

Effects of System Rotation on Turbulent Thermal Flow in a Circular Pipe

by

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Abstract

This thesis reports a set of numerical studies designed to advance the understanding of turbulent flow and heat transfer in a circular pipe under either radial or axial system rotation. To achieve this, direct numerical simulation (DNS) is used, enabling full resolution of the spatial and temporal scales inherent to the turbulent flow. The first DNS focuses on the evolution of vortical structures and enstrophy in a turbulent thermal pipe flow under radial system rotation, and the findings are systematically compared with those from the non-rotating case. The dynamics of vortical structures are examined through analyses of the instantaneous axial vorticity field, swirling strength, vorticity fluctuation intensities and fluctuating enstrophy, vorticity spectra, and the transport of fluctuating enstrophy in both physical and spectral spaces. It is observed that, in response to radial system rotation, large-scale secondary vortices develop near the suction side of the pipe. At higher rotation numbers, the radial rotation strongly suppresses the turbulent production of fluctuating enstrophy by weakening the vortex stretching mechanisms. The influence of the large-scale secondary flow structures on the heat convection is analysed based on statistical moments of the temperature field and budget balance of the transport equation of turbulent heat fluxes.

The second DNS investigates the effects of axial system rotation on turbulent thermal pipe flows. To ensure that energetic turbulent eddies are properly captured at high rotation numbers, very long pipe lengths significantly exceeding those used in previous studies are employed. Two distinct types of energy-containing flow structures are observed. The first type is streaks (or hairpin structures), which are characteristic of the turbulent boundary layer developing over the pipe wall for both non-rotating and axially rotating flows. The second type is Taylor columns, which emerge at moderate and high rotation numbers. Because the DNS spans a wide range of rotation numbers and utilises extended pipe lengths, a large dataset of velocity fields is analysed to elucidate the influence of axial rotation on turbulent structures. This

analysis includes examinations of the mean velocity field, Reynolds stresses, two-point autocorrelations, velocity spectra, skewness and flatness factors, and joint probability density functions (JPDF). To further study the effects of axial system rotation on turbulent transport in a circular pipe, a detailed investigation of the Reynolds stress and turbulence kinetic energy (TKE) transport processes is conducted in both physical and spectral spaces. Finally, the turbulent heat transfer in a circular pipe under axial system rotation is investigated through the analyses of the mean temperature field, Nusselt number, temperature variance and turbulent heat flux.

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Nomenclature

English Symbols

A_ω	combined anisotropy invariant of b_{ij}^ω
b_{ij}^ω	vorticity anisotropy tensor
C_i	convection term of turbulent heat flux $\langle u'_i \theta' \rangle$
C_{ij}	convection term of Reynolds stresses
$\tilde{C}_{ij}$	convection term of velocity-spectrum tensor
C_p	specific heat of the fluid
C_{TKE}	convection term of TKE
$\tilde{C}_{TKE}$	convection term of TKE spectrum
$C_{\omega\omega}$	convection term of fluctuating enstrophy
$\tilde{C}_{\omega\omega}$	convection term of fluctuating enstrophy spectrum
D	diameter of the circular pipe
D_i^m	molecular diffusion-dissipation term of turbulent heat flux $\langle u'_i \theta' \rangle$
D_i^p	pressure diffusion-redistribution term of turbulent heat flux $\langle u'_i \theta' \rangle$
D_{ij}^p	pressure diffusion term of Reynolds stresses
$\tilde{D}_{ij}^p$	pressure diffusion term of velocity-spectrum tensor
D_{TKE}^p	pressure diffusion term of TKE
$\tilde{D}_{TKE}^p$	pressure diffusion term of TKE spectrum

D_i^t	turbulent diffusion term of turbulent heat flux $\langle u'_i \theta' \rangle$
D_{ij}^t	turbulent diffusion term of Reynolds stresses
$\tilde{D}_{ij}^t$	turbulent diffusion term of velocity-spectrum tensor
D_{TKE}^t	turbulent diffusion term of TKE
$\tilde{D}_{TKE}^t$	turbulent diffusion term of TKE spectrum
$D_{\omega\omega}^t$	turbulent diffusion term of fluctuating enstrophy
$\tilde{D}_{\omega\omega}^t$	turbulent diffusion term of fluctuating enstrophy spectrum
D_{ij}^v	viscous diffusion term of Reynolds stresses
$\tilde{D}_{ij}^v$	viscous diffusion term of velocity-spectrum tensor
D_{TKE}^v	viscous dissipation term of TKE
$\tilde{D}_{TKE}^v$	viscous dissipation term of TKE spectrum
$D_{\omega\omega}^v$	viscous diffusion term of fluctuating enstrophy
$\tilde{D}_{\omega\omega}^v$	viscous diffusion term of fluctuating enstrophy spectrum
$\hat{\mathbf{e}}_i$	base unit vector
$\tilde{E}_{ij}$	velocity-spectrum tensor (i.e. autospectrum or co-spectrum of velocity fluctuations)
$\tilde{E}_{TKE}$	TKE spectrum
$\tilde{E}_{u_i\theta}$	co-spectrum of velocity and temperature fluctuations
$\tilde{E}_{\theta\theta}$	spectrum of temperature fluctuations
$\tilde{E}_{\omega_i}$	spectra of vorticity fluctuations
$\tilde{E}_{\omega\omega}$	spectrum of fluctuating enstrophy
F_i	Coriolis force component
$F(u'_i)$	flatness factor
h	convective heat transfer coefficient
i, j and k	indices for a vector or tensor: 1, 2 or 3 (corresponding to r , β or z , respectively)
$\underline{i}$	imaginary unit: $\sqrt{-1}$
Π_ω	second-order invariant of b_{ij}^ω

III_ω	third-order invariant of b_{ij}^ω
k_m	volume-averaged TKE
k_z	axial wavenumber: $n_z k_{z0}$
k_{z0}	the lowest wavenumber of the axial direction: $2\pi/L_z$
k_θ	volume-averaged TSE
K	thermal conductivity: $\alpha\rho C_p$
L_z	axial pipe length
n_z	an integer in interval $[-N_z/2, N_z/2-1]$
Nu	Nusselt number: hD/K
N_e	number of finite elements
N_{tot}	total number of grid points
N_z	number of Fourier modes in the axial direction
R	radius of circular pipe
C_{R^2}	coefficients of determination
p	pressure
P_i	turbulent production term of turbulent heat flux $\langle u'_i \theta' \rangle$
P_{ij}	turbulent production term of Reynolds stresses
$\tilde{P}_{ij}$	turbulent production term of velocity-spectrum tensor
P_{TKE}	turbulent production term of TKE
$\tilde{P}_{TKE}$	turbulent production term of TKE spectrum
P_i^c	Coriolis production term of turbulent heat flux $\langle u'_i \theta' \rangle$
P_{ij}^c	Coriolis production term of Reynolds stresses
$\tilde{P}_{ij}^c$	Coriolis production term of velocity-spectrum tensor
$P_{\omega\omega}^c$	Coriolis production term of fluctuating enstrophy
$\tilde{P}_{\omega\omega}^c$	Coriolis production term of fluctuating enstrophy spectrum
$P_{\omega\omega}^g$	production term of fluctuating enstrophy by mean vorticity gradients
$\tilde{P}_{\omega\omega}^g$	production term of fluctuating enstrophy spectrum by mean

	vorticity gradients
$P_{\omega\omega}^m$	production term of fluctuating enstrophy by mean vortex stretching
$\tilde{P}_{\omega\omega}^m$	production term of fluctuating enstrophy spectrum by mean vortex stretching
$P_{\omega\omega}^s$	production term of fluctuating enstrophy by mean shear flow stretching
$\tilde{P}_{\omega\omega}^s$	production term of fluctuating enstrophy spectrum by mean shear flow stretching
P_i^t	turbulent production term of turbulent heat flux $\langle u'_i \theta' \rangle$ generated from the
$P_{\omega\omega}^t$	production term of fluctuating enstrophy by turbulent vortex stretching
$\tilde{P}_{\omega\omega}^t$	production term of fluctuating enstrophy spectrum by turbulent vortex stretching
	constant mean axial temperature gradient
Pr	Prandtl number: ν/α
Pr_t	turbulent Prandtl number: ν_t/α_t
$P(u'_z, -u'_r)$	JPDF of axial and radial velocity fluctuations
$P(\theta', u'_z)$	JPDF of temperature and axial velocity fluctuations
$P(\theta', -u'_r)$	JPDF of temperature and radial velocity fluctuations
q_r^{tot}	total heat flux
q_r^{vis}	viscous heat flux
q_w''	local axially-averaged wall heat flux
q_{wa}''	peripherally-averaged wall heat flux
$\dot{Q}_w$	total heat rate through the pipe wall
R	radius of circular pipe
$R_{\theta\theta}$	two-point autocorrelation coefficient for temperature

	fluctuations θ'
Re_b	Reynolds number based on bulk mean velocity: $U_b D/\nu$
Re_τ	Reynolds number based on wall friction velocity: $u_\tau R/\nu$
Re_τ^A	actual Reynolds number obtained by averaging local Reynolds numbers: $\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} Re_\tau^L d\beta$
Re_τ^L	local Reynolds number based on wall friction velocity at a specific azimuthal position
Ro_b	rotation number based on bulk mean velocity: $2\Omega R/U_b$
Ro_τ	rotation number based on wall friction velocity: $2\Omega R/u_\tau$
r, β and z	radial, azimuthal and axial coordinates, respectively
t	time
$S(u'_i)$	skewness factor
t_{avg}	averaging time
T	temperature
T_b	bulk mean temperature
$\tilde{T}_{ij}$	interscale transport term of velocity-spectrum tensor
$\tilde{T}_{TKE}$	interscale transport term of TKE spectrum
$\tilde{T}_{\omega\omega}$	interscale transport term of fluctuating enstrophy spectrum
T_w	local wall temperature
T_τ	nominal wall friction temperature determined by the peripherally averaged wall heat flux: $q''_{wa}/(\rho C_p u_\tau)$
U_b	bulk mean velocity
u_i	velocity component
u_τ	nominal wall friction velocity: $\sqrt{-\Pi R/2\rho}$
y	dimensionless radial coordinate measured from the wall: $1 - r/R$

Greek Letters

α	molecular thermal diffusivity
α_t	eddy thermal diffusivity
Δ	grid size
δ_{ij}	Kronecker delta
ε	dissipation rate of TKE
ε_{ij}	viscous dissipation term of Reynolds stresses
$\tilde{\varepsilon}_{ij}$	viscous dissipation term of velocity-spectrum tensor
ε_{TKE}	viscous dissipation term of TKE
$\tilde{\varepsilon}_{TKE}$	viscous dissipation term of TKE spectrum
$\varepsilon_{\omega\omega}$	viscous dissipation term of fluctuating enstrophy
$\tilde{\varepsilon}_{\omega\omega}$	viscous dissipation term of fluctuating enstrophy spectrum
η	Kolmogorov length scale: $(\nu^3/\varepsilon)^{1/4}$
η_B	Batchelor length scale: $\eta/\sqrt{Pr}$
θ	deficit temperature with reference to the local wall temperature: $\theta = T - T_w$
θ_b	bulk mean deficit temperature
κ	von Kármán constant
λ_{ci}	swirling strength
λ_z	axial wavelength: $2\pi/k_z$
ν	molecular kinematic viscosity
ν_t	eddy viscosity
Π	mean axial pressure gradient
Π_{ij}	pressure-strain term of Reynolds stresses
$\tilde{\Pi}_{ij}$	pressure-strain term of velocity-spectrum tensor
ρ	density
ρ_{ij}	Reynolds shear-stress correlation coefficients
τ_η	Kolmogorov time scale: $\sqrt{\nu/\varepsilon}$

ω_i	vorticity
Ω	angular speed of system rotation
∇	differential operator del or nabla operator

Subscripts

$(\cdot)_0$	quantity of the non-rotating pipe flow case at $Ro_\tau = 0$
$(\cdot)_i$	vector for $i = 1, 2$ or 3 (corresponding to r, β or z , respectively)
$(\cdot)_{ij}$	second-order tensor for $i, j = 1, 2$ or 3
$(\cdot)_{rms}$	root-mean-square

Superscripts

$\vec{(\cdot)}$	vector
$\langle \cdot \rangle$	averaging over time and over the homogeneous direction
$(\cdot)^+$	dimensionless wall coordinates
$\overline{(\cdot)}$	time-averaging
$\widehat{(\cdot)}$	Fourier transform
$(\cdot)^*$	conjugate of a complex number
$(\cdot)'$	fluctuating component
$\max(\cdot)$	maximum value
$\text{Re}\{\cdot\}$	real part of a complex number

Miscellaneous

1C	one-component
1-D	one-dimensional
2C	two-component
2C-AXI	two-component axisymmetric
2-D	two-dimensional
3-D	three-dimensional
AIM	anisotropy invariant map
CFL	Courant-Friedrichs-Lewy
DNS	direct numerical simulation
DOF	degree of freedom
FIK	Fukagata-Iwamoto-Kasagi
GB	gigabyte
GLLL	Gauss-Lobatto-Legendre-Lagrange
ISO	isotropic
JPDF	joint probability density function
LDA	laser Doppler anemometry
LDV	laser Doppler velocimetry
LES	large-eddy simulation
LETOT	large-eddy turnover time: R/u_τ
LSE	linear stochastic estimation
MPI	message passing interface
PCG	preconditioned-conjugate-gradient
RANS	Reynolds-averaged Navier-Stokes
RHS	right-hand-side
RMS	root-mean-square
SEM	spectral-element method
TB	Terabyte

TKE	turbulence kinetic energy
TSE	turbulence scalar energy
TP	Taylor-Proudman

Chapter 1

Introduction

1.1 Background and motivation

Turbulent flows confined within a circular pipe represent an important subject with engineering application in rotary machinery, heat exchangers and centrifugal separators. The flow and thermal characteristics that develop over curved pipe walls have been extensively studied through both experimental and numerical approaches. Nevertheless, in many industrial applications, the corresponding turbulent flow in circular pipes are managed by some flow-control techniques, such as system rotation. For instance, system rotation affects the performance of centrifugal separators through separation efficiency and pressure loss, alters the heat transfer rates and mitigates hot spots that cause thermal fatigue of gas turbine blades. Although the circular pipe flow under the influences of system rotation is generally encountered in engineering applications, associated research is still lacking in the academic sector. Therefore, the goal of this thesis is to conduct a systematic investigation of turbulent circular pipe flows subjected to the radial system rotation and axial system rotation, respectively, in order to understand the complex turbulent flow and thermal characteristics of the circular pipe under this form of flow control.

1.2 Literature review

1.2.1 DNS study of vortical structures and fluctuating enstrophy in a turbulent pipe flow under radial rotation

In comparison with a conventional stationary circular pipe flow, the appearance of Coriolis force in a radially rotating pipe flow changes the characteristics of the turbulence field fundamentally. The central symmetry of the statistical moments of the velocity field, typical of stationary circular pipe flow, is destroyed as soon as radial system rotation is imposed. Ito and Nanbu (1971) conducted an experimental investigation into the effects of radial system rotation on fully developed turbulent flow in circular pipes by measuring friction factors, axial velocity profiles, and azimuthal wall pressure distributions. They summarised semi-empirical friction factor equations for both laminar and turbulent flows under radial system rotation. Lei et al. (1994) measured the axial velocity of a radially rotating laminar pipe flow using laser Doppler anemometry (LDA). The influence of rotation number on the mean flow and turbulence statistics was investigated. Ishigaki (1994, 1996) analysed the similarity between laminar flows in a curved pipe and a radially rotating pipe, and extended their research to the turbulent flow in these two types of pipes based on analyses of experimental and numerical data. Zhang and Wang (2019) studied the radially rotating circular pipe flows for a wide range of rotation numbers using DNS. They investigated large-scale mean secondary motions in the cross-stream direction, and analysed coherent structures within the rotating flow. They showed that the Coriolis production term played a key role in turbulence energy transfer between axial and radial Reynolds normal stress components. Recently, Ceci and Pirozzoli (2025) performed DNS to investigate the influences of Reynolds numbers and rotation numbers on turbulent circular pipe flows. They decomposed the bulk mean velocity and the total wall friction into viscous, turbulent, cross-stream convection, and rotation-induced components. They showed that as the rotation number increased, the impact of the

cross-stream secondary-flow motion on the wall friction drag varied non-monotonically and decreased significantly at high rotation numbers.

To understand the mechanisms underlying turbulent transport of fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle$ in a pipe flow subjected to either radial or axial system rotation, it is useful to study its transport equation. Orlandi and Ebstein (2000) conducted DNS of turbulent pipe flow under axial system rotation to investigate the rotating effects on turbulent transport processes. In their study, the transport equation of fluctuating enstrophy was briefly analysed in physical space, providing new insights into the mechanisms of turbulent dissipation. They showed that axial rotation strongly influenced the fluctuating enstrophy transport in the near-wall region. As the rotation number increased, the dominant budget terms varied considerably in magnitude, indicating that axial system rotation significantly modified the vorticity dynamics and dissipation behaviour in a turbulent pipe flow.

To develop a deeper understanding of mechanisms of turbulent transport of fluctuating enstrophy, the budget balance of its spectrum can be further studied. However, thus far, the number of studies on turbulent transport of fluid tensor spectra has been very scarce in the literature in general, and the few available studies of a similar type have been limited only to investigations into turbulent transport of the spectrum tensor of Reynolds stresses since the seminal work of Lumley (1964), who derived the spectral energy equation to study turbulent transport of TKE in spectral space in wall turbulence. Furthermore, this type of advanced study of turbulent transport phenomena of TKE or Reynolds stresses in spectral space has been primarily limited to non-rotating turbulent boundary-layer flows developing over a flat plate or in a plane channel (Aulery et al., 2015, 2017; Mizuno, 2016; Lee and Moser, 2019), which have simple domain geometries. Yang et al. (2020) performed DNS to study the transport of Reynolds normal stress spectra in the context of streamwise-rotating plane-channel flow. They discovered four key processes that are responsible for sustaining the motion of the Taylor-Görtler-like vortices which appear as two layers of streamwise-elongated

roll cells in turbulent plane-channel flows subjected to streamwise system rotation. Very recently, Xiong et al. (2025) conducted a spectral analysis of the transport process of TKE in a channel roughened with spanwise-aligned circular-arc ribs of varying pitch-to-height ratios. They discovered that a higher pitch-to-height ratio leads to shortening of the characteristic spanwise wavelength of the dissipation process but prolongation of those of the production, interscale-transport and turbulent diffusion processes in the internal shear layer triggered by the rib crests. To the authors' knowledge, spectral analysis of turbulent transport of fluctuating enstrophy of any type of flow (including turbulent plane-channel flows and circular pipe flows) is still absent in the current literature. In fact, even the transport equations of the fluctuating enstrophy spectrum have yet to be derived and documented.

Based on a careful literature review, it is confirmed that the number of detailed DNS studies of radially rotating pipe flows is still very limited. In particular, the interaction between TKE production and fluctuating enstrophy, as well as the relationship between dissipation and vorticity stretching rates under radial system rotation has yet to be thoroughly investigated. Furthermore, notwithstanding the contributions of the limited number of studies of turbulent transport of TKE and Reynolds-stresses spectra reviewed above, it is recognised that a spectral study of turbulent transport of fluctuating enstrophy is lacking in the literature, and its application to a systematic analysis of turbulent circular pipe flow under radial system rotation is new and interesting. In view of these knowledge gaps, I aim to conduct a comprehensive study of vorticity fluctuations and turbulence structures in radially rotating pipe flows. I will derive and document the transport equations for fluctuating enstrophy in both physical and spectral spaces, and use them to study the budget balance of fluctuating enstrophy, and physical mechanisms of local vortex filament stretching and compression.

1.2.2 DNS of turbulent heat transfer in a radially rotating circular pipe flow

Turbulent heat and fluid flows through a stationary (non-rotating) circular pipe is a classical topic, which has been well studied using a variety of experimental and numerical approaches. In the current literature, it is seen that DNS has been applied to numerical studies of turbulent flows (Eggels et al., 1994; Wu and Moin, 2008; Chin et al., 2010; Wu et al., 2012) and heat convection (Piller, 2005; Redjem-Saad et al., 2007; Saha et al., 2011) in stationary circular pipes.

In contrast to various studies of turbulent heat transfer in the stationary pipe flows briefly mentioned above, the number of experimental and numerical studies of heat convection in radially rotating pipe flows is still very limited. In their pioneering study of rotating turbulent heat and fluid flow in a circular pipe, Mori et al. (1971) compared experimental data acquired using the naphthalene-sublimation method with their theoretical solutions. They observed large-scale secondary flows induced by radial system rotation in a pipe, which destroyed the axial symmetry of the mean velocity and temperature fields, and led to enhancement of both wall friction coefficient and Nusselt number as the rotation number increased. Morris and Rahmat-Abadi (1996) conducted an experiment to investigate the effect of radial system rotation on turbulent heat transfer in a ribbed pipe flow. In their study, radial system rotation significantly enhanced heat convection rate, especially at the pipe inlet. Cheng and Zhang (2003) performed DNS of mixed convection in a curved circular pipe and studied the effects of radial system rotation on the formation of secondary flows, wall friction factor and Nusselt number.

Based on a thorough literature view, it is understood that the number of detailed DNS studies of turbulent heat convection in radially rotating pipe flows is still very limited. In view of this knowledge gap, the first objective of this research is to perform detailed DNS to systematically investigate the effects of radial rotation on turbulent

heat convection in a cooling pipe. To this purpose, 11 test cases (i.e., cases 0–10) covering a wide range of rotation numbers (from $Ro_\tau = 0$ to 1) have been tested. To identify the rotation effects on the temperature field, DNS of heat convection in a stationary circular pipe flow (i.e., case 0 with $Ro_\tau = 0$) of the same Reynolds number has also been conducted, which serves as a baseline case of comparison.

Ever since the pioneering work of Jiménez and Moin (1991) on DNS of plane-channel flows, it has become known to the community that the accuracy of the DNS results is influenced by the computational domain size. Unless a pipe is sufficiently long to allow the most energetic eddies to develop and to be captured by DNS, the statistical moments of the turbulence field obtained may not be physically accurate. The cause for this problem as identified by Jiménez and Moin (1991) stems from the physics of the flow rather than numerical accuracy of the algorithm or grid resolution. Thus far, study on the impact of pipe length on the predictive accuracy of DNS of turbulent heat convection in radially rotating pipe flows is still absent in the current literature. This observation leads to the second objective of this research that I aim to perform a sideline study of the minimum pipe length that is required for conducting DNS of turbulent heat convection in radially rotating pipe flows. To this purpose, five additional DNS test cases (D1–D5) are considered with varying pipe lengths of $L_z = 2\pi R$ – $20\pi R$.

In total, 15 test cases (i.e., cases 0–10 and D1–D5) are considered for the two research objectives mentioned above. The Coriolis effects on the mean and turbulent thermal structures are examined in both physical and spectral spaces. Furthermore, the effects of radial system rotation on the turbulent transport of thermal energy has been studied through a budget analysis of the transport equations of turbulent heat fluxes.

1.2.3 DNS of turbulent flow and structures in a circular pipe subjected to axial system rotation

In comparison with the streamwise-rotating plane-channel flows, the presence of circumferential curvature imposes additional complexity to the secondary-flow structures in an axially rotating circular pipe flow. In their pioneering work, Murakami and Kikuyama (1980) conducted an experiment to investigate the effects of axial system rotation on a turbulent pipe flow and observed that turbulence level of the flow was suppressed as the rotation number rose. Experimental measurements of axially rotating pipe flows were also conducted using hotwires by Kikuyama et al. (1983a), three-hole pressure probes by Reich and Beer (1989) and laser Doppler velocimetry (LDV) by Kikuyama et al. (1983b), Imao et al. (1996) and Facciolo et al. (2007). It should be indicated that in these experiments, typically the pipe rotated while the flow measurement sensors were kept still (relative to the inertial coordinate system fixed to the ground). This treatment method for the system rotation does not explicitly show the Coriolis effects, and is different from the approach based on a rotating coordinate frame (an non-inertial coordinate system), in which Coriolis force appears explicitly in the momentum equation. The latter method of having a Coriolis force term in the momentum equation is often used in the areas of studies such as turbomachinery or meteorology. To be clear, the observations based on the absolute and rotating frames (or inertial and non-inertial frames, respectively) are equivalent, as both reflect the same physical process. For the two velocity fields $\vec{c}$ and $\vec{u}$ observed with respect to the inertial and non-inertial coordinate systems, respectively, they are related by $\vec{c} = \vec{u} + \vec{\Omega} \times \vec{r}$, where $\vec{\Omega}$ is the angular speed vector of system rotation and $\vec{r}$ is the position vector relative to the central axis of rotation.

In addition to the experimental approaches, axially rotating pipe flow has also been investigated numerically using Reynolds-averaged Navier-Stokes (RANS) approaches (Hirai et al., 1988; Speziale et al., 2000; Jakirlić et al., 2002), and using large-eddy

simulation (LES) (Feiz et al., 2003). In these RANS studies, the focus was on the test of turbulence models and analysis of the first- and second-order statistical moments of the velocity field. For example, in the study of Speziale et al. (2000), both linear and nonlinear explicit algebraic stress models and second-order closure models were investigated in terms of their predictive performances in the context of an axially rotating flow. In the LES study of Feiz et al. (2003), a relatively short pipe length of $L_z = 20R$ was used for testing axially rotating pipe flows at low rotation numbers of $Ro_b = 0-4$, and the obtained LES data were used for studying secondary turbulence structures based on the contours of the instantaneous axial velocity fluctuations and axial vorticity field. Here, $Ro_b = 2\Omega_z R/U_b$, U_b is the bulk mean velocity defined as $U_b = \int_0^{2\pi} \int_0^R \langle u_z \rangle r dr d\beta / (\pi R^2)$, and R is the radius of the pipe.¹ Oberlack (1999) studied alternative scaling laws of the mean velocity of an axially rotating pipe flow based on a Lie group theory (also referred to as ‘symmetry analysis’) in contrast to the classical wall-friction-based scaling laws. Their scaling law resulted from symmetry analysis has been validated using the DNS data of Orlandi and Fatica (1997) in the context of axially rotating pipe flows.

In their pioneering DNS studies of axially rotating pipe flows, Orlandi and Fatica (1997), Orlandi (1997) and Ebstein (1998) observed large-scale secondary vortical structures in the pipe centre, which became increasingly elongated as the rotation number increased. The bulk Reynolds number was maintained constant at $Re_b = 2U_b R/\nu = 4900$ and rotation number ranged from $Ro_b = 0$ to 20 in these earlier DNS studies. Here, ν is the kinematic viscosity of the fluid. Orlandi (1997) performed DNS to compare axially rotating and non-rotating pipe flows, and observed a correlation between the helicity density and the TKE dissipation rate. In the PhD thesis of Ebstein (1998), the statistical moments of the velocity field and budget balances of

¹In the literature on rotating plane-channel and pipe flows (see, e.g., Kristoffersen and Andersson, 1993; Grundestam et al., 2008; Zhang and Wang, 2019), it is popular to use the rotation number to quantify the non-dimensional angular speed of system rotation. However, in the literature on rotating isotropic turbulence, it is popular to use Rossby number, which is simply an inverse of the rotation number and can be defined as $1/Ro_b$.

Reynolds stresses were systematically examined. In their follow-up study, Orlandi and Ebstein (2000) examined the impact of axial system rotation on the budget balances of TKE, Reynolds stresses and enstrophy of the turbulent pipe flow using DNS.

Based on a thorough literature review, I note that detailed DNS studies of the axially rotating circular pipe flow are still very limited, and an in-depth understanding of the Coriolis force effects on the flow physics and coherent structures needs to be developed. In view of this, I aim to conduct a systematic DNS study of turbulent pipe flows subjected to axial system rotation for a wide range of rotation numbers.

For a steady-state fully developed axially rotating pipe flow, it is driven by a mean axial pressure gradient and Coriolis forces. Thus, it is anticipated that the energetic flow structures include near-wall streaks and hairpins that are characteristics of a shear-driven boundary layer (whether the flow is subjected to a system rotation or not), and Taylor columns (appearing at moderate and high rotation numbers, as widely observed in rotating isotropic turbulence). It is understood that if the pipe length used in DNS is not long enough to capture the characteristic length scales of energy-containing eddies (such as streaks and Taylor columns), the velocity spectra of the turbulence field at low wavenumbers would be either artificially distorted or bluntly chopped off. Among the a few DNS studies available in the literature, the longest pipe was that used in Ebstein (1998) and Orlandi and Ebstein (2000), who tested pipe lengths of $L_z = 15R$ – $25R$ (or $L_z = 4.775\pi R$ – $7.958\pi R$) for axially rotating flows of a fixed Reynolds number of $Re_b = 4900$ and varying rotation numbers of $Ro_b = 0$ – 20 . Owing to the use of small pipe lengths in the DNS studies of axially rotating pipe flows in the current literature, the actual effects of axial system rotation on the characteristic axial length scale of energetic eddies is still unknown. In view of this, as my first research objective, I aim to study flow physics based on precise statistical moments and coherent structures obtained in DNS for a wide range of rotation numbers varying from $Ro_b = 0$ to 20 at a fixed Reynolds number of $Re_\tau = 180$ using much longer pipes than those in the literature. Here, $Re_\tau = u_\tau R/\nu$ is the friction

Reynolds number and u_τ denotes the wall friction velocity. In order to capture Taylor columns at high rotation numbers, eight DNS cases of very long pipes are considered, which have pipe lengths of $L_z = 30\pi R$ – $180\pi R$ as the rotation number is increased from $Ro_b = 0$ to 20.

To precisely demonstrate the impact of pipe lengths on the accuracy of DNS results, DNS-based short pipe lengths needs to be conducted. It is anticipated that use of overly short pipes to perform DNS can lead to spurious results of the statistical moments and spectra of a velocity field. To prove the concept, a complementary comparative study of the pipe length effects on the accuracy of DNS results is conducted, which encompasses eight additional DNS cases of short pipe lengths at two rotation numbers $Ro_b = 2$ and 20. The short pipe lengths tested vary from $L_z = \pi R$ to $7.958\pi R$ for $Ro_b = 2$ and from $L_z = \pi R$ to $80\pi R$ for $Ro_b = 20$. The DNS results obtained based on eight short pipes are compared with the accurate results of the longest pipes to determine the minimum pipe length that is required for performing physically accurate DNS of an axially rotating pipe flow, and this constitutes the second objective of this research.

1.2.4 DNS of axial rotation effects on Reynolds stress and TKE transport in turbulent pipe flow

The transport equations of the Reynolds stresses and TKE in physical space have been widely studied in turbulent circular pipe flows, regardless of whether the flow is in a rotating reference frame or not. In their DNS study of a non-rotating pipe flow, Eggels et al. (1994) systematically investigated the wall curvature effects on turbulence statistics and on the transport processes of TKE and Reynolds stresses. By comparing the corresponding budget terms with those in plane-channel flows, they found that the budget profiles in pipe flows were similar to those in channel flows in the near wall region. The main influence of wall curvature appeared in the

pressure-strain and pressure diffusion terms in the transport equation of the radial Reynolds normal stress. El Khoury et al. (2013) subsequently performed a series DNS to explore the influence of Reynolds number on turbulence statistics and on the TKE transport equation in turbulent pipe flow. In their study, the mechanisms associated with each TKE budget term and the dominant contributions across different regions of the pipe were examined in detail. They also identified both the similarities and the differences in TKE transport as the Reynolds number increased. Orlandi and Ebstein (2000) conducted DNS to study the effects of axial system rotation on the transport of Reynolds stresses, TKE, and enstrophy in turbulent pipe flow. In their formulation of the Reynolds stress transport equations, the pressure-velocity correlation was not decomposed into pressure-strain and pressure diffusion components, and the Coriolis production term was incorporated into the convection term to simplify the equations. However, such a treatment does not clearly isolate the Coriolis force effects on the Reynolds stress transport, even though the convection term remains relatively small in both axially rotating and non-rotating pipe flows. Later, a similar method was employed by Satake and Kunugi (2002), who performed DNS to investigate the effects of axial system rotation on the transport equations of the Reynolds stresses, TKE, temperature variance, and turbulent heat fluxes in turbulent heat convection of a circular pipe flow. Liu et al. (2005) conducted LES of turbulent flow in an axially rotating pipe to investigate the influence of axial rotation on near-wall turbulence. They analysed the mechanisms of turbulence transport under the system rotation by investigating the transport equations of the resolved Reynolds stresses.

In addition to analysing the Reynolds stress and TKE budgets in physical space, exploring energy transport process in spectral space can offer valuable insights into the characteristic length scales of the physical mechanisms. Lumley (1964) developed the spectral energy equation to study the scale dependence of TKE transport within inhomogeneous turbulent flows. Mizuno (2016) investigated the interscale and wall-normal transport of TKE through an energy spectral analysis in DNS study of

turbulent channel flows, and showed a self-similarity in the fluctuation structures involved in TKE transport across different scales. Yang et al. (2020) performed DNS of turbulent channel flows in a streamwise-rotating frame to study the mechanism of Taylor column vortices. They analysed various budget terms in the transport equations of velocity-spectrum tensors to elucidate the underlying dynamics. Recently, Xiong et al. (2025) conducted DNS of turbulent channel flows with spanwise-aligned circular-arc ribs to study the effects of rib geometry and Reynolds number on the TKE transport process in spectral space. They analysed the role of each budget term in the TKE transport equation across different wavelength scales.

Based on a thorough literature review, it is observed that studies of axial rotation effects on the transport processes of Reynolds stresses and TKE in circular pipe flows are still lacking, and an in-depth understanding of the Coriolis force effects on the budget balance of Reynolds stresses and TKE needs to be developed. In view of this knowledge gap, I aim to conduct a detailed DNS study of axial rotation effects on the budget balances of Reynolds stresses and TKE in both physical and spectral spaces.

1.2.5 DNS of turbulent heat transfer in a circular pipe subjected to axial system rotation

Compared with the study of turbulent heat transfer through a stationary (non-rotating) circular pipe flow, the number of experimental and numerical investigations of turbulent heat convection in an axially rotating pipe flow is rather limited in the literature. Cannon and Kays (1969) conducted the first experiment to investigate the effects of axial rotation on turbulent heat transfer in a circular pipe flow. They observed that the system rotation suppressed turbulent velocity fluctuations in the near-wall region of the pipe. As the rotation number increased, the turbulent flow became increasingly laminarised as the convective heat transfer rate decreased. Reich and Beer (1989) measured turbulent heat and fluid flow in an axially rotating pipe

flow using LDV. They observed that the friction factor and Nusselt number reduced considerably with an increasing rotation number. Satake and Kunugi (2002) performed DNS to study a heated axially rotating pipe flow under a uniform peripheral wall heat flux condition. In their research, the effects of axial system rotation were investigated through various turbulence statistics of the velocity and temperature fields. Ould-Rouiss et al. (2010) compared DNS and LES results of the turbulent heat transfer in an axially rotating pipe flow under identical thermal boundary conditions. They observed that the axial and azimuthal turbulent heat fluxes significantly reduced and augmented, respectively, as the rotation number increased. By contrast, the axial system rotation had only a slight influence on the radial component of turbulent heat flux. Bousbai et al. (2013) conducted LES to investigate the effects of axial system rotation on the turbulent heat convection in a pipe flow. They observed that the degree of intermittency as indicated by the skewness and flatness factors of the temperature fluctuations tended to increase near the pipe wall as the rotation number increased.

Based on a thorough literature review, it is noticed that detailed DNS studies of turbulent heat convection in an axially rotating turbulent pipe flow are still rather scarce, and in-depth understanding of the effects of Coriolis forces on the temperature statistics and thermal structures needs to be developed. In view of this knowledge gap, I aim to conduct a systematic DNS study of turbulent heat transfer in a circular pipe subjected to axial system rotation for a wide range of rotation numbers, which is the first objective of this research.

Furthermore, among a few available DNS studies of axially rotating pipe flows in the literature, the longest pipe was that used in Orlandi and Ebstein (2000), which had a pipe length of $L_z = 25R$ (or, $7.96\pi R$), where R is the pipe radius. In order to reasonably capture the energetic axial eddy motions and precisely reproduce the physical process of heat convection, very long pipes are used in my DNS, which constitutes the second objective of this research. The range of pipe lengths of this

research is $L_z = 30\pi R$ – $180\pi R$, such that the shortest and longest pipes are 3.77 and 22.62 times of that used in the DNS of Orlandi and Ebstein (2000). Comparatively speaking, the long pipes used in my current DNS studies demand a significant increase of the computational efforts.

1.3 Objectives

Major objectives of this thesis is presented as follows:

1. Systematically investigate the effects of radial system rotation on vortical structures and fluctuating enstrophy in a circular pipe. This objective aims to clarify how radial rotation influences the turbulent statistical moments of the vorticity field and the transport of vortical energy.
2. Systematically investigate the effects of radial system rotation on turbulent heat transfer in a circular pipe. This objective aims to clarify how the Coriolis force induced by radial rotation influences heat-transfer coefficients, turbulent heat fluxes, and their transport processes in turbulent circular-pipe flows.
3. Systematically investigate the effects of axial system rotation on the turbulent flow and structures in a circular pipe. This objective is to understand the influence of axial system rotation on the different order statistical moment of the velocity field and turbulent flow structures in a circular pipe.
4. Systematically investigate the effects of axial system rotation on the turbulent transport process in a circular pipe. This objective is to understand the influences of Coriolis force induced by the axial rotation on the transport equations of Reynolds stresses and TKE in both physical and spectral spaces.
5. Systematically investigate the effects of axial system rotation on the heat convection in the turbulent circular pipe flows. This objective is to understand

the influence of Coriolis force induced by the axial rotation on the turbulent statistical moments of the temperature field.

1.4 Outline of thesis

The remainder of this thesis is organised as follows:

- In Chapter 2, a DNS study of vortical structures and fluctuating enstrophy in a turbulent pipe flow under radial rotation is presented.
- In Chapter 3, a DNS study of turbulent heat transfer in a radially rotating circular pipe flow is presented.
- In Chapter 4, a DNS study of turbulent flow and structures in a circular pipe subjected to axial system rotation is presented.
- In Chapter 5, a DNS study of axial rotation effects on Reynolds stress transport in turbulent pipe flow is presented.
- In Chapter 6, a DNS study of turbulent heat transfer in a circular pipe subjected to axial system rotation is presented.
- In Chapter 7, major conclusions of this thesis and the recommended future works are presented.
- In Appendix A, a detailed description for the numerical algorithm of the SEM code is presented.
- In Appendix B, the transport equations for fluctuating enstrophy and its spectrum in a radially rotating reference frame are presented.
- In Appendix C, the transport equation of turbulent heat flux in a rotating reference frame is presented.

- In Appendix D, the impact of pipe length on the accuracy of DNS results of axially rotating turbulent circular pipe flow is presented.
- In Appendix E, the transport equations for Reynolds stresses and TKE, together with those for the corresponding velocity-spectrum tensor and TKE spectrum, in an axially rotating reference frame are presented.

Chapter 2

DNS study of vortical structures and fluctuating enstrophy in a turbulent pipe flow under radial rotation

In this chapter, the DNS study of radial system rotation on vortical structures and fluctuating enstrophy in a circular pipe is conducted using eleven rotating cases with rotation numbers ranging from $Ro_\tau = 0$ to 1. The pipe length is fixed at $L_z = 20\pi R$. It is worth noting that the longest pipe dimension of $20\pi R$ is also the longest in the current literature of radially rotating pipe flows.

2.1 Test cases and numerical algorithm

Figure 2.1 displays the computational domain and cross-sectional mesh configuration of a circular pipe with respect to a radially rotating reference frame in a cylindrical coordinate system. In figure 2.1(a), the radial, azimuthal and axial directions are

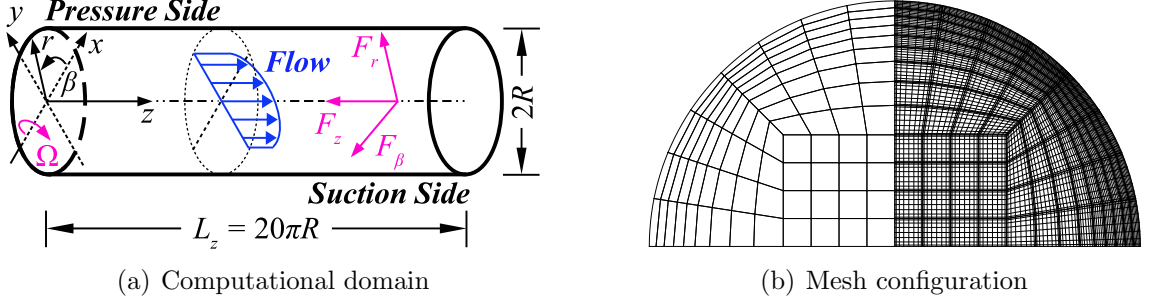


Figure 2.1: Diagrams of computational domain and cross-sectional mesh configuration for a circular pipe under radial system rotation with angular velocity Ω . In panel (a), the radial, azimuthal and axial components of the Coriolis force are denoted by F_r , F_β and F_z , respectively, in a cylindrical coordinate system. The top side of the pipe that the radial Coriolis force F_r points to is referred to as the pressure side, while the bottom side is designated as the suction side. The full circular cross-section is discretised into 352 quadrilateral elements, and within each finite element, an 8th-order GLL polynomial is further used for spatial interpolation. In panel (b), the left-half panel illustrates the finite-element mesh while the right-half panel shows the grid refinement using the GLL polynomial in a quarter domain. In the streamwise direction, a Fourier spectral expansion with 960 modes is applied to all 11 test cases (see table 2.1 for details).

denoted as r , β and z , and three corresponding velocity components are u_r , u_β and u_z , respectively. The pipe length is specified as $L_z = 20\pi R$, where R denotes the pipe radius. This pipe length is among the longest in the current literature on radially rotating turbulent pipe flows (Zhang and Wang, 2019; Ceci and Pirozzoli, 2025) to ensure that axial energetic flow structures are fully captured in DNS. The flow is fully developed with a fixed Reynolds number of $Re_\tau = u_\tau R/\nu = 180$, where ν is the kinematic viscosity of the fluid and $u_\tau = \sqrt{-\Pi R/2\rho}$ is the wall friction velocity. Here, Π is a constant mean axial pressure gradient imposed to drive the flow. A periodic boundary condition is employed in the streamwise (z -) direction, and a no-slip condition is assigned on the pipe wall. The radial system rotation features a constant angular speed (Ω) about the positive x -direction, varying for a wide range of rotation numbers from a non-rotating state at $Ro_\tau = 0$ to a fully laminarised state at $Ro_\tau = 1$. Here, the rotation number is defined as $Ro_\tau = 2\Omega R/u_\tau$. Owing to the radial system rotation imposed, three components of the Coriolis force $\vec{F}$ are

induced, which are $F_r = 2\Omega u_z \sin\beta$, $F_\beta = 2\Omega u_z \cos\beta$ and $F_z = -2\Omega(u_r \sin\beta + u_\beta \cos\beta)$, appearing in the radial, azimuthal and axial directions, respectively. In response to the radial component of the Coriolis force F_r , the pressure on the top and bottom sides of the pipe increases and decreases, and therefore, they are referred to as the pressure and suction sides of the pipe, respectively. The continuity and Navier-Stokes equations that govern a radially rotating pipe flow take the following forms

$$\nabla \cdot \vec{u} = 0 \quad , \quad (2.1)$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{\Pi}{\rho} \hat{\mathbf{e}}_z - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} + \vec{F} \quad , \quad (2.2)$$

where $\vec{u}$, ρ and p denote the velocity vector, density and pressure of the fluid, respectively, and $\hat{\mathbf{e}}_z$ is the base unit vector of the axial direction.

To conduct DNS, the above governing equations were solved numerically using a spectral-element code called ‘Semtex’, originally developed by Blackburn and Sherwin (2004) and further described by Blackburn et al. (2019). The code was written in C++ and FORTRAN, and parallelised using message passing interface (MPI) libraries. Time integration is conducted through a three-step second-order time-splitting method developed by Karniadakis et al. (1991). Specifically, the convection and body force (including the mean pressure gradient and Coriolis force) terms are integrated in the first time substep to yield an intermediate velocity using a second-order backward time-differencing scheme. Subsequently, the intermediate velocity is used in the second time substep to determine the pressure field in order to satisfy the continuity equation. In the last time substep, the viscous term of the momentum equation is implicitly integrated with the preset boundary conditions. It is noted that the last two time substeps rely on solving 2-D Helmholtz equations in the spectral space based on a static condensation technique introduced by Karniadakis and Sherwin (2005). This spectral-element code offers spectral accuracy and is highly accurate for conducting DNS of flow cases with at least one statistically homogeneous

Table 2.1: Summary of the mesh design and resolutions for eleven test cases.

Case#	L_z	Ro_τ	N_e	N_z	N_{tot}	Δz^+	Δr^+		$R\Delta\beta^+$		Δt^+
							$r = 0$	$r = R$	min	max	
1	$20\pi R$	0.0	352	960	22,096,320	11.776	1.025	0.246	1.771	6.393	0.0492
2		0.1				11.768	1.024	0.246	1.769	6.389	0.0492
3		0.2				11.732	1.021	0.245	1.764	6.369	0.0489
4		0.3				11.690	1.017	0.244	1.758	6.346	0.0485
5		0.4				11.684	1.017	0.244	1.757	6.343	0.0485
6		0.5				11.628	1.012	0.243	1.748	6.313	0.0480
7		0.6				11.606	1.010	0.242	1.745	6.301	0.0478
8		0.7				11.574	1.007	0.242	1.740	6.283	0.0476
9		0.8				11.549	1.005	0.241	1.737	6.270	0.0474
10		0.9				11.510	1.002	0.240	1.731	6.249	0.0470
11		1.0				11.405	0.992	0.238	1.715	6.192	0.0462

direction. It has been used by our group to perform DNS studies of rotating elliptical pipe flow and heat convection (Rosas et al., 2021; Rosas and Wang, 2022), turbulent square duct flows and heat convection (Fang et al., 2017; Fang and Wang, 2018), turbulent flow and heat transfer in radially rotating circular pipes (Zhang and Wang, 2019, 2026), axially rotating turbulent circular pipe flows (Zhang and Wang, 2024), and turbulent flows in a square duct roughened with wall-mounted longitudinal ribs (Tachie et al., 2025), in a wavy-walled channel (Opperman et al., 2026) and in a channel roughened using circular-arc ribs (Xiong et al., 2023, 2025).

Figure 2.1(b) shows a cross-sectional view of the mesh in a radial-azimuthal (r - β) plane. To maintain fine grid resolution demanded by DNS, the cross-section of the mesh was discretised into $N_e = 352$ quadrilateral finite elements. Additionally, an 8th-order Gauss-Lobatto-Legendre-Lagrange (GLLL) polynomial was further used for nodal-based interpolation in each finite element. Given the axial symmetry of the pipe domain, only a quarter of its cross section is plotted to show the finite-element mesh (left half panel) and the total mesh refined with GLLL interpolant points (right half panel) in figure 2.1(b). The mesh is expanded into 960 Fourier modes along the axial (z -) direction which is statistically homogeneous. Table 2.1 summarises the

parameters of the eleven test cases in terms of their grid and temporal resolutions. The performances of ten radially rotating pipe flow cases with rotation numbers $Ro_\tau = 0.1$ –1 (cases 2–11) are compared with that of the baseline non-rotating pipe flow case at $Ro_\tau = 0$ (case 1). The total number of grid points N_{tot} (based on the GLLL interpolant refinements and Fourier modes) exceeds 22 million. For all eleven test cases, the grid resolutions in all directions are comparable to those in benchmark DNS studies of non-rotating turbulent pipe flows (Wu and Moin, 2008; Wagner et al., 2001; Chin et al., 2010). In case 1, the axial grid spacing is $\Delta z^+ = 11.776$. The radial grid spacing is $\Delta r^+ = 1.025$ at the pipe centre but reduces to 0.246 at the first node off the wall. The azimuthal grid resolution varies, and at the first node off the wall, it ranges within $R\Delta\beta^+ \in [1.771, 6.393]$. For time marching, the Courant-Friedrichs-Lewy (CFL) number is restricted to 0.081. As such, the largest temporal resolution occurring in the first test case is limited to $\Delta t^+ = 0.0492$.

To compute the statistical moments of the flow field, 300 instantaneous flow fields were collected over 80 large-eddy turnover times (LETOTs, defined as R/u_τ) for each simulated case. All simulations were performed on the supercomputers of the Digital Research Alliance of Canada.

For each simulated case, 305 gigabytes (GB) of data have been collected for computing the statistical moments. In the following analysis, a pair of angular brackets $\langle \cdot \rangle$ is used to denote quantities that are averaged over time and over the homogeneous (z -) direction. A prime is used to denote a fluctuating component relative to the mean, i.e., $u'_i = u_i - \langle u_i \rangle$. To facilitate a near-wall analysis of the pipe flow in analogy to the classical turbulent boundary layer developing over a flat plate, a wall coordinate is introduced, defined as $y^+ \stackrel{\text{def}}{=} (R - r)u_\tau/\nu$.

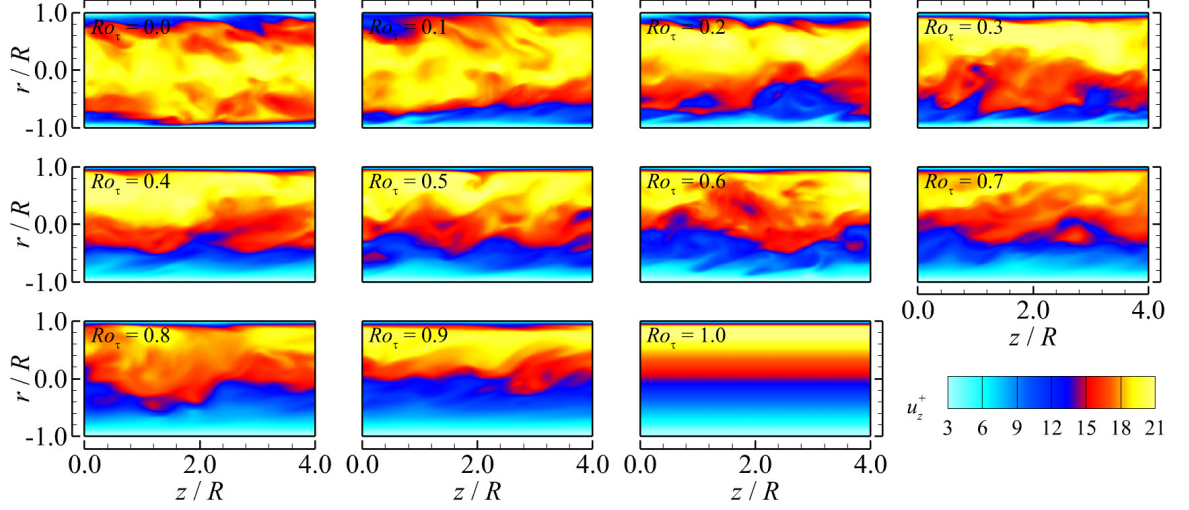


Figure 2.2: Central-vertical distributions of instantaneous axial velocity u_z^+ in the plane position at $\beta = \pi/2$ at eleven different rotation numbers from $Ro_\tau = 0$ to 1, extracted within axial range of $z/R \in [0, 4]$ and at the same time instant of $t = 8.21$ LETOTs. The values of u_z^+ have been non-dimensionalised by u_τ .

2.2 Results and discussion

In this section, the influence of radial system rotation on flow statistics is examined through a comparison of DNS results of non-rotating and radially rotating circular pipe flows. Section 2.2.1 provides a brief overview of the rotating effects on instantaneous flow fields and coherent structures. Detailed analyses of vorticity stretching, fluctuating enstrophy and its transport equation are presented in sections 2.2.2-2.2.4.

2.2.1 Turbulent flow structures

Figure 2.2 compares the instantaneous axial velocity u_z^+ in the central vertical plane ($\beta = \pi/2$) for all 11 rotation numbers. For visual clarity, only a short segment of the pipe is shown within axial range $z/R \in [0, 4]$. In the non-rotating pipe flow, the high- and low-momentum fluids (as indicated by the contour value of u_z^+) are distributed in the central and near-wall regions of the pipe, respectively. However, it is seen that as the rotation number increases from $Ro_\tau = 0$ to 0.2, the radial Coriolis

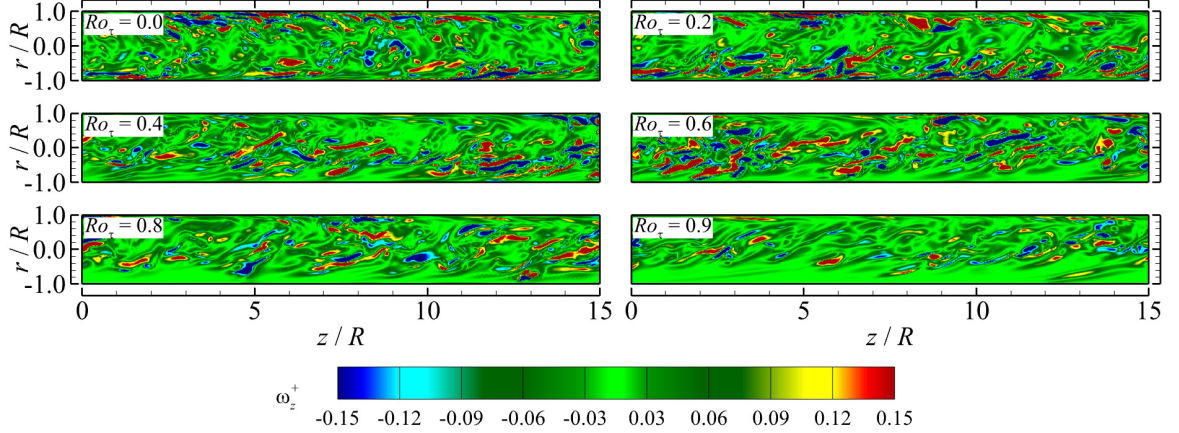


Figure 2.3: Central-vertical distributions of instantaneous axial vorticity ω_z^+ in the plane positioned azimuthally at $\beta = \pi/2$ at six different rotation numbers $Ro_\tau = 0, 0.2, 0.4, 0.6, 0.8$ and 0.9 , extracted within axial range of $z/R \in [0, 15]$ and at the same time instant of $t = 8.21$ LETOTs as shown in figure 2.2. The values of ω_z^+ have been non-dimensionalised by u_τ^2/ν .

force component F_r drives the high-momentum fluid towards the pressure side (or, top wall) and draws the low-momentum fluid towards the suction side (or, bottom wall) of the pipe. As Ro_τ further increases to 0.9 , the tendency of laminarisation becomes increasingly evident on the suction side. Finally, at $Ro_\tau = 1$, the flow field turns into a fully laminar state, with the high- and low-momentum fluids distinctly occupying the pressure and suction sides of the pipe, respectively.

The instantaneous vorticity vector field $\vec{\omega} = \nabla \times \vec{u}$ consists of three components in a cylindrical coordinate system, i.e. $\omega_z = u_\beta/r + \partial u_\beta/\partial r - (\partial u_r/\partial \beta)/r$, $\omega_r = (\partial u_z/\partial \beta)/r - \partial u_\beta/\partial z$ and $\omega_\beta = \partial u_r/\partial z - \partial u_z/\partial r$. To highlight the effects of radial system rotation on vortical structures in the central vertical plane ($\beta = \pi/2$), figure 2.3 compares the instantaneous axial vorticity ω_z^+ at six rotation numbers ranging from $Ro_\tau = 0$ to 0.9 , over a quarter of the pipe length. The positively- and negatively-valued ω_z^+ , visualised through red and blue colour patterns, respectively, correspond to high- and low-momentum vortices of the flow field. In the non-rotating pipe flow, high- and low-momentum vortices dominate the near-wall region of the pipe and appear in an alternating pattern. In response to the radial system rotation,

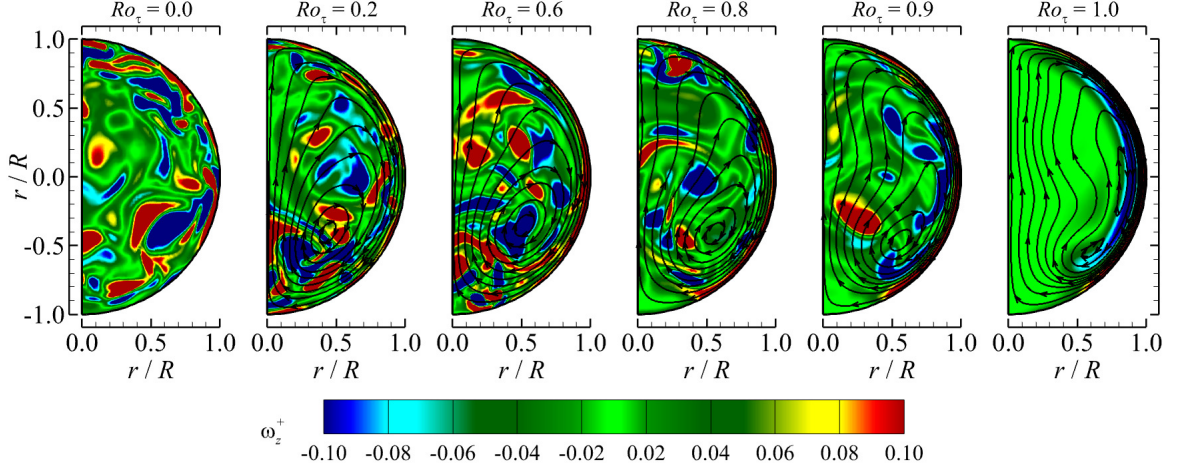


Figure 2.4: Half cross-stream views of instantaneous axial vorticity ω_z^+ superimposed with mean streamlines at the same axial position of $z/R = 2\pi$ and the same time instant of $t = 8.21$ LETOTs for six different rotation numbers $Ro_\tau = 0, 0.2, 0.6, 0.8, 0.9$ and 1 . The values of ω_z^+ have been non-dimensionalised using u_τ^2/ν .

these structures become tilted and elongated on the suction side due to the influence of the axial and radial Coriolis force components F_z and F_r . As the rotation number increases, the instantaneous vortices become the most intense at $Ro_\tau = 0.2$, implying that the TKE level reaches its maximum at this rotation number. Furthermore, at $Ro_\tau = 0.2$, a considerable accumulation of the vortical structures is observed near the suction wall due to the azimuthal Coriolis force component F_β , which delivers near-wall vortices from the pressure to the suction side along the peripheral wall. However, beyond $Ro_\tau = 0.2$, this accumulation behaviour diminishes because the rotation enhances the radial Coriolis force component F_r , which transports these structures from the suction side towards the pressure side of the pipe. At $Ro_\tau = 0.9$, the intensity of the instantaneous vortices, along with the TKE level of the flow field, is significantly reduced on the suction side of the pipe. Consequently, laminarisation initiates from the suction side and spreads towards the pressure side of the pipe.

Figure 2.4 compares the contours of the instantaneous axial vorticity ω_z^+ in a fixed cross-stream plane (located axially at $z/R = 2\pi$) for six rotation numbers from $Ro_\tau = 0$ to 1 . Due to the axial symmetry of the flow field, only one half of the domain

is plotted. The streamlines of the mean velocity field have been superimposed onto the panels of the five rotating flow cases to show the mean secondary flow structures induced by system rotation. All six snapshots are captured at the same time instant in the simulations. From figure 2.4, it is clear that for the non-rotating pipe flow, the instantaneous energetic eddies mostly appear as counter-rotating vortex pairs in the near-wall region, uniformly distributed in the circumferential direction. It is seen that as the rotation number increases to $Ro_\tau = 0.2$, a pair of large-scale secondary flow vortices (as indicated by the streamlines of the mean flow) appears in the cross-stream plane, with their vortex centres located towards the suction side of the pipe. The accumulation behaviour of vortices (as indicated by the contours of ω_z^+) at $Ro_\tau = 0.2$, as shown in figure 2.3, is also observed in figure 2.4. However, as Ro_τ further increases to 0.9, it becomes evident that the intensity of ω_z^+ decreases drastically as only a few instantaneous vortices are observable. Eventually, at $Ro_\tau = 1$, a second pair of large-scale secondary vortices emerges near the lateral side wall, contributing to sustaining the fully laminarised steady-state flow pattern. As shown in figure 2.4, at $Ro_\tau = 1$, the number of the instantaneous vortex pairs reduces precisely to two (over the full cross-section), located near both side walls, where ω_z^+ is negatively valued.

Since the centre locations of the large-scale mean secondary flows are clearly visualised by the streamlines of the mean velocity field, figures 2.5(a,b) focus on tracking their evolution with an increasing rotation number in the radial (r -) and azimuthal (β -) directions, respectively. As shown in figures 2.5(a,b), the centre locations of the mean secondary flows exhibit non-monotonic variations in both the radial and azimuthal directions as the rotation number increases from $Ro_\tau = 0$ to 0.4. Once Ro_τ is beyond this threshold, the vortex centres strictly follow linear regression trends, given by

$$r/R = 0.314Ro_\tau + 0.444 \quad (\text{with } C_{R^2} = 1) \quad (2.3)$$

and

$$\beta/\pi = -0.0894Ro_\tau + 1.863 \quad (\text{with } C_{R^2} = 0.991) \quad , \quad (2.4)$$

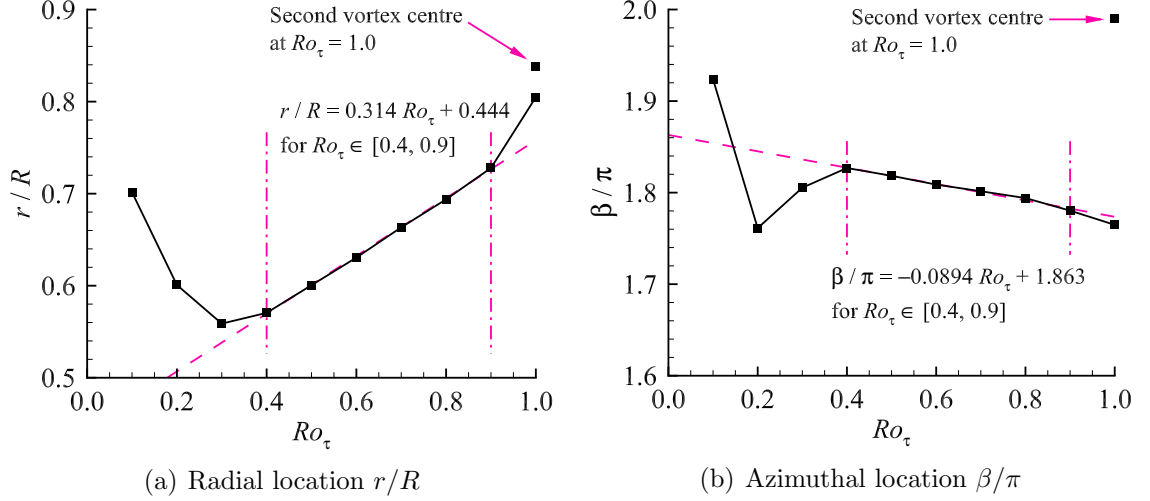


Figure 2.5: Profiles of centre locations of the mean secondary flow as a function of rotation number Ro_τ in the radial (r -) and azimuthal (β -) directions, respectively. The two dashed lines show linear regression variations for $Ro_\tau \in [0.4, 0.9]$. In panel (b), only the azimuthal locations β/π on the right half of the cross-stream plane are shown here given the axial symmetry of the mean circular pipe flow field.

where $C_{R^2} = 1 - \frac{RSS}{TSS}$ is the coefficient of determination of linear regression, with RSS and TSS standing for the sum of squared residuals (errors) and the total sum of squares, respectively. These two regression equations are valid within $Ro_\tau \in [0.4, 0.9]$. The values of C_{R^2} are close to unity, demonstrating a strong linear dependence of the centre locations of the mean secondary flows on the rotation number. At $Ro_\tau = 1$, owing to the full laminarisation of the pipe flow under the radial system rotation, it is interesting to observe a second centre of the mean secondary flow motions emerging in the cross-stream direction (see the last panel of figure 2.4), which is located significantly farther from the first one in both r - and β -directions. Apparently, as Ro_τ increases beyond 0.4, the azimuthal Coriolis force component F_β gradually pushes the first centre of the mean secondary flows towards the suction side of the pipe.

To demonstrate the effect of system rotation on the near-wall streaky structures, figure 2.6 compares the contours of the instantaneous axial velocity fluctuation $u_z'^+$ in the azimuthal-axial (β - z) plane, located radially at $r/R = 0.921$ (or, $y^+ = 14.24$) for six rotation numbers $Ro_\tau = 0, 0.2, 0.3, 0.5, 0.8$ and 0.9 . This particular wall-normal

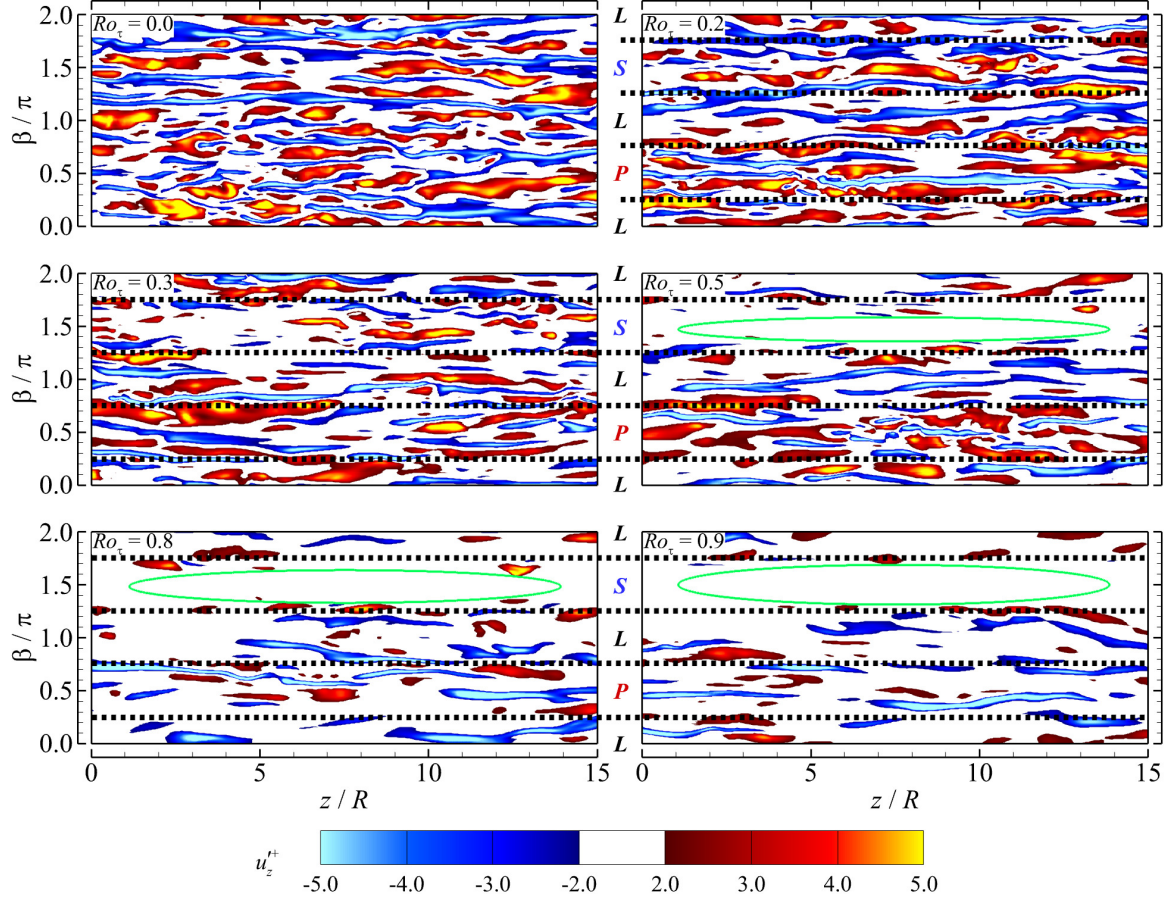


Figure 2.6: Azimuthal-axial views of instantaneous axial velocity fluctuation $u_z'^+$ in the cylindrical plane located radially at $r/R = 0.921$ (or, $y^+ = 14.24$) for six different rotation numbers $Ro_\tau = 0, 0.2, 0.3, 0.5, 0.8$ and 0.9 . All contours are extracted within axial range of $z/R \in [0, 15]$ at the same time instant of $t = 8.21$ LETOTs. Letters ‘P’, ‘S’ and ‘L’ denote the pressure (or, top), suction (or, bottom) and lateral sides of the pipe, respectively, which are segmented by dotted lines along the azimuthal (β -) direction. The green elliptical circles indicate an expanding laminar region in the β -direction on the suction side of the pipe with an increasing rotation number. The values of $u_z'^+$ have been non-dimensionalised using u_τ .

position is where the maximum value of the axial velocity fluctuation intensity occurs (on the pressure side) at $Ro_\tau = 0.2$. A detailed discussion of the axial RMS velocity fluctuation $u_{z,rms}'^+$ under the influence of different rotation numbers will be presented later in section 2.2.2. In the contours, the pressure, suction, and lateral sides of the pipe are labelled by ‘P’, ‘S’ and ‘L’, respectively, and delineated by dotted lines

along the azimuthal (β -) direction. Among the five azimuthal zones separated by the dotted lines, the pressure and suction sides are signified by $\beta/\pi \in [0.25, 0.75]$ and $[1.25, 1.75]$, respectively, while the lateral side involves three azimuthal regions of $\beta/\pi \in [0, 0.25]$, $[0.75, 1.25]$, and $[1.75, 2]$. As shown in figure 2.6, alternating red and blue streaks, as represented by positively- and negatively-valued $u_z'^+$, correspond to the high- and low-momentum eddy motions, respectively. In the non-rotating pipe flow, streaks are uniformly distributed along the β -direction. The blue streaks are much longer than the red ones in the z -direction, while the red streaks have a greater width in the near-wall region compared to their blue counterparts. As the rotation number increases beyond $Ro_\tau = 0.2$, the streaks significantly shorten and become less dense within the range $\beta/\pi \in [1.25, 1.75]$ on the suction side of the pipe. Moreover, the strength of eddy motions as indicated by the contour levels of $u_z'^+$ begins to weaken on the suction side. As a result, it is observed that at $Ro_\tau = 0.5$, an apparent blank region emerges within the same β interval, demonstrating that the flow field on the suction side tends towards a laminar state. As the rotation number continues to increase beyond $Ro_\tau = 0.5$, as shown in figure 2.6, the blank region steadily expands, eventually occupying the entire suction side at $y^+ = 14.24$ at $Ro_\tau = 0.9$. Simultaneously, the shrinking dimensions and growing sparsity of the streaks indicate that laminarisation is gradually supplanting turbulence within the circular pipe, consistent with the previous observations in figures 2.3 and 2.4.

Figure 2.7 compares the coherent flow structures of the non-rotating ($Ro_\tau = 0$) and radially rotating ($Ro_\tau = 0.2$ and 0.9) pipe flows. These structures are visualised using the swirling strength criterion (λ_{ci}) introduced by Zhou et al. (1999), which is defined as the imaginary part of a complex eigenvalue of the velocity gradient tensor, invariant under a change of reference frame. For clarity, the figure shows 3-D coherent structures over only a quarter of the pipe length. As is clear in figure 2.7, arch-like vortical structures, commonly known as hairpin structures, populate the entire domain of the non-rotating pipe flow (Baltzer et al., 2013; Wu et al., 2012).

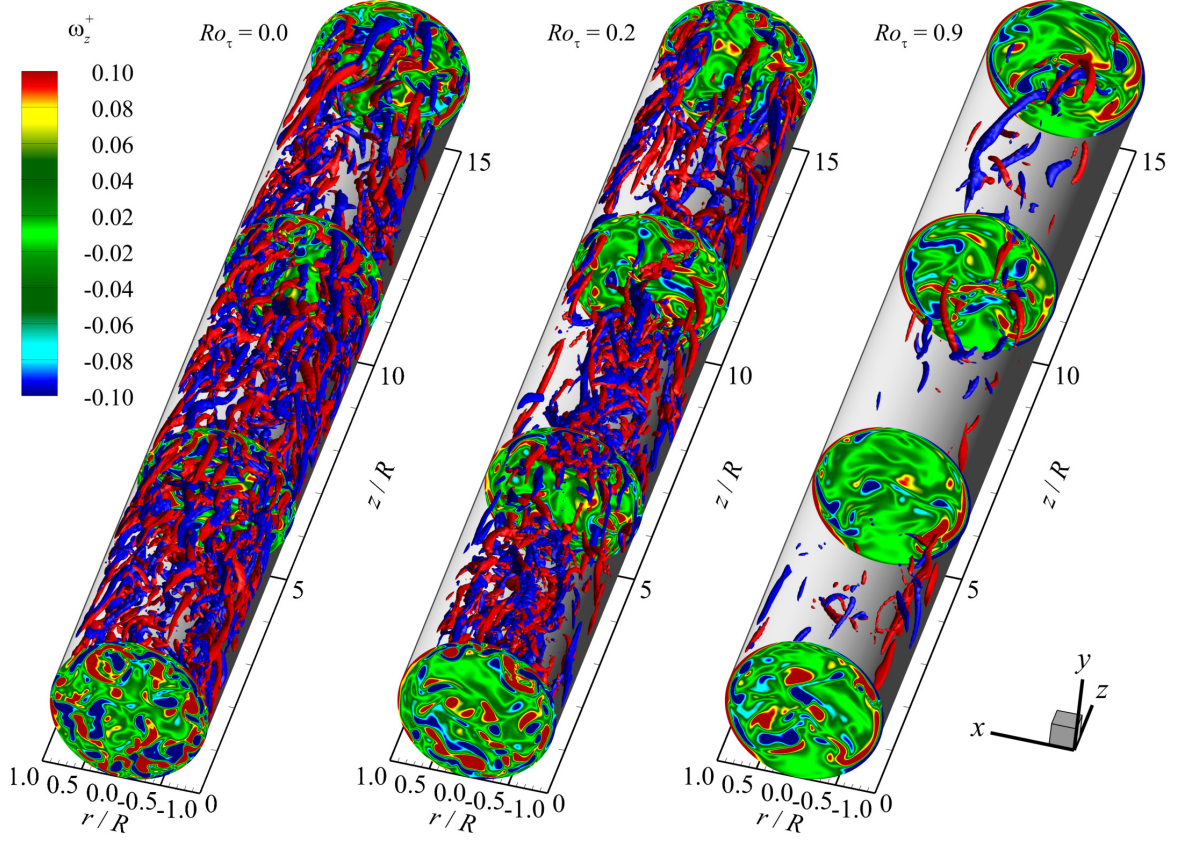


Figure 2.7: Iso-surfaces of the swirling strength criterion (with $\lambda_{ci} = 1.3$) coloured by instantaneous axial vorticity ω_z^+ for three different rotation numbers $Ro_\tau = 0, 0.2$ and 0.9 . All contours are extracted within the axial range of $z/R \in [0, 15]$ at the same time instant of $t = 8.21$ LETOTs and separated by four cross-stream views of ω_z^+ of different axial (z -) positions to show how the contour patterns of ω_z^+ vary streamwise. The values of ω_z^+ have been non-dimensionalised using u_τ^2/ν .

These coherent structures present the characteristic “head-neck-legs” formation as described in Robinson (1991), Zhou et al. (1999) and Adrian (2007). The formation of the hairpin structures is linked to the turbulent bursting mechanism, wherein a pair of counter-rotating streamwise vortices interact to shape an arch-shaped head in the outer layer of the flow through the ejection ($Q2$) and sweep ($Q4$) events. In response to the radial system rotation, the hairpin vortical structures become asymmetrical in the central vertical plane, consistent with the vortex distributions shown in figures 2.3 and 2.4. At $Ro_\tau = 0.2$, as shown in figure 2.7, the hairpin structures are still visible

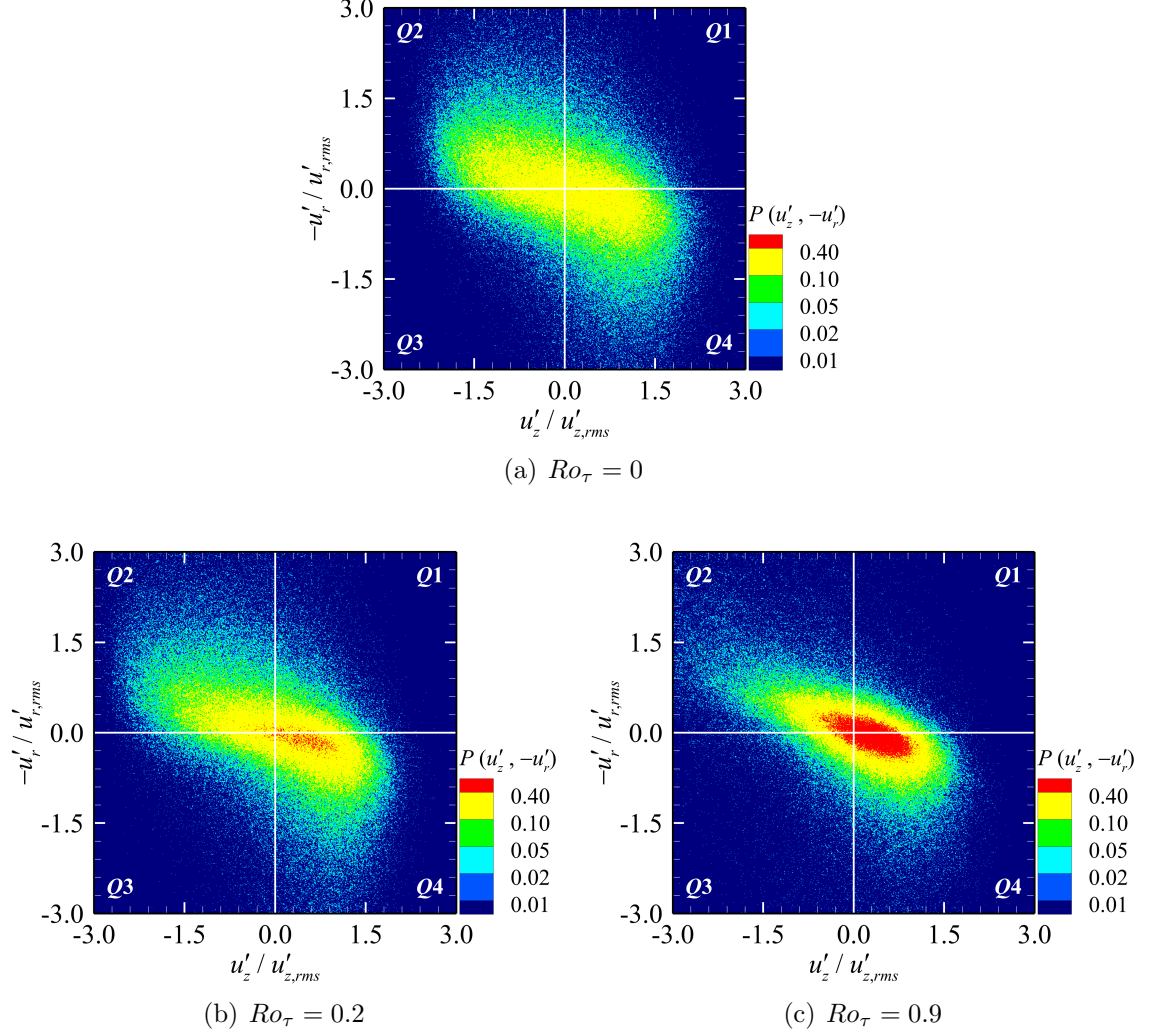


Figure 2.8: JPDF $P(u'_z, -u'_r)$ measured from the wall-normal position $y^+ = 14.24$ on the pressure side in the central vertical plane (located at $\beta = \pi/2$) of the pipe at $Ro_\tau = 0, 0.2$ and 0.9 . All contours are partitioned into four quadrants ($Q1-4$) by white solid lines.

within the axial range $z/R \in [10, 15]$ on the pressure side of the pipe. However, as the rotation number continues to increase, the system rotation disrupts the formation mechanism of these structures. At $Ro_\tau = 0.9$, the hairpin structures have completely vanished, leaving behind only a few streamwise vortices in the highly laminarised flow field.

To further examine the effects of radial system rotation on the coherent structures,

figure 2.8 compares the JPDF contours $P(u'_z, -u'_r)$ at three rotation numbers $Ro_\tau = 0, 0.2$ and 0.9 . These contours are evaluated at the wall-normal position $y^+ = 14.24$ on the pressure side in the central vertical plane (located at $\beta = \pi/2$) of the pipe. In figure 2.8, the negative of radial velocity fluctuation (i.e., $-u'_r$) is used in the calculation of the JPDF, which provides a direct analogy in quadrant analysis between the current pipe flow and the classical boundary-layer flow developing over a flat plate. The second quadrant ($Q2$) corresponds to ejection events, featuring $-u'_r > 0$ and $u'_z < 0$, while the fourth quadrant ($Q4$) corresponds to sweep events, featuring $-u'_r < 0$ and $u'_z > 0$. From figure 2.8, the pattern of $P(u'_z, -u'_r)$ shows pronounced distributions in the $Q2$ and $Q4$ quadrants for both the non-rotating and rotating flow cases. A comparison of figures 2.8(a-c) indicates that the sweep ($Q4$) events intensify within regions characterised by lower magnitudes of u'_z and $-u'_r$, while their occurrence is suppressed in ranges of larger magnitudes of these two velocity fluctuations at $Ro_\tau = 0.9$ relative to the non-rotating pipe flow at $Ro_\tau = 0$. Furthermore, the contour pattern of $P(u'_z, -u'_r)$ in the first and second quadrants shrinks with an increasing rotation number, indicating that the ejection events tend to possess high probabilities over a narrower distribution range. The rotating effects on the ejection and sweep behaviours are mainly attributed to the radial Coriolis force component F_r , which arises from system rotation and points to the pressure side of the pipe.

2.2.2 Turbulence statistics and fluctuating enstrophy

Figure 2.9 compares the profiles of axial RMS velocity fluctuation $u'_{z,rms}^+$ for all eleven rotation numbers, measured in the half central vertical plane on the pressure side of the pipe (located at $\beta = \pi/2$). For the purpose of comparison, the figure also includes benchmark DNS datasets of a plane-channel flow (Kim et al., 1987) and a non-rotating pipe flow (Eggels et al., 1994). From figure 2.9, it is clear that the influence of radial system rotation on the magnitude of $u'_{z,rms}^+$ is non-monotonic. In the non-rotating

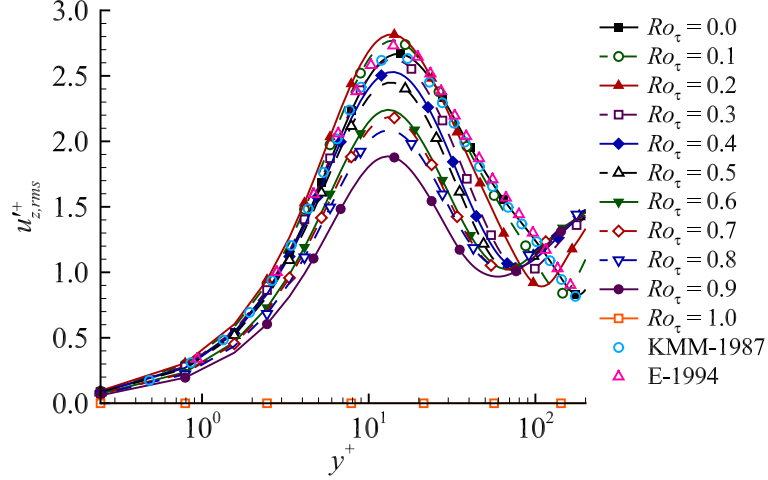


Figure 2.9: Profiles of axial RMS velocity fluctuation $u'_{z,rms}^+$ in the half central vertical plane on the pressure side of the pipe (located at $\beta = \pi/2$) for all 11 rotation numbers. The values have been non-dimensionalised by u_τ . The DNS results of Kim et al. (1987, labelled as ‘KMM-1987’) and Eggels et al. (1994, labelled as ‘E-1994’) are displayed for the purpose of comparison.

pipe flow, the axial RMS velocity fluctuation reaches a peak value of approximately $u'_{z,rms}^+ = 2.67$ at the wall-normal position $y^+ = 15.48$, consistent with prior DNS studies of stationary pipe flow (Wu and Moin, 2008; Chin et al., 2010; Ahn et al., 2013; El Khoury et al., 2013). By contrast, in a radially rotating flow at $Ro_\tau = 0.2$, a higher peak of $u'_{z,rms}^+ = 2.82$ is observed at $y^+ = 14.24$, consistent with the strong turbulent motions observed previously based on the contours of instantaneous axial vorticity ω_z^+ in figures 2.3 and 2.4 at the same rotation number. As Ro_τ rises from 0.2 to 0.9, as shown in figure 2.9, the peak value reduces to $u'_{z,rms}^+ = 1.89$ at $y^+ = 12.99$. Clearly, as the rotation number increases, the peak location of $u'_{z,rms}^+$ is pushed deeper into the buffer layer by the enhanced radial rotation. Meanwhile, the minimum value of $u'_{z,rms}^+$ shifts away from the pipe centre (at $y^+ = 180$) to $y^+ = 59.79$. Eventually, as Ro_τ reaches 1, the magnitude of $u'_{z,rms}^+$ becomes trivial, indicating that the flow becomes fully laminarised.

In contrast to a non-rotating pipe flow, the mean flow varies azimuthally in a radially rotating flow under the influence of the Coriolis forces. The local wall friction

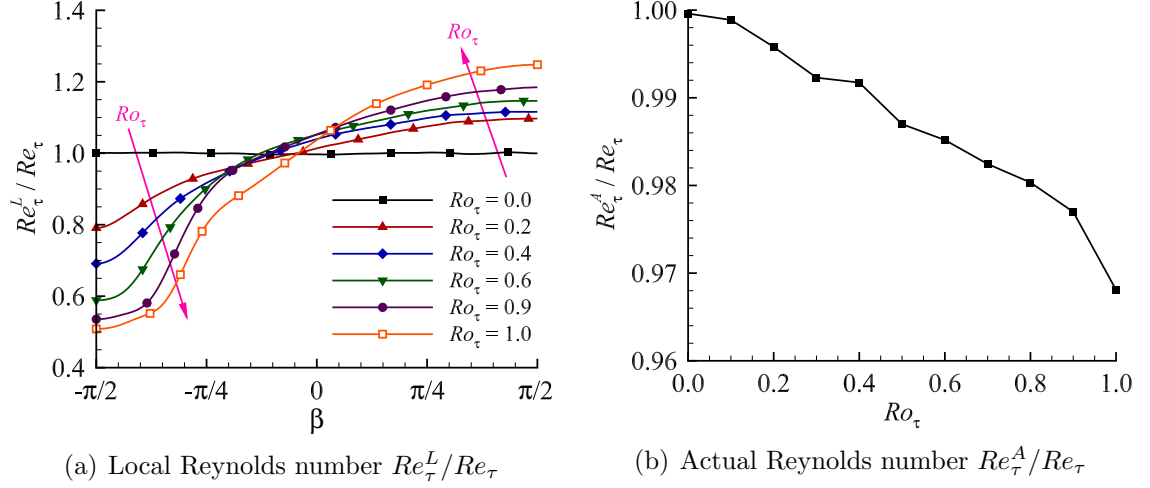


Figure 2.10: Profiles of normalised local and actual Reynolds numbers, Re_τ^L / Re_τ and Re_τ^A / Re_τ . In panel (a), arrows point to the local monotonic trend of Re_τ^L / Re_τ with an increasing rotation number Ro_τ .

velocity is a function of the azimuthal angle β and rotation number Ro_τ , calculated as $u_\tau^L(\beta, Ro_\tau) = \sqrt{\nu(\partial\langle u_z \rangle / \partial r)_{wall}}$. To examine the effect of radial system rotation on the wall friction, figure 2.10 presents the profiles of Re_τ^L / Re_τ and Re_τ^A / Re_τ . Here, $Re_\tau^L = u_\tau^L R / \nu$ is the local friction Reynolds number, $Re_\tau^A = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} Re_\tau^L d\beta$ is the actual friction Reynolds number calculated from the DNS data by azimuthally averaging the obtained value of Re_τ^L , and Re_τ is the nominal Reynolds number determined by the constant mean axial pressure gradient Π of the flow. In figure 2.10(a), the profiles of Re_τ^L / Re_τ are plotted as a function of the azimuthal coordinate, ranging from $\beta = -\pi/2$ (on the bottom suction side) to $\beta = \pi/2$ (on the top pressure side of the pipe), for six different rotation numbers. In figure 2.10(b), the value of Re_τ^A / Re_τ at all eleven rotation numbers is shown. It is evident that the value reduces monotonically with an increasing rotation number. As is evident in figures 2.10(a,b), both $Re_\tau^L / Re_\tau \equiv 1$ and $Re_\tau^A / Re_\tau \equiv 1$ hold strictly for the non-rotating circular pipe flow, confirming that the wall friction velocity calculated from the mean velocity gradient is precisely the same as its nominal value predetermined by the constant mean axial pressure gradient Π . By contrast, as soon as the radial system rotation is imposed, the

values of Re_τ^L/Re_τ and Re_τ^A/Re_τ deviate considerably from the unity value characteristic of the non-rotating pipe flow. From figure 2.10(a), it is interesting to observe that the magnitude of Re_τ^L/Re_τ decreases monotonically on the suction side (around $\beta = -\pi/2$), while increasing monotonically on the pressure side (around $\beta = \pi/2$) of the pipe. The deviation from unity is the most significant on the suction side due to the fact that laminarisation first starts in this region under enhanced rotating effects. By contrast, the variation in the magnitude of Re_τ^L/Re_τ remains comparatively stable near the two ends of the rotating axis (or, around $\beta = 0$ and π) even as the rotation number increases.

Figure 2.11 compares the profiles of RMS vorticity fluctuations, $\omega'_{z,rms}^+$, $\omega'_{r,rms}^+$ and $\omega'_{\beta,rms}^+$, and fluctuating enstrophy (or vorticity variance) $\langle \omega'_i \omega'_i \rangle^+$ in the central vertical plane (located at $\beta = \pi/2$) of the pipe for six rotation numbers Ro_τ . The DNS data of the three RMS vorticity fluctuations in the plane-channel flow of Kim et al. (1987) are also included in figures 2.11(a-c). Clearly, all three RMS vorticity components and the fluctuating enstrophy are axisymmetric about the pipe centre in the non-rotating pipe flow case ($Ro_\tau = 0$). From figures 2.11(a-c), it is evident that these three components approximately coincide with the results of Kim et al. (1987) in the near-wall region. However, owing to the presence of surface curvature and secondary flows in the pipe flow, the RMS vorticity values of the non-rotating pipe and plane-channel flows deviate slightly from each other in the central regions. In figure 2.11(a), the profile of $\omega'_{z,rms}^+$ shows a complex pattern along the r -direction in the non-rotating pipe flow (at $Ro_\tau = 0$). A local minimum is observed within the viscous sublayer at $|r/R| = 0.974$ (or, $y^+ = 4.66$) with $\omega'_{z,rms}^+ = 0.093$, while a local peak appears in the buffer layer at $|r/R| = 0.881$ (or, $y^+ = 21.35$), where $\omega'_{z,rms}^+ = 0.15$. These radial coordinate measurements show good agreement with the plane-channel flow results of Kim et al. (1987), who introduced a Rankine vortex model to characterise the streamwise vortices. This model is built upon a simplified vortical structure without considering the viscous effects at the wall, including a solid-body rotating core and

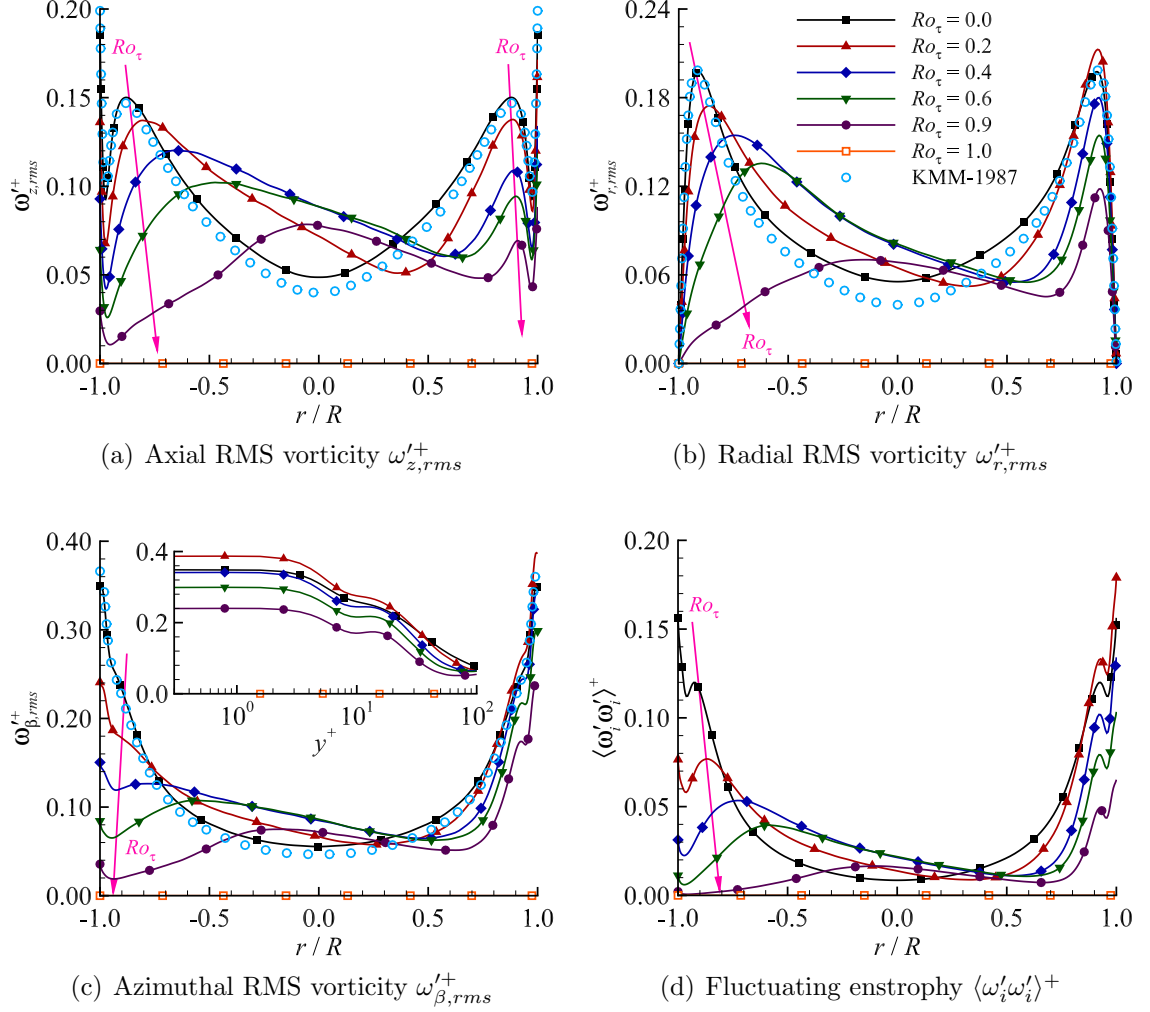


Figure 2.11: Profiles of RMS vorticity fluctuations ($\omega'_{z,rms}{}^+$, $\omega'_{r,rms}{}^+$ and $\omega'_{\beta,rms}{}^+$) and fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ in the central vertical plane (located at $\beta = \pi/2$) of the pipe for six different rotation numbers Ro_τ . Pink arrows point to the direction of an increasing rotation number. The RMS vorticity fluctuations are non-dimensionalised by u_τ^2/ν and the fluctuating enstrophy is non-dimensionalised by u_τ^4/ν^2 . The DNS results of Kim et al. (1987, labelled as ‘KMM-1987’) for a plane-channel flow are displayed for the purpose of comparison.

a free vortex outer region. At the centre of the Rankine vortex, the axial RMS vorticity $\omega'_{z,rms}{}^+$ reaches a local peak value. As depicted in figure 2.11(a), in response to the radial system rotation imposed, the magnitudes of $\omega'_{z,rms}{}^+$ in the central vertical plane (located at $\beta = \pi/2$) on both sides of the pipe decrease monotonically with an increasing rotation number. The streamwise vortices, as identified by the peaks of

$\omega'_{z,rms}^+$, are shifted along the direction of the radial Coriolis force component as the rotation number increases, consistent with the trend of vortical structures visualised using the instantaneous axial vorticity contours in figures 2.3 and 2.4.

