Examples of Minimum Uniform Compactifications

by

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Abstract

Given a metric space (X, d), one can construct the "minimum uniform compactification" $u_d X$ of X. $u_d X$ is minimum in the sense that it is the smallest compactification of the Tychonoff space X (in the partially ordered set of compactifications of X) to which every bounded uniformly continuous real-valued function with domain X can be continuously extended . A substantial amount of theory on $u_d X$ has been developed in the case where (X, d) is a locally compact separable metric space. We will describe the properties of $u_d X$ using the language of clusters, as employed in the construction of $u_d X$. Also, we intend to analyze the structure of $u_d X \setminus X$ for some locally compact separable spaces other than the Euclidean space $\mathbb{R}^n$. Finally, we will analyze the structure of $u_d X \setminus X$ where (X, d) is one of several important nowhere locally compact complete separable metric spaces, in particular the irrationals with a compatible complete metric.

Acknowledgements

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Contents

Chapter 1

Introduction

1.1 Introduction

Given a metric space (X, d), one can construct the "minimum uniform compactification" u_dX of X. u_dX is minimum in the sense that it is the smallest compactification of a Tychonoff space X (in the partially ordered set of compactifications of X) to which every bounded uniformly continuous real valued function with domain X can be extended. In this thesis, we investigate some of the properties of u_dX. We begin by building u_dX using maximal clans. The construction of u_dX is similar to the construction of the Stone-Čech compactification of a Tychonoff space. In [Wo], the structure of u_dX is investigated in the case where (X, d) is a locally compact separable metric space. We apply results from [Wo] to construct the minimum uniform compactification of the locally compact separable metric space $X = \cup_{n\in\mathbb{N}}[n-1, n] \times [0, 1]^{n-1}$ with the subspace metric inherited from the "standard" metric on $\mathbb{R}^\omega$. We show that the outgrowth $u_dX \setminus X$ can be written as the union of two regular closed sets each homeomorphic to $[0, 1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$, whose intersection is nowhere dense in $u_dX \setminus X$. Lastly, we use inverse limit systems to build the the minimum uniform compactification of a certain class of metric spaces. We apply this technique to construct the minimum uniform compactification of the irrationals. Some results on the minimum uniform compactification of the irrationals are given.

1.2 Organization of the Thesis

The main goal of this thesis is to provide some examples of minimum uniform compactifications and how they are constructed.

In Chapter 2 we formally define compactifications and state some well known results concerning compactifications. In addition, we define the minimum uniform compactification u_dX of a metric space (X, d). We briefly state some results known about u_dX.

The construction of the minimum uniform compactification u_dX of a metric space (X, d) is described in Chapter 3. Following arguments similar to [Ra], u_dX is constructed from maximal clans. The construction of u_dX is similar to the construction of the Stone-Cěch compactification of a Tychonoff space. Some properties of u_dX derived from the construction are also stated.

In Chapter 4 we analyze the minimum uniform compactification of a locally compact $\sigma-$compact complete metric space. Using results from [Wo], a detail analysis of the compactification is given. The motivation for this chapter comes from the analysis of $u\mathbb{R} \setminus \mathbb{R}$ in [Wo]. The approach taken in this chapter follows that of the one taken in [Wo].

Chapter 5 states some well known results on inverse limit systems. These results are then used in Chapter 6 in constructing minimum uniform compactifications. A technique is given in Chapter 6 on how to construct the minimum uniform compactification for certain metric spaces.

In Chapter 7, we apply the results from Chapter 6 to build the minimum uniform compactification of the irrationals. Some properties of this compactification are given. The main result gives a characterization of the clopen subsets of this compactification.

1.3 Frequently used Symbols

Some of the symbols used frequently throughout this thesis are defined as follows:

- $\mathbb{R}$ - the real line
- $\mathbb{Q}$ - the rationals
- $\mathbb{N}$ - the positive integers
- βX - the Stone-Cĕch compactification of a space X
- $u_d X$ - the minimum uniform compactification of the metric space (X, d)
- I - the subspace $[0, 1]$ of $\mathbb{R}$
- $C(X)$ - the ring of continuous real-valued continuous functions with domain X
- $C^*(X)$ - the ring of bounded continuous real-valued functions with domain X
- $U_d^*(X)$ - the ring of bounded uniformly continuous real-valued functions with domain (X, d)
- A^u - $cl_{u_d X} A \setminus X$ where A is a subset of the metric space (X, d)
- $\mathcal{P}(X)$ - power set of the set X

Chapter 2

Introduction to Compactifications

Most of the material in this chapter can be found in [PW]. The results pertaining to the minimum uniform compactification of a metric space can be found in [Wo]. For proofs of the results in this chapter and a more definitive treatment of the subject, the interested reader should consult the references.

2.1 Some Basic Results on Compactification

Let X be a Tychonoff space. Then a *compactification* of X is a compact Hausdorff space αX that contains X as a dense subspace. Two compactifications αX and γX of X are said to be *equivalent* if there is a homeomorphism h from αX onto γX such that for each x in X, $h(x) = x$. We denote this by writing $\alpha X \cong \gamma X$. Equivalent compactifications are considered to be the "same". Let $\mathcal{K}(X)$ denote the class of all compactifications of X. If we identify equivalent compactifications, then $\mathcal{K}(X)$ can be regarded as a set and partially ordered as follows : $\alpha X \leq \gamma X$ if there is a continuous function $f : \gamma X \to \alpha X$ such that for each x in X, $f(x) = x$. It is a well known result that $(\mathcal{K}(X), \leq)$ is a complete upper semilattice whose largest member is the Stone-Cěch compactification βX. It is also well known that $(\mathcal{K}(X), \leq)$ is a complete lattice if and only if X is locally compact (see 4.3.e of [PW]). If X is locally compact, then the one-point compactification X^* is the smallest member of $(\mathcal{K}(X), \leq)$.

Taimonov's theorem gives a characterization for when $\alpha X \leq \gamma X$ where αX and γX are two compactifications of X. The theorem can be found as 4.2.h of [PW] and is stated below for reference.

Theorem 2.1 : *Let X be a Tychonoff space and let αX, $\gamma X \in \mathcal{K}(X)$. The following are equivalent :*

1. $\alpha X \geq \gamma X$.
2. *If A and B are disjoint closed subsets of X and if $cl_{\gamma X} A \cap cl_{\gamma X} B = \emptyset$, then $cl_{\alpha X} A \cap cl_{\alpha X} B = \emptyset$.*

The following result is well known (see 4.6.b2 of [PW]).

Theorem 2.2 : *Let X be a Tychonoff space, αX a compactification of X. Then $\{cl_{\alpha X} C : C \text{ closed in } X\}$ is a base for the closed subsets of αX.*

For a space X, the projective maximum X is denoted by βX and is called the *Stone-Cěch compactification* of X. There is a great deal of results on βX (see [GJ], [W], [PW]). Some useful results for the Stone-Cěch compactification of a Tychonoff space X will now be stated.

We give a characterization of βX that we shall use in this thesis. See 4.6.g of [PW] for a reference.

Theorem 2.3 : *Let X be a Tychonoff space. The following are equivalent for $\alpha X \in \mathcal{K}(X)$:*

1. $\alpha X \cong \beta X$ (as compactifications of X).

2. disjoint zero-sets in X have disjoint closures in αX.

A space X is said to be *zero-dimensional* if it has a base of open-closed (clopen) subsets. We denote the collection of all clopen subsets of X by $\mathcal{B}(X)$. A Tychonoff space X is called *strongly zero-dimensional* if βX is zero-dimensional. We now give a result concerning Lindelof zero-dimensional spaces (see 4.8.i of [PW]).

Theorem 2.4 : *Let A and B be disjoint closed subsets of the zero-dimensional Lindelof space X. Then there exists $C \in \mathcal{B}(X)$ such that $A \subseteq C$ and $B \cap C = \emptyset$ (that is, X is strongly zero-dimensional).*

2.2 Boolean Algebras and Stone Spaces

In this section , we give a brief introduction to Boolean algebras and Stone spaces. Lattices are mentioned but not defined. The reader is assumed to have a knowledge of lattices. For a thorough treatment of lattices, the interested reader should refer to Chapter 2 of [PW]. The material in this section have been taken from Chapters 3 and 4 of [PW]. We begin with some basic definitions.

A *Boolean Algebra* is a complemented distributive lattice. Boolean algebras are important in the study of Stone spaces. We shall be most concerned with the Boolean algebra of the clopen subsets of a topological space X. We denote this Boolean algebra by $\mathcal{B}(X)$. If $(B, \vee, \wedge, ', 0, 1)$ is a Boolean algebra and $A \subset B$, A is called a *Boolean subalgebra* if $\{0,1\} \subseteq A$ and $a, b \in A$ implies $a \wedge b$, $a \vee b$, $a' \in A$.

Let A, B be Boolean algebras. A function $f : A \rightarrow B$ is a *Boolean homomorphism* if $a, b \in A$ implies $f(a \vee b) = f(a) \vee f(b)$ and for $a \in A$, $f(a') = (f(a))'$. The function

f is called a *Boolean isomorphism* if f is a bijection and both f and $f^{\leftarrow}$ are Boolean homomorphisms.

We now define the Stone space of a Boolean algebra. Let B be a Boolean algebra and let $S(B) = \{\mathcal{U} : \mathcal{U}$ is a $B-$ultrafilter$\}$. For $a \in B$ let $\lambda(a) = \{\mathcal{U} \in S(B) : a \in \mathcal{U}\}$. The set $S(B)$ equipped with the topology for which $\{\lambda(a) : a \in B\}$ is an open base is called the *Stone space of* B.

The following is Stone's representation theorem. This can be found as theorem 3.2.d of [PW].

Theorem 2.5 : *Let B be a Boolean algebra. Then :*

1. *$S(B)$ is a compact zero-dimensional space,*
2. *$\{\lambda(a) : a \in B\} = \mathcal{B}(S(B))$, and*
3. *λ is a Boolean isomorphism from B onto $\mathcal{B}(S(B))$.*

We now restrict our discussion of Boolean algebras to the Boolean algebra $\mathcal{B}(X)$ of the clopen subsets of a zero-dimensional space X.

Let X be a zero-dimensional space and let $\mathcal{B} \subseteq \mathcal{B}(X)$ be a Boolean subalgebra such that $\mathcal{B}$ is a clopen base for X. By Stone's representation theorem, $S(\mathcal{B})$ is a compact zero-dimensional space and for each $A \in \mathcal{B}$, the mapping $\lambda(A) = cl_{S(\mathcal{B})}A$ is a Boolean isomorphism from $\mathcal{B}$ onto $\mathcal{B}(S(\mathcal{B}))$. The following is Proposition 4.7.b of [PW].

Theorem 2.6 : *Let X be a zero-dimensional space and let $\mathcal{B} \subseteq \mathcal{B}(X)$ be a Boolean subalgebra such that $\mathcal{B}$ is an open base for X. For $x \in X$, let $U_x = \{B \in \mathcal{B} : x \in B\}$. Then :*

1. *For* $x \in X$, $U_x \in S(\mathcal{B})$,

2. *The function* $f : X \to S(\mathcal{B})$ *defined by* $f(x) = U_x$ *is a dense embedding, and*

3. *If* $B_1, B_2 \in \mathcal{B}$ *are disjoint, then* $cl_{S(\mathcal{B})}B_1 \cap cl_{S(\mathcal{B})}B_2 = \emptyset$.

By identifying each $x \in X$ with U_x, we may think of X as a subset of $S(\mathcal{B})$. Thus $S(\mathcal{B})$ is a compactification of X.

2.3 The Minimum Uniform Compactification

Let (X, d) be a metric space. We may regard X as a topological space with the metric topology τ_d inducted by d. It is well known (see 7.7 of [NW]) that there exists a compactification of X, called the *Smirnov compactification* of X. In Chapter 3, we will give the actual construction of the Smirnov compactification of X. Properties of this compactification are stated in the following theorem (see 1.1 of [Wo]).

Theorem 2.7 : *Let* (X, d) *be a metric space. Then the topological space* (X, τ_d) *has a compactification* u_dX *with these properties :*

1. *If* $A, B \in \mathcal{P}(X)$ *then* $cl_{u_dX}A \cap cl_{u_dX}B \neq \emptyset$ *if and only if* $d(A, B) = 0$

2. *If* (X, d) *and* (Y, e) *are metric spaces and* $f : X \to Y$ *is uniformly continuous, then there is a continuous function* $f^u : u_dX \to u_eY$ *such that* $f^u|X = f$.

3. *Let* $U_d^*(X)$ *denote the ring of all bounded real-valued uniformly continuous functions with domain* (X, d). *If* $f \in U_d^*(X)$ *then there is a (necessarily unique) continuous function* $f^* : u_dX \to \mathbb{R}$ *such that* $f^*|X = f$.

We call u_dX the *minimum uniform compactification* of the metric space (X, d). u_dX is minimum in the sense that it is the smallest compactification of X to which each member of $U^*(X)$ can be extended continuously. The mininum uniform compactification can be characterized in several ways. The following is 2.5 of [Wo]:

Theorem 2.8 : *Let (X,d) be a metric space and let $\alpha X \in \mathcal{K}(X)$. The following are equivalent :*

1. *$\alpha X \cong u_d X$ (as compactifications of X).*
2. *If $A, B \subseteq X$ then $cl_{\alpha X} A \cap cl_{\alpha X} B \neq \emptyset$ if and only if $d(A,B) = 0$.*
3. *$\{f \in C^*(X) : f$ can be continously extended to $\alpha X\} = U_d^*(X)$.*

A useful consequence of theorem 2.8 is the following. It is stated as corollary 2.6 of [Wo].

Corollary 2.9 : *Let (X,d) be a metric space. If $A \subseteq X$ and $x \in u_d X \setminus cl_{u_d X} A$, then there exists a closed subset B of X such that $x \in cl_{u_d X} B$ and $d(A,B) > 0$.*

The following theorem is 2.9 of [Wo] :

Theorem 2.10 : *If (X,d) is a metric space , and if $S \subseteq X$, then $cl_{u_d X} S \cong uS$ (up to equivalence), where uS is the minimum uniform compactification of the metric space $(S, d|S)$.*

If (X,d) and (Y,e) are two metric spaces, then a bijection $f : X \rightarrow Y$ is called a *uniform isomorphism* if f and $f^{\leftarrow}$ are uniformly continuous. If a uniform isomorphism exists between X and Y, we say that the metric spaces (X,d) and (Y,e) are *uniformly equivalent.* If $X = Y$ then the metrics d and e are said to be uniformly equivalent. Clearly uniformly isomorphic spaces are homeomorphic but the converse fails. The following theorem (see 2.10 of [Wo]) states a sufficient condition for the minimum uniform compactifications of two metric spaces to be homeomorphic.

Theorem 2.11 : *If (X,d) and (Y,e) are metric spaces and if $f : X \rightarrow Y$ is a uniform isomorphism, then f extends to a homeomorphism $F : u_d X \rightarrow u_e Y$; in particular, $u_d X \setminus X$ is homeomorphic to $u_s Y \setminus Y$.*

Let (X, d) be a metric space. The *metric completion* of (X, d) will be denoted by (X^*, d^*). The following theorem (see 3.2 of [Wo]) relates the minimum uniform compactification of (X, d) and the minimum uniform compactification of (X^*, d^*).

Theorem 2.12 : *Let (X, d) be a metric space. Then $u_{d^*}X^*$ is equivalent (as a compactification of X) to u_dX.*

The theorem states we may deal exclusively with complete metric spaces when analyzing the minimum uniform compactifications of metric spaces.

If $\epsilon > 0$ is given , a subset D of X is called *ϵ–discrete* if $S(x, \epsilon) \cap D = \{x\}$ for each $x \in D$. We now give a characterization of those metric spaces (X, d) such that $\beta X \cong u_dX$ (see 3.4 of [Wo]).

Theorem 2.13 : *The following are equivalent for a metric space (X, d) :*

1. $u_dX \cong \beta X$.
2. $C^*(X) = U^*(X)$.
3. $C(X) = U(X)$.
4. *There is a compact subset K of X such that $X \setminus K$ consists of isolated points of X and for each $r > 0$ there exists $\epsilon_r > 0$ such that $\{x \in X : d(x, K) >\}$ is ϵ_r–discrete.*

Let X and Y be Tychonoff spaces. Glicksberg's theorem (see 4AG of [PW]) states that $\beta(X \times Y) \cong \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact. If (X, d) and (Y, e) are metric spaces, it is natural to ask whether there is a necessary and sufficient conditions on X and Y such that $u_dX \times u_eY \cong u_t(X \times Y)$ where the metric t on $X \times Y$ is defined as follows :

$$t((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + e(y_1, y_2).$$

Such conditions are known and are stated in the following theorem (see 3.6 of [Wo]) .

Theorem 2.14 : *The following are equivalent for two metric spaces (X, d) and (Y, e) :*

1. *$u_t(X \times Y) \cong u_d X \times u_e Y$ (where t is as described above).*
2. *At least one of (X, d) and (Y, e) is totally bounded.*

If a metric space (X, d) is locally compact and $\sigma-$compact, then [Wo] contain results that can be used to analyze $u_d X$. If X is a locally compact $\sigma-$compact noncompact Hausdorff space, it is well known (see 11.7.2 of [D]) that there exists a sequence $\{K_n : n \in \mathbb{N}\}$ of compact subsets of X such that :

1. K_n is a proper subset of $int_X K_{n+1}$,
2. $K_n = cl_X int_X K_n$,
3. $X = \cup_{n \in \mathbb{N}} K_n$.

We state this formally in the next theorem .

Theorem 2.15 : *If X is a locally compact $\sigma-$compact noncompact Hausdorff space, then there exists a sequence $\{K_n : n \in \mathbb{N}\}$ of compact subsets of X such that :*

1. *K_n is a proper subset of $int_X K_{n+1}$,*
2. *$K_n = cl_X int_X K_n$,*
3. *$X = \cup_{n \in \mathbb{N}} K_n$.*

Suppose C is a subset of a metric space (X, d). Then let C^u denote $cl_{u_d X} C \setminus X$. Clearly, for $A \subseteq X$, $(cl_X A)^u = A^u$.

The following theorem is 4.2 of [Wo].

