A Model of Dilute Magnets

by

Javid Ashraff

A thesis
presented to the University of Manitoba
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ABSTRACT

This work is concerned with modelling a dilute ferromagnet and antiferromagnet using an exact fractal known as the Viscek snowflake. Employing real-space renormalization group techniques we derive the recursion relations for both the ferromagnet and the antiferromagnet and obtain the spectral dimension characterizing the form of the low frequency density of states. We find that the spectral dimension of the antiferromagnet is twice that of the ferromagnet, as is the case for regular Euclidean lattices with integral dimension. In addition, both the ferromagnetic and antiferromagnetic density of states are calculated numerically using functional integral techniques and it is shown that the associated spectral dimensions obtained from ln-ln plots of the integrated density of states, are in agreement with those obtained from the corresponding fixed point analysis.

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Chapter I

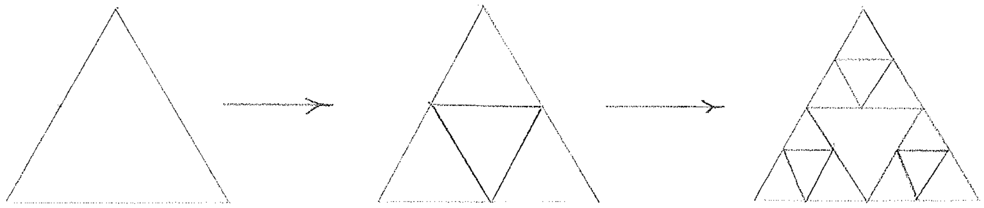
INTRODUCTION

"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line". This quotation, taken from Mandelbrot's most recent book [14] expresses the obvious inadequacies in the scope of the regular geometrical structures of Euclid to describe the 'real world'. In order to describe such shapes Mandelbrot has conceived and developed a new geometry of nature, for which the basic geometrical shapes are fractals. Thus we are more than justified in regarding such shapes as the fundamental objects in any lattice related study in physics.

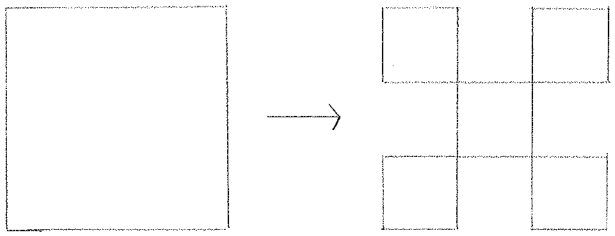
Fractals are fully explicitly described geometrical shapes of fractional dimensionality that may be viewed as hybrids between ordinary Euclidean structures such as lines and planes [7]. It is necessary to distinguish between two basic types of fractals

1. deterministic or exact fractals
2. random or statistical fractals

The difference in these two types of fractals lie in their construction and their invariance properties under dilation transformations. Deterministic fractals, such as the Sierpinski gasket or the Viscek snowflake, are exactly self similar, a direct result of the very proceedure used to generate them. Typically they are constructed by starting with some given shape called an initiator (a segment,square,triangle,etc.). The decoration process then proceeds by replacing the initiator by N copies of a shape (possibly the same shape as the initiator itself) each scaled down by a factor b. For example, the Sierpinski gasket which formed the basis of the study in reference [4] is constructed by dividing an equilateral triangle of side a into four smaller triangles of side a/2. The central triangle is then discarded and the procedure is repeated in the three remaining triangles ad infinitum. Pictorially this decoration process corresponds to the following situation:



The basis for the present study is yet another exact fractal known as the Viscek snowflake, which is constructed by taking a square of side a and dividing it into nine smaller squares of side a/3. Of these nine smaller squares four are then discarded, the process then being repeated in each of the five remaining squares, giving rise to the following structure:



Having described how such lattices are generated, we are now faced with the task of defining for them an effective dimensionality (possibly fractional). The most important constraint of any (generalized) definition of dimensionality must be that it agree with the well known integral values obtained for regular Euclidean lattices. The first attempt to define such a generalized dimensionality, applicable to the above mentioned type of fractal lattices, appears to be due to Mandelbrot [13] who adopts the Hausdorff definition whereby d, the fractal dimension, is defined by the relation

$$N = b^{\bar{d}} \tag{1.1-1}$$

or

$$\bar{d} = \ln(N)/\ln(b) \tag{1.1-2}$$

Thus, in the case of the Sierpinski gasket where each stage of the decoration process corresponds to replacing all upward pointing triangles by three copies of the same triangle scaled down by a factor of two, we have $\bar{d} = \ln(3)/\ln(2)$.

Similarly, for the Viscek snowflake, every stage involves replacing only five of the original nine squares, each scaled down by a factor of three, and so $\bar{d} = \ln(5)/\ln(3)$ A short time later, Dhar [3] proposed an alternate definition of effective dimensionality based upon the exponent characterizing the low frequency density of states. Dhar shows that for an assembly of harmonic oscillators the low frequency density of states, in d dimensions (d integral) takes the form

$$\rho(\omega) \sim \omega^{d}$$

and proposes this to be a defining relation even for irregular lattices where d could be non-integral. As we shall see, such an exponent is none other than the spectral dimension $\tilde{d}$, introduced by Alexander and Orbach five years later. In contrast to exact fractals, random or statistical fractals are constructed by some random process, such as a fractional Brownian motion, and are only self similar in a statistical sense.

It is generally accepted now, that a complete characterization of a fractal structure requires the specification of a minimum of three dimensionalities, the embedding Euclidean dimension,d, the fractal or Hausdorff dimension, $\bar{d}$, a measure of how much volume the object occupies, and the spectral dimension, $\tilde{d}$ [1,19], an exponent characterizing the low energy density of states of the elementary excitations in this geometry. These three dimensionalities satisfy the inequality

$$\tilde{d} < \bar{d} < d \qquad (1.1\text{-}3)$$

As one passes from the self-similar regime which exhibits dilational invariance to the Euclidean regime which exhibits translational invariance, $\tilde{d}$ and $\bar{d}$ both collapse to d, the ordinary Euclidean dimension, a situation which is often construed to correspond to a degenerate case [19].

In recent years much work has been done on modelling disordered systems using these geometric objects of fractional dimension. Fractals were first introduced into physics by Stanley [22] within the context of percolation, and since then have served as models for such inhomogenous systems as metal insulator thin films, gels and dilute spin systems [24]. Not only are such shapes non- translationally invariant and self similar, a property akin to all random systems, but in many cases can be treated exactly using real- space renormalization group techniques. The most extensively studied fractal of late is the Sierpinski gasket, an exact fractal which has served as a model for the backbone of the infinite cluster at the percolation threshold (c.f. reference [23] for a review of percolation), a description believed to be valid due to the closeness in the fractal dimensionalities of these two structures. In addition this same object has served as the basis for an exact study of the so-called phonon-fracton crossover [20]. Such a study was motivated by recent effective medium calculations of Orbach and co-workers [15] of the dynamical properties of percolating networks. They found that in the long wavelength or phonon regime, the vibrational density of states was proportional to ω^{d-1} whereas at short length scales $\rho(\omega)$ became proportional to $\omega^{\tilde{d}-1}$ where $\tilde{d}$ is the spectral dimension. Furthermore, it was found that there was a sharp crossover between these two regions, indicating the existence of a qualitatively new type of excitation at small length scales, subsequently termed fractons. Since the calculation predicting these excitations was based upon an approximation, a natural question to ask was whether or not the presence of these fractons is simply a manifestation of the particular approximation used. The work of reference [20] addressed such a question. Using a Sierpinski gasket embedded in a triangular lattice as a model of a disordered system, they calculated the vibrational density of states in both regions and found no such sharp cross-over.

A natural extension of such a calculation is a generalization to magnetic systems which will allow more overlap with experiment. Orbach and Yu [33,34] have recently performed an effective medium calculation on both a percolating ferromagnet and antiferromagnet. They found that in the case of the ferromagnet, the low frequency spin wave density of states obeyed the scaling law $\rho(\omega) \sim \omega^{d/2-1}$ in the magnon regime but crossed over sharply to fracton behaviour where the density of states scaled like $\rho(\omega) \sim \omega^{\tilde{d}/2 - 1}$. In particular their results indicate that in the fracton region the density of states scales like $\rho(\omega) \sim \omega^{-.5}$ independent of dimension, leading to a spectral dimension of 1. For the antiferromagnet they found that for small length scales the density of spin wave states behaved like $\rho(\omega) \sim \omega^{d-1}$ in the magnon regime but crossed over smoothly to the fracton regime at smaller length scales where the density of states scaled like $\rho(\omega) \sim \omega^{\tilde{d}-1}$. Furthermore they found that in three dimensions the fracton density of states was constant indicating a spectral dimension of 1, as in the case of the ferromagnet.

The aim of this present work is to propose as a model for a dilute magnetic system, the Viscek snowflake, an exact fractal. This structure first appeared in the literature within the context of aggregation phenomena where it was proposed as a model for diffusion controlled deposition on a surface [29]. As a model for a dilute magnetic system, the Viscek snowflake has two favourable attributes. Firstly, its ability to support both ferromagnetic and antiferromagnet order makes it an ideal structure with which to compare experimental results and secondly, the presence of both first and second neighbour interactions makes it a much more realistic model of a dilute magnet than that proposed by Orbach and Yu whose calculations only accounted for nearest neighbour exchange. In addition it is shown that the low frequency ferromagnetic or antiferromagnetic spin wave density of states obeys the same scaling laws as obtained by Orbach and Yu.

This thesis is organized as follows. Chapter 2 contains a brief review of some basic results and techniques which form the basis of the calculations presented in the later chapters. In particular we include a brief discussion of real space renormalization group techniques and their application to the solution of various problems on fractal lattices. In chapter 3 we present the phenomena to be studied and to illustrate the technique of real space rescaling we give a detailed account of the solution of an exactly solvable model, namely a linear chain of ferromagnetic spins. We then proceed to define a model of a dilute ferromagnet and and to calculate the associated eigenvalue spectrum and the exponent characterizing the form of the low frequency density of states. Chapter 4 contains a similar analysis to that already treated in chapter 3 except now we consider the case of an antiferromagnet. Finally in chapter 5 we present our conclusions and indicate the directions in which future research will follow.

Chapter II

SOME PRELIMINARY RESULTS

2.1 REAL SPACE RESCALING

A theory of thermal critical phenomena was put forth in the early 1970's by K.G. Wilson [31,32], to provide a systematic method for calculating the various critical exponents characterizing the power law behaviour of the thermodynamic functions near the critical point.

The difficulty in describing a system near its critical point lies in the fact that near this point there are a large number of degrees of freedom correlated over a multitude of length scales [10]. The spatial extent over which these degrees of freedom are coupled is called the correlation length and in fact, right at the critical point this characteristic length diverges. The divergence of the correlation length has a very profound impact on the physics of the system since it is this (divergent) length which induces the singular behaviour in the Helmholtz free energy and hence in all other thermodynamic variables. Furthermore, it is this very same idea of a divergent correlation length which forms the basis for the renormalization group analysis of Wilson.

At the critical point the system is invariant under dilations, or equivalently, the system is self similar on all length scales, while in the vicinty of the critical point this dilational symmetry is broken, to an extent which increases as the system moves further away from criticality [16]. The reason for this dilational invariance at the critical point is related to the fact that the correlation length ξ, the only relevant length in the problem, is infinite; to see how a diverging correlation length can give rise to self similarity consider the following argument. Suppose the system is critical and we perform some transformation which corresponds to increasing the lattice spacing by some factor b. Now the only length with which the system has to compare such a change is the correlation length ξ, which is infinite, and so the system does not notice the change. In other words, at Tc the system is insensitve to changes in length scale since the only other length with which to compare this change has become infinite.

Length scaling techniques generally involve exploring the consequences of changing some reference length in the system, while maintaining the properties of the system on some gross scale [25]. For example one can visualize increasing the lattice spacing by some scaling factor b while maintaining the absolute correlation length ξ. Applying such a transformation recursively then, corresponds to increasing the lattice spacing (from 'a') up to about the size of the correlation length (assumed large but nevertheless finite) at which point the problems encountered initially with the large range of length scales have been eliminated [27]. Such methods were originally designed within the context of thermal critical phenomena to exploit the diverging correlation length and associated self similarity. The formalism, however, can equally well be applied to obtain the non - critical dynamic properties of fractals which are intrinsically self similar (exact or statistically) and may be considered to have an infinite correlation length by definition.

The Hamiltonian $H[\{P(i)\}]$ describing the dynamics of a system is a function of a set of parameters $\{P(i)\}$, which are characteristic of the particular system under consideration. This set of parameters defines a point in an abstract space called parameter space. Under successive applications of a renormalization group transformation $R(b)$ of scaling factor b these parameters are modified, which can be visualized as describing a trajectory in this space. Symbolically, this translates into

$$\{P'(i)\} = R(b)\ \{P(i)\} \tag{2.1-1}$$

Such a transformation $R(b)$ then defines a set of recursion relations between the various parameters $P(i)$. Furthermore, these parameters will in general possess certain special values $\{P^*(i)\}$, called fixed points, which remain invariant under the length scaling transformation R(b). This invariance of {P(i)} under R(b) is equivalent to the statement that the system (which is described by {P(i)}) is insensitive to changes in length scale, ie. it is self similar. This however is precisely the situation which arises at the critical point and thus it is clear that the fixed points of our renormalization group transformation are intimately related to the critical point. In the case of simple one dimensional systems the fixed points do in fact correspond to the critical point, whereas in more complicated systems in higher dimensions the fixed point only determines a point on a critical manifold, the actual critical point then lying somewhere on this critical surface.