Figure 2.11(b) compares the profiles of the radial RMS vorticity $\omega'_{r,rms}^+$ at six rotation numbers. In the non-rotating flow, the profile exhibits a dual-peak pattern, with two peaks occurring at $|r/R| = 0.908$ (or, $y^+ = 16.56$) where $\omega'_{r,rms}^+ = 0.198$. As the rotation number increases, the magnitude of $\omega'_{r,rms}^+$ decreases monotonically on the suction side of the pipe, a trend that is similar to the decay in $\omega'_{z,rms}^+$, whereas on the pressure side, the variation is non-monotonic. At $Ro_\tau = 0.2$, a maximum peak occurs at $r/R = 0.914$ (or, $y^+ = 15.48$) with $\omega'_{r,rms}^+ = 0.213$ due to enhanced vortex stretching under the condition of radial system rotation. Given that $\omega'_r = (\partial u'_z / \partial \beta) / r - \partial u'_\beta / \partial z$ and the flow is homogeneous in the axial (z -) direction, the variations in $\omega'_{r,rms}^+$ are mainly caused by $(\partial u'_z / \partial \beta) / r$. Thus, the increased level of $\omega'_{r,rms}^+$ on the pressure side at $Ro_\tau = 0.2$ directly corresponds to an enhancement in the azimuthal gradient of $u'_{z,rms}$ in that region, around the peak value of $u'_{z,rms}^+$ in figure 2.9.

The profiles of the azimuthal RMS vorticity $\omega'_{\beta,rms}^+$ for different rotation numbers are shown in figure 2.11(c). In the non-rotating pipe flow, the profile peaks at the wall ($|r/R| = 1.0$) with $\omega'_{\beta,rms}^+ = 0.349$, featuring azimuthal homogeneity and axisymmetry. As soon as the system rotation is imposed, the magnitude of $\omega'_{\beta,rms}^+$ decreases monotonically on the suction side, but exhibits a non-monotonic trend on the pressure side with the largest peak occurring at $Ro_\tau = 0.2$. Given that $\omega'_\beta = \partial u'_r / \partial z - \partial u'_z / \partial r$ and the flow is homogeneous in the axial direction, the intensification of $\omega'_{\beta,rms}^+$ on the pressure side at $Ro_\tau = 0.2$ directly relates to the peak value of $u'_{z,rms}$ at the same rotation number and wall-normal distance, similar to our previous analysis of the profile of $\omega'_{r,rms}^+$. As Ro_τ further increases, the magnitude of $\omega'_{\beta,rms}^+$ decreases steadily on both sides of the pipe. In comparison with the non-rotating pipe flow, large-scale secondary flows are induced by the Coriolis forces in radially rotating pipe flows, which generate azimuthal wall shear flow and reinforce asymmetry in both radial

and azimuthal directions. Consequently, it is seen from figure 2.11(c) that beyond $Ro_\tau = 0.2$, azimuthal vortices, as identified by the local peaks and valleys, rise in the near-wall region of the pipe. At $Ro_\tau = 0.9$, a local peak and a local valley of $\omega'_{\beta,rms}{}^+$ occur at $r/R = 0.928$ (or, $y^+ = 12.99$) and $r/R = 0.945$ (or, $y^+ = 9.85$) on the pressure side of the pipe, with $\omega'_{\beta,rms}{}^+ = 0.174$ and 0.17 , respectively. Furthermore, it is interesting to observe that the azimuthal vortices grow larger in size on the suction side due to increasingly strong secondary flow motions near the suction wall. As is evident in figures 2.11(a-c), the suppression of all three RMS vorticity components is more significant on the suction side than on the pressure side, indicating that laminarisation begins on the suction side and gradually extends towards the pressure side of the pipe. At $Ro_\tau = 1$, the RMS vorticity fluctuations become negligibly small as the flow field reaches a fully laminarised state.

Figure 2.11(d) compares the profile of fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ in the central vertical plane (located at $\beta = \pi/2$) of the pipe for different rotation numbers. As shown in figure 2.11(d), in the non-rotating pipe flow, the small-scale dissipative eddies are concentrated near the pipe wall, where the fluctuating enstrophy peaks. Slightly off the wall, a local valley and peak of $\langle \omega'_i \omega'_i \rangle^+$ are observed at $|r/R| = 0.962$ (or, $y^+ = 6.79$) and $|r/R| = 0.928$ (or, $y^+ = 12.99$), with values of $\langle \omega'_i \omega'_i \rangle^+ = 0.112$ and 0.12 , respectively. Once the rotation number rises to $Ro_\tau = 0.2$, the profile of fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ becomes asymmetrical in the r -direction, as its value increases significantly on the pressure side ($r/R = 1.0$) of the pipe. This suggests enhanced local vortex stretching, attributed to amplified radial and azimuthal vorticity fluctuation intensities $\langle \omega'_r \omega'_r \rangle$ and $\langle \omega'_\beta \omega'_\beta \rangle$, as evidenced by the profiles of $\omega'_{r,rms}{}^+$ and $\omega'_{\beta,rms}{}^+$ shown in figures 2.11(b,c). As Ro_τ continues to increase, the magnitude of $\langle \omega'_i \omega'_i \rangle^+$ decreases drastically on both the pressure and suction sides of the pipe, indicating that the radial system rotation tends to suppress the eddy motions and reduce the TKE dissipation rate by weakening the vortex stretching mechanisms. At $Ro_\tau = 1$, as shown in figure 2.11(d), the flow becomes completely laminarised,

evidenced by the trivial values of $\langle \omega'_i \omega'_i \rangle^+$.

So far, the effect of radial system rotation on the vorticity fluctuations has been examined in physical space. To gain a deeper insight into the characteristics of vortical structures, it is necessary to extend the analysis of vorticity fluctuations to spectral space. The axial wavelengths and vorticity energy levels of eddy motions can be identified precisely through the premultiplied vorticity spectra $k_z^+ \tilde{E}_{\omega_i}^+$ of vorticity fluctuations ω'_i , where the one-dimensional (1-D) axial spectrum of vorticity fluctuations is defined as

$$\tilde{E}_{\omega_i}(r, \beta, k_z) = \text{Re} \left\{ \overline{\hat{\omega}'_i{}^*(r, \beta, k_z) \hat{\omega}'_i(r, \beta, k_z)} \right\} \quad , \quad (2.5)$$

where subscript runs as $i = 1, 2$ and 3 , denoting the radial, azimuthal and axial (r -, β - and z -) directions, respectively (no summation convention over subscript “ i ” is implied) and an overline operation $\overline{(\cdot)}$ represents time averaging. Operators $\text{Re}\{\cdot\}$ and $(\cdot)^*$ denote the real part and conjugate of a complex number, respectively. A hat operation $\widehat{(\cdot)}$ denotes axial Fourier transform of an arbitrary variable $\phi(r, \beta, k_z, t)$, i.e.

$$\hat{\phi}(r, \beta, k_z, t) = \frac{1}{L_z} \int_0^{L_z} \phi(r, \beta, z, t) e^{-ik_z z} dz \quad , \quad (2.6)$$

where $\underline{i} = \sqrt{-1}$ is the imaginary unit and $k_z = n_z k_{z0}$ is the axial wavenumber, with $n_z \in [-N_z/2, N_z/2 - 1]$ being an integer and $k_{z0} = 2\pi/L_z$ being the smallest positive wavenumber.

Figure 2.12 compares the premultiplied energy spectra profiles for the three components of vorticity fluctuations, $k_z^+ \tilde{E}_{\omega_z}^+$, $k_z^+ \tilde{E}_{\omega_r}^+$ and $k_z^+ \tilde{E}_{\omega_\beta}^+$, at six different rotation numbers. Here, all profiles are plotted with respect to the axial wavelength λ_z^+ at the wall-normal position $y^+ = 12.99$ on the pressure side (in the central vertical plane located at $\beta = \pi/2$). This specific position corresponds to the fluctuating enstrophy peak in the non-rotating pipe flow (at $Ro_\tau = 0$). The axial wavelength is defined as $\lambda_z = 2\pi/k_z$, non-dimensionalised as $\lambda_z^+ = \lambda_z u_\tau / \nu$. From figure 2.12(a), it is seen that

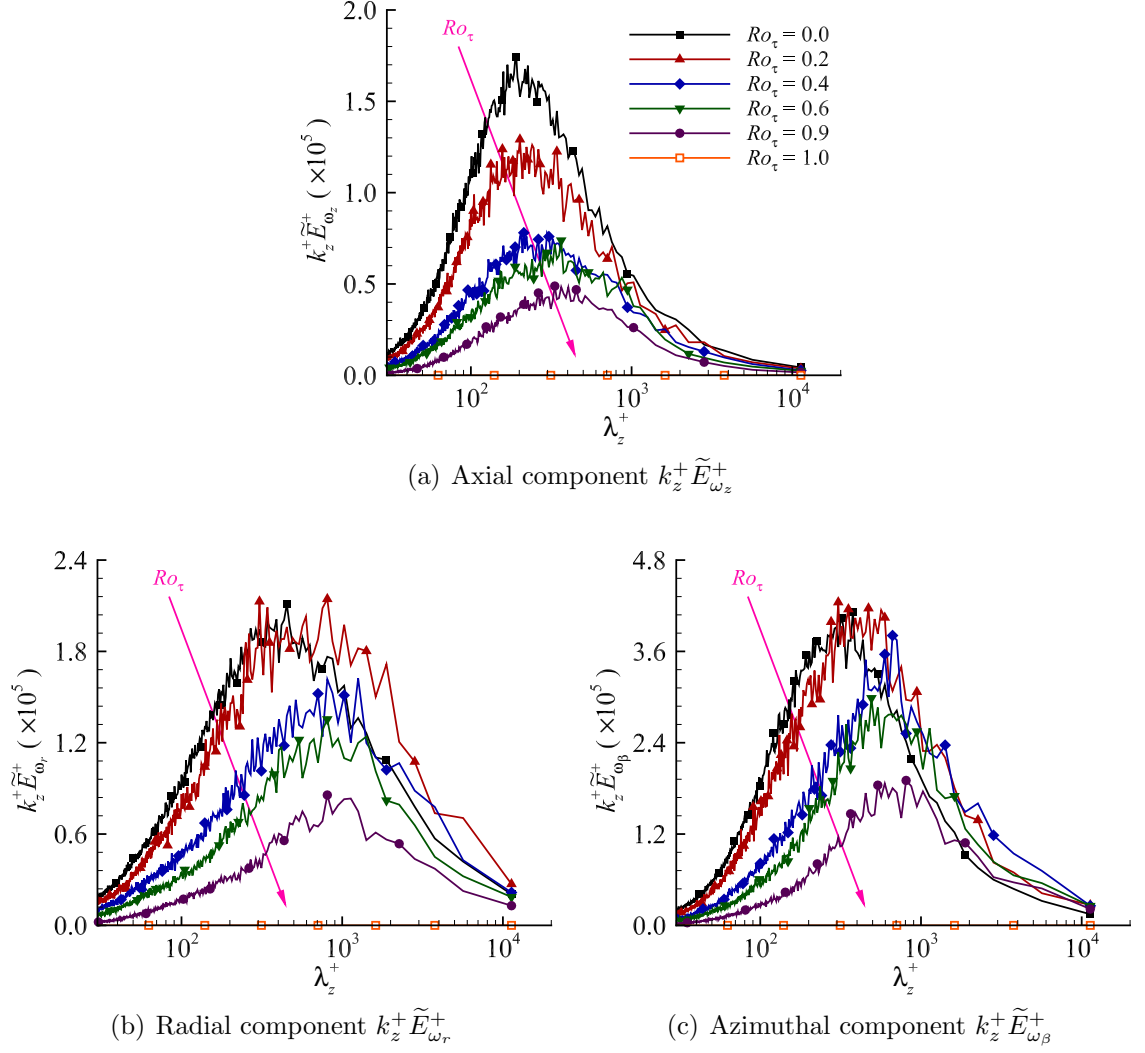


Figure 2.12: Premultiplied axial spectra of vorticity fluctuations $k_z^+ \tilde{E}_{\omega_z}^+$, $k_z^+ \tilde{E}_{\omega_r}^+$ and $k_z^+ \tilde{E}_{\omega_\beta}^+$ along the streamwise line of the pipe at $r/R = 0.928$ (or, $y^+ = 12.99$) on the pressure side (located at $\beta = \pi/2$) for six different rotation numbers within $Ro_\tau \in [0, 1]$. The premultiplied vorticity spectra have been non-dimensionalised using u_τ^5/ν^3 . For visual clarity, their magnitudes are amplified by a factor of 10^5 as labelled in round brackets.

in the non-rotating pipe flow ($Ro_\tau = 0$), the peak mode of the premultiplied spectrum of axial vorticity fluctuations $k_z^+ \tilde{E}_{\omega_z}^+$ (corresponding to the characteristic length scale of the streamwise vortices) is confined to a small wavelength range of $\lambda_z^+ \in [155, 230]$. As the rotation number increases, the magnitude of $k_z^+ \tilde{E}_{\omega_z}^+$ decreases monotonically

at the sampling position on the pressure side of the pipe. At $Ro_\tau = 0.9$, the peak slightly shifts towards larger axial wavelengths of $\lambda_z^+ \in [260, 455]$, indicating that the radial system rotation tends to elongate the characteristic length scales of the streamwise vortices. Finally, at $Ro_\tau = 1$, the fluctuating enstrophy energy level as indicated by the value of $k_z^+ \tilde{E}_{\omega_z}^+$ vanishes completely. By comparing figures 2.12(b,c) with 2.12(a), it is seen that the magnitudes of $k_z^+ \tilde{E}_{\omega_r}^+$ and $k_z^+ \tilde{E}_{\omega_\beta}^+$ are slightly higher than that of $k_z^+ \tilde{E}_{\omega_z}^+$ in the non-rotating pipe flow. As Ro_τ increases, the magnitudes of both radial and azimuthal premultiplied vorticity spectra reduce monotonically at small wavelengths, with the largest peak occurring at $Ro_\tau = 0.2$, an observation that is consistent with the profile trends of $\omega_{r,rms}^+$ and $\omega_{\beta,rms}^+$ shown in figures 2.11(b,c). As the rotation number goes beyond $Ro_\tau = 0.2$, the turbulent energy level of both radial and azimuthal vortices is suppressed markedly, indicating the damping effect of the radial system rotation on turbulence. Meanwhile, the axial wavelength scales for the radial and azimuthal vortical structures expand from $\lambda_z^+ \in [305, 630]$ and $\lambda_z^+ \in [210, 435]$ at $Ro_\tau = 0$ to $\lambda_z^+ \in [565, 1135]$ and $\lambda_z^+ \in [450, 1030]$ at $Ro_\tau = 0.9$, respectively. Figures 2.12(a-c) clearly reveal that the radial system rotation systematically enlarges the characteristic length scales (or the modal wavelengths) of all three vorticity components in the streamwise direction near the pressure wall of the pipe.

Figure 2.13 demonstrates the rotation number effects on the premultiplied axial spectra of TKE ($k_z^+ \tilde{E}_{TKE}^+$, for $\langle u_i' u_i' \rangle / 2$) and fluctuating enstrophy ($k_z^+ \tilde{E}_{\omega\omega}^+$, for $\langle \omega_i' \omega_i' \rangle$). As is evident in figure 2.13(a), the peak of $k_z^+ \tilde{E}_{TKE}^+$ occurs within the wavelength range $\lambda_z^+ \in [400, 755]$ near the pressure wall at $y^+ = 12.99$ in the non-rotating pipe flow. When the rotation number increases, the magnitude of $k_z^+ \tilde{E}_{TKE}^+$ attenuates gradually at small wavelengths and the peak mode moves increasingly towards larger wavelengths. At $Ro_\tau = 0.9$, the modal wavelength is extended to $\lambda_z^+ \in [940, 1415]$. As Ro_τ reaches 1, the premultiplied TKE spectrum collapses to zero, indicating complete suppression of turbulent fluctuations (and full laminarisation) of the flow field. By comparing figure 2.13(b) with figure 2.12(c), it is evident that the profile of the

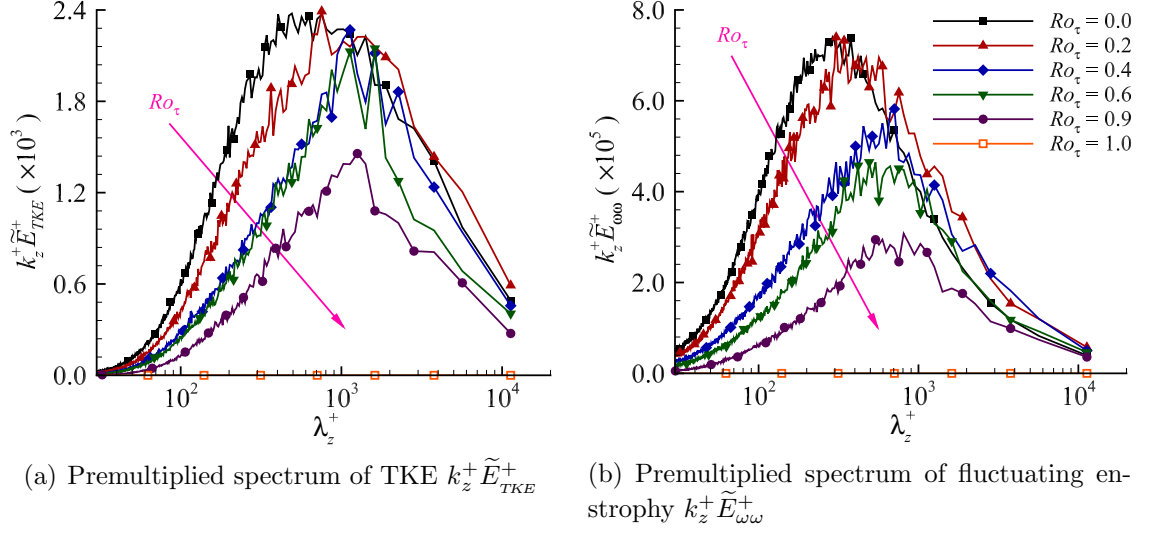


Figure 2.13: Premultiplied axial spectra of TKE and fluctuating enstrophy, $k_z^+ \tilde{E}_{TKE}^+$ and $k_z^+ \tilde{E}_{\omega\omega}^+$, along the streamwise line of the pipe at $r/R = 0.928$ (or, $y^+ = 12.99$) on the pressure side (located at $\beta = \pi/2$) for six different rotation numbers within $Ro_\tau \in [0, 1]$. The premultiplied spectra of TKE and fluctuating enstrophy have been non-dimensionalised using u_τ^3/ν and u_τ^5/ν^3 , respectively. For visual clarity, their magnitudes are amplified by a factor of 10^3 or 10^5 as labelled in round brackets.

premultiplied spectrum of fluctuating enstrophy $k_z^+ \tilde{E}_{\omega\omega}^+$ shown in figure 2.13(b) resembles its dominant (azimuthal) component $k_z^+ \tilde{E}_{\omega_\beta}^+$. In figure 2.13(b), the premultiplied fluctuating enstrophy spectrum shows that in the non-rotating pipe flow, dissipative eddy motions are distributed in smaller wavelength ranges of $\lambda_z^+ \in [225, 435]$ at the sampling position on the pressure side of the pipe. As the rotation number increases, the vorticity energy level decreases monotonically at small wavelengths, while exhibiting non-monotonic behaviour at larger wavelengths due to vortex stretching under the system rotation. At $Ro_\tau = 0.9$, the vortex filaments are elongated to larger axial scales, ranging from $\lambda_z^+ = 450$ to 1030 . At $Ro_\tau = 1$, the fluctuating enstrophy spectrum vanishes entirely, indicating a complete disappearance of dissipative vortices of turbulence.

The above analysis of the premultiplied spectra, $k_z^+ \tilde{E}_{TKE}^+$ and $k_z^+ \tilde{E}_{\omega\omega}^+$, is conducted at a fixed radial position of $y^+ = 12.99$ on the pressure side, corresponding to the local

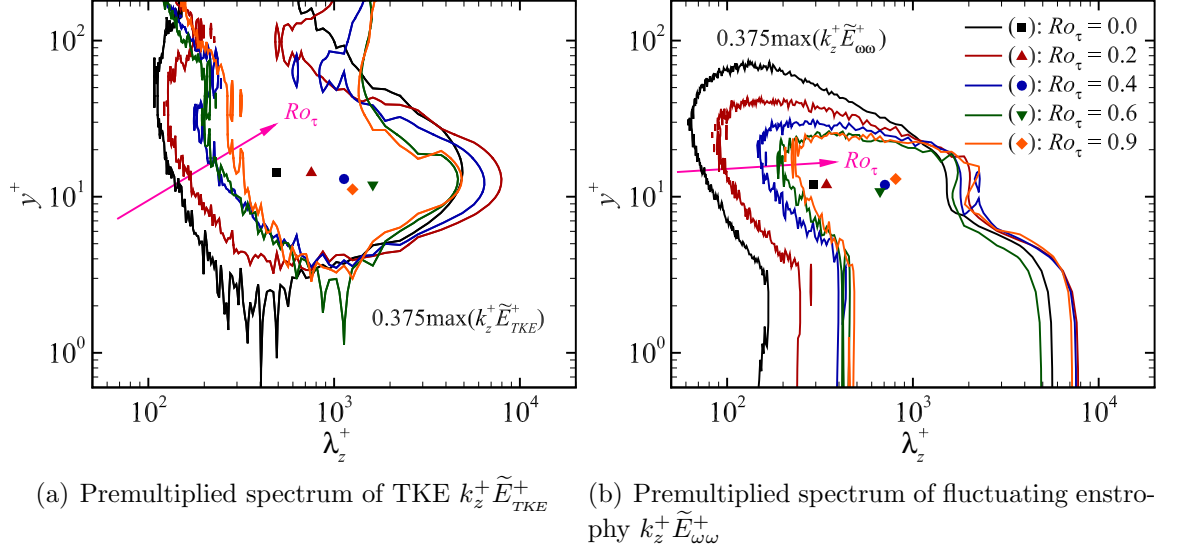


Figure 2.14: Isopleths of premultiplied axial spectrum of TKE and fluctuating enstrophy ($k_z^+ \tilde{E}_{TKE}^+$ and $k_z^+ \tilde{E}_{\omega\omega}^+$) in the half central vertical plane on the pressure side (located at $\beta = \pi/2$) of the pipe with respect to axial wavelength λ_z^+ and wall coordinate y^+ for five different rotation numbers of $Ro_\tau \in [0, 1]$. The isopleths in panels (a,b) correspond to $0.375 \max(k_z^+ \tilde{E}_{TKE}^+)$ and $0.375 \max(k_z^+ \tilde{E}_{\omega\omega}^+)$, respectively. Symbols demarcate the peak position at each of the five rotation numbers. Pink arrows indicate the direction of an increasing rotation number. The premultiplied spectra of TKE and fluctuating enstrophy have been non-dimensionalised using u_τ^3/ν and u_τ^5/ν^3 , respectively.

peak position of the fluctuating enstrophy in the non-rotating pipe flow. To refine our study of the premultiplied spectra, it is beneficial to consider how they vary across the entire radial direction. For this purpose, figure 2.14 depicts the isopleths of $k_z^+ \tilde{E}_{TKE}^+$ and $k_z^+ \tilde{E}_{\omega\omega}^+$ in the half central vertical plane on the pressure side (located at $\beta = \pi/2$) as a function of both wavelength λ_z^+ and wall coordinate y^+ for five rotation numbers, which vividly shows the near-wall effects on the vortex scales following the approach of Guala et al. (2006) and Hutchins and Marusic (2007). In the figure, the isopleths of $k_z^+ \tilde{E}_{TKE}^+$ and $k_z^+ \tilde{E}_{\omega\omega}^+$ are plotted at $3/8$ of their respective modal values, i.e., $0.375 \max(k_z^+ \tilde{E}_{TKE}^+)$ and $0.375 \max(k_z^+ \tilde{E}_{\omega\omega}^+)$, at each rotation number.

In figure 2.14(a), the isopleth of $k_z^+ \tilde{E}_{TKE}^+$ of the non-rotating pipe flow ($Ro_\tau = 0$) encompasses the largest area, implying the broadest distribution of the most energetic

eddy scales in the absence of radial system rotation. As the rotation number increases from $Ro_\tau = 0$ to 0.9, the isopleth apparently narrows down in the radial direction and shifts towards the pressure wall of the pipe (associated with a reduction in the y^+ value). Meanwhile, the range of axial wavelengths shrinks in spite of a slight stretch at large wavelengths at $Ro_\tau = 0.2$. Clearly, energetic turbulent motions of large length scales are favoured at low rotation numbers. As Ro_τ reaches 1, the pipe flow is completely laminarised, and the value of $k_z^+ \tilde{E}_{TKE}^+$ becomes identically zero (which cannot be shown in the figure). As is evident in figure 2.14(a), the radial system rotation has a limited impact on the radial position (in terms of the value of y^+) of the peak of $k_z^+ \tilde{E}_{TKE}^+$ as the rotation number increases from $Ro_\tau = 0$ to 0.9. However, the modal characteristic wavelength (in terms of the value of λ_z^+) varies substantially. The modes of $k_z^+ \tilde{E}_{TKE}^+$ range within $y^+ \in [11.14, 14.24]$ and $\lambda_z^+ \in [491.73, 1615.68]$.

Figure 2.14(b) compares the isopleths of the premultiplied fluctuating enstrophy spectrum $k_z^+ \tilde{E}_{\omega\omega}^+$ across various rotation numbers. Similar to the premultiplied TKE spectrum shown in figure 2.14(a), the fluctuating enstrophy spectrum also exhibits the largest isopleth-enclosed area in the non-rotating pipe flow. With increasing rotation, the isopleth gradually shrinks towards the pressure wall, especially at small axial wavelengths. It is also interesting to observe that the left isopleth bound of $k_z^+ \tilde{E}_{\omega\omega}^+$ shifts towards larger wavelengths as the rotation number increases, indicating an enhancement in the axial length scales of dissipative turbulent eddies (or instantaneous fine vortex filaments associated with ω'_i). In addition, two distinct peak values appear in the isopleth of $k_z^+ \tilde{E}_{\omega\omega}^+$. The first is located within the buffer layer as shown in figure 2.14(b), whereas the second occurs directly at the pipe wall ($y^+ = 0$), which cannot be shown here due to the use of the double logarithmic coordinate system. This interesting dual-peak distribution of $k_z^+ \tilde{E}_{\omega\omega}^+$ in the near-wall region on the pressure side of the pipe is consistent with the profile pattern of fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ in physical space presented previously in figure 2.11(d). As Ro_τ increases, the first peak remains nearly fixed in the wall-normal (radial) direction but under-

goes a shift towards larger axial wavelengths. By contrast, the second peak shows non-systematic variation under radial rotation (see also figure 2.11(d)). As the rotation number varies, the first mode of $k_z^+ \tilde{E}_{\omega\omega}^+$ is confined within $y^+ \in [10.8, 12.99]$ and $\lambda_z^+ \in [289.99, 807.84]$, while the second mode ranges within $\lambda_z^+ \in [706.86, 1615.68]$ at the wall on the pressure side of the pipe.

The anisotropy invariant map (AIM) of the vorticity anisotropy tensor b_{ij}^ω is a useful tool for diagnosing the rotating effects on fine turbulent vortical structures associated with vorticity fluctuations. The method of AIM was originally introduced to study the invariants and topology of the Reynolds stress anisotropy tensor (Lumley and Newman, 1977). By analogy, the vorticity anisotropy tensor can be defined (in terms of the vorticity fluctuation covariance tensor $\langle \omega'_i \omega'_j \rangle$) as

$$b_{ij}^\omega = \frac{\langle \omega'_i \omega'_j \rangle}{\langle \omega'_k \omega'_k \rangle} - \frac{\delta_{ij}}{3} \quad , \quad (2.7)$$

where δ_{ij} represents Kronecker delta. The tensor b_{ij}^ω has three independent invariants. The first invariant relates to the zero-trace property of an anisotropy tensor, i.e. $I_\omega = b_{ii}^\omega = 0$, while the second and third invariants are non-trivial and are expressed as

$$II_\omega = \frac{b_{ij}^\omega b_{ji}^\omega}{2} \quad \text{and} \quad III_\omega = \frac{b_{ij}^\omega b_{jk}^\omega b_{ki}^\omega}{3} \quad . \quad (2.8)$$

Based on the definitions II_ω and III_ω of b_{ij}^ω , a combined anisotropy invariant can be further defined as

$$A_\omega = \frac{III_\omega}{2} \left(\frac{II_\omega}{3} \right)^{-\frac{3}{2}} \quad . \quad (2.9)$$

Figure 2.15 compares the AIM of the vorticity anisotropy tensor b_{ij}^ω in the central vertical plane (located at $\beta = \pi/2$) of the pipe at five rotation numbers. The AIM of the vorticity fluctuations on the suction and pressure sides is displayed in figures 2.15(a) and (b), respectively. In the figures, the distribution of the tensor invariants is naturally bounded by the so-called Lumley triangle, which consists of three distinct boundaries. The upper boundary represents two-component (2C) tur-

bulence, strictly obeying the linear relation $II_\omega = 3III_\omega + 1/9$. The 2C turbulence state occurs mainly in the near-wall region of the pipe where $\langle \omega'_r \omega'_r \rangle \rightarrow 0$, as shown in figure 2.11(b). The left and right boundaries of the Lumley triangle are defined by $A_\omega = -1.0$ and 1.0 , respectively, featuring axisymmetric turbulence states when one of the three eigenvalues of b_{ij}^ω is larger ($III_\omega > 0$) or smaller ($III_\omega < 0$) than the other two which are equal in magnitude. As such, on the left boundary of the Lumley triangle, the shape of the ellipsoid formed by the eigenvalues of b_{ij}^ω is characterised by an oblate spheroid, featuring an axisymmetric expansion (disk-shaped) topology. By contrast, on the right boundary of the Lumley triangle, the shape of the ellipsoid of the eigenframe is characterised by a prolate spheroid, featuring an axisymmetric contraction (rod-shaped) topology (Pope, 2000). As labelled in figure 2.15, the three limiting points of the Lumley triangle correspond to the isotropic (ISO) turbulence state at $(III_\omega, II_\omega) = (0, 0)$, two-component axisymmetric (2C-AXI) turbulence state at $(III_\omega, II_\omega) = (-1/108, 1/12)$, and one-component (1C) turbulence state at $(III_\omega, II_\omega) = (2/27, 1/3)$. It should be noted that all trajectories within the AIM originate at the pipe wall (where $\langle \omega'_r \omega'_r \rangle = 0$ and $|r| = R$), corresponding to the upper boundary of the Lumley triangle, and progressively advance inward towards the pipe centre, ending at the lower limiting point ISO, where the turbulent state is isotropic such that $\langle \omega'_z \omega'_z \rangle = \langle \omega'_r \omega'_r \rangle = \langle \omega'_\beta \omega'_\beta \rangle$ (at $r = 0$), an identity that is evidenced by the DNS data shown previously in figure 2.11. In each panel of figures 2.15(a) and (b), a zoom-in inset is included to show more detailed trajectories of invariants at different rotation numbers as the ISO point is approached.

As shown in figures 2.15(a,b), in regions very close to the pipe wall of $|r/R| \in [0.986, 1.0]$ (or, $y^+ < 2.45$), the flow exhibits characteristics of 2C turbulence (close to the upper boundary of the Lumley triangle) at all rotation numbers. However, as the wall-normal distance from the pipe wall increases, the trajectory advances inward, getting closer to the right boundary of $A_\omega = 1.0$. As such, the AIM transitions from 2C turbulence to rod-like structures, followed by a further shift towards

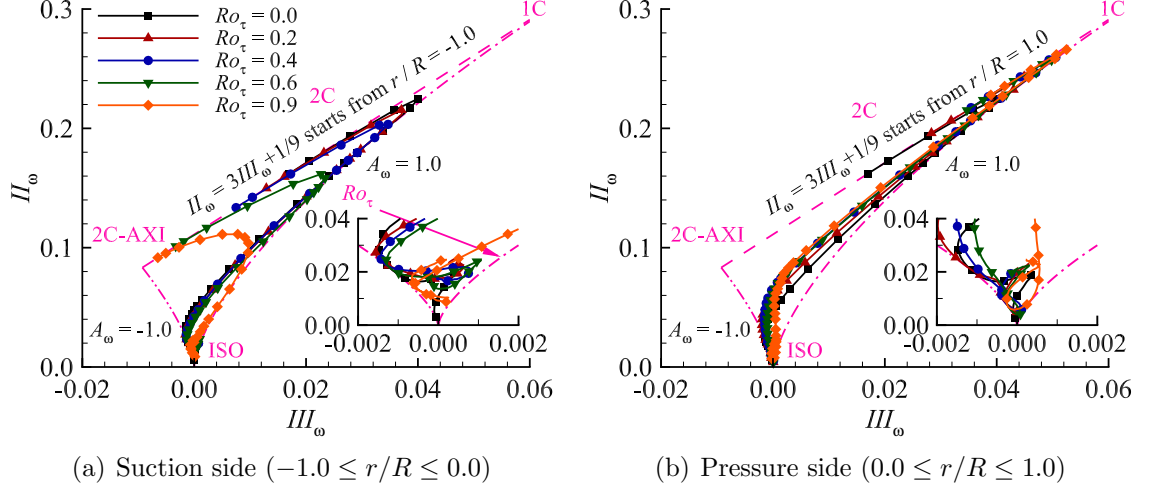


Figure 2.15: AIM of the anisotropy tensor of vorticity fluctuations (b_{ij}^ω) in the central vertical plane (located at $\beta = \pi/2$) of the pipe for five different rotation numbers Ro_τ . Panels (a,b) display the distributions on the suction and pressure sides of the pipe, respectively. The dashed lines indicate 2C turbulence, characterised by the relation $II_\omega = 3III_\omega + 1/9$, while the dot-dashed and dot-dot-dashed lines represent axisymmetric turbulence, corresponding to the axisymmetry parameters $A_\omega = 1.0$ and $A_\omega = -1.0$, respectively.

disk-like structures as the pipe centre is approached. At the pipe centre, all invariant trajectories converge precisely to the ISO point. From figure 2.15(a), it is clear that the AIM near the suction wall shows an increasing tendency towards the 2C-AXI turbulence point $(-1/108, 1/12)$ as the rotation number increases. By contrast, as shown in figure 2.15(b), the AIM near the pressure wall favours increasingly the 1C point $(2/27, 1/3)$ as Ro_τ increases, corresponding to the enhanced axisymmetric expansion. From the inset of figure 2.15(a), it is interesting to see that turbulence deviates increasingly from the axisymmetric contraction ($A_\omega = -1.0$) to the axisymmetric expansion ($A_\omega = 1.0$) at high elevations from the wall (before reaching the ISO point at the pipe centre) on the suction side of the pipe as the rotation number increases. However, such a monotonic trend is not observed on the pressure side of the pipe as shown in the inset panel of figure 2.15(b), because the invariant trajectory at $Ro_\tau = 0.2$ is closer to the left boundary of the Lumley triangle than that at $Ro_\tau = 0$. Clearly, this observation of a subtle expansion of the region associated with disk-like

turbulence state on the pressure side of the pipe at $Ro_\tau = 0.2$ coincides with the observation of the peak value of $u'_{z,rms}+$ in figure 2.9 and that of fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ in figure 2.11(d).

2.2.3 Transport of fluctuating enstrophy

So far, the effects of radial rotation on the turbulent vortical structures have been investigated using RMS vorticity fluctuations, fluctuating enstrophy and premultiplied vorticity spectra. To gain a deeper understanding of how vorticity intensity varies in radially rotating pipe flows, it is crucial to examine the individual budget terms of the fluctuating enstrophy transport equation, which takes the following form in a steady-state pipe flow

$$\frac{\partial \langle \omega'_i \omega'_i \rangle}{\partial t} = C_{\omega\omega} + P_{\omega\omega}^g + P_{\omega\omega}^t + P_{\omega\omega}^m + P_{\omega\omega}^s + \varepsilon_{\omega\omega} + D_{\omega\omega}^t + D_{\omega\omega}^v + P_{\omega\omega}^c = 0 \quad . \quad (2.10)$$

In total, there are nine budget terms. In sequence, they represent convection $C_{\omega\omega}$, production by mean vorticity gradients $P_{\omega\omega}^g$, production by turbulent vortex stretching $P_{\omega\omega}^t$, production by mean vortex stretching $P_{\omega\omega}^m$, production by mean shear flow stretching $P_{\omega\omega}^s$, viscous dissipation $\varepsilon_{\omega\omega}$, turbulent diffusion $D_{\omega\omega}^t$, viscous diffusion $D_{\omega\omega}^v$, and Coriolis production $P_{\omega\omega}^c$. Notably, because the rotating axis is aligned with the x -direction (i.e., at $\beta = 0$), the Coriolis production term further simplifies to

$$P_{\omega\omega}^c = 4\Omega \left\langle \omega'_i \frac{\partial u'_i}{\partial r} \right\rangle \quad . \quad (2.11)$$

The transport equation of the fluctuating enstrophy, given by equation (2.10), is not directly available in the literature with respect to a cylindrical coordinate system or in the context of a radially rotating flow. To facilitate the analysis, its derivation and detailed form are provided in Appendix B.1 and in equation (B.11).

Figure 2.16 compares the budget balance of the fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ in

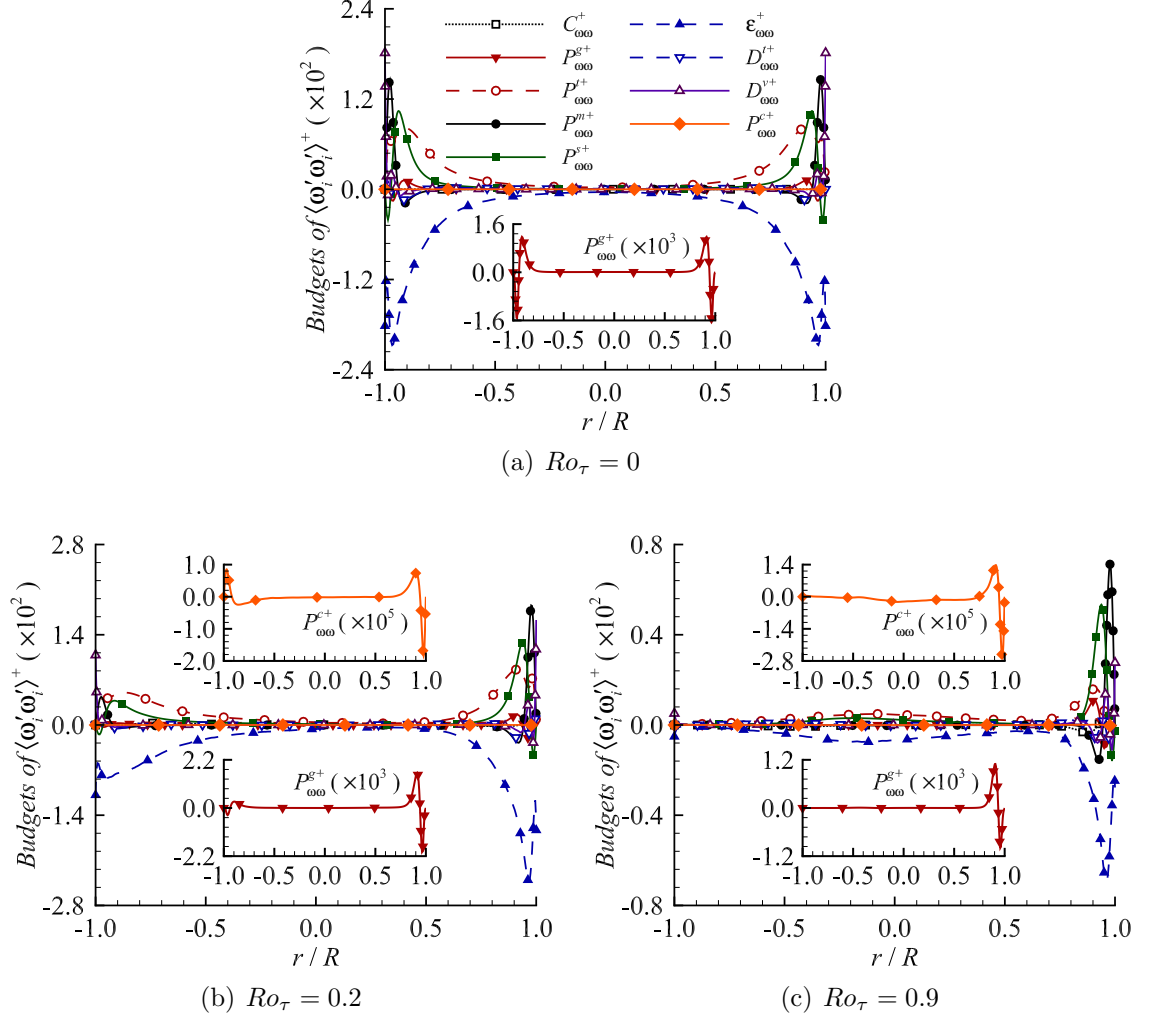


Figure 2.16: Budget balance of fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ in the central vertical plane (located at $\beta = \pi/2$) of the pipe at $Ro_\tau = 0, 0.2$ and 0.9 . All budget terms have been non-dimensionalised by u_τ^6/ν^3 . For visual clarity, their magnitudes are amplified by a factor of 10^2 as labelled in round brackets.

the central vertical plane (located at $\beta = \pi/2$) of the pipe for three different rotation numbers $Ro_\tau = 0, 0.2$ and 0.9 . To show clearly the radial system rotation effects on the production terms $P_{\omega\omega}^{g+}$ and $P_{\omega\omega}^{c+}$ (due to the mean vorticity gradients and Coriolis forces, respectively), their profiles are plotted individually in inset panels. As is evident in figure 2.16(a), in the non-rotating pipe flow, the budget balance of $\langle \omega'_i \omega'_i \rangle^+$ is dominated by the viscous diffusion $D_{\omega\omega}^{v+}$ as the source term and viscous dissipation $\epsilon_{\omega\omega}^+$ as the sink term near the wall, while away from the wall, it is mainly balanced

between three production terms, $P_{\omega\omega}^{t+}$, $P_{\omega\omega}^{m+}$ and $P_{\omega\omega}^{s+}$, and viscous dissipation term $\varepsilon_{\omega\omega}^+$. It is evident that the viscous dissipation term functions as a sink throughout the entire radial direction. In figures 2.16(a-c), the budget terms on the suction side significantly decrease in magnitude with an increasing rotation number. Clearly, as soon as the radial system rotation is imposed, the profiles of all budget terms become asymmetrical in the radial direction. As the rotation number increases from $Ro_\tau = 0.2$ to 0.9, the magnitudes of all budget terms attenuate significantly, especially on the suction side of the pipe where the laminarisation process first begins. A significant reduction in the value of the turbulent vortex stretching term $P_{\omega\omega}^{t+}$ at a high rotation number indicates that the radial system rotation strongly suppresses turbulent production of fluctuating enstrophy through the mechanism of fluctuating vortex stretching. As illustrated in figure 2.16(c), at $Ro_\tau = 0.9$, the budget of $\langle\omega'_i\omega'_i\rangle^+$ on the pressure side is balanced mainly by the two production terms $P_{\omega\omega}^{m+}$ and $P_{\omega\omega}^{s+}$ as the sources and by the viscous dissipation term $\varepsilon_{\omega\omega}^+$ as the sink for the fluctuating enstrophy, while the influences from other budget terms (including $P_{\omega\omega}^{t+}$) are relatively small. Clearly, at high rotation numbers, the fluctuating enstrophy production is mainly attributed to $P_{\omega\omega}^{m+}$ through the mechanism of the mean vortex stretching of turbulence, indicated by $\langle\omega'_i\nabla_j u'_i\rangle$, acting on the mean vorticity $\langle\omega_i\rangle$, and $P_{\omega\omega}^{s+}$, through the mechanism of the mean shear flow stretching, indicated by mean velocity gradients $\nabla_j\langle u_i\rangle$, under the turbulent “stresses” of the vorticity fluctuation covariance tensor $\langle\omega'_i\omega'_j\rangle$. At the highest rotation number tested ($Ro_\tau = 1$), all budget terms vanish and the pipe flow becomes completely laminarised.

As shown in the inset panels of figures 2.16(b,c), the Coriolis production term $P_{\omega\omega}^{c+}$ (explicitly featuring the radial system rotation speed Ω) on the suction side contributes positively to the fluctuating enstrophy $\langle\omega'_i\omega'_i\rangle^+$. As the rotation number increases from $Ro_\tau = 0.2$ to 0.9, the magnitude of $P_{\omega\omega}^{c+}$ decreases and increases on the suction and pressure sides of the pipe, respectively. On the pressure side of the pipe, as shown in figures 2.16(b,c), it is interesting to observe that $P_{\omega\omega}^{c+}$ acts as a sink term

in the vicinity of the pipe wall, but rapidly transitions to a source term as the distance from the wall increases. Nevertheless, as is evident in figure 2.16, the magnitude of the Coriolis production term $P_{\omega\omega}^{c+}$ is much smaller than those of the other four production terms ($P_{\omega\omega}^{g+}$, $P_{\omega\omega}^{m+}$, $P_{\omega\omega}^{t+}$ and $P_{\omega\omega}^{s+}$). This indicates that the contribution of Coriolis forces to the production of fluctuating enstrophy is mostly by altering the fluctuating vortex stretching vector, mean shear-flow velocity gradient, mean vorticity and its gradient, and coupling of vorticity-velocity and vorticity-vorticity fluctuations of the flow field instead of their direct participation in production through the $P_{\omega\omega}^{c+}$ term alone.

2.2.4 Transport of fluctuating enstrophy spectrum

In section 2.2.3, the budget balance of the transport equation of fluctuating enstrophy $\langle\omega'_i\omega'_i\rangle$ has been analysed in physical space. To advance our understanding of the subject, it is important to further examine its corresponding turbulent transport mechanisms in spectral space. In the context of a steady-state radially rotating flow, the transport equation of fluctuating enstrophy spectrum $\tilde{E}_{\omega\omega}$ is expressed as

$$\frac{\partial \tilde{E}_{\omega\omega}}{\partial t} = \tilde{C}_{\omega\omega} + \tilde{P}_{\omega\omega}^g + \tilde{P}_{\omega\omega}^t + \tilde{P}_{\omega\omega}^m + \tilde{P}_{\omega\omega}^s + \tilde{T}_{\omega\omega} + \tilde{\varepsilon}_{\omega\omega} + \tilde{D}_{\omega\omega}^t + \tilde{D}_{\omega\omega}^v + \tilde{P}_{\omega\omega}^c = 0 \quad . \quad (2.12)$$

In total, there are 10 budget terms ($\tilde{C}_{\omega\omega}$, $\tilde{P}_{\omega\omega}^g$, ..., $\tilde{P}_{\omega\omega}^c$) in this equation. These budget terms for $\tilde{E}_{\omega\omega}$ in equation (2.12) in spectral space studied in this subsection directly correspond to those for $\langle\omega'_i\omega'_i\rangle$ in equation (2.10) in physical space described in subsection 2.2.3, except for one additional term, known as the interscale transport term $\tilde{T}_{\omega\omega}$, which is a unique feature in transport equation of fluctuating enstrophy spectrum and characterises the redistribution of turbulence energy across different wavelengths. Therefore, the names of the remaining nine budget terms of equation (2.12) are not repeated to keep the discussion concise. Equation (2.12) represents an advanced analysis of the transport phenomenon of fluctuating vorticity. Its usefulness in the

analysis of the vorticity field is comparable to recent studies of the transport equation of TKE spectrum (or broadly speaking, velocity-spectrum tensor) of the velocity field (Aulery et al., 2017; Mizuno, 2016; Lee and Moser, 2019; Yang et al., 2020; Xiong et al., 2025). Despite its usefulness, equation (2.12) has not been studied in the literature due to its challenges in mathematical expression and physical interpretation of the budget terms. In fact, equation (2.12) is new to the literature, and its derivation and definitions of all ten budget terms are presented in Appendix B.2 and in equation (B.20).

Figure 2.17 compares the budget balance of the premultiplied axial spectra of fluctuating enstrophy $k_z^+ \tilde{E}_{\omega\omega}^+$ at three rotation numbers $Ro_\tau = 0, 0.2$ and 0.9 . In this and the following figures for the study of the budget balance of equation (2.12), the premultiplied spectra of the budget terms are non-dimensionalised in wall units, e.g. $k_z^+ \tilde{C}_{\omega\omega}^+ = k_z \tilde{C}_{\omega\omega} / (u_\tau^7 / \nu^4)$. All profiles are plotted along the axial direction of the pipe at $r/R = 0.928$ (or, $y^+ = 12.99$) on the pressure side (located at $\beta = \pi/2$), where a local peak is observed in the fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ at $Ro_\tau = 0$, as shown in figure 2.11(d). In figures 2.17(a-c), the wavelength range enclosed by two vertical dot-dashed lines highlights the dome peak of $k_z^+ \tilde{E}_{\omega\omega}^+$ shown previously in figure 2.13(b). As is evident in figure 2.17(a), the budget balance of $k_z^+ \tilde{E}_{\omega\omega}^+$ in the non-rotating pipe flow is primarily balanced by the two production terms, $k_z^+ \tilde{P}_{\omega\omega}^{t+}$ and $k_z^+ \tilde{P}_{\omega\omega}^{s+}$, interscale transport term $k_z^+ \tilde{T}_{\omega\omega}^+$, and viscous dissipation term $k_z^+ \tilde{\varepsilon}_{\omega\omega}^+$, at the radial sampling position of $r/R = 0.928$ (or, $y^+ = 12.99$) near the pressure wall of the pipe. The production terms arising from turbulent vortex stretching $k_z^+ \tilde{P}_{\omega\omega}^{t+}$ and mean shear flow stretching $k_z^+ \tilde{P}_{\omega\omega}^{s+}$ peak in wavelength ranges $\lambda_z^+ \in [105, 195]$ and $\lambda_z^+ \in [245, 345]$, respectively. Apparently, among the two premultiplied axial spectrum terms of production due to turbulent vortex stretching ($k_z^+ \tilde{P}_{\omega\omega}^{t+}$) and mean shear flow stretching ($k_z^+ \tilde{P}_{\omega\omega}^{s+}$), term $k_z^+ \tilde{P}_{\omega\omega}^{t+}$ mainly contributes to the budget balance of $k_z^+ \tilde{E}_{\omega\omega}^+$ at smaller wavelengths, whereas term $k_z^+ \tilde{P}_{\omega\omega}^{s+}$ is more dominant at moderate wavelengths. It is observed that the peak mode of $k_z^+ \tilde{P}_{\omega\omega}^{s+}$ coincides with that of

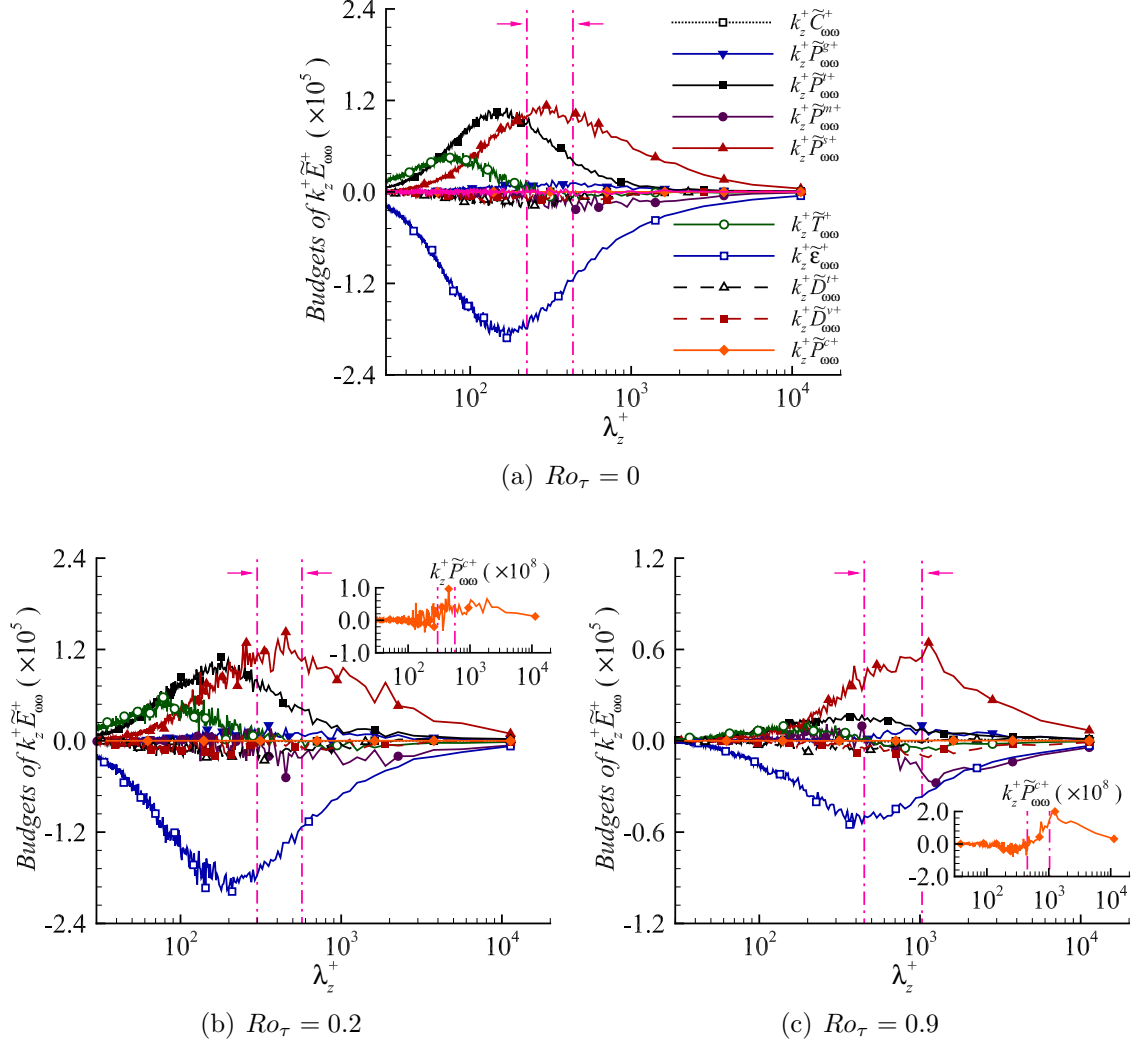


Figure 2.17: Budget balance of premultiplied axial spectra of fluctuating enstrophy $k_z^+ \tilde{E}_{\omega\omega}^+$ along the streamwise line of the pipe at $r/R = 0.928$ (or, $y^+ = 12.99$) on the pressure side (located at $\beta = \pi/2$) for three different rotation numbers $Ro_\tau = 0, 0.2$ and 0.9 . The modal wavelength range corresponds to the dome peak of $k_z^+ \tilde{E}_{\omega\omega}^+$, shown previously in figure 2.13(b), is enclosed by two vertical dash-dotted lines. The premultiplied spectra of all ten budget terms of equation (2.12) have been non-dimensionalised by u_τ^7/ν^4 . For visual clarity, their magnitudes are amplified by a factor of 10^5 as labelled in round brackets.

$k_z^+ \tilde{E}_{\omega\omega}^+$ (enclosed by two vertical dash-dotted lines), implying that the mean shear flow stretching directly empowers the fluctuating enstrophy spectrum at the identical wavelength scales. By contrast, the interscale transport term $k_z^+ \tilde{T}_{\omega\omega}^+$ is negatively-

and positively-valued in large and small wavelengths, facilitating the redistribution of turbulent vorticity energy from large-scale to small-scale vortices for dissipation. The peak of the interscale transport term $k_z^+ \tilde{T}_{\omega\omega}^+$ occurs at small wavelengths within $\lambda_z^+ \in [55, 85]$, where the viscous dissipation term $k_z^+ \tilde{\varepsilon}_{\omega\omega}^+$, acting as an energy sink, counterbalances the combined influences of $k_z^+ \tilde{P}_{\omega\omega}^{t+}$, $k_z^+ \tilde{P}_{\omega\omega}^{s+}$, and $k_z^+ \tilde{T}_{\omega\omega}^+$, with its peak located in the relatively small wavelength range of $\lambda_z^+ \in [130, 225]$.

The magnitudes of the four dominant budget terms ($k_z^+ \tilde{P}_{\omega\omega}^{t+}$, $k_z^+ \tilde{P}_{\omega\omega}^{s+}$, $k_z^+ \tilde{T}_{\omega\omega}^+$ and $k_z^+ \tilde{\varepsilon}_{\omega\omega}^+$) reach their maxima (as indicated by their peaks and valleys) at $Ro_\tau = 0.2$. Similar to its behaviour in physical space analysed previously in section 2.2.3, the premultiplied axial spectrum term $k_z^+ \tilde{P}_{\omega\omega}^{c+}$ due to Coriolis production emerges as a small supplemental source of turbulent vorticity energy that is significantly smaller than the other budget terms, as shown in figures 2.17(b,c). This again indicates that the contribution of Coriolis forces to the production of fluctuating enstrophy spectrum is mostly through physical mechanisms of altering the values of fluctuating vortex stretching vector, mean velocity and vorticity gradients, and correlations of vorticity-velocity and vorticity-vorticity fluctuation tensors rather than their direct participation in production through the Coriolis production term $k_z^+ \tilde{P}_{\omega\omega}^{c+}$. As shown in figure 2.17(c), the magnitudes of all dominant terms decrease drastically at $Ro_\tau = 0.9$. Interestingly, at this high rotation number, the premultiplied axial spectrum term $k_z^+ \tilde{P}_{\omega\omega}^{m+}$ caused by mean vortex stretching becomes one of the leading budget terms. It acts as a source at small wavelengths but as a sink that consumes energy at large wavelengths. By comparing figures 2.17(a-c), it is clear that the radial system rotation consistently shifts the modes of all budget terms towards larger wavelengths with an increasing rotation number. This observation implies that as the rotation number increases, the small-scale dissipative eddies tend to be stretched into rod-like structures by enhanced axisymmetric expansion on the pressure wall of the pipe, a conclusion that is consistent with the observations of the AIM in figure 2.15(b).

Chapter 3

DNS of turbulent heat transfer in a radially rotating circular pipe flow

In this chapter, turbulent heat transfer in a circular pipe subjected to radial system rotation is tested for a wide range of rotation number of $Ro_\tau \in [0, 1]$. For clarity, the heat convection features are investigated through a comparison with those of a non-rotating case. The Coriolis effects on the mean and turbulent thermal structures are examined in both physical and spectral spaces. In addition, by analysing the budget balance of the turbulent heat fluxes transport equations, the radially rotating effects on the turbulent thermal energy transfer are studied as well. The content of this chapter has recently been published as a journal article (Zhang and Wang, 2026).

3.1 Tests Cases and Numerical Algorithm

Figure 3.1 shows the side and front views of a circular cooling pipe subjected to radial system rotation. In this figure, r , β and z represent the radial, azimuthal and axial coordinates, and the three corresponding velocity components are denoted as u_r , u_β and u_z , respectively. The radial system rotation is implemented by im-

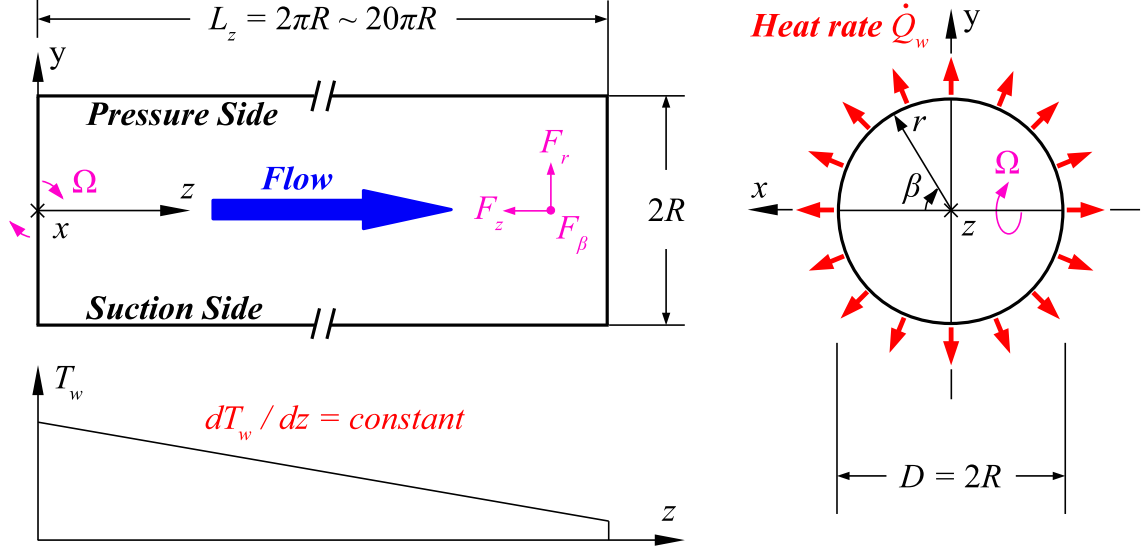


Figure 3.1: Side and front views of the computational domain of a circular pipe subjected to radial system rotation of constant counterclockwise angular speed Ω about the positive x -axis, strictly following the convention for a right-handed coordinate. The radial, azimuthal and axial Coriolis force components are denoted as F_r , F_β and F_z , respectively. A cross symbol “ $\times$ ” denotes the tail of an arrow pointing into the page for the x -axis in the left panel and the z -axis in the right panel, while a dot symbol “ $\bullet$ ” denotes the head of an arrow pointing out of the page for the Coriolis force component F_β . The pipe flow is cooled along the axial (or, z -) direction with a total heat rate $\dot{Q}_w$ drained from the wall.

posing a constant counterclockwise angular speed Ω around the positive x -direction, leading to three Coriolis force components $F_r = 2\Omega u_z \sin\beta$, $F_\beta = 2\Omega u_z \cos\beta$ and $F_z = -2\Omega (u_r \sin\beta + u_\beta \cos\beta)$ appearing in the r -, β - and z -directions, respectively. As shown in side view of figure 3.1, the radial Coriolis force component F_r points to the top side but away from the bottom side of the circular pipe, respectively. Therefore, the top side of the circular pipe is referred to as the ‘pressure side’, while the bottom side of the pipe is referred to as the ‘suction side’ to reflect the effects of the radial Coriolis force component.

The flow is fully developed both hydraulically and thermally in the pipe. Periodic and no-slip boundary conditions are imposed along the axial direction and at the solid wall, respectively. The flow is cooled down along the axial direction by wall

Table 3.1: Summary of test cases, flow parameters, and spatial and temporal resolutions (in comparison with the Batchelor length scale η_B and Kolmogorov time scale τ_η) at the friction Reynolds number of $Re_\tau = 180$.

Case #	L_z	Ro_τ	Re_b	N_z	N_{tot}	$\max(\Delta/\eta_B)$	$\max(\Delta t/\tau_\eta)$
D1	$2\pi R$	0.2	5366	96	2,209,632	2.71	0.0210
D2	$5\pi R$		5326	240	5,524,080	2.73	0.0210
D3	$8\pi R$		5342	384	8,838,528	2.70	0.0208
D4	$12\pi R$		5316	576	13,257,792	2.68	0.0210
0	$20\pi R$	0.0	5261	960	22,096,320	2.62	0.0199
1		0.1	5270			2.68	0.0205
2 (D5)		0.2	5332			2.71	0.0209
3		0.3	5417			2.63	0.0193
4		0.4	5407			2.63	0.0180
5		0.5	5343			2.59	0.0177
6		0.6	5252			2.53	0.0159
7		0.7	5152			2.40	0.0154
8		0.8	5036			2.30	0.0153
9		0.9	4899			2.13	0.0126
10		1.0	4615			0.00	0.0000

heat flux q''_w . Both temperature T and q''_w are homogeneous in the axial (z) direction but vary azimuthally due to the radial system rotation. Thus, the value of q''_w can be considered as axially-averaged. As such, $q''_w = q''_w(\beta)$. The peripherally-averaged wall heat flux is calculated as

$$q''_{wa} = \frac{1}{2\pi} \int_0^{2\pi} q''_w d\beta \quad , \quad (3.1)$$

which is kept constant, leading to a linear variation of the local wall temperature T_w along the z -direction. Apparently, in the non-rotating flow case, $q''_{wa} = q''_w$ holds. To examine the effect of radial system rotation on the turbulent transport of thermal energy, a total of 11 rotation numbers varying from $Ro_\tau = 0$ to 1 with an increment of 0.1 are tested at a fixed nominal Reynolds number of $Re_\tau = 180$ and Prandtl number of $Pr = 1$. The Prandtl number is defined as $Pr = \nu/\alpha$, where ν and α represent the kinematic viscosity and thermal diffusivity, respectively. In the context of a circular pipe flow, the rotation number and friction Reynolds number are defined as $Ro_\tau = 2\Omega R/u_\tau$ and $Re_\tau = u_\tau R/\nu$, respectively, where R is the pipe radius and u_τ

is the nominal wall friction velocity given by $u_\tau = \sqrt{-\Pi R/2\rho}$. Here, Π denotes the constant mean axial pressure gradient that drives the flow and ρ is the fluid density. Table 3.1 summarises fifteen test cases in terms of their key flow parameters and spatial and temporal resolutions.

In the radially rotating reference frame, the continuity, momentum and thermal energy equations of an incompressible flow are expressed as

$$\nabla \cdot \vec{u} = 0 \quad , \quad (3.2)$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{\Pi}{\rho} \hat{\mathbf{e}}_z - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} + \vec{F} \quad , \quad (3.3)$$

$$\frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta = -u_z \frac{dT_w}{dz} + \alpha \nabla^2 \theta \quad , \quad (3.4)$$

respectively. Here, $\vec{u}$ and p denote the velocity and pressure of the fluid, respectively, $\vec{F}$ represents the Coriolis force vector, and $\hat{\mathbf{e}}_z$ is the unit vector in the z -direction, with $|\hat{\mathbf{e}}_z| \equiv 1$. Following the approach of Johnston et al. (1972) and Kristoffersen and Andersson (1993), the effective pressure is defined as $p = p_s - \rho \Omega^2 r^2 / 2$, which combines the static pressure p_s and the centrifugal force. In equation (3.4), $\theta = \theta(r, \beta, z)$ is the local deficit temperature defined in reference to the local wall temperature T_w , as $\theta = T - T_w$. By definition, $\theta(R, \beta, z) \equiv 0$ holds at the wall (or, at $r = R$). For convenience, without causing confusion, θ will be referred to as the ‘temperature’ in the remainder of this paper. As shown in figure 3.1, the pipe flow is cooled down steadily along the axial direction under a constant peripherally-averaged wall heat flux q''_{wa} condition such that dT_w/dz is constant along the axial direction. Consequently, both the local wall temperature and the bulk mean temperature T_b decrease linearly along the axial direction, i.e.

$$\frac{dT_w}{dz} = \frac{dT_b}{dz} = -\frac{2q''_{wa}}{\rho C_p U_b R} \quad , \quad (3.5)$$

where C_p is the specific heat of the fluid. The total heat rate drained from the entire

pipe wall is $\dot{Q}_w = -\dot{m}C_p(T_{bo} - T_{bi})$, where $\dot{m}$ is the mass flow rate, and T_{bo} and T_{bi} denote the bulk mean temperatures at the inlet and outlet of the pipe, respectively. It relates to the peripherally-averaged wall heat flux q''_{wa} as $\dot{Q}_w = 2\pi RL_z q''_{wa}$. This treatment method for the temperature field and boundary conditions is similar to that used in the DNS studies of turbulent heat transfer in a concentric annular pipe flow by Chung and Sung (2003) and Bagheri and Wang (2021), LES and DNS studies of turbulent heat transfer in rotating square duct flows by Pallares and Davidson (2002) and Fang and Wang (2018), and DNS study of turbulent heat convection in elliptical pipe flows by Rosas and Wang (2022). In equation (3.5), U_b is the bulk mean velocity of the pipe flow, defined as

$$U_b = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \langle u_z \rangle r dr d\beta \quad . \quad (3.6)$$

All simulations were performed with a spectral-element method (SEM) code called ‘Semtex’ developed by Blackburn and Sherwin (2004). In order to apply the Semtex code to DNS of convective heat transfer problems, an additional SEM solver has been developed by our group for computing the temperature field by solving thermal energy equation (3.4). Specifically, the thermal energy equation (3.4) is treated by extending the first temporal substep to include the thermal energy advection term, discretised using a backward-facing differencing scheme. In addition, the third temporal substep is extended to implicitly advance the molecular diffusion term under the given boundary conditions (Fang and Wang, 2018; Blackburn et al., 2019). The new temperature solver has the same spectral accuracy as that of the original flow solver, and has been successfully applied to DNS studies of turbulent heat convection in spanwise rotating duct flows (Fang and Wang, 2018), radially rotating elliptical pipe flows (Rosas and Wang, 2022) and in rib-roughened channel flows (Xiong et al., 2024).

In order to utilise this SEM code for numerical solution of the current problem, a

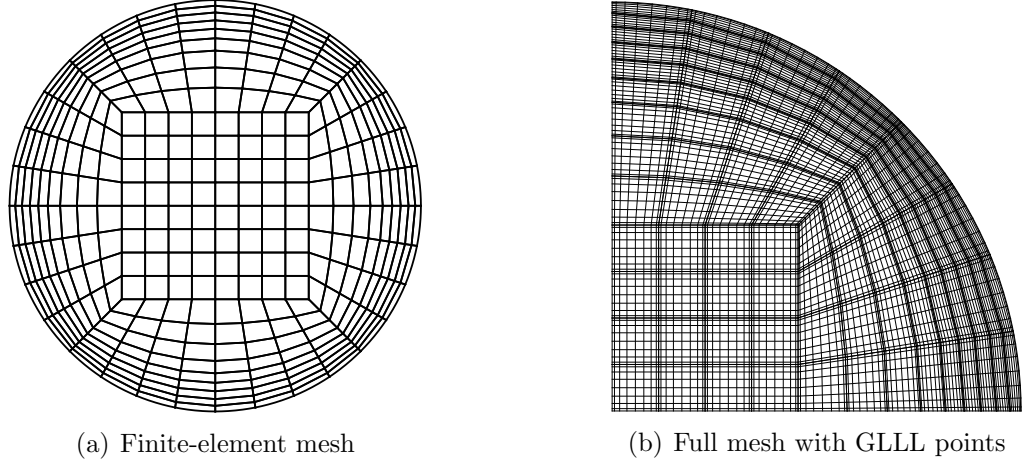


Figure 3.2: Cross-sectional view of the computational domain and mesh. The domain is first discretised into 352 quadrilateral finite elements as shown in panel (a). Then, within each element, an 8th-order GLLL polynomial is further used for refining the spatial resolution as shown in panel (b). For clarity, only a quarter of the cross section is plotted in panel (b).

Fourier expansion has been applied to the homogeneous axial (z) direction, while the cross-sectional (r - β) plane has been first discretised using quadrilateral elements, and then, each finite element is further discretised using an 8th-order GLLL orthogonal polynomial. As shown in figure 3.2(a), the cross-section (i.e., the r - β plane) of the pipe was divided into 352 finite elements. In figure 3.2(b), the full mesh based on further refinement using 8th-order GLLL interpolants within each finite element is demonstrated (for visual clarify, only a quarter of the cross section is shown). In the axial direction, all flow variables are expanded into the spectral space using Fourier series with $N_z = 96$ –960 modes, where N_z denotes the number of axial Fourier modes, depending on the pipe length of a specific test case. As indicated by table 3.1, the total number of ‘nodes’ varies from $N_{tot} = 2.2$ to 22.1 million for the 15 test cases. The calculation of the total number of ‘nodes’ N_{tot} is directly based on the number of Fourier modes in the axial direction and the number of the 8th-order GLLL interpolants (within each of the 352 finite elements) in the cross-sectional plane. The spatial resolutions in these 15 test cases range as $\Delta r^+ = 0.25$ –1.03, $R\Delta\beta^+ = 1.78$ –6.4,

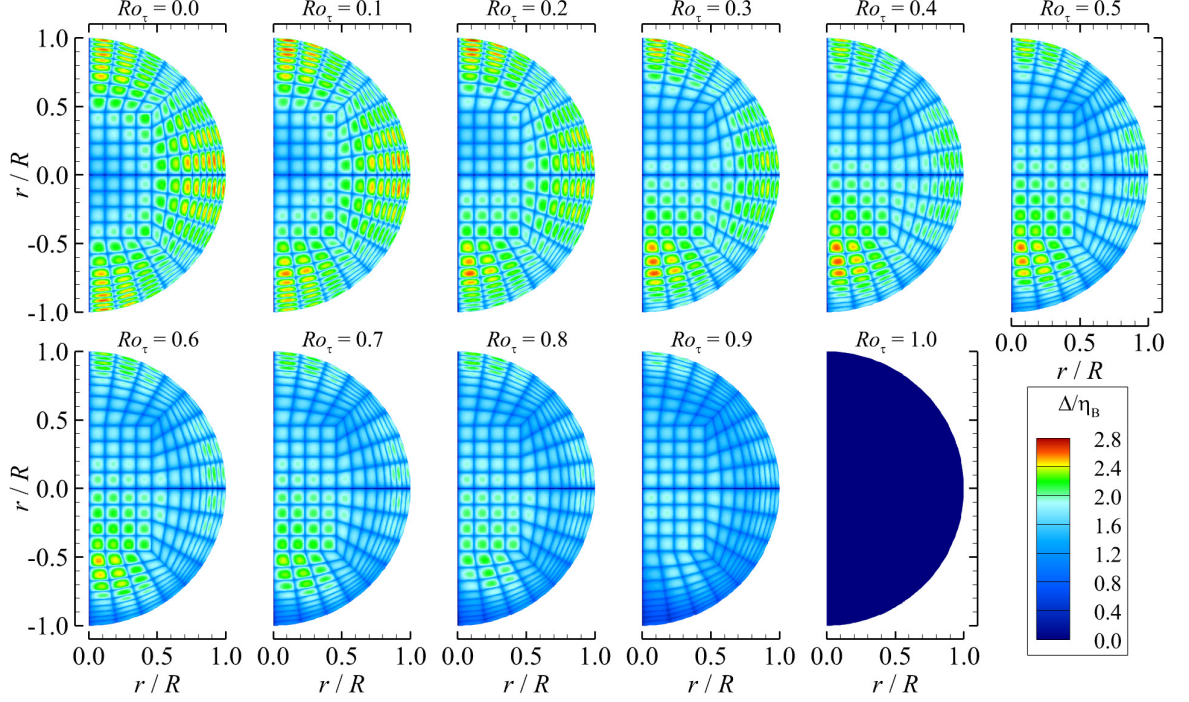


Figure 3.3: Contours of the ratio of the grid size to the Batchelor length scale Δ/η_B . Owing to the axial symmetry of the pipe domain, only one half of the pipe cross-section is shown for each rotation number of $Re_\tau = 0-1$. The results are based on cases 0–10, which possess the longest pipe length of $L_z = 20\pi R$, as listed in table 3.1.

and $\Delta z^+ = 11.78$. The current axial and azimuthal grid resolutions are comparable to those used in the DNS studies of stationary pipe flows at $Re_\tau = 180$ of Wagner et al. (2001), Redjem-Saad et al. (2007), Wu and Moin (2008) and Saha et al. (2011). However, the radial grid resolution of the current research is much finer than those reported in these classical DNS papers, which were up to $\max(\Delta r^+) = 4.32$ in Wagner et al. (2001) and $\max(\Delta r^+) = 1.647$ in Wu and Moin (2008).

To refine my assessment of the grid resolution, the ratio of the grid size to the Batchelor length scale Δ/η_B can be further examined. The grid size is calculated as $\Delta = [(\Delta r) \times (r\Delta\beta) \times (\Delta z)]^{1/3}$ whereas the Batchelor length scale for the turbulent scalar field is defined by $\eta_B = \eta/\sqrt{Pr}$, where $\eta = (\nu^3/\varepsilon)^{1/4}$ denotes the Kolmogorov length scale and ε is the TKE dissipation rate. Given that $Pr = 1$, $\eta_B = \eta$. To meet the stringent resolution requirement of DNS on resolving the smallest scales of

turbulent thermal flows, the grid spacing must be kept on the order of the Batchelor length scale, i.e., $O(\Delta/\eta_B) \sim O(1)$. Figure 3.3 shows the distribution of Δ/η_B for 11 rotating cases corresponding to the longest pipe of $L_z = 20\pi R$, based on cases 0–10 listed in table 3.1. From the figure, it is confirmed that the DNS criterion of $O(\Delta/\eta_B) \sim O(1)$ is strictly satisfied for all cases. The maximum values of Δ/η_B for all 15 test cases are summarised in table 3.1, which range from 0 to 2.73. Furthermore, it should be indicated that the Semtex code developed based on the SEM algorithm is of a spectral precision, highly accurate for performing DNS and free from any non-physical numerical dissipation.

The simulations were initiated from a laminar flow field perturbed by additional small random fluctuations, and statistical data were collected only after the flow had evolved into a fully-developed and statistically-stationary state. All DNS runs were conducted with a constant time step of $\Delta t^+ = u_\tau^2 \Delta t / \nu = 0.0492$, and the statistical moments were sampled for every 1,000 time steps. In total, 300 snapshots containing instantaneous velocity and temperature fields (approximately 310 GB data) over an averaging time of $t_{avg} = 82.128$ LETOTs were collected for computing turbulence statistics once the flow field became fully developed and statistically stationary after a precursor simulation for each of the 15 test cases. The CFL number was kept less than 0.5 during the simulations. The temporal resolution can be further assessed by comparing the time step with the Kolmogorov time scale defined as $\tau_\eta = \sqrt{\nu/\varepsilon}$. The maximum values of the ratio $\Delta t/\tau_\eta$ for all test cases are listed in table 3.1, which clearly indicate that the smallest turbulence time scales are well resolved, as the maximum value of the ratio remains very small, with $\Delta t/\tau_\eta \leq 0.021$. All DNS runs and data storage were performed on the supercomputer of the Digital Research Alliance of Canada.

To assess the temporal convergence of the temperature field, figures 3.4(a,b) show the central vertical profiles (at $\beta = \pi/2$) of the temperature variance $\langle \theta' \theta' \rangle^+$ (or, TSE) at a low rotation number of $Ro_\tau = 0.2$ and a high rotation number of $Ro_\tau = 0.8$,

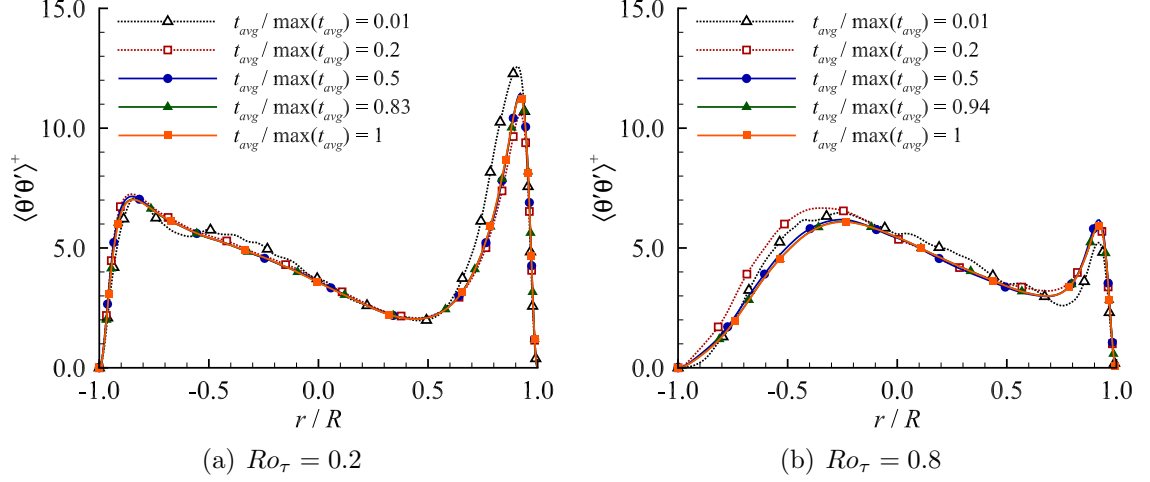


Figure 3.4: Influence of the averaging time t_{avg} on the accuracy of the temperature variance $\langle \theta' \theta' \rangle^+$ at two rotation numbers, $Ro_\tau = 0.2$ and 0.8 , evaluated in the central vertical plane (located at $\beta = \pi/2$) of the pipe. Quantity $\max(t_{avg})$ represents the sampling time duration of 82.128 LETOTs actually used for performing DNS at each rotation number of cases 0–10.

respectively. The central vertical position is examined because the follow-up analysis highlights the influence of radial system rotation on the statistical moments on the top pressure side ($r/R = 1$) and bottom suction side ($r/R = -1$) of the pipe. In the figure, the averaging time t_{avg} has been normalised by the actual sampling time duration used for performing DNS at all rotation numbers of cases 0–10, i.e., $\max(t_{avg}) = 82.128$ LETOTs. As is evident in figures 3.4(a,b), the results obtained at the two smallest time durations of $t_{avg}/\max(t_{avg}) = 0.01$ and 0.2 are unsatisfactory, as the profiles of $\langle \theta' \theta' \rangle^+$ deviate from the converged ones. However, the results of $\langle \theta' \theta' \rangle^+$ quickly become converged once the time duration goes beyond one half the actual sampling duration (i.e., $t_{avg}/\max(t_{avg}) \geq 0.5$). Clearly, the adopted sampling time duration of $t_{avg} = 82.128$ LETOTs is sufficient to ensure statistically converged results for the present investigation of turbulent heat convection in a radially rotating pipe flow.

In the subsequent investigation of the wall-scaling behaviours of the velocity and temperature fields for test cases of different rotation numbers, the wall coordinates and non-dimensional variables (indicated by superscript ‘+’) are calculated through

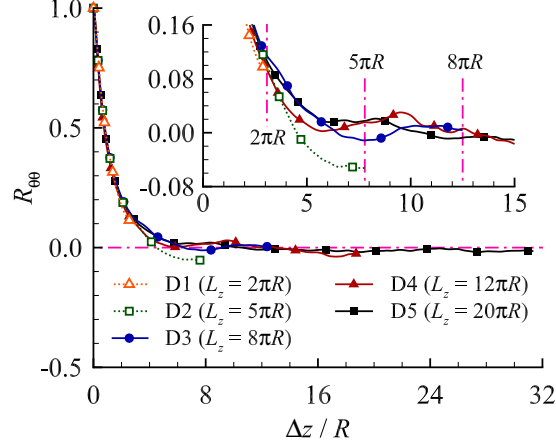


Figure 3.5: Influence of pipe length on the DNS results with respect to the two-point autocorrelation coefficient $R_{\theta\theta}$ at the same rotation number $Ro_\tau = 0.2$. All values are calculated along the streamwise direction at $r/R = 0.921$ (or at wall-normal position $y^+ = 14.24$) on the pressure side (located at $\beta = \pi/2$) of test cases D1–D5 (of pipe lengths $L_z = 2\pi R$, $5\pi R$, $8\pi R$, $12\pi R$ and $20\pi R$) listed in table 3.1.

non-dimensionalisation based on the nominal wall friction velocity u_τ and nominal wall friction temperature defined as $T_\tau = q''_{wa}/(\rho C_p u_\tau)$.

3.2 Results and Discussion

3.2.1 Minimum pipe length for performing DNS of heat convection

As mentioned, the primary objective of this research is to examine the effects of radial system rotation on turbulent pipe flows based on 11 test cases (i.e., cases 0–10) of different rotation numbers varying from $Ro_\tau = 0$ to 1. The pipe length for these 11 test cases is $L_z = 20\pi R$ following the recommendation of Zhang and Wang (2019) based on their DNS study of the turbulent flows in a circular pipe subjected to radial system rotations.

To ensure that this pipe length ($L_z = 20\pi R$) is suitable for conducting DNS of

turbulent heat convection in radially rotating pipe flows, a sideline comparative study of pipe length effects on DNS of turbulent convection has been conducted based on five additional test cases (i.e., cases D1–D5) at rotation number $Ro_\tau = 0.2$. The reason this rotation number is selected for performing these five DNS runs is that the turbulence scalar energy (TSE) or temperature variance $\langle \theta' \theta' \rangle^+$ is the highest at this rotation number. As shown in table 3.1, for these five DNS test cases D1–D5, the pipe length varies from $L_z = 2\pi R$ to $20\pi R$. Figure 3.5 compares the two-point autocorrelation coefficient $R_{\theta\theta}$ for five different pipe lengths, respectively. All profiles are calculated along the streamwise direction at $r/R = 0.921$ (or at wall-normal position $y^+ = 14.24$) on the pressure side (located at $\beta = \pi/2$) of the pipe at $Ro_\tau = 0.2$. It will be shown later that the temperature variance $\langle \theta' \theta' \rangle$ reaches its peak value at this specific radial position and rotation number. In the figure, the two-point correlation $R_{\theta\theta}$ tends to approach zero with an increasing two-point separation $\Delta z/R$. However, it is evident that only the profiles obtained by the three longer pipes of $L_z = 8\pi R$, $12\pi R$ and $20\pi R$ (associated with cases D3–D5) fully reach zero at $\Delta z/R \approx 10$, indicating that the minimum pipe length has to be larger than $L_z = 20R$ or $6.4\pi R$. This conclusion is consistent with the DNS of turbulent heat transfer in a stationary pipe flow by Saha et al. (2011).

Figure 3.6 displays the isopleths of premultiplied spectrum of temperature fluctuations $k_z^+ \tilde{E}_{\theta\theta}^+$ with respect to axial wavelength λ_z^+ and wall coordinate y^+ along the central vertical plane (for $\beta = \pi/2$) on the pressure side of the pipe. Here, the axial energy spectrum of temperature fluctuations is defined as $\tilde{E}_{\theta\theta}(k_z) = \text{Re}\{\widehat{\theta'}^* \widehat{\theta'}\}$. Operator $\text{Re}\{\cdot\}$ and superscript $(\cdot)^*$ denote the real part and conjugate of a complex number, respectively. A hat denotes Fourier transform in the axial direction of a variable (θ) . The wavelength is defined as $\lambda_z = 2\pi/k_z$, non-dimensionalised as $\lambda_z^+ = \lambda_z u_\tau / \nu$. Here, the axial wavenumber is calculated as $k_z = n_z k_{z0}$, $n_z \in [-N_z/2, N_z/2 - 1]$ is an integer, and $k_{z0} = 2\pi/L_z$ is the lowest wavenumber of the axial direction. It should be indicated that it is neither necessary nor realistic to require for a zero percentage

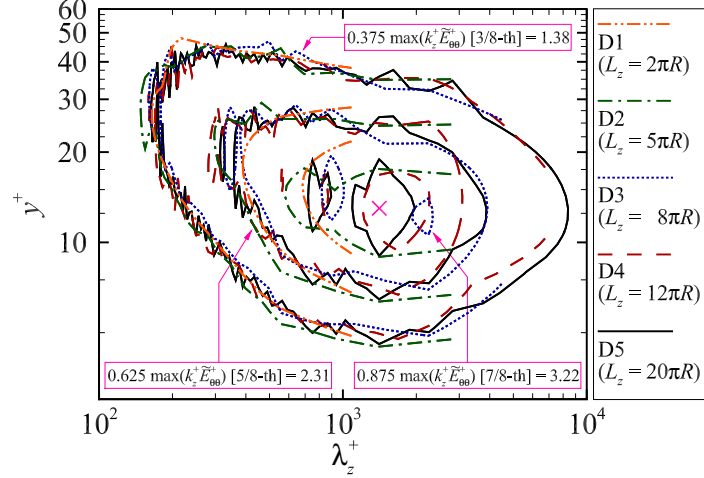


Figure 3.6: Isopleths of premultiplied axial spectrum of temperature fluctuations $k_z^+ \tilde{E}_{\theta\theta}^+$ as functions of wavelength λ_z^+ and wall-normal coordinate y^+ along the central vertical plane (located at $\beta = \pi/2$) on the pressure side of pipes of lengths $L_z = 2\pi R, 5\pi R, 8\pi R, 12\pi R$ and $20\pi R$ (based on test cases D1–D5 listed in table 3.1) at the same rotation number $Ro_\tau = 0.2$. The innermost, middle and outermost isopleths correspond to $0.875 \max(k_z^+ \tilde{E}_{\theta\theta}^+)$, $0.625 \max(k_z^+ \tilde{E}_{\theta\theta}^+)$ and $0.375 \max(k_z^+ \tilde{E}_{\theta\theta}^+)$, respectively. The cross symbol ‘x’ marks the peak location of $k_z^+ \tilde{E}_{\theta\theta}^+$, identified as $\lambda_z^+ = 1407.83$, and $y^+ = 12.99$ based on case D5.

of premultiplied spectrum at the cutoff wavelength in a DNS or a transient physical experiment. There are two good reasons for not pursuing this idealistic requirement: firstly, it is simply impossible as it demands use of an infinite-long pipe; and secondly, the energy level of turbulence structures becomes trivial at the extremely large wavelengths (far deviating from modal wavelength). A practical method in DNS is to select threshold values of premultiplied spectrum for defining energy-containing eddy motions. For example, in the DNS studies of turbulent plane-channel flow by Hoyas and Jiménez (2006), turbulent Couette flow by Avsarkisov et al. (2014) and turbulent streamwise-rotating channel flow by Yang and Wang (2018), the energetic eddy motions are defined with three TKE levels, corresponding to the 7/8-th, 5/8-th and 3/8-th (or 87.5%, 62.5%, and 37.5%) of the peak value of the premultiplied spectrum of streamwise velocity fluctuations. These three energy levels of the premultiplied spectrum correspond to the high-, intermediate- and low-intensity cores of

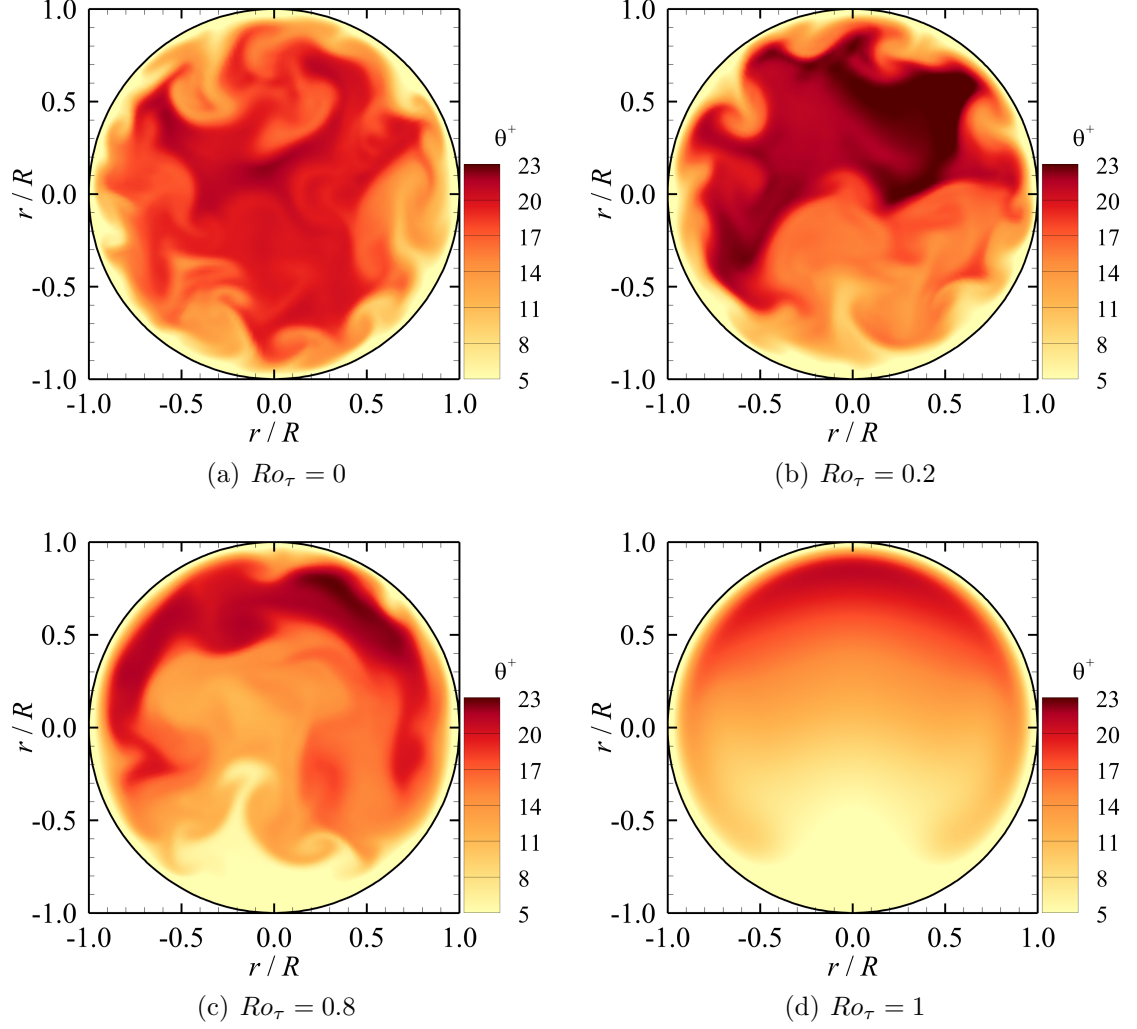


Figure 3.7: Visualisation of the instantaneous temperature field θ^+ for cases of different rotation numbers, extracted at the same streamwise location ($z/R = 10\pi$) and time instant (40.0 LETOTs).

turbulence structures, which all need to be captured in DNS. This method of analysis can be further extended to the study of turbulence structures associated with temperature fluctuations.

In figure 3.6, the peak value of $k_z^+ \tilde{E}_{\theta\theta}^+$ occurs at approximately $\lambda_z^+ = 1407.83$ and $y^+ = 12.99$, indicated using a cross symbol ‘ $\times$ ’. Three levels of isopleths are plotted for $k_z^+ \tilde{E}_{\theta\theta}^+ = 1.38, 2.31$ and 3.22 , corresponding to 3/8-th, 5/8-th, and 7/8-th of $\max(k_z^+ \tilde{E}_{\theta\theta}^+)$, respectively. It is evident that the isopleth of $k_z^+ \tilde{E}_{\theta\theta}^+$ obtained based on

case D1 (of pipe length $L_z = 2\pi R$) is incomplete in the sense that it appears only on the left side of the cross symbol ‘ $\times$ ’, insufficient to enclose the peak of $k_z^+ \tilde{E}_{\theta\theta}^+$. The only test case that can provide complete isopleths around the peak of $k_z^+ \tilde{E}_{\theta\theta}^+$ at all three levels is case D5 (of $L_z = 20\pi R$). A careful perusal of this figure indicates case D4 (of $L_z = 12\pi R$) is capable of generating complete innermost and intermediate isopleths at levels of $0.875 \max(k_z^+ \tilde{E}_{\theta\theta}^+)$ and $0.625 \max(k_z^+ \tilde{E}_{\theta\theta}^+)$. However, at a lower TSE level, a small portion of the outermost isopleth, corresponding to $0.375 \max(k_z^+ \tilde{E}_{\theta\theta}^+)$, is missed at large wavelengths in the DNS based on case D4, since 40.42% of $\max(k_z^+ \tilde{E}_{\theta\theta}^+)$ is truncated at the cutoff wavelength. This observation indicates that a minimum pipe length of $L_z = 12\pi R$ is generally required in DNS of radially rotating pipe flows in order to capture the axial energetic eddy motions. However, if the criterion is to strictly capture all eddy motions that are capable of contributing to TSE at the level higher than $0.375 \max(k_z^+ \tilde{E}_{\theta\theta}^+)$, the pipe length must be kept at $L_z = 20\pi R$. Therefore, to maintain a high accuracy in simulation of radially rotating pipe flows, I use a strict criterion of $L_z = 20\pi R$ in cases 0–10, which however, increases the computational expenses (due to use of additional grid points in the axial direction).

