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SOME SYNTHESIS METHODS FOR
MULTIVARIABLE NETWORK FUNCTIONS

BY

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A dissertation submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

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To my parents

ABSTRACT

This thesis is concerned with the synthesis of multivariable network functions.

The realizations of multivariable network functions by simple decomposition techniques are investigated. The conditions for a multivariable rational function to be realizable in certain simple structures with constituent building blocks involving functions of reduced complexity are derived. Three different configurations are considered:

(1) A sum connection of immittances which are functions of mutually disjoint sets of variables.

(2) A cascade connection of single-variable passive lumped networks, the cascaded subnetworks are also assumed lossless except the last termination.

(3) An extended Bott-Duffin type structure.

Apart from the general formulations in terms of the multivariable positive reality condition, more direct and explicit alternative approaches are also presented.

The synthesis of independent zeros of the even part of a multivariable positive real function is studied. In addition to the usual cascade extraction by the basic sections, viz., the Richards', Brune, type C, type E and type D sections, removal methods without resorting to gyrators and transformers are presented. The developments of the latter are primarily based on Miyata's separation concept of the even part function in single variable synthesis theory.

The problem of synthesizing a class of networks composed of cascaded noncommensurate transmission lines separated by passive lumped lossless two-ports and terminated by a passive lumped network is considered. A new set of realizability conditions is presented. The proposed set of conditions, which is simple in application, circumvents the difficulty associated with the test of multivariable positive reality. Several interesting special cases are also considered and the realizability conditions are accordingly modified to produce much simpler synthesis procedures.

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CONTENTS

CHAPTER I

INTRODUCTION

The concept of multivariable networks was first introduced by Ozaki and Kasami [15] arising from their work on variable-parameter networks. The theory has since been investigated extensively in the study of analysis and synthesis of many other classes of networks [20,27,33], in particular, the class of mixed lumped-distributed networks.

One of the prominant features of a multivariable formulation is that the resulting network functions are rational functions of a set of complex variables. Each of these variables characterizes a special type of component. For example, consider a class of networks made up of mixed lumped elements and noncommensurate transmission lines [25,27]. The associated network functions are not rational in the complex frequency variable p. However, such functions may be conveniently expressed as multivariable rational functions by characterizing the lumped RLC elements by the frequency variable p and the i-th type transmission lines by its Richards' [25] variable $\mu_i = \tanh \tau_i p$, where τ_i is the basic electrical length of the i-th type line.

It should be noted that the independent variables in a multivariable formulation, in general, are not necessarily physically independent. For instance, in the example given above, the Richards' variables μ_i's are in fact functions of the frequency variable p.

Furthermore, it is also not necessary to require that each variable be a function of the frequency p; as in the case of variable-parameter networks where some of the variables could be functions of some outside factors such as temperature, a control setting, etc..

Similar to single variable theory, the concept of multivariable positive reality is of paramount importance in multivariable synthesis theory. The following are the fundamental definitions:

A multivariable rational function $Z(\underline{p})$ of a set of complex variables $\underline{p} = (p_1,p_2,\ldots,p_n)$ is said to be a multivariable positive (m.p.) function if and only if (iff)

$$\text{Re } Z(\underline{p}) \geq 0 \quad \text{for} \quad \text{Re } p_i \geq 0, \quad i = 1,2,\ldots,n,$$

where Re denotes "The real part of". An m.p. function $Z(\underline{p})$ is said to be multivariable positive and real (m.p.r.) iff $Z(\underline{p})$ is real when all the variables are real.

An m.p. function $Z(\underline{p})$ is said to be multivariable para-odd iff

$$Z(\underline{p}) + Z_*(\underline{p}) = 0,$$

where $Z_*(\underline{p})$ is the para-conjugate of $Z(\underline{p})$ and is defined as $Z_*(\underline{p}) = Z^*(-\underline{p}^*)$, where the upper asterisk denotes the conjugate operation. For real rational functions, one has $Z_*(\underline{p}) = Z(-\underline{p})$, and an m.p.r. function $Z(\underline{p})$ is said to be a multivariable reactance function iff

$$Z(\underline{p}) + Z(-\underline{p}) = 0.$$

A multivariable rational function $s(\underline{p})$ of a set of complex variables $\underline{p} = (p_1,p_2,\ldots,p_n)$ is said to be a multivariable bounded function iff

$$|s(\underline{p})| \leq 1 \quad \text{for} \quad \text{Re } p_i \geq 0, \quad i = 1,2,\ldots,n.$$

A multivariable bounded function $s(\underline{p})$ is said to be multivariable bounded real iff $s(\underline{p})$ is also real when all the variables are real.

Since its introduction [15] in 1960, a substantial amount of work on multivariable synthesis has been reported in the literature. Detailed reviews of the early developments have been given by Scanlan [27] and Youla [33], and recently a comprehensive bibliography has been presented by Ramachandran and Rao [20]. In spite of the extensive developments in the past, the multivariable synthesis proved to be unwieldy, and consequently more straightforward synthesis techniques are expected to emerge. This study is concerned with the development of simple and straightforward special synthesis methods. Furthermore, it is noted that although the multivariable positive reality is a compact gauge for the measurement of the realizability of a multivariable function, the verification of such a property is difficult and laborious. In this study, a special emphasis is also placed upon deriving possible alternative explicit realizability conditions, which would replace this prerequisite condition by some simpler conditions.

In Chapter II, the realizations of multivariable rational functions, in the forms of certain simple structures with component building blocks involving functions of reduced complexity, are investigated. Three different configurations are considered:

(1) A sum connection of immittances which are functions of mutually disjoint sets of variables.

(2) A cascade of single-variable blocks.

(3) A Bott-Duffin type structure.

The realizability conditions are formulated in terms of the decomposability of the given function into certain special forms. In addition to the general formulations based on the multivariable positive reality condition, more direct and explicit approaches are also discussed.

Chapter III is the study of the removal of independent zeros of the even part of a multivariable positive real function. Apart from the discussion of the cascade extraction by the basic sections, viz., the Richards', Brune, type C, type E and type D sections, realization methods without resorting to gyrators and transformers are also presented.

In Chapter IV, the problem of synthesizing a class of networks comprising cascaded noncommensurate transmission lines separated by passive lumped lossless two-ports and terminated by a passive lumped network is considered. A new set of realizability conditions is presented. The advantage of the proposed set of conditions is that it replaces the multivariable reality test and facilitates the synthesis procedure in a straightforward manner. Several interesting special cases are also considered and the realizability conditions are duly modified into much simpler forms.

CHAPTER II

SYNTHESIS OF MULTIVARIABLE NETWORK FUNCTIONS BY SIMPLE DECOMPOSITION METHODS

The synthesis of general m.p.r. functions was first proposed by Koga [13]. He proved that the multivariable positive reality is a sufficient condition for realizability. However, his approach, involving certain factorization processes of multivariable matrices, is known to be difficult and laborious. Furthermore, the validity of his result has been questioned recently by Bose [37], who provides a counter example indicating that his method does not always work. To circumvent the inherent difficulties of the general synthesis problem, some workers [3,4,30] have recently developed special techniques for certain classes of functions. The essential idea of these developments is to derive simple criteria for the decomposition of a given m.p.r. function into a sum of single variable p.r. functions so that the synthesis may be performed by the well-established single variable methods. In this chapter, we consider a more general aspect of synthesizing multivariable network functions in the forms of certain simple structures with constituent building blocks involving functions of reduced complexity. Three different configurations are considered:

(1) A sum connection of immittances being functions of mutually disjoint sets of variables.

(2) A cascade connection of single variable subnetworks, which are lossless except the last termination.

(3) An extended Bott-Duffin type structure.

The realizability conditions are formulated in terms of the decomposability of the given function into certain special forms. Furthermore, since the verification of the multivariable positive reality, in general, is rather unwieldy and intricate, we shall, in the following, also develope possible alternative explicit formulations which remove this prerequisite condition in favour of some one-variable type conditions.

2.1 SUM DECOMPOSITION

The concept of realizing a class of multivariable reactance functions in terms of single variable reactance functions in a sum form was first advanced by Soliman and Bose [30]. Recently, Bose [3] extended the method by presenting a revised version for the previous result. However, the above work mainly dealt with the complete decomposability of an m.p.r. function into a sum of single variable p.r. functions; moreover, the decomposition algorithm for the case of reactance functions involves laborious steps of extracting various constants. In this section, we consider the more general problem of decomposing a class of m.p.r. functions into a sum of such functions each having a smaller number of variables than the original one. In particular, the decomposition of the class of multivariable reactance functions into a sum of single variable reactance functions is reinvestigated. Results are presented in Section 2.1.1 and illustrated by examples in Section 2.1.3. Explicit formulations are discussed in Section 2.1.2.

2.1.1 General MPR Approach

In the following, we first establish a simple criterion, stated in the form of Theorem 2.1, for the decomposability of a given m.p.r. function into a sum of m.p.r. functions with fewer variables. Several interesting consequences of the theorem are then discussed. For the special class of multivariable reactance functions, an extremely simple decomposition method is given in Theorem 2.2[1] [16] which eliminates the laborious steps of constant extractions as required by the algorithm given in [3].

Theorem 2.1

Let $Z(\underline{p})$ be a multivariable positive real function of a set of complex variables $\underline{p} = (p_1,p_2,\ldots,p_n)$. Then $Z(\underline{p})$ can be decomposed as

$$Z(\underline{p}) = Z_1(p_1,p_2,\ldots,p_\ell) + Z_2(p_{\ell+1},p_{\ell+2},\ldots,p_n) \;, \quad \ell<n \qquad (2.1)$$

where $Z_1(p_1,p_2,\ldots,p_\ell)$ is m.p.r. in $p_1,p_2,\ldots,p_\ell$ and $Z_2(p_{\ell+1},p_{\ell+2},\ldots,p_n)$ is m.p.r. in $p_{\ell+1},p_{\ell+2},\ldots,p_n$, if and only if

$$Z(\underline{p}) - Z(p_1,p_2,\ldots,p_\ell,1,1,\ldots,1) \qquad (2.2)$$

is not a function of $p_1,p_2,\ldots,p_\ell$.

Proof: The necessity is evident. We shall show the sufficiency.

Since $Z(\underline{p})$ is m.p.r., $Z(p_1,p_2,\ldots,p_\ell,1,1,\ldots,1)$ is also m.p.r. Let $p_i = j\omega_{io}$, $i = 1,2,\ldots,\ell$, be the minimum point of $\mathrm{Re}\, Z(j\omega_1,j\omega_2,\ldots,j\omega_\ell,1,1,\ldots,1)$ with the minimum value K, where $\mathrm{Re}\, Z$ denotes

1 Independently, a similar result was also reported recently in [23].

the real part of Z. By repeated applications of the maximum modulus theorem of a function of a complex variable, it can be shown that

$$\text{Re } Z(p_1,p_2,\ldots,p_\ell,1,1,\ldots,1) - K \geq 0 \tag{2.3}$$

$$\text{for} \quad \text{Re } p_i \geq 0, \quad i = 1,2,\ldots,\ell.$$

Let Z_1 be defined as

$$Z_1 = Z(p_1,p_2,\ldots,p_\ell,1,\ldots,1) - K , \tag{2.4}$$

then with (2.3), it follows immediately from the definition of an m.p.r. function that Z_1 is m.p.r. in $p_1,p_2,\ldots,p_\ell$.

Now, let Z_2 be defined as

$$Z_2 = Z(\underline{p}) - Z_1(p_1,p_2,\ldots,p_\ell) . \tag{2.5}$$

By hypothesis (2.2), it is apparent that Z_2 so defined is not a function of $p_1,p_2,\ldots,p_\ell$. Hence, by selecting $p_i = j\omega_{io}$, $i=1,2,\ldots,\ell$, and taking the real part of both sides of (2.5) it is seen that

$$\text{Re } Z_2 = \text{Re } Z(j\omega_{10},j\omega_{20},\ldots,j\omega_{\ell 0},p_{\ell+1},\ldots,p_n) \geq 0 ,$$

$$\text{for} \quad \text{Re } p_i \geq 0, \quad i = \ell+1,\ell+2,\ldots,n.$$

Therefore, Z_2 is also m.p.r. in $p_{\ell+1},p_{\ell+2},\ldots,p_n$. The sufficiency thus follows from (2.5).

Note that apart from an additive constant the component functions Z_1 and Z_2 are completely defined by (2.4) and (2.5). For $\ell = 1$, Theorem 2.1 yields the following useful corollary which enables us to detect the possibility of extracting a single variable p.r. function from a given m.p.r. function while leaving the remaining function still m.p.r. and having one variable less.

Corollary 2.1.1

A necessary and sufficient condition for an m.p.r. function $Z(\underline{p})$ to be decomposed as

$$Z(\underline{p}) = Z_1(p_1) + Z_2(p_2,p_3,\ldots,p_n) ,$$

where $Z_1(p_1)$ is a single variable p.r. function in p_1 and $Z_2(p_2,p_3,\ldots,p_n)$ is m.p.r. in $p_2,p_3,\ldots,p_n$, is that

$$Z(\underline{p}) - Z(p_1,1,1,\ldots,1)$$

is not a function of p_1. (The component function $Z_1(p_1)$ may be determined according to (2.4) as follows

$$Z_1(p_1) = Z(p_1,1,1,\ldots,1) - \min_{\omega_1} \operatorname{Re} Z(j\omega_1,1,1,\ldots,1)) \quad . \qquad (2.6)$$

By repeated applications of Corollary 2.1.1, we obtain the following corollary, which corresponds to the special case considered in [3].

Corollary 2.1.2

A necessary and sufficient condition for an m.p.r. function $Z(\underline{p})$ to be decomposed as

$$Z(\underline{p}) = \sum_{i=1}^{n} Z_i(p_i) ,$$

where $Z_i(p_i)$ is a single variable p.r. function in p_i, is that

$$Z(\underline{p}) - Z(1,\ldots,1,p_i,1,\ldots,1)$$

is not a function of p_i, for $i = 1,2,\ldots,n-1$.

As in (2.6), the sub-functions $Z_i(p_i)$ may be derived from (2.4) as follows

$$Z_i(p_i) = Z(1,\ldots,1,p_i,1,\ldots,1) - K_i, \quad i = 1,2,\ldots,n-1, \tag{2.7}$$

and

$$Z_n(p_n) = Z(\underline{p}) - \sum_{i=1}^{n-1} Z_i(p_i), \tag{2.8}$$

where

$$K_i = \underset{\omega_i}{\mathrm{Min}}\ \mathrm{Re}\ Z(1,\ldots,1,j\omega_i,1,\ldots,1) . \tag{2.9}$$

Note that Corollary 2.1.2 is essentially equivalent to the main theorem given in [3]. However, it may be noted that the condition that $F^{(n-1)}(p_n)$ in [3] ($Z_n(p_n)$ of (2.8) above) be p.r. is superfluous as far as the realizability is concerned. The fact is that the p.r. nature of $F^{(n-1)}(p_n)$ is automatically satisfied from the hypothesis that the given function is m.p.r.. As evident from Corollary 2.1.2 above, no further p.r. test at any stage is necessary provided that the given function is m.p.r..

From (2.7), it is observed that the determination of the sub-functions $Z_i(p_i)$'s involves the extraction of a maximum possible positive constant from a p.r. function. For reactance functions, however, such laborious steps may be avoided by fully exploiting the property of reactance functions. The improved result is summarized in the following theorem.

Theorem 2.2

A necessary and sufficient condition for a multivariable reactance function

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})}$$

to be decomposed as

$$Z(\underline{p}) = \sum_{i=1}^{n} Z_i(p_i) ,$$

where $Z_i(p_i)$ is a single variable reactance function in p_i, for $i = 1,2,\ldots,n$, is that the denominator $Q(\underline{p})$ can be factored as

$$Q(\underline{p}) = \prod_{i=1}^{n} q_i(p_i) , \tag{2.10}$$

where $q_i(p_i)$ is a single variable polynomial in p_i, $i = 1,2,\ldots,n$. Furthermore,

(i) If $q_i(p_i)$ does not vanish at the origin, then

$$Z_i(p_i) = Z(0,\ldots,0,p_i,0,\ldots,0). \tag{2.11}$$

(ii) If $q_i(p_i)$ vanishes at the origin, defining a new function

$$\hat{Z}(\underline{p}) = Z(\underline{p}) - \sum_{i=1}^{n} \frac{A_i}{p_i} , \tag{2.12}$$

where

$$A_i = p_i Z(\underline{p})\big|_{p_i=0}$$

is the residue of $Z(\underline{p})$ at $p_i = 0$, then

$$Z_i(p_i) = \hat{Z}(0,\ldots,p_i,0,\ldots,0) + \frac{A_i}{p_i} . \tag{2.13}$$

Proof: The necessity is evident. We shall prove the sufficiency.

It is observed that a multivariable reactance function $Z(\underline{p})$ is an m.p.r. function satisfying the following additional condition

$$Z(\underline{p}) + Z(-\underline{p}) \equiv 0 . \tag{2.14}$$

With (2.14), it can be shown that a pole of a multivariable reactance function in a p_i-plane, independent of all other variables, lies on the imaginary axis of the p_i-plane.

Also, according to a result due to Ozaki and Kasami [15], the residue of an imaginary axis independent pole including the origin and infinity of a m.p.r. function is a positive constant, and the removal of such a pole yields a remaining function which is also m.p.r..

By hypothesis, the denominator of $Z(\underline{p})$ can be factored in the form of (2.10), therefore all the poles of $Z(\underline{p})$ are independent poles. Consequently, it follows from the above two results that $Z(\underline{p})$ can be expressed in the form

$$Z(\underline{p}) = A_{1\infty}p_1 + \frac{A_{10}}{p_1} + \sum_{\ell} \frac{A_{1\ell}p_1}{p_1^2 + \omega_{1\ell}^2} + Z_2(\underline{p}) \ , \tag{2.15}$$

where $A_{1\infty}$, A_{10} and $A_{1\ell}$'s are non-negative and $Z_2(\underline{p})$ is m.p.r..

It is apparent from (2.15) that the denominator of $Z_2(\underline{p})$ is free of the variable p_1. Therefore, it can be shown from the degree property of m.p.r. functions that $Z_2(\underline{p})$ is no more a function of p_1.

Applying the analogous procedure successively, we can decompose $Z(\underline{p})$ as

$$Z(\underline{p}) = \sum_{i=1}^{n} \left(A_{i\infty}p_i + \frac{A_{io}}{p_i} + \sum_{\ell} \frac{A_{i\ell}p_i}{p_i^2 + \omega_{i\ell}^2} \right)$$

$$= \sum_{i=1}^{n} Z_i(p_i) \ , \tag{2.16}$$

where the $Z_i(p_i)$'s are obviously single variable reactance functions.

Now, if $q_i(p_i)$ does not vanish at the origin, then $A_{io} = 0$ for every i. By setting $p_k = 0$ for every k except $k = i$, it follows from (2.16) that

$$Z_i(p_i) = Z(0,\ldots,0,p_i,0,\ldots,0) \ .$$

Moreover, if $q_i(p_i)$ has p_i as a factor, we can always remove the terms $\frac{A_{io}}{p_i}$'s by inspection in advance to yield a new function which satisfies the same condition as the previous one. Hence, we can write (2.13) as a consequence of (2.11) and (2.12). Q.E.D.

2.1.2 Explicit Approach

Due to the particular nature of the problems considered in Corollary 2.1.2 and Theorem 2.2, the requirement of the multivariable positive reality on the given function can therefore be relaxed by reformulating the propositions into the following alternative forms.

Theorem 2.3

Let $Z(\underline{p})$ be a multivariable rational function. The following two conditions

(i) $Z(1,\ldots,1,p_i,1,\ldots,1)$ is a single variable p.r. function of p_i, $i = 1,2,\ldots,n-1$.

(ii) The function

$$Z_n = Z(\underline{p}) - \sum_{i=1}^{n-1} Z_i(p_i) \tag{2.17}$$

is a single variable p.r. function in p_n, where

$$Z_i(p_i) = Z(1,\ldots,1,p_i,1,\ldots,1) - K_i \tag{2.18}$$

with

$$K_i = \min_{\omega_i} \operatorname{Re} Z(1,\ldots,1,j\omega_i,1,\ldots,1) \ .$$

are necessary and sufficient for $Z(\underline{p})$ to be a member of a subclass of m.p.r. functions, which can be decomposed as a sum of single variable p.r. functions.

Proof: The proof is straightforward.

Necessity: Since $Z(\underline{p})$ is m.p.r., condition (i) is obviously satisfied. Next, we shall show that Z_n defined by (2.17) is a single variable p.r. function of p_n.

Since $Z(1,\ldots,1,p_i,1,\ldots,1)$ is p.r. in p_i, the functions $Z_i(p_i)$'s defined by (2.18) are also p.r. and minimum, viz.,

$$\text{Re } Z_i(j\omega_{io}) = 0 , \quad i = 1,2,\ldots,n-1 \tag{2.20}$$

where ω_{io} is the minimum point of (2.19).

Since $Z(\underline{p})$ is decomposable into a sum of single variable p.r. functions, it can be shown that Z_n defined by (2.17) is solely a function of p_n. Consequently, by selecting $p_i = j\omega_{io}$ for $i = 1,2,\ldots,n-1$ and taking the real parts of both sides of (2.17) then comparing with (2.20), we have

$$\text{Re } Z_n(p_n) = \text{Re } Z(j\omega_{10},j\omega_{20},\ldots,j\omega_{n-10},p_n) . \tag{2.21}$$

Since $Z(\underline{p})$ is m.p.r.,

$$\text{Re } Z(j\omega_{10},j\omega_{20},\ldots,j\omega_{n-10},p_n) \geq 0 \quad \text{for} \quad \text{Re } p_n \geq 0 . \tag{2.22}$$

Therefore,

$$\text{Re } Z_n(p_n) \geq 0 \quad \text{for} \quad \text{Re } p_n \geq 0 . \tag{2.23}$$

With (2.23), it is apparent from the definition of p.r. function that $Z_n(p_n)$ is p.r. in p_n.

Sufficiency: As shown above condition (i) coupled with equation (2.18) indicates that the $Z_i(p_i)$'s, $i = 1,2,\ldots,n-1$, are also p.r..

The sufficiency is thus evident from (2.17). Q.E.D.

Since independent poles at the origin of p_i-plane can easily be removed by inspection in advance, in the following theorem, we shall assume without lack of generality that the given function does not possess poles at the origin.

Theorem 2.4

Let

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})}$$

be an irreducible multivariable rational function having no poles at the origin. The necessary and sufficient conditions for $Z(\underline{p})$ to be decomposed into a sum of single variable reactance functions in the form of

$$Z(\underline{p}) = \sum_{i=1}^{n} \frac{\alpha_i(p_i)}{q_i(p_i)}$$

are

(i) The denominator can be factored as

$$Q(\underline{p}) = \prod_{i=1}^{n} q_i(p_i),$$

where $q_i(p_i)$ are even polynomials of p_i.

(ii) The numerator can be expressed as

$$P(\underline{p}) = \sum_{i=1}^{n} \alpha_i(p_i) \prod_{\ell \neq i} q_\ell(p_\ell) ,$$

where

$$\alpha_i(p_i) = \frac{P(0,\ldots,0,p_i,0,\ldots,0)}{\prod_{\ell \neq i} q_\ell(0)} .$$

(iii) $\alpha_i(p_i)/q_i(p_i)$ is a reactance function of p_i.

Proof: It is noted that the first two conditions assure the separability of the given function into a sum form; while the third condition attests to the reactance nature of the component functions. The proof is straightforward and thus omitted for brevity.

2.1.3 Examples

Example 2.1.1: Consider the following m.p.r. function in $\underline{p} = (p_1,p_2,p_3,p_4)$

$$Z(\underline{p}) = \frac{p_1p_2p_3p_4 + p_1p_3p_4 + 3p_1p_2p_3 + 2p_1p_2 + p_3p_4 + 2p_1 + 3p_3 + 2}{(p_1p_2 + 1)\ (p_3p_4 + 2)} . \tag{2.24}$$

It is required to determine the decomposability of (2.24) into a sum of m.p.r. functions having fewer variables.

It is observed that

$$Z(\underline{p}) - Z(p_1,p_2,1,1) = \frac{3p_3 - p_3p_4 - 2}{p_3p_4 + 2}$$

is not a function of p_1 and p_2, therefore, it follows from Theorem 2.1 that $Z(\underline{p})$ can be decomposed as

$$Z(\underline{p}) = Z_1(p_1,p_2) + Z_2(p_3,p_4) . \tag{2.25}$$

Using (2.4) we obtain

$$\begin{aligned} Z_1(p_1,p_2) &= Z(p_1,p_2,1,1) - \min_{\omega_1,\omega_2} \text{Re } Z(j\omega_1,j\omega_2,1,1) \\ &= \frac{p_1}{p_1p_2 + 1} . \end{aligned}$$

Hence from (2.25) the desired decomposition is

$$Z(\underline{p}) = \frac{p_1}{p_1 p_2 + 1} + \frac{p_3 p_4 + 3p_3 + 2}{p_3 p_4 + 2} .$$

Example 2.1.2: It is desired to determine whether the following multivariable reactance function[1] in $\underline{p} = (p_1, p_2, p_3)$

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})} = \frac{5p_1^2 p_2^2 p_3 + p_2^2 p_1 + p_1^2 p_2 + 20p_2^2 p_3 + 30p_1^2 p_3 + 6p_1 + 4p_2 + 120p_3}{p_1^2 p_2^2 + 4p_2^2 + 6p_1^2 + 24}$$

can be decomposed into a sum of single variable reactance functions.

We see that the denominator $Q(\underline{p})$ is factorable as

$$\begin{aligned} Q(\underline{p}) &= (p_2^2 + 6)p_1^2 + 4p_2^2 + 24 \\ &= (p_2^2 + 6)(p_1^2 + 4) . \end{aligned}$$

Therefore, using Theorem 2.2, the given function is decomposable as

$$Z_1(p_1) = Z(p_1, 0, 0) = \frac{p_1}{p_1^2 + 4}$$

$$Z_2(p_2) = Z(0, p_2, 0) = \frac{p_2}{p_2^2 + 6}$$

$$Z_3(p_3) = Z(0, 0, p_3) = 5p_3$$

$$Z(\underline{p}) = \frac{p_1}{p_1^2 + 4} + \frac{p_2}{p_2^2 + 6} + 5p_3 .$$

The proposed technique, as evidenced above, is much simpler than the method given in [30] where laborious steps of constant extractions are required in the process of deriving the component functions.

1 For comparison, the identical problem given in [30] is considered.

2.2 CASCADE DECOMPOSITION

The synthesis of an m.p.r. function as an input impedance of the cascade structure shown in Fig. 2.2 was first considered in [6] and recently in [2]. However, Reference [6] considers only the very special case that the extraction order of the p_i-variable two-ports is s interchangeable. For interchangeable configurations, the input impedances are more restricted. Furthermore, their conditions assume certain special forms for the even part functions, and this consequently limits their applications to non-reactance functions. The conditions given in [2] are more relaxed and include reactance functions. However, their development is based on the single variable Darlington theory [7], and the results are therefore restricted to reciprocal realizations. In addition, as it is pointed out in the Appendix that the formulation presented in [2] leads to faulty conclusion under certain circumstances. In this section, we tackle the problem with a different approach. The realizability conditions are derived from the chain parameter characterization of passive lumped lossless two-ports. As a consequence, the results obtained are simpler and more general.