Theorem 2.16 : *Let (X, d) be a locally compact $\sigma-$compact metric space, and let A and B be two closed noncompact subsets of X. The following are equivalent :*

1. $lim_{n \to \infty} sup\{d(x, A) : x \in B \setminus K_n\} = 0$,

2. $B^u \subseteq A^u$.

For a more definitive presentation of the minimum uniform compactification, the reader is asked to refer to [Wo].

Chapter 3

Building the Minimum Uniform Compactification

The minimum uniform compactification $u_d X$ of a metric space (X, d) can be built abstractly much like the Stone-Cěch compactification of a Tychonoff space. The method for the construction of both these compactifications of a metric space (X, d) are quite similar. The main difference is that the underlying "points" of the two compactifications come from different sources. The points of the Stone-Cěch compactification are z-ultrafilters of X whereas the points of the minimum uniform compactification are the " $d-$ clusters" of X. We proceed to build the minimum uniform compactification $u_d X$ of a metric space (X, d) in this chapter. This compactification is a special case of the Smirnov Compactification of a proximity space. The Smirnov Compactification can be found in [NW] or [Ra]. Most of the material in this chapter can be traced to the manuscript by Dr. M. Rayburn [Ra].

3.1 Definitions

Several terms need to be defined in this section. As we are dealing only with metric spaces, all terms will be defined in terms of metric space. However, most of these definitions can be generalized to a wider range of spaces (in particular, proximity spaces).

Definition 3.1 : *Let (X, d) be a metric space. A clan σ is a non-empty collection σ of subsets of X such that :*

1. *if $A \in \sigma$ and $A \subseteq B$ then $B \in \sigma$,* (3.1)

2. *if $A \cup B \in \sigma$ then $A \in \sigma$ or $B \in \sigma$,* (3.2)

3. *if $A, B \in \sigma$ then $d(A, B) = 0$.* (3.3)

Using a standard Zorn's Lemma argument, one easily sees that any clan is contained in some maximal clan. Maximal clans are clans which are not strictly contained in any other clan. If A is a subset of a space X and σ is collection of subsets of X, then we write $d(A, \sigma) = 0$ to denote

$$\forall B \in \sigma, d(A, B) = 0.$$

Definition 3.2 : *A clan σ on a space (X, d) is called a d-cluster if*

$$A \subseteq X \text{ and } d(A, \sigma) = 0 \text{ implies } A \in \sigma. \tag{3.4}$$

If there is no ambiguity about which metric is under discussion, we will say "cluster" rather than "d-cluster".

3.2 Some Results on Clusters

This section will list some results for clusters that will be useful in the construction of $u_d X$. The main result in this section is that maximal clans and clusters are the same objects. This will be a useful fact since results are sometimes easier to obtain using maximal clans instead of clusters and vice versa.

Lemma 3.3 : *Let (X, d) be a metric space and let $A, B \subseteq X$. Then $d(A, B) = 0$ if and only if there exist sequences $\{a_n\} \subseteq A, \{b_n\} \subseteq B$ such that $lim_{n\to\infty} d(a_n, b_n) = 0$.*

Proof :
If $d(A, B) = 0$, then for each $n \in \mathbb{N}$ there exists $a_n \in A$, $b_n \in B$ such that $d(a_n, b_n) \leq \frac{1}{n}$. Thus $lim_{n\to\infty} d(a_n, b_n) = 0$. □
The other implication follows directly from the definition. □

Lemma 3.4 : *Let (X, d) be a metric space with $A, B \subseteq X$. If $d(A \cup B, C) = 0$ then $d(A, C) = 0$ or $d(B, C) = 0$*

Proof : If $d(A \cup B, C) = 0$; then there exists sequences $\{a_n\} \subseteq A \cup B$ and $\{c_n\} \subseteq C$ such that $lim_{n\to\infty} d(a_n, c_n) = 0$. If $d(A, C) \neq 0$ then there exists some $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $a_n \in B$. Hence $d(B, C) = 0$. □

Definition 3.5 : *For each point x in a metric space (X, d), define $\sigma_x = \{A \subseteq X : d(A, x) = 0\}$. σ_x is called a point-cluster.*

Clearly $\{x\} \in \sigma_x$.

Theorem 3.6 : *For each $x \in X$, σ_x is a cluster.*

Proof : Let $x \in X$ be given.

1. Suppose $A \in \sigma_x$ and $A \subseteq B$. As $d(A, x) = 0$, then $d(B, x) = 0$. Thus $B \in \sigma_x$ and hence σ_x satisfies (3.1).

2. Let $A, B \in \sigma_x$. Then $d(A, x) = 0$ and $d(B, x) = 0$, so there exists a sequence $\{a_n\} \subseteq A$ and $\{b_n\} \subseteq B$ such that $\lim_{n\to\infty} d(a_n, x) = 0$ and $\lim_{n\to\infty} d(b_n, x) = 0$. By the triangle inequality, we see that for each $n \in \mathbb{N}$

$$d(a_n, b_n) \leq d(a_n, x) + d(x, b_n).$$

This implies

$$\lim_{n\to\infty} d(a_n, b_n) = 0.$$

Hence $d(A, B) = 0$ and σ_x satisfies (3.3).

3. If $d(A, C) = 0$ for all $C \in \sigma_x$, then in particular $d(A, \{x\}) = 0$ as $\{x\} \in \sigma_x$. Thus $d(A, x) = 0$ and so $A \in \sigma_x$. Hence σ_x satisfies (3.4).

4. Suppose that $A \cup B \in \sigma_x$; then $d(A \cup B, x) = 0$. By lemma 3.4 $d(A, x) = 0$ or $d(B, x) = 0$. Thus σ_x satisfies (3.2). □

The following is an example of a collection σ of subsets of a space (X, d) that satisfies (3.1), (3.3), and (3.4) but doesn't satisfy (3.2).

Example 3.7 : *Let $(\mathbb{R}, d)$ be the Euclidean metric space. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, $B = \{-\frac{1}{n} : n \in \mathbb{N}\}$, and $C = \{1 - \frac{1}{n} : n \geq 2\} \cup \{-1 + \frac{1}{n} : n \geq 2\}$. Clearly*

$$d(A, B) = 0, d(A, C) = 0 \text{ and } d(B, C) = 0 \tag{3.5}$$

Let $\lambda = \{S \subseteq X : A \subseteq S$ or $B \subseteq S$ or $C \subseteq S\}$. It is clear from (3.5) that, if $S, T \in \lambda$ then $d(S, T) = 0$. Hence λ satisfy (3.1). By the definition of λ, if $S \in \lambda$ and $S \subseteq T \subseteq X$, then $T \in \lambda$. Thus λ satisfy (3.3).

Let $\mathcal{P} = \{\sigma \subset \mathcal{P}(X) : \sigma$ satisfies (3.1) and (3.3)$\}$. Then clearly, $\lambda \in \mathcal{P}$. Now order $\mathcal{P}$ be inclusion . $(\mathcal{P}, \subseteq)$ is a partially order set (POSET). By Zorn's lemma, there is a maximal chain $\mathcal{C} \subseteq \mathcal{P}$ such that $\lambda \in \mathcal{C}$. Let $\gamma = \cup\mathcal{C}$.

Claim : *γ satisfies (3.1), (3.3) and (3.4).*

Proof of Claim :

1. If $S \in \gamma$ and $S \subseteq T \subseteq X$, then there exists some $\alpha \in \mathcal{C}$ such that $S \in \alpha$. As α satisfies (3.1), $T \in \alpha$. Thus $T \in \gamma$.

2. If $S, T \in \gamma$, then there exists $\alpha_1, \alpha_2 \in \mathcal{C}$ such that $S \in \alpha_1$ and $T \in \alpha_2$. Since $\mathcal{C}$ is a chain, either $\alpha_1 \leq \alpha_2$ or $\alpha_2 \leq \alpha_1$. Without loss of generality, assume $\alpha_1 \leq \alpha_2$. Then $S, T \in \alpha_2$. But as α_2 satisfies (3.3), then $d(S, T) = 0$.

3. Suppose γ doesn't satisfy (3.4). Then there exists $S \subseteq X$ such that for all $T \in \gamma$, $d(S, T) = 0$ but $S \notin \gamma$.
   Let $\gamma^* = \gamma \cup \{U \subseteq X : S \subseteq U\}$. Clearly γ^* satisfies (3.1),(3.3) and γ is strictly contained in γ^*. This contradicts the maximality of γ. Hence γ must satisfy (3.4) and the claim holds.

It was just shown that γ satisfies (3.1), (3.3) and (3.4). We now proceed to show that γ fails (3.2).

Recall $C = \{1 - \frac{1}{n} : n \geq 2\} \cup \{-1 + \frac{1}{n} : n \geq 2\}$ and $C \in \gamma$. Let $C_1 = \{1 - \frac{1}{n} : n \geq 2\}$ and $C_2 = \{-1 + \frac{1}{n} : n \geq 2\}$. Clearly, $C = C_1 \cup C_2$. Now $d(A, C_2) = \frac{1}{2} > 0$, hence $C_2 \notin \gamma$. Similarly $d(B, C_1) = \frac{1}{2} > 0$, hence $C_1 \notin \gamma$. This shows γ doesn't satisfy (3.2). □

In a metric space, distinct points give rise to distinct point clusters. This will be important when we build the minimum uniform compactification of a metric space. Namely, the set of point clusters is identified with the metric space in the minimum uniform compactification.

Theorem 3.8 : *If $x \neq y$ then $\sigma_x \neq \sigma_y$.*

Proof : If $x \neq y$ then $d(x, y) \neq 0$ and hence $\{x\} \notin \sigma_y$ and $\{y\} \notin \sigma_x$. This shows $\sigma_x \neq \sigma_y$. □

Lemma 3.9 : *If σ is a cluster in X then $X \in \sigma$.*

Proof: If $A \in \sigma$, then $d(A, X) = 0$ since $A \subseteq X$. As σ is a cluster, (3.4) implies $X \in \sigma$. □

Corollary 3.10 : *Given $A \subseteq X$ and a cluster σ, either $A \in \sigma$ or $X \setminus A \in \sigma$.*

Proof : Let A be a subset of X . We see that $X = A \cup (X \setminus A)$ and by lemma 3.9, $X \in \sigma$. Since σ is a cluster, either $A \in \sigma$ or $X \setminus A \in \sigma$. □

Lemma 3.11 : *Let (X, d) be a metric space and let A, B be subsets of X. If $d(A, B) \neq 0$ then there exists $E \subseteq X$ such that $d(A, E) \neq 0$ and $d(X \setminus E, B) \neq 0$.*

Proof :Suppose $d(A, B) = \epsilon$ where $\epsilon > 0$. Let $E = \{x \in X : d(x, B) \leq \frac{\epsilon}{2}\}$. Then $d(A, E) \geq \frac{\epsilon}{2}$ and $d(X \setminus E, B) \geq \frac{\epsilon}{2}$. □

It is a well known result [Wi] that if A, B are subsets of a space (X, d) then

$$d(A, B) = 0 \text{ if and only if } d(cl_X A, cl_X B) = 0 \tag{3.6}$$

Theorem 3.12 : *If $A \subseteq X$ and σ is a cluster on X, then $A \in \sigma$ if and only if $cl_X A \in \sigma$.*

Proof : If $A \in \sigma$, then by (3.1), $cl_X A \in \sigma$. On the other hand, if $A \notin \sigma$, then by (3.4), there exists $C \in \sigma$ such that $d(A, C) \neq 0$. By (3.6), $d(cl_X A, cl_X C) \neq 0$. As $C \in \sigma$ and $C \subseteq cl_X C$, by (3.1), $cl_X C \in \sigma$. Hence by (3.3), $cl_X A \notin \sigma$. □

The following theorem shows that clusters and maximal clans on a metric space are the same.

Theorem 3.13 : *The following are equivalent for a clan σ on a metric space (X, d)*

1. σ is a maximal clan,

2. σ is a cluster.

Proof :

2) $\Longrightarrow$ 1) Let σ be a cluster on X, and suppose that β is a maximal clan such that $\sigma \subseteq \beta$. Let $A \in \beta$. Since for all $S \in \beta$, $d(A, S) = 0$, we see that for all $S \in \sigma$, $d(A, S) = 0$, so $A \in \sigma$. Thus $\beta = \sigma$. Hence σ is a maximal clan as required.

1) $\Longrightarrow$ 2) Let σ be a maximal clan. We show that σ is a cluster.

Let $\lambda = \{C \subseteq X : \forall B \in \sigma, d(C, B) = 0\}$.

1. Let $A \in \lambda$ and let C be a set containing A. Then ; for all $B \in \sigma$, $d(A, B) = 0$ which implies $d(C, B) = 0$. Hence $C \in \lambda$.

2. Let $A, B \in \lambda$ and suppose $d(A, B) \neq 0$. Then; by lemma 3.11 , there exist $C \subseteq X$ such that $d(A, C) \neq 0$ and $d(B, X \setminus C) \neq 0$. As σ is a clan and $X = C \cup (X \setminus C) \in \sigma$, then either $C \in \sigma$ or $X \setminus C \in \sigma$. In either case, it contradicts the definition of λ. Hence $A, B \in \lambda$ implies $d(A, B) = 0$.

3. Let $A, B \subseteq X$ such that $A \notin \lambda$ and $B \notin \lambda$.

   Claim : $A \cup B \notin \lambda$.

   Proof of claim : Since $A \notin \lambda$, there exist $D_1 \in \sigma$ such that $d(A, D_1) \neq 0$. Thus, there exists C_1 such that $d(A, C_1) \neq 0$ and $d(D_1, X \setminus C_1) \neq 0$. Similarly, since $B \notin \lambda$ there exists $D_2 \in \sigma$ such that $d(B, D_2) \neq 0$. Thus, there exist C_2 such that $d(B, C_2) \neq 0$ and $d(D_2, X \setminus C_2) \neq 0$.

   Note $X = (C_1 \cap C_2) \cup (X \setminus (C_1 \cap C_2)) \in \sigma$. Hence $C_1 \cap C_2 \in \sigma$ or $(X \setminus (C_1 \cap C_2)) \in \sigma$. If $(X \setminus (C_1 \cap C_2)) = ((X \setminus C_1) \cup (X \setminus C_2)) \in \sigma$, then $X \setminus C_1 \in \sigma$ or $X \setminus C_2 \in \sigma$. But $X \setminus C_1$ cannot belong to σ since $d(D_1, X \setminus C_1) \neq 0$. Similarly, But $X \setminus C_2$ cannot belong to σ since $d(D_2, X \setminus C_2) \neq 0$. Thus $(X \setminus (C_1 \cap C_2)) \notin \sigma$ and hence $C_1 \cap C_2 \in \sigma$.

Now $d(C_1 \cap C_2, A) \neq 0$ because $d(C_1, A) \neq 0$. Similarly $d(C_1 \cap C_2, B) \neq 0$ because $d(C_2, B) \neq 0$. By lemma 3.4, $d(C_1 \cap C_2, A \cup B) \neq 0$. Hence $A \cup B \notin \lambda$.

The 3 items above imply λ is a clan. By definition of λ, λ satisfies (3.4). Since σ is a maximal clan, $\sigma = \lambda$. Hence σ is a cluster. □

The above theorem provides a characterization of clusters that will be used in subsequent proofs.

Theorem 3.14 : *If $x \in X$ and σ is a cluster containing $\{x\}$, then $\sigma = \sigma_x$.*

Proof : Suppose $x \in X, \sigma$ is a cluster and $\{x\} \in \sigma$. If $A \in \sigma$, then $d(A, x) = 0$ which by definition means $A \in \sigma_x$. Thus $\sigma \subseteq \sigma_x$. By the maximality of σ_x, $\sigma = \sigma_x$. □

We now give a useful characterization of T_2 space . This characterization involves closed sets instead of open sets.

Theorem 3.15 : *The following are equivalent :*

1. *A topological space X is T_2,*
2. *Given two distinct points x and y in X, there are 2 closed sets F_x, F_y of X such that $x \in F_x \subseteq (X \setminus \{y\})$ and $y \in F_y \subseteq (X \setminus \{x\})$ and $F_x \cup F_y = X$.*

Proof : $(1 \Rightarrow 2)$ Suppose X is T_2. Let x and y be distinct points of X. Then there exist open disjoint subsets U_x, U_x of X such that $x \in U_x$ and $y \in U_y$. Let $F_x = cl_X U_x$, $F_y = X \setminus U_x$. It is clear that $F_x \cup F_y = X$. It is also clear that $x \in F_x$. Suppose $y \in cl_X U_X = F_x$. Then as U_y is a neighborhood of y, we have $U_x \cap U_y \neq \emptyset$ which is a contradiction. Hence $y \notin F_x$ which implies $F_x \subseteq (X \setminus \{y\})$. Using a similar argument, we see $y \in F_y$ and $F_y \subseteq (X \setminus \{x\})$

$(2 \Rightarrow 1)$: Let x and y be distinct points of X.Let F_x and F_y be closed subsets of X such that $x \in F_x \subseteq (X \setminus \{y\})$, $y \in F_y \subseteq (X \setminus \{x\})$ and $F_x \cup F_y = X$. Then $X \setminus F_x \cap X \setminus F_y = \emptyset$ and both $X \setminus F_x$ and $X \setminus F_y$ are open subsets of X. Since $x \notin F_y$, we have $x \in X \setminus F_y$. Similarly, as $y \notin F_x$, we have $y \in X \setminus F_x$. Hence X is T_2 and we are done. □

3.3 Construction of $u_d X$

We now proceed to building of the minimum uniform compactification of the metric space (X, d). The construction is similar to that of the construction of the Stone-Cěch compactification.

Let $u_d X$ be the collection of all clusters on (X, d). For a closed subset C of X, we define $\overline{C} = \{\sigma \in u_d X : C \in \sigma\}$.