3.2.2 Instantaneous temperature field

Figure 3.7 compares the cross-stream distributions of instantaneous temperature field θ^+ at four different rotation numbers, extracted at the same streamwise location ($z/R = 10\pi$) and time instant (40.0 LETOTs). As is evident in figure 3.7(a), the thermal flow structures are distributed uniformly along the azimuthal direction in a stationary turbulent pipe flow. Given the surface cooling condition, large-scale high-temperature (in red colour) and low-temperature (in faint yellow colour) flow structures are mostly concentrated in the central region and the near-wall region of the pipe, respectively. From figures 3.7(b-d), it is seen that as the rotation number increases, the temperature field distribution becomes increasingly asymmetrical in

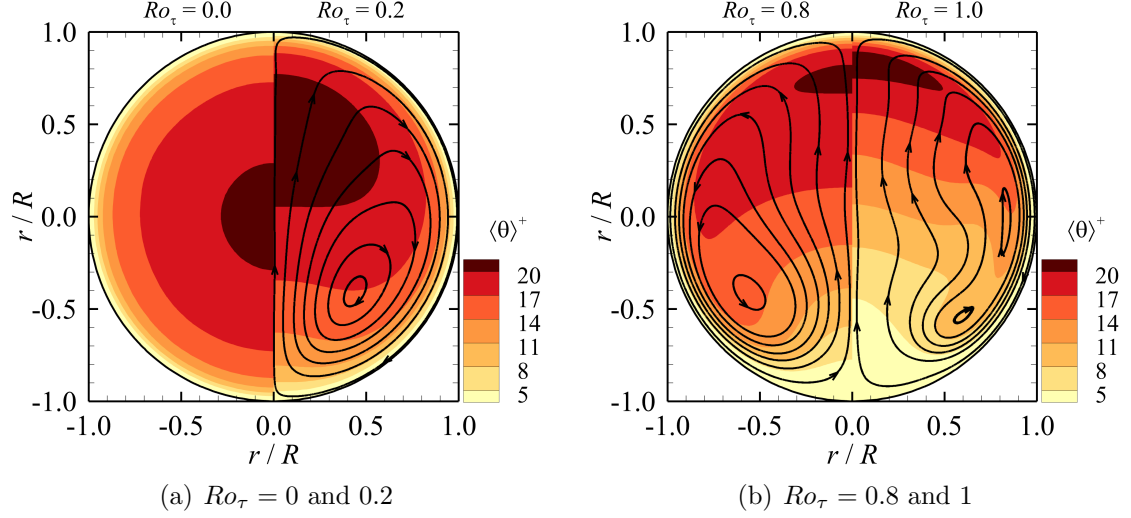


Figure 3.8: Cross-stream distribution of the time- and axial-averaged temperature $\langle\theta\rangle^+$ superimposed with time- and axial-averaged cross-stream streamlines at four different rotation numbers.

the vertical direction. The instantaneous high-temperature structures move towards the pressure (or top) side while the low-temperature structures are pushed towards the suction (or bottom) side. From figure 3.7(b), it is also interesting to observe that the instantaneous high-temperature structures are reinforced at $Ro_\tau = 0.2$, implying that the temperature field of the flow maintains a higher level of turbulent activities at low rotation numbers. However, as shown in figure 3.7(c), these structures are significantly reduced as the rotation number increases to $Ro_\tau = 0.8$. Eventually, at the highest rotation number $Ro_\tau = 1$ tested as shown in figure 3.7(d), the thermal flow field becomes fully laminarised.

3.2.3 Mean temperature fields

Figure 3.8 compares the contours of mean temperature field $\langle\theta\rangle^+$ superimposed with the mean streamlines in the cross-stream direction at four different rotation numbers. Given the axial symmetry of the mean temperature field, only one half of the cross-section is shown for each test case, which facilitates a direct comparison of two test

cases of different rotation numbers. From figure 3.8(a), a concentric characteristic is clearly observed in the mean temperature distribution in the cross-stream plane of the non-rotating flow case. At $Ro_\tau = 0.2$, mean secondary flows appear in the cross-sectional plane as a pair of large-scale counter-rotating vortices. Directly influenced by the radial component of the Coriolis force, the vortex centre is near the suction side of the pipe, resulting in an upward shift of the high temperature region towards the pressure side of the pipe. From figure 3.8(b), it is seen that this upshifting trend continues as the rotation number increases. Eventually, at $Ro_\tau = 1$, a full laminarisation state is reached and two pairs of the large-scale vortices can be observed in the cross-stream plane.

Figure 3.9 compares the profiles of mean temperature $\langle\theta\rangle^+$ of six different rotation numbers in the central vertical ($\beta = \pi/2$) and central horizontal ($\beta = 0$) planes, respectively. From figure 3.9(a), it is seen that the profile of $\langle\theta\rangle^+$ is symmetrical about the axial centre for the non-rotating pipe flow (at $Ro_\tau = 0$). However, this symmetrical pattern breaks as soon as the radial system rotation is imposed. The magnitude of $\langle\theta\rangle^+$ weakens monotonically near the suction side of the pipe as the rotation number increases, and its peak position is pushed towards the pressure side by the radial component of the Coriolis force. From figure 3.9(b), it is evident that as Ro_τ increases, the magnitude of $\langle\theta\rangle^+$ is suppressed monotonically near the pipe centre. The physical mechanism underlying this monotonic trend lies in the mean secondary flows shown previously in figure 3.8. The mean secondary flows in form of large vortex pairs carry low-temperature fluid near the pipe wall (along the entire periphery of the pipe) to the bottom (or the suction side) of the pipe, and then further to the top (or the pressure side) of the pipe. As such, the mean temperature of the fluid drops in the central region of the pipe.

Figure 3.10 shows the overall effects of radial system rotation on volume-averaged TKE (k_m) and TSE (k_θ), and the bulk mean velocity and deficit temperature (U_b^+ and θ_b^+ , respectively). All these four quantities result from the mean temperature and

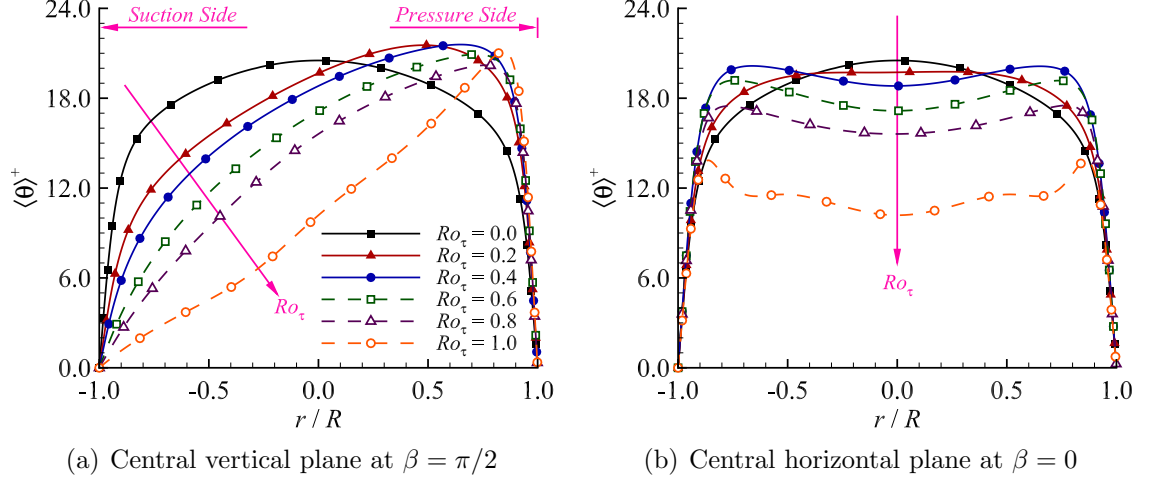


Figure 3.9: Comparison of mean temperature $\langle \theta \rangle^+$ at six different rotation numbers along central vertical ($\beta = \pi/2$) and central horizontal ($\beta = 0$) planes, respectively. Arrows point to the direction of an increasing rotation number.

velocity fields, and reflect the overall thermohydraulic performance of the rotating pipe flows. The definitions of k_θ and k_m are given by

$$k_\theta = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R \langle \theta' \theta' \rangle r dr d\beta \quad , \quad (3.7)$$

and

$$k_m = \frac{1}{2\pi R^2} \int_0^{2\pi} \int_0^R \langle u'_i u'_i \rangle r dr d\beta \quad , \quad (3.8)$$

respectively. In figure 3.10(a), these two quantities have been non-dimensionalised by their corresponding values of the non-rotating pipe flow ($Ro_\tau = 0$), i.e. k_{m0} and $k_{\theta 0}$, respectively. It is clear that ratio k_m/k_{m0} decreases monotonically with an increasing rotation number Ro_τ , whereas ratio $k_\theta/k_{\theta 0}$ increases slightly in the low rotation number range and reaches a maximum at $Ro_\tau = 0.3$. Going beyond this rotation number, both ratios decrease gradually. At $Ro_\tau = 0.9$, the values of $k_\theta/k_{\theta 0}$ and k_m/k_{m0} reduce to 0.67 and 0.5, respectively, indicating that the flow retains only 67% of the volume-averaged TSE and 50% of the volume-averaged TKE relative to the non-rotating pipe flow. As Ro_τ further increases from 0.9 to 1, both ratios became

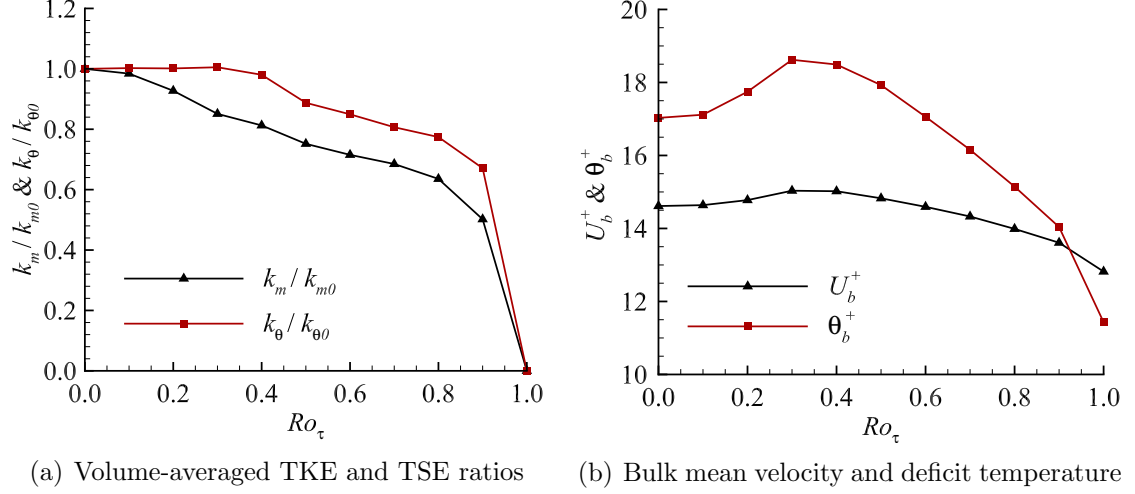


Figure 3.10: Comparison of the profiles of volume-averaged TKE (k_m) and TSE (k_θ), and bulk mean velocity and deficit temperature (U_b^+ and θ_b^+ , respectively) with respect to varying rotation number Ro_τ . The volume-averaged TKE and TSE are non-dimensionalised by their corresponding values of the non-rotating pipe flow ($Ro_\tau = 0$), i.e. k_{m0} and $k_{\theta0}$, respectively. The bulk mean velocity and deficit temperature are non-dimensionalised by u_τ and T_τ , respectively.

attenuated rapidly, reaching zero at $Ro_\tau = 1$ when the flow becomes fully laminarised.

In figure 3.10(b), the effect of radial system rotation on the bulk mean deficit temperature θ_b^+ and the bulk mean velocity U_b^+ are similar. The bulk mean deficit temperature is defined as

$$\theta_b = \frac{1}{U_b \pi R^2} \int_0^{2\pi} \int_0^R \langle \theta \rangle \langle u_z \rangle r dr d\beta \quad . \quad (3.9)$$

Both profiles of U_b^+ and θ_b^+ are non-monotonic with an increasing rotation number, reaching their maxima at $Ro_\tau = 0.3$. However, at the highest rotation number $Ro_\tau = 1$, the magnitudes of both U_b^+ and θ_b^+ reduce dramatically. Ceci and Pirozzoli (2025) recently used a generalised Fukagata-Iwamoto-Kasagi (FIK) identity to decompose the bulk mean velocity and the total wall friction into viscous, turbulent, cross-stream convection (secondary-flow motion), and rotation-induced components. Their results showed that as the rotation number increases, the effect of the secondary-

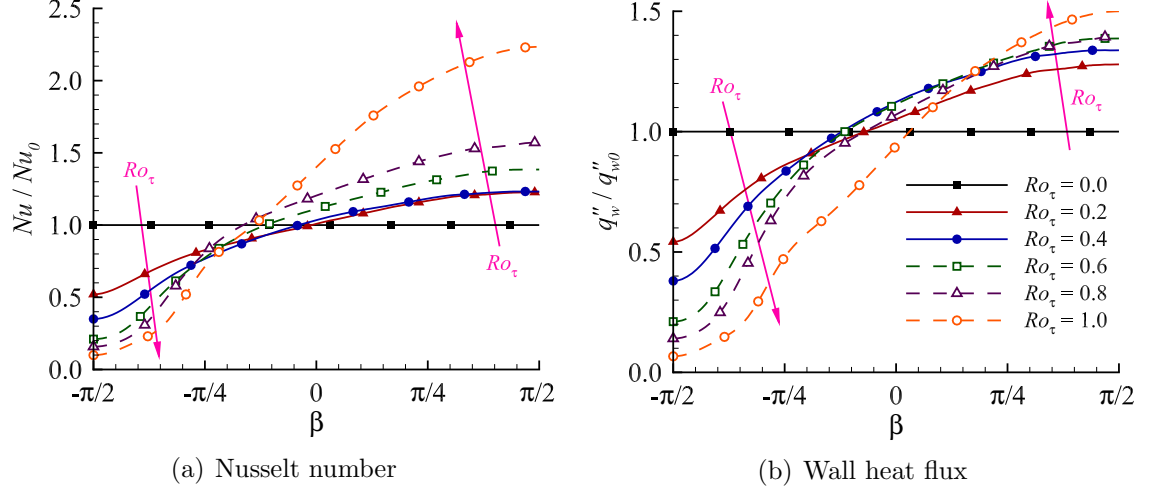


Figure 3.11: Azimuthal distributions of the Nusselt number Nu and the wall heat flux q''_w for six different rotation numbers within the range of $Ro_\tau = 0$ –1. Both Nu and q''_w have been non-dimensionalised by their corresponding values (Nu_0 and q''_{w0}) of the non-rotating pipe flow ($Ro_\tau = 0$). Arrows indicate the direction of an increasing rotation number.

flow motion on the wall friction is non-monotonic and reduces dramatically at high rotation numbers. From previous analysis of figures 3.7 and 3.8, it is understood that compared with the non-rotating case, strong secondary flow motions occur in the rotating flow cases, which makes the flow asymmetrical azimuthally, generally enhancing the mean streamwise velocity and turbulence level on the pressure side of the pipe while reducing them on the suction side of the pipe. From figures 3.10(a,b), it is clear that as the rotation number increases from $Ro_\tau = 0.3$ to 1, both U_b^+ and θ_b^+ reduce gradually transitioning towards laminarisation.

Figure 3.11 compares the profiles of the local time- and axially-averaged Nusselt number Nu and wall heat flux q''_w as a function of the azimuthal coordinate (β) for six cases of different rotation numbers. To effectively identify the rotating effects on heat transfer, both Nu and q''_w have been non-dimensionalised by their corresponding values (Nu_0 and q''_{w0} , respectively) of the non-rotating pipe flow ($Ro_\tau = 0$). Here, the

Nusselt number is defined as

$$Nu = \frac{hD}{K} = \frac{D}{\theta_b} \frac{\partial \langle \theta \rangle}{\partial r} \Big|_{wall} \quad , \quad (3.10)$$

where h denotes the convective heat transfer coefficient, K is the thermal conductivity of the fluid. The time- and axially-averaged local wall heat flux is calculated as $q''_w = \alpha \rho C_p (\partial \langle \theta \rangle / \partial r)_{wall}$. Given the axial symmetry of the flow about the y -axis, only one half the pipe domain is shown in figure 3.11, with the azimuthal coordinate varying from $\beta = -\pi/2$ on the suction side of the pipe to $\beta = \pi/2$ on the pressure side of the pipe. In figure 3.11, it is evident that both ratios Nu/Nu_0 and q''_w/q''_{w0} deviate considerably from the unity value as a direct consequence of radial system rotation. Furthermore, it is seen from figure 3.11(a) that as the rotation number increases, the value of Nu/Nu_0 reduces monotonically on the suction side but increases monotonically on the pressure side of the pipe. Clearly, the deviation of Nu/Nu_0 from the unity is the most pronounced at $Ro_\tau = 1$, when the flow becomes completely laminarised. This observation indicates that the radial system rotation tends to enhance the heat transfer efficiency on the pressure side while simultaneously suppressing it on the suction side the pipe. It is interesting to observe that the profile of Nu/Nu_0 crosses the unity line near $\beta = 0$ at low rotation numbers of $Ro_\tau = 0.2$ and 0.4 . However, as the rotation number keeps increasing till $Ro_\tau = 0.8$ (shortly before the flow becomes fully laminarised at $Ro_\tau = 1$), the unity-crossing point shifts increasingly towards the suction side, indicating a steady expansion of enhanced heat transfer zone in the circular pipe. The wall heat flux q''_w provides an additional measure of the heat transfer performance, which is directly proportional to the Nusselt number, i.e., $q''_w = (K\theta_b/D)Nu$. Although θ_b varies with Ro_τ (see figure 3.10(b)), the profile of q''_w/q''_{w0} in figure 3.11(b) resembles that of Nu/Nu_0 . Similarly, a clear monotonic decrease in the value of q''_w/q''_{w0} is observed on the suction side of the pipe. Apparently, as the rotation number increases, the heat transfer rate enhances and reduces monotonically on the pressure and suction sides of the pipe, respectively.

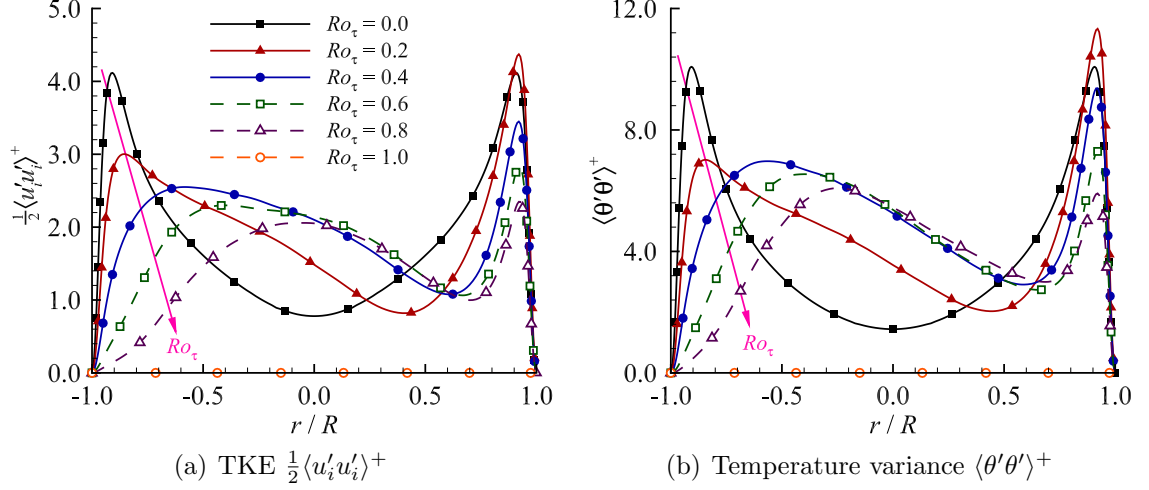


Figure 3.12: Central vertical profiles (located at $\beta = \pi/2$) of TKE $\frac{1}{2}\langle u'_i u'_i \rangle^+$ and temperature variance $\langle \theta' \theta' \rangle^+$ for six different rotation numbers. Arrows points to the direction of an increasing rotation number.

3.2.4 Temperature fluctuations and turbulent heat fluxes

The profiles of non-dimensionalised TKE $\frac{1}{2}\langle u'_i u'_i \rangle^+$ and TSE (or temperature variance) $\langle \theta' \theta' \rangle^+$ in the central vertical ($\beta = \pi/2$) plane for six different rotation numbers are shown in figures 3.12(a,b), respectively. For the non-rotating pipe flow, it is evident that there is a characteristic dual-peak pattern in the profiles of both $\frac{1}{2}\langle u'_i u'_i \rangle^+$ and $\langle \theta' \theta' \rangle^+$, which are symmetrical about the axial centre of the pipe. However, as the rotation number increases, both TKE and TSE decreases considerably on the suction side of the pipe (for $-1.0 \leq r/R < 0.0$), clearly indicating a trend of laminarisation. The profiles of $\frac{1}{2}\langle u'_i u'_i \rangle^+$ and $\langle \theta' \theta' \rangle^+$ varies non-monotonically on the pressure side of the pipe (for $0.0 < r/R \leq 1.0$), and reach their maxima at $r/R = 0.921$ (or $y^+ = 14.24$) and $Ro_\tau = 0.2$. This interesting observation demonstrates that the levels of TKE and TSE can be enhanced by the system rotation at relatively low rotation numbers for $Ro_\tau \leq 0.2$. When the rotation number increases to $Ro_\tau = 1$, the magnitudes of $\frac{1}{2}\langle u'_i u'_i \rangle^+$ and $\langle \theta' \theta' \rangle^+$ fall down to zero, implying that the flow has been completely laminarised.

As important second-order statistical moments of the velocity and temperature

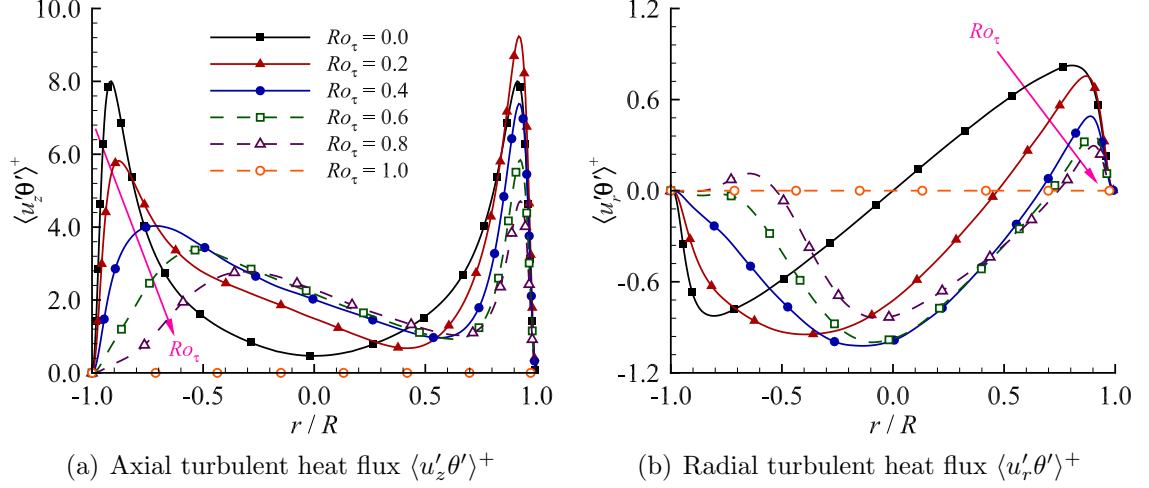


Figure 3.13: Central vertical profiles (located at $\beta = \pi/2$) of axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ and radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ for six different rotation numbers. Arrows points to the direction of an increasing rotation number.

fields, both Reynolds stresses $\langle u'_i u'_j \rangle$ and turbulent heat fluxes $\langle u'_i \theta' \rangle$ play critical roles in the processes of turbulent convection, production and diffusion. The knowledge gained on turbulent heat fluxes can be applied to validation turbulence models for RANS simulations, and development of fundamental physical understanding of turbulent transport processes. Figure 3.13 compares the profiles of axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ and radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ in the central vertical direction (located at $\beta = \pi/2$) for six different rotation numbers. The azimuthal turbulent heat flux $\langle u'_\beta \theta' \rangle^+$ is not plotted here because it is zero identically in the central vertical plane regardless of the rotation number value. As is evident from figure 3.13(a), the profile of $\langle u'_z \theta' \rangle^+$ is axially-symmetrical in the non-rotating pipe flow. However, as Ro_τ starts to increase, the axial symmetry in the profile of $\langle u'_z \theta' \rangle^+$ is destroyed. The magnitude of $\langle u'_z \theta' \rangle^+$ reduces monotonically near the suction side of the pipe, but reaches its maximum at $r/R = 0.921$ (or $y^+ = 14.24$) on the pressure side of the pipe at $Ro_\tau = 0.2$. Overall, the profile of axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ is similar to those of TKE and TSE shown previously in figure 3.12. From figure 3.13(b), it is seen that the profile of $\langle u'_r \theta' \rangle^+$ exhibits a linear distribution along the radial direction,

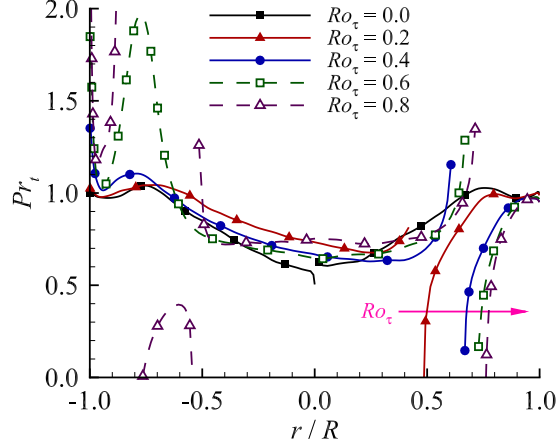


Figure 3.14: Central vertical profile (located at $\beta = \pi/2$) of turbulent Prandtl number Pr_t with respect to rotation number Ro_τ .

symmetrical about the axial centreline of the pipe in the non-rotating pipe flow. As rotation number increases, the peak magnitude of $\langle u'_r \theta' \rangle^+$ decreases monotonically, and the peak position moves increasingly towards to the pressure side of the pipe. In addition, it is also observed that as the rotation number increases, the zero-crossing point moves from the pipe centre ($r/R = 0.0$) increasingly towards the pressure side of the pipe. At a high rotation number of $Ro_\tau = 0.8$, a second zero-crossing point in the profile $\langle u'_r \theta' \rangle^+$ appears at $r/R = -0.526$ on the suction side of the pipe. However, as the rotation number continues to increase to $Ro_\tau = 1$, both $\langle u'_z \theta' \rangle^+$ and $\langle u'_r \theta' \rangle^+$ become completely vanished, indicating that the flow has become fully laminarised under the most rapid system rotation condition.

Figure 3.14 shows the profiles of turbulent Prandtl number Pr_t in the central vertical plane (located at $\beta = \pi/2$) for five turbulent flow cases of different rotation numbers ranging from $Ro_\tau = 0$ to 0.8. The turbulent Prandtl number describes the relative strengths of turbulent diffusion of momentum and thermal energy through the ratio of eddy viscosity (ν_t) to eddy thermal diffusivity (α_t) of turbulence, i.e.

$$Pr_t = \frac{\nu_t}{\alpha_t} = \frac{\langle u'_r u'_z \rangle}{\langle u'_r \theta' \rangle} \frac{d\langle \theta \rangle / dr}{d\langle u_z \rangle / dr} . \quad (3.11)$$

If the Reynolds analogy holds, then $Pr_t = 1$. The characteristics of Pr_t are relatively well understood in the classical context of turbulent thermal boundary layers over a flat plate, and its value is affected by factors such as the molecular Prandtl number Pr , Reynolds number, and wall normal distance. Although there have been DNS studies of turbulent Prandtl number for turbulent heat transfer in stationary pipe flows (Piller, 2005; Redjem-Saad et al., 2007), the knowledge of Pr_t in the context of heat convection in rotating pipe flows is still absent in the literature. As such, it would be very interesting to develop new insights into the wall-scaling behaviour of Pr_t as a function of Ro_τ .

From figure 3.14, it is seen that for the non-rotating pipe flow, the profile of Pr_t shows a quasi-parabolic shape. The magnitude of Pr_t is close to unity at $|r/R| \in [0.677, 1]$ (or $y^+ \leq 58.17$) near the top and bottom sides of the pipe but drops to approximately 0.6 at the pipe centre (at $Ro_\tau = 0$). The value of Pr_t is comparable with the results of Piller (2005) and Redjem-Saad et al. (2007) on DNS studies of turbulent heat convection in stationary pipes. In the DNS studies of turbulent heat convection in stationary plane-channel flows conducted by Antonia et al. (2009), Pirozzoli et al. (2016) and Abe and Antonia (2017), the magnitude of Pr_t is approximately 0.85 in the outer region of the turbulent boundary layer, which is slightly larger than that of stationary turbulent pipe flows. Clearly, the turbulent Prandtl number is very sensitive to the wall geometry. As is evident in figure 3.14, the magnitude of Pr_t is also very sensitive to the rotation number as it increases from $Ro_\tau = 0$ to 0.8. The minimum value of Pr_t remains relatively stable in the central region of the pipe, varying from $Pr_t = 0.6$ to 0.74. However, its value increases significantly in the near-wall region of the suction side at $Ro_\tau = 0.6$ and 0.8, indicating that turbulent transport of momentum is more efficient than that of the thermal energy on the suction side of the pipe at relatively high rotation numbers. By contrast, the magnitude of Pr_t reduces rapidly in the vicinity of the wall on the pressure side of the pipe once the radial system rotation is imposed. From its definition equation (3.11),

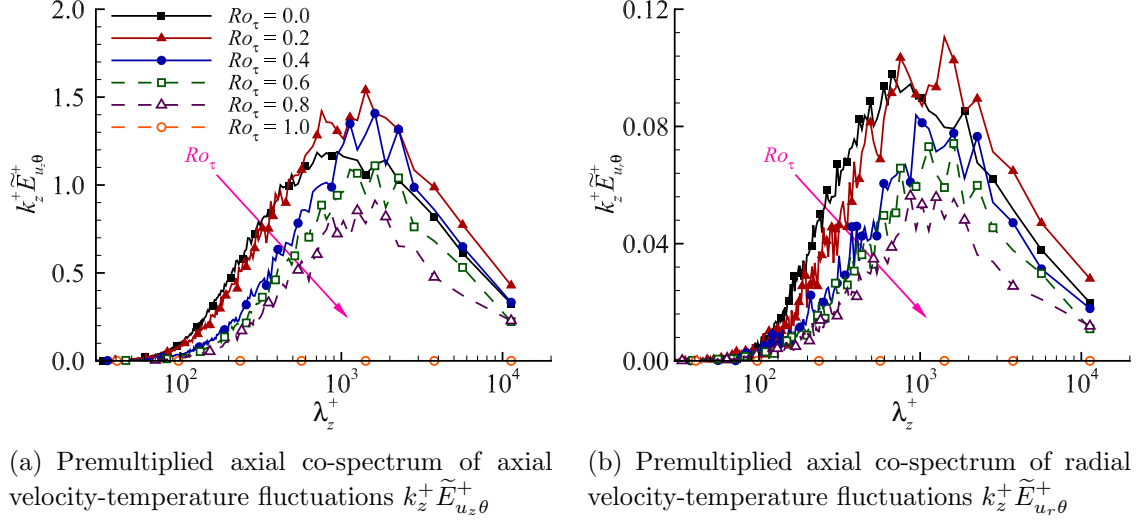


Figure 3.15: Effects of radial system rotation on premultiplied axial co-spectra of axial and radial velocity-temperature fluctuations $k_z^+ \tilde{E}_{u_z\theta}^+$ and $k_z^+ \tilde{E}_{u_r\theta}^+$ calculated along the axial line at $r/R = 0.921$ (or, at wall-normal position $y^+ = 14.24$) on the pressure side (located at $\beta = \pi/2$) of the pipe. Arrow indicates a local monotonic variation with an increasing rotation number.

it is understood that the value of Pr_t is singular at the radial position where the mean axial velocity is the maximum (such that $d\langle u_z \rangle / dr = 0$ holds locally). For the non-rotating pipe flow, this discontinuity point is located at the pipe centre exactly. However, as the rotation number increases shown in figure 3.14, it shifts from the pipe centre increasingly towards the pressure side.

The effects of radial system rotation on turbulent heat fluxes have been analysed through their profiles in figure 3.13 and the turbulent Prandtl number in figure 3.14. In order to gain deeper insights on this subject, figure 3.15 compares the premultiplied co-spectra of axial and radial velocity-temperature fluctuations ($k_z^+ \tilde{E}_{u_z\theta}^+$ and $k_z^+ \tilde{E}_{u_r\theta}^+$) of six rotation numbers (all with $L_z = 20\pi R$ based on cases 0, 2, 4, 6, 8 and 10, see table 3.1). Here, the co-spectrum of velocity-temperature fluctuations is defined as $\tilde{E}_{u_i\theta}(k_z) = \text{Re}\{\widehat{u_i^*} \widehat{\theta'}\}$. The profiles are extracted along the steamwise direction at $r/R = 0.921$ (or $y^+ = 14.24$) near the pressure wall, where both TKE and TSE peak. From figures 3.15(a,b), it is seen that the magnitude of $k_z^+ \tilde{E}_{u_z\theta}^+$ is much higher than

that of $k_z^+ \tilde{E}_{u_r\theta}^+$ by two orders. The magnitudes of $k_z^+ \tilde{E}_{u_z\theta}^+$ and $k_z^+ \tilde{E}_{u_r\theta}^+$ reduce monotonically as rotation number increases at small wavelengths for $\lambda_z^+ \leq 300$ and 700, respectively. In this regard, the axial and radial turbulent heat fluxes are suppressed by the radial system rotation. Clearly, the turbulent thermal energy as indicated by $k_z^+ \tilde{E}_{u_z\theta}^+$ and $k_z^+ \tilde{E}_{u_r\theta}^+$ is the largest at $Ro_\tau = 0.2$. This is consistent with previous observation of figure 3.12 which shows that both TKE and TSE reach their maxima at $Ro_\tau = 0.2$. However, as Ro_τ further increases beyond 0.2, these two premultiplied co-spectra decrease rapidly, reflecting the trend of laminarisation. Eventually, at the highest rotation number $Ro_\tau = 1$, the magnitudes of $k_z^+ \tilde{E}_{u_z\theta}^+$ and $k_z^+ \tilde{E}_{u_r\theta}^+$ both become trivial as the flow has been completely laminarised. From figure 3.15(a), the characteristic length scales corresponding to the modal wavelengths of $k_z^+ \tilde{E}_{u_z\theta}^+$ are quite consistent at different rotation numbers, ranging within $\lambda_z^+ \in [600, 2300]$. However, as shown in figure 3.15(b), the characteristic length-scale range of $k_z^+ \tilde{E}_{u_r\theta}^+$ increases from $\lambda_z^+ \in [400, 1300]$ to $[800, 2000]$ as the rotation number increases from $Ro_\tau = 0$ to 0.8. In the following subsections, the transport processes of turbulent heat fluxes will be precisely analysed through the budget balance of their transport equations, and furthermore, the turbulent thermal-fluid structures responsible for the generation of turbulent heat fluxes will also be investigated.

3.2.5 Transport of turbulent heat fluxes

The transport equation of turbulent heat fluxes $\langle u'_i \theta' \rangle$ is expressed as

$$\frac{\partial \langle u'_i \theta' \rangle}{\partial t} = C_i + P_i + P_i^t + D_i^t + D_i^p + D_i^m + P_i^c = 0 \quad , \quad (3.12)$$

where C_i , P_i , D_i^t , D_i^p , D_i^m , P_i^c represent the convection, turbulent production, turbulent diffusion, pressure diffusion-redistribution, molecular diffusion-dissipation and Coriolis production terms, respectively. Additionally, P_i^t denotes the turbulent production term generated from the constant mean axial temperature gradient. The

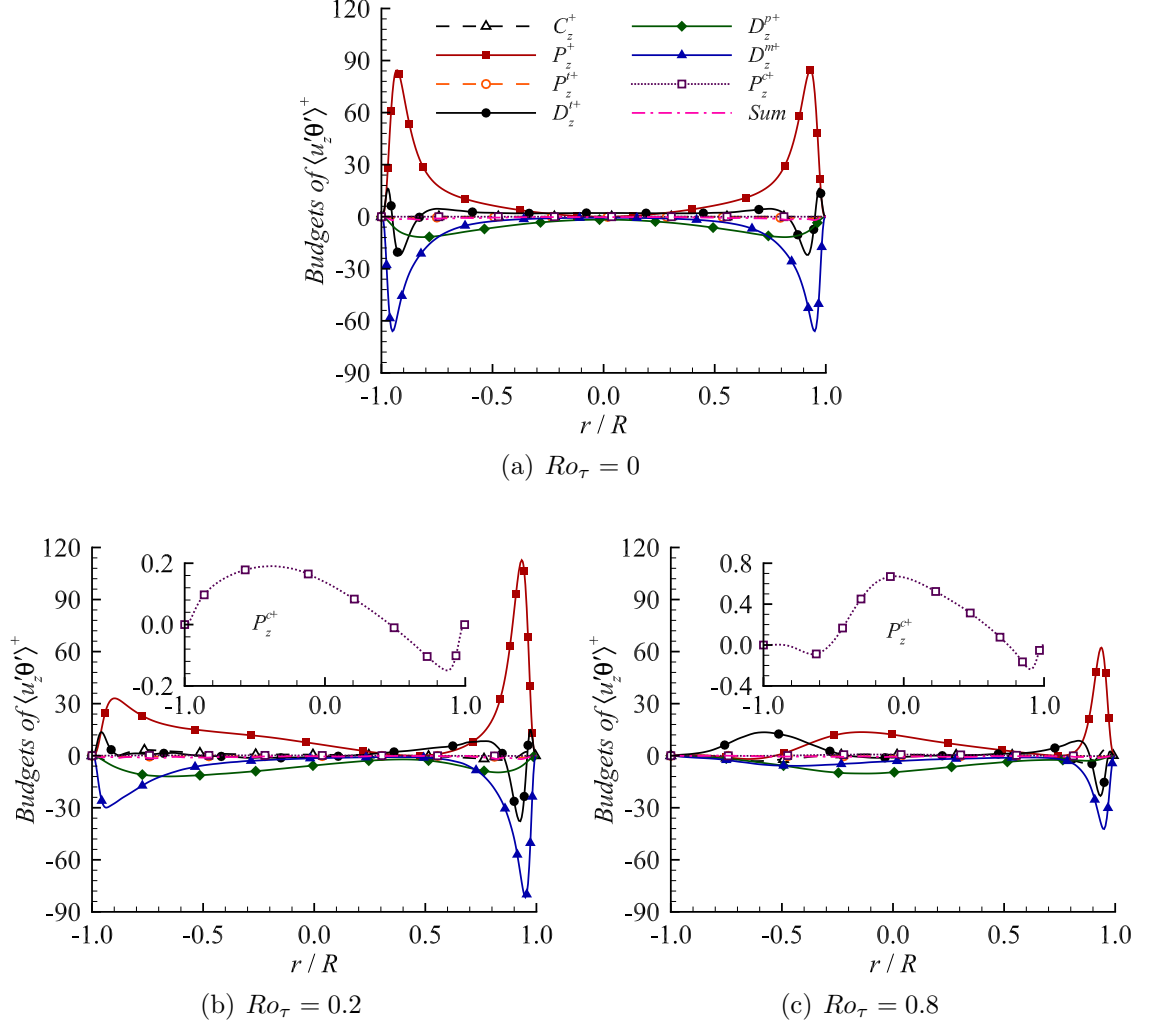


Figure 3.16: Budget balance of axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ in the central vertical plane (located at $\beta = \pi/2$) of the pipe at $Ro_\tau = 0, 0.2$ and 0.8 . All budget terms are non-dimensionalised by $u_\tau^2 T_\tau / R$.

detailed definitions of these budget terms are given in Appendix C.

Figures 3.16 and 3.17 compare the budget balances of $\langle u'_z \theta' \rangle^+$ and $\langle u'_r \theta' \rangle^+$ for the non-rotating ($Ro_\tau = 0$) and radially rotating ($Ro_\tau = 0.2$ and 0.8) pipe flows, respectively. All profiles are plotted in the central vertical plane (located at $\beta = \pi/2$) of the pipe. The reason that the azimuthal turbulent heat flux $\langle u'_\beta \theta' \rangle^+$ is not included is because it is zero identically. In response to the radial system rotation, the Coriolis production term P_i^c appears in the transport equation of $\langle u'_i \theta' \rangle$. According

to equations (C.1) and (C.2), the axial and radial Coriolis production terms are expressed as

$$P_z^c = -2\Omega (\langle u_r' \theta' \rangle \sin\beta + \langle u_\beta' \theta' \rangle \cos\beta) \quad , \quad (3.13)$$

and

$$P_r^c = 2\Omega \langle u_z' \theta' \rangle \sin\beta \quad , \quad (3.14)$$

respectively. At $\beta = \pi/2$, the above two equations are simplified to

$$P_z^c = -2\Omega \langle u_r' \theta' \rangle \quad , \quad (3.15)$$

and

$$P_r^c = 2\Omega \langle u_z' \theta' \rangle \quad . \quad (3.16)$$

respectively. As is clear in figure 3.16(a), all budget profiles are perfectly symmetrical about the pipe centre in the non-rotating pipe flow (at $Ro_\tau = 0$). The turbulent production P_z^+ , turbulent diffusion D_z^{t+} and molecular diffusion-dissipation D_z^{m+} are the leading budget terms for $\langle u_z' \theta' \rangle^+$ in the near-wall regions of the pipe. However, as the rotation number increases to $Ro_\tau = 0.2$ as shown in figure 3.16(b), the symmetry of the profiles is destroyed by the radial system rotation imposed. The peak values of those three dominant terms are enlarged remarkably on the pressure side, while reduced significantly on the suction side of the pipe. This observation reflects a fact that the turbulence level is enhanced and suppressed on the pressure and suction sides of the pipe, respectively. As the rotation number continues to increase to $Ro_\tau = 0.8$ as shown in figure 3.16(c), the trend of laminarisation becomes apparent, especially on the suction side of the pipe. As shown in figures 3.16(b,c), although the Coriolis production term P_z^{c+} (for the axial turbulent heat flux $\langle u_z' \theta' \rangle^+$) strengthens with an increasing rotation number, its magnitude remains small in the central vertical plane of $\beta = \pi/2$. From figures 3.13(b) and 3.16(b), it is seen that both $\langle u_r' \theta' \rangle^+$ and P_z^{c+} change their signs simultaneously at $r/R = 0.47$ at $Ro_\tau = 0.2$ because of their

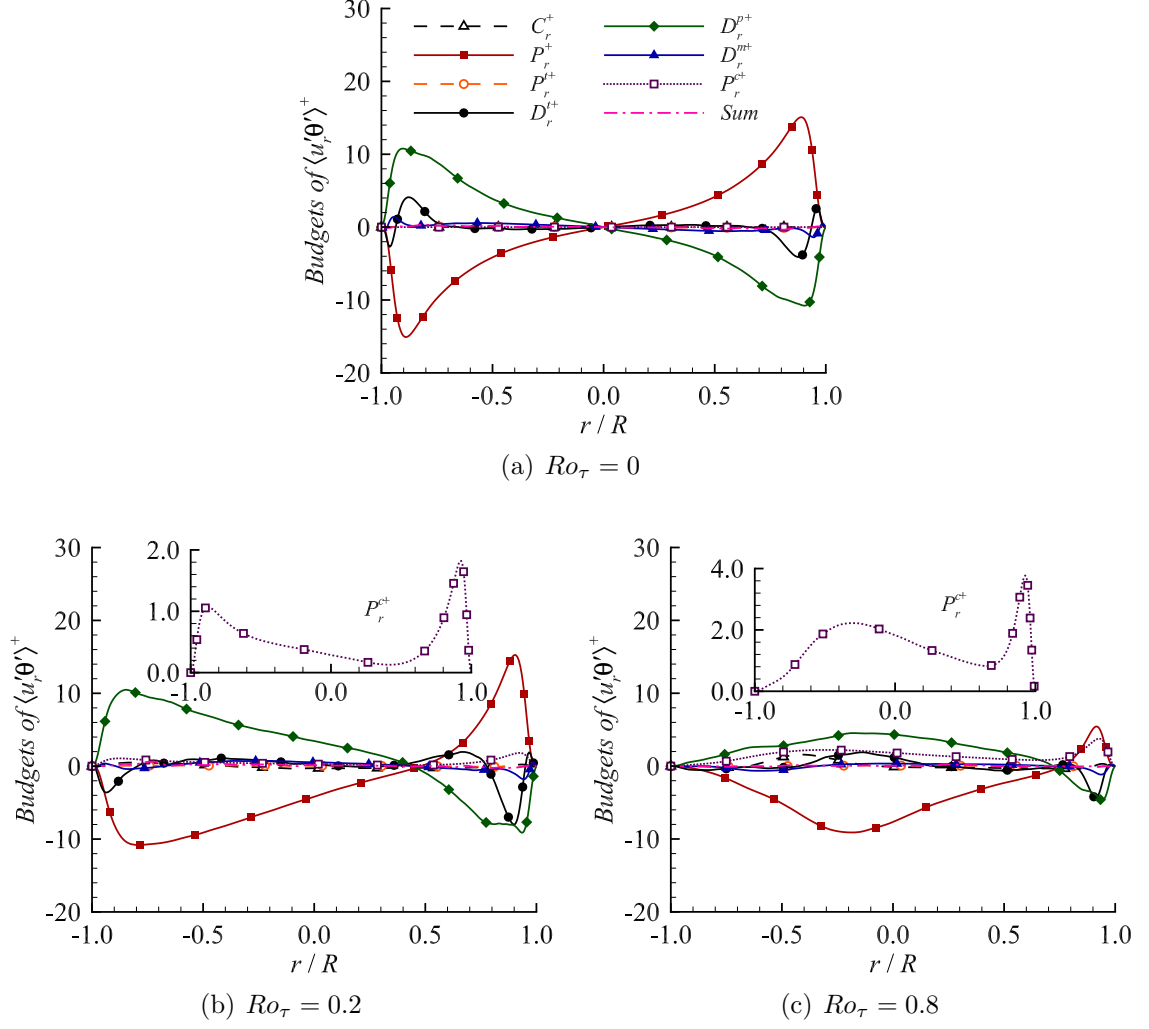


Figure 3.17: Budget balance of radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ in the central vertical plane (located at $\beta = \pi/2$) of the pipe at $Ro_\tau = 0, 0.2$ and 0.8 . All budget terms are non-dimensionalised by $u_\tau^2 T_\tau / R$.

linear relationship given by equation (3.15). Thus, the Coriolis production term P_z^{c+} acts as a source of TSE to empower axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ within range $r/R \in [-1.0, 0.47]$ but acts as a sink of TSE for $\langle u'_z \theta' \rangle^+$ in range $r/R \in [0.47, 1.0]$.

As is evident in figure 3.17(a), the budget balance of radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ is dominated by the turbulent production term P_r^+ and pressure diffusion-redistribution term D_r^{p+} in the near-wall region of the non-rotating pipe. The profiles of these two dominant terms are central symmetrical about the axial centreline. As

the rotation number increases from $Ro_\tau = 0$ to 0.8, it is clear from figures 3.17(a-c) that the zero-crossing points of these two dominant terms shift from the pipe centre to the pressure side of the pipe. Meanwhile, their peak positions near the suction side of the pipe are shifted increasingly towards the pipe centre. At $Ro_\tau = 0.8$, the magnitudes of all budget terms of $\langle u'_r \theta' \rangle^+$ become significantly attenuated, showing a clear trend towards laminarisation of the flow. From figures 3.17(b,c), it is seen that the contribution from the Coriolis production term P_r^{c+} becomes increasingly important as the rotation number rises, acting as a thermal energy source for the radial turbulent heat flux $\langle u'_r \theta' \rangle^+$. At $Ro_\tau = 0.8$ as shown in figure 3.17(c), the Coriolis production term P_r^{c+} possesses the same order of the magnitude as the turbulent production term P_r^+ , especially in range $r/R \in [0.76, 1.0]$ on the pressure side of the pipe, both acting as local dominant thermal energy sources for $\langle u'_r \theta' \rangle^+$.

3.2.6 Turbulent thermal-flow structures

Figures 3.18 and 3.19 show the distributions of JPDFs $P(\theta', u'_z)$ and $P(\theta', -u'_r)$ of the non-rotating ($Ro_\tau = 0$) and radially rotating ($Ro_\tau = 0.2$ and 0.8) pipe flows, respectively. All JPDF contours are extracted at the wall-normal position of $r/R = 0.921$ (or $y^+ = 14.24$) of turbulent heat fluxes $\langle u'_z \theta' \rangle^+$ and $\langle u'_r \theta' \rangle^+$. In figures 3.18 and 3.19, two crossing black lines divide the domain into four quadrants ($Q1-4$) and the diagonal line through the first and third quadrants (i.e., $Q1$ and $Q3$) is shown using a black dashed line. Given the surface cooling condition of the pipe, quadrants $Q1$ and $Q3$ correspond to hot sweep and cold ejection events, respectively. The negative of the radial velocity fluctuation $-u'_r$ is used in figure 3.19 in order to perform quadrant analysis based on the radial coordinate measured from the pipe wall.

From figures 3.18(a-c), it is seen that the distribution of $P(\theta', u'_z)$ always spreads along the diagonal line at 45° , indicating that the temperature fluctuations θ' strongly synchronise with the axial velocity fluctuations u'_z at all rotation numbers tested. Be-

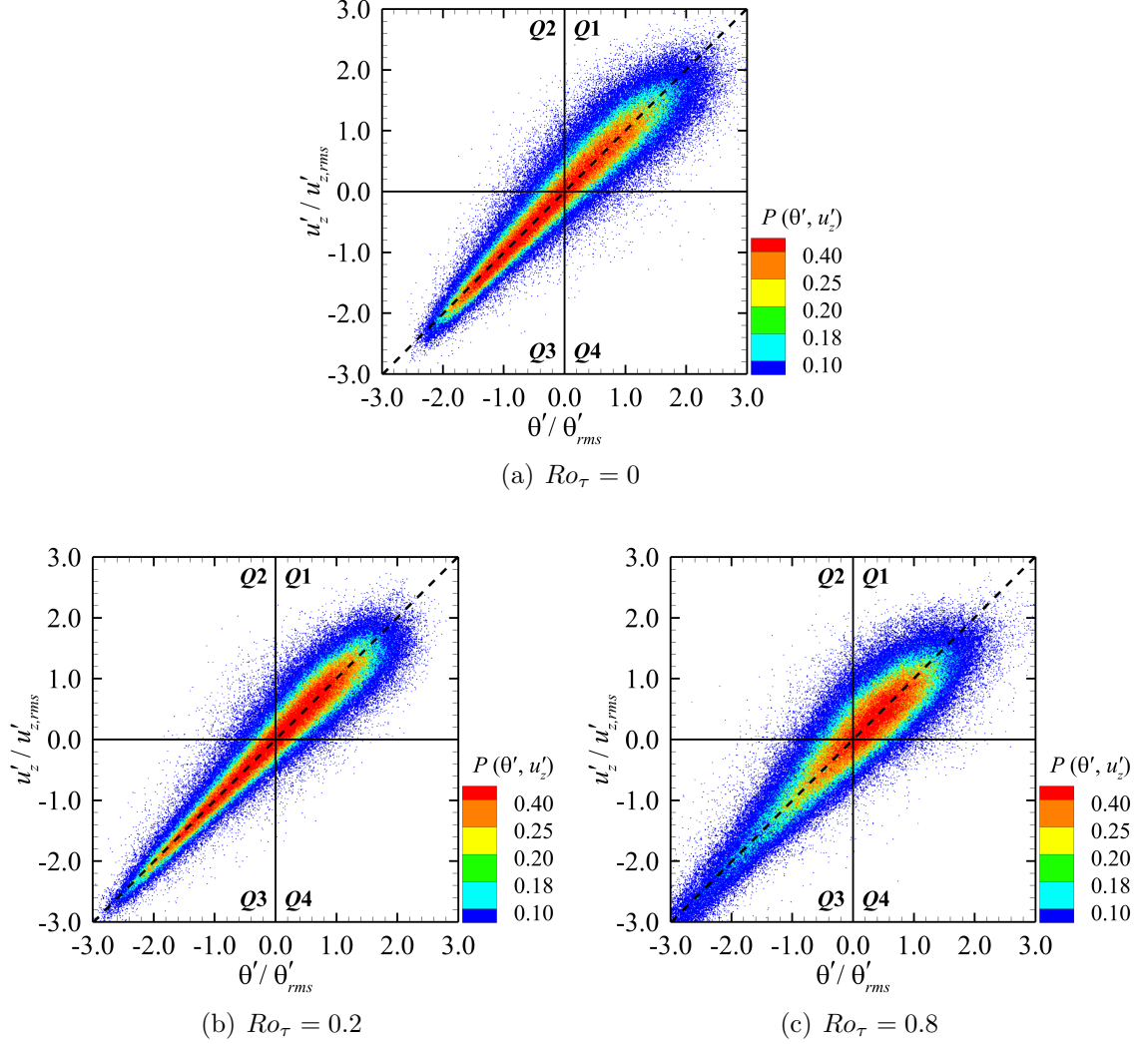


Figure 3.18: JPDF $P(\theta', u'_z)$ at $r/R = 0.921$ (or at wall-normal position $y^+ = 14.24$) on the pressure side (located at $\beta = \pi/2$) of the pipe for $Ro_\tau = 0, 0.2$ and 0.8 . The black dashed lines highlight the events of $u'_z/u'_{z,rms} = \theta'/\theta'_{rms}$ along the slope of 45° . All contours are partitioned into four quadrants ($Q1$ – 4) by black lines.

cause $P(\theta', u'_z)$ is mostly concentrated in $Q1$ and $Q3$, the axial turbulent heat flux $\langle u'_z \theta' \rangle$ are dominated by hot sweep and cold ejection events, and must be positively valued at the sampling position (i.e., at the wall-normal position $y^+ = 14.24$ on the pressure side of the pipe), which is evidenced by figure 3.13(a). It is interesting to observe from figures 3.18(a-c) that the modal position of $P(\theta', u'_z)$ becomes progressively concentrated in $Q3$ as the rotation number increases.

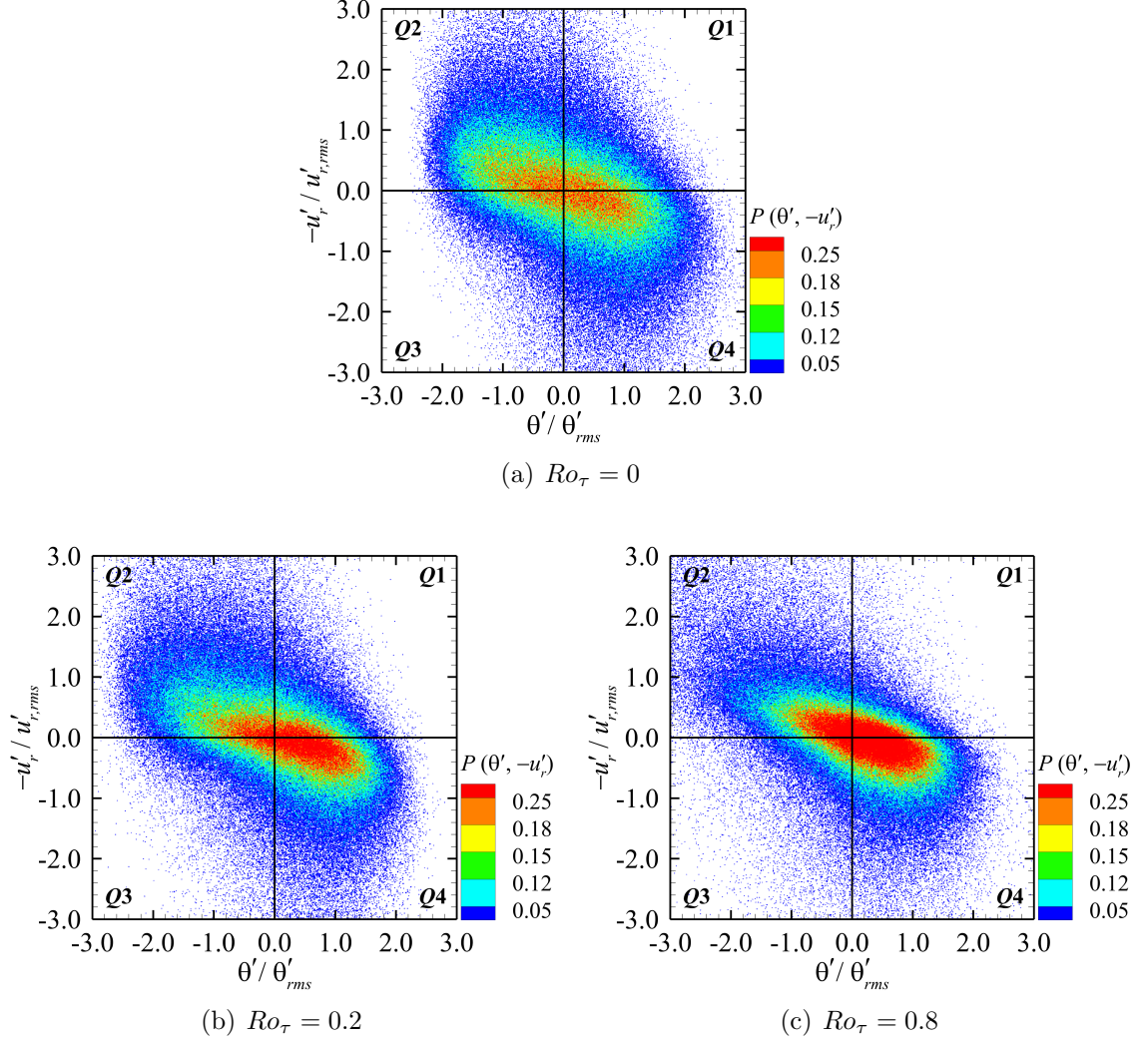


Figure 3.19: JPDF $P(\theta', -u'_r)$ at $r/R = 0.921$ (or at wall-normal position $y^+ = 14.24$) on the pressure side (located at $\beta = \pi/2$) of the pipe for $Ro_\tau = 0, 0.2$ and 0.8 . All contours are partitioned into four quadrants (Q1–4) by black solid lines.

Figures 3.19(a-c) compare the distributions of $P(\theta', -u'_r)$ of three rotation numbers, as illustrated in figure 3.18. Clearly, the JPDF of θ' and $-u'_r$ is preferentially distributed in Q2 and Q4 (corresponding to cold ejection and hot sweep events, respectively), resulting in a positively-valued radial turbulent heat flux $\langle u'_r \theta' \rangle$ at the sampling position (i.e., $y^+ = 14.24$), a conclusion that is directly supported by figure 3.13(b). Following Pope (2000), it is understood that near-wall ejection events feature $u'_z < 0$ and $-u'_r > 0$ and sweep events feature $u'_z > 0$ and $-u'_r < 0$. From

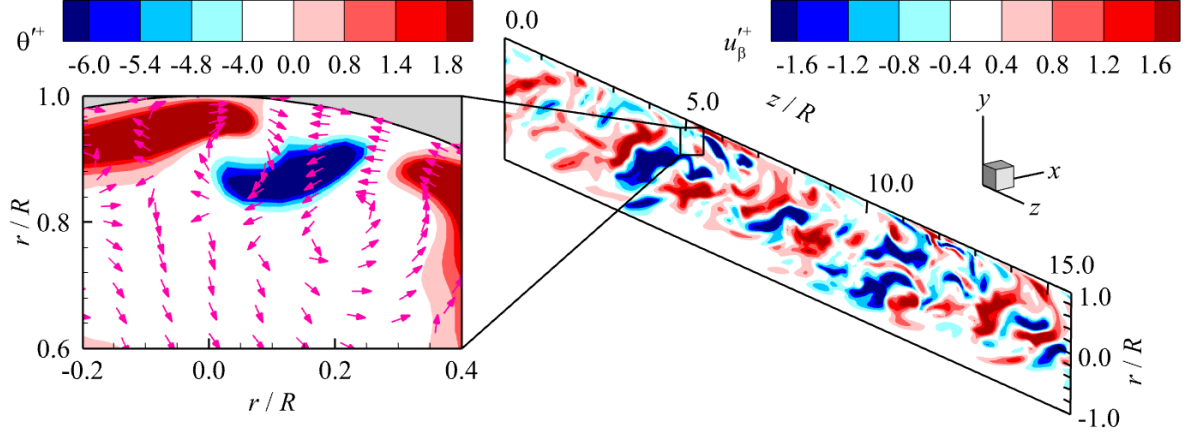


Figure 3.20: Central vertical view of turbulence structures (located at $\beta = \pi/2$) based on instantaneous velocity and temperature fluctuations at a relatively high rotation number of $Ro_\tau = 0.8$. The right panel displays the streamwise vortices demonstrated with the instantaneous azimuthal velocity fluctuation $u_\beta'^+$ in the central vertical plane (for clarity, only one-fourth axial domain is plotted). The left panel shows an arbitrary cross-stream plane extracted from the right panel, which is partially enlarged to show the contours of instantaneous fluctuating temperature θ'^+ , superimposed with the instantaneous cross-stream velocity fluctuation vectors $(u_r'^+, u_\beta'^+)$.

figures 3.18 and 3.19, it is seen that the probabilities of negatively-valued temperature fluctuations ($\theta' < 0$) and positively-valued temperature fluctuations ($\theta' > 0$) are decreased and increased with an increasing rotation number, respectively. It is apparent that the JPDF zones of cold ejection featuring $u_z' < 0$ and $\theta' < 0$ in $Q3$ of figure 3.18 and that featuring $-u_r' > 0$ and $\theta' < 0$ in $Q2$ of figure 3.19 shrink in area as the rotation number increases, indicating that the ejection events have a strong tendency of possessing higher probabilities within a smaller range of distribution. Meanwhile, the hot sweep events of $u_z' > 0$ and $\theta' > 0$ in $Q1$ of figure 3.18 and that featuring $-u_r' < 0$ and $\theta' > 0$ in $Q4$ of figure 3.19 are enhanced in ranges of smaller magnitudes of u_z' , $-u_r'$ and θ' , but are suppressed in ranges of larger magnitudes of u_z' , $-u_r'$ and θ' . Thus, as the rotation number increases, large-magnitude temperature fluctuations through either ejection or sweep events are suppressed at the sampling position. This is consistent with the general trend of laminarisation of the flow at higher rotation numbers.

To develop a deeper insight into the turbulent thermal structures in regions around the sampling position of figures 3.18 and 3.19, the right panel of figure 3.20 shows the central vertical view of the contour of instantaneous azimuthal velocity fluctuations $u_\beta'^+$ at a high rotation number of $Ro_\tau = 0.8$. It is observed that instantaneous streamwise eddy motions featuring positively- and negatively-valued $u_\beta'^+$, alternate in the radial direction and meander downstream along the axial direction. In response to the radial system rotation imposed, the vortical structures are pushed towards the pressure side of the pipe, meanwhile resulting in laminarisation on the suction side of the pipe. As is evident in the left panel of figure 3.20, the instantaneous cross-stream velocity fluctuation vectors ($u_r'^+$ and $u_\beta'^+$) show a pair of counter-rotating roll cells in the vicinity of the sampling point of figures 3.18 and 3.19 (i.e., around $r/R = 0.921$ or $y^+ = 14.24$ on the pressure side of the pipe wall). It is noted that the cold ejection and hot sweep events are coupled with the negatively-valued and positively-valued temperature fluctuations θ'^+ , respectively. The counter-rotating roll cells play an important role in the development of the ejection and sweep events and in turbulent transport of thermal energy (coupled with θ'). Clearly, the overall effects of radial system rotation is to suppress sweep events (of large temperature fluctuations θ') in the neighborhood of $r/R = 0.921$ at a high rotation number of $Ro_\tau = 0.8$. Moreover, both the ejection and sweep events (coupled with negatively and positively-valued θ' , shown in blue and red colours, respectively) make a contribution to the positively-valued radial turbulent heat flux $\langle u_r' \theta' \rangle$ near the sampling position, which provides a physical explanation to the relevant observations of figures 3.13(b).

To refine our understanding of the role of roll cells appearing at the sampling position of figures 3.18-3.20 in near-wall heat convection, the linear stochastic estimation (LSE) method (Adrian, 1994; Fang and Wang, 2018) can be considered, which is expressed as

$$\langle g' | E \rangle = g_{LSE} \cdot E \quad , \quad (3.17)$$

where g' , E and g_{LSE} represent a general turbulence quantity (such as u_z' and θ'),

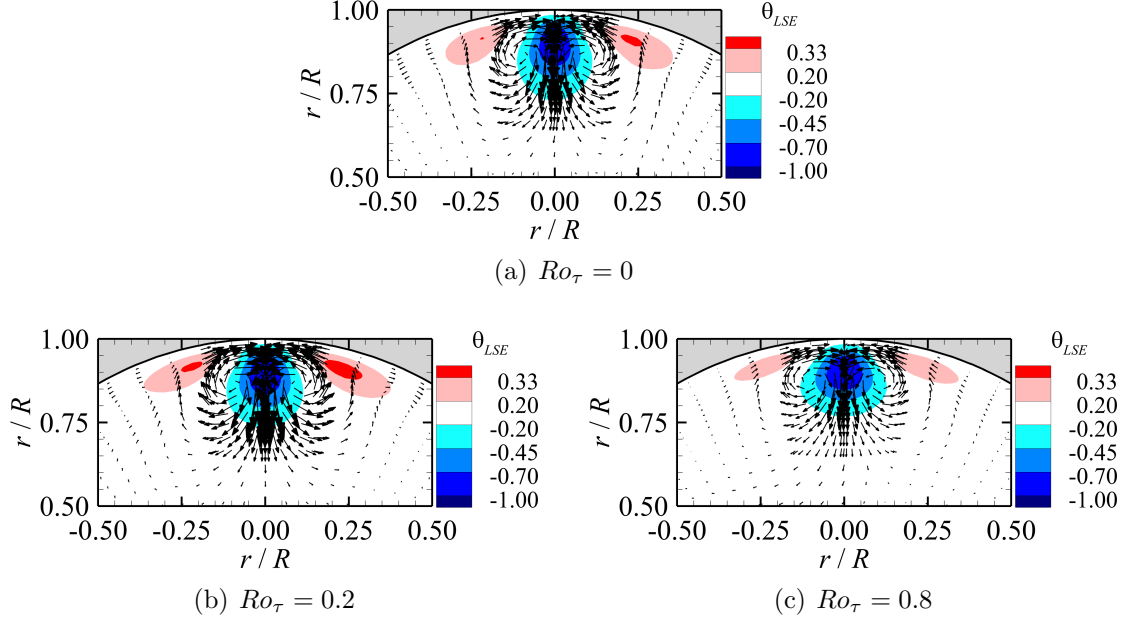


Figure 3.21: Contours of temperature θ_{LSE} superimposed with velocity vectors ($u_{r,LSE}$, $u_{\beta,LSE}$) using the LSE method in the cross-stream direction for the non-rotating ($Ro_\tau = 0$) and radially rotating ($Ro_\tau = 0.2$ and 0.8) pipe flows under a surface cooling condition. The reference point is at $r/R = 0.921$ (or at wall-normal position $y^+ = 14.24$) from the pressure side (located at $\beta = \pi/2$) of the pipe. All LSE variables are normalised by their absolute maximum values.

a targeted event and an estimated coefficient for g' corresponding to event E , respectively. By applying the least square approach to the estimation, g_{LSE} can be determined as

$$g_{LSE}(\Delta z) = \frac{\langle E(z)g'(z + \Delta z) \rangle}{\langle E(z)^2 \rangle} . \quad (3.18)$$

From the previous analysis of figures 3.18 and 3.19, it is understood that the radial system rotation tends to suppress the large magnitudes of the cold ejection and hot sweep events at the sampling position of $y^+ = 14.24$ on the pressure side of the pipe. The cold ejection events at this position can be selected as the target event (E), defined as

$$E = \begin{cases} |u'_z u'_r| & \text{if } u'_z < -u'_{z,rms} \text{ and } -u'_r > u'_{r,rms} \\ 0 & \text{otherwise.} \end{cases} \quad (3.19)$$

Here, the thresholds for defining the strong ejection events are based on the RMS velocities, $-u'_{z,rms}$ and $u'_{r,rms}$.

Figure 3.21 compares the contours of LSE of temperature θ_{LSE} superimposed with the LSE of velocity vectors $(u_{r,LSE}, u_{\beta,LSE})$ in the cross-stream direction for the non-rotating ($Ro_\tau = 0$) and radially rotating ($Ro_\tau = 0.2$ and 0.8) pipe flows under the surface cooling boundary condition. All LSE variables have been normalised by their absolute maximum values such that their magnitudes vary within $[-1.0, 1.0]$. Since the LSE method only captures the vortical structures around the reference position by means of conditional average, only one pair of counter-rotating vortices around $y^+ = 14.24$ are captured on the pressure side of the pipe as shown in figure 3.21. Under the influence of the counter-rotating vortices, low temperature fluid of negatively-valued θ' is ejected downwards from the pipe wall. As such, through coupling with negatively-valued u'_r , the radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ becomes positively-valued near the pressure side of the pipe, a conclusion that is known based on previous observation of figure 3.13(b). As is clear in figure 3.21, the magnitude of positively-valued θ_{LSE} (indicated by red colour, featuring hot sweep) at $y^+ = 14.24$ reaches its maximum at $Ro_\tau = 0.2$. This is because the system rotation can slightly enhance the hot sweep events near the pressure side of the pipe at a relatively low rotation number, resulting in a higher local TSE level in turbulent heat transfer. However, as the rotation number further increases to $Ro_\tau = 0.8$, such hot sweep events coupling with larger magnitude of θ_{LSE} reduce significantly.

Chapter 4

DNS of turbulent flow and structures in a circular pipe subjected to axial system rotation

In this chapter, turbulent circular pipe flows in a circular pipe subjected to axial rotation are tested for a wide range of rotation numbers from $Ro_b = 0$ to 20 at a fixed Reynolds number of $Re_\tau = 180$ using much longer pipes than those in the literature. In order to capture Taylor columns at high rotation numbers, eight DNS cases of very long pipes are considered, which have pipe lengths of $L_z = 30\pi R$ – $180\pi R$ as the rotation number is increased from $Ro_b = 0$ to 20. The content of this chapter has been published as a journal article (Zhang and Wang, 2024).

4.1 Test cases and numerical algorithm

Figure 4.1(a) illustrates schematically a circular pipe under axial system rotation at a constant clockwise angular speed Ω (about the z -direction). The radial, azimuthal and axial coordinates of the cylindrical coordinate system are denoted using r , β and

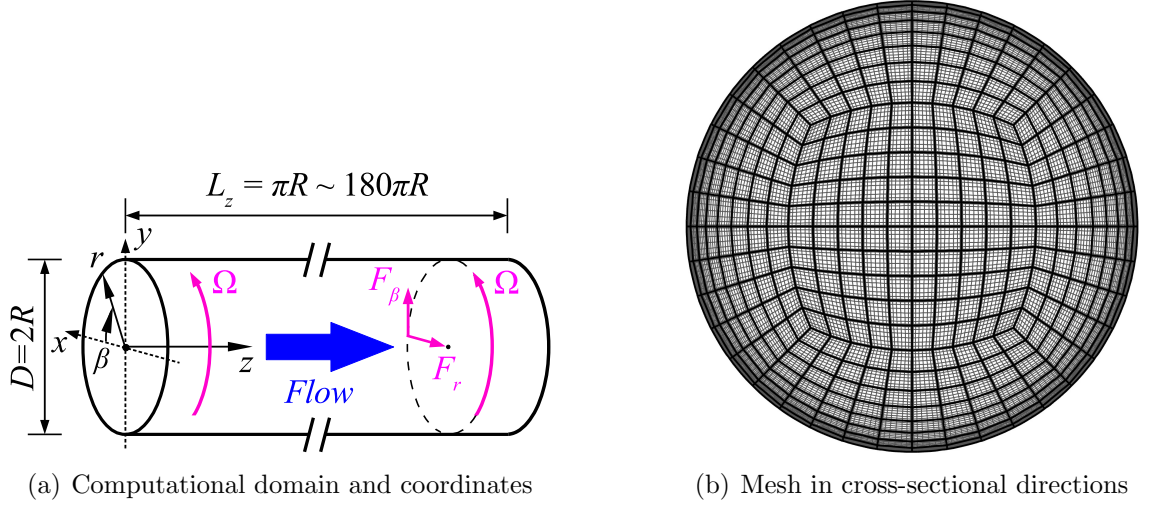


Figure 4.1: Schematic of turbulent pipe flow subjected to axial system rotation in a cylindrical coordinate system. The radial and azimuthal Coriolis force components are F_r and F_β , respectively. The cross-sectional mesh of the pipe consists of 420 quadrilateral-structural elements based on a SEM. Each element is refined using an eighth-order GLLL polynomial.

z , and the corresponding velocity components are u_r , u_β and u_z , respectively. In order to study the axially rotating effect, a wide range of rotation numbers varying from $Ro_b = 0$ (non-rotating case) to 20 are compared at a fixed Reynolds number of $Re_\tau = 180$. The pipe flow is fully developed such that a periodic boundary condition is applied to the axial direction. No-slip condition is imposed on the pipe surface.

Since the pioneering work of Jiménez and Moin (1991) on the ‘minimum domain’ for DNS of a plane-channel flow, it has become well known that the results of DNS of a wall-bounded flow can be physically incorrect if the computational domain is smaller than the characteristic length scale of the most energetic eddies. Turbulence contains a cascade of wavelengths. If the domain is smaller than the wavelengths of the most energetic eddies, physical phenomena related to those most energetic eddy motions would be missing in DNS, and consequently, a DNS does not reflect the physical reality (even if it is carried out numerically with high-order discretisation schemes and with good convergence). For a steady-state and axially fully developed turbulent pipe flow, the statistical moments (denoted using ϕ_u) of the turbulent velocity field

should be statistically stationary, independent of time t , axial location z and pipe length L_z . Otherwise, if ϕ_u depends on the pipe length (i.e. $\phi_u = f(L_z)$), the DNS results are spurious and unphysical, because the axially fully developed flow condition is violated, which demands $\partial\phi_u/\partial z \equiv 0$ and $\partial\phi_u/\partial L_z \equiv 0$. The goal here is to run high-fidelity DNS that is physically realistic and mathematically accurate, such that the statistical moments of the velocity field are independent of axial pipe length L_z . Because the flow structures (streaks and Taylor columns) in an axially rotating pipe become increasingly elongated as the rotation number Ro_b increases, the pipe length L_z needs to be increased accordingly.

This study includes 16 DNS tests cases listed in table 4.1 and table D.1 of Appendix D, in conformity with the two research objectives aforementioned. For the eight rotation numbers tested, the statistical moments obtained from DNS runs based on the longest pipes are all independent of L_z . The flow parameters of these eight longest-pipe cases (for the eight rotation numbers tested) are summarised in table 4.1. Depending upon the rotation number, the pipe length varies from $L_z = 30\pi R$ to $180\pi R$. The choice of these pipe lengths is to ensure that energetic turbulent eddy motions (dominated by streaks at low rotation numbers, and Taylor columns at moderate and high rotation numbers) are reasonably captured in the axial direction at each rotation number. To achieve this goal, even the shortest pipe length of $L_z = 30\pi R$ is much longer than those used in the current literature, that is, $L_z = 25R$ or $7.958\pi R$ in Ebstein (1998) and Orlandi and Ebstein (2000). In addition to the eight longer pipe flow cases summarised in table 4.1, a complementary comparative study of the pipe length effects on the predictive accuracy of DNS is also conducted in Appendix D, which includes 8 additional DNS cases of short pipes, that is, cases 3A–3C at $Ro_b = 2$ and 8A–8E at $Ro_b = 20$, as summarised in table D.1. Through this complementary comparative study, it is proven that statistical moments obtained from DNS runs based on short pipes can be sensitive to L_z , artificially violating the axially fully developed flow assumption.

Table 4.1: Summary of eight test cases of the longest pipe at each rotation number.

Case #	1	2	3	4	5	6	7	8
Ro_b	0	1	2	4	6	10	14	20
Ro_b^A	0	0.92	1.83	3.67	5.53	9.82	14.77	22.97
Ro_τ^A	0	14.73	29.45	58.88	88.36	147.23	206.09	294.55
Re_b^A	5262	5746	5805	5773	5749	5397	5022	4614
Re_τ^A	179.96	179.94	179.97	180.03	179.95	179.99	180.02	179.94
L_z	$30\pi R$	$30\pi R$	$30\pi R$	$60\pi R$	$60\pi R$	$90\pi R$	$120\pi R$	$180\pi R$
N_z	3600	3600	3600	5400	5400	6000	6400	7200
N_{tot}	97,347,600	97,347,600	97,347,600	146,021,400	146,021,400	162,246,000	173,062,400	194,695,200
Δz^+	4.712	4.712	4.712	6.283	6.283	8.482	10.603	14.137
$(\Delta r^+)_{\max}$				3.595				
$(\Delta r^+)_{\min}$				0.123				
$(r\Delta\beta^+)_{\max}$				5.133				
$(r\Delta\beta^+)_{\min}$				0.813				
$(\Delta/\eta)_{\max}$	1.92	1.952	1.904	2.044	2.028	2.277	2.492	2.740

The governing equations for an incompressible flow with respect to an axially rotating reference frame are

$$\nabla \cdot \vec{u} = 0 \quad , \quad (4.1)$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{\Pi}{\rho} \hat{e}_z - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} + \vec{F} \quad , \quad (4.2)$$

where $\vec{u}$ is the velocity, ρ is the density of the fluid and $p = p_s - \rho\Omega^2 r^2/2$ represents the effective pressure that has absorbed both static pressure p_s and the centrifugal force. Here Π represents the constant mean axial pressure gradient and $\hat{e}_z$ is the base unit vector of the z -direction, with $|\hat{e}_z| \equiv 1$. In response to the axial rotation, two components of the Coriolis force ($\vec{F}$) appear in the radial and azimuthal directions, i.e. $F_r = 2\Omega u_\beta$ and $F_\beta = -2\Omega u_r$. Clearly, the instantaneous Coriolis force (F_r and F_β , defined based on the instantaneous velocities) impacts directly on the velocity field itself (u_r and u_β , respectively) in a nonlinear manner through equation (4.2). The simulations were performed with a spectral-element code ‘Semtex’ made available by Blackburn and Sherwin (2004), which is highly accurate in algorithm suitable for conducting DNS. As shown in figure 4.1(b), a quadrilateral-element method was used to divide the cross-section of the pipe into 420 finite elements with each element further discretised spatially using an 8th-order GLLL polynomial.

The pipe lengths and grid resolutions of the eight test cases of the longest pipes

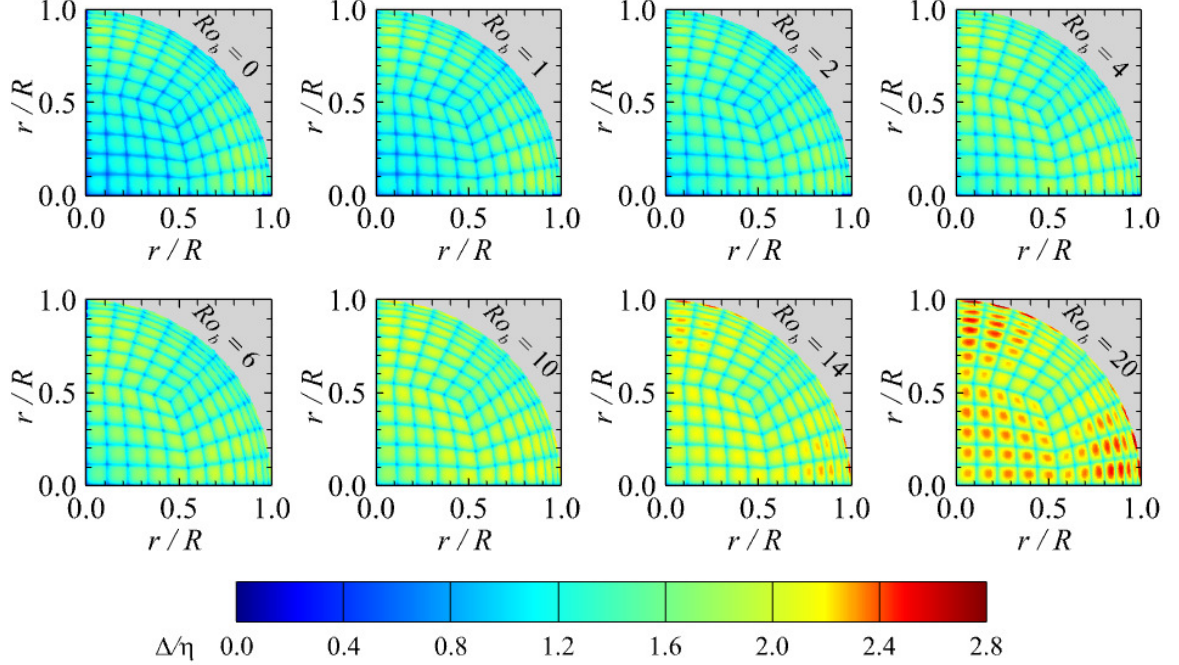


Figure 4.2: Contours of the ratio of the grid size to the Kolmogorov length scale Δ/η . Given the axial symmetry of the flow field, only a quarter cross-section of the pipe is plotted for each rotation number $Ro_b = 0$ –20. Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

are shown in table 4.1. All physical quantities are expanded into the spectral space using Fourier series with 3600–7200 modes (N_z) in the z -direction for pipes of different lengths. As indicated by table 4.1, the total number of nodes varies from $N_{tot} = 97.3$ to 194.7 million for the 8 test cases. The calculation of N_{tot} is directly based on the number of Fourier modes in the axial direction and the number of the eighth-order GLLL orthogonal polynomial interpolants (within each of the 420 finite elements) in the cross-stream directions. Both Fourier series and the GLLL orthogonal polynomial offer very high numerical discretisation accuracies. In each test case, the grid spacing is uniform in the streamwise direction with $\Delta z^+ = 4.712$ –14.137, and varies in the radial and azimuthal directions with $\Delta r^+ = 0.123$ –3.595 and $r\Delta\beta^+ = 0.813$ –5.133. Here, superscript ‘+’ denotes the wall coordinate calculated through non-dimensionalisation based on the kinematic viscosity of the fluid ν and wall friction velocity u_τ (defined as $u_\tau = \sqrt{-\Pi R/2\rho}$). To satisfy the demanding

requirement of DNS on grid resolution for capturing the smallest scale of turbulence, the grid size needs to be kept at the same order as the Kolmogorov length scale, i.e. $O(\Delta/\eta) \sim O(1)$. Figure 4.2 shows the ratio Δ/η for the eight test cases of the longest pipes listed in table 4.1. The grid size is defined as $\Delta = [(\Delta r) \times (r\Delta\beta) \times (\Delta z)]^{1/3}$ and the Kolmogorov length scale is determined by $\eta = (\nu^3/\varepsilon)^{1/4}$, where ε is the dissipation rate of TKE. From the figure, it is clear that the strict grid resolution requirement of $\Delta/\eta = O(1)$ for accurately performing DNS is satisfied. The maximum value of Δ/η of each of all 16 test cases is given in tables 4.1 and D.1, which varies from 1.904 to 2.740.

The axial system rotation has a direct impact on the bulk mean velocity U_b such that the actual calculated rotation number Ro_b^A deviates considerably from its nominal value Ro_b (see table 4.1). Here, superscript ‘A’ denotes the actual result calculated using the DNS data. By contrast, the value of the wall friction velocity u_τ is insensitive to the axial system rotation. Consequently, the wall-friction-velocity-based rotation number $Ro_\tau = 2\Omega R/u_\tau$ varies linearly with the angular speed Ω and the value of the wall-friction-velocity-based Reynolds number Re_τ^A varies little from its nominal value of $Re_\tau = 180$. The values of rotation numbers Ro_b , Ro_b^A and Ro_τ^A , and Reynolds numbers Re_b^A and Re_τ^A of the 16 test cases associated with the two research objectives are given in tables 4.1 and D.1.

I started simulations with a laminar solution added with arbitrary perturbations, and statistics were collected after the pipe flow became statistically stationary. All DNS calculations were conducted on the Alliance (Digital Research Alliance of Canada) supercomputers. For the 8 test cases of the longest pipes listed in table 4.1, 300 instantaneous snapshots of the flow field over 40 LETOTs were collected for cases 1–7, whereas 600 instantaneous snapshots were collected over 80 LETOTs for case 8. For each simulated case, approximately 1.4–4.5 Terabytes (TB) data are stored on the server. However, for the cases of short pipe lengths (of $L_z = \pi R$ – $7.958\pi R$) listed in table D.1, 900–1200 instantaneous snapshots of the flow fields over

122–163 LETOTs were collected for computing the flow statistics. In my analysis, an instantaneous turbulence variable ϕ is decomposed as $\phi = \langle \phi \rangle + \phi'$, where $\langle \phi \rangle$ is the temporally and spatially averaged component over the homogeneous (β and z) directions, and ϕ' represents the residual fluctuating component.

4.2 Results and discussion

This result analysis includes two parts to examine: (i) the impact of axial system rotation on the velocity field of the pipe flow based on a comparative study of eight rotation numbers of $Ro_b = 0$ –20 (with the longest pipe at each rotation number) given in table 4.1; and (ii) the impact of pipe length on the predictive accuracy of DNS based on 10 cases of varying pipe lengths at 2 rotation numbers $Ro_b = 2$ and 20 given in table D.1. Part (ii) represents a complementary study for part (i) by establishing a knowledge foundation to ensure that the DNS results of part (i) are physically accurate obtained using sufficiently long pipes. In this section, the results of part (i) are analysed, whereas those of part (ii) are presented in Appendix D. In the following, the characteristics of the instantaneous flow are first discussed, then followed by an analysis of statistical moments of the velocity field, premultiplied energy spectra, high-order turbulence statistics and turbulence structures.

4.2.1 Instantaneous flow fields

Figure 4.3 shows contours of the instantaneous axial velocity u_z^+ (left half of each panel) and instantaneous axial vorticity ω_z^+ (right half of each panel) in the cross-stream plane of flows at different rotation numbers. Here, the axial vorticity is defined as $\omega_z = [\partial(ru_\beta)/\partial r - \partial u_r/\partial \beta]/r$. All contours are depicted at the same axial stream-wise location of $z/R = 20\pi$ and the same time instant of $t = 20.38$ LETOTs. As

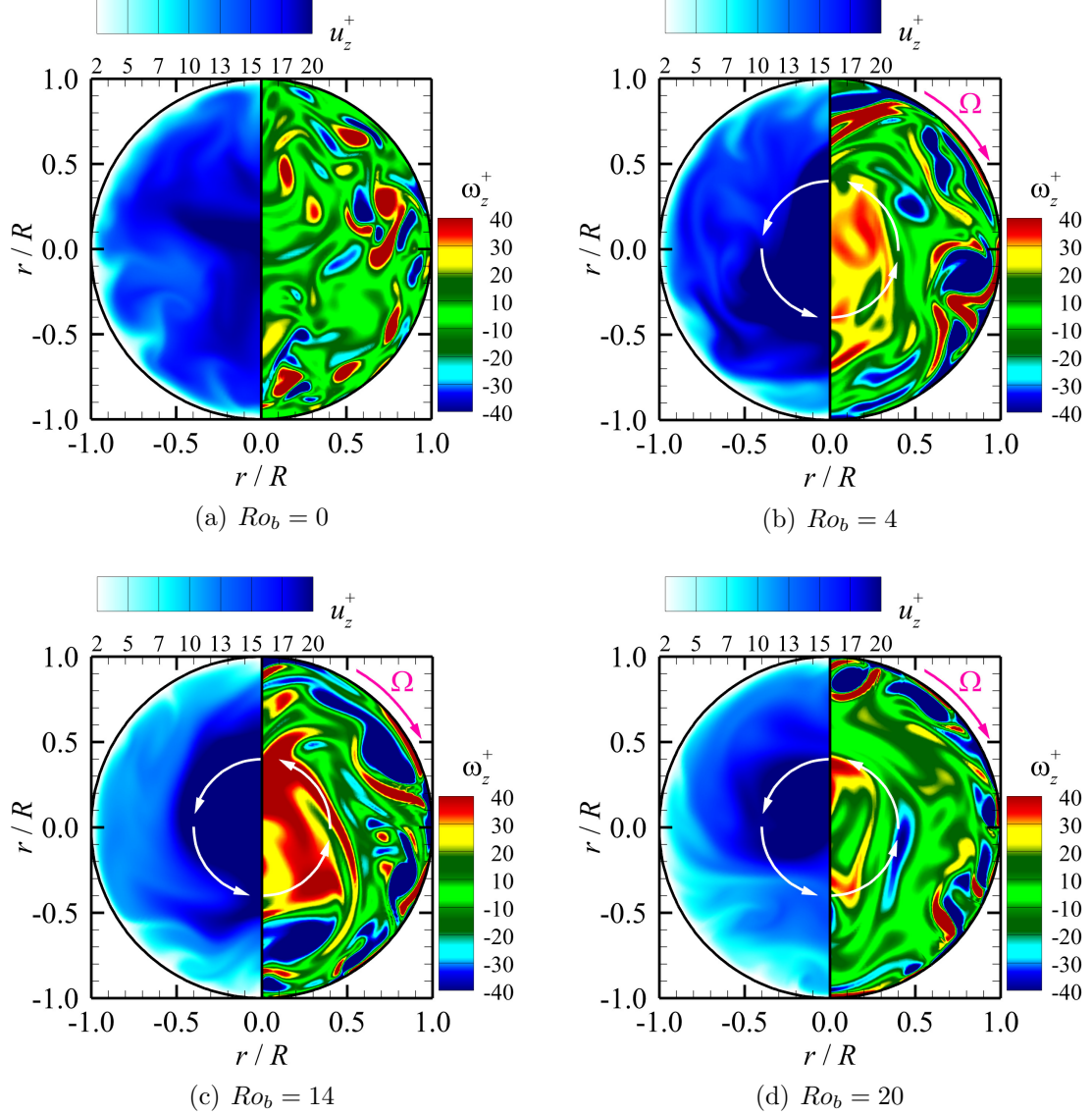


Figure 4.3: Cross-stream view of instantaneous axial velocity u_z^+ (left half of each panel) and instantaneous axial vorticity ω_z^+ (right half of each panel) of the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 4, 14$ and 20) pipe flows, respectively. White arrows indicate the counterclockwise rotating direction of the secondary flow. All contours are plotted at the same axial location (at $z/R = 20\pi$) and the same time instant (at $t = 20.38$ LETOTs). The values of u_z^+ and ω_z^+ have been non-dimensionalised using u_τ . Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

is clear from the left half of figure 4.3(a), in a non-rotating pipe, the flow structures show ‘mushroom patterns’ in the near-wall region where several pairs of small

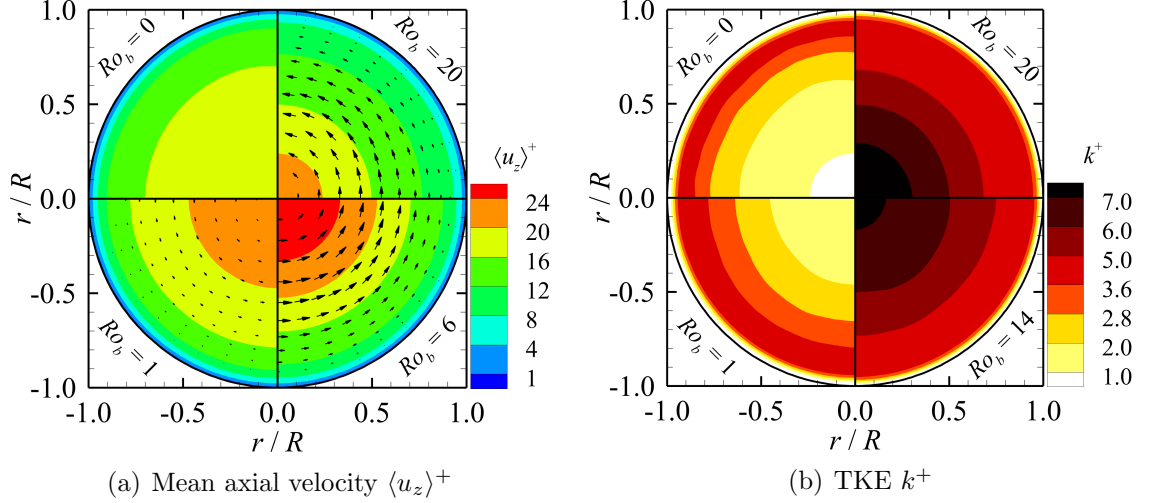


Figure 4.4: Cross-stream view of mean axial velocity $\langle u_z \rangle^+$ and TKE k^+ at four different rotation numbers for $Ro_b = 0$ –20. The direction of the system rotation is clockwise, identical to that shown in figure 4.3. In panel (a), the contours are superimposed with mean cross-stream velocity vectors indicated by black arrows. The values of $\langle u_z \rangle^+$ and k^+ have been non-dimensionalised based on u_τ . Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

counter-rotating vortices are observed. From figure 4.3(b), it is observed that because of the axial system rotation imposed (at $Ro_b = 4$), a large counterclockwise-rotating secondary-flow structure appears at the pipe centre, clearly indicated by the contours of positively valued ω_z^+ (shown in red). As the rotation number continues to increase to $Ro_b = 14$ as shown in figure 4.3(c), the secondary flow structures indicated by the positively valued ω_z^+ become the most intense, significantly stretched in the azimuthal direction. The counterclockwise-rotating secondary-flow structures observed at relatively high rotation numbers are the so-called Taylor columns, which become increasingly elongated in the axial direction as the rotation number is increased.

4.2.2 Mean flow fields

Figure 4.4 compares the cross-stream distributions of the mean axial velocity $\langle u_z \rangle^+$ and TKE ($k^+ = \frac{1}{2} \langle u'_i u'_i \rangle^+$) of four different rotation numbers. Given the axial symmetry of the flow field, only a quarter of the domain is plotted for each rotation number to facilitate a direct comparison. In figure 4.4(a), the contours of $\langle u_z \rangle^+$ are superimposed with mean cross-stream velocity vectors for $Ro_b = 0, 1, 6$ and 20 . In figure 4.4(b), the contour plots of k^+ are based on rotation numbers $Ro_b = 0, 1, 14$ and 20 . By comparing the mean flow patterns at $Ro_b = 0$ and 1 shown in figure 4.4(a), it can be seen that as soon as the system rotation is imposed, mean secondary-flow motion occurs, behaving as counterclockwise rotations as indicated by the mean cross-stream velocity vectors. From figure 4.4(a), it is observed that in general, the magnitude of $\langle u_z \rangle^+$ increases monotonically as the pipe centre is approached for both rotating and non-rotating flow cases. The magnitude of $\langle u_z \rangle^+$ at the pipe centre reaches its maximum at $Ro_b = 6$. However, as the rotation number further increases from $Ro_b = 6$ to 20 , the magnitude of $\langle u_z \rangle^+$ reduces. This non-monotonic trend in the value of $\langle u_z \rangle^+$ with an increasing rotation number is interesting, and will be studied further by examining the evolution of the bulk mean velocity U_b^+ with Ro_b . The variation of the TKE distribution in response to an increasing rotation number also exhibits an interesting trend. As shown in figure 4.4(b), for a non-rotating pipe flow ($Ro_b = 0$), the peak value of k^+ occurs in the buffer layer (at approximately $y^+ = 15$). Clearly, as the value of Ro_b increases, the magnitude of k^+ increases in general, and furthermore, the radial distribution of the k^+ value varies. More specifically, as the rotation number is increased to $Ro_b = 14$, the peak value of k^+ shifts to the pipe centre, a pattern that is in sharp contrast to that of the non-rotating flow ($Ro_b = 0$). As the rotation number continues to increase from $Ro_b = 14$ to 20 , the region of large k^+ values expands slightly at the pipe centre.

