Approximate Identities for Banach Algebras

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Approximate Identities for Banach Algebras

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Xuan Li

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Of

Master of Science

To Mom and Dad.

Contents

Abstract

We examine the properties of approximate identities for Banach algebras and survey the known results involving bounded and unbounded approximate identities for Banach algebras, in particular, those related to the factorization theory for Banach algebras.

Special attention is also paid to condition (U) for unbounded approximate identities. We prove that every sequential approximate identity for a Banach algebra satisfies condition (U), we construct an example of a commutative separable normed algebra having a sequential approximate identity that does not satisfy condition (U), and we show that the closed ideal $\mathcal{I}(E) = \{f \in L^2(G) : \widehat{f} = 0 \text{ on } E\}$ of $L^2(G)$ with G a commutative locally compact group has an approximate identity satisfying condition (U), where E is a subset of the dual group Σ of G.

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Xuan Li

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Chapter 1

Introduction

Approximate identity is an interesting topic in the field of functional analysis. The theory about it is useful in solving many problems in the field. It is closely related to the factorization theory for Banach algebras and their modules.

The concept of "approximate identity" was first described by L. H. Loomis [27] in 1953. Years later, P. J. Cohen [5] gave an explicit definition and started to establish the connection between the existence of a bounded approximate identity and the factorization theory. However, the study of unbounded approximate identities was beyond the scope of the mathematicians at that time.

Our goal in this thesis is to survey some known results about bounded and unbounded approximate identities for Banach algebras.

In Chapter 2, we present some elementary properties of approximate identities for normed algebras. We show that if S is an uncountable set then the algebra $l^1(S)$ does not have a sequential approximate identity. Chapter 3 concerns bounded approximate identities for Banach algebra. We discuss the concept of "weak factorization of bounded sequences" and summarize the relations among various factorizations of sequences. In Chapter 4, we study unbounded approximate identities for Banach algebras. The investigation is around condition (U). We prove that every sequential approximate identity for a Banach algebra satisfies condition (U). We then give an example of a commutative separable normed algebra with a sequential approximate identity that does not satisfy condition (U). Finally, we show that the closed ideal $\mathcal{I}(E) = \{f \in L^2(G) : \widehat{f} = 0 \text{ on } E\}$ of $L^2(G)$ for a commutative locally compact group G has an approximate identity satisfying condition (U), where E is any subset of the dual group Σ of G. Questions for further investigation are raised in Chapter 5.

Chapter 2

Approximate Identities in Normed Algebras

Throughout this Chapter the symbol $\mathbb{F}$ will be used to denote a field that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

2.1 Preliminaries

Here we will first recall some elementary definitions and notations which are consistent with those in [3] and [8].

An associative *algebra* over $\mathbb{F}$ is a linear space $\mathcal{A}$ over $\mathbb{F}$ together with a multiplication mapping $(x, y) \to xy$ of $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ that satisfies the following axioms (for all $x, y, z \in \mathcal{A}, \alpha \in \mathbb{F}$):

(i) $x(yz) = (xy)z$,

(ii) $x(y + z) = xy + xz$, $(x + y)z = xz + yz$,

(iii) $(\alpha x)y = \alpha(xy) = x(\alpha y)$.

An *algebra-norm* on an algebra $\mathcal{A}$ is a mapping $\|\cdot\|\colon \mathcal{A} \mapsto \mathbb{R}$, with which $\mathcal{A}$ is a normed space and the following inequality holds

$$\|xy\| \leq \|x\| \, \|y\| \quad (x, y \in \mathcal{A}).$$

If $\mathcal{A}$ has an algebra-norm defined on it, we call it a normed algebra. A complete normed algebra is called a *Banach algebra.*

A *directed set* is a partially ordered set Λ (admitting Reflexivity, Antisymmetry and Transitivity) such that, given $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda \in \Lambda$ with $\lambda \geq \lambda_k \quad (k = 1, 2)$.

Let E be a topological space. A *net* in E is a mapping from a directed set Λ into E. A net $\{x_\lambda\}_{\lambda \in \Lambda}$ in E is said to *converge* to $x \in E$, denoted by

$$\lim_{\lambda \in \Lambda} x_\lambda = x,$$

if, for every neighborhood U of x, there exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geq \lambda_0$.

2.2 Approximate identities

Definition 2.2.1. Let $\mathcal{A}$ be a normed algebra. A *left approximate identity* for $\mathcal{A}$ is a net $\{e_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{A}$ such that for all $x \in \mathcal{A}$,

$$\lim_{\lambda \in \Lambda} e_\lambda x = x \tag{2.1}$$

Right approximate identities are similarly defined by replacing $e_\lambda x$ with xe_λ in Equation (2.1). A *two-sided approximate identity* is a net that is both a left and a right approximate identity.

An approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ is called *sequential* if Λ is identical to positive integers with the usual order, and is said to be *commutative* (or *abelian*) if e_{λ_1} and e_{λ_2} commute for all $\lambda_1, \lambda_2 \in \Lambda$.

An approximate identity is called *bounded* if there exists a positive constant k such that

$$\|e_\lambda\| \leq k \qquad (\lambda \in \Lambda).$$

In this case, we define the *bound* k of $\{e_\lambda\}_{\lambda\in\Lambda}$ by

$$k = \sup_\lambda \|e_\lambda\|,$$

and the *norm* by

$$\|\{e_\lambda\}\| = \lim_\lambda sup \|e_\lambda\|.$$

Remark 2.2.1. The norm of an approximate identity is always no less than 1. In fact, for all $x \in \mathcal{A}$

$$\|x\| = \lim_\lambda \|e_\lambda x\| \leq \limsup_\lambda \|e_\lambda\| \cdot \|x\|.$$

So $1 \leq \limsup_\lambda \|e_\lambda\| = \|\{e_\lambda\}\|$.

The following proposition asserts an equivalent definition of approximate identity. The proof is straightforward and can be found in book [8] by Doran and Wichmann.

Proposition 2.2.1. *A normed algebra A has a left approximate identity if and only if for every finite set $\{x_1, x_2, \cdots, x_n\}$ of elements in A and every $\epsilon > 0$ there exists an element $e \in A$ such that $\|x_i - ex_i\| < \epsilon$ for $i = 1, 2, \cdots, n$.*

Remark 2.2.2. One can draw the right side and the two-side versions of the above proposition. In general, if a result holds for left approximate identities, it will be true also for right approximate identities and two-sided approximate identities with minor respective changes.

For bounded approximate identities we have the following Dixon's Theorem. See [14].

Proposition 2.2.2. *A normed algebra $\mathcal{A}$ has a left approximate identity bounded by k if and only if for every $x \in \mathcal{A}$ and every $\epsilon > 0$ there exists an element $e \in \mathcal{A}$ such that $\|e\| \leq k$ and $\|x - ex\| < \epsilon$.*

2.3 Properties of approximate identities

We start from the relationship between left, right and two-sided approximate identity. It is clear that if a normed algebra has both a bounded left approximate identity and a bounded right approximate identity, then it has a bounded two-sided approximate identity. Naturally, the following questions arise:

(i) Does the existence of a left approximate identity and a right approximate identity imply the existence of a two-sided approximate identity?

(ii) What happens when there is a bounded approximate identity on one side and an unbounded approximate identity on the other side?

