}$$

and

$$\sum_{j=0}^{s}\left[2^{s-j}\rho^2\binom{\alpha-j}{s-j}+\sum_{l=j}^{s}\rho\binom{\alpha-j}{l-j}a_{s-l}\right]c_j$$

$$+\sum_{j=0}^{s-K}\left[\sum_{l=j}^{s-K}(-1)^{l-j}\binom{-M_a}{l-j}b_{s-K-l}\right]c_j=0, \tag{4.28}$$

for s=K,K+1,....

When $s = 0$, (4.27) reduces to the equation

$$(\rho + a_0)c_0 = 0.$$

By assuming $c_0 = 1$, we obtain the characteristic equation (4.12).

To determine α and c_s, it should be considered whether $K = 1$ or not. When $K = 1$, α may be obtained from (4.28) for $s = 1$, which is

$$(2\rho^2\alpha + \rho a_1 + \rho\alpha a_0 + b_0)c_0 = 0.$$

In view of (4.12), we obtain

$$\alpha = \frac{a_1}{a_0} - \frac{b_0}{a_0^2}. \tag{4.29}$$

We note that the coefficients of c_s and c_{s-1} in (4.28) are, respectively, $\rho^2 + \rho a_0$ and

$$2\rho^2(\alpha - s + 1) + \rho\left[a_1 + (\alpha - s + 1)a_0\right] + b_0.$$

In view of (4.12) and (4.29), these are in turn equal to 0 and $(1-s)a_0^2$. Therefore (4.28) can be written as

$$\begin{aligned} c_{s-1} = \frac{1}{(s-1)a_0^2}\Bigg\{\sum_{j=0}^{s-2}\Bigg[2^{s-j}\begin{pmatrix}\alpha - j\\ s-j\end{pmatrix}a_0^2 - \sum_{l=j}^{s}\begin{pmatrix}\alpha - j\\ l-j\end{pmatrix}a_0 a_{s-l} \\ + \sum_{l=j}^{s-1}(-1)^{l-j}\begin{pmatrix}-M_a\\ l-j\end{pmatrix}b_{s-1-l}\Bigg]c_j\Bigg\}, \qquad s = 2, 3, \ldots. \end{aligned} \tag{4.30}$$

When $K > 1$, α can be obtained from the relation (4.27) for $s = 1$, i.e.,

$$(2\rho\alpha + a_1 + \alpha a_0)c_0 + (\rho + a_0)c_1 = 0.$$

In view of (4.12), we have

$$\alpha = \frac{a_1}{a_0}. \tag{4.31}$$

Observe that the coefficients of c_s and c_{s-1} in (4.27) are $\rho + a_0$ and

$$2\rho(\alpha - s + 1) + a_1 + (\alpha - s + 1)a_0.$$

These are in turn equal to 0 and $a_0(s-1)$ on account of (4.12) and (4.31). Therefore from (4.27) and the fact that $\rho = -a_0$, we obtain

$$\begin{aligned} c_{s-1} = \frac{1}{a_0(s-1)} \Bigg\{ \sum_{j=0}^{s-2} \Bigg[2^{s-j} a_0 \binom{\alpha - j}{s-j} \\ - \sum_{l=j}^{s} \binom{\alpha-j}{l-j} a_{s-l} \Bigg] c_j \Bigg\}, \\ s = 2, 3, \ldots, K-1. \end{aligned} \tag{4.32}$$

Similarly, observe that the coefficients of c_s and c_{s-1} in (4.28) are $\rho^2 + \rho a_0$ and

$$2\rho^2(\alpha - s + 1) + \rho a_1 + \rho(\alpha - s + 1)a_0.$$

These are in turn equal to 0 and $a_0^2(1-s)$. Thus in view of (4.28) and (4.12), we obtain

$$\begin{aligned} c_{s-1} = \frac{1}{(s-1)a_0^2} \Bigg\{ \sum_{j=0}^{s-2} \Bigg[2^{s-j} a_0^2 \binom{\alpha-j}{s-j} - \sum_{l=j}^{s} \binom{\alpha-j}{l-j} a_0 a_{s-l} \Bigg] c_j \\ + \sum_{j=0}^{s-K} \Bigg[\sum_{l=j}^{s-K} (-1)^{l-j} \binom{-M_a}{l-j} b_{s-K-l} \Bigg] c_j \Bigg\} \\ s = K, K+1, \ldots. \end{aligned} \tag{4.33}$$

Thus we obtain one formal series solution by inserting (4.14) into (4.18).