Figure 4.5 compares the profiles of the mean axial velocity $\langle u_z \rangle^+$, mean swirl velocity $r \langle u_\beta \rangle^+ / R$ and mean axial vorticity $\langle \omega_z \rangle^+$ with respect to the non-dimensional

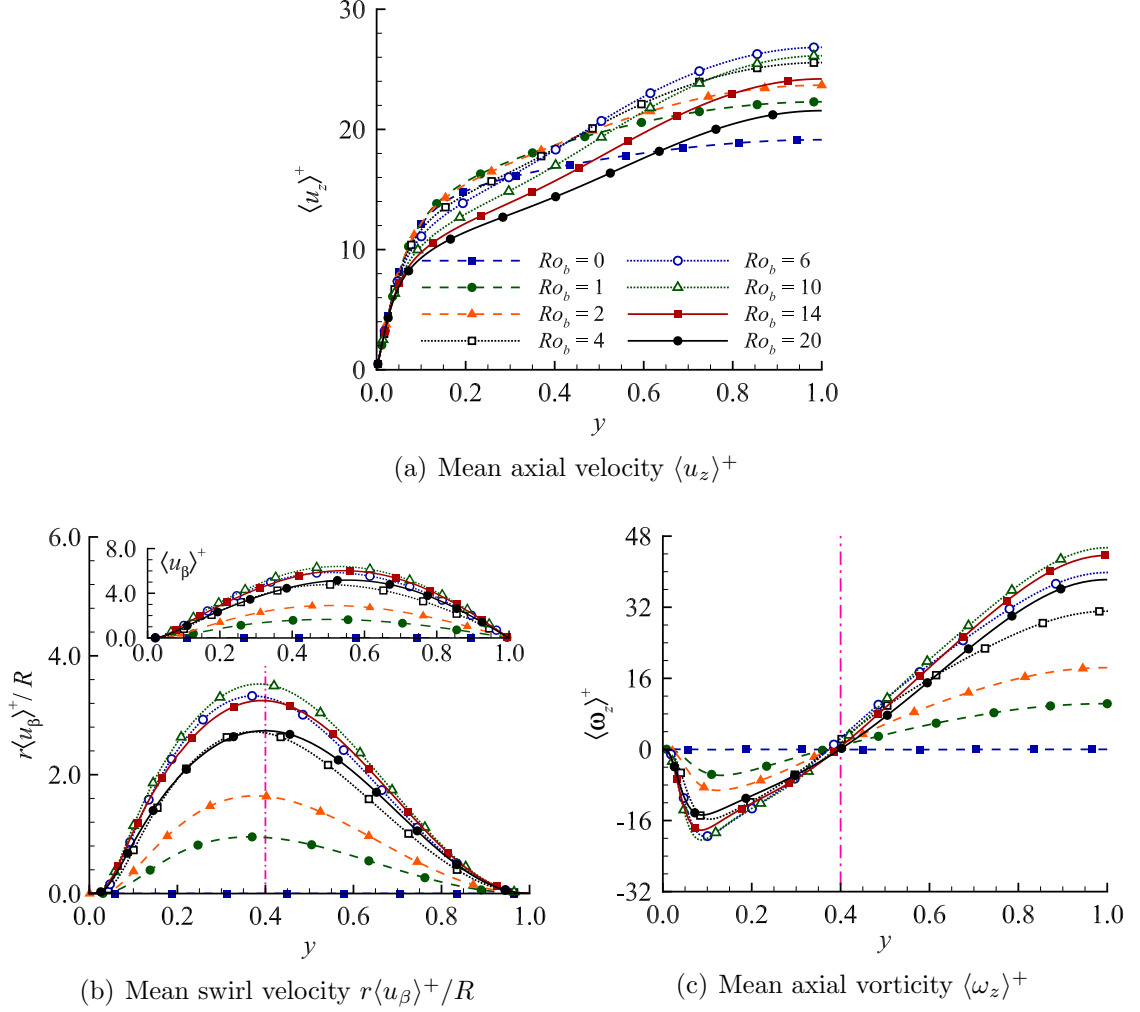


Figure 4.5: Profiles of mean axial velocity $\langle u_z \rangle^+$, mean swirl velocity $r\langle u_\beta \rangle^+ / R$ and mean axial vorticity $\langle \omega_z \rangle^+$ at eight rotation numbers for $Ro_b = 0$ –20 based on test cases listed in table 4.1. In panel (b), the profiles of $\langle u_\beta \rangle^+$ at different rotation numbers are displayed using a subpanel. In panels (b,c), the pink vertical dash-dotted line demarcates the location of zero value of $\langle \omega_z \rangle^+$ occurring at $y = 0.4$. All values have been non-dimensionalised by u_τ .

wall-normal distance (y) of the eight test cases of table 4.1. Given its axial symmetry, only one half the profile of $\langle u_z \rangle^+$ is plotted in figure 4.5(a), which varies from the pipe wall to the pipe centre (for $y \in [0, 1]$). From figure 4.5(a), it is evident that the magnitude of $\langle u_z \rangle^+$ varies non-monotonically with an increasing Ro_b value. At the pipe centre, the peak value of $\langle u_z \rangle^+$ increases when the rotation number rises from

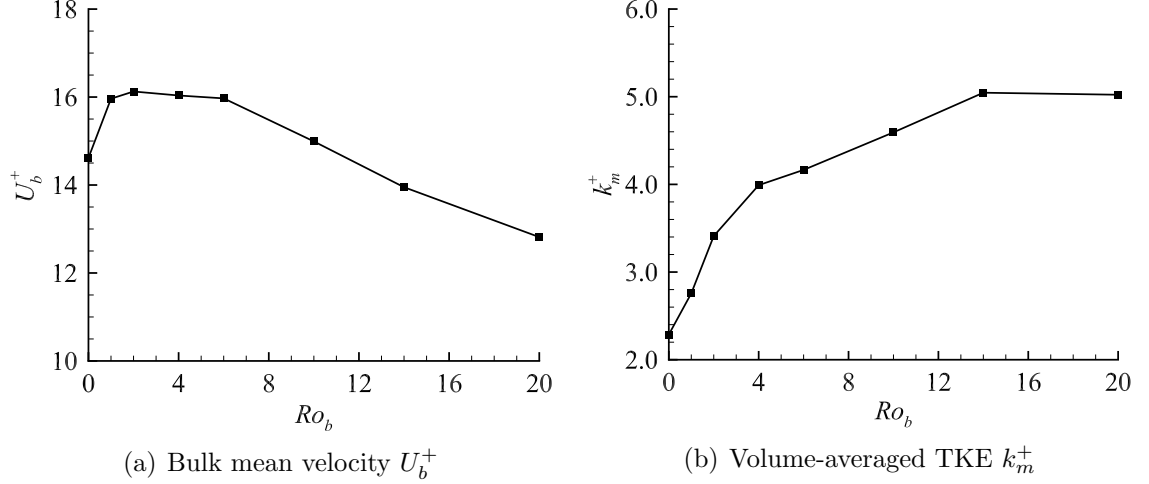


Figure 4.6: Profiles of bulk mean velocity U_b^+ and volume-averaged TKE k_m^+ with respect to the rotation number Ro_b . The values of U_b^+ and k_m^+ have been non-dimensionalised based on u_τ . Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

$Ro_b = 0$ to 6, and reaches its maximum at $Ro_b = 6$ with a magnitude that is 40.19% higher than that of the non-rotating pipe flow. As Ro_b continues to increase from 6 to 20, the peak value of $\langle u_z \rangle^+$ reduces, which however is still larger than that of the non-rotating pipe flow ($Ro_b = 0$). These observations are qualitatively consistent with figure 4.4(a). From the subpanel of figure 4.5(b), it is clear that the profile of $\langle u_\beta \rangle^+$ is zero at both the pipe wall and pipe centre. In addition, it is interesting to observe from figures 4.5(b,c) that both $r\langle u_\beta \rangle^+/R$ and $\langle \omega_z \rangle^+$ reach their maxima at $Ro_b = 10$, and the peak of $r\langle u_\beta \rangle^+/R$ is located at $y \approx 0.4$ where the mean axial vorticity $\langle \omega_z \rangle^+$ is zero identically.

Figure 4.6 compares the bulk mean velocity U_b^+ and volume-averaged TKE k_m^+ at varying rotation numbers based on the test cases listed in table 4.1. In figure 4.6(a), the profile of U_b^+ peaks at $Ro_b = 2$ (or generally in the range of $Ro_b \in [1, 6]$), with a magnitude that is 10.33% higher than that of the non-rotating pipe flow. As the rotation number is increased beyond 6, the magnitude of U_b^+ decreases significantly. At the highest rotation number $Ro_b = 20$, the value of U_b^+ is 12.32% lower than that of the non-rotating flow. The monotonic decreasing trend of U_b^+ with respect to an

increasing value of Ro_b (at high rotation numbers of $Ro_b \geq 6$) is vividly demonstrated in figure 4.4(a), and is also consistent with the trend of the mean axial velocity profiles of $\langle u_z \rangle^+$ shown in figure 4.5(a) and the result of Orlandi and Ebstein (2000). From figure 4.6(b), the value of k_m^+ increases as the rotation number increases in general, and reaches its maximum at $Ro_b = 14$. However, as the rotation number is increased from $Ro_b = 14$ to 20, the value of k_m^+ varies little. I explain later that axial rotation enhances the magnitudes of $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$, which subsequently make a positive contribution to the value of k_m^+ .

4.2.3 Reynolds stresses

Figure 4.7 compares the profiles of six Reynolds stresses $\langle u'_z u'_z \rangle^+$, $\langle u'_r u'_r \rangle^+$, $\langle u'_\beta u'_\beta \rangle^+$, $\langle u'_r u'_z \rangle^+$, $\langle u'_r u'_\beta \rangle^+$ and $\langle u'_\beta u'_z \rangle^+$ at eight rotation numbers of $Ro_b = 0$ –20 based on test cases of table 4.1. All these Reynolds stress profiles shown are axially symmetrical about the pipe centre (located at $y = 1.0$). As shown in figure 4.7(a), the profile of $\langle u'_z u'_z \rangle^+$ peaks at $y = 0.083$ (or $y^+ \approx 15$) in the near-wall region of the non-rotating pipe flow. As the rotation number is increased, the magnitude of $\langle u'_z u'_z \rangle^+$ reduces monotonically in the near-wall region of the pipe. Furthermore, the peak of $\langle u'_z u'_z \rangle^+$ moves towards the pipe centre as Ro_b increases. The lowest and highest peak values occur at $Ro_b = 4$ and 14, respectively, which are 15.01% and 4.57% lower than that of the non-rotating pipe flow ($Ro_b = 0$). In addition, it is interesting to observe that the magnitude of $\langle u'_z u'_z \rangle^+$ increases monotonically with an increasing rotation number at the pipe centre. At $Ro_b = 20$, the magnitude of $\langle u'_z u'_z \rangle^+$ increases by more than fivefold in comparison with the non-rotating flow ($Ro_b = 0$). The observations of the apparent migration of the peak of $\langle u'_z u'_z \rangle^+$ towards the pipe centre at higher rotation numbers (for $Ro_b \geq 6$) and the monotonic increasing trend of the magnitude of $\langle u'_z u'_z \rangle^+$ with an increasing rotation number at the pipe centre are the result of Taylor columns. The appearance of the Taylor columns at moderate and high rotation

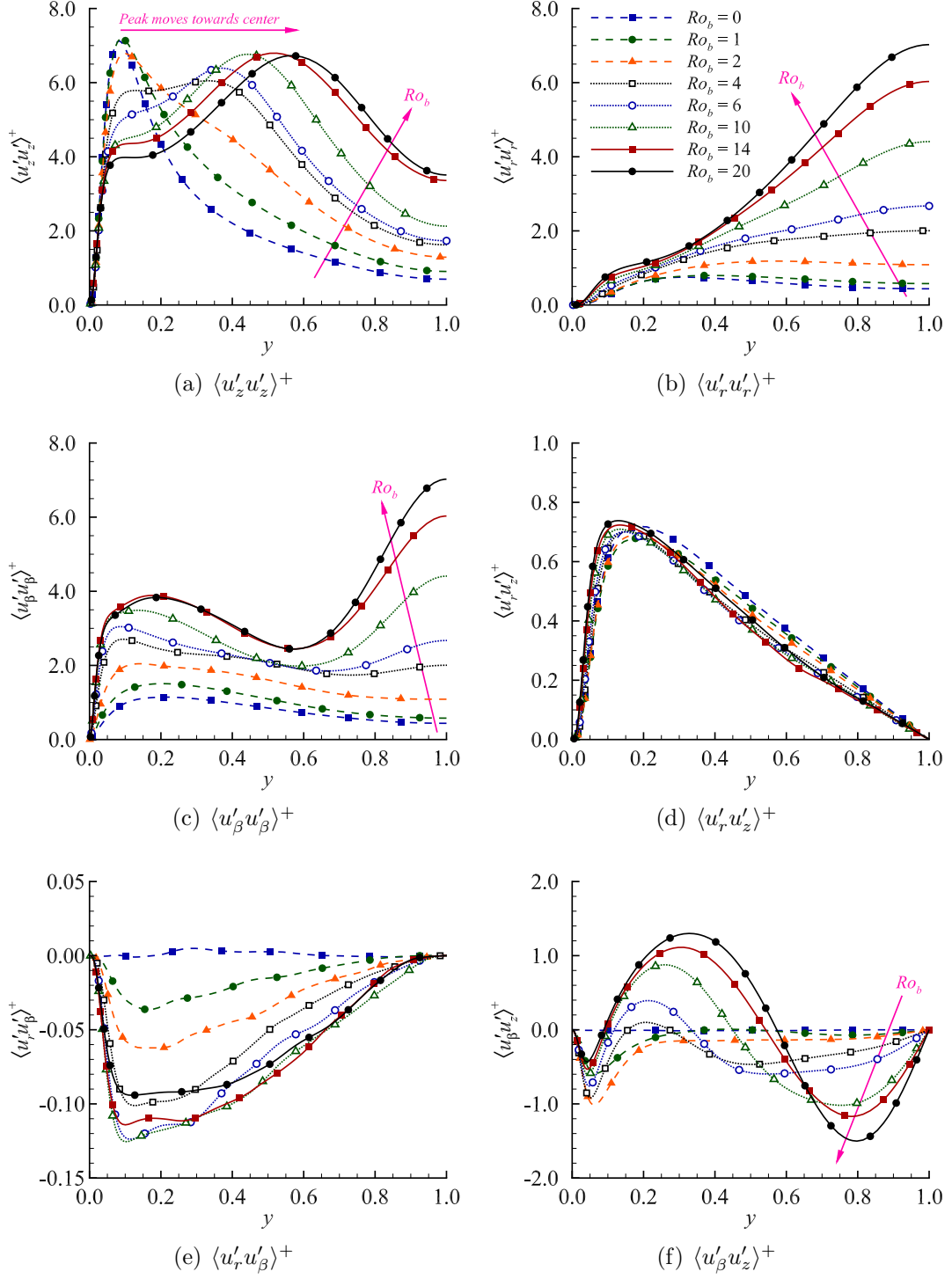


Figure 4.7: Profiles of six Reynolds stresses $\langle u'_z u'_z \rangle^+$, $\langle u'_r u'_r \rangle^+$, $\langle u'_\beta u'_\beta \rangle^+$, $\langle u'_r u'_z \rangle^+$, $\langle u'_r u'_\beta \rangle^+$ and $\langle u'_\beta u'_z \rangle^+$ at eight different rotation numbers for $Ro_b = 0-20$ based on test cases of table 4.1. Pink arrows point to the direction of an increasing rotating effect. All values are non-dimensionalised by u_τ^2 .

numbers also significantly affects the physical features of the mean flow field, e.g. the reduction of the bulk mean velocity U_b^+ with an increasing rotation number shown previously in figure 4.6(a). The characteristics of these two types of coherent flow structures, hairpins and Taylor columns, are studied in detail in both physical and spectral spaces in sections 4.2.4 and 4.2.6.

As is clear in figures 4.7(b,c), the magnitudes of $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$ increase monotonically in the pipe centre as the rotation number is increased. Furthermore, as the rotation number rises, the profile of $\langle u'_r u'_r \rangle^+$ evolves from a dual-peak pattern to a single-peak pattern (over the entire radial direction along a diameter across the pipe), but that of $\langle u'_\beta u'_\beta \rangle^+$ turns into a triple-peak pattern over the full range of a diameter. In response to the system rotation imposed, secondary flows are induced, which tend to enhance the general levels of $\langle u'_z u'_z \rangle^+$, $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$, especially in the central region of the pipe. As a result, the level of the volume-averaged TKE k_m^+ , shown previously in figure 4.6(b), enhances as the rotation number increases within the range of this study (for $Ro_b \in [0, 20]$).

From figure 4.7(d), it is clear that the profile of $\langle u'_r u'_z \rangle^+$ is approximately linear and symmetrical in the central region of the circular pipe for both non-rotating and rotating flows. For the non-rotating flow case, the profile of $\langle u'_r u'_z \rangle^+$ peaks at $y = 0.177$ (or $y^+ \approx 32$). However, as the rotation number is increased, the peak position of $\langle u'_r u'_z \rangle^+$ shifts slightly towards the wall, with its profile being relatively insensitive to the rotation number. For a non-rotating flow, it is well-known that $\langle u'_r u'_\beta \rangle^+ \equiv 0$ and $\langle u'_\beta u'_z \rangle^+ \equiv 0$ hold strictly due to the axial symmetry of the flow. However, this is not the case for an axially rotating pipe flow. As is evident in figures 4.7(e,f), both these Reynolds shear stress components are non-trivial in an axially rotating pipe flow, albeit their magnitudes are one order of magnitude smaller than those of the other four Reynolds normal and shear stress components. Furthermore, as shown in figure 4.7(e), an approximately linear behaviour can be observed in the profile of $\langle u'_r u'_\beta \rangle^+$ as soon as the axial system rotation is imposed (for $Ro_b \geq 1$), which is a

consequence of clockwise system rotation of the pipe. This observation is consistent with the finding by Orlandi and Fatica (1997). Figure 4.7(f) shows the radial profile of $\langle u'_\beta u'_z \rangle^+$. Clearly, the value of $\langle u'_\beta u'_z \rangle^+$ is zero identically at the pipe wall ($y = 0$) and at the pipe centre ($y = 1.0$), due to the no-slip and axial-symmetry flow conditions, respectively. As the rotation number is increased, the amplitude of $\langle u'_\beta u'_z \rangle^+$ increases monotonically. At high rotation numbers of $Ro_b \geq 14$, it is interesting to observe that the value of $\langle u'_\beta u'_z \rangle^+$ changes sign three times within half a radial domain (for $y \in [0, 1]$), such that there are five zero-crossing points along the entire radial direction of a diameter.

The behaviours of the three components of Reynolds shear stresses associated with figures 4.7(d-f) can be further analysed through their corresponding shear stress balance equations. Assuming that the flow is fully developed in the axial direction, and statistically homogeneous in the axial and azimuthal directions, the following equations can be obtained from the z -, β - and r -components of the momentum equation (4.2) after applying turbulence decomposition (i.e. $u_i = \langle u_i \rangle + u'_i$) to the velocity (Speziale et al., 2000)

$$\frac{\nu}{r} \frac{d}{dr} \left(r \frac{d\langle u_z \rangle}{dr} \right) - \frac{1}{r} \frac{d(r\langle u'_r u'_z \rangle)}{dr} = \Pi \quad , \quad (4.3)$$

$$\nu \frac{d}{dr} \left[\frac{1}{r} \frac{d(r\langle u_\beta \rangle)}{dr} \right] - \frac{1}{r} \frac{d(r\langle u'_r u'_\beta \rangle)}{dr} - \frac{\langle u'_r u'_\beta \rangle}{r} = 0 \quad , \quad (4.4)$$

$$\langle u'_\beta u'_\beta \rangle - \langle u'_r u'_r \rangle = -\langle u_\beta \rangle^2 + r \left(\frac{1}{\rho} \frac{d\langle p \rangle}{dr} + \frac{d\langle u'_r u'_r \rangle}{dr} - 2\Omega \langle u_\beta \rangle \right) \quad . \quad (4.5)$$

Note that in the derivation of equation (4.4) from the β -component of the momentum equation, the mean azimuthal Coriolis force $\langle F_\beta \rangle$ is zero identically because $\langle u_r \rangle \equiv 0$. Both viscous shear and Reynolds shear stresses vanish from equation (4.5), the simplified r -component of the momentum equation. From equation (4.5), it is clear that at pipe centre (where $r = 0$ or $y = 1.0$), $\langle u'_\beta u'_\beta \rangle - \langle u'_r u'_r \rangle = 0$. This explains the observation in figures 4.7(b,c) that the trends and magnitudes of $\langle u'_\beta u'_\beta \rangle$ and $\langle u'_r u'_r \rangle$

become increasingly similar as the pipe centre is approached. In a fully developed pipe flow, $\tau_{\beta z}^{vis} = \tau_{z\beta}^{vis} \equiv 0$. In equations (4.3) and (4.4), the remaining two non-trivial mean viscous shear stresses are defined as (Reich and Beer, 1989)

$$\tau_{rz}^{vis} = \nu \frac{\partial \langle u_z \rangle}{\partial r} \quad \text{and} \quad \tau_{r\beta}^{vis} = \nu r \frac{\partial}{\partial r} \left(\frac{\langle u_\beta \rangle}{r} \right) \quad , \quad (4.6)$$

respectively. The total shear stresses balance as

$$\tau_{ri}^{tot} = \tau_{ri}^{vis} - \langle u'_r u'_i \rangle \quad , \quad (4.7)$$

where i denotes z or β . The budget balance of total shear stresses in non-dimensional forms (i.e. τ_{rz}^{tot+} and $\tau_{r\beta}^{tot+}$) is displayed in figure 4.8. The linear behaviour of τ_{rz}^{tot+} is evident from equation (4.3), whose non-dimensional form expresses as

$$\tau_{rz}^{tot+} = \tau_{rz}^{vis+} - \langle u'_r u'_z \rangle^+ = y - 1 \quad , \quad (4.8)$$

in the context of an axially rotating pipe flow. From figure 4.8(a), the profile of $-\langle u'_r u'_z \rangle^+$ changes slightly in its shape as the rotation number is increased from $Ro_b = 2$ to 20. The reason the profile of $-\langle u'_r u'_z \rangle^+$ is insensitive to Ro_b is that its value is linearly related to the viscous shear stress such that the total shear stress varies as $\tau_{rz}^{tot+} = y - 1$ for both rotating and non-rotating pipe flows. From equation (4.8), it is understood that at the wall ($y = 0$), $\langle u'_r u'_z \rangle^+|_w \equiv 0$, and so the non-dimensional wall-shear stress becomes $\tau_w^+ = \tau_{rz}^{vis+}|_w \equiv -1$. Therefore, for axially rotating pipe flows of different rotation numbers, the wall shear stress τ_w is expected to be constant, which is evidenced by the stable values of the calculated wall friction number Re_τ^A listed in table 4.1. Furthermore, because the wall shear stress τ_w is constant, it is inferred that the skin friction coefficient, defined as $C_f = \tau_w / (\rho U_b^2 / 2)$, is proportional to U_b^{-2} with the value of U_b given in figure 4.6(a).

From figure 4.8(b), it is interesting to observe that the profile of $\tau_{r\beta}^{vis+}$ is almost a

mirror reflection of that of Reynolds stress $-\langle u'_r u'_\beta \rangle^+$ such that the total shear stress is zero, i.e. $\tau_{r\beta}^{tot+} = 0$. This is an expected feature, which can be strictly proven as follows. Equation (4.4) is derived from the β -component of the momentum equation, which can also be written as $\nabla \cdot \tau_{r\beta}^{tot} = 0$, or

$$\frac{\partial \tau_{r\beta}^{tot}}{\partial r} + \frac{2\tau_{r\beta}^{tot}}{r} = 0 \quad . \quad (4.9)$$

The solution to the above equation is

$$\tau_{r\beta}^{tot} = \frac{C}{r^2} \quad , \quad (4.10)$$

where C is a constant. The regularity condition at the pipe centre requires that $C \equiv 0$. As such,

$$\tau_{r\beta}^{tot} \equiv 0 \quad \text{and} \quad \tau_{r\beta}^{vis} \equiv \langle u'_r u'_\beta \rangle \quad , \quad (4.11)$$

hold rigorously in an axially rotating pipe flow. This is an interesting result, which indicates that the total r - β shear stress is zero identically, such that the viscous shear stress is always balanced by the turbulent shear stress (in an r - β plane) at all rotation numbers. This mechanism gives an analytical explanation to the mirror effect between the profiles of $\tau_{r\beta}^{vis+}$ and $-\langle u'_r u'_\beta \rangle^+$ as observed in figure 4.8(b).

4.2.4 Two-point correlations and spectral analysis

Figure 4.9 compares the two-point autocorrelation coefficients of three velocity components R_{zz} , R_{rr} and $R_{\beta\beta}$ at different rotation numbers based on the eight test cases of table 4.1. All coefficients are determined along the axial direction at wall-normal position $y^+ \approx 15$, where Reynolds stress component $\langle u'_z u'_z \rangle^+$ peaks in the context of

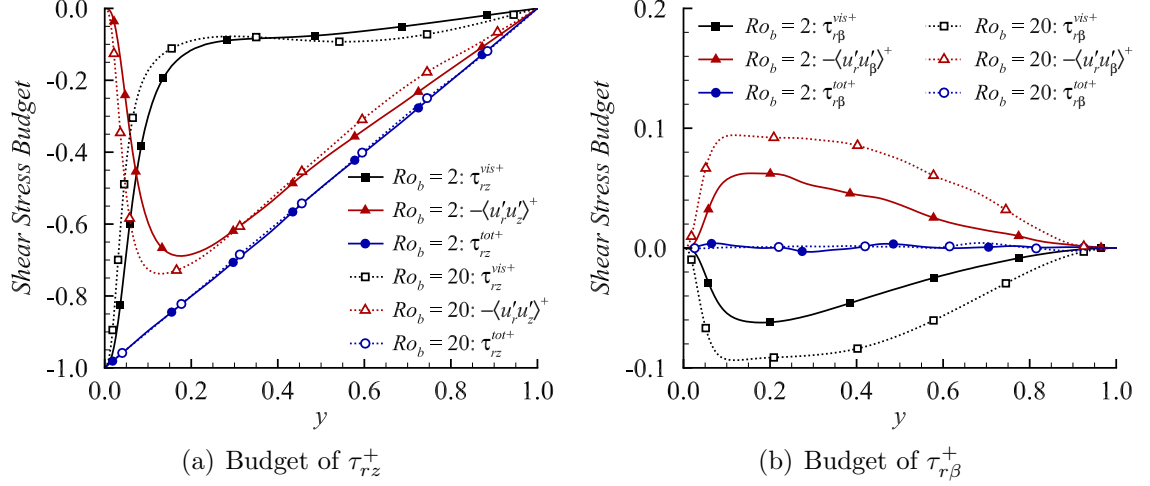


Figure 4.8: Budget profiles of mean shear stress τ_{rz}^+ and $\tau_{r\beta}^+$ at two rotation numbers $Ro_b = 2$ and 20 based on test cases of the longest pipes of table 4.1. All shear stress budget terms are non-dimensionalised by u_τ^2 . Panel (a) shows radial-axial shear stress components (τ_{rz}^{vis+} and $-\langle u'_r u'_z \rangle^+$). Panel (b) shows radial-azimuthal shear stress components ($\tau_{r\beta}^{vis+}$ and $-\langle u'_r u'_\beta \rangle^+$).

a non-rotating pipe flow. The two-point autocorrelation coefficient is defined as

$$R_{ii}(\Delta z) = \frac{\langle u'_i(r, \beta, z, t) u'_i(r, \beta, z + \Delta z, t) \rangle}{\langle u'_i(r, \beta, z, t) u'_i(r, \beta, z, t) \rangle} \quad , \quad (4.12)$$

where i indicates the three directions (i.e. r , β and z) of the cylindrical coordinate system, and no summation convention is implied in this equation. In figures 4.9(a,c), a partially enlarged panel is also displayed in a semi-logarithmic coordinate in order to clearly display the variations of R_{zz} and $R_{\beta\beta}$ for smaller two-point separations ($\Delta z/R$). From figure 4.9(b), it is apparent that the profiles of the radial two-point autocorrelation coefficient R_{rr} at all rotation numbers approach zero rather rapidly as the two-point separation $\Delta z/R$ increases. By contrast, the axial and azimuthal two-point correlation coefficients R_{zz} and $R_{\beta\beta}$ shown in figures 4.9(a,c) are more sensitive to the rotation number, as their values increase monotonically with an increasing value of Ro_b for small two-point separations. Furthermore, it is very interesting to observe a quasi-periodical pattern in the profiles of R_{zz} and $R_{\beta\beta}$ in figures 4.9(a) and

4.9(c), respectively. The spatial periods become larger as the rotation number Ro_b increases. It is evident that the axial scales of the most energetic secondary flow structures (i.e. Taylor columns) induced by the Coriolis force increase monotonically as the rotation number is increased. The quasi-periodical patterns in the profiles of R_{zz} and $R_{\beta\beta}$ are most apparent at high rotation numbers.

As mentioned earlier, there are two main types of flow structure in an axially rotating pipe flow. The first type is streaky structures near the wall (or hairpins) that are typical of the turbulent boundary layer of a pipe flow (rotating or not). The second type is Taylor columns similar to those observed in rotating isotropic turbulence (see, e.g., Bartello et al., 1994; Gallet, 2015; van Kan and Alexakis, 2020; Pestana and Hickel, 2020). However, compared with rotating isotropic turbulence, a turbulent pipe flow is bounded peripherally by the pipe wall of curvature $1/R$. From the profiles of R_{zz} and $R_{\beta\beta}$, it is inferred that the Taylor columns in an axially rotating pipe flow exhibit a spiral pattern, which elongate in the axial direction and spin in the azimuthal direction. In fact, the axial quasi-periodicity of the two-point autocorrelation ($R_{\beta\beta}$) observed in figure 4.9(c) is a consequence of the counterclockwise spinning motion of Taylor columns demonstrated previously in figures 4.3, 4.4(a) and 4.5(b). Furthermore, both axial and azimuthal motions of Taylor columns are quasi-periodic in the axial direction, which are evident in figure 4.9(a,c). A vivid analogy for describing the instantaneous shape of Taylor columns in an axially rotating pipe flow is ‘a bundle of hemp ropes’, twisted together with many smaller individual ropes of different brand names (as such, each can have its own axial and azimuthal periods). As turbulence structures, small Taylor columns can be destroyed by dissipation and transported by diffusion and convection in a viscous flow.

The analysis of Taylor columns can be refined through a spectral analysis, which will show that Taylor columns have two characteristic axial length scales. To demonstrate, figure 4.10 compares the profiles of the premultiplied axial spectra of axial, radial and azimuthal velocity fluctuations (i.e. $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$, respectively)

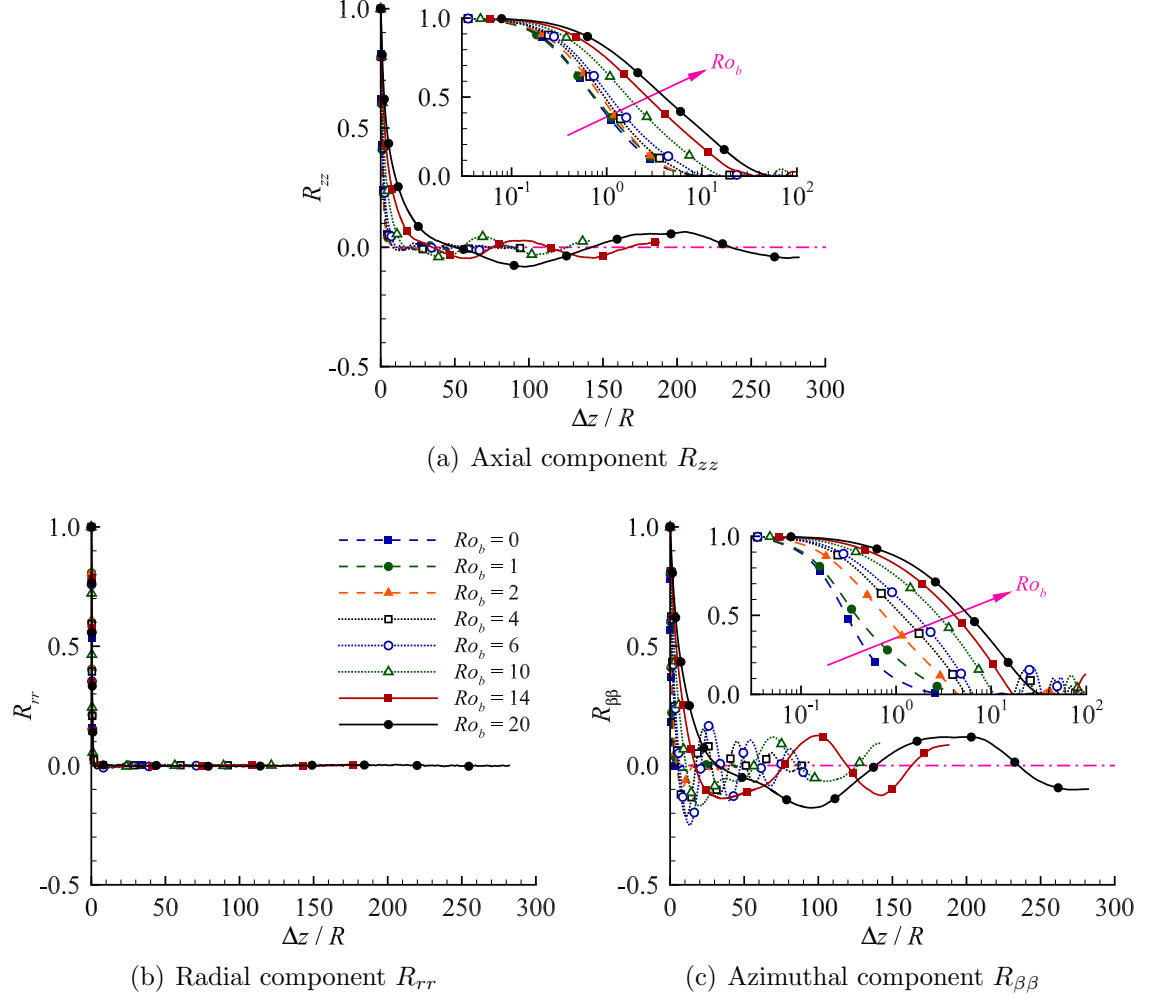


Figure 4.9: Profiles of two-point autocorrelation coefficients R_{zz} , R_{rr} and $R_{\beta\beta}$ for the eight test cases of different rotation numbers Ro_b , calculated along the streamwise direction at wall-normal position $y = 0.083$ (or $y^+ \approx 15$). Pink arrows in the partially enlarged panels (a,c) indicate the direction of an increasing rotation number. Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

of the eight test cases of the longest pipes of table 4.1. All profiles are calculated at wall-normal position $y^+ \approx 15$, where the axial Reynolds normal stress peaks in the context of a non-rotating pipe flow. The 1-D axial energy spectrum of velocity fluctuations is defined as

$$\tilde{E}_{ii}(k_z) = \text{Re} \left\{ \overline{\widehat{u}_i'^* \widehat{u}_i'} \right\} \quad , \quad (4.13)$$

where index i denotes r, β or z of the cylindrical coordinate system, and no summation convention is implied. An overbar represents temporal averaging. The operator $\text{Re}\{\cdot\}$ and superscript $*$ denote the real part and conjugate of a complex number, respectively. A hat denotes Fourier transform in the axial direction of an arbitrary variable $\phi(r, \beta, k_z, t)$, i.e.

$$\hat{\phi}(r, \beta, k_z, t) = \frac{1}{L_z} \int_0^{L_z} \phi(r, \beta, z, t) e^{-\mathbf{i} k_z z} dz, \quad (4.14)$$

where $\mathbf{i} = \sqrt{-1}$ is the imaginary unit and $k_z = n_z k_{z0}$ is the axial wavenumber, with $n_z \in [-N_z/2, N_z/2 - 1]$ being an integer and $k_{z0} = 2\pi/L_z$ being the smallest positive wavenumber. The axial wavelength is defined as $\lambda_z = 2\pi/k_z$, non-dimensionalised as $\lambda_z^+ = \lambda_z u_\tau / \nu$.

For the premultiplied axial velocity spectrum $k_z^+ \tilde{E}_{zz}^+$, it is evident in figure 4.10(a) that the mode of its dome peak (corresponding to the characteristic length scale of the most energetic eddies) of the non-rotating pipe flow ($Ro_b = 0$) is located within $\lambda_z^+ \in [530, 2450]$ at $y^+ \approx 15$ in the near-wall region, where axial Reynolds normal stress peaks. This peak of $k_z^+ \tilde{E}_{zz}^+$ is a consequence of near-wall streaks of the boundary layer developing over the pipe wall. The formation of this peak does not rely on axial system rotation, but rather, it is a characteristic of a turbulent boundary layer commonly found in wall-bounded flows over flat plates (see, e.g., Zhou et al., 1999; Adrian, 2007; Wu and Moin, 2009) or in non-rotating pipe flows (see, e.g., Wu and Moin, 2008; Wu et al., 2012; Baltzer et al., 2013). This peak of the premultiplied axial velocity spectrum $k_z^+ \tilde{E}_{zz}^+$ is denoted as peak ‘ I ’ hereafter, and for a specific rotation number (e.g. $Ro_b = 14$ or 20), it is indicated using the rotation number as the subscript (as I_{14} or I_{20} , respectively). As the rotation number is increased from $Ro_b = 0$ to 20 , the magnitude of the first peak (I) of $k_z^+ \tilde{E}_{zz}^+$ decreases monotonically. Meanwhile, the mode of the first peak moves towards larger wavelengths, leading to an increasingly broadband vortical scales in the axial direction as the rotation number is increased. At the highest rotation number $Ro_b = 20$, the wavelengths of the first

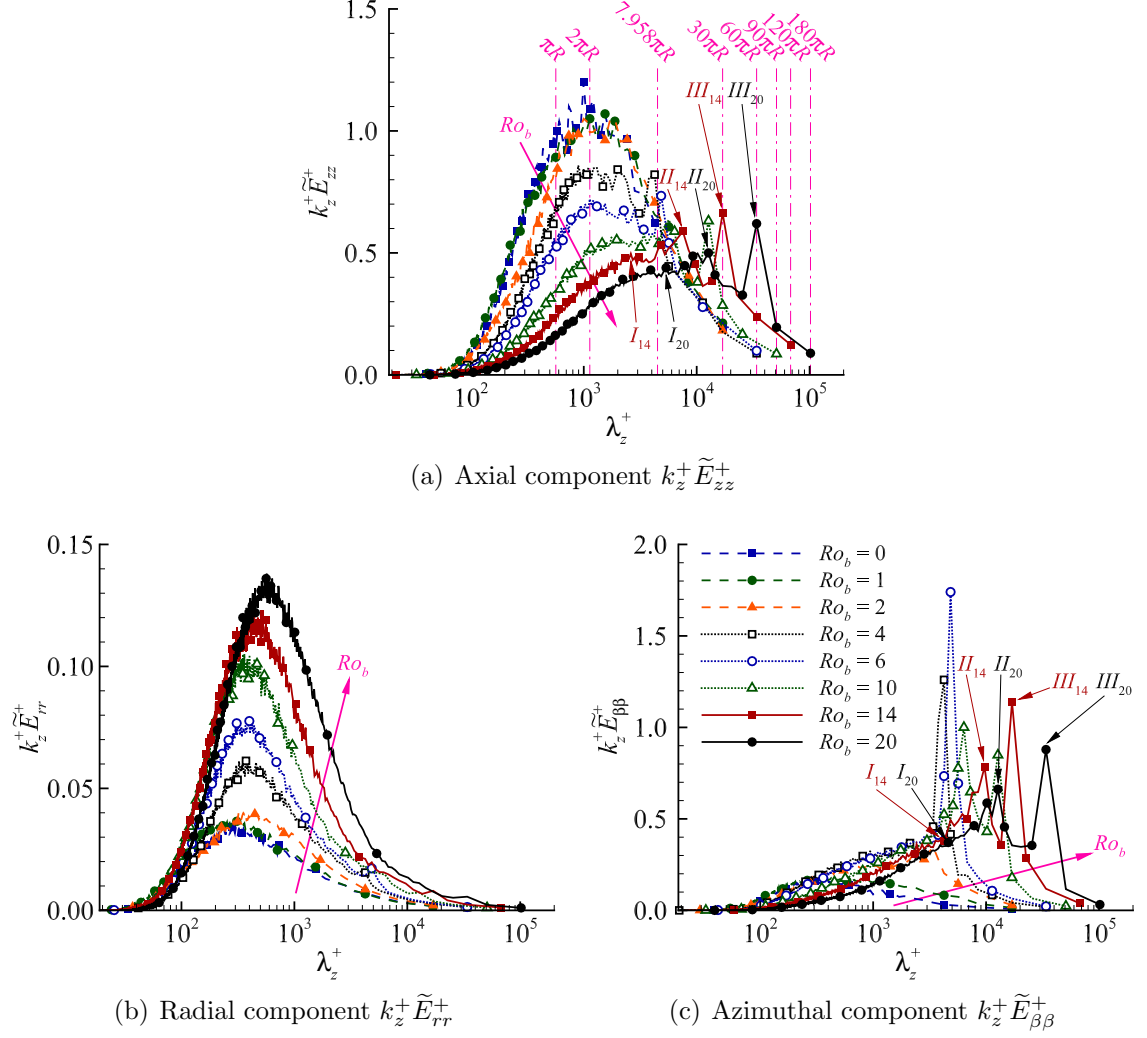


Figure 4.10: Profiles of premultiplied axial, radial and azimuthal velocity spectra $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ at eight different rotation numbers for $Ro_b = 0-20$, calculated along the streamwise direction at wall-normal position $y = 0.083$ (or $y^+ \approx 15$). Pink arrow points to the direction of an increasing rotation number. For clarity, the peaks of premultiplied spectra (*I*, *II* and *III*) are labelled only for the two highest rotation numbers $Ro_b = 14$ and 20 . Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

peak range within $\lambda_z^+ \in [4840, 5660]$.

From figure 4.10(a), it is interesting to see that at high rotation numbers (for $Ro_b \geq 10$), there are two additional peaks (denoted as peaks ‘*II*’ and ‘*III*’) in addition to peak *I* in the profiles of the premultiplied axial velocity spectrum $k_z^+ \tilde{E}_{zz}^+$.

Similarly, peaks of different rotation numbers are differentiated symbolically by using rotation numbers as the subscripts (e.g. II_{20} or III_{20} for the case of $Ro_b = 20$). The appearance of peaks II and III in the profiles of $k_z^+ \tilde{E}_{zz}^+$ is due to the presence of Taylor columns, which are elongated in the axial direction and spin in the azimuthal direction, as discussed previously through the analysis of two-point autocorrelation coefficients R_{zz} and $R_{\beta\beta}$ associated with figures 4.9(a) and 4.9(c), respectively. Using the examples of $Ro_b = 14$ and 20 , the peak magnitudes follow an ascending order, i.e. ' $I_{14} < II_{14} < III_{14}$ ' and ' $I_{20} < II_{20} < III_{20}$ '. This indicates that at high rotation numbers, the share of TKE held by Taylor columns associated with peaks II and III (and especially with peak III) is much larger than that held by streaks at this radial position. Within the range of the rotation numbers tested here, all three peaks of $k_z^+ \tilde{E}_{zz}^+$ have been well captured. The premultiplied energy spectrum loss (in terms of the percentage of its peak value) is less than 20% at their cutoff wavelengths. It is also interesting to observe that there is no apparent distinction between the second and third peaks at moderate rotation numbers of $Ro_b = 4$ and 6 , as the two types (axial and azimuthal) of structures of Taylor columns are still evolving temporally and spatially. This indicates that Taylor columns start to occur at moderate rotation numbers with similar characteristic axial wavelengths. However, as the rotation number continues to increase, Taylor columns evolve spatially to feature two distinct axial characteristic wavelengths, corresponding to modal values of peaks II and III of $k_z^+ \tilde{E}_{zz}^+$. Furthermore, both characteristic axial wavelengths of Taylor columns increase as the rotation number is increased.

Given that the characteristic axial length scale of energetic eddy motions grows with an increasing rotation number, the pipe length is extended from $L_z = 30\pi R$ to $180\pi R$ as rotation number increases from $Ro_b = 0$ to 20 (see table 4.1). As shown in figure 4.10(a), the cutoff wavelength varies from $(\lambda_z^+)_{CF} = 16964.6$ at $Ro_b = 0$ to $(\lambda_z^+)_{CF} = 101787.6$ at $Ro_b = 20$. At the cutoff wavelength, the premultiplied energy spectrum loss varies from 10.33% at $Ro_b = 4$ to 19.81% at $Ro_b = 1$. Clearly, if the

short pipe of $L_z = 25R$ (or $7.958\pi R$) is used (as in Orlandi and Ebstein, 2000), the prediction of the largest peak III_{20} of $k_z^+ \tilde{E}_{zz}^+$ that is characteristic of Taylor columns would be missed entirely. It should be indicated that it is neither necessary nor realistic to fully prevent premultiplied energy spectrum loss at the cutoff wavelength in either a DNS or an experiment, as it implies usage of a pipe of infinite length. Furthermore, flow structures of large wavelengths far beyond the mode of the premultiplied spectrum typically possess very low TKE. However, it is important to ensure that all dominant modes of premultiplied spectra are captured in a DNS, LES or physical experiment, which correspond to the characteristic length scales of the most energetic eddy motions. In practice, for example, the premultiplied energy spectrum loss at the cutoff wavelength is often kept lower than $5/8$ or $3/8$ of its modal value (i.e. $0.625 \max(k_z^+ \tilde{E}_{ii}^+)$ or $0.375 \max(k_z^+ \tilde{E}_{ii}^+)$ for moderate and relatively high-precision requirements, respectively) in DNS studies of turbulent plane-channel flows (Hoyas and Jiménez, 2006; Avsarkisov et al., 2014; Yang and Wang, 2018; Bagheri et al., 2020).

In contrast to the general trend of the premultiplied axial velocity spectrum $k_z^+ \tilde{E}_{zz}^+$ shown in figure 4.10(a), the magnitude of the premultiplied radial velocity spectrum $k_z^+ \tilde{E}_{rr}^+$ increases monotonically as the rotation number is increased, a pattern that is apparent in figure 4.10(b). The presence of the dominant peak of $k_z^+ \tilde{E}_{rr}^+$ at this near-wall position of approximately $y^+ = 15$ results from the streaky structures. As shown in figure 4.10(b), the modal position of $k_z^+ \tilde{E}_{rr}^+$ is relatively stable at all eight rotation numbers tested. A careful perusal of the figure indicates that the characteristic axial length scale of the most energetic radial eddy motions as indicated by the mode of $k_z^+ \tilde{E}_{rr}^+$ increases from $\lambda_z^+ = 282.74$ to 568.65 as the rotation number is increased from $Ro_b = 0$ to 20 .

Figure 4.10(c) compares the premultiplied azimuthal velocity spectrum $k_z^+ \tilde{E}_{\beta\beta}^+$ at eight rotation numbers. Similar to figure 4.10(a), the number of peaks in the profile of $k_z^+ \tilde{E}_{\beta\beta}^+$ varies with the rotation number. At low rotation numbers $Ro_b = 0$ and 1 , the profile of $k_z^+ \tilde{E}_{\beta\beta}^+$ shows only one dominant peak (denoted as I_0 and I_1 ,

respectively, not labelled in the figure) within the range of $\lambda_z^+ \in [220, 600]$, which is a signature of near-wall streaks, shown also at higher rotation numbers at slightly larger wavelengths. However, at moderately high rotation numbers of $Ro_b = 4$ and 6 as shown in figure 4.10(c), a second sharp peak (denoted as II_4 and II_6 , respectively, not labelled in the figure) appears around $\lambda_z^+ = 4241$ and 4847 , which is an indication of the occurrence of Taylor columns. As the rotation number continues to increase, Taylor columns further develop in both axial and azimuthal directions leading to two characteristic axial wavelengths. At higher rotation numbers for $Ro_b \geq 10$, two dominant peaks (II and III) of Taylor columns (besides the existing reduced peak I of the streaks) are identified. At the two highest rotation numbers tested, these two characteristic peaks of Taylor columns are labelled as II_{14} and III_{14} (at $Ro_b = 14$) and II_{20} and III_{20} (at $Ro_b = 20$) in figure 4.10(c). By comparing figures 4.10(a) and 4.10(c), it is clear that the pipe flow motion is dominated by streaks (or hairpins, signified by peak I) at low rotation numbers for $Ro_b \in [0, 2]$, Taylor columns of one single characteristic wavelength (signified by peak II) at moderate rotation numbers for $Ro_b \in [4, 6]$ and Taylor columns of two characteristic wavelengths (signified by peaks II and III) at high rotation numbers for $Ro_b \in [10, 20]$. The two peaks (II and III) of $k_z^+ \tilde{E}_{zz}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ observed at the high rotation numbers for $Ro_b \geq 10$ are consistent with the quasi-periodic behaviours of R_{zz} and $R_{\beta\beta}$ observed previously in figures 4.9(a,c).

Thus far, the analysis of the axial premultiplied velocity spectra has been conducted at $y = 0.083$ (or $y^+ \approx 15$) in the near-wall region of the pipe in figure 4.10. From the previous analysis of figure 4.7(a), it was observed that the value of $\langle u'_z u'_z \rangle^+$ peaks at this wall-normal position at $Ro_b = 0$, but at $y = 0.566$ (or $y^+ \approx 102$) at $Ro_b = 20$. It would be interesting to further examine the behaviours of the three components of the axial premultiplied velocity spectra at this special wall-normal position of $y = 0.566$, which is demonstrated in figure 4.11. From this figure, it is seen that all dominant peaks of the axial premultiplied velocity spectra (e.g. peaks I_{14} ,

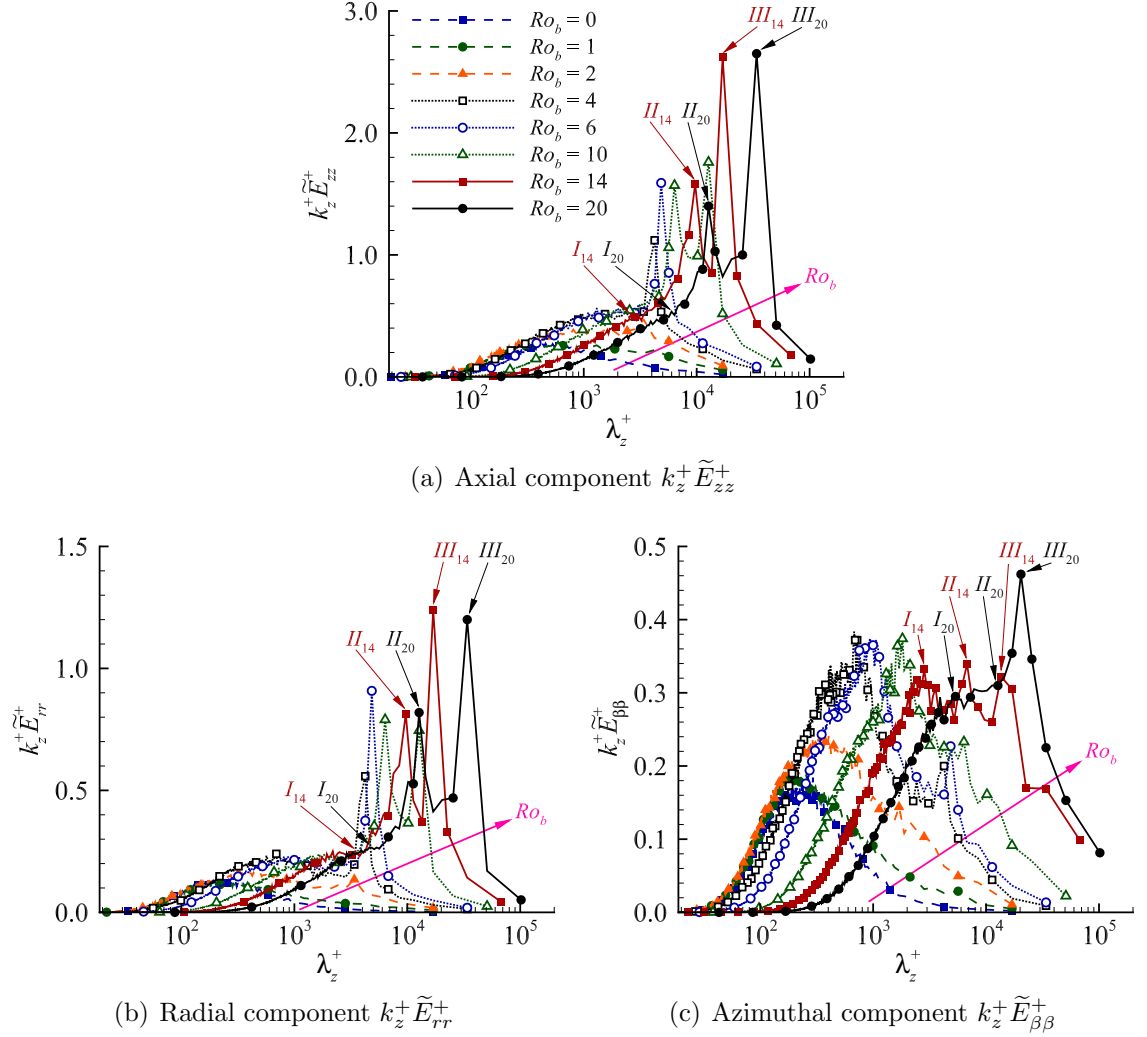


Figure 4.11: Profiles of premultiplied axial, radial and azimuthal velocity spectra $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ at eight different rotation numbers for $Ro_b = 0-20$, calculated along the streamwise direction at wall-normal position $y = 0.566$ (or $y^+ \approx 102$). For clarity, the peaks of premultiplied spectra (*I*, *II* and *III*) are labelled only for the two highest rotation numbers $Ro_b = 14$ and 20 . Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

II_{14} and III_{14} for $Ro_b = 14$, peaks I_{20} , II_{20} and III_{20} for $Ro_b = 20$) have been well captured based on the eight test cases of the longest pipes listed in table 4.1. As shown in figure 4.11(a), the largest premultiplied energy spectrum loss is about 21.37% (in terms of the percentage of its peak value) at $Ro_b = 1$ at the cutoff wavelength.

By comparing figure 4.11 with figure 4.10, it is clear that the profile patterns of

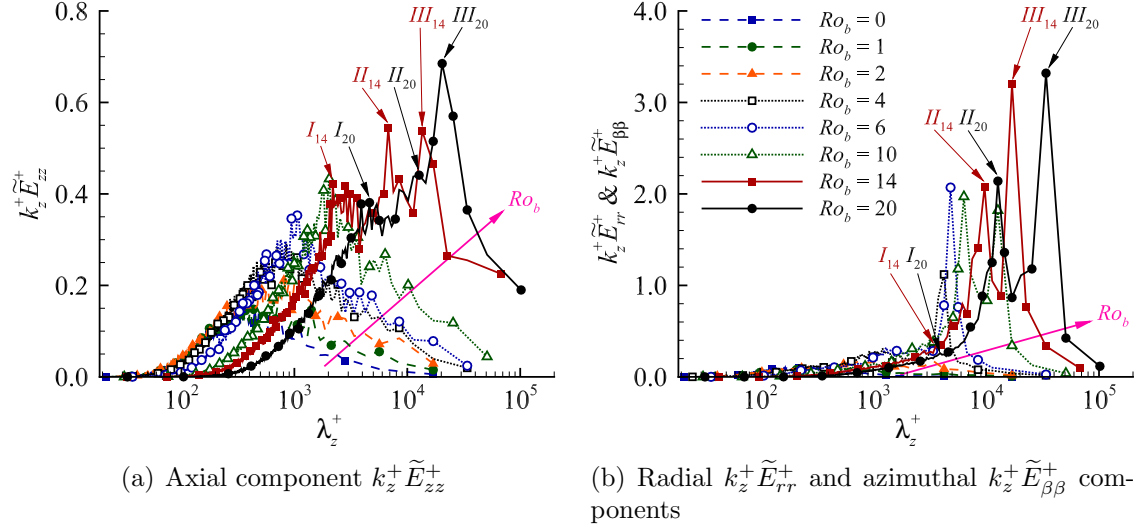


Figure 4.12: Profiles of premultiplied axial, radial and azimuthal velocity spectra $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ at eight different rotation numbers for $Ro_b = 0$ –20, calculated along the streamwise direction at the pipe centre $y = 1.0$. For clarity, the peaks of premultiplied spectra (I , II and III) are labelled only for the two highest rotation numbers $Ro_b = 14$ and 20 . Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

all three components of the axial premultiplied velocity spectra change considerably as the elevation increases from $y = 0.083$ to 0.566 . As shown in figure 4.11(a), the magnitudes of peaks II and III of $k_z^+ \tilde{E}_{zz}^+$ are significantly larger than that of peak I at higher rotation numbers of $Ro_b \geq 10$. This indicates that Taylor columns become increasingly energetic as the distance from the pipe wall increases. Such trend of Taylor columns is more clearly reflected in the profiles of all three axial premultiplied spectra. As shown in figure 4.10(b), the profile of $k_z^+ \tilde{E}_{rr}^+$ has only one dominant peak in the near-wall region (at $y = 0.083$), which is peak I resulted from near-wall streaks of the boundary layer developing over the pipe wall. By contrast, as is evident in figure 4.11(b), the profile of $k_z^+ \tilde{E}_{rr}^+$ is dominated by peaks II and III of Taylor columns at moderate and high rotation numbers of $Ro_b \geq 4$ at a higher elevation of $y = 0.566$. In comparison with the profiles of $k_z^+ \tilde{E}_{\beta\beta}^+$ in the near-wall region shown in figure 4.10(c), those shown in figure 4.11(c) indicate that the magnitudes of peaks II and III reduce relative to that of peak I as the elevation increases from $y = 0.083$ to

0.566.

Figure 4.12 shows the profiles of the three components of the axial premultiplied velocity spectra at the pipe centre ($y = 1.0$) where the mean axial velocity value reaches the maximum and the magnitudes of Reynolds normal stresses $\langle u'_r u'_r \rangle$ and $\langle u'_\beta u'_\beta \rangle$ are the maximum. At the pipe centre, $k_z^+ \tilde{E}_{rr}^+ \equiv k_z^+ \tilde{E}_{\beta\beta}^+$. Clearly, all dominant peaks of $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ have been well captured at eight rotation numbers based on test cases of the longest pipes listed in table 4.1. In comparison with figures 4.10 and 4.11, it is clear that the peak magnitude of $k_z^+ \tilde{E}_{zz}^+$ shown in figure 4.12 drops as the elevation increases from $y = 0.083$ to 1.0. Furthermore, it is clear that the profile of $k_z^+ \tilde{E}_{\beta\beta}^+$ features peak *I* of hairpins in non-rotating or slowly rotating pipe flows (of low rotation numbers $Ro_b = 0-2$). However, as the rotation number increases, the profile of $k_z^+ \tilde{E}_{\beta\beta}^+$ begins to signify Taylor columns, featuring one single dominant peak *II* at moderate rotation numbers (of $Ro_b = 4-6$) but two dominant peaks *II* and *III* at high rotation numbers (of $Ro_b \geq 10$).

Figure 4.13 compares the contour patterns of the non-dimensionalised premultiplied 1-D spectrum of axial velocity fluctuations $k_z^+ \tilde{E}_{zz}^+$ as a function of λ_z^+ and y^+ at two rotation numbers $Ro_b = 0$ and 20 based on the longest-pipe cases of table 4.1. The effects of axial system rotation on the generation of Taylor columns are evident by directly contrasting figures 4.13(a) against 4.13(b). For the non-rotating pipe flow, there is only one mode located at $(\lambda_z^+, y^+) = (997.4, 12.9)$, which corresponds to peak I_0 . This peak is a consequence of near-wall streaks. However, at the highest rotation number $Ro_b = 20$ as shown in figure 4.13(b), the presence of all three peaks (I_{20} , II_{20} and III_{20}) are evident, located at $(\lambda_z^+, y^+) = (7268.5, 63.8)$, $(12719.9, 114.3)$ and $(33919.8, 97.6)$, respectively. In figure 4.13(a), three regions of high-, intermediate- and low-intensity cores are distinguished by different colours enclosed within three isopleths, corresponding to $0.875 \max(k_z^+ \tilde{E}_{zz}^+)$, $0.625 \max(k_z^+ \tilde{E}_{zz}^+)$ and $0.375 \max(k_z^+ \tilde{E}_{zz}^+)$ (or $7/8$, $5/8$ and $3/8$ of the peak value of $k_z^+ \tilde{E}_{zz}^+$, respectively). The high-intensity core within the innermost isopleth of $0.875 \max(k_z^+ \tilde{E}_{zz}^+)$ are associated with the most

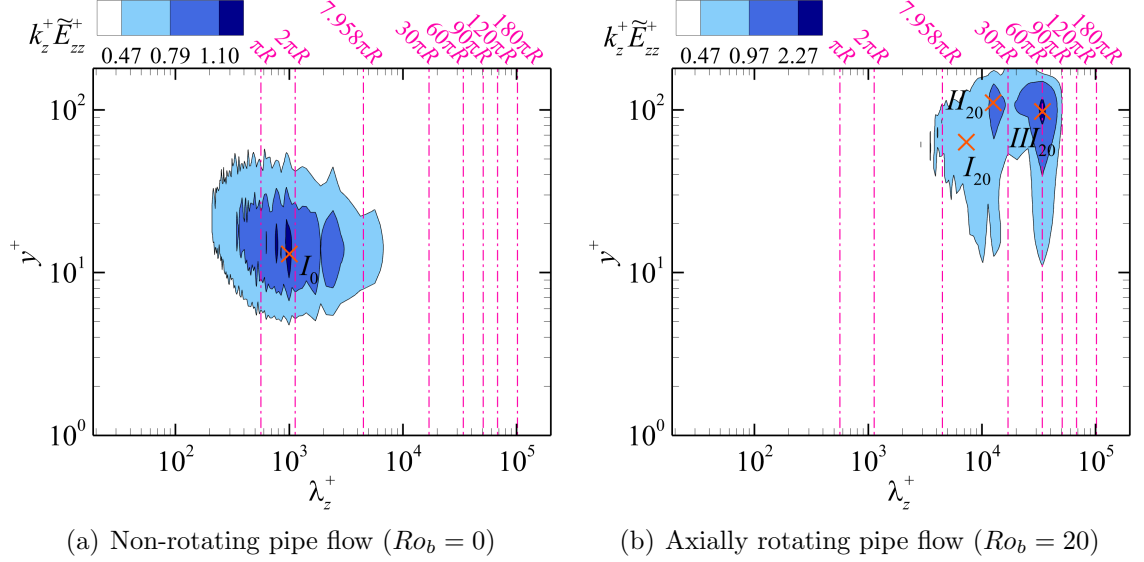


Figure 4.13: Isopleths of premultiplied axial velocity spectra $k_z^+ \tilde{E}_{zz}^+$ with respect to wall-normal coordinate y^+ and wavelength λ_z^+ for the non-rotating ($Ro_b = 0$) and axially rotating ($Ro_b = 20$) pipe flows. The cross symbols ‘ $\times$ ’ demarcate the locations of the three peaks I , II and III . The three energy levels are distinguished by the innermost, intermediate and outermost isopleths of 87.5%, 62.5% and 37.5% of the peak value $\max(k_z^+ \tilde{E}_{zz}^+)$ in panel (a), and 87.5%, 37.5% and 18.0% of the peak value $\max(k_z^+ \tilde{E}_{zz}^+)$ in panel (b), respectively.

energetic eddies of the turbulent flow field. Although the low-intensity core enclosed by the outermost isopleth of $0.375 \max(k_z^+ \tilde{E}_{zz}^+)$ corresponds to less-dominant energetic eddies consisting of a wide range wavelengths, it still makes a considerable contribution to the total TKE. Because the TKE is mostly held by Taylor columns at the high rotation number $Ro_b = 20$, the magnitude of peak I_{20} is much less than those of peaks II_{20} and III_{20} . Therefore, the value of the outermost isopleth is reduced to $0.18 \max(k_z^+ \tilde{E}_{zz}^+)$ in order to have a clear view of the location of peak I_{20} . By comparing figures 4.13(a) and 4.13(b), it is also seen that the modal position of all three peaks (I_{20} , II_{20} and III_{20}) are near the pipe centre. At the rotation number is increased from $Ro_b = 0$ to 20, the modal position of peak I increases from $y^+ = 12.9$ to 63.8.

Figure 4.14 compares the profiles of $k_z^+ \tilde{E}_{zz}^+$ of $Ro_b = 20$ at five wall-normal dis-

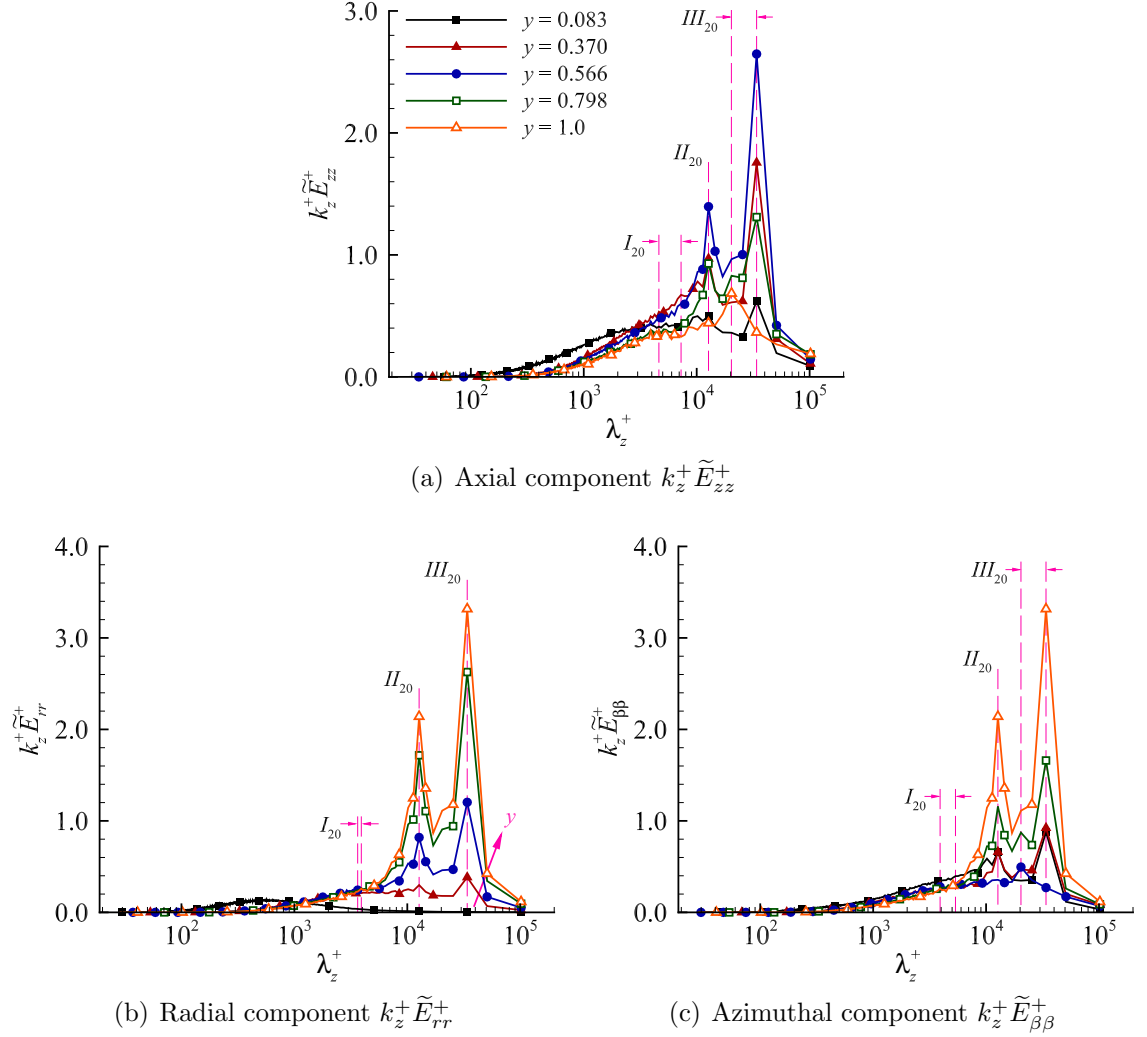


Figure 4.14: Profiles of premultiplied axial, radial and azimuthal velocity spectra $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ at the highest rotation number $Ro_b = 20$, calculated along the streamwise direction at different wall-normal (y) positions, based on the test case of the longest pipe $L_z = 180\pi R$.

tances ranging from $y = 0.083$ (near the wall) to $y = 1.0$ (at the pipe centre). The results are based on the case of the longest pipe $L_z = 180\pi R$. At this high rotation number, it is clear that turbulent flow structures of an axially rotating pipe flow feature predominantly Taylor columns (as indicated by peaks II_{20} and III_{20}). From the previous analysis of figures 4.10-4.12, it is understood that the modal wavelengths corresponding to peaks I , II and III of profiles of $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ may vary

with wall-normal distance y . As is evident in figures 4.14(a-c), the characteristic wavelength corresponding to peak II_{20} is rather stable (with $\lambda_z^+ \approx 12723$) as the elevation (y value) varies. However, the characteristic wavelengths of peaks I_{20} and III_{20} range slightly, which are indicated by a pair of dashed lines and arrows in figures 4.14(a-c). A careful perusal of figures 4.14(a,c) indicates that the magnitudes of $k_z^+ \tilde{E}_{zz}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ are the largest and the smallest, respectively, at $y = 0.566$, which are consistent with the previous observations of a local maximum value of $\langle u'_z u'_z \rangle^+$ and a local minimum value of $\langle u'_\beta u'_\beta \rangle^+$ in figures 4.7(a,c) at a similar wall-normal position, respectively. In figure 4.14(b), it is shown that the magnitude of $k_z^+ \tilde{E}_{rr}^+$ increases as y increases. Such general monotonic trend is consistent with that of $\langle u'_r u'_r \rangle^+$ observed in figure 4.7(b). This is an expected feature, simply because the Reynolds stresses relate directly to their premultiplied spectra $k_z \tilde{E}_{ij}$ through integration $\langle u'_i u'_j \rangle = 2 \int_0^\infty k_z \tilde{E}_{ij} d(\ln k_z)$.

4.2.5 Higher-order turbulence statistics

So far, the influence of the Coriolis force on an axially rotating pipe flow has been studied thoroughly through investigations into the mean flow and second-order turbulence statistics. In order to develop a deeper insight into the flow, the third- and fourth-order statistical moments can be further examined. Figure 4.15 shows the radial profiles of the skewness and flatness factors of three velocity fluctuations, defined as

$$S(u'_i) = \frac{\langle u_i'^3 \rangle}{\langle u_i'^2 \rangle^{3/2}} \quad \text{and} \quad F(u'_i) = \frac{\langle u_i'^4 \rangle}{\langle u_i'^2 \rangle^2} \quad , \quad (4.15)$$

respectively. Here, indices $i = 1, 2$ and 3 , correspond to cylindrical coordinates r , β and z , respectively, and no summation convention is implied. To study the ejection and sweep events in a circular pipe flow, it is convenient to consider the analogy of a boundary-layer flow developing over a flat plate. Thus, the negative of instantaneous radial velocity fluctuation ($-u'_r$) is used in the calculation of the radial skewness factor $S(-u'_r)$. This greatly facilitates an intuitive near-wall analysis of physical phenomena

associated with the skewness factor based on the non-dimensional wall coordinate y .

Figure 4.15(a) compares the profiles of the axial skewness factor $S(u'_z)$ at different rotation numbers. Clearly, the value of $S(u'_z)$ drops rapidly in the near-wall region. For a non-rotating pipe flow, the axial skewness factor has a zero-crossing point at $y = 0.068$ (in the near-wall region of the pipe), and then holds its magnitude almost constant with $S(u'_z) > -0.6$ as the wall-normal distance increases, which is consistent with the observation of Eggels et al. (1994) on non-rotating pipe flows. As soon as the axial rotation is imposed, this pattern of $S(u'_z)$ is broken and the zero-crossing position is shifted towards the pipe centre. At $Ro_b = 20$, the $S(u'_z)$ profile crosses zero at a much high elevation (of $y = 0.422$) from the wall. In figure 4.15, although a strict general monotonic trend in the profiles of skewness and flatness factors with respect to a varying rotation number is not observed, arrows are still plotted wherever a clear local monotonic trend can be identified in the profiles. As is evident in figure 4.15(c), the radial skewness factor $S(-u'_r)$ has two zero-crossing points at $y = 0.023$ and $y = 0.203$ in the near-wall region. A negative peak value can be observed at $y = 0.078$ with $S(-u'_r) = -0.386$ at $Ro_b = 0$. However, as the rotating effect intensifies, the negative peak of $S(-u'_r)$ transitions progressively to become a positive peak. At $Ro_b = 10$, the radial skewness factor reaches its maximum of $S(-u'_r) = 1.066$ at $y = 0.093$, and the two zero-crossing points occur at $y = 0.003$ and $y = 0.468$. At the highest rotation number $Ro_b = 20$, the profile of $S(-u'_r)$ crosses zero at $y = 0.005$ and the profile of $S(-u'_r)$ maintains positively valued in most of the regions. From figure 4.15(e), it is evident that the azimuthal skewness factor basically follows the ideal Gaussian distribution of $S(u'_\beta) = 0$ in the non-rotating pipe flow. However, as soon as the system rotation is imposed, this behaviour of $S(u'_\beta)$ is broken, exhibiting a complex profile pattern in the radial direction.

Figures 4.15(b,d,f) compare the flatness factors of three velocity components at different rotation numbers. Apparently, all three flatness factors peak at the wall for both non-rotating and rotating pipe flows. As shown in figure 4.15(b), the profile of

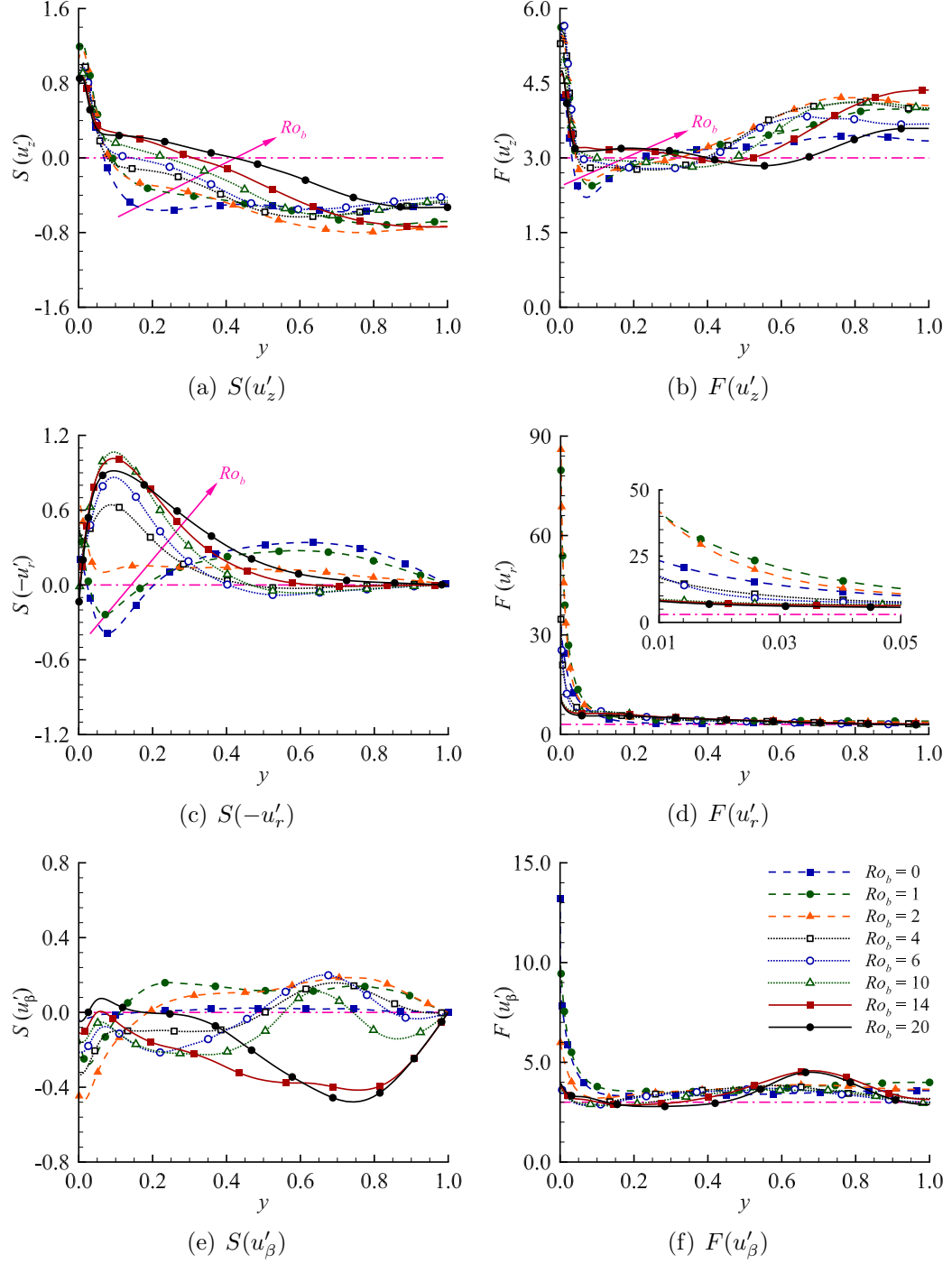


Figure 4.15: Profiles of skewness (a,c,e) and flatness factors (b,d,f) of three velocity fluctuations at various rotation numbers of $Ro_b = 0-20$. The pink dash-dotted line indicates the theoretical values of an ideal Gaussian distribution, which are 0 and 3 for skewness and flatness, respectively. The pink arrow shows a local trend of an increasing rotation number Ro_b . Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

the axial flatness factor $F(u'_z)$ has two zero-crossing points in the near-wall region of the pipe (such that it changes sign five times along a diameter). The two zero-crossing points are located at $y = 0.033$ and $y = 0.221$, respectively, for the non-rotating flow. As the rotation number is increased, the profile of $F(u'_z)$ deviates more apparently from the ideal Gaussian distribution showing an enhanced level of turbulence intermittency. Moreover, the two zero-crossing points tend to shift towards the pipe centre as the rotation number is increased. At the highest rotation number $Ro_b = 20$, the zero-crossing points are located at $y = 0.399$ and at $y = 0.667$. From figure 4.15(d), it is seen that the value of $F(u'_r)$ reaches its maximum at the wall at $Ro_b = 2$, which is almost threefold of that at $Ro_b = 0$. It is evident that the profile of $F(u'_r)$ approaches the theoretical value of 3 of an ideal Gaussian distribution rapidly as the distance from the wall increases. As is evident from figure 4.15(f), the wall value of $F(u'_\beta)$ is the highest in the non-rotating flow case ($Ro_b = 0$), which drops rather rapidly as the rotation number is increased. At the two highest rotation numbers tested ($Ro_b = 14$ and 20), the profile of $F(u'_\beta)$ shows a complex pattern and deviates significantly from a Gaussian distribution at radial positions slightly off the pipe centre ($0.5 < y < 0.9$).

In the analysis of the skewness factor associated with figure 4.15, attention has been paid to the locations of its zero-crossing points. This is because of the sign of the skewness factor is critical in defining near-wall sweep and ejection events. In the context of a circular pipe flow, the ejection events feature $S(u'_z) < 0$ and $S(-u'_r) > 0$, whereas the sweep events are characterised by $S(u'_z) > 0$ and $S(-u'_r) < 0$. The ejection and sweep events correspond to the $Q2$ (featuring $u'_z < 0$ and $-u'_r > 0$) and $Q4$ (featuring $u'_z > 0$ and $-u'_r < 0$) events in the quadrant analysis of the radial-axial Reynolds shear stress, respectively.

Figure 4.16 compares the ejection and sweep events of the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 20$) pipe flows based on the skewness factors of u'_z and $-u'_r$. The intervals corresponding to the ejection and sweep events are labelled in blue and

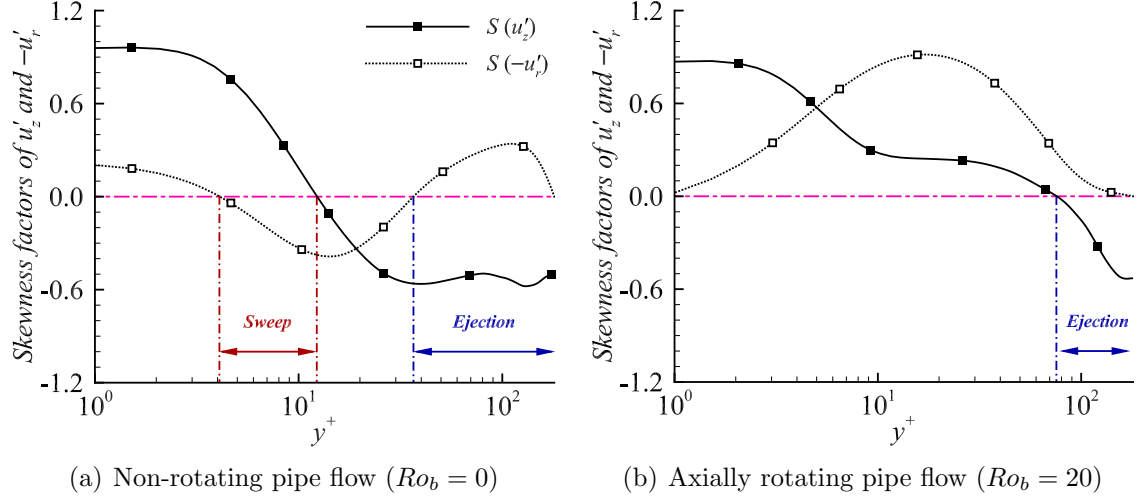


Figure 4.16: Profiles of skewness factors of u'_z and $-u'_r$ of the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 20$) pipe flow cases. Pink horizontal dash-dotted line indicates the zero value of an ideal Gaussian distribution. The intervals corresponding to ejection and sweep events are labelled in blue and red colours, respectively. Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

red colours, respectively. These intervals are identified based on the zero-crossing points of $S(u'_z)$ and $S(-u'_r)$. To ensure a clear view of the events, all profiles are plotted semi-logarithmically in the radial direction with respect to the wall coordinate y^+ . For the non-rotating case shown in figure 4.16(a), the ejection events occur at $y^+ \in [36.885, 180]$ (or, $y \in [0.205, 1]$) in the log-law region, whereas the sweep events are observed at $y^+ \in [4.101, 12.203]$ (or, $y \in [0.023, 0.068]$) in the viscous sublayer. From figure 4.16(b), it is seen that as the rotation number reaches $Ro_b = 20$, the ejection interval is shrunk by 27.30% and shifts towards the pipe centre with $y^+ \in [75.96, 180]$ (or, $y \in [0.422, 1]$). In contrast to the non-rotating pipe flow, the disappearance of the sweep interval in the near-wall region of the pipe indicates that the axial system rotation impedes the formation of the near-wall turbulence structures by suppressing the sweep events. The analysis of sweep and ejection events is conducted solely based on the sign of the skewness factors, $S(u'_z)$ and $S(-u'_r)$, which can only provide a partial explanation of the axially rotating effects on these

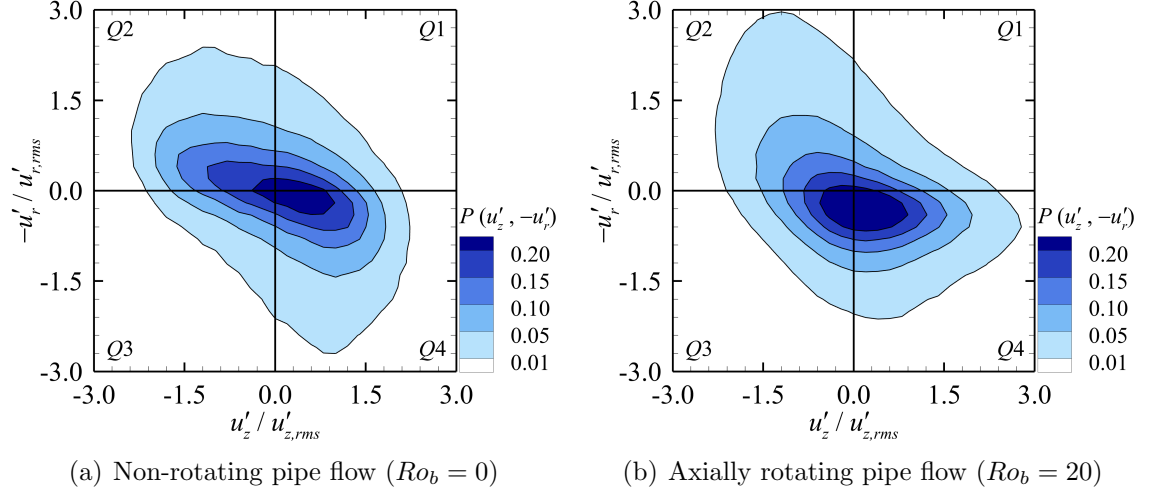


Figure 4.17: Contours of JPDF $P(u'_z, -u'_r)$ at wall-normal position $y^+ \approx 15$ of the pipe for the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 20$) pipe flows. All contours have been split into four quadrants (Q1–4) by black solid lines. Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

two events. To refine the research, the JPDF can be further employed to perform a quadrant analysis of sweep and ejection events, and near-wall turbulence structures.

Figure 4.17 compares the contours of JPDF of u'_z and $-u'_r$, i.e. $P(u'_z, -u'_r)$, at the wall-normal position $y^+ \approx 15$ of the pipe for the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 20$) pipe flows. This particular near-wall position is selected because the Reynolds stress component $\langle u'_z u'_z \rangle^+$ reaches its maximum value at this radial position. From figures 4.17(a,b), it is evident that the Q2 and Q4 events are preferred, corresponding to the ejection and sweep events, respectively. However, as the rotation number is increased from $Ro_b = 0$ to 20, the Q2 events tend to produce high intensities with a low probability, whereas the Q4 events show an opposite trend.

4.2.6 Turbulent flow structures

Figure 4.18 compares instantaneous axial turbulence structures (visualised using the contours of $u'_z{}^+$) in the β - z plane at the wall-normal position $y^+ \approx 15$ for the non-

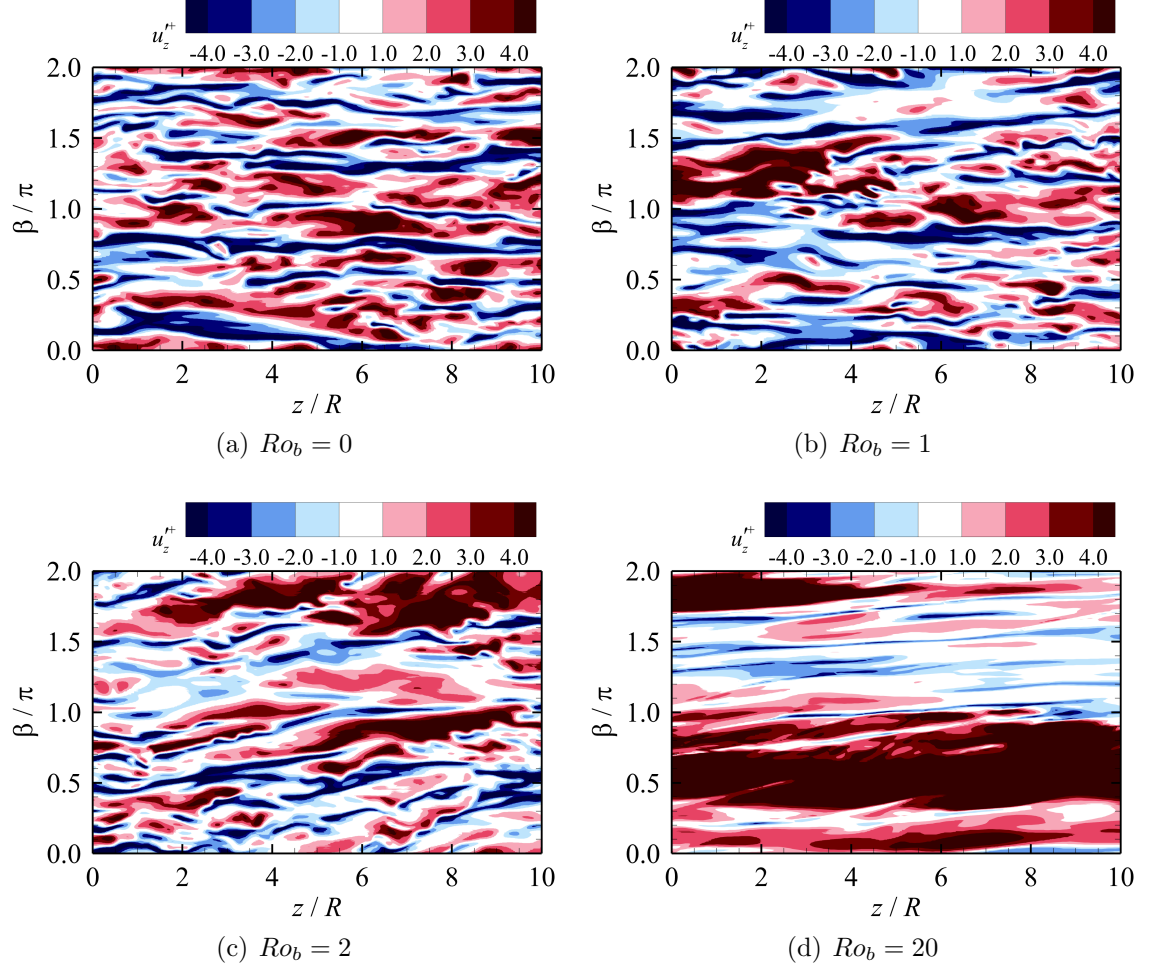


Figure 4.18: Contours of instantaneous axial velocity fluctuations ($u_z'^+$) in the β - z plane at wall-normal position $y^+ \approx 15$ for the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 1, 2$ and 20) flows. The axial domain is arbitrarily extracted within range of $z/R \in [0, 10]$. Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

rotating ($Ro_b = 0$) and rotating ($Ro_b = 1, 2$ and 20) pipe flows. The axial domain is arbitrarily selected within the range of $z/R \in [0, 10]$. At $Ro_b = 0$ as shown in figure 4.18(a), the axial turbulence structures are purely streaks, visualised using contours of positively (indicated by red) and negatively valued (indicated by blue) instantaneous axial velocity fluctuations $u_z'^+$, which alternate in the azimuthal direction in the non-rotating pipe flow. At low rotation numbers $Ro_b = 1$ and 2 shown in figures 4.18(b,c), it is interesting to see that the streaky structures in the near-

wall region tilt slightly, a physical feature that relates to the secondary flows shown previously in figure 4.3(b). At $Ro_b = 20$ as shown in figure 4.18(d), the patterns of axial turbulence structures are qualitatively different from those at zero or very low rotation numbers shown in figures 4.18(a-c) in the sense that they are now mixed with streaks and Taylor columns (at this high rotation number). Clearly, the axial turbulence structures are more intensified and their azimuthal period becomes greater at $Ro_b = 20$. From the previous discussion of figure 4.10, it is further understood that peak magnitudes of II_{20} and III_{20} are larger than that of I_{20} , indicating that turbulence structures are dominated by Taylor columns (over the streaks) at the near-wall position of $y^+ = 15$ at the highest rotation number tested ($Ro_b = 20$).

Figure 4.19 compares instantaneous axial vorticity fluctuation ($\omega_z'^+$) patterns in the β - z plane at $y^+ \approx 15$ for the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 1, 2$ and 20) pipe flows. For the non-rotating pipe flow shown in figure 4.19(a), numerous near-wall small-scale vortical structures are seen in the β - z plane. However, in response to the axial system rotation, these small-scale structures tend to stretch along the axial direction and tilt slightly along the azimuthal direction at $Ro_b = 1$ and 2 , a pattern that is consistent with the observation of figures 4.18(b,c). At the highest rotation number $Ro_b = 20$ as shown in figure 4.19(d), the vortical structures become greatly stretched in the axial direction. The instantaneous vortical structures at zero or very low rotation numbers ($Ro_b = 0-2$) mainly reflect the characteristics of the turbulent boundary layer of a viscous flow. However, at the highest rotation number $Ro_b = 20$, Taylor columns become dominant, a mechanism that is described by the Taylor-Proudman (TP) theorem of the inviscid flow theory and is fundamentally different from that of a boundary layer.

Figure 4.20 compares the coherent structures of the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 2, 6$ and 14) pipe flows. To visualise the coherent structures, the swirling strength (λ_{ci}) criterion of Zhou et al. (1999) is used, which is defined as the imaginary part of the complex eigenvalue of the velocity gradient tensor. The

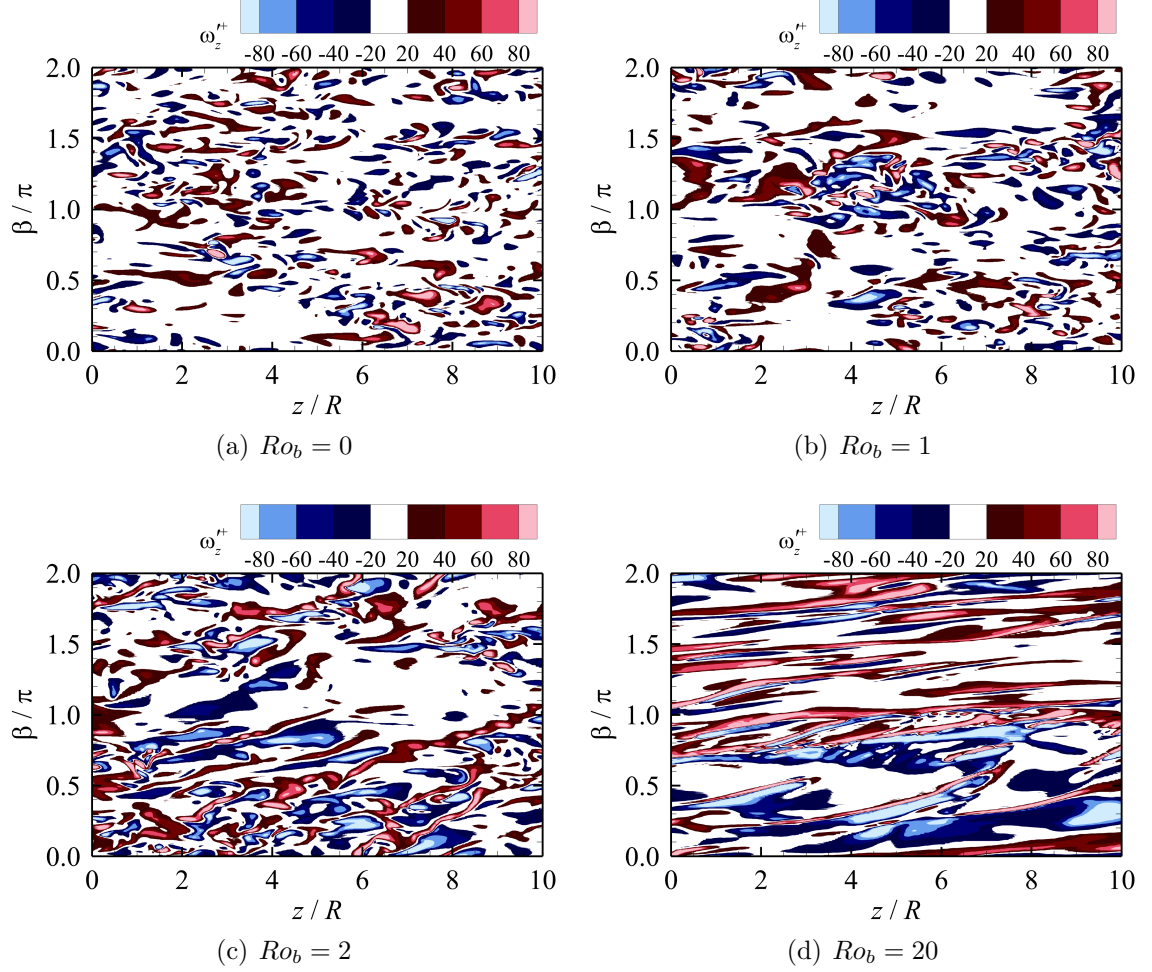


Figure 4.19: Contours of instantaneous axial vorticity fluctuations ($\omega_z'^+$) in the β - z plane at wall-normal position $y^+ \approx 15$ for the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 1, 2$ and 20) pipe flows. The axial domain is arbitrarily extracted within range of $z/R \in [0, 10]$. Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

iso-surfaces of the flow structures are further coloured by non-dimensionalised instantaneous axial vorticity ω_z^+ . Therefore, the blue (representing negatively valued ω_z^+) and red (representing positively valued ω_z^+) flow structures correspond to the clockwise and counterclockwise spiral motions with respect to the axial direction, respectively. Furthermore, in order to provide a clear view of the hairpin structures at zero or low rotation numbers and Taylor columns at high rotation numbers (see also figure 4.3), only one half of the azimuthal domain is displayed, arbitrarily extracted

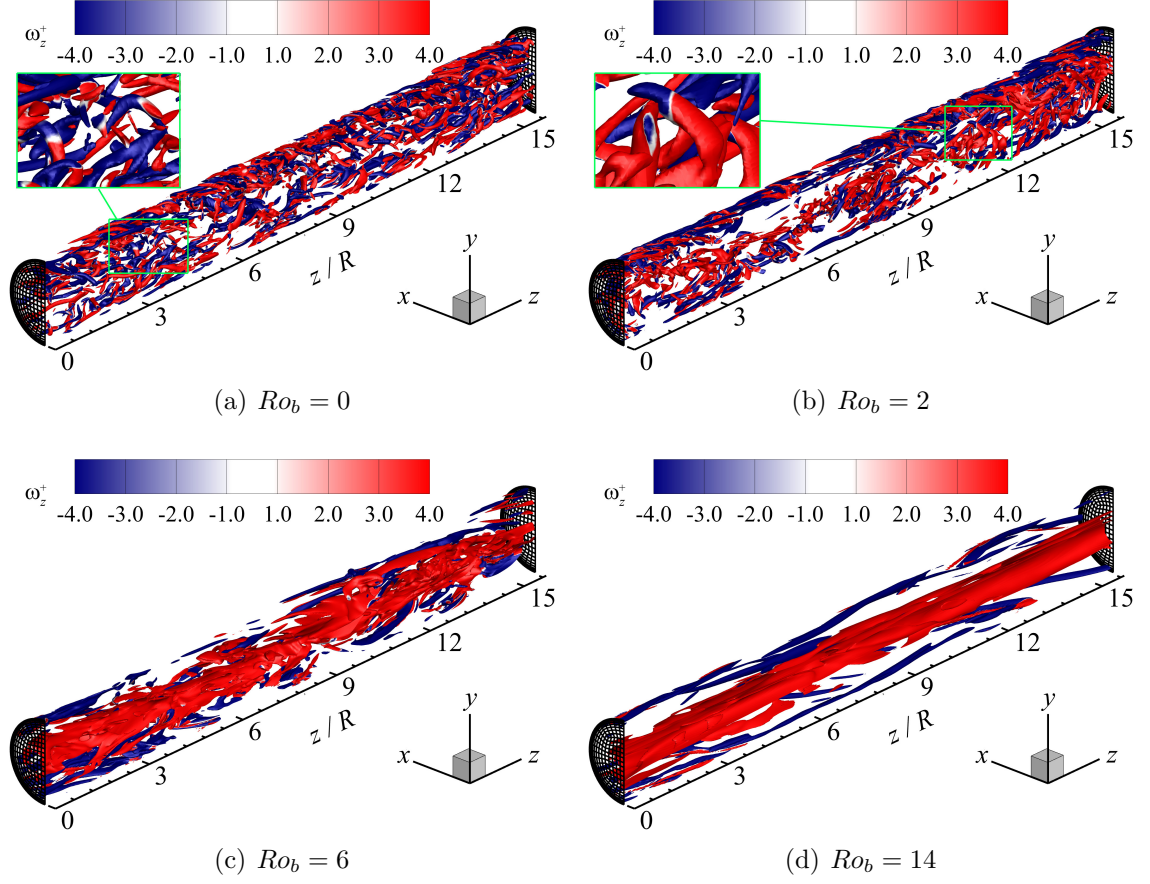


Figure 4.20: Contours of vortical structures of the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 2, 6$ and 14) flows. For clarify, only one half of the azimuthal domain is displayed. The axial domain (iso-surfaces are plotted with $\lambda_{ci} = 1.1$) is arbitrarily extracted within range of $z/R \in [0, 15]$. All contours are coloured by non-dimensionalised instantaneous axial vorticity ω_z^+ . The blue (representing negatively valued ω_z^+) and red (representing positively valued ω_z^+) flow structures correspond to the clockwise and counterclockwise spiral motions, respectively. Results are based on test cases of the longest pipes at corresponding rotation numbers listed in table 4.1.

within the axial range of $z/R \in [0, 15]$. In figure 4.20(a), the hairpin structures are clearly observed in the non-rotating pipe flow, which are qualitatively similar to those reported in Wu et al. (2015). In the literature, the discussion of hairpin structures is often made based on turbulent boundary-layer flows developing over flat plates, and the formation of a standard complete hairpin structure model of a ‘head-neck-legs’ pattern is often attributed to near-wall bursting events, spreading to the outer region

through the $Q2$ and $Q4$ events (Robinson, 1991; Adrian, 2007). The presence of hairpin structures is also evident at $Ro_b = 2$ shown in figure 4.20(b). It is seen that in response to the system rotation imposed (at $Ro_b = 2$), a number of counterclockwise vortical structures (aligned in the z -direction, corresponding to positively valued ω_z^+) begin to become populated at the pipe centre, whereas some clockwise vortical structures (corresponding to negatively valued ω_z^+) are shifted away from the pipe centre. Because of the low axial system rotating speed at $Ro_b = 2$, the secondary flows are not strong enough to result in any significant migration of the flow structures. However, as the rotation number continues to increase to $Ro_b = 6$ and 14, such spatial evolution of the flow structures becomes apparent. The effects of rotation number on the instantaneous flow structures observed here are consistent with the profile trend of the mean axial vorticity $\langle \omega_z \rangle^+$ demonstrated previously in figure 4.5(c), which clearly shows that as Ro_b increases, the value of $\langle \omega_z \rangle^+$ becomes increasingly negative (corresponding to local clockwise rotation) in the near-wall region (for $0 \leq y < 0.4$) and positive (corresponding to local counterclockwise rotation) in the pipe centre (for $0.4 < y \leq 1.0$).