These two questions above have been answered in [9] with the following two results.

Example 2.3.1. *Let $\mathcal{A}_0$ be the complex associative algebra generated by $\{e_i : i = 1, 2, 3, \cdots\}$ subject to the relations $e_i e_j = e_{\min\{i,j\}}$, unless i is odd and j is even. Then it can be checked that a typical element of $\mathcal{A}_0$ is uniquely expressible in the form*

$$x = \sum_{r=1}^{\infty} \lambda_r e_r + \sum_{i,j} \lambda_{ij} e_i e_j, \tag{2.2}$$

where all but finitely many of the scalars λ_r, λ_{ij} are zero, and the second summation is taken over odd values of i and even values of j. We define a norm for $\mathcal{A}_0$ by

$$\|x\| = \sum_{r=1}^{\infty} |\lambda_r| 2^r + \sum_{i,j} |\lambda_{ij}| 2^{i+j}, \tag{2.3}$$

and form the completion $\mathcal{A}$. A typical element x of $\mathcal{A}$ is of the form (2.2), without the restriction that all but finitely many of the scalars λ_r, λ_{ij} should vanish, but with $\|x\|$, as in (2.3), being finite.

By some technical computation, one can check that $\{e_{2i+1}\}$ forms a right approximate identity, and $\{e_{2i}\}$ a left approximate identity for $\mathcal{A}$. But $\mathcal{A}$ does not have any two-sided approximate identity.

Proposition 2.3.2. *Let $\mathcal{A}$ be a normed algebra. If $\mathcal{A}$ has a bounded left approximate identity and a right approximate identity, then it has a two-sided* (unbounded, possibly) *approximate identity.*

For the boundedness of sequential approximate identities, we have the following proposition. See [9].

Proposition 2.3.3. *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ has a bounded left approximate identity, then every sequential right approximate identity is also bounded* (not necessarily by the same bound though).

The existence of sequential approximate identity implies some interesting properties of the structure of the underlying space. First, regarding the renorming, P. G. Dixon proved the following result in [9].

Theorem 2.3.4. *Let $\mathcal{A}$ be a commutative normed algebra with a bounded sequential approximate identity $\{e_n\}$. Then there is an equivalent algebra norm on $\mathcal{A}$ for which there exists a sequential approximate identity $\{f_n\}$ of norm 1.*

The above theorem was also generalized to non-commutative case in [9], which extended a work by A. M. Sinclair [41].

Theorem 2.3.5. *If $\mathcal{A}$ is a normed algebra with a bounded sequential two-sided approximate identity $\{e_n\}$, then there is an equivalent norm in which $\mathcal{A}$ has a bounded sequential two-sided approximate identity $\{f_n\}$ of norm 1. Furthermore, if $\mathcal{A}$ is complete, then $\{f_n\}$ may be chosen commuting.*

But we can not expect a Banach algebra having a bounded approximate identity to have a sequential approximate identity. In particular, we have the following proposition about the sequential approximate identities for l^1 algebra.

Proposition 2.3.6. *If S is an uncountable set, then $l^1(S)$ does not have a sequential approximate identity.*

Proof. Suppose not. Let $\{\mathbf{e_n}\}_{n=1}^{\infty}$ be a sequential approximate identity in $l^1(S)$ for an uncountable set S. Then for $n = 1, 2, \cdots$,

$$\|\mathbf{e_n}\|_1 = \sum_{s \in S} |e_n(s)| < \infty$$

For each $n = 1, 2, \cdots$ and integer k, the set

$$S_{k,n} = \{s \in S \,|\, |e_n(s)| \geq \frac{1}{k}\}$$

is finite. Hence,

$$S_1 = \{s \in S \,|\, e_n(s) \neq 0 \text{ for some } n\} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} S_{k,n}$$

is countable.

So for any element $s \in S \backslash S_1$, $e_n(s) = 0$ for $n = 1, 2, \cdots$. Then for δ_{s} in $l^1(S)$ defined by

$$\delta_{\mathrm{s}}(t) = \begin{cases} 1, & \text{if } t = s \\ 0, & \text{if } t \neq s \end{cases}$$

we have that

$$\|\mathbf{e_n}\delta_{\mathrm{s}} - \delta_{\mathrm{s}}\|_1 = |e_n(s) - 1| = 1 \neq 0$$

which is impossible because $\{\mathbf{e_n}\}_{n=1}^{\infty}$ is a approximate identity.

Therefore, if S is an uncountable set then $l^1(S)$ does not have a sequential approximate identity. $\square$

We will discuss more properties of a sequential approximate identity in Section 4.3.

2.4 Approximate units

Motivated by the criteria in Proposition 2.2.2 for the existence of a bounded approximate identity in a normed algebra, we introduce the concept of approximate units. See [8].

A normed algebra $\mathcal{A}$ is said to have *left approximate units* if for every $x \in \mathcal{A}$ and $\epsilon > 0$ there exists an element $u \in A$ (depending on x and ϵ) such that $\|x - ux\| < \epsilon$. The right and two-sided approximate units for a normed algebra can be defined similarly.

If in addition, u can be chosen to be bounded by a fixed constant k, we say that $\mathcal{A}$ has *bounded left approximate units.*

We say that a normed algebra $\mathcal{A}$ has *pointwise-bounded left approximate units* if for each $x \in \mathcal{A}$ there exists a constant $k(x)$ such that for every $\epsilon > 0$ there exists an element $u \in \mathcal{A}$ such that

$$\|u\| \le k(x) \quad \text{and} \quad \|x - ux\| < \epsilon.$$

It is obvious that every normed algebra with a left approximate identity has left approximate units. The converse is true for bounded case. In fact, Proposition 2.2.2 can be restated as follows. See [8].

Theorem 2.4.1. *A normed algebra $\mathcal{A}$ has left, right or two-sided approximate units bounded by a constant k if and only if $\mathcal{A}$ has left, right or two-sided approximate identity bounded by the same constant k.*

The following Proposition from [43] gives a relation between the existence of an approximate identity and the existence of pointwise-bounded approximate units for the commutative algebras.

Proposition 2.4.2. *A commutative normed algebra with pointwise-bounded approximate units has an approximate identity.*

Unfortunately, we can not in general assert the existence of a bounded approximate identity under the conditions of Propositon 2.4.2. The following example also from [43] serves as a counterexample.

Example 2.4.3. *Consider the commutative normed algebra over $\mathbb{C}$*

$$\mathcal{A} = \big\{(\lambda_1, \lambda_2, \cdots) : \quad \lambda_i = 0 \quad \textit{for all but finite indices} \quad i\,\big\}$$

with coordinatewise algebraic operations and the norm defined by

$$\|(\lambda_1, \lambda_2, \cdots)\| = \max_i |\, i\lambda_i \,|.$$

Then $\mathcal{A}$ has pointwise-bounded approximate units of the form

$$u_i = (1, \cdots, 1, 0, \cdots).$$

Obviously, $\mathcal{A}$ has no bounded approximate identity.

However, if the normed algebra is complete then we have a stronger result still from [43] as follows.

Theorem 2.4.4. *A commutative Banach algebra has pointwise-bounded approximate units if and only if it has a bounded approximate identity.*

Another characterization for the existence of left approximate units is given in [8] by the next Proposition, from which we see that for a normed algebra $\mathcal{A}$, if there is a constant $q : 0 < q < 1$, such that each $x \in \mathcal{A}$ with $\|x\| = 1$ associates an element $u \in \mathcal{A}$ such that $\|x - ux\| \leq q$ then the algebra $\mathcal{A}$ has left approximate units.

