

**LOT SIZING AND DISTRIBUTION REQUIREMENTS
PLANNING WITH FUZZY INFORMATION (DATA)**

BY

KULVIR BHATHAL

A Thesis

**Submitted to the Faculty of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree of**

MASTER OF SCIENCE

**Department of Mechanical and Industrial Engineering
University of Manitoba
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**DEDICATED TO MY PARENTS,
WIFE BOBBY
AND DAUGHTER GURLEENE**

ABSTRACT

The subject of the present research is to investigate, model and solve the problems of Lot Sizing and Distribution Inventory Management when the available data is given in the form of fuzzy numbers. Subject to a variety of assumptions, main contribution of this thesis is included in Chapters 3 to 5.

Chapters 1 and 2 provide introduction and literature survey, respectively, related to the problems considered in the thesis. Chapter 3 deals with the distribution inventory management problem with variable demand rate and both without and with backorders, under fuzzy information. Similarly, Chapters 4 and 5 consider the lot sizing inventory problem under variable demand rate and both without and with backorders, respectively, under fuzzy information with a finite planning horizon. Finally, the conclusion and the discussion on the contributions made in the thesis, along with some recommendations for further research, are given in Chapter 6.

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Chapter 1

INTRODUCTION

The present thesis deals with the analysis and modeling of two important problems of lot sizing and distribution inventory management in a fuzzy environment.

Lot Sizing is a significant aspect of Materials Requirement Planning (MRP) production planning process. Although its perceived importance has declined as a result of system development, wide spread Just in Time (JIT) orientations and development of satisfactory heuristics, lot sizing is still a major component of a balanced MRP operation.

After a product is manufactured, the emphasis of an organization shifts towards distributing the good at a reasonable cost to establish or maintain a competitive advantage. Distribution Requirements Planning (DRP) helps achieving this objective by effectively managing the required flow of goods and inventories between a firm and a market. It provides planners with the visibility of future requirement and substantially reduces unexpected demands. DRP also provides the basis for managing logistic system.

The present study is motivated by the realization that data for these problems normally exists in an ambiguous (vague or imprecise) form. Therefore, it is natural to deal with such problems through fuzzy systems. Under such circumstances, using the fuzzy approach yields a relatively "more satisfactory and flexible solution". The fuzzy numbers, for example can be obtained from experts who, instead of one but possibly forecasted estimate, provide three or four imprecise (vague) estimates of some important parameters involved. We suggest that, under such circumstances, the related problem may be handled using special type of fuzzy numbers (for example, triangle fuzzy

numbers or trapezoidal fuzzy numbers). The approach used in the present thesis is a small step in this direction.

Under fuzzy information, we consider inventory problem with variable demand rate and a finite-planning horizon, and solve it using Wagner-Whittin algorithm. Also, an attempt is made to model the problem by incorporating back orders. Furthermore, the distribution inventory management problem with fuzzy information is presented using DRP approach.

We now give a brief introduction to these problems.

1.1 Lot Sizing

1.1.1 Basic Inventory Concepts

1 Inventory

Inventory includes all those goods and materials that are used in the production and distribution process. Raw materials, component parts, subassemblies, and finished parts are all part of inventory, as are the various supplies required in the production and distribution process. Good inventory management is important to all firms, whether manufacturing or service. Four reasons for its importance are:

- Inventories can be a major commitment of monetary resources.
- Inventories affect virtually every aspect of daily operations.
- Inventories can be a major competitive weapon.
- Inventories are the major control problem in many companies.

2 Independent Demand Items

These are shipped as end items to customers and may be finished goods or spare/repair parts. Demand is market-based, and is independent of the demand for other items.

3 Dependent Demand Items

These are used in the production of a finished product. Such items may be raw materials, component parts or subassemblies. Demand is based on the number needed in each higher-level (parent) item where the part is used. Dependent demand items are frequently managed by certain inventory replenishment systems such as MRP or JIT systems.