Another solution may be derived by choosing $\lambda = M_a - M_b$. Equation (4.19) changes to

$$x(n+2) + n^K a^*(n)x(n+1) + n^K b^*(n)x(n) = 0, \tag{4.34}$$

where

$$\begin{aligned} a^*(n) &= a(n), & a_0^* = a_0 \neq 0, \quad &(4.35)\\ b^*(n) &= \sum_{s=0}^{\infty}\left\{\sum_{l=0}^{s}(-1)^l \binom{M_a - M_b}{l} b_{s-l}\right\} n^{-s}, & b_0^* = b_0 \neq 0 \quad &(4.36) \end{aligned}$$

for large values of n. Similarly, substituting (4.14) and (4.36) and (4.35) into (4.34) gives

$$\begin{aligned} \mathcal{L}\{X(n), M_a - M_b\} &= \sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s} 2^{s-j}\binom{\alpha - j}{s - j} c_j\right\}\rho^{n+2} n^{\alpha - s} \\ &+ \sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s}\left[\sum_{l=j}^{s}\left(\rho a_{s-l}\binom{\alpha - j}{l - j}\right.\right.\right. \\ &\left.\left.\left. +(-1)^{l-J}\binom{M_a - M_b}{l - j} b_{s-l}\right)\right] c_j\right\}\rho^n n^{\alpha + K - s}. \end{aligned} \tag{4.37}$$

Letting the coefficient of the leading term in (4.37) be equal to zero, we obtain the characteristic equation (4.13). Generally, equating the coefficients of terms in (4.37) with like powers of n to zero, we have

$$\sum_{j=0}^{s}\left\{\sum_{l=j}^{s}\left[\rho a_{s-l}\binom{\alpha - j}{l - j} + (-1)^{l-j}\binom{M_a - M_b}{l - j} b_{s-l}\right]\right\} c_j = 0 \qquad s = 0, 1, \ldots, K-1, \tag{4.38}$$

and

$$\rho^2 \sum_{j=0}^{s-K} 2^{s-K-j} \binom{\alpha - j}{s-K-j} c_j + \sum_{j=0}^{s} \left\{ \sum_{l=j}^{s} \left[\rho a_{s-l} \binom{\alpha - j}{l-j} \right. \right.$$
$$\left. \left. + (-1)^{l-j} \binom{M_a - M_b}{l-j} b_{s-l} \right] \right\} c_j = 0,$$
$$(4.39) \qquad s = K, K+1, \ldots.$$

When $K = 1$, α can be obtained from (4.39) for $s = 1$ which is

$$\left[\rho^2 + \rho a_1 + b_1 + \rho a_0 \alpha - (M_a - M_b) b_0\right] c_0 + \left[\rho a_0 + b_0\right] c_1 = 0.$$

In view of (4.13), this produces

$$(4.40) \qquad \alpha = \frac{b_1}{b_0} + \frac{b_0}{a_0^2} - \frac{a_1}{a_0} + M_b - M_a.$$

Since the coefficients of c_s and c_{s-1} in (4.39) are $\rho a_0 + b_0$ and

$$\rho^2 + \rho a_1 + b_1 + \rho a_0(\alpha - s + 1) - (M_a - M_b) b_0,$$

and consequently, these are in turn zero and $(s-1)b_0$ on account of (4.13) and (4.40), we may obtain the recurrence relation from (4.39), i. e. ,

$$c_{s-1} = \frac{-1}{b_0(s-1)} \sum_{j=0}^{s-2} \left\{ \rho^2 2^{s-1-j} \binom{\alpha - j}{s-1-j} \right.$$
$$\left. + \left[\sum_{l=j}^{s} \rho a_{s-l} \binom{\alpha - j}{l-j} + (-1)^{l-j} \binom{M_a - M_b}{l-j} b_{s-l} \right] \right\} c_j,$$
$$(4.41) \qquad s = 2, 3, \ldots.$$