Proposition 2.4.5. *A normed algebra $\mathcal{A}$ has left approximate units if and only if there exists $q \in (0, 1)$ with the following property: for every $x \in \mathcal{A}$ there exists an element $u \in \mathcal{A}$ such that*

$$\|x - ux\| \leq q \|x\|.$$

Chapter 3

Factorization and Bounded Approximate Identities

Factorization is a very important and useful property of algebras and modules. For a normed algebra and its modules, this property is closely related to whether the algebra has an approximate identity. In this chapter, we discuss these relations.

3.1 Various factorization of sequences

Definition 3.1.1. We say that $\mathcal{A}$ has *factorization of sequences* (**FS** in short) if, for every sequence $\{x_i\} \subset \mathcal{A}$ there exist $a \in \mathcal{A}$ and a sequence $\{y_i\} \subset \mathcal{A}$ such that $x_i = ay_i$ for each i.

Definition 3.1.2. We say that $\mathcal{A}$ has *factorization of bounded sequences* (**FBS** in short) if, for every bounded sequence $\{x_i\} \subset \mathcal{A}$ there exist $a \in \mathcal{A}$ and bounded sequence $\{y_i\} \subset \mathcal{A}$ such that $x_i = ay_i$ for each i.

Definition 3.1.3. We say that $\mathcal{A}$ has *factorization of null sequences* (**FNS** in short) if, for every null sequence $\{x_i\} \subset \mathcal{A}$ there exist $a \in \mathcal{A}$ and a null sequence $\{y_i\} \subset \mathcal{A}$ such that $x_i = ay_i$ for each i.

We say $\mathcal{A}$ that has *factorization of elements* (**FE** in short) if, for every element x in $\mathcal{A}$, there exist a and y in $\mathcal{A}$ such that $x = ay$.

The following Proposition gives the relation among **FS**, **FBS**, **FNS** and **FE**.

Proposition 3.1.1. *For a normed algebra $\mathcal{A}$, **FBS** implies **FNS**, **FNS** implies **FS** and **FS** implies **FE**.*

Proof. **FBS** $\Rightarrow$ **FNS**: This is because for every null sequence $\{x_i\}$, $\left\{\frac{x_i}{\|x_i\|}\right\}$ is a bounded sequence.

FNS $\Rightarrow$ **FS**: Let $\{x_i\}_{i=1}^{\infty}$ be any sequence in $\mathcal{A}$. Then the sequence $y_i = \frac{x_i}{i\|x_i\|}$ is a null sequence in $\mathcal{A}$. From **FNS**, there exist $a \in \mathcal{A}$ and a null sequence $\{z_i\} \subset \mathcal{A}$ such that $y_i = az_i$ $(i = 1, 2, 3, \ldots)$. Letting $w_i = i\,\|x_i\|\,z_i$, we then have $x_i = aw_i$ $(i = 1, 2, 3, \ldots)$. Therefore $\mathcal{A}$ has **FS**.

FS $\Rightarrow$ **FE** : Trivial. $\square$

Let $\mathcal{A}$ be an algebra. Denote $\mathcal{A}^2 = \{ab : a, b \in \mathcal{A}\}$ and $\mathcal{A}^{[2]} = span(A^2)$. Then $\mathcal{A}$ has **FE** if $\mathcal{A} = \mathcal{A}^2$. We say that $\mathcal{A}$ has weak factorization of elements (**WFE** in short) if $\mathcal{A} = \mathcal{A}^{[2]}$. See also [11], [15] and [2] for references. Similarly, we have other weak versions of factorizations as follows:

Definition 3.1.4. We say that $\mathcal{A}$ has *weak factorization of sequences* (**WFS** in short) if, for every sequence $\{x_i\} \subset \mathcal{A}$ there exist an integer N, elements $a^{(n)}$ of $\mathcal{A}$ and sequences $\{{y_i}^{(n)}\}_{i=1}^{\infty}$ in $\mathcal{A}$, where $n = 1, 2, \ldots, N$, such that $x_i = \sum_{n=1}^{N} a^{(n)}\, {y_i}^{(n)}$ for each i.

Definition 3.1.5. We say $\mathcal{A}$ has *weak factorization of bounded sequences* (**WFBS** in short) if, for every bounded sequence $\{x_i\} \subset \mathcal{A}$ there exist an integer N, elements $a^{(n)}$ of $\mathcal{A}$ and bounded sequences $\{{y_i}^{(n)}\}_{i=1}^{\infty}$ in $\mathcal{A}$, where $n = 1, 2, \ldots, N$, such that $x_i = \sum_{n=1}^{N} a^{(n)}\, {y_i}^{(n)}$ for each i.

Definition 3.1.6. We say that $\mathcal{A}$ has *weak factorization of null sequences* (**WFNS** in short) if, for every null sequence $\{x_i\} \subset \mathcal{A}$ there exist an integer N, element $a^{(n)} \in \mathcal{A}$, $n = 1, 2, \ldots, N$, and null sequences $\{{y_i}^{(n)}\}_{i=1}^{\infty}$ in $\mathcal{A}$ such that $x_i = \sum_{n=1}^{N} a^{(n)}\, {y_i}^{(n)}$ for each i.

Analogously, we have the following weakness version of Proposition 3.1.1.

Proposition 3.1.2. ***WFBS* $\Rightarrow$ *WFNS* $\Rightarrow$ *WFS* $\Rightarrow$ *WFE***

We therefore have the following implications diagram:

$$(*) \qquad \begin{array}{ccccccc} \textbf{FBS} & \Rightarrow & \textbf{FNS} & \Rightarrow & \textbf{FS} & \Rightarrow & \textbf{FE} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \textbf{WFBS} & \Rightarrow & \textbf{WFNS} & \Rightarrow & \textbf{WFS} & \Rightarrow & \textbf{WFE} \end{array}$$

The converses of these implications are not true. Counterexamples have been given by J. P. R. Christense [4], P. G. Dixon [11] [13], M. Leinert [23], R. J. Loy [26], S. I. Ouzomgi [29], W. L. Paschke [32] and G. A. Willis [44] [46] [47]. See also [45] for a summary.

3.2 Cohen's Factorization Theorem

First, we state the famous Cohen's Factorization Theorem (See [5]).

Theorem 3.2.1. *If $\mathcal{A}$ is a Banach algebra with a bounded approximate identity, then for each element $x \in \mathcal{A}$ and $\delta > 0$, there exist elements y and z in $\mathcal{A}$ such that*

(i) $x = yz$;

(ii) *z belongs to the closed left ideal generated by x* ;

(iii) $\|x - z\| < \delta$.

The beauty and deepness of the Cohen's Factorization Theorem is that its hypothesis is of topological nature while the conclusion is mainly algebraic. In this sense, people call it a characteristic theorem of Banach algebra. For instant, when Palmer reviewed the book [8] by Doran and Wichmann, he commented that "*the theory of approximate identity reached maturity with the discovery by P. J. Cohen of his famous factorization theorem*".