1.1.2 Purpose Of Inventories

Some of the major reasons for holding inventories are:

- 1 Decouple demand from immediate dependence on the source of supply.
- 2 Service customer with variable demands.
- 3 Serve as a shock absorber between successive stages in operations.
- 4 Help level production activities, stabilize employment, and improve labor relations.
- 5 Provide a means of obtaining and handling materials in economic lot sizes and of gaining discounts.
- 6 Facilitate the production of different products on the same facilities.

1.1.3 Methods for Measurement of Inventory

There are three accounting categories, or types, of inventories:

- Raw materials
- Work in process
- Finished goods

There are at least three methods for measuring inventory

1. **Aggregate inventory value** (average or maximum) gives answer to the question of HOW MUCH is the stock of inventory?
2. **Weeks of supply** (or other time unit) provide answer to the question of HOW LONG will inventory last?
3. **Inventory turnover or turns** (ratio of sales to inventory) gives answer to the question of HOW MANY times inventory is sold?

1.1.4 The Objectives of Inventory Management

The primary function of inventory management is to have items available to maintain the flow of goods through the production process to the customer while minimizing the investment required achieving this purpose.

An organization's Inventory Management System must carry out objectives set by upper management and must perform in such a way that the organization's profit or performance is enhanced. The objectives set by management will frequently fall into either of two categories:

- customer service objectives, and
- inventory investment objectives.

The first category includes such concepts as service level and stock-out rate, and the second category includes such items as number of inventory turnovers per time period. Generally, the achievement of higher levels of customer service, however defined, is accomplished with larger amounts of inventory, and is subject to diminishing returns. The achievement of higher levels of the investment objectives is generally met with smaller inventories. Thus, we see the basic conflict of inventory management: some objectives call for economizing on inventory levels, while other objectives call for increasing inventories. These objectives may create conflict along departmental lines: finance wants

smaller sums tied up in inventory, while marketing wants larger amounts so that customer orders can be more promptly satisfied.

1.1.5 Functional Classifications of Inventories

1 Anticipation Inventories

Anticipation inventories are additional inventories that are either produced or purchased in anticipation of infrequent events, such as vacations shutdowns, strikes, peak sales periods, and sales promotions. Hedge inventory is a special type of anticipation inventory, which is built or produced to take advantage of present costs or to avoid anticipated substantial price increases. Ideally, anticipation inventory should consist of items that have high labor content and a low material content.

2 Lot Sizing Inventory

Lot sizing is the purchasing or producing items in large enough lots to take advantage of cost efficiencies, quantity discounts, learning curves, scale economies, etc.

3 Fluctuation Inventories

These are carried to absorb the variations in demand and lead-time because it is not realistic in most cases to expect the demand for products to be perfectly predictable. Fluctuation inventories are also known as safety stock. They enable an organization to service its customers when the demand for that service is above average or when delivery of replenishment stocks takes longer than usual.

4 Transportation Inventories

Items in movement from one stage to the next are called transportation inventories. They are also referred to as pipeline inventory. This is a type of inventory often neglected.

Such inventories are important since they do reflect moneys tied up for periods of time, and hence do incur inventory-holding costs.

5 Service Parts

These are the items carried in inventory as replacement parts for operating equipment or other needs.

1.1.6 Classification of Inventory Models

There are several ways of classifying the inventory models. Some of the attributes useful in distinguishing between various inventory models are given in this section (Gill, 1992).

1 Number of Items

- Single Item – This type of model recognizes one type of product at a time. If the demand rate changes from period to period, and then the problem becomes that of a dynamic lot-sizing problem.
- Multi Item – This type of model considers a number of products simultaneously. These products must have at least one interrelating or binding factor such as budget or capacity constraint or a common setup.

2 Stocking Points

- Single Echelon Models – Only one stocking location is considered.
- Multi Echelon Models – More than one interconnected stocking locations are considered.

3 Frequency of Review

This is the frequency of assessment of the current stock position of the system and the implementation of the ordering decision.

- Periodic – Placement of orders is done at discrete points in time, with a given periodicity.
- Continuous – Order placement can occur at any time.

4 Order Quantity

- Fixed – Order quantity is fixed to the same amount each time.
- Variable – Order quantity can be variable.