When $K > 1$, α can be obtained from (4.38) for $s = 1$ which is

$$[\rho a_1 + b_1 + \rho a_0 \alpha - (M_a - M_b) b_0]\, c_0 + [\rho a_0 + b_0]\, c_1 = 0.$$

In view of (4.13), this leads to

$$\alpha = \frac{b_1}{b_0} - \frac{a_1}{a_0} + M_b - M_a. \tag{4.42}$$

Since the coefficients of c_s and c_{s-1} in (4.38) and (4.39) are zero and $b_0(s-1)$, we obtain the recurrence relation from (4.38),

$$\begin{aligned} c_{s-1} = \frac{-1}{b_0(s-1)} \sum_{j=0}^{s-2} \Bigg\{ \sum_{l=j}^{s} \Bigg[\rho a_{s-l} \begin{pmatrix} \alpha - j \\ l - j \end{pmatrix} \\ + (-1)^{l-j} \begin{pmatrix} M_a - M_b \\ l - j \end{pmatrix} b_{s-l} \Bigg] \Bigg\} c_j, \\ s = 2, 3, \ldots, K-1, \end{aligned} \tag{4.43}$$

and from (4.39),

$$\begin{aligned} c_{s-1} = \frac{-1}{b_0(s-1)} \Bigg\{ \sum_{j=0}^{s-K} \rho^2 2^{s-K-j} \begin{pmatrix} \alpha - j \\ s - K - j \end{pmatrix} c_j \\ + \sum_{j=0}^{s-2} \Bigg[\sum_{l=j}^{s} \Bigg(\rho a_{s-l} \begin{pmatrix} \alpha - j \\ l - j \end{pmatrix} \\ + (-1)^{l-j} \begin{pmatrix} M_a - M_b \\ l - j \end{pmatrix} b_{s-l} \Bigg) \Bigg] c_j \Bigg\}, \\ s = K, K+1, \ldots. \end{aligned} \tag{4.44}$$

Thus from (4.18) and (4.14) we obtain the other formal series solution.

4.3 Case 2, $K < 0$ and Odd

We next consider the case $K < 0$ and odd. Condition (3) in §4.1 tells us that when choose

$$\lambda_1 = \lambda_2 = -\frac{M_b}{2},$$

(4.19) becomes

$$x(n+2) + n^{\frac{1}{2}K} a^\star(n) x(n+1) + b^\star(n) x(n) = 0, \tag{4.45}$$

where

$$\begin{aligned} (4.46)\quad a^\star(n) &= a(n), & a_0^\star = a_0 \neq 0,\\ (4.47)\quad b^\star(n) &= \sum_{s=0}^{\infty}\left\{\sum_{l=0}^{s}(-1)^l \binom{-M_b/2}{l} b_{s-l}\right\} n^{-s}, & b_0^\star = b_0 \neq 0, \end{aligned}$$

for large values of n. We shall show that (4.17) is indeed the formal solution of (4.45) and $\gamma = 0$ when $K < -1$.

Upon substituting (4.46), (4.47) and (4.17) into (4.45) and making use of the notation $G_s^{(\mu)}(\gamma)$ of (2.7) and $F_s^{(\mu)}(\alpha)$ of (2.11) in chapter 2, we have

$$\begin{aligned} \mathcal{L}\left\{X(n), -\frac{M_a}{2}\right\} = e^{\gamma\sqrt{n}}\rho^n n^\alpha & \left\{\sum_{s=0}^{\infty}\rho^2\left[\sum_{l=0}^{s} G_{s-l}^{(2)}(\gamma) F_l^{(2)}(\alpha)\right] n^{-\frac{s}{2}}\right.\\ & + \sum_{s=0}^{\infty}\rho\left[\sum_{t=0}^{s}\frac{1+(-1)^t}{2} a_{\frac{t}{2}}^\star \sum_{q=t}^{s} G_{s-q}^{(1)}(\gamma) F_{q-t}^{(1)}(\alpha)\right] n^{\frac{K-s}{2}}\\ & \left. + \sum_{s=0}^{\infty}\left[\sum_{j=0}^{s}\frac{1+(-1)^{s-j}}{2} b_{\frac{s-j}{2}}^\star c_j\right] n^{-\frac{s}{2}}\right\}. \end{aligned} \tag{4.48}$$