Before Cohen proved his factorization theorem, W. Rudin had already shown in [35] and [37] that every function in $L^1(G)$ is the convolution of two other functions from $L^1(G)$ where G is the additive group of Euclidean n-space or the n-dimensional torus. Cohen's Factorization Theorem has been widely used in studying the structure of Banach algebra. For example, it was used by B. E. Johnson to answer many questions in cohomology theory in [21]. The theorem was improved by N. Th. Varopoulous in [42] to the following:

Theorem 3.2.2. *Every Banach algebra with a bounded approximate identity has **FNS**.*

In [45], G. Willis observed the following result.

Proposition 3.2.3. *If $\mathcal{A}$ has **FNS**, then there is a constant $M > 0$ such that for every null sequence $\{x_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ there are an element a and a null sequence $\{y_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ such that $x_i = ay_i$ for each i, $\sup_i \|y_i\| \leq \sup_i \|x_i\|$ and $\|a\| < M$.*

Hence we have the following theorem:

Theorem 3.2.4. *If Banach algebra $\mathcal{A}$ has a bounded approximate identity, then there is a constant $M > 0$ such that for every null sequence $\{x_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ there are an element a and a null sequence $\{y_i\}_{i=1}^{\infty}$ in $\mathcal{A}$ such that $x_i = ay_i$ for each i, $\sup_i \|y_i\| \leq \sup_i \|x_i\|$ and $\|a\| < M$.*

We note that in all the factorization theorems cited in this section, the existence of a bounded two-sided approximate identity is not necessary and can be replaced by the existence of a bounded one-sided approximate identity. Eventually, we have the following Proposition. See [8].

Proposition 3.2.5. *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*

(i) *there exists a constant* $K \geq 1$ *such that for every* $\epsilon > 0$ *and every* $x \in \mathcal{A}$ *there are elements* $a, y \in \mathcal{A}$ *with*

$$x = ay, \qquad \|a\| \leq K, \qquad \text{and} \quad \|y - x\| < \epsilon$$

(ii) $\mathcal{A}$ *has a left bounded approximate identity.*

In general, factorization of elements does not imply that the algebra has a bounded approximate identity. We will continue this discussion in Section 4.1.

Factorization theorems for Banach algebras can naturally be generalized to Banach modules.

Let $\mathcal{A}$ be a normed algebra over $\mathbb{F}$ and let X be a normed linear space over $\mathbb{F}$. X is said to be a *normed left* $\mathcal{A}$*-module* if X is a left $\mathcal{A}$-module and also satisfies that there exists a constant $k > 0$ such that

$$\|a\,x\| \leq k\,\|a\|\;\|x\| \qquad \text{for all } a \in \mathcal{A} \text{ and } x \in X.$$

A mormed left $\mathcal{A}$-module is called a *Banach left* $\mathcal{A}$*-module* if it is complete as a normed linear space. We denote $\mathcal{A}^\sharp$ the unitization of $\mathcal{A}$. Then every left $\mathcal{A}$-module X can be viewed as a left $\mathcal{A}^\sharp$-module.

Definition 3.2.1. A (*bounded*) *approximate identity* in $\mathcal{A}$ for X is a (bounded) net $\{e_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{A}$ such that for all $x \in X$,

$$\lim_{\lambda\in\Lambda} e_\lambda\, x = x$$

E. Hewitt and K. A. Ross [20] gave the following theorem:

Theorem 3.2.6. *Let $\mathcal{A}$ be a Banach algebra and X be a Banach left $\mathcal{A}$-module. If $\mathcal{A}$ has a bounded approximate identity for X then for $z \in X$ and $\delta > 0$ there exist $a \in \mathcal{A}$, $y \in X$ such that $z = ay$ and $\|z - y\| \leq \delta$.*

We point out that the proof of Theorem 3.2.2 given in [3, §11] also works to conclude the following:

Proposition 3.2.7. *Let $\mathcal{A}$ be a Banach algebra and X be a Banach left $\mathcal{A}$-module. If $\mathcal{A}$ has a bounded approximate identity for X then for $z_n \in X$ with $\lim_{n\to\infty} z_n = 0$, there exist $a \in \mathcal{A}$ and $y_n \in X$ with $\lim_{n\to\infty} y_n = 0$ such that $z_n = ay_n$ $(n = 1, 2, \cdots)$.*

3.3 Banach algebras having a bounded approximate identity

In this section we concern with basic results involving approximate identities for various concrete classes of Banach algebras. Among them are two important classes: group algebras $L^1(G)$ of a locally compact group G and amenable Banach algebras.

Let G be a locally compact (Hausdorff) topological group and μ be the left invariant Haar measure on G. The space $L^1(G)$ of all Haar integrable functions on G is a Banach algebra under the norm defined by

$$\|f\|_1 = \int_G |f| \, d\mu, \qquad f \in L^1(G)$$

and the convolution multiplication given by

$$(f*g)(x) = \int_G f(xy)g(y^{-1})\, d\mu(y) = \int_G f(y)g(y^{-1}x)\, d\mu(y), \qquad f, g \in L^1(G).$$

This algebra is called $L^1(G)$ *group algebra.*

If G is discrete, $L^1(G) = l^1(G)$. In this case, it has an identity. If G is not discrete, then $L^1(G)$ does not have an identity in general but it always admits a bounded approximate identity.

Theorem 3.3.1 ([36] and [20]). *The group algebra $L^1(G)$ of a locally compact group G has a two-sided approximate identity bounded by 1.*

In fact, if $\{U_\alpha\}$ is a neighborhood basis at e_G, the unit of G, then $\frac{\chi_{U_\alpha}}{|U_\alpha|}$ gives a bounded approximate identity of bound equal to 1, where χ_{U_α} is the characteristic function of U_α and $|U_\alpha|$ is the Haar measure of U_α.

More generally, we have the following Proposition [38]:

Proposition 3.3.2. *Let G be a locally compact group, let $\mathfrak{U}$ be a basis of neighborhoods of e_G and let $(e_U)_{u \in \mathfrak{U}}$ be a net in $L^1(G)$ satisfying the following properties:*

(i) $e_U \geq 0$ *for all* $U \in \mathfrak{U}$;

(ii) *supp* $(e_U) = \{x \in U : e_U(x) \neq 0\} \subset U$ *for all* $U \in \mathfrak{U}$;

(iii) $\|e_U\|_1 = 1$ *for all* $U \in \mathfrak{U}$.

Then $(e_U)_{u \in \mathfrak{U}}$ is a bounded approximate identity for $L^1(G)$.

An approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ in a normed algebra $\mathcal{A}$ is called *central* if $e_\lambda x = xe_\lambda$ for all $x \in \mathcal{A}$ and $\lambda \in \Lambda$. It is not difficult to see the following stronger version of Theorem 3.3.1 for compact groups as follows [18].

Proposition 3.3.3. *The group algebra $L^1(G)$ of a compact group G has a central approximate identity bounded by 1.*

Let G be a locally compact group. A subset of G is called *invariant* if it is invariant under all the inner automorphisms. The group G is said to have *small invariant neighborhoods* (denoted by $G \in [SIN]$) if every neighborhood of the identity contains a compact invariant neighborhood of the identity. For example, locally compact commutative groups, compact groups and discrete groups are all $[SIN]$ group. See W. Palmer's book [31] for details and also H. Rindler's paper [34] for a well-organized summary.