Such length scaling transformations can be implemented in either momentum space as in Wilson's original formulation, or in position space, which for our purposes will be the only suitable method. The simplest real space renormalization scheme is that of decimation, which consists of eliminating a fraction $N' = (b - 1)/b$ of the original N sites, where b is called the scaling factor, the result being a new lattice with spacing $a' = ba$. In order that the physics be preserved we impose the restriction of covariance on the equations of motion, as determined from the specific form of the Hamiltonian. Normally one only works with the linearized form of the recursion relations, valid only near the critical point. In thermal critical phenomena this allows a determination of the critical exponents whereas in the case of fractals it will provide a basis for the calculation of the exponent characterizing the low frequency density of states. Linearizing our recursion relations

$$P' = R(b)\ P \qquad (2.1\text{-}2)$$

about the fixed point P*, gives [7]

$$[\ P'(\alpha) - P^*(\alpha)\ ] = \sum_{\alpha} T(\alpha,\beta)\ [\ P(\beta) - P^*(\beta)\ ] \qquad (2.1\text{-}3)$$

where

$$T(\alpha,\beta) = [dP'(\alpha)/dP(\beta)]\ P^* \qquad (2.1\text{-}4)$$

Let $\lambda(i)$ denote an eigenvalue of $T(\alpha,\beta)$ and $\phi(i,\alpha)$ the corresponding eigenvector. ie.

$$\sum_{\alpha} T(\alpha,\beta)\ \phi(i,\alpha) = \lambda(i)\ \phi(i,\beta) \qquad (2.1\text{-}5)$$

We now define normal coordinates u(i), also called Wegner's scaling fields [30] by

$$u(i) = \sum_{\alpha} \phi(i,\alpha)\ [\ P(\alpha) - P^*(\alpha)\ ] \qquad (2.1\text{-}6)$$

noting that at the fixed point they vanish identically. It then follows that

$$\begin{aligned} u'(i) &= \sum_{\alpha} \phi(i,\alpha)\ [\ P'(\alpha) - P^*(\alpha)\ ] \\ &= \lambda(i)\ u(i) \end{aligned} \qquad (2.1\text{-}7)$$

There are three cases to consider

1. $|\lambda(i)| > 1$, u(i) relevant
2. $|\lambda(i)| < 1$, u(i) irrelevant
3. $|\lambda(i)| = 1$, u(i) marginal

of which case 1 will be our only concern. To see how such a formalism provides us with exponents characterizing the divergence of various thermodynamic quantities consider the following: Since the scaling field is identically zero at the critical point, Tc, we assume that away from this point u(i) is proportional to (T-Tc). Now, it is know experimentally that as one approaches the critical point, the correlation length ξ diverges like $(T-Tc)^{-\nu}$, where ν is a critical exponent. The rescaled system, away from the critical point Tc, has a new temperature T' and a new correlation length ξ'. Taking the ratio of ξ to ξ' and using the fact that under the renormalization group transformation $\xi'=\xi/b$ and $u(i)'=\lambda\ u(i)$ it follows that $b=\lambda^{\nu}$, and so $\nu = \ln(b)/\ln(\lambda)$. It may happen that a particular matrix $T(\alpha,\beta)$ has more than one eigenvalue which is relevant. In this case we will consider only the largest eigenvalue since it follows from (2.1-7) that the largest eigenvalue corresponds to moving away from criticality the fastest under successive renormalization group transfomations.

2.2 GREEN FUNCTIONS AND FUNCTIONAL INTEGRALS

The spectral properties of a system are most easily obtained if one works within the functional integral formalism. By spectral properties we shall have in mind the density of states(cf. section 2.3)

The Green function operator is defined to be a solution of the following operator equation

$$(zI - H) G = I \tag{2.2-1}$$

where I is the identity operator and $z = E + i\eta$ is the (complex) energy. We shall eventually be interested in calculating the average density of states per site which is related to the imaginary part of the trace of the following diagonal matrix element divided by N:

$$G(i,i) = < i| (zI - H)^{-1} |i > \tag{2.2-2}$$

It will prove useful, in discussing the Green function technique, to make reference to a particular model Hamiltonian , which, for historical purposes we choose to be of the tight-binding form (TBH), commonly used to describe simple electronic systems [5]. Such a Hamiltonian is rather universal, in the sense that by re-interpreting the parameters defining the TBH we will be able to describe a wide range of physical systems including lattice vibrations, Ising Glauber dynamics and both ferromagnetic as well as antiferromagnetic spin wave dynamics. The TBH is formally given by

$$H = \sum_{j} |j> \epsilon(j) < j| + \sum_{j,k} |j> V(j,k) < k| \tag{2.2-3}$$

where $|j>$ represents an atomic-like orbital centered at site j and $\epsilon(j)$ and $V(j,k)$ represent the diagonal and off-diagonal matrix elements of H, respectively, in this (Wannier) basis. Inserting the identity operator into (1), between (zI - H) and G, in the form

$$I = \sum_{k} |k><k| \tag{2.2-4}$$

and taking matrix elements between states $|l>$ and $|m>$ on both sides leads to the following "equation of motion" description of the matrix elements of the Green function operator for a Hamiltonian of the tight-binding form:

$$(z - \epsilon(l)) G(l,m) = \delta(l,m) + \sum_{k} V(l,k) G(k,m) \tag{2.2-5}$$

In the case of a one dimensional ordered system the decimation scheme of real space renormalization can be applied exactly, in which case the calculation of the site averaged Green function

$$<G> = (1/N) \sum_{i} G(i,i) \tag{2.2-6}$$

involves only the calculation of one G(i,i), say G(0,0), since translational invariance implies

$$G(i,i) = G(j,j) \quad \text{for all } i \text{ and } j \ .$$

On the other hand, in the case of a fractal structure, which is not translationally invariant, the sites are not all equivalent and so a similar approach based on the calculation of all the G(i,i) is clearly impractical. Thus we seek an alternate method which will enable us to calculate <G> directly, without having to calculate each G(i,i) individually. Such a method is embodied in the so-called functional integral formalism.

It is instructive, at this point, to make contact with the field of statistical mechanics where one encounters a similar problem. For example, in statistical mechanics one often wishes to calculate the average internal energy of a system. Now, the internal energy U of the system as a whole is determined by the energies e(i) associated with the various degrees of freedom of the constituent particles. Since the number of degrees of freedom is generally of the order of 10^{23}, a calculation of U based on

$$U = (1/N) \sum_i e(i) \qquad (2.2\text{-}7)$$

is clearly not feasible. However, by calculating the partition function Z(N,H), one can obtain U by performing a simple derivative of the logarithm of Z(N,H).

In complete analogy with statistical mechanics then, we define a quantity $\Phi(N,H;z)$ which is formally the equivalent of the Helmholtz free energy $F(N,H)$ [12]. However, whereas $F(N,H)$ is a strictly real-valued functional of its arguments, $\Phi(N,H;z)$ may, in general assume complex values [11].

$$\Phi(N,H;z) = \ln \Xi(N,H;z) \qquad (2.2\text{-}8)$$

where

$$\Xi(N,H;z) = \int Du \exp[\ i\ S(N,H;z)\ ] \qquad (2.2\text{-}9)$$

and

$$S(N,H;z) = (1/2)\ U^{\dagger}(\ zI - H\ )\ U \qquad (2.2\text{-}10)$$

and

$$\int Du = \prod_{j} \int_{-\infty}^{\infty} du(j) \qquad (2.2\text{-}11)$$

where the $u(j)$ are the expansion coefficients of U in our tight-binding basis.

We now claim that within this functional integral formalism a consistent definition of the Green function is

$$G(l,m) = i\ \frac{\int Du\ u(l)u(m) \exp[\ i\ S(N,H;z)\ ]}{\int Du \exp[\ i\ S(N,H;z)\ ]} \qquad (2.2\text{-}12)$$

To prove our claim we must show that such a definition of $G(l,m)$ is consistent with equation (2.2-5). Let

$$I(l,m) = i \frac{\int Du\ u(l)\ [d/du(m)]\ \exp[\ i\ S(N,H;z)\ ]}{\int Du\ \exp[\ i\ S(N,H;z)\ ]} \qquad (2.2\text{-}13)$$

First integrate the right hand side by parts. The surface term vanishes since the integrand is odd. We obtain then

$$I(l,m) = -\delta(l,m) \qquad (2.2\text{-}14)$$

Now we evaluate I(l,m) by performing the derivative within the integrand before performing the u(i) integrals. First note that from the definition (2.2-10) of S(N,H;z) and by simple matrix multiplication we have

$$\begin{aligned} S(N,H;z) &= (1/2) \sum_{j} u(j)^2\ (\ z - \epsilon(j)\ ) \\ &\quad - (1/2) \sum_{jk} u(j)\ u(k)\ V(j,k) \end{aligned} \qquad (2.2\text{-}15)$$

Thus we can write I(l,m) as

$$\begin{aligned} I(l,m) = i\ & \frac{\int Du\ u(l)\ u(m)\ [z - \epsilon(m)]\ \exp[\ i\ S(N,H;z)\ ]}{\int Du\ \exp[\ i\ S(N,H;z)\ ]} \\ - i\ & \frac{\int Du\ u(l)\ u(k)\ V(l,k)\ \exp[\ i\ S(N,H;z)\ ]}{\int Du\ \exp[\ i\ S(N,H;z)\ ]} \end{aligned} \qquad (2.2\text{-}16)$$

Equating (2.2-14) and (2.2-16) leads to the previous result of (2.2-5) if the definition (2.2-12) for G(l,m) is adopted.

In addition, we can obtain G(l,m) from $\Phi(N,H;z)$ by adding to S(N,H;z) the quantity ϕU

$$\phi = (\ \phi(1),\phi(2),\ldots,\phi(N)\ ) \qquad (2.2\text{-}17)$$

ϕ is commonly referred to as a ghost field and is inserted so we can differentiate with respect to it and in the end set $\phi = 0$. So

$$G(l,m) = \frac{d^2\Phi}{d\phi(l)d\phi(m)} \qquad (2.2\text{-}18)$$

Thus it is clear why $\Phi(N,H;z)$ is a generating function. Also, note that

$$\frac{d\Phi}{dz} = \frac{1}{2} \frac{\int Du\ u(j)^2 \exp[\ i\ S(N,H;z)\ ]}{\int Du \exp[\ i\ S(N,H;z)\ ]} \qquad (2.2\text{-}19)$$

and so we arrive at the important result

$$\langle G \rangle = 2(d\Phi/dz) \qquad (2.2\text{-}20)$$

Finally, the average density of states (DOS) is defined by

$$\rho(E,\eta) = (1/N\pi)\ \mathrm{Im}\ \langle G \rangle \qquad (2.2\text{-}21)$$

or using the result (2.2-20) it follows that

$$\rho(E,\eta) = (2/N\pi)\ \mathrm{Im}\ (d\Phi/dz) \qquad (2.2\text{-}22)$$

It should be noted that (2.2-22) only yields the true density of states in the limit η tending to zero. For η finite but small, we will still refer to $\rho(E,\eta)$ as the density of states, the value of η then determining the precision to which each eigenvalue can be obtained from the DOS [28].

Having described the formalism, we now proceed to illustrate how it may be implemented using real space techniques. Since our Green function is defined in terms of $\Xi(N,H;z)$, it is clear that a determination of $G(l,m)$ must begin with a calculation of $\Xi(N,H;z)$. The proceedure we adopt is not unlike the Wilson momentum space method used in critical phenomena where one evaluates the partition function in steps, successively integrating out the high momentum (short wavelength) components until one has accounted for all length scales up to the correlation length with having faced all the length scales at once. In the real space method we calculate $\Xi(N,H;z)$ by integrating over a fraction of the u(i)'s on the smallest length scale, progressively.