2.2.1 General MPR Approach

First, we establish a fundamental lemma which is essential to the subsequent developments.

Lemma 2.1

The 2x2 real polynomial matrix

$$\frac{1}{f(p)}\begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} \tag{2.26}$$

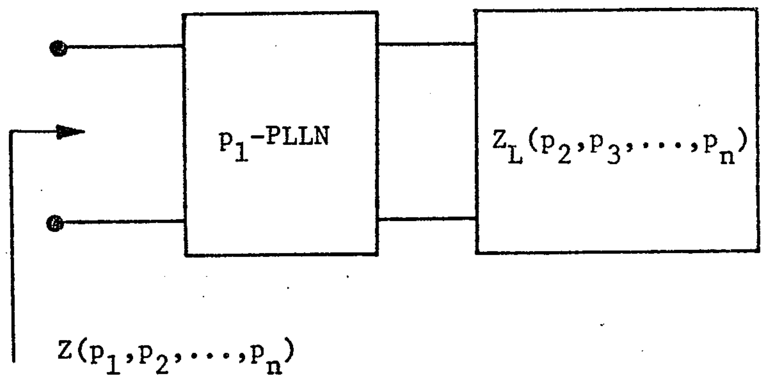


Fig. 2.1 Realization of $Z(p_1,p_2,\ldots,p_n)$ in the form of a p_1-variable passive lumped lossless two-port closed on an (n-1)-variable impedance $Z_L(p_2,p_3,\ldots,p_n)$.

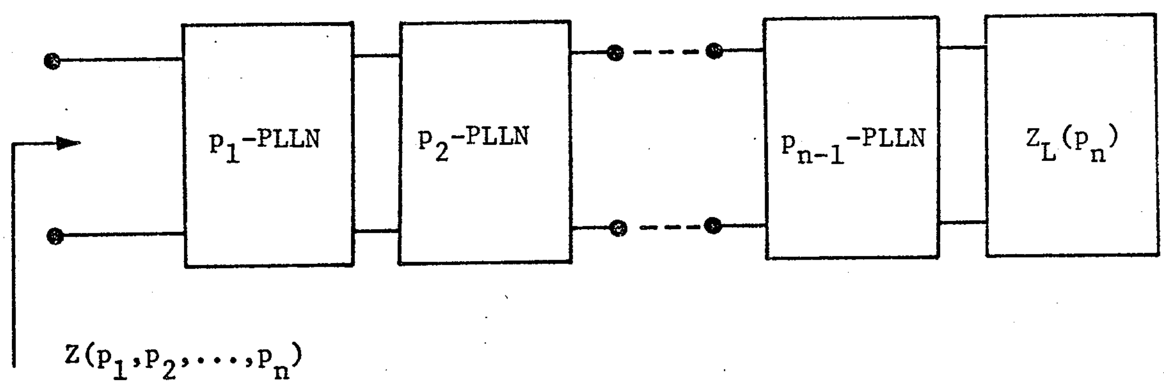


Fig. 2.2 A cascade of single-variable passive lumped lossless two-ports terminated by a single variable passive impedance.

represents the chain matrix of a passive lumped lossless two-port iff

(1) $A(p)$ and $D(p)$ are either both even or odd, while oppositely, $B(p)$ and $C(p)$ are either both odd or even.

(2) $A(p)+B(p)+C(p)+D(p) = G(p)H(p)$, where $H(p)$ is strictly Hurwitz, and $G(p)$ is either even or odd and is a common divisor of $A(p)+B(p)$ and $C(p)+D(p)$.

(3) $A(p)D(p)-B(p)C(p) = \varepsilon f(p)f(-p)$, where $f(p)$ is a real polynomial, $\varepsilon=1$ if $A(p)$ is even and $\varepsilon=-1$ if $A(p)$ is odd.

In particular, the two-port is reciprocal iff the following additional condition is satisfied.

(4) $A(p)D(p)-B(p)C(p) = f^2(p)$, i.e., $f(p)$ is even iff $A(p)$ is even and $f(p)$ is odd iff $A(p)$ is odd.

The above lemma is a classical result and a brief proof may be found in [32].

Lemma 2.2

A necessary and sufficient condition for the n-variable p.r. function $Z(\underline{p})$ to be realizable as an input impedance in the form of a p_1-variable passive lumped lossless two-port terminated by an $(n-1)$-variable p.r. impedance as shown in Fig. 2.1 is that $Z(\underline{p})$ can be decomposed as

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})} = \frac{\alpha(p_1)h(p_2,p_3,\ldots,p_n) + \beta(p_1)g(p_2,p_3,\ldots,p_n)}{\gamma(p_1)h(p_2,p_3,\ldots,p_n) + \delta(p_1)g(p_2,p_3,\ldots,p_n)}, \quad (2.27)$$

where $P(\underline{p})$ and $Q(\underline{p})$ are assumed to be relatively prime in $\operatorname{Re} p_i \geq 0$ for every i, and $\alpha(p_1)$ and $\delta(p_1)$ are both even polynomials of p_1, whereas, $\beta(p_1)$ and $\gamma(p_1)$ are odd polynomials of p_1.

In particular, the p_1-variable two-port is reciprocal iff

$$\alpha(p_1)\delta(p_1) - \beta(p_1)\gamma(p_1) = \varepsilon f^2(p_1) \ ,$$

where $f(p_1)$ is either even (for $\varepsilon=1$) or odd (for $\varepsilon=-1$).

Proof:

Necessity: It is noted that if the output terminals of a two-port characterized by the chain matrix

$$\frac{1}{f}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

are terminated by an impedance Z_L, then the input impedance seen looking into the input terminals is given by

$$Z = \frac{AZ_L + B}{CZ_L + D} \ . \tag{2.28}$$

The necessity is therefore evident from (2.28) and Lemma 2.1.

Sufficiency: First we show that $h(p_2,p_3,\ldots,p_n)/g(p_2,p_3,\ldots,p_n)$ is an $(n-1)$-variable p.r. function.

As a consequence of a result due to Ramachandran et al. [22], we have that if

$$Z(\underline{p}) = \frac{c_k p_1^k + c_{k+1}p_1^{k+1} + \ldots + c_r p_1^r}{d_{k'}p_1^{k'} + d_{k'+1}p_1^{k'+1} + \ldots + d_{r'}p_1^{r'}} \tag{2.29}$$

is an n-variable p.r. function, where the c_i's and d_i's are polynomials of $p_2,p_3,\ldots,p_n$ and $c_k,d_{k'},c_r,d_{r'} \not\equiv 0$, then $k-k' = \pm1$ or 0, $r-r' = \pm1$ or 0 and the quotions c_i/d_i for $\max(k,k') \leq i \leq \min(r,r')$ are $(n-1)$-variable p.r. functions of $p_2,p_3,\ldots,p_n$.

By hypothesis, we have that $\alpha(p_1)$ and $\delta(p_1)$ are both even and $\beta(p_1)$ and $\gamma(p_1)$ are both odd, therefore by writing (2.27) in the form of (2.29) and taking any i within the defined range, it follows from the above result that h/g is $(n-1)$-variable p.r. in $p_2, p_3, \ldots, p_n$.

Now, let σ be an arbitrary positive constant, then it is clear from the definition of m.p.r. functions that

$$\frac{h(\sigma,\sigma,\ldots,\sigma)}{g(\sigma,\sigma,\ldots,\sigma)} = K > 0 \tag{2.30}$$

and

$$Z(p_1,\sigma,\sigma,\ldots,\sigma) = \frac{\alpha(p_1)K + \beta(p_1)}{\gamma(p_1)K + \delta(p_1)} \tag{2.31}$$

is p.r. in p_1.

By assumption, $P(\underline{p})$ and $Q(\underline{p})$ are relatively prime in Re $p_i \geq 0$ for every i, therefore, $\alpha(p_1)K+\beta(p_1)$ and $\gamma(p_1)K+\delta(p_1)$ are also relatively prime in Re $p_1 \geq 0$. Consequently, it can be shown from the properties of single variable p.r. functions that

$$\alpha(p_1) + \beta(p_1) + \gamma(p_1) + \delta(p_1) \quad \text{is strictly Hurwitz}$$

and $\alpha(p_1)\delta(p_1) - \beta(p_1)\gamma(p_1)$ can be factored as

$$\alpha(p_1)\delta(p_1) - \beta(p_1)\gamma(p_1) = f(p)f(-p) ,$$

where $f(p)$ is a real polynomial.

Hence by virtue of Lemma 2.1, the following matrix

$$[T]_{p_1} = \frac{1}{f(p_1)} \begin{bmatrix} \alpha(p_1) & \beta(p_1) \\ \gamma(p_1) & \delta(p_1) \end{bmatrix} \tag{2.32}$$

represents the chain-parameter matrix of a lumped passive lossless two-port in the variable p_1.

Therefore, by letting

$$Z_L = \frac{h}{g} , \tag{2.33}$$

it is apparent from (2.27), (2.28) and (2.32) that $Z(\underline{p})$ is realizable in the form of Fig. 2.1.

Now, for the case of reciprocal realization, it may be shown from condition (4) of Lemma 2.1 that

(i) If f is even, then

$$[T]_{p_1} = \frac{1}{f}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{2.34a}$$

$$Z_L = \frac{h}{g} . \tag{2.34b}$$

(ii) If f is odd, then

$$[T]_{p_1} = \frac{1}{f}\begin{bmatrix} \beta & \alpha \\ \delta & \gamma \end{bmatrix} \tag{2.35a}$$

$$Z_L = \frac{g}{h} . \tag{2.35b}$$

Theorem 2.5

Let $Z(\underline{p})$ be an n-variable p.r. function. Then, $Z(\underline{p})$ is realizable as an input impedance of the cascade structure shown in Fig. 2.2 iff $Z(\underline{p})$ is decomposable into the following form

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})} = \frac{\alpha_1(p_1)h_2(p_2,p_3,\ldots,p_n) + \beta_1(p_1)g_2(p_2,p_3,\ldots,p_n)}{\gamma_1(p_1)h_2(p_2,p_3,\ldots,p_n) + \delta_1(p_1)g_2(p_2,p_3,\ldots,p_n)} ,$$

$$\frac{h_2(p_2,p_3,\ldots,p_n)}{g_2(p_2,p_3,\ldots,p_n)} = \frac{\alpha_2(p_2)h_3(p_3,p_4,\ldots,p_n) + \beta_2(p_2)g_3(p_3,p_4,\ldots,p_n)}{\gamma_2(p_2)h_3(p_3,p_4,\ldots,p_n) + \delta_2(p_2)g_3(p_3,p_4,\ldots,p_n)} ,$$

$$\vdots \qquad (2.36)$$

$$\frac{h_{n-1}(p_{n-1},p_n)}{g_{n-1}(p_{n-1},p_n)} = \frac{\alpha_{n-1}(p_{n-1})h_n(p_n) + \beta_{n-1}(p_{n-1})g_n(p_n)}{\gamma_{n-1}(p_{n-1})h_n(p_n) + \delta_{n-1}(p_{n-1})g_n(p_n)} ,$$

where $P(\underline{p})$ and $Q(\underline{p})$ are assumed to be relatively prime in $\mathrm{Re}\ p_i \geq 0$ for every i, and both $\alpha_i(p_i)$ and $\delta_i(p_i)$ are even polynomials of p_i, while oppositely, both $\beta_i(p_i)$ and $\gamma_i(p_i)$ are odd polynomials of p_i for $i = 1,2,\ldots,n-1$.

In addition, the p_i-variable two-ports are reciprocal iff

$$\alpha_i(p_i)\delta_i(p_i) - \beta_i(p_i)\gamma_i(p_i) = \varepsilon_i f_i^2(p_i) , \quad i = 1,2,\ldots,n-1,$$

where $f_i(p_i)$ is either even (for $\varepsilon_i=1$) or odd (for $\varepsilon_i=-1$).

Proof: The theorem follows immediately by repeated applications of Lemma 2.2 to (2.36), successively.

It may be noted from (2.32) and (2.33) that, in general, a realization may be obtained with the chain matrices of the p_i-two-ports given by

$$[T]_{p_i} = \frac{1}{f_i}\begin{bmatrix} \alpha_i(p_i) & \beta_i(p_i) \\ \gamma_i(p_i) & \delta_i(p_i) \end{bmatrix} \text{ for } \quad i = 1,2,\ldots,n-1 \qquad (2.37a)$$

and the terminating impedance given by

$$Z_L = \frac{h_n(p_n)}{g_n(p_n)} \quad . \tag{2.37b}$$

As for reciprocal realizations, it can be shown from (2.34) and (2.35) that

$$[T]_{p_i} = \frac{1}{f_i(p_i)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\nu_{i-1}} \begin{bmatrix} \alpha_i(p_i) & \beta_i(p_i) \\ \gamma_i(p_i) & \delta_i(p_i) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\nu_i} \tag{2.38a}$$

for $i = 1,2,\ldots,n-1$, and

$$Z_L = \begin{cases} \dfrac{h_n(p_n)}{g_n(p_n)} & \text{if} \quad \nu_{n-1} = 0 \\[2ex] \dfrac{g_n(p_n)}{h_n(p_n)} & \text{if} \quad \nu_{n-1} = 1 \, , \end{cases} \tag{2.38b}$$

where ν_k (with $\nu_0 = 0$) is introduced for convenience and is defined as

$$\nu_k = \begin{cases} 0 & \text{if} \quad \prod_{\ell=1}^{k} f_\ell \quad \text{is even.} \\[2ex] 1 & \text{if} \quad \prod_{\ell=1}^{k} f_\ell \quad \text{is odd.} \end{cases} \tag{2.39}$$

2.2.2 Explicit Approach

Similar to the previous section, we first present a preliminary lemma then establish the main theorem from the lemma.

Lemma 2.3

The n-variable rational function $Z(\underline{p})$ may be realized as a driving-point impedance of the structure shown in Fig. 2.1 iff $Z(\underline{p})$ can be decomposed in the form of (2.27) with the following conditions.

(1) $\alpha(p_1)+\beta(p_1)+\gamma(p_1)+\delta(p_1)$ is strictly Hurwitz.

(2) $\alpha(p_1)\delta(p_1)-\beta(p_1)\gamma(p_1) = f(p_1)f(-p_1)$, where $f(p_1)$ is a real polynomial.

(3) h/g is an (n-1)-variable p.r. function of $p_2,p_3,\ldots,p_n$.

Moreover, the p_1-two-port is reciprocal iff

(4) $f(p_1)$ is either even or odd.

Proof: The lemma is a direct consequence of (2.28) and Lemma 2.1.

Theorem 2.6

A necessary and sufficient condition for the n-variable rational function $Z(\underline{p})$ to be realizable as a driving point impedance of the cascade structure of Fig. 2.2 is that $Z(\underline{p})$ is decomposable into the form of (2.36) with the following conditions.

(1) $\alpha_i(p_i)+\beta_i(p_i)+\gamma_i(p_i)+\delta_i(p_i)$ is strictly Hurwitz, $i=1,2,\ldots,n-1$.

(2) $\alpha_i(p_i)\delta_i(p_i)-\beta_i(p_i)\gamma_i(p_i) = f_i(p_i)f_i(-p_i)$, $i = 1,2,\ldots,n-1$, where $f_i(p_i)$ is a real polynomial of p_i.

(3) h_n/g_n is a p.r. function of p_n.

Furthermore, the p_i-two-ports are reciprocal iff

(4) $f_i(p_i)$, $i = 1,\ldots,n-1$, is either even or odd.

Proof: The proof follows by repeated applications of the above lemma to the set of expressions (2.36) starting from the last expression to the first one, succesively. Realizations may be obtained with the same set of equations as (2.37) and (2.38).

2.2.3 Examples

Example 2.2.1: Consider the feasibility of realizing the following m.p.r. function $Z(\underline{p})$ in $\underline{p} = (p_1,p_2,p_3,p_4)$ as a driving point impedance of the cascade structure of Fig. 2.2

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})} \quad , \tag{2.40}$$

where

$$\begin{aligned} P(\underline{p}) = {} & 2p_1^2p_2p_3^2p_4 + 5p_1^2p_2p_3^2 + p_1^2p_3^2 + 2p_1^2p_2p_4 + 2p_1^2p_3p_4 \\ & + 5p_1^2p_2 + 5p_1^2p_3 + p_1^2 + 2p_1p_2p_3^2 + 4p_1p_2p_3p_4 + 10p_1p_2p_3 \\ & + 2p_1p_2 + 4p_2p_3^2p_4 + 10p_2p_3^2 + 2p_3^2 + 4p_2p_4 + 4p_3p_4 \\ & + 10p_2 + 10p_3 + 2 \end{aligned}$$

and

$$\begin{aligned} Q(\underline{p}) = {} & 4p_1^2p_2p_3^2 + 8p_1^2p_2p_3p_4 + 20p_1^2p_2p_3 + 4p_1^2p_2 + 2p_1p_2p_3^2p_4 \\ & + 5p_1p_2p_3^2 + p_1p_3^2 + 2p_1p_2p_4 + 2p_1p_3p_4 + 5p_1p_2 + 5p_1p_3 \\ & + p_1 + 2p_2p_3^2 + 4p_2p_3p_4 + 10p_2p_3 + 2p_2 \quad . \end{aligned}$$

Rewriting $P(\underline{p})$ and $Q(\underline{p})$ as polynomials of p_1, it can be shown that $Z(\underline{p})$ can be decomposed as

$$Z(\underline{p}) = \frac{(p_1^2+2)h_2 + 2p_1g_2}{p_1h_2 + (4p_1^2+2)g_2} \quad , \text{ where} \tag{2.41a}$$

$$\frac{h_2}{g_2} = \frac{h_3 + p_2g_3}{p_2h_3} \quad , \text{ where} \tag{2.41b}$$

$$\frac{h_3}{g_3} = \frac{(p_3^2+1)h_4 + p_3g_4}{(p_3^2+1)g_4} \quad , \text{ where} \tag{2.41c}$$

$$\frac{h_4}{g_4} = \frac{1}{2p_4 + 5} \quad . \tag{2.41d}$$

Therefore, by virtue of Theorem 2.5, $Z(\underline{p})$ is realizable in the form of Fig. 2.2.

Now, from (2.36) and (2.41) it can be identified that

$$\alpha_1 = p_1^2 + 2 \qquad \beta_1 = 2p_1$$
$$\gamma_1 = p_1 \qquad \delta_1 = 4p_1^2 + 2$$

$$\alpha_2 = 1 \qquad \beta_2 = p_2$$
$$\gamma_2 = p_2 \qquad \delta_2 = 0$$

$$\alpha_3 = p_3^2 + 1 \qquad \beta_3 = p_3$$
$$\gamma_3 = 0 \qquad \delta_3 = p_3^2 + 1 \quad .$$

Therefore, we have

$$\alpha_1\delta_1 - \beta_1\gamma_1 = 4(p_1^2+1)^2 \implies f_1 = 2(p_1^2+1)$$
$$\alpha_2\delta_2 - \beta_2\gamma_2 = -p_2^2 \implies f_2 = p_2$$
$$\alpha_3\delta_3 - \beta_3\gamma_3 = (p_3^2+1)^2 \implies f_3 = (p_3^2+1) \quad ,$$

and from (2.39), we have

$$\nu_1 = 0, \quad \nu_2 = 1 \quad \text{and} \quad \nu_3 = 1.$$

Consequently from (2.38), we have

$$[T]_{p_1} = \frac{1}{f_1}\begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} = \frac{1}{2(p_1^2+1)}\begin{bmatrix} p_1^2+2 & 2p_1 \\ p_1 & 4p_1^2+2 \end{bmatrix}$$

$$[T]_{p_2} = \frac{1}{f_2}\begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{p_2}\begin{bmatrix} p_2 & 1 \\ 0 & p_2 \end{bmatrix}$$

$$[T]_{p_3} = \frac{1}{f_3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha_3 & \beta_3 \\ \gamma_3 & \delta_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{p_3^2+1}\begin{bmatrix} p_3^2+1 & 0 \\ p_3 & p_3^2+1 \end{bmatrix}$$

and

$$Z_L = 2p_4 + 5 .$$

A complete realization is given in Fig. 2.3.

Example 2.2.2: Consider the following 2-variable rational function

$$Z(p_1,p_2) = \frac{(p_1^2+1)(6p_2^2+3) + 10p_1p_2}{p_1(6p_2^2+3) + 5(p_1^2+1)p_2} . \tag{2.42}$$

Comparing (2.42) and (2.27), we see that

$$\alpha(p_1) = p_1^2 + 1 \qquad \beta(p_1) = 2p_1$$
$$\gamma(p_1) = p_1 \qquad \delta(p_1) = p_1^2 + 1$$

$$h(p_2) = 6p_2^2 + 3$$
$$g(p_2) = 5p_2 .$$

Thus, it can be seen that

(1) $\alpha+\beta+\gamma+\delta = 2p_1^2+3p_1+2$ is strictly Hurwitz.

(2) $\alpha\delta-\beta\gamma = (p_1^2+1)^2-2p_1^2 = p_1^4+1 = f(p_1)f(-p_1)$, where $f(p_1) = (p_1^2+\sqrt{2}p_1+1)$.

(3) $h/g = (6p_2^2+3)/5p_2$ is obviously p.r..

Therefore, by virtue of Lemma 2.3, $Z(p_1,p_2)$ is realizable in the form of Fig. 2.1 with

$$[T]_{p_1} = \frac{1}{p_1^2+\sqrt{2}p_1+1}\begin{bmatrix} p_1^2+1 & 2p_1 \\ p_1 & p_1^2+1 \end{bmatrix}$$

and

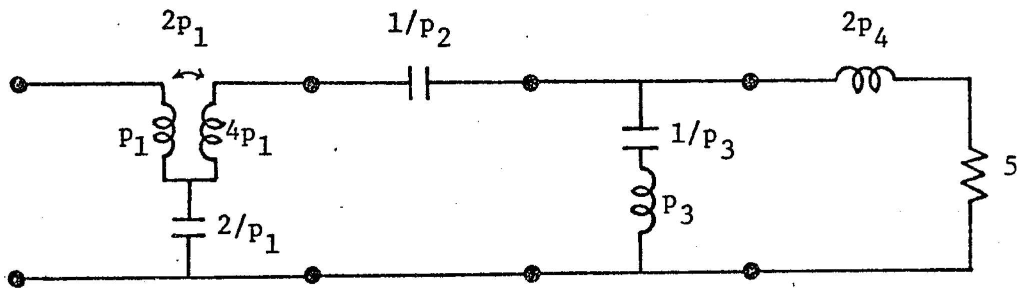


Fig. 2.3 Example 2.2.1.

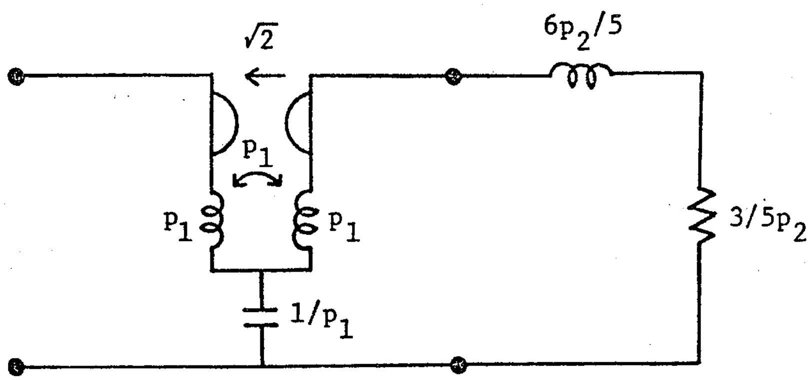


Fig. 2.4 Example 2.2.2.

$$Z_L = \frac{6p_2^2 + 3}{5p_2} \quad .$$

A complete realization is given in Fig. 2.4. It is noted that (2.42) is excerpted from Example 2 of [2] which is shown not to be realizable in the form of Fig. 2.1. However, the reverse is true as evident above. This is due to the fact that the formulation given in [2] is not general enough to cater for non-reciprocal two-ports.

2.3 BOTT-DUFFIN TYPE DECOMPOSITION

In this section, we present a different type of realization. As far as the mathematical decomposition of the given function is concerned, it bears a similar form to that discussed in Section 2.2. However, as to network configuration, it resembles the cyclic bridge form of a realization obtained by the Bott-Duffin process [31].

The classical Bott-Duffin synthesis process [7] is developed from Richards' Theorem [24] which was first introduced to multivariable synthesis by Saito [26]. Soliman and Bose [31], later, extended Saito's version to a more general case. Based on the multivariable Richards' theorem, they derived a set of sufficient conditions for the realization of a class of m.p.r. functions bilinear in all variables except one by repeated applications of the Bott-Duffin process. In essence, at each cycle, an impedance $Z(p_1,p_2,\ldots,p_n)$ bilinear in p_1 is realized in terms of a p_1-capacitor, a p_1-inductor and two impedances Z_1 and Z_2 which are free of p_1; the four components are connected in a bridge form as shown in Fig. 2.5. In the following, we consider a more general case shown in Fig. 2.6, where a

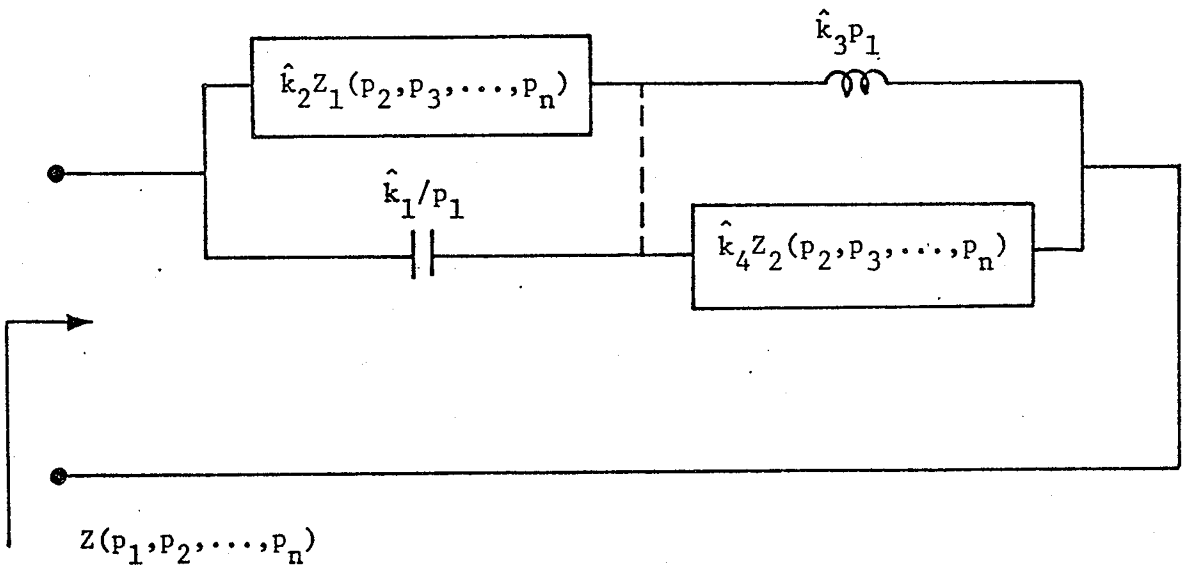


Fig. 2.5 The Bott-Duffin realization of $Z(p_1,p_2,\ldots,p_n)$ bilinear in p_1 in terms of a p_1-capacitor, a p_1-inductor and two (n-1)-variable impedances $Z_1(p_2,p_3,\ldots,p_n)$ and $Z_2(p_2,p_3,\ldots,p_n)$.