From figure 4.20(c), it is seen that as the rotation number is increased to $Ro_b = 6$, stabler and stronger large-scale secondary motion occurs at the pipe centre, as a result of the occurrence of Taylor columns. Meanwhile, hairpin structures are considerably destructed by the axial system rotation. As the rotation number further increases to $Ro_b = 14$ as shown in figure 4.20(d), it is seen that the vortical flow structures corresponding to positively and negatively valued ω_z^+ become visibly separated as most of the structures of positively valued ω_z^+ cluster together at the pipe centre to form very long secondary vortical structures of Taylor columns which spin counterclockwise (see also figure 4.3).

Chapter 5

DNS of axial rotation effects on Reynolds stress and TKE transport in turbulent pipe flow

In this chapter, turbulent transport associated with the Reynolds stresses and TKE is investigated using the DNS database presented in Chapter 4. The analysis focuses on the effects of the Coriolis force on these transport equations in both physical and spectral spaces.

5.1 Test Case and Numerical Algorithm

Figure 5.1(a) shows configuration of a circular pipe subjected to axial system rotation in the cylindrical coordinate system. The radial, azimuthal and axial coordinates are denoted by r , β and z , respectively. Accordingly, the corresponding velocity components are represented by u_r , u_β and u_z . The turbulent pipe flows are fully developed with a periodic boundary condition applied to the axial direction and a no-slip condition imposed on the pipe wall. To impose the axial system rotation, a

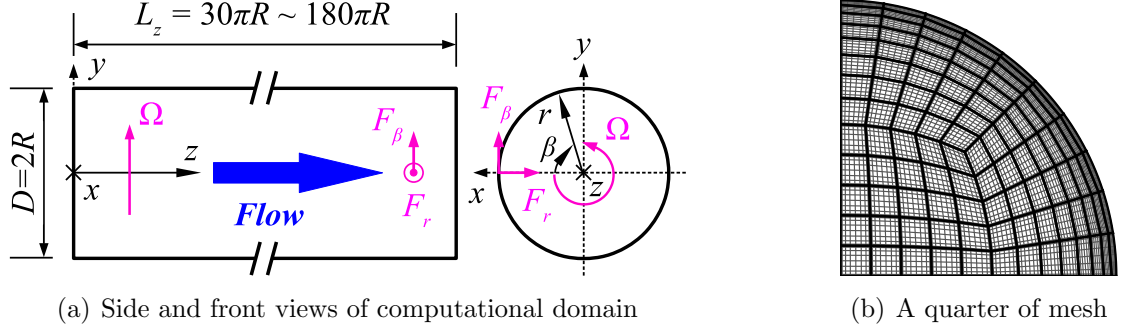


Figure 5.1: Schematic of a circular pipe subjected to axial system rotation in a cylindrical coordinate system. The radial and azimuthal components of the Coriolis force are denoted by F_r and F_β , respectively. The pipe cross-section is discretised using 420 quadrilateral structural elements based on the SEM, with each element refined using an eighth-order GLLL polynomial.

constant clockwise angular velocity Ω is prescribed about the z -axis. Consequently, two components of Coriolis force arise in the radial r - and β -directions, i.e., $F_r = 2\Omega u_\beta$ and $F_\beta = -2\Omega u_r$. Eight rotation numbers (defined as $Ro_b = 2\Omega R/U_b$) were widely selected, ranging from 0 to 20 at a fixed Reynolds number of $Re_\tau = u_\tau R/\nu = 180$. Here, R , U_b , u_τ and ν represent the radius of the pipe, bulk mean velocity, wall friction velocity and kinematic viscosity of the fluid, respectively. Depending upon the value of Ro_b , five different pipe lengths were employed, i.e. $L_z = 30\pi R$ for $Ro_b = 0, 1$ and 2 ; $L_z = 60\pi R$ for $Ro_b = 4$ and 6 ; $L_z = 90\pi R$ for $Ro_b = 10$; $L_z = 120\pi R$ for $Ro_b = 14$; and $L_z = 180\pi R$ for $Ro_b = 20$. The governing equations for an incompressible flow in a rotating reference frame are

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\beta}{\partial \beta} + \frac{\partial u_z}{\partial z} = 0 \quad , \quad (5.1)$$

$$\begin{aligned} \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\beta}{r} \frac{\partial u_r}{\partial \beta} - \frac{u_\beta^2}{r} + u_z \frac{\partial u_r}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \\ + \nu \left(\frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \beta^2} - \frac{2}{r^2} \frac{\partial u_\beta}{\partial \beta} + \frac{\partial^2 u_r}{\partial z^2} \right) + 2\Omega u_\beta \quad , \end{aligned} \quad (5.2)$$

$$\begin{aligned}
\frac{\partial u_\beta}{\partial t} + u_r \frac{\partial u_\beta}{\partial r} + \frac{u_\beta}{r} \frac{\partial u_\beta}{\partial \beta} + \frac{u_r u_\beta}{r} + u_z \frac{\partial u_\beta}{\partial z} = -\frac{1}{\rho r} \frac{\partial p}{\partial \beta} \\
+ \nu \left(\frac{1}{r} \frac{\partial u_\beta}{\partial r} - \frac{u_\beta}{r^2} + \frac{\partial^2 u_\beta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_\beta}{\partial \beta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \beta} + \frac{\partial^2 u_\beta}{\partial z^2} \right) - 2\Omega u_r \quad ,
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\beta}{r} \frac{\partial u_z}{\partial \beta} + u_z \frac{\partial u_z}{\partial z} = -\frac{\Pi}{\rho} - \frac{1}{\rho} \frac{\partial p}{\partial z} \\
+ \nu \left(\frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \beta^2} + \frac{\partial^2 u_z}{\partial z^2} \right) \quad ,
\end{aligned} \tag{5.4}$$

where equations (5.1)-(5.4) denote the continuity equation and the momentum equations in the axial, radial and azimuthal directions, respectively. In equations (5.2)-(5.4), ρ and p represent the fluid density and pressure, respectively. In equation (5.4), Π denotes the constant mean axial pressure gradient driving the flow.

Figure 5.1(b) only shows a quarter of the cross-sectional view of the mesh of each rotating pipe flow case which is divided by 420 finite elements, and each element is discretised spatially by an 8th-order GLL polynomial. All physical quantities are expanded into the spectral space by a Fourier transform with 3600–7200 modes in the z -direction based on different rotation numbers and pipe lengths. The number of grid points for conducting DNS varied from $N_{tot} = 97.4$ million for the shortest pipe case of $L_z = 30\pi R$ to $N_{tot} = 194.7$ million for the longest pipe case of $L_z = 180\pi R$. The maximum ratio of the grid size to the Kolmogorov length scale is limited to $\max(\Delta/\eta) \leq 2.74$ in all test cases. All simulations were performed with a spectral-element code called ‘Semtex’ by Blackburn and Sherwin (2004). More details about each rotating case and the numerical method can be found in Zhang and Wang (2024).

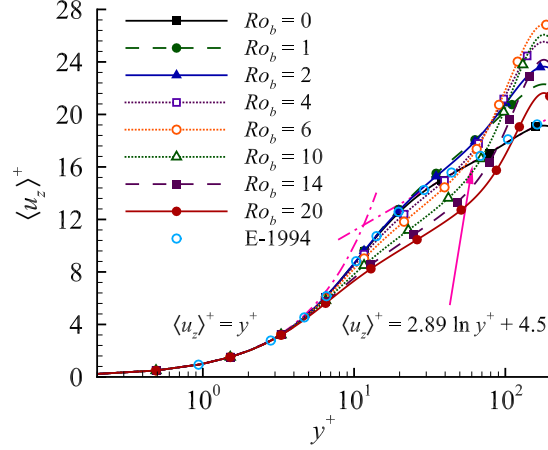


Figure 5.2: Profiles of mean axial velocity $\langle u_z \rangle^+$ as a function of the dimensionless wall coordinate y^+ for eight different rotation numbers $Ro_b = 0$ –20. The dot-dashed lines indicate the classical law of the wall. The DNS result of Eggels et al. (1994, labelled as ‘E-1994’) is displayed for the purpose of comparison.

5.2 Results and Discussion

5.2.1 Basic flow statistics

Figure 5.2 compares the profiles of mean axial velocity $\langle u_z \rangle^+$ with respect to the dimensionless wall coordinate y^+ at eight different rotation numbers $Ro_b = 0$ –20. Apparently, all profiles show the expected linear velocity distribution within the viscous sublayer ($y^+ < 5$). In contrast, for $y^+ > 30$ in the overlap region, the variation of $\langle u_z \rangle^+$ becomes non-monotonic. In the non-rotating pipe flow, the profile of $\langle u_z \rangle^+$ follows the logarithmic law of the wall and shows very close agreement with the DNS data of Eggels et al. (1994). Based on the present DNS measurements, the von Kármán constant is determined as $\kappa = 0.346$, with an additive constant of 4.5. In figure 5.2, once the axial rotation is imposed, the logarithmic velocity distribution of $\langle u_z \rangle^+$ is disrupted by the secondary flow motions induced along the pipe centreline. At $Ro_b = 1$, the magnitude of $\langle u_z \rangle^+$ reaches its maximum in the overlap region, after which it decreases with further increases in rotation number. At the pipe centre, $\langle u_z \rangle^+$ reaches its peak at $Ro_b = 6$. As the rotation number continues to rise to $Ro_b = 20$,

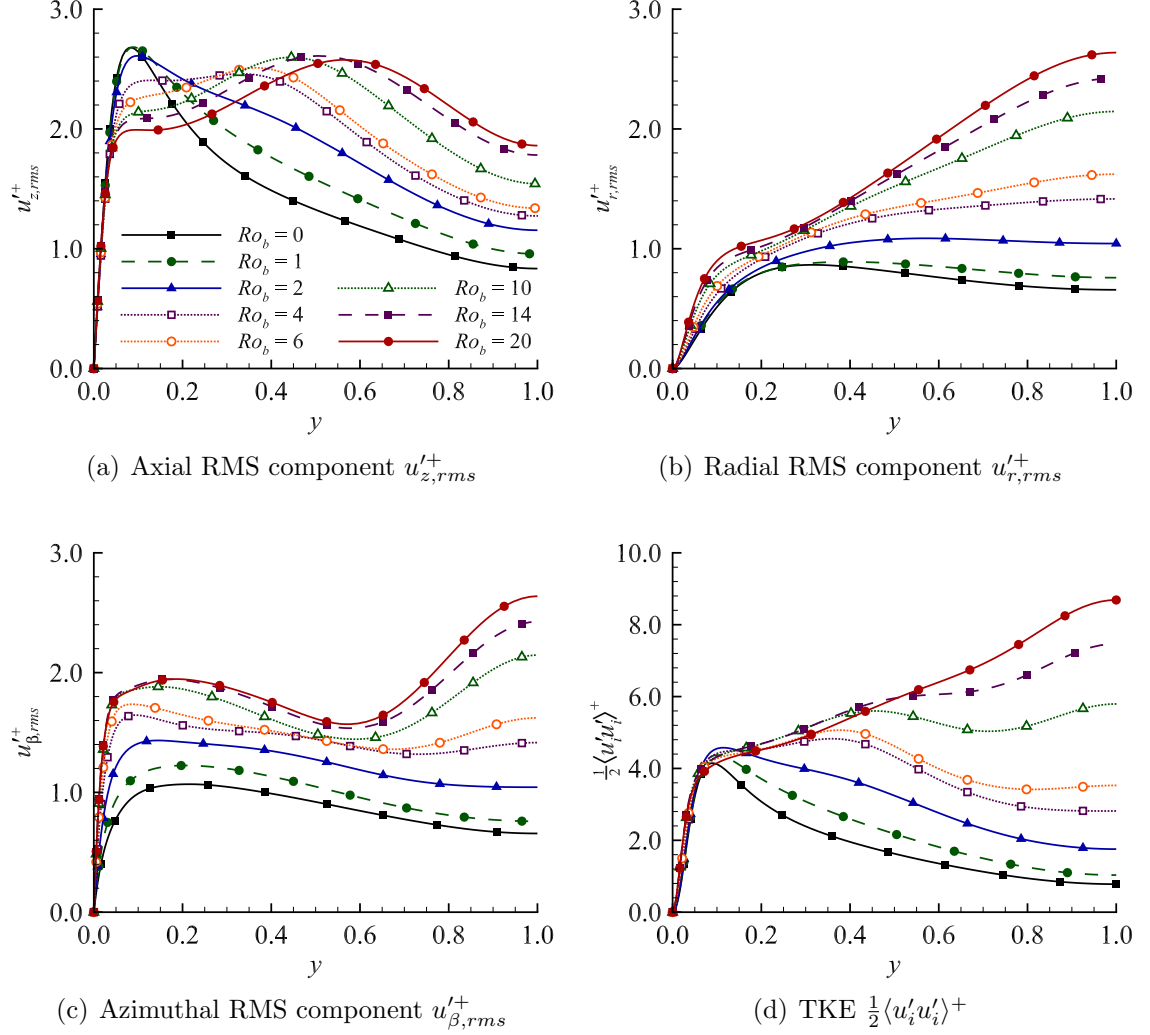


Figure 5.3: Profiles of RMS velocity fluctuations $u'_{z,rms}$, $u'_{r,rms}$, $u'_{\beta,rms}$ and TKE $\frac{1}{2} \langle u'_i u'_i \rangle^+$ for eight tested rotation number varying from $Ro_b = 0$ to 20. All RMS velocity components have been non-dimensionalised by u_τ , and the TKE by u_τ^2 .

the value of $\langle u_z \rangle^+$ reduces but remains larger than that observed in the non-rotating pipe flow.

Figure 5.3 compares the profiles of three RMS velocity fluctuations ($u'_{z,rms}$, $u'_{r,rms}$ and $u'_{\beta,rms}$) and TKE ($\frac{1}{2} \langle u'_i u'_i \rangle^+$) as a function of the dimensionless wall-normal coordinate y for eight rotation numbers in the range of $Ro_b = 0$ –20. In the definition of TKE, the subscript $(\cdot)_i$ denotes the radial, azimuthal and axial directions (r , β and z). Therefore, $\frac{1}{2} \langle u'_i u'_i \rangle = (\langle u'_r u'_r \rangle + \langle u'_\beta u'_\beta \rangle + \langle u'_z u'_z \rangle)/2$. From figures 5.3(a-c), it is evident

that the secondary flow motions induced by the axial rotation considerably enhances the magnitudes of all three RMS velocity fluctuations in the central region of the pipe (at $y = 1$) as the rotation number Ro_b increases. In the near-wall region, however, this trend is reversed for the axial component $u'_{z,rms}{}^+$, whereas the magnitudes of the radial component $u'_{r,rms}{}^+$ and azimuthal component $u'_{\beta,rms}{}^+$ continue to increase. As a result, axial rotation causes the peak of $u'_{z,rms}{}^+$ to shift towards the pipe centre. Meanwhile, the profiles of $u'_{r,rms}{}^+$ and $u'_{\beta,rms}{}^+$ evolve from a double-peak pattern into single-peak and triple-peak patterns, respectively. As shown in figure 5.3(d), the peak of the TKE profile in the non-rotating pipe flow occurs at $y = 0.093$ (or $y^+ = 16.7$) with $\frac{1}{2}\langle u'_i u'_i \rangle^+ = 4.15$. As Ro_b increases, the enhancement of all three RMS velocity fluctuations within the central region of the pipe leads to a huge increase in the TKE magnitude in that region. By contrast, the variation of $\frac{1}{2}\langle u'_i u'_i \rangle^+$ remains subtle in the near-wall region, indicating that axial rotation primarily influences the TKE level through its impact on the pipe centre. Therefore, the profile of $\frac{1}{2}\langle u'_i u'_i \rangle^+$ tends to become a single-peak pattern with an increasing Ro_b . These impacts arise directly from the large-scale secondary vortical structures formed by the Taylor columns.

Figure 5.4 compares the profiles of three Reynolds shear-stress correlation coefficients ρ_{rz} , $\rho_{r\beta}$ and $\rho_{\beta z}$ for different rotation numbers. The coefficients are defined as

$$\rho_{ij} = \frac{\langle u'_i u'_j \rangle^+}{u'_{i,rms}{}^+ u'_{j,rms}{}^+} \quad . \quad (5.5)$$

Here, the subscript $(\cdot)_j$ also denotes r -, β - and z -directions, and no summation convention over the subscripts “ i ” and “ j ” is implied. In figure 5.4(a), the profile of the axial Reynolds shear-stress correlation coefficient ρ_{rz} exhibits a non-monotonic variation in the non-rotating pipe flow. Two distinct peaks are observed at $y = 0.058$ (or $y^+ = 10.4$) and $y = 0.402$ (or $y^+ = 72.4$), respectively, with a local valley at $y = 0.177$ (or $y^+ = 31.9$). Clearly, although the ejection-sweep events mechanism gradually weakens as the dimensionless coordinate approaches the pipe centre, it also has a slight intensification between $y = 0.177$ and 0.402 at $Ro_b = 0$. With an increas-

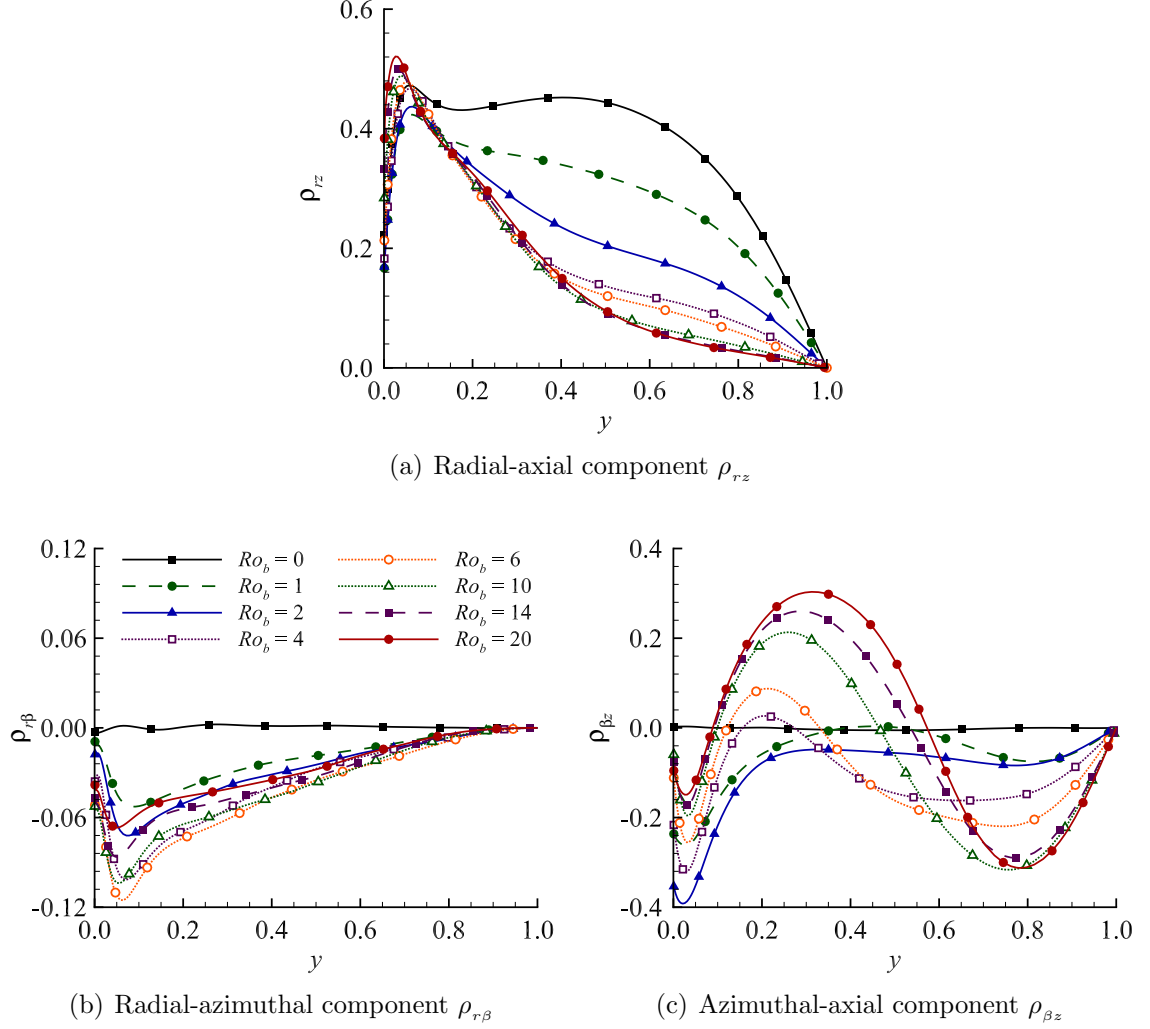


Figure 5.4: Profiles of Reynolds shear-stress correlation coefficients ρ_{rz} , $\rho_{r\beta}$ and $\rho_{\beta z}$ for eight tested rotation number varying from $Ro_b = 0$ to 20.

ing rotation number Ro_b , the profile of ρ_{rz} reduces significantly in the magnitude after reaching its peak in the near-wall region of the pipe. This observation indicates that axial rotation tends to weaken the ejection-sweep events mechanism within the pipe flow. By contrast with the profile of ρ_{rz} , figures 5.4(b,c) display that the magnitudes of the radial-azimuthal component $\rho_{r\beta}$ and the azimuthal-axial component $\rho_{\beta z}$ are trivial in the non-rotating case. In response to the axial rotation imposed, the profile of $\rho_{r\beta}$ develops a clear peak in the near-wall region of the pipe, reaching its minimum at $y = 0.065$ (or $y^+ = 11.6$) and $Ro_b = 6$. The variation of $\rho_{\beta z}$, however, is more

intricate. A near-wall valley appears at $y = 0.021$ (or $y^+ = 3.9$) and $Ro_b = 2$. As Ro_b increases beyond this value, the profile cross zero three times along the y -direction. At $Ro_b = 20$, the profile of $\rho_{\beta z}$ shows the smallest near-wall valley among all rotating cases, while attaining its maximum at $y = 0.312$ (or $y^+ = 56.2$).

The foregoing analyses of Reynolds stresses and TKE have been confined to the physical space. It is well established, however, that a Reynolds stress $\langle u'_i u'_j \rangle$ can also be obtained as the integral of the velocity-spectrum tensor $\tilde{E}_{ij}$ (including both the autospectrum and co-spectrum of the velocity fluctuations) along the homogeneous directions over which the Fourier transform is applied. To gain deeper insight into the effect of axial rotation on the six Reynolds stresses and the TKE, it is therefore essential to investigate their behaviours in spectral space. To this end, figures 5.5 and 5.6 compare the profiles of the six premultiplied velocity-spectrum tensor components of the three velocity fluctuations and the premultiplied TKE spectrum for eight rotating cases. All spectra are evaluated at the wall-normal position $y = 0.093$ (or $y^+ = 16.7$), corresponding to the location of the TKE peak in the non-rotating pipe flow. The 1-D autospectrum or co-spectrum of velocity fluctuations about the axial direction is defined as

$$\tilde{E}_{ij}(k_z) = \text{Re} \left\{ \overline{\widehat{u'_i}^* \widehat{u'_j}} \right\} \quad , \quad (5.6)$$

where an overbar $\overline{(\cdot)}$ represents temporal averaging, the operator $\text{Re}\{\cdot\}$ denotes the real part, the superscript $(\cdot)^*$ indicates the complex conjugation, and the subscripts do not imply summation convention. A hat symbol $\widehat{(\cdot)}$ denotes the Fourier transform in the axial direction of an arbitrary variable $\phi(r, \beta, k_z, t)$, i.e.,

$$\hat{\phi}(r, \beta, k_z, t) = \frac{1}{L_z} \int_0^{L_z} \phi(r, \beta, z, t) e^{-ik_z z} dz \quad , \quad (5.7)$$

where $\underline{i} = \sqrt{-1}$ is the imaginary unit, and $k_z = n_z k_{z0}$ is the axial wavenumber. Here, $n_z \in [-N_z/2, N_z/2 - 1]$ is an integer index, and $k_{z0} = 2\pi/L_z$ denotes the

smallest positive wavenumber associated with the pipe length L_z . Therefore, the axial wavelength is given by $\lambda_z = 2\pi/k_z$, and its non-dimensionalised form is expressed as $\lambda_z^+ = \lambda_z u_\tau / \nu$. As mentioned by Zhang and Wang (2024), when the rotation number increases beyond a certain threshold, multiple peaks emerge in the premultiplied velocity spectra. These peaks are associated with two distinct types of coherent flow structures, streaks and Taylor columns, in the axially rotating pipe flow. From small to large wavelengths, the first dome peak arises from streaks, which are characteristic of the wall-bounded turbulence, while the subsequent peaks are induced by the axial rotation and correspond to the formation of the Taylor columns. In order to capture these peak values and minimise energy loss in the premultiplied velocity spectra, it is crucial to elongate the pipe length as the rotation number increases.

As is evident in figures 5.5(a-d) that, in the non-rotating pipe flow, the profiles of both the premultiplied autospectra of the velocity fluctuations ($k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$) and the premultiplied TKE spectrum ($k_z^+ \tilde{E}_{TKE}^+$, where $\tilde{E}_{TKE} = (\tilde{E}_{rr} + \tilde{E}_{\beta\beta} + \tilde{E}_{zz})/2$) show clear dome-shaped peaks at $y = 0.093$ (or $y^+ = 16.7$) in the near-wall region of the pipe. Specifically, the premultiplied axial autospectrum attains a peak at $\lambda_z^+ = 997.9$ with $k_z^+ \tilde{E}_{zz}^+ = 0.69$; the radial and azimuthal components peak at $\lambda_z^+ = 212.1$ with $k_z^+ \tilde{E}_{rr}^+ = 0.11$, and at $\lambda_z^+ = 238.9$ with $k_z^+ \tilde{E}_{\beta\beta}^+ = 0.18$, respectively; and the mode of the premultiplied TKE spectrum is located at $\lambda_z^+ = 547.2$, corresponding to the characteristic length scale of the most energetic eddies, with a magnitude of $k_z^+ \tilde{E}_{TKE}^+ = 0.44$. By comparing figures 5.5(a,c) with figure 5.5(b), it becomes apparent that, with an increasing rotation number, the magnitudes of $k_z^+ \tilde{E}_{zz}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ consistently surpass that of $k_z^+ \tilde{E}_{rr}^+$. Therefore, in figure 5.5(d), the profile of $k_z^+ \tilde{E}_{TKE}^+$ mainly follows the behaviour of $k_z^+ \tilde{E}_{zz}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ at $y^+ = 16.7$. As Ro_b rises, an additional sharp peak emerges at $\lambda_z^+ = 3392.9$ (corresponding to the axial wavelength scales of Taylor column structures) in the profiles of $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{\beta\beta}^+$, and $k_z^+ \tilde{E}_{TKE}^+$ at $Ro_b = 2$, and at $\lambda_z^+ = 3769.9$ in the profile of $k_z^+ \tilde{E}_{rr}^+$ at $Ro_b = 4$, respectively, owing to the formation of Taylor columns induced by the axial rotation.

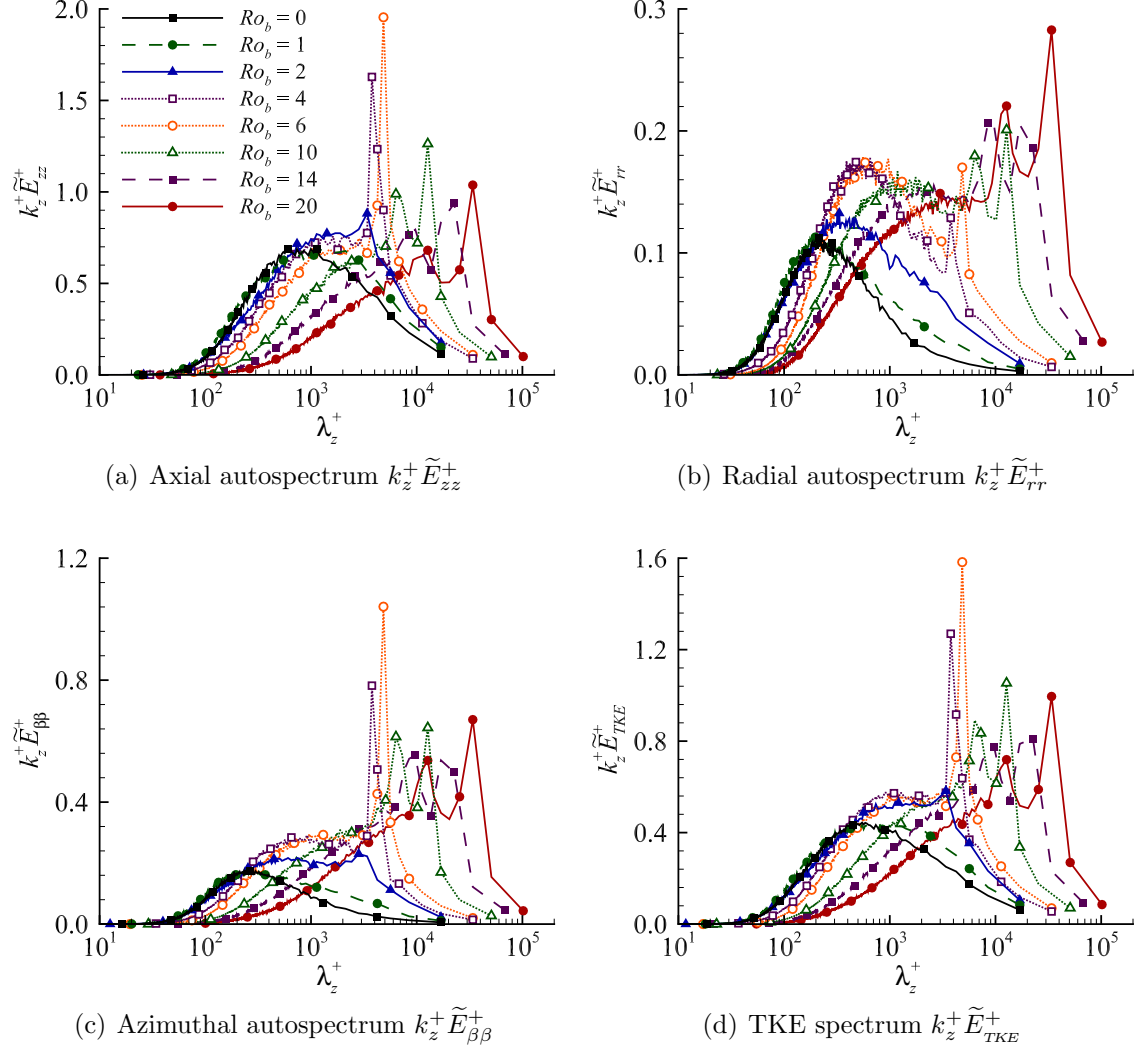


Figure 5.5: Profiles of premultiplied autospectra $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{rr}^+$, $k_z^+ \tilde{E}_{\beta\beta}^+$, and premultiplied TKE spectrum $k_z^+ \tilde{E}_{TKE}^+$ for eight rotation numbers ranging from $Ro_b = 0$ to 20. These spectra are calculated along the axial direction at a fixed wall-normal position $y = 0.093$ (or $y^+ = 16.7$). All spectra are non-dimensionalised using u_τ^2/R .

Notably, at $Ro_b = 6$, the second peak reaches its maximum at $\lambda_z^+ = 4847$ in the profiles of $k_z^+ \tilde{E}_{zz}^+$, $k_z^+ \tilde{E}_{\beta\beta}^+$, and $k_z^+ \tilde{E}_{TKE}^+$. As the rotation number increases to $Ro_b = 10$, it is interesting to observe that another sharp peak appears at $\lambda_z^+ = 12723.5$ in all four spectral profiles. Clearly, figures 5.5(a-d) demonstrate that the modes of the premultiplied autospectra and the premultiplied TKE spectrum experience a considerable shift towards larger wavelengths with an increasing Ro_b . At the highest tested

rotation number of $Ro_b = 20$, both the second and third peaks of the premultiplied axial spectra are located at $\lambda_z^+ = 12723.5$ and $\lambda_z^+ = 33929.3$, respectively, although the modes of their first dome-shaped peak are different. Meanwhile, in figure 5.5(d), the share of TKE associated with the Taylor columns (indicated by the sharp peaks of $k_z^+ \tilde{E}_{TKE}^+$) is much larger than that held by the streaks (indicated by the dome-shaped peak of $k_z^+ \tilde{E}_{TKE}^+$) at this wall-normal position of $y^+ = 16.7$. Furthermore, the adjustment of pipe length for different rotation numbers ensures that the loss of $k_z^+ \tilde{E}_{TKE}^+$ (defined as a percentage of its maximum peak value) at the cut-off wavelength remains below 20% at $y^+ = 16.7$ in the near-wall region of the pipe. The maximum loss of $k_z^+ \tilde{E}_{TKE}^+$ is 18.9% at $Ro_b = 1$.

Figure 5.6 compares the profiles of the premultiplied co-spectra $k_z^+ \tilde{E}_{rz}^+$, $k_z^+ \tilde{E}_{r\beta}^+$ and $k_z^+ \tilde{E}_{\beta z}^+$ at eight different rotation numbers at the same wall-normal position. From figure 5.6(a), in the non-rotating pipe flow, the premultiplied radial-axial co-spectrum $k_z^+ \tilde{E}_{rz}^+$ exhibits a dome-shaped peak within the range $\lambda_z^+ \in [580, 1130]$. As the rotation number increases, the profile of $k_z^+ \tilde{E}_{rz}^+$ shows only subtle variation in its peak value, and the mode has little shift until $Ro_b = 20$. Moreover, a secondary sharp peak is observed at $Ro_b = 4$, whose magnitude remains smaller than that of the dome-shaped peak for all rotation numbers. For the premultiplied radial-azimuthal and azimuthal-axial co-spectra $k_z^+ \tilde{E}_{r\beta}^+$ and $k_z^+ \tilde{E}_{\beta z}^+$ in figures 5.6(b,c), their magnitudes are nearly zero across the entire wavelength range at $Ro_b = 0$, consistent with the zero value of $\rho_{r\beta}$ and $\rho_{\beta z}$ shown in figures 5.4(b,c). At $Ro_b = 1$, the profiles of $k_z^+ \tilde{E}_{r\beta}^+$ and $k_z^+ \tilde{E}_{\beta z}^+$ show dome-shaped valleys located at $\lambda_z^+ = 547.2$ and $\lambda_z^+ = 1131$, respectively. With an increasing Ro_b , the magnitudes of these dome valleys in $k_z^+ \tilde{E}_{r\beta}^+$ and $k_z^+ \tilde{E}_{\beta z}^+$ vary in a non-monotonic manner, reaching minima at $\lambda_z^+ = 302.9$ for $Ro_b = 10$ and $\lambda_z^+ = 848.2$ for $Ro_b = 2$, respectively. The modes of $k_z^+ \tilde{E}_{r\beta}^+$ and $k_z^+ \tilde{E}_{\beta z}^+$ likewise display non-monotonic variation, confined to the ranges $\lambda_z^+ \in [300, 550]$ and $\lambda_z^+ \in [490, 1140]$, respectively. Meanwhile, A clear sharp peak is also observed in both $k_z^+ \tilde{E}_{r\beta}^+$ and $k_z^+ \tilde{E}_{\beta z}^+$, appearing at $\lambda_z^+ = 4847$ for $Ro_b = 6$ and at $\lambda_z^+ = 3769.9$ for $Ro_b = 4$,

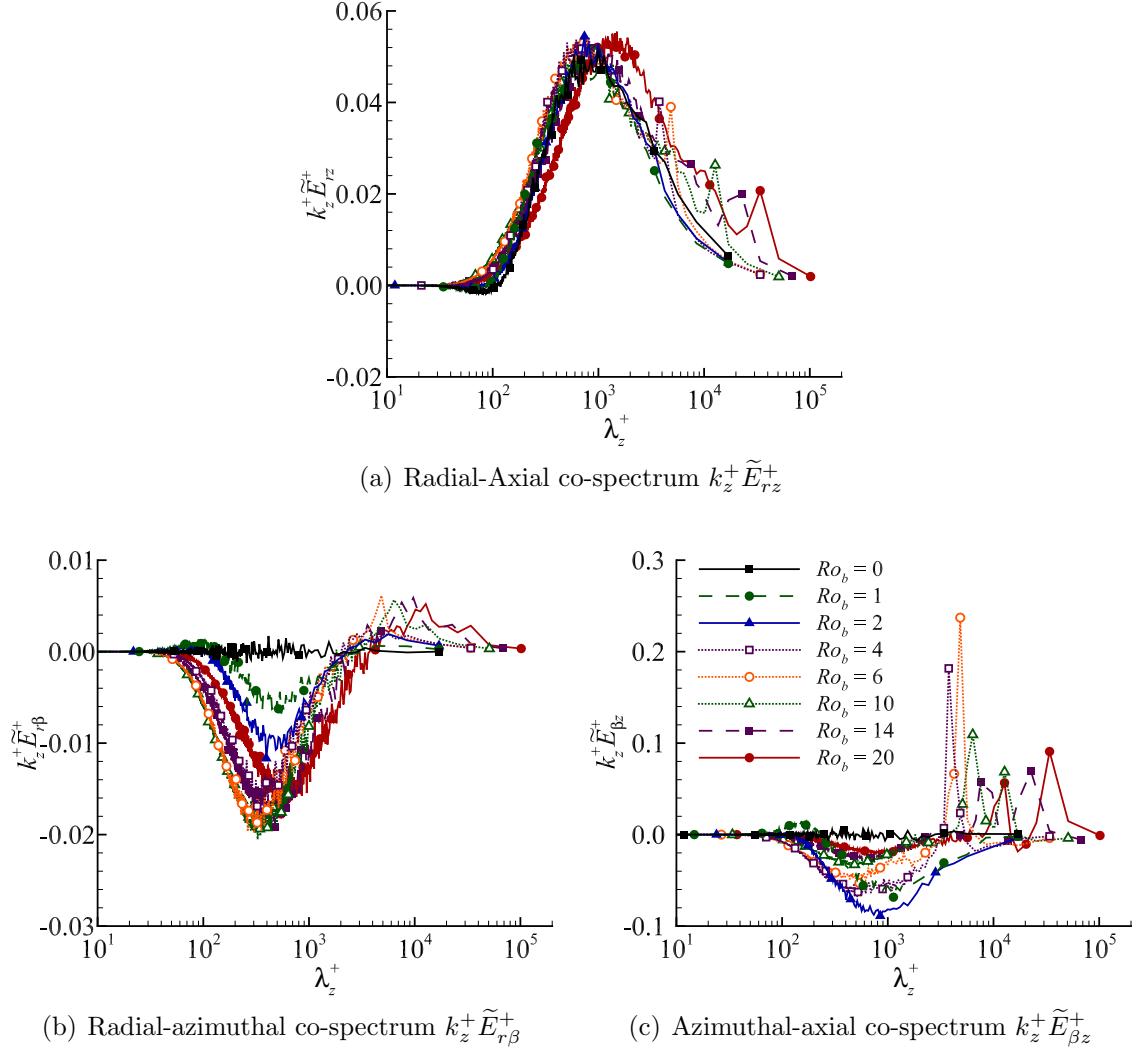


Figure 5.6: Profiles of premultiplied co-spectra $k_z^+ \tilde{E}_{rz}^+$, $k_z^+ \tilde{E}_{r\beta}^+$, $k_z^+ \tilde{E}_{\beta z}^+$ for eight rotation numbers ranging from $Ro_b = 0$ to 20. These spectra are calculated along the axial direction at a fixed wall-normal position $y = 0.093$ (or $y^+ = 16.7$). All spectra are non-dimensionalised by u_τ^2/R .

respectively. In contrast to figures 5.6(a,b), figure 5.6(c) shows the emergence of a third peak in the profile of $k_z^+ \tilde{E}_{\beta z}^+$ at $Ro_b = 10$. At higher rotation numbers, the sharp peaks induced by the Taylor columns exceed the dome peak associated with the streaks in magnitude. It is further evident from figures 5.6(a-c) that these sharp peaks gradually move towards larger wavelengths as Ro_b increases.

5.2.2 Transport of Reynolds stresses and TKE

To refine my investigation into the axial rotation impact on the six Reynolds stresses $\langle u'_i u'_j \rangle$ and the TKE $\frac{1}{2} \langle u'_i u'_i \rangle$, their respective transport equations can be further studied. For a statistically steady flow, the transport equation of Reynolds stress $\langle u'_i u'_j \rangle$ is given by

$$\frac{\partial \langle u'_i u'_j \rangle}{\partial t} = C_{ij} + P_{ij} + \Pi_{ij} + \varepsilon_{ij} + D_{ij}^t + D_{ij}^p + D_{ij}^v + P_{ij}^c = 0 \quad , \quad (5.8)$$

where C_{ij} , P_{ij} , Π_{ij} , ε_{ij} , D_{ij}^t , D_{ij}^p , D_{ij}^v and P_{ij}^c represent the convection, turbulent production, pressure-strain, viscous dissipation, turbulent diffusion, pressure diffusion, viscous diffusion and Coriolis production terms, respectively. No summation convention over the subscripts is implied in equation (5.8). To facilitate the analysis of all six Reynolds stress transport equations, the simplified forms of these equations are provided in Appendix E.1, see equations (E.5)-(E.10). Moreover, the six Coriolis production terms P_{ij}^c are summarised in table 5.1. In response to the axial system rotation, no Coriolis production term P_{zz}^c arises in the transport equation of the axial Reynolds normal stress $\langle u'_z u'_z \rangle$, owing to the absence of a Coriolis force in the z -direction. By contrast, the terms of P_{rr}^c and $P_{\beta\beta}^c$ in the transport equations of $\langle u'_r u'_r \rangle$ and $\langle u'_\beta u'_\beta \rangle$, respectively, function to transfer turbulence energy between these two stresses, a consequence of their opposite signs. Meanwhile, the Coriolis production terms in the transport equations of the three Reynolds shear stresses act only as energy source or sink terms.

Figure 5.7 compares the budget balance of the axial Reynolds normal stress $\langle u'_z u'_z \rangle^+$ for the non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows, plotted as a function of the wall-normal (y) coordinate. From the simplified transport equation E.5, it is evident that the convection term C_{zz}^+ and the pressure diffusion term D_{zz}^{p+} vanish identically, independent of the axial system rotation. For the non-rotating pipe flow, in figure 5.7(a), the turbulent production term P_{zz}^+ acts as the dominant

Table 5.1: Coriolis production term P_{ij}^c in the transport equation of Reynolds stress $\langle u'_i u'_j \rangle$.

Reynolds stress	Component $\langle u'_i u'_j \rangle$	Coriolis production term P_{ij}^c
Axial Reynolds normal stress	$\langle u'_z u'_z \rangle$	0
Radial Reynolds normal stress	$\langle u'_r u'_r \rangle$	$4\Omega \langle u'_r u'_\beta \rangle$
Azimuthal Reynolds normal stress	$\langle u'_\beta u'_\beta \rangle$	$-4\Omega \langle u'_r u'_\beta \rangle$
Radial-axial Reynolds shear stress	$\langle u'_r u'_z \rangle$	$2\Omega \langle u'_\beta u'_z \rangle$
Radial-azimuthal Reynolds shear stress	$\langle u'_r u'_\beta \rangle$	$2\Omega (\langle u'_\beta u'_\beta \rangle - \langle u'_r u'_r \rangle)$
Azimuthal-axial Reynolds shear stress	$\langle u'_\beta u'_z \rangle$	$-2\Omega \langle u'_r u'_z \rangle$

source in the budget balance of $\langle u'_z u'_z \rangle^+$, reaching its peak value at $y = 0.072$, where energy transfer from the mean flow field to the axial velocity fluctuations is most efficient. In contrast, the viscous dissipation term ε_{zz}^+ constitutes as the major sink, consuming energy from $\langle u'_z u'_z \rangle^+$. The pressure-strain term Π_{zz}^+ also removes energy, although it acts only as a comparatively minor sink. Moreover, the turbulent diffusion term D_{zz}^{t+} and the viscous diffusion term D_{zz}^{v+} act to redistribute energy from the regions $y \in [0.048, 0.167]$ and $y \in [0.029, 0.131]$, respectively, towards both sides of these regions. This redistribution results in a local counterbalance between D_{zz}^{v+} and ε_{zz}^+ near the pipe wall. From figure 5.7(b), it is seen that as the rotation number increases to $Ro_b = 20$, the budget balance of $\langle u'_z u'_z \rangle^+$ exhibits a similar pattern to that at $Ro_b = 0$. However, Π_{zz}^+ plays a greater role as a sink term while D_{zz}^{t+} becomes relatively more pronounced in the pipe centre. It is expected that, in response to the axial rotation, the peak positions of all budget terms in the near-wall region shift closer to the pipe wall.

Figure 5.8 shows a comparison of the budget balance of the radial Reynolds normal

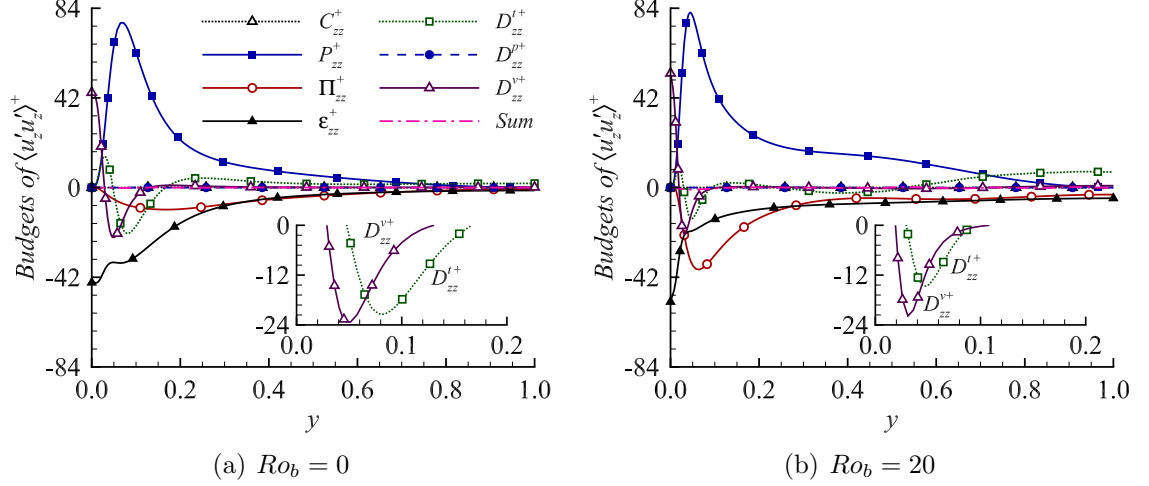


Figure 5.7: Budget balance of axial Reynolds normal stress $\langle u'_z u'_z \rangle^+$ for non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. The inset panels show the variation of the turbulent diffusion D_{zz}^{t+} and viscous diffusion D_{zz}^{v+} terms for the two rotating cases. All budget terms are non-dimensionalised by u_τ^3/R .

stress $\langle u'_r u'_r \rangle^+$ at four different rotation numbers, $Ro_b = 0, 1, 4$ and 20 . It is evident from figure 5.8(a) that, in the non-rotating pipe flow, the convection term C_{rr}^+ and the turbulent production term P_{rr}^+ are nearly zero. This observation is consistent with their simplified forms in equation (E.6), where the radial-azimuthal Reynolds shear stress $\langle u'_r u'_\beta \rangle$ is trivial, in agreement with the correlation $\rho_{r\beta}$ shown in figure 5.4(b). In the near-wall region, the transport of $\langle u'_r u'_r \rangle^+$ is mainly balanced by the pressure-strain term Π_{rr}^+ , the pressure diffusion term D_{rr}^{p+} and the viscous dissipation term ϵ_{rr}^+ . Among these terms, ϵ_{rr}^+ plays a central role as a major sink, reaching its valley at $y = 0.197$. The behaviour of D_{rr}^{p+} is more complicated, crossing zero three times along the y -direction; nevertheless, its most significant contribution is to redistribute energy towards the pipe wall within $y \in [0, 0.08]$. It is also clear that Π_{rr}^+ acts as a major source for $y > 0.069$, transferring energy from $\langle u'_z u'_z \rangle^+$, whereas within $y \in [0, 0.069]$ it switches to a sink, transferring energy to $\langle u'_\beta u'_\beta \rangle^+$ to maintain the overall balance of the turbulent field. In contrast to the leading budget terms, the contributions of the turbulent diffusion term D_{rr}^{t+} and the viscous diffusion term D_{rr}^{v+} are comparatively small. However, D_{rr}^{t+} facilitates the redistribution of energy from the region of $y \in$

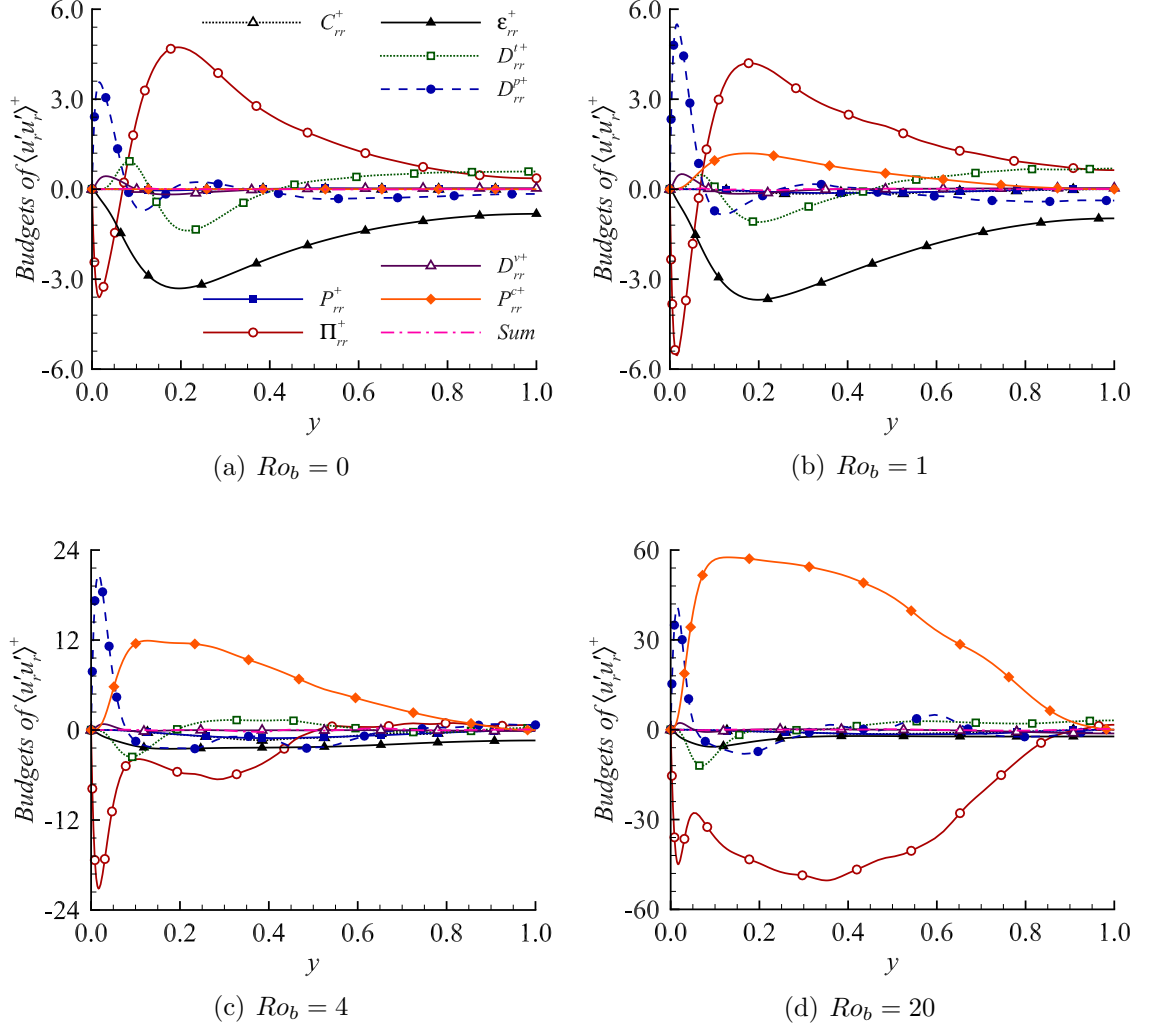


Figure 5.8: Budget balance of radial Reynolds normal stress $\langle u'_r u'_r \rangle^+$ in axially rotating pipe flows at $Ro_b = 0, 1, 4$ and 20 . All budget terms are non-dimensionalised by u_τ^3/R .

[0.131, 0.419] towards both of its outer boundaries, becoming increasingly dominant as the wall-normal coordinate y approaches the pipe centre. In figure 5.8(b), in response to the axial system rotation, the Coriolis production term P_{rr}^{c+} rises as an energy source and attains a peak at $y = 0.177$ at $Ro_b = 1$. This increase in P_{rr}^{c+} is accompanied by enhanced magnitudes of both Π_{rr}^+ and D_{rr}^{p+} near the pipe wall. As is evident from figure 5.8(c), as the rotation number increase to $Ro_b = 4$, P_{rr}^{c+} becomes the leading source term. Meanwhile, the contribution of Π_{rr}^+ as an energy source becomes nearly

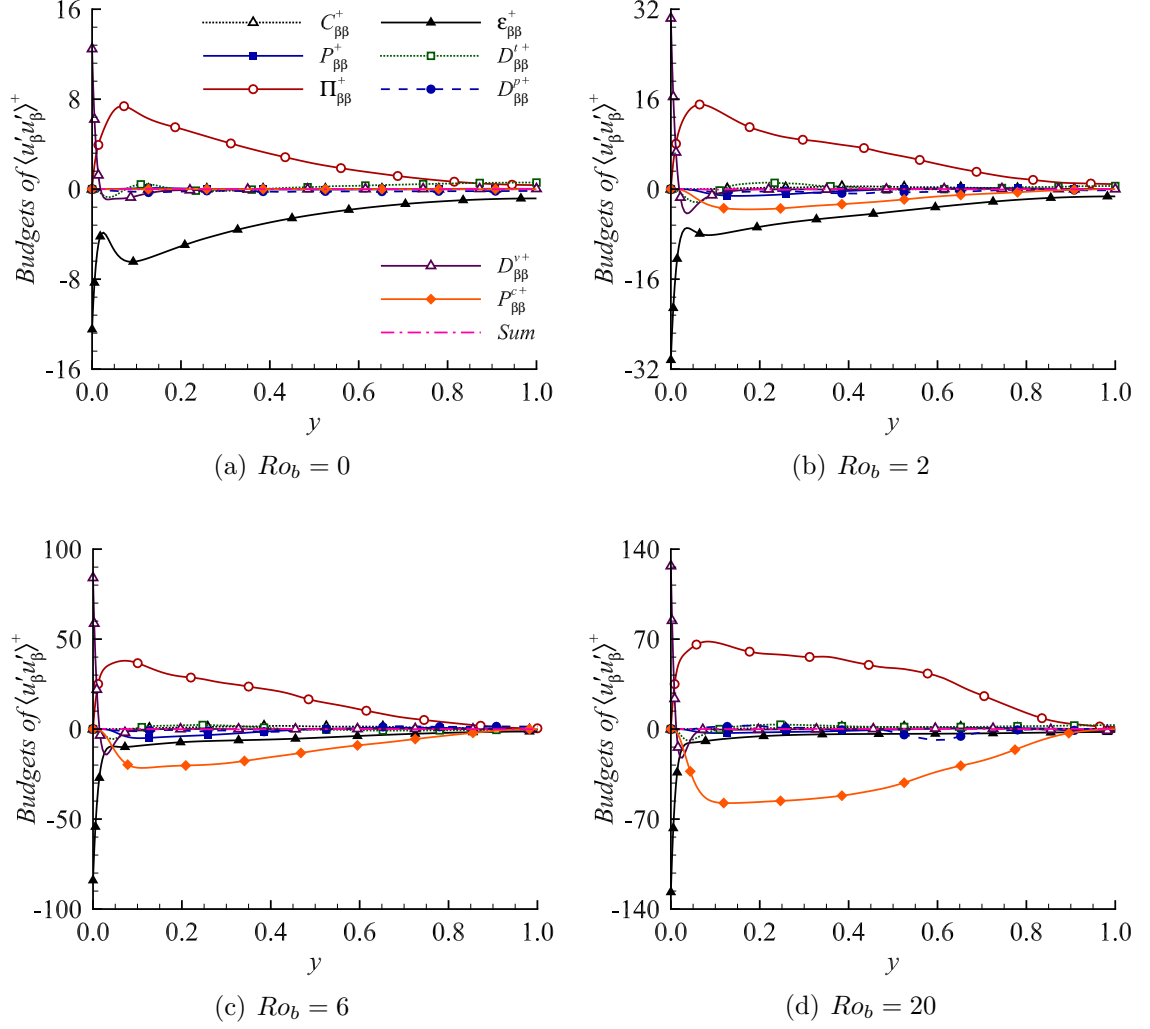


Figure 5.9: Budget balance of azimuthal Reynolds normal stress $\langle u'_\beta u'_\beta \rangle^+$ in axially rotating pipe flows at $Ro_b = 0, 2, 6$ and 20 . All budget terms are non-dimensionalised by u_τ^3/R .

negligible, and it acts as an energy sink throughout the near-wall region, thereby replacing ϵ_{rr}^+ as the dominant sink term. Finally, in figure 5.8(d), at $Ro_b = 20$, Π_{rr}^+ functions almost entirely as the dominant energy sink along the y -coordinate, counterbalancing D_{rr}^{p+} and P_{rr}^{c+} as the leading source terms. Apparently, P_{rr}^{c+} pumps energy into $\langle u'_r u'_r \rangle^+$ due to the fact that $P_{rr}^c = 4\Omega \langle u'_r u'_\beta \rangle$.

Figure 5.9 compares the budget balance of the azimuthal Reynolds normal stress $\langle u'_\beta u'_\beta \rangle^+$ for four different rotation numbers, $Ro_b = 0, 2, 6$ and 20 . In figure 5.9(a),

the transport of $\langle u'_\beta u'_\beta \rangle^+$ in the non-rotating pipe flow is dominated at the wall by the viscous diffusion term $D_{\beta\beta}^{v+}$ and viscous dissipation term $\varepsilon_{\beta\beta}^+$. Away from the wall, however, the pressure-strain term $\Pi_{\beta\beta}^+$ and $\varepsilon_{\beta\beta}^+$ are the major source and sink for $\langle u'_\beta u'_\beta \rangle^+$, with their respective peak and valley occurring at $y = 0.065$ and $y = 0.087$. As is evident in equation (E.7), both the convection term $C_{\beta\beta}^+$ and turbulent production term $P_{\beta\beta}^+$ remain nearly zero because $\langle u'_r u'_\beta \rangle$ is trivial at $Ro_b = 0$, while the pressure diffusion term $D_{\beta\beta}^{p+}$ vanishes due to the axial symmetry of the flow. In addition, the turbulent diffusion term $D_{\beta\beta}^{t+}$ mainly behaves to redistribute energy towards the pipe centre. As soon as the axial rotation is imposed, the Coriolis production term $P_{\beta\beta}^{c+}$ emerges as an additional energy sink along the y -direction. From figure 5.9(b), at $Ro_b = 2$, $P_{\beta\beta}^{c+}$ becomes one of the major sink terms, together with $\varepsilon_{\beta\beta}^+$ neutralizing energy provided by $\Pi_{\beta\beta}^+$ away from the wall. As the rotation number increases to $Ro_b = 6$, figure 5.9(c) clearly shows that the dominant sink term shifts from $\varepsilon_{\beta\beta}^+$ to $P_{\beta\beta}^{c+}$, indicating that the Coriolis production becomes the primary mechanism of removing energy from $\langle u'_\beta u'_\beta \rangle^+$. Eventually, at $Ro_b = 20$ in figure 5.9(d), the contribution of $\varepsilon_{\beta\beta}^+$ is negligible off the wall. As shown in figures 5.9(d) and 5.8(d), the Coriolis production terms P_{rr}^{c+} and $P_{\beta\beta}^{c+}$ display opposite trends, and their profiles are mirror reflections of each other, clearly indicating an energy transfer from $\langle u'_\beta u'_\beta \rangle^+$ to $\langle u'_r u'_r \rangle^+$. Furthermore, the leading budget terms in both figures are approximately an order of magnitude larger than those in the non-rotating pipe flow, demonstrating a significantly intensified transport activity in both $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$. Thus, it is understood that the TKE level is mainly enhanced by system rotation through the budget terms of $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$.

For the three Reynolds shear stresses, it is appropriate to begin by examining the effect of axial rotation on the transport of the radial-axial Reynolds shear stresses $\langle u'_r u'_z \rangle^+$, as illustrated in figure 5.10 for four different rotation numbers at $Ro_b = 0, 1, 6$ and 20 . For the non-rotating pipe flow, in figure 5.10(a), $\langle u'_r u'_z \rangle^+$ is supplied with energy by the pressure diffusion term D_{rz}^{p+} near the wall from the coordinate

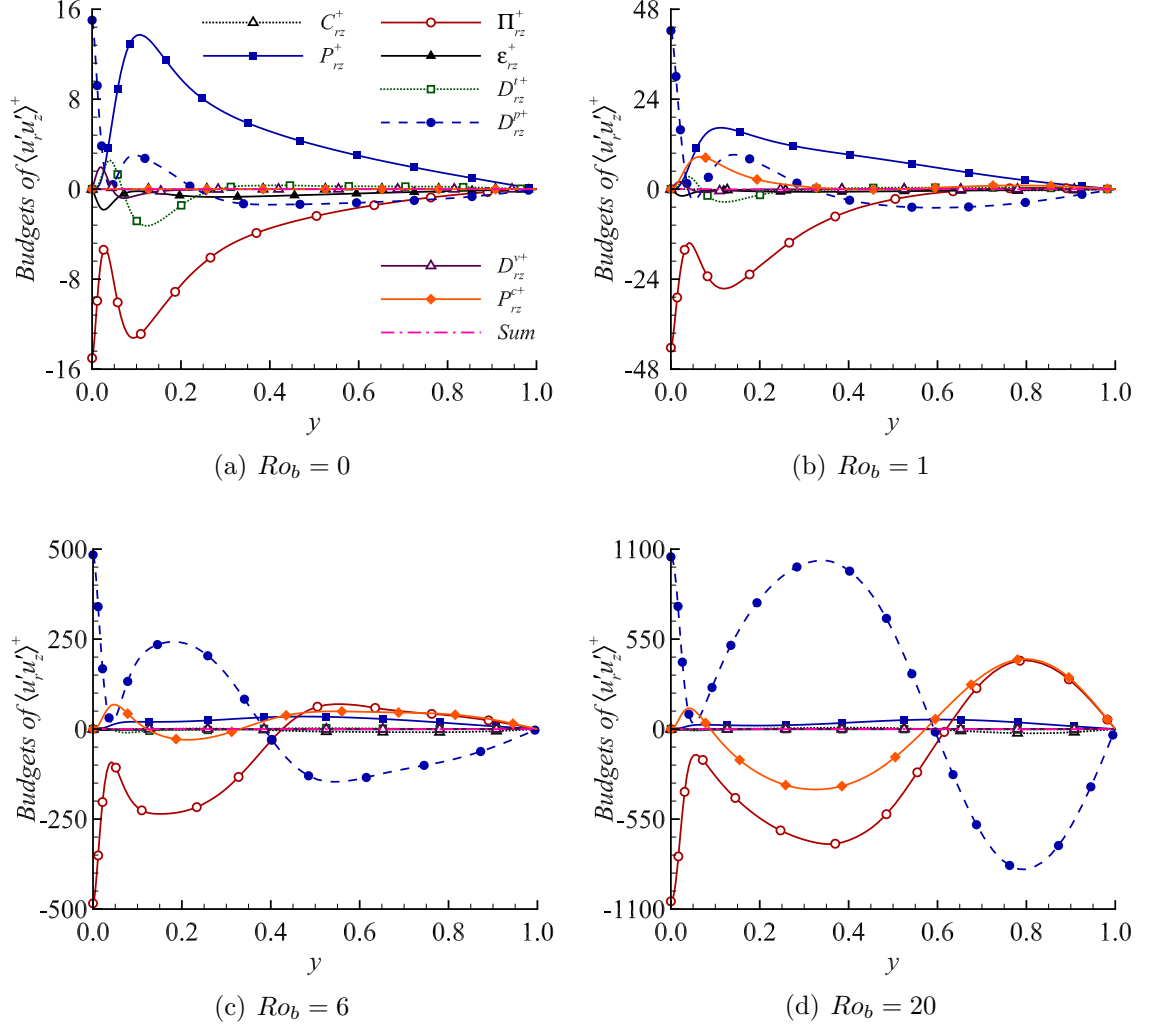


Figure 5.10: Budget balance of radial-axial Reynolds shear stress $\langle u'_r u'_z \rangle^+$ in axially rotating pipe flows at $Ro_b = 0, 1, 6$ and 20 . All budget terms are non-dimensionalised by u_τ^3/R .

interval of $y \in [0.232, 1]$. Away from the wall, the turbulent production term P_{rz}^+ is the major energy source, reaching its peak at $y = 0.11$. The pressure-strain term Π_{rz}^+ remains the major energy sink along the y -direction, reaching its valley at $y = 0.093$. The turbulent diffusion term D_{rz}^{t+} transports energy outwards within the region of $y \in [0.068, 0.284]$. The viscous dissipation term ε_{rz}^+ and viscous diffusion term D_{rz}^{v+} make only minor contributions to the budget balance due to their comparatively small magnitudes. In addition, according to equation (E.8), the con-

vection term C_{rz}^+ identically vanishes, and the turbulent production term simplifies to $P_{rz} = -\langle u'_r u'_r \rangle \partial \langle u_z \rangle / \partial r$, since $\langle u'_\beta u'_z \rangle = 0$ at $Ro_b = 0$. All budget terms diminish to zero at the pipe centre ($y = 1$) as required by the axial symmetry condition. It is clear from figure 5.10(b) that, in response to the axial rotation, the Coriolis production term P_{rz}^{c+} rises as a comparable source term in the near-wall region at $Ro_b = 1$. However, from figure 5.10(c), as the rotation number increases to $Ro_b = 6$, the profiles of Π_{rz}^+ and P_{rz}^{c+} become increasingly complex. In particular, the profile of P_{rz}^{c+} crosses zero three times along the y -direction. This behaviour indicates that the role of P_{rz}^{c+} alternates between acting as a source and a sink for $\langle u'_r u'_z \rangle^+$ with an increasing wall-normal distance. Such alternation closely follows the variation of the azimuthal-axial Reynolds shear stress $\langle u'_\beta u'_z \rangle$, as displayed in table 5.1 and observed by the correlation coefficient $\rho_{r\beta}$ in figure 5.4(c). At $Ro_b = 20$, in figure 5.10(d), the budget balance of $\langle u'_r u'_z \rangle^+$ is dominated by three major budgets, i.e., Π_{rz}^+ , D_{rz}^{p+} and P_{rz}^{c+} . A comparison between figures 5.10(d) and 5.10(a) shows that the magnitudes of Π_{rz}^+ and D_{rz}^{p+} increase drastically, by nearly two orders of magnitude. As the leading source and sink terms, their roles switch around $y = 0.6$ at $Ro_b = 20$.

Given that the radial-azimuthal and azimuthal-axial Reynolds shear stresses, $\langle u'_r u'_\beta \rangle^+$ and $\langle u'_\beta u'_z \rangle^+$, are trivial in the non-rotating pipe flow, their associated budget balances are presented only for rotating cases with $Ro_b \geq 1$. Accordingly, the budget balances of the radial-azimuthal Reynolds shear stress $\langle u'_r u'_\beta \rangle^+$ for four different rotation numbers, $Ro_b = 1, 2, 4$ and 20 , are shown in figures 5.11(a-d). As evident from figure 5.11(a), the turbulent production term $P_{r\beta}^+$ and Coriolis production term $P_{r\beta}^{c+}$ play single and opposite roles along the wall-normal direction at $Ro_b = 1$, acting as an energy source and an energy sink, respectively. These two terms reach their peak and valley at $y = 0.258$ and $y = 0.1$, respectively. In contrast, the budget profiles indicate that the pressure-strain term $\Pi_{r\beta}^+$ and pressure diffusion term $D_{r\beta}^{p+}$ exhibit dual roles. Specifically, $\Pi_{r\beta}^+$ acts as an energy source over the region of $y \in [0.037, 1]$, but becomes an energy sink within $y \in [0, 0.037]$. Meanwhile, $D_{r\beta}^{p+}$ redistributes energy from the

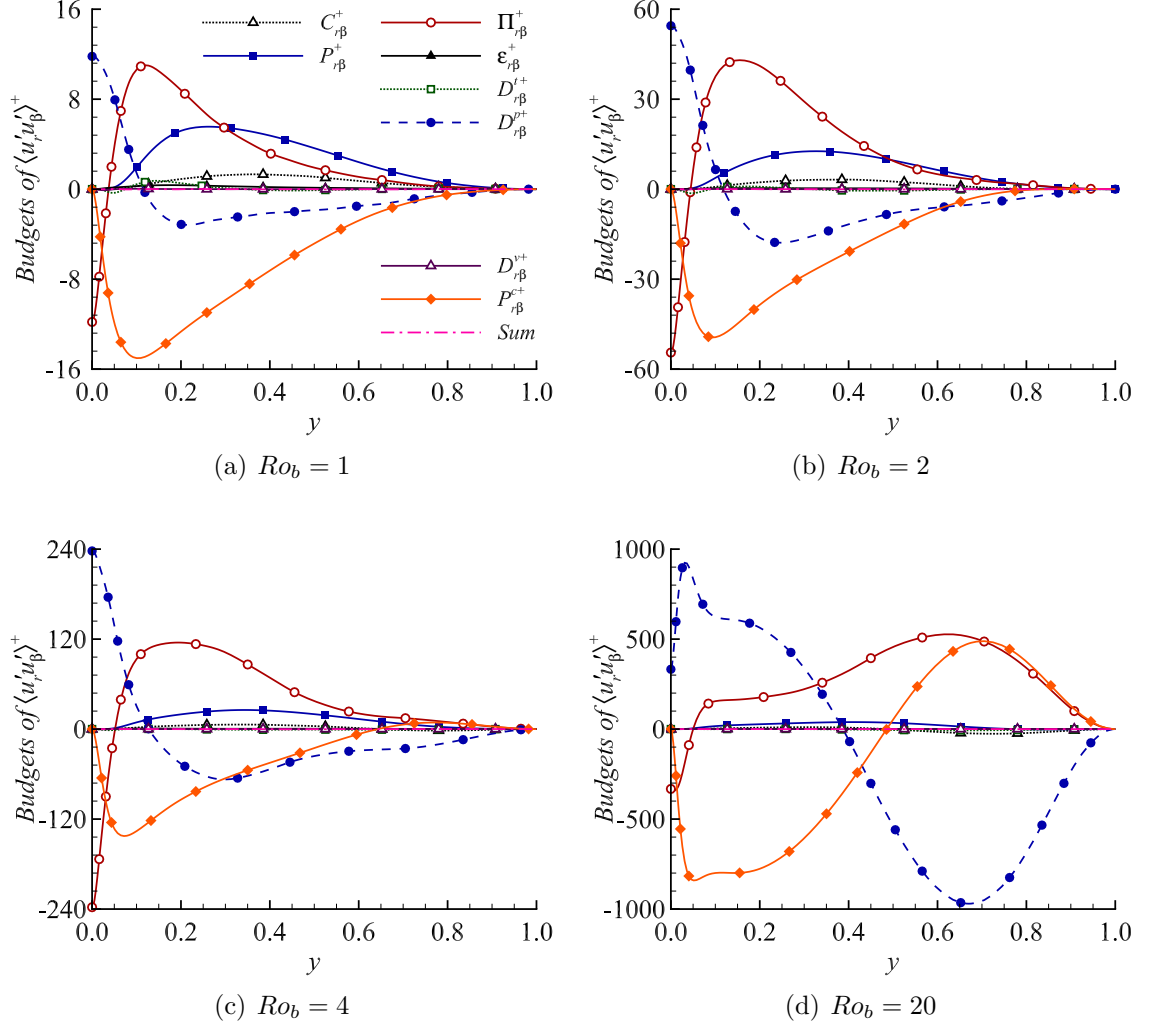


Figure 5.11: Budget balance of radial-azimuthal Reynolds shear stress $\langle u'_r u'_\beta \rangle^+$ in axially rotating pipe flows at $Ro_b = 1, 2, 4$ and 20 . All budget terms are non-dimensionalised by u_τ^3/R .

outer region of $y \in [0.116, 1]$ towards the inner region of $y \in [0, 0.116]$. Furthermore, the convection term $C_{r\beta}^+$ acts as a minor source along the y -direction. The viscous dissipation term $\varepsilon_{r\beta}^+$, turbulent diffusion term $D_{r\beta}^{t+}$ and viscous diffusion term $D_{r\beta}^{v+}$ remain very small in comparison with the dominant budget terms. In figure 5.11(b), as the rotation number increases to $Ro_b = 2$, the axial system rotation significantly amplifies the magnitudes of $\Pi_{r\beta}^+$, $D_{r\beta}^{p+}$ and $P_{r\beta}^{c+}$, whereas the increases in $P_{r\beta}^+$ and $C_{r\beta}^+$ are comparatively small. In the pipe core region, $D_{r\beta}^{p+}$, acting as a sink term, exceeds

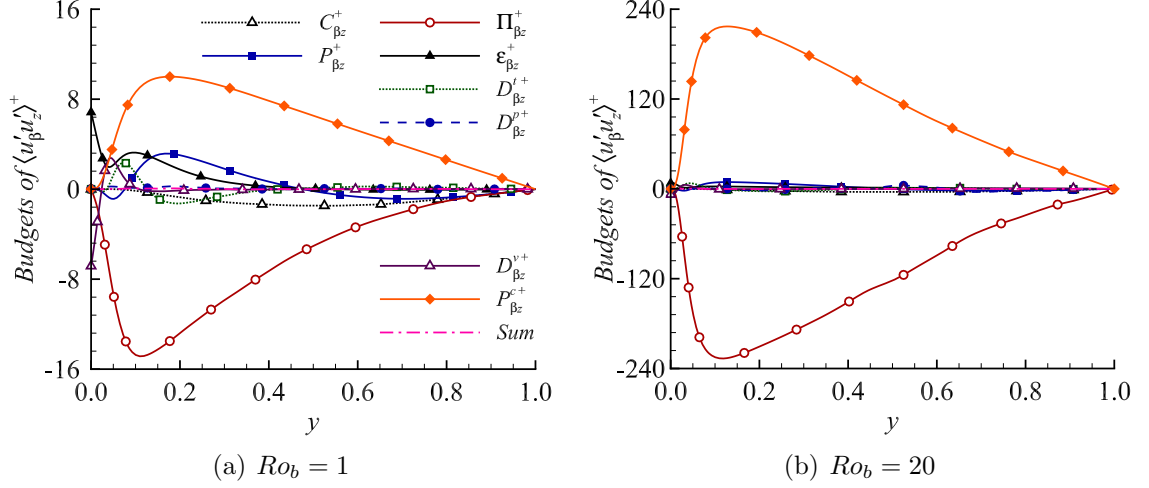


Figure 5.12: Budget balance of azimuthal-axial Reynolds shear stress $\langle u'_\beta u'_z \rangle^+$ for low rotating ($Ro_b = 1$) and high rotating ($Ro_b = 20$) pipe flows. All budget terms are non-dimensionalised by u_τ^3/R .

$P_{r\beta}^{c+}$ in magnitude over $y \in [0.616, 1]$. As shown in figure 5.11(c), at $Ro_b = 4$, these trends become more apparent. The sink contribution of $D_{r\beta}^{p+}$ is intensified and extended by the axial rotation over a broader region of $y \in [0.294, 1]$. As a result, the peak and valley values of $D_{r\beta}^{p+}$ and $\Pi_{r\beta}^+$ at the pipe wall are much larger than those of the budgets away from the wall. Moreover, it is interesting that $P_{r\beta}^{c+}$ starts to act as an energy source at $Ro_b = 4$ within a narrow region of the pipe core, over $y \in [0.697, 1]$. From figure 5.11(d), the budget balance of $\langle u'_r u'_\beta \rangle^+$ at $Ro_b = 20$ is dominated by $\Pi_{r\beta}^+$, $D_{r\beta}^{p+}$ and $P_{r\beta}^{c+}$. The term $\Pi_{r\beta}^+$ functions as a major energy source through most of the pipe, while acting as a weak sink within the near-wall region of $y \in [0, 0.05]$. Among these three budgets, $D_{r\beta}^{p+}$ redistributes energy from $y \in [0.387, 1]$ towards $y \in [0, 0.387]$. Meanwhile, $P_{r\beta}^{c+}$ supplies energy over $y \in [0.493, 1]$ and consumes it within $y \in [0, 0.493]$. It is clear from table 5.1 that the behaviour of the Coriolis production term $P_{r\beta}^{c+}$ is based on the imbalance between the azimuthal and radial Reynolds normal stresses, i.e., $\langle u'_\beta u'_\beta \rangle - \langle u'_r u'_r \rangle$. Accordingly, when $\langle u'_\beta u'_\beta \rangle > \langle u'_r u'_r \rangle$, $P_{r\beta}^{c+}$ acts as a sink term for $\langle u'_r u'_\beta \rangle^+$. Conversely, when $\langle u'_\beta u'_\beta \rangle < \langle u'_r u'_r \rangle$, it functions as a source term for $\langle u'_r u'_\beta \rangle^+$.

Figure 5.12 compares the budget balance of the azimuthal-axial Reynolds shear stress $\langle u'_\beta u'_z \rangle^+$ for the low rotating ($Ro_b = 1$) and high rotating ($Ro_b = 20$) pipe flows. As indicated by equation (E.10), the pressure diffusion term $D_{\beta z}^{p+}$ identically vanishes owing to the axisymmetric and fully developed conditions of the circular pipe flow. In figure 5.12(a), for the low rotating flow case at $Ro_b = 1$, the budget balance of $\langle u'_\beta u'_z \rangle^+$ near the wall is dominated by the viscous dissipation term $\varepsilon_{\beta z}^+$ and the viscous diffusion term $D_{\beta z}^{v+}$. Away from the wall, the balance is primarily governed by the Coriolis production term $P_{\beta z}^{c+}$ and pressure-strain term $\Pi_{\beta z}^+$. Across the wall-normal direction, $\varepsilon_{\beta z}^+$ and $P_{\beta z}^{c+}$ act as energy sources, whereas $\Pi_{\beta z}^+$ serves as the major sink term. The role of $D_{\beta z}^{v+}$ is to redistribute energy from both outer regions towards the interval of $y \in [0.024, 0.108]$. By contrast, the convection term $C_{\beta z}^+$, turbulent production term $P_{\beta z}^+$ and turbulent diffusion term $D_{\beta z}^{t+}$ contribute comparatively small to $\langle u'_\beta u'_z \rangle^+$. Specifically, $C_{\beta z}^+$ acts as a minor sink along the y -direction. The budget $P_{\beta z}^+$ generates energy within $y \in [0.075, 0.476]$ and consumes energy in the regions of $y \in [0, 0.075]$ and $y \in [0.476, 1]$. The turbulent diffusion term $D_{\beta z}^{t+}$ shows a more complex redistribution mechanism along the y -direction, crossing zero four times and mainly transferring energy from the region of $y \in [0.127, 0.444]$ towards $y \in [0.035, 0.127]$. It is evident from figure 5.12(b) that as the rotation number increases to $Ro_b = 20$, the magnitudes of $\Pi_{\beta z}^+$, acting as the major sink, and $P_{\beta z}^{c+}$, acting as the major source, increase drastically compared with the other budget terms. Their magnitudes are approximately fifteen times larger than those at $Ro_b = 1$. These two budget terms closely counterbalance each other, showing nearly symmetric profiles along the y -direction.

By summing each budget term of the three Reynolds normal stresses and dividing the result by two, the TKE transport equation for the statistically steady flow is obtained and can be expressed as

$$\frac{\partial \frac{1}{2} \langle u'_i u'_i \rangle}{\partial t} = C_{TKE} + P_{TKE} + \varepsilon_{TKE} + D_{TKE}^t + D_{TKE}^p + D_{TKE}^v = 0 \quad . \quad (5.9)$$

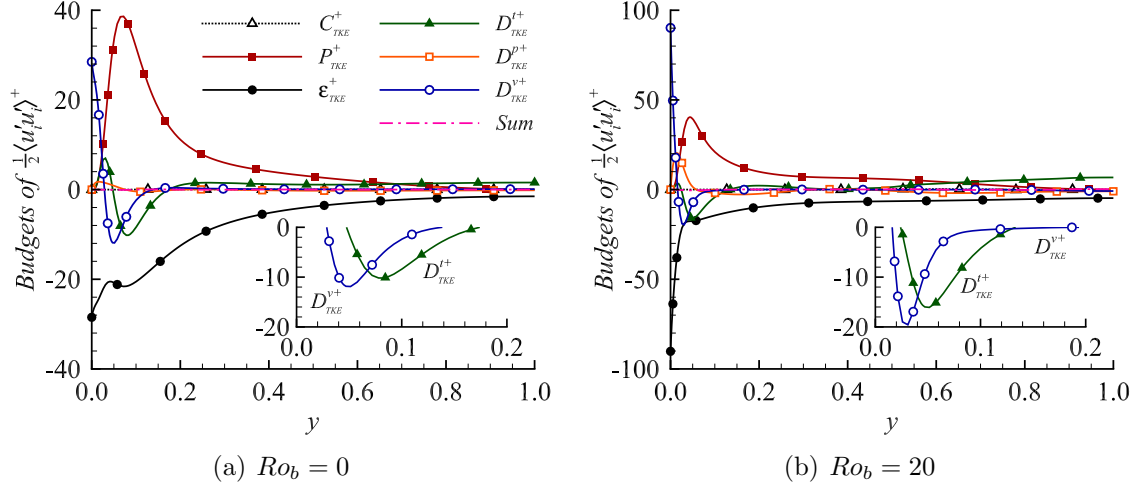


Figure 5.13: Budget balance of TKE $\frac{1}{2}\langle u'_i u'_i \rangle^+$ for non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. The inset panels show the variation of the turbulent diffusion D_{TKE}^{t+} and viscous diffusion D_{TKE}^{v+} terms for the two rotating cases. All budget terms are non-dimensionalised by u_τ^3/R .

Here, C_{TKE} , P_{TKE} , ε_{TKE} , D_{TKE}^t , D_{TKE}^p and D_{TKE}^v denote the convection, turbulent production, viscous dissipation, turbulent diffusion, pressure diffusion and viscous diffusion terms, respectively. Apparently, in equation (5.9), there is no explicit Coriolis contribution to the TKE transport equation, because the sum of the Coriolis production terms appearing in the transport equations of the three Reynolds normal stresses is zero. For convenience in analysing the TKE transport equation, its simplified form is given in equation (E.11) in Appendix E.1.

Figure 5.13 compares the budget balance of the TKE $\frac{1}{2}\langle u'_i u'_i \rangle^+$ in the non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. As shown in figure 5.13(a), for the non-rotating pipe flow, the TKE $\frac{1}{2}\langle u'_i u'_i \rangle^+$ is primarily influenced by the turbulent production term P_{TKE}^+ , viscous dissipation term ε_{TKE}^+ and viscous diffusion term D_{TKE}^{v+} in the near-wall region of the pipe. Among these three budgets, P_{TKE}^+ and ε_{TKE}^+ act as the TKE source and sink terms along the y -direction, respectively. The turbulent diffusion D_{TKE}^{t+} and pressure diffusion D_{TKE}^{p+} terms make small contributions to the TKE budget, while the convection term C_{TKE}^+ is identically zero. At the wall, the TKE transport is balanced solely by D_{TKE}^{v+} and ε_{TKE}^+ . It is seen from figure 5.13(a) that D_{TKE}^{v+}

and D_{TKE}^{t+} transfer TKE from the regions of $y \in [0.029, 0.139]$ and $y \in [0.047, 0.174]$, respectively, towards both sides of each interval. These regions coincide with the location of the peak turbulent production at $y = 0.072$. Compared with the turbulent diffusion term, the viscous diffusion term transports a larger fraction of TKE towards the wall, where it counterbalances the highest viscous dissipation. This indicates that viscous diffusion plays a more important role than turbulent diffusion in a low-Reynolds-number turbulent pipe flow at $Re_\tau = 180$. From figure 5.13(b), as the rotation number increases to $Ro_b = 20$, the magnitudes of ε_{TKE}^+ , D_{TKE}^{p+} and D_{TKE}^{v+} increase drastically in the near-wall region of the pipe, in clear contrast to the slight changes in the other budget terms. At the wall, the peaks of ε_{TKE}^+ and D_{TKE}^{v+} are nearly three times larger than those in the non-rotating pipe flow. To sustain the modified TKE budget balance under strong rotation, it is observed that the pressure diffusion term D_{TKE}^{p+} becomes a more dominant mechanism for the TKE transport. Its peak value increases by more than an order of magnitude compared with the non-rotating pipe flow. In addition, due to the axial rotation, the peak locations of these budget terms consistently move towards the pipe wall.

5.2.3 Transport of velocity-spectrum tensor and TKE spectrum

The transport equations of the six Reynolds stresses $\langle u'_i u'_j \rangle$ and the TKE $\frac{1}{2} \langle u'_i u'_i \rangle$ focus on their wall-normal distribution within the pipe. To gain deeper insight into the interscale energy transport mechanisms, it is necessary to examine the transport equations of the velocity-spectrum tensor $\tilde{E}_{ij}$ and the TKE spectrum $\tilde{E}_{TKE}$, which provide the spectral counterparts to the Reynolds stress and TKE transport equations in physical space. Therefore, the transport equation of the velocity-spectrum tensor

$\tilde{E}_{ij}$ is written as

$$\frac{\partial \tilde{E}_{ij}}{\partial t} = \tilde{C}_{ij} + \tilde{P}_{ij} + \tilde{T}_{ij} + \tilde{\Pi}_{ij} + \tilde{\varepsilon}_{ij} + \tilde{D}_{ij}^t + \tilde{D}_{ij}^p + \tilde{D}_{ij}^v + \tilde{P}_{ij}^c = 0 \quad . \quad (5.10)$$

Here, $\tilde{C}_{ij}$, $\tilde{P}_{ij}$, $\tilde{T}_{ij}$, $\tilde{\Pi}_{ij}$, $\tilde{\varepsilon}_{ij}$, $\tilde{D}_{ij}^t$, $\tilde{D}_{ij}^p$, $\tilde{D}_{ij}^v$ and $\tilde{P}_{ij}^c$ represent the convection, turbulent production, interscale transport, pressure-strain, viscous dissipation, turbulent diffusion, pressure diffusion, viscous diffusion and Coriolis production terms, respectively. The subscripts do not imply the summation convention. For clarity, a detailed derivation of the transport equation of the velocity-spectrum tensor is provided in Appendix E.2. The corresponding simplified forms of its six components are shown in equations (E.21)-(E.26). Moreover, table 5.2 summarises the six Coriolis production terms $\tilde{P}_{ij}^c$ of the velocity-spectrum tensor $\tilde{E}_{ij}$, as defined in equation (5.10). It is clear in table 5.2 that $\tilde{P}_{ij}^c$ exhibits a direct one-to-one correspondence with their physical-space counterparts P_{ij}^c . For instance, $\tilde{P}_{zz}^c$ associated with $\tilde{E}_{zz}$ is zero in spectral space, consistent with the corresponding physical-space term P_{zz}^c in $\langle u'_z u'_z \rangle$. In addition, $\tilde{P}_{rr}^c$ and $\tilde{P}_{\beta\beta}^c$, associated with $\tilde{E}_{rr}$ and $\tilde{E}_{\beta\beta}$, respectively, are equal in magnitude but opposite in sign. This antisymmetric relationship is likewise observed in the physical-space Coriolis production terms P_{rr}^c and $P_{\beta\beta}^c$ that appear in the transport equations of $\langle u'_r u'_r \rangle$ and $\langle u'_\beta u'_\beta \rangle$.

Figures 5.14-5.19 compare the budget balances of the six premultiplied velocity-spectrum tensor components at the wall-normal position of $y = 0.093$ (or $y^+ = 16.7$) in the high rotating ($Ro_b = 20$) pipe flow with those in the non-rotating ($Ro_b = 0$) or low rotating ($Ro_b = 1$) pipe flows. This wall-normal position corresponds to the peak TKE location at $Ro_b = 0$. Since the axial pipe length at $Ro_b = 20$ is significantly larger than at $Ro_b = 0$ or $Ro_b = 1$, the range of the non-dimensionalised axial wavelength λ_z^+ is kept consistent in figures 5.14-5.19 to facilitate a direct comparison of the characteristic length scales of the budget terms at these three rotation numbers. In order to more clearly characterise the wavelength dependence of each budget term,

Table 5.2: Coriolis production term $\tilde{P}_{ij}^c$ in the transport equation of velocity-spectrum tensor $\tilde{E}_{ij}$.

Velocity spectrum	Component $\tilde{E}_{ij}$	Coriolis production term $\tilde{P}_{ij}^c$
Axial autospectrum	$\tilde{E}_{zz}$	0
Radial autospectrum	$\tilde{E}_{rr}$	$\text{Re} \left\{ 4\Omega \widehat{u_r'}^* \widehat{u_\beta'} \right\}$
Azimuthal autospectrum	$\tilde{E}_{\beta\beta}$	$\text{Re} \left\{ -4\Omega \widehat{u_r'}^* \widehat{u_\beta'} \right\}$
Radial-axial co-spectrum	$\tilde{E}_{rz}$	$\text{Re} \left\{ 2\Omega \widehat{u_\beta'}^* \widehat{u_z'} \right\}$
Radial-azimuthal co-spectrum	$\tilde{E}_{r\beta}$	$\text{Re} \left\{ 2\Omega \left(\widehat{u_\beta'}^* \widehat{u_\beta'} - \widehat{u_r'}^* \widehat{u_r'} \right) \right\}$
Azimuthal-axial co-spectrum	$\tilde{E}_{\beta z}$	$\text{Re} \left\{ -2\Omega \widehat{u_r'}^* \widehat{u_z'} \right\}$

the full wavelength range in figures 5.14-5.19 is divided into four intervals, defined as follows:

- wavelengths with $\lambda_z^+ < 10^2$ are classified as ‘small’ wavelength scales;
- wavelengths within $\lambda_z^+ \in [10^2, 10^3]$ are referred to as ‘intermediate’ wavelength scales;
- wavelengths within $\lambda_z^+ \in [10^3, 10^4]$ are referred to as ‘large’ wavelength scales;
- wavelengths with $\lambda_z^+ > 10^4$ are classified as ‘very large’ wavelength scales.

Moreover, in figures 5.14-5.19, the label ‘*I*’ represents the first peak, or dome peak, in the premultiplied velocity spectra, indicating the most energetic eddies associated with streaks (or hairpin structures) in pipe flows. At a relatively high rotation number ($Ro_b = 20$), two additional peaks, labeled ‘*II*’ and ‘*III*’, appear in the premultiplied velocity spectra and are attributed to Taylor column structures induced by the axial system rotation. These peak locations are marked by vertical dash-dotted lines as references that highlight the characteristic length scales of flow structures of the two types.