Substituting the expansion of $F_s^{(\mu)}(\alpha)$ into (4.48) and changing the order of summation signs yields

$$\mathcal{L}\left\{X(n), -\frac{M_a}{2}\right\} = e^{\gamma\sqrt{n}}\rho^n n^\alpha$$

$$\cdot\left\{\sum_{s=0}^{\infty}\left[\sum_{j=0}^{s}\left(\rho^2\sum_{l=j}^{s}\frac{1+(-1)^{l-j}}{2}2^{\frac{l-j}{2}}\binom{\alpha-j/2}{(l-j)/2}G^{(2)}_{s-l}(\gamma)\right.\right.\right.$$

$$\left.\left.+\frac{1+(-1)^{s-j}}{2}b^*_{\frac{s-j}{2}}\right)c_j\right]n^{-\frac{s}{2}}$$

$$+\rho\sum_{s=0}^{\infty}\left[\sum_{j=0}^{s}\left(\sum_{t=0}^{s-j}\frac{1+(-1)^t}{2}a^*_{\frac{t}{2}}\right.\right.$$

$$(4.49)\qquad \left.\left.\left.\cdot\sum_{q=j+t}^{s}\frac{1+(-1)^{q-t-j}}{2}\binom{\alpha-j/2}{(q-t-j)/2}G^{(1)}_{s-q}(\gamma)\right)c_j\right]n^{\frac{K-s}{2}}\right\}.$$

Substituting (4.46) and (4.47) into (4.49), changing variables and then equating coefficients of $n^{-s/2}$ to zero, we obtain

$$\sum_{j=0}^{s}\left\{\rho^2\sum_{l=0}^{s-j}\frac{1+(-1)^l}{2}2^{\frac{l}{2}}\binom{\alpha-j/2}{l/2}G^{(2)}_{s-l-j}(\gamma)\right.$$

$$(4.50)\qquad \left.+\frac{1+(-1)^{s-j}}{2}\sum_{l=0}^{(s-j)/2}(-1)^l\binom{-M_b/2}{l}b_{\frac{s-j}{2}-l}\right\}c_j = 0,$$

for $s = 0, 1, \ldots, -K-1$, and

$$\sum_{j=0}^{s}\left\{\rho^2\sum_{l=0}^{s-j}\frac{1+(-1)^l}{2}2^{\frac{l}{2}}\binom{\alpha-j/2}{l/2}G^{(2)}_{s-l-j}(\gamma)\right.$$

$$(4.51)\qquad \left.+\frac{1+(-1)^{s-j}}{2}\sum_{l=0}^{(s-j)/2}(-1)^l\binom{-M_b/2}{l}b_{\frac{s-j}{2}-l}\right\}c_j$$

$$+\rho\sum_{j=0}^{s+K}\left\{\sum_{t=0}^{s+K-j}\frac{1+(-1)^t}{2}a_{\frac{t}{2}}\right.$$

$$\cdot \sum_{q=0}^{s+K-j-t} \frac{1+(-1)^q}{2} \binom{\alpha - j/2}{q/2} G^{(1)}_{s+K-q-j-t}(\gamma) \Bigg\} c_j = 0,$$

$$s = -K, -K+1, \ldots .$$

Since $G_0^{(2)}(\gamma) = 1$ and $c_0 = 1$, (4.50) for $s = 0$ reduces to the characteristic equation (4.16). Similarly, the determination of parameters γ and α are dependent on K. When $K = -1$, γ and α should necessarily satisfy (4.51) for $s = 1$ and $s = 2$ which, since $G_1^{(2)}(\gamma) = \gamma$, $G_0^{(1)}(\gamma) = 1$, $G_2^{(2)}(\gamma) = \frac{1}{2}\gamma^2$, $G_1^{(2)}(\gamma) = \gamma$ and $G_1^{(1)}(\gamma) = \frac{1}{2}\gamma$ are, respectively,

$$(\rho^2\gamma + \rho a_0)c_0 + (\rho^2 + b_0)c_1 = 0$$

and

$$\left(\frac{1}{2}\rho^2\gamma^2 + 2\rho^2\alpha + b_1 + \frac{M_b b_0}{2} + \frac{1}{2}\rho a_0 \gamma\right) c_0$$
$$+ \left(\rho^2\gamma + \rho a_0\right) c_1 + \left(\rho^2 + b_0\right) c_2 = 0.$$