Lemma 3.16 : *Let $\mathcal{B}$ be the family of closed sets of X. Then $\{\overline{B} : B \in \mathcal{B}\}$ is a base of closed sets for a topology τ on $u_d X$. That is, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.*

Proof : Let A, B be closed subsets of X. Then by (3.2),

$$\overline{A \cup B} = \{\sigma : A \cup B \in \sigma\} = \{\sigma : A \in \sigma \text{ or } B \in \sigma\} = \overline{A} \cup \overline{B}.$$

Hence we are done.□

Lemma 3.17 : *$(u_d X, \tau)$ is T_2.*

Proof : Let σ_1 an d σ_2 be distinct clusters. By the maximality of σ_1 and σ_2, there exists $A \in \sigma_1, B \in \sigma_2$ such that $d(A, B) \neq 0$. By lemma 3.11 there exists $C \subseteq X$ such that $d(A, C) \neq 0$ and $d(B, X \setminus C) \neq 0$. Hence $A \subseteq X \setminus C \subseteq cl_X(X \setminus C) = X \setminus int_X(C)$ which implies $X \setminus int_X(C) \in \sigma_1$. Let F_1 denote $\overline{X \setminus int_X(C)}$.Then $\overline{X \setminus int_X(C)}$ is a closed subset of u_dX containing σ_1. By symmetry, we have $B \subseteq cl_X(C)$, so $cl_X(C) \in \sigma_2$. Let F_2 denote $\overline{cl_X(C)}$. Then $\overline{cl_X(C)}$ is a closed subset of u_dX containing σ_2.

If $\sigma_2 \in F_1$, then by theorem 3.12, $X \setminus int_X(C) \in \sigma_2$. This implies $d(B, X \setminus int_X(C)) = 0$ and thus $d(B, X \setminus C) = 0$.This is a contradiction. So we have $\sigma_2 \notin F_1$, and similarly $\sigma_1 \notin F_2$. Thus

$$\overline{cl_X(C)} \cup \overline{X \setminus int_X(C)} = \overline{cl_X(C) \cup (X \setminus int_X(C))} = \overline{X} = u_dX.$$

Hence by theorem 3.15, u_dX is T_2. □

Lemma 3.18 *u_dX is compact.*

Proof : Let $\{\overline{F_j} : j \in \Lambda, F_j$ is a closed subset of $X\}$ be a collection $\mathcal{M}$ of basic closed subsets of u_dX with the finite intersection property. To prove u_dX is compact, it suffices to show that

$$\bigcap_{j\in\Lambda} \overline{F_j} \neq \emptyset.$$

Let $\mathcal{F} = \{F_j : j \in \Lambda\}$. Choose a finite subfamily of $\mathcal{F}$, say $\{A_i\}_{i=1}^n \subseteq \mathcal{F}$. By the finite intersection property, we can choose

$$\sigma \in \bigcap_{i=1}^{n} \overline{A_i}.$$

This implies $A_1, A_2, \cdots, A_n \in \sigma$. Thus each finite sumbfamily of $\mathcal{F}$ belongs to some cluster on X.

Define

$$\mathcal{A} = \{\lambda \subset \mathcal{P}(X) : \mathcal{F} \subset \lambda \text{ and } \{G_i\}_{i=1}^n \subset \lambda \Longrightarrow \exists \sigma \in u_dX \ni \{G_i\}_{i=1}^n \subset \sigma\}.$$

As seen above, $\mathcal{F} \in \mathcal{A}$, hence $\mathcal{A} \neq \emptyset$. Partially order $\mathcal{A}$ by inclusion. Then by Zorn's lemma, there is a maximal element $\mu \in \mathcal{A}$.

We proceed to show that $\mu \in u_dX$. Since $\mathcal{F} \subseteq \mu$, it follows that $\mu \in \cap_{j\in\Lambda}\overline{F_j}$ and so $\cap_{j\in\Lambda}\overline{F_j} \neq \emptyset$. To do this, we now show that μ is a maximal clan and invoke theorem 3.13.

1. $\emptyset \notin \mu$; for if not, there exists $\sigma \in u_d X$ such that $\emptyset \in \sigma$, a contradiction.

2. Let $G_0 \in \mu$ and $G_0 \subseteq H \subseteq X$. Now $\mathcal{F} \subseteq \mu \cup \{H\}$. Let $G_1, G_2, .., G_n$ be arbitrarily chosen from μ. Then $\{G_1, G_2, ..., G_n, H\} \subseteq \mu \cup \{H\}$. By definition of μ, there exists a $\sigma \in u_d X$ such that $\{G_0, G_1, ..., G_n\} \subseteq \sigma$. But as $G_0 \in \sigma$, then $H \in \sigma$ (as $\sigma \in u_d X$). Thus $\mu \subseteq \mu \cup \{H\} \in \mathcal{A}$. So as μ is a maximal element in $\mathcal{A}$ we must have $\mu = \mu \cup \{H\}$. Hence $H \in \mu$.

3. Suppose that $A \cup B \in \mu$ and $A \notin \mu$. We need to show that $B \in \mu$. As $A \notin \mu$ and by the maximality of μ, there exists a finite family $\{G_i\}_{i=1}^n \subseteq \mu$ such that $\{G_1, G_2, .., G_n, A\}$ is not contained in any maximal clan. Consider $\mu \cup \{B\}$ and take any finite subfamily $\{H_1, H_2, ...H_m, B\}$. Then there exists $\sigma \in u_d X$ such that $\{G_1, G_2, ..., G_n, H_1, H_2, ..., H_m, A \cup B\} \subseteq \sigma$. As $A \notin \sigma$, we have $B \in \sigma$ (as σ satisfy (3.2)).Thus $\{H_1, H_2, ..., H_m, B\} \subseteq \sigma$. Hence $\mu \cup \{B\} \in \mathcal{A}$. But as μ is maximal in A,we have $B \in \mu$.

4. If $A, B \in \mu$ then by 5.14 of [NW], there exists some $\sigma \in u_d X$ with $\{A, B\} \subseteq \sigma$. Since σ is a clan, we see that $d(A, B) = 0$ and thus μ is a clan. Now there will be a maximal clan α that contains μ. Clearly $\alpha \in \mathcal{A}$. But as μ is a maximal element of $\mathcal{A}$, we have $\alpha = \mu$.

This shows that μ is a maximal clan and hence by theorem 3.13, it is an element of $u_d X$. Hence $u_d X$ is compact and we are done. □

Let $S : X \to u_d X$ be defined as :

$$S(x) = \sigma_x.$$

Proposition 3.19 : *S embeds X homeomorphically into $u_d X$.*

Proof :

1. If x and y be distinct points of X. Then by theorem 3.8, $\sigma_x \neq \sigma_y$. Hence S is one-to-one.

2. Let A be a closed subset of X. Now $\overline{A}$ is a basic closed subset of u_dX . We see

$$S^{\leftarrow}[\overline{A}] = \{x : \sigma_x \in \overline{A}\} = \{x : A \in \sigma_x\} = \{x : x \in A\} = A$$

which is closed in X. Hence S is continuous.

3. If A is closed in X we claim that

$$S[A] = \overline{A} \cap S[X].$$

To see this; note that if $\sigma_x \in S[A]$, then $\sigma_x \in \{\sigma : A \in \sigma\} = \overline{A}$ as $x \in A$. Hence $\sigma_x \in \overline{A} \cap S[X]$.

Conversely, if $\sigma \in S[X] \cap \overline{A}$ then $A \in \sigma$ and $\sigma = \sigma_x$ for some $x \in X$. This implies $x \in A$ and hence $\sigma \in S[A]$. This immediately imply that $S : X \to S[X]$ is a closed map, and hence a homeomorphism.

□

Proposition 3.20 : *$S[X]$ is dense in u_dX.*

Proof : We note by defintion that

$$cl_{u_dX}S[X] = \bigcap\{\overline{A} : S[X] \subseteq \overline{A}\}.$$

Suppose $S[X] = \{\sigma_x : x \in X\}$ is contained in $\overline{A}$. Then for each fixed $x \in X$, $\sigma_x \in \overline{A}$ which implies $A \in \sigma_x$.Thus $x \in A$, as $\cap\sigma_x = \{x\}$. We see that $A = X$, hence

$$cl_{u_dX}S[X] = \overline{X} = u_dX.$$

Thus $S[X]$ is dense in u_dX. □

We have shown that u_dX is a compactification of (X, d). We proceed to investigate the properties of u_dX .

Proposition 3.21 : *For any closed $F \subseteq X$, $cl_{u_dX} S[F] = \overline{F}$.*

Proof : Note that $S(F) \subseteq \overline{F}$ which is closed in u_dX, so $cl_{u_dX} S(F) \subseteq \overline{F}$. Conversely if $S(F) \subseteq \overline{B}$ where B is a closed subset of X. Then $F \subseteq B$. For if $x \in F$ then $\sigma_x \in S(F) \subseteq \overline{B}$ which implies $\sigma_x \in \overline{B}$, and hence $x \in B$. Thus $\overline{F} \subseteq \overline{B}$; because if $\sigma \in \overline{F}$ then $F \in \sigma$, hence $B \in \sigma$, and so $\sigma \in \overline{B}$. Hence we have:

$$\overline{F} \subseteq \bigcap \{\overline{B} : S(F) \subseteq \overline{B}\} = cl_{u_dX} S(F).$$

Together we have :

$$\overline{F} = cl_{u_dX} S(F) = cl_{u_dX} F. \quad \square$$

Theorem 3.22 : *Let $A, B \subseteq X$. Then $d(A, B) = 0$ iff $cl_{u_dX} A \cap cl_{u_dX} B \neq \emptyset$.*

Proof : ($\Leftarrow$) Suppose $d(A, B) \neq \emptyset$. Then for any cluster $\sigma \in u_dX$, one of the following is true :

- $A \in \sigma,\ B \notin \sigma$,
- $A \notin \sigma,\ B \in \sigma$,
- $A \notin \sigma,\ B \notin \sigma$.

In any case, $\sigma \notin \overline{A} \cap \overline{B} = cl_{u_dX} A \cap cl_{u_dX} B$. Since this is true for all $\sigma \in u_dX$, $cl_{u_dX} A \cap cl_{u_dX} B = \emptyset$.

$(\Rightarrow)$: By theorem 5.14 [NW], there exist a cluster σ such that both A and B belong to σ. Thus $\sigma \in \overline{A} \cap \overline{B} = cl_{u_d X} A \cap cl_{u_d X} B$. So, $cl_{u_d X} A \cap cl_{u_d X} B \neq \emptyset$. □

Theorem 3.23 : *Let (X, d) and (Y, e) be two metric spaces. Let $f : X \to Y$ be a uniformly continuous function. Then f extends continuously to a function $f^u : u_d X \to u_e Y$.*

Proof : Let $\sigma \in u_d X$ and define

$$\alpha_\sigma = \{A \subseteq Y : \forall B \in \sigma, e(A, f[B]) = 0\}.$$

Claim : *α_σ is a cluster.*

Proof of Claim :

1. Clearly if $A \in \alpha_\sigma$ and $A \subseteq D \subseteq Y$ then $D \in \alpha_\sigma$.

   We claim that if $P, Q \in \alpha_\sigma$ then $e(P, Q) = 0$. To see this, suppose that $e(P, Q) \neq 0$. Then by lemma 3.11, there exist $S \subseteq Y$ such that $e(Q, S) \neq 0$ and $e(P, Y \setminus S) \neq 0$. Let $E = f^{\leftarrow}[S]$. For $B \in \sigma$, $B = (B \cap E) \cup (B \setminus E)$. As $B \in \sigma$ and σ is a cluster, either $B \cap E \in \sigma$ or $B \setminus E \in \sigma$.

   - If $B \cap E \in \sigma$, then since $e(Q, S) \neq 0$ and $f[B \cap E] \subseteq S$, we have $Q \notin \alpha_\sigma$.
   - If $B \setminus E \in \sigma$, then since $e(P, Y \setminus S) \neq 0$ and $f[B \setminus E] \subseteq Y \setminus S$, we have $P \notin \alpha_\sigma$.

   So if $e(P, Q) \neq 0$, then either $P \notin \alpha_\sigma$ or $Q \notin \alpha_\sigma$.

2. We claim that if $e(P, Q) = 0$ for all $Q \in \alpha_\sigma$, then $P \in \alpha_\sigma$. To see this; suppose $C \in \sigma$. Then as σ is a cluster, we have that $d(B, C) = 0$ for all $B \in \sigma$. As f is uniformly continuous, we have $e(f[B], f[C]) = 0$ for all $B \in \sigma$. This shows that $f[C] \in \alpha_\sigma$. Thus for any $C \in \sigma$, $f[C] \in \alpha_\sigma$. By hypothesis, $e(f[C], P) = 0$. As C is arbitarily choosen from σ, then definition of α_σ, $P \in \alpha_\sigma$.

3. We claim that if $P \cup Q \in \alpha_\sigma$ then $P \in \alpha_\sigma$ or $Q \in \alpha_\sigma$. To see this, suppose $P \notin \alpha_\sigma$. Then there exists $B_0 \in \sigma$ such that $e(P, f[B_0]) \neq 0$. By lemma 3.11, there exists $S \subseteq Y$ such that $e(P, S) \neq 0$ and $e(f[B_0], Y \setminus S) \neq 0$. Let $E = f^{\leftarrow}[S]$. For any $B \in \sigma$ write $B = (B \cap E) \cup (B \setminus E)$. As σ is a clan, either $B \cap E \in \sigma$ or $B \setminus E \in \sigma$. As$f[B \setminus E] \subseteq Y \setminus S$ it follows that $e(f[B \setminus E], f[B_0]) \neq 0$. Hence $B \setminus E \notin \sigma$. Thus $B \cap E \in \sigma$. But $f[B \cap E] \subseteq S$ which implies $e(P, f[B \cap E]) \neq 0$. As $e(P \cup Q, f[B \cap E]) = 0$ we see that $e(Q, f[B \cap E]) = 0$ and thus $e(Q, f[B]) = 0$. This means $Q \in \alpha_\sigma$.

By theorem 3.13, it follows from the above that α_σ is a cluster in Y. This proves our claim.

Now we define $f^u : u_d X \to u_e Y$ as $f^u(\sigma) = \alpha_\sigma$ for each $\sigma \in u_d X$.

Claim : $f^u(\sigma_x) = \sigma_{f(x)}$.

Proof of Claim :

If $A \in \sigma_{f(x)}$ then $f(x) \in A$.But for each $C \in \sigma_x$,$x \in C$ so $f(x) \in f[C]$ which implies $e(f[C], A) = 0$. Thus $A \in \alpha_{\sigma_x} = f^u(\sigma_x)$. Hence $\sigma_{f(x)} \subseteq f^u(\sigma_x)$. By the maximality of $\sigma_{f(x)}$, it follows that $\sigma_{f(x)} = f^u(\sigma_x)$. □

If we identify X with $\{\sigma_x : x \in X\}$, this shows that $f^u|X = f$.

Claim : f^u *is continuous.*

Proof of Claim: It suffices to show that if $\mathcal{A} \subseteq u_d X$, then $f^u[cl_{u_d X} \mathcal{A}] \subseteq cl_{u_e Y} f^u[\mathcal{A}]$. Suppose not. Then there exists $\sigma \in cl_{u_d X} \mathcal{A}$ such that $f^u(\sigma) \notin cl_{u_e Y} f^u[\mathcal{A}]$. Note that $cl_{u_d X} \mathcal{A} = \cap\{\overline{P} : \mathcal{A} \subseteq \overline{P}\} = \cap\{\overline{P} : \forall\gamma \in \mathcal{A}, \gamma \in \overline{P}\} = \cap\{\overline{P} : \forall\gamma \in \mathcal{A}, P \in \gamma\}$. So, $\sigma \in cl_{u_d X} \mathcal{A}$ if and only if

$$\forall\gamma \in \mathcal{A}, P \in \gamma \Longrightarrow P \in \sigma. \tag{3.7}$$

If $f^u(\sigma) \notin cl_{u_e Y} f^u[\mathcal{A}]$, then there is some closed subset Q of Y such that for all $f^u(\gamma) \in$

$f^u(\mathcal{A}), Q \in f^u(\gamma)$ (that is for each $\gamma \in \mathcal{A}$ and every $B \in \gamma, e(Q, f[B]) = 0$) but yet there is some $B_0 \in \sigma \ni e(Q, f[B_0]) \neq 0$). Then by lemma 3.11, there exist $C \subseteq Y$ such that $e(C, f[B_0]) \neq 0$ and $e(Q, Y \setminus C) \neq 0$. Let $S = f^{-}[C]$ which is closed in X. For each $\gamma \in \mathcal{A}$ and $B \in \gamma$, we see $B = (B \cap S) \cup (B \setminus S)$. As γ is a cluster, $B \cap S \in \gamma$ or $B \setminus S \in \gamma$. Now $f[B \setminus S] \subseteq Y \setminus C$ implies $e(f[B \setminus S], Q) \neq 0$. Since for all $B \in \gamma, e(Q, f[B]) = 0$, then $B \setminus S \notin \gamma$. Hence $B \cap S \in \gamma$, which implies $S \in \gamma$ (as γ is a cluster and $B \cap S \subseteq S$). So S is closed in X such that for all $\gamma \in \mathcal{A}$, $S \in \gamma$. Now $S \notin \sigma$ for if $S \in \sigma$ then as $B_0 \in \sigma$, we have $d(S, B_0) = 0$. As f is uniformly continuous, $e(f[S], f[B_0]) = 0$ which implies $e(C, f[B_0]) = 0$. This is a contradiction, hence $S \notin \sigma$ as require.

This ultimately give :

$$\forall \gamma \in \mathcal{A}, S \in \gamma, S \notin \sigma$$

which contradicts (by (3.7)), the fact $\sigma \in cl_{u_dX}\mathcal{A}$. We have shown that f^u is continuous and our claim holds .

Since all the requirement for the theorem are proved, then we are done. $\square$

An immediate consequence of theorem 3.22 is the following result that characterizes u_dX.

Theorem 3.24 : *The following is equivalent for a metric space (X, d) and a compactification γX of X*

1. $\gamma X \cong u_dX$,
2. $d(A, B) = 0$ *if and only if* $cl_{\gamma X}A \cap cl_{\gamma X}B \neq \emptyset$.

We have shown in this chapter the existence of the minimum uniform compactification u_dX of any metric space (X, d) and described how it is built using clusters.