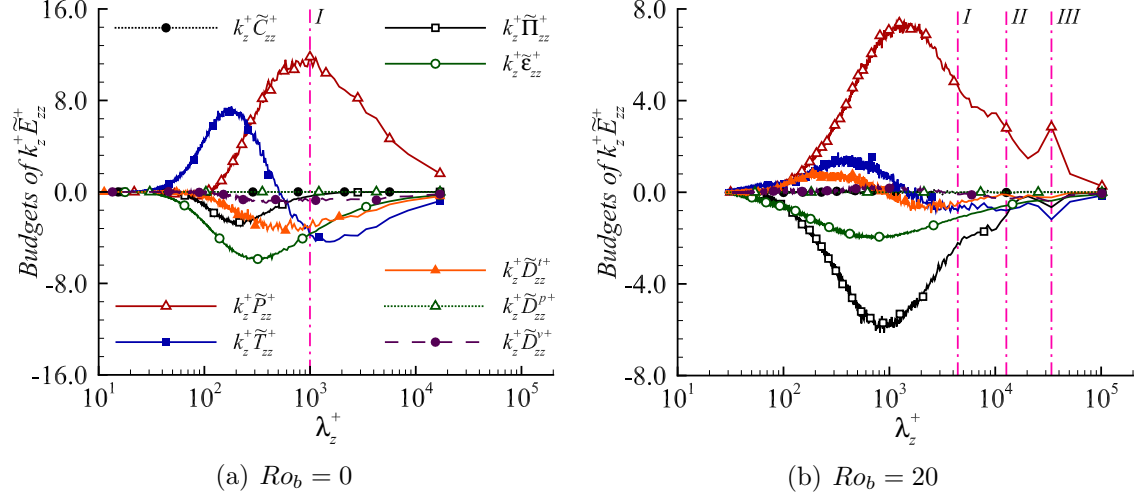


Figure 5.14: Budget balance of premultiplied axial autospectrum $k_z^+ \tilde{E}_{zz}^+$ for non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. The spectra are calculated along the axial direction at a fixed wall-normal position $y = 0.093$ (or $y^+ = 16.7$). All budget terms are non-dimensionalised by u_τ^3/R^2 .

In figures 5.14-5.19, all premultiplied budget terms have been non-dimensionalised by u_τ^3/R^2 .

Figure 5.14 compares the budget balance of the premultiplied axial autospectrum $k_z^+ \tilde{E}_{zz}^+$ at the wall-normal coordinate of $y = 0.093$ for the non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. As shown in figure 5.14(a), the budget profiles of $k_z^+ \tilde{E}_{zz}^+$ in the non-rotating pipe flow show a mechanism similar to that of the axial Reynolds normal stress $\langle u'_z u'_z \rangle^+$ in physical space. The balance is primarily dominated by the turbulent production term $k_z^+ \tilde{P}_{zz}^+$, which acts as an energy source and reaches its peak at $\lambda_z^+ = 997.9$, and by the viscous dissipation term $k_z^+ \tilde{\epsilon}_{zz}^+$, which acts as an energy sink and attains its valley at $\lambda_z^+ = 346.2$. The pressure-strain term $k_z^+ \tilde{\Pi}_{zz}^+$ and turbulent diffusion term $k_z^+ \tilde{D}_{zz}^{t+}$ behave as secondary energy sinks and contribute comparatively small over the entire wavelength range. Their valley values occur at $\lambda_z^+ = 197.3$ and $\lambda_z^+ = 771.1$, respectively. In addition, the interscale transport term $k_z^+ \tilde{T}_{zz}^+$ plays a crucial role in redistributing energy from vortices at $\lambda_z^+ > 508.4$ to those at $\lambda_z^+ < 508.4$. It is noted that the peak position of $k_z^+ \tilde{P}_{zz}^+$

coincides with that of $k_z^+ \tilde{E}_{zz}^+$ (indicated by *I*) at a intermediate wavelength scale of $\lambda_z^+ = 997.9$, indicating that $k_z^+ \tilde{P}_{zz}^+$ directly contributes to the generation of $\langle u'_z u'_z \rangle^+$. It is clear from figure 5.14(b) that as the rotation number increases to $Ro_b = 20$, the axial system rotation significantly suppresses the magnitudes of all spectral budget terms. The pressure-strain term $k_z^+ \tilde{\Pi}_{zz}^+$ becomes the major energy sink, replacing $k_z^+ \tilde{\varepsilon}_{zz}^+$ and thereby counterbalancing $k_z^+ \tilde{P}_{zz}^+$. In figure 5.14(b), $k_z^+ \tilde{P}_{zz}^+$ attains its dome peak at $\lambda_z^+ = 1375.5$, whereas $k_z^+ \tilde{\Pi}_{zz}^+$ and $k_z^+ \tilde{\varepsilon}_{zz}^+$ reach their valley values at $\lambda_z^+ = 892.9$ and $\lambda_z^+ = 743$, respectively. Moreover, the turbulent diffusion term $k_z^+ \tilde{D}_{zz}^{t+}$ supports $k_z^+ \tilde{T}_{zz}^+$ in energy transfer at different wavelength scales. Specifically, $k_z^+ \tilde{T}_{zz}^+$ redistribute energy from wavelengths larger than $\lambda_z^+ \approx 1590$ towards wavelengths shorter than this scale. Likewise, $k_z^+ \tilde{D}_{zz}^{t+}$ redistributes energy from $\lambda_z^+ \gtrsim 900$ towards smaller wavelengths of $\lambda_z^+ \lesssim 900$. At very large wavelength scales, $k_z^+ \tilde{P}_{zz}^+$ has another peak that aligns with the peak *III* of $k_z^+ \tilde{E}_{zz}^+$ at $\lambda_z^+ = 33929.3$, indicating that the turbulent production term also enhances energy level through Taylor columns. However, at $Ro_b = 20$, the influence of Taylor columns is clearly weaker than that of hairpin structures at $y = 0.093$ in the near-wall region of the pipe.

Figure 5.15 shows a comparison of the budget balance of the premultiplied radial autospectrum $k_z^+ \tilde{E}_{rr}^+$ between the non-rotating case ($Ro_b = 0$) and the strongly rotating case ($Ro_b = 20$), determined at the same wall-normal location. As shown in figure 5.15(a), for the non-rotating pipe flow, the transport of $k_z^+ \tilde{E}_{rr}^+$ is predominantly influenced by the pressure-strain term $k_z^+ \tilde{\Pi}_{rr}^+$, viscous dissipation term $k_z^+ \tilde{\varepsilon}_{rr}^+$ and pressure diffusion term $k_z^+ \tilde{D}_{rr}^{p+}$. The pressure-strain term $k_z^+ \tilde{\Pi}_{rr}^+$ acts as the major source for wavelengths larger than $\lambda_z^+ \approx 140$, where it receives energy transferred from the axial pressure-strain term $k_z^+ \tilde{\Pi}_{zz}^+$ associated with the premultiplied axial autospectrum $k_z^+ \tilde{E}_{zz}^+$ in figure 5.14(a). For wavelengths smaller than $\lambda_z^+ \approx 140$, however, $k_z^+ \tilde{\Pi}_{rr}^+$ reverses its role and becomes an energy sink. The variation of $k_z^+ \tilde{D}_{rr}^{p+}$ exhibits an inverse trend relative to $k_z^+ \tilde{\Pi}_{rr}^+$, redistributing energy from vortices with $\lambda_z^+ \gtrsim 180$ towards those at smaller wavelengths of $\lambda_z^+ \lesssim 180$. The viscous dissipation term

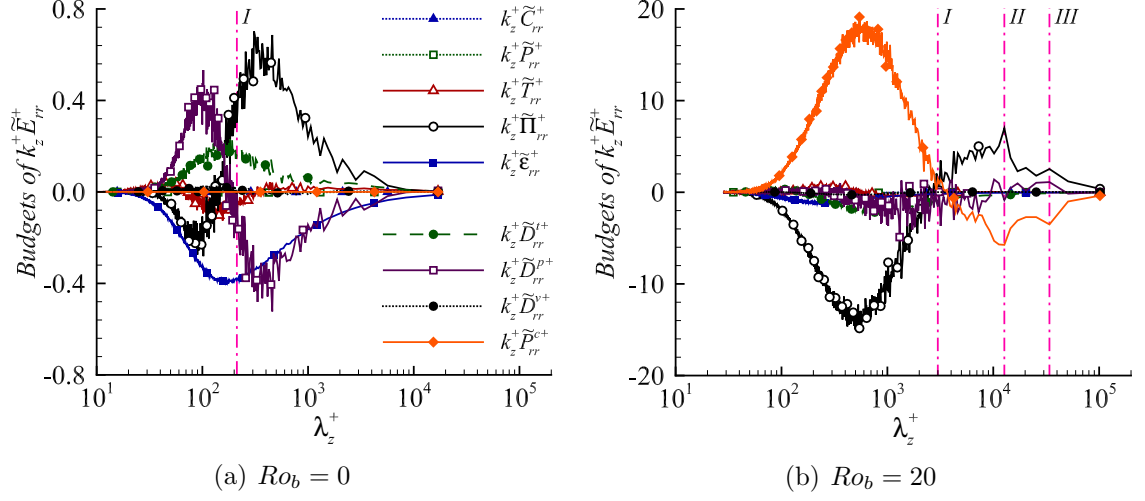


Figure 5.15: Budget balance of premultiplied radial autospectrum $k_z^+ \tilde{E}_{rr}^+$ for non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. The spectra are calculated along the axial direction at a fixed wall-normal position $y = 0.093$ (or $y^+ = 16.7$). All budget terms are non-dimensionalised by u_τ^3/R^2 .

$k_z^+ \tilde{\epsilon}_{rr}^+$ acts as the major energy sink and reaches its valley at $\lambda_z^+ = 176.7$, whereas the turbulent diffusion term $k_z^+ \tilde{D}_{rr}^{t+}$, which peaks at $\lambda_z^+ = 190.6$, serves as a secondary energy source across all wavelength scales. Furthermore, the interscale transport term $k_z^+ \tilde{T}_{rr}^+$ exhibits a bidirectional energy-transfer mechanism, redistributing only a limited amount of energy from the wavelength interval of $\lambda_z^+ \in [70, 355]$ towards both larger and smaller scales. From figure 5.15(b), it is seen that at $Ro_b = 20$, the budget balance of $k_z^+ \tilde{E}_{rr}^+$ is predominantly governed by $k_z^+ \tilde{\Pi}_{rr}^+$ and the Coriolis production term $k_z^+ \tilde{P}_{rr}^{c+}$. Owing to the axial system rotation, the values of these two leading budgets increase by more than two orders of magnitude compared with those at $Ro_b = 0$. In addition, they have nearly symmetrical profiles, appearing as mirror reflections of one another, and both cross zero near the dome peak I of $k_z^+ \tilde{E}_{rr}^+$ at $\lambda_z^+ = 2993.8$. Therefore, $k_z^+ \tilde{P}_{rr}^{c+}$ functions as an energy sink for $\lambda_z^+ > 2993.8$, whereas it becomes an energy source at shorter wavelengths of $\lambda_z^+ < 2993.8$. It is also evident from figure 5.15(b) that the two valleys of $k_z^+ \tilde{P}_{rr}^{c+}$ at very large wavelengths coincide with the spectral peaks II and III of $k_z^+ \tilde{E}_{rr}^+$, which are relative to the Taylor column structures.

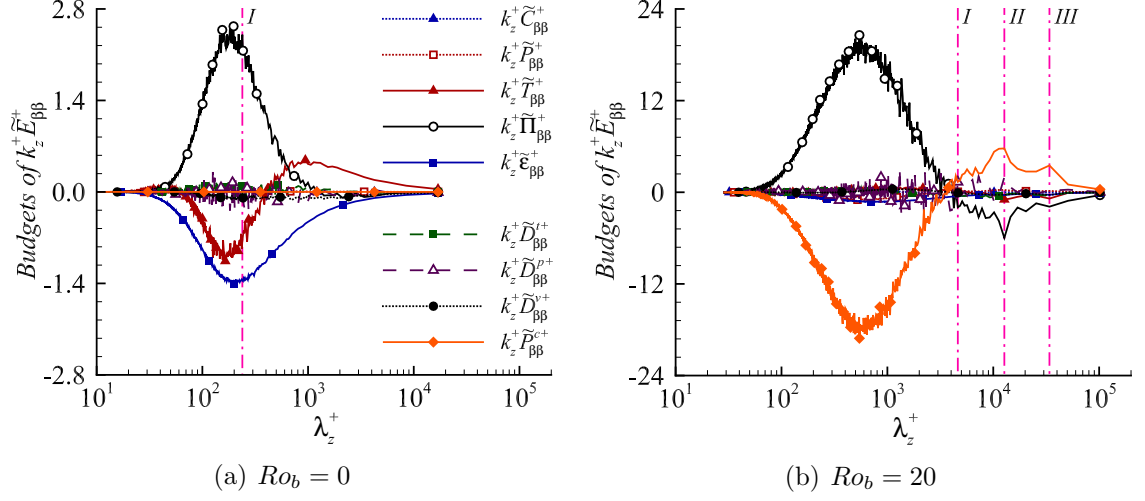


Figure 5.16: Budget balance of premultiplied azimuthal autospectrum $k_z^+ \tilde{E}_{\beta\beta}^+$ for non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. The spectra are calculated along the axial direction at a fixed wall-normal position $y = 0.093$ (or $y^+ = 16.7$). All budget terms are non-dimensionalised by u_τ^3/R^2 .

This alignment indicates that the Coriolis production term attenuates energy level at the characteristic wavelengths of Taylor columns at $y = 0.093$ for $Ro_b = 20$.

Figure 5.16 illustrates the budget profiles of the premultiplied azimuthal autospectrum $k_z^+ \tilde{E}_{\beta\beta}^+$ for both the non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. In figure 5.16(a), the pressure-strain term $k_z^+ \tilde{\Pi}_{\beta\beta}^+$, interscale transport term $k_z^+ \tilde{T}_{\beta\beta}^+$ and viscous dissipation term $k_z^+ \tilde{\varepsilon}_{\beta\beta}^+$ dominate the transport of $k_z^+ \tilde{E}_{\beta\beta}^+$ at $y = 0.093$ in the non-rotating pipe flow. Among these three budgets, $k_z^+ \tilde{\Pi}_{\beta\beta}^+$ and $k_z^+ \tilde{\varepsilon}_{\beta\beta}^+$ act as energy source and sink terms, respectively, across all wavelength scales, with their peak and valley occurring at $\lambda_z^+ = 195$ and $\lambda_z^+ = 199.6$, respectively. Apparently, their extrema are located near the dome peak I of $k_z^+ \tilde{E}_{\beta\beta}^+$ at $\lambda_z^+ = 238.94$, indicating that the pressure-strain and the viscous dissipation terms directly govern the production and dissipation of $k_z^+ \tilde{E}_{\beta\beta}^+$, respectively. In comparison with figures 5.14(a) and 5.15(a), it becomes evident that $k_z^+ \tilde{\Pi}_{\beta\beta}^+$ gains energy exclusively from $k_z^+ \tilde{\Pi}_{zz}^+$ at wavelength scales larger than $\lambda_z^+ \approx 140$, whereas at smaller scales it draws energy from both $k_z^+ \tilde{\Pi}_{zz}^+$ and $k_z^+ \tilde{\Pi}_{rr}^+$. The interscale transport term $k_z^+ \tilde{T}_{\beta\beta}^+$ redistributes en-

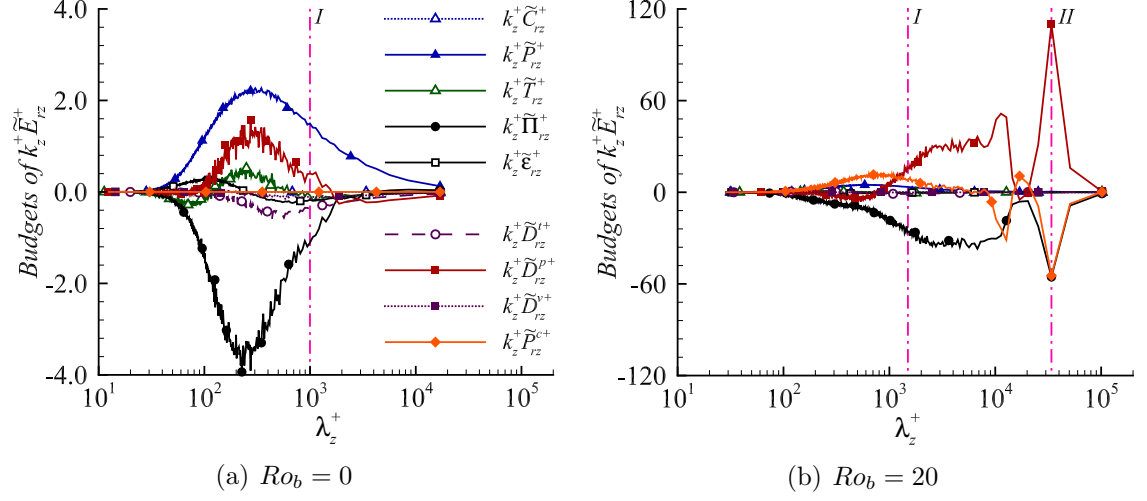


Figure 5.17: Budget balance of premultiplied radial-axial co-spectrum $k_z^+ \tilde{E}_{rz}^+$ for non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. The spectra are calculated along the axial direction at a fixed wall-normal position $y = 0.093$ (or $y^+ = 16.7$). All budget terms are non-dimensionalised by u_τ^3/R^2 .

ergy in much the same way as the radial interscale transport term $k_z^+ \tilde{T}_{rr}^+$ of $k_z^+ \tilde{E}_{rr}^+$, transferring energy from vortices within the wavelength range of $\lambda_z^+ \in [60, 410]$ towards both smaller- and larger-scale vortices. It is clear in figure 5.16(b) that at the highest rotation number of $Ro_b = 20$, the budget balance of $k_z^+ \tilde{E}_{\beta\beta}^+$ is dominated by $k_z^+ \tilde{P}_{\beta\beta}^{c+}$ and $k_z^+ \tilde{\Pi}_{\beta\beta}^+$. A comparison with figure 5.15(b) shows that the profiles of $k_z^+ \tilde{P}_{\beta\beta}^{c+}$ and $k_z^+ \tilde{\Pi}_{\beta\beta}^+$ are nearly inverted relative to those of $k_z^+ \tilde{P}_{rr}^{c+}$ and $k_z^+ \tilde{\Pi}_{rr}^+$ about the zero point. This inversion arises from the opposite signs of $k_z^+ \tilde{P}_{rr}^{c+}$ and $k_z^+ \tilde{P}_{\beta\beta}^{c+}$, as listed in table 5.2. As a result, $k_z^+ \tilde{P}_{\beta\beta}^{c+}$ also reverses its behaviour compared with $k_z^+ \tilde{P}_{rr}^{c+}$ over the entire wavelength range. At $Ro_b = 20$, the Taylor column structures induced by the axial rotation at $y = 0.093$ elevate the energy level of $k_z^+ \tilde{E}_{\beta\beta}^+$ at large and very large wavelengths by directly strengthening the positive contribution of $k_z^+ \tilde{P}_{\beta\beta}^{c+}$. Furthermore, as evident in figure 5.16(b), the leading budgets of $k_z^+ \tilde{E}_{\beta\beta}^+$ increase by approximately an order of magnitude in the rotating pipe flow at $Ro_b = 20$ compared to those in the non-rotating pipe flow at $Ro_b = 0$.

Figure 5.17 compares the budget balance of the premultiplied radial-axial co-

spectrum $k_z^+ \tilde{E}_{rz}^+$ for the non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) cases. Clearly, in figure 5.17(a), the budget balance of $k_z^+ \tilde{E}_{rz}^+$ at $y = 0.093$ in the non-rotating case is primarily governed by the turbulent production term $k_z^+ \tilde{P}_{rz}^+$, pressure-strain term $k_z^+ \tilde{\Pi}_{rz}^+$ and pressure diffusion term $k_z^+ \tilde{D}_{rz}^{p+}$. Across all wavelengths, $k_z^+ \tilde{P}_{rz}^+$ and $k_z^+ \tilde{\Pi}_{rz}^+$ act as the major source and sink terms, respectively, while $k_z^+ \tilde{D}_{rz}^{p+}$ redistributes energy from $\lambda_z^+ > 1595.9$ towards $\lambda_z^+ < 1595.9$. The peak of $k_z^+ \tilde{P}_{rz}^+$ and the valley of $k_z^+ \tilde{\Pi}_{rz}^+$ are located at $\lambda_z^+ = 302.9$ and $\lambda_z^+ = 226.2$, respectively. In addition, the contributions of the interscale transport term $k_z^+ \tilde{T}_{rz}^+$, viscous dissipation term $k_z^+ \tilde{\varepsilon}_{rz}^+$, turbulent diffusion term $k_z^+ \tilde{D}_{rz}^{t+}$ and viscous diffusion term $k_z^+ \tilde{D}_{rz}^{v+}$ remain comparatively limited. The function of $k_z^+ \tilde{T}_{rz}^+$ is to redistribute energy from both the smaller- and larger-scale motions towards vortices at intermediate scales within the range of $\lambda_z^+ \in [120, 635]$. In contrast, $k_z^+ \tilde{\varepsilon}_{rz}^+$ acts as an energy sink for $\lambda_z^+ > 256$ and as an energy source for $\lambda_z^+ < 256$. Both $k_z^+ \tilde{D}_{rz}^{t+}$ and $k_z^+ \tilde{D}_{rz}^{v+}$ remove energy across the entire wavelength range. As the rotating speed reaches its highest tested value at $Ro_b = 20$, it is apparent from figure 5.17(b) that the magnitudes of the leading budgets are approximately thirty times larger than those in the non-rotating case. In addition to the three leading terms, the transport of $k_z^+ \tilde{E}_{rz}^+$ at $Ro_b = 20$ is drastically affected by the Coriolis production term $k_z^+ \tilde{P}_{rz}^{c+}$ as well. From figure 5.17(b), $k_z^+ \tilde{P}_{rz}^+$ and $k_z^+ \tilde{\Pi}_{rz}^+$ retain the same roles across all wavelengths. For wavelengths λ_z^+ shorter than the very-large-scale range, $k_z^+ \tilde{P}_{rz}^{c+}$ acts as an energy source for $\lambda_z^+ \lesssim 8480$, while $k_z^+ \tilde{D}_{rz}^{p+}$ acts as an energy sink for $\lambda_z^+ \lesssim 790$. However, in the very-large-wavelength interval, the roles of $k_z^+ \tilde{P}_{rz}^{c+}$ and $k_z^+ \tilde{D}_{rz}^{p+}$ are reversed, although each term briefly remains its original behaviour in the vicinity of $\lambda_z^+ = 16964.6$. Moreover, the extrema of $k_z^+ \tilde{D}_{rz}^{p+}$, $k_z^+ \tilde{\Pi}_{rz}^+$ and $k_z^+ \tilde{P}_{rz}^{c+}$ occur at $\lambda_z^+ = 50893.8$; these extrema coincide with the second peak *II* of $k_z^+ \tilde{E}_{rz}^+$, indicating that the budget balance of $k_z^+ \tilde{E}_{rz}^+$ is directly and strongly influenced by these three budgets and is more closely associated with the Taylor column structures than with the streaky structures at $y = 0.093$ and $Ro_b = 20$.

The distributions of the premultiplied radial-azimuthal co-spectrum $k_z^+ \tilde{E}_{r\beta}^+$ at

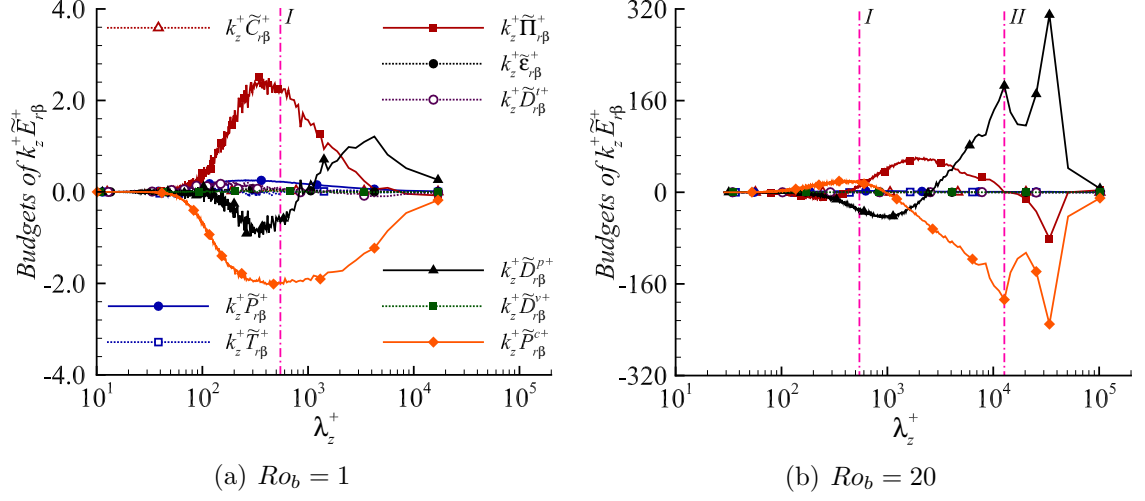


Figure 5.18: Budget balance of premultiplied radial-azimuthal co-spectrum $k_z^+ \tilde{E}_{r\beta}^+$ for low rotating ($Ro_b = 1$) and high rotating ($Ro_b = 20$) pipe flows. The spectra are calculated along the axial direction at a fixed wall-normal position $y = 0.093$ (or $y^+ = 16.7$). All budget terms are non-dimensionalised by u_τ^3/R^2 .

$y = 0.093$ for the low rotating ($Ro_b = 1$) and high rotating ($Ro_b = 20$) pipe flows are illustrated in figure 5.18. As evident from figure 5.18(a), in the low rotating case at $Ro_b = 1$, the transport of $k_z^+ \tilde{E}_{r\beta}^+$ is predominantly governed by the pressure-strain term $k_z^+ \tilde{\Pi}_{r\beta}^+$, pressure diffusion term $k_z^+ \tilde{D}_{r\beta}^{p+}$ and Coriolis production term $k_z^+ \tilde{P}_{r\beta}^{c+}$. Clearly, $k_z^+ \tilde{\Pi}_{r\beta}^+$ and $k_z^+ \tilde{P}_{r\beta}^{c+}$ act as source and sink terms, respectively, across all wavelength scales, whereas $k_z^+ \tilde{D}_{r\beta}^{p+}$ redistributes energy from the intermediate scales ($\lambda_z^+ \lesssim 900$) towards the large scales. The extrema of $k_z^+ \tilde{\Pi}_{r\beta}^+$ and $k_z^+ \tilde{P}_{r\beta}^{c+}$ occur at $\lambda_z^+ = 346.2$ and $\lambda_z^+ = 403.9$, respectively. By contrast, the turbulent production term $k_z^+ \tilde{P}_{r\beta}^+$ provides only a minor contribution as an energy source, and the role of the interscale transport term $k_z^+ \tilde{T}_{r\beta}^+$ can be negligible. It is seen from figure 5.18(b) that at $Ro_b = 20$, the budget balance of $k_z^+ \tilde{E}_{r\beta}^+$ is still dominated by $k_z^+ \tilde{\Pi}_{r\beta}^+$, $k_z^+ \tilde{D}_{r\beta}^{p+}$ and $k_z^+ \tilde{P}_{r\beta}^{c+}$. However, under the influence of the axial system rotation, $k_z^+ \tilde{D}_{r\beta}^{p+}$ transfers energy from wavelengths shorter than $\lambda_z^+ = 2498.2$ towards those larger than this scale. In addition, $k_z^+ \tilde{\Pi}_{r\beta}^+$ and $k_z^+ \tilde{P}_{r\beta}^{c+}$ no longer exhibit a single role across the wavelength range. At $Ro_b = 20$, $k_z^+ \tilde{\Pi}_{r\beta}^+$ functions as an energy source over the interval

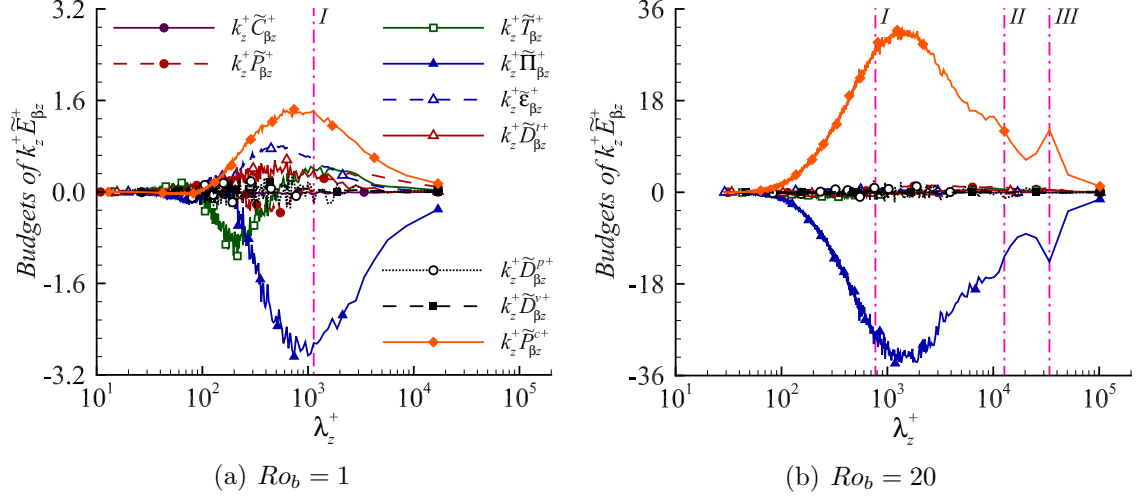


Figure 5.19: Budget balance of premultiplied azimuthal-axial co-spectrum $k_z^+ \tilde{E}_{\beta z}^+$ for low rotating ($Ro_b = 1$) and high rotating ($Ro_b = 20$) pipe flows. The spectra are calculated along the axial direction at a fixed wall-normal position $y = 0.093$ (or $y^+ = 16.7$). All budget terms are non-dimensionalised by u_τ^3/R^2 .

of $\lambda_z^+ \in [413, 12703.2]$, while acting as a sink at both smaller and larger wavelength scales. In contrast, $k_z^+ \tilde{P}_{r\beta}^{c+}$ serves as a source for $\lambda_z^+ < 956.1$ and becomes a sink for $\lambda_z^+ > 956.1$ (corresponding to the large and very large wavelength scales, respectively). From figure 5.18(b), the extrema of these three leading budgets occur at very large wavelength scales and coincide with the the second peak *II* of $k_z^+ \tilde{E}_{r\beta}^+$. Therefore, this observation indicates that at the highest rotating rate of $Ro_b = 20$, the Taylor columns exert a much stronger influence than the near-wall streaks in reinforcing the transport of $k_z^+ \tilde{E}_{r\beta}^+$ at $y = 0.093$.

Figure 5.19 compares the budget balance of the premultiplied azimuthal-axial co-spectrum $k_z^+ \tilde{E}_{\beta z}^+$ for the low rotating ($Ro_b = 1$) and high rotating ($Ro_b = 20$) pipe flows. As shown in figure 5.19(a), in the low rotating case at $Ro_b = 1$, the budget balance of $k_z^+ \tilde{E}_{\beta z}^+$ is dominated by the interscale transport term $k_z^+ \tilde{T}_{\beta z}^+$, pressure-strain term $k_z^+ \tilde{\Pi}_{\beta z}^+$ and Coriolis production term $k_z^+ \tilde{P}_{\beta z}^{c+}$. The mechanism of $k_z^+ \tilde{T}_{\beta z}^+$ is to redistribute energy from vortical motions at the intermediate wavelength scales within $\lambda_z^+ \in [85, 515]$ towards both smaller and larger scales. Meanwhile, $k_z^+ \tilde{\Pi}_{\beta z}^+$ and $k_z^+ \tilde{P}_{\beta z}^{c+}$

act as energy sink and source terms, respectively, across all wavelengths, although $k_z^+ \tilde{\Pi}_{\beta z}^+$ briefly becomes a source within the narrow interval of $\lambda_z^+ \in [100, 175]$. Both budget terms reach their extremal values at $\lambda_z^+ = 737.6$. Furthermore, the viscous dissipation term $k_z^+ \tilde{\varepsilon}_{\beta z}^+$ and turbulent diffusion term $k_z^+ \tilde{D}_{\beta z}^{t+}$ serves as secondary source terms, supplying additional energy to $k_z^+ \tilde{E}_{\beta z}^+$. Their peak values occur at $\lambda_z^+ = 484.7$ and $\lambda_z^+ = 628.3$, respectively. It is also noted that the contribution of the turbulent production term $k_z^+ \tilde{P}_{\beta z}^+$ to $k_z^+ \tilde{E}_{\beta z}^+$ is relatively limited. It removes energy at the intermediate wavelength scales within $\lambda_z^+ \in [185, 910]$ and provide energy outside this interval. In figure 5.19(b), as the rotation number increases to $Ro_b = 20$, the transport of $k_z^+ \tilde{E}_{\beta z}^+$ is balanced only by $k_z^+ \tilde{\Pi}_{\beta z}^+$ and $k_z^+ \tilde{P}_{\beta z}^{c+}$. Compared with their magnitudes at $Ro_b = 1$, those at $Ro_b = 20$ increase by more than a factor of eleven. Meanwhile, the profiles of these two budget terms nearly reflect one another about the zero point, and their extrema occur at their dome-shaped peaks, located at $\lambda_z^+ = 1183.6$ for $k_z^+ \tilde{\Pi}_{\beta z}^+$ and $\lambda_z^+ = 1375.5$ for $k_z^+ \tilde{P}_{\beta z}^{c+}$. This observation indicates that in the near-wall region at $y = 0.093$, the influences of the pressure-strain and Coriolis production terms are more amplified by the streaky structures, thereby enhancing the transport of $k_z^+ \tilde{E}_{\beta z}^+$ at $Ro_b = 20$.

To further analyse the influence of the axial system rotation on the transport of TKE in spectral space, the transport equation of the TKE spectrum $\tilde{E}_{TKE}$ is derived and written as

$$\frac{\partial \tilde{E}_{TKE}}{\partial t} = \tilde{C}_{TKE} + \tilde{P}_{TKE} + \tilde{T}_{TKE} + \tilde{\varepsilon}_{TKE} + \tilde{D}_{TKE}^t + \tilde{D}_{TKE}^p + \tilde{D}_{TKE}^v = 0 \quad , \quad (5.11)$$

where $\tilde{C}_{TKE}$, $\tilde{P}_{TKE}$, $\tilde{T}_{TKE}$, $\tilde{\varepsilon}_{TKE}$, $\tilde{D}_{TKE}^t$, $\tilde{D}_{TKE}^p$ and $\tilde{D}_{TKE}^v$ denote the convection, turbulent production, interscale transport, viscous dissipation, turbulent diffusion, pressure diffusion and viscous diffusion terms, respectively. Similarly to the absence of a Coriolis production term in the TKE transport equation in physical space, no Coriolis production appears in the transport equation of the TKE spectrum. This is because the summation of the Coriolis production terms associated with the three autospectral

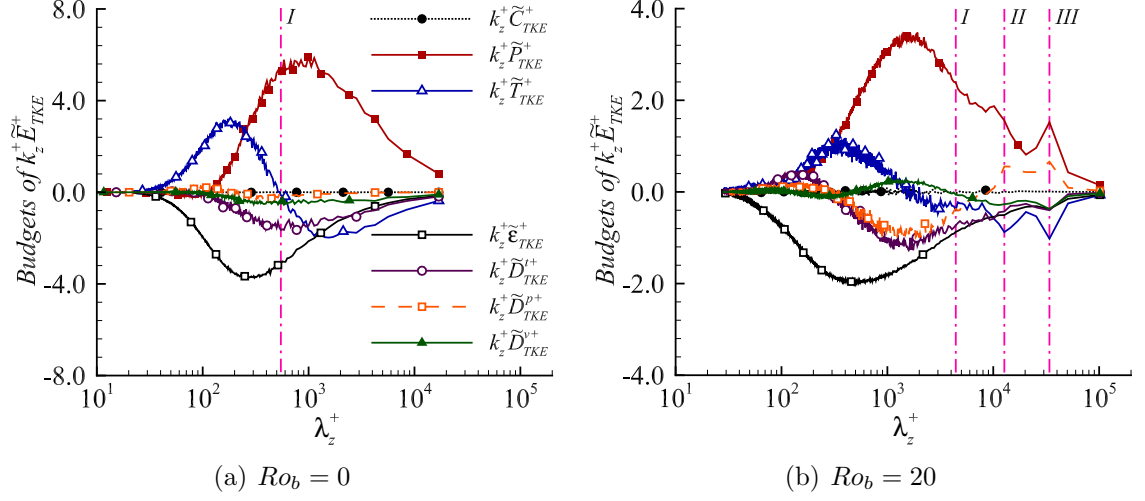


Figure 5.20: Budget balance of premultiplied TKE spectrum $k_z^+ \tilde{E}_{TKE}^+$ for non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. The spectra are calculated along the axial direction at a fixed wall-normal position $y = 0.093$ (or $y^+ = 16.7$). All budget terms are non-dimensionalised by u_τ^3/R^2 .

components (listed in table 5.2) is identically zero. Moreover, the simplified form of this transport equation is provided in equation (E.27) in Appendix E.2.

Figure 5.20 compares the budget balance of the premultiplied TKE spectrum $k_z^+ \tilde{E}_{TKE}^+$ for the non-rotating ($Ro_b = 0$) and high rotating ($Ro_b = 20$) pipe flows. As evident from figure 5.20(a), in the near-wall region at $y = 0.093$, the budget balance of $k_z^+ \tilde{E}_{TKE}^+$ at $Ro_b = 0$ is predominately influenced by the turbulent production term $k_z^+ \tilde{P}_{TKE}^+$, interscale transport term $k_z^+ \tilde{T}_{TKE}^+$ and viscous dissipation term $k_z^+ \tilde{\epsilon}_{TKE}^+$. The budget $k_z^+ \tilde{P}_{TKE}^+$ acts as the major TKE source across the entire wavelength scales and attain its peak at $\lambda_z^+ = 997.9$, whereas $k_z^+ \tilde{\epsilon}_{TKE}^+$ serves as the major TKE sink and reaches its valley at $\lambda_z^+ = 269.3$. The interscale transport term $k_z^+ \tilde{T}_{TKE}^+$ behaves to transfer TKE from vortices at wavelengths larger than $\lambda_z^+ = 531.8$ towards those at smaller scales. In addition, the turbulent diffusion term $k_z^+ \tilde{D}_{TKE}^{t+}$ acts as a secondary sink that removes energy from $k_z^+ \tilde{E}_{TKE}^+$, reaching its valley value at $\lambda_z^+ = 547.2$; meanwhile, the pressure diffusion term $k_z^+ \tilde{D}_{TKE}^{p+}$ and viscous diffusion term $k_z^+ \tilde{D}_{TKE}^{v+}$ are very small in magnitude. From figure 5.20(b), as the rotating rate increases

to $Ro_b = 20$, the transport of $k_z^+ \tilde{E}_{TKE}^+$ is still dominated by $k_z^+ \tilde{P}_{TKE}^+$, $k_z^+ \tilde{T}_{TKE}^+$ and $k_z^+ \tilde{\varepsilon}_{TKE}^+$; however, the three diffusion terms become increasingly important. It is clear in figure 5.20(b) that the axial system rotation tends to suppress the magnitudes of the budget terms and shifts their spectral distributions towards larger wavelength scales. As a result, the dome peak of $k_z^+ \tilde{P}_{TKE}^+$ moves to the large-scale range located at $\lambda_z^+ = 1920.5$, whereas $k_z^+ \tilde{\varepsilon}_{TKE}^+$ attains its valley at the intermediate scale of $\lambda_z^+ = 544.3$. Meanwhile, $k_z^+ \tilde{D}_{TKE}^{t+}$ plays a role similar to that of $k_z^+ \tilde{T}_{TKE}^+$, as both terms redistribute TKE energy from larger to smaller wavelength scales. Specifically, $k_z^+ \tilde{D}_{TKE}^{t+}$ transfer energy from wavelengths larger than $\lambda_z^+ \approx 275$, whereas $k_z^+ \tilde{T}_{TKE}^+$ redistributes energy from wavelengths larger than $\lambda_z^+ \approx 1610$ towards smaller scales. Furthermore, $k_z^+ \tilde{D}_{TKE}^{p+}$ transfers energy from the wavelength interval of $\lambda_z^+ \in [295, 9073.2]$ towards both smaller and larger wavelength scales. The behaviour of $k_z^+ \tilde{D}_{TKE}^{v+}$ is more complex, as it alternates between acting as a TKE source and a TKE sink, with its profile crossing zero three times along the entire wavelength range. As seen in figure 5.20(b), although the Taylor columns contribute directly to the transport of $k_z^+ \tilde{E}_{TKE}^+$ through the turbulent production term, indicated by the second peak of $k_z^+ \tilde{P}_{TKE}^+$ coinciding with the third peak *III* of $k_z^+ \tilde{E}_{TKE}^+$ at the very large wavelength of $\lambda_z^+ = 33929.3$, the larger magnitude of the dome peak of $k_z^+ \tilde{P}_{TKE}^+$ indicates that the near-wall streaks have a stronger influence on the turbulent production of $k_z^+ \tilde{E}_{TKE}^+$ at $y = 0.093$ in the high rotating pipe flow at $Ro_b = 20$.

Chapter 6

DNS of turbulent heat transfer in a circular pipe subjected to axial system rotation

In this chapter, turbulent heat convection of axially rotating circular pipe flows is investigated by DNS for a wide range of rotation number of $Ro_b \in [0, 20]$. In order to reasonably capture the energetic axial eddy motions and precisely reproduce the physical process of heat convection, very long pipes are used in my DNS. The range of pipe lengths of this research is $L_z = 30\pi R - 180\pi R$. The Coriolis effects on the turbulent heat flux transport equations will be studied.

6.1 Test cases and numerical algorithm

Figure 6.1 shows the computational domain under axial system rotation at a clockwise angular speed Ω . The radial, azimuthal and axial coordinates of the cylindrical coordinate system are denoted using r , β and z , and the corresponding velocity components are denoted as u_r , u_β and u_z , respectively. Depending upon the rotation

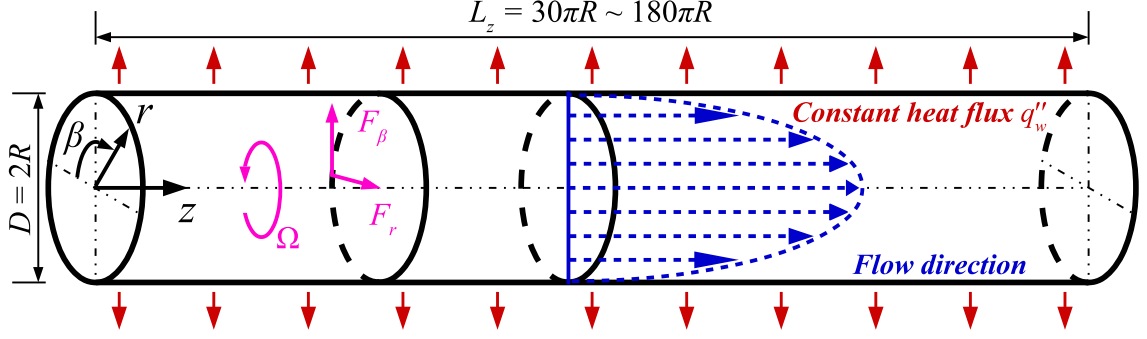


Figure 6.1: Schematic of the computational domain of an axially rotating circular pipe in cylindrical coordinate system. F_r and F_β are two Coriolis force components in radial and azimuthal directions, respectively. The pipe flow is cooled along the axial direction by imposing a constant wall heat flux (q''_w) on the curved wall.

number, the pipe length varies from $L_z = 30\pi R$ (at $Ro_b = 0$) to $180\pi R$ (at $Ro_b = 20$) to ensure that energetic turbulent eddy motions are reasonably captured in the z -direction. The specific pipe lengths in regard to different rotation numbers are given in table 6.1. The pipe flow is fully developed at Reynolds number of $Re_\tau = 180$ and Prandtl number of $Pr = 1$. In total, eight test cases of rotation numbers ranging from $Ro_b = 0$ to 20 are compared. Periodic boundary condition and no-slip condition are applied to the z -direction and on the pipe surface, respectively. A uniform peripheral wall heat flux (q''_w) boundary condition is imposed on the pipe wall for cooling the flow. Here, $Ro_b = 2\Omega R/U_b$ and $Re_\tau = u_\tau R/\nu$, where U_b is the bulk mean velocity, u_τ is the wall friction velocity, and ν is the kinematic viscosity of the fluid. The governing equations for an incompressible flow with respect to an axial rotating reference frame are

$$\nabla \cdot \vec{u} = 0 \quad , \quad (6.1)$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{\Pi}{\rho} \hat{\mathbf{e}}_z - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} + \vec{F} \quad , \quad (6.2)$$

$$\frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta = -u_z \frac{dT_w}{dz} + \alpha \nabla^2 \theta \quad , \quad (6.3)$$

Table 6.1: Summary of eight rotating cases tested.

Case #	1	2	3	4	5	6	7	8
Ro_b	0	1	2	4	6	10	14	20
L_z	$30\pi R$	$30\pi R$	$30\pi R$	$60\pi R$	$60\pi R$	$90\pi R$	$120\pi R$	$180\pi R$
N_z	3600	3600	3600	5400	5400	6000	6400	7200
Δz^+	4.712	4.712	4.712	6.283	6.283	8.482	10.603	14.137

where $\vec{u}$, ρ , α , p and θ are the velocity, density, thermal diffusivity, pressure and deficit temperature (defined as $\theta = T - T_w$) of the fluid, respectively. Π and dT_w/dz represent the constant mean axial pressure and temperature gradients, respectively. Specifically, $dT_w/dz = dT_b/dz = -2q_w''/(\rho C_p U_b R)$, where T_w is the local wall temperature, T_b is the bulk mean temperature, and C_p is the specific heat capacity of the fluid. As shown schematically in figure 6.1, in response to the axial rotation, two components of the Coriolis force $\vec{F}$ appear in the r - and β -directions, i.e., $F_r = 2\Omega u_\beta$ and $F_\beta = -2\Omega u_r$. The DNS was performed with a spectral-element code so-called “Semtex” by Blackburn and Sherwin (2004). The quadrilateral SEM was applied which divides the cross-section of the pipe into 420 finite elements, with each element further discretised spatially with 8th-order GLLL interpolants. As is shown in table 6.1, depending upon the rotation number, the physical quantities were expanded into the spectral space using Fourier series with $N_z = 3600$ –7200 modes in the z -direction. Therefore, the total grid points varied from $N_{tot} = 97$ to 194 million. The grid spacing in three directions are $\Delta z^+ = 4.712$ –14.137, $\Delta r^+ = 0.123$ –3.595, and $R\Delta\beta^+ = 0.813$ –5.133, respectively. All simulations were held at a constant time step of $\Delta t^+ = u_\tau^2 \Delta t / \nu = 0.0122$. Here, the friction velocity is defined as $u_\tau = \sqrt{-\Pi R / 2\rho}$. More details about the numerical method can be found in Zhang and Wang (2026). Moreover, in order to non-dimensionalise the temperature statistics, the wall friction temperature needs to be introduced which is defined as $T_\tau = q_w''/(\rho C_p u_\tau)$.

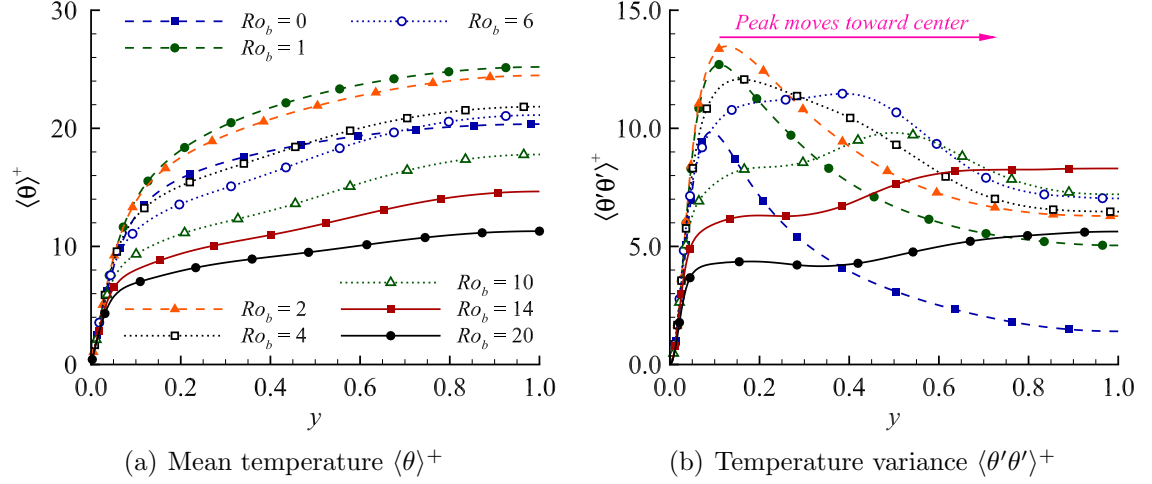
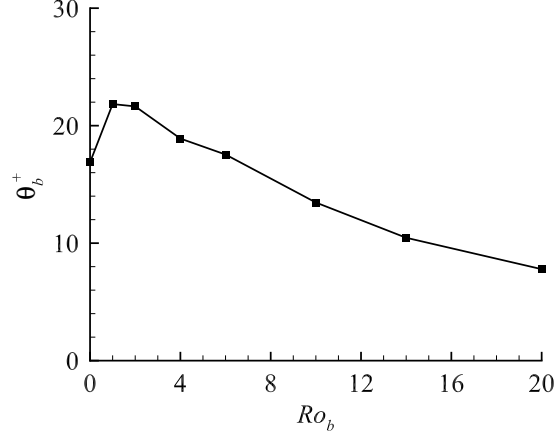


Figure 6.2: Profiles of mean temperature $\langle \theta \rangle^+$ and temperature variance $\langle \theta' \theta' \rangle^+$ at eight different rotation numbers for $Ro_b = 0-20$. All values are non-dimensionalised by T_τ .

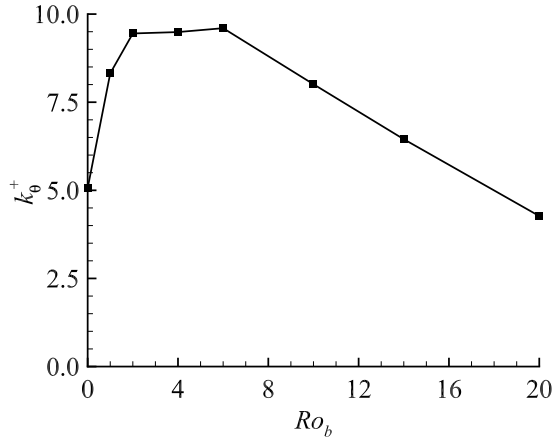
6.2 Results and discussion

Figure 6.2 compares the profiles of the mean temperature $\langle \theta \rangle^+$ and temperature variance $\langle \theta' \theta' \rangle^+$ with respect to the wall normal distance y at eight different rotation numbers. Clearly, the profiles of $\langle \theta \rangle^+$ and $\langle \theta' \theta' \rangle^+$ are axial-symmetric, varying non-monotonically with Ro_b . As shown in figure 6.2(a), the peak value of $\langle \theta \rangle^+$ augments by 23.77% at the pipe centre as the rotation number increases from $Ro_b = 0$ to 1. As Ro_b further increases to 20, the peak magnitude decreases considerably, which is 44.54% lower than that of $Ro_b = 0$. In figure 6.2(b), the profile of $\langle \theta' \theta' \rangle^+$ peaks at $y = 0.093$ for the non-rotating pipe flow. Owing to the system rotation imposed, the peak of $\langle \theta' \theta' \rangle^+$ increases at $y = 0.127$ at $Ro_b = 2$, and then reduces monotonically at the pipe centre at $Ro_b = 20$. It is evident that the peak position tends to shift toward the pipe centre when as the rotation number increases from $Ro_b = 0$ to 20. Meanwhile, the profile of $\langle \theta' \theta' \rangle^+$ turns into a single-peak pattern from a dual-peak pattern in the entire r -direction. Clearly, the system rotation behaves to suppress the TSE level in the near-wall region of the pipe.

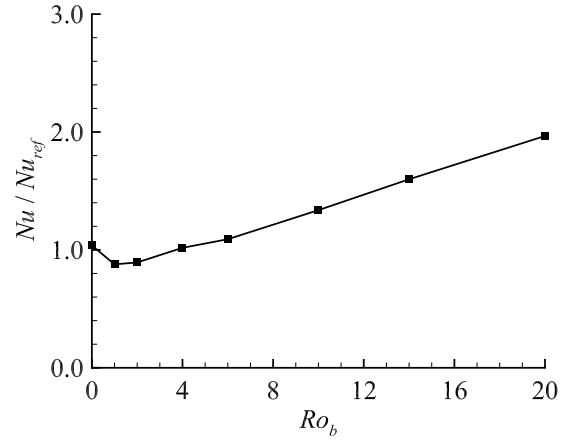
Figure 6.3 shows the bulk mean deficit temperature θ_b^+ , volumed-averaged TSE



(a) Bulk mean temperature θ_b^+



(b) Volume-averaged TSE k_θ^+



(c) Nusselt number Nu/Nu_{ref}

Figure 6.3: Profiles of bulk mean temperature θ_b^+ , volume-averaged TSE k_θ^+ and Nusselt number Nu/Nu_{ref} with respect to rotation number Ro_b . In panels (a,b), the values of θ_b^+ and k_θ^+ have been non-dimensionalised by T_τ . In panel (c), all values are further treated by dividing Nu_{ref} determined from the Gnielinski correlation.

k_θ^+ and Nusselt number Nu/Nu_{ref} as a function of rotation number Ro_b . Their definitions read as

$$\theta_b = \frac{2}{U_b R^2} \int_0^R \langle \theta \rangle \langle u_z \rangle r dr \quad , \quad (6.4)$$

$$k_\theta = \frac{2}{R^2} \int_0^R \langle \theta' \theta' \rangle r dr \quad , \quad (6.5)$$

$$Nu = \frac{hD}{K} = \frac{2R}{\theta_b} \left. \frac{\partial \langle \theta \rangle}{\partial r} \right|_{wall} \quad , \quad (6.6)$$

respectively. Here, h is the convective heat transfer coefficient, and K is the thermal conductivity of the fluid. In figure 6.3(a), the profile of θ_b^+ peaks at $Ro_b = 1$, with a magnitude that is 29.38% higher than that of $Ro_b = 0$. As Ro_b increases beyond 1, the value of θ_b^+ reduces dramatically. From figure 6.3(b), it is seen that the profile of k_θ^+ increases considerably with Ro_b and reaches its maximum at $Ro_b = 6$. As Ro_b continues to increase to 20, the value of k_θ^+ decreases by 15.62% in comparison with that of $Ro_b = 0$. It is clear from figure 6.3(b) that the mechanism of the axial system rotation is to inject TSE into the pipe flow if $Ro_b \leq 6$. However, once Ro_b is higher than 6, the system rotation tends to suppress the TSE level. In figure 6.3(c), the Nusselt number Nu is further treated by dividing Nu_{ref} determined from the Gnielinski correlation. The minimum value of Nu is observed at $Ro_b = 1$, which is 15.5% lower than that of $Ro_b = 0$. When Ro_b increases beyond 1, the value of Nu enhances dramatically, and reaches its maximum at $Ro_b = 20$, with a magnitude that is 89.4% higher than that of $Ro_b = 0$. From Figs. 6.3(a-c), it is evident that the axial system rotation tends to promote the thermal energy transport at high rotation numbers.

Figure 6.4 compares the turbulent heat fluxes $\langle u'_z \theta' \rangle^+$, $\langle u'_r \theta' \rangle^+$, and $\langle u'_\beta \theta' \rangle^+$ at eight rotation numbers tested. All profiles are axial-symmetric about the pipe centre ($y = 1.0$). In figure 6.4(a), the magnitude of $\langle u'_z \theta' \rangle^+$ varies non-monotonically in the near-wall region with Ro_b , and reaches its maximum at $y = 0.093$ at $Ro_b = 1$. As Ro_b further increases to 20, the system rotation tends to lessen the magnitude of $\langle u'_z \theta' \rangle^+$. Meanwhile, the profile of $\langle u'_z \theta' \rangle^+$ evolves from a dual-peak to a quadruple-peak pattern. From figure 6.4(b), the profile of $\langle u'_r \theta' \rangle^+$ is strictly linear and symmetric about the pipe centre at $Ro_b = 0$. However, as soon as the system rotation is imposed, the profile of $\langle u'_r \theta' \rangle^+$ deviates from this linear characteristic. The maximum value of $\langle u'_r \theta' \rangle^+$ is observed at $y = 0.187$ at $Ro_b = 6$. In figure 6.4(c), $\langle u'_\beta \theta' \rangle^+$ is trivial at $Ro_b = 0$, but its magnitude varies non-monotonically with Ro_b . At $Ro_b = 4$, the magnitude of $\langle u'_\beta \theta' \rangle^+$ reaches its maximum at $y = 0.058$. Moreover, the shape

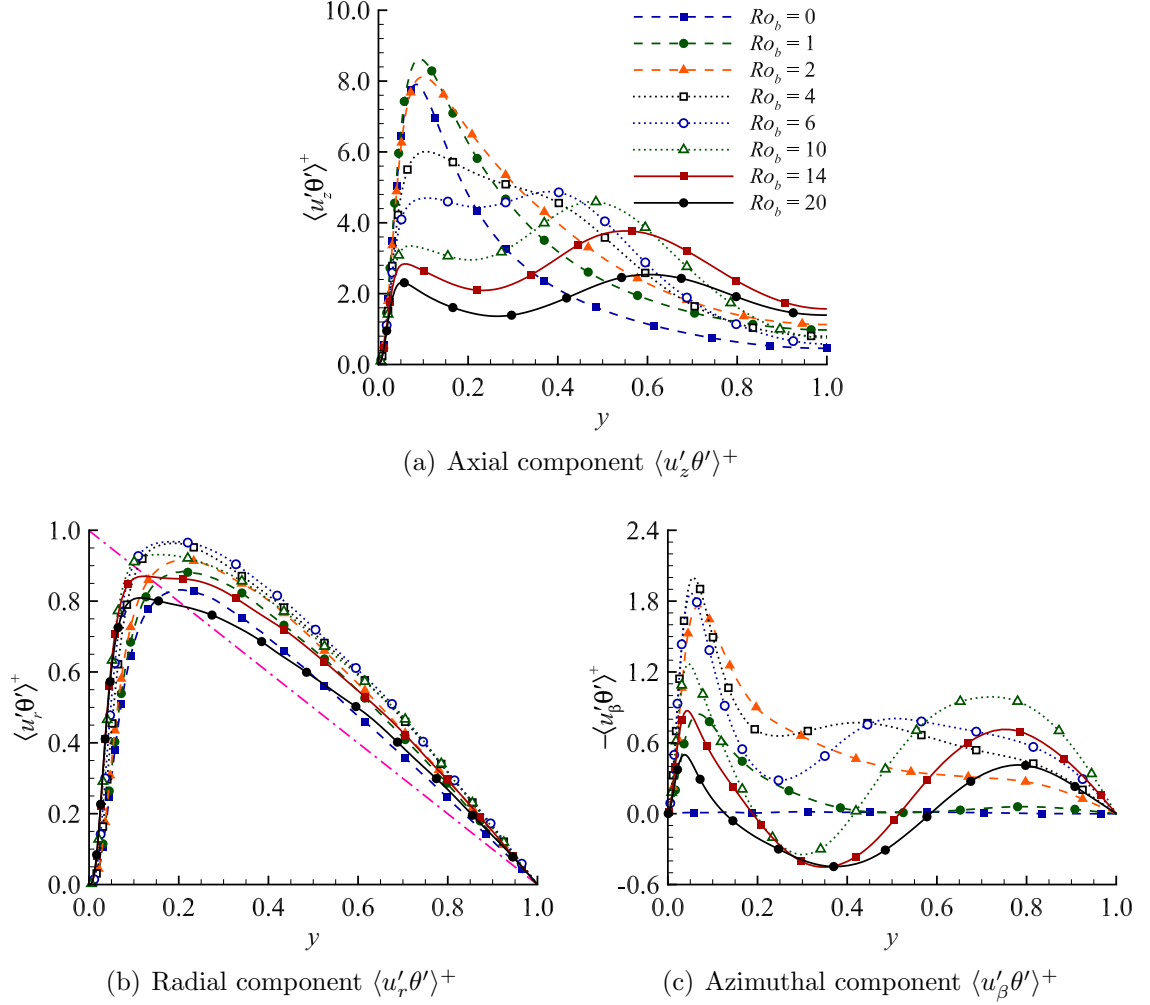


Figure 6.4: Profiles of turbulent heat fluxes $\langle u'_z \theta' \rangle^+$, $\langle u'_r \theta' \rangle^+$ and $\langle u'_\beta \theta' \rangle^+$ at eight different rotation numbers for $Ro_b = 0$ –20. All values are non-dimensionalised by u_τ and T_τ .

of $\langle u'_r \theta' \rangle^+$ remains stable and crosses zero five times over a diameter (in the entire r -direction) if $Ro_b \geq 10$.

By applying turbulence decomposition (i.e., $\theta = \langle \theta \rangle + \theta'$) to the thermal energy equation (6.3), the following radial heat flux decomposition can be obtained

$$q_r^{tot+} = q_r^{vis+} - \langle u'_r \theta' \rangle^+ = \frac{2}{1-y} \int_y^1 \frac{\langle u_z \rangle}{U_b} (y-1) dy \quad , \quad (6.7)$$

where q_r^{tot} and q_r^{vis} represent total heat flux and viscous heat flux (expressed as

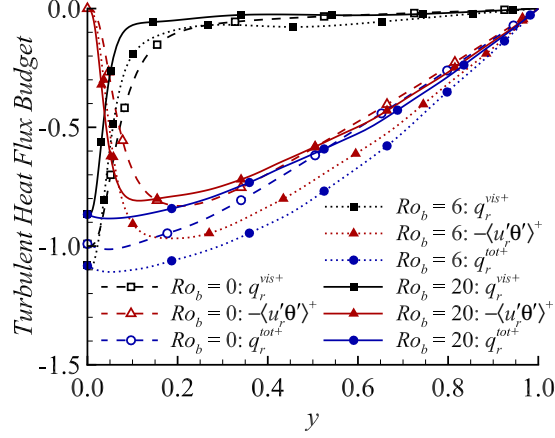


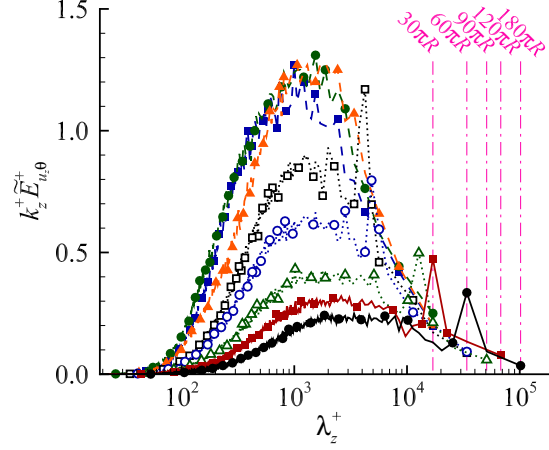
Figure 6.5: Budget profiles of radial heat flux q_r^+ at three rotation numbers $Ro_b = 0, 6$ and 20 . q_r^{vis+} and $-\langle u'_r \theta' \rangle^+$ represent viscous and turbulent heat fluxes, respectively. All radial heat fluxes are non-dimensionalised by u_τ and T_τ .

$\alpha \partial \langle \theta \rangle / \partial r$) of the radial direction, respectively. Figure 6.5 shows the decomposition of the radial heat flux at three different rotation numbers $Ro_b = 0, 6$ and 20 . Clearly, all three profiles of $-\langle u'_r \theta' \rangle^+$, q_r^{tot+} and q_r^{vis+} are sensitive to the rotation number, although the Coriolis force (as indicated by Ω) does not explicitly show in equation (6.7). The influence of the axial system rotation on the transport of thermal energy is through the convection of the fluid flow.

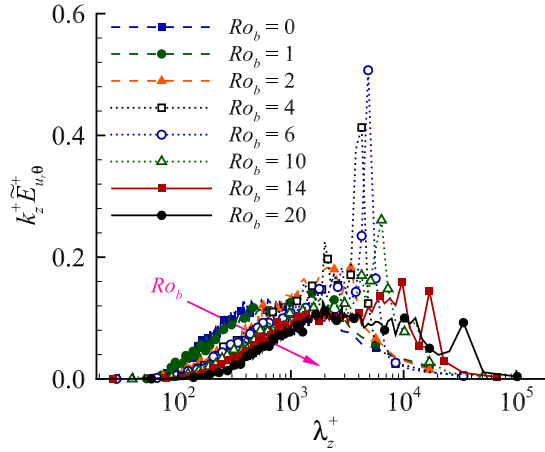
Figure 6.6 compares the premultiplied co-spectrum of axial, radial and azimuthal velocity-temperature fluctuations $k_z^+ \tilde{E}_{u_z \theta}^+$, $k_z^+ \tilde{E}_{u_r \theta}^+$ and $k_z^+ \tilde{E}_{u_\beta \theta}^+$ calculated along the streamwise direction at wall-normal positions $y = 0.093, 0.187$ and 0.058 , respectively. The 1-D axial co-spectrum is defined as

$$\tilde{E}_{u_i \theta}(k_z) = \text{Re} \left\{ \overline{\widehat{u'_i}^* \widehat{\theta'}} \right\} \quad , \quad (6.8)$$

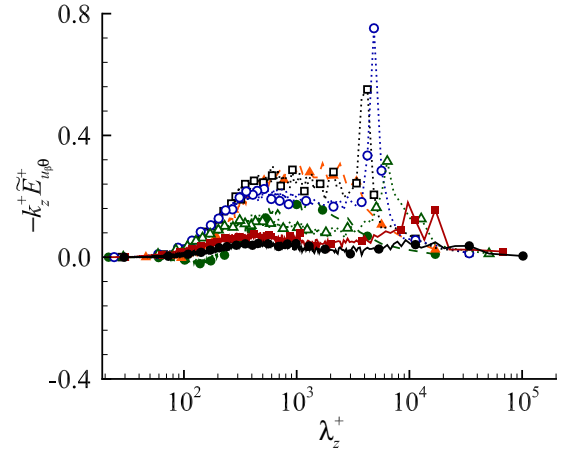
where an overbar represents temporal averaging, and operator $\text{Re}\{\cdot\}$ and superscript “*” denote the real part and conjugate of a complex number, respectively. A hat denotes Fourier transform in the axial direction of an arbitrary variable $\phi(r, \beta, k_z, t)$,



(a) Axial component at $y = 0.093$



(b) Radial component at $y = 0.187$



(c) Azimuthal component at $y = 0.058$

Figure 6.6: Profiles of premultiplied co-spectrum of axial, radial and azimuthal velocity-temperature fluctuations $k_z^+ \tilde{E}_{u_z\theta}^+$, $k_z^+ \tilde{E}_{u_r\theta}^+$ and $k_z^+ \tilde{E}_{u_\beta\theta}^+$ at eight different rotation numbers for $Ro_b = 0$ –20, calculated along the streamwise direction at wall-normal position $y = 0.093$, 0.187 and 0.058 , respectively.

i.e.

$$\hat{\phi}(r, \beta, k_z, t) = \frac{1}{L_z} \int_0^{L_z} \phi(r, \beta, z, t) e^{-\mathbf{i} k_z z} dz, \quad (6.9)$$

where $\mathbf{i} = \sqrt{-1}$ is the imaginary unit and $k_z = n_z k_{z0}$ is the axial wavenumber, with $n_z \in [-N_z/2, N_z/2 - 1]$ being an integer and $k_{z0} = 2\pi/L_z$ being the smallest positive wavenumber. The axial wavelength is defined as $\lambda_z = 2\pi/k_z$, non-dimensionalised as $\lambda_z^+ = \lambda_z u_\tau / \nu$.

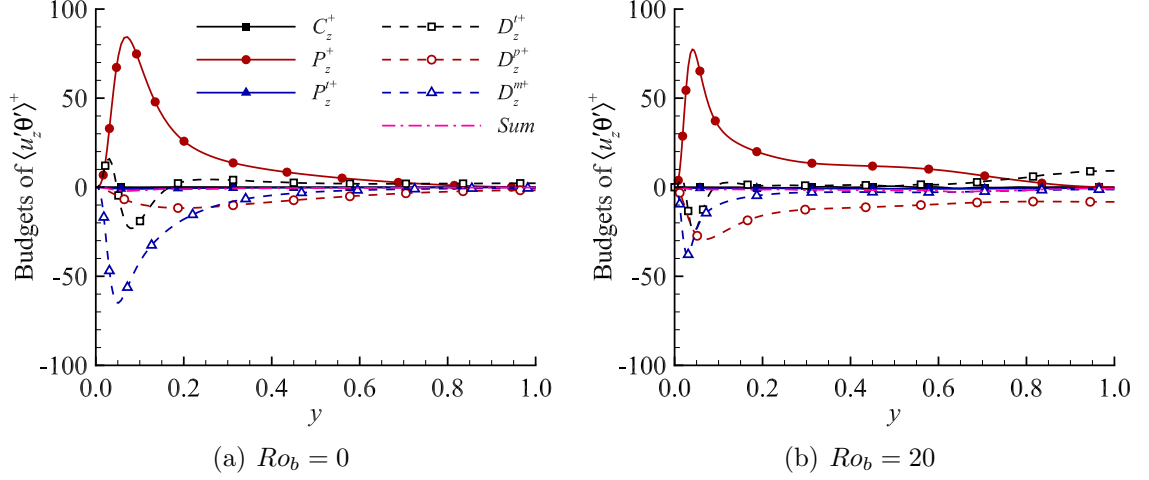


Figure 6.7: Budget balance of axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ of the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 20$) flows. All budget terms are non-dimensionalised by $u_\tau^2 T_\tau / R$.

It is clear from figure 6.6(a) that the mode of the dome peak of $k_z^+ \tilde{E}_{u_\beta \theta}^+$ (corresponding to the characteristic length scale of the most energetic eddies) is located within $\lambda_z^+ \in [450, 2400]$ in the near-wall region of the non-rotating pipe flow. It results from the hairpin structures of the boundary layer developing over the pipe wall. In response to the axial system rotation, the dome peak reaches its maximum at $Ro_b = 1$ within $\lambda_z^+ \in [610, 2400]$. As Ro_b increases from 1 to 20, the dome peak reduces dramatically, and its mode further moves toward the larger wavelengths. It is known that the axial system rotation induces counterclockwise-rotating secondary-flow structures in the pipe flows known as Taylor columns (at relatively high rotation numbers), which elongate in the axial direction and spin in the azimuthal direction. Consequently, additional peaks appear at the large wavelengths as Ro_b increases above 4. Clearly, from figure 6.6(a), the axial system rotation tends to weaken the TSE level at high rotation numbers. Similar observations also made in the profiles of $k_z^+ \tilde{E}_{u_r \theta}^+$ and $k_z^+ \tilde{E}_{u_\beta \theta}^+$ shown in Figs. 6.6(b,c), respectively. The additional peaks of $k_z^+ \tilde{E}_{u_r \theta}^+$ and $k_z^+ \tilde{E}_{u_\beta \theta}^+$ occur for $Ro_b \geq 4$. At very high rotation numbers, the magnitudes of $k_z^+ \tilde{E}_{u_r \theta}^+$ and $k_z^+ \tilde{E}_{u_\beta \theta}^+$ become substantially attenuated.

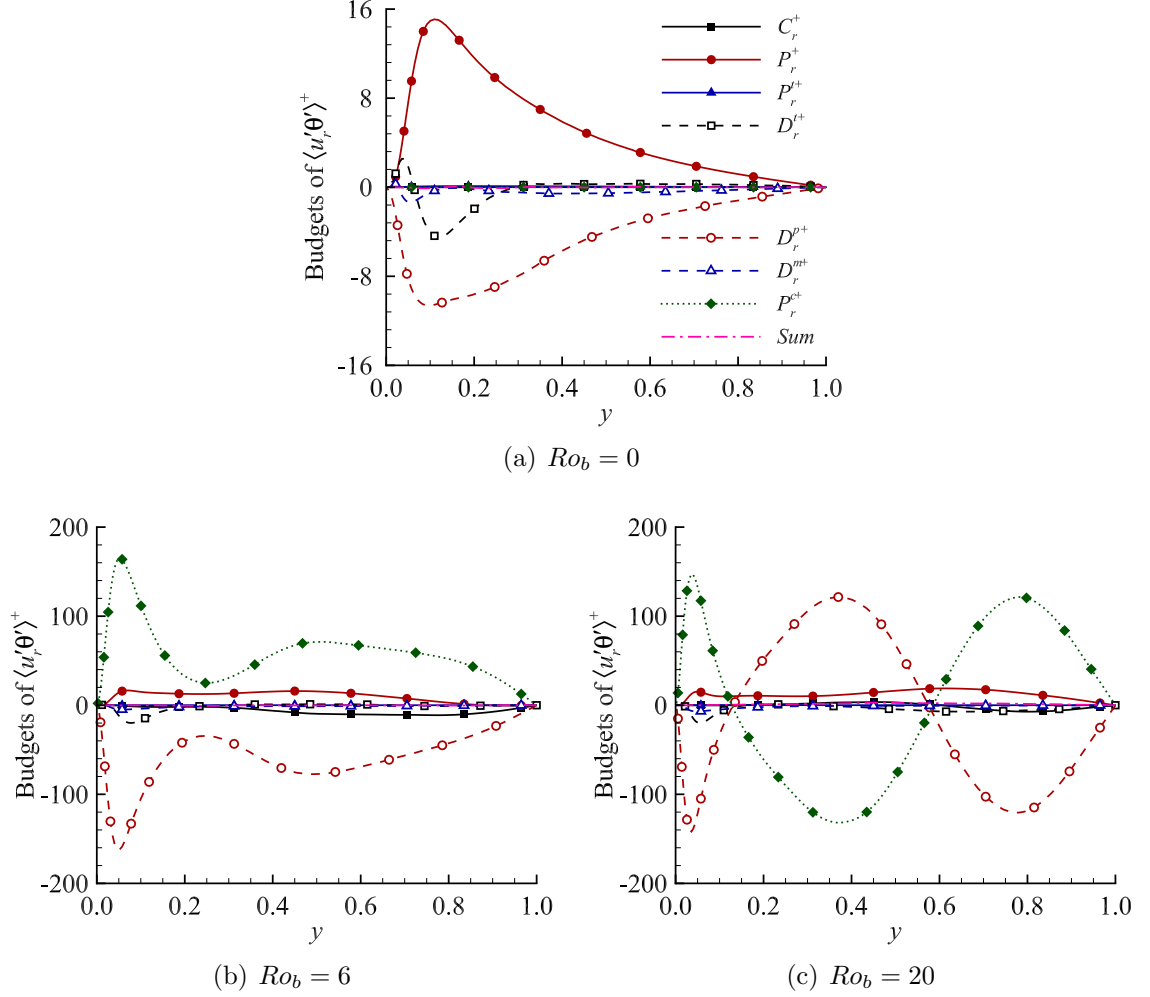


Figure 6.8: Budget balance of radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ of the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 6$ and 20) flows. All budget terms are non-dimensionalised by $u_\tau^2 T_\tau / R$.

To refine my investigation into the rotating impacts on the turbulent heat fluxes $\langle u'_i \theta' \rangle$, its transport equation can be further studied, viz.

$$\frac{\partial \langle u'_i \theta' \rangle}{\partial t} = C_i + P_i + P_i^t + D_i^t + D_i^p + D_i^m + P_i^c = 0 \quad , \quad (6.10)$$

where C_i , P_i , P_i^t , D_i^t , D_i^p , D_i^m and P_i^c represent the convection, turbulent production, turbulent production generated from the constant mean axial driving temperature gradient, turbulent diffusion, pressure diffusion-redistribution, molecular diffusion-

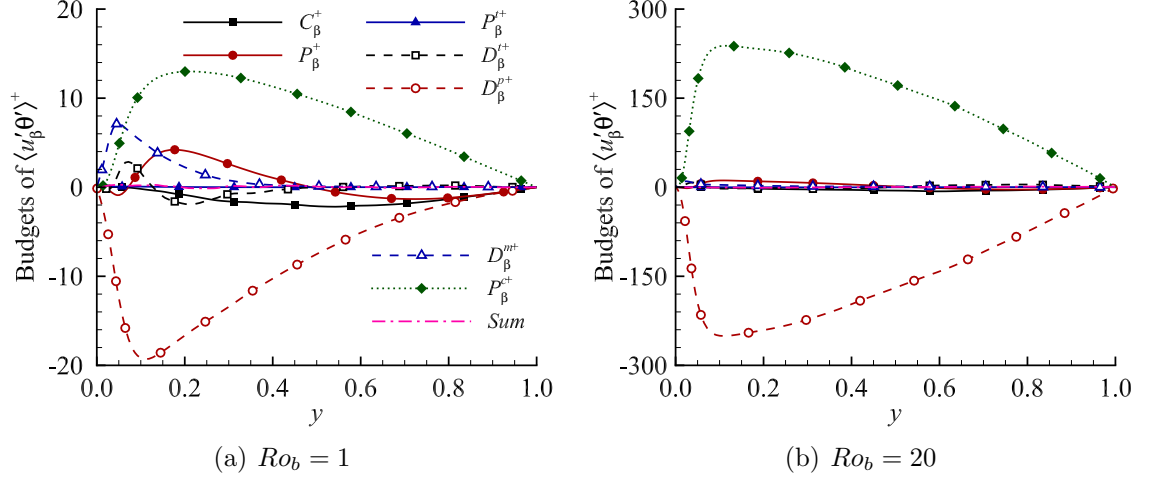


Figure 6.9: Budget balance of azimuthal turbulent heat flux $\langle u'_\beta \theta' \rangle^+$ for two flow cases ($Ro_b = 1$ and 20). All budget terms are non-dimensionalised by $u_\tau^2 T_\tau / R$.

dissipation and Coriolis production terms, respectively. The detailed definitions of these budget terms are provided in Appendix C. As is shown in figure 6.1, there are only two Coriolis force components in the radial and azimuthal directions. Therefore, only the transport equations of $\langle u'_r \theta' \rangle$ and $\langle u'_\beta \theta' \rangle$ have the Coriolis production terms, i.e., $P_r^c = 2\Omega \langle u'_\beta \theta' \rangle$ and $P_\beta^c = -2\Omega \langle u'_r \theta' \rangle$.

Figure 6.7 shows the budget balance of the axial turbulent flux $\langle u'_z \theta' \rangle^+$ of the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 2$) pipe flows. As is shown in figure 6.7(a), the budget balance of $\langle u'_z \theta' \rangle^+$ is mostly dominated by turbulent production P_z^+ and molecular diffusion-dissipation D_z^{m+} in the near-wall region of the non-rotating pipe flow. More specifically, P_z^+ and D_z^{m+} behave as source and sink terms, respectively. By contrast, at $Ro_b = 20$, the mechanisms of molecular diffusion-dissipation D_z^{m+} and pressure diffusion-redistribution D_z^{p+} are weakened and strengthened, respectively, by the axial system rotation. Consequently, D_z^{m+} and D_z^{p+} become comparable in magnitude and both function as sink terms in the near-wall region of the pipe.

Figure 6.8 compares the budget balance of the radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ at three different rotation numbers $Ro_b = 0, 6$ and 20. For the non-rotating pipe flow, turbulent production P_r^+ and pressure diffusion-redistribution D_r^{p+} dominate

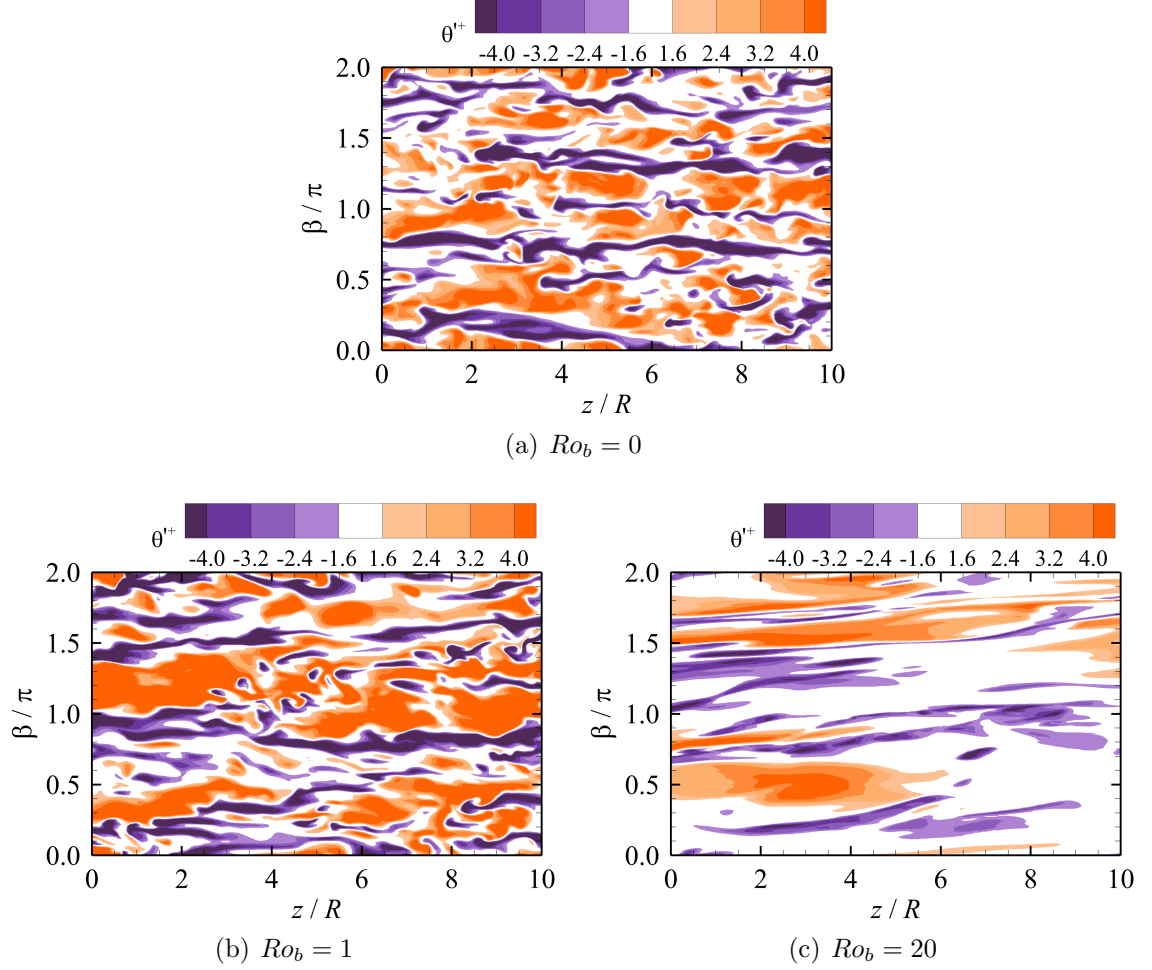


Figure 6.10: Contours of instantaneous temperature fluctuation θ'^+ in the β - z plane at wall normal position $y = 0.127$ for the non-rotating ($Ro_b = 0$) and rotating ($Ro_b = 1$ and 20) flows.

the budget balance of $\langle u'_z \theta' \rangle^+$ in the y -direction, acting as source and sink terms, respectively. However, as Ro_b increases to 6, Coriolis production P_r^{c+} acts as the most dominant source term (surpassing P_r^+) to balance with D_r^{p+} . As Ro_b further increases to 20, it is interesting to observe that the functions of P_r^{c+} and D_r^{p+} are reversed in region $y \in [0.127, 0.578]$. It is interesting to observe that at $Ro_b = 6$ and 20, P_r^{c+} and D_r^{p+} balance each other so well as if they are a mirror reflection of each other. It is clear that the magnitude of D_r^{p+} increases almost ninefold in the near-wall region of the pipe at $Ro_b = 6$ and 20 compared with those of the non-rotating pipe

flow in order to neutralise the significant impact of P_r^{c+} .

Figure 6.9 compares the budget balance of the azimuthal turbulent heat flux $\langle u'_\beta \theta' \rangle^+$ for two rotating flow cases ($Ro_b = 1$ and 20). At $Ro_b = 1$, the budget balance of $\langle u'_\beta \theta' \rangle^+$ is mostly dominated by Coriolis production P_β^{c+} and pressure diffusion-redistribution D_β^{p+} . As the rotation number increases to $Ro_b = 20$, the dominance of P_β^{c+} and D_β^{p+} is strongest. Their magnitudes increase almost fifteenfold in the near-wall region in comparison with those of the rotating pipe flow at $Ro_b = 1$.

Figure 6.10 compares streaky thermal structures in the β - z plane at $y = 0.127$ for $Ro_b = 0, 1$ and 20, in a domain that is arbitrarily selected with $z/R \in [0, 10]$. At $Ro_b = 0$, the thermal streaks shown by the positively- and negatively-valued θ'^+ alternate in the β -direction. As Ro_b increases to 6, the strengths of those streaks enhance in response to the axial system rotation. However, as Ro_b reaches 20, the positively-valued thermal streaks reduce significantly, while the negatively-valued thermal streaks stretch along the z -direction and tilt slightly along the β -direction, leading to an interruption of the formation of hairpin structures.

Chapter 7

Conclusions and future studies

This chapter integrates the contributions presented in the preceding chapters to demonstrate how they collectively fulfill the objectives of the thesis. It concludes with a summary of the principal findings and provides recommendations for future research.

7.1 Conclusions on DNS study of vortical structures and fluctuating enstrophy in a turbulent pipe flow under radial rotation

The effects of radial system rotation on vortical structures and fluctuating enstrophy in a turbulent circular pipe flow have been investigated using DNS. To systematically examine the vortex stretching mechanisms under the rotation, a total of eleven test cases were considered in the comparative study with rotation numbers ranging from $Ro_\tau = 0$ to 1, at a fixed Reynolds number of $Re_\tau = 180$ and a pipe length of $L_z = 20\pi R$. The pipe length applied in this study is among the longest in the literature on radially rotating pipe flows, which facilitates fully capturing energetic

streamwise turbulence structures in DNS. In response to the radial system rotation, a pair of large-scale mean secondary vortices emerge in the cross-stream plane, leading to tilted and elongated vortical structures near the suction wall of the pipe at low rotation numbers. It is observed that instantaneous vortices become the most intense at $Ro_\tau = 0.2$. Once Ro_τ increases beyond 0.2, this accumulation of vortical structures weakens due to the enhanced radial Coriolis force component F_r , which transports these structures from the suction side towards the pressure side of the pipe. At $Ro_\tau = 0.9$, both the vorticity intensity and TKE reduce dramatically, along with the destruction of the hairpin structures due to the suppression of near-wall ejection ($Q2$) and sweep ($Q4$) events. At $Ro_\tau = 1$, a second pair of mean secondary vortices appear near the lateral side wall of the pipe, and the flow field becomes fully laminarised.

In the non-rotating pipe flow, the RMS vorticity fluctuations $\omega'_{i,rms}{}^+$ and fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ are axisymmetric about the pipe centre. The small-scale dissipative eddies are primarily concentrated near the pipe wall, where $\langle \omega'_i \omega'_i \rangle^+$ attains its peak value. In response to radial rotation, all three RMS vortices $\omega'_{i,rms}{}^+$ reduce significantly on the suction side of the pipe. On the pressure side, the axial RMS vorticity $\omega'_{z,rms}{}^+$ also decreases monotonically with an increasing rotation number. As Ro_τ increases beyond 0.2, the magnitude of fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle^+$ decreases drastically on both the pressure and suction sides of the pipe, indicating that the radial system rotation tends to suppress the eddy motions and reduce the TKE dissipation rate by weakening the vortex stretching mechanisms. At $Ro_\tau = 1$, both $\omega'_{i,rms}{}^+$ and $\langle \omega'_i \omega'_i \rangle^+$ become negligibly small, indicating that the flow field has transitioned to a fully laminar state. Through an analysis of the premultiplied spectra of vorticity fluctuations $k_z^+ \tilde{E}_{\omega_i}^+$, it is seen that the radial system rotation systematically enlarges the characteristic length scales (or the modal wavelengths) of all three vorticity components in the streamwise direction near the pressure wall of the pipe. As the rotation number increases, the vorticity energy level as indicated by the premultiplied axial spectrum of fluctuating enstrophy $k_z^+ \tilde{E}_{\omega\omega}^+$ decreases monotonically at small wavelengths, while

exhibiting non-monotonic behaviour at larger wavelengths due to the vortex stretching under system rotation. At $Ro_\tau = 0.9$, the vortex filaments drop considerably in strength while their characteristic length scales become significantly enlarged (as indicated by the magnitude and modal wavelength of $k_z^+ \tilde{E}_{\omega\omega}^+$), reflecting the trend of strong laminarisation at a high rotation number. The effects of radial system rotation on fine turbulent vortical structures associated with vorticity fluctuations can be studied using the AIM of the vorticity anisotropy tensor b_{ij}^ω constructed based on the vorticity fluctuation covariance tensor $\langle \omega'_i \omega'_j \rangle$. The AIM near the suction wall shows an increasing tendency towards the two-component axisymmetric (2C-AXI) turbulence state as the rotation number increases. By contrast, the flow near the pressure wall increasingly favours the one-component (1C) turbulence state featuring enhanced axisymmetric expansion with an increasing value of Ro_τ . At the pipe centre, all invariant trajectories converge precisely to the isotropic (ISO) state.

This research highlights a detailed study of turbulent transport of fluctuating enstrophy in both physical and spectral spaces. To this purpose, both sets of transport equations (for $\langle \omega'_i \omega'_i \rangle$ and $\tilde{E}_{\omega\omega}$) are derived and assessed using DNS data from eleven test cases at different rotation numbers. Based on a budget analysis of the transport of $\langle \omega'_i \omega'_i \rangle^+$, all budget terms decrease monotonically near the suction wall as Ro_τ increases. By contrast, near the pressure wall, the budget terms vary non-monotonically and reach their maximum values at $Ro_\tau = 0.2$. At $Ro_\tau = 0.9$, the influences of the production term by turbulent vortex stretching $P_{\omega\omega}^{t+}$ and viscous diffusion term $D_{\omega\omega}^{v+}$ attenuate on the pressure side, while turbulent transport of $\langle \omega'_i \omega'_i \rangle^+$ becomes dominated by the two production terms $P_{\omega\omega}^{m+}$ and $P_{\omega\omega}^{s+}$ as the sources and by the viscous dissipation term $\varepsilon_{\omega\omega}^+$ as the sink. As the rotation number increases from $Ro_\tau = 0.2$ to 0.9, the magnitude of the Coriolis production term $P_{\omega\omega}^{c+}$ decreases and increases on the suction and pressure sides of the pipe, respectively. Overall, the magnitude of $P_{\omega\omega}^{c+}$ is much smaller than those of the other four production terms ($P_{\omega\omega}^{g+}$, $P_{\omega\omega}^{m+}$, $P_{\omega\omega}^{t+}$ and $P_{\omega\omega}^{s+}$), indicating that the effect of radial system rotation on the production of

fluctuating enstrophy is mostly achieved through mechanisms that alter the fluctuating vortex stretching vector, mean shear-flow velocity gradient, mean vorticity and its gradient, and coupling of vorticity-velocity and vorticity-vorticity fluctuations of the flow field.

There are nine budget terms in the transport equation of $\langle \omega'_i \omega'_i \rangle$ in physical space, but ten in its counterpart transport equation of $\tilde{E}_{\omega\omega}$ in spectral space. The additional budget term is $\tilde{T}_{\omega\omega}$, which is known as the interscale transport term appearing only in the transport equation of fluctuating enstrophy spectrum. The function of $\tilde{T}_{\omega\omega}$ is to facilitate the transfer of turbulence energy from large-scale to small-scale vortices, thereby sustaining the cascade of fluctuating enstrophy. An investigation into the effects of radial system rotation on the budget balance of $k_z^+ \tilde{E}_{\omega\omega}^+$ at $y^+ = 12.99$ and $\beta = \pi/2$ shows that $k_z^+ \tilde{P}_{\omega\omega}^{t+}$ and $k_z^+ \tilde{\varepsilon}_{\omega\omega}^+$ act as an energy source and sink of $k_z^+ \tilde{E}_{\omega\omega}^+$, respectively. Among the two premultiplied axial spectrum terms of production due to turbulent vortex stretching ($k_z^+ \tilde{P}_{\omega\omega}^{t+}$) and mean shear flow stretching ($k_z^+ \tilde{P}_{\omega\omega}^{s+}$), term $k_z^+ \tilde{P}_{\omega\omega}^{t+}$ mainly contributes to the budget balance of $k_z^+ \tilde{E}_{\omega\omega}^+$ at small wavelengths, whereas $k_z^+ \tilde{P}_{\omega\omega}^{s+}$ is more dominant at moderate wavelengths. At a high rotation number of $Ro_\tau = 0.9$, all budget terms decrease markedly in magnitude, and interestingly, the premultiplied axial spectrum term $k_z^+ \tilde{P}_{\omega\omega}^{m+}$ caused by mean vortex stretching becomes one of the leading budget terms, which acts as an energy source at small wavelengths but as an energy sink at large wavelengths. Meanwhile, consistent with the behaviour of its counterpart in physical space, the premultiplied spectrum of the Coriolis production term $k_z^+ \tilde{P}_{\omega\omega}^{c+}$ is also comparatively small.

7.2 Conclusions on DNS of turbulent heat transfer in a radially rotating circular pipe flow

Turbulent heat transfer confined within a circular pipe flow subjected to radial system rotation has been investigated systematically using DNS over a wide range of rotation numbers varying from $Ro_\tau = 0$ to 1. To ensure that the computational domain is sufficient for capturing dominant TSE-containing eddy motions, a comparative study of five pipe lengths of $L_z = 2\pi R$ – $20\pi R$ has been conducted. It is observed that a minimum pipe length of $L_z = 12\pi R$ is generally required for conducting DNS of radially rotating pipe flows. However, if the criterion is to capture all energetic eddy motions that are capable of contributing to TSE at the level higher than 3/8-th of $\max(k_z^+ \tilde{E}_{\theta\theta}^+)$, the pipe length must be kept at $L_z = 20\pi R$. With this conclusion, 11 test cases have been considered in my comparative study of the effects of radial rotation on turbulent heat convection based on a fixed pipe length of $L_z = 20\pi R$, which is the largest in the current literature on numerical study of radially rotating pipe flows.

In response to the radial system rotation imposed, mean secondary flows appear in the cross-stream direction of the pipe as a pair of large-scale counter-rotating vortices under the influence of the radial component of the Coriolis force, breaking the characteristic concentric pattern of the mean temperature $\langle\theta\rangle^+$ field of the non-rotating pipe flow. As the rotation number increases from $Ro_\tau = 0$ to 1, the high- and low-temperature flow structures progressively dominate the near-wall regions of the pressure and suction sides of the pipe, respectively. As such, the magnitude of $\langle\theta\rangle^+$ becomes suppressed monotonically near the pipe centre. Meanwhile, the distribution of $\langle\theta\rangle^+$ becomes increasingly asymmetrical, with its peak shifted towards the pressure side of the pipe. It is observed that both Nusselt number Nu and wall heat flux q_w'' increase on the pressure side but decrease on the suction of the pipe. This indicates that the radial system rotation acts to enhance the convective heat transfer rate on

the pressure side while reducing it on the suction side of the pipe. As the rotation number increases, the strength of the mean secondary flows increases, which helps to expand the enhanced heat transfer zone from the pressure side to the suction side of the pipe.

For the non-rotating pipe flow, the profile of radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ exhibits a strict linear distribution along the radial direction while that of axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ is strictly axially-symmetrical about the axial centreline of the pipe. As soon as the radial system rotation is imposed, these physical features of turbulent heat fluxes break down. As the rotation number Ro_τ increases, the magnitude of $\langle u'_r \theta' \rangle^+$ decreases monotonically on the pressure side, with its zero-crossing point moving from the pipe centre progressively towards the pressure side of the pipe. By contrast, the TKE $\frac{1}{2} \langle u'_i u'_i \rangle^+$, temperature variance $\langle \theta' \theta' \rangle^+$ and axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ decrease monotonically on the suction side but vary non-monotonically on the pressure side of the pipe, reaching their maxima at radial position $r/R = 0.921$ (or $y^+ = 14.24$) at rotation number $Ro_\tau = 0.2$.

In spectral space, the TSE level (indicated by premultiplied co-spectra of temperature and velocity fluctuations, i.e., $k_z^+ \tilde{E}_{u_z\theta}^+$ and $k_z^+ \tilde{E}_{u_r\theta}^+$) becomes intensified at large wavelengths at $r/R = 0.921$ (or $y^+ = 14.24$) as the rotation number increases from $Ro_\tau = 0$ to 0.2. As Ro_τ continues to increase beyond 0.2, these two premultiplied co-spectra decrease rapidly. Eventually, at the highest rotation number $Ro_\tau = 1$, the magnitudes of $k_z^+ \tilde{E}_{u_z\theta}^+$ and $k_z^+ \tilde{E}_{u_r\theta}^+$ both become trivial as the flow has been completely laminarised.