In view of (4.16), we obtain successively

$$\gamma = -\frac{a_0}{\rho} \tag{4.52}$$

and

$$\alpha = \frac{b_1}{2b_0} + \frac{M_b}{4}. \tag{4.53}$$

As $b_0 \neq 0$ and $a_0 \neq 0$, we find that ρ, γ do not vanish.

Finally we note that the coefficients of c_s, c_{s-1} and c_{s-2} in (4.51) for $s > 2$

are $\rho^2 + b_0$, $\rho^2\gamma + \rho a_0$ and

$$\rho^2\left[\frac{1}{2}\gamma^2 + 2\alpha - (s-2)\right] + b_1 + \frac{M_b}{2}b_0 + \frac{1}{2}\gamma\rho a_0,$$

respectively. In view of (4.16), (4.52) and (4.53), it is easy to see that the first two of these are zero and the last is $-\rho^2(s-2)$. Hence, (4.51) can be written as

$$\begin{aligned} c_{s-2} = \frac{1}{\rho^2(s-2)} &\left\{ \sum_{j=0}^{s-3} \left[\rho^2 \sum_{l=0}^{s-j} \frac{1+(-1)^l}{2} 2^{\frac{l}{2}} \binom{\alpha - j/2}{l/2} G^{(2)}_{s-l-j}(\gamma) \right.\right. \\ &+ \frac{1+(-1)^{s-j}}{2} \sum_{l=0}^{(s-j)/2} (-1)^l \binom{-M_b/2}{l} b_{\frac{s-j}{2}-l} \\ &+ \rho \sum_{t=0}^{s-1-j} \frac{1+(-1)^t}{2} a_{\frac{t}{2}} \\ &\left.\left. \cdot \sum_{q=0}^{s-1-j-t} \frac{1+(-1)^q}{2} \binom{\alpha - j/2}{q/2} G^{(1)}_{s-1-q-j-t}(\gamma) \right] c_j \right\}, \\ & s = 3, 4, \ldots. \end{aligned} \tag{4.54}$$

The coefficients c_1, c_2, $\ldots$ in (4.17) are now determined recursively by (4.54), and $c_0 = 1$. In particular, we have

$$\begin{aligned} c_1 &= \frac{1}{6}\gamma^3 - \frac{1}{2}\gamma + 2\gamma\alpha + \frac{1}{\rho}\left(\frac{1}{8}\gamma^2 a_0 + \alpha a_0 + a_1\right), \\ c_2 &= \frac{1}{2\rho^2}\left\{\left[\rho^2\left(\frac{1}{24}\gamma^4 - \frac{1}{2}\gamma^2 + \alpha\gamma^2 + 2\alpha^2 - 2\alpha\right) + b_2 + \frac{1}{8}M_b(M_2+2)b_0\right.\right. \\ &\quad \left. \frac{1}{2}M_b b_1 + \rho\left(\left(\frac{1}{48}\gamma^3 - \frac{1}{8}\gamma\right)a_0 + \frac{1}{2}\alpha\gamma a_0\right) + \frac{1}{2}\rho\gamma a_1\right] \\ &\quad \left.\left[\rho^2\left(\frac{1}{6}\gamma^3 - \frac{1}{2}\gamma + (2\alpha - 1)\gamma\right) + \rho\left(\frac{1}{8}\gamma^2 + \alpha - \frac{1}{2}\right)a_0 + \rho a_1\right] c_1\right\}. \end{aligned}$$

When $K < -1$, γ should necessarily satisfy (4.50) for $s = 1$ rather than (4.51). We find that when $s = 1$, (4.50) becomes, in view of (4.16) and