We begin by dividing up our lattice into two sublattices. Let $W = (w(1),w(2),...,w(N))$ represent the variables which will be integrated over during the initial decimation, and $V = (v(1),v(2),...,v(N))$ represent the variables which will remain unintegrated. Then, in this new basis $(zI - H)$ can be represented as

$$\begin{bmatrix} (zI-H_{11}) & -H_{12} \\ -H_{21} & (zI-H_{22}) \end{bmatrix} \qquad (2.2\text{-}23)$$

where $(zI-H_{11})$ only involves sites remaining and $(zI-H_{22})$ only involves sites which are to be eliminated. Then

$$S(N,H;z) = (1/2)\ \{\ V^{\dagger}(\ zI - H_{11}\ )\ V - V^{\dagger}H_{12}\ W\ \}$$
$$+(1/2)\ \{\ W^{\dagger}(\ zI - H_{22}\ )\ W - W^{\dagger}H_{21}\ V\ \}$$
$$(2.2\text{-}24)$$

Now we perform the following transformation (whose Jacobian is unity)

$$W' = W + (\ zI - H_{22}\ )^{-1}\ H_{21}\ V \qquad (2.2\text{-}25)$$

Then it follows that S(N,H;z) can be decomposed into the sum of two distinct parts, namely

$$S(N,H;z) = (1/2)\ [\ V^{\dagger}(\ zI - H'\ )\ V + W^{\dagger}(\ zI - H_{22}\ )\ W\ ]$$
$$(2.2\text{-}26)$$

where

$$H' = H_{11} + H_{12}(\ zI - H_{22}\ )^{-1}\ H_{21}\ . \qquad (2.2\text{-}27)$$

It should be noted that the transformation (2.2-25) defines a new Hamiltonian given by (2.2-27) which itself defines a set of recursion relations for the parameters appearing in the original Hamiltonian. Furthermore, this set of recursion relations is (and must be) identical to the set obtained by performing decimations on equation (2.2-5) directly. Using expression (2.2-26) for S(N,H;z) we now proceed to perform a partial sum in (2.2-9) by integrating over the variables w(j) and employing the identity

$$\int Du\ \exp[\ i(1/2)\ U\ A\ U\ ] = (2\pi i)^{N/2}\ /[\det(A)]^{1/2}$$
$$(2.2\text{-}28)$$

valid for any symmetric N by N matrix A [6,17]. It then follows that

$$\Phi(N,H;z) = \ln \{ \int Du \exp [(i/2) V^{\dagger} (zI - H') V] \}$$

$$+ (N'/2) \ln (2\pi i) - (1/2) \ln [\det (zI - H_{22})] \tag{2.2-29}$$

where N' is the number of sites being eliminated. We drop the constant term since it doesn't affect the density of states (which is related to a derivative of (2.2-29) with respect to z). The procedure outlined above can be summarized as

1. Divide the sites into two groups V and W
2. Perform the corresponding blocking of the Hamiltonian
3. Transform to new variables W'
4. Integrate over the w'(j)

Now this process is iterated until suitable convergence is obtained. It is clear that each time steps 1 to 4 are repeated we obtain a formally identical expression for $\Phi(N,H;z)$ (but with N scaled down by some finite fraction b) as well as a term proportional to

$$C(l) = \ln \det [zI - H_{22}^{(\ell)}] . \tag{2.2-30}$$

Thus as we iterate the transformation (2.2-27) n times we generate a finite series C(n) given by

$$C(n) = \sum_{l=1}^{n} C(l) \qquad (2.2\text{-}31)$$

In the limit n going to infinity this series converges due to the finite imaginary part, η, and having eliminated all the sites we then obtain $\Phi(N,H;z)$ trivially as

$$\Phi(N,H;z) = \lim_{n\to\infty} \sum_{l=1}^{n} C(l) \qquad (2.2\text{-}32)$$

It should be emphasised that one doesn't really have to physically calculate the integrals appearing in the definition of $\Phi(N,H;z)$ since it follows from (2.2-29) that this is equivalent to being able to calculate the determinant of $(zI-H_{22})$ at each stage of the iteration process, which may be determined directly by decimating the equations of motion.

2.3 DENSITY OF STATES

Consider a periodic Euclidean lattice of integer dimensionality d. The low frequency density of states, $\rho(\omega)$, for excitations of frequency ω, is determined by the low frequency limiting form of the dispersion relation $\omega(k)$ and from a proper mode counting in reciprocal space. Since this present work is devoted to a study of spin wave dynamics we will confine our discussions of the density of states within the context of magnetism. The formalism, however, will be equally applicable to the similar concept in electron or phonon-like systems.

The ground state of a simple ferromagnet has all spins parallel [10] and corresponds to an energy of Eo. The next highest state in energy, Eo+Δ, is one corresponding to a single spin deviation which is shared among the N spins of the lattice, giving rise to a low energy excitation termed a spin wave. The corresponding quantum counterpart, namely quantized spin waves or magnons, have a single polarization direction for each value of the wavevector k, and so it follows that in d dimensions the number of modes with wavevector less than k is proportional to the volume in k-space.

ie.

$$N(k) \sim k^{d} \qquad (2.3\text{-}1)$$

Now, the spin wave density of states, the number of spin waves with frequencies in the range ω to $\omega + d\omega$, is defined by

$$\rho(\omega) = dN/d\omega \tag{2.3-2}$$

or using the chain rule of differentiation

$$\rho(\omega) = (dN/dk)\,(dk/d\omega) \tag{2.3-3}$$

But for small k the spin wave dispersion relation for a ferromagnet has the following form [2]

$$\omega(k) \sim k^2 \tag{2.3-4}$$

and so it follows from (2.3-3) and (2.3-4) that

$$\rho(\omega) \sim \omega^{d/2-1} \tag{2.3-5}$$

which describes the low frequency density of ferromagnetic spin waves. In the case of an antiferromagnet the small k dispersion relation has the form

$$\omega(k) \sim k \tag{2.3-6}$$

and so the expression corresponding to (2.3-5) is

$$\rho(\omega) \sim \omega^{d-1} \tag{2.3-7}$$

Now consider a non-translationally invariant object such as a fractal lattice. It is natural to ask whether or not there exits a relation analogous to (2.3-5) or (2.3-7), ie. does the density of states in a disordered system obey a scaling law at low frequency. The answer to this question turns out to be yes, and in order to provide a theoretical basis for why such a scaling law should be followed and to predict the form of the exponent we proceed as follows: It is clear that to take into account the reduced dimensionality of the disordered system we must replace the Euclidean dimensionality d by fractal dimensionality $\tilde{d}$. Furthermore we make the scaling ansatz

$$\omega(q) \sim q^{z} \tag{2.3-8}$$

where q is a reciprocal length and z is the exponent characterising the form of the dispersion relation for small q. Again using our definition (2.3-3) for the density of states (with k replaced by q), it follows that

$$\rho(\omega) \sim \omega^{\tilde{d}-1} \tag{2.3-9}$$

It is useful to see how such a scaling form for the density of states translates into a scaling law for a bulk property such as the specific heat C(T)=(dE/dT). Using the fact that the number of magnons, n(q), excited at temperature T is determined by a Bose- Einstein distribution function and that each magnon contributes an energy $\epsilon(q) = h\omega(q)$ to the total, it follows that the total energy E(T) is given by

$$E(T) = \sum_{q} \epsilon(q)\, n(q) = \int \omega\, \rho(\omega)\, n(\omega)\, d\omega \tag{2.3-10}$$

where

$$n(\omega) = [\ \exp(\ \beta h\omega) - 1\ ]^{-1} \qquad (2.3\text{-}11)$$

Finally, performing the derivative with respect to T leads to a form for the low temperature specific heat given by leading to a form of the low temperature specific heat given by

$$C(T) \sim T^{\tilde{d}} \qquad (2.3\text{-}12)$$

where we have defined the spectral dimension d to be

$$\tilde{d} = \bar{d}/z \qquad (2.3\text{-}13)$$

In a homogeneous system, the fractal dimension and the Euclidean dimension coincide. Furthermore the exponent z can be derived from a fixed point analysis of a rescaling transformation and in chapters 3 and 4 we show explicitly that for the homogeneous linear ferromagnet and antiferromagnet z is equal to 2 and 1 respectively. Thus we recover our previous results of (2.3-5) and (2.3-7).

Chapter III

FERROMAGNETIC SPIN WAVES

3.1 DESCRIPTION

We begin our description of ferromagnetic spin waves, the low-lying excitations characteristic of a magnetic system, with the specification of the Heisenberg Hamiltonian given by

$$H = -(1/2)\sum_{R,R'} J(R-R') \, S(R,t) \, S(R',t) \qquad (3.1\text{-}1)$$

where $J(R-R') \geqslant 0$ for all R,R' is the exchange constant favoring alignment of adjacent spins, and $S(R,t)$ is the spin operator acting at a lattice site described by the lattice vector R. At absolute zero the state of the ferromagnet is the one of lowest energy and in fact corresponds to a situation where all spins are aligned. It is convenient to introduce the spin deviation operators $S^{\dagger}(R,t)$ and $S^{-}(R,t)$ defined by

$$\begin{aligned} S^{\dagger}(R,t) &= S^{x}(R,t) + iS^{y}(R,t) \\ S^{-}(R,t) &= S^{x}(R,t) - iS^{y}(R,t) \end{aligned} \qquad (3.1\text{-}2)$$

Written in terms of these spin deviation operators, expression (3.1-1) for H becomes

$$\begin{aligned} H = &-(1/4)\sum_{R,R'} J(R-R')\ S^{+}(R,t)S^{-}(R',t) \\ &-(1/4)\sum_{R,R'} J(R-R')\ S^{-}(R,t)S^{+}(R',t) \qquad (3.1\text{-}3) \\ &-(1/2)\sum_{R,R'} J(R-R')\ S^{z}(R,t)S^{z}(R',t) \end{aligned}$$

The dynamics of the transverse component of spin, $S\ (R'',t)$, is determined by the following Heisenberg equation of motion

$$i\hbar\ \dot{S}^{+}(R'',t) = [\ S^{+}(R'',t)\ ,\ H\ ] \qquad (3.1\text{-}4)$$

where $[A,B]=AB-BA$ denotes the commutator of two arbitrary operators A and B, and the dot denotes differentiation with respect to time. Inserting expression (3.1-3) for H into (3.1-4) and setting $\hbar=1$ gives

$$\begin{aligned} i\ \dot{S}^{+}(R'',t) = &-(1/4)\sum_{R,R'} J(R-R')[\ S^{+}(R'',t)\ ,\ S^{+}(R,t)S^{-}(R',t)\ ] \\ &-(1/4)\sum_{R,R'} J(R-R')[\ S^{+}(R'',t)\ ,\ S^{-}(R,t)S^{+}(R',t)\ ] \\ &-(1/2)\sum_{R,R'} J(R'R')[\ S^{+}(R'',t)\ ,\ S^{z}(R,t)S^{z}(R',t)\ ] \end{aligned}$$

(3.1-5)

Then using the fact that

$$\begin{aligned} &[\ S^{+}(R,t)\ ,\ S^{-}(R',t)\ ] = 2\ S^{z}(R,t)\ \delta(\ R-R') \\ & \qquad\qquad\qquad\qquad\qquad\qquad (3.1\text{-}6) \\ &[\ S^{z}(R,t)\ ,\ S^{+}(R',t)\ ] = S^{+}(R,t)\ \delta(\ R-R') \end{aligned}$$

it follows that (3.1-5) may be written as

$$
\begin{aligned}
i\,\dot{S}^{+}(R,t) = & -(1/2)\sum_{R,R'} J(R-R')\ S^{+}(R',t)S^{z}(R,t) \\
& - (1/2)\sum_{R,R'} J(R-R')\ S^{z}(R,t)S^{+}(R',t) \\
& + (1/2)\sum_{R,R'} J(R-R')\ S^{z}(R',t)S^{+}(R,t) \\
& + (1/2)\sum_{R,R'} J(R-R')\ S^{+}(R,t)S^{z}(R',t)
\end{aligned}
\tag{3.1-7}
$$

Since we are interested in the low lying excitations (ie. the spin waves) it is natural to let (3.1-7), an operator equation for $S^{+}(R,t)$, act on the ground state $|0>$ which is given (exactly) by the tensor product of eigenstates of the operator $S^{z}(R,t)$ with maximum eigenvalue S.

$$
|\ 0\ > = \bigotimes_{R} |\ S\ > \tag{3.1-8}
$$

Then by what amounts to replacing $S^{z}(R,t)$ by its average in the ground state, we obtain the linearized form of (3.1-7) given by

$$
i\,\dot{S}^{+}(R,t) = S\sum_{R'} J(R-R')\ \{\ S^{+}(R,t) - S^{+}(R',t)\ \}
\tag{3.1-9}
$$

Finally, Fourier transforming with respect to time leads to

$$
\{\ S\sum_{R'} J(R-R') - \omega\ \}\ S^{+}(R,\omega) = S\sum_{R'} J(R-R')\ S^{+}(R',\omega)
\tag{3.1-10}
$$

which is the starting point for our real space rescaling. If the lattice vectors R lie on a Bravais lattice, then the usual method of solution is to perform a (spatial) Fourier transform of (3.1-10), immediately diagonalizing the problem [5]. In the case of non-Bravais lattices such as the Viscek snowflake or Sierpinski gasket, such a method will not work, and we are forced to use the real-space rescaling approach.