5 Planning Horizon

- Finite – Demands are recognized over a limited number of periods.
- Infinite – Demands are recognized over an unlimited number of periods.

6 Demand

- Deterministic – Demands are known with certainty over the planning horizon.
 - a) Static – Demand rate is constant over every period.
 - b) Dynamic – Demand rate is not necessarily constant.
- Stochastic (Probabilistic) – Demand is unknown, and must be estimated. The demand probability distribution may be known or unknown.

The emphasis in the present thesis is to deal with an inventory lot size problem with ambiguous or imprecisely known demand.

7 Lead Time

- Zero – No time elapses between placement and receipt of orders.
- Non-Zero – Significant time elapses between the placement and receipt of orders. This time may be constant or random.

8 Capacity

- Capacitated – There are capacity restrictions on the amount produced or ordered.

- Un-capacitated – Capacity is assumed to be unlimited.

9 Unsatisfied Demand

- Not allowed – In this case, all demand is met and no shortages are allowed.
- Allowed – Demand not satisfied in a particular period may be retained and satisfied in a future period (backlogging), partially retained and partially lost or completely lost (no backlogging).

1.1.7 Lot Size Inventory Problem

Most of the inventory situations need answers to the following two basic questions:

- when to order (the reorder point) and,
- how much to order (the lot size).

Answers to above questions can be found out by using the Classical Economic Order Quantity Model (EOQ Model), when the demand rate is constant over time. But when the demand rate varies over time, i.e. not necessarily constant from one period to another, the associated problem of planning is a bit more challenging and is said to be dynamic in nature. The problem considered for this study is uncapacitated single item lot sizing problem with dynamic demand.

1.2 Distribution Inventory Management

1.2.1 Distribution Inventory Management Systems

The objective of distribution inventory management is to have inventory in the right place at the right time at reasonable cost. In brief, the objective is to achieve a desired

level of customer service at or below a specified cost. Distribution Inventory Management systems can be divided into two categories.

- Pull System
- Push System

Pull System

In a pull system, inventory is pulled into the warehouse. The warehouse determines its requirements and orders from the factory. The standard pull system orders without regard for the needs of other warehouses, the inventory available at the central warehouse, or the production schedule. Five types of traditional pull systems are (Fogarty 1991)

1 The Order Point System

In the order point system, the branch warehouse establishes a order point based on the normal demand during the average time required to obtain the order from the central warehouse plus the safety stock. The order is placed with the central warehouse when the quantity in stock at branch reaches its order point.

2 The Periodic Review System

With this system, branch warehouse inventory status is determined at a regular interval, and the warehouse orders the quantity required to bring inventory to the target level. All other things being equal, branch warehouse safety stock must be greater in this system than in order point system because it cover variations in demand during the cycle as well as during lead time.

3 The Double Order Point System

As the name suggests, the second order point is established based on the normal order point plus the normal demand during manufacturing lead-time. This enables the

central warehouse to examine its inventory position relative to anticipated warehouse orders and take appropriate action. Theoretically, the central warehouse need not carry safety stock since it is forewarned of pending orders and adds inventory required to meet those orders.

4 The Sales Replacement System

In this system each warehouse periodically establishes a stocking level based on local demand. Sales at each warehouse are reported to the central warehouse at periods shorter than the normal order interval. Shipments replacing the quantities sold are sent to each warehouse at the end of replenishment periods.

5 The Base Stock System

This is also a sales replacement system. But, in this system, the base stock level is established at each stocking location based on the normal demand during replenishment lead-time and the interval between sales report and the safety stock. Sales are reported on a weekly or, preferably, a daily basis to all inventory-holding facilities rather than only when ordering. The primary advantage of this system is that it enables manufacturing, the central warehouse, and regional warehouses to plan and react on the basis of actual customer demand rather than on the basis of the replenishment orders filled at secondary stock points, such as regional warehouse.