Chapter 4

Analysis of a Locally Compact Space

4.1 The Locally compact $\sigma-$compact Complete Metric Space (X, σ)

Let $X = \cup_{n=0}^{\infty}[n, n+1] \times [0,1]^n$. For $\vec{x} = (x_1, x_2, ..., x_n)$, $\vec{y} = (y_1, y_2, ..., y_m) \in X$, define $x_i = 0$ if $i > n$ and $y_i = 0$ if $i > m$. As $\vec{x}$ has n components, it lies in $[n-1, n] \times [0,1]^{n-1}$, and similarly $\vec{y} \in [m-1, m] \times [0,1]^{m-1}$. We define $\sigma : X \times X \to \mathbb{R}$ by

$$\sigma(\vec{x}, \vec{y}) = \sum_{j=1}^{\infty} \frac{1}{2^j}|x_j - y_j|.$$

It is easily verified that σ is a metric on X, and a point has n components if and only if it belongs to $[n-1, n] \times [0,1]^{n-1}$. Let τ_σ denote the topology induced on X by σ. A useful fact to note is that $[n, n+1] \times [0,1]^n$ inherits the usual product topology from τ_σ and hence is compact. Also note that $[n, n+1] \times [0,1]^n$ is regular closed in X. By convention , $[0,1]$ will denote $[0,1] \times [0,1]^0$. In this chapter, we shall analyze the structure of $u_\sigma X \setminus X$. We now set forth the major steps taken to achieve this goal.

- Show that (X, σ) is a locally compact σ-compact complete metric space.
- Write X as the union of two sets M and L.
- Define two metric spaces M_E and M_O that are uniformly equivalent to M and L respectively. This is done because it is easier to work with M_E and M_O.
- Show that $u_{\sigma|M}M \setminus M$ and $u_{\sigma|L}L \setminus L$ are homeomorphic to $[0,1]^\omega \times \beta\mathbb{N} \setminus \mathbb{N}$ and whose union is $u_\sigma X \setminus X$.
- Show that $u_{\sigma|M}M \setminus M$ and $u_{\sigma|L}L \setminus L$ are regular closed subsets of the outgrowth of X and the interior (in the outgrowth of X) of their intersection is empty.

Theorem 4.1 : *(X, σ) is a locally compact $\sigma-$ compact complete metric space.*

Proof :

1. Since $[n, n+1] \times [0,1]^n$ is compact for each $n \in \mathbb{N}$, X is $\sigma-$compact.

2. Let $\{\vec{x_r}\}$ be a Cauchy sequence in X. We need to show that $\{\vec{x_r}\}$ converges in X. Suppose $n, m \in \mathbb{N}$. If $|m - n| \geq 2$, then for $\vec{x} \in [n, n+1] \times [0,1]^n, \vec{y} \in [m, m+1] \times [0,1]^m$ we have $d(\vec{x}, \vec{y}) \geq |x_1 - y_1| \geq 1$. So as $\{\vec{x_r}\}$ is Cauchy, there exists an $n_0 \in \mathbb{N}$ such that for all $r \geq n_0$ there exists an n such that $\vec{x_r} \in [n, n+1] \times [0,1]^n \cup [n+1, n+2] \times [0,1]^{n+1}$. But $[n, n+1] \times [0,1]^n \cup [n+1, n+2] \times [0,1]^{n+1}$ is compact, and thus complete. Thus the sequence $\{\vec{x_r}\}$ must converge to a point in $[n, n+1] \times [0,1]^n \cup [n+1, n+2] \times [0,1]^{n+1}$. That means $\{\vec{x_r}\}$ converges in X, implying X is complete.

3. Let $\vec{x} = (x_1, x_2, .., x_n) \in X$. We consider 3 cases

   (a) : If $x_1 = 0$, the we see that $\vec{x} \in [0,1) \subseteq [0,1]$ so in this case $[0,1]$ is a compact neighborhood of x.

(b) : If $x_1 = n - 1$ where $n \neq 1$, then

$$\vec{x} \in \cup_{k=0}^{n}[k, k+1) \times [0,1]^k \subseteq \cup_{k=0}^{n}[k, k+1] \times [0,1]^k$$

and the latter set is a compact neighborhood of $\vec{x}$.

(c) : If x_1 is not an integer, then $\vec{x} \in (n-1, n) \times [0,1]^{n-1} \subseteq [n-1, n] \times [0,1]^{n-1}$, and the latter set is again a compact neighborhood of $\vec{x}$.

Thus in all cases X is locally compact at $\vec{x}$, so X is locally compact.

We have shown x is a locally compact $\sigma-$compact complete metric space, so we are done. $\square$

Let $M = \cup_{n=1,3,5,\ldots}^{\infty}[n, n+1] \times [0,1]^n \subset X$ and let $d = \sigma|M$. Here is a partial picture of M

Lemma 4.2 *:M is a closed subset of X.*

Proof : Let $\vec{x} = (x_1, x_2, ..., x_{2n}, x_{2n+1}) \in X \setminus M$. Let

$$U_1 = (2n, 2n+1) \cap (\frac{x_1 + 2n}{2}, \frac{2n+1+x_1}{2})$$

and let

$$U = U_1 \times [0,1]^{2n}.$$

Clearly $\vec{x} \in U$,U is an open subset of X, and $U \subseteq X \setminus M$. That is, $X \setminus M$ is open in X, so M is closed in X. □

Theorem 4.3 : *(M, d) is a locally compact $\sigma-$ compact complete metric space.*

Proof : As local compactness, $\sigma-$compactness and (for metric spaces) completeness are all inherited by closed subspaces, the result follows immediately from the previous lemma. □

Let $K = [0,1]^\omega$ be given the metric e it naturally inherits from $\mathbb{R}^\omega$. That is, if $\vec{x} = (x_1, x_2, ...), \vec{y} = (y_1, y_2, ...)$ then :

$$e(\vec{x}, \vec{y}) = \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i - y_i|.$$

Let $\pi_i : K \to [0,1]$ be the i^{th} projection map from K onto $[0,1]$. Denote $K \times \mathbb{N}$ by X_ω and define a metric ϕ on X_ω by : $\phi((\vec{x}, n), (\vec{y}, m)) = e(\vec{x}, \vec{y}) + |n - m|$.

Definition 4.4 : *We denote M_E by*

$$\{(\vec{m}, j) \in X_\omega : j \text{ is even, and for } i > j, \pi_i(\vec{m}) = 0\}.$$

Theorem 4.5 : *(X_ω, ϕ) is a complete metric space.*

Proof : If $n \neq m$, the distance from $K \times \{n\}$ to $K \times \{m\}$ is at least 1, so any Cauchy sequence must eventually be contained in $K \times \{n\}$ for some $n \in \mathbb{N}$. But $K \times \{n\}$ is compact hence complete, so we are done. □

Theorem 4.6 : *M_E is a closed subset of X_ω.*

Proof :First suppose j is even. Let $(\vec{x}, j) \in X_\omega \setminus M_E$, then without loss of generality suppose $x_{j+1} \neq 0$. For $i = 1$ to j, define $U_i = (\frac{x_i}{2}, \frac{x_i+1}{2})$, and define $U_{j+1} = (\frac{x_{j+1}}{2}, \frac{x_{j+1}+1}{2})$. Each U_i is open in $[0,1]$and we see $0 \notin U_{j+1}$. Let U denote the open subset $(\prod_{i=1}^{j+1} U_i) \times (\prod_{j+2}^{\infty}[0,1]) \times \{j\}$ of X_ω. One easily sees that $\vec{x} \in U$ and

$$U \cap M_E = \emptyset.$$

Note that U is an open neigborhood of $(\vec{x}, j)$.

Now suppose that j is odd. Then $K \times \{j\}$ is an open neighborhood of $(\vec{x}, j)$ such that $(\vec{x}, j) \in K \times \{j\} \subseteq X_\omega \setminus M_E$.

Thus if $(\vec{x}, j) \in X_\omega \setminus M_E$, there is an $X_\omega-$ neighborhood of $(\vec{x}, j)$ disjoint from M_E. Hence M_E is closed. □

Corollary 4.7 : *$(M_E, \phi | M_E)$ is a complete metric space.*

Proof : Closed subspaces of complete metric spaces are complete, and X_ω is complete. Thus by theorem 4.6, M_E is complete. □

Theorem 4.8 : *$(M_E, \phi | M_E)$ is uniformly equivalent to (M, d).*

Proof : Define $f : (M, d) \to (M_E, \phi | M_E)$ as follows : Let $\vec{x} \in M$; then there is a unique n such that $\vec{x} \in [n-1, n] \times [0,1]^{n-1}$ where n is even. Define

$$f(\vec{x}) = (x_1, x_2, ..., x_n) = (x_1 - n + 1, x_2, ..., x_n, 0, 0, 0, ..., n).$$

Clearly, f is a bijection.

We claim that f is uniformly continuous. To prove this, let $\epsilon > 0$ be arbitarily chosen. Let $\delta = min\{\frac{1}{2}, \frac{\epsilon}{2}\}$. Suppose $\vec{x} = (x_1, ..., x_n)$, $\vec{y} = (y_1, ..., y_m) \in M$ and $d(\vec{x}, \vec{y}) < \delta$. Thus $d(\vec{x}, \vec{y}) < \frac{1}{2}$ which means that $m = n$, hence $|m - n| = 0$. Thus $\phi(f(\vec{x}), f(\vec{y})) = d(\vec{x}, \vec{y}) = \delta < \epsilon$. So f is uniformly continuous.

We now claim that $f^{\leftarrow}$ is uniformly continuous. To prove this, let $f^{\leftarrow}$ be denoted by g to simplify notation. Let $\epsilon > 0$ be arbitarily given, $\delta = min\{\frac{\epsilon}{2}, \frac{1}{2}\}$, and let $\vec{x}, \vec{y} \in M_E$. Now $\vec{x} = (\vec{m}, i) \in K \times \mathbb{N}$, and $\vec{y} = (\vec{n}, j) \in K \times \mathbb{N}$ where i, j are even numbers (i, j are even by definition of M_E), $\pi_k(\vec{m}) = 0$ for $k > i$ and $\pi_k(\vec{n}) = 0$ for $k > j$. Let $\vec{m} = (m_1, m_2, m_3, ...)$ and $\vec{n} = (n_1, n_2, n_3, ...)$. By the definition of f we have

$$g(\vec{x}) = (m_1 + i - 1, m_2, m_3, m_4, ..., m_i)$$

and

$$g(\vec{y}) = (n_1 + j - 1, n_2, n_3, ..., n_j).$$

If $\phi(\vec{x}, \vec{y}) < \delta \leq \frac{1}{2}$, then $i = j$ which implies

$$g(\vec{x}) = (m_1 + i - 1, m_2, m_3, m_4, ..., m_i)$$

and

$$g(\vec{y}) = (n_1 + i - 1, n_2, n_3, ..., n_i).$$

This gives the following :

$$d(g(\vec{x}), g(\vec{y})) = \sum_{k=1}^{i} \frac{1}{2^k}|m_k - n_k| \leq (\sum_{k=1}^{i} \frac{1}{2^k}|m_k - n_k|) + |i - j| = \phi(\vec{x}, \vec{y}) < \delta < \epsilon.$$

This shows that $f^{\leftarrow}$ is uniformly continuous.

Thus to investigate $u_d M$, it is enough to investigate $u_{\phi|M_E} M_E$ since theorem 4.8 and theorem 2.11 together imply that $u_d M$ is homeomorphic to $u_{\phi|M_E} M_E$.

A useful well known theorem that we now state and prove here is that a uniform isomorphism preserves completeness. A proof of this cannot be readily found so it is included for completeness.

Theorem 4.9 *A uniform isomorphism $f : (X,d) \to (Y,e)$ from a metric space (X,d) onto another space (Y,e) preserves completeness.*

Proof : Suppose that (X,d) is complete. We show that (Y,e) is also complete. Let $\{y_n\}$ be a Cauchy sequence in Y. We first show that $\{f^{\leftarrow}(y_n)\}$ is a Cauchy sequence in X. To do this, let $\epsilon > 0$ be arbitarily chosen. As $f^{\leftarrow}$ is uniformly continuous, there exists $\delta > 0$ such that if $x, y \in Y$ and $e(x,y) < \delta$ then $d(f^{\leftarrow}(x), f^{\leftarrow}(y)) < \epsilon$. Now as $\{y_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that if $m,n > N$ then $e(y_n, y_m) < \delta$. Thus if $m,n > N$ we have $e(y_n, y_m) < \delta$ which implies $d(f^{\leftarrow}(y_n), f^{\leftarrow}(y_m)) < \epsilon$. This shows that the sequence $\{f^{\leftarrow}(y_n)\}$ is Cauchy.
As X is complete $\{f^{\leftarrow}(y_n)\}$ must converge to a point $x \in X$. By the continuity of f it follows that the sequence $\{f(f^{\leftarrow}(y_n))\} = \{y_n\}$ converges in Y. Hence Y is complete and our theorem holds. $\Box$

By theorem 2.14,$u_{\phi|M_E} M_E \subseteq u_\phi X_\omega \cong u_d K \times u_s\mathbb{N} = [0,1]^\omega \times u_s\mathbb{N}$, where s is the metric on $\mathbb{N}$ given by $s(n,m) = |n-m|$. By theorem 2.13, we have $u_s\mathbb{N} \cong \beta\mathbb{N}$,and thus $u_\phi X_\omega = [0,1]^\omega \times \beta\mathbb{N}$. Now, one can easily see that $u_\phi X_\omega \setminus X_\omega \cong [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. By theorem 2.10 , we have $u_{\phi|M_E} M_E \cong cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E$.

Because M_E is closed in X_ω, we have the following.

Theorem 4.10 : $u_{\phi|M_E} M_E \setminus M_E = u_{\phi|M_E} M_E \setminus X_\omega$.

The following lemma is a well known fact whose proof has been included for completeness.

Lemma 4.11 : *If U is open in $\beta\mathbb{N}$ and $U \cap (\beta\mathbb{N} \setminus \mathbb{N}) \neq \emptyset$ then $|U \cap \mathbb{N}| = \infty$.*

Proof : Suppose $U \cap \mathbb{N} = \{x_1, x_2, .., x_n\}$. But we know that since $\mathbb{N}$ is dense in $\beta\mathbb{N}$, $cl_{\beta\mathbb{N}} U = cl_{\beta\mathbb{N}}(U \cap \mathbb{N}) = \{x_1, x_2, ..., x_n\}$ as $\beta\mathbb{N}$ is Hausdorff. This is a contradiction, hence our result holds. □

Theorem 4.12 : $cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E \setminus M_E = [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$.

Proof :

We first show that $cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E \setminus M_E \subseteq [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. Let $\vec{x} \in cl_{u,\phi[0,1]^\omega \times \beta\mathbb{N}} M_E \setminus M_E$. Then $\vec{x} \notin M_E$ and $\vec{x} \in cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E$. Suppose $\vec{x} \notin [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. Then $\vec{x} = (\vec{m}, j)$ where $j \in \mathbb{N}$ and $\vec{m} \in [0,1]^\omega$. Denote $\vec{m}$ by $(m_1, m_2, ...)$.We will derive get a contradiction. We consider the 2 cases :

1. If j is odd, then $[0,1]^\omega \times \{j\}$ is an open neighborhood of $\vec{x}$ in $[0,1]^\omega \times \beta\mathbb{N}$. But $M_E \cap ([0,1]^\omega \times \{j\}) = \emptyset$ which contradicts the assumption that $\vec{x} \in cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E$.

2. If j is even, we can assume without loss of generality that $m_{j+1} \neq 0$ since if for all $k > j, m_k \neq 0$ then $\vec{x}$ would belong to M_E, contrary to hypothesis. Now

$$(\sum_{k=1}^{j}[0,1]) \times (\frac{m_{j+1}}{2}, 1] \times (\sum_{k=j+2}^{\infty}[0,1]) \times \{j\}$$

is an open neighborhood of $\vec{x}$ that has empty intersection with M_E. This contradicts the assumption that $\vec{x} \in cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E$.

So we have shown that $cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E \setminus M_E \subseteq [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$.

We now show $cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E \setminus M_E \supseteq [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. Let

$\vec{x} = (x_1, x_2, ..., \alpha) \in [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$.

Define

$$\vec{x_2} = (x_1, x_2, 0, 0, ..., 2)$$
$$\vec{x_4} = (x_1, x_2, x_3, x_4, ..., 4)$$
$$\vdots$$
$$\vec{x_{2n}} = (x_1, x_2, ..., x_{2n}, 0, ..., 2n)$$
$$\vdots$$

If i is an even number, $\vec{x_i} \in M_E$. Let

$$U = (\prod_{i=1}^{\infty} U_i) \times V$$

be an open neighborhood of $\vec{x}$ where each U_i is open in $[0,1]$, V is open in $\beta\mathbb{N}$, and all but finitely many of the U_i's are all of $[0,1]$. Let

$$F = \{i \in \mathbb{N} : U_i \neq [0,1]\}.$$

Then F is a finite set. Let $m = max(F)$. By the lemma 4.11, we see that $|V \cap \mathbb{N}| = \infty$. Hence there exists $m^* \in V \cap \mathbb{N}$ such that $m^* > m$ and m^* is an even number. Now $\vec{x_{m^*}} \in M_E \cap U$, so $(U \cap M) \setminus \{\vec{x}\} \neq \emptyset$. As this is true for arbitary neighborhoods U of $\vec{x}$, we see that $\vec{x} \in cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E \setminus M_E$. But this holds true for arbitary $\vec{x} \in [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$ we have

$$[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N}) \subseteq cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E \setminus M_E.$$

We have shown that $cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E \setminus M_E = [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. □

So we have $u_{\phi|M_E} M_E \cong cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E$ and $cl_{[0,1]^\omega \times \beta\mathbb{N}} M_E \setminus M_E = [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. Recall $(M_E, \phi|M_E)$ is uniformly equivalent to (M, d) by theorem 4.8. Thus $cl_{u_\sigma X} M \setminus M$ is homeomorphic to $[0,1]^\omega \times \beta\mathbb{N} \setminus \mathbb{N}$.