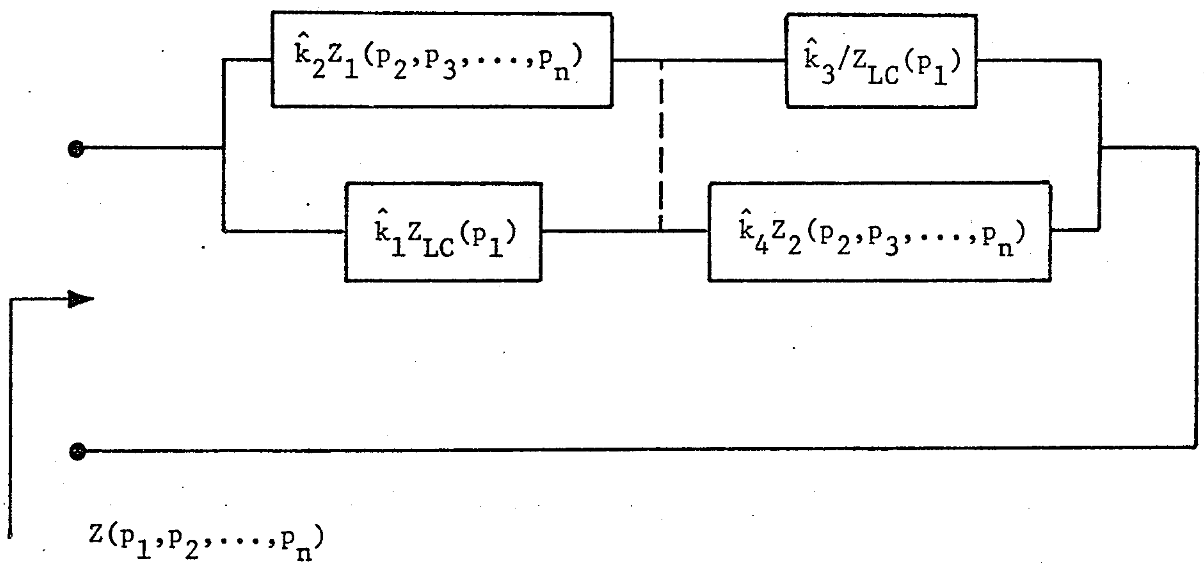


Fig. 2.6 Realization of $Z(p_1,p_2,\ldots,p_n)$ in terms of a p_1-variable reactance function $Z_{LC}(p_1)$, its reciprocal $1/Z_{LC}(p_1)$ and two (n-1)-variable impedances $Z_1(p_2,p_3,\ldots,p_n)$ and $Z_2(p_2,p_3,\ldots,p_n)$; the four components are connected in the Bott-Duffin bridge form.

general p_1-variable reactance function $Z_{LC}(p_1)$ and its reciprocal have replaced the p_1-capacitor and p_1-inductor in Fig. 2.5, respectively. As a consequence, the class of functions to be covered is wider than the one considered in [31], i.e., the bilinear constraint on the given function is now relaxed.

2.3.1 General MPR Approach

As previously, we first derive a lemma which gives the realizability condition for an m.p.r. function to be realizable in the form of Fig.2.6 and then establish the main result from the lemma.

Lemma 2.4

A necessary and sufficient condition for the n-variable p.r. function $Z(\underline{p})$ to be realizable as a driving point impedance in the form of Fig. 2.6 is that $Z(\underline{p})$ can be decomposed as

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})} = \frac{k_1\alpha(p_1)h(p_2,p_3,\ldots,p_n) + k_2\beta(p_1)g(p_2,p_3,\ldots,p_n)}{k_3\beta(p_1)h(p_2,p_3,\ldots,p_n) + k_4\alpha(p_1)g(p_2,p_3,\ldots,p_n)} , \quad (2.43)$$

where $P(\underline{p})$ and $Q(\underline{p})$ are expressed as relatively prime in $\mathrm{Re}\ p_i \geq 0$ for every i, $\alpha(p_1)$ is a monic even polynomial of p_1 and $\beta(p_1)$ is a monic odd polynomial of p_1 (The k_i's ≥ 0, $i = 1,2,3,4$ are introduced to take care of the degenerate cases).

Proof: Necessity is evident. We shall show the sufficiency.

Since the cases with more than one of the k_i's equal to zero are trivial, in the following, we assume that only one of the four k_i's may vanish.

First, we shall establish that h/g is (n-1)-variable p.r. in $p_2,p_3,\ldots,p_n$ and $\alpha(p_1)/\beta(p_1)$ is a reactance function of p_1.

It is observed that (2.43) and (2.27) are of similar forms, therefore it follows from the proof of Lemma 2.2 that

h/g is $(n-1)$-variable p.r. in $p_2, p_3, \ldots, p_n$

and

$$k_1\alpha(p_1)K + k_2\beta(p_1) \tag{2.44a}$$

$$k_3\beta(p_1)K + k_4\alpha(p_1) \tag{2.44b}$$

are Hurwitz polynomials of p_1, where K is given by (2.30).

Since it is assumed that only one of the four k_i's may vanish, by taking either (2.44a) or (2.44b) it can be shown from the property of Hurwitz polynomials that

$\dfrac{\alpha(p_1)}{\beta(p_1)}$ is a reactance function of p_1.

Now, from (2.43), it can be seen that $Z(\underline{p})$ is decomposable into the following form

$$Z(\underline{p}) = \frac{1}{\dfrac{k_3}{k_1}\dfrac{\beta(p_1)}{\alpha(p_1)} + \dfrac{k_4}{k_1}\dfrac{g}{h}} + \frac{1}{\dfrac{k_3}{k_2}\dfrac{h}{g} + \dfrac{k_4}{k_2}\dfrac{\alpha(p_1)}{\beta(p_1)}} \quad . \tag{2.45}$$

Therefore, by letting

$$Z_{LC}(p_1) = \frac{\alpha(p_1)}{\beta(p_1)} \quad ,$$

$$Z_1(p_2,p_3,\ldots,p_n) = \frac{h(p_2,p_3,\ldots,p_n)}{g(p_2,p_3,\ldots,p_n)} \quad ,$$

$$Z_2(p_2,p_3,\ldots,p_n) = \frac{g(p_2,p_3,\ldots,p_n)}{h(p_2,p_3,\ldots,p_n)} \quad ,$$

$$\hat{k}_1 = \frac{k_1}{k_3}\,, \quad \hat{k}_2 = \frac{k_1}{k_4}\,, \quad \hat{k}_3 = \frac{k_2}{k_4} \quad \text{and} \quad \hat{k}_4 = \frac{k_2}{k_3} \quad ,$$

the realization of $Z(\underline{p})$ in the form of Fig. 2.6 with the dotted line replaced by a solid line is apparent from (2.45).

Next, it is observed that

$$\hat{k}_1 Z_{LC}(p_1) \cdot \frac{\hat{k}_3}{Z_{LC}(p_1)} = \frac{k_1 k_2}{k_3 k_4}$$

and

$$\hat{k}_2 Z_1 \cdot \hat{k}_4 Z_2 = \left(\frac{k_1}{k_4}\frac{h}{g}\right)\left(\frac{k_2}{k_3}\frac{g}{h}\right) = \frac{k_1 k_2}{k_3 k_4} \quad .$$

Therefore, similar to the Bott-Duffin cycle, the structure of Fig. 2.6 is a balanced bridge. Hence, the dotted line may be replaced by any impedance without altering the input impedance $Z(\underline{p})$. Q.E.D.

Theorem 2.7

A sufficient condition for the n-variable p.r. function $Z(\underline{p})$ to be completely realizable by successive applications of the process depicted in Fig. 2.6 is that $Z(\underline{p})$ can be decomposed in the following form

$$Z(\underline{p}) = \frac{P(\underline{p})}{Q(\underline{p})} = \frac{k_{11}\alpha_1(p_1)h_2(p_2,p_3,\dots,p_n) + k_{12}\beta_1(p_1)g_2(p_2,p_3,\dots,p_n)}{k_{13}\beta_1(p_1)h_2(p_2,p_3,\dots,p_n) + k_{14}\alpha_1(p_1)g_2(p_2,p_3,\dots,p_n)}$$

$$\frac{h_2}{g_2} = \frac{k_{21}\alpha_2(p_2)h_3(p_3,p_4,\dots,p_n) + k_{22}\beta_2(p_2)g_3(p_3,p_4,\dots,p_n)}{k_{23}\beta_2(p_2)h_3(p_3,p_4,\dots,p_n) + k_{24}\alpha_2(p_2)g_3(p_3,p_4,\dots,p_n)}$$

$$\vdots \qquad (2.46)$$

$$\frac{h_{n-1}}{g_{n-1}} = \frac{k_{n-1,1}\alpha_{n-1}(p_{n-1})h_n(p_n) + k_{n-1,2}\beta_{n-1}(p_{n-1})g_n(p_n)}{k_{n-1,3}\beta_{n-1}(p_{n-1})h_n(p_n) + k_{n-1,4}\alpha_{n-1}(p_{n-1})g_n(p_n)} ,$$

where $P(\underline{p})$ and $Q(\underline{p})$ are expressed as relatively prime in $\mathrm{Re}\ p_i \geq 0$ for every i, $\alpha_i(p_i)$ is a monic even polynomial of p_i, $\beta_i(p_i)$ is a monic odd polynomial of p_i and $k_{ij} \geq 0$ for $i = 1,2,\ldots,n-1$ and $j = 1,2,3,4$.

Proof: The proof may be shown by repeated applications of the above lemma to the set of expressions (2.46), successively.

2.3.2 Explicit Approach

First, we establish a basic lemma and then derive the main result from the lemma.

Lemma 2.5

The n-variable rational function $Z(\underline{p})$ may be realized as a driving point impedance in the structure of Fig. 2.6 iff $Z(\underline{p})$ can be decomposed in the form of (2.43) with the following conditions

(1) $\alpha(p_1)/\beta(p_1)$ is a reactance function of p_1.

(2) h/g is an (n-1)-variable p.r. function of $p_2,p_3,\ldots,p_n$.

Proof: Apart from the fact that it is now given that α/β is a reactance function of p_1 and h/g is m.p.r. in $p_2,p_3,\ldots,p_n$,, the proof follows along the same lines as those of Lemma 2.4 and is thus omitted for brevity.

Theorem 2.8

The n-variable rational function $Z(\underline{p})$ may be completely realized by successive applications of the process described in Fig. 2.6 if $Z(\underline{p})$ can be decomposed in the form of (2.46) with the following conditions.

(1) $\alpha_i(p_i)/\beta_i(p_i)$ is a reactance function of p_i, $i=1,2,\ldots,n-1$.

(2) $h_n(p_n)/g_n(p_n)$ is p.r..

Proof: The proof follows directly by repeated applications of the above lemma to (2.46) starting from the last expression to the first one, successively.

If the $\deg_{p_i} Z(\underline{p}) = 1$ for $i = 1,2,\ldots,n-1$, we have the following corollary.

Corollary 2.8.1

A sufficient condition for the n-variable rational function $Z(\underline{p})$ bilinear in $p_1,p_2,\ldots,p_{n-1}$ to be completely realizable by successive applications of the process described in Fig. 2.5 is that $Z(\underline{p})$ can be decomposed in the following form

$$
\begin{aligned}
Z(\underline{p}) &= \frac{k_{11}h_2(p_2,p_3,\ldots,p_n) + k_{12}p_1g_2(p_2,p_3,\ldots,p_n)}{k_{13}p_1h_2(p_2,p_3,\ldots,p_n) + k_{14}g_2(p_2,p_3,\ldots,p_n)} \\
\frac{h_2}{g_2} &= \frac{k_{21}h_3(p_3,p_4,\ldots,p_n) + k_{22}p_2g_3(p_3,p_4,\ldots,p_n)}{k_{23}p_2h_3(p_3,p_4,\ldots,p_n) + k_{24}g_3(p_3,p_4,\ldots,p_n)} \\
&\ \ \vdots \\
\frac{h_{n-1}}{g_{n-1}} &= \frac{k_{n-1,1}h_n(p_n) + k_{n-1,2}p_{n-1}g_n(p_n)}{k_{n-1,3}p_{n-1}h_n(p_n) + k_{n-1,4}g_n(p_n)} ,
\end{aligned}
\tag{2.47}
$$

with the following conditions.

(1) $k_{ij} \geq 0$ for $i = 1,2,\ldots,n-1$ and $j = 1,2,3,4$.

(2) $h_n(p_n)/g_n(p_n)$ is p.r..

Note that apart from a few degenerate cases it may be shown that Corollary 2.8.1 is equivalent to Theorem 3 given in [31]. Nevertheless, the simplicity of the former over the latter could be easily noted.

Firstly, it replaces the multivariable p.r. test by the simple verification of the nonnegativeness of a set of constants and one 1-variable p.r. test, viz., conditions (1) and (2) of the corollary. Secondly, it replaces the computational work involved in the testing of the applicability of the multivariable Richards' theorem by the simple rearrangements of polynomials in the form of (2.47).

2.3.3 Examples

Example 2.3.1: Consider the following multivariable rational function

$$Z(p_1,p_1,p_3) = \frac{2p_1^2p_3+12p_1^2+p_1^2p_2+p_1p_2p_3+6p_1p_2+5p_1+2p_3+p_2+12}{p_1^2p_2p_3+6p_1^2p_2+5p_1^2+4p_1p_3+2p_1p_2+24p_1+p_2p_3+6p_2+5} . \quad (2.48)$$

Rewriting $Z(p_1,p_2,p_3)$ as a function of p_1, it can be easily seen that

$$Z(p_1,p_2,p_3) = \frac{(p_1^2+1)h_2 + p_1g_2}{2p_1h_2 + (p_1^2+1)g_2} ,$$

where

$$\frac{h_2}{g_2} = \frac{2(p_3+6) + p_2}{p_2(p_3+6) + 5} .$$

Thus we can identify

$$k_{11} = k_{12} = k_{14} = 1 \quad k_{13} = 2 \quad \alpha_1(p_1) = p_1^2+1 \quad \beta_1(p_1) = p_1$$

$$k_{21} = 2 \quad k_{22} = k_{23} = 1 \quad k_{24} = 5 \quad \alpha_2(p_2) = 1 \quad \beta_2(p_2) = p_2$$

$$h_3(p_3) = p_3+6 \quad g_3(p_3) = 1 .$$

Clearly, α_1/β_1 and α_2/β_2 are single variable reactance functions, and h_3/g_3 is p.r.. Therefore by virtue of Theorem 2.8, $Z(p_1,p_2,p_3)$ is realizable by the process of Fig. 2.6.

To realize $Z(p_1,p_2,p_3)$, we write (2.48) in the form of (2.45)

$$Z(p_1,p_2,p_3) = \frac{1}{\dfrac{2p_1}{p_1^2+1} + \dfrac{1}{Z_1}} + \frac{1}{\dfrac{p_1^2+1}{p_1} + \dfrac{1}{Z_2}} ,$$

where

$$Z_1 = \frac{h_2}{g_2} = \frac{1}{\dfrac{p_2}{2} + \dfrac{5}{2}\dfrac{g_3}{h_3}} + \frac{1}{\dfrac{5}{p_2} + \dfrac{h_3}{g_3}}$$

$$Z_2 = \frac{g_2}{2h_2} = \frac{1}{\dfrac{4}{p_2} + 2\dfrac{g_3}{h_3}} + \frac{1}{\dfrac{2p_2}{5} + \dfrac{4}{5}\dfrac{h_3}{g_3}} .$$

A complete realization is given in Fig. 2.7.

Example 2.3.2: Consider the following 2-variable rational function

$$Z(p_1,p_2) = \frac{4p_1p_2^2 + 4p_1p_2 + p_2^2 + p_2 + 16p_1 + 1}{p_1p_2^2 + p_1p_2 + p_2^2 + p_2 + p_1 + 4} . \tag{2.49}$$

It is observed that $Z(p_1,p_2)$ is bilinear in p_1. By rewriting $Z(p_1,p_2)$ as a function of p_1, we have

$$Z(p_1,p_2) = \frac{4(p_2^2+p_2+4)p_1 + p_2^2 + p_2 + 1}{(p_2^2+p_2+1)p_1 + p_2^2 + p_2 + 4} . \tag{2.50}$$

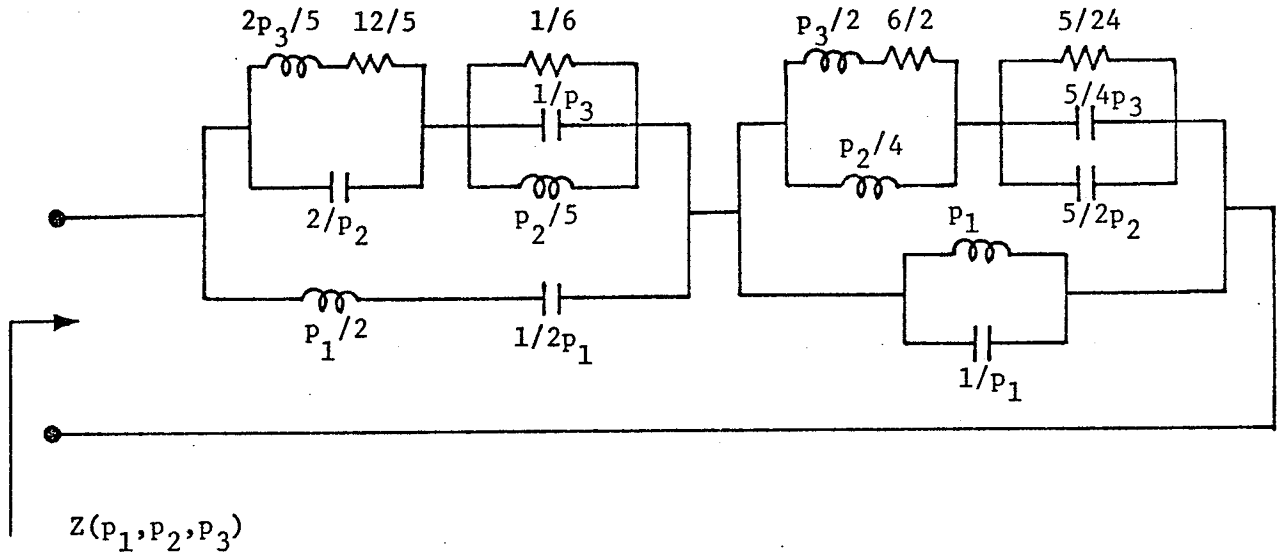


Fig. 2.7 Example 2.3.1.

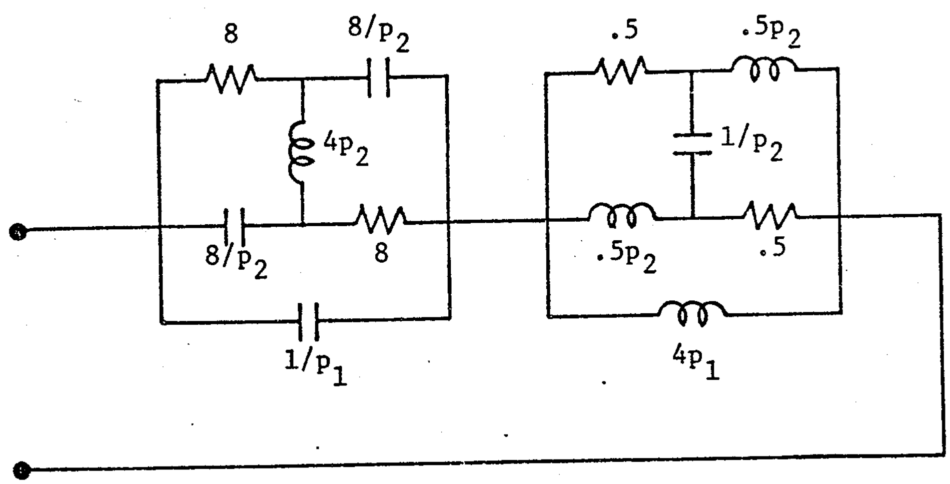


Fig. 2.8 Example 2.3.2.

Comparing (2.50) and (2.47), we see that

$$
\begin{aligned}
k_{11} &= 1 > 0 \\
k_{12} &= 4 > 0 \\
k_{13} &= 1 > 0 \\
k_{14} &= 1 > 0
\end{aligned}
\qquad (2.51)
$$

and

$$\frac{h_2}{g_2} = \frac{p_2^2 + p_2 + 1}{p_2^2 + p_2 + 4} \quad \text{is clearly p.r.. ,} \qquad (2.52)$$

Thus, by virtue of Corollary 2.8.1 $Z(p_1,p_2)$ is realizable in the form of Fig. 2.5. To realize $Z(p_1,p_2)$, we write (2.50) in the form of (2.45),

$$Z(p_1,p_2) = \frac{1}{p_1 + \frac{1}{Z_1}} + \frac{1}{\frac{1}{4p_1} + \frac{1}{Z_2}} \quad ,$$

where

$$Z_1 = \frac{p_2^2 + p_2 + 1}{p_2^2 + p_2 + 4}$$

$$Z_2 = \frac{4(p_2^2 + p_2 + 4)}{(p_2^2 + p_2 + 1)} \quad .$$

A complete realization is given in Fig. 2.8.

The example (2.49) is excerpted from the example of [31]. In contrast to the method presented in [31], here we have demonstrated that

(1) The given function does not have to be known to be m.p.r. to begin with. Instead, the 2 simple tests of (2.51) and (2.52) replace

this prerequisite condition.

(2) Besides the two tests mentioned above, no further computational work is required for testing of the applicability of the multivariable Richards' Theorem.

CHAPTER III

SYNTHESIS OF INDEPENDENT ZEROS OF THE EVEN PART OF A MULTIVARIABLE POSITIVE REAL FUNCTION

It is well-known that the even part zero removal techniques play an important role in the synthesis of single variable p.r. functions. For multivariable functions, the even part of an m.p.r. function $Z(\underline{p})$ with respect to all the variables $\underline{p} = (p_1, p_2, \ldots, p_n)$ is defined as

$$\mathrm{Ev}_{\underline{p}}\, Z(\underline{p}) = \frac{1}{2} [\, Z(\underline{p}) + Z_*(\underline{p}) \,] \,. \tag{3.1}$$

Some aspects of the synthesis of the p_i-variable independent zeros[1] of (3.1) have been reported [9,26,34] for certain classes of mixed lumped-distributed networks, and also a special development has been given recently in [6]. However, the essence of these contributions is the extension of Youla's [32] single variable cascade theory to multivariable functions. As to the treatment of realization without gyrators and transformers, only meager results have been reported [10,31,36]. In this chapter, we present a more general development for the synthesis of independent zeros of the even part of m.p.r. functions. Special emphasis will be placed on the development of realization methods without using gyrators and transformers. In the following, Section 3.1

---

1 Let $f(\underline{p})$ be an irreducible multivariable rational function. If $f(\underline{p})$ can be expressed as $f(\underline{p}) = (p_i - p_{io})^r \hat{f}(\underline{p})$, where $\hat{f}(\underline{p})$ does not possess the $(p_i - p_{io})$ factor, then $p_i = p_{io}$ is called an p_i-independent zero of multiplicity r of $f(\underline{p})$.

is devoted to the fundamental development of various forms of Richards' theorem [24,32]. Section 3.2 deals with general realizations with no constraint on the use of coupling elements. Section 3.3 is concerned with realizations without transformers.

3.1 DEVELOPMENT OF VARIOUS FORMS OF RICHARDS' THEOREM

The well-known Richards' theorem [24] corresponding to a positive real constant of single variable theory was first introduced to multivariable synthesis by Saito [26], and subsequently Saito's version was extended to a more general case by Soliman et al. [31]. The Richards' theorem [32] corresponding to a conjugate pair of complex constants with positive real part was first applied to the multivariable synthesis of mixed lumped-distributed networks by Kamp [9] and Youla et al. [34], and a more formal extension was given recently by Fujimoto et al. [6]. Since these theorems form the fundamental core for the subsequent developments, we shall, in this section, give a detailed presentation together with their proofs. Our derivations here, which are based on Belevitch's [1] arguments on one-port synthesis by all-pass extraction, are different from those given in [6,34] and are more compact and systematic.

We shall first establish a lemma which can be considered as the fundamental version of Richards' theorem for a multivariable positive (not necessarily real) function corresponding to a complex constant with positive real part. The other forms of Richards' theorem are then derived from this lemma.

Lemma 3.1

Let $Z(\underline{p})$ be a multivariable rational positive (not necessarily real) function of a set of complex variables $\underline{p} = (p_1, p_2, \ldots, p_n)$. Let $p_1 = p_{10} = \sigma_{10} + j\omega_{10}$ $(\sigma_{10} > 0)$ be a fixed point in the right-half p_1-plane. If $Z(\underline{p})$ at $p_1 = p_{10}$ is a constant independent of all other variables

$$Z(\underline{p})\Big|_{p_1=p_{10}} = Z_0 = R_0 + jX_0 \quad , \tag{3.2}$$

then $Z(\underline{p})$ can be expressed as

$$Z(\underline{p}) = \frac{A_Z(p_1)\; W(\underline{p}) + B_Z(p_1)}{C_Z(p_1)\; W(\underline{p}) + D_Z(p_1)} \quad , \tag{3.3}$$

where

$$\begin{aligned} A_Z(p_1) &= \sigma_{10}R_0 + \omega_{10}X_0 + jp_1X_0 \\ B_Z(p_1) &= p_1R_0 + j(\sigma_{10}X_0 - \omega_{10}R_0) \\ C_Z(p_1) &= p_1 - j\omega_{10} \\ D_Z(p_1) &= \sigma_{10} \quad , \end{aligned} \tag{3.4}$$

and the residual function

$$W(\underline{p}) = \frac{D_Z(p_1)\; Z(\underline{p}) - B_Z(p_1)}{-C_Z(p_1)\; Z(\underline{p}) + A_Z(p_1)} \tag{3.5}$$

possesses the following properties.

(1) $W(\underline{p})$ is an m.p. function.

(2) $\deg_{p_i} W(\underline{p}) \le \deg_{p_i} Z(\underline{p})$ for every i. (3.6)

(3) In particular, if p_{10} is a p_1-independent zero of the even part function $Ev_{\underline{p}}\, Z(\underline{p})$, then

$$\deg_{p_1} W(\underline{p}) = \deg_{p_1} Z(\underline{p}) - 1 \quad . \tag{3.7}$$

($\deg_{p_i} Z(\underline{p})$ is the degree of $Z(\underline{p})$ with respect to p_i and is defined as the maximum degree of $Z(\underline{p})$ in p_i when all the variables except p_i are held constant, whereas the degree of a single variable rational function is defined as the totality of its poles including poles at the origin and infinity and counting multiplicities.)