Push System

In a push system, inventory is pushed into the warehouses. This system considers total projected requirements (all warehouse and direct sales replenishments), inventory available at the regional warehouses and the central warehouse, inventory in transit, and schedule receipt from the supplier and determines the quantity available for each

warehouse and direct factory sale. This allocation is controlled centrally on the basis of such criteria as equal day coverage, shipping schedule and competitive factors.

1.2.2 Distribution Requirements Planning (DRP)

Distribution requirements planning (DRP) is a push system. In DRP, the central warehouse decides what to send to the regional warehouse. It provides the basis for tying the physical distribution system to the manufacturing planning and control (MPC) system. DRP relates current inventory positions and forecast of field demand to master production schedule (MPS) and material planning modules. When the items on the MPS are not the final product and require finishing, packaging, or fabrication into a final assembly, these final operations can be viewed as the first stage in distribution. James Heskett (1977) points out that automobile plants are distribution centers equipped to receive orders, fabricate individually designed final assembly configurations from standard components, and deliver them in a reasonable time.

DRP is best conceived as one part of demand management. It is a link between the market place, demand management and master production scheduling. The link is effected through time-phased information on inventories and through material and shipping planes that coordinates activities in these modules. Finished good inventories are often positioned in a complicated physical system, consisting of field warehouses, intermediate distribution centers, and a central supply. In such a system, a key task is effectively managing the required flow of material and inventories between the market and the firm. In performing this task, DRP has a central coordinating role similar to material requirements planning's role in coordinating materials in manufacturing. DRP's

role is to provide the necessary data for matching customer demand with the supply of products at various stages in the physical distribution system and product being manufactured by manufacturing. Key elements of these data are the planned timings and quantities for replenishing inventories throughout the physical distribution system. These data take into account currently available field inventories and forecasts.

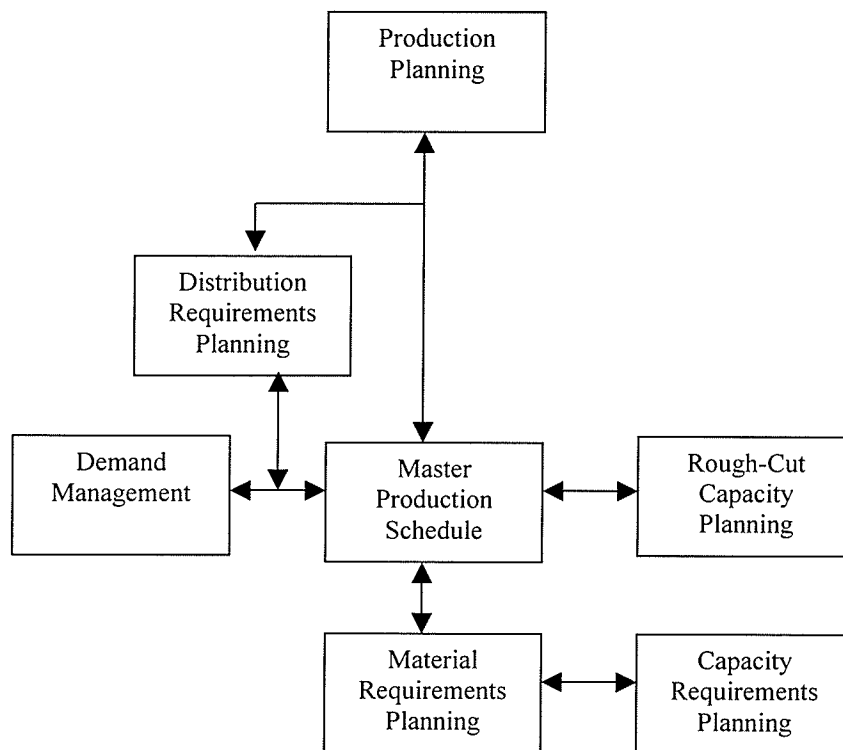


Figure 1.1. Information Flows in Distribution and Production Planning

1.2.3 DRP and Demand Management

According to Berry (1992), the demand management module is gateway between the manufacturing facility and the market place. In some systems with field inventories, it is where information on demand is taken in and where product for the field warehouse is sent out. This process requires detailed matching of supply to demand in every location – and requires providing supply to meet all sources of demand. DRP is a method for

managing the resultant large volume of dynamic information and for generating the set of plans for manufacturing and replenishment. As actual field demands vary around the forecasts, adjustments to plans are required. DRP continually make these adjustments, sending the inventories from central warehouse to those distribution centers where they're most needed. In circumstances where insufficient total inventories exist, DRP provides the basis for deciding on allocations.