R. D. Mosak [28] has characterized this type of groups in terms of a bounded approximate identity for its group algebra.

Theorem 3.3.4. *A locally compact group G is an SIN-group if and only if $L^1(G)$ has a central approximate identity.*

Let $\mathcal{A}$ be a Banach algebra and E be a Banach $\mathcal{A}$-module. A linear map $D : \mathcal{A} \to E$ is called a *derivation* if

$$D(ab) = a \cdot Db + (Da) \cdot b \qquad \text{for } a, b \in \mathcal{A}.$$

Let $x \in E$. The mapping $ad_x : \mathcal{A} \to E$ given by $ad_x(a) = ax - xa$ $(a \in A)$ is a bounded derivation, called *an inner derivation.*

Denote $\mathcal{Z}^1(\mathcal{A}, E)$ the space of all derivations from $\mathcal{A}$ into E and $\mathcal{B}^1(\mathcal{A}, E)$ the space of all inner derivations from $\mathcal{A}$ into E.

Definition 3.3.1. A Banach algebra $\mathcal{A}$ is called *amenable* if

$$\mathcal{Z}^1(\mathcal{A}, E^*) = \mathcal{B}^1(\mathcal{A}, E^*)$$

or in terms of quotient

$$\mathcal{H}^1(\mathcal{A}, E^*) := \frac{\mathcal{Z}^1(\mathcal{A}, E^*)}{\mathcal{B}^1(\mathcal{A}, E^*)} = \{0\}$$

for every Banach $\mathcal{A}$-bimodule E, where E^* is the dual module of E.

We list here several classes of Banach algebras and their amenability.

Group Algebra $L^1(G)$: B. E. Johnson showed in [21] that *the group algebra $L^1(G)$ for a locally compact group G is amenable if and only if G is an amenable group.*

Measure Algebra $M(G)$: Let G be a locally compact group. The *measure algebra $M(G)$* is the unital Banach algebra of all (finite) complex regular Borel measures on G with the convolution product defined by

$$\langle f, \mu * \nu \rangle := \int_G \left(\int_G f(gh) d\mu(g) \right) d\nu(h), \quad \mu, \nu \in M(G) \text{ and } f \in C_0(G)$$

where $C_0(G)$ is the space of all continuous functions on G vanishing at infinity.

H. G. Dales, F. Ghahramani and A. Ya. Helemskii proved in [7] that *a measure algebra $M(G)$ is amenable if and only if G is a discrete and amenable group.*

Uniform Algebra: A *uniform algebra* on a locally compact Hausdorff space X is a uniformly closed subalgebra of $C_b(X)$ which contains the constants and separates the points of X. When endowed with the supremum norm $\|f\|_X = \sup_{x \in X} |f(x)|$, the uniform algebra $\mathcal{A}$ becomes a Banach algebra, called *Banach uniform algebra.*

M. V. Sheinberg proved in [40] that *the uniform Banach algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ is isometrically isomorphic to $C_0(X)$ for some locally compact space X.*

Fourier Algebra $A(G)$ and Fourier–Stieltjes algebra $B(G)$: B. E. Forrest and V. Runde recently showed in [17] that *the Fourier algebra $A(G)$ on a locally compact group G is amenable if and only if G has an abelian subgroup of finite index*, and that *the Fourier–Stieltjes algebra $B(G)$ is amenable if and only if G has a compact, abelian subgroup of finite index.* In [24], on the other hand, H. Leptin proved that *Fourier algebra $A(G)$ has an approximate identity if and only if G is an amenable group.*

B. E. Johnson [21] revealed the following general implication theorem:

Theorem 3.3.5. *If a Banach algebra $\mathcal{A}$ is amenable then $\mathcal{A}$ has a bounded approximate identity.*

Moreover, for ideals of an amenable Banach algebra, we need the following notation to characterize the existence of a bounded approximate identity as follows.

Definition 3.3.2 (Weak Complementation). let E be a Banach space. A closed subspace F of E is called *weakly complemented* in E if

$$F^{\perp} = \{\phi \in E^* : \quad \langle x, \phi \rangle = 0 \text{ for all } x \in F\}$$

is complemented in E^*, where E^* is the dual of E.

In [6], Jr, P. C. Curtis and R. J. Loy showed the following result:

Theorem 3.3.6. *Let $\mathcal{A}$ be an amenable Banach algebra, and let $\mathcal{I}$ be a closed (two-sided) ideal of $\mathcal{A}$. Then the following are equivalent:*

(i) *$\mathcal{I}$ is amenable.*

(ii) *$\mathcal{I}$ has a bounded approximate identity.*

(iii) *$\mathcal{I}$ is weakly complemented.*

We mention here that a C^*-algebra also belongs to the list of Banach algebras having a bounded approximate identity. We refer to [22] for details.

Chapter 4

Unbounded Approximate Identities

4.1 More factorization theorems

In this section we deal with two questions: (1) how to weaken the hypothesis of boundedness on the approximate identity in the condition of Cohen's Factorization Theorem; and (2) how to construct counter–examples of a Banach algebra with factorization but without a bounded approximate identity.

For (1), H. G. Feichtinger and M. Leinert [15] showed that for a fixed element x in a Banach algebra $\mathcal{A}$, x can be expressed as a product if the following two assumptions hold:

(i) there exist constants $K > 0$ and $0 < \alpha < 1$ such that, for every $\epsilon > 0$, there exists $u \in \mathcal{A}$ with $\|u\| \leq K\epsilon^{-\alpha}$ and $\|ux - x\| < \epsilon$;

(ii) these $u = u(\epsilon)$ may be chosen so that $u(\epsilon_1) \cdot u(\epsilon_2) = u(\epsilon_2) \cdot u(\epsilon_1) = u(\epsilon_1)$

whenever $\epsilon_2 < \epsilon_1$.

P. G. Dixon further showed that for a commutative Banach algebra the above condition (i) is somehow as strong as requiring the existence of a bounded approximate identity in [11].

Theorem 4.1.1. *Let $\mathcal{A}$ be a commutative Banach algebra. If for every $x \in \mathcal{A}$, there exist constants $K > 0$, $\alpha \in (0, \frac{1}{2})$ such that, for each $0 < \epsilon < 1$, there is an element $u \in \mathcal{A}$ with $\|u\| \le K\epsilon^{-\alpha}$ and $\|ux - x\| \le \epsilon \|x\|$. Then $\mathcal{A}$ has a bounded approximate identity.*

For (2), here we give a non–commutative Banach algebra which does not have an approximate identity but does have factorization property, using the idea of P. G. Dixon in [11].

Example 4.1.2. *There is a four–dimensional Banach algebra that factorizes, but does not have approximate identities.*

Consider the algebra $\mathcal{A} = (\mathbb{C}^4, \|\cdot\|_1)$ with the product

$$(a_1, a_2, a_3, a_4)(b_1, b_2, b_3, b_4) = (a_2b_1, a_2b_2, a_3b_4, a_4b_4)$$

Obviously, it is a non–commutative Banach algebra. For any $(a_1, a_2, a_3, a_4) \in \mathcal{A}$, it can be written as

$$(a_1, a_2, a_3, a_4) = (c, 1, a_3, a_4)(a_1, a_2, d, 1)$$

where c, d are two arbitrary complex numbers.