3.2 HEISENBERG DYNAMICS OF A PURE FERROMAGNETIC LINEAR XHAIN

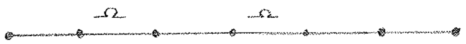
In order to illustrate how length scaling methods can be applied to obtain recursion relations between the parameters, fixed points, eigenvalues and exponents, it is instructive to consider a specific example where these may be calculated analytically. The simplest such example concerns the dynamics of a pure ferromagnetic chain, and for simplicity we restrict interactions to nearest neighbour exhange only. Since the chain is homogeneous J(R-R') = J for all R and R' (3.1-10) may be written as [25]

$$(2 - \Omega) S^{+}(R,\Omega) = S^{+}(R+a,\Omega) + S^{+}(R-a,\Omega) \qquad (3.2\text{-}1)$$

where $\Omega = \omega/J$ is a reduced energy, and 'a' is the lattice constant and we have set S=1 for convenience. The procedure is to eliminate $S^{+}(R+a,\Omega)$ and $S^{+}(R-a,\Omega)$ by using (3.2-1) with R replaced by R+ba and R-ba where b is the rescaling factor. We are, however, at liberty to choose the particular rescaling factor to be used and in the case of an ordered system this choice is dictated purely by convenience whereas in the case of an exact fractal structure we do not necessarily have the luxury of choosing b since it is uniquely determined once we have specified the structure itself. Suppose we were to choose a scaling factor of 2. Then we would eliminate $S^{+}(R+a,\Omega)$ and $S^{+}(R-a,\Omega)$ from (3.2-1) in favour of $S^{+}(R+2a,\Omega)$ and $S^{+}(R-2a,\Omega)$ and find that $S^{+}(R,\Omega)$ is now described by

$$\{ 2 - (4\Omega - \Omega^2) \} S^{+}(R,\Omega) = S^{+}(R+2a,\Omega) + S^{+}(R-2a,\Omega) \tag{3.2-2}$$

Pictorially the elimination process corresponds to the following



It is clear that we may write (3.2-2) in a covariant form by defining

$$\Omega' = 4\Omega - \Omega^2 \tag{3.2-3}$$

This transformation has two qualitatively different types of behaviour. For $\Omega>4$ and $\Omega<0$ Ω' tends to infinity under iteration whereas for all other values of Ω the transformation behaves chaotically.

For small wavevector k, we make the scaling ansatz, namely

$$\Omega = C\, k^{z} \tag{3.2-4}$$

where C is a constant and z is the dynamic exponent. When (3.2-4) is valid we are justified in working with the linearized form of (3.2-3) which is obtained by expanding Ω' in a Taylor series about its fixed point. Then

$$(\Omega' - \Omega^*) = \lambda\,(\Omega - \Omega^*) \tag{3.2-5}$$

where

$$\lambda = (d\Omega'/d\Omega) \tag{3.2-6}$$

is called an eigenvalue and is evaluated at a fixed point. The recursion relation (3.2-3) has two fixed points, $\Omega^*=0$ and $\Omega^*=3$, with corresponding eigenvalues 4 and -2 respectively. The $\Omega^*=0$ fixed point describes how the excitation energy depends on length scale at low ω and can be used to extract the exponent z; the $\Omega^*=3$ fixed point, however, is only an artifact of our b=2 scaling factor and hence is not physically meaningful. Furthermore, if we denote the wavevector of our rescaled lattice by k' then

$$k' = 2\,k \tag{3.2-7}$$

and taking the ratio of (3.2-4) with its corresponding counterpart in the rescaled system and using (3.2-5) and (3.2-7) it follows that

$$\lambda = 2^{z} \tag{3.2-8}$$

and so $z = \ln(4)/\ln(2) = 2$, the well known quadratic dispersion for a linear ferromagnetic chain. The similarity with critical phenomena should now be obvious, since the fixed point $\Omega^*=0$ and the dynamic exponent z are entirely analogous to the critical point Tc and the critical exponent ν.

We could equally well have chosen a rescaling factor of 3 in which case we would have found that the expression corresponding to (2.7) was given by

$$\{ 2 - (9\Omega - 6\Omega^2 + \Omega^3) \} S^+(R,\omega) = S^+(R+3a,\omega) + S^+(R-3a,\omega) \tag{3.2-9}$$

and so the restriction of covariance implies the transformation

$$\Omega' = 9\Omega - 6\Omega^2 + \Omega^3 \tag{3.2-10}$$

which has the fixed points $\Omega^* = 0$, $\Omega^*= 2$, and $\Omega^* = 4$, yielding the eigenvalues 9, -3 and 9 respectively. Again the $\Omega^*=0$ fixed point is the only physical one and proceeding in exactly the same fashion, one can show that the dynamic exponent z is given by $z = \ln(9)/\ln(3) = 2$ as before. Thus we see that the particular choice of rescaling factor is not crucial. Choosing different rescaling factors simply means that our recursion relations will be different and hence in general will have different fixed points. The physics, however, which is related to the largest eigenvalue, does not change.

The form of the full dispersion relation [26]

$$\Omega = g(k) \tag{3.2-11}$$

can be obtained from the recursion relation for Ω as given by (3.2-3) with $k'=2k$, and is a solution to the following non- linear functional equation

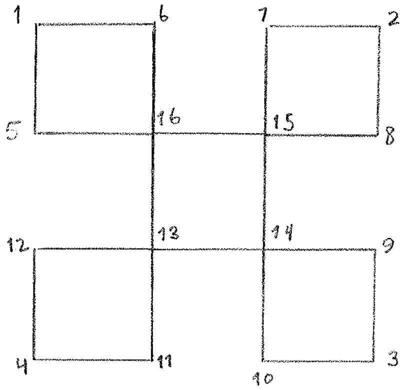
$$g(2k) = 4\ g(k) - g(k)^2 \tag{3.2-12}$$

It can be easily verified that the solution is given by

$$g(k) = 2\ [\ 1 - \cos(\alpha k)\ ] \tag{3.2-13}$$

3.3 THE VISCEK SNOWFLAKE - A DILUTE FERROMAGNET

As mentioned in the introduction, the Viscek snowflake, pictured below, is an example of an exact fractal which is constructed in a recursive fashion. It is, in fact, this same recursive nature that makes such objects ideally suited to the decimation method of real space renormalization. The indicated numbering scheme is entirely arbitrary and serves only as an aid when discussing the elimination process.



Such a lattice can be viewed as an ordinary square lattice with sites and bonds absent, the number removed being proportional to the stage of decoration, and corresponds to a structure with reduced dimensionality. As a model of a dilute ferromagnetic system, we are to picture at each of the sites a spin coupled to its nearest and next nearest neighbours by exchange interactions J_1 and J_2 respectively. The reason we include the second neighbour interactions from the very outset is that even if they were absent to start with, the decimation procedure generates them during the first stage. This is not unlike the usual difficulty of proliferation of interactions characteristic of two and higher dimensions [21], however in this example such a proliferation does not extend beyond second neighbours. Thus to obtain a truly general set of recursion relations we must allow for these addtional interactions from the very beginning. It follows then that our expression (3.1-10) for the transverse Fourier component of the spin operator $S^+(R,t)$ can be written as

$$[\ z_1 J_1 + z_2 J_2 - \omega\ ]\ S^+(R,\omega) = \sum_{R'} J(R-R')\ S^+(R',\omega) \tag{3.3-1}$$

where z_1 and z_2 denote the number of nearest and next nearest neighbours, respectively, and R' now denotes a first or second neighbour of R. Note that there are two basic types of sites, those that have two first neighbours and one second neighbour, and those that have four first neighbours and two second neighbours. Adopting the notation

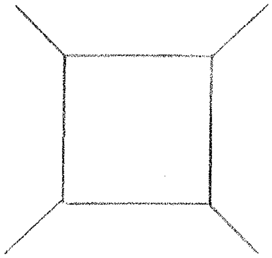
$$\begin{aligned} S^+(R,\omega) &= u(R) \\ \epsilon_2 &= 2J_1 + J_2 \\ \epsilon_4 &= 4J_1 + 2J_2 \\ V_1 &= -J_1 \\ V_2 &= -J_2 \\ E &= \omega \end{aligned} \tag{3.3-2}$$

it follows that these two types of sites are described by

$$(E - \epsilon_2) u(R) = \sum_{R'} V(R') u(R')$$

$$(E - \epsilon_4) u(R'') = \sum_{R'} V(R') u(R') \qquad (3.3-3)$$

The decimation procedure will be carried out in two stages. First we will perform the (simple) elimination of sites 5-12 inclusive leading to the following structure



To illustrate we take sites 5 and 6 as being representative of the generic case. These in turn interact with sites 1 and 16 and the complete set is described by the following matrix equation

$$\begin{bmatrix} (E-\epsilon_4) & -V_2 & -V_1 & -V_1 \\ -V_2 & (E-\epsilon_4) & -V_1 & -V_1 \\ -V_1 & -V_1 & (E-\epsilon_2) & -V_2 \\ -V_1 & -V_1 & -V_2 & (E-\epsilon_2) \end{bmatrix} \begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ u(4) \end{bmatrix} = 0 \qquad (3.3-4)$$

where we have performed the following site re-labelling

16 → 2 , 5 → 3 , 6 → 4

Then performing the blocking transformation followed by site elimination gives rise to a new Hamiltonian H' given by

$$H' = H_{11} + H_{12} (E - H_{22})^{-1} H_{21} . \qquad (3.3\text{-}5)$$

The correspondence between H and H' defines the following set of recursion relations

$$\begin{aligned} \epsilon' &= \epsilon_4 + \Delta \\ V' &= V_2 + \Delta \end{aligned} \qquad (3.3\text{-}6)$$

where

$$\Delta = (2 V_1^2)/(E - \epsilon_2 - V_2) \qquad (3.3\text{-}7)$$

and so performing the same set of steps for sites 7-12 leads to the following matrix equation for the remaining sites.

$$\begin{bmatrix} (E-\epsilon') & -V_1 & -V_2 & -V_1 \\ -V_1 & (E-\epsilon') & -V_1 & -V_2 \\ -V_2 & -V_1 & (E-\epsilon') & -V_1 \\ -V_1 & -V_2 & -V_1 & (E-\epsilon') \end{bmatrix} \begin{bmatrix} u(13) \\ u(14) \\ u(15) \\ u(16) \end{bmatrix} = V' \begin{bmatrix} u(4) \\ u(3) \\ u(2) \\ u(1) \end{bmatrix} \qquad (3.3\text{-}8)$$

We now come to the second and final stage of the elimination procedure which will involve inverting the above 4 by 4 matrix. Defining

$$a = (E-\epsilon') , \; b = -V_1 , \; c = -V_2 \qquad (3.3\text{-}9)$$

we find that

$$u(13) = V' [\ \alpha\ u(4) + \beta(\ u(1) + u(3)\ ) + \gamma\ u(2)\ ]$$

$$u(14) = V' [\ \alpha\ u(3) + \beta(\ u(2) + u(4)\ ) + \gamma\ u(1)\ ]$$

$$u(15) = V' [\ \alpha\ u(2) + \beta(\ u(1) + u(3)\ ) + \gamma\ u(4)\ ]$$

$$u(16) = V' [\ \alpha\ u(1) + \beta(\ u(2) + u(4)\ ) + \gamma\ u(3)\ ]$$

(3.3-10)

where

$$\alpha = \frac{c\ (\ a + c\ ) - 2\ b^2}{(\ a - c\ )[\ (\ a + c\ )^2 - 4\ b^2\ ]}$$

$$\beta = \frac{-\ b}{(\ a + c\ )^2 - 4\ b^2} \qquad (3.3\text{-}11)$$

$$\gamma = \frac{2\ b^2 - c\ (\ a + c\ )}{(\ a - c\ )[\ (\ a + c\ )^2 - 4\ b^2\ ]}$$

Eliminating sites 13,14,15 and 16 leads to the following set of equations for the corner sites 1,2,3 and 4.

$$(\ E - \epsilon_2'\ )\ u(1) = V_1'\ [\ u(2) + u(4)\ ] + V_2'\ u(3)$$

$$(\ E - \epsilon_2'\ )\ u(2) = V_1'\ [\ u(1) + u(3)\ ] + V_2'\ u(4)$$

$$(\ E - \epsilon_2'\ )\ u(3) = V_1'\ [\ u(2) + u(4)\ ] + V_2'\ u(1)$$

$$(\ E - \epsilon_2'\ )\ u(3) = V_1'\ [\ u(1) + u(3)\ ] + V_2'\ u(2)$$

(3.3-12)

where

$$\begin{aligned} \epsilon_2' &= \epsilon_2 + \Delta + \alpha V'^2 \\ \epsilon_4' &= \epsilon_4 + 2\Delta + 2\alpha V'^2 \\ V_1' &= \beta V'^2 \\ V_2' &= \gamma V'^2 \end{aligned} \tag{3.3-13}$$

Note that ϵ_2' in (3.3-12) is ϵ_2' or ϵ_4' depending on whether the corresponding site has a coordination number of 2 or 4 respectively. Defining the reduced variables

$$X = V_1/(E-\epsilon_2)\ ,\ Y = V_2/(E-\epsilon_2)\ ,\ Z = (E-\epsilon_4)/(E-\epsilon_2) \tag{3.3-14}$$

it follows that our recursion relations can be written in the dimensionless form

$$\begin{aligned} X' &= f(X,Y,Z)/[\ 1 - h(X,Y,Z)\ ] \\ Y' &= g(X,Y,Z)/[\ 1 - h(X,Y,Z)\ ] \\ Z' &= [\ Z - 2h(X,Y,Z)\ ]/[\ 1 - h(X,Y,Z)\ ] \end{aligned} \tag{3.3-15}$$

where f, g, and h are complicated functions of their arguments (c.f. appendix A). This set of recursion relations possesses six fixed points. However, since we are interested in excitations from the ground state, E=0, the only relevant fixed point will be the fixed point (X^*,Y^*,Z^*), corresponding to E=0, which is given by

$$(X^*,Y^*,Z^*) = (1/3,1/3,2) \tag{3.3-16}$$

Linearizing about this fixed point we find that the largest eigenvalue λ is 15, and the associated eigenvector is (2/ 2,2/ 2,0). Expanding the scaling field, u, in terms of this eigenvector leads to an expression for the scaling field with leading order term linear in E. The dynamic exponent is given by z=ln(15)/ln(3), the fractal dimension is d=ln(5)/ln(3) and so the spectral dimension is ln(5)/ln(15) = .59 . Thus we expect the low frequency density of states to obey the the scaling form

$$\rho(E) \sim E^{-.41} \qquad (3.3\text{-}17)$$

and so should diverge at E=0. Also it follows that the low frequency integrated density of states, N(E), should behave like

$$N(E) \sim E^{.59} \qquad (3.3\text{-}18)$$

and so should go to zero as E goes to zero. In addition, a ln-ln plot of the integrated density of states should be linear and have slope .59.