1.2.4 DRP and Master Production Schedule

Master Production Schedule (MPS) is a list of all the items that the firm anticipates to manufacture each period. DRP provides the MPS better information, in a format consistent with the MRP records, to matching manufacturing output with the shipping needs. Requirements based on shipments to the distribution centers can be quite different from the demand in the field. DRP data also provides the basis for adjusting MPS to reflect changes in the demand or product mix. If manufacturing and shipping priorities can't be adjusted to respond to these requirements, the implications can be evaluated and communicated to customer in a timely fashion. Common records and system integration means there is complete visibility to see how best to use available inventories and to adjust future schedules. DRP provides a solid base of information to make these decisions, instead of relying on political negotiations between field and factory.

1.2.5 DRP and the Logistics System

Plans derived from the DRP information and the resultant shipping requirements are the basis for managing the logistics system. Shipping requirements are used to determine vehicle loads, vehicle dispatching, vehicle capacity planning and warehouse receipt

planning. Vehicle capacity planning is the process of planning the vehicle availability for the set of future shipments as generated by DRP. By planning future replenishment needs, DRP establishes the basis for more effective vehicle decisions. These decisions are continually adjusted to reflect current conditions. Long-term plans help to determine the necessary transportation capacity. Warehouses' near-term needs are used to efficiently load a vehicle without compromising customer service levels. Data on the planned resupply of the warehouses can be used for scheduling the labor force in the warehouses.

1.2.5 Benefits of Distribution Requirements Planning

Distribution requirements planning

- enables us to capture data, including local demand conditions, for modifying the forecast and to report current inventory positions,
- avoids stock outs at the central warehouse by projecting warehouse requirements by period and generating planned orders on the central warehouse,
- improves customer service levels through deliveries consistent with promises,
- reduces inventories investment by effectively allocating inventory,
- quickly determines shortage and avoids unnecessarily premature commitment to customers,
- provides the database at warehouse level for consistent communication with customers and the rest of the company, and
- encourages significant logistic savings through better planning of aggregate transportation capacity needs and dispatching of shipments.

1.3 Fuzzy Set Theory

In this section we introduce some of the basic concepts and terminology of fuzzy set theory. Theory of fuzzy sets is basically a theory of graded concepts (Zimmerman, 1991). A central concept of fuzzy set theory is that it is permissible for an element to belong partly to a fuzzy set.

1.3.1 Fuzzy Set

Let X be a classical set of objects, called the universe, whose generic elements are denoted by x . The membership in a crisp subset of X is viewed as a characteristic function μ_A from X to $[0, 1]$ such that:

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

where $[0, 1]$ is called a valuation set (Lai and Hwang, (1992)).

If the valuation set is allowed to be the real interval $[0, 1]$, A is called a fuzzy set proposed by Zadeh (1996). $\mu_A(x)$ is the degree of membership of x in A . The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A . Therefore, A is completely characterized by the set of ordered pairs:

$$A = \{(x, \mu_A(x)) / x \in X\}$$

where $\mu_A(x)$ maps X to the membership space $[0, 1]$. Elements with zero degree of membership are usually not listed. If $\sup \mu(x) = 1, \forall x \in R$, then the fuzzy set A is called a normal fuzzy set in R . A fuzzy set that is not normal is called subnormal fuzzy set.