Proof:

(1) Since $Z(\underline{p})$ is m.p. and $\sigma_{10} > 0$, R_0 is positive. Therefore the reflectance s of $Z(\underline{p})$ with respect to $Z_0 = R_0+jX_0$

$$s = \frac{Z_0 - Z(\underline{p})}{Z_0^* + Z(\underline{p})} \tag{3.8}$$

is a multivariable bounded function, i.e.,

$$s \text{ is regular in } \operatorname{Re} p_i > 0 \quad \text{and} \tag{3.9}$$

$$|s| \leq 1 \quad \text{for} \quad \operatorname{Re} p_i = 0 \,, \tag{3.10}$$

for every i.

Now, let s_W be defined as

$$s_W = \frac{Z_0 - Z(\underline{p})}{Z_0^* + Z(\underline{p})} \, \frac{p_{10}^* + p_1}{p_{10} - p_1} \,. \tag{3.11}$$

Since at $p_1 = p_{10}$, $Z(\underline{p}) = Z_0$ which is a constant independent of all other variables, the numerator of s possesses the factor $p_{10} - p_1$. Consequently, the factor $p_{10} - p_1$ in the denominator of s_W is cancelled out by the same factor in $Z_0 - Z(\underline{p})$. Therefore, it follows from (3.9) that s_W is regular in $\operatorname{Re} p_i > 0$, for every i. It is noted that

$$\left| \frac{p_{10}^* + p_1}{p_{10} - p_1} \right| = 1 \quad \text{for} \quad \operatorname{Re} p_1 = 0 \,. \tag{3.12}$$

Thus, it follows from (3.10), (3.11) and (3.12) that $|s_W| \le 1$ for $\mathrm{Re}\, p_i = 0$ for every i, and consequently s_W is a multivariable bounded function. Therefore, the corresponding impedance

$$W(\underline{p}) = \frac{1 - s_W}{1 + s_W} \tag{3.13}$$

is a multivariable positive function.

Substituting (3.11) into (3.13) with $Z_0 = R_0 + jX_0$ and $p_{10} = \sigma_{10} + j\omega_{10}$, we obtain (3.5) with the parameters $A_Z(p_1)$, $B_Z(p_1)$, $C_Z(p_1)$ and $D_Z(p_1)$ satisfying (3.4). By solving (3.5) for $Z(\underline{p})$, (3.3) is obtained.

(2) Condition 2 is evident from the cancellation of the factor $p_{10} - p_1$ up and down in s_W.

(3) If p_{10} is an independent zero of $\mathrm{Ev}_{\underline{p}}\, Z(\underline{p})$, it can be shown that

$$Z(-p_{10}^*, p_2, \ldots, p_n) = -Z_0^* \quad .$$

Thus, $Z_0^* + Z(\underline{p})$ possesses the factor $p_{10}^* + p_1$. The degree property of (3.7) results from the cancellation of the factors $(p_{10} - p_1)$ and $(p_{10}^* + p_1)$ up and down in s_W. Q.E.D.

It is observed that if $Z(\underline{p})$ is multivariable positive and real, then, from the real-property of m.p.r. functions, $X_0 = 0$ for $\omega_{10} = 0$. This fact implemented in (3.4) yields

$$W(\underline{p}) = \frac{\sigma_{10} Z(\underline{p}) - p_1 R_0}{-p_1 Z(\underline{p}) + \sigma_{10} R_0} \tag{3.14}$$

which is now real when all the p_i's are real. By setting $F(\underline{p}) = R_0 W(\underline{p})$, we have the following theorem:

Theorem 3.1

Let $Z(\underline{p})$ be an m.p.r. function. If at $p_1 = \sigma_{10} > 0$

$$Z(\underline{p})\Big|_{p_1=\sigma_{10}} = R_0 \quad ,$$

where R_0 is a constant independent of all other variables, then $Z(\underline{p})$ can be expressed as

$$Z(\underline{p}) = \frac{\sigma_{10}F(\underline{p}) + p_1 R_0}{\frac{1}{R_0}\, p_1 F(\underline{p}) + \sigma_{10}} \quad , \tag{3.15}$$

and the residual function $F(\underline{p})$ satisfies the following conditions.

(1) $F(\underline{p})$ is m.p.r..

(2) $F(\underline{p})$ possesses the identical degree properties as given in Lemma 3.1 for $W(\underline{p})$.

Corollary 3.1.1

Let $Z(\underline{p})$ be an m.p.r. function. If at $p_1 = \sigma_{10} > 0$

$$Z(\underline{p}) = R_0 \tag{3.16}$$

$$\frac{\partial}{\partial p_1} Z(\underline{p}) = Z_0' \quad , \tag{3.17}$$

where R_0 and Z_0' are constants independent of all other variables, then $Z(\underline{p})$ can be expressed as

$$Z(\underline{p}) = \frac{A(p_1)\,\hat{F}(\underline{p}) + B(p_1)}{C(p_1)\,\hat{F}(\underline{p}) + D(p_1)} \quad , \tag{3.18}$$

where

$$
\begin{aligned}
A(p_1) &= a_2p_1^2 + a_0 = [(R_0+\sigma_{10}Z_0')/(R_0-\sigma_{10}Z_0')]p_1^2 + \sigma_{10}^2 \\
B(p_1) &= a_1p_1 \qquad = [2\sigma_{10}R_0^2/(R_0+\sigma_{10}Z_0')]p_1 \\
C(p_1) &= b_1p_1 \qquad = [2\sigma_{10}/(R_0-\sigma_{10}Z_0')]p_1 \\
D(p_1) &= b_2p_1^2 + b_0 = [(R_0-\sigma_{10}Z_0')/(R_0+\sigma_{10}Z_0')]p_1^2 + \sigma_{10}^2 \quad .
\end{aligned}
\tag{3.19}
$$

The residual function $\hat{F}(\underline{p})$ satisfies the following conditions.

(1) $\hat{F}(\underline{p})$ is m.p.r..

(2) $\deg_{p_i} \hat{F}(\underline{p}) \leq \deg_{p_i} Z(\underline{p})$ for every i. (3.20)

(3) In particular, if $p_1 = \sigma_{10}$ is a p_1-independent zero of $Ev_{\underline{p}}\ Z(\underline{p})$, then

$$
\deg_{p_1} \hat{F}(\underline{p}) = \begin{cases} \deg_{p_1} Z(\underline{p}) - 1 & \text{if } p_1 = \sigma_{10} \text{ is a simple zero.} \\ \deg_{p_1} Z(\underline{p}) - 2 & \text{if the multiplicity is 2 or higher.} \end{cases}
\tag{3.21}
$$

Proof:

(1) Since (3.16) holds, Theorem 3.1 is applicable to $Z(\underline{p})$. By solving (3.15) for $F(\underline{p})$, we have

$$
F(\underline{p}) = \frac{\sigma_{10}Z(\underline{p}) - p_1R_0}{-\frac{1}{R_0}\, p_1Z(\underline{p}) + \sigma_{10}} \quad . \tag{3.22}
$$

At $p_1 = \sigma_{10}$, (3.22) is of indeterminate form. Applying L'Hospital's rule to (3.22), we have

$$
F(\underline{p})\Big|_{p_1=\sigma_{10}} = R_0\,\frac{R_0 - \sigma_{10}Z_0'}{R_0 + \sigma_{10}Z_0'} = F_0 \quad . \tag{3.23}
$$

Since F_0 is a constant, we have

$$F(\underline{p}) = \frac{\sigma_{10}\hat{F}(\underline{p}) + p_1 F_0}{\frac{1}{F_0} p_1 \hat{F}(\underline{p}) + \sigma_{10}} , \tag{3.24}$$

where $\hat{F}(\underline{p})$ is m.p.r..

Substituting $F(\underline{p})$ of (3.24) into (3.15), we have

$$Z(\underline{p}) = \frac{[(R_0/F_0)p_1^2 + \sigma_{10}^2]\hat{F}(\underline{p}) + \sigma_{10}(F_0+R_0)p_1}{[\sigma_{10}(F_0+R_0)/R_0F_0]p_1\hat{F}(\underline{p}) + (F_0/R_0)p_1^2 + \sigma_{10}^2} ; \tag{3.25}$$

and by substituting F_0 of (3.23) into (3.25), we obtain (3.18) with the parameters $A(p_1)$, $B(p_1)$, $C(p_1)$ and $D(p_1)$ satisfying (3.19).

(2) Condition 2 is evident when (3.6) is applied to $F(\underline{p})$ and $\hat{F}(\underline{p})$:

$$\deg_{p_i} \hat{F}(\underline{p}) \leq \deg_{p_i} F(\underline{p}) \leq \deg_{p_i} Z(\underline{p}) \quad \text{for every } i.$$

(3) It is observed that if $p_1 = \sigma_{10}$ is a simple zero of $\mathrm{Ev}_{\underline{p}}\, Z(\underline{p})$, then $\mathrm{Ev}_{\underline{p}}\, F(\underline{p})$ does not have the same zero at $p_1 = \sigma_{10}$. However, if the multiplicity of the zero is equal to 2 or higher, then this same zero is retained in $\mathrm{Ev}_{\underline{p}}\, F(\underline{p})$ with multiplicity reduced by 1. The degree property (3.21) is therefore apparent from (3.7). Q.E.D.

Theorem 3.2:

Let $Z(\underline{p})$ be an m.p.r. function. If at $p_1 = p_{10} = \sigma_{10} + j\omega_{10}$ $(\sigma_{10} > 0)$

$$Z(\underline{p})\Big|_{p_1=p_{10}} = Z_0 = R_0 + jX_0 ,$$

where Z_0 is constant independent of all other variables, then $Z(\underline{p})$

may be expressed as

$$Z(\underline{p}) = \frac{A(p_1)F(\underline{p}) + B(p_1)}{C(p_1)F(\underline{p}) + D(p_1)} \quad , \tag{3.26}$$

where

$$\begin{aligned}
A(p_1) &= a_2p_1^2 + a_0 = [(\omega_{10}R_0+\sigma_{10}X_0)/(\omega_{10}R_0-\sigma_{10}X_0)]p_1^2 + \sigma_{10}^2+\omega_{10}^2 \\
B(p_1) &= a_1p_1 \qquad = [2\sigma_{10}\omega_{10}(R_0^2+X_0^2)/(\omega_{10}R_0+\sigma_{10}X_0)]p_1 \\
C(p_1) &= b_1p_1 \qquad = [2\sigma_{10}\omega_{10}/(\omega_{10}R_0-\sigma_{10}X_0)]p_1 \\
D(p_1) &= b_2p_1^2 + b_0 = [(\omega_{10}R_0-\sigma_{10}X_0)/(\omega_{10}R_0+\sigma_{10}X_0)]p_1^2 + \sigma_{10}^2+\omega_{10}^2 \quad .
\end{aligned} \tag{3.27}$$

The residual function $F(\underline{p})$ satisfies

(1) $F(\underline{p})$ is m.p.r..

(2) $\deg_{p_i} F(\underline{p}) \leq \deg_{p_i} Z(\underline{p})$ for every i. (3.28)

(3) In particular, if $p_1 = p_{10}$ is an independent zero of $Ev_{\underline{p}}\, Z(\underline{p})$, then

$$\deg_{p_1} F(\underline{p}) = \deg_{p_1} Z(\underline{p}) - 2 \quad . \tag{3.29}$$

Proof:

Since $Z(\underline{p})$ at $p_1 = p_{10}$ is constant independent of all other variables, it follows from Lemma 3.1 that $Z(\underline{p})$ may be expressed in terms of an m.p. function $W(\underline{p})$ as follows

$$Z(\underline{p}) = \frac{(\sigma_{10}R_0+\omega_{10}X_0+jp_1X_0)W(\underline{p}) + p_1R_0+j(\sigma_{10}X_0-\omega_{10}R_0)}{(p_1-j\omega_{10})W(\underline{p}) + \sigma_{10}} . \tag{3.30}$$

Since the reciprocal of an m.p. function is also m.p.,

$$G(\underline{p}) = \frac{1}{W(\underline{p})} = \frac{-(p_1 - j\omega_{10})Z(\underline{p}) + \sigma_{10}R_0 + \omega_{10}X_0 + jp_1X_0}{\sigma_{10}Z(\underline{p}) - p_1R_0 - j(\sigma_{10}X_0 - \omega_{10}R_0)} \tag{3.31}$$

is m.p..

Evaluating $G(\underline{p})$ at $p_1 = p_{10}^*$, we have

$$G(\underline{p})\Big|_{p_1 = p_{10}^*} = \frac{\omega_{10}R_0 + \sigma_{10}X_0}{\omega_{10}R_0 - \sigma_{10}X_0} + j\,\frac{-2\omega_{10}X_0}{\omega_{10}R_0 - \sigma_{10}X_0} = R_G + jX_G \quad . \tag{3.32}$$

which is clearly a constant. Applying Lemma 3.1 again to $G(\underline{p})$, we have

$$G(\underline{p}) = \frac{A_G(p_1)W_G(\underline{p}) + B_G(p_1)}{C_G(p_1)W_G(\underline{p}) + D_G(p_1)} \quad , \tag{3.33}$$

where $W_G(\underline{p})$ is an m.p. function and

$$\begin{aligned} A_G(p_1) &= \sigma_{10}R_G - \omega_{10}X_G + jp_1X_G \\ B_G(p_1) &= p_1R_G + j(\sigma_{10}X_G + \omega_{10}R_G) \\ C_G(p_1) &= p_1 + j\omega_{10} \\ D_G(p_1) &= \sigma_{10} \quad . \end{aligned} \tag{3.34}$$

If we let K_1 be a real constant and K_2 be a positive real constant, then it is apparent that $F(\underline{p})$ defined as

$$F(\underline{p}) = K_2\left(\frac{1}{W_G(\underline{p})} - jK_1\right) \tag{3.35}$$

is also an m.p. function.

From (3.33) and (3.35), it can be shown that

$$G(\underline{p}) = \frac{B_G(p_1)F(\underline{p}) + [A_G(p_1) + jK_1B_G(p_1)]K_2}{D_G(p_1)F(\underline{p}) + [C_G(p_1) + jK_1D_G(p_1)]K_2} \quad . \tag{3.36}$$

By letting

$$K_1 = X_0/R_0 \tag{3.37}$$

$$K_2 = (\omega_{10}R_0 - \sigma_{10}X_0) \; / \; (\omega_{10}R_0 + \sigma_{10}X_0) \tag{3.38}$$

and substituting $W(\underline{p}) = 1/G(\underline{p})$ into (3.30), $Z(\underline{p})$ is expressed in the form of (3.26) where the parameters $A(p_1)$, $B(p_1)$, $C(p_1)$ and $D(p_1)$ assume (3.27).

With (3.35), it has been shown that $F(\underline{p})$ is m.p.. By hypothesis, $Z(\underline{p})$ is m.p.r., therefore $Z(\underline{p})$ is real when all the p_i's are real. Also, $A(p_1)$, $B(p_1)$, $C(p_1)$ and $D(p_1)$ as given by (3.27) are obviously real for real p_1. Therefore, it can be seen from (3.26) that $F(\underline{p})$ is real when all the variables are real. $F(\underline{p})$ is thus multivariable positive and real.

The degree properties are evident from (3.6) and (3.7).

Corollary 3.2.1

Let $Z(\underline{p})$ be an m.p.r. function. If at $p_1 = p_{10}$

$$Z(\underline{p}) = Z_0 = R_0 + jX_0$$

$$\frac{\partial}{\partial p_1} Z(\underline{p}) = Z_0' = R_0' + jX_0'$$

where Z_0 and Z_0' are constants independent of all other variables, then $Z(\underline{p})$ may be expressed as

$$Z(\underline{p}) = \frac{\hat{A}(p_1)\hat{F}(\underline{p}) + \hat{B}(p_1)}{\hat{C}(p_1)\hat{F}(\underline{p}) + \hat{D}(p_1)} \tag{3.39}$$

with

$$\hat{A}(p_1) = a_2\hat{a}_2p_1^4 + [a_0(a_2+\hat{a}_2)+a_1\hat{b}_1]p_1^2 + a_0^2$$

$$\hat{B}(p_1) = (a_2\hat{a}_1+a_1\hat{b}_2)p_1^3 + a_0(a_1+\hat{a}_1)p_1$$

$$\hat{C}(p_1) = (b_1\hat{a}_2+b_2\hat{b}_1)p_1^3 + a_0(b_1+\hat{b}_1)p_1 \tag{3.40}$$

$$\hat{D}(p_1) = b_2\hat{b}_2p_1^4 + [a_0(b_2+\hat{b}_2)+b_1\hat{a}_1]p_1^2 + a_0^2 \quad ,$$

where

$$a_2 = 1/b_2 = (\omega_{10}R_0 + \sigma_{10}X_0) \;/\; (\omega_{10}R_0 - \sigma_{10}X_0)$$

$$a_1 = 2\sigma_{10}\omega_{10}(R_0^2 + X_0^2) \;/\; (\omega_{10}R_0 + \sigma_{10}X_0)$$

$$b_1 = 2\sigma_{10}\omega_0 \;/\; (\omega_{10}R_0 - \sigma_{10}X_0) \tag{3.41}$$

$$a_0 = \sigma_{10}^2 + \omega_{10}^2$$

and $\hat{a}_2$, $\hat{b}_2$, $\hat{a}_1$ and $\hat{b}_1$ are obtained from a_2, b_2, a_1 and b_1 by replacing R_0 and X_0 by R_F and X_F, respectively, where $R_F + jX_F = F(\underline{p})|_{p_1=p_{10}}$ with $F(\underline{p})$ defined in (3.26). The residual function $\hat{F}(\underline{p})$ exhibits the following properties.

(1) $\hat{F}(\underline{p})$ is m.p.r..

(2) $\deg_{p_i} \hat{F}(\underline{p}) \leq \deg_{p_i} Z(\underline{p})$ for every i.

(3) In particular, if $p_1 = p_{10}$ is an independent zero of $Ev_{\underline{p}}\, Z(\underline{p})$, then

$$\deg_{p_1} \hat{F}(\underline{p}) = \begin{cases} \deg_{p_1} Z(\underline{p}) - 2 & \text{if } p_1 = p_{10} \text{ is a simple zero.} \\ \deg_{p_1} Z(\underline{p}) - 4 & \text{if the multiplicity is 2 or higher.} \end{cases}$$

The corollary may be shown in a similar manner as Corollary 3.1.1 and is thus omitted for brevity.

The development thus far deals with the points in the strict right half p_1-plane. For the particular case of $p_1 = j\omega_{10}$, we have the following Brune [7,15] result.

Theorem 3.3

Let $Z(\underline{p})$ be an m.p.r. function. If $Z(\underline{p})$ at $p_1 = j\omega_{10}$ is a purely imaginary constant independent of all other variables

$$Z(\underline{p})\Big|_{p_1=j\omega_{10}} = jX_0 \quad ,$$

then $Z(\underline{p})$ may be expressed as

$$Z(\underline{p}) = \frac{A(p_1)F(\underline{p}) + B(p_1)}{C(p_1)F(\underline{p}) + D(p_1)} \tag{3.42}$$

with

$$\begin{aligned} A(p_1) &= a_2p_1^2 + a_0 = [(\omega_{10}X_0'+X_0)/(\omega_{10}X_0'-X_0)]p_1^2 + \omega_{10}^2 \\ B(p_1) &= a_1p_1 \qquad = [2\omega_{10}X_0^2/(\omega_{10}X_0'+X_0)]p_1 \\ C(p_1) &= b_1p_1 \qquad = [2\omega_{10}/(\omega_{10}X_0'-X_0)]p_1 \\ D(p_1) &= b_2p_1^2 + b_0 = [(\omega_{10}X_0'-X_0)/(\omega_{10}X_0'+X_0)]p_1^2 + \omega_{10}^2 \quad , \end{aligned} \tag{3.43}$$

where $X_0' = \frac{\partial}{\partial p_1} Z(\underline{p})\Big|_{p_1=j\omega_{10}}$. The residual function $F(\underline{p})$ possesses the following properties.

(1) $F(\underline{p})$ is m.p.r..

(2) $\deg_{p_i} F(\underline{p}) \le \deg_{p_i} Z(\underline{p}) \qquad i \neq 1$

$\deg_{p_1} F(\underline{p}) = \deg_{p_1} Z(\underline{p}) - 2$.

The theorem can be proved by the direct step by step realization of a Brune cycle [15] and is thus omitted for brevity.

3.2 GENERAL REALIZATIONS WITH NO CONSTRAINT ON THE USE OF TRANSFORMERS

The development of the various forms of Richards' theorem presented in the previous section resulted in extended applications [5, 26,34] of Youla's [32] cascade theory with regard to multivariable functions. In this section, we shall summarize previous results in the form of the following two theorems and proceed to develop realization methods eliminating the use of gyrators.

Theorem 3.4

Let $Z(\underline{p})$ be an m.p.r. function and let $p_1 = p_{10} = \sigma_{10} + j\omega_{10}$ be an independent zero of $Ev_{\underline{p}}\ Z(\underline{p})$. Then the even part zero $p_1 = p_{10}$ may be removed from $Z(\underline{p})$ by a p_1-variable Richards' section[1] (for $\omega_{10} = 0$, Theorem 3.1) or Brune section (for $\sigma_{10} = 0$, Theorem 3.3) or type E section (for $\sigma_{10} \neq 0$, $\omega_{10} \neq 0$, Theorem 3.2) if

$$Z(\underline{p})\Big|_{p_1=p_{10}} = Z_0 = R_0 + jX_0$$

is a constant independent of all other variables. Furthermore, the terminating impedance after the extraction is the same residual function $F(\underline{p})$ of the corresponding theorem.

Theorem 3.5

Let $Z(\underline{p})$ be an m.p.r. function and let $p_1 = p_{10} = \sigma_{10} + j\omega_{10}$ $(\sigma_{10} \neq 0)$ be an independent zero of $Ev_{\underline{p}}\ Z(\underline{p})$. Then the even part zero $p_1 = p_{10}$ may be realized by a p_1-variable type C section (for $\omega_{10} = 0$, Corollary 3.1.1) or type D section (for $\omega_{10} \neq 0$, Corollary 3.2.1) if at $p_1 = p_{10}$

---

1 A detailed description for the Richards', Brune, type C, type E and type D sections may be found in [32].

$$Z(\underline{p}) = Z_0 = R_0 + jX_0$$

$$\frac{\partial}{\partial p_1} Z(\underline{p}) = Z_0' = R_0' + jX_0'$$

where Z_0 and Z_0' are constants independent of all other variables. Furthermore, the terminating impedance after the extraction is the same residual function $\hat{F}(p)$ of the corresponding corollary.

It is observed that the Brune, type C and type D sections are reciprocal two-ports, whereas the Richards' and type E sections are not. In single variable theory, the Richards' and type E sections can always be augmented by appropriate non zero-producing sections to yield the reciprocal type C and type D sections, respectively. Nevertheless, such a process is not always possible for multivariable functions. As is seen from Theorem 3.5, the removal of an independent even part zero $p_1 = \sigma_{10} + j\omega_{10}$ with $\sigma_{10} \neq 0$ by the type C or type D sections requires an additional condition that the partial derivative with respect to p_1 of $Z(\underline{p})$ evaluated at $p_1 = \sigma_{10} + j\omega_{10}$ is constant independent of all other variables. However, in the following, we shall show that gyratorless realization without resorting to this additional condition can be made possible by relaxing the cascade constraint. Using a separation approach similar to that of Miyata [7], we decompose $Z(\underline{p})$ into two m.p.r. functions

$$Z(\underline{p}) = Z_1(\underline{p}) + Z_2(\underline{p}) \tag{3.44}$$

such that each individual component function is amenable to realization without gyrators. We shall discuss the two cases: $\omega_{10} = 0$ and $\omega_{10} \neq 0$, separately.

Case 1: $\omega_{10} = 0$. This corresponds to the familiar 1st order real-axis Bott-Duffin separation. From (3.15), we have

$$Z(\underline{p}) = \frac{\sigma_{10}F(\underline{p})}{\frac{1}{R_0}\,p_1F(\underline{p})+\sigma_{10}} + \frac{p_1R_0}{\frac{1}{R_0}\,p_1F(\underline{p})+\sigma_{10}} \tag{3.45}$$

$$= Z_1(\underline{p}) + Z_2(\underline{p}) \quad .$$

The realization of $Z(\underline{p})$ in terms of $Z_1(\underline{p})$ and $Z_2(\underline{p})$ is given in Fig. 3.1.

Case 2: $\omega_{10} \neq 0$.

It may be noted from Lemma 2.1 of the previous chapter that a p_1-variable two-port characterized by a chain matrix

$$\frac{1}{f(p_1)}\begin{bmatrix} A(p_1) & B(p_1) \\ C(p_1) & D(p_1) \end{bmatrix}$$

where $A(p_1)$, $B(p_1)$, $C(p_1)$, $D(p_1)$ and $f(p_1)$ satisfying conditions 1, 2 and 3 of Lemma 2.1, is reciprocal iff

$$e(p_1^2) = A(p_1)D(p_1) - B(p_1)C(p_1)$$

is a perfect (or the negative of a perfect) square.

From (3.27), we have

$$e(p_1^2) = A(p_1)D(p_1) - B(p_1)C(p_1)$$

$$= p_1^4 + 2(\omega_{10}^2-\sigma_{10}^2)p_1^2 + \sigma_{10}^2+\omega_{10}^2 \tag{3.46}$$

It is seen that $e(p_1^2)$ is neither a perfect nor the negative of a perfect square. However, we can decompose $e(p_1^2)$ into two perfect square terms and then construct two m.p.r. component functions

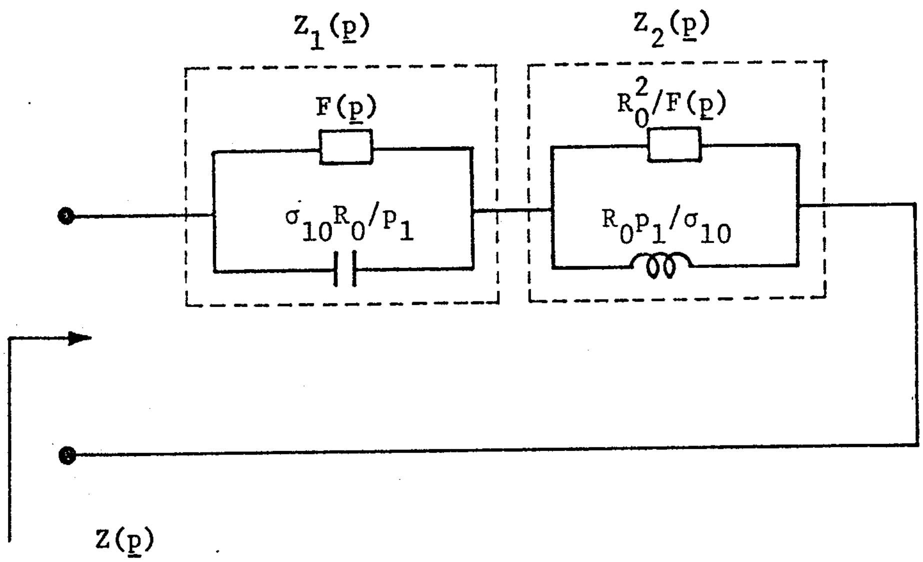


Fig. 3.1 The 1st order Bott-Duffin realization of equation (3.45).