As mentioned in section 2 of chapter 2, a determination of the density of states is based upon our ability to calculate the quantity $\det[(E-H_{22})]$ at any stage l of the elimination process. It follows from (2.3-4) and (2.3-8) that the required determinant is of the form

$$\det[(E-H_{22})] = (E-\epsilon'+V_2)^2[(E-\epsilon'-V_2)^2-4V_1{}^2]$$

$$[(E-\epsilon_2)^2-V_2^2]^4$$

(3.3-19)

Furthermore, since the density of states is related to a derivative of the logarithm of this quantitiy, and in anticipation of the fact that the resulting spectra will be highly singular (making numerical differentiation impractical) it is clear that we should determine a set of recursion relations for the following derivatives:

$$d(E-\epsilon_2)/dE \ , \ dV_1/dE \ , \ dV_2/dE$$

Denoting the free energy $\Phi(N,H;z)$ per site as $f(N,H;z)$ and noting that the number of sites we eliminate at any stage l depends upon how many iterations k of the decorating process were performed in constructing the fractal, it follows that

$$f(l+1) = f(l) - R(k,l) \ln [\det (E - H_{22}^{(l)})]$$

(3.3-20)

where

$$R(k,l) = [N(k,l) - N(k,l+1)]/[2 n(B) N(k,0)]$$

(3.3-21)

with $n(B)$ denoting the number of sites in the block being eliminated ($n(B)=12$) and $N(k,l)$ the number of sites on the Viscek snowflake given that it has been decorated up to some level k after which l decimations have been performed. It is straightforward to show that

$$N(k,l) = 3*5^{k-l} + 1 \qquad (3.3-22)$$

Then in the limit k going to infinity (when we have a true fractal structure) we obtain

$$f(l+1) = f(l) -(1/6*5^{l+1}) \ln [\det (E - H_{22}^{(l)})] \qquad (3.3-23)$$

Numerically, the density of states is now obtained by iterating the recursion relations given in (3.3-13), accumulating f in (3.3-22) at each stage. Typically only about 20 iterations are required to obtain a precision of one part in a million, however more iterations are required as the imaginary part of E is made smaller. Finally, taking the imaginary part of f and multiplying by $2/\pi$ leads to the spectrum in figure 3.1. The intitial conditions used to generate this spectrum were $V_1=-1.0$, $V_2=0.0$, $\epsilon_2=2.0$, $\epsilon_4=4.0$ and Im(E)=1.D-03. Qualitatively the spectrum is very similar to that found for other exact fractal lattices [4,28], notably the Sierpinski gasket in two dimensions. A characteristic feature of such lattices appears to be that their associated spectra is highly singular and consists of gaps and peaks. Referring now to figure 3.1, it appears that the only truly localized mode occurs at E=2 since it becomes a delta function in the limit of zero imaginary part. Furthermore, we find that as the imaginary part is made to vanish, the density of states at E=0 appears to diverge, supporting our previous findings obtained from the fixed point analysis. A good check on the calculation involves a determination of the integrated density of states, $N(\omega)$, obtained by numerically integrating the spectrum appearing in figure 3.1. If properly normalized, the density of states should integrate to unity, a result which rests upon the analytic properties of the Green function [28]. Figure 3.2 shows a plot of the integrated density of states, confirming that it does indeed integrate to one. Figure 3.3 is a ln-ln plot of the low frequency integrated density of states. The linear portion has a slope of 0.68, yielding an exponent in reasonable agreement with the one obtained from the fixed point analysis.