1.3.2 α – Level Set or α – Cut

One of the most important concepts of fuzzy sets is the concept of an α -cut or α -level set. An α -cut denoted by A_α is the crisp set of elements x in R whose degree of belonging to the fuzzy set A is at least $\alpha \in [0, 1]$. This means

$$A_\alpha = \{x \in R \mid \mu_A(x) \geq \alpha, \alpha \in [0, 1]\}$$

that is, the α -cut or α -level set of a fuzzy set is the crisp set A_α that contains all elements of the universal set $X \in R$ whose membership grades in A are greater than or equal to the specified value of α , $\alpha \in [0, 1]$.

1.3.3 Support of a Fuzzy Set

Support of a fuzzy set A is a set $S(A)$ such that $X \in S(A)$ if and only if $\mu_A(x) > 0$.

1.3.4 Intersection of Fuzzy Sets

Intersection of two fuzzy sets A and B is a fuzzy set C denoted by $C = A \cap B$, whose membership function is related to those of A and B by

$$\mu_C(x) = \min [\mu_A(x), \mu_B(x)], \quad \forall x \in X$$

1.3.5 Algebraic Operations on Fuzzy Sets

In addition to the set theoretic operations, we can also define a number of other ways of forming combinations of fuzzy sets and relating them to one another. Here we present some more important operations among those:

1. Algebraic product of two fuzzy sets A and B , is $A (\cdot) B$, whose membership function is

$$\mu_{A(\cdot)B}(x) = \mu_A(x) (\cdot) \mu_B(x), \quad \forall x \in X$$

2. The algebraic sum of A and B is $A + B$ whose membership function is defined as

$$\mu_{(A+B)}(x) = \mu_A(x) (+) \mu_B(x), \quad \forall x \in X$$

$$\text{provided } \mu_A(x) (+) \mu_B(x) \leq 1, \quad \forall x \in X$$

1.3.6 Convexity of Fuzzy Set

The notion of convexity can be extended to fuzzy sets in such a way as to preserve many of the properties that it has in case of crisp sets. In what follows, we assume that the set X is the n -dimensional space R^n . We now have the following two equivalent definitions of convexity of a fuzzy set.

A fuzzy set A is convex if and only if every set $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ for all

$\alpha \in [0, 1]$ is a convex set.

The second definition of convexity of a fuzzy set is as follows:

A fuzzy set A is said to be a convex set if

$$\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu(x_1), \mu(x_2)), \quad x_1, x_2 \in X, \lambda \in [0, 1].$$

1.3.7 Fuzzy Arithmetic

The first definition of a fuzzy set allows us to extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts.

An ordinary number 'a' can be characterized by using the membership function

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

Fuzzy Number

A fuzzy number A is a fuzzy set on the real line R , which possesses the following properties:

- A is a normal, convex fuzzy set on R
- The α -level set A_α must be a closed interval for every $\alpha \in [0, 1]$
- The support of A, $S(A) = \{x \mid \mu_A(x) > 0\}$, must be bounded.

Fuzzy arithmetic is based on the following two properties of fuzzy numbers:

1. Each fuzzy set and thus, each fuzzy number can be fully and uniquely represented by its α -level sets.
2. α -level sets of each fuzzy number are closed intervals of real numbers for all $\alpha \in [0, 1]$.

These properties enable us to define an arithmetic operation on fuzzy numbers in terms of arithmetic operations on their α -level sets (i.e. arithmetic operations on closed intervals).

1.3.8 Fuzzy Arithmetic Based On Operations On Closed Intervals

A fuzzy number can be characterized by an interval of confidence at level α ,

$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ which has the property $\alpha \leq \alpha' \Rightarrow A_{\alpha'} \subset A_\alpha$.

According to Kaufmann and Gupta (1985, 1988), let $A = [a, b] \in R$ and $B = [c, d] \in R$ be two fuzzy numbers, then the arithmetic operations on them are as follows:

| | |
|--------------------------------------|--|
| Addition | $A + B = [a + c, b + d]$. |
| Subtraction | $A - B = [a - d, b - c]$. |
| Multiplication | $AB = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$. |
| Division | $A/B = [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]$. |
| Minimum (\wedge) | $A \wedge B = [a \wedge c, b \wedge d]$. |
| Maximum (\vee) | $A \vee B = [a \vee c, b \vee d]$. |

Let A and B be two fuzzy numbers, $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ be the α -level set of A , and

$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α -level set of B .