$G_1^{(2)}(\gamma) = \gamma,$

$$\rho^2 \gamma c_0 = 0.$$

This implies $\gamma = 0$ as $\rho \neq 0$. We note, from (2.7) in Chapter 2, that $G_s^{(\mu)}(0) = 0$, $\mu = 1, 2$ for allpositive integers s except for $G_0^{(\mu)}(0) = 1$, $\mu = 1, 2$. Therefore (4.50) and (4.51) may be simplified to

$$\sum_{j=0}^{s} \frac{1+(-1)^{s-j}}{2} \left\{ \rho^2 2^{\frac{s-j}{2}} \binom{\alpha - j/2}{(s-j)/2} + \sum_{l=0}^{(s-j)/2} (-1)^l \binom{-M_b/2}{l} b_{\frac{s-j}{2}-l} \right\} c_j = 0, \tag{4.55}$$
$$s = 2, 3, \ldots, -K-1,$$

and

$$\sum_{j=0}^{s} \frac{1+(-1)^{s-j}}{2} \left\{ \rho^2 2^{\frac{s-j}{2}} \binom{\alpha - j/2}{(s-j)/2} + \sum_{l=0}^{(s-j)/2} (-1)^l \binom{-M_b/2}{l} b_{\frac{s-j}{2}-l} \right\} c_j \tag{4.56}$$
$$+\rho \sum_{j=0}^{s+K} \frac{1+(-1)^{s+k-j}}{2} \cdot \left\{ \sum_{t=0}^{s+K-j} \frac{1+(-1)^t}{2} a_{\frac{t}{2}} \binom{\alpha - j/2}{(s+K-j-t)/2} \right\} c_j = 0,$$
$$s = -K, -K+1, \ldots.$$

Noting that K is odd implies that $K \leq -3$, α should be determined from (4.55) with $s = 2$ which is

$$\left(2\rho^2 \alpha + b_1 + \frac{M_b b_0}{2} \right) c_0 + (\rho^2 + b_0) c_2 = 0.$$

In view of (4.16), α is the same as (4.53). We find that the coefficients of c_j in (4.55) vanish for such j that $s-j$ is odd, and the coefficients of c_s and c_{s-2} are ρ^2+b_0 and

$$2\rho^2\alpha + b_1 + \frac{M_b b_0}{2} - \rho^2(s-2),$$

and, consequently, equal to zero and $-\rho^2(s-2)$ respectively. Hence we have the recurrence relation for c_{s-2},

$$\begin{aligned} c_{s-2} = {} & \frac{1}{\rho^2(s-2)} \sum_{j=0}^{s-4} \frac{1+(-1)^{s-j}}{2} \\ & \cdot \left\{ \rho^2 2^{\frac{s-j}{2}} \binom{\alpha - j/2}{(s-j)/2} + \sum_{l=0}^{(s-j)/2} (-1)^l \binom{-M_b/2}{l} b_{\frac{s-j}{2}-l} \right\} c_j, \\ & s = 4, 5, \ldots, -K-1, \end{aligned} \tag{4.57}$$

where the coefficient of c_{s-3} vanishes, and $c_1 = 0$ if $K < -3$. Thus, by the method of induction, we find the fact that all $c_{s-2} = 0$ when s is odd but not larger than $-K-1$.

By a similar method applied to (4.56), the coefficients of c_s c_{s-1} and c_{s-2} are zero, zero and $-\rho^2(s-2)$. Hence from this follows

$$\begin{aligned} c_{s-2} = {} & \frac{1}{\rho^2(s-2)} \left\{ \sum_{j=0}^{s-4} \frac{1+(-1)^{s-j}}{2} \right. \\ & \cdot \left[\rho^2 2^{\frac{s-j}{2}} \binom{\alpha - j/2}{(s-j)/2} + \sum_{l=0}^{(s-j)/2} (-1)^l \binom{-M_b/2}{l} b_{\frac{s-j}{2}-l} \right] c_j \\ & + \rho \sum_{j=0}^{s+K} \frac{1+(-1)^{s+K-j}}{2} \\ & \left. \cdot \left[\sum_{t=0}^{s+K-j} \frac{1+(-1)^t}{2} a_{\frac{t}{2}} \binom{\alpha - j/2}{(s+K-j-t)/2} \right] c_j \right\}, \end{aligned} \tag{4.58}$$

$$s = -K, -K+1, \ldots,$$

where the coefficient of c_{s-3} in the first summation vanishes.