FERROMAGNETIC DOS

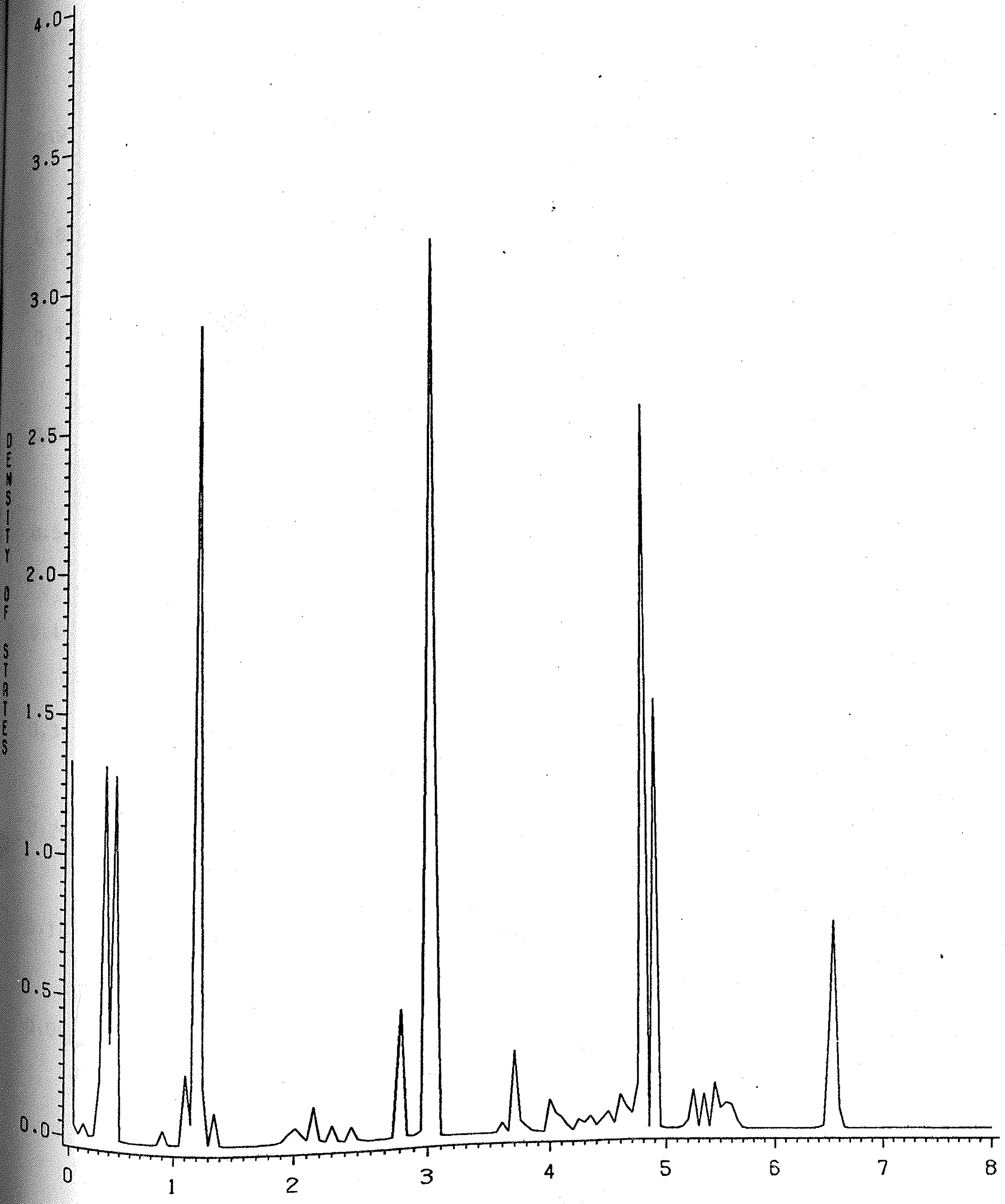


figure 3.1

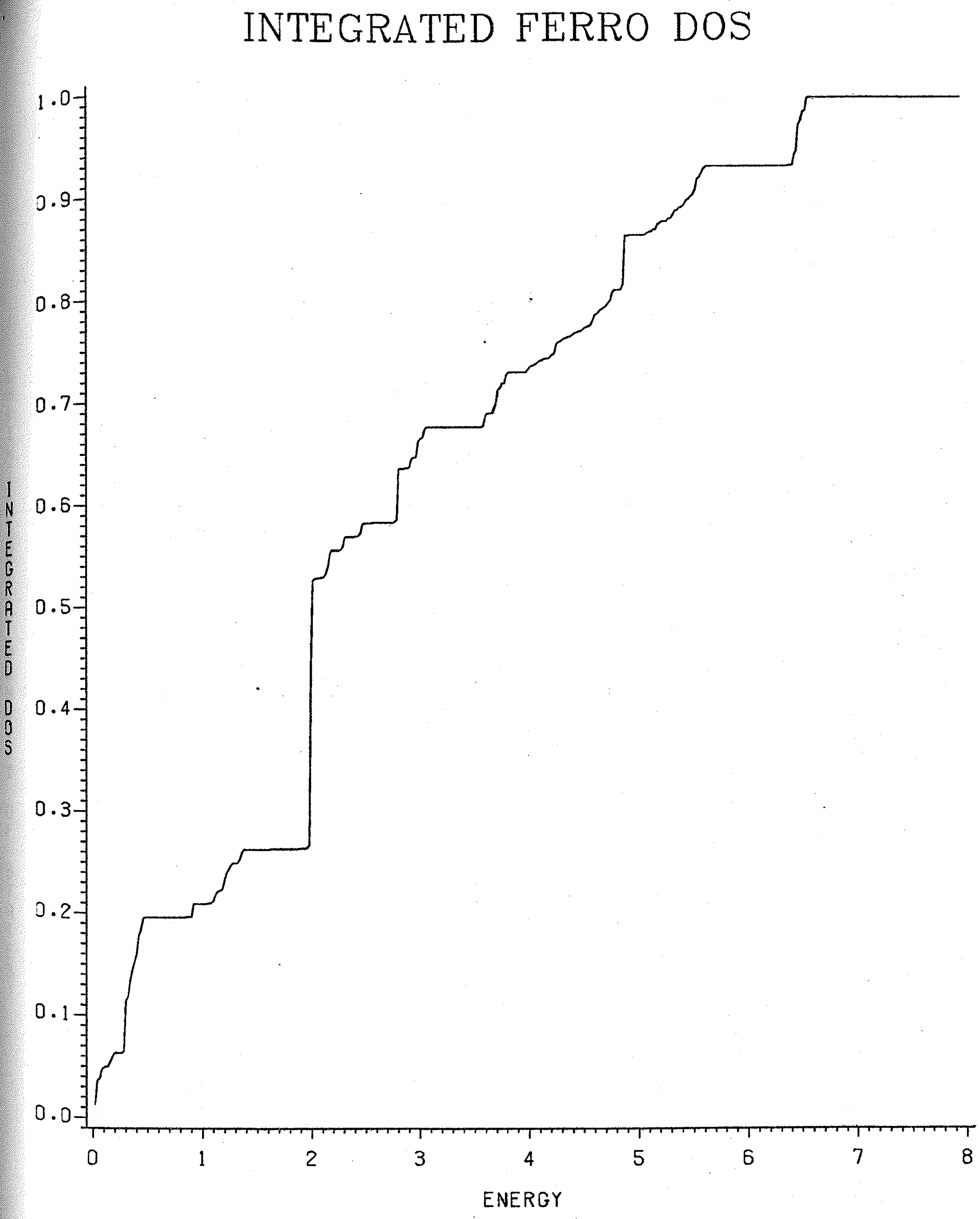

INTEGRATED FERRO DOS
INTEGRATED DOS
ENERGY
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0
1
2
3
4
5
6
7
8

Figure 3.2

LOW E FERRO INTEGRATED DOS

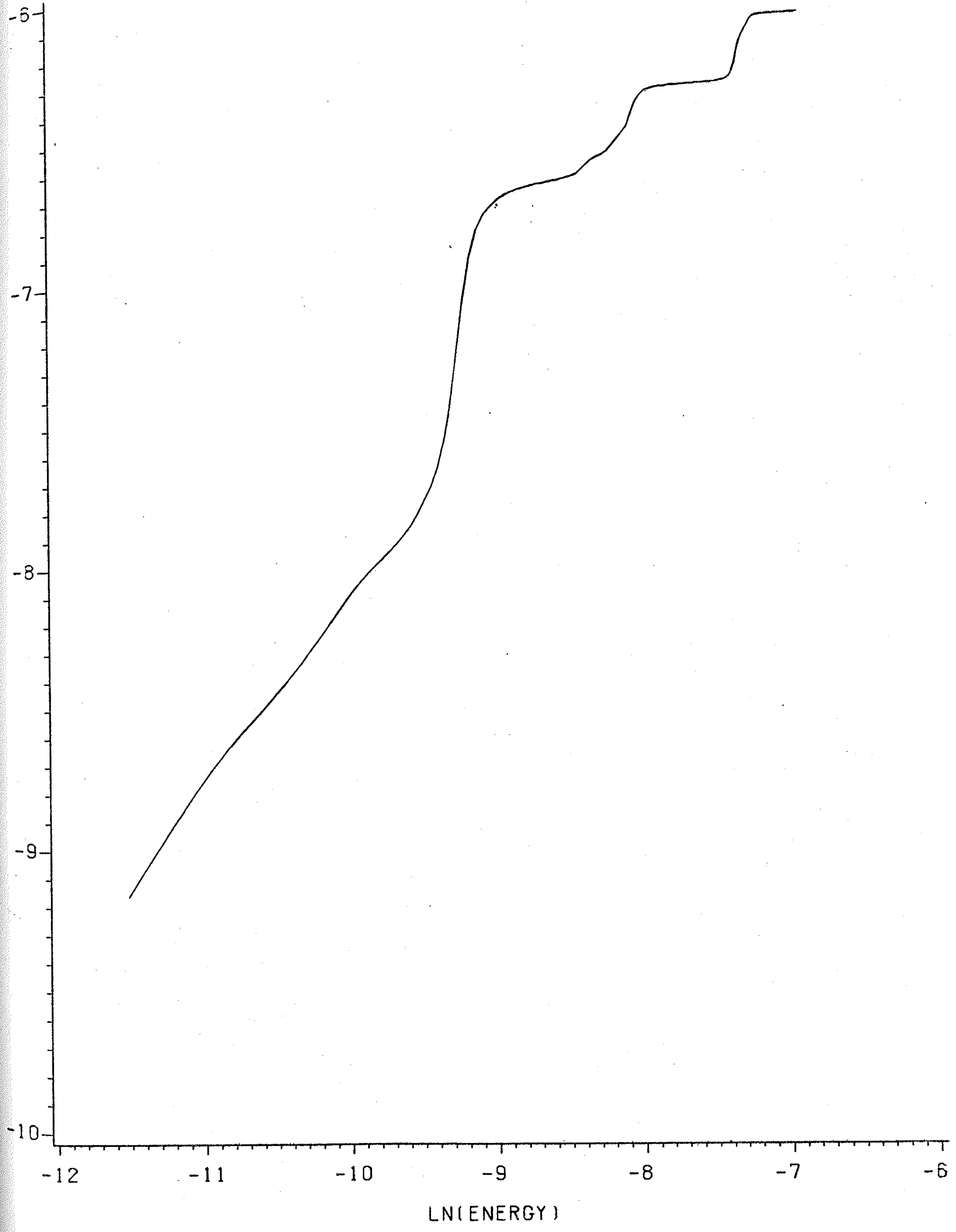


figure 3.3

Chapter IV

ANTIFERROMAGNETIC SPIN WAVES

4.1 DESCRIPTION

In our description of antiferromagnetic spin waves we shall picture our lattice as being composed of two interpenetrating sublattices having equal and opposite values of magnetization. The Hamiltonian describing such a collection of spins is given by

$$H = (1/2)\sum_{R,R'} J(R-R')\ S(R,t)\ S(R',t) \qquad (4.1\text{-}1)$$

where $J(R-R') \geqslant 0$ for all R,R'. Proceeding in exactly the same way as in section (3.1) we introduce spin raising and lowering operators and after evaluating the various commutators, we arrive at the following equation of motion for $S^{+}(R,t)$

$$\begin{aligned} i\,\dot{S}^{+}(R,t) = {} & (1/2)\sum_{R,R'} J(R-R')\ S^{+}(R',t)S^{z}(R,t) \\ & + (1/2)\sum_{R,R'} J(R-R')\ S^{z}(R,t)S^{+}(R',t) \\ & - (1/2)\sum_{R,R'} J(R-R')\ S^{z}(R',t)S^{+}(R,t) \\ & - (1/2)\sum_{R,R'} J(R-R')\ S^{+}(R,t)S^{z}(R',t) \end{aligned}$$

$$(4.1\text{-}2)$$

which is the analogous expression to (3.1-7). At this point in our discussion of the ferromagnet, we replaced S (R,t) by its average in the ground state, |0> , which is well defined. However in the case of the antiferromagnet the ground state is not known and what is commonly done is to replace S (R,t) by its average value in the Neel state. Following this convention, we make the following substitution

$$S^{z}(R,t) = (-1)^{\alpha_R} S \tag{4.1-3}$$

where

$\alpha_R = 1$ if R refers to a site on sublattice B

$\alpha_R = 2$ if R refers to a site on sublattice A

We then arrive at the following

$$i\,\dot{S}^{+}(R,t) = S \sum_{R'} J(R-R')\ \{\ (-1)^{\alpha_R} S^{+}(R',t) - (-1)^{\alpha_{R'}} S^{+}(R,t)\ \} \tag{4.1-4}$$

Finally, performing a Fourier transform with respect to time leads to the form

$$\{\ \omega + S \sum_{R'} J(R-R')(-1)^{\alpha_{R'}} \}\ S^{+}(R,\omega) = S(-1)^{\alpha_R} \sum_{R'} J(R-R')\ S^{+}(R',\omega) \tag{4.1-5}$$

which is the starting point for our real space rescaling.

4.2 HEISENBERG DYNAMICS OF A PURE ANTIFERROMAGNETIC CHAIN

Specializing (4.1-5) to the case of nearest neighbour exchange only it follows that

$$\{ \omega + JS \sum_{R'} (-1)^{\alpha_R} \} S^{+}(R,\omega) = JS(-1)^{\alpha_R} \sum_{R'} S^{+}(R',\omega) \tag{4.2-1}$$

where now R' represents a nearest neighbour of R. It is necessary to distinguish the following two cases:

1. R corresponds to a site on sublattice A (hence R' is on sublattice B). Then (4.2-1) becomes

$$(\omega - 2JS) S_A^{+}(R,\omega) = JS\{ S_B^{+}(R-a,\omega) + S_B^{+}(R+a,\omega) \} \tag{4.2-2}$$

2. R corresponds to a site on sublattice B (hence R' is on sublattice A). Then (4.2-1) becomes

$$(-\omega - 2JS) S_B^{+}(R,\omega) = JS\{ S_A^{+}(R-a,\omega) + S_A^{+}(R+a,\omega) \} \tag{4.2-3}$$

There are two basic ways to perform a rescaling transformation on the antiferromagnet. One way is to perform a b = 2n decimation (n = 1,2,3,...) transforming the antiferromagnet into a ferromagnet. Alternatively we can perform a b = 2n + 1 decimation in which case the dual sublattice structure is preserved. Consider first a b = 2 decimation. Proceeding as in chapter 3, we define a reduced (dimensionless) frequency Ω and setting $S=1$ for convenience we eliminate $S^{+}_{B}(R-a,\omega)$ and $S^{+}_{B}(R+a,\omega)$ from (4.2-2) in favour of $S^{+}_{A}(R-2a,\omega)$ and $S^{+}_{A}(R+2a,\omega)$ using (4.2-3). We obtain

$$(2 - \Omega^2) \; S^{+}_{A}(R,\Omega) = S^{+}_{A}(R-2a,\Omega) + S^{+}_{A}(R+2a,\Omega) \tag{4.2-4}$$

This equation has the same form as (3.2-1) which describes the ferromagnet, except that Ω is now replaced by Ω^2, and so the results of section 3.2 apply. Letting $\Lambda = \Omega^2$ it follows that the appropriate rescaling transformation has the form

$$\Lambda' = 4\,\Lambda - \Lambda^2 \tag{4.2-5}$$

with fixed points $\Lambda^*=0$ and $\Lambda^*=3$, and linearizing about the fixed point $\Lambda^*=0$ yields an eigenvalue of 4. Noting that the scaling field corresponding to the eigenvalue 4 is proportional to $(\Lambda-\Lambda^*) \sim \Omega^2$, we deduce that the associated dynamic exponent is 1.

Suppose instead, we performed a $b=3$ decimation. This would involve eliminating $S^{+}_{B}(R-a,\Omega)$ and $S^{+}_{B}(R+a,\Omega)$ in favour of $S^{+}_{B}(R-3a,\Omega)$ and $S^{+}_{B}(R+3a,\Omega)$, and as we shall see is somewhat more complicated than the corresponding $b=2$ decimation already treated. Suppose we focus our attention on a particular A site which we arbitrarily assume is located at site R. Then eliminating sites R+a, R-a, R+2a, and R-2a in favour of sites R+3a and R-3a using equation (4.2-2) we would find that the equation describing the site R was now given by

$$\{ (\Omega + 2 \Omega^2 - \Omega^3) - 2 \} S_A^+(R,\Omega) = S_B^+(R-3a,\Omega) + S_B^+(R+3a,\Omega) \tag{4.2-6}$$

and so our rescaling transformation would have the form

$$\Omega' = \Omega + 2 \Omega^2 - \Omega^3 \tag{4.2-7}$$

Now let us focus our attention on a particular B site located at R+3a. Then eliminating sites R+a, R+2a, R+4a and R+5a using (4.2-3) it follows that the equation describing the site R+3a is now given by

$$\{ -(\Omega - 2\Omega^2 - \Omega^3) - 2 \} S_B^+(R+3a,\Omega) = S_A^+(R,\Omega) + S_A^+(R+6a,\Omega) \tag{4.2-8}$$

and so the corresponding rescaling transformation is now given by

$$\Omega' = \Omega - 2 \Omega^2 - \Omega^3 \tag{4.2-9}$$

Notice that the rescaling transformations (4.2-7) and (4.2-9) are different. In other words under a b=3 rescaling transformation the Ω corresponding to the A sites is different from that for the B sites, and hence we must start off with a situation where these are both different initially. Thus the most general set of equations to begin with are given by

$$(\Omega_A - 2) S^+_A(R,\Omega) = S^+_B(R-a,\Omega) + S^+_B(R+a,\Omega) \qquad (4.2\text{-}10)$$

$$(-\Omega_B - 2) S (R,\Omega) = S^+_A(R-a,\Omega) + S^+_A(R+a,\Omega) \qquad (4.2\text{-}11)$$

Then proceeding in exactly the same way as just indicated but using equations (4.2-9) and (4.2-10) instead of (4.2-2) and (4.2-3) we find that our recursion relations now take the form

$$\Omega_A' = -\Omega_A^2 \Omega_B + 4 \Omega_A \Omega_B - 2 \Omega_A^2 + 5 \Omega_A - 4 \Omega_B \qquad (4.2\text{-}12)$$

$$\Omega ' = -\Omega_A \Omega_B^2 - 4 \Omega_A \Omega_B + 2 \Omega_B^2 + 5 \Omega_B - 4 \Omega_A$$

Linearizing about the fixed point (0,0) yields the following matrix T (c.f. eq.(2.1-4)):

$$\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} \qquad (4.2\text{-}13)$$

for which the largest eigenvalue is 9, with an associated eigenvector of (1/ 2) (1,-1). It now follows from (2.1-6) that the corresponding scaling field, u, is given by

$$u = (1/ 2) (\Omega_A - \Omega_B) \qquad (4.2\text{-}14)$$

and under iteration scales like

$$u' = (1/ 2) (\Omega_A' - \Omega_B') \qquad (4.2\text{-}15)$$

which has a leading order term quadratic in $\Omega_A = \Omega_B = \Omega$. Thus the dynamic exponent z is given by $(1/2)\ln(9)/\ln(3) = 1$ as before.