Let $*$ denote any of the arithmetic operations $+$, $-$, \cdot , $/$, \wedge and \vee on fuzzy numbers.

Then, we define a fuzzy set $A * B$ in R , by defining its α -level sets $(A * B)_\alpha$ as

$$(A * B)_\alpha = A_\alpha * B_\alpha \text{ for any } \alpha \in [0, 1]$$

Since $(A * B)_\alpha$ is a closed interval for each $\alpha \in [0, 1]$ and A and B are fuzzy numbers,

$A * B$ is also a fuzzy number.

The multiplication of fuzzy number $A \subset R$ by an ordinary number $k \in R^+$ can also be defined as

$$k(\cdot) A_\alpha = [ka_1^{(\alpha)}, ka_2^{(\alpha)}]$$

or equivalently, $\mu_{k.A}(x) = \mu_A(x/k) \quad \forall x \in R$.

1.3.9 Triangular Fuzzy Number

A triangular fuzzy number (T.F.N.), A , is denoted by the triplet (a_1, a_2, a_3) and its membership function is written as

$$\mu_A(x) = \begin{cases} 0 & x \leq a_1 \\ \frac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ \frac{a_3 - x}{a_3 - a_2} & a_2 \leq x \leq a_3 \\ 0 & x \geq a_3 \end{cases}$$

The α -level set of a triangular fuzzy number is

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3] \quad \forall \alpha \in [0, 1]$$

Below, in Figure 1.2, we give a graphic representation of a fuzzy number.

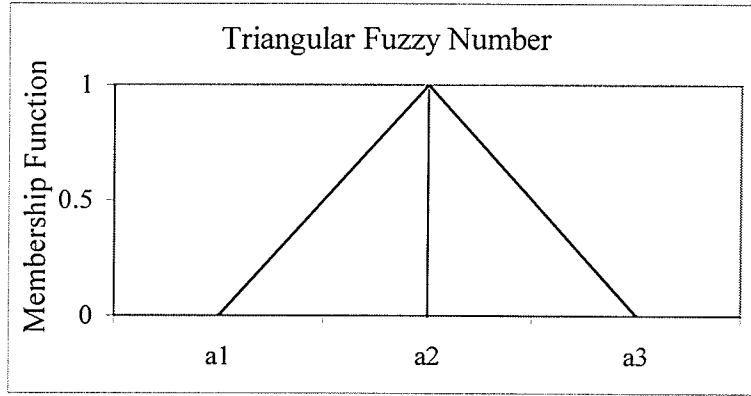


Figure 1.2 Graphic representation of a Triangular Fuzzy Number

Algebraic Operations on T.F.N.

Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two T.F.Ns then,

- Addition $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
- Subtraction $A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

For the following two operations, we assume that a_i and b_i , $i = 1, 2, 3$ are positive.

- Multiplication $A (.) B = (a_1 b_1, a_2 b_2, a_3 b_3)$
- Division $A (:) B = \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1} \right)$

1.3.10 Special Triangular Fuzzy Number

A special triangular fuzzy number (S.T.F.N.), A , is a triangular fuzzy number represented by the triplet $[a_1, a_2, (a_2 + \delta)]$, where $\delta \rightarrow 0$, and its membership function is written as

$$\mu_A(x) = \begin{cases} 0 & x \leq a_1 \\ \frac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ \frac{(a_2 + \delta) - x}{\delta} & a_2 \leq x \leq (a_2 + \delta) \\ 0 & x \geq (a_2 + \delta) \end{cases}$$

The α -level set of a special fuzzy number is

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [(a_2 - a_1)\alpha + a_1, -(\delta)\alpha + (a_2 + \delta)] \quad \forall \alpha \in [0, 1]$$