But $\mathcal{A}$ has neither left approximate identity nor right approximate units. To see this, consider $\mathbf{a} = (1,0,1,0)$ then for all $\mathbf{x} = (a_1, a_2, a_3, a_4) \in \mathcal{A}$,

$$\|\mathbf{ax} - \mathbf{a}\|_1 = \|(1, 0, 1 - a_4, 0)\|_1 \geq 1$$

and

$$\|\mathbf{xa} - \mathbf{a}\|_1 = \|(1 - a_2, 0, 1, 0)\|_1 \geq 1.$$

So $\mathcal{A}$ does not have any left or right approximate units and hence does not have left or right approximate identities.

M. Leinert gave an example in [23] of a commutative semisimple algebra with factorization but without approximate units. Another example of a commutative semisimple Banach algebra which has factorization but does not have a bounded approximate identity was given in [30].

G. Willis constructed some examples of separable and non–separable Banach algebras which factorize but do not have bounded approximate identities in [45]. We state some of them here.

Example 4.1.3. The following gives *a commutative separable semisimple Banach algebra $\mathcal{A}$ which does not have bounded approximate identity but in which null sequences factor.*

Let $\mathbb{Q}$ be the set of rational numbers. For each positive integer n, define a weight function ω_n on $\mathbb{Q}$ by

$$\omega_n(t) = \begin{cases} n & \text{if } t \geq 0, \\ 1 & \text{if } t < 0. \end{cases}$$

Define a norm $\|\cdot\|_n$ on $l^1(\mathbb{Q})$ by

$$\|f\|_n = \sum_{t\in\mathbb{Q}} |f(t)|\omega_n(t) = \sum_{t<0} |f(t)| + n\sum_{t\geq 0} |f(t)|.$$

Thus $(l^1(\mathbb{Q}), \|\cdot\|_n)$ is a sequence of Banach algebras. Define the commutative Banach algebra

$$\mathcal{A} = \left\{ \{f_n\}_{n=1}^{\infty} \middle| \quad f_n \in l^1(\mathbb{Q}) \text{ and } \lim_{n\to\infty} \|f_n\|_n = 0 \right\}$$

with pointwise sum and product and with the norm defined by $\|\{f_n\}_{n=1}^{\infty}\| = \sup_n \|f_n\|_n$. Then $\mathcal{A}$ has the properties we wanted.

Example 4.1.4. The following gives *a non–separable commutative Banach algebra which does not have approximate units but in which bounded sequences can be factored.*

Let $\mathbb{Q}^+$ denote the additive semigroup of positive rational numbers and let $l^1(\mathbb{Q}^+)$ be a commutative Banach algebra with the convolution product

$$(f * g)(t) = \sum_{0<s<t} f(t-s)g(s) \qquad (t \in \mathbb{Q}^+,\ f, g \in l^1(\mathbb{Q}^+)).$$

Now let $l^\infty(l^1(\mathbb{Q}^+))$ be the Banach algebra consisting of bounded sequences, $\mathbf{f} = (f_n)_{n=1}^{\infty}$, of functions in $l^1(\mathbb{Q}^+)$ with the product, $(\mathbf{f}\cdot\mathbf{g})_n = f_n * g_n$ and norm given by $\|\mathbf{f}\| = \sup_n \|f_n\|_1$. The closed subspace, $c_0(l^1(\mathbb{Q}^+))$, consisting of those sequences $\mathbf{f}$ such that $\lim_{n\to\infty} \|f_n\|_1 = 0$ is an ideal in $l^\infty(l^1(\mathbb{Q}^+))$. Then $\mathcal{A} = \dfrac{l^\infty(l^1(\mathbb{Q}^+))}{c_0(l^1(\mathbb{Q}^+))}$ will be a non–separable Banach algebra which does not have approximate units but in which bounded sequences can be factored.

4.2 More about factorization of sequences

From Theorem 3.2.2 and the diagram $(*)$ in Section 3.1, we have the following diagram:

$$(**) \qquad \begin{array}{ccccccc} & & \text{existence of a bounded} & & & & \\ & & \text{approximate identity} & & & & \\ & & \Downarrow & & & & \\ \textbf{FBS} & \Rightarrow & \textbf{FNS} & \Rightarrow & \textbf{FS} & \Rightarrow & \textbf{FE} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \textbf{WFBS} & \Rightarrow & \textbf{WFNS} & \Rightarrow & \textbf{WFS} & \Rightarrow & \textbf{WFE} \end{array}$$

Some converses of these implications are known to be untrue. However, the relation between **FBS** and the existence of a bounded approximate identity is still open.

In fact, it seems that **FBS** is much more restricted than **FNS** as suggested in the following two results that are due to P. G. Dixon in [11].

Theorem 4.2.1. *If $\mathcal{A}$ is a commutative, separable Banach algebra with **FBS**, then $\mathcal{A}$ has an identity.*

Theorem 4.2.2. *If $\mathcal{A}$ is a commutative, separable Banach algebra with **FNS**, then $\mathcal{A}$ has a* (possibly unbounded) *approximate identity.*

Furthermore, the approximate identity asserted in Theorem 4.2.2 can be chosen so that its Gelfand transform is bounded by arbitrarily slowly growing functions on the maximal ideal space. Precisely, from [11], we have:

Theorem 4.2.3. *Let $\mathcal{A}$ be a commutative separable Banach algebra with maximal ideal space X. If $\mathcal{A}$ has **FNS**, then*

(i) *for every positive real–valued function* $\beta \in C_0(X)$, *there exists* $a \in \mathcal{A}$ *with* $|\hat{a}(t)| > \beta(t)$ $(t \in X)$;

(ii) *for every positive real–valued function* $\gamma \in C(X)$ *with* $\gamma(t) \to \infty$ *as* $t \to \infty$, $\mathcal{A}$ *has an approximate identity whose elements* u *satisfy* $|1 - \hat{u}(t)| < \gamma(t)$ $(t \in X)$;

Also, G. Willis constructed a plenty of counter–examples dealing with the relations between the existence of bounded approximate identity and various factorizations of sequences in [45].

4.3 Condition (U)

Condition (U) is a relatively new concept concerning unbounded approximate identities, which was first introduced by Y. Zhang. See [49] and [48].

Definition 4.3.1. Let $\mathcal{A}$ be a normed algebra and (e_α) be a left (right) approximate identity for $\mathcal{A}$. We say that (e_α) satisfies *condition* (U) if, for every compact set K of $\mathcal{A}$, $\|e_\alpha x - x\|$ (resp. $\|x - e_\alpha x\|$) converges to 0 uniformly for $x \in K$.

If (e_α) is a two-sided approximate identity for $\mathcal{A}$, by condition (U) we mean that, for every compact set K of $\mathcal{A}$, both $\|e_\alpha x - x\|$ and $\|x - e_\alpha x\|$ converge to 0 uniformly for $x \in K$.

It is easy to see that a bounded left, right or two–sided approximate identity always satisfies condition (U). All the Banach algebras discussed in [45] have approximate identities satisfying condition (U), but none of them has a bounded approximate identity.

It is also known that there is a Banach algebra which has a bounded left approximate identity and a right approximate identity satisfying condition (U) but does not have a bounded right approximate identity [9].

For sequential approximate identity, we have the following theorem.

Theorem 4.3.1. *Every left sequential approximate identity for a Banach algebra satisfies condition* (U).