4.3 A DILUTE ANTIFERROMAGNET

As with the ferromagnet, already treated in chapter 2, we picture our lattice occupied with spins. Since we have adopted the two sublattice structure for our description of the antiferromagnet, the direction of the z-component of spin will alternate from +S to -S from site to site. Furthermore, whereas in section 3.3 it was only necessary to have one second neighbour interaction and to distinguish between two types of sites, it will be necessary in the case of the antiferromagnet to distinguish between two types of second neighbour interactions and between four types of sites. In particular we shall have to distinguish between second neighbour interactions between sites on sublattice A and those on sublattice B. Also, we shall have to distinguish between sites on sublattice A which have coordination numbers of 2 and 4 with similarly coordinated sites on sublattice B. Arbitrarily specifying site 1 to lie on sublattice A (which then uniquely specifies on which sublattice all other sites lie), it follows from (3.3-5) that these four types of sites can be described by the following set of equations:

$$(E - \epsilon_{2A}) u(1) = V_1 [u(5) + u(6)] + V_2 u(16)$$

$$(-E - \epsilon_{2B}) u(2) = V_1 [u(7) + u(8)] + V_2 u(15)$$

$$(E - \epsilon_{4A}) u(15) = V_1 [u(7) + u(8) + u(14) + u(16)] + V_{2A} [u(2) + u(13)]$$

$$(-E - \epsilon_{4B}) u(16) = V_1 [u(5) + u(6) + u(13) + u(15)] + V_{2B} [u(1) + u(14)]$$

(4.3-1)

where

$$\epsilon_{2A} = 2J_1S - J_{2A} S , \quad \epsilon_{2B} = 2J_1S - J_{2B} S$$

$$\epsilon_{4A} = 2\epsilon_{2A} , \quad \epsilon_{4B} = 2\epsilon_{2B} \qquad (4.3\text{-}2)$$

$$V_1 = J_1S , \quad V_{2A} = J_{2A} S , \quad V_{2B} = J_{2B} S$$

Then proceeding in exactly the same way as in section 2.3 and performing the partial elimination of sites 5-12 followed by the elimination of sites 13-16 yields the following four equations describing the remaining sites:

$$(E - \epsilon_{2A}') u(1) = V_1' [u(2) + u(4)] + V_{2A}' u(3)$$

$$(-E - \epsilon_{2B}') u(2) = V_1' [u(1) + u(3)] + V_{2B}' u(4)$$

$$(E - \epsilon_{2A}') u(3) = V_1' [u(2) + u(4)] + V_{2A}' u(1)$$

$$(-E - \epsilon_{2B}') u(4) = V_1' [u(1) + u(3)] + V_{2B}' u(2)$$

(4.3-3)

where

$$\epsilon_{2A}' = \eta_{2A} + a_2 \nu_{2A}^2 , \quad \epsilon_{2B}' = \eta_{2B} + a_1 \nu_{2B}^2$$

$$V_1' = \beta \nu_{2A} \nu_{2B} \qquad (4.3\text{-}4)$$

$$V_{2A}' = \gamma_2 \nu_{2A}^2 , \quad V_{2B}' = \gamma_1 \nu_{2B}^2$$

and

$$\eta_{2A} = \epsilon_{2A} + \Delta_{B} \; , \; \eta_{2B} = \epsilon_{2B} + \Delta_{A}$$

(4.3-5)

$$\nu_{2A} = V_{2A} + \Delta_{B} \; ; \; \nu_{2B} = V_{2B} + \Delta_{A}$$

and

$$\Delta_{A} = 2V_1{}^2/(-E-\epsilon_{2A} - V_{2A}) \; , \; \Delta_{B} = 2V_1{}^2/(E-\epsilon_{2B} - V_{2B})$$

(4.3-6)

and

$$\alpha_1 = \frac{a_1 (a_2 + c_2) - 2 b^2}{(a_1 - c_1) [(a_2 + c_2)(a_1 + c_2) - 4 b^2]}$$

$$\alpha_2 = \frac{a_2 (a_1 + c_1) - 2 b^2}{(a_2 - c_2) [(a_1 + c_2)(a_2 + c_2) - 4 b^2]}$$

$$\beta = \frac{- b}{(a_1 + c_1)(a_2 + c_2) - 4 b^2} \qquad (4.3\text{-}7)$$

$$\gamma_1 = \frac{2 b^2 - c_1 (a_2 + c_2)}{(a_1 - c_1) [(a_1 + c_1)(a_2 + c_2) - 4 b^2]}$$

$$\gamma_2 = \frac{2 b^2 - c_2 (a_1 + c_1)}{(a_2 - c_2) [(a_1 + c_1)(a_2 + c_2) - 4 b^2]}$$

with

$$a_1 = - E - \eta_{4B} \; , \; a_2 = E - \eta_{4A} \; , \; b = - V_1$$

$$c_1 = -V_{2A} \; , \; c_2 = -V_{2B} \qquad (4.3\text{-}8)$$

Definining the reduced parameters

$$X = V_1/(E-\epsilon_{2A}) \ , \ Y_1 = V_{2A}/(E-\epsilon_{2A}) \ , \ Y_2 = V_{2B}/(E-\epsilon_{2A}) \quad (4.3\text{-}9)$$

$$Z_1 = (E-\epsilon_{4A})/(E-\epsilon_{2A}) \ , \ Z_2 = (E-\epsilon_{2B})/(E-\epsilon_{2A})$$

$$Z_3 = (E-\epsilon_{4B})/(E-\epsilon_{2A})$$

it follows that the above set of recursion relations can be written in the following form:

$$X' = f/(1 - h_1)$$

$$Y_1' = g_1/(1 - h_1)$$

$$Y_2' = g_2/(1 - h_1)$$

$$Z_1' = (Z_1 - 2h_1)/(1 - h_1)$$

$$Z_2' = (Z_2 - h_2)/(1 - h_2)$$

$$Z_3' = (Z_3 - 2h_2)/(1 - h_1)$$

where f, g_1, g_2, h_1, h_2, h_3 are complicated of their arguments [c.f. appendix B]. The above six parameter map describes the trajectory of a point (X,Y_1,Y_2,Z_1,Z_2,Z_3) in a six-dimensional parameter space. Again, seeking the E=0 fixed point which governs the behaviour of the density of states at low energy we find that it is given by (1/3,-1/3,-1/3,2,1,2). Linearizing about this fixed point we find that the largest eigenvalue is given by $\lambda=15$, as obtained in the ferromagnetic case treated in chapter 3. The associated scaling field, however, which is defined in terms of the corresponding eigenvector (1/ 3)(1,-1,-1,0,0,0) has a leading order term which under iteration is quadratic in E, in contrast to that for the ferromagnet which was linear in E. Thus the dynamic exponent for the antiferromagnet is half that for the ferromagnet and hence the spectral dimension is double. It follows then that the low frequency density of states should behave like

$$\rho(\omega) \sim \omega^{.19} \qquad (4.3\text{-}10)$$

and so should tend to zero in the limit ω 0. In addition, we expect that a ln-ln plot of the low frequency integrated density of states should be linear and have slope 1.19. Figure 4.1 is a plot of the antiferromagnetic density of states. The initial conditions used to generate this plot were $V_1=1.0$, $V_{2A}=V_{2B}=0$, $\epsilon_{2A}=\epsilon_{2B}=2.0$, $\epsilon_{4A}=\epsilon_{4B}=4.0$, and Im(E)=1.d-03. Clearly the spectra has much in common with that for the ferromagnet, including the presence of localized modes in the spectrum. Also, we find that in the limit of vanishing imaginary part that the density of states at E=0 tends to zero , confirming our previous prediction from the fixed point analysis. Figure 4.2 is a plot of the integrated antiferromagnetic density of states indicating that the spectra is properly normalized. Finally, figure 4.3 is a ln-ln plot of the low frequency integrated density of states. The linear portion has a slope of 1.01 which is in reasonable agreement with that predicted from the fixed points.

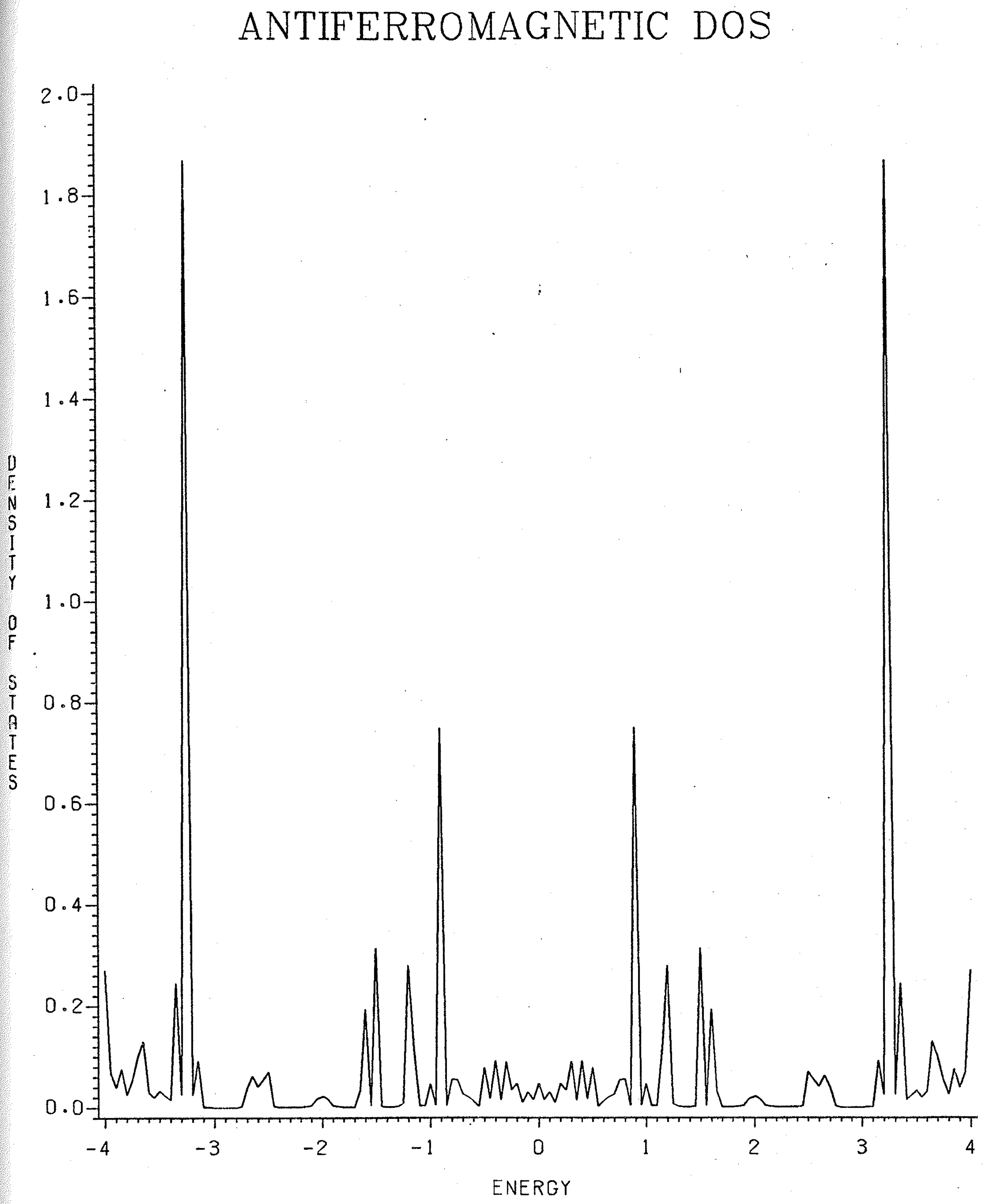

ANTIFERROMAGNETIC DOS
DENSITY OF STATES
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-4
-3
-2
-1
0
1
2
3
4
ENERGY

figure 4.1

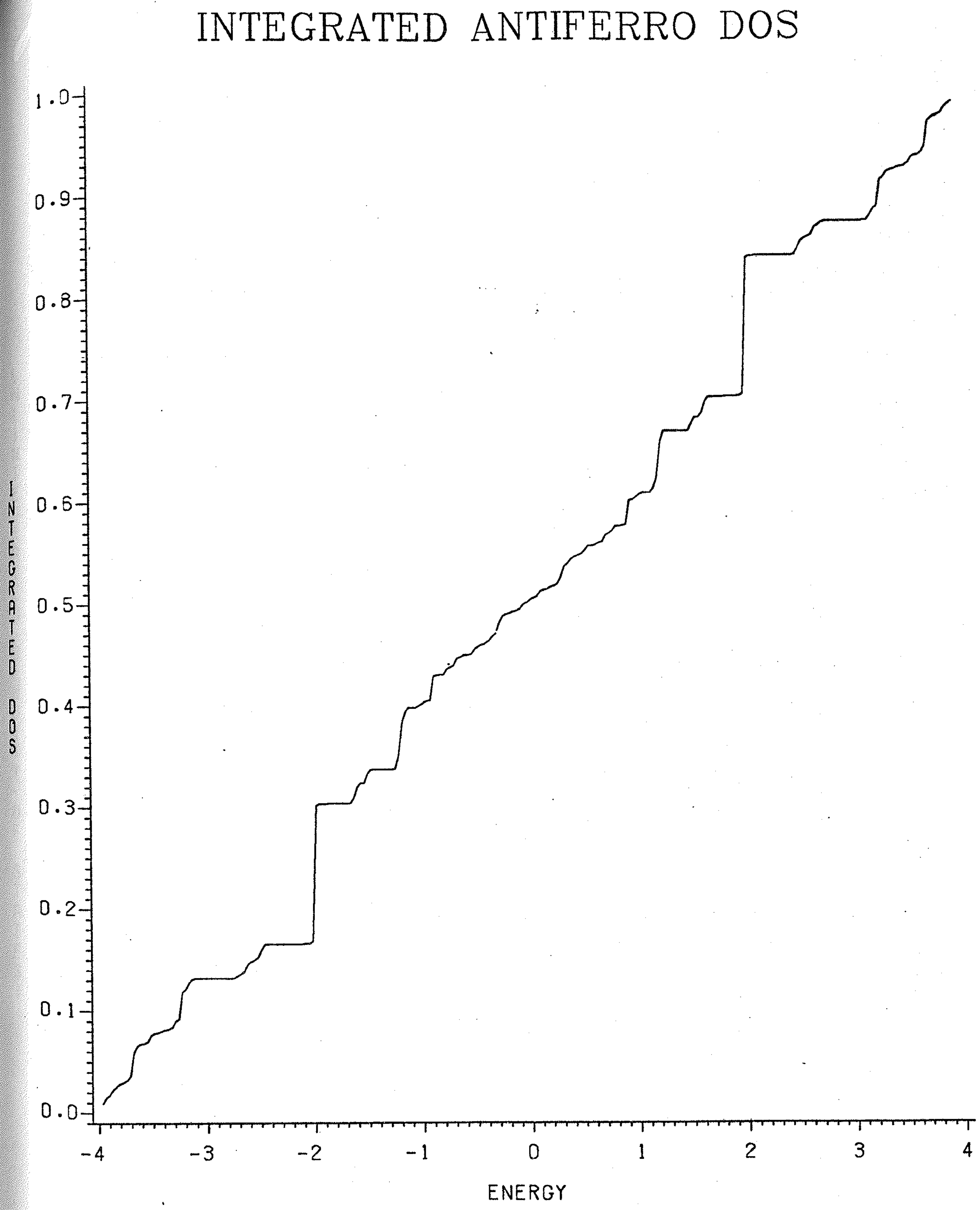

INTEGRATED ANTIFERRO DOS
INTEGRATED DOS
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-4
-3
-2
-1
0
1
2
3
4
ENERGY