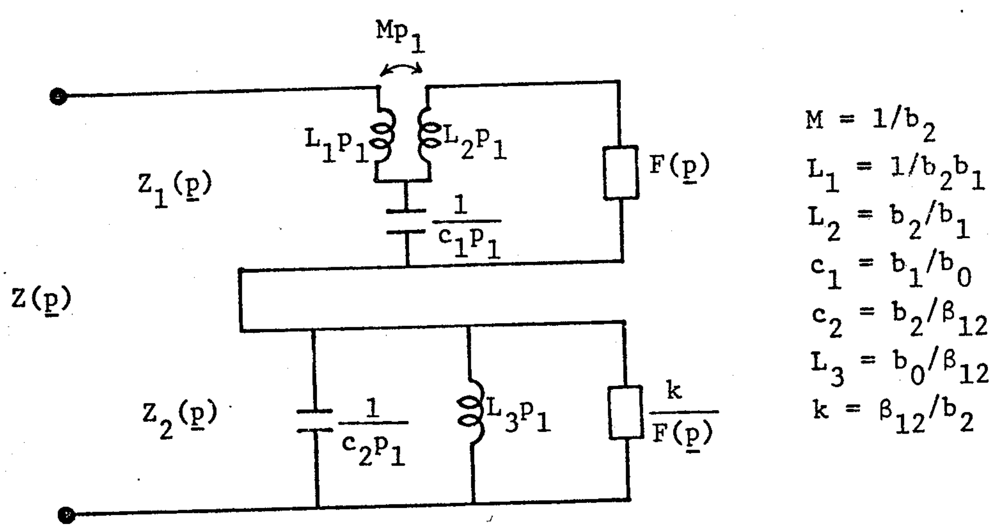


Fig. 3.2 A gyratorless realization of a complex p_1-variable independent even part zero.

associated with these two terms so as to make each individual function amenable to reciprocal realizations. Let $Z(\underline{p})$ of (3.26) be decomposed as

$$Z(\underline{p}) = \frac{A(p_1)F(\underline{p}) + B(p_1)}{C(p_1)F(\underline{p}) + D(p_1)} = \sum_{i=1}^{2} \frac{A_i(p_1)F(\underline{p}) + B_i(p_1)}{C(p_1)F(\underline{p}) + D(p_1)}$$

$$= Z_1(\underline{p}) + Z_2(\underline{p}) \quad . \tag{3.47}$$

Therefore ,

$$e(p_1^2) = \sum_{i=1}^{2} [A_i(p_1)D(p_1) - B_i(p_1)C(p_1)] = e_1(p_1^2) + e_2(p_1^2) . \tag{3.48}$$

An examination of (3.46) indicates that the decomposition is not unique, here we present a simple convenient possibility.

$$e(p_1^2) = [p_1^2 + \sigma_{10}^2 + \omega_{10}^2]^2 - 4\sigma_{10}^2 p_1^2 = e_1(p_1^2) + e_2(p_1^2) \tag{3.48}$$

By assigning $e_1(p_1^2) = (p_1^2 + \sigma_{10}^2 + \omega_{10}^2)^2$ to $Z_1(\underline{p})$, we have

$$A_1(p_1)D(p_1) - B_1(p_1)C(p_1) = (p_1^2 + \sigma_{10}^2 + \omega_{10}^2)^2 \quad .$$

With $D(p_1)$ and $C(p_1)$ as given in (3.27), we can solve for

$$A_1(p_1) = \frac{1}{b_2} p_1^2 + b_0 = \frac{\omega_{10}R_0 + \sigma_{10}X_0}{\omega_{10}R_0 - \sigma_{10}X_0} p_1^2 + \sigma_{10}^2 + \omega_{10}^2 \tag{3.49a}$$

$$B_1(p_1) = \beta_{11}p_1 = \frac{2\sigma_{10}X_0^2(\sigma_{10}^2 + \omega_{10}^2)}{\omega_{10}(\omega_{10}R_0 + \sigma_{10}X_0)} p_1 \quad . \tag{3.49b}$$

Thus ,

$$Z_1(\underline{p}) = \frac{(\frac{1}{b_2} p_1^2 + b_0)F(\underline{p}) + \beta_{11}p_1}{b_1p_1F(\underline{p}) + b_2p_1^2 + b_0} \quad . \tag{3.49c}$$

Similarly, by assigning $e_2(p_1^2) = -2\sigma_{10}^2 p_1^2$ to $Z_2(\underline{p})$, we have

$$A_2(p_1) = 0 \ , \tag{3.50a}$$

$$B_2(p_1) = \beta_{12}p_1 = \frac{2\sigma_{10}(\omega_{10}R_0 - \sigma_{10}X_0)}{\omega_{10}} \tag{3.50b}$$

and

$$Z_2(\underline{p}) = \frac{\beta_{12}p_1}{b_1p_1F(\underline{p}) + b_2p_1^2 + b_0} \ . \tag{3.50c}$$

It is observed that $Z_1(\underline{p})$ of (3.49c) satisfies Theorem 3.3 and is thus realizable by a Brune section terminated in $F(\underline{p})$. $Z_2(\underline{p})$ as given by (3.50c) may be realized by a simple parallel connection. The complete realization is given in Fig. 3.2.

Example 3.1: Consider the following 2-variable p.r. impedance

$$Z(p_1,p_2) = \frac{P(p_1,p_2)}{Q(p_1,p_2)} = \frac{2p_1^2p_2 + p_1p_2 + 3p_1 + p_2 + 1}{p_1^2p_2 + p_1^2 + 3p_1p_2 + p_1 + 2} \ . \tag{3.51}$$

First, we compute

$$\mathrm{Ev}_{p_1,p_2}\ Z(p_1,p_2) = \frac{2(p_1p_2 + 1)^2(1 - p_1^2)}{Q(p_1,p_2)\ Q(-p_1,-p_2)} \ .$$

It is seen that $p_1 = 1$ is a real independent even part zero. Evaluating $Z(p_1,p_2)$ at $p_1 = 1$, we have

$$Z(1,p_2) = R_0 = 1 \ .$$

Thus, by virtue of Theorem 3.1, $Z(p_1,p_2)$ may be expressed as

$$Z(p_1,p_2) = \frac{F(p_1,p_2) + p_1}{p_1F(p_1,p_2) + 1} \ ,$$

where

$$F(p_1,p_2) = \frac{Z(p_1,p_2) - p_1}{-p_1Z(p_1,p_2) + 1} = \frac{p_1p_2 + p_1 + p_2 + 1}{2p_1p_2 + 2} .$$

Therefore, $Z(p_1,p_2)$ may be realized by a Richards' section terminated on $F(p_1,p_2)$ as shown in Fig. 3.3a.

Alternatively, if a realization without gyrator is desired, we try the partial derivative condition

$$\frac{\partial}{\partial p_1} Z(p_1,p_2)\Big|_{p_1=1} = Z_0' = 0 .$$

It is seen that Z_0' is constant. Therefore Corollary 3.1.1 is satisfied and $Z(p_1,p_2)$ may be expressed as

$$Z(p_1,p_2) = \frac{(p_1^2 + 1)\hat{F}(p_1,p_2) + 2p_1}{2p_1\hat{F}(p_1,p_2) + p_1^2 + 1} ,$$

where

$$\hat{F}(p_1,p_2) = \frac{(p_1^2 + 1)Z(p_1,p_2) - 2p_1}{-2p_1Z(p_1,p_2) + p_1^2 + 1} = \frac{2p_1p_2 + p_2 + 1}{p_1p_2 + p_1 + 2} .$$

Therefore, the even part zero at $p_1 = 1$ may be removed by the reciprocal type C section terminated on $\hat{F}(p_1,p_2)$ as shown in Fig. 3.3b.

Example 3.2: Consider the following 2-variable p.r. impedance

$$Z(p_1,p_2) = \frac{P(p_1,p_2)}{Q(p_1,p_2)} = \frac{12p_1^3p_2+4p_1^3+13p_1^2p_2+12p_1^2+6p_1p_2+7p_1+4p_2+6}{p_1^3p_2+6p_1^2p_2+3p_1^2+6p_1p_2+6p_1+2} . \tag{3.52}$$

We have,

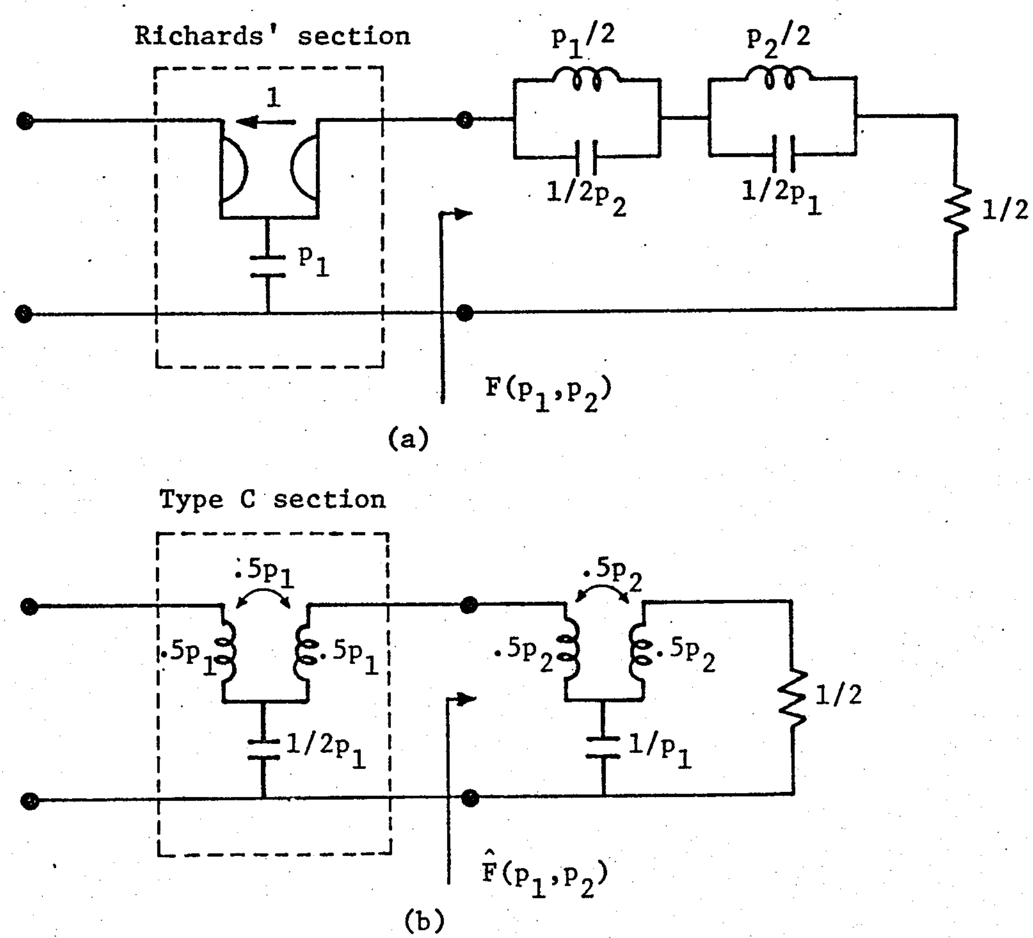


Fig. 3.3 (a) A realization of Example 3.1.
(b) An alternative gyratorless realization of (a).

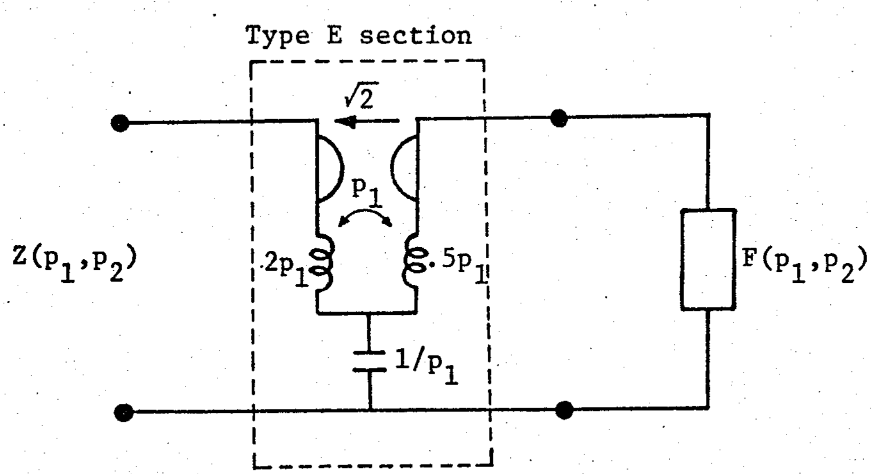


Fig. 3.4 Example 3.2.

$$\mathrm{Ev}_{p_1,p_2}\, Z(p_1,p_2) = \frac{12(p_1^4+1)(p_1p_2+1)^2}{Q(p_1,p_2)Q(-p_1,-p_2)} \quad .$$

Therefore, it is seen that $p_1 = p_{10} = \sqrt{2}/2 + j\sqrt{2}/2$ is a complex independent even part zero. Evaluating $Z(p_1,p_2)$ at $p_1 = p_{10}$, we have

$$Z(p_{10},p_2) = R_0 + jX_0 = \frac{3\sqrt{2}}{2} + j\,\frac{\sqrt{2}}{2}$$

which is a constant independent of p_2. Thus by virtue of Theorem 3.2, $Z(p_1,p_2)$ may be expressed as

$$Z(p_1,p_2) = \frac{(2p_1^2+1)F(p_1,p_2) + 2.5p_1}{p_1F(p_1,p_2) + .5p_1^2 + 1} \quad , \tag{3.53}$$

where

$$\begin{aligned} F(p_1,p_2) &= \frac{(.5p_1^2+1)Z(p_1,p_2) - 2.5p_1}{-p_1Z(p_1,p_2) + 2p_1^2 + 1} \\ &= \frac{3p_1p_2 + p_1 + 2p_2 + 3}{p_1p_2 + 1} \quad . \end{aligned} \tag{3.54}$$

Therefore, $Z(p_1,p_2)$ is realizable by a type E section terminated in $F(p_1,p_2)$ as shown in Fig. 3.4.

Next, an evaluation of the partial derivative of $Z(p_1,p_2)$ with respect to p_1 at $p_1 = p_{10}$ shows that it is not a constant. Therefore Theorem 3.5 is not satisfied. Consequently, the complex even part zero $p_1 = p_{10}$ can not be removed by a reciprocal type D section. However, to obtain a gyratorless realization we may employ the second approach described above.

According to (3.48), we have

$$e(p_1^2) = (2p_1^2 + 1)(.5p_1^2 + 1) - 2.5p_1^2 = p_1^4 + 1$$

$$= (p_1^2 + 1)^2 - 2p_1^2 = e_1(p_1^2) + e_2(p_1^2) \ .$$

From (3.49) and (3.50), we have

$$A_1(p_1) = 2p_1^2 + 1$$

$$B_1(p_1) = .5p_1$$

$$A_2(p_1) = 0$$

$$B_2(p_1) = 2p_1 \ .$$

Therefore, $Z(p_1,p_2)$ is decomposable as

$$Z(p_1,p_2) = \frac{(2p_1^2+1)F(p_1,p_2) + .5p_1}{p_1F(p_1,p_2) + (.5p_1^2+1)} + \frac{2p_1}{p_1F(p_1,p_2) + (.5p_1^2+1)}$$

$$= Z_1(p_1,p_2) + Z_2(p_1,p_2) \ .$$

The complete network is then given by Fig. 3.2 with the following set of element values: $M = 1$, $L_1 = 2$, $L_2 = .5$, $c_1 = 1$, $c_2 = .25$, $L_3 = .5$, $k = 4$ and $F(p_1,p_2)$ as given by (3.54).

3.3 REALIZATIONS WITHOUT TRANSFORMERS

In the previous section, we have presented how independent zeros of the even part function $Ev_{\underline{p}}\ Z(\underline{p})$ may be realized by the basic cascade sections as well as by a separation method without using gyrators. In this section, we develope realization methods without using coupling elements. The following presentation is divided into three cases according to the location of the even part zero.

Case 1: Real-axis zero.

Real-axis zero may be realized in the same manner as discussed in Case 1 of Section 3.2.

Case 2: Imaginary-axis zero.

The classical Bott-Duffin method for the realization of imaginary-axis even part zero has been extended to multivariable synthesis by Soliman et al. [31]. Here, we establish a different method which is based on Seshu's [29] minimum bridge network.

In the article [29], Seshu demonstrated that if a single variable biquadratic minimum function $Z(p)$ satisfies either one of the following two conditions

$$Z(\infty) = 4Z(0)$$

$$Z(0) = 4Z(\infty) ,$$

then $Z(p)$ is realizable by a simple bridge network without coupling elements. This particular result may be easily generalized as follows.

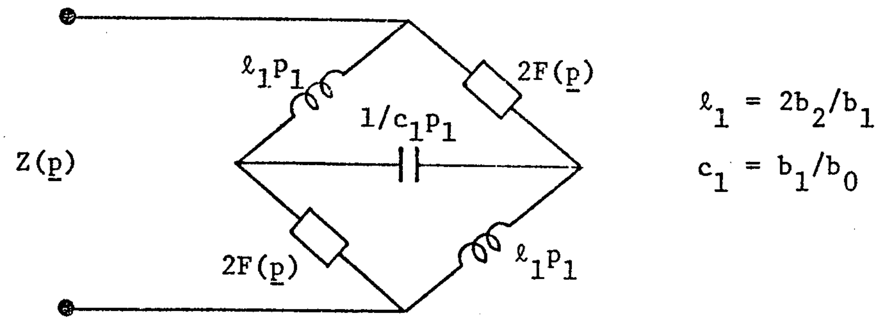
Lemma 3.2

Let $Z(\underline{p})$ be an m.p.r. function. If $Z(\underline{p})$ is expressable in either one of the following two forms

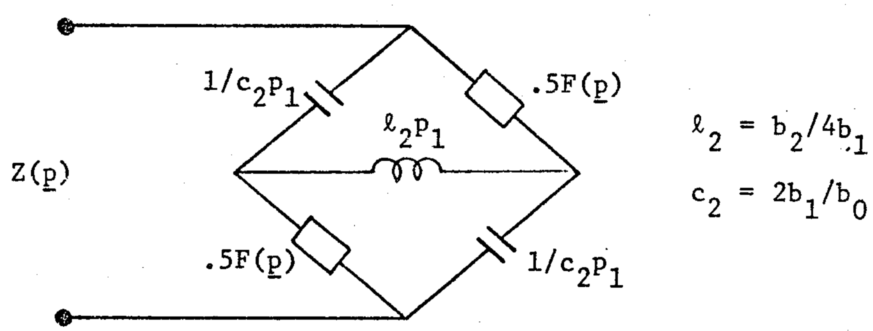
$$Z(\underline{p}) = \frac{(4b_2p_1^2 + b_0)F(\underline{p}) + (\frac{b_2b_0}{b_1})p_1}{b_1p_1F(\underline{p}) + (b_2p_1^2 + b_0)} \tag{3.55a}$$

$$Z(\underline{p}) = \frac{(\frac{1}{4}b_2p_1^2 + b_0)F(\underline{p}) + (\frac{b_2b_0}{4b_1})p_1}{b_1p_1F(\underline{p}) + (b_2p_1^2 + b_0)} , \tag{3.55b}$$

then $Z(\underline{p})$ is realizable as an input impedance of one of the two structures shown in Fig. 3.5.



(a)



(b)

Fig. 3.5 (a) Network configuration for expression (3.55a)
(b) Network configuration for expression (3.55b).

Now, let $p_1 = j\omega_{10}$ be an independent zero of $Ev_{\underline{p}}\, Z(\underline{p})$ and let

$$Z(\underline{p})\Big|_{p_1=j\omega_{10}} = jX_0 \quad ,$$

where X_0 is a real constant independent of all other variables. Then by virtue of Theorem 3.3, $Z(\underline{p})$ may be expressed as

$$Z(\underline{p}) = \frac{\left(\dfrac{\omega_{10}X_0'+X_0}{\omega_{10}X_0'-X_0}\, p_1^2 + \omega_{10}^2 \right)F(\underline{p}) + \dfrac{2\omega_{10}X_0^2}{\omega_{10}X_0'+X_0}\, p_1}{\dfrac{2\omega_{10}}{\omega_{10}X_0'-X_0}\, p_1 F(\underline{p}) + \dfrac{\omega_{10}X_0'-X_0}{\omega_{10}X_0'+X_0}\, p_1^2 + \omega_{10}^2} \quad . \tag{3.56}$$

In order for $Z(\underline{p})$ of (3.56) to be amenable to (3.55), we set

(i) $\left(\dfrac{\omega_{10}X_0' + X_0}{\omega_{10}X_0' - X_0} \right)^2 = 4$. This yields $X_0' = 3X_0/\omega_{10}$.

Thus, (3.56) reduces to

$$Z(\underline{p}) = \frac{(2p_1^2 + \omega_{10}^2)F(\underline{p}) + .5\omega_{10}X_0 p_1}{(\omega_{10}/X_0)p_1 F(\underline{p}) + .5p_1^2 + \omega_{10}^2} \quad . \tag{3.57}$$

With $b_2 = .5$, $b_0 = \omega_{10}^2$ and $b_1 = \omega_{10}/X_0$, it is apparent that $Z(\underline{p})$ satisfies (3.55a).

(ii) $\left(\dfrac{\omega_{10}X_0' + X_0}{\omega_{10}X_0' - X_0} \right)^2 = \dfrac{1}{4}$. This yields $X_0' = -3X_0/\omega_{10}$.

Thus, (3.56) reduces to

$$Z(\underline{p}) = \frac{(.5p_1^2 + \omega_{10}^2)F(\underline{p}) - \omega_{10}X_0 p_1}{(-\omega_{10}/2X_0)p_1 F(\underline{p}) + 2p_1^2 + \omega_{10}^2} \quad . \tag{3.58}$$

With $b_2 = 2$, $b_0 = \omega_{10}^2$ and $b_1 = -\omega_{10}/2X_0$, $Z(\underline{p})$ now assumes the form of (3.55b).

Hence, we obtain the following theorem.

Theorem 3.6

Let $Z(\underline{p})$ be an m.p.r. function and let $p_1 = j\omega_{10}$ be an independent zero of $Ev_{\underline{p}}\, Z(\underline{p})$. If

(i) $Z(\underline{p})\Big|_{p_1=j\omega_{10}} = jX_0$, where X_0 is a real constant independent of all other variables

(ii) $X_0' = \pm 3X_0/\omega_{10}$, where $X_0' = \frac{\partial}{\partial p_1} Z(\underline{p})\Big|_{p_1=j\omega_{10}}$,

then the even part zero $p_1=j\omega_{10}$ may be realized by Fig. 3.5a for $X_0' = 3X_0/\omega_{10}$ or Fig. 3.5b for $X_0' = -3X_0/\omega_{10}$.

Example 3.3: Consider the following 2-variable p.r. function

$$Z(p_1,p_2) = \frac{P(p_1,p_2)}{Q(p_1,p_2)} = \frac{4p_1^3p_2+5p_1^2p_2+4p_1^2+2p_1p_2+p_1+2p_2+2}{p_1^3p_2+2p_1^2p_2+p_1^2+4p_1p_2+2p_1+2} \quad . \quad (3.59)$$

We have

$$Ev_{p_1,p_2}\, Z(p_1,p_2) = \frac{4(p_1^2+1)^2(p_1p_2+1)^2}{Q(p_1,p_2)Q(-p_1,-p_2)} \quad .$$

Therefore, $p_1 = j1$ is an imaginary-axis even part zero. Evaluating $Z(p_1,p_2)$ at $p_1 = j1$, we have

$$Z(j1,p_2) = jX_0 = j1 \ .$$

Since $X_0 = 1 > 0$, with reference to [31], we set $p_1 = \sigma$ and solve for σ from

$$Z(\sigma,p_2) = \sigma X_0/\omega_{10} \quad . \qquad (3.60)$$

With $\omega_{10} = 1$ and $X_0 = 1$, (3.59) and (3.60) yield

$$p_2\sigma^2 + (1 - 2p_2)\sigma + 2(p_2 + 1) = 0 \tag{3.61}$$

Solving (3.61), we see that there exists no positive real σ independent of p_2. Therefore, according to [31], the even part zero $p_1 = j1$ of this example can not be removed by the multivariable Bott-Duffin process. However, evaluating

$$X_0' = \frac{\partial}{\partial p_1} Z(\underline{p})\Big|_{p_1=j1} = 3 \ ,$$

we see that $X_0' = 3X_0/\omega_{10}$. Therefore, Theorem 3.6 is satisfied. From (3.57), we have

$$Z(p_1,p_2) = \frac{(2p_1^2 + 1)F(p_1,p_2) + .5p_1}{p_1F(p_1,p_2) + .5p_1^2 + 1} \ ,$$

where

$$F(p_1,p_2) = \frac{(.5p_1^2 + 1)Z(p_1,p_2) - .5p_1}{-p_1Z(p_1,p_2) + 2p_1^2 + 1} = \frac{p_1p_2 + p_2 + 1}{p_1p_2 + 1} \ , \tag{3.62}$$

and $Z(p_1,p_2)$ is thus realizable in the form of Fig. 3.5a with $\ell_1 = c_1 = 1$ and $F(p_1,p_2)$ as given by (3.62).

Case 3: Complex zero.

In single-variable synthesis theory, a complex even part zero may be realized by first shifting the zero to the imaginary axis then applying the Bott-Duffin procedure to remove this newly generated imaginary-axis zero. In a broader sense than the Brune's process of minimum resistance extraction, the complex zero may be shifted to the imaginary-axis by decomposing the given function into a sum of two sub-functions one of which may be realized by inspection and the other possessing the desired imaginary-axis even part zero. Such a preliminary extraction step is also possible for multivariable functions, however, the resulting imaginary-axis even part zero might not meet the realizability conditions required by the methods discussed previously. As to the Fialkow-Gerst's [1] method, on the contrary, the difficulty associated with multivariable functions arises at the first preliminary process of deriving a new impedance which is purely real at the complex even part zero.

To circumvent all these difficulties, we develope, in the following, a simple separation approach which is based on Kim's [11,12] work on transformerless synthesis.