Proof. Let $\mathcal{A}$ be a Banach algebra and $(e_n)_{n=1}^{\infty}$ be a left sequential approximate identities for $\mathcal{A}$. Then for every $x \in \mathcal{A}$, there exists $N > 0$ such that

$$\|e_n x - x\| < 1, \qquad \text{for } n > N.$$

This implies that, corresponding to each $x \in \mathcal{A}$, there is a constant M_x, such that

$$\|e_n x - x\| \leq M_x, \qquad \text{for all } n.$$

In fact, we can take $M_x = \max_{1 \leq i \leq N} \{\|e_i x - x\|, 1\}$.

Define $T_n : \mathcal{A} \to \mathcal{A}$ by

$$T_n(x) = e_n x - x \qquad (x \in \mathcal{A}).$$

Then for each n, T_n is a bounded linear operator on $\mathcal{A}$. Moreover, for each $x \in \mathcal{A}$,

$$\|T_n x\| = \|e_n x - x\| \leq M_x, \qquad \text{for all } n.$$

From the Principle of Uniform Boundedness, T_n is uniformly bounded, i.e. there exists a constant M such that

$$\|T_n\| \leq M \qquad \text{for all } n.$$

Let K be any compact subset of $\mathcal{A}$. Then there are $\{x_1, x_2, \ldots, x_m\} \subset K$ with the following property: for any $x \in K$ there is some $i \in \{1, 2, \ldots, m\}$ such that

$$\|x - x_i\| < \frac{\epsilon}{2(M+2)}$$

Since (e_n) is left approximate identity, for each $x_i \in \{x_1, x_2, \ldots, x_m\}$ there exist, correspondingly, N_i such that

$$\|e_n x_i - x_i\| < \epsilon/2 \qquad \text{for } n > N_i,\ i = 1, 2, \ldots, m.$$

Now we take $N_0 = \max\{N_1, N_2, \ldots, N_m\}$. Then for all $x \in K$ and $n > N_0$,

$$\begin{aligned}
\|e_n x - x\| &\leq \|e_n x - e_n x_i\| + \|e_n x_i - x_i\| + \|x_i - x\| \\
&= \|T_n(x) + x - T_n(x_i) - x_i\| + \|e_n x_i - x_i\| + \|x_i - x\| \\
&= \|T_n(x - x_i) + (x - x_i)\| + \|e_n x_i - x_i\| + \|x_i - x\| \\
&\leq (\|T_n\| + 2)\|x_i - x\| + \|e_n x_i - x_i\| \\
&< (M+2) \cdot \frac{\epsilon}{2(M+2)} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Therefore, the left sequential approximate identity (e_n) satisfies condition (U). The proof is complete. $\square$

Similarly, we can conclude that a sequential right or two–sided approximate identity for a Banach algebra satisfy condition (U). The converse of Theorem 4.3.1 is not true. It is easy to check that $l^1(S)$ for an infinite set S has an approximate identity satisfying condition (U). In fact, let (V) be the collection of all finite subsets of S with the inclusive partial order, then (χ_V) is such an approximate identity, where (χ_V) denotes the characteristic function of V. However, Proposition 2.3.6 tells us that $l^1(S)$ does not have a sequential approximate identity for an uncountable set S.

The completeness of the algebra is not removable in the above theorem. We use an algebra constructed in [14] to illustrate our assertion.

Example 4.3.2. *There exists a commutative separable normed algebra with a sequential approximate identity that does not satisfy condition* (U).

Let $\mathcal{A}_0$ be the commutative separable normed algebra with generators e_n, x_n $(n = 1, 2, 3, \cdots)$ satisfying the following conditions for $i, j \in N$:

$$e_i e_j = e_{\min\{i,j\}},$$

$$e_i x_j = x_j, \qquad \text{if } i \geq j,$$

$$x_i x_j = 0.$$

Then any element $x \in \mathcal{A}_0$ is of the form

$$x = \sum_i \alpha_i e_i + \sum_j \beta_j x_j + \sum_{i<j} \gamma_{ij} e_i x_j, \tag{4.1}$$

the sum being finite.

Define the norm of $x \in \mathcal{A}_0$ by

$$\|x\| = \sum_i |\alpha_i| 2^i + \sum_j |\beta_j| + \sum_{i<j} |\gamma_{ij}| 2^i.$$

Thus $\mathcal{A}_0$ is a commutative normed algebra that admits a sequential unbounded approximate identity (e_n). In fact, if x is as in (4.1) and n is greater than all the i, j occurring in the (finite) sums in (4.1), then $e_n x = x$.

Now consider the compact subset of $\mathcal{A}_0$

$$\left\{ \frac{x_j}{2^{j-2}} \right\}_{j=1}^{\infty}$$

For any $j > n$,

$$\begin{aligned} \left\| e_n \frac{x_j}{2^{j-2}} - \frac{x_j}{2^{j-2}} \right\| &\geq \left\| e_n \frac{x_j}{2^{j-2}} \right\| - \left\| \frac{x_j}{2^{j-2}} \right\| \\ &= \frac{2^n - 1}{2^{j-2}} \end{aligned}$$

Let $j = n + 1$,

$$\left\| e_n \frac{x_{n+1}}{2^{n-1}} - \frac{x_{n+1}}{2^{n-1}} \right\| \geq \frac{2^n - 1}{2^{n-1}} > 3$$

This shows that this sequential approximate identity (e_n) for normed algebra $\mathcal{A}_0$ does not satisfy condition (U).

Y. Zhang studied in [49] the existence of an approximate identity satisfying condition (U) for the closed ideals of group algebras on a compact group. He proved the following:

Proposition 4.3.3. *Let G be a compact group. Then every closed ideal $\mathcal{I}$ of $L^1(G)$ has an approximate identity that lies in the center of $L^1(G)$ and satisfies condition* (U).

Using the similar technique, we obtain the following proposition for the closed ideal of $L^2(G)$.

Proposition 4.3.4. *Let G be a locally compact commutative group, and let Σ be its dual group. For $E \subset \Sigma$, let*

$$\mathcal{I}(E) = \{f \in L^2(G) : \hat{f}(\sigma) = 0 \; for \, \sigma \in E\},$$

where $\hat{f}$ is the Fourier transform of f. Then $\mathcal{I}(E)$ is a closed ideal of $L^2(G)$ and it has an approximate identity that satisfies condition (U).

Proof. The first assertion is clear. Denote $\mathcal{I} = \mathcal{I}(E)$. Let (U_α) be a net of compact neighborhoods of e the unit of G and denote $u_\alpha = \dfrac{\chi_{U_\alpha}}{|U_\alpha|}$, where χ_{U_α} is the characteristic function of U_α and $|U_\alpha|$ is the Haar measure of U_α. From the commutativity of G, (u_α) is an approximate identity for $L^2(G)$ with L^1-norm bounded.