Similarly, let

$$L = \cup_{n=1,2|n}^{\infty} [n, n+1] \times [0,1]^n \subset X,$$

and let $e = \sigma|L$. An argument similar to that for (M, d) shows that (L, e) will be a locally compact $\sigma-$compact complete metric space. Also L is a closed subset of X.
Define

$$M_O = \{(\vec{m}, j) \in X_\omega : j \text{ is odd and }, \; i > j \Rightarrow \pi_i(\vec{m}) = 0\}.$$

Using an argument like the one for M_E, we can show that $(M_O, \phi|M_O)$ is uniformly equivalent to (L, d) and that $cl_{u_\sigma X} L \setminus L$ (as a subspace of $u_\sigma X$) is homeomorphic to $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. Since $X = L \cup M$ and L, M are closed subsets of X, it follows that $u_\sigma X \setminus X$ can be written as the union of two copies of $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$.

Proposition 4.13 : *$cl_{u_\sigma X} M \setminus M = cl_{u_\sigma X} M \setminus X$ and $cl_{u_\sigma X} L \setminus L = cl_{u_\sigma X} L \setminus X$*

Proof This result is follows immediately from fact M, L are closed subsets of X, as $(cl_{u_\sigma X} M) \cap X = cl_X M = M$ and $(cl_{u_\sigma X} L) \cap X = cl_X L = L$ □

Following the steps that we used to show $u_d M \setminus M$ is homeomorphic to $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$, we now proceed to show that $u_{\sigma|(M \cap L)}(M \cap L) \setminus (M \cap L)$ is homeomorphic to $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$.

Let $T = (M \cap L) \setminus \{1\} = \cup_{n=1}^{\infty}(\{n+1\} \times [0,1]^n)$. We see that $M_O \cup M_E = \{(\vec{x}, j) \in X_\omega : j \in \mathbb{N}, i > j \Longrightarrow \pi_i(\vec{x}) = 0\}$.

Proposition 4.14 : *The function $f : (M_O \cup M_E, \phi|(M_O \cup M_E)) \to (T, \sigma|T)$ defined as : if $\vec{x} = (\vec{m}, j) \in M_O \cup M_E$ where $\vec{m} = (m_1, m_2, ...)$ and for $i > j$, $\pi_i(\vec{m}) = 0$, then*

$$f(\vec{x}) = (j + 1, m_1, m_2, ..., m_j)$$

is a uniform isomorphism.

Proof : It is clear that f is one-to-one and onto. To show f is uniformly continuous, let $\epsilon > 0$ be given. If $\delta = min\{\frac{1}{2}, \frac{\epsilon}{2}\}$, and $\vec{x} = (\vec{m}, j), \vec{y} = (\vec{n}, i) \in M_O \cup M_E$ such that $\phi(\vec{x}, \vec{y}) < \delta$, then $\phi(\vec{x}, \vec{y}) < \frac{1}{2}$ which implies $i = j$. We have

$$\phi(\vec{x}, \vec{y}) = \sum_{k=1}^{\infty} \frac{1}{2^k}|m_k - n_k| = \sum_{k=1}^{i} \frac{1}{2^k}|m_k - n_k|.$$

Thus we conclude that

$$\sigma(f(\vec{x}), f(\vec{y})) = \frac{1}{2}|i - j| + \sum_{k=1}^{i} \frac{1}{2^{k+1}}|m_k - n_k| = \sum_{k=1}^{i} \frac{1}{2^{k+1}}|m_k - n_k| = \frac{1}{2}\phi(\vec{x}, \vec{y}) < \epsilon.$$

This shows that f is uniformly continuous.

We now show that $f^{\leftarrow}$ is uniformly continuous. Let g denote $f^{\leftarrow}$, and let $\epsilon > 0$ be arbitarily given and let $\delta = min\{\frac{1}{2}, \frac{\epsilon}{4}\}$. If $\vec{x} = (n, x_1, x_2, ..., x_{n-1}), \vec{y} = (m, y_1, y_2, ..., y_{m-1}) \in T$ such that $\sigma(\vec{x}, \vec{y}) < \delta$, then

$$\sigma(\vec{x}, \vec{y}) < \frac{1}{2}.$$

This implies that $m = n$. Now we see that $g(\vec{x}) = (\vec{s}, n - 1)$ where $\pi_i(\vec{s}) = x_i$ for $0 < i \leq n - 1$ and $\pi_i(\vec{s}) = 0$ for $i \geq n$. Similarly, $g(\vec{y}) = (\vec{t}, n - 1)$ where $\pi_i(\vec{t}) = y_i$ for $0 < i \leq n - 1$ and $\pi_i(\vec{t}) = 0$ for $i \geq n$. Since

$$\sigma(\vec{x}, \vec{y}) = \sum_{k=1}^{i} \frac{1}{2^{k+1}}|x_k - y_k| < \frac{\epsilon}{4},$$

we see that

$$\phi(g(\vec{x}), g(\vec{y})) = \sum_{k=1}^{i} \frac{1}{2^k}|x_k - y_k| = 2\sigma(\vec{x}, \vec{y}) < \epsilon.$$

This shows that $f^{\leftarrow}$ is also uniformly continuous. □

We conclude from the above that $(M_O \cup M_E, \phi|(M_O \cup M_E))$ and $(T, \sigma|T)$ are uniformly equivalent.Observe that $T^u = (M \cap L)^u$. We now proceed to show that $(M \cap L)^u$ is a homeomorphic copy of $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$ in X^u. To show this it is enough to show that $cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E) \setminus (M_O \cup M_E) = [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$ since $cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E) \setminus (M_O \cup M_E) \cong (M \cap L)^u$. Recall that $u_\phi([0,1]^\omega \times \mathbb{N}) = [0,1]^\omega \times \beta\mathbb{N}$.

Proposition 4.15 : $cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E) \setminus (M_O \cup M_E) = [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$

Proof :

Let $\vec{x} \in cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E) \setminus (M_O \cup M_E)$. So $\vec{x} \notin M_O \cup M_E$ and $\vec{x} \in cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E)$. Suppose that $\vec{x} \notin [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. We want a contradiction. Since $\vec{x} \notin [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$, then we write $\vec{x} = (\vec{m}, j)$ where $j \in \mathbb{N}$. Without loss of generality assume that $m_{j+1} \neq 0$. Now the set

$$\sum_{k-1}^{j} [0,1] \times (\frac{m_{j+1}}{2}, 1] \times \sum_{k=j+2}^{\infty} [0,1] \times \{j\}$$

is an open neighborhood of $\vec{x}$ that has empty intersection with $M_O \cup M_E$ which contradicts the assumption that $\vec{x} \in cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E)$. This shows that $cl_{u_\phi[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E) \setminus (M_O \cup M_E) \subseteq [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$.

We now show that $cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E) \setminus (M_O \cup M_E) \supseteq [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. Let

$$\vec{x} = (x_1, x_2, ..., \alpha) \in [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N}).$$

Define

$$\vec{x_1} = (x_1, 0, 0, ..., 1) \tag{4.1}$$

$$\vec{x_2} = (x_1, x_2, 0, 0, ..., 2)$$

$$\vdots$$

$$\vec{x_n} = (x_1, x_2, ..., x_n, 0, 0, ..., n)$$

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Clearly each $\vec{x_i} \in M_O \cup M_E$. Let

$$U = (\prod_{i=1}^{\infty} U_i) \times V$$

be a basic open neighborhood of $\vec{x}$; thus each U_i is open in $[0,1]$, V is open in $\beta\mathbb{N}$, and all but finitely many of the U_i's are all of $[0,1]$. Let

$$F = \{i \in \mathbb{N} : U_i \neq [0,1]\}.$$

Then F is a finite set, so let $m = max(F)$. By lemma 4.11, we see the $V \cap \mathbb{N}$ is infinite. So there exist $m^* \in V \cap \mathbb{N}$ such that $m^* > m$. Now $\vec{x_{m^*}} \in (M_E \cup M_O) \cap U$, so $(U \cap (M_E \cup M_O)) \setminus \{\vec{x}\} \neq \emptyset$. As this is true for arbitary neighborhoods U of $\vec{x}$, we see that $\vec{x} \in cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E) \setminus (M_O \cup M_E)$. But this holds true for arbitary $\vec{x} \in [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$ we have

$$[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N}) \subseteq cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E) \setminus (M_O \cup M_E).$$

We have shown that $cl_{[0,1]^\omega \times \beta\mathbb{N}}(M_O \cup M_E) \setminus (M_O \cup M_E) = [0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. $\square$

Define $K_1 = [0,1], K_2 = K_1 \cup ([1,2] \times [0,1]), K_3 = K_2 \cup ([2,3] \times [0,1]^2)$ and so on for each $n \in \mathbb{N}$. That is, $K_n = K_{n-1} \cup ([n-1,n] \times [0,1]^{n-1})$. It is clearly obvious that K_n is a compact subset of X and K_n is a proper subset of K_{n+1} for each $n \in \mathbb{N}$ and $\cup_{n \in \mathbb{N}} K_n = X$.

Proposition 4.16 *: $cl_X int_X K_n = K_n$ for each $n \in \mathbb{N}$.*

Proof : Let $n > 1 \in \mathbb{N}$. It is easy to see that

$$int_X K_n = K_{n-1} \cup ([n-1, n) \times [0,1]^{n-1})$$

and hence we have

$$\begin{aligned} cl_X int_X K_n &= cl_X K_{n-1} \cup ([n-1, n) \times [0,1]^{n-1}) \\ &= K_{n-1} \cup cl_X([n, n-1) \times [0,1]^{n-1}) \\ &= K_{n-1} \cup [n-1, n] \times [0,1]^{n-1} = K_n. \end{aligned}$$

So this proves our claim. □

The following is a well known theorem(28.2 of [Wi]) that will allow us to show that $u_\sigma X$ is not homeomorphic to $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$.

Theorem 4.17 *: If $(X_n)_{n\in\mathbb{N}}$ is a collection of compact connected subspaces of a Hausdorff space X, and $\forall n \in \mathbb{N}, X_{n+1} \subset X_n$ then $\cap_{n\in\mathbb{N}} X_n$ is a compact connected subspace of X.*

We now proceed to prove that

$$u_\sigma X \setminus X = \cap_{n\in\mathbb{N}}(cl_{u_\sigma X}(\cup_{k\geq n}[k-1, k] \times [0,1]^{k-1})).$$

Theorem 4.18 *: $u_\sigma X \setminus X = \cap_{n\in\mathbb{N}}(cl_{u_\sigma X}(\cup_{k\geq n}[k-1, k] \times [0,1]^{k-1}))$.*

Proof : Let $T_n = \bigcup_{k\geq n}([k-1, k] \times [0,1]^{k-1})$. We see that $X = K_n \cup T_n$, so $u_\sigma X = cl_{u_\sigma X} K_n \cup cl_{u_\sigma X} T_n = K_n \cup cl_{u_\sigma X} T_n$ (as K_n is compact). Thus $u_\sigma X \setminus X = (K_n \setminus X) \cup (cl_{u_\sigma X} T_n \setminus X) \subseteq cl_{u_\sigma X} T_n$ as $K_n \subseteq X$. This holds for each $n \in \mathbb{N}$, so $u_\sigma X \setminus X \subseteq \cap_{n\in\mathbb{N}} cl_{u_\sigma X} T_n$. But $X = \cup_{n\in\mathbb{N}} K_n$ and $K_n \cap cl_{u_\sigma X} T_{n+1} = \emptyset$. Thus $\cap_{n\in\mathbb{N}} cl_{u_\sigma X} T_n \subseteq u_\sigma X \setminus X$. Hence our theorem holds. □

Now for each $n \in \mathbb{N}$,

$$cl_{u_\sigma X}(\cup_{k \geq n}[k, k+1] \times [0,1]^{k-1})$$

is a compact connected subset of $u_\sigma X$ because $(\cup_{k \geq n}[k, k+1] \times [0,1]^{k-1})$ is connected. So by theorem 4.17, $u_\sigma X \setminus X$ is a compact connected subspace of $u_\sigma X$. Hence it is not homeomorphic to $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$. Observe that since $\beta\mathbb{N} \setminus \mathbb{N}$ is not connected, $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$ cannot connected. We summarize these remarks in the following theorem.

Theorem 4.19 : *$u_\sigma X \setminus X$ is not homeomorphic to $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$.*

We record for future use the following well-known theorem:

Theorem 4.20 : $\{cl_{u_\sigma X} C \setminus X : C$ closed in $X\}$ *is a closed base for $u_\sigma X \setminus X$.*

In the next section, a more detailed description of $u_\sigma X \setminus X$ is given.

4.2 Properties of $u_\sigma X \setminus X$

Theorem 4.21 : *$u_d M \setminus M, u_e L \setminus L$ are regular closed subsets of $u_\sigma X \setminus X$.*

Proof : We show this for $u_e L \setminus L$. A similar argument will yield the same result for $u_d M \setminus M$. To show that $u_e L$ is a regular closed subset of $u_\sigma X \setminus X$ it suffices to show that if A is a closed subset of X, $p \in (u_e L \setminus L) \setminus A^u$, then

$$((u_\sigma X \setminus X) \setminus A^u) \cap ((u_\sigma X \setminus X) \setminus M^u) \neq \emptyset$$

since clearly

$$(u_\sigma X \setminus X) \setminus M^u \subseteq int_{u_\sigma X \setminus X} L^u.$$

By theorem 2.16 it suffices to show if $p \in L^u \setminus A^u$ then

$$lim_{n \to \infty} sup\{\sigma(x, A \cup M) : x \notin K_n\} \neq 0. \tag{4.2}$$

Suppose 4.2 fails. Since $p \notin A^u$ by theorem 2.9 there exists $G \subset X$ and $\delta \in (0, \frac{1}{4})$ such that $p \in G^u$ and $d(G, A) \geq \delta$. Since (4.2) fails there exists $n_\delta \in \mathbb{N}$ such that if $x \notin K_{n_\delta}$ then $\sigma(x, A \cup M) < \frac{\delta}{4}$.

Claim : *Let* $B = \bigcup\{[2n + \frac{\delta}{2}, 2n + 1 - \frac{\delta}{2}] \times [0, 1]^{2n} : n \geq n_\delta\}$. *Then*

$$B \subseteq \{x : \sigma(x, A) < \frac{\delta}{4}\}.$$

Proof of Claim : If $p \in B$, and $n \geq n_\delta$ then $p \notin K_{n_\delta}$ which implies $\sigma(p, A \cup M) < \frac{\delta}{4}$. But $d(p, M) \geq \frac{\delta}{4}$ hence we must have $\sigma(p, A) < \frac{\delta}{4}$. So our claim holds.

Claim :

$$G \setminus K_{n_\delta} \subseteq \bigcup\{[2n - 1 + \delta, 2n - \delta] \times [0, 1]^{2n-1} : n \in \mathbb{N}\}. \tag{4.3}$$

Proof of Claim : if $p \notin K_{n_\delta}$ and $\sigma(p, B) < \frac{3\delta}{4}$. Then p is within δ of A which is a contradiction since $p \in G$ and $d(A, G) \geq \delta$. So, in order for it not to be within $\frac{3\delta}{4}$ of B, then clearly we must have

$$p \in \bigcup\{[2n - 1 + \delta, 2n - \delta] \times [0, 1]^{2n-1} : n \in \mathbb{N}\}.$$

So, the claim holds.

But as $p \in X^u$ and $p \in G^u \cap L^u$ it is clear that

$$p \in (G \setminus K_{n_\delta})^u \cap (L \setminus K_{n_\delta})^u$$

and thus

$$\sigma((G \setminus K_{n_\delta}), (L \setminus K_{n_\delta}) = 0.$$

But from (4.3), we see that

$$\sigma((G \setminus K_{n_\delta}), (L \setminus K_{n_\delta}) \geq \frac{3\delta}{4},$$

which is a contradiction. Hence (4.2) must hold and thus L^u is a regular closed subset of X^u. □

Theorem 4.22 : $(M \cap L)^u = M^u \cap L^u$.

Proof : Clearly $(M \cap L)^u \subseteq M^u \cap L^u$. On the other hand; suppose $p \in X^u \setminus (M \cap L)^u$. By theorem 2.9 there exists $D \subseteq X$ such that $p \in D^u$ and

$$\sigma(D, M \cap L) = r > 0. \tag{4.4}$$

Note that $r < 1$. For each $n \in \mathbb{N}$, define $C_n = [n - \frac{r}{2}, n] \times [0,1]^{n-1}$. Clearly the collection $\{C_n\}$ is a locally finite collection of subsets of X, since for $n \neq m$, $\sigma(C_m, C_n) \leq \frac{r}{4}$. Each C_n is also closed in X. Define C be to

$$C = \cup_{n \in \mathbb{N}} C_n.$$

As $\{C_n\}_{n \in \mathbb{N}}$ is a locally finite collection of closed subsets of X, C is closed in X. Here is a partial diagram of how C looks like :

Claim : $\sigma(C, D) \geq \frac{r}{4}$.

Proof of Claim : Suppose not. Then there exists $\vec{c} \in C, \vec{d} \in D$ such that $\sigma(\vec{c}, \vec{d}) < \frac{r}{4}$.

By definition of C, $sup\{\sigma(\vec{c}, M \cap L)\} \leq \frac{r}{4}$. Let $\vec{x}$ be an arbitrary point of $M \cap L$. Then

$$\sigma(\vec{c}, \vec{x}) \leq \frac{r}{4}$$

which implies

$$\sigma(\vec{d}, \vec{x}) \leq \sigma(\vec{d}, \vec{c}) + \sigma(\vec{c}, \vec{x}) < \frac{r}{4} + \frac{r}{4} < r.$$

This contradicts (4.4). Hence our claim holds.

By theorem 2.8 and the fact C,D are closed subsets of X , $C^u \cap D^u = \emptyset$. Thus $p \notin C^u$ since $p \in D^u$.

Claim: $p \notin M^u \cap L^u$.

Proof of Claim : If $p \in M^u \cap L^u$, then $p \in (M \setminus C)^u \cap (L \setminus C)^u$. But we saw that $\sigma(M \setminus C, L \setminus C) = \frac{r}{2} > 0$ which is a contradiction to theorem 2.8. Thus our claim holds.