It is noted that the input impedance $Z(\underline{p})$ of a p_1-variable lossless reciprocal two-port closed on an impedance $Z_L(\underline{p})$ may be expressed in terms of the z-parameter of the two-port as

$$Z(\underline{p}) = z_{11}(p_1)\,\frac{Z_L(\underline{p}) + [z_{11}(p_1)z_{22}(p_1) - z_{12}^2(p_1)]/z_{11}(p_1)}{Z_L(\underline{p}) + z_{22}(p_1)} \; . \tag{3.63}$$

From (3.26), the impedance $Z(\underline{p})$ can be rewritten as

$$Z(\underline{p}) = \frac{A(p_1)}{C(p_1)}\,\frac{F(\underline{p}) + B(p_1)/A(p_1)}{F(\underline{p}) + D(p_1)/C(p_1)} \tag{3.64a}$$

$$= \frac{B(p_1)}{D(p_1)}\,\frac{1/F(\underline{p}) + A(p_1)/B(p_1)}{1/F(\underline{p}) + C(p_1)/D(p_1)} \; . \tag{3.64b}$$

Comparing (3.63) and (3.64), we have the following two distinct identifications:

$$z_{11}(p_1) = A(p_1)/C(p_1) \tag{3.65a}$$

$$z_{22}(p_1) = D(p_1)/C(p_1) \tag{3.65b}$$

$$z_{12}(p_1) = \sqrt{e(p_1^2)}/C(p_1) \tag{3.65c}$$

$$Z_L(\underline{p}) = F(\underline{p}) \tag{3.65d}$$

and

$$z_{11}(p_1) = B(p_1)/D(p_1) \tag{3.66a}$$

$$z_{22}(p_1) = C(p_1)/D(p_1) \tag{3.66b}$$

$$z_{12}(p_1) = \sqrt{-e(p_1^2)}/D(p_1) \tag{3.66c}$$

$$Z_L(\underline{p}) = 1/F(\underline{p}) \quad , \tag{3.66d}$$

where $e(p_1^2) = A(p_1)D(p_1) - B(p_1)C(p_1)$.

It may be noted that the p_1-variable two-port is realizable by simple structures without transformers if either one of the following two cases are satisfied:

Case a: $e(p_1^2)$ is a perfect square and the residues of $z_{12}(p_1)$ of (3.65c) are not greater than those of the corresponding poles of $z_{11}(p_1)$ and $z_{22}(p_1)$ of (3.65a) and (3.65b), respectively.

Case b: $e(p_1^2)$ is the negative of a perfect square and the residues of $z_{12}(p_1)$ of (3.66c) are not greater than those of the corresponding poles of (3.66a) and (3.66b).

As in (3.46), we see that

$$e(p_1^2) = p_1^4 + 2(\omega_{10}^2 - \sigma_{10}^2)p_1^2 + \sigma_{10}^2 + \omega_{10}^2$$

which is neither a perfect nor the negative of a perfect square. However, we may decompose $e(p_1^2)$ in the manner shown in (3.68) so as to make each individual term satisfy either case a or case b.

Let $Z(\underline{p})$ of (3.26) be decomposed as

$$Z(\underline{p}) = \frac{A(p_1)F(\underline{p}) + B(p_1)}{C(p_1)F(\underline{p}) + D(p_1)} = \sum_i \alpha_i \frac{A_i(p_1)F(\underline{p}) + B_i(p_1)}{C(p_1)F(p) + D(p_1)}$$

$$= \sum_i \alpha_i Z_i(\underline{p}) \ . \qquad (3.67)$$

Therefore, we have

$$e(p_1^2) = \sum_i \alpha_i [A_i(p_1)D(p_1) - B_i(p_1)C(p_1)] = \sum_i \alpha_i e_i(p_1^2) \ . \qquad (3.68)$$

Case a and case b actually amount to the following simple situations:

$$e_1(p_1^2) = A_1(p_1)D(p_1) - B_1(p_1)C(p_1) = b_0^2 \qquad (3.69a)$$

$$e_2(p_1^2) = A_2(p_1)D(p_1) - B_2(p_1)C(p_1) = -(b_1p_1)^2 \qquad (3.69b)$$

$$e_3(p_1^2) = A_3(p_1)D(p_1) - B_3(p_1)C(p_1) = (b_2p_1^2)^2 \qquad (3.69c)$$

$$e_4(p_1^2) = A_4(p_1)D(p_1) - B_4(p_1)C(p_1) = (b_2p_1^2 + b_0)^2 \ . \qquad (3.69d)$$

Solving for $A_1(p_1)$ and $B_1(p_1)$ from (3.69a), we have

$$Z_1(\underline{p}) = \frac{b_0F(\underline{p}) + (b_2b_0/b_1)p_1}{b_1p_1F(\underline{p}) + (b_2p_1^2+b_0)} \ . \qquad (3.70)$$

Since case a holds, we have

$$z_{11}^{(1)} = z_{12}^{(1)} = \frac{b_0}{b_1p_1} \ ,$$

$$z_{22}^{(1)} = \frac{b_2}{b_1} p_1 + \frac{b_0}{b_1p_1} \quad \text{and} \quad Z_L^{(1)}(\underline{p}) = F(\underline{p}) \ .$$

Similarly, we have

$$Z_2(\underline{p}) = \frac{b_1p_1}{b_1p_1F(\underline{p}) + b_2p_1^2 + b_0} , \tag{3.71}$$

$$z_{11}^{(2)} = z_{12}^{(2)} = z_{22}^{(2)} = \frac{b_1p_1}{b_2p_1^2 + b_0} \quad \text{and} \quad Z_L^{(2)}(\underline{p}) = 1/F(\underline{p}),$$

$$Z_3(\underline{p}) = \frac{b_2p_1^2F(\underline{p}) + (b_2b_0/b_1)p_1}{b_1p_1F(\underline{p}) + b_2p_1^2 + b_0} , \tag{3.72}$$

$$z_{11}^{(3)} = z_{12}^{(3)} = \frac{b_2}{b_1}p_1 ,$$

$$z_{22}^{(3)} = \frac{b_2}{b_1}p_1 + \frac{b_2}{b_1p_1} \quad \text{and} \quad Z_L^{(3)}(\underline{p}) = F(\underline{p})$$

and

$$Z_4(\underline{p}) = \frac{(b_2p_1^2 + b_0)F(\underline{p})}{b_1p_1F(\underline{p}) + b_2p_1^2 + b_0} , \tag{3.73}$$

$$z_{11}^{(4)} = z_{12}^{(4)} = z_{22}^{(4)} = \frac{b_2}{b_1}p_1 + \frac{b_0}{b_1p_1} \quad \text{and} \quad Z_L^{(4)}(\underline{p}) = F(\underline{p}) .$$

Network configurations for the Z_i's are given in Fig. 3.6.

To extend the range of applicability, we may also add to (3.67) the two simple bridge structures given by (3.55a) and (3.55b).

By assigning Z_5 and Z_6 corresponding to the impedances of (3.55a) and (3,55b), respectively, we derive the following set of equations with respect to (3.67).

$$\begin{aligned} &\alpha_3 + \alpha_4 + 4\alpha_5 + .25\alpha_6 = a_2/b_2 \\ &\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6 = 1 \\ &\alpha_1 + (b_1^2/b_0b_2)\alpha_2 + \alpha_3 + \alpha_5 + .25\alpha_6 = a_1b_1/b_0b_2 \quad . \end{aligned} \tag{3.74}$$

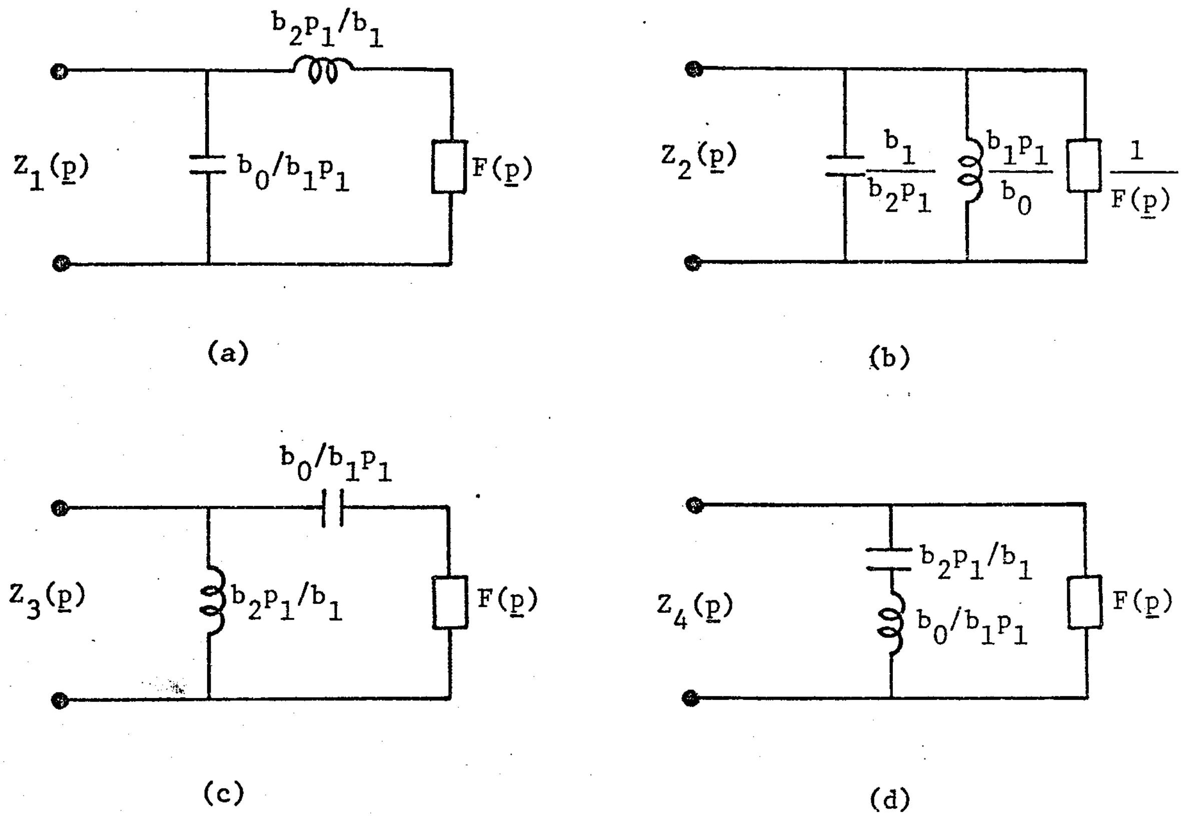


Fig. 3.6 (a) Network for $Z_1(\underline{p})$ of (3.70).
(b) Network for $Z_2(\underline{p})$ of (3.71).
(c) Network for $Z_3(\underline{p})$ of (3.72).
(d) Network for $Z_4(\underline{p})$ of (3.73).

The synthesis problem is now reduced to the simple problem of solving for a set of positive α_i's satisfying system (3.74). Usually the α_i's may be obtained by simple inspections; nevertheless, in general, the problem can be handled systematically and easily on a digital computer. In the following, for illustration, we show some special cases.

Example 3.4: In this example, we consider the solution of a combination composed of only the first three component functions. By setting $\alpha_4 = \alpha_5 = \alpha_6 = 0$, from (3.74), we have

$$\alpha_1 = 1$$

$$\alpha_2 = [a_1 b_1 - b_0(a_2 + b_2)]/b_1^2$$

$$\alpha_3 = a_2/b_2 \quad .$$

Substituting the a_i's and the b_i's of (3.27) into the α_i's above, we have

$$\begin{aligned}
\alpha_1 &= 1 \\
\alpha_2 &= 2(\sigma_{10}^2 - \omega_{10}^2)[(\omega_{10}R_0 - \sigma_{10}X_0)/(\omega_{10}R_0 + \sigma_{10}X_0)]^2 \\
\alpha_3 &= [(\omega_{10}R_0 + \sigma_{10}X_0)/(\omega_{10}R_0 - \sigma_{10}X_0)]^2 \quad .
\end{aligned} \tag{3.75}$$

It is clear that α_1 and α_3 are positive, and $\alpha_2 \geq 0$ if

$$\sigma_{10}^2 - \omega_{10}^2 \geq 0$$

or equivalently

$$|\angle P_{10}| \leq 45^\circ \ .$$

Therefore, we can conclude that if the complex independent even part zero p_{10} lies within the sector $|\underline{/p_{10}}| \leq 45^\circ$, then the zero can always be removed by a combination of the first three component functions.

For instance, the complex even part zero $p_1 = p_{10} = \sqrt{2}/2+j\sqrt{2}/2$ of the example of (3.52) is located at $\underline{/p_{10}} = \tan^{-1}1 = 45^\circ$. Therefore, it is realizable by this combination. From (3.75), we have

$$\alpha_1 = 1, \quad \alpha_2 = 0 \quad \text{and} \quad \alpha_3 = 4 \ .$$

Since $\alpha_2 = 0$, only $Z_1(\underline{p})$ and $Z_3(\underline{p})$ are necessary. A complete realization is given in Fig. 3.7.

Example 3.5: Consider the following 2-variable p.r. impedance

$$Z(p_1,p_2) = \frac{P(p_1,p_2)}{Q(p_1,p_2)} = \frac{4p_1^3p_2+8p_1^3+4.5p_1^2p_2+4p_1^2+p_1p_2+6.5p_1+1}{p_1^3p_2+.4p_1^2p_2+1.8p_1^2+p_1p_2+.4p_1+1} \ .$$

We have

$$Ev_{p_1,p_2} Z(p_1,p_2) = \frac{(4p_1^4 + 3.2p_1^2 + 1)(p_1p_2 + 1)^2}{Q(p_1,p_2)Q(-p_1,-p_2)} \ .$$

Solving the equation $4p_1^4 + 3.2p_1^2 + 1 = 0$, we see that $p_1 = p_{10} = .223 + j.672$ is a complex even part zero. Evaluating $Z(p_1,p_2)$ at $p_1 = p_{10}$, we have

$$Z(p_{10},p_2) = 3.36 + j3.36$$

which is a constant. Therefore, by virtue of Theorem 3.2, $Z(p_1,p_2)$ may be expressed as

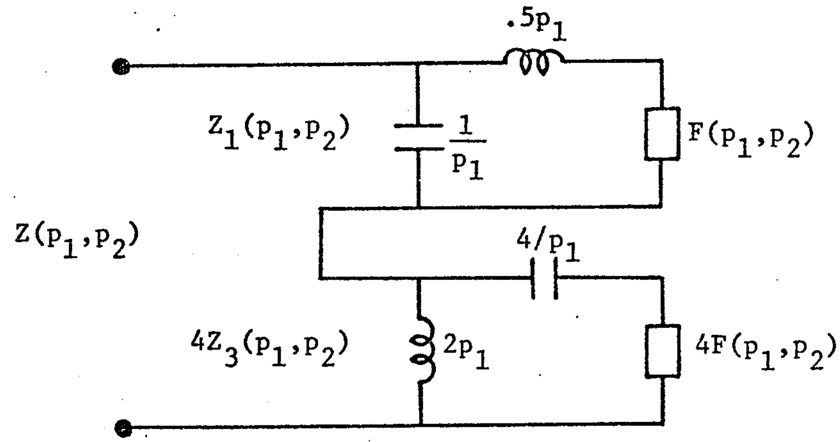


Fig. 3.7 A transformerless realization of expression (3.53).

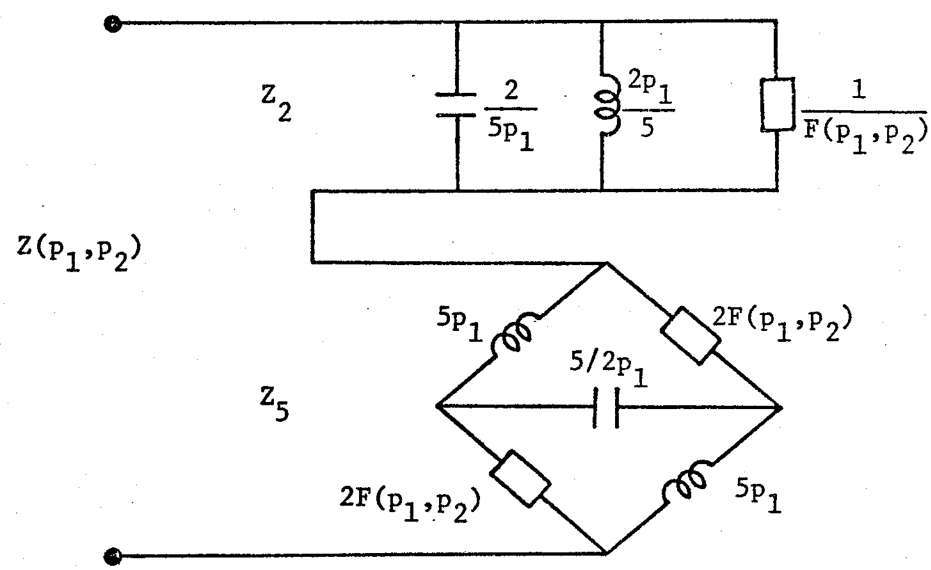


Fig. 3.8 Example 3.5.

$$Z(p_1,p_2) = \frac{(2p_1^2 + .5)F(p_1,p_2) + 2.25p_1}{.2p_1F(p_1,p_2) + .5p_1^2 + .5} \quad , \tag{3.76}$$

where

$$F(p_1,p_2) = \frac{(.5p_1^2 + .5)Z(p_1,p_2) - 2.25p_1}{-.2p_1Z(p_1,p_2) + 2p_1^2 + .5}$$

$$= \frac{p_1p_2 + 2p_1 + 1}{p_1p_2 + 1} \quad .$$

The argument of $p_{10} = .223 + j.672$ is

$$\underline{/ p_{10}} = \tan^{-1}(.672/.223) = 71.56^\circ > 45^\circ.$$

Thus p_{10} lies outside the realizability region discussed in the previous example. Therefore, it is not possible to remove p_{10} by any combination of the first three component functions. However, from (3.76), it is easily seen that $Z(p_1,p_2)$ can be decomposed as

$$Z(p_1,p_2) = \frac{(2p_1^2+.5)F(p_1,p_2)+1.25p_1}{.2p_1F(p_1,p_2)+.5p_1^2+.5} + \frac{p_1}{.2p_1F(p_1,p_2)+.5p_1^2+.5}$$

$$= Z_5(\underline{p}) + Z_2(\underline{p}) \quad .$$

Hence, $Z(p_1,p_2)$ is realizable by a combination of Z_2 and Z_5 as shown in Fig. 3.8.

CHAPTER IV

SYNTHESIS OF A CLASS OF MIXED LUMPED-DISTRIBUTED NETWORKS

It is well-recognized [27] that much of the multivariable theory was motivated by its practical applications to mixed lumped-distributed networks. In this chapter, we consider the synthesis of a class of networks made up of m cascaded noncommensurate transmission lines (also referred to as unit elements) separated by passive lumped lossless two-ports and terminated on a passive lumped network. A similar problem was first tackled in [28]; however, here we employ a more general and explicit approach which includes the possibility of a reactive termination and eliminates the multivariable positive-reality condition in favor of some simpler one-variable type condition. Explicit formulae for the chain matrices of the lumped passive lossless two-ports and the terminating impedance are derived. Several interesting special cases are also considered and the realizability conditions are accordingly modified to produce much simpler synthesis procedures.

For convenience, in this chapter, we denote a set of m complex variables by $(\mu) = (\mu_1, \mu_2, \ldots, \mu_m)$; and the symbol $(\mu)^{i,j,\ldots}_{k,l,\ldots}$ will represent a particular point (μ) with the variables corresponding to the upper indices assuming a value of 1, the lower indices assuming 0 and the variables that are not indicated assuming any arbitray values. For instance, $(\mu)^{1}_{3,5}$ implies that $\mu_1 = 1$, $\mu_3 = \mu_5 = 0$ and the remaining variables μ_2, μ_4, etc. may assume any arbitrary values.

4.1 REALIZABILITY CONDITIONS

The synthesis of the class of networks (Fig. 4.1) composed of m-cascaded noncommensurate transmission lines, separated by passive lumped lossless two-ports and terminated by a positive resistor was first advanced in [28]. It is shown that necessary and sufficient conditions under which the multivariable rational function $Z[p,(\mu)]$ may be realized as a driving point impedance of the structure shown in Fig. 4.1 are

(a) $Z[p,(\mu)]$ is a bilinear function[1] in the set of variables $(\mu) = (\mu_1,\mu_2,\ldots,\mu_m)$.

(b) $Z[p,(\mu)]$ is a multivariable positive real function.

(c) $Z[p,(\mu)] + Z_*[p,(\mu)] = 0$ at $\mu_i = 1$ for $i = 1,2,\ldots,m$.

(d) $Z[p,(\mu)] + Z_*[p,(\mu)] \not\equiv 0$,

where the lower asterisk denotes the reversal of sign of all the independent variables.

It is noted that condition b is impracticable and unwieldy to verify. To circumvent this difficulty, we take a more direct and explicit approach and derive an alternative new set of realizability conditions, stated in the form of the following main theorem, which replaces condition b by some simpler one-variable type conditions. Also, in order to be able to cater for the possibility of a reactive termination. The more general structure shown in Fig. 4.2 is considered, where the terminating network is lumped and passive but otherwise arbitrary. Furthermore, since the extraction order of

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1 $Z[p,(\mu)]$ is said to be bilinear in the set of variables (μ) if $\deg_{\mu_i} Z[p,(\mu)] = 1$ for every i.

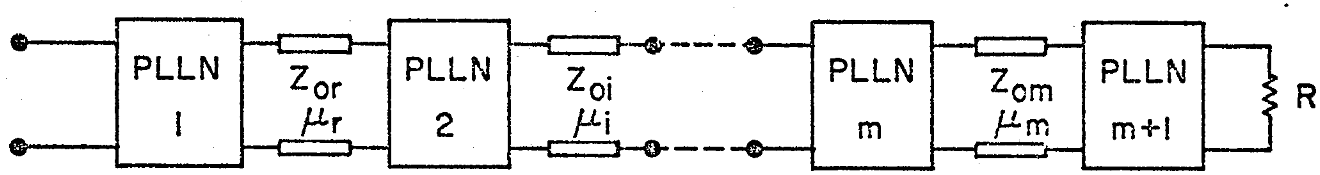


Fig. 4.1 A resistively-terminated cascade of m noncommensurate transmission lines separated by passive lumped lossless two-ports.

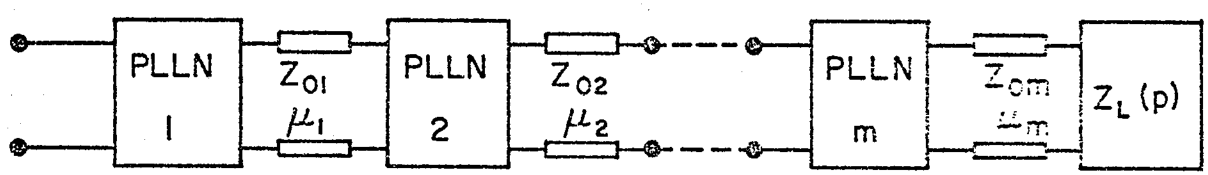


Fig. 4.2 A passive-lumped-terminated cascade of m noncommensurate transmission lines separated by passive lumped lossless two-ports.

the transmission lines can always be determined in advance [18], in the following, we shall assume without loss of generality that the extraction order is $\{\mu_1,\mu_2,\ldots,\mu_m\}$.

4.1.1 The Main Theorem

The necessary and sufficient conditions for the multivariable rational function $Z[p,(\mu)]$ bilinear in the set of variables $(\mu) = (\mu_1,\mu_2,\ldots,\mu_m)$ to be realizable as a driving point impedance of the structure shown in Fig. 4.2 are as follows

$$\text{(1)}\quad Z[p,(\mu)^i_{1,2,\ldots,i-1}] = -Z_*[p,(\mu)^i_{1,2,\ldots,i-1}] = \frac{n_i(p)}{d_i(p)} \tag{4.1}$$

for $i = 1,2,\ldots,m$, where $n_i(p)$ and $d_i(p)$ are polynomials of p only and are expressed as being relatively prime.

(2) With $n_o = d_o = 1$ and

$$Z[p,(\mu)_{1,2,\ldots,m}] = \frac{n_{m+1}}{d_{m+1}} \quad ,$$

where n_{m+1} and d_{m+1} are relatively prime, formulate

$$\begin{aligned}
A_i &= \alpha_i\Phi_i = (d_{i-1}+d_{i-1*})(n_i+n_{i*}) - (n_{i-1}-n_{i-1*})(d_i-d_{i*}) \\
B_i &= \beta_i\Phi_i = (d_{i-1}+d_{i-1*})(n_i-n_{i*}) - (n_{i-1}-n_{i-1*})(d_i+d_{i*}) \\
C_i &= \gamma_i\Phi_i = -(d_{i-1}-d_{i-1*})(n_i+n_{i*}) + (n_{i-1}+n_{i-1*})(d_i-d_{i*}) \\
D_i &= \delta_i\Phi_i = -(d_{i-1}-d_{i-1*})(n_i-n_{i*}) + (n_{i-1}+n_{i-1*})(d_i+d_{i*})
\end{aligned} \tag{4.2}$$

for $i = 1,2,\ldots,m+1$, where Φ_i is the greatest common factor of A_i, B_i, C_i and D_i. Then

(2a) $\alpha_i+\beta_i+\gamma_i+\delta_i$ is strictly Hurwitz.

(2b) $\alpha_i\delta_i-\beta_i\gamma_i$ can be factored as

$$\alpha_i\delta_i - \beta_i\gamma_i = \varepsilon_i f_i f_{i*} \tag{4.3}$$

where f_i is a real polynomial and ε_i is a constant which is equal to +1 if α_i is even and −1 if α_i is odd.

Synthesis Procedure: A realization in which all the passive lumped lossless two-ports are all-pass free on their output sides may be obtained by the following

(i) The passive lumped lossless two-ports are characterized by the following chain matrices

$$[T]_i = \frac{1}{f_i}\begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix} \quad \text{for } i = 1,2,\ldots,m. \tag{4.4}$$

(ii) The characteristic impedances of the unit elements are normalized to unity,

$$Z_{0i} = 1 \quad \text{for } i = 1,2,\ldots,m. \tag{4.5}$$

(iii) The terminating impedance is given by

$$Z_L = \frac{\alpha_{m+1} + \beta_{m+1}}{\gamma_{m+1} + \delta_{m+1}} \quad . \tag{4.6}$$

Proof of the Main Theorem: The necessity is evident from simple analysis. In the following, we shall show the sufficiency.

By (4.2), it is observed that A_i and D_i are even polynomials whereas B_i and C_i are odd. Consequently, both α_i and δ_i are of the same parity (that is, both are even or odd depending on whether the greatest common factor Φ_i is even or odd), while β_i and γ_i are of parity opposite to that of α_i and δ_i. This fact together with conditions 2a and 2b show that the matrices defined by (4.4) represent the chain matrices of realizable lumped lossless two-ports.

On extracting from $Z[p,(\mu)]$ a two-port characterized by $[T]_1$ and an unit element corresponding to μ_1 with unity characteristic impedance in succession, the remaining impedance is of the following form

$$Z_1[p,(\mu)] = \frac{(\delta_1 Z[p,(\mu)]-\beta_1) - \mu_1(-\gamma_1 Z[p,(\mu)]+\alpha_1)}{-\mu_1(\delta_1 Z[p,(\mu)]-\beta_1) + (-\gamma_1 Z[p,(\mu)]+\alpha_1)} \quad . \tag{4.7}$$

From (4.2), we have

$$\alpha_1 = n_1 + n_{1*}$$
$$\beta_1 = n_1 - n_{1*}$$
$$\gamma_1 = d_1 - d_{1*}$$
$$\delta_1 = d_1 + d_{1*} \quad ;$$

and from (4.1), we have

$$Z[p,(\mu)^1] = - Z_*[p,(\mu)^1] = \frac{n_1}{d_1} \quad .$$

Therefore, it can be shown that both the numerator and the denominator of (4.7) possess the common factor $(1-\mu_1^2)$. Hence,

$$\deg_{\mu_1} Z_1[p,(\mu)] = \deg_{\mu_1} Z[p,(\mu)] - 1 \quad .$$

By hypothesis, $Z[p,(\mu)]$ is bilinear in μ_1, it therefore follows that the remaining impedance $Z_1[p,(\mu)]$ is free of the variable μ_1.