In the central vertical plane, the transport of axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ is primarily dominated by turbulent production P_z^+ , turbulent diffusion D_z^{t+} and molecular diffusion-dissipation D_z^{m+} in the near-wall regions of the non-rotating pipe flow. However, as soon as the radial system rotation is imposed, all budget profiles transition from being perfectly symmetrical about the pipe centreline to increasingly asymmetrical along the radial direction. These three dominant budgets reach their

maxima near the pressure side of the pipe at $Ro_\tau = 0.2$. The budget balance of radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ is predominantly balanced by turbulent production P_r^+ and pressure diffusion-redistribution D_r^{p+} in the near-wall region of the non-rotating pipe flow, with their corresponding profiles intersecting at the pipe centre. However, as the rotation number rises, this intersection point shifts from the pipe centre towards the pressure side. It is observed that the Coriolis production term P_z^{c+} functions as a source of TSE to empower the axial turbulent heat flux $\langle u'_z \theta' \rangle^+$ near the suction side but acts as a sink of TSE for $\langle u'_z \theta' \rangle^+$ near the pressure side of the pipe. Although the Coriolis production term P_z^{c+} strengthens with an increasing rotation number, its magnitude remains small in the central vertical plane. By contrast, the contribution from the Coriolis production term P_r^{c+} to the radial turbulent heat flux $\langle u'_r \theta' \rangle^+$ becomes increasingly important as the rotation number rises, acting as a thermal energy source for $\langle u'_r \theta' \rangle^+$.

In order to investigate the effects of radial system rotation on turbulent thermal-flow structures, the JPDF and LSE analyses of the velocity and temperature fluctuations have been conducted on the pressure side of the pipe. It is concluded that the radial system rotation consistently suppresses the cold ejection events, resulting in higher probabilities with lower magnitudes in the JPDF calculations. It is interesting to observe that although the hot sweep events are significantly suppressed in large magnitudes with low probabilities near the pressure side of the pipe at relatively high rotation numbers, the maximum value of positively-valued LSE of temperature θ_{LSE} at $Ro_\tau = 0.2$ in the cross-stream direction suggests that the secondary flows induced by the radial system rotation are capable of enhancing the hot sweep events and result in a higher local TSE level in turbulent heat transfer at a relatively low rotation number. However, once the rotation number increases beyond $Ro_\tau = 0.8$, both hot sweep events and magnitude of positively-valued θ_{LSE} reduce dramatically to reflect the trend of laminarisation.

7.3 Conclusions on DNS of turbulent flow and structures in a circular pipe subjected to axial system rotation

Turbulent flow confined within a circular pipe subjected to axial system rotation has been studied using DNS for a wide range of rotation numbers varying from $Ro_b = 0$ to 20. To ensure that energetic turbulent eddy motions in form of hairpins and Taylor columns are reasonably captured at different rotation numbers, pipes of length up to $L_z = 180\pi R$ are used in DNS.

The bulk mean velocity U_b^+ increases as soon as the axial system rotation is imposed, and reaches its maximum at $Ro_b = 2$. However, at higher rotation numbers for $Ro_b \geq 6$, the value of U_b^+ decreases significantly. It is interesting to observe that the peak position of the mean swirl (azimuthal) velocity $r\langle u_\beta \rangle^+/R$ is independent of Ro_b , which occurs at $y \approx 0.4$ where the mean axial vorticity $\langle \omega_z \rangle^+$ is zero identically. For a non-rotating pipe flow, $\langle u'_r u'_\beta \rangle^+ \equiv 0$ and $\langle u'_\beta u'_z \rangle^+ \equiv 0$. By contrast, in response to the axial system rotation imposed, all six Reynolds stress components are non-trivial. Under an axial system rotation, secondary flows are induced, which tend to suppress $\langle u'_z u'_z \rangle^+$ in the near-wall region but enhance the magnitudes of $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$ in most pipe regions. This leads to a significant increase in the TKE level as the rotation number is increased. However, as the rotation number further increases from $Ro_b = 14$ to 20, the TKE value becomes stable. In the context of an axially rotating pipe flow, the total shear stress τ_{rz}^{tot+} in the r - z plane exhibits a linear profile in the radial direction. By contrast, it is proven that the total shear stress $\tau_{r\beta}^{tot+}$ in the r - β plane is zero identically such that the profiles of $\tau_{r\beta}^{vis}$ and $-\langle u'_r u'_\beta \rangle$ are mirror reflections of each other.

As the rotation number is increased, the profile of flatness factor $F(u'_z)$ deviates apparently from the ideal Gaussian distribution showing an enhanced level of tur-

bulence intermittency. It is observed that the azimuthal skewness factor generally follows the ideal Gaussian distribution of $S(u'_\beta) = 0$ in the non-rotating pipe flow. However, as soon as the system rotation is imposed, this behaviour of $S(u'_\beta)$ is broken, exhibiting a complex profile pattern in the radial direction. Furthermore, based on the analysis of the signs of $S(u'_z)$ and $S(-u'_r)$, it is observed that the axial system rotation tends to impede the formation of hairpin structures by suppressing the sweep events.

There are two types of energetic axial flow structures in an axially rotating pipe flow. The first type is hairpin structures (associated with streaks in the near-wall region), typical of a turbulent boundary layer over a solid wall. The second type is the Taylor columns, which spin in the azimuthal direction and elongate in the axial direction at moderate and high rotation numbers. The streaks and Taylor columns are distinctively different in terms of their characteristic wavelengths corresponding to the peaks (*I* for hairpin and streaks, and peaks *II* and *III* for Taylor columns) of the premultiplied spectra $k_z \tilde{E}_{ii}$. It is interesting to observe that Taylor columns feature one single dominant peak *II* at moderate rotation numbers, but two dominant peaks *II* and *III* at high rotation numbers. It is also observed that as the rotation number is increased, Taylor columns become increasingly populated in the central regions of the pipe, and furthermore, TKE held by Taylor columns enhances rapidly associated with significant increases in their axial length scales. As such, both the magnitude and characteristic wavelength of peak *III* increases drastically as the rotation number is increased. In view of this, a rather long pipe is needed for performing DNS of an axially rotating pipe flow at high rotation numbers in order to capture Taylor columns. Through a complementary comparative study of test cases of different pipe lengths, it is observed that use of short pipes can lead to physically inaccurate flow statistics and velocity spectra at high rotation numbers.

Although a very long pipe of $L_z = 180\pi R$ has been used for conducting DNS at the highest rotation number $Ro_b = 20$ in this study, a longer pipe would be

needed if a higher rotation number was further pursued. It is also recognised that my comparative study of eight rotation numbers has been conducted based on a fixed low Reynolds number. The combined effects of both Reynolds and rotation numbers on axially rotating turbulent pipe flow and structures are still unknown, which need to be explored in future studies.

7.4 Conclusions on DNS of axial rotation effects on Reynolds stress and TKE transport in turbulent pipe flow

The transport processes of the Reynolds stresses and TKE in turbulent pipe flow subjected to axial system rotation have been studied in both physical and spectral spaces using DNS for a wide range of rotation numbers of $Ro_b = 0$ –20. Through an analysis of premultiplied velocity spectra, two types of turbulent flow structures have been identified. The first type is streaks (or hairpin structures), which develop in the turbulent boundary layer over the pipe wall, irrespective of system rotation. The second type consists of Taylor columns, induced by the axial rotation at moderate and high rotation numbers.

In physical space, as the rotation number increases to $Ro_b = 20$, the budget profiles of the axial Reynolds normal stress $\langle u'_z u'_z \rangle^+$ remain comparable to those of the non-rotating case. The main distinction is that the pressure-strain term Π_{zz}^+ becomes a more prominent energy sink in the near-wall region. By contrast, the transports of the radial and azimuthal Reynolds normal stresses, $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$, are predominantly governed by the pressure-strain term (Π_{rr}^+ and $\Pi_{\beta\beta}^+$) and Coriolis production term (P_{rr}^{c+} and $P_{\beta\beta}^{c+}$). The magnitudes of these major contributions increase drastically, leading to significantly intensified transport activity in both $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$, especially in the near-wall region. Meanwhile, P_{rr}^{c+} and $P_{\beta\beta}^{c+}$ exhibit opposite

trends, indicating a transfer of turbulence energy between $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$. As a result, compared with the non-rotating pipe flow, the axial system enhances the TKE level primarily through $\langle u'_r u'_r \rangle^+$ and $\langle u'_\beta u'_\beta \rangle^+$, rather than through $\langle u'_z u'_z \rangle^+$. Moreover, at $Ro_b = 20$, the near-wall TKE transport is dominated by the viscous diffusion term ε_{TKE}^+ and viscous dissipation term D_{TKE}^{v+} , both of which increases significantly in magnitude under the axial system rotation.

For the three Reynolds shear stresses, the budget balances of the radial-axial and radial-azimuthal components, $\langle u'_r u'_z \rangle^+$ and $\langle u'_r u'_\beta \rangle^+$, are primarily influenced by the pressure-strain term (Π_{rz}^+ and $\Pi_{r\beta}^+$), the pressure diffusion term (D_{rz}^{v+} and $D_{r\beta}^{v+}$) and the Coriolis production term (P_{rz}^{c+} and $P_{r\beta}^{c+}$) at $Ro_b = 20$. These major contributions act as either energy sources or sinks depending on the wall-normal location, with their roles changing as the coordinate moves from the pipe wall towards the pipe centre. However, at the same rotation number, the transport of the azimuthal-axial Reynolds shear stress $\langle u'_\beta u'_z \rangle^+$ is dominated only by the pressure-strain term $\Pi_{\beta z}^+$ and Coriolis production term $P_{\beta z}^{c+}$, whose profiles appear as near-mirror reflections of one another. Furthermore, it is seen that the dominant budget terms of all three Reynolds shear stresses increase drastically in magnitude at $Ro_b = 20$ compared with those in the non-rotating or low-rotating cases.

The transports of the velocity-spectrum tensor and the TKE spectrum are analysed at $y = 0.093$ (or $y^+ = 16.7$) in the near-wall region of the pipe, because this wall-normal location corresponds to the peak of the premultiplied TKE spectrum $k_z^+ \tilde{E}_{TKE}^+$ in the non-rotating pipe flow. It is seen that as the rotation number increases to $Ro_b = 20$, the budget terms of the six premultiplied velocity-spectrum tensor components and the premultiplied TKE spectrum consistently have similar dominance to those of the corresponding Reynolds stresses and TKE in physical space. The interscale transport term facilitates energy redistribution across different wavelength scales in both the non-rotating and rotating pipe flows.

For the premultiplied axial autospectrum $k_z^+ \tilde{E}_{zz}^+$ and premultiplied TKE spectrum

$k_z^+ \tilde{E}_{TKE}^+$, their transports are primarily governed by the turbulent production term ($k_z^+ \tilde{P}_{zz}^+$ and $k_z^+ \tilde{P}_{TKE}^+$), interscale transport term ($k_z^+ \tilde{T}_{zz}^+$ and $k_z^+ \tilde{T}_{TKE}^+$) and pressure-strain term ($k_z^+ \tilde{\Pi}_{zz}^+$ and $k_z^+ \tilde{\Pi}_{TKE}^+$) across all wavelengths in both the non-rotating and high rotating cases. In contrast, the budget balances of the premultiplied radial and azimuthal autospectra, $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$, at $Ro_b = 20$ are dominated by the pressure-strain term ($k_z^+ \tilde{\Pi}_{rr}^+$ and $k_z^+ \tilde{\Pi}_{\beta\beta}^+$) and Coriolis production term ($k_z^+ \tilde{P}_{rr}^{c+}$ and $k_z^+ \tilde{P}_{\beta\beta}^{c+}$). Similarly, the profiles of these two Coriolis production terms $k_z^+ \tilde{P}_{rr}^{c+}$ and $k_z^+ \tilde{P}_{\beta\beta}^{c+}$ also exhibit a nearly mirror-symmetric behaviour. Under strong axial system rotation, the Taylor column structures complicate the roles of $k_z^+ \tilde{P}_{rr}^{c+}$ and $k_z^+ \tilde{P}_{\beta\beta}^{c+}$, resulting in these two budgets acting simultaneously as sources and sinks of turbulence energy across various scales. Compared with the non-rotating case, the dominant budgets in $k_z^+ \tilde{E}_{rr}^+$ and $k_z^+ \tilde{E}_{\beta\beta}^+$ are considerably amplified at $Ro_b = 20$, whereas those associated with $k_z^+ \tilde{E}_{zz}^+$ and $k_z^+ \tilde{E}_{TKE}^+$ are reduced. In addition, a comparison of the extremal values of these major budgets across different wavelengths at $Ro_b = 20$ indicates that, although the Taylor column structures influence these spectral transports at the large scales, the near-wall streaks exert a stronger control over the transfer of turbulence energy at shorter wavelength scales.

For the three premultiplied co-spectra of the velocity fluctuations, their leading budget terms increase significantly at $Ro_b = 20$ compared with those in the non-rotating pipe flow. The effect of the interscale transport terms weakens drastically with an increasing rotation number. At $Ro_b = 20$, the transports of the premultiplied radial-axial and radial-azimuthal co-spectra, $k_z^+ \tilde{E}_{rz}^+$ and $k_z^+ \tilde{E}_{r\beta}^+$, are predominantly influenced by the pressure-strain term ($k_z^+ \tilde{\Pi}_{rz}^+$ and $k_z^+ \tilde{\Pi}_{r\beta}^+$), pressure diffusion term ($k_z^+ \tilde{D}_{rz}^{p+}$ and $k_z^+ \tilde{D}_{r\beta}^{p+}$) and Coriolis production term ($k_z^+ \tilde{P}_{rz}^{c+}$ and $k_z^+ \tilde{P}_{r\beta}^{c+}$). As Ro_b increases, the mechanisms associated with these major budget become increasingly complex across the entire wavelength range. In contrast, the budget balance of the premultiplied azimuthal-axial co-spectrum $k_z^+ \tilde{E}_{\beta z}^+$ at $Ro_b = 20$ is dominated solely by the pressure-strain term $k_z^+ \tilde{\Pi}_{\beta z}^+$ and Coriolis production term $k_z^+ \tilde{P}_{\beta z}^{c+}$. Moreover,

unlike the transport of $k_z^+ \tilde{E}_{\beta z}^+$, which is strongly influenced by the near-wall streaks at the intermediate and large scales, the Taylor columns exert a more pronounced influence on the transport processes of $k_z^+ \tilde{E}_{rz}^+$ and $k_z^+ \tilde{E}_{r\beta}^+$ at the very large scales.

7.5 Conclusions on DNS of turbulent heat transfer in a circular pipe subjected to axial system rotation

Turbulent heat transfer confined within a circular pipe flow subjected to axial rotation has been studied using DNS. In order to thoroughly investigate the Coriolis force impacts on the temperature field, a wide range of rotation numbers varying from $Ro_b = 0$ to 20 have been examined. In response to the axial system rotation, a maximum of $\langle \theta \rangle^+$ is observed at $Ro_b = 1$. As Ro_b further increases, the system rotation tends to reduce $\langle \theta \rangle^+$ in magnitude. Furthermore, the magnitudes of both $\langle \theta' \theta' \rangle^+$ and $\langle u'_z \theta' \rangle^+$ decrease while the Nusselt number increases with an increasing value of Ro_b , indicating that the system rotation functions to suppress the TSE level of the flow.

For the premultiplied turbulent thermal spectra, the dome peak occurs due to the hairpin structures of the boundary layer developing over the pipe wall. The magnitude of the premultiplied co-spectrum of axial velocity-temperature fluctuations $k_z^+ \tilde{E}_{u_z \theta}^+$ reaches its maximum at $Ro_b = 1$. As Ro_b continues to increase, its peak value reduces dramatically and its mode moves toward larger wavelengths. It is interesting that additional peaks occur when $Ro_b \geq 4$, which is created by the Taylor columns associated with the axial system rotation imposed.

In the budget balance of turbulent heat fluxes of the axially rotating pipe flows, the Coriolis force (as indicated by Ω_z) does not directly influence $\langle u'_z \theta' \rangle^+$. For both $\langle u'_r \theta' \rangle^+$ and $\langle u'_z \beta' \rangle^+$, the most dominant terms are the Coriolis production (as a

source exceeding turbulent production) and pressure-diffusion redistribution (as a sink). These terms in $\langle u'_r \theta' \rangle^+$ and $\langle u'_\beta \theta' \rangle^+$ balance each other so well as if they are a mirror reflection of each other.

It is interesting to observe that in response to the system rotation imposed, the near-wall thermal streaky structures extend in the z -direction and tilt in the β -direction, resulting in a restriction on the formation of the hairpin structures.

7.6 Future studies

To continue this current study, the following research topics suggested:

(1) Turbulent transport of velocity-temperature co-spectra and TSE spectrum in axially rotating pipe flows

This proposed study will focus on spectral analyses of the turbulent heat flux and temperature variance (or TSE) using the DNS database of axially rotating thermal pipe flows for rotation numbers ranging from $Ro_b = 0$ to 20. Given the complexity of the equations and advanced turbulent transport analysis in the special space, no such analysis is available in the current literature. In view of this knowledge gap, the proposed research aims to

- (i) examine how turbulence influences heat transfer across different wavelength scales in the non-rotating pipe flow, and
- (ii) to understand how axial system rotation affects turbulent transport of velocity-temperature co-spectra and the TSE spectrum, which are seen as the spectral counterparts of the turbulent heat flux and temperature variance in physical space.

(2) Turbulent flow and heat transfer in stationary and radially rotating circular pipes roughened with surface grooves

This thesis has focused on studying turbulent flow and heat transfer in a smooth circular pipe subjected to system rotation. In comparison with a smooth pipe, grooved

pipes can significantly alter turbulent flow structure and heat convection, which can potentially improve the thermo-hydraulic performance with potential applications to centrifugal rotatory machinery. Thus far, a comprehensive DNS study of the grooved pipe flow and heat transfer in either stationary and radially rotating pipes is still absent in the current literature. To achieve the research goal, the following sub-projects can be considered:

- (i) DNS study of turbulent flow and heat convection in stationary grooved pipes,
- (ii) DNS study turbulent flow and heat convection in grooved pipes subjected to radial system rotation, in comparison with stationary grooved pipe flows.
- (iii) To advance the knowledge of turbulent heat and fluid flow in stationary and radially rotating pipes, spectral analysis should be also further conducted to understand the turbulent transport processes of momentum and thermal energy in the Fourier space.

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Appendix A

Spectral-element method

In this thesis, all direct numerical simulations (DNS) are conducted using a spectral-element method (SEM) code “Semtex” primarily written with C++ and FORTRAN programming languages. The SEM code is based on a high-order finite element technique and is capable of offering geometric flexibility of finite elements while maintaining a high spectral accuracy. Prior to the use of the code, the underlying algorithm needs to be studied. After getting familiar with Semtex and its mesh requirements, I developed a quadrilateral-structural grid generator with MATLAB using Transfinite Interpolation method in order to deal with the circular pipe geometry. In simulations, the function of Semtex is to conduct the instantaneous files containing instantaneous flow field in a binary form. Therefore, to ensure that the instantaneous flow field data can be used in post-processing, several standalone post-processing codes have been developed using the C++ and Linux scripting programming languages to calculate turbulence statistics of instantaneous flows, including the mean velocity and temperature fields, turbulence intensities (RMS), two point autocorrelation coefficients, Reynolds stresses, turbulent heat flux, premultiplied energy spectra, and swirling strength criterion for flow structure visualisations. Furthermore, the budget terms of the transport equations in both physical and spectral space are determined for the fluctuating enstrophy, Reynolds stresses, TKE, and turbulent heat flux. This

appendix describes the temporal and spatial discretisation in details for Semtex code.

A.1 Time-splitting Algorithm

The governing equations containing the continuity and incompressible Navier-Stokes equations can be written as

$$\nabla \cdot \mathbf{u} = 0 \quad , \quad (\text{A.1})$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{N}(\mathbf{u}) = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f} \quad , \quad (\text{A.2})$$

where $\mathbf{N}(\mathbf{u})$ is the nonlinear convective term and can be represented in several different forms as follows

$$\mathbf{N}(\mathbf{u}) = \mathbf{u} \cdot \nabla \mathbf{u} \quad , \quad (\text{A.3})$$

$$\mathbf{N}(\mathbf{u}) = \nabla \cdot (\mathbf{u}\mathbf{u}) \quad , \quad (\text{A.4})$$

$$\mathbf{N}(\mathbf{u}) = \frac{1}{2} [\mathbf{u} \cdot \nabla \mathbf{u} + \nabla \cdot (\mathbf{u}\mathbf{u})] \quad , \quad (\text{A.5})$$

$$\mathbf{N}(\mathbf{u}) = (\nabla \times \mathbf{u}) \times \mathbf{u} + \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u}) \quad , \quad (\text{A.6})$$

which in order are the convective, divergence, skew-symmetric and rotational forms. Although those forms of $\mathbf{N}(\mathbf{u})$ are identical in the continuous mathematics, they possess significantly different properties after discretisation. For a low-order spatial scheme, such as the finite volume and finite difference methods, the divergence form is widely used to satisfy the momentum conservation. For a spectral accurate spatial scheme, all four schemes can conduct almost identical results after employing the de-aliasing technique. Nevertheless, de-aliasing can be expensive to implement and is commonly deactivated for the fine enough grid resolution. In comparison with other forms of the nonlinear term, although the skew-symmetric form is more expensive in computation, it is also more tolerable in deal with the aliasing error. In order to maintain a balance between stability and computational cost, the SEM code in this

thesis employed the convective and divergence forms to calculate the nonlinear term alternatively (Zang, 1991).

A “stiffly stable” integrating method used for the temporal discretisation reads

$$\frac{\partial \mathbf{u}^{(n+1)}}{\partial t} = \frac{1}{\Delta t} \sum_{q=0}^{J_e} \alpha_q \mathbf{u}^{(n+1-q)}, \quad (\text{A.7})$$

where α_q are some chosen weights for J_e -order scheme. In this method, the region of stability shrinks as J_e increases. More specifically, equation (A.7) represents backward Euler equation at $J_e = 1$. As J_e reaches 2, it is known as A-stable method; and it has a good balance between CFL stability and spatial-temporal accuracy.

Three sub-steps of a time-splitting scheme for decoupling the velocity and pressure can be read as follows

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \sum_{q=0}^{J_e-1} \beta_q [\mathbf{N}(\mathbf{u}^{n-q}) + \mathbf{f}^{n-q}] \quad , \quad (\text{A.8})$$

$$\frac{\mathbf{u}^{**} - \mathbf{u}^*}{\Delta t} = -\frac{1}{\rho} \nabla p^{n+1} \quad , \quad (\text{A.9})$$

$$\frac{\alpha_0 \mathbf{u}^{n+1} - \mathbf{u}^{**}}{\Delta t} = \nu \nabla^2 \mathbf{u}^{n+1} \quad , \quad (\text{A.10})$$

where β_q are some chosen weights for J_e -order scheme. In equations (A.8) and (A.9), $\mathbf{u}^*$ and $\mathbf{u}^{**}$ are defined as intermediate velocity fields. No boundary conditions are employed in the first sub-step. To ensure that $\mathbf{u}^{**}$ satisfies the incompressible constraint in the second sub-step, it has to be solved by following equation

$$\frac{1}{\rho} \nabla^2 p^{n+1} = \frac{\nabla \cdot \mathbf{u}^*}{\Delta t} \quad , \quad (\text{A.11})$$

with subjected velocity boundary condition

$$\frac{1}{\rho} \frac{\partial p^{n+1}}{\partial n} = \mathbf{n} \cdot \sum_{q=0}^{J_e-1} \beta_q [\mathbf{N}(\mathbf{u}^{n-q}) - \nu \nabla \times (\nabla \times \mathbf{u}^{n-q})] \quad . \quad (\text{A.12})$$

For the third sub-step as equation (A.10), the prescribed boundary conditions are imposed.

A.2 Spatial discretisation

A.2.1 Weighted residual method

In a 3-D simulation, Semtex transforms all physical quantities into a Fourier space in z -direction (assumed as homogenous direction). Meanwhile, it interpolates the quantities with the quadrilateral SEM based on Gauss-Lobatto-Legendre-Lagrange (GLLL) polynomial in x - and y -directions. Semtex also uses the scheme called “isoparametric projection” to map every quadrilateral element into the canonical domain $(\xi, \gamma) \in [-1, 1] \times [-1, 1]$. Therefore, equations (A.11) and (A.10) can be further expressed respectively as

$$\frac{1}{\rho} [\nabla_{xy}^2 \hat{p}^{n+1} - k_z^2 \hat{p}^{n+1}] = \nabla \cdot \left(\frac{\hat{\mathbf{u}}^*}{\Delta t} \right) \quad , \quad (\text{A.13})$$

$$\nabla_{xy}^2 \hat{\mathbf{u}}^{n+1} - \left(\frac{1}{\nu \Delta t} + k_z^2 \right) \hat{\mathbf{u}}^{n+1} = -\frac{\hat{\mathbf{u}}^{**}}{\nu \Delta t} \quad , \quad (\text{A.14})$$

where $\nabla_{xy}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator in cross-stream directions, k_z denotes the wavenumber in z -direction and $(\hat{\cdot})$ represents the quantities in Fourier space.

In order to simplify above equations (A.14) and (A.13), a two-dimensional (2-D)

Helmholtz equation can be employed in general form as

$$\nabla^2 u - \lambda^2 u = f \quad , \quad (\text{A.15})$$

where λ and f represent real and general constants, respectively. The subscript in ∇_{xy} is omitted for convenience. In a bounded domain of Ω , equation (A.15) is required to be integrated and multiplied by a weight function v , i.e.,

$$\int_{\Omega} (\nabla^2 u - \lambda^2 u - f) v d\Omega = 0 \quad . \quad (\text{A.16})$$

The above equation can be further expressed using the integration by parts on ∇^2 as

$$\int_{\Omega} \nabla u \nabla v d\Omega + \lambda^2 \int_{\Omega} u v d\Omega = - \int_{\Omega} f v d\Omega + \oint_{\partial\Omega} h v d(\partial\Omega) \quad , \quad (\text{A.17})$$

where h represents the Neumann boundary condition defined as $\mathbf{n} \cdot \nabla \mathbf{u}$. In order to partition the whole bounded domain Ω into sub-domain Ω_e for each element, a mathematical definition has to be introduced, which is a integral of sum equals a sum of integral as $\int_{\Omega} (\cdot) d\Omega = \sum_{e=1}^{N_e} \int_{\Omega_e} (\cdot) d\Omega_e$. Therefore, the above equation can be rearranged as

$$\sum_{e=1}^{N_e} \left[\int_{\Omega_e} \nabla u \nabla v d\Omega_e + \lambda^2 \int_{\Omega_e} u v d\Omega_e + \int_{\Omega_e} f v d\Omega_e - \oint_{\partial\Omega_e} h v d(\partial\Omega_e) \right] = 0 \quad , \quad (\text{A.18})$$

where N_e denotes the total number of elements.

A.2.2 Discretisation in a 1-D quadrilateral element

To ensure that each of the integrations of sub-domains in equation (A.18) can be discretised, some important concepts in one-dimensional (1-D) SEM have to be introduced.

A continuous base function $u(\xi) \in [-1, 1]$ in a variational form reads as

$$u(\xi) = \sum_{i=0}^N u_i \Psi_i(\xi) \quad , \quad (\text{A.19})$$

where $\Psi_i(\xi)$ is defined as the nodal base functions and u_i is the coefficients. In the canonical domain, the nodal base function is chosen as Lagrange polynomial satisfying $\Psi_i(\xi_j) = \delta_{ij}$ for $\{j = 0, 1, \dots, N\}$ where ξ_j is the Gauss-Lobatto-Legendre quadrature points, corresponding to the zeros of $(1 - \xi^2)L'_N(\xi)$ with $L'_N(\xi)$ being the Legendre polynomial of order N .

Because any linear operation on continuous base function $u(\xi)$ in equation (A.19) can be discretised, the numerical integration on $u(\xi)$ with the Gaussian quadrature rule reads

$$\int_{-1}^1 u(\xi) d\xi = \sum_{i=0}^N u_i \int_{-1}^1 \Psi_i(\xi) d\xi = \sum_{i=0}^N w_i u_i = \hat{\mathbf{w}}^T \mathbf{u} \quad , \quad (\text{A.20})$$

where the quadrature weight for the i -th base function is defined as $w_i = \int_{-1}^1 \Psi_i(\xi) d\xi$. To simplify the above equation, matrices $\hat{\mathbf{w}} = [w_0, \dots, w_N]^T$ and $\mathbf{u} = [u_0, \dots, u_N]^T$ are introduced.

The spatial derivative on $u(\xi_j)$ reads

$$\frac{\partial u(\xi_j)}{\partial \xi} = \sum_{i=0}^N u_i \frac{\partial \Psi_i(\xi_j)}{\partial \xi} = \hat{\mathbf{D}} \mathbf{u} \quad , \quad (\text{A.21})$$

where $\hat{\mathbf{D}} = [\partial \Psi_i(\xi_j) / \partial \xi]$ represents the differentiation matrix.

In order to employ Galerkin weighted residual method, the nodal base function has also been used to represent the weight function, i.e., $v_j = \Psi_j(\xi)$. As such,

$$(u, v_j) = \sum_{i=0}^N u_i \int_{-1}^1 \Psi_i(\xi) \Psi_j(\xi) d\xi \quad . \quad (\text{A.22})$$

By using the Gaussian quadrature rule and $\Psi_i(\xi_j) = \delta_{ij}$, which can also be expressed

as

$$(u, v_j) = \sum_{i=0}^N u_i \sum_{k=0}^N w_k \Psi_i(\xi_k) \Psi_j(\xi_k) = \sum_{i=0}^N u_i \sum_{k=0}^N w_k \delta_{ik} \delta_{jk} = u_j w_j = \mathbf{v}_j^T \hat{\mathbf{W}} \mathbf{u} \quad . \quad (\text{A.23})$$

Here, the mass matrix is defined as $\hat{\mathbf{W}} = \hat{\mathbf{I}} \hat{\mathbf{w}}$ where $\hat{\mathbf{I}}$ is the identical matrix of rank $N + 1$. $\mathbf{v}_j$ is the j -th weight function in the variational form of equation (A.19). Because only the j -th row is identical in $\mathbf{v}_j$, i.e., $\mathbf{v}_j = [0, \dots, 1, \dots, 0]^T$. Therefore, equation (A.22) can be further expanded for all weight functions with a simplified matrix notation as

$$(u, v) = \hat{\mathbf{W}} \mathbf{u} \quad , \quad (\text{A.24})$$

where $\hat{\mathbf{I}}$ is related with the weight functions as $\hat{\mathbf{I}} = [\hat{\mathbf{v}}_0, \dots, \hat{\mathbf{v}}_N]^T$.

Moreover, in order to discretise $(\nabla u, \nabla v)$ of equation (A.18) in a 1-D quadrilateral element, $(\nabla u, \nabla v)$ can be expressed as

$$\left(\frac{\partial u}{\partial \xi}, \frac{\partial v_j}{\partial \xi} \right) = (\hat{\mathbf{D}} \mathbf{u}, \hat{\mathbf{D}} \mathbf{v}_j) = (\hat{\mathbf{D}} \mathbf{v}_j)^T \hat{\mathbf{W}} \hat{\mathbf{D}} \mathbf{u} = \mathbf{v}_j^T \hat{\mathbf{D}}^T \hat{\mathbf{W}} \hat{\mathbf{D}} \mathbf{u} \quad . \quad (\text{A.25})$$

Thus, for all weight function, the discretisation of $(\nabla u, \nabla v)$ can be further derived as follows,

$$\left(\frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \xi} \right) = \hat{\mathbf{D}}^T \hat{\mathbf{W}} \hat{\mathbf{D}} \mathbf{u} \quad . \quad (\text{A.26})$$

Meanwhile, the last term in equation (A.18) in 1-D quadrilateral element can be derived as

$$\int_{\partial \Omega_e} h v_j d\tau = \int_{-1}^1 h v_j \frac{\partial \tau}{\partial \xi} d\xi = \sum_{i=0}^N w_i h_i \Psi_j(\xi_i) \frac{\partial \tau}{\partial \xi}(\xi_i) = w_j h_j \frac{\partial \tau}{\partial \xi}(\xi_j) \quad . \quad (\text{A.27})$$

where τ denotes the tangential direction. Therefore, for all weight functions, the last component of equation (A.18) reads

$$\int_{\partial \Omega_e} h v d\tau = \mathbf{B} \mathbf{h} \quad , \quad (\text{A.28})$$

where $\mathbf{B} = \text{diag}(w_0 \frac{\partial \tau}{\partial \xi}(\xi_0), \dots, w_N \frac{\partial \tau}{\partial \xi}(\xi_N))$ and $\mathbf{h} = [h_0, h_1, \dots, h_N]^T$. Among these two parts, $\partial \tau / \partial \xi$ can be determined with

$$\frac{\partial \tau}{\partial \xi} = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} . \quad (\text{A.29})$$

A.2.3 Discretisation in a 2-D quadrilateral element

For a 2-D quadrilateral spectral-element, a similar method mentioned in subsection A.2.2 has been used for mapping $u(x, y)$. A continuous base function $u(\xi_1, \xi_2) \in [-1, 1] \times [-1, 1]$ can be expressed as follows

$$u(\xi_1, \xi_2) = \sum_{i=0}^N \sum_{j=0}^N u_{ij} \Psi_i(\xi_1) \Psi_j(\xi_2) \quad , \quad (\text{A.30})$$

where $\Psi_i(\xi)$ are the nodal base functions using Gauss-Lobatto-Legendre interpolants. And then, the spatial derivatives of $u(\xi_1, \xi_2)$ can be derived as

$$\frac{\partial u(\xi_1, \xi_2)}{\partial \xi_1} = \sum_{i=0}^N \sum_{j=0}^N u_{ij} \frac{\partial \Psi_i(\xi_1)}{\partial \xi_1} \Psi_j(\xi_2) = \hat{\mathbf{D}} \otimes \hat{\mathbf{I}} \mathbf{u} = \mathbf{D}_1 \mathbf{u} \quad , \quad (\text{A.31})$$

$$\frac{\partial u(\xi_1, \xi_2)}{\partial \xi_2} = \sum_{i=0}^N \sum_{j=0}^N u_{ij} \Psi_i(\xi_1) \frac{\partial \Psi_j(\xi_2)}{\partial \xi_2} = \hat{\mathbf{I}} \otimes \hat{\mathbf{D}} \mathbf{u} = \mathbf{D}_2 \mathbf{u} \quad . \quad (\text{A.32})$$

Here, the symbol $\otimes$ represents the Kronecker product operator. $\mathbf{u}$ is defined as $\mathbf{u} = [u_{00}, u_{10}, \dots, u_{N0}, u_{01}, u_{11}, \dots, u_{NN}]^T$.

In order to determine the spatial derivatives of a function $u(x, y)$, the employed chain rule is presented as

$$\frac{\partial}{\partial x} = \frac{\partial \xi_1}{\partial x} \frac{\partial}{\partial \xi_1} + \frac{\partial \xi_2}{\partial x} \frac{\partial}{\partial \xi_2} \quad , \quad (\text{A.33})$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi_1}{\partial y} \frac{\partial}{\partial \xi_1} + \frac{\partial \xi_2}{\partial y} \frac{\partial}{\partial \xi_2} \quad . \quad (\text{A.34})$$

Meanwhile, the transformation tensor can be expressed as

$$\begin{bmatrix} \frac{\partial \xi_1}{\partial x} & \frac{\partial \xi_1}{\partial y} \\ \frac{\partial \xi_2}{\partial x} & \frac{\partial \xi_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} \end{bmatrix}^{-1} = \frac{1}{J} \begin{bmatrix} \frac{\partial y}{\partial \xi_2} & \frac{\partial y}{\partial \xi_1} \\ \frac{\partial x}{\partial \xi_2} & \frac{\partial x}{\partial \xi_1} \end{bmatrix}, \quad (\text{A.35})$$

where the transformation Jacobian is defined as $J = \partial x / \partial \xi_1 \partial y / \partial \xi_2 - \partial x / \partial \xi_2 \partial y / \partial \xi_1$.

Therefore, the spatial derivatives as a function of x and y read

$$\mathbf{D}_x = \Lambda \left(\frac{\partial \xi_1}{\partial x} \right) \mathbf{D}_1 + \Lambda \left(\frac{\partial \xi_2}{\partial x} \right) \mathbf{D}_2, \quad (\text{A.36})$$

$$\mathbf{D}_y = \Lambda \left(\frac{\partial \xi_1}{\partial y} \right) \mathbf{D}_1 + \Lambda \left(\frac{\partial \xi_2}{\partial y} \right) \mathbf{D}_2, \quad (\text{A.37})$$

where function $\Lambda(f_{ij})$ returns a matrix of rank $(N+1)^2$ with the diagonal components being the vector f_{ij} .

By employing the transformation Jacobian J , the integration in a 2-D quadrilateral element can be transformed into the canonical domain $\Omega_{st} \in [-1, 1] \times [-1, 1]$ as follows

$$\begin{aligned} \int_{\Omega} u(x, y) dx dy &= \int_{\Omega_{st}} u(\xi_1, \xi_2) J d\xi_1 d\xi_2 \\ &= \sum_{i=0}^N \sum_{j=0}^N u_{ij} \int_{-1}^1 \int_{-1}^1 J(\xi_1, \xi_2) \Psi_i(\xi_1) \Psi_j(\xi_2) d\xi_1 d\xi_2 \\ &= \sum_{i=0}^N \sum_{j=0}^N u_{ij} \sum_{p=0}^N w_p \int_{-1}^1 J(\xi_{1p}, \xi_2) \Psi_i(\xi_{1p}) \Psi_j(\xi_2) d\xi_2 \\ &= \sum_{i=0}^N \sum_{j=0}^N u_{ij} \sum_{p=0}^N w_p \sum_{q=0}^N w_q J(\xi_{1p}, \xi_{2q}) \Psi_i(\xi_{1p}) \Psi_j(\xi_{2q}) \\ &= \sum_{i=0}^N \sum_{j=0}^N u_{ij} \sum_{p=0}^N w_p \sum_{q=0}^N w_q J(\xi_{1p}, \xi_{2q}) \delta_{ip} \delta_{jq} \\ &= \sum_{i=0}^N \sum_{j=0}^N u_{ij} w_i w_j J_{ij} \\ &= \mathbf{w}^T \mathbf{u}, \end{aligned} \quad (\text{A.38})$$

where $\mathbf{w} = [w_0 w_0 J_{00}, w_1 w_0 J_{10}, \dots, w_N w_N J_{NN}]^T$.

The weight functions in a 2-D quadrilateral element are also defined with the

nodal base functions, which is as same as the method in subsection A.2.2, i.e.,

$$v_{mn} = \Psi_m(\xi_1)\Psi_n(\xi_2) \quad , \quad (\text{A.39})$$

where $m, n = 0, 1, 2, \dots, N$. As such,

$$\begin{aligned} (u, v_{mn}) &= \sum_{i=0}^N \sum_{j=0}^N u_{ij} \int_{-1}^1 \int_{-1}^1 J(\xi_1, \xi_2) \Psi_i(\xi_1) \Psi_j(\xi_2) \Psi_m(\xi_1) \Psi_n(\xi_2) d\xi_1 d\xi_2 \\ &= \sum_{i=0}^N \sum_{j=0}^N u_{ij} \sum_{p=0}^N \sum_{q=0}^N J(\xi_{1p}, \xi_{2q}) w_p w_q \delta_{ip} \delta_{mq} \delta_{jp} \delta_{nq} \\ &= \mathbf{v}_{pq}^T \mathbf{W} \mathbf{u} \quad . \end{aligned} \quad (\text{A.40})$$

Here, $\mathbf{W} = \mathbf{I} \mathbf{w}$ where $\mathbf{I}$ is the identical matrix of rank $(N+1)^2$. The value in vector $\mathbf{v}_{pq}$ is one at the pq -th component and zero at any other components. Therefore, for all weight functions in a 2-D quadrilateral element, the above expression can be further expanded into a matrix form as follows

$$(u, v) = \mathbf{W} \mathbf{u} \quad (\text{A.41})$$

The discretisation of $(\nabla u, \nabla v)$ in a 2-D quadrilateral element for equation (A.18) can be generally written as

$$\begin{aligned} (\nabla u, \nabla v) &= \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right) + \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y} \right) \\ &= (\mathbf{D}_x \mathbf{u}, \mathbf{D}_x \mathbf{v}) + (\mathbf{D}_y \mathbf{u}, \mathbf{D}_y \mathbf{v}) \\ &= \mathbf{D}_x^T \mathbf{W} \mathbf{D}_x \mathbf{u} + \mathbf{D}_y^T \mathbf{W} \mathbf{D}_y \mathbf{u} \\ &= (\mathbf{D}_x^T \mathbf{W} \mathbf{D}_x + \mathbf{D}_y^T \mathbf{W} \mathbf{D}_y) \mathbf{u} \quad . \end{aligned} \quad (\text{A.42})$$

Therefore, the local elemental linear operator (the first two-components of equation (A.18)) can be presented as

$$\int_{\Omega_e} \nabla u \nabla v d\Omega_e + \lambda^2 \int_{\Omega_e} u v d\Omega_e = (\mathbf{D}_x^T \mathbf{W} \mathbf{D}_x + \mathbf{D}_y^T \mathbf{W} \mathbf{D}_y + \lambda \mathbf{W}) \mathbf{u} \equiv \mathbf{A}_e \mathbf{u} \quad , \quad (\text{A.43})$$

where $\mathbf{D}_x^T \mathbf{W} \mathbf{D}_x + \mathbf{D}_y^T \mathbf{W} \mathbf{D}_y$ and $\mathbf{W}$ represent stiffness and mass matrices, respectively.

A.2.4 Global operations

In order to express the local $\mathbf{u}_l$ using the global degree of freedom (or global DOF) $\mathbf{u}_g$, an assembly matrix $\mathbf{G}$ has to be introduced, which reads

$$\mathbf{u}_l = \mathbf{G} \mathbf{u}_g \quad . \quad (\text{A.44})$$

Therefore, each component of equation (A.17) can be discretised as

$$\int_{\Omega} \nabla u \nabla v d\Omega + \lambda^2 \int_{\Omega} u v d\Omega = \mathbf{G}^T (\mathbf{A}_e \otimes \mathbf{I}_e) \mathbf{G} \mathbf{u}_g \quad , \quad (\text{A.45})$$

$$- \int_{\Omega} f v d\Omega = \mathbf{G}^T (\mathbf{W}_e \otimes \mathbf{I}_e) \mathbf{f}_e \quad , \quad (\text{A.46})$$

and

$$\oint_{\partial\Omega} h v d(\partial\Omega) = \mathbf{G}^T (\mathbf{B}_e \otimes \mathbf{I}_e) \mathbf{h}_e \quad , \quad (\text{A.47})$$

respectively, where $\mathbf{I}_e$ is an identical matrix of rank as same as the total element number. Therefore, equation (A.17) becomes

$$\mathbf{G}^T (\mathbf{A}_e \otimes \mathbf{I}_e) \mathbf{G} \mathbf{u}_g = \mathbf{G}^T (\mathbf{W}_e \otimes \mathbf{I}_e) \mathbf{f}_e + \mathbf{G}^T (\mathbf{B}_e \otimes \mathbf{I}_e) \mathbf{h}_e \quad , \quad (\text{A.48})$$

then it can be expressed as $\mathbf{A}_g \mathbf{u}_g = \mathbf{b}_g$.

The final expression can also be further presented employing the Dirichlet boundary condition, $\mathbf{u}_g = [\mathbf{u}_g^u, \mathbf{u}_g^D]^T$, where the superscripts $(\cdot)^u$ and $(\cdot)^D$ are the unknown and Dirichlet DOF, respectively, as follows

$$\begin{bmatrix} \mathbf{A} & \mathbf{A}^D \\ 0 & \mathbf{I}^D \end{bmatrix} \begin{bmatrix} \mathbf{u}_g^u \\ \mathbf{u}_g^D \end{bmatrix} = \begin{bmatrix} \mathbf{b}_g^u \\ \mathbf{b}_g^D \end{bmatrix} \quad . \quad (\text{A.49})$$

Therefore, the linear equation must be solved with

$$\mathbf{A}\mathbf{u}_g^u = \mathbf{b}_g^u - \mathbf{A}^D\mathbf{u}_g^D \equiv \mathbf{b} \quad . \quad (\text{A.50})$$

A.3 Linear solver

The above 2-D Helmholtz equation, in equation (A.50), needs to be employed to solve both the pressure and viscous substeps at each wavenumber. Particularly, for the viscous substep, a large value of $1/(\nu\Delta t)$ results in a strong domination on the diagonal components of the matrix $\mathbf{A}$, indicating that a Jacobian preconditioned-conjugate-gradient (PCG) solver is enough for solving the equation. In contrast, for the pressure substep, due to a poor Jacobian-preconditioned effect on the matrix $\mathbf{A}$, the 2-D Helmholtz equation has to be calculated by a multigrid solver. It is noticed that Semtex allows the parallel computation in the homogeneous (or Fourier) direction, and each 2-D Helmholtz equation has to be solved in a sequence. A direct solver is possibly required to avoid the complex multigrid implementation in the pressure substep.

A.3.1 Iterative solver for the viscous substep

In order to solve 2-D Helmholtz equation for the viscous substep, such as $\mathbf{A}\mathbf{u}^u = \mathbf{b}$, a basic iterative solver can be presented as follows

$$\mathbf{u}_{k+1}^u = \mathbf{u}_k^u + \mathbf{P}^{-1}r_n \quad \text{for } k = 0, 1, 2, \dots \quad , \quad (\text{A.51})$$

where $\mathbf{P}$ is a preconditioning matrix for relaxation, and r_n is a residual matrix defined as $r_n = \mathbf{b} - \mathbf{A}\mathbf{u}_k^u$. To avoid a slow convergence rate, $\mathbf{P}$ needs to satisfy a requirement that it has to be close to $\mathbf{A}$ as much as possible. Therefore, a Jacobian preconditioner is introduced in the viscous substep, i.e. $\mathbf{P} \equiv \text{diag}(\mathbf{A})$.

The PCG solver begins with an initial guess $\mathbf{u}_0^u$ and performs the following iterating steps:

```

 $r_0 = \mathbf{b} - \mathbf{A}\mathbf{u}_0^u, z_0 = \mathbf{P}^{-1}r_0$ 
 $w_0 = z_0$  and  $j = 0$ 
Loop  $j$ :
     $\alpha_j = (r_j, z_j) / (\mathbf{A}w_j, w_j)$ 
     $\mathbf{u}_{j+1}^u = \mathbf{u}_j^u + \alpha_j w_j$ 
     $r_{j+1} = r_j - \alpha_j \mathbf{A}w_j$ 
     $z_{j+1} = \mathbf{P}^{-1}r_{j+1}$ 
     $\beta_j = (r_{j+1}, z_{j+1}) / (r_j, z_j)$ 
     $w_{j+1} = z_{j+1} + \beta_j w_j$ 
break if converged
End

```

In fact, $\mathbf{A}w_j$ is solved employing a local elemental matrix $\mathbf{A}_e$ and globally assembled afterward in the PCG solver.

A.3.2 Direct solver for the pressure substep

For the pressure substep, a direct solver can avoid a complication in use of multigrid solver. However, the inverse matrix $\mathbf{A}^{-1}$ from direct solver results in a huge cost on computational resources. Therefore, a so-called static condensation technique is involved in order to avoid this situation. The function of the static condensation is to split $\mathbf{u}^u$ into element-interior DOF $\mathbf{u}_i^u$ and element-boundary DOF $\mathbf{u}_b^u$. Therefore, $\mathbf{A}\mathbf{u}^u = \mathbf{b}$ can be expressed as follows

$$\begin{bmatrix} \mathbf{A}_b & \mathbf{A}_c \\ \mathbf{A}_c^T & \mathbf{A}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_b^u \\ \mathbf{u}_i^u \end{bmatrix} = \begin{bmatrix} \mathbf{b}_b \\ \mathbf{b}_i \end{bmatrix} . \quad (\text{A.52})$$

By simple manipulation, the above expression can be further written as

$$\begin{bmatrix} \mathbf{A}_b - \mathbf{A}_c \mathbf{A}_i^{-1} \mathbf{A}_c^T & 0 \\ \mathbf{A}_c^T & \mathbf{A}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_b^u \\ \mathbf{u}_i^u \end{bmatrix} = \begin{bmatrix} \mathbf{b}_b - \mathbf{A}_c \mathbf{A}_i^{-1} \mathbf{b}_i \\ \mathbf{b}_i \end{bmatrix} . \quad (\text{A.53})$$

From above equation, it is straightforward to determine boundary unknowns with

$$\mathbf{u}_b^u = (\mathbf{A}_b - \mathbf{A}_c \mathbf{A}_i^{-1} \mathbf{A}_c^T)^{-1} (\mathbf{b}_b - \mathbf{A}_c \mathbf{A}_i^{-1} \mathbf{b}_i) . \quad (\text{A.54})$$

Here, the rank of $(\mathbf{A}_b - \mathbf{A}_c \mathbf{A}_i^{-1} \mathbf{A}_c^T)^{-1}$ is significantly lower than it of $\mathbf{A}^{-1}$, indicating that the usage of memory for the element-boundary matrix is not much expensive. Moreover, the interior unknowns can be determine directly by

$$\mathbf{u}_i^u = \mathbf{A}_i^{-1} \mathbf{b}_i - \mathbf{A}_i^{-1} \mathbf{A}_c^T \mathbf{u}_b^u . \quad (\text{A.55})$$

Appendix B

Transport equations for fluctuating enstrophy and its spectrum

In the appendix, the transport equations of fluctuating enstrophy in the context of a radially rotating pipe flow are derived in both physical and spectral spaces. Both equations (especially that for the fluctuating enstrophy spectrum) are not readily available in the literature, and need to be documented.

B.1 Transport equation for fluctuating enstrophy

To obtain the transport equations of the fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle$ and its spectrum $\tilde{E}_{\omega\omega}$ in a radially rotating circular pipe flow, the derivation begins with the incompressible Navier-Stokes equations with Coriolis body forces in the r -, β -, and z -directions, viz.

$$\begin{aligned}
\frac{\partial u_r}{\partial t} = & -u_r \frac{\partial u_r}{\partial r} - \frac{u_\beta}{r} \frac{\partial u_r}{\partial \beta} + \frac{u_\beta^2}{r} - u_z \frac{\partial u_r}{\partial z} - \frac{1}{\rho} \frac{\partial p}{\partial r} \\
& + \nu \left(\frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \beta^2} - \frac{2}{r^2} \frac{\partial u_\beta}{\partial \beta} + \frac{\partial^2 u_r}{\partial z^2} \right) + 2\Omega u_z \sin \beta \quad , \quad (B.1)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u_\beta}{\partial t} = & -u_r \frac{\partial u_\beta}{\partial r} - \frac{u_\beta}{r} \frac{\partial u_\beta}{\partial \beta} - \frac{u_r u_\beta}{r} - u_z \frac{\partial u_\beta}{\partial z} - \frac{1}{\rho r} \frac{\partial p}{\partial \beta} \\
& + \nu \left(\frac{1}{r} \frac{\partial u_\beta}{\partial r} - \frac{u_\beta}{r^2} + \frac{\partial^2 u_\beta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_\beta}{\partial \beta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \beta} + \frac{\partial^2 u_\beta}{\partial z^2} \right) + 2\Omega u_z \cos \beta \quad , \quad (B.2)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u_z}{\partial t} = & -u_r \frac{\partial u_z}{\partial r} - \frac{u_\beta}{r} \frac{\partial u_z}{\partial \beta} - u_z \frac{\partial u_z}{\partial z} - \frac{\Pi}{\rho} - \frac{1}{\rho} \frac{\partial p}{\partial z} \\
& + \nu \left(\frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \beta^2} + \frac{\partial^2 u_z}{\partial z^2} \right) - 2\Omega (u_r \sin \beta + u_\beta \cos \beta) \quad . \quad (B.3)
\end{aligned}$$

Taking the curl of equations (B.1)-(B.3) (i.e., $\vec{\omega} = \nabla \times \vec{u}$) yields the corresponding instantaneous vorticity equations, expressed as

$$\begin{aligned}
\frac{\partial \omega_r}{\partial t} = & -u_r \frac{\partial \omega_r}{\partial r} - \frac{u_\beta}{r} \frac{\partial \omega_r}{\partial \beta} + \frac{u_\beta \omega_\beta}{r} - u_z \frac{\partial \omega_r}{\partial z} + \omega_r \frac{\partial u_r}{\partial r} + \frac{\omega_\beta}{r} \frac{\partial u_r}{\partial \beta} - \frac{\omega_\beta u_\beta}{r} + \omega_z \frac{\partial u_r}{\partial z} \\
& + \nu \left(\frac{1}{r} \frac{\partial \omega_r}{\partial r} - \frac{\omega_r}{r^2} + \frac{\partial^2 \omega_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \omega_r}{\partial \beta^2} - \frac{2}{r^2} \frac{\partial \omega_\beta}{\partial \beta} + \frac{\partial^2 \omega_r}{\partial z^2} \right) \\
& + 2\Omega \left(\frac{\partial u_r}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial u_r}{\partial \beta} \sin \beta + \frac{u_\beta}{r} \sin \beta \right) \quad , \quad (B.4)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \omega_\beta}{\partial t} = & -u_r \frac{\partial \omega_\beta}{\partial r} - \frac{u_\beta}{r} \frac{\partial \omega_\beta}{\partial \beta} - \frac{u_\beta \omega_r}{r} - u_z \frac{\partial \omega_\beta}{\partial z} + \omega_r \frac{\partial u_\beta}{\partial r} + \frac{\omega_\beta}{r} \frac{\partial u_\beta}{\partial \beta} + \frac{\omega_\beta u_r}{r} + \omega_z \frac{\partial u_\beta}{\partial z} \\
& + \nu \left(\frac{1}{r} \frac{\partial \omega_\beta}{\partial r} - \frac{\omega_\beta}{r^2} + \frac{\partial^2 \omega_\beta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \omega_\beta}{\partial \beta^2} + \frac{2}{r^2} \frac{\partial \omega_r}{\partial \beta} + \frac{\partial^2 \omega_\beta}{\partial z^2} \right) \\
& + 2\Omega \left(\frac{\partial u_\beta}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial u_\beta}{\partial \beta} \sin \beta - \frac{u_r}{r} \sin \beta \right) ,
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
\frac{\partial \omega_z}{\partial t} = & -u_r \frac{\partial \omega_z}{\partial r} - \frac{u_\beta}{r} \frac{\partial \omega_z}{\partial \beta} - u_z \frac{\partial \omega_z}{\partial z} + \omega_r \frac{\partial u_z}{\partial r} + \frac{\omega_\beta}{r} \frac{\partial u_z}{\partial \beta} + \omega_z \frac{\partial u_z}{\partial z} \\
& + \nu \left(\frac{1}{r} \frac{\partial \omega_z}{\partial r} + \frac{\partial^2 \omega_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \omega_z}{\partial \beta^2} + \frac{\partial^2 \omega_z}{\partial z^2} \right) \\
& + 2\Omega \left(\frac{\partial u_z}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial u_z}{\partial \beta} \sin \beta \right) .
\end{aligned} \tag{B.6}$$

Then, by subtracting the ensemble-averaged form from equations (B.4)-(B.6), such as $\omega'_r = \omega_r - \langle \omega_r \rangle$, the fluctuating vorticity equations in the three directions are written as

$$\begin{aligned}
\frac{\partial \omega'_r}{\partial t} = & -\langle u_r \rangle \frac{\partial \omega'_r}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \omega'_r}{\partial \beta} + \frac{\langle u_\beta \rangle \omega'_\beta}{r} - \langle u_z \rangle \frac{\partial \omega'_r}{\partial z} \\
& - u'_r \frac{\partial \langle \omega_r \rangle}{\partial r} - \frac{u'_\beta}{r} \frac{\partial \langle \omega_r \rangle}{\partial \beta} + \frac{u'_\beta \langle \omega_\beta \rangle}{r} - u'_z \frac{\partial \langle \omega_r \rangle}{\partial z} \\
& + \omega'_r \frac{\partial u'_r}{\partial r} + \frac{\omega'_\beta}{r} \frac{\partial u'_r}{\partial \beta} - \frac{\omega'_\beta u'_\beta}{r} + \omega'_z \frac{\partial u'_r}{\partial z} \\
& + \langle \omega_r \rangle \frac{\partial u'_r}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \frac{\partial u'_r}{\partial \beta} - \frac{\langle \omega_\beta \rangle u'_\beta}{r} + \langle \omega_z \rangle \frac{\partial u'_r}{\partial z} \\
& + \omega'_r \frac{\partial \langle u_r \rangle}{\partial r} + \frac{\omega'_\beta}{r} \frac{\partial \langle u_r \rangle}{\partial \beta} - \frac{\omega'_\beta \langle u_\beta \rangle}{r} + \omega'_z \frac{\partial \langle u_r \rangle}{\partial z} \\
& - u'_r \frac{\partial \omega'_r}{\partial r} - \frac{u'_\beta}{r} \frac{\partial \omega'_r}{\partial \beta} + \frac{u'_\beta \omega'_\beta}{r} - u'_z \frac{\partial \omega'_r}{\partial z} \\
& + \nu \left(\frac{1}{r} \frac{\partial \omega'_r}{\partial r} - \frac{\omega'_r}{r^2} + \frac{\partial^2 \omega'_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \omega'_r}{\partial \beta^2} - \frac{2}{r^2} \frac{\partial \omega'_\beta}{\partial \beta} + \frac{\partial^2 \omega'_r}{\partial z^2} \right) \\
& + 2\Omega \left(\frac{\partial u'_r}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial u'_r}{\partial \beta} \sin \beta + \frac{u'_\beta}{r} \sin \beta \right) \\
& - \frac{\partial \langle u'_r \omega'_r \rangle}{\partial r} - \frac{\langle u'_r \omega'_r \rangle}{r} - \frac{1}{r} \frac{\partial \langle u'_r \omega'_\beta \rangle}{\partial \beta} + \frac{\langle u'_\beta \omega'_\beta \rangle}{r} - \frac{\partial \langle u'_r \omega'_z \rangle}{\partial z} \\
& + \frac{\partial \langle \omega'_r u'_r \rangle}{\partial r} + \frac{\langle \omega'_r u'_r \rangle}{r} + \frac{1}{r} \frac{\partial \langle \omega'_r u'_\beta \rangle}{\partial \beta} - \frac{\langle \omega'_\beta u'_\beta \rangle}{r} + \frac{\partial \langle \omega'_r u'_z \rangle}{\partial z} ,
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
\frac{\partial \omega'_\beta}{\partial t} = & -\langle u_r \rangle \frac{\partial \omega'_\beta}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \omega'_\beta}{\partial \beta} - \frac{\langle u_\beta \rangle \omega'_r}{r} - \langle u_z \rangle \frac{\partial \omega'_\beta}{\partial z} \\
& - u'_r \frac{\partial \langle \omega_\beta \rangle}{\partial r} - \frac{u'_\beta}{r} \frac{\partial \langle \omega_\beta \rangle}{\partial \beta} - \frac{u'_\beta \langle \omega_r \rangle}{r} - u'_z \frac{\partial \langle \omega_\beta \rangle}{\partial z} \\
& + \omega'_r \frac{\partial \langle u_\beta \rangle}{\partial r} + \frac{\omega'_\beta}{r} \frac{\partial \langle u_\beta \rangle}{\partial \beta} + \frac{\omega'_\beta \langle u_r \rangle}{r} + \omega'_z \frac{\partial \langle u_\beta \rangle}{\partial z} \\
& + \langle \omega_r \rangle \frac{\partial u'_\beta}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \frac{\partial u'_\beta}{\partial \beta} + \frac{\langle \omega_\beta \rangle u'_r}{r} + \langle \omega_z \rangle \frac{\partial u'_\beta}{\partial z} \\
& + \omega'_r \frac{\partial u'_\beta}{\partial r} + \frac{\omega'_\beta}{r} \frac{\partial u'_\beta}{\partial \beta} + \frac{\omega'_\beta u'_r}{r} + \omega'_z \frac{\partial u'_\beta}{\partial z} \\
& - u'_r \frac{\partial \omega'_\beta}{\partial r} - \frac{u'_\beta}{r} \frac{\partial \omega'_\beta}{\partial \beta} - \frac{u'_\beta \omega'_r}{r} - u'_z \frac{\partial \omega'_\beta}{\partial z} \\
& + \nu \left(\frac{1}{r} \frac{\partial \omega'_\beta}{\partial r} - \frac{\omega'_\beta}{r^2} + \frac{\partial^2 \omega'_\beta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \omega'_\beta}{\partial \beta^2} + \frac{2}{r^2} \frac{\partial \omega'_r}{\partial \beta} + \frac{\partial^2 \omega'_\beta}{\partial z^2} \right) \\
& + 2\Omega \left(\frac{\partial u'_\beta}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial u'_\beta}{\partial \beta} \sin \beta - \frac{u'_r}{r} \sin \beta \right) \\
& - \frac{\partial \langle u'_\beta \omega'_r \rangle}{\partial r} - \frac{\langle u'_\beta \omega'_r \rangle}{r} - \frac{1}{r} \frac{\partial \langle u'_\beta \omega'_\beta \rangle}{\partial \beta} - \frac{\langle u'_r \omega'_\beta \rangle}{r} - \frac{\partial \langle u'_\beta \omega'_z \rangle}{\partial z} \\
& + \frac{\partial \langle \omega'_\beta u'_r \rangle}{\partial r} + \frac{\langle \omega'_\beta u'_r \rangle}{r} + \frac{1}{r} \frac{\partial \langle \omega'_\beta u'_\beta \rangle}{\partial \beta} + \frac{\langle \omega'_r u'_\beta \rangle}{r} + \frac{\partial \langle \omega'_\beta u'_z \rangle}{\partial z} ,
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
\frac{\partial \omega'_z}{\partial t} = & -\langle u_r \rangle \frac{\partial \omega'_z}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \omega'_z}{\partial \beta} - \langle u_z \rangle \frac{\partial \omega'_z}{\partial z} \\
& - u'_r \frac{\partial \langle \omega_z \rangle}{\partial r} - \frac{u'_\beta}{r} \frac{\partial \langle \omega_z \rangle}{\partial \beta} - u'_z \frac{\partial \langle \omega_z \rangle}{\partial z} \\
& + \omega'_r \frac{\partial \langle u_z \rangle}{\partial r} + \frac{\omega'_\beta}{r} \frac{\partial \langle u_z \rangle}{\partial \beta} + \omega'_z \frac{\partial \langle u_z \rangle}{\partial z} \\
& + \langle \omega_r \rangle \frac{\partial u'_z}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \frac{\partial u'_z}{\partial \beta} + \langle \omega_z \rangle \frac{\partial u'_z}{\partial z} \\
& + \omega'_r \frac{\partial u'_z}{\partial r} + \frac{\omega'_\beta}{r} \frac{\partial u'_z}{\partial \beta} + \omega'_z \frac{\partial u'_z}{\partial z} \\
& - u'_r \frac{\partial \omega'_z}{\partial r} - \frac{u'_\beta}{r} \frac{\partial \omega'_z}{\partial \beta} - u'_z \frac{\partial \omega'_z}{\partial z} \\
& + \nu \left(\frac{1}{r} \frac{\partial \omega'_z}{\partial r} + \frac{\partial^2 \omega'_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \omega'_z}{\partial \beta^2} + \frac{\partial^2 \omega'_z}{\partial z^2} \right) \\
& + 2\Omega \left(\frac{\partial u'_z}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial u'_z}{\partial \beta} \sin \beta \right) \\
& - \frac{\partial \langle u'_z \omega'_r \rangle}{\partial r} - \frac{\langle u'_z \omega'_r \rangle}{r} - \frac{1}{r} \frac{\partial \langle u'_z \omega'_\beta \rangle}{\partial \beta} - \frac{\partial \langle u'_z \omega'_z \rangle}{\partial z} \\
& + \frac{\partial \langle \omega'_z u'_r \rangle}{\partial r} + \frac{\langle \omega'_z u'_r \rangle}{r} + \frac{1}{r} \frac{\partial \langle \omega'_z u'_\beta \rangle}{\partial \beta} + \frac{\partial \langle \omega'_z u'_z \rangle}{\partial z} .
\end{aligned} \tag{B.9}$$

The next step is to derive the transport equation for each component of the fluctuating enstrophy, i.e., $\langle \omega'_r \omega'_r \rangle$, $\langle \omega'_\beta \omega'_\beta \rangle$, and $\langle \omega'_z \omega'_z \rangle$. For illustration, consider the radial component $\langle \omega'_r \omega'_r \rangle$. This equation is obtained by multiplying equation (B.7) by ω'_r , multiplying the resulting expression by two, and then applying ensemble averaging. Under this operation, the left-hand side of equation (B.7) yields the time derivative term associated with $\langle \omega'_r \omega'_r \rangle$, viz.

$$\left\langle 2\omega'_r \frac{\partial \omega'_r}{\partial t} \right\rangle = \left\langle \omega'_r \frac{\partial \omega'_r}{\partial t} + \omega'_r \frac{\partial \omega'_r}{\partial t} \right\rangle = \frac{\partial \langle \omega'_r \omega'_r \rangle}{\partial t} . \tag{B.10}$$

It should be noted that, under the above operation, the results associated with the terms in the last two rows of equations (B.7)-(B.9) vanish, since the ensemble average of the vorticity fluctuations is identically zero, such as $\langle \omega'_r \rangle = 0$. These terms can therefore be omitted in the subsequent derivation.

The final step is to obtain the transport equation of the fluctuating enstrophy by summing the equations of the three components, i.e., $\langle \omega'_i \omega'_i \rangle = \langle \omega'_r \omega'_r \rangle + \langle \omega'_\beta \omega'_\beta \rangle + \langle \omega'_z \omega'_z \rangle$. Meanwhile, because the flow field is fully developed in the homogeneous axial (z -) direction, the corresponding mean gradient vanishes, that is $\partial \langle \cdot \rangle / \partial z = 0$. Following these procedures yield the transport equation of the fluctuating enstrophy

$\langle \omega'_i \omega'_i \rangle$, with each budget term written as

$$\begin{aligned}
\frac{\partial \langle \omega'_i \omega'_i \rangle}{\partial t} = & \underbrace{- \langle u_r \rangle \frac{\partial \langle \omega'_i \omega'_i \rangle}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \langle \omega'_i \omega'_i \rangle}{\partial \beta}}_{C_{\omega\omega}} \\
& + 2 \underbrace{\left(- \langle \omega'_i u'_r \rangle \frac{\partial \langle \omega_i \rangle}{\partial r} - \frac{\langle \omega'_i u'_\beta \rangle}{r} \frac{\partial \langle \omega_i \rangle}{\partial \beta} - \langle \omega'_\beta u'_\beta \rangle \frac{\langle \omega_r \rangle}{r} \right)}_{P_{\omega\omega}^g} \\
& + 2 \underbrace{\left(\left\langle \omega'_i \omega'_r \frac{\partial u'_i}{\partial r} \right\rangle + \frac{1}{r} \left\langle \omega'_i \omega'_\beta \frac{\partial u'_i}{\partial \beta} \right\rangle + \left\langle \omega'_i \omega'_z \frac{\partial u'_i}{\partial z} \right\rangle \right.}_{P_{\omega\omega}^t} \\
& \quad \left. - \frac{\langle \omega'_r \omega'_\beta u'_\beta \rangle}{r} + \frac{\langle \omega'_\beta \omega'_\beta u'_r \rangle}{r} \right) \\
& + 2 \underbrace{\left(\langle \omega_r \rangle \left\langle \omega'_i \frac{\partial u'_i}{\partial r} \right\rangle + \frac{\langle \omega_\beta \rangle}{r} \left\langle \omega'_i \frac{\partial u'_i}{\partial \beta} \right\rangle + \langle \omega_z \rangle \left\langle \omega'_i \frac{\partial u'_i}{\partial z} \right\rangle + \langle \omega_\beta \rangle \frac{\langle \omega'_\beta u'_r \rangle}{r} \right)}_{P_{\omega\omega}^m} \\
& + 2 \underbrace{\left(\langle \omega'_i \omega'_r \rangle \frac{\partial \langle u_i \rangle}{\partial r} + \frac{\langle \omega'_i \omega'_\beta \rangle}{r} \frac{\partial \langle u_i \rangle}{\partial \beta} - \langle \omega'_r \omega'_\beta \rangle \frac{\langle u_\beta \rangle}{r} + \langle \omega'_\beta \omega'_\beta \rangle \frac{\langle u_r \rangle}{r} \right)}_{P_{\omega\omega}^s} \\
& + 2\nu \underbrace{\left(- \left\langle \frac{\partial \omega'_i}{\partial r} \frac{\partial \omega'_i}{\partial r} \right\rangle - \frac{1}{r^2} \left\langle \frac{\partial \omega'_i}{\partial \beta} \frac{\partial \omega'_i}{\partial \beta} \right\rangle - \left\langle \frac{\partial \omega'_i}{\partial z} \frac{\partial \omega'_i}{\partial z} \right\rangle \right.}_{\varepsilon_{\omega\omega}} \\
& \quad \left. - \frac{2}{r^2} \left\langle \omega'_r \frac{\partial \omega'_\beta}{\partial \beta} \right\rangle + \frac{2}{r^2} \left\langle \omega'_\beta \frac{\partial \omega'_r}{\partial \beta} \right\rangle - \frac{\langle \omega'_r \omega'_r \rangle}{r^2} - \frac{\langle \omega'_\beta \omega'_\beta \rangle}{r^2} \right) \\
& + \underbrace{\left(- \frac{\partial \langle \omega'_i \omega'_i u'_r \rangle}{\partial r} - \frac{\langle \omega'_i \omega'_i u'_r \rangle}{r} - \frac{1}{r} \frac{\partial \langle \omega'_i \omega'_i u'_\beta \rangle}{\partial \beta} \right)}_{D_{\omega\omega}^t} \\
& + \nu \underbrace{\left(\frac{\partial^2 \langle \omega'_i \omega'_i \rangle}{\partial r^2} + \frac{1}{r} \frac{\partial \langle \omega'_i \omega'_i \rangle}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \langle \omega'_i \omega'_i \rangle}{\partial \beta^2} \right)}_{D_{\omega\omega}^v} \\
& + 4\Omega \underbrace{\left(\left\langle \omega'_i \frac{\partial u'_i}{\partial r} \right\rangle \cos \beta - \frac{1}{r} \left\langle \omega'_i \frac{\partial u'_i}{\partial \beta} \right\rangle \sin \beta \right.}_{P_{\omega\omega}^c} \\
& \quad \left. + \frac{\langle \omega'_r u'_\beta \rangle}{r} \sin \beta - \frac{\langle \omega'_\beta u'_r \rangle}{r} \sin \beta \right) = 0 \quad .
\end{aligned}$$

(B.11)

B.2 Transport equation for fluctuating enstrophy spectrum

By applying the Fourier transform ($\mathcal{F}$), as defined in equation (2.6), to equations (B.7)-(B.9), the Fourier-transformed equations of the vorticity fluctuations in the three directions are obtained as

$$\begin{aligned}
\frac{\partial \widehat{\omega}'_r}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{\omega}'_r}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{\omega}'_r}{\partial \beta} + \frac{\langle u_\beta \rangle \widehat{\omega}'_\beta}{r} - \underline{i} k_z \langle u_z \rangle \widehat{\omega}'_r \\
& - \widehat{u}'_r \frac{\partial \langle \omega_r \rangle}{\partial r} - \frac{\widehat{u}'_\beta}{r} \frac{\partial \langle \omega_r \rangle}{\partial \beta} + \frac{\widehat{u}'_\beta \langle \omega_\beta \rangle}{r} - \widehat{u}'_z \frac{\partial \langle \omega_r \rangle}{\partial z} \\
& + \frac{\partial \widehat{\omega}'_r \widehat{u}'_r}{\partial r} + \frac{\widehat{\omega}'_r \widehat{u}'_r}{r} + \frac{1}{r} \frac{\partial \widehat{\omega}'_\beta \widehat{u}'_r}{\partial \beta} - \frac{\widehat{\omega}'_\beta \widehat{u}'_\beta}{r} + \underline{i} k_z \widehat{\omega}'_z \widehat{u}'_r \\
& + \langle \omega_r \rangle \frac{\partial \widehat{u}'_r}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \frac{\partial \widehat{u}'_r}{\partial \beta} - \frac{\langle \omega_\beta \rangle \widehat{u}'_\beta}{r} + \underline{i} k_z \langle \omega_z \rangle \widehat{u}'_r \\
& + \widehat{\omega}'_r \frac{\partial \langle u_r \rangle}{\partial r} + \frac{\widehat{\omega}'_\beta}{r} \frac{\partial \langle u_r \rangle}{\partial \beta} - \frac{\widehat{\omega}'_\beta \langle u_\beta \rangle}{r} + \widehat{\omega}'_z \frac{\partial \langle u_r \rangle}{\partial z} \\
& - \frac{\partial \widehat{u}'_r \widehat{\omega}'_r}{\partial r} - \frac{\widehat{u}'_r \widehat{\omega}'_r}{r} - \frac{1}{r} \frac{\partial \widehat{u}'_\beta \widehat{\omega}'_r}{\partial \beta} + \frac{\widehat{u}'_\beta \widehat{\omega}'_\beta}{r} - \underline{i} k_z \widehat{u}'_z \widehat{\omega}'_r \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{\omega}'_r}{\partial r} - \frac{\widehat{\omega}'_r}{r^2} + \frac{\partial^2 \widehat{\omega}'_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{\omega}'_r}{\partial \beta^2} - \frac{2}{r^2} \frac{\partial \widehat{\omega}'_\beta}{\partial \beta} - k_z^2 \widehat{\omega}'_r \right) \\
& + 2\Omega \left(\frac{\partial \widehat{u}'_r}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial \widehat{u}'_r}{\partial \beta} \sin \beta + \frac{\widehat{u}'_\beta}{r} \sin \beta \right) , \tag{B.12}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \widehat{\omega}'_\beta}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{\omega}'_\beta}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{\omega}'_\beta}{\partial \beta} - \frac{\langle u_\beta \rangle \widehat{\omega}'_r}{r} - \underline{i} k_z \langle u_z \rangle \widehat{\omega}'_\beta \\
& - \widehat{u}'_r \frac{\partial \langle \omega_\beta \rangle}{\partial r} - \frac{\widehat{u}'_\beta}{r} \frac{\partial \langle \omega_\beta \rangle}{\partial \beta} - \frac{\widehat{u}'_\beta \langle \omega_r \rangle}{r} - \widehat{u}'_z \frac{\partial \langle \omega_\beta \rangle}{\partial z} \\
& + \frac{\partial \widehat{\omega}'_r \widehat{u}'_\beta}{\partial r} + \frac{\widehat{\omega}'_r \widehat{u}'_\beta}{r} + \frac{1}{r} \frac{\partial \widehat{\omega}'_\beta \widehat{u}'_\beta}{\partial \beta} + \frac{\widehat{\omega}'_\beta \widehat{u}'_r}{r} + \underline{i} k_z \widehat{\omega}'_z \widehat{u}'_\beta \\
& + \langle \omega_r \rangle \frac{\partial \widehat{u}'_\beta}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \frac{\partial \widehat{u}'_\beta}{\partial \beta} + \frac{\langle \omega_\beta \rangle \widehat{u}'_r}{r} + \underline{i} k_z \langle \omega_z \rangle \widehat{u}'_\beta \\
& + \widehat{\omega}'_r \frac{\partial \langle u_\beta \rangle}{\partial r} + \frac{\widehat{\omega}'_\beta}{r} \frac{\partial \langle u_\beta \rangle}{\partial \beta} + \frac{\widehat{\omega}'_\beta \langle u_r \rangle}{r} + \widehat{\omega}'_z \frac{\partial \langle u_\beta \rangle}{\partial z} \\
& - \frac{\partial \widehat{u}'_r \widehat{\omega}'_\beta}{\partial r} - \frac{\widehat{u}'_r \widehat{\omega}'_\beta}{r} - \frac{1}{r} \frac{\partial \widehat{u}'_\beta \widehat{\omega}'_\beta}{\partial \beta} - \frac{\widehat{u}'_\beta \widehat{\omega}'_r}{r} - \underline{i} k_z \widehat{u}'_z \widehat{\omega}'_\beta \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{\omega}'_\beta}{\partial r} - \frac{\widehat{\omega}'_\beta}{r^2} + \frac{\partial^2 \widehat{\omega}'_\beta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{\omega}'_\beta}{\partial \beta^2} + \frac{2}{r^2} \frac{\partial \widehat{\omega}'_r}{\partial \beta} - k_z^2 \widehat{\omega}'_\beta \right) \\
& + 2\Omega \left(\frac{\partial \widehat{u}'_\beta}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial \widehat{u}'_\beta}{\partial \beta} \sin \beta - \frac{\widehat{u}'_r}{r} \sin \beta \right) \quad ,
\end{aligned} \tag{B.13}$$

$$\begin{aligned}
\frac{\partial \widehat{\omega'_z}}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{\omega'_z}}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{\omega'_z}}{\partial \beta} - \underline{i} k_z \langle u_z \rangle \widehat{\omega'_z} \\
& - \widehat{u'_r} \frac{\partial \langle \omega_z \rangle}{\partial r} - \frac{\widehat{u'_\beta}}{r} \frac{\partial \langle \omega_z \rangle}{\partial \beta} - \widehat{u'_z} \frac{\partial \langle \omega_z \rangle}{\partial z} \\
& + \frac{\partial \widehat{\omega'_r u'_z}}{\partial r} + \frac{\widehat{\omega'_r u'_z}}{r} + \frac{1}{r} \frac{\partial \widehat{\omega'_\beta u'_z}}{\partial \beta} + \underline{i} k_z \widehat{\omega'_z u'_z} \\
& + \langle \omega_r \rangle \frac{\partial \widehat{u'_z}}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \frac{\partial \widehat{u'_z}}{\partial \beta} + \underline{i} k_z \langle \omega_z \rangle \widehat{u'_z} \\
& + \widehat{\omega'_r} \frac{\partial \langle u_z \rangle}{\partial r} + \frac{\widehat{\omega'_\beta}}{r} \frac{\partial \langle u_z \rangle}{\partial \beta} + \widehat{\omega'_z} \frac{\partial \langle u_z \rangle}{\partial z} \\
& - \frac{\partial \widehat{u'_r \omega'_z}}{\partial r} - \frac{\widehat{u'_r \omega'_z}}{r} - \frac{1}{r} \frac{\partial \widehat{u'_\beta \omega'_z}}{\partial \beta} - \underline{i} k_z \widehat{u'_z \omega'_z} \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{\omega'_z}}{\partial r} + \frac{\partial^2 \widehat{\omega'_z}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{\omega'_z}}{\partial \beta^2} - k_z^2 \widehat{\omega'_z} \right) \\
& + 2\Omega \left(\frac{\partial \widehat{u'_z}}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial \widehat{u'_z}}{\partial \beta} \sin \beta \right) .
\end{aligned} \tag{B.14}$$

Notably, due to the independence of the terms in the last two rows of equations (B.7)-(B.9) from z in physical space, their Fourier transforms yield nontrivial contributions only at $k_z = 0$. Meanwhile, the Fourier transforms of the vorticity fluctuations ($\widehat{\omega'_r}$, $\widehat{\omega'_\beta}$ and $\widehat{\omega'_z}$) are zero at $k_z = 0$; therefore, these terms do not appear in equations (B.12)-(B.14). Moreover, the following differentiation property of the Fourier transform for a quantity $\phi(r, \beta, z, t)$ in the homogeneous (z -) direction is utilised, viz.

$$\mathcal{F} \left[\frac{\partial^n \phi(r, \beta, z, t)}{\partial z^n} \right] = (\underline{i} k_z)^n \widehat{\phi}(r, \beta, k_z, t) \quad , \tag{B.15}$$

where n represents the order of the derivative. Then, by taking the complex conjugates of equations (B.12)-(B.14), the conjugate forms of the Fourier-transformed

fluctuating-vorticity equations can be written as

$$\begin{aligned}
\frac{\partial \widehat{\omega}_r^*}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{\omega}_r^*}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{\omega}_r^*}{\partial \beta} + \frac{\langle u_\beta \rangle \widehat{\omega}_\beta'}{r} + \underline{i} k_z \langle u_z \rangle \widehat{\omega}_r^* \\
& - \widehat{u}_r^* \frac{\partial \langle \omega_r \rangle}{\partial r} - \frac{\widehat{u}_\beta'}{r} \frac{\partial \langle \omega_r \rangle}{\partial \beta} + \frac{\widehat{u}_\beta'}{r} \langle \omega_\beta \rangle - \widehat{u}_z^* \frac{\partial \langle \omega_r \rangle}{\partial z} \\
& + \frac{\partial \widehat{\omega}_r' u_r^*}{\partial r} + \frac{\widehat{\omega}_r' u_r^*}{r} + \frac{1}{r} \frac{\partial \widehat{\omega}_\beta' u_r^*}{\partial \beta} - \frac{\widehat{\omega}_\beta' u_r^*}{r} - \underline{i} k_z \widehat{\omega}_z' u_r^* \\
& + \langle \omega_r \rangle \frac{\partial \widehat{u}_r^*}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \frac{\partial \widehat{u}_r^*}{\partial \beta} - \frac{\langle \omega_\beta \rangle \widehat{u}_\beta^*}{r} - \underline{i} k_z \langle \omega_z \rangle \widehat{u}_r^* \\
& + \widehat{\omega}_r^* \frac{\partial \langle u_r \rangle}{\partial r} + \frac{\widehat{\omega}_\beta^*}{r} \frac{\partial \langle u_r \rangle}{\partial \beta} - \frac{\widehat{\omega}_\beta^* \langle u_\beta \rangle}{r} + \widehat{\omega}_z^* \frac{\partial \langle u_r \rangle}{\partial z} \\
& - \frac{\partial \widehat{u}_r' \widehat{\omega}_r^*}{\partial r} - \frac{\widehat{u}_r' \widehat{\omega}_r^*}{r} - \frac{1}{r} \frac{\partial \widehat{u}_\beta' \widehat{\omega}_r^*}{\partial \beta} + \frac{\widehat{u}_\beta' \widehat{\omega}_\beta^*}{r} + \underline{i} k_z \widehat{u}_z' \widehat{\omega}_r^* \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{\omega}_r^*}{\partial r} - \frac{\widehat{\omega}_r^*}{r^2} + \frac{\partial^2 \widehat{\omega}_r^*}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{\omega}_r^*}{\partial \beta^2} - \frac{2}{r^2} \frac{\partial \widehat{\omega}_\beta^*}{\partial \beta} - k_z^2 \widehat{\omega}_r^* \right) \\
& + 2\Omega \left(\frac{\partial \widehat{u}_r^*}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial \widehat{u}_r^*}{\partial \beta} \sin \beta + \frac{\widehat{u}_\beta^*}{r} \sin \beta \right) , \tag{B.16}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \widehat{\omega}'_\beta}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{\omega}'_\beta}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{\omega}'_\beta}{\partial \beta} - \frac{\langle u_\beta \rangle \widehat{\omega}'_r}{r} + \underline{i} k_z \langle u_z \rangle \widehat{\omega}'_\beta \\
& - \widehat{u}'_r \frac{\partial \langle \omega_\beta \rangle}{\partial r} - \frac{\widehat{u}'_\beta}{r} \frac{\partial \langle \omega_\beta \rangle}{\partial \beta} - \frac{\widehat{u}'_\beta \langle \omega_r \rangle}{r} - \widehat{u}'_z \frac{\partial \langle \omega_\beta \rangle}{\partial z} \\
& + \frac{\partial \widehat{\omega}'_r u'_\beta}{\partial r} + \frac{\widehat{\omega}'_r u'_\beta}{r} + \frac{1}{r} \frac{\partial \widehat{\omega}'_\beta u'_\beta}{\partial \beta} + \frac{\widehat{\omega}'_\beta u'_r}{r} - \underline{i} k_z \widehat{\omega}'_z u'_\beta \\
& + \langle \omega_r \rangle \frac{\partial \widehat{u}'_\beta}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \frac{\partial \widehat{u}'_\beta}{\partial \beta} + \frac{\langle \omega_\beta \rangle \widehat{u}'_r}{r} - \underline{i} k_z \langle \omega_z \rangle \widehat{u}'_\beta \\
& + \widehat{\omega}'_r \frac{\partial \langle u_\beta \rangle}{\partial r} + \frac{\widehat{\omega}'_\beta}{r} \frac{\partial \langle u_\beta \rangle}{\partial \beta} + \frac{\widehat{\omega}'_\beta \langle u_r \rangle}{r} + \widehat{\omega}'_z \frac{\partial \langle u_\beta \rangle}{\partial z} \\
& - \frac{\partial \widehat{u}'_r \omega'_\beta}{\partial r} - \frac{\widehat{u}'_r \omega'_\beta}{r} - \frac{1}{r} \frac{\partial \widehat{u}'_\beta \omega'_\beta}{\partial \beta} - \frac{\widehat{u}'_\beta \omega'_r}{r} + \underline{i} k_z \widehat{u}'_z \omega'_\beta \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{\omega}'_\beta}{\partial r} - \frac{\widehat{\omega}'_\beta}{r^2} + \frac{\partial^2 \widehat{\omega}'_\beta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{\omega}'_\beta}{\partial \beta^2} + \frac{2}{r^2} \frac{\partial \widehat{\omega}'_r}{\partial \beta} - k_z^2 \widehat{\omega}'_\beta \right) \\
& + 2\Omega \left(\frac{\partial \widehat{u}'_\beta}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial \widehat{u}'_\beta}{\partial \beta} \sin \beta - \frac{\widehat{u}'_r}{r} \sin \beta \right) ,
\end{aligned} \tag{B.17}$$

$$\begin{aligned}
\frac{\partial \widehat{\omega'_z}^*}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{\omega'_z}^*}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{\omega'_z}^*}{\partial \beta} + \underline{i} k_z \langle u_z \rangle \widehat{\omega'_z}^* \\
& - \widehat{u'_r}^* \frac{\partial \langle \omega_z \rangle}{\partial r} - \frac{\widehat{u'_\beta}^*}{r} \frac{\partial \langle \omega_z \rangle}{\partial \beta} - \widehat{u'_z}^* \frac{\partial \langle \omega_z \rangle}{\partial z} \\
& + \frac{\partial \widehat{\omega'_r u'_z}^*}{\partial r} + \frac{\widehat{\omega'_r u'_z}^*}{r} + \frac{1}{r} \frac{\partial \widehat{\omega'_\beta u'_z}^*}{\partial \beta} - \underline{i} k_z \widehat{\omega'_z u'_z}^* \\
& + \langle \omega_r \rangle \frac{\partial \widehat{u'_z}^*}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \frac{\partial \widehat{u'_z}^*}{\partial \beta} - \underline{i} k_z \langle \omega_z \rangle \widehat{u'_z}^* \\
& + \widehat{\omega'_r}^* \frac{\partial \langle u_z \rangle}{\partial r} + \frac{\widehat{\omega'_\beta}^*}{r} \frac{\partial \langle u_z \rangle}{\partial \beta} + \widehat{\omega'_z}^* \frac{\partial \langle u_z \rangle}{\partial z} \\
& - \frac{\partial \widehat{u'_r \omega'_z}^*}{\partial r} - \frac{\widehat{u'_r \omega'_z}^*}{r} - \frac{1}{r} \frac{\partial \widehat{u'_\beta \omega'_z}^*}{\partial \beta} + \underline{i} k_z \widehat{u'_z \omega'_z}^* \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{\omega'_z}^*}{\partial r} + \frac{\partial^2 \widehat{\omega'_z}^*}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{\omega'_z}^*}{\partial \beta^2} - k_z^2 \widehat{\omega'_z}^* \right) \\
& + 2\Omega \left(\frac{\partial \widehat{u'_z}^*}{\partial r} \cos \beta - \frac{1}{r} \frac{\partial \widehat{u'_z}^*}{\partial \beta} \sin \beta \right) .
\end{aligned} \tag{B.18}$$

The next step is to derive the transport equation for each fluctuating vorticity spectrum, i.e., $\widetilde{E}_{\omega_r}$, $\widetilde{E}_{\omega_\beta}$ and $\widetilde{E}_{\omega_z}$. Taking the radial component $\widetilde{E}_{\omega_r}$ as an example, the derivation proceeds by multiplying equations (B.16) and (B.12) by $\widehat{\omega'_r}$ and $\widehat{\omega'_r}^*$, respectively, followed by time averaging and taking the real part of the resulting sum. For clarity, consider the time derivative term of $\widetilde{E}_{\omega_r}$. Applying the above procedure to the left-hand sides of equations (B.16) and (B.12) yields

$$\text{Re} \left\{ \overline{\left(\frac{\partial \widehat{\omega'_r}^*}{\partial t} \right) \widehat{\omega'_r}} + \overline{\left(\frac{\partial \widehat{\omega'_r}}{\partial t} \right) \widehat{\omega'_r}^*} \right\} = \text{Re} \left\{ \frac{\partial \widehat{\omega'_r}^* \widehat{\omega'_r}}{\partial t} \right\} = \frac{\partial \widetilde{E}_{\omega_r}}{\partial t} . \tag{B.19}$$

The transport equations of $\widetilde{E}_{\omega_\beta}$ and $\widetilde{E}_{\omega_z}$ are derived in the same manner. Note that only the real parts of these three transport equations are considered. Terms such as $\overline{\widehat{\omega'_r}^* \widehat{u'_\beta}}$ and $\overline{\widehat{\omega'_r} \widehat{u'_\beta}^*}$ are complex conjugates of each other and therefore yield identical results when their real parts are taken. Accordingly, in order to simplify the transport

equations, only one of the two conjugated ‘equivalent’ terms is kept and multiplied by two.