Figure 4.2

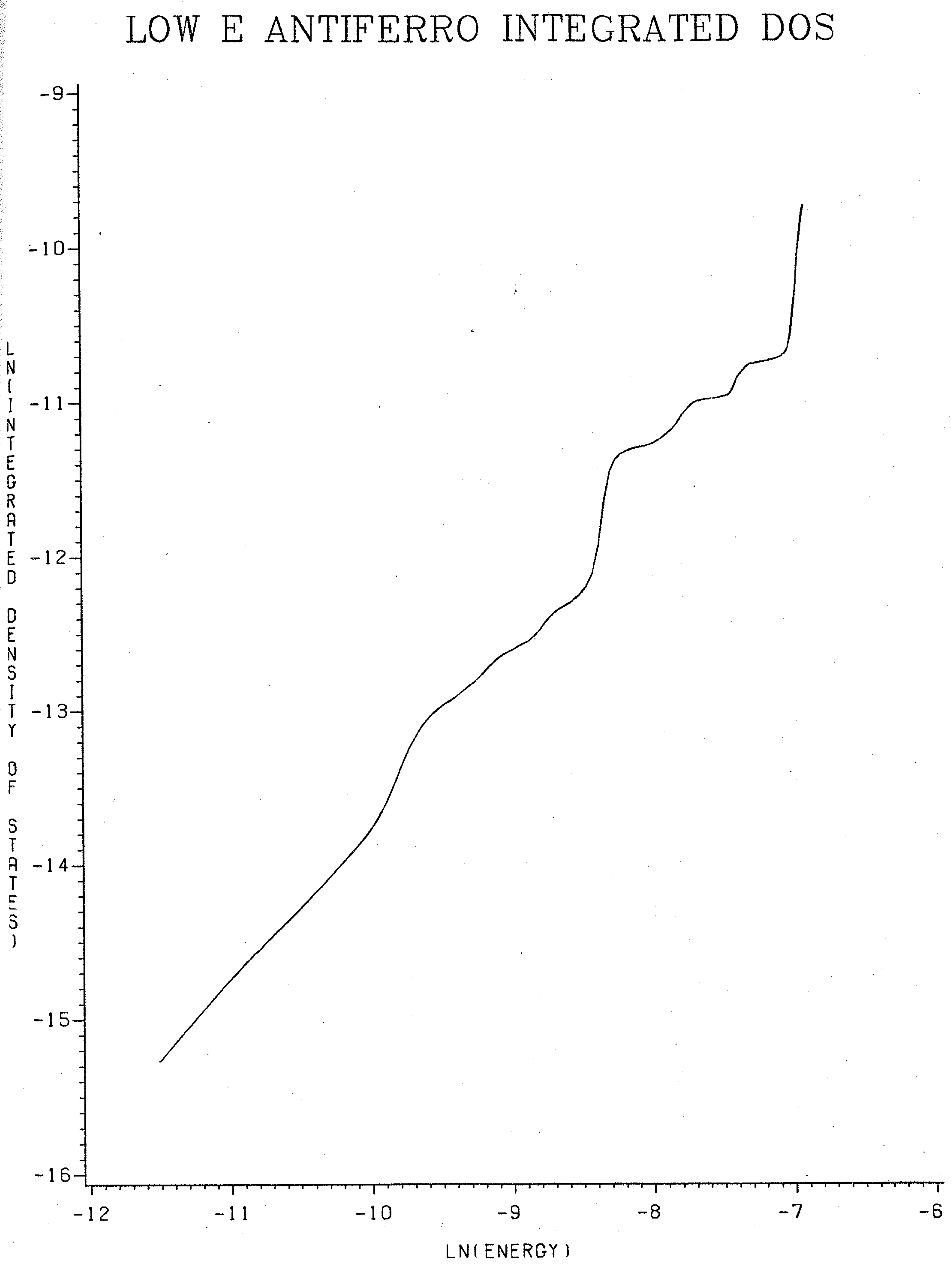

LOW E ANTIFERRO INTEGRATED DOS
LN(INTEGRATED DENSITY OF STATES)
-9
-10
-11
-12
-13
-14
-15
-16
-12
-11
-10
-9
-8
-7
-6
LN(ENERGY)

figure 4.3

Chapter V

CONCLUSIONS

In conclusion then, we have proposed a model of a dilute ferromagnet and antiferromagnet that is based upon a fractal known as the Viscek snowflake. The appearance of second neighbour interactions in the recursion relations for the parameters describing the Heisenburg Hamiltonian, while making the model more realistic, has led to an enlarged parameter space of three and six dimensions respectively. Previously used exact fractals have had a parameter space of only one dimension and the corresponding renormalization group transformations have been well studied where it was found that extremely complicated behaviour could occur. This increase to three and six dimensional maps thus offers incredible scope for the possibility of observing qualitatively new types of behaviour.

In addition, using real space renormalization group techniques we have obtained the spectral dimension for both the ferromagnet and antiferromagnet and have shown that they are in agreement with the numerical calculation of the same using a functional integral method. This, together with the fact that the low frequency behaviour of our spectra shows the same qualitative features as the model proposed by Orbach and Yu indicates that we are justified in modelling a dilute magnet using the Viscek snowflake. In particular, our results for the ferromagnet indicate a spectral dimension of 1.18 whereas Orbach and Yu obtained the result d=1, independent of dimension. Such a comparison is not possible in the case of the antiferromagnet since the (numerical) results of Orbach and Yu are confined to three dimensions only. Furthermore our results indicate that the effective dimension d (characterizing the scaling behaviour of the low frequency density of states) for the antiferromagnet is twice that for the ferromagnet, similar to what occurs in ordered magnetic systems. It thus appears that the effect of diluting a ferromagnet and antiferromagnet is to preserve this relationship between exponents.

One avenue of further work involves investigating the special case of the situation already presented, where we begin with a configuration for which all nearest neighbour interactions are zero and second neighbour interactions having some finite value. Since the nearest neighbour interactions are zero initially they remain so under iteration and the resulting situation is then one of an assembly of non-interacting chains with lengths that are powers of three. Thus in this limit of zero V_1 our model could be thought of as describing a polymer suspension and finite V_1 then corresponding to some interaction between the polymer chains.

The primary direction of research, however, will be one which (hopefully) will shed additional light on the presently controversial topic of the existence (or non-existence) of a qualitatively new type of excitation called fractons. As mentioned in the introduction, the recent work by Orbach and Yu indicates that in a dilute ferromagnet there is a sharp crossover to fracton excitations at small length scales whereas no such behaviour was found for the antiferromagnet. By embedding the Viscek snowflake into a square lattice in the same way as was done in reference [ ] we hope to address such a question. Thus the results which were presented here are just a first step in the investigation of this more general problem. The presence of next nearest neighbour interactions poses additional difficulties since no analytic results exist for the associated density of states.

Appendix A

FERROMAGNETIC RECURSION RELATIONS

In terms of the reduced variables X, Y, and Z our recursion relations can be written in the form

$$X' = f/(1 - h) \tag{A.1}$$

$$Y' = g/(1- h) \tag{A.2}$$

$$Z' = (z - 2 h)/(1 - h) \tag{A.3}$$

where the functions f, g and h are defined as follows:

$$f = A_1/B_1 \tag{A.4}$$

with

$$A_1 = X[Y(1-Y) + 2X^2]^2 \tag{A.5}$$

$$B_1 = [(Z-Y)(1-Y) - 2X^2]^2 - 4X^2(1-Y)^2 \tag{A.6}$$

and

$$g = A_2/B_2 \tag{A.7}$$

with

$$A_2 = \{ 2X^2(1-Y) + Y[(Z-Y)(1-Y) - 2X^2] \} [Y(1-Y) + 2X^2]^2 \tag{A.8}$$

$$B_2 = [(Z+Y)(1-Y) - 2X^2] B_1 \tag{A.9}$$

and

$$h = 2X^2/(1-Y) + A_3/B_3 \tag{A.10}$$

with

$$\begin{aligned} A_3 = \{ [\; Z(1-Y) - 2X^2 \;][\; (Z-Y)(1-Y) - 2X^2 \;] - 2X^2(1-Y)^2 \} \\ [\; Y(1-Y) + 2X^2 \;]^2 \end{aligned} \qquad (A.11)$$

$$B_3 = B_2 \qquad (A.12)$$

It follows from (A.3) that $Z=2$ (provided h is non-zero) and so eliminating $(1-h)$ from (A.1) using (A.2) yields the result $X^* = Y^*$. Then, substituting $X=Y$ and $Z=2$ into (A.1) gives

$$X' = X^3/[\; (1-2X)(3X^2-12X+4) \;] \qquad (A.13)$$

from which one obtains $X^*=1/3$. Also, if $X=0$ then $X^*=0$ ($Z=2$ always) and so the recursion relation for Y reduces to

$$Y' = Y^3/(4-3Y^2) \qquad (A.14)$$

and changing variables to Ω using

$$Y = 2/(2-\Omega) \qquad (A.15)$$

we recover (3.2-10).

Appendix B

ANTIFERROMAGNETIC RECURSION RELATIONS

In terms of the reduced variables the recursion relations for the antiferromagnet can be written as

$$X' = f/(1 - h \quad) \qquad (B.1)$$

$$Y_1' = g_1/(1 - h \quad) \ , \ Y_2' = g_2/(1 - h \quad) \qquad (B.2)$$

$$Z_1 = (Z_1 - 2 h_1)/(1 - h_1) \qquad (B.3)$$

$$Z_2 = (Z_2 - h_2)/(1 - h_1) \qquad (B.4)$$

$$Z_3 = (Z_3 - 2 h_2)/(1- h_1) \qquad (B.5)$$

where the functions f, g_1, g_2, h_1, and h_2 are given as follows:

$$f = A_1/B_1 \qquad (B.6)$$

with

$$A_1 = [Y_2(1-Y_1) + 2X^2][Y_1(Z_2-Y_2) + 2X^2] \qquad (B.7)$$

$$B_1 = [(Z_3-Y_2)(1-Y_1) - 2X^2][(Z_1-Y_1)(Z_2-Y_2) - 2X^2] - 4X^2(1-Y_1)(Z_2-Y_2) \qquad (B.8)$$

and

$$g_1 = A_2/B_2 \qquad (B.9)$$

with

$$A_2 = \{ 2X^2(1-Y_1) + Y_2[(Z_3-Y_2)(1-Y_1) - 2X^2] \}$$

$$[\ Y_1(Z_2-Y_2) + 2X^2\ ]^2$$

$$\tag{B.10}$$

$$B_2 = [\ (Z_1+Y_1)(Z_2-Y_2) - 2X^2\ ]\ B_1 \tag{B.11}$$

and

$$g_2 = A_3/B_3 \tag{B.12}$$

with

$$A_3 = \{\ 2X^2(Z_2-Y_2) + Y_2[\ (Z_1-Y_1)(Z_2-Y_2) - 2X^2\ ]\ \}$$

$$[\ Y_2(1-Y_1) + 2X^2\ ]^2 \tag{B.13}$$

$$B_3 = [\ (Z_3+Y_2)(1-Y_1) - 2X^2\ ]\ B_1 \tag{B.15}$$

and

$$h_1 = 2X^2/(Z_2-Y_2) - A_4/B_4 \tag{B.16}$$

with

$$A_4 = \{\ [\ Z_1(Z_2-Y_2) - 2X^2\ ][\ (Z_3-Y_2)(1-Y_1) - 2X^2\ ]$$

$$- 2X^2(1-Y_1)(Z_2-Y_2)\ \}\ [\ Y_1(Z_2-Y_2) + 2X^2\ ]^2 \tag{B.17}$$

$$B_4 = [\ (Z_1+Y_1)(Z_2-Y_2) - 2X^2\ ]\ (Z_2-Y_2)\ B_1 \tag{B.18}$$

and

$$h_2 = 2X^2/(1-Y_1) - A_5/B_5 \tag{B.19}$$

with

$$A_5 = \{\ [Z_3(1-Y_1) - 2X^2\ ][\ (Z_1-Y_1)(Z_2-Y_2) - 2X^2\ ]$$

$$- 2X^2(1-Y_1)(Z_2-Y_2)\ \}\ [\ Y_2(1-Y_1) + 2X^2\ ]^2 \tag{B.20}$$

$$B_5 = [\ (Z_3+Y_2)(1-Y_1) - 2X^2\ ]\ (1-Y_1)\ B_1 \tag{B.21}$$

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