It follows from the two claims that $M^u \cap L^u = (M \cap L)^u$ and hence we are done. □

Theorem 4.23 *: $int_{X^u}(M \cap L)^u = 0$.*

Proof : Suppose not; then there exists A closed in X such that

$$\emptyset \neq X^u \setminus A^u \subseteq (M \cap L)^u.$$

We see that $(M \cap L)^u \cup A^u = X^u$ and $X^u \not\subseteq A^u$. By theorem 2.16 we have :

$$\exists \delta > 0, \text{ where } \{n \in \mathbb{N} : \sigma(x, A) > \delta \text{ and } x \notin K_n\} \text{ is infinite.}$$

Let T be an infinite subset of $\mathbb{N}$ such that for all $n \in T$, there exists $\vec{x_n} \in X$ such that $\vec{x_n} \notin K_n$ and $\sigma(\vec{x_n}, A) \geq \delta$. Without loss of generality assume that if $n, k \in T$ where $n \neq k$,then $\sigma(\vec{x_n}, \vec{x_k}) > 2$. For each $n \in T$, define S_n as follows :

$$S_n = (\prod_{k=1}^{\infty} [x_{n,k} - \frac{\delta}{8}, x_{n,k} - \frac{\delta}{8}]) \cap X.$$

Clearly $\{S_n\}$ is a locally finite collection of closed subsets of X and

$$sup\{\sigma(\vec{x_n}, \vec{s}) : \vec{s} \in S_n\} < \frac{\delta}{4}.$$

Let $S = \cup_{n>k_\delta} S_n$.

Claim : $d(A, S) \neq 0$.

Proof of Claim : Suppose $d(A, S) = 0$. Then there exists sequences $\{\vec{a_n}\} \subseteq A, \{\vec{s_n}\} \subseteq S$ where there exists an $N \in \mathbb{N}$ such that for all $n > N$, $\sigma(\vec{a_n}, \vec{s_n}) < \frac{\delta}{4}$. Let n_0 be a fixed integer greater than N . Then $\sigma(\vec{a_{n_0}}, \vec{s_{n_0}}) < \frac{\delta}{4}$. There exists $k \in T$ such that $\vec{s_{n_0}} \in S_k$. Thus $\sigma(\vec{s_{n_0}}, \vec{x_k}) \leq \frac{\delta}{4}$. By the triangle inequality, we have the following :

$$\sigma(\vec{x_k}, \vec{a_{n_0}}) \leq \sigma(\vec{x_k}, \vec{s_{n_0}}) + \sigma(\vec{s_{n_0}}, \vec{a_{n_0}}) \leq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2} < \delta.$$

This contradicts the assumption that $d(\vec{x_k}, A) > \delta$. Hence $\sigma(A, S) \neq 0$ and our claim holds.

By theorem 2.8, $A^u \cup S^u = \emptyset$. Hence $S^u \subseteq (M \cap L)^u$. Now for each $n \in T$, choose $\vec{y_n} \in S_n$ such that $\sigma(\vec{y_n}, M \cap L) \geq \frac{\delta}{32}$ (this is possible because of the definition of S_n and $M \cap L$). Let $W = \{\vec{y_n} : n \in \mathbb{N}\}$. Since $\{S_n : n \in \mathbb{N}\}$ is a locally finite collection of subsets of X and S is closed in X, we have W is closed in X. Since $\sigma(W, M \cap L) \geq \frac{\delta}{32}$, then by theorem 2.8 $W^u \cap (M \cap L)^u = \emptyset$. But as W is closed, $W^u \subseteq S^u$. This is a contradicts the fact $S^u \subseteq (M \cap L)^u$. Hence we must have $int_{X^u}(M \cap L)^u = \emptyset$. □.

The result of these 2 theorems and proposition 4.15 immediately yield the following theorem.

Theorem 4.24 : *$u_\sigma X \setminus X$ is the union of two regular closed homeomorphic copies of $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$ whose intersection is nowhere dense and is homeomorphic to $[0,1]^\omega \times (\beta\mathbb{N} \setminus \mathbb{N})$.*

In this chapter, the space that was analyzed was a space that was a locally compact σ-compact non-compact complete metric space. The analysis was aided by theorems in

[Wo] that apply to metric spaces that are locally compact σ-compact non-compact and complete. In the chapter 7 , we will be analyzing a space that is a nowhere locally compact non-compact complete metric space.

Chapter 5

Inverse Systems and Their Limits

5.1 Some Results on Inverse Systems

This section is a brief overview of inverse systems and inverse limits. Most of the results stated in this chapter can be found in [EK]. We talk about spaces indexed by the positive integers $\mathbb{N}$ and state results based on this assumption. However, most of the results can be extended to spaces in an inverse system indexed by an arbitrary directed set. For our purposes, spaces indexed by $\mathbb{N}$ will be sufficient. In actuality, when spaces in a inverse system are indexed by $\mathbb{N}$, it is called an inverse sequence. Our goal is to state here the results which we use in the next chapter to develop theories about the minimum uniform compactification of a specific class of metric spaces.

Suppose $\{X_n : n \in \mathbb{N}\}$ is a sequence of topological spaces, and for $n \leq m$ there exists a continuous function $\pi_n^m : X_m \to X_n$ such that it satisfies the following 2 conditions:

1. if $k \leq n \leq m$ then $\pi_k^n \circ \pi_n^m = \pi_k^m$,
2. $\pi_n^n = id_{X_n}$.

If these two conditions are satisfied, we say that the family $\mathcal{F} = \{X_n, \pi_n^m, \mathbb{N}\}$ is an *inverse* *system* of the spaces X_n. The mappings π_n^m are called *bonding maps* of the inverse system $\mathcal{F}$. This is a special case of more general inverse systems that are described in [EK].

Let $\{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$. We say $\{x_n\}$ is a *thread* of $\mathcal{F}$ if $\pi_n^m(x_m) = x_n$ for all $n \leq m$. The subspace of $\prod_{n\in\mathbb{N}} X_n$ consisting of all threads of $\mathcal{F}$ is called the *limit of the inverse system* $\mathcal{F}$. This subspace of $\prod_{n\in\mathbb{N}} X_n$ will be denoted by X_∞ or by $InvLim\ \mathcal{F}$. We may also denote the limit of the inverse system $\mathcal{F}$ by $InvLim\ X_n$ if there is no ambiguity about the bonding maps and the indexing set.

It is clear (see 2.5.1 and 2.5.2 of [EK]) that the limit of an inverse system $\mathcal{F}$ is a closed (possibly empty) subspace of $\prod_{n\in\mathbb{N}} X_n$, and that the limit of an inverse system of T_1, T_2, T_3, and $T_{3\frac{1}{2}}$ spaces is a T_1, T_2, T_3, and $T_{3\frac{1}{2}}$ space respectively. Furthermore (by 3.2.13 of [EK]), the limit of an inverse system of compact spaces is compact non-empty. Another way to ensure that our inverse limit is non-empty is to assume each of the bonding maps is onto. It can easily be shown that if the bonding maps are onto, then the inverse limit will be non-empty(Exercise 2.5.A of [EK]). We formalize this in the following theorem.

Theorem 5.1 : *The limit of an inverse system* $(X_n, \pi_n^m, \mathbb{N})$ *of non-empty compact* T_2 *spaces is non-empty compact* T_2.

Let p_n denote the n^{th} projection from $\prod_{n\in\mathbb{N}} X_n$ to X_n. Let $\pi_n = p_n|X_\infty$. Clearly, $\pi_n = \pi_n^m \circ \pi_m$ for all $n \leq m$.

The following is exercise 2.5B.(a) of [EK].

Theorem 5.2 : *If the bonding maps of the inverse system* $(X_n, \pi_n^m, \mathbb{N})$ *are all onto, then all the projections* π_i *are onto.*

The following theorem gives a base for the open sets of X_∞ (see 2.5.5 of [EK]).

Theorem 5.3 : *Let $X_\infty = InvLim\{X_n, \pi_n^m, \mathbb{N}\}$ and let $k \in \mathbb{N}$. Both*

$$\{\pi_n^{\leftarrow}[U_n] : n \in \mathbb{N}, U_n \text{ open in } X_n\}$$

and

$$\{\pi_n^{\leftarrow}[U_n] : n \in \mathbb{N}, U_n \text{ open in } X_n, n \geq k\}$$

are bases for the open sets of X_∞.

A very useful fact is that the closure of any subset of an inverse limit is itself the inverse limit of some inverse system. This result is stated in the following theorem which is 2.5.6 of [EK].

Theorem 5.4 : *Let A be a subset of a inverse limit X_∞ of an inverse system $\mathcal{F} = \{X_n, \pi_n^m, \mathbb{N}\}$, and let $\mathcal{F}_A$ denote the family $\{cl_{X_n}A_n, \rho_n^m, \mathbb{N}\}$ (where $A_n = \pi_n[A]$ and $\rho_n^m(x) = \pi_n^m(x)$ for $x \in cl_{X_n}A_n$).Then $\mathcal{F}_A$ is an inverse system and $InvLim\mathcal{F}_A = cl_{X_\infty}A \subseteq X_\infty$.*

Theorem 5.5 : *The limit of an inverse system $(X_n, \pi_n^m)$of zero dimensional spaces is zero dimensional.*

Proof : Let $\{X_n : n \in \mathbb{N}\}$ be zero dimensional spaces and let X_∞ be the limit of those spaces. Then X_∞ is a subspace of $\prod_{n\in\mathbb{N}} X_n$. But this product is zero dimensional, and hence X_∞ is zero dimensional as subspaces of zero dimensional spaces are zero dimensional. □

We note that the metric d, given by

$$d(\vec{x}, \vec{y}) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i)$$

is put on $\prod_{n\in\mathbb{N}} X_n$. Clearly the metric on $\prod_{n\in\mathbb{N}} X_n$ whose induced topology is the product topology. If we start with an inverse system of complete metric space, then the inverse limit of this system is complete because the inverse limit is a closed subset of the product space which is a complete metric space and thus the limit is a complete metric space. We summarize the theorem here.

Theorem 5.6 : *The limit of an inverse system $(X_n, \pi_n^m, \mathbb{N})$ of complete metric spaces is a complete metric space.*

Suppose that we are given two inverse limit systems $\mathcal{F} = (X_n, \pi_n^m, \mathbb{N})$ and $\mathcal{G} = (Y_n, \rho_n^m, \mathbb{N})$. A mapping of the system $\mathcal{F}$ to $\mathcal{G}$ is a family of continuous mappings $\mu_n : X_n \to Y_n$ such that

$$\mu_n \circ \pi_n^m = \rho_n^m \circ \mu_m \quad \text{for each } n, m \in \mathbb{N} \text{ with } n \leq m.$$

In other words, the following diagram commutes.

These mappings from $\mathcal{F}$to $\mathcal{G}$ induce a continuous mapping μ from $InvLim\ \mathcal{F} = X_\infty$ to $InvLim\ \mathcal{G} = Y_\infty$ in a natural way. For any thread $\{x_n\}_{n\in\mathrm{N}} \in X_\infty$, define $y_n = \mu_n(x_n)$, then the commutativity condition imposed above implies that $\{y_n\}_{n\in\mathrm{N}}$ is a thread in Y_∞ and we define μ by letting $\mu(\{x_n\}_{n\in\mathrm{N}}) = \{y_n\}_{n\in\mathrm{N}}$. This function is known to be continuous (see page 139 of [EK]). The following result is 2.5.9 of [EK].

Theorem 5.7 : *If for each $n \in N$, μ_n is a one-to-one continuous function, then the continuous function μ will also be a one-to-one function.*

Chapter 6

Constructing $u_d X$ Using Inverse Systems

6.1 Inverse Limit Construction of $u_d X$

In this chapter, we use inverse systems to construct the minimum uniform compactification of a metric space for a certain class of metric spaces.

For each $n \in \mathbb{N}$, let (X_n, d_n) be a complete metric space where the metric d_n is bounded by 1. That is, for any $x, y \in X_n$, $d_n(x, y) \leq 1$. Suppose also for each $n \in \mathbb{N}$ $f_n^{n+1} : X_{n+1} \to X_n$ is a uniformly continuous surjection from X_{n+1} onto X_n. Then $\{X_n, f_n^{n+1}, \mathbb{N}\}$ is an inverse limit system. Note if $m > n$, we define $f_n^m : X_m \to X_n$ to be $f_n^m = f_{m-1}^m \circ f_{m-2}^{m-1} \circ \ldots \circ f_n^{n+1}$. Clearly f_n^m is a uniformly continuous surjection from X_m onto X_n. This shows that it is enough to just talk about the bonding maps f_n^{n+1} where $n \in \mathbb{N}$ when dealing with inverse systems that are indexed by $\mathbb{N}$. As $f_n^{n+1} : X_{n+1} \to X_n$ is a uniformly continuous surjection, by theorem 2.7, there exists a continuous surjection $g_n^{n+1} : u_{d_{n+1}} X_{n+1} \to u_{d_n} X_n$ such that $g_n^{n+1}|X_{n+1} = f_n^{n+1}$. We denote each $u_{d_n} X_n$ by Y_n. We can see that $\{Y_n, g_n^{n+1}, \mathbb{N}\}$ is also an inverse limit system. Let X_∞ be the inverse limit of the system $\{X_n, f_n^{n+1}, \mathbb{N}\}$, and let Y_∞ be the inverse limit of the system $\{Y_n, g_n^{n+1}, \mathbb{N}\}$

. For each $n \in \mathbb{N}$ let $\varphi_n : X_n \to Y_n$ be the embedding map of X_n into Y_n. Each φ_n is continuous and clearly $f_n^{n+1} \circ \varphi_n = \varphi_{n+1} \circ g_n^{n+1}$. Hence there is a continuous function $\varphi : X_\infty \to Y_\infty$ defined as :

$$\varphi((x_1, x_2, x_3, ...)) = (\varphi_1(x_1), \varphi_2(x_2), \varphi_3(x_3), ...),$$

where $(x_1, x_2, ...) \in X_\infty$. This is a special case of the construction of μ from the previous chapter.

Let $p_{X_n} : \prod_{n\in\mathbb{N}} X_n \to X_n$ be the n^{th} projection map from $\prod_{n\in\mathbb{N}} X_n$ to X_n. We define $\pi_{X_n} : X_\infty \to X_n$ by $\pi_{X_n} = p_{X_n}|X_\infty$. Similarly let $p_{Y_n} : \prod_{n\in\mathbb{N}} Y_n \to X_n$ be the n^{th} projection map from $\prod_{n\in\mathbb{N}} Y_n$ onto Y_n. Define $\pi_{Y_n} : Y_\infty \to Y_n$ by $\pi_{Y_n} = p_{Y_n}|Y_\infty$.To simplify notation, let g_{n+1} denote g_n^{n+1} and let f_{n+1} denote f_n^{n+1}. We now proceed to show that Y_∞ is a T_2 compactification of X_∞

Proposition 6.1 *: $\varphi : X_\infty \to \varphi[X_\infty]$ is a homeomorphism and Y_∞ is a compactification of $\varphi[X_\infty]$.*

Proof : By definition ,φ is the embedding map from X_∞ into Y_∞, so it is one to one, and onto $\varphi[X_\infty]$. By theorem 5.7, φ is continuous. Since each Y_n is T_2, we have that Y_∞ is T_2. As each Y_n is compact, Y_∞ will also be compact and non-empty . Hence it remains to show $\varphi : X_\infty \to \varphi[X_\infty]$ is open and that $\varphi[X_\infty]$ is dense in Y_∞. To show $\varphi : X_\infty \to \varphi[X_\infty]$ is an open map,let U_n be an open subset of X_n. By theorem 5.3,it suffices to show that $\varphi[\pi^{\leftarrow}_{X_n}[U_n]]$ is open in $\varphi[X_\infty]$. Note that if $A \subseteq \varphi_n[X_n]$ then $\pi^{\leftarrow}_{X_n}[\varphi^{\leftarrow}_n[A]] = \varphi^{\leftarrow}[\pi^{\leftarrow}_{Y_n}[A]]$. Let $A = \varphi_n[U_n]$. Then

$$\pi^{\leftarrow}_{X_n}[U_n] = \pi^{\leftarrow}_{X_n}[\varphi^{\leftarrow}_n[\varphi_n[U_n]]] = \varphi^{\leftarrow}[\pi^{\leftarrow}_{Y_n}[\varphi_n[U_n]]],$$

as φ_n is one to one. Now $\varphi_n[U_n] = V_n \cap X_n$ where V_n is open in Y_n, since φ_n is an embedding of X_n in Y_n. Hence,

$$\varphi[\pi^{\leftarrow}_{X_n}[U_n]] = \pi^{\leftarrow}_{Y_n}[V_n \cap X_n] = (\pi^{\leftarrow}_{Y_n}[V_n]) \cap \varphi[X_\infty].$$

Thus, the map is open. Next we show that X_∞ is dense in Y_∞. Let U_n be an open subset of Y_n. Now $\pi_{Y_n}^{\leftarrow}[U_n]$ is a basic open subset of Y_∞. We need to show $\pi_{Y_n}^{\leftarrow}[U_n] \cap X_\infty \neq \emptyset$. As U_n is open in Y_n then $X_n \cap U_n \neq \emptyset$ as X_n is dense in Y_n. Let $x \in U_n \cap X_n$. By theorem 5.2, there exists $(a_1, a_2, a_3, ...) \in X_\infty$ such that $a_n = x$. Also $(a_1, a_2, a_3, ...) \in \pi_{Y_n}^{\leftarrow}[U_n]$ from the definition of g_{n+1} and the fact $(a_1, a_2, ..., a_n) \in U_n$. Hence X_∞ is dense in Y_∞. □

Now note that if $A \subseteq X_\infty$, then $\pi_{Y_n}[A] = \pi_{X_n}[A]$. We next show that Y_∞ is the minimum uniform compactification of (X_∞, d_∞). To do this, we prove that Y_∞ satisfies the conditions of αX stated in (2) of theorem 2.8. The following theorem is well known.

Theorem 6.2 : *If X is compact Hausdorff and Y is Hausdorff and $f : X \to Y$ is a continuous function then f is a closed function.*

It immediately follows that the map $\pi_{Y_n} : Y_\infty \to Y_n$ is a continuous closed map.