Following the final step used to obtain the transport equation of the fluctuating enstrophy $\langle \omega'_i \omega'_i \rangle$ in Appendix B.1, the transport equation of the fluctuating enstrophy spectrum $\tilde{E}_{\omega\omega}$ is derived by summing the transport equations of the three fluctuating-vorticity spectral components, i.e., $\tilde{E}_{\omega\omega} = \tilde{E}_{\omega_r} + \tilde{E}_{\omega_\beta} + \tilde{E}_{\omega_z}$, and can be written as

$$\begin{aligned}
\frac{\partial \tilde{E}_{\omega\omega}}{\partial t} = & \underbrace{\operatorname{Re} \left\{ -\langle u_r \rangle \frac{\overline{\widehat{\omega}'_i}^* \widehat{\omega}'_i}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{\omega}'_i}^* \widehat{\omega}'_i}{\partial \beta} \right\}}_{\tilde{C}_{\omega\omega}} \\
& + \underbrace{\operatorname{Re} \left\{ 2 \left(-\overline{\widehat{\omega}'_i}^* \widehat{u}'_r \frac{\partial \langle \omega_i \rangle}{\partial r} - \frac{\overline{\widehat{\omega}'_i}^* \widehat{u}'_\beta}{r} \frac{\partial \langle \omega_i \rangle}{\partial \beta} - \overline{\widehat{\omega}'_\beta}^* \widehat{u}'_\beta \frac{\langle \omega_r \rangle}{r} \right) \right\}}_{\tilde{P}_{\omega\omega}^g} \\
& + \underbrace{\operatorname{Re} \left\{ 2 \left(\overline{\widehat{\omega}'_i}^* \widehat{\omega}'_r \frac{\partial \widehat{u}'_i}{\partial r} + \frac{1}{r} \overline{\widehat{\omega}'_i}^* \widehat{\omega}'_\beta \frac{\partial \widehat{u}'_i}{\partial \beta} - \frac{\overline{\widehat{\omega}'_r}^* \widehat{\omega}'_\beta \widehat{u}'_\beta}{r} + \frac{\overline{\widehat{\omega}'_\beta}^* \widehat{\omega}'_\beta \widehat{u}'_r}{r} \right) \right\}}_{\tilde{P}_{\omega\omega}^t} \\
& + \underbrace{\operatorname{Re} \left\{ 2 \left(\langle \omega_r \rangle \overline{\widehat{\omega}'_i}^* \frac{\partial \widehat{u}'_i}{\partial r} + \frac{\langle \omega_\beta \rangle}{r} \overline{\widehat{\omega}'_i}^* \frac{\partial \widehat{u}'_i}{\partial \beta} + i k_z \langle \omega_z \rangle \overline{\widehat{\omega}'_i}^* \widehat{u}'_i + \langle \omega_\beta \rangle \frac{\overline{\widehat{\omega}'_\beta}^* \widehat{u}'_r}{r} \right) \right\}}_{\tilde{P}_{\omega\omega}^m} \\
& + \underbrace{\operatorname{Re} \left\{ 2 \left(\overline{\widehat{\omega}'_i}^* \widehat{\omega}'_r \frac{\partial \langle u_i \rangle}{\partial r} + \frac{\overline{\widehat{\omega}'_i}^* \widehat{\omega}'_\beta}{r} \frac{\partial \langle u_i \rangle}{\partial \beta} - \overline{\widehat{\omega}'_r}^* \widehat{\omega}'_\beta \frac{\langle u_\beta \rangle}{r} + \overline{\widehat{\omega}'_\beta}^* \widehat{\omega}'_\beta \frac{\langle u_r \rangle}{r} \right) \right\}}_{\tilde{P}_{\omega\omega}^s} \\
& + \underbrace{\operatorname{Re} \left\{ 2 \left(-\frac{\partial \overline{\widehat{\omega}'_i}^* \widehat{u}'_r}{\partial r} \widehat{\omega}'_i - \frac{1}{r} \frac{\partial \overline{\widehat{\omega}'_i}^* \widehat{u}'_\beta}{\partial \beta} \widehat{\omega}'_i + i k_z \overline{\widehat{\omega}'_i}^* \widehat{u}'_z \widehat{\omega}'_i \right) - \frac{\overline{\widehat{\omega}'_i}^* \widehat{u}'_r \widehat{\omega}'_i}{r} + \frac{\partial \overline{\widehat{\omega}'_i}^* \widehat{u}'_i}{\partial r} + \frac{1}{r} \frac{\partial \overline{\widehat{\omega}'_\beta}^* \widehat{u}'_i}{\partial \beta} \right. \right. \\
& \quad \left. \left. + 2 \left(\overline{\widehat{\omega}'_i}^* \widehat{u}'_i \frac{\partial \widehat{\omega}'_r}{\partial r} + \frac{\overline{\widehat{\omega}'_i}^* \widehat{u}'_\beta \widehat{\omega}'_r}{r} + \frac{1}{r} \overline{\widehat{\omega}'_i}^* \widehat{u}'_i \frac{\partial \widehat{\omega}'_\beta}{\partial \beta} + i k_z \overline{\widehat{\omega}'_i}^* \widehat{u}'_i \widehat{\omega}'_z \right) \right\}}_{\tilde{T}_{\omega\omega}} \\
& + \underbrace{\operatorname{Re} \left\{ 2\nu \left(-\frac{\partial \overline{\widehat{\omega}'_i}^*}{\partial r} \frac{\partial \widehat{\omega}'_i}{\partial r} - \frac{1}{r^2} \frac{\partial \overline{\widehat{\omega}'_i}^*}{\partial \beta} \frac{\partial \widehat{\omega}'_i}{\partial \beta} - k_z^2 \overline{\widehat{\omega}'_i}^* \widehat{\omega}'_i - \frac{2}{r^2} \overline{\widehat{\omega}'_r}^* \frac{\partial \widehat{\omega}'_\beta}{\partial \beta} + \frac{2}{r^2} \overline{\widehat{\omega}'_\beta}^* \frac{\partial \widehat{\omega}'_r}{\partial \beta} - \frac{\overline{\widehat{\omega}'_r}^* \widehat{\omega}'_r}{r^2} - \frac{\overline{\widehat{\omega}'_\beta}^* \widehat{\omega}'_\beta}{r^2} \right) \right\}}_{\tilde{\varepsilon}_{\omega\omega}} \\
& + \underbrace{\operatorname{Re} \left\{ -\frac{\partial \overline{\widehat{\omega}'_i}^* \widehat{u}'_r}{\partial r} \widehat{\omega}'_i - \frac{\overline{\widehat{\omega}'_i}^* \widehat{u}'_\beta \widehat{\omega}'_i}{r} - \frac{1}{r} \frac{\partial \overline{\widehat{\omega}'_i}^* \widehat{u}'_\beta}{\partial \beta} \widehat{\omega}'_i + 2 \left(\frac{\overline{\widehat{\omega}'_\beta}^* \widehat{u}'_r \widehat{\omega}'_r}{r} - \frac{\overline{\widehat{\omega}'_r}^* \widehat{u}'_\beta \widehat{\omega}'_\beta}{r} \right) \right\}}_{\tilde{D}_{\omega\omega}^t} \\
& + \underbrace{\operatorname{Re} \left\{ \nu \left(\frac{\partial^2 \overline{\widehat{\omega}'_i}^* \widehat{\omega}'_i}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{\widehat{\omega}'_i}^* \widehat{\omega}'_i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \overline{\widehat{\omega}'_i}^* \widehat{\omega}'_i}{\partial \beta^2} \right) \right\}}_{\tilde{D}_{\omega\omega}^v} \\
& + \underbrace{\operatorname{Re} \left\{ 4\Omega \left(\overline{\widehat{\omega}'_i}^* \frac{\partial \widehat{u}'_i}{\partial r} \cos \beta - \frac{1}{r} \overline{\widehat{\omega}'_i}^* \frac{\partial \widehat{u}'_i}{\partial \beta} \sin \beta + \frac{\overline{\widehat{\omega}'_r}^* \widehat{u}'_\beta}{r} \sin \beta - \frac{\overline{\widehat{\omega}'_\beta}^* \widehat{u}'_r}{r} \sin \beta \right) \right\}}_{\tilde{P}_{\omega\omega}^c} = 0 \quad .
\end{aligned}$$

(B.20)

Appendix C

Transport equations of turbulent heat flux

In this appendix, the exact transport equations for the turbulent heat fluxes $\langle u'_z \theta' \rangle$, $\langle u'_r \theta' \rangle$ and $\langle u'_\beta \theta' \rangle$ in a cylindrical coordinate system are derived in the context of turbulent heat transfer in a circular pipe subjected to system rotation. It should be noted that the Coriolis production terms corresponding to the two rotation directions are distinguished using the labels “**Radial**” for radial system rotation and “**Axial**” for axial system rotation. Based on a through literature study, it is confirmed that this set of transport equations are not readily available in the current literature. Here, without showing the lengthy derivation process, they are directly given as follows:

$$\begin{aligned}
\frac{\partial \langle u'_z \theta' \rangle}{\partial t} = & \underbrace{-\langle u_r \rangle \frac{\partial \langle u'_z \theta' \rangle}{\partial r} - \langle u_\beta \rangle \frac{1}{r} \frac{\partial \langle u'_z \theta' \rangle}{\partial \beta} - \langle u_z \rangle \frac{\partial \langle u'_z \theta' \rangle}{\partial z}}_{C_z} \\
& + \underbrace{\left(\begin{aligned} & -\langle u'_r \theta' \rangle \frac{\partial \langle u_z \rangle}{\partial r} - \langle u'_\beta \theta' \rangle \frac{1}{r} \frac{\partial \langle u_z \rangle}{\partial \beta} - \langle u'_z \theta' \rangle \frac{\partial \langle u_z \rangle}{\partial z} \\ & -\langle u'_r u'_z \rangle \frac{\partial \langle \theta \rangle}{\partial r} - \langle u'_\beta u'_z \rangle \frac{1}{r} \frac{\partial \langle \theta \rangle}{\partial \beta} - \langle u'_z u'_z \rangle \frac{\partial \langle \theta \rangle}{\partial z} \end{aligned} \right)}_{P_z} \\
& + \underbrace{\left(-\langle u'_z u'_z \rangle \frac{dT_w}{dz} \right)}_{P_z^t} \\
& + \underbrace{\left[-\frac{1}{r} \frac{\partial r \langle u'_r u'_z \theta' \rangle}{\partial r} - \frac{1}{r} \frac{\partial \langle u'_\beta u'_z \theta' \rangle}{\partial \beta} - \frac{\partial \langle u'_z u'_z \theta' \rangle}{\partial z} \right]}_{D_z^t} \\
& + \underbrace{\frac{1}{\rho} \left(-\left\langle \frac{\partial p'}{\partial z} \theta' \right\rangle \right)}_{D_z^p} \\
& + \underbrace{\left\{ \begin{aligned} & \nu \left[\left\langle \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u'_z}{\partial r} \right) \theta' \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial^2 u'_z \theta'}{\partial \beta^2} \right\rangle + \left\langle \frac{\partial^2 u'_z \theta'}{\partial z^2} \right\rangle \right] + \\ & \alpha \left[\left\langle u'_z \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta'}{\partial r} \right) \right\rangle + \frac{1}{r^2} \left\langle u'_z \frac{\partial^2 \theta'}{\partial \beta^2} \right\rangle + \left\langle u'_z \frac{\partial^2 \theta'}{\partial z^2} \right\rangle \right] \end{aligned} \right\}}_{D_z^m} \\
& + \underbrace{\begin{aligned} & \text{Radial: } -2\Omega (\langle u'_r \theta' \rangle \sin \beta + \langle u'_\beta \theta' \rangle \cos \beta) \text{ or} \\ & \text{Axial : } 0 \end{aligned}}_{P_z^c} = 0 \quad , \tag{C.1}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \langle u'_r \theta' \rangle}{\partial t} = & \underbrace{-\langle u_r \rangle \frac{\partial \langle u'_r \theta' \rangle}{\partial r} - \langle u_\beta \rangle \frac{1}{r} \left(\frac{\partial \langle u'_r \theta' \rangle}{\partial \beta} - \langle u'_\beta \theta' \rangle \right) - \langle u_z \rangle \frac{\partial \langle u'_r \theta' \rangle}{\partial z}}_{C_r} \\
& + \underbrace{\left[-\langle u'_r \theta' \rangle \frac{\partial \langle u_r \rangle}{\partial r} - \langle u'_\beta \theta' \rangle \frac{1}{r} \left(\frac{\partial \langle u_r \rangle}{\partial \beta} - \langle u_\beta \rangle \right) - \langle u'_z \theta' \rangle \frac{\partial \langle u_r \rangle}{\partial z} \right.} \\
& \quad \left. - \langle u'_r u'_r \rangle \frac{\partial \langle \theta \rangle}{\partial r} - \langle u'_r u'_\beta \rangle \frac{1}{r} \frac{\partial \langle \theta \rangle}{\partial \beta} - \langle u'_r u'_z \rangle \frac{\partial \langle \theta \rangle}{\partial z} \right]}_{P_r} \\
& + \underbrace{\left(-\langle u'_r u'_z \rangle \frac{dT_w}{dz} \right)}_{P_r^t} \\
& + \underbrace{\left[-\frac{1}{r} \frac{\partial r \langle u'_r u'_r \theta' \rangle}{\partial r} - \frac{1}{r} \left(\frac{\partial \langle u'_r u'_\beta \theta' \rangle}{\partial \beta} - \langle u'_\beta u'_\beta \theta' \rangle \right) - \frac{\partial \langle u'_r u'_z \theta' \rangle}{\partial z} \right]}_{D_r^t} \\
& + \underbrace{\frac{1}{\rho} \left(-\left\langle \frac{\partial p'}{\partial r} \theta' \right\rangle \right)}_{D_r^p} \\
& + \underbrace{\left\{ \nu \left[\left\langle \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial r u'_r}{\partial r} \right) \theta' \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial^2 u'_r}{\partial \beta^2} \theta' \right\rangle - \frac{2}{r^2} \left\langle \frac{\partial u'_\beta}{\partial \beta} \theta' \right\rangle + \left\langle \frac{\partial^2 u'_r}{\partial z^2} \theta' \right\rangle \right] + \right.} \\
& \quad \left. \alpha \left[\left\langle u'_r \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta'}{\partial r} \right) \right\rangle + \frac{1}{r^2} \left\langle u'_r \frac{\partial^2 \theta'}{\partial \beta^2} \right\rangle + \left\langle u'_r \frac{\partial^2 \theta'}{\partial z^2} \right\rangle \right] \right\}}_{D_r^m} \\
& + \underbrace{\begin{array}{l} \text{Radial: } 2\Omega \langle u'_z \theta' \rangle \sin \beta \text{ or} \\ \text{Axial : } 2\Omega \langle u'_\beta \theta' \rangle \end{array}}_{P_r^c} = 0 \quad ,
\end{aligned}$$

(C.2)

and

$$\begin{aligned}
\frac{\partial \langle u'_\beta \theta' \rangle}{\partial t} = & \underbrace{-\langle u_r \rangle \frac{\partial \langle u'_\beta \theta' \rangle}{\partial r} - \langle u_\beta \rangle \frac{1}{r} \left(\frac{\partial \langle u'_\beta \theta' \rangle}{\partial \beta} + \langle u'_r \theta' \rangle \right) - \langle u_z \rangle \frac{\partial \langle u'_\beta \theta' \rangle}{\partial z}}_{C_\beta} \\
& + \underbrace{\left[-\langle u'_r \theta' \rangle \frac{\partial \langle u_\beta \rangle}{\partial r} - \langle u'_\beta \theta' \rangle \frac{1}{r} \left(\frac{\partial \langle u_\beta \rangle}{\partial \beta} + \langle u_r \rangle \right) - \langle u'_z \theta' \rangle \frac{\partial \langle u_\beta \rangle}{\partial z} \right.} \\
& \quad \left. - \langle u'_r u'_\beta \rangle \frac{\partial \langle \theta \rangle}{\partial r} - \langle u'_\beta u'_\beta \rangle \frac{1}{r} \frac{\partial \langle \theta \rangle}{\partial \beta} - \langle u'_\beta u'_z \rangle \frac{\partial \langle \theta \rangle}{\partial z} \right]}_{P_\beta} \\
& + \underbrace{\left(-\langle u'_\beta u'_z \rangle \frac{dT_w}{dz} \right)}_{P_\beta^t} \\
& + \underbrace{\left[-\frac{1}{r} \frac{\partial r \langle u'_r u'_\beta \theta' \rangle}{\partial r} - \frac{1}{r} \left(\frac{\partial \langle u'_\beta u'_\beta \theta' \rangle}{\partial \beta} + \langle u'_r u'_\beta \theta' \rangle \right) - \frac{\partial \langle u'_\beta u'_z \theta' \rangle}{\partial z} \right]}_{D_\beta^t} \\
& + \underbrace{\frac{1}{\rho r} \left(-\left\langle \frac{\partial p'}{\partial \beta} \theta' \right\rangle \right)}_{D_\beta^p} \\
& + \underbrace{\left\{ \nu \left[\left\langle \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial r u'_\beta}{\partial r} \right) \theta' \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial^2 u'_\beta}{\partial \beta^2} \theta' \right\rangle + \frac{2}{r^2} \left\langle \frac{\partial u'_r}{\partial \beta} \theta' \right\rangle + \left\langle \frac{\partial^2 u'_\beta}{\partial z^2} \theta' \right\rangle \right] + \right.} \\
& \quad \left. \alpha \left[\left\langle u'_\beta \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \theta'}{\partial r} \right) \right\rangle + \frac{1}{r^2} \left\langle u'_\beta \frac{\partial^2 \theta'}{\partial \beta^2} \right\rangle + \left\langle u'_\beta \frac{\partial^2 \theta'}{\partial z^2} \right\rangle \right] \right\}}_{D_\beta^m} \\
& + \underbrace{\begin{array}{l} \text{Radial: } 2\Omega \langle u'_z \theta' \rangle \cos \beta \text{ or} \\ \text{Axial : } -2\Omega \langle u'_r \theta' \rangle \end{array}}_{P_\beta^c} = 0 \quad .
\end{aligned} \tag{C.3}$$

Appendix D

Effects of pipe length on the accuracy of DNS results

A turbulence field consists of eddies of different temporal and spatial scales. Thus, in a high-fidelity numerical or experimental study of a transient turbulent flow problem, the frequency and wavelength ranges of energetic eddy motions need to be carefully considered. Since the pioneering work of Jiménez and Moin (1991), it has become understood that a minimum computational domain is needed in DNS in order to capture major physical processes associated with large energetic turbulence structures such as near-wall hairpins and streaks in a plane-channel flow. In the context of an axially rotating turbulent pipe flow, the axial turbulent motions are dominated by streaks at low rotation numbers, and by Taylor columns at moderate and high rotation numbers. Because the characteristic length scales of both streaks and Taylor columns increase as the rotation number Ro_b is increased, the pipe length L_z needs to be increased accordingly in DNS. Previous studies of the domain size effects on the accuracy of the DNS results have focused primarily on boundary-layer flows over flat plates. In this appendix, I study the effects of pipe length on the accuracy of DNS results in the context of an axially rotating pipe flow. A comparative study of 10 DNS cases of varying pipe lengths has been conducted at low and high rotation

Table D.1: Test cases for the comparative study of the pipe length effects at $Ro_b = 2$ and 20.

Ro_b	2				20					
Case #	3A	3B	3C	3	8A	8B	8C	8D	8E	8
Ro_b^A	1.84	1.86	1.83	1.83	16.86	17.96	19.4	21.9	23	22.97
Ro_τ^A	29.32	29.41	29.47	29.45	293.91	294.39	294.39	295.11	294.50	294.55
Re_b^A	5770	5684	5799	5805	6287	5902	5463	4840	4608	4614
Re_τ^A	180.74	180.2	179.87	179.97	180.33	180.03	180.03	179.6	179.96	179.94
L_z	πR	$2\pi R$	$7.958\pi R$	$30\pi R$	πR	$2\pi R$	$7.958\pi R$	$30\pi R$	$80\pi R$	$180\pi R$
N_z	120	240	960	3600	120	240	960	3600	4800	7200
N_{tot}	3,244,920	6,489,840	25,959,360	97,347,600	3,244,920	6,489,840	25,959,360	97,347,600	129,796,800	194,695,200
Δz^+	4.712	4.712	4.688	4.712	4.712	4.712	4.688	4.712	9.425	14.137
$(\Delta r^+)_{\max}$						3.595				
$(\Delta r^+)_{\min}$						0.123				
$(r\Delta\beta^+)_{\max}$						5.133				
$(r\Delta\beta^+)_{\min}$						0.813				
$(\Delta/\eta)_{\max}$	1.979	1.945	1.917	1.904	2.025	1.977	1.92	1.895	2.4	2.740

numbers ($Ro_b = 2$ and 20). The pipe lengths, Reynolds numbers and grid resolutions of the 10 DNS test cases are summarised in table D.1, in which the pipe length varies drastically from $L_z = \pi R$ to $180\pi R$.

D.1 Impact on the mean velocity

Figure D.1 compares the mean axial velocity $\langle u_z \rangle^+$ computed based on a variety of pipe lengths at two rotation numbers $Ro_b = 2$ and 20. For the purpose of comparison, the DNS data of Orlandi and Fatica (1997, based on the pipe length of $L_z = 10R$ or $3.183\pi R$) and Ebstein (1998, based on the pipe length of $L_z = 25R$ or $7.958\pi R$) at a similar Reynolds number of $Re_b = 4900$ are also displayed in the figures. In addition, the classical law of the wall for a non-rotating turbulent circular pipe flow based on von Kármán's two-layer boundary-layer model (i.e. $\langle u_z \rangle^+ = y^+$ for $y^+ \leq 10.8$ and $\langle u_z \rangle^+ = 2.5 \ln y^+ + 5.5$ for $y^+ > 10.8$) is shown to highlight the influence of axial system rotation on the mean axial velocity profile. By comparing figures D.1(a) and D.1(b), it is clear that the values of $\langle u_z \rangle^+$ of test cases of different pipe lengths agree better at the lower rotation number $Ro_b = 2$ than at the higher rotation number $Ro_b = 20$. In other words, the effect of pipe length on the prediction of $\langle u_z \rangle^+$ manifests as the

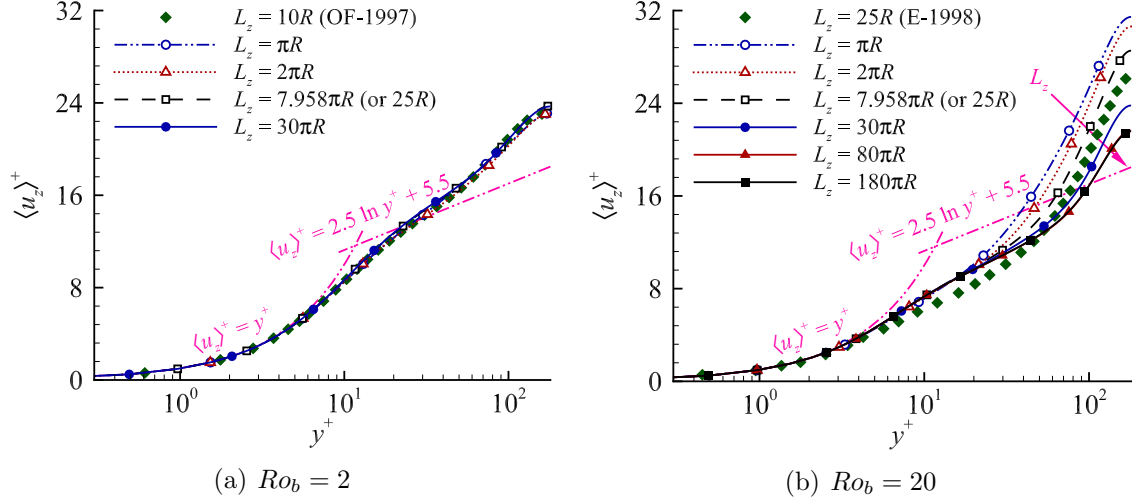


Figure D.1: Profiles of the mean axial velocity $\langle u_z \rangle^+$ computed based on varying pipe lengths L_z at two rotation numbers $Ro_b = 2$ and 20 , based on test cases listed in table D.1. For the purpose of comparison, the DNS results of Orlandi and Fatica (1997, labelled as ‘OF-1997’) and Ebstein (1998, labelled as ‘E-1998’) are displayed. The pink arrow points in the direction of increasing pipe length.

rotation number is increased. At the highest rotation number shown in figure D.1(b), it is clear that the profiles of $\langle u_z \rangle^+$ of all seven cases become increasingly different in values as the pipe centre is approached. The results from the short pipes (of $L_z = \pi R$ – $7.958\pi R$) based on either my simulations or those of Ebstein (1998) all deviate considerably from those of longer pipes (of $L_z = 30\pi R$ – $180\pi R$). The pipe length effect is evident in terms of the calculation of the mean axial velocity $\langle u_z \rangle^+$, and will be further demonstrated based on the calculations of RMS velocities, Reynolds shear stresses, two-point autocorrelation coefficients and premultiplied spectra of velocity fluctuations.

D.2 Impact on the second-order statistical moments

Figure D.2 compares the profiles of the second-order statistical moments of $Ro_b = 2$ and 20 obtained based on test cases of varying pipe lengths described in table D.1.

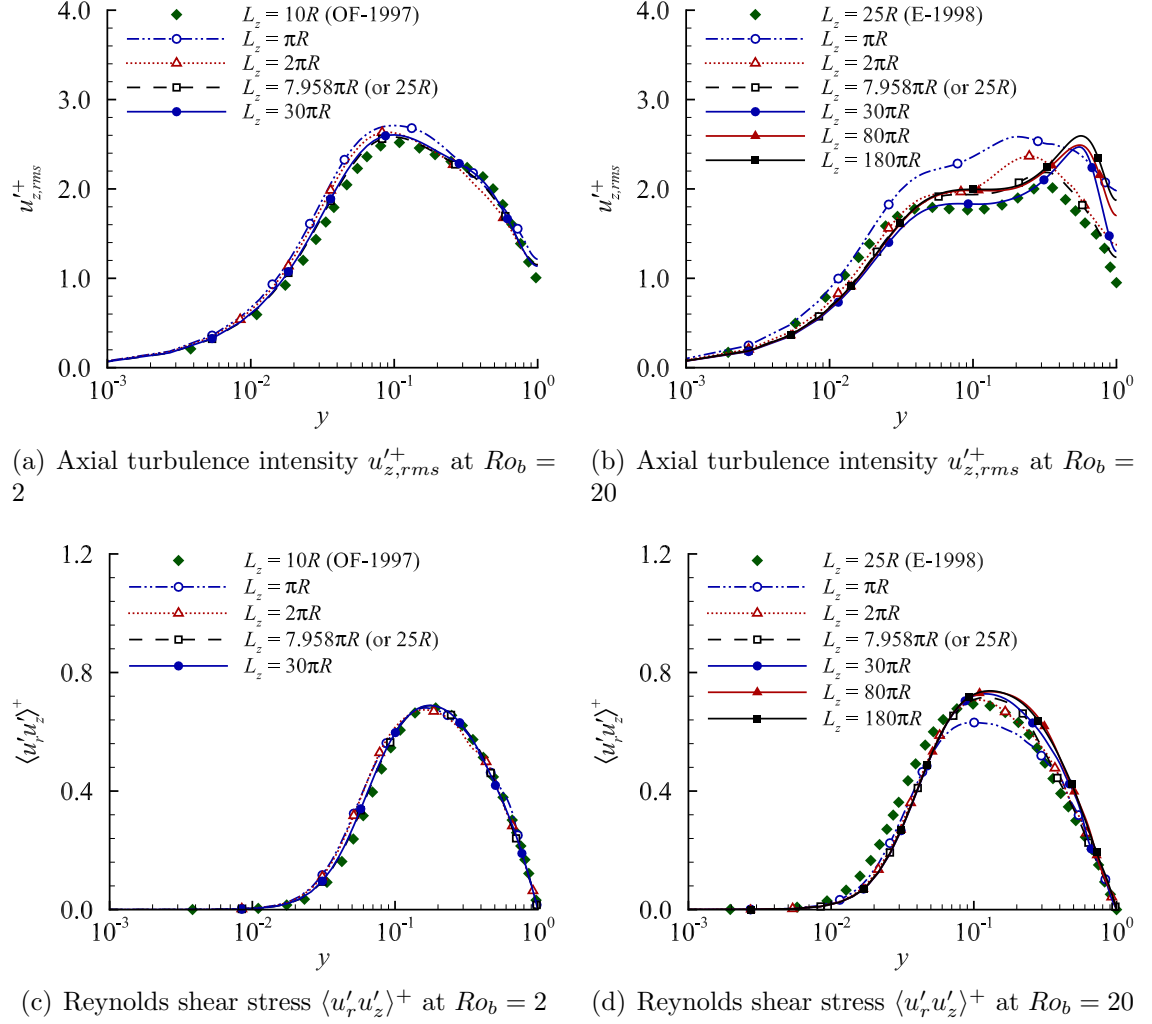


Figure D.2: Profiles of non-dimensionalised axial turbulence intensity $u'_{z,rms}{}^+$ and Reynolds shear stress $\langle u'_r u'_z \rangle^+$ for different pipe lengths (L_z) at two rotation numbers $Ro_b = 2$ and 20 (based on the test cases of table D.1). The DNS results of Orlandi and Fatica (1997, labelled as ‘OF-1997’) and Ebstein (1998, labelled as ‘E-1998’) are displayed for the purpose of comparison.

For the purpose of comparison, the DNS data of Orlandi and Fatica (1997) and Ebstein (1998) are also displayed in the figure. The second-order statistical moments of the velocity field are non-dimensionalised using the mean axial velocity at the pipe centreline U_{CL} in Orlandi and Fatica (1997) and are presented in a dimensional form in Orlandi and Ebstein (2000). The Reynolds stresses and RMS velocities non-dimensionalised using the wall friction velocity u_τ are available in Ebstein (1998).

From figures D.2(a,c), it is clear that predictions of $u'_{z,rms}^+$ and $\langle u'_r u'_z \rangle^+$ based on cases of different pipe lengths are generally similar at the lower rotation number $Ro_b = 2$, although the value of $u'_{z,rms}^+$ obtained using the shortest pipe length ($L_z = \pi R$) is slightly over-predicted in comparison with those obtained using longer pipes. However, as the rotation number is increased to $Ro_b = 20$, it shows clearly in figures D.2(b,d) that the DNS results of both $u'_{z,rms}^+$ and $\langle u'_r u'_z \rangle^+$ based on the three shortest pipe lengths (of $L_z = \pi R - 7.958R$) of either my current study or those of Ebstein (1998) deviate from those of longer pipes (of $L_z = 30\pi R - 180\pi R$). From figures D.2(b,d), it is seen that the performance of the shortest-pipe-length case of $L_z = \pi R$ is the least satisfactory. Furthermore, the impact of pipe length on the calculation of the second-order statistical moments is the most apparent with respect to the profiles of $u'_{z,rms}^+$ at $Ro_b = 20$ shown in figure D.2(b). Clearly, in order to calculate the values of $u'_{z,rms}^+$ and $\langle u'_r u'_z \rangle^+$ accurately at the highest rotation number $Ro_b = 20$, the minimum pipe length needs to be kept at $L_z = 30\pi R$.

D.3 Impact on the two-point correlations and energy spectra

The pipe length effects on the predictive accuracy of DNS with respect to the value of the axial two-point correlation R_{zz} and that of the non-dimensionalised premultiplied one-dimensional (1-D) spectrum of axial velocity fluctuations $k_z^+ \tilde{E}_{zz}^+$ are demonstrated in figure D.3. As indicated by table D.1, the largest pipe length is $L_z = 30\pi R$ and $180\pi R$ at $Ro_b = 2$ and 20 , respectively, which ensure the value of R_{zz} approaches zero and that of $k_z^+ \tilde{E}_{zz}^+$ is sufficiently low at the cutoff wavelength. From figures D.3(a,b), it is clear that the profile R_{zz} truncates at a high level if short pipes are used for $L_z \leq 2\pi R$ (at $Ro_b = 2$) and for $L_z \leq 7.958\pi R$ (at $Ro_b = 20$). In these short-pipe-length cases, the value of R_{zz} is considerably above zero and, therefore, they

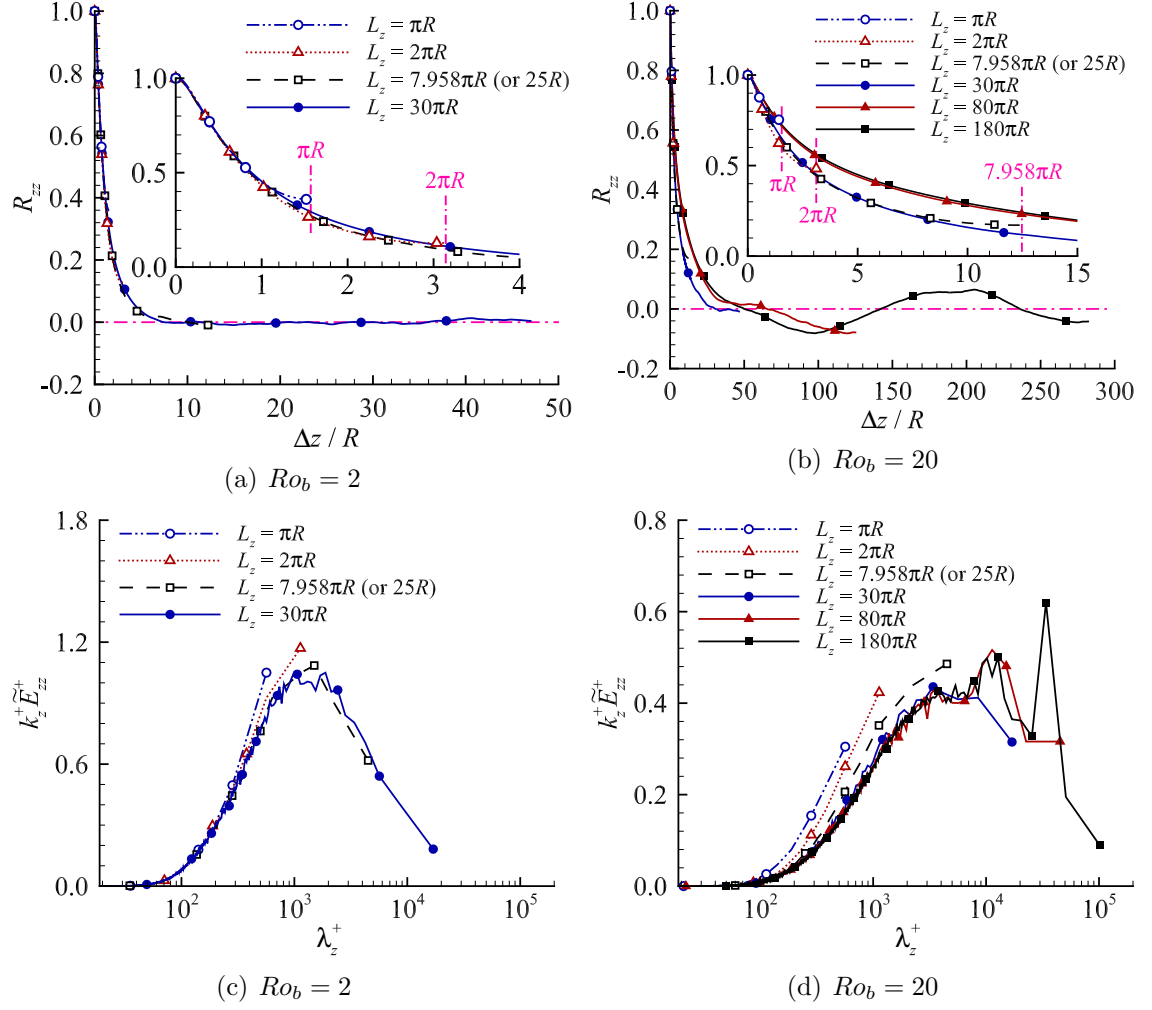


Figure D.3: Influence of pipe length on the predictive accuracy of DNS with respect to the axial two-point autocorrelation coefficient R_{zz} and premultiplied axial velocity spectra $k_z^+ \tilde{E}_{zz}^+$ at two rotation numbers $Ro_b = 2$ and 20 , based on test cases of different pipe lengths listed in table D.1. The values of R_{zz} and $k_z^+ \tilde{E}_{zz}^+$ are calculated along the streamwise direction at wall-normal position $y = 0.083$ (or $y^+ \approx 15$).

are incapable of capturing the signature point of the maximum negative correlation that corresponds to the characteristic length scale of the energetic flow structures. A similar situation holds in the profiles of $k_z^+ \tilde{E}_{zz}^+$ shown in figures D.3(c,d). From figure D.3(c), it is seen that the two short-pipe-length cases of $L_z \leq 2\pi R$ are unable to capture the full dome peak (i.e., peak I_2) of $k_z^+ \tilde{E}_{zz}^+$ at $Ro_b = 2$. At the high rotation number of $Ro_b = 20$, the two short-pipe-length cases of $L_z \leq 2\pi R$ completely miss

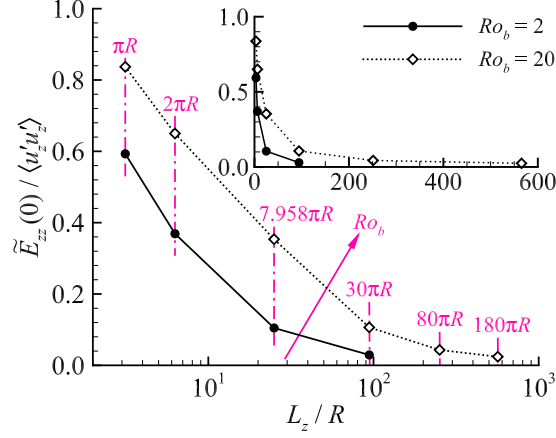


Figure D.4: Influence of the pipe length on DNS prediction of the axial velocity spectra $\tilde{E}_{zz}$ (non-dimensionalised by $\langle u'_z u'_z \rangle$) at zero mode ($k_z = 0$) with respect to axial pipe length L_z/R at two rotation numbers $Ro_b = 2$ and 20 , calculated along the streamwise direction at wall-normal position $y = 0.083$ (or $y^+ \approx 15$). The calculations are based on test cases listed in table D.1. Pink arrow points to the direction of an increasing rotation number.

the prediction of all three peaks (i.e. peak I_{20} , II_{20} and III_{20}) of $k_z^+ \tilde{E}_{zz}^+$. Although the case of $L_z = 30\pi R$ is able to capture peak I_{20} , it misses peaks II_{20} and III_{20} in its prediction. Even in the case of a larger pipe length $L_z = 80\pi R$, peak III_{20} is still missed in the DNS prediction. Therefore, in order to capture the energetic eddy motions of Taylor columns (as indicated by its two signature modal wavelengths of peaks II_{20} and III_{20}), the pipe length needs to be kept at $L_z = 180\pi R$ (which has a relatively small premultiplied energy spectrum loss that is 14.32% of its peak value at its cutoff wavelength).

D.4 Impact on the ratio of axial integral scale over pipe length

From the comparative study based on cases of different pipe lengths of table D.1, it is clear that an overly short pipe can give spurious calculations of the first- and

second-order flow statistics, and the values of the axial two-point correlation R_{zz} and premultiplied spectrum of axial velocity fluctuations $k_z^+ \tilde{E}_{zz}^+$ in DNS. To further demonstrate, the values of the zero mode of $k_z^+ \tilde{E}_{zz}^+$ calculated based on test cases of different pipe lengths can be compared. At a fixed radial position r , $\tilde{E}_{zz}(r, k_z)$ varies with k_z only and, therefore, can be denoted using $\tilde{E}_{zz}(k_z)$ for simplicity. At the zero mode ($k_z = 0$), the axial velocity spectrum relates to the axial Reynolds normal stress as

$$\tilde{E}_{zz}(0) = \frac{l_{int,z}}{L_z} \langle u'_z u'_z \rangle \quad , \quad (\text{D.1})$$

where $l_{int,z} = \int_0^{L_z} R_{zz} dz$ is the integral length scale of the axial (z) direction. I can rearrange this equation to obtain

$$\frac{\tilde{E}_{zz}(0)}{\langle u'_z u'_z \rangle} = \frac{l_{int,z}}{L_z} \quad , \quad (\text{D.2})$$

which has clear physical meaning that at the zero mode, the axial velocity spectra $\tilde{E}_{zz}(0)$ non-dimensionalised by $\langle u'_z u'_z \rangle$ is the ratio of the axial integral length $l_{int,z}$ over the pipe length L_z . The value of $l_{int,z}/L_z$ would approach zero, if the pipe length is significantly larger than the axial integral length scale. Figure D.4 shows the value of $l_{int,z}/L_z$ computed based on test cases of varying pipe lengths listed in table D.1. From figure D.4, it is clear that the axial integral length is much larger at $Ro_b = 20$ than at $Ro_b = 2$. This is expected, because both streaky structures and Taylor columns become increasingly elongated as the rotation number is increased. The percentage value of $l_{int,z}/L_z$ is 59.3% and 36.9% for the shortest pipe lengths of $L_z = \pi R$ and $2\pi R$ at $Ro_b = 2$. Both these percentage values are too large to be acceptable in DNS, because they indicate that the axial integral length scale are of the same order of the pipe length (as such, there is insufficient room for large turbulence structures to move and develop axially). At the high rotation number $Ro_b = 20$, the percentage value of $l_{int,z}/L_z$ is as high as 83.7% in the shortest-pipe-length case of $L_z = \pi R$, a scenario that is also physically unacceptable for DNS. By contrast, in the

two longest-pipe cases of $L_z = 80\pi R$ and $180\pi R$ for $Ro_b = 20$, the percentage value of $l_{int,z}/L_z$ drops to 4.3% and 2.4%, respectively, indicating the pipes are sufficiently long to allow large eddy motions (of integral length scales) to develop spatially.

Appendix E

Transport equations of velocity fluctuation statistics in physical and spectral spaces

In this appendix, the transport equations of all six Reynolds stresses and the TKE, $\langle u'_z u'_z \rangle$, $\langle u'_r u'_r \rangle$, $\langle u'_\beta u'_\beta \rangle$, $\langle u'_r u'_z \rangle$, $\langle u'_r u'_\beta \rangle$, $\langle u'_\beta u'_z \rangle$ and $\frac{1}{2} \langle u'_i u'_i \rangle$, are derived in physical space. The transport equations of the velocity-spectrum tensor $\tilde{E}_{ij}$, including the autospectra and co-spectra of the velocity fluctuations, and the TKE spectrum $\tilde{E}_{TKE}$ are subsequently derived in spectral space for axially rotating circular pipe flow in a cylindrical coordinate system. A thorough review of the literature indicates that simplified forms of these equations are not readily available. Therefore, they are presented here to facilitate a detailed study of the budget balances of the corresponding velocity fluctuation statistics in both physical and spectral spaces in the context of turbulent pipe flow subjected to axial system rotation.

E.1 Transport equations for Reynolds stresses and TKE

To obtain the transport equations of the Reynolds stresses and the TKE, the derivation begins with the equations governing the velocity fluctuations u'_r , u'_β and u'_z . These equations are obtained by subtracting the ensemble-averaged momentum equations from their instantaneous counterparts in equations (5.2)-(5.4), such as $u'_r = u_r - \langle u_r \rangle$. Therefore, the resulting equations in the r -, β - and z -directions are expressed as

$$\begin{aligned}
\frac{\partial u'_r}{\partial t} = & -\langle u_r \rangle \frac{\partial u'_r}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial u'_r}{\partial \beta} + \frac{\langle u_\beta \rangle}{r} u'_\beta - \langle u_z \rangle \frac{\partial u'_r}{\partial z} \\
& - u'_r \frac{\partial \langle u_r \rangle}{\partial r} - \frac{u'_\beta}{r} \frac{\partial \langle u_r \rangle}{\partial \beta} + \frac{u'_\beta}{r} \langle u_\beta \rangle - u'_z \frac{\partial \langle u_r \rangle}{\partial z} \\
& - \frac{u'_r u'_r}{r} - \frac{\partial u'_r u'_r}{\partial r} - \frac{1}{r} \frac{\partial u'_r u'_\beta}{\partial \beta} + \frac{u'_\beta u'_\beta}{r} - \frac{\partial u'_r u'_z}{\partial z} \\
& - \frac{1}{\rho} \frac{\partial p'}{\partial r} \\
& + \nu \left(\frac{1}{r} \frac{\partial u'_r}{\partial r} - \frac{u'_r}{r^2} + \frac{\partial^2 u'_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u'_r}{\partial \beta^2} - \frac{2}{r^2} \frac{\partial u'_\beta}{\partial \beta} + \frac{\partial^2 u'_r}{\partial z^2} \right) \\
& + 2\Omega u'_\beta \\
& + \frac{\langle u'_r u'_r \rangle}{r} + \frac{\partial \langle u'_r u'_r \rangle}{\partial r} + \frac{1}{r} \frac{\partial \langle u'_r u'_\beta \rangle}{\partial \beta} - \frac{\langle u'_\beta u'_\beta \rangle}{r} + \frac{\partial \langle u'_r u'_z \rangle}{\partial z} ,
\end{aligned} \tag{E.1}$$

$$\begin{aligned}
\frac{\partial u'_\beta}{\partial t} = & -\langle u_r \rangle \frac{\partial u'_\beta}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial u'_\beta}{\partial \beta} - \frac{\langle u_\beta \rangle}{r} u'_r - \langle u_z \rangle \frac{\partial u'_\beta}{\partial z} \\
& - u'_r \frac{\partial \langle u_\beta \rangle}{\partial r} - \frac{u'_\beta}{r} \frac{\partial \langle u_\beta \rangle}{\partial \beta} - \frac{u'_\beta}{r} \langle u_r \rangle - u'_z \frac{\partial \langle u_\beta \rangle}{\partial z} \\
& - \frac{u'_r u'_\beta}{r} - \frac{\partial u'_r u'_\beta}{\partial r} - \frac{1}{r} \frac{\partial u'_\beta u'_\beta}{\partial \beta} - \frac{u'_r u'_\beta}{r} - \frac{\partial u'_\beta u'_z}{\partial z} \\
& - \frac{1}{\rho r} \frac{\partial p'}{\partial \beta} \\
& + \nu \left(\frac{1}{r} \frac{\partial u'_\beta}{\partial r} - \frac{u'_\beta}{r^2} + \frac{\partial^2 u'_\beta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u'_\beta}{\partial \beta^2} + \frac{2}{r^2} \frac{\partial u'_r}{\partial \beta} + \frac{\partial^2 u'_\beta}{\partial z^2} \right) \\
& - 2\Omega u'_r \\
& + \frac{\langle u'_r u'_\beta \rangle}{r} + \frac{\partial \langle u'_r u'_\beta \rangle}{\partial r} + \frac{1}{r} \frac{\partial \langle u'_\beta u'_\beta \rangle}{\partial \beta} + \frac{\langle u'_r u'_\beta \rangle}{r} + \frac{\partial \langle u'_\beta u'_z \rangle}{\partial z} ,
\end{aligned} \tag{E.2}$$

$$\begin{aligned}
\frac{\partial u'_z}{\partial t} = & -\langle u_r \rangle \frac{\partial u'_z}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial u'_z}{\partial \beta} - \langle u_z \rangle \frac{\partial u'_z}{\partial z} \\
& - u'_r \frac{\partial \langle u_z \rangle}{\partial r} - \frac{u'_\beta}{r} \frac{\partial \langle u_z \rangle}{\partial \beta} - u'_z \frac{\partial \langle u_z \rangle}{\partial z} \\
& - \frac{u'_r u'_z}{r} - \frac{\partial u'_r u'_z}{\partial r} - \frac{1}{r} \frac{\partial u'_\beta u'_z}{\partial \beta} - \frac{\partial u'_z u'_z}{\partial z} \\
& - \frac{1}{\rho} \frac{\partial p'}{\partial z} \\
& + \nu \left(\frac{1}{r} \frac{\partial u'_z}{\partial r} + \frac{\partial^2 u'_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u'_z}{\partial \beta^2} + \frac{\partial^2 u'_z}{\partial z^2} \right) \\
& + \frac{\langle u'_r u'_z \rangle}{r} + \frac{\partial \langle u'_r u'_z \rangle}{\partial r} + \frac{1}{r} \frac{\partial \langle u'_\beta u'_z \rangle}{\partial \beta} + \frac{\partial \langle u'_z u'_z \rangle}{\partial z} .
\end{aligned} \tag{E.3}$$

By using the three fluctuating velocity equations, the transport equations of the six Reynolds stress components can be derived. To illustrate the derivation in detail, the radial-axial Reynolds shear stress $\langle u'_r u'_z \rangle$ is considered as an example. Its transport equation is obtained by multiplying equation (E.1) by u'_z , multiplying equation (E.3) by u'_r , adding the resulting equations, and then apply ensemble averaging. Through these operations, the time derivative term on the left-hand side of the transport

equation of $\langle u'_r u'_z \rangle$ is written as

$$\left\langle \frac{\partial u'_r}{\partial t} u'_z + \frac{\partial u'_z}{\partial t} u'_r \right\rangle = \frac{\partial \langle u'_r u'_z \rangle}{\partial t} \quad . \quad (\text{E.4})$$

It needs to be clarified that, following the operations described above, the terms in the last row of equations (E.1)-(E.3) vanish from the resulting transport equations due to the zero ensemble averages of the velocity fluctuations, i.e., $\langle u'_r \rangle = 0$ and $\langle u'_z \rangle = 0$.

As presented in equation (5.8), the general expression for the Reynolds stress transport equations given in the main text comprises eight budget terms. However, under the four flow conditions considered here, some of these terms need not be maintained in the budget balance. These conditions are as follows:

- the flow is statistically stationary, such that $\frac{\partial \langle \cdot \rangle}{\partial t} = 0$ and $\frac{\partial \overline{(\cdot)}}{\partial t} = 0$;
- the flow is fully developed in the streamwise direction, such that $\frac{\partial \langle \cdot \rangle}{\partial z} = 0$;
- the flow is axisymmetric, such that $\frac{\partial \langle \cdot \rangle}{\partial \beta} = 0$ and $\frac{\partial \overline{(\cdot)}}{\partial \beta} = 0$; and
- the flow satisfies the no-slip boundary condition, such that $\langle u_r \rangle \equiv 0$.

After applying these four flow conditions, the simplified forms of all six Reynolds stress transport equations are obtained. Therefore, the transport equation of axial Reynolds normal stress $\langle u'_z u'_z \rangle$ is

$$\underbrace{-2\langle u'_r u'_z \rangle \frac{\partial \langle u_z \rangle}{\partial r}}_{P_{zz}} + \underbrace{\frac{2}{\rho} \left\langle p' \frac{\partial u'_z}{\partial z} \right\rangle}_{\Pi_{zz}} - \underbrace{2\nu \left(\left\langle \frac{\partial u'_z}{\partial r} \frac{\partial u'_z}{\partial r} \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial u'_z}{\partial \beta} \frac{\partial u'_z}{\partial \beta} \right\rangle + \left\langle \frac{\partial u'_z}{\partial z} \frac{\partial u'_z}{\partial z} \right\rangle \right)}_{\varepsilon_{zz}} \\ - \underbrace{\frac{\partial \langle u'_r u'_z u'_z \rangle}{\partial r} - \frac{\langle u'_r u'_z u'_z \rangle}{r}}_{D_{zz}^t} + \underbrace{\nu \left(\frac{\partial^2 \langle u'_z u'_z \rangle}{\partial r^2} + \frac{1}{r} \frac{\partial \langle u'_z u'_z \rangle}{\partial r} \right)}_{D_{zz}^v} = 0 \quad . \quad (\text{E.5})$$

The transport equation of radial Reynolds normal stress $\langle u'_r u'_r \rangle$ is

$$\begin{aligned}
& \underbrace{2 \frac{\langle u_\beta \rangle}{r} \langle u'_r u'_\beta \rangle}_{C_{rr}} + \underbrace{2 \frac{\langle u'_r u'_\beta \rangle}{r} \langle u_\beta \rangle}_{P_{rr}} + \underbrace{\frac{2}{\rho} \left\langle p' \frac{\partial u'_r}{\partial r} \right\rangle}_{\Pi_{rr}} \\
& - 2\nu \underbrace{\left(\left\langle \frac{\partial u'_r}{\partial r} \frac{\partial u'_r}{\partial r} \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial u'_r}{\partial \beta} \frac{\partial u'_r}{\partial \beta} \right\rangle + \left\langle \frac{\partial u'_r}{\partial z} \frac{\partial u'_r}{\partial z} \right\rangle - \frac{2}{r^2} \left\langle u'_\beta \frac{\partial u'_r}{\partial \beta} \right\rangle + \frac{\langle u'_\beta u'_\beta \rangle}{r^2} \right)}_{\varepsilon_{rr}} \\
& - \underbrace{\frac{\partial \langle u'_r u'_r u'_r \rangle}{\partial r} - \frac{\langle u'_r u'_r u'_r \rangle}{r}}_{D_{rr}^t} + \underbrace{2 \frac{\langle u'_r u'_\beta u'_\beta \rangle}{r} - \frac{2}{\rho} \frac{\partial \langle p' u'_r \rangle}{\partial r}}_{D_{rr}^p} \\
& + \nu \underbrace{\left(\frac{\partial^2 \langle u'_r u'_r \rangle}{\partial r^2} + \frac{1}{r} \frac{\partial \langle u'_r u'_r \rangle}{\partial r} - 2 \frac{\langle u'_r u'_r \rangle}{r^2} + 2 \frac{\langle u'_\beta u'_\beta \rangle}{r^2} \right)}_{D_{rr}^v} + \underbrace{4\Omega \langle u'_r u'_\beta \rangle}_{P_{rr}^c} = 0 \quad .
\end{aligned} \tag{E.6}$$

The transport equation of azimuthal Reynolds normal stress $\langle u'_\beta u'_\beta \rangle$ is

$$\begin{aligned}
& \underbrace{-2 \frac{\langle u_\beta \rangle}{r} \langle u'_r u'_\beta \rangle}_{C_{\beta\beta}} - \underbrace{2 \langle u'_r u'_\beta \rangle \frac{\partial \langle u_\beta \rangle}{\partial r}}_{P_{\beta\beta}} + \underbrace{\frac{2}{\rho r} \left\langle p' \frac{\partial u'_\beta}{\partial \beta} \right\rangle}_{\Pi_{\beta\beta}} \\
& - 2\nu \underbrace{\left(\left\langle \frac{\partial u'_\beta}{\partial r} \frac{\partial u'_\beta}{\partial r} \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial u'_\beta}{\partial \beta} \frac{\partial u'_\beta}{\partial \beta} \right\rangle + \left\langle \frac{\partial u'_\beta}{\partial z} \frac{\partial u'_\beta}{\partial z} \right\rangle + \frac{2}{r^2} \left\langle u'_r \frac{\partial u'_\beta}{\partial \beta} \right\rangle + \frac{\langle u'_r u'_r \rangle}{r^2} \right)}_{\varepsilon_{\beta\beta}} \\
& - \underbrace{\frac{\partial \langle u'_r u'_\beta u'_\beta \rangle}{\partial r} - 3 \frac{\langle u'_r u'_\beta u'_\beta \rangle}{r}}_{D_{\beta\beta}^t} + \underbrace{\nu \left(\frac{\partial^2 \langle u'_\beta u'_\beta \rangle}{\partial r^2} + \frac{1}{r} \frac{\partial \langle u'_\beta u'_\beta \rangle}{\partial r} + 2 \frac{\langle u'_r u'_r \rangle}{r^2} - 2 \frac{\langle u'_\beta u'_\beta \rangle}{r^2} \right)}_{D_{\beta\beta}^v} \\
& - \underbrace{4\Omega \langle u'_r u'_\beta \rangle}_{P_{\beta\beta}^c} = 0 \quad .
\end{aligned} \tag{E.7}$$

The transport equation of radial-axial Reynolds shear stress $\langle u'_r u'_z \rangle$ is

$$\begin{aligned}
& \underbrace{\frac{\langle u_\beta \rangle}{r} \langle u'_\beta u'_z \rangle}_{C_{rz}} - \underbrace{\langle u'_r u'_r \rangle \frac{\partial \langle u_z \rangle}{\partial r}}_{P_{rz}} + \underbrace{\frac{\langle u'_\beta u'_z \rangle}{r} \langle u_\beta \rangle}_{\Pi_{rz}} + \frac{1}{\rho} \left(\left\langle p' \frac{\partial u'_z}{\partial r} \right\rangle + \left\langle p' \frac{\partial u'_r}{\partial z} \right\rangle \right) \\
& - 2\nu \left(\left\langle \frac{\partial u'_r}{\partial r} \frac{\partial u'_z}{\partial r} \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial u'_r}{\partial \beta} \frac{\partial u'_z}{\partial \beta} \right\rangle + \left\langle \frac{\partial u'_r}{\partial z} \frac{\partial u'_z}{\partial z} \right\rangle - \frac{1}{r^2} \left\langle u'_\beta \frac{\partial u'_z}{\partial \beta} \right\rangle \right) \\
& - \underbrace{\frac{\partial \langle u'_r u'_r u'_z \rangle}{\partial r}}_{D_{rz}^t} - \underbrace{\frac{\langle u'_r u'_r u'_z \rangle}{r}}_{D_{rz}^p} + \underbrace{\frac{\langle u'_\beta u'_\beta u'_z \rangle}{r}}_{D_{rz}^p} - \underbrace{\frac{1}{\rho} \frac{\partial \langle p' u'_z \rangle}{\partial r}}_{D_{rz}^p} \\
& + \nu \left(\underbrace{\frac{\partial^2 \langle u'_r u'_z \rangle}{\partial r^2} + \frac{1}{r} \frac{\partial \langle u'_r u'_z \rangle}{\partial r} - \frac{\langle u'_r u'_z \rangle}{r^2}}_{D_{rz}^v} \right) + \underbrace{2\Omega \langle u'_\beta u'_z \rangle}_{P_{rz}^c} = 0 \quad .
\end{aligned} \tag{E.8}$$

The transport equation of radial-azimuthal Reynolds shear stress $\langle u'_r u'_\beta \rangle$ is

$$\begin{aligned}
& \underbrace{\frac{\langle u_\beta \rangle}{r} \langle u'_\beta u'_\beta \rangle}_{C_{r\beta}} - \underbrace{\frac{\langle u_\beta \rangle}{r} \langle u'_r u'_r \rangle - \langle u'_r u'_r \rangle \frac{\partial \langle u_\beta \rangle}{\partial r}}_{P_{r\beta}} + \frac{\langle u'_\beta u'_\beta \rangle}{r} \langle u_\beta \rangle \\
& + \frac{1}{\rho} \left(\left\langle p' \frac{\partial u'_\beta}{\partial r} \right\rangle + \frac{1}{r} \left\langle p' \frac{\partial u'_r}{\partial \beta} \right\rangle \right) \\
& - 2\nu \left(\left\langle \frac{\partial u'_r}{\partial r} \frac{\partial u'_\beta}{\partial r} \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial u'_r}{\partial \beta} \frac{\partial u'_\beta}{\partial \beta} \right\rangle + \left\langle \frac{\partial u'_r}{\partial z} \frac{\partial u'_\beta}{\partial z} \right\rangle \right. \\
& \left. + \frac{1}{r^2} \left\langle u'_r \frac{\partial u'_r}{\partial \beta} \right\rangle - \frac{1}{r^2} \left\langle u'_\beta \frac{\partial u'_\beta}{\partial \beta} \right\rangle - \frac{\langle u'_r u'_\beta \rangle}{r^2} \right) \\
& - \underbrace{\frac{\partial \langle u'_r u'_r u'_\beta \rangle}{\partial r}}_{D_{r\beta}^t} - \underbrace{2 \frac{\langle u'_r u'_r u'_\beta \rangle}{r}}_{D_{r\beta}^p} + \underbrace{\frac{\langle u'_\beta u'_\beta u'_\beta \rangle}{r}}_{D_{r\beta}^p} - \underbrace{\frac{1}{\rho} \frac{\partial \langle p' u'_\beta \rangle}{\partial r}}_{D_{r\beta}^p} \\
& + \nu \left(\underbrace{\frac{\partial^2 \langle u'_r u'_\beta \rangle}{\partial r^2} + \frac{1}{r} \frac{\partial \langle u'_r u'_\beta \rangle}{\partial r} - 4 \frac{\langle u'_r u'_\beta \rangle}{r^2}}_{D_{r\beta}^v} \right) + \underbrace{2\Omega (\langle u'_\beta u'_\beta \rangle - \langle u'_r u'_r \rangle)}_{P_{r\beta}^c} = 0 \quad .
\end{aligned} \tag{E.9}$$

The transport equation of azimuthal-axial Reynolds shear stress $\langle u'_\beta u'_z \rangle$ is

$$\begin{aligned}
& \underbrace{-\frac{\langle u_\beta \rangle}{r} \langle u'_r u'_z \rangle}_{C_{\beta z}} - \underbrace{\langle u'_r u'_z \rangle \frac{\partial \langle u_\beta \rangle}{\partial r}}_{P_{\beta z}} - \underbrace{\langle u'_r u'_\beta \rangle \frac{\partial \langle u_z \rangle}{\partial r}}_{\Pi_{\beta z}} + \frac{1}{\rho} \left(\frac{1}{r} \left\langle p' \frac{\partial u'_z}{\partial \beta} \right\rangle + \left\langle p' \frac{\partial u'_\beta}{\partial z} \right\rangle \right) \\
& - 2\nu \underbrace{\left(\left\langle \frac{\partial u'_\beta}{\partial r} \frac{\partial u'_z}{\partial r} \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial u'_\beta}{\partial \beta} \frac{\partial u'_z}{\partial \beta} \right\rangle + \left\langle \frac{\partial u'_\beta}{\partial z} \frac{\partial u'_z}{\partial z} \right\rangle + \frac{1}{r^2} \left\langle u'_r \frac{\partial u'_z}{\partial \beta} \right\rangle \right)}_{\varepsilon_{\beta z}} \\
& - \underbrace{\frac{\partial \langle u'_r u'_\beta u'_z \rangle}{\partial r}}_{D_{\beta z}^t} - 2 \underbrace{\frac{\langle u'_r u'_\beta u'_z \rangle}{r}}_{D_{\beta z}^v} + \nu \left(\frac{\partial^2 \langle u'_\beta u'_z \rangle}{\partial r^2} + \frac{1}{r} \frac{\partial \langle u'_\beta u'_z \rangle}{\partial r} - \frac{\langle u'_\beta u'_z \rangle}{r^2} \right) \\
& - \underbrace{2\Omega \langle u'_r u'_z \rangle}_{P_{\beta z}^c} = 0 \quad .
\end{aligned} \tag{E.10}$$

The next step is to derive the transport equation of the TKE (or $\frac{1}{2}\langle u'_i u'_i \rangle$) by summing the transport equations of the three Reynolds normal stresses and dividing the result by two, i.e., $\frac{1}{2}\langle u'_i u'_i \rangle = (\langle u'_r u'_r \rangle + \langle u'_\beta u'_\beta \rangle + \langle u'_z u'_z \rangle)/2$. Therefore, the simplified form of the TKE transport equation can be expressed as

$$\begin{aligned}
& \underbrace{-\langle u'_r u'_\beta \rangle \frac{\partial \langle u_\beta \rangle}{\partial r} - \langle u'_r u'_z \rangle \frac{\partial \langle u_z \rangle}{\partial r} + \frac{\langle u'_r u'_\beta \rangle}{r} \langle u_\beta \rangle}_{P_{TKE}} \\
& - \nu \underbrace{\left(\left\langle \frac{\partial u'_i}{\partial r} \frac{\partial u'_i}{\partial r} \right\rangle + \frac{1}{r^2} \left\langle \frac{\partial u'_i}{\partial \beta} \frac{\partial u'_i}{\partial \beta} \right\rangle + \left\langle \frac{\partial u'_i}{\partial z} \frac{\partial u'_i}{\partial z} \right\rangle + \frac{2}{r^2} \left\langle u'_r \frac{\partial u'_\beta}{\partial \beta} \right\rangle - \frac{2}{r^2} \left\langle u'_\beta \frac{\partial u'_r}{\partial \beta} \right\rangle + \frac{\langle u'_r u'_r \rangle}{r^2} + \frac{\langle u'_\beta u'_\beta \rangle}{r^2} \right)}_{\varepsilon_{TKE}} \\
& - \underbrace{\frac{1}{2} \left(\frac{\partial \langle u'_r u'_i u'_i \rangle}{\partial r} + \frac{\langle u'_r u'_i u'_i \rangle}{r} \right)}_{D_{TKE}^t} - \underbrace{\frac{1}{\rho} \left(\frac{\partial \langle p' u'_r \rangle}{\partial r} + \frac{\langle p' u'_r \rangle}{r} \right)}_{D_{TKE}^p} \\
& + \underbrace{\frac{\nu}{2} \left(\frac{\partial^2 \langle u'_i u'_i \rangle}{\partial r^2} + \frac{1}{r} \frac{\partial \langle u'_i u'_i \rangle}{\partial r} \right)}_{D_{TKE}^v} = 0 \quad .
\end{aligned} \tag{E.11}$$

E.2 Transport equations for velocity-spectrum tensor and TKE spectrum

The derivation of the transport equations for the velocity-spectrum tensor and the TKE spectrum starts with the equations of the velocity fluctuations, u'_r , u'_β and u'_z , given in equations (E.1)-(E.3). The first step is to apply the Fourier transform, introduced in equation (5.7), to these equations, thereby reformulating them in spectral space. The resulting equations are written as

$$\begin{aligned}
\frac{\partial \widehat{u'_r}}{\partial t} = & - \langle u_r \rangle \frac{\partial \widehat{u'_r}}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{u'_r}}{\partial \beta} + \frac{\langle u_\beta \rangle}{r} \widehat{u'_\beta} - \underline{i} k_z \langle u_z \rangle \widehat{u'_r} \\
& - \widehat{u'_r} \frac{\partial \langle u_r \rangle}{\partial r} - \frac{\widehat{u'_\beta}}{r} \frac{\partial \langle u_r \rangle}{\partial \beta} + \frac{\widehat{u'_\beta}}{r} \langle u_\beta \rangle - \widehat{u'_z} \frac{\partial \langle u_r \rangle}{\partial z} \\
& - \frac{\widehat{u'_r u'_r}}{r} - \frac{\partial \widehat{u'_r u'_r}}{\partial r} - \frac{1}{r} \frac{\partial \widehat{u'_r u'_\beta}}{\partial \beta} + \frac{\widehat{u'_\beta u'_\beta}}{r} - \underline{i} k_z \widehat{u'_r u'_z} \\
& - \frac{1}{\rho} \frac{\partial \widehat{p'}}{\partial r} \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{u'_r}}{\partial r} - \frac{\widehat{u'_r}}{r^2} + \frac{\partial^2 \widehat{u'_r}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{u'_r}}{\partial \beta^2} - \frac{2}{r^2} \frac{\partial \widehat{u'_\beta}}{\partial \beta} - k_z^2 \widehat{u'_r} \right) \\
& + 2\Omega \widehat{u'_\beta} \quad ,
\end{aligned} \tag{E.12}$$

$$\begin{aligned}
\frac{\partial \widehat{u'_\beta}}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{u'_\beta}}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{u'_\beta}}{\partial \beta} - \frac{\langle u_\beta \rangle}{r} \widehat{u'_r} - \underline{i} k_z \langle u_z \rangle \widehat{u'_\beta} \\
& - \widehat{u'_r} \frac{\partial \langle u_\beta \rangle}{\partial r} - \frac{\widehat{u'_\beta}}{r} \frac{\partial \langle u_\beta \rangle}{\partial \beta} - \frac{\widehat{u'_\beta}}{r} \langle u_r \rangle - \widehat{u'_z} \frac{\partial \langle u_\beta \rangle}{\partial z} \\
& - \frac{\widehat{u'_r} \widehat{u'_\beta}}{r} - \frac{\partial \widehat{u'_r} \widehat{u'_\beta}}{\partial r} - \frac{1}{r} \frac{\partial \widehat{u'_\beta} \widehat{u'_\beta}}{\partial \beta} - \frac{\widehat{u'_r} \widehat{u'_\beta}}{r} - \underline{i} k_z \widehat{u'_\beta} \widehat{u'_z} \\
& - \frac{1}{\rho r} \frac{\partial \widehat{p'}}{\partial \beta} \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{u'_\beta}}{\partial r} - \frac{\widehat{u'_\beta}}{r^2} + \frac{\partial^2 \widehat{u'_\beta}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{u'_\beta}}{\partial \beta^2} + \frac{2}{r^2} \frac{\partial \widehat{u'_r}}{\partial \beta} - k_z^2 \widehat{u'_\beta} \right) \\
& - 2\Omega \widehat{u'_r} \quad ,
\end{aligned} \tag{E.13}$$

$$\begin{aligned}
\frac{\partial \widehat{u'_z}}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{u'_z}}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{u'_z}}{\partial \beta} - \underline{i} k_z \langle u_z \rangle \widehat{u'_z} \\
& - \widehat{u'_r} \frac{\partial \langle u_z \rangle}{\partial r} - \frac{\widehat{u'_\beta}}{r} \frac{\partial \langle u_z \rangle}{\partial \beta} - \widehat{u'_z} \frac{\partial \langle u_z \rangle}{\partial z} \\
& - \frac{\widehat{u'_r} \widehat{u'_z}}{r} - \frac{\partial \widehat{u'_r} \widehat{u'_z}}{\partial r} - \frac{1}{r} \frac{\partial \widehat{u'_\beta} \widehat{u'_z}}{\partial \beta} - \underline{i} k_z \widehat{u'_z} \widehat{u'_z} \\
& - \frac{1}{\rho} \underline{i} k_z \widehat{p'} \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{u'_z}}{\partial r} + \frac{\partial^2 \widehat{u'_z}}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{u'_z}}{\partial \beta^2} - k_z^2 \widehat{u'_z} \right) \quad .
\end{aligned} \tag{E.14}$$

It should be emphasised that the terms in the last row of equations (E.1)-(E.3) represent turbulent inertia. Because these terms are independent of z in physical space, their Fourier transforms are nonzero only at $k_z = 0$. Moreover, since $\widehat{u'_r}$, $\widehat{u'_\beta}$ and $\widehat{u'_z}$ vanish at $k_z = 0$, these transformed terms do not show in equations (E.12)-(E.14). Subsequently, taking the complex conjugates of equations (E.12)-(E.14) yields the governing equations of $\widehat{u'_r}^*$, $\widehat{u'_\beta}^*$ and $\widehat{u'_z}^*$, expressed as

$$\begin{aligned}
\frac{\partial \widehat{u}_r'^*}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{u}_r'^*}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{u}_r'^*}{\partial \beta} + \frac{\langle u_\beta \rangle}{r} \widehat{u}_\beta'^* + \underline{i}k_z \langle u_z \rangle \widehat{u}_r'^* \\
& - \widehat{u}_r'^* \frac{\partial \langle u_r \rangle}{\partial r} - \frac{\widehat{u}_\beta'^*}{r} \frac{\partial \langle u_r \rangle}{\partial \beta} + \frac{\widehat{u}_\beta'^*}{r} \langle u_\beta \rangle - \widehat{u}_z'^* \frac{\partial \langle u_r \rangle}{\partial z} \\
& - \frac{\widehat{u}_r' \widehat{u}_r'^*}{r} - \frac{\partial \widehat{u}_r' \widehat{u}_r'^*}{\partial r} - \frac{1}{r} \frac{\partial \widehat{u}_r' \widehat{u}_\beta'^*}{\partial \beta} + \frac{\widehat{u}_\beta' \widehat{u}_\beta'^*}{r} + \underline{i}k_z \widehat{u}_r' \widehat{u}_z'^* \\
& - \frac{1}{\rho} \frac{\partial \widehat{p}'^*}{\partial r} \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{u}_r'^*}{\partial r} - \frac{\widehat{u}_r'^*}{r^2} + \frac{\partial^2 \widehat{u}_r'^*}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{u}_r'^*}{\partial \beta^2} - \frac{2}{r^2} \frac{\partial \widehat{u}_\beta'^*}{\partial \beta} - k_z^2 \widehat{u}_r'^* \right) \\
& + 2\Omega \widehat{u}_\beta'^* ,
\end{aligned} \tag{E.15}$$

$$\begin{aligned}
\frac{\partial \widehat{u}_\beta'^*}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{u}_\beta'^*}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{u}_\beta'^*}{\partial \beta} - \frac{\langle u_\beta \rangle}{r} \widehat{u}_r'^* + \underline{i}k_z \langle u_z \rangle \widehat{u}_\beta'^* \\
& - \widehat{u}_r'^* \frac{\partial \langle u_\beta \rangle}{\partial r} - \frac{\widehat{u}_\beta'^*}{r} \frac{\partial \langle u_\beta \rangle}{\partial \beta} - \frac{\widehat{u}_\beta'^*}{r} \langle u_r \rangle - \widehat{u}_z'^* \frac{\partial \langle u_\beta \rangle}{\partial z} \\
& - \frac{\widehat{u}_r' \widehat{u}_\beta'^*}{r} - \frac{\partial \widehat{u}_r' \widehat{u}_\beta'^*}{\partial r} - \frac{1}{r} \frac{\partial \widehat{u}_\beta' \widehat{u}_\beta'^*}{\partial \beta} - \frac{\widehat{u}_r' \widehat{u}_\beta'^*}{r} + \underline{i}k_z \widehat{u}_\beta' \widehat{u}_z'^* \\
& - \frac{1}{\rho r} \frac{\partial \widehat{p}'^*}{\partial \beta} \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{u}_\beta'^*}{\partial r} - \frac{\widehat{u}_\beta'^*}{r^2} + \frac{\partial^2 \widehat{u}_\beta'^*}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{u}_\beta'^*}{\partial \beta^2} + \frac{2}{r^2} \frac{\partial \widehat{u}_r'^*}{\partial \beta} - k_z^2 \widehat{u}_\beta'^* \right) \\
& - 2\Omega \widehat{u}_r'^* ,
\end{aligned} \tag{E.16}$$

$$\begin{aligned}
\frac{\partial \widehat{u'_z}^*}{\partial t} = & -\langle u_r \rangle \frac{\partial \widehat{u'_z}^*}{\partial r} - \frac{\langle u_\beta \rangle}{r} \frac{\partial \widehat{u'_z}^*}{\partial \beta} + i k_z \langle u_z \rangle \widehat{u'_z}^* \\
& - \widehat{u'_r}^* \frac{\partial \langle u_z \rangle}{\partial r} - \frac{\widehat{u'_\beta}^*}{r} \frac{\partial \langle u_z \rangle}{\partial \beta} - \widehat{u'_z}^* \frac{\partial \langle u_z \rangle}{\partial z} \\
& - \frac{\widehat{u'_r u'_z}^*}{r} - \frac{\partial \widehat{u'_r u'_z}^*}{\partial r} - \frac{1}{r} \frac{\partial \widehat{u'_\beta u'_z}^*}{\partial \beta} + i k_z \widehat{u'_z}^* \widehat{u'_z}^* \\
& + \frac{1}{\rho} i k_z \widehat{p'}^* \\
& + \nu \left(\frac{1}{r} \frac{\partial \widehat{u'_z}^*}{\partial r} + \frac{\partial^2 \widehat{u'_z}^*}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \widehat{u'_z}^*}{\partial \beta^2} - k_z^2 \widehat{u'_z}^* \right) .
\end{aligned} \tag{E.17}$$

The next step is to derive the transport equations for the six velocity-spectrum tensor components $\widetilde{E}_{ij}$. To illustrate this procedure, the derivation is demonstrated for the radial-axial co-spectrum $\widetilde{E}_{rz}$. Equations (E.15) and (E.14) are multiplied by $\widehat{u'_z}$ and $\widehat{u'_r}^*$, respectively, and the results are subsequently combined. After applying time averaging and taking the real part of the resulting expression, the transport equation of $\widetilde{E}_{rz}$ is obtained. For ease of understanding, the time derivative term of $\widetilde{E}_{rz}$ is considered as an example. Applying the aforementioned operations to the left-hand side of equations (E.15) and (E.14) yields

$$\text{Re} \left\{ \left(\frac{\partial \widehat{u'_r}^*}{\partial t} \right) \widehat{u'_z} + \left(\frac{\partial \widehat{u'_z}^*}{\partial t} \right) \widehat{u'_r}^* \right\} = \text{Re} \left\{ \frac{\partial \widehat{u'_r}^* \widehat{u'_z}}{\partial t} \right\} = \frac{\partial \widetilde{E}_{rz}}{\partial t} . \tag{E.18}$$

To simplify the transport equations of the autospectra and co-spectra, the four flow conditions listed in Appendix E.1 are applied. In addition, since only the real parts of these equations are retained, certain terms can be reduced when they take the following forms:

$$\text{Re} \left\{ \widehat{A^* B} + \widehat{A B^*} \right\} \quad \text{and} \quad \text{Re} \left\{ i k_z \left(\widehat{A^* B} - \widehat{A B^*} \right) \right\} . \tag{E.19}$$

The above two expressions simplify respectively to

$$\text{Re} \left\{ 2\widehat{A^*B} \right\} \quad \text{and} \quad \text{Re} \left\{ 2\underline{i}k_z\widehat{A^*B} \right\} . \quad (\text{E.20})$$

Following the operations and considerations described above, the simplified transport equations of the six velocity-spectrum tensor components $\widetilde{E}_{ij}$ are obtained. Therefore, the transport equation of axial autospectrum $\widetilde{E}_{zz}$ is

$$\begin{aligned} & \underbrace{\text{Re} \left\{ -2\widehat{u_r'^* u_z'} \frac{\partial \langle u_z \rangle}{\partial r} \right\}}_{\widetilde{P}_{zz}} \\ & + \underbrace{\text{Re} \left\{ \frac{\partial \widehat{u_r' u_z'^*}}{\partial r} + \underline{i}k_z 2\widehat{u_z' u_z'^*} - \frac{\widehat{u_r' u_z'^*}}{r} - 2 \left(\frac{\partial \widehat{u_r' u_z'^*}}{\partial r} \widehat{u_z'} + \frac{1}{r} \frac{\partial \widehat{u_\beta' u_z'^*}}{\partial \beta} \widehat{u_z'} \right) \right\}}_{\widetilde{T}_{zz}} \\ & + \underbrace{\text{Re} \left\{ \underline{i}k_z \frac{2}{\rho} \widehat{p'^* u_z'} \right\}}_{\widetilde{\Pi}_{zz}} + \underbrace{\text{Re} \left\{ -2\nu \left(\frac{\partial \widehat{u_z'^*}}{\partial r} \frac{\partial \widehat{u_z'}}{\partial r} + \frac{1}{r^2} \frac{\partial \widehat{u_z'^*}}{\partial \beta} \frac{\partial \widehat{u_z'}}{\partial \beta} + k_z^2 \widehat{u_z'^* u_z'} \right) \right\}}_{\widetilde{\varepsilon}_{zz}} \\ & + \underbrace{\text{Re} \left\{ -\frac{\partial \widehat{u_r' u_z'^*}}{\partial r} - \frac{\widehat{u_r' u_z'^*}}{r} \right\}}_{\widetilde{D}_{zz}^t} + \underbrace{\text{Re} \left\{ \nu \left(\frac{\partial^2 \widehat{u_z'^*}}{\partial r^2} + \frac{1}{r} \frac{\partial \widehat{u_z'^*}}{\partial r} \right) \right\}}_{\widetilde{D}_{zz}^v} = 0 . \end{aligned} \quad (\text{E.21})$$

The transport equation of radial autospectrum $\tilde{E}_{rr}$ is

$$\begin{aligned}
& \underbrace{\operatorname{Re} \left\{ 2 \frac{\langle u_\beta \rangle}{r} \widehat{u_r'}^* \widehat{u_\beta'} \right\}}_{\tilde{C}_{rr}} + \underbrace{\operatorname{Re} \left\{ 2 \frac{\widehat{u_r'}^* \widehat{u_\beta'}}{r} \langle u_\beta \rangle \right\}}_{\tilde{P}_{rr}} \\
& + \underbrace{\operatorname{Re} \left\{ \frac{\partial \widehat{u_r'}^* \widehat{u_r'}}{\partial r} + i k_z 2 \widehat{u_r'}^* \widehat{u_z'} - \frac{\widehat{u_r'}^* \widehat{u_r'}}{r} - 2 \left(\frac{\partial \widehat{u_r'}^*}{\partial r} \widehat{u_r'} + \frac{1}{r} \frac{\partial \widehat{u_r'}^*}{\partial \beta} \widehat{u_r'} \right) \right\}}_{\tilde{T}_{rr}} \\
& + \underbrace{\operatorname{Re} \left\{ \frac{2}{\rho} \widehat{p'}^* \frac{\partial \widehat{u_r'}}{\partial r} \right\}}_{\tilde{\Pi}_{rr}} \\
& + \underbrace{\operatorname{Re} \left\{ -2\nu \left(\frac{\partial \widehat{u_r'}^*}{\partial r} \frac{\partial \widehat{u_r'}}{\partial r} + \frac{1}{r^2} \frac{\partial \widehat{u_r'}^*}{\partial \beta} \frac{\partial \widehat{u_r'}}{\partial \beta} + k_z^2 \widehat{u_r'}^* \widehat{u_r'} - \frac{2}{r^2} \widehat{u_\beta'}^* \frac{\partial \widehat{u_r'}}{\partial \beta} + \frac{\widehat{u_\beta'}^* \widehat{u_\beta'}}{r^2} \right) \right\}}_{\tilde{\epsilon}_{rr}} \\
& + \underbrace{\operatorname{Re} \left\{ -\frac{\partial \widehat{u_r'}^* \widehat{u_r'}}{\partial r} - \frac{\widehat{u_r'}^* \widehat{u_r'}}{r} + 2 \frac{\widehat{u_\beta'}^* \widehat{u_r'}}{r} \right\}}_{\tilde{D}_{rr}^t} + \underbrace{\operatorname{Re} \left\{ -\frac{2}{\rho} \frac{\partial \widehat{p'}^*}{\partial r} \widehat{u_r'} \right\}}_{\tilde{D}_{rr}^p} \\
& + \underbrace{\operatorname{Re} \left\{ \nu \left(\frac{\partial^2 \widehat{u_r'}^* \widehat{u_r'}}{\partial r^2} + \frac{1}{r} \frac{\partial \widehat{u_r'}^* \widehat{u_r'}}{\partial r} - 2 \frac{\widehat{u_r'}^* \widehat{u_r'}}{r^2} + 2 \frac{\widehat{u_\beta'}^* \widehat{u_\beta'}}{r^2} \right) \right\}}_{\tilde{D}_{rr}^v} + \underbrace{\operatorname{Re} \left\{ 4\Omega \widehat{u_r'}^* \widehat{u_\beta'} \right\}}_{\tilde{P}_{rr}^c} = 0 \quad .
\end{aligned} \tag{E.22}$$

The transport equation of azimuthal autospectrum $\tilde{E}_{\beta\beta}$ is

$$\begin{aligned}
& \underbrace{\operatorname{Re} \left\{ -2 \frac{\langle u_\beta \rangle}{r} \widehat{u_r'^* u'_\beta} \right\}}_{\tilde{C}_{\beta\beta}} + \underbrace{\operatorname{Re} \left\{ -2 \widehat{u_r'^* u'_\beta} \frac{\partial \langle u_\beta \rangle}{\partial r} \right\}}_{\tilde{P}_{\beta\beta}} \\
& + \underbrace{\operatorname{Re} \left\{ \frac{\partial \widehat{u_r' u'_\beta^*}}{\partial r} + i k_z 2 \widehat{u'_\beta u_z'^* u'_\beta} - \frac{\widehat{u_r' u'_\beta^* u'_\beta}}{r} - 2 \left(\frac{\partial \widehat{u_r' u'_\beta^*}}{\partial r} \widehat{u'_\beta} + \frac{1}{r} \frac{\partial \widehat{u'_\beta u'_\beta^*}}{\partial \beta} \widehat{u'_\beta} \right) \right\}}_{\tilde{T}_{\beta\beta}} \\
& + \underbrace{\operatorname{Re} \left\{ \frac{2}{\rho r} \widehat{p'^*} \frac{\partial \widehat{u'_\beta}}{\partial \beta} \right\}}_{\tilde{\Pi}_{\beta\beta}} \\
& + \underbrace{\operatorname{Re} \left\{ -2\nu \left(\frac{\partial \widehat{u'_\beta^*}}{\partial r} \frac{\partial \widehat{u'_\beta}}{\partial r} + \frac{1}{r^2} \frac{\partial \widehat{u'_\beta^*}}{\partial \beta} \frac{\partial \widehat{u'_\beta}}{\partial \beta} + k_z^2 \widehat{u'_\beta^* u'_\beta} + \frac{2}{r^2} \widehat{u_r'^*} \frac{\partial \widehat{u'_\beta}}{\partial \beta} + \frac{\widehat{u_r'^* u'_r}}{r^2} \right) \right\}}_{\tilde{\varepsilon}_{\beta\beta}} \\
& + \underbrace{\operatorname{Re} \left\{ -\frac{\partial \widehat{u_r' u'_\beta^*}}{\partial r} - 3 \frac{\widehat{u_r' u'_\beta^* u'_\beta}}{r} \right\}}_{\tilde{D}_{\beta\beta}^t} \\
& + \underbrace{\operatorname{Re} \left\{ \nu \left(\frac{\partial^2 \widehat{u'_\beta^* u'_\beta}}{\partial r^2} + \frac{1}{r} \frac{\partial \widehat{u'_\beta^* u'_\beta}}{\partial r} + 2 \frac{\widehat{u_r'^* u'_r}}{r^2} - 2 \frac{\widehat{u'_\beta^* u'_\beta}}{r^2} \right) \right\}}_{\tilde{D}_{\beta\beta}^v} \\
& + \underbrace{\operatorname{Re} \left\{ -4\Omega \widehat{u_r'^* u'_\beta} \right\}}_{\tilde{P}_{\beta\beta}^c} = 0 \quad .
\end{aligned} \tag{E.23}$$

The transport equation of radial-axial co-spectrum $\tilde{E}_{rz}$ is

$$\begin{aligned}
& \underbrace{\operatorname{Re} \left\{ \frac{\langle u_\beta \rangle}{r} \overline{\widehat{u'_\beta}^* \widehat{u'_z}} \right\}}_{\tilde{C}_{rz}} + \underbrace{\operatorname{Re} \left\{ -\overline{\widehat{u'_r} \widehat{u'_r}^*} \frac{\partial \langle u_z \rangle}{\partial r} + \frac{\overline{\widehat{u'_\beta}^* \widehat{u'_z}}}{r} \langle u_\beta \rangle \right\}}_{\tilde{P}_{rz}} \\
& + \operatorname{Re} \left\{ \underbrace{\begin{aligned} & \frac{1}{2} \left(\frac{\overline{\partial \widehat{u'_r} \widehat{u'_r}^* \widehat{u'_z}}}{\partial r} + \frac{\overline{\partial \widehat{u'_r} \widehat{u'_z} \widehat{u'_r}^*}}{\partial r} \right) + \underline{i} k_z \left(\overline{\widehat{u'_r} \widehat{u'_z}^* \widehat{u'_z}} - \overline{\widehat{u'_z} \widehat{u'_z} \widehat{u'_r}^*} \right) \\ & - \frac{1}{2} \left(\frac{\overline{\widehat{u'_r} \widehat{u'_r}^* \widehat{u'_z}}}{r} - \frac{\overline{\widehat{u'_r} \widehat{u'_z} \widehat{u'_r}^*}}{r} \right) \\ & - \frac{\overline{\partial \widehat{u'_r} \widehat{u'_r}^*}}{\partial r} \widehat{u'_z} - \frac{\overline{\partial \widehat{u'_r} \widehat{u'_z}}}{\partial r} \widehat{u'_r}^* - \frac{1}{r} \frac{\overline{\partial \widehat{u'_r} \widehat{u'_\beta}^*}}{\partial \beta} \widehat{u'_z} - \frac{1}{r} \frac{\overline{\partial \widehat{u'_\beta} \widehat{u'_z}}}{\partial \beta} \widehat{u'_r}^* \end{aligned}}_{\tilde{T}_{rz}} \right\} \\
& + \underbrace{\operatorname{Re} \left\{ \frac{1}{\rho} \left(\overline{\widehat{p}'^* \frac{\partial \widehat{u'_z}}{\partial r}} - \underline{i} k_z \overline{\widehat{p}' \widehat{u'_r}^*} \right) \right\}}_{\tilde{\Pi}_{rz}} \\
& + \underbrace{\operatorname{Re} \left\{ -2\nu \left(\frac{\overline{\partial \widehat{u'_r}^*}}{\partial r} \frac{\partial \widehat{u'_z}}{\partial r} + \frac{1}{r^2} \frac{\overline{\partial \widehat{u'_r}^*}}{\partial \beta} \frac{\partial \widehat{u'_z}}{\partial \beta} + k_z^2 \overline{\widehat{u'_r}^* \widehat{u'_z}} - \frac{1}{r^2} \overline{\widehat{u'_\beta}^* \frac{\partial \widehat{u'_z}}{\partial \beta}} \right) \right\}}_{\tilde{\varepsilon}_{rz}} \\
& + \underbrace{\operatorname{Re} \left\{ -\frac{1}{2} \left(\frac{\overline{\partial \widehat{u'_r} \widehat{u'_r}^* \widehat{u'_z}}}{\partial r} + \frac{\overline{\widehat{u'_r} \widehat{u'_r}^* \widehat{u'_z}}}{r} + \frac{\overline{\partial \widehat{u'_r} \widehat{u'_z} \widehat{u'_r}^*}}{\partial r} + \frac{\overline{\widehat{u'_r} \widehat{u'_z} \widehat{u'_r}^*}}{r} \right) + \frac{\overline{\widehat{u'_\beta} \widehat{u'_\beta}^* \widehat{u'_z}}}{r} \right\}}_{\tilde{D}_{rz}^t} \\
& + \underbrace{\operatorname{Re} \left\{ -\frac{1}{\rho} \frac{\partial \overline{\widehat{p}'^* \widehat{u'_z}}}{\partial r} \right\}}_{\tilde{D}_{rz}^p} + \underbrace{\operatorname{Re} \left\{ \nu \left(\frac{\partial^2 \overline{\widehat{u'_r}^* \widehat{u'_z}}}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{\widehat{u'_r}^* \widehat{u'_z}}}{\partial r} - \frac{\overline{\widehat{u'_r}^* \widehat{u'_z}}}{r^2} \right) \right\}}_{\tilde{D}_{rz}^v} \\
& + \underbrace{\operatorname{Re} \left\{ 2\Omega \overline{\widehat{u'_\beta}^* \widehat{u'_z}} \right\}}_{\tilde{P}_{rz}^c} = 0 \quad .
\end{aligned} \tag{E.24}$$

The transport equation of radial-azimuthal co-spectrum $\tilde{E}_{r\beta}$ is

$$\begin{aligned}
& \underbrace{\operatorname{Re} \left\{ \frac{\langle u_\beta \rangle}{r} \overline{\widehat{u'_\beta}^* \widehat{u'_\beta}} - \frac{\langle u_\beta \rangle}{r} \overline{\widehat{u'_r}^* \widehat{u'_r}} \right\}}_{\tilde{C}_{r\beta}} + \underbrace{\operatorname{Re} \left\{ -\overline{\widehat{u'_r}^* \widehat{u'_r}} \frac{\partial \langle u_\beta \rangle}{\partial r} + \frac{\overline{\widehat{u'_\beta}^* \widehat{u'_\beta}}}{r} \langle u_\beta \rangle \right\}}_{\tilde{P}_{r\beta}} \\
& + \operatorname{Re} \left\{ -\frac{1}{2} \left(\frac{\overline{\partial \widehat{u'_r}^* \widehat{u'_r}}}{\partial r} + \frac{\overline{\partial \widehat{u'_\beta}^* \widehat{u'_\beta}}}{\partial r} \right) + ik_z \left(\overline{\widehat{u'_r}^* \widehat{u'_\beta}} - \overline{\widehat{u'_\beta}^* \widehat{u'_r}} \right) \right. \\
& \quad \left. - \frac{1}{2} \left(\frac{\overline{\widehat{u'_r}^* \widehat{u'_\beta}}}{r} + \frac{\overline{\widehat{u'_\beta}^* \widehat{u'_r}}}{r} \right) \right. \\
& \quad \left. - \frac{\overline{\partial \widehat{u'_r}^* \widehat{u'_r}}}{\partial r} \widehat{u'_\beta} - \frac{\overline{\partial \widehat{u'_\beta}^* \widehat{u'_\beta}}}{\partial r} \widehat{u'_r} - \frac{1}{r} \frac{\overline{\partial \widehat{u'_r}^* \widehat{u'_\beta}}}{\partial \beta} \widehat{u'_\beta} - \frac{1}{r} \frac{\overline{\partial \widehat{u'_\beta}^* \widehat{u'_r}}}{\partial \beta} \widehat{u'_r} \right\}}_{\tilde{T}_{r\beta}} \\
& + \underbrace{\operatorname{Re} \left\{ \frac{1}{\rho} \left(\widehat{p'}^* \frac{\partial \widehat{u'_\beta}}{\partial r} + \frac{1}{r} \widehat{p'} \frac{\partial \widehat{u'_r}^*}{\partial \beta} \right) \right\}}_{\tilde{\Pi}_{r\beta}} \\
& + \operatorname{Re} \left\{ -2\nu \left(\frac{\overline{\partial \widehat{u'_r}^* \widehat{u'_\beta}}}{\partial r \partial r} + \frac{1}{r^2} \frac{\overline{\partial \widehat{u'_r}^* \widehat{u'_\beta}}}{\partial \beta \partial \beta} + k_z^2 \overline{\widehat{u'_r}^* \widehat{u'_\beta}} \right. \right. \\
& \quad \left. \left. + \frac{1}{r^2} \widehat{u'_r} \frac{\partial \widehat{u'_r}^*}{\partial \beta} - \frac{1}{r^2} \widehat{u'_\beta}^* \frac{\partial \widehat{u'_\beta}}{\partial \beta} - \frac{\overline{\widehat{u'_\beta}^* \widehat{u'_r}}}{r^2} \right) \right\}}_{\tilde{\varepsilon}_{r\beta}} \\
& + \underbrace{\operatorname{Re} \left\{ -\frac{1}{2} \left(\frac{\overline{\partial \widehat{u'_r}^* \widehat{u'_\beta}}}{\partial r} + \frac{\overline{\widehat{u'_r}^* \widehat{u'_\beta}}}{r} + \frac{\overline{\partial \widehat{u'_r}^* \widehat{u'_\beta}}}{\partial r} + 3 \frac{\overline{\widehat{u'_r}^* \widehat{u'_\beta}}}{r} \right) + \frac{\overline{\widehat{u'_\beta}^* \widehat{u'_\beta}}}{r} \right\}}_{\tilde{D}_{r\beta}^t} \\
& + \underbrace{\operatorname{Re} \left\{ -\frac{1}{\rho} \frac{\overline{\partial \widehat{p'}^* \widehat{u'_\beta}}}{\partial r} \right\}}_{\tilde{D}_{r\beta}^p} + \underbrace{\operatorname{Re} \left\{ \nu \left(\frac{\overline{\partial^2 \widehat{u'_r}^* \widehat{u'_\beta}}}{\partial r^2} + \frac{1}{r} \frac{\overline{\partial \widehat{u'_r}^* \widehat{u'_\beta}}}{\partial r} - 4 \frac{\overline{\widehat{u'_r}^* \widehat{u'_\beta}}}{r^2} \right) \right\}}_{\tilde{D}_{r\beta}^v} \\
& + \underbrace{\operatorname{Re} \left\{ 2\Omega \left(\overline{\widehat{u'_\beta}^* \widehat{u'_\beta}} - \overline{\widehat{u'_r}^* \widehat{u'_r}} \right) \right\}}_{\tilde{P}_{r\beta}^c} = 0 \quad .
\end{aligned} \tag{E.25}$$

The transport equation of azimuthal-axial co-spectrum $\tilde{E}_{\beta z}$ is

$$\begin{aligned}
& \underbrace{\text{Re} \left\{ -\frac{\langle u_\beta \rangle}{r} \widehat{u'_r}^* \widehat{u'_z} \right\}}_{\tilde{C}_{\beta z}} + \underbrace{\text{Re} \left\{ -\widehat{u'_r}^* \widehat{u'_z} \frac{\partial \langle u_\beta \rangle}{\partial r} - \widehat{u'_r} \widehat{u'_\beta}^* \frac{\partial \langle u_z \rangle}{\partial r} \right\}}_{\tilde{P}_{\beta z}} \\
& + \underbrace{\text{Re} \left\{ -\frac{1}{2} \left(\frac{\widehat{\partial u'_r u'_\beta}^* \widehat{u'_z}}{\partial r} + \frac{\widehat{\partial u'_r u'_z} \widehat{u'_\beta}^*}{\partial r} \right) + i k_z \left(\widehat{u'_\beta u'_z}^* \widehat{u'_z} - \widehat{u'_z u'_z} \widehat{u'_\beta}^* \right) \right.}_{\tilde{T}_{\beta z}} \\
& \quad \left. - \frac{1}{2} \left(\frac{\widehat{u'_r u'_\beta}^* \widehat{u'_z}}{r} + \frac{\widehat{u'_r u'_z} \widehat{u'_\beta}^*}{r} \right) \right. \\
& \quad \left. - \frac{\widehat{\partial u'_r u'_\beta}^*}{\partial r} \widehat{u'_z} - \frac{\widehat{\partial u'_r u'_z} \widehat{u'_\beta}^*}{\partial r} - \frac{1}{r} \frac{\widehat{\partial u'_\beta u'_\beta}^*}{\partial \beta} \widehat{u'_z} - \frac{1}{r} \frac{\widehat{\partial u'_\beta u'_z} \widehat{u'_\beta}^*}{\partial \beta} \right\}}_{\tilde{T}_{\beta z}} \\
& + \underbrace{\text{Re} \left\{ \frac{1}{\rho} \left(\frac{1}{r} \widehat{p'}^* \frac{\partial \widehat{u'_z}}{\partial \beta} - i k_z \widehat{p'} \widehat{u'_\beta}^* \right) \right\}}_{\tilde{\Pi}_{\beta z}} \\
& + \underbrace{\text{Re} \left\{ -2\nu \left(\frac{\widehat{\partial u'_\beta}^*}{\partial r} \frac{\partial \widehat{u'_z}}{\partial r} + \frac{1}{r^2} \frac{\widehat{\partial u'_\beta}^*}{\partial \beta} \frac{\partial \widehat{u'_z}}{\partial \beta} + k_z^2 \widehat{u'_\beta}^* \widehat{u'_z} + \frac{1}{r^2} \widehat{u'_r}^* \frac{\partial \widehat{u'_z}}{\partial \beta} \right) \right\}}_{\tilde{\mathcal{E}}_{\beta z}} \\
& + \underbrace{\text{Re} \left\{ -\frac{1}{2} \left(\frac{\widehat{\partial u'_r u'_\beta}^* \widehat{u'_z}}{\partial r} + 3 \frac{\widehat{u'_r u'_\beta}^* \widehat{u'_z}}{r} + \frac{\widehat{\partial u'_r u'_z} \widehat{u'_\beta}^*}{\partial r} + \frac{\widehat{u'_r u'_z} \widehat{u'_\beta}^*}{r} \right) \right\}}_{\tilde{D}_{\beta z}^t} \\
& + \underbrace{\text{Re} \left\{ \nu \left(\frac{\partial^2 \widehat{u'_\beta}^* \widehat{u'_z}}{\partial r^2} + \frac{1}{r} \frac{\widehat{\partial u'_\beta}^* \widehat{u'_z}}{\partial r} - \frac{\widehat{u'_\beta}^* \widehat{u'_z}}{r^2} \right) \right\}}_{\tilde{D}_{\beta z}^v} + \underbrace{\text{Re} \left\{ -2\Omega \widehat{u'_r}^* \widehat{u'_z} \right\}}_{\tilde{P}_{\beta z}^c} = 0 \quad .
\end{aligned} \tag{E.26}$$

The final step is to derive the transport equation of the TKE spectrum. It is obtained by summing the budget terms of the three autospectra transport equations and dividing the result by two, i.e., $\tilde{E}_{TKE} = (\tilde{E}_{rr} + \tilde{E}_{\beta\beta} + \tilde{E}_{zz})/2$. Therefore, the transport equation of the turbulence kinetic energy (TKE) spectrum $\tilde{E}_{TKE}$ is

$$\begin{aligned}
& \underbrace{\text{Re} \left\{ -\widehat{u'_r}^* \widehat{u'_\beta} \frac{\partial \langle u_\beta \rangle}{\partial r} - \widehat{u'_r}^* \widehat{u'_z} \frac{\partial \langle u_z \rangle}{\partial r} + \frac{\widehat{u'_r}^* \widehat{u'_\beta}}{r} \langle u_\beta \rangle \right\}}_{\tilde{P}_{TKE}} \\
& + \underbrace{\text{Re} \left\{ \frac{1}{2} \frac{\partial \widehat{u'_r}^* \widehat{u'_i}}{\partial r} + i k_z \widehat{u'_i}^* \widehat{u'_i} - \frac{1}{2} \frac{\widehat{u'_r}^* \widehat{u'_i}}{r} - \frac{\partial \widehat{u'_r}^* \widehat{u'_i}}{\partial r} \widehat{u'_i} - \frac{1}{r} \frac{\partial \widehat{u'_\beta}^* \widehat{u'_i}}{\partial \beta} \widehat{u'_i} \right\}}_{\tilde{T}_{TKE}} \\
& + \underbrace{\text{Re} \left\{ -\nu \left(\frac{\partial \widehat{u'_i}^*}{\partial r} \frac{\partial \widehat{u'_i}}{\partial r} + \frac{1}{r^2} \frac{\partial \widehat{u'_i}^*}{\partial \beta} \frac{\partial \widehat{u'_i}}{\partial \beta} + k_z^2 \widehat{u'_i}^* \widehat{u'_i} \right. \right.}{\tilde{\varepsilon}_{TKE}} \\
& \quad \left. \left. + \frac{2}{r^2} \widehat{u'_r}^* \frac{\partial \widehat{u'_\beta}}{\partial \beta} - \frac{2}{r^2} \widehat{u'_\beta}^* \frac{\partial \widehat{u'_r}}{\partial \beta} + \frac{\widehat{u'_r}^* \widehat{u'_r}}{r^2} + \frac{\widehat{u'_\beta}^* \widehat{u'_\beta}}{r^2} \right) \right\}}_{\tilde{D}_{TKE}^t} \\
& + \underbrace{\text{Re} \left\{ -\frac{1}{2} \left(\frac{\partial \widehat{u'_r}^* \widehat{u'_i}}{\partial r} + \frac{\widehat{u'_r}^* \widehat{u'_i}}{r} \right) + \frac{\widehat{u'_\beta}^* \widehat{u'_r}}{r} - \frac{\widehat{u'_r}^* \widehat{u'_\beta}}{r} \right\}}_{\tilde{D}_{TKE}^p} \\
& + \underbrace{\text{Re} \left\{ -\frac{1}{\rho} \left(\frac{\partial \widehat{p'}^* \widehat{u'_r}}{\partial r} + \frac{\widehat{p'}^* \widehat{u'_r}}{r} \right) \right\}}_{\tilde{D}_{TKE}^p} + \underbrace{\text{Re} \left\{ \frac{\nu}{2} \left(\frac{\partial^2 \widehat{u'_i}^* \widehat{u'_i}}{\partial r^2} + \frac{1}{r} \frac{\partial \widehat{u'_i}^* \widehat{u'_i}}{\partial r} \right) \right\}}_{\tilde{D}_{TKE}^v} = 0 \quad .
\end{aligned} \tag{E.27}$$