Thus we have fulfilled all the derivations of formal solutions.

4.4 Examples

Example 1 *Behavior of the number of involutions $t(n)$ on $\{1, 2, \ldots, n\}$, which satisfies*

$$t(n+2) - t(n+1) - (n+1)t(n) = 0. \tag{4.59}$$

Comparing (4.59) to the standard form (4.1) gives

$$M_a = 0, \qquad M_b = 1,$$

$$a(n) = -1; \qquad a_0 = -1, \qquad a_n = 0, \quad n \geq 1,$$

$$b(n) = -\left(1 + \frac{1}{n}\right); \qquad b_0 = -1, \qquad b_1 = -1, \; b_n = 0, \quad n \geq 2.$$

As $K = -1$, the formal solutions are given by inserting (4.18) into (4.17), the parameters λ, ρ, γ, α and coefficients c_s, $s = 1, 2, \ldots$, may be calculated by recurrence relations (4.15), (4.16), (4.52), (4.53), and (4.54). We find $\lambda_i = -\frac{1}{2}$, $\rho_i = \pm 1$, $\gamma_i = \pm 1$, and $\alpha_i = \frac{3}{4}$, $i = 1, 2$. The two subnormal series solutions of (4.59) are

$$t_1(n) = \sqrt{(n-2)!}\, e^{\sqrt{n}} n^{\frac{3}{4}} \left\{1 + \frac{7}{24} n^{-\frac{1}{2}} - \frac{743}{1152} n^{-1} + O\left(n^{-\frac{3}{2}}\right)\right\}$$

and

$$t_2(n) = (-1)^n \sqrt{(n-2)!} e^{-\sqrt{n}} n^{\frac{3}{4}} \left\{1 - \frac{7}{24} n^{-\frac{1}{2}} - \frac{743}{1152} n^{-1} + O\left(n^{-\frac{3}{2}}\right)\right\}.$$

By making use of the *Stirling* formula

$$(4.60) \qquad (n-2)! = (2\pi)^{\frac{1}{2}} e^{-n} n^n n^{-\frac{3}{2}} \left\{1 + \frac{13}{12} n^{-1} + \frac{313}{288} n^{-2} + O\left(n^{-3}\right)\right\},$$

we obtain

$$t_1(n) = e^{\sqrt{n}} n^{\frac{n}{2}} e^{-\frac{n}{2}} \left\{1 + \frac{7}{24} n^{-\frac{1}{2}} - \frac{119}{1152} n^{-1} + O\left(n^{-\frac{3}{2}}\right)\right\},$$

and

$$t_2(n) = (-1)^n e^{-\sqrt{n}} n^{\frac{n}{2}} e^{-\frac{n}{2}} \left\{1 - \frac{7}{24} n^{-\frac{1}{2}} - \frac{119}{1152} n^{-1} + O\left(n^{-\frac{3}{2}}\right)\right\}.$$

Since $t(n)$ is monotone, increasing to ∞, we have

$$t(n) = K t_1(n),$$

for some constant $K > 0$. The same results can be seen in *J. Wimp* and *D. Zeilberger* [13].

Example 2 *Construct formal solutions for the equation*

$$(4.61) \qquad y_{n+1}(x) - (2n+1) x y_n(x) - y_{n-1}(x) = 0.$$

Changing the equation into the form (4.1) by means of replacing n by $n+1$ gives

$$M_a = 1, \qquad M_b = 0,$$

$$a(n) = -(2x + 3xn^{-1}); \qquad a_0 = -2x, \quad a_1 = -3x, \quad a_n = 0 \quad n \geq 2,$$