Now let χ_E be the characteristic function of $E \subset \Sigma$, we have that $\chi_{\Sigma\setminus E} \cdot L^p(\Sigma) \subset L^p(\Sigma)$ for $p \geq 1$. In particular, for $p = 2$, $\chi_{\Sigma\setminus E} \cdot \widehat{u_\alpha} \in L^2(\Sigma)$ for each α, since $\widehat{u_\alpha} \in L^2(\Sigma)$ from the Plancherel Theorem. Using the Plancherel Theorem again, we have that there is a $p_\alpha \in L^2(G)$ such that $\widehat{p_\alpha} = \chi_{\Sigma\setminus E} \cdot \widehat{u_\alpha}$. For each σ in E, $\widehat{p_\alpha}(\sigma) = \chi_{\Sigma\setminus E}(\sigma) \cdot \widehat{u_\alpha}(\sigma) = 0$, so $p_\alpha \in \mathcal{I} = \mathcal{I}(E)$. If $f \in \mathcal{I}$,

$$(p_\alpha * f)\hat{} = (f * p_\alpha)\hat{} = \hat{f} \cdot \widehat{p_\alpha} = \hat{f} \cdot \chi_{\Sigma\setminus E} \cdot \widehat{u_\alpha} = \hat{f} \cdot \widehat{u_\alpha} = (f * u_\alpha)\hat{} = (u_\alpha * f)\hat{}$$

Thus, $p_\alpha * f = u_\alpha * f$ for $f \in \mathcal{I}$.

Given a compact set K of $\mathcal{I}$, for every $\epsilon > 0$, there exist $f_1, f_2, \cdots, f_n \in K$ satisfying that for every $f \in K$, $\|f - f_i\|_2 < \epsilon/3$ for some $i \in \{1, 2, \cdots, n\}$.

Since (u_α) is an approximate identity for $L^2(G)$ with L^1-norm bounded,

$$\|u_\alpha * f_i - f_i\|_2 < \epsilon/3$$

for $i = 1, 2, \cdots, n$ and $\alpha \succ \alpha_0$.

Let $f \in K$ with $\|f - f_i\|_2 < \epsilon/3$, we have from Proposition 2.39 a, [16] that for above $\epsilon > 0$

$$\begin{aligned}\|u_\alpha * f - u_\alpha * f_i\|_2 &= \|u_\alpha * (f - f_i)\|_2 \\ &\leq \|u_\alpha\|_1 \|f - f_i\|_2 = \|f - f_i\|_2 \\ &< \epsilon/3.\end{aligned}$$

Hence for $\alpha \succ \alpha_0$,

$$\begin{aligned}\|u_\alpha * f - f\|_2 &\leq \|u_\alpha * f - u_\alpha * f_i\|_2 + \|u_\alpha * f_i - f_i\|_2 + \|f_i - f\|_2 \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \qquad \text{for all } f \in K\end{aligned}$$

i.e.,

$$\|p_\alpha * f - f\|_2 < \epsilon \qquad \text{for all } f \in K.$$

This shows that, for every compact set K of $\mathcal{I}$, $p_\alpha * f \xrightarrow{\alpha} f$ uniformly on K. Therefore (p_α) is an approximate identity for $\mathcal{I}$ that satisfies condition (U). $\square$

4.4 Approximate identities in algebras of compact operators

In this section we assume that $\mathcal{A}$ is an operator algebra on a Banach space X, containing finite rank operators and being contained in the algebra of compact operators. We collect some results about unbounded approximate identities in $\mathcal{A}$.

Let X be a Banach space and K be a subset of X. Let φ be a set of operators on X. We say that *the identity is approximable on K by operators in φ* if, for every $\epsilon > 0$, there is an $S \in \varphi$ with $\|Sx - x\| < \epsilon$ $(x \in K)$. We say that X has the *approximation property* (**AP** in short) if, for every compact set $K \subseteq X$, the identity is approximable on K by finite–rank operators; X has the *compact approximation property* (**CAP** in short) if, for every compact set $K \subseteq X$, the identity is approximable on K by compact operators.

Suppose that $\mathfrak{B}(X)$ is the operator algebra of continuous operators on X. Denote by $F(X)$, $\mathcal{F}(X)$ and $\mathcal{K}(X)$ the ideals of $\mathfrak{B}(X)$ of respectively finite–rank operators, operators that are uniformly approximable by finite operators and compact operators on X. One can check that $F(X) \subset \mathcal{F}(X) = \mathrm{cl}(F(X)) \subset \mathcal{K}(X)$.

The question of whether the Banach algebra of all compact operators on a Banach space always has a bounded approximate identity was first stated as an open problem in [8] and P. G. Dixon examined both bounded and unbounded cases in [12], showing that

Theorem 4.4.1. *Let X be a Banach space. Then*

(i) $F(X)$ has a bounded left approximate identity if and only if X has bounded **AP**;

(ii) $\mathcal{K}(X)$ has a bounded left approximate identity if and only if X has bounded **CAP**.

Theorem 4.4.2. *Let X be a Banach space. Then*

(i) *$F(X)$ has a left approximate identity;*

(ii) *if X has* **AP** *then $\mathcal{F}(X)$ has a left approximate identity;*

(iii) *if X has* **CAP** *then $\mathcal{K}(X)$ has a left approximate identity.*

In [19] [39], N. Grønbæk, G. A. Willis and C. Samuel extended the approach of [12] to consider further right approximate identities as follows.

Theorem 4.4.3. *Let X be a Banach space. Then*

(i) *$\mathcal{F}(X)$ has a bounded right approximate identity if and only if X^*, the dual space of X, has bounded* **AP***;*

(ii) *$\mathcal{K}(X)$ has a bounded right approximate identity if and only if the identity operator on X^* is uniformly approximable on compact sets of X^* by a bounded net of adjoint operators of compact operators on X.*

In light of condition (U), Y. Zhang finally clarified the relation between **AP**/**CAP** of X and the existence of approximate identities for $F(X)/\mathcal{K}(X)$. He showed the following three theorems in his paper [51].

Theorem 4.4.4. *Let X be a Banach space. The following are equivalent:*

(i) *X has* **AP***;*

(ii) *$F(X)$ has a left approximate identity that satisfies condition* (U);

(iii) *$\mathcal{F}(X)$ has a left approximate identity that satisfies condition* (U).

Theorem 4.4.5. *For a Banach space X, $\mathcal{K}(X)$ has a left approximate identity that satisfies condition* (U) *if and only if X has* **CAP**.

Theorem 4.4.6. *Let X be a Banach space. The following are equivalent:*

(i) *the dual space X^* has* **AP**;

(ii) *$F(X)$ has a right approximate identity that satisfies condition* (U);

(iii) *$\mathcal{F}(X)$ has a right approximate identity that satisfies condition* (U);

(iv) *each of $F(X)$ and $\mathcal{F}(X)$ has a two-sided approximate identity that satisfies condition* (U).

Chapter 5

Further Questions

From Theorem 4.2.1,Theorem 4.2.2 and (**) in Section 4.2, **FBS** seems a very strong condition on a Banach algebra.

Question 1. Is there any relation between **FBS** and the existence of a bounded approximate identity for a Banach algebra? In particular, does the existence of a bounded approximate identity imply **FBS** for a Banach algebra?

Question 2. What can one say about the reverse of any implication in diagram (**) except for those that have known?

For condition (U),

Question 3. Under what conditions does a closed ideal of $L^2(G)$ for a general locally compact abelian group G have an approximate identity satisfying condition (U)? What if for a closed ideal of group algebra $L^1(G)$ for a general locally compact abelian group G?

Question 4. Does there exist a Banach algebra that has an approximate identity but does not have an approximate identity satisfying condition (U)?

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