Lemma 6.3 : *Let $A, B \subseteq X_\infty$ If there exists $n \in \mathbb{N}$ such that $d_n(\pi_{X_n}[A], \pi_{X_n}[B]) \neq 0$ then $d_\infty(A, B) \neq 0$.*

Proof: Suppose $d_\infty(A, B) = 0$, then by lemma 3.3, there exists sequences $\{(a_{i,1}, a_{i,2}, ...)\}_{i\in\mathbb{N}} \subseteq A$, $\{(b_{i,1}, b_{i,2}, ...)\}_{i\in\mathbb{N}} \subseteq B$ such that

$$lim_{i\to\infty} d_\infty((a_{i,1}, a_{i,2}, ...), (b_{i,1}, b_{i,2}, ...)) = 0.$$

So for fixed $n > 0$ $lim_{i\to\infty} \frac{1}{2^n} d_n(a_{i,n}, b_{i,n}) = 0$ from the definition of d_∞. Hence we have $lim_{i\to\infty} d_n(a_{i,n}, b_{i,n}) = 0$ which implies $d_n(\pi_{X_n}[A], \pi_{X_n}[B]) = 0$ by lemma 3.3. We have shown the contrapositive of the theorem. Hence our claim holds. □

Lemma 6.4 : *Let $A, B \subseteq X_\infty$. If $\forall n > 0$, $d_n(\pi_{X_n}[A], \pi_{X_n}[B]) = 0$ then $d_\infty(A, B) = 0$.*

Proof : Suppose that $\forall n \geq 0$ $d_n(\pi_{X_n}[A], \pi_{X_n}[B]) = 0$. Let $\epsilon > 0$ be arbitarily given, then for fixed $n \geq 0$, choose $x_n \in \pi_{X_n}[A]$ and $y_n \in \pi_{X_n}[B]$, such that $d_n(x_n, y_n) < \epsilon$. By theorem 5.2, there exists $\vec{a_n} = (a_{n,1}, a_{n,2}, \ldots)$ and $\vec{b_n} = (b_{n,1}, b_{n,2}, ..)$ in A and B respectively such that $a_{n,n} = x_n$ and $b_{n,n} = y_n$. Now let $A^* = \{\vec{a_n} : n \in \mathbb{N}\} \subseteq A$ and let $B^* = \{\vec{b_n} : n \in \mathbb{N}\} \subseteq B$. Choose $n_0 \in N \ni \frac{1}{2^{n_0}} < \epsilon$. If $n > n_0$ then (recalling our earlier assumption that $d_n(x, y) \leq 1$ for each $n \in \mathbb{N}$ and each $x, y \in X_n$) we have

$$\begin{aligned} d_\infty(\vec{a_n}, \vec{b_n}) &= \sum_{k=1}^{\infty} \frac{1}{2^k} d_k(\pi_{X_k}(\vec{a_n}), \pi_{X_k}(\vec{b_n})) && (6.1) \\ &\leq \sum_{k=1}^{n_0} \frac{1}{2^k} d_k((\pi_{X_k}(\vec{a_n}), \pi_{X_k}(\vec{b_n})) + \sum_{k=n_0+1}^{\infty} \frac{1}{2^k} && (6.2) \\ &< (\epsilon \sum_{k=1}^{n_0} \frac{1}{2^k}) + \epsilon < 2\epsilon && (6.3) \end{aligned}$$

So, given any $\epsilon > 0$ we can find a sequences $\{\vec{a_n}\}$ and $\{\vec{b_n}\}$ of points of A and B respectively such that $d_\infty(\vec{a_n}, \vec{b_n}) < \epsilon$ for large enough n (which depends only on ϵ). Hence we see that $d_\infty(A, B) = 0$ as required. □

Theorem 6.5 *: If $A, B \subseteq X_\infty$ then $cl_{Y_\infty} A \cap cl_{Y_\infty} B \neq \emptyset$ if and only if $d_\infty(A, B) = 0$.*

Proof : $\Rightarrow$: Suppose $cl_{Y_\infty} A \cap cl_{Y_\infty} B \neq \emptyset$. Then $cl_{Y_n}\pi_{Y_n}[A] \cap cl_{Y_n}\pi_{Y_n}[B] \neq \emptyset$ for each $n \in \mathbb{N}$ since

$$\begin{aligned} \emptyset &\neq \pi_{Y_n}[cl_{Y_\infty} A \cap cl_{Y_\infty} B] \\ &\subseteq \pi_{Y_n}[cl_{Y_\infty} A] \cap \pi_{Y_n}[cl_{Y_\infty} B] \\ &= cl_{Y_n}\pi_{Y_n}[A] \cap cl_{Y_n}\pi_{Y_n}[B] \\ &= cl_{Y_n}\pi_{X_n}[A] \cap cl_{Y_n}\pi_{X_n}[B]. \end{aligned}$$

Thus $\forall n \geq 0$ $d(\pi_{Y_n}[A], \pi_{Y_n}[B]) = 0$. By lemma 6.4we see $d_\infty(A, B) = 0$.

$\Leftarrow$: Suppose $d_\infty(A, B) = 0$. Then by lemma 6.3 we have that $\forall n \in \mathbb{N}$, $d_n(\pi_{X_n}[A], \pi_{X_n}[B]) =$

0. Hence, as Y_n is the minimum uniform compactification of X_n, by theorem 2.8, we have :

$$cl_{Y_n}\pi_{X_n}[A] \cap cl_{Y_n}\pi_{X_n}[B] \neq \emptyset.$$

Let $A_n = cl_{Y_n}\pi_{X_n}[A]$ and $B_n = cl_{Y_n}\pi_{X_n}[B]$. Let $\mathcal{F} = \{A_n, g_n|A_n, \mathbb{N}\}$ and $\mathcal{G} = \{B_n, g_n|B_n, \mathbb{N}\}$. By theorem 5.4, these are inverse systems and

$$InvLimA_n = cl_{Y_\infty}A$$

and

$$InvLimB_n = cl_{Y_\infty}B.$$

Claim :$\mathcal{H} = \{A_n \cap B_n, g_n|A_n \cap B_n, \mathbb{N}\}$ *is an inverse system and* $InvLimA_n \cap B_n \neq \emptyset$.

Proof of Claim : First we need to show $\{A_n \cap B_n, g_n|A_n \cap B_n, \mathbb{N}\}$ is an inverse system. Let us let $h_n = g_n|(A_n \cap B_n)$. We see that for each $n \in \mathbb{N}$, each h_n is well-defined and continuous. What needs to be shown is that for each $n \in \mathbb{N}$, $h_n[A_n \cap B_n] \subseteq A_{n-1} \cap B_{n-1}$. Let $n \in \mathbb{N}$, then

$$\begin{aligned}
h_n[A_n \cap B_n] &\subseteq g_n[A_n] \cap g_n[B_n] \\
&= g_n[cl_{Y_n}(\pi_{Y_n}[A])] \cap g_n[cl_{Y_n}(\pi_{Y_n}[B])] \\
&= cl_{Y_{n-1}}(g_n[\pi_{Y_n}[A]]) \cap cl_{Y_{n-1}}(g_n[\pi_{Y_n}[B]]) \text{ (as } g_n \text{is continuous and closed)} \\
&= cl_{Y_{n-1}}(\pi_{Y_{n-1}}[A]) \cap cl_{Y_{n-1}}(\pi_{Y_{n-1}}[B]) \text{ (as } g_n \circ \pi_{Y_n} = \pi_{Y_{n-1}}) \\
&= A_{n-1} \cap B_{n-1}.
\end{aligned}$$

This shows that the function $h_n = g_n|(A_n \cap B_n)$ is a well defined continuous function from $A_n \cap B_n$ to $A_{n-1} \cap B_{n-1}$. So, $\mathcal{H}$ is an inverse system. It remains to show the inverse limit of this system is non-empty. Note each $A_n \cap B_n$ is non-empty. So as these sets are non-empty closed subsets of the compact space Y_n for each $n \in \mathbb{N}$, we see that each

$A_n \cap B_n$ is a compact subset of Y_n and hence $InvLim(A_n \cap B_n) \neq \emptyset$. By theorem 5.1 our claim holds.

Claim : $InvLim(A_n \cap B_n) \subseteq InvLim A_n \cap InvLim B_n$.

Proof of Claim : let $\{x_n\} \in InvLim(A_n \cap B_n)$, then $g_{n+1}|(A_{n+1} \cap B_{n+1})(x_{n+1}) = x_n$ for all n. Hence $\forall n \in \mathbb{N}$, $g_{n+1}|A_n(x_{n+1}) = x_n$, $g_{n+1}|B_n(x_{n+1}) = x_n$ which implies that $\{x_n\}_{n\in\mathbb{N}} \in InvLim A_n \cap InvLim B_n$ and our claim holds.

By the previous claim and theorem 5.4, we have :

$$\emptyset \neq InvLim(A_n \cap B_n) \subseteq InvLim A_n \cap InvLim B_n = cl_{Y_\infty} A \cap cl_{Y_\infty} B. \Box$$

Corollary 6.6 : $u_{d_\infty} X_\infty = Y_\infty$.

Proof : This follows immediately from theorem 6.5 above and the characterization of $u_d X$ given in theorem 2.8. $\Box$

We have constructed the minimum uniform compactification of an inverse limit of any inverse system of the form $(X_n, f_n^{n+1}, \mathbb{N})$. This technique in constructing the minimum uniform compactification of spaces will aid us in analyzing certain minimum uniform compactifications.

Chapter 7

The Irrationals with a Compatible Complete Metric

7.1 Constructing the Minimum Uniform Compactification of the Irrationals

In this chapter, we analyze the minimum uniform compactification of the irrationals with a compatible complete metric. The results from the last chapter will be used to build the minimum uniform compactification of the irrationals as the inverse limit of a certain inverse system. The following is excercise 24K of [Wi].

Theorem 7.1 : *The space of irrationals with the subspace metric inherited from $\mathbb{R}$ is homeomorphic to $\mathbb{N}^\omega$ equipped with the canonical product metric d where the metric d_n on $\mathbb{N}$ is as follows*

$$d_{\mathbb{N}}(n, m) = \begin{cases} 1 & if\ m \neq n \\ 0 & if\ m = n \end{cases}$$

,

and

$$d((x_1, x_2, ...), (y_1, y_2, ...)) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} d_{\mathbb{N}}(x_n, y_n).$$

Clearly, $(\mathbb{N}^\omega, d)$ is a nowhere locally compact complete metric space.

Let $X_n = \mathbb{N}^n$ and define the metric d_n on X_n by:

$$d_n(\vec{a}, \vec{b}) = \sum_{i=1}^{n} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i).$$

Observe that with this metric, disjoint subsets of X_n are a distance at least $\frac{1}{2^n}$ apart. Thus $u_{d_n} X_n = \beta X_n$. Now define $f_{n+1} : X_{n+1} \to X_n$ by

$$f_{n+1}((a_1, a_2, ..., a_n, a_{n+1})) = (a_1, a_2, ...a_n).$$

Proposition 7.2 : *f_{n+1} is uniformly continuous.*

Proof : Let $\epsilon > 0$ be given, let $\delta = \epsilon$. If $(a_1, a_2, .., a_{n+1}), (b_1, b_2, ..., b_{n+1}) \in X_{n+1}$ such that

$$d_{n+1}((a_1, a_2, ..., a_{n+1}), (b_1, b_2, ..., b_{n+1})) < \delta$$

then

$$\sum_{i=1}^{n+1} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) < \delta$$

which implies

$$\sum_{i=1}^{n} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) < \delta = \epsilon.$$

Hence f_{n+1} is uniformly continuous. □

As the $f_n's$ are uniformly continuous, we see that $\{X_n, f_n, \mathbb{N}\}$ is an inverse system of the sort discussed in the chapter 6 .We denote its limit by X_∞

We can clearly see that $X_\infty = \{((a_1), (a_1, a_2), ...) : a_i \in \mathbb{N}, i \in \mathbb{N}\}$. As before, let d_∞ denote the metric on X_∞. Now define a mapping $f : \mathbb{N}^\omega \to X_\infty$ by

$$f((a_1, a_2, ...)) = ((a_1), (a_1, a_2), ...).$$

Clearly f is one -to -one and onto. We now show that f is a uniform isomorphism.

Proposition 7.3 : *f is uniformly continuous.*

Proof : Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. Suppose that $d((a_1, a_2, ...), (b_1, b_2, ...)) < \delta$. Then we have that

$$\sum_{i=1}^{\infty} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) < \delta$$

which implies that

$$\sum_{i=1}^{n} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) < \delta \ \forall n \in \mathbb{N}.$$

Hence we have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{i=1}^{n} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) < \sum_{n=1}^{\infty} \frac{1}{2^n} \delta = \delta = \epsilon.$$

That is $d_\infty(f((a_1, a_2, ...)), f((b_1, b_2, ...))) < \epsilon$ so f is uniformly continuous. □

Proposition 7.4 : *$f^{\leftarrow}$ is uniformly continuous.*

Proof : Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$ and so

$$\sum_{i=N+1}^{\infty} \frac{1}{2^i} < 2\epsilon.$$

Let $\delta = \frac{\epsilon}{2^N}$. If $((a_1), (a_1, a_2), , ,), ((b_1, (b_1, b_2), ...) \in X_\infty$ such that the distance between these 2 points is less than δ, we would have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{i=1}^{n} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) < \delta.$$

Thus

$$\frac{1}{2^N} \sum_{i=1}^{N} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) < \delta$$

which implies

$$\sum_{i=1}^{N} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) < 2^N \delta.$$

Now

$$d((a_1, a_2, ..), (b_1, b_2, ...)) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) \leq \sum_{i=1}^{N} \frac{1}{2^i} d_{\mathbb{N}}(a_i, b_i) + \sum_{i=N+1}^{\infty} \frac{1}{2^i} < 2^N \delta + 2\epsilon < 3\epsilon.$$

Hence $f^{\leftarrow}$ is uniformly continuous. □

Hence (X_∞, d_∞) is uniformly equivalent to $(\mathbb{N}^\omega, d)$.

Now for each $n \in \mathbb{N}$ $\beta\mathbb{N}^n$ is the minimum uniform compactification for $\mathbb{N}^n$. Using the results from the previous chapter we can generate the following diagram.

We see $u_{d_\infty} X_\infty = Y_\infty \subseteq \prod_{n \in \mathbb{N}} \beta\mathbb{N}^n$. That is , $u_d \mathbb{N}^\omega = Y_\infty$.

7.2 The Structure of $u_d(\mathbb{N}^\omega)$

In this section, we analyze $u_d(\mathbb{N}^\omega)$. We begin by analyzing the "size" of $u_d\mathbb{N}^\omega$.

As $\mathbb{N}^\omega$ is a zero-dimensional Lindelof space, theorem 2.4 states we may replace "If A and B are disjoint closed subsets of X" of theorem 2.1 by "If A and B are complementary clopen subsets of $\mathbb{N}^\omega$". We use this to achieve the following result.

Theorem 7.5 *: $(\beta\mathbb{N})^\omega < u_d\mathbb{N}^\omega$.*

Proof : Let A be a clopen subset of $\mathbb{N}^\omega$. Suppose that $cl_{(\beta\mathbb{N})^\omega}A \cap cl_{(\beta\mathbb{N})^\omega}(\mathbb{N}^\omega \setminus A) = \emptyset$. Then $cl_{(\beta\mathbb{N})^\omega}A$ and $cl_{(\beta\mathbb{N})^\omega}(\mathbb{N}^\omega \setminus A)$ are complementary clopen subsets of $(\beta\mathbb{N})^\omega$. Let C denote the set $cl_{(\beta\mathbb{N})^\omega}A$. Note since A is a clopen subset of $\mathbb{N}^\omega$, then $C \cap \mathbb{N}^\omega = A$ and $((\beta\mathbb{N})^\omega \setminus C) \cap \mathbb{N}^\omega = \mathbb{N}^\omega \setminus A$. As C is open, C can be wrriten as:

$$C = \cup_{i \in I}[\cap_{j=1}^{n_i} \pi_j^{\leftarrow}[A_{i,j}]]$$

where I is an indexing set and the $A_{i,j}$'s are open subsets of $\beta\mathbb{N}$. As C is closed in the compact space $(\beta\mathbb{N})^\omega$, then C is compact. Thus there exist a finite subset F of I such that :

$$C = \cup_{i \in F}[\cap_{j=1}^{n_i} \pi_j^{\leftarrow}[A_{i,j}]] \tag{7.1}$$

From (7.1), it is easy to see there exist a smallest $k \in \mathbb{N}$ such that for every $n > k$ and for each $i \in F$,

$$\pi_n[\cap_{j=1}^{n_i} \pi_j^{\leftarrow}[A_{i,j}]] = \beta\mathbb{N}. \tag{7.2}$$

Claim : $d(C \cap \mathbb{N}^\omega, ((\beta\mathbb{N})^\omega \setminus C) \cap \mathbb{N}^\omega) \neq 0$.
Proof of Claim : Let $\vec{x} = (x_1, x_2, ..., x_k, x_{k+1}, ...) \in C \cap \mathbb{N}^\omega$ and let $\vec{y} = (y_1, y_2, ..., y_k, y_{k+1}, ...) \in ((\beta\mathbb{N})^\omega \setminus C) \cap \mathbb{N}^\omega$.
If for each $i = 1$ to k, $x_i = y_i$ then $\vec{y} = (x_1, x_2, ...x_k, y_{k+1}, ...)$. By (7.2), we have $\vec{y} \in C \cap \mathbb{N}^\omega$ which is a contradiction. Hence there exists some $i \leq k$ such that $x_i \neq y_i$. This implies $d(\vec{x}, \vec{y}) \geq \frac{1}{2^i} \geq \frac{1}{2^k}$. Since this holds for arbitary $\vec{x} \in C \cap \mathbb{N}^\omega$ and $\vec{y} \in ((\beta\mathbb{N})^\omega \setminus C) \cap \mathbb{N}^\omega$, then $d(C \cap \mathbb{N}^\omega, ((\beta\mathbb{N})^\omega \setminus C) \cap \mathbb{N}^\omega) \neq 0$ and our claim holds.