$$b(n) = -1; \qquad b_0 = -1, \qquad b_n = 0, \quad n \geq 1.$$

As $K = 2$, when $x \neq 0$, the formal series solutions are given by inserting (4.14) into (4.18). From formulas (4.9), (4.12), (4.31), (4.33), (4.13), (4.42) and (4.44) we have $\lambda_1 = -1$, $\rho_1 = 2x$, $\alpha_1 = \frac{3}{2}$, $\lambda_2 = 1$, $\rho_2 = -\frac{1}{2x}$, $\alpha_2 = -\frac{5}{2}$. The two formal series solutions of (4.61) are

$$y_n^1(x) = (n-2)!(2x)^n n^{\frac{3}{2}}$$
$$\cdot\left\{1 - \left(\frac{9}{8} + \frac{1}{4}x^{-2}\right)\right) n^{-1} + \left(\frac{9}{64}x^2 + \frac{21}{128} + \frac{1}{64}x^{-2}\right) n^{-2} + O\left(n^{-3}\right)\right\},$$

and

$$y_n^2(x) = \frac{(-1)^2}{(n-2)!(2x)^n} n^{-\frac{5}{2}}$$
$$\cdot\left\{1 + \left(\frac{5}{8} + \frac{1}{4}x^{-2}\right)\right) n^{-1} + \left(\frac{105}{128} - \frac{7}{32}x^{-2} + \frac{1}{32}x^{-4}\right) n^{-2} + O\left(n^{-3}\right)\right\}.$$

By making use of the *Stirling* formula (4.60), we have

$$Y_n^1(x) = e^{-n}n^n(2x)^n$$
$$\cdot\left\{1 - \left(\frac{1}{24} + \frac{1}{4}x^{-2}\right) n^{-1} + \left(\frac{9}{64}x^2 + \frac{111}{3456} - \frac{49}{192}x^{-2}\right) n^{-2} + O\left(n^{-3}\right)\right\},$$

and

$$Y_n^2(x) = (-1)^n e^n n^{-n}(2x)^{-n} n^{-4}$$
$$\cdot\left\{1 + \left(-\frac{11}{24} + \frac{1}{4}x^{-2}\right)\right) n^{-1} + \left(-\frac{65}{1152} - \frac{47}{96}x^{-2} + \frac{1}{32}x^{-4}\right) n^{-2} + O\left(n^{-3}\right)\right\}.$$

Example 3 *Find the formal solutions for the equation*

$$C^{(a)}_{n+1}(x) + (n + a - x)C^{(a)}_n(x) + anC^{(a)}_{n-1}(x) = 0.$$

Replacing n by $n + 1$ gives

$$M_a = 1, \qquad M_b = 1,$$

$$a(n) = 1 + (1 + a - x)n^{-1}; \qquad a_0 = 1, \qquad a_1 = 1 + a - x,$$

$$b(n) = a + an^{-1}; \qquad b_0 = a, \qquad b_1 = a,$$

and $a_n = b_n = 0$ when $n \geq 2$. As $K = 1$, we can use formulas (4.9), (4.12), (4.29), (4.30) to form one solution and (4.9), (4.13) (4.40) and (4.41) to form another solution, i.e.,

$$\begin{aligned} C^{(a)}_{n,1}(x) &= (n-2)!(-1)^n n^{1-x} \\ &\cdot \left\{1 + \left[\frac{1}{2}x^2 + \left(a + \frac{1}{2}\right)x + a - 1\right]n^{-1} + O\left(n^{-2}\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} C^{(a)}_{n,2}(x) &= (-a)^n n^x \\ &\cdot \left\{1 - \left[\frac{1}{2}x^2 + \left(a - \frac{1}{2}\right)x\right]n^{-1} + O\left(n^{-2}\right)\right\}. \end{aligned}$$

In terms of the *Stirling* formula (4.60), $C^{(a)}_{n,1}(x)$ can be written as

$$\begin{aligned} C^{(a)}_{n,1}(x) &= (-1)^n e^{-n} n^n n^{-(\frac{1}{2}+x)} \\ &\cdot \left\{1 + \left[\frac{1}{2}x^2 + \left(a + \frac{1}{2}\right)x + a + \frac{1}{12}\right]n^{-1} + O\left(n^{-2}\right)\right\}. \end{aligned}$$

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