Following a similar argument, we can extract from $Z_1[p,(\mu)]$ a two-port characterized by $[T]_2$ and a μ_2-unit-element with $Z_{02} = 1$ in succession to obtain a remaining impedance further free of the variable μ_2.

In short, condition 2 guarantees the successful extractions of the passive lumped lossless two-ports, and condition 1 assures the degree reduction in the μ_i's variables. The same aforementioned procedure can be repeated m times until all the unit elements are extracted. The synthesis is then completed by a final cascade realization of the terminating impedance Z_L which is given by (4.6) and is p.r. as evident from conditions 2a and 2b with $i = m + 1$.

It may be noted that the passive lumped lossless two-ports defined by (4.4) are all-pass free [1,35] on their output sides. This is due to the assumption that the polynomial pair n_i and d_i is expressed as a relatively prime pair. The all-pass section, if it exists, on the output side of the i-th two-port is passed over the μ_i-unit-element section to the (i+1)-th two-port and becomes a front all-pass of the latter. Such an all-pass section can be easily identified [1] by recognizing the common factors between the two polynomials $(\alpha_i+\beta_i+\gamma_i+\delta_i)$ and $(\alpha_i-\beta_i+\gamma_i-\delta_i)$. Different syntheses may be generated by re-distribution of the all-pass sections.

It is also worthwhile to note that although (4.5) yields a realization with unity characteristic impedances, the denormalization of the characteristic impedances can always be achieved by appropriate eliminations of transformers. This process is carried out in the proofs of the following corollaries and also demonstrated in Example 4.1.

4.1.2 Special Cases

The Main Theorem is established for a general structure. In the following, we derive some interesting corollaries for several special cases. According to the particular natures of the passive lumped lossless two-ports of these special cases, condition 2 of the Main Theorem is modified into much simpler forms to greatly facilitate the synthesis procedures.

Corollary 1a

The multivariable rational function $Z[p,(\mu)]$ bilinear in the set of variables $(\mu) = (\mu_1,\mu_2,\ldots,\mu_m)$ may be realized as a driving-point impedance of the resistively terminated cascade of m noncommensurate transmission lines separated by series-arm parallel-LC sections as shown in Fig. 4.3a if and only if $Z[p,(\mu)]$ satisfies condition 1 of the Main Theorem plus the following condition:

(2c) The function n_i/d_i defined in the Main Theorem may be expressed in the form

$$\frac{n_i}{d_i} = \sum_{k=1}^{i} \frac{G_k}{g_k} + Z_{0i} \tag{4.8}$$

for $i = 1,2,\ldots,m+1$, where

$$Z_{0i} = \lim_{p\to\infty} \frac{n_i}{d_i} \quad \text{is a positive constant,} \tag{4.9}$$

and G_k and g_k assume one of the following cases:

Case 1: $g_k = 1$ and $G_k = \ell_k p$ $(\ell_k \geq 0)$. This corresponds to the case that the k-th parallel-LC connection degenerates to a single inductor arm with inductance ℓ_k.

Case 2: $g_k = p$ and $G_k = 1/c_k \geq 0$. This corresponds to the case that the k-th LC section degenerates to a single capacitor arm with capacitance c_k.

Case 3: $g_k = p^2 + \omega_k^2$ $(\omega_k^2 > 0)$ and $G_k = \ell_k p$ $(\ell_k > 0)$. Both

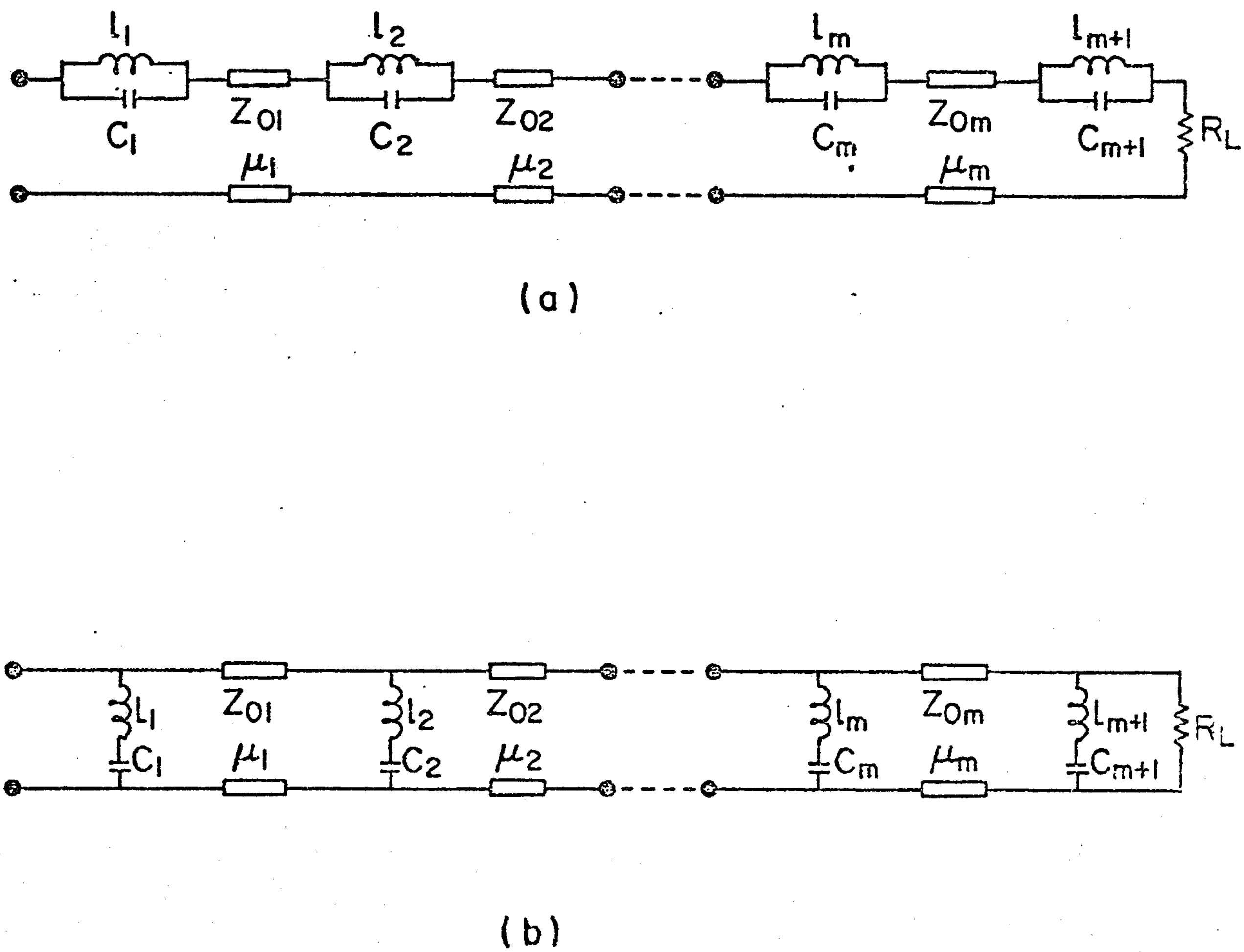


Fig. 4.3 A resistively-terminated cascade of m noncommensurate transmission lines separated by (a) series-arm parallel-LC sections, (b) shunt-arm series-LC sections.

elements of the k-th parallel LC section exist with inductance ℓ_k and capacitance $c_k = 1/\omega_k^2 \ell_k$.

The characteristic impedances Z_{0i}'s of the transmission lines are given by (4.9) for $i = 1,2,\ldots,m$ and the terminating resistance is $R_L = Z_{0m+1}$ (also from (4.9)).

Proof: The necessity is obvious from simple analysis. We shall prove that condition 2c implies condition 2 of the Main Theorem and the set of formulae (4.4), (4.5) and (4.6) leads to the desired realization of Fig. 4.3a.

With $Z_{00} = 1$, $g_0 = 1$ and $G_0 = 1$, it can be shown from (4.2) and (4.8) that the set of parameters α_i, β_i, γ_i and δ_i, $i = 1,2,\ldots,m+1$, assume one of the following four cases depending on whether g_i and the product polynomial $\prod_{k=1}^{i-1} g_k$ are even or odd:

(i) Both g_i and $\prod_{k=1}^{i-1} g_k$ are even,

$$\begin{aligned} \alpha_i &= Z_{0i} g_i \\ \beta_i &= G_i \\ \gamma_i &= 0 \\ \delta_i &= Z_{0i-1} g_i \quad . \end{aligned} \tag{4.10a}$$

(ii) g_i is even and $\prod_{k=1}^{i-1} g_k$ is odd,

$$\begin{aligned} \alpha_i &= 0 \\ \beta_i &= Z_{0i-1} g_i \\ \gamma_i &= Z_{0i} g_i \\ \delta_i &= G_i \quad . \end{aligned} \tag{4.10b}$$

(iii) g_i is odd and $\prod_{k=1}^{i-1} g_k$ is even,

$$\begin{aligned} \alpha_i &= G_i \\ \beta_i &= Z_{0i} g_i \\ \gamma_i &= Z_{0i-1} g_i \\ \delta_i &= 0 \ . \end{aligned} \tag{4.10c}$$

(iv) Both g_i and $\prod_{k=1}^{i-1} g_k$ are odd,

$$\begin{aligned} \alpha_i &= Z_{0i-1} g_i \\ \beta_i &= 0 \\ \gamma_i &= G_i \\ \delta_i &= Z_{0i} g_i \ . \end{aligned} \tag{4.10d}$$

Therefore, we always have

$$\alpha_i+\beta_i+\gamma_i+\delta_i = G_i+(Z_{0i}+Z_{0i-1})g_i \tag{4.11}$$

and

$$\alpha_i\delta_i - \beta_i\gamma_i = \varepsilon_i f_i f_{i*} \tag{4.12}$$

for $i = 1,2,\ldots,m+1$, where $f_i = \sqrt{Z_{0i}Z_{0i-1}}\, g_i$, $\varepsilon_i = 1$ if g_i is even and $\varepsilon_i = -1$ if g_i is odd.

Consequently, from (4.12), it is obvious that condition 2b is satisfied. Also, from (4.11) and the definitions of G_i, g_i and Z_{0i} given in the corollary, condition 2a is satisfied. Hence, according to the Main Theorem, $Z[p,(\mu)]$ is realizable in the general structure of Fig. 4.2. Next, we show that the set of formulae (4.4), (4.5) and (4.6) leads to the particular structure of Fig. 4.3a.

From (4.4), (4.6) and (4.10), it can be shown that

$$[T]_i = \frac{1}{\sqrt{Z_{0i}Z_{0i-1}}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\nu_{i-1}} \begin{bmatrix} Z_{0i} & \frac{G_i}{g_i} \\ 0 & Z_{0i-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\nu_i} \tag{4.13}$$

for $i = 1,2,\ldots,m$, and

$$Z_L = \begin{cases} \dfrac{g_{m+1}R_L + G_{m+1}}{Z_{0m}\, g_{m+1}} & \text{if } \nu_m = 0 \\[2ex] \dfrac{Z_{0m}\, g_{m+1}}{g_{m+1}R_L + G_{m+1}} & \text{if } \nu_m = 1 \quad , \end{cases} \tag{4.14}$$

where the parameter ν_k is introduced for convenience and is defined as follows

$$\nu_k = \begin{cases} 1 & \text{if } \prod_{\ell=0}^{k} g_\ell \text{ is odd.} \\[2ex] 0 & \text{if } \prod_{\ell=0}^{k} g_\ell \text{ is even.} \end{cases} \tag{4.15}$$

It is noted that if the parameters ν_{i-1} and ν_i in (4.13) are not equal to 0, the factor

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

indicates the presence of gyrators in the i-th two-port. However, we shall show that such gyrators can always be eliminated. To do this, let us consider two adjacent two-ports $[T]_i$ and $[T]_{i+1}$

$$\cdots \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\nu_{i-1}} \begin{bmatrix} Z_{0i} & \frac{G_i}{g_i} \\ 0 & Z_{0i-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\nu_i} \begin{bmatrix} 1 & \mu_i \\ \mu_i & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\nu_i} \begin{bmatrix} Z_{0i+1} & \frac{G_{i+1}}{g_{i+1}} \\ 0 & Z_{0i} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\nu_{i+1}} \cdots$$

It is observed that the factor

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\nu_i}$$

of the i-th two-port can always be passed over the μ_i-unit-element section and cancels out the same factor of the (i+1)-th two-port. By a similar argument, all the gyrators can be combined and thus be eliminated. Hence, the chain matrix representation of the overall network can be shown to be

$$\frac{1}{\prod_{i=1}^{m} Z_{0i}\sqrt{1-\mu_i^2}} \begin{bmatrix} Z_{01} & \frac{G_1}{g_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu_1 \\ \mu_1 & 1 \end{bmatrix} \begin{bmatrix} Z_{02} & \frac{G_2}{g_2} \\ 0 & Z_{01} \end{bmatrix} \cdots \begin{bmatrix} Z_{0m} & \frac{G_m}{g_m} \\ 0 & Z_{0m-1} \end{bmatrix} \begin{bmatrix} 1 & \mu_m \\ \mu_m & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{G_{m+1}}{g_{m+1}} \\ 0 & Z_{0m} \end{bmatrix} \begin{bmatrix} R_L \\ 1 \end{bmatrix} .$$

Upon denormalization of the characteristic impedances of the unit elements, we have

$$\frac{1}{\prod_{i=1}^{m} \sqrt{1-\mu_i^2}} \begin{bmatrix} 1 & \frac{G_1}{g_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_{01}\mu_1 \\ \frac{1}{Z_{01}}\mu_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{G_2}{g_2} \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & \frac{G_m}{g_m} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_{0m}\mu_m \\ \frac{1}{Z_{0m}}\mu_m & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{G_{m+1}}{g_{m+1}} \\ 0 & Z_m \end{bmatrix} \begin{bmatrix} R_L \\ 1 \end{bmatrix} . \tag{4.16}$$

The desired realization of Fig. 4.3a is apparent from (4.16) above.

By a dual consideration, we have the following corollary for the structure of Fig. 4.3b.

Corollary 1b

The multivariable rational function $Y[p,(\mu)]$ bilinear in the set of variables $(\mu) = (\mu_1,\mu_2,\ldots,\mu_m)$ may be realized as a driving-point admittance of the structure shown in Fig. 4.3b iff $Y[p,(\mu)]$ satisfies condition 1 of the Main Theorem (with Z replaced by Y) plus the following condition:

(2d) The function n_i/d_i defined in the Main Theorem (with Z replaced by Y) can be expressed as

$$\frac{n_i}{d_i} = \sum_{k=1}^{i} \frac{G_k}{g_k} + Y_{0i} \tag{4.17}$$

for $i = 1,2,\ldots,m+1$, where

$$Y_{0i} = \lim_{p\to\infty} \frac{n_i}{d_i} > 0 \tag{4.18}$$

and G_k and g_k assume one of the following cases:

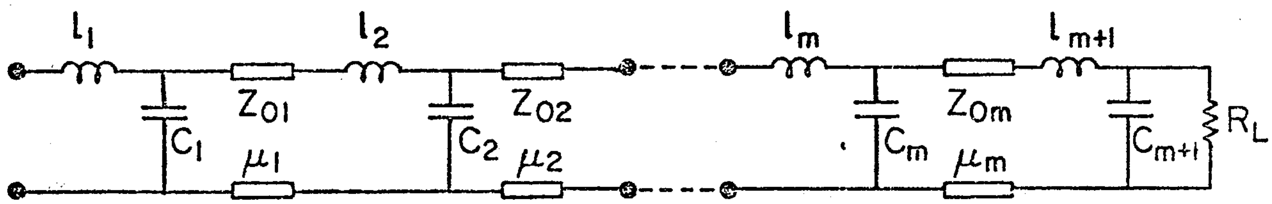
Case 1: $g_k = 1$ and $G_k = c_k p$ $(c_k \geq 0)$. This corresponds to the case that the k-th series-LC connection degenerates to a single capacitor shunt arm with capacitance c_k.

Case 2: $g_k = p$ and $G_k = 1/\ell_k \geq 0$. This corresponds to the case that the k-th series-LC section degenerates to a single inductor shunt arm with inductance ℓ_k.

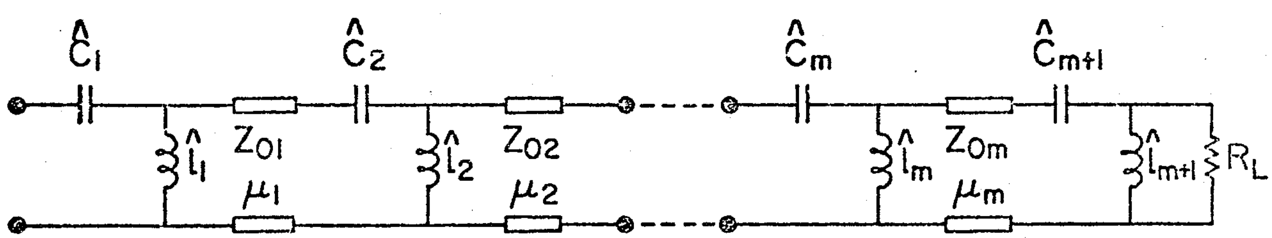
Case 3: $g_k = p^2+\omega_k^2$ $(\omega_k^2 > 0)$ and $G_k = g_k p$ $(c_k > 0)$. Both elements of the k-th series-LC connection exists with a capacitance c_k and inductance $\ell_k = 1/\omega_k^2 c_k$.

Corollary 2a

The multivariable rational function $Z[p,(\mu)]$ bilinear in the set of variables $(\mu) = (\mu_1,\mu_2,\ldots,\mu_m)$ may be realized as an input impedance of the resistively terminated cascade of m noncommensurate transmission lines separated by low-pass LC sections as shown in Fig. 4.4a iff $Z[p,(\mu)]$ satisfies condition 1 of the Main Theorem



(a)



(b)

Fig. 4.4 A resistively-terminated cascade of m noncommensurate transmission lines separated by (a) low-pass LC sections, (b) high-pass LC sections.

plus the following condition:

(2e) The function n_i/d_i defined in the Main Theorem can be expressed in the following cumulant[1] forms

$$\frac{n_i}{d_i} = \frac{\sum_{k=0}^{2i} a_{ik}p^k}{\sum_{k=0}^{2i-1} b_{ik}p^k} = \frac{(\ell_1 p, c_1 p \text{ ——— } \ell_i p, c_i p + Y_{0i})}{(c_1 p, \ell_2 p \text{ ——— } \ell_i p, c_i p + Y_{0i})}, \tag{4.19}$$

where

$$Y_{0i} = \frac{b_{i0}}{a_{i0}} > 0 \tag{4.20a}$$

$$c_i = \left(\frac{b_{i1}}{a_{i0}} - \sum_{j=1}^{i-1} c_j \right) \geqslant 0 \tag{4.20b}$$

$$\ell_i = \left(\frac{a_{i1}}{b_{i0}} - \sum_{j=1}^{i-1} \ell_j \right) \geqslant 0 \tag{4.20c}$$

for $i = 1,2,\ldots,m+1$.

---

1 Cumulant is a convenient mathematical tool in dealing with ladder networks, detail on the subject may be found in [8]. A cumulant denoted by $(a_1,a_2 \text{ ——— } a_{n-1},a_n)$ is defined as the determinant of the matrix

$$\begin{bmatrix} a_1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -1 & a_2 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & a_3 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & a_{n-2} & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & a_{n-1} & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 & a_n \end{bmatrix}$$

Simple rules for evaluating cumulants are given in [8].

The ℓ_i's and c_i's are the element values of the inductors and capacitors, respectively, $Z_{0i} = 1/Y_{0i}$ for $i = 1,2,\ldots,m$ are the characteristic impedances of the transmission lines and $R_L = 1/Y_{0m+1}$ is the terminating resistance.

Proof: The necessity is evident. We shall show that condition 2e implies condition 2 of the Main Theorem and the set of formulae (4.4), (4.5) and (4.6) leads to the particular realization of Fig. 4.4a.

From the even and odd parts of n_i and d_i, it can be easily verified taht

$$n_i + n_{i*} = 2(\ell_1 p, c_1 p \text{ —— } \ell_i p, c_i p) \tag{4.21a}$$

$$n_i - n_{i*} = 2Y_{0i}(\ell_1 p, c_1 p \text{ —— } c_{i-1} p, \ell_i p) \tag{4.21b}$$

$$d_i + d_{i*} = \begin{cases} 2Y_{01} & \text{for } i=1 \\ 2Y_{0i}(c_1 p, \ell_2 p \text{ —— } c_{i-1} p, \ell_i p) & \text{otherwise} \end{cases} \tag{4.21c}$$

$$d_i - d_{i*} = 2(c_1 p, \ell_2 p \text{ —— } \ell_i p, c_i p) \tag{4.21d}$$

for $i = 1,2,\ldots,m+1$.

Using the following cumulant identity [8]

$$(a_1 \text{ —— } a_{n-2}, a_{n-1}, a_n) = (a_1 \text{ —— } a_{n-2}, a_{n-1}) a_n + (a_1 \text{ —— } a_{n-2}) ,$$

it can be shown from (4.2) and (4.21) that

$$A_i = Y_{0i-1}(\ell_i c_i p^2 + 1)\Phi_i$$

$$B_i = Y_{0i-1} Y_{0i} \ell_i p \Phi_i$$

$$C_i = c_i p \Phi_i$$

$$D_i = Y_{0i} \Phi_i$$

for $i = 1,2,\ldots,m+1$, where $Y_{00} = 1$ and

$$\Phi_i = 2\{(c_1p - \ell_{i-1}p)(\ell_1p - c_{i-1}p) - (\ell_1p - \ell_{i-1}p)(c_1p - c_{i-1}p)\} .$$

Therefore,

$$\begin{aligned} \alpha_i &= Y_{0i-1}(\ell_i c_i p^2+1) \\ \beta_i &= Y_{0i-1}Y_{0i}\ell_i p \\ \gamma_i &= c_i p \\ \delta_i &= Y_{0i} ; \end{aligned} \tag{4.22}$$

and we have

$$\alpha_i+\beta_i+\gamma_i+\delta_i = Y_{0i-1}\ell_i c_i p^2 + (Y_{0i-1}Y_{0i}\ell_i+c_i)p + Y_{0i-1}+Y_{0i} \tag{4.23}$$

and

$$\alpha_i\delta_i - \beta_i\gamma_i = f_i f_{i*} , \tag{4.24}$$

where $f_i = \sqrt{Y_{0i}Y_{0i-1}}$.

By hypothesis, $Y_{0i} > 0$, $\ell_i \geq 0$ and $c_i \geq 0$, therefore, it is apparent from (4.23) that condition 2a is satisfied. Also, condition 2b is satisfied as evident from (4.24). Hence, according to the Main Theorem, $Z[p,(\mu)]$ is realizable in the general form of Fig. 4.2. Next, we prove that the set of formulae (4.4), (4.5) and (4.6) leads to the particular structure of Fig. 4.4a.

From (4.4), (4.6) and (4.22), we have

$$[T]_i = \frac{1}{\sqrt{Y_{0i}Y_{0i-1}}} \begin{bmatrix} Y_{0i-1}(\ell_i c_i p^2+1) & Y_{0i-1}Y_{0i}\ell_i p \\ c_i p & Y_{0i} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{Z_{0i-1}}} & 0 \\ 0 & \sqrt{Z_{0i-1}} \end{bmatrix} \begin{bmatrix} \ell_i c_i p^2+1 & \ell_i p \\ c_i p & 1 \end{bmatrix} \begin{bmatrix} \sqrt{Z_{0i}} & 0 \\ 0 & \frac{1}{\sqrt{Z_{0i}}} \end{bmatrix} \tag{4.25}$$

for $i = 1,2,\ldots,m,$ and

$$Z_L = Y_{0m} \frac{(\ell_{m+1} c_{m+1} p^2+1) + R_L \ell_{m+1} p}{c_{m+1} p + R_L} \ . \tag{4.26}$$

From (4.25) and (4.26), it can be shown that the chain matrix representation of the overall network is given by

$$\frac{1}{\prod_{i=1}^{m} \sqrt{1-\mu_i^2}} \begin{bmatrix} \ell_1 c_1 p^2+1 & \ell_1 p \\ c_1 p & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_{01}\mu_1 \\ \frac{1}{Z_{01}}\mu_1 & 1 \end{bmatrix} \begin{bmatrix} \ell_2 c_2 p^2+1 & \ell_2 p \\ c_2 p & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_{02}\mu_2 \\ \frac{1}{Z_{02}}\mu_1 & 1 \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} \ell_m c_m p^2+1 & \ell_m p \\ c_m p & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_{0m}\mu_m \\ \frac{1}{Z_{0m}}\mu_m & 1 \end{bmatrix} \begin{bmatrix} \ell_{m+1} c_{m+1} p^2+1 & \ell_{m+1} p \\ c_{m+1} p & 1 \end{bmatrix} \begin{bmatrix} R_L \\ 1 \end{bmatrix} . \tag{4.27}$$

Therefore, the desired realization of Fig. 4.4a is apparent from (4.27) above.

On replacing p by $1/\hat{p}$, ℓ_i by $1/\hat{c}_i$ and c_i by $1/\hat{\ell}_i$ in Corollary 2a, we have the following corollary for the structure of Fig. 4.4b.

Corollary 2b

The multivariable rational function $\hat{Z}[\hat{p},(\mu)]$ bilinear in the set of variables $(\mu) = (\mu_1,\mu_2,\ldots,\mu_m)$ may be realized as a driving point impedance of the structure shown in Fig. 4.4b if and only if the function $Z[p,(\mu)]$ obtained by

$$Z[p,(\mu)] = \hat{Z}[\hat{p},(\mu)]\Big|_{\hat{p}=\frac{1}{p}}$$

satisfies Corollary 2a with ℓ_i and c_i being replaced by $1/\hat{c}_i$ and $1/\hat{\ell}_i$, respectively.