By theorem 2.8,

$$cl_{u_d\mathbb{N}^\omega}(C \cap \mathbb{N}^\omega) \cap cl_{u_d\mathbb{N}^\omega X}(((\beta\mathbb{N})^\omega \setminus C) \cap \mathbb{N}^\omega) = \emptyset.$$

Hence,

$$cl_{u_d\mathbb{N}^\omega} A \cap cl_{u_d\mathbb{N}^\omega}(\mathbb{N}^\omega \setminus A) = \emptyset.$$

By theorem 2.1, $(\beta\mathbb{N})^\omega \leq u_d\mathbb{N}^\omega$.

We proceed to show that $(\beta\mathbb{N})^\omega \neq u_d\mathbb{N}^\omega$. Let $A = \cup_{n\in\mathbb{N}}(\{n\}\times\{n\}\times(\prod_{i>2}^{\infty}\mathbb{N}))$. Clearly A is a clopen subset of $\mathbb{N}^\omega$. Let $B = \mathbb{N}^\omega \setminus A$. If $\vec{x} = (x_1, x_2, ...) \in A, \vec{y} = (y_1, y_2, ...) \in B$, then there exists $n \in \mathbb{N}$ such that $x_1 = x_2 = n$ and $y_1 \neq y_2$. We consider the following two possible cases :

1. If $y_1 \neq n$, then $d(\vec{x}, \vec{y}) \geq \frac{1}{2}$.
2. If $y_2 \neq n$, then $d(\vec{x}, \vec{y}) \geq \frac{1}{2}$.

Thus, $d(A, B) > 0$ and hence by theorem 2.8, $cl_{u_d\mathbb{N}^\omega} A \cap cl_{u_d\mathbb{N}^\omega} B = \emptyset$.

Claim : $cl_{(\beta\mathbb{N})^\omega} A \cap cl_{(\beta\mathbb{N})^\omega} B \neq \emptyset$.

Proof of Claim : Let $\alpha \in \beta\mathbb{N} \setminus \mathbb{N}$. We now proceed to show $\vec{x} = (\alpha, \alpha, n, n, n, ...)$ $\in cl_{(\beta\mathbb{N})^\omega} A \cap cl_{(\beta\mathbb{N})^\omega} B$ for any $n \in \mathbb{N}$. A basic neighborhood of $\vec{x}$ is given by $cl_{\beta\mathbb{N}}C \times cl_{\beta\mathbb{N}}D \times \prod_{i=3}^{\infty} U_i$ where $C, D \in \alpha$ and U_i is a neighborhood of n in $\beta\mathbb{N}$ for each $i > 2$. As α is a filter, $A \cap B \neq \emptyset$. Let $m \in A \cap B$. Then

$$(m, m, n, n, n, n, ...) \in (cl_{\beta\mathbb{N}}C \times cl_{\beta\mathbb{N}}D \times \prod_{i=3}^{\infty} U_i) \cap A.$$

Hence $\vec{x} \in cl_{(\beta\mathbb{N})^\omega} A$.

As $\alpha \in \beta\mathbb{N} \setminus \mathbb{N}$, then C and D are infinite subsets of $\mathbb{N}$. For if C (respetively D) is finite, then $cl_{\beta\mathbb{N}}C = C$(respectively $cl_{\beta\mathbb{N}}D = D$) which contradicts the fact $C \in \alpha$ (respectively

$D \in \alpha$). Let $m \in C, l \in D$ such that $m \neq l$. Then

$$(m, l, n, n, n, n, ...) \in (cl_{\beta\mathbb{N}}C \times cl_{\beta\mathbb{N}}D \times \prod_{i=3}^{\infty} U_i) \cap B.$$

Hence $\vec{x} \in cl_{(\beta\mathbb{N})^\omega}B$. Thus $cl_{(\beta\mathbb{N})^\omega}A \cap cl_{(\beta\mathbb{N})^\omega}B \neq \emptyset$ and our claim holds. By theorem 2.1, $(\beta\mathbb{N})^\omega \ncong u_d\mathbb{N}^\omega$. □

The following illustrates that $\beta(\mathbb{N}^\omega) \ncong u_d\mathbb{N}^\omega$.

Proposition 7.6 : $\beta\mathbb{N}^\omega \ncong u_d\mathbb{N}^\omega$.

Proof : Let $A = \{(n, n, n, ...) : n \in \mathbb{N}\}$ and $B = \{(n, n, n, ..., n, n+1, n+2, n+3, ...) : n \in \mathbb{N}\}$.

Claim : *A is a closed subset of $\mathbb{N}^\omega$.*

Proof of claim : Let $\vec{x} \in \mathbb{N}^\omega \setminus A$. Then there exists $n < m$ such that $x_n \neq x_m$. For every $i \neq n, m$, define $A_i = \mathbb{N}$. Define $A_n = \{x_n\}$ and $A_m = \{x_m\}$. Then $\prod_{n\in\mathbb{N}} A_n$ is a basic open neighborhood of $\vec{x}$ such that $(\prod_{n\in\mathbb{N}} A_n) \cap A = \emptyset$. Thus $\vec{x} \in \prod_{n\in\mathbb{N}} A_n \subseteq \mathbb{N}^\omega \setminus A$. Hence A is closed in $\mathbb{N}^\omega$.

Clearly B is also a closed subset of $\mathbb{N}^\omega$; since B has no convegent sequences. Obviously $A \cap B = \emptyset$.

Claim : $d(A, B) = 0$.

Proof of claim : Denote $(n, n, n, n, ...)$ by $\vec{a_n}$ and $(n, n, n, ..., n, n+1, n+2, n+3, ...)$ by $\vec{b_n}$. Observe that for each $n \in \mathbb{N}$, $\vec{a_n} \in A$ and $\vec{b_n} \in B$. It suffices to show $lim_{n\to\infty}d(a_n, b_n) = 0$. To this end, we see

$$d(\vec{a_n}, \vec{b_n}) = \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}}.$$

Thus, $lim_{n\to\infty}d(\vec{a_n}, \vec{b_n}) = 0$.

By theorem 2.8, as $d(A,B)=0$ we see that $cl_{u_d\mathbb{N}^\omega}A \cap cl_{u_d\mathbb{N}^\omega}B \neq \emptyset$. By theorem 2.3 and the fact that A and B are disjoint closed (hence zero-sets) of $\mathbb{N}^\omega$, we have $cl_{\beta(\mathbb{N}^\omega)}A \cap cl_{\beta(\mathbb{N}^\omega)}B = \emptyset$. By theorem 2.1 $\beta(\mathbb{N}^\omega) \ncong u_d\mathbb{N}^\omega$. □

Lemma 7.7 : *Let C and D be disjoint subsets of N^ω. Then $d(C,D)>0$ if and only if there exists $k>0$ such that for each $\vec{c}\in C$ and $\vec{y}\in D$, there exists $i\in\{1,2,...,k\}$ such that $c_i \neq y_i$.*

Proof : Let C,D be disjoint subsets of $\mathbb{N}^\omega$.
($\Leftarrow$) : This is obvious since for any $\vec{c}\in C$ and $\vec{y}\in D$, $d(\vec{c},\vec{y}) \geq \frac{1}{2^k}$.
($\Rightarrow$) : We prove the contrapositive. Suppose for every $k>0$, there exists $\vec{c_k}\in C$ and $\vec{y_k}\in D$ such that for each $i\in\{1,2,...,k\}$, $y_{k_i}=c_{k_i}$. Then $d(\vec{c_k},\vec{y_k}) \leq \frac{1}{2^k}$. Thus $lim_{k\to\infty} d(\vec{c_k},\vec{y_k})=0$ which implies $d(C,D)=0$. □

We now give a characterization of those clopen subsets C of $\mathbb{N}^\omega$ for which $cl_{u_d\mathbb{N}^\omega}C$ is clopen in $u_d\mathbb{N}^\omega$.

Theorem 7.8 : *The following are equivalent for a clopen subset C of $\mathbb{N}^\omega$:*

1. *There exists $k\in\mathbb{N}$ such that C can be written as $\cup_{i\in J}A_i$ where each A_i is a basic open subset of $\mathbb{N}^\omega$ of the form $\cap_{m\in F}\pi_m^{\leftarrow}[B_{i,m}]$ where $B_{i,m}\subseteq\mathbb{N}$ and $F\subseteq\{1,2,3,...,k\}$.*
2. *$d(C,\mathbb{N}^\omega\setminus C)>0$.*

Proof :
$(1\Rightarrow 2)$: Let $\vec{c}\in C$ and $\vec{y}\in\mathbb{N}^\omega\setminus C$. Suppose for $i\in\{1,2,....,k\}$, $c_i=y_i$. Then $\vec{y}\in C$ since by hypothesis, for each $n>k, i\in J$, $\pi_n[A_i]=\mathbb{N}$. This contradicts the hypothesis $\vec{y}\in\mathbb{N}^\omega\setminus C$. Thus there exists $i\in\{1,2,...,k\}$ such the $c_i\neq y_i$. By the previous lemma

7.7,$d(C, \mathbb{N}^\omega \setminus C) > 0$.

$(2 \Rightarrow 1)$: We prove the contrapositive. Suppose condition 1 fails. Then given any representation of C as the union of basic open subsets of $\mathbb{N}^\omega$ and for each $k \in \mathbb{N}$ there exists $j_k \in J$ such that A_{j_k} is a basic open subset of $\mathbb{N}^\omega$ and there exists $m_k > k$ such that $\pi_{m_k}[A_{j_k}] \neq \mathbb{N}$. Let $f : J \to \mathbb{N}$ be a mapping defined as follows :

$$f(j) = n,$$

where $\pi_n[A_j] \neq \mathbb{N}$ and for all $m > n, \pi_m[A_j] = \mathbb{N}$.

For each $j \in J$, denote A_j as $\prod_{i=1}^{\infty} B_{j,i}$ where each $B_{j,i}$ is a (open) subset of $\mathbb{N}$ and for $m > f(j)$, $\pi_m[B_{j,m}] = \mathbb{N}$. For each $k > 0$ let T_k denote all $j \in J$ such that $f(j) > k$. By hypothesis, $T_k \neq \emptyset$.

Claim : *Let $k \in \mathbb{N}$. Then there exists $j \in T_k$ such that $\prod_{i=1}^{k} B_{j,i} \times \prod_{i=k+1}^{\infty} \mathbb{N} \not\subseteq C$.*

Proof of Claim : Suppose not. Suppose there exists $k \in \mathbb{N}$ such that for all $j \in T_k$, $\prod_{i=1}^{k} B_{j,i} \times \prod_{i=k+1}^{\infty} \mathbb{N} \subseteq C$. Then we could replace each A_j, $j \in T_k$ with $C_j = \prod_{i=1}^{k} B_{j,i} \times \prod_{i=k+1}^{\infty} \mathbb{N} \subseteq C$ and write C as

$$C = (\cup_{j \in J \setminus T_k} A_j) \cup (\cup_{j \in T_k} C_j).$$

This contradicts our hypothesis that condition 1 fails. Hence our claim holds.

Thus for any $k > 0$ choose $j \in T_k$ such that $\prod_{i=1}^{k} B_{j,i} \times \prod_{i=k+1}^{\infty} \mathbb{N} \not\subseteq C$. Let $\vec{y} \in (\prod_{i=1}^{k} B_{j,i} \times \prod_{i=k+1}^{\infty} \mathbb{N}) \setminus A_j$. We have $(\prod_{i=1}^{k} B_{j,i} \times \prod_{i=k+1}^{\infty} \mathbb{N}) \setminus A_j \neq \emptyset$ since $f(j) > k$. Note that $\vec{y} \notin C$. Clearly there exists $\vec{c} \in A_j$ such that for each $i \in \{1, 2, 3, ..., k\}$, $c_i = y_i$. By lemma 7.7, we have $d(C, \mathbb{N}^\omega \setminus C) = 0$. Thus we have shown the contrapositive. □

We conclude with the main result of the thesis.

Theorem 7.9 : *$u_d\mathbb{N}^\omega$ is equivalent (as a compactification) to the Stone space of the Boolean subalgebra $\mathcal{A} = \{C \in \mathcal{B}(\mathbb{N}^\omega) : d(C, \mathbb{N}^\omega \setminus C) > 0\}$ of $\mathcal{B}(\mathbb{N}^\omega)$.*

Proof :

Claim : *$\mathcal{A}$ is a Boolean subalgebra of $\mathcal{B}(\mathbb{N}^\omega)$ and is a clopen base for $\mathbb{N}^\omega$.*

Proof of Claim : Suppose $C_1, C_2 \in \mathcal{A}$. Then by lemma 7.7, there exists $k_1 > 0$ such that for each $\vec{c} \in C_1$ and $\vec{y} \in \mathbb{N}^\omega \setminus C_1$, there exists $i \in \{1, 2, 3, ..., k_1\}$ such that $c_i \neq y_i$. Similarly, there exists $k_2 > 0$ such that for each $\vec{c} \in C_2$ and $\vec{y} \in \mathbb{N}^\omega \setminus C_2$, there exists $i \in \{1, 2, 3, ..., k_2\}$ such that $c_i \neq y_i$. Let $k = max\{k_1, k_2\}$. Then for each $\vec{c} \in C_1 \cup C_2$ and $\vec{y} \in (\mathbb{N}^\omega \setminus C_1) \cap (\mathbb{N}^\omega \setminus C_2)$, there exists $i \in \{1, 2, 3, ..., k\}$ such that $c_i \neq y_i$. Thus by lemma 7.7, $d(C_1 \cup C_2, \mathbb{N}^\omega \setminus (C_1 \cup C_2)) > 0$. This means $C_1 \cup C_2 \in \mathcal{A}$.

If $C \in \mathcal{A}$ then $d(C, \mathbb{N}^\omega \setminus C) > 0$ which means $d(\mathbb{N}^\omega \setminus C, C) > 0$. Hence $\mathbb{N}^\omega \setminus C \in \mathcal{A}$. Hence $\mathcal{A}$ is a Boolean subalgebra of $\mathcal{B}(\mathbb{N}^\omega)$

Sets of the form $\cap_{m \in F} \pi_m^{\leftarrow}[B_m]$ where $B_m \subseteq \mathbb{N}$ and $F \subseteq \{1, 2, 3, ...k\}$ for some $k \in \mathbb{N}$ belong to $\mathcal{A}$ by theorem 7.8. But these sets form a base for the open subsets of $\mathbb{N}^\omega$. Hence $\mathcal{A}$ is a clopen base for $\mathbb{N}^\omega$. Our claim holds .

By theorem 2.5, $S(\mathcal{A})$ is a zero-dimensional compactification of $\mathbb{N}^\omega$.

Claim : *Suppose $B \in \mathcal{B}(\mathbb{N}^\omega)$. Then $cl_{S(\mathcal{A})} B \cap cl_{S(\mathcal{A})}(\mathbb{N}^\omega \setminus B) = \emptyset$ if and only if $B \in \mathcal{A}$.*

Proof of Claim : If $B \in \mathcal{A}$, then by theorem 2.6, $cl_{S(\mathcal{A})} B \cap cl_{S(\mathcal{A})}(\mathbb{N}^\omega \setminus B) = \emptyset$.

Conversely, suppose $cl_{S(\mathcal{A})} B \cap cl_{S(\mathcal{A})}(\mathbb{N}^\omega \setminus B) = \emptyset$. Clearly $cl_{S(\mathcal{A})} B \cup cl_{S(\mathcal{A})}(\mathbb{N}^\omega \setminus B) = S(\mathcal{A})$. Thus $cl_{S(\mathcal{A})} B \in \mathcal{B}(S(\mathcal{A}))$. But the mapping $\lambda : \mathcal{A} \to \mathcal{B}(S(\mathcal{A}))$ defined by $\lambda(A) = cl_{S(\mathcal{A})} A$ is a Boolean algebra isomorphism from $\mathcal{A}$ onto $\mathcal{B}(S(\mathcal{A}))$. Thus, as $cl_{S(\mathcal{A})} B \in \mathcal{B}(S(\mathcal{A}))$, we see that $B \in \mathcal{A}$. Hence our claim holds.

Claim : *Suppose $M, L \in \mathbb{N}^\omega$. Then $cl_{S(\mathcal{A})} M \cap cl_{S(\mathcal{A})} L = \emptyset$ if and only if $d(M, L) > 0$.*

Proof of Claim : Suppose $cl_{S(\mathcal{A})} M \cap cl_{S(\mathcal{A})} L = \emptyset$. By theorem 2.4, there exists $G \in \mathcal{B}(S(\mathcal{A}))$ such that $cl_{S(\mathcal{A})} M \subseteq G$ and $(cl_{S(\mathcal{A})} L) \cap G = \emptyset$. Hence $G \cap \mathbb{N}^\omega \in \mathcal{A}$ which implies $d(M, L) > 0$.

Conversly, suppose $d(M, L) > 0$. Then by lemma 7.7, there exists $k > 0$ such that for each $\vec{c} = (c_1, c_2, ...) \in M$ and $\vec{y} = (y_1, y_2, ...) \in L$, there exists $i \in \{1, 2, ..., k\}$ such that

$c_i \neq y_i$. Define the set G as

$$G = \bigcup_{\vec{c} \in M} (\{c_1\} \times \{c_2\} \times \cdots \times \{c_k\} \times \mathbb{N} \times \mathbb{N} \times \cdots).$$

Clearly $G \in \mathcal{A}$ and $M \subseteq G$. Now $L \subseteq \mathbb{N}^\omega \setminus G$; for if $\vec{y} \in L$ then given $\vec{c} \in M$, there exists $i \in \{1, 2, 3, ..., k\}$ such that $c_i \neq y_i$. Thus $\vec{y} \notin (\{c_1\} \times \{c_2\} \times ... \times \{c_k\} \times \mathbb{N} \times \mathbb{N} \times \cdots)$. Since this is true for each $\vec{c} \in M$, we have $\vec{y} \notin G$. Hence $L \subseteq \mathbb{N}^\omega \setminus G$. Now by the above claim, we have $cl_{S(\mathcal{A})} G \cap cl_{S(\mathcal{A})} \mathbb{N}^\omega \setminus G = \emptyset$. This implies $cl_{S(\mathcal{A})} M \cap cl_{S(\mathcal{A})} L = \emptyset$. Thus our claim holds.

By theorem 2.8 and the above claim, $u_d \mathbb{N}^\omega \cong S(\mathcal{A})$ (as compactifications of $\mathbb{N}^\omega$). $\square$

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