Corollary 3a

The multivariable rational function $Z[p,(\mu)]$ bilinear in the set of variables $(\mu) = (\mu_1,\mu_2,\ldots,\mu_m)$ may be realized as a driving point impedance of the resistively terminated cascade of m noncommensurate transmission lines separated by series-arm series-LC sections as shown in Fig. 4.5a if and only if $Z[p,(\mu)]$ satisfies condition 1 of the Main Theorem plus the following condition:

(2f) The function n_i/d_i defined in the Main Theorem can be expressed as

$$\frac{n_i}{d_i} = L_i p + \frac{S_i}{p} + Z_{0i} \ , \tag{4.28}$$

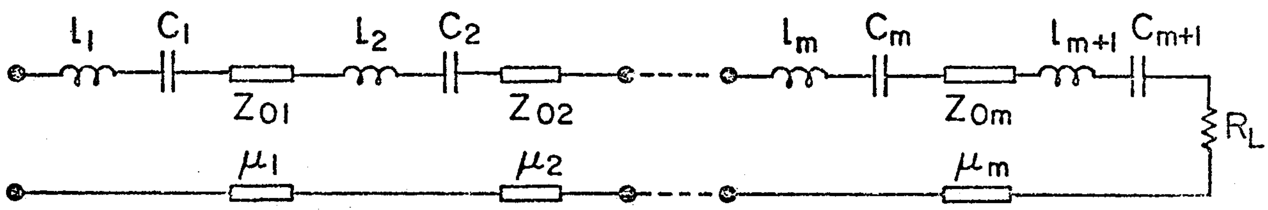
where

$$Z_{0i} > 0 \ ,$$

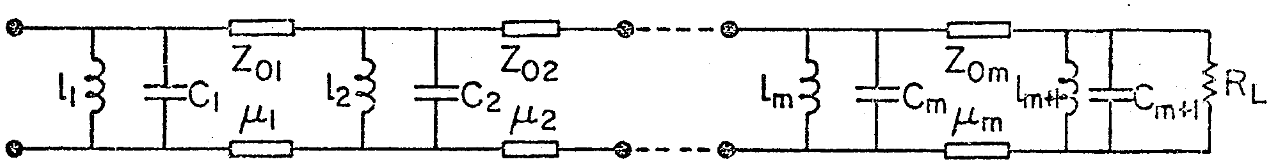
$$L_i \geq L_{i-1} \geq 0 \quad \text{with } L_0 = 0 \ ,$$

$$S_i \geq S_{i-1} \geq 0 \quad \text{with } S_0 = 0 \ ,$$

for $i = 1,2,\ldots,m+1$.



(a)



(b)

Fig. 4.5 A resistively-terminated cascade of m noncommensurate transmission lines separated by (a) series-arm series-LC sections, (b) shunt-arm parallel-LC sections.

The characteristic impedances, the terminating resistance, the inductances ℓ_i's and the capacitances c_i's are given by

$$Z_{0i} \quad \text{for} \quad i = 1,2,\ldots,m,$$

$$R_L = Z_{0m+1} \quad ,$$

$$\ell_i = L_i - L_{i-1} \quad \text{and}$$

$$1/c_i = S_i - S_{i-1} \quad ,$$

respectively.

Corollary 3b

The multivariable rational function $Y[p,(\mu)]$ bilinear in the set of variables $(\mu) = (\mu_1,\mu_2,\ldots,\mu_m)$ may be realized as an input admittance of the structure of Fig. 4.5b if and only if $Y[p,(\mu)]$ satisfies condition 1 of the Main Theorem (with Z replaced by Y) plus the following condition:

(2g) The function n_i/d_i defined in the Main Theorem (with Z replaced by Y) can be expressed as

$$\frac{n_i}{d_i} = C_i p + \frac{\Gamma_i}{p} + Y_{0i} \tag{4.30}$$

where

$$Y_{0i} > 0 \ ,$$

$$C_i \geq C_{i-1} \geq 0 \quad \text{with} \quad C_0 = 0 \ ,$$

$$\Gamma_i \geq \Gamma_{i-1} \geq 0 \quad \text{with} \quad \Gamma_0 = 0 \ ,$$

for $i = 1,2,\ldots,m+1$.

The characteristic impedances, the terminating resistance, the inductances ℓ_i's and the capacitances c_i's are given by

$$Z_{0i} = 1/Y_{0i} \quad i = 1,2,\ldots,m,$$

$$R_L = 1/Y_{0m+1} ,$$

$$1/\ell_i = \Gamma_i - \Gamma_{i-1} \quad \text{and}$$

$$c_i = C_i - C_{i-1} ,$$

respectively.

The proofs of these two corollaries may be shown in a similar manner as the previous ones and are thus omitted for brevity.

4.2 EXAMPLES

Example 4.1: Consider the following multivariable function

$$Z[p,(\mu)] = \frac{N[p,(\mu)]}{D[p,(\mu)]} ,$$

where

$$\begin{aligned} N[p,(\mu)] = {} & (11p^3+11p^2+13p+4)\mu_1\mu_2\mu_3 + (6p^4+6p^3+21p^2+15p+12)\mu_1\mu_2 \\ & + (9p^4+9p^3+29p^2+20p+18)\mu_1\mu_3 + (20p^2+9p+4)\mu_2\mu_3 \\ & + (9p^3+9p^2+12p+6)\mu_1 + (12p^3+6p^2+27p+12)\mu_2 \\ & + (18p^3+9p^2+38p+18)\mu_3 + (15p^2+6p+6) \end{aligned}$$

and

$$\begin{aligned} D[p,(\mu)] = {} & (18p^4+18p^3+29p^2+11p+9)\mu_1\mu_2\mu_3 + (12p^3+12p^2+9p+3)\mu_1\mu_2 \\ & + (13p^3+13p^2+11p+2)\mu_1\mu_3 + (36p^3+18p^2+20p+9)\mu_2\mu_3 \\ & + (12p^4+12p^3+21p^2+9p+6)\mu_1 + (18p^2+6p+3)\mu_2 \\ & + (22p^2+9p+2)\mu_3 + (24p^3+12p^2+15p+6) . \end{aligned}$$

First, we shall determine the order by which the unit elements are extracted. Following similar lines described in [18], we compute

$$Z[p,(\mu)^{1,2,3}] = \frac{p^2+p+2}{2p^2+p+1} ,$$

$$Z[p,(\mu)^{1}_{2,3}] = \frac{3p^2+2p+2}{4p^3+4p^2+3p+2} ,$$

$$Z[p,(\mu)^{2}_{1,3}] = \frac{p^2+p+2}{2p^2+p+1}$$

and

$$Z[p,(\mu)^{3}_{1,2}] = \frac{3p^2+2p+6}{4p^2+3p+2} .$$

Since

$$Z[p,(\mu)^{1,2,3}] = Z[p,(\mu)^{2}_{1,3}] ,$$

the first unit element to be extracted corresponds to μ_2.

Next, we compute

$$Z[p,(\mu)^{1,3}_{2}] = \frac{3p^2+2p+6}{4p^2+3p+2} .$$

It is observed that

$$Z[p,(\mu)^{1,3}_{2}] = Z[p,(\mu)^{3}_{2,1}] .$$

Therefore, the second unit element to be extracted corresponds to μ_3. Hence, it follows that the order of extraction is $\{\mu_2,\mu_3,\mu_1\}$.

Knowing the extraction order, we are now in a position to verify the realizability conditions.

(i) Condition 1 of the Main Theorem is satisfied, since

$$Z[p,(\mu)^2] = -Z_*[p,(\mu)^2] = \frac{n_1}{d_1} = \frac{p^2+p+2}{2p^2+p+1}$$

$$Z[p,(\mu)_2^3] = -Z_*[p,(\mu)_2^3] = \frac{n_2}{d_2} = \frac{3p^2+2p+6}{4p^2+3p+2}$$

$$Z[p,(\mu)_{2,3}^1] = -Z_*[p,(\mu)_{2,3}^1] = \frac{n_3}{d_3} = \frac{3p^2+2p+2}{4p^3+4p^2+3p+2} \quad .$$

(ii) From the set of equations (4.2), we have

$$\left.\begin{aligned} \alpha_1 &= p^2+2 \\ \beta_1 &= p \\ \gamma_1 &= p \\ \delta_1 &= 2p^2+1 \end{aligned}\right\} \qquad \begin{aligned} \alpha_1+\beta_1+\gamma_1+\delta_1 &= 3p^2+2p+3 \\ \alpha_1\delta_1 - \beta_1\gamma_1 &= 2(p^2+1)^2 \end{aligned}$$

$$\left.\begin{aligned} \alpha_2 &= 3 \\ \beta_2 &= 0 \\ \gamma_2 &= 0 \\ \delta_2 &= 2 \end{aligned}\right\} \qquad \begin{aligned} \alpha_2+\beta_2+\gamma_2+\delta_2 &= 5 \\ \alpha_2\delta_2 - \beta_2\gamma_2 &= 6 \end{aligned}$$

$$\left.\begin{aligned} \alpha_3 &= 1 \\ \beta_3 &= 0 \\ \gamma_3 &= 3p \\ \delta_3 &= 3 \end{aligned}\right\} \qquad \begin{aligned} \alpha_3+\beta_3+\gamma_3+\delta_3 &= 3p+4 \\ \alpha_3\delta_3 - \beta_3\gamma_3 &= 3 \end{aligned}$$

$$\left.\begin{aligned} \alpha_4 &= 1 \\ \beta_4 &= 0 \\ \gamma_4 &= p \\ \delta_4 &= 1 \end{aligned}\right\} \qquad \begin{aligned} \alpha_4+\beta_4+\gamma_4+\delta_4 &= p+2 \\ \alpha_4\delta_4 - \beta_4\gamma_4 &= 1 \end{aligned} \quad .$$

Therefore, it is obvious that conditions 2a and 2b are satisfied. Hence, according to the Main Theorem $Z[p,(\mu)]$ is realizable as an input impedance of the structure of Fig. 4.2 with the extraction order $\{\mu_2,\mu_3,\mu_1\}$.

Using the set of formulae (4.4), (4.5) and (4.6), a complete realization is given in Fig. 4.6a. Upon eliminating of transformers, we have an alternative realization shown in Fig. 4.6b in which the characteristic impedances are denormalized.

Example 4.2: Realize

$$Z[p,(\mu)] = \frac{N[p,(\mu)]}{D[p,(\mu)]} ,$$

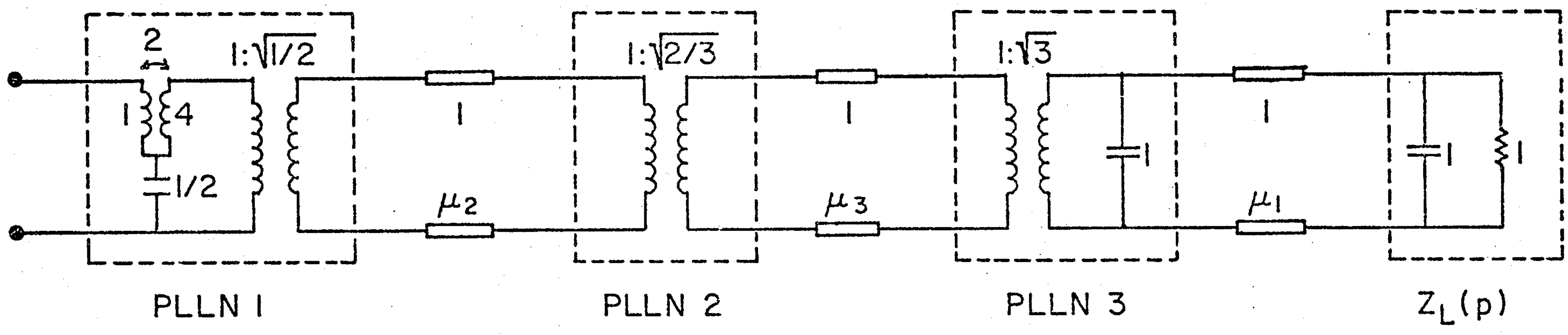
where

$$\begin{aligned} N[p,(\mu)] = {} & (p^5+4p^4+9p^3+24p^2+5p+8)\mu_1\mu_2\mu_3 + (5p^4+16p^3+8p^2+20p+2)\mu_1\mu_2 \\ & + (2p^5+2p^4+18p^3+12p^2+10p+4)\mu_1\mu_3 + (4p^4+12p^3+9p^2+12p+2)\mu_2\mu_3 \\ & + (10p^4+8p^3+16p^2+10p+4)\mu_1 + (p^5+4p^4+7p^3+12p^2+7p+8)\mu_2 \\ & + (8p^4+6p^3+18p^2+6p+4)\mu_3 + (2p^5+2p^4+14p^3+6p^2+14p+4) \end{aligned}$$

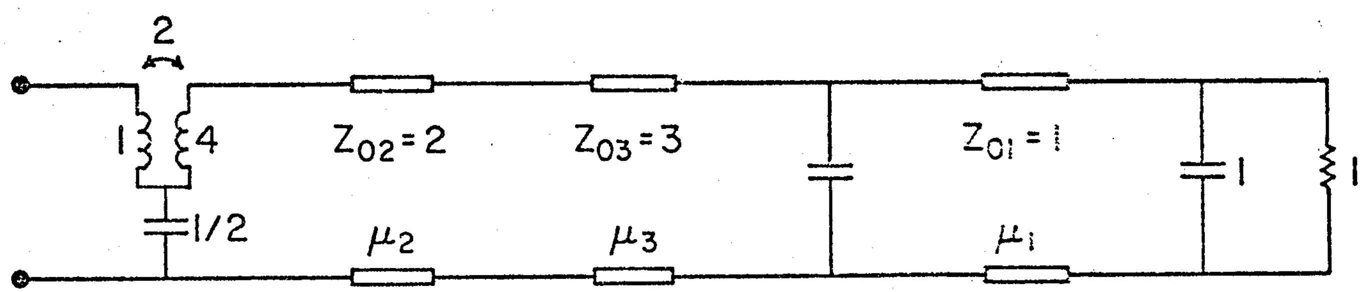
and

$$\begin{aligned} D[p,(\mu)] = {} & (2p^4+4p^3+5p^2+8p+2)\mu_1\mu_2\mu_3 + (p^5+4p^4+3p^3+12p^2+2p+8)\mu_1\mu_2 \\ & + (4p^4+2p^3+10p^2+4p+4)\mu_1\mu_3 + (p^5+4p^4+4p^3+12p^2+4p+8)\mu_2\mu_3 \\ & + (2p^5+2p^4+6p^3+6p^2+4p+4)\mu_1 + (p^4+3p^2+2)\mu_2 \\ & + (2p^5+2p^4+8p^3+6p^2+8p+4)\mu_3 + (2p^4+6p^2+4) . \end{aligned}$$

First, following the same steps as in Example 4.1, the extraction order is determined as $\{\mu_3,\mu_1,\mu_2\}$.



(a)



(b)

Fig. 4.6 (a) A realization of Example 4.1.
(b) An alternate realization of (a) with the characteristic impedances of the unit elements being denormalized.

Next, we see that condition 1 is satisfied, since

$$Z[p,(\mu)^3] = -Z_*[p,(\mu)^3] = \frac{n_1}{d_1} = \frac{p^2+3p+2}{p^2+2}$$

$$Z[p,(\mu)_3^1] = -Z_*[p,(\mu)_3^1] = \frac{n_2}{d_2} = \frac{p^4+4p^3+3p^2+5p+2}{(p^2+1)(p^2+2)}$$

$$Z[p,(\mu)_{3,1}^2] = -Z_*[p,(\mu)_{3,1}^2] = \frac{n_3}{d_3} = \frac{p^5+2p^4+7p^3+6p^2+7p+4}{(p^2+1)(p^2+2)} \quad .$$

Upon performing partial fraction expansions for the functions n_i/d_i for $i = 1,2,3,4$, we have

$$\frac{n_1}{d_1} = \frac{3p}{p^2+2} + 1 \tag{4.32a}$$

$$\frac{n_2}{d_2} = \frac{3p}{p^2+2} + \frac{p}{p^2+1} + 1 \tag{4.32b}$$

$$\frac{n_3}{d_3} = \frac{3p}{p^2+2} + \frac{p}{p^2+1} + p + 2 \tag{4.32c}$$

$$\frac{n_4}{d_4} = \frac{3p}{p^2+2} + \frac{p}{p^2+1} + p + 1 \quad . \tag{4.32d}$$

Therefore, condition 2c is satisfied. Hence, according to Corollary 1a, $Z[p,(\mu)]$ is realizable as an input impedance of the structure of Fig. 4.2a with the extraction order $\{\mu_3,\mu_1,\mu_2\}$. The element values can be easily identified from (4.32). A complete realization is given in Fig. 4.7.

Example 4.3: Realize

$$Z[p,(\mu)] = \frac{N[p,(\mu)]}{D[p,(\mu)]} \quad ,$$

where

$$N[p,(\mu)] = (20p^4+40p^3+13p^2+18p+1)\mu_1\mu_2\mu_3 + (20p^5+40p^4+27p^3+46p^2+7p+8)\mu_1\mu_2$$
$$+(40p^4+20p^3+26p^2+9p+2)\mu_1\mu_3 + (8p^5+16p^4+20p^3+24p^2+6p+4)\mu_2\mu_3$$
$$+(40p^5+20p^4+54p^3+23p^2+14p+4)\mu_1 + (8p^6+16p^5+26p^4+36p^3+18p^2+16p+2)\mu_2$$
$$+(16p^5+8p^4+40p^3+12p^2+12p+2)\mu_3 + (16p^6+8p^5+52p^4+18p^3+36p^2+8p+4)$$

and

$$D[p,(\mu)] = (20p^3+40p^2+8p+8)\mu_1\mu_2\mu_3 + (20p^4+40p^3+17p^2+26p+1)\mu_1\mu_2$$
$$+(40p^3+20p^2+16p+4)\mu_1\mu_3 + (8p^4+16p^3+18p^2+20p+2)\mu_2\mu_3$$
$$+(40p^4+20p^3+34p^2+13p+2)\mu_1 + (8p^5+16p^4+22p^3+28p^2+8p+4)\mu_2$$
$$+(16p^4+8p^3+36p^2+10p+4)\mu_3 + (16p^5+8p^4+44p^3+14p^2+16p+2) \; .$$

The extraction order of this example can be shown to be $\{\mu_3,\mu_1,\mu_2\}$. Condition 1 of the Main Theorem is satisfied, since

$$Z[p,(\mu)^3] = -Z_*[p,(\mu)^3] = \frac{n_1}{d_1} = \frac{2p^2+p+1}{2p+1} \tag{4.33a}$$

$$Z[p,(\mu)_3^1] = -Z_*[p,(\mu)_3^1] = \frac{n_2}{d_2} = \frac{4p^4+2p^3+5p^2+1.5p+1}{4p^3+2p^2+3p+.5} \tag{4.33b}$$

$$Z[p,(\mu)_{3,1}^2] = -Z_*[p,(\mu)_{3,1}^2] = \frac{n_3}{d_3} = \frac{4p^6+4p^5+13p^4+9p^3+9p^2+4p+1}{4p^5+4p^4+11p^3+7p^2+4p+1} \; . \tag{4.33c}$$

Now, let us assume that the given function is realizable in the form of Fig. 4.4a, then the element values can be easily identified from (4.20), (4.33) and the following function

$$\frac{n_4}{d_4} = Z[p,(\mu)_{1,2,3}] = \frac{4p^6+2p^5+13p^4+4.5p^3+9p^2+2p+1}{4p^5+2p^4+11p^3+3.5p^2+4p+.5} \quad . \tag{4.34}$$

We have

$$Y_{01} = 1, \quad Y_{02} = .5, \quad Y_{03} = 1, \quad R_L = 1/Y_{04} = 2,$$

$$\ell_1 = 1, \quad \ell_2 = 2, \quad \ell_3 = 1, \quad \ell_4 = 0,$$

$$c_1 = 2, \quad c_2 = 1, \quad c_3 = 1, \quad c_4 = 0.$$

To show that $Z[p,(\mu)]$ is indeed realizable in the form of Fig. 4.4a, we need to verify that condition 2e of Corollary 2a is satisfied. Evaluating the cumulants defined on the right hand side of (4.19) and comparing with (4.33) and (4.34), we have

$$(\ell_1 p, c_1 p+Y_{01}) = (p,2p+1) = 2p^2+p+1 = n_1$$

$$(c_1 p+Y_{01}) = (2p+1) = 2p+1 = d_1$$

$$(\ell_1 p, c_1 p, \ell_2 p, c_2 p+Y_{02}) = (p,2p,2p,p+.5) = 4p^4+2p^3+5p^2+1.5p+1 = n_2$$

$$(c_1 p, \ell_2 p, c_2 p+Y_{02}) = (2p,2p,p+.5) = 4p^3+2p^2+3p+.5 = d_2$$

$$(\ell_1 p, c_1 p, \ell_2 p, c_2 p, \ell_3 p, c_3 p+Y_{03}) = (p,2p,2p,p,p,p+1)$$
$$= 4p^6+4p^5+13p^4+9p^3+9p^2+4p+1 = n_3$$

$$(c_1 p, \ell_2 p, c_2 p, \ell_3 p, c_3 p+Y_{03}) = (2p,2p,p,p,p+1)$$
$$= 4p^5+4p^4+11p^3+7p^2+4p+1 = d_3$$

$$(\ell_1 p, c_1 p, \ell_2 p, c_2 p, \ell_3 p, c_3 p, \ell_4 p, c_4 p+Y_{04}) = (p,2p,2p,p,p,p,0,0+.5)$$
$$= 4p^6+2p^5+13p^4+4.5p^3+9p^2+2p+1 = n_4$$

$$(c_1 p, \ell_2 p, c_2 p, \ell_3 p, c_3 p, \ell_4 p, c_4 p+Y_{04}) = (2p,2p,p,p,p,0,0+.5)$$
$$= 4p^5+2p^4+11p^3+3.5p^2+4p+.5 = d_4 \; .$$

Therefore, Corollary 2a is satisfied. A complete realization is given in Fig. 4.8.

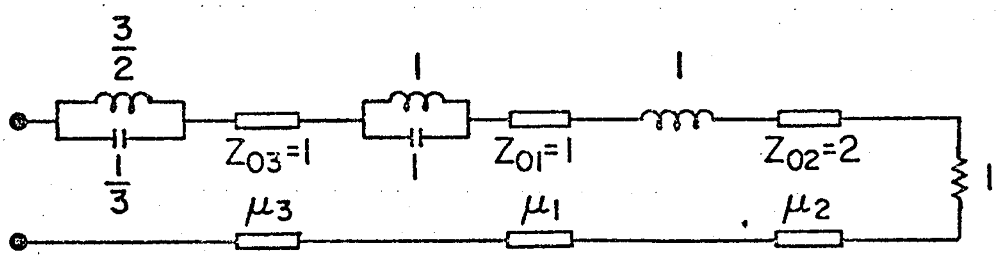


Fig. 4.7 Example 4.2.

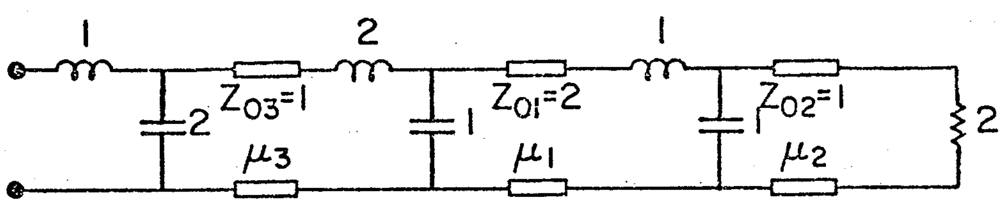


Fig. 4.8 Example 4.3.

CHAPTER V

CONCLUSIONS

The realizations of multivariable network functions in the forms of certain simple structures with constituent building blocks involving functions of reduced complexity have been investigated. Three different configurations have been considered:

(1) A sum connection of immittances which are functions of mutually disjoint sets of variables.

(2) A cascade connection of single-variable passive lumped networks, the cascaded subnetworks are also assumed lossless except the last termination.

(3) An extended Bott-Duffin type structure.

The realizability conditions have been formulated in terms of the decomposability of the given function in certain special forms. Apart from the general formulations based on the multivariable positive reality condition, more direct and explicit alternative approaches have also been discussed. The improvements and generalizations of the presented results over the existing ones have been illustrated by examples.

The synthesis of independent zeros of the even part of a multivariable positive real function has been studied. Besides the discussion of the usual cascade extraction by the basic sections, viz., the Richards', Brune, type C, type E and type D sections, removal methods without resorting to gyrators and transformers have been presented. The developments of the latter were primarily based on Miyata's separation concept of the even part function in single variable synthesis theory. In particular, Seshu's results on the minimal realization of biquadratic single variable minimum functions have also been generalized to multivariable functions as a means to realize imaginary-axis independent even part zeros without using transformers.

The problem of the synthesis of a class of networks made up of m cascaded noncommensurate transmission lines separated by passive lumped lossless two-ports and closed on a passive lumped network has been considered. A new set of realizability conditions has been presented. The advantage of the proposed set of conditions is that it requires no multivariable p.r. test and is simple to apply. Explicit formulae for the chain matrices of the passive lumped lossless two-ports and the terminating impedance have been derived. Several interesting special cases have also been considered and the realizability conditions have accordingly been modified in such a way that the synthesis is carried out almost by simple inspection.

APPENDIX

In a recent article [2], the synthesis of a two-variable driving point impedance $Z_1(s,p)$ in the form of Fig. A1 is considered.

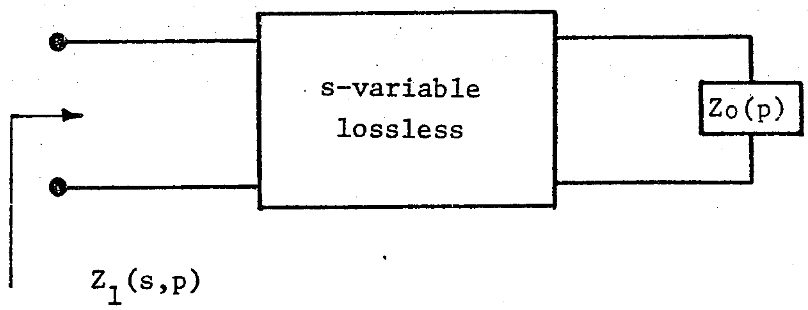


Fig. A1 The basic network configuration considered in [2].

It was shown that the necessary and sufficient condition for $Z_1(s,p)$, a two-variable p.r. function of two complex variable s and p, to be a driving point impedance realizable by an s-variable lossless two-port network with a p-variable driving point impedance $Z_0(p)$ termination are that $Z_1(s,p)$ can be written as

$$Z_1(s,p) = \frac{m_1(s)f(p) + n_1(s)g(p)}{m_2(s)g(p) + n_2(s)f(p)}$$

where m_1 and m_2 are even and n_1 and n_2 are odd polynomials in s with the following conditions:

(i) $F(s) = [m_1(s)+n_1(s)]/[m_2(s)+n_2(s)]$ is p.r..

(ii) $G(s) = m_1(s)m_2(s)-n_1(s)n_2(s)$ is a perfect (or negative of a perfect) square.

(iii) The rational function $f(p)/g(p)$ is p.r..

In the following, we observe that

(1) The 2-variable positive reality requirement is superfluous. Nevertheless, if it is indeed the author's intention to regard it as a prerequisite condition on the given function, then conditions (i) and (ii) of the theorem are redundant.

(2) The theorem has not been formulated in a proper manner and consequently leads to faulty result under certain circumstances. For example, consider the following function

$$Z_1(s,p) = \frac{(s^2+1)p - 2s}{-2sp + (s^2+1)} \quad .$$

Here, we see that

(i) $\dfrac{m_1 + n_1}{m_2 + n_2} = \dfrac{s^2+1 - 2s}{s^2+1 - 2s} = 1$ is obviously p.r..

(ii) $m_1 m_2 - n_1 n_2 = (s^2+1)^2 - 4s^2 = (s^2-1)^2$ is clearly a perfect square.

(iii) $\dfrac{f(p)}{g(p)} = p$ is evidently p.r..

Although all the three conditions of the theorem are satisfied, $Z_1(s,p)$ is not a realizable function.

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