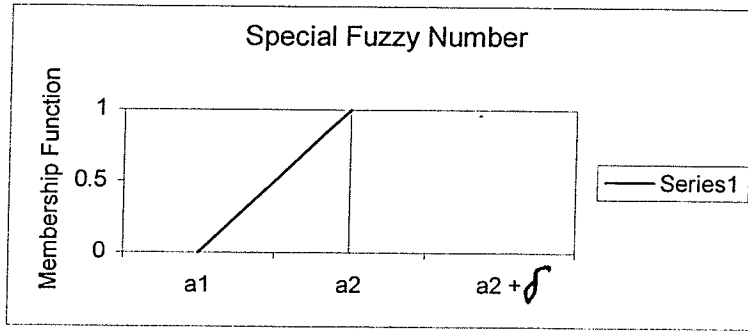


Figure 1.3 Graphic representation of a Special Triangular Fuzzy

Number

Algebraic Operations on S.T.F.N.

Let $A = [a_1, a_2, (a_2 + \delta)]$ and $B = [b_1, b_2, (b_2 + \delta)]$ be two S.T.F.N.'s, where $\delta \rightarrow 0$, then,

- Addition $A + B = [a_1 + b_1, a_2 + b_2, (a_2 + b_2 + 2\delta)]$
- Subtraction $A - B = [(a_1 - (b_2 + \delta), a_2 - b_2, (a_2 + \delta) - b_1]$

1.3.11 Trapezoidal Fuzzy Number

A trapezoidal fuzzy number (Tr.F.N.), A , is denoted by a quadruplet (a_1, a_2, a_3, a_4) whose membership function is written as

$$\mu_A(x) = \begin{cases} 0 & x \leq a_1 \\ \frac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \frac{a_4 - x}{a_4 - a_3} & a_3 \leq x \leq a_4 \\ 0 & x \geq a_4 \end{cases}$$

The α -level set of a trapezoidal fuzzy number A is

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4] \quad \forall \alpha \in [0, 1]$$

Graphically, a trapezoidal fuzzy number is shown as in Figure 1.4 below.

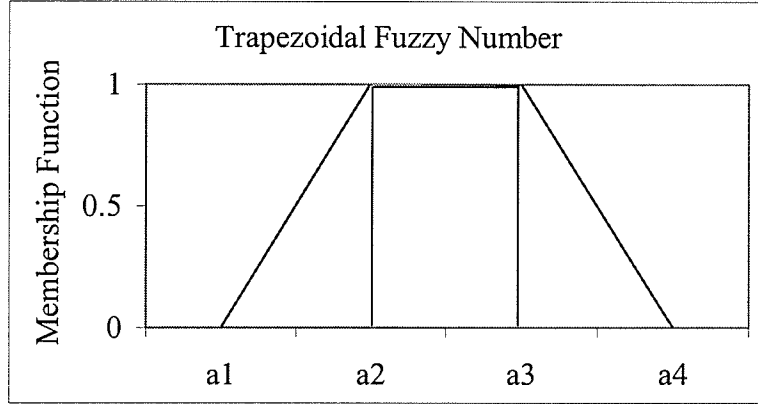


Figure 1.4 Graphic representation of a Trapezoidal Fuzzy Number

Algebraic Operations on Tr.F.N.

Let $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ be two Tr.F.Ns then,

- Addition $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$
- Subtraction $A - B = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$

For the following two operations, we assume that a_i and b_i , $i = 1, 2, 3$ are positive.

- Multiplication $A (.) B = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4)$
- Division $A (:) B = \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1} \right)$

1.3.12 Ranking of Triangular Fuzzy Numbers

One can rank the triangular fuzzy numbers using a variety of criteria by using various weights on the different features of the fuzzy numbers. We shall consider three different criteria of ordering triangular fuzzy numbers. If the first criterion does not give a unique linear order, then the second and the third criteria should be used.

Criterion 1. Linear Ordering using Associated Ordinary Number (A.O.N.)

Ordinary number associated with a T.F.N. $A = (a_1, a_2, a_3)$, is given by

$$\text{A.O.N. } a = \left(\frac{a_1 + 2a_2 + a_3}{4} \right)$$

If the values of A.O.N. for some T.F.Numbers happen to be equal, then form the classes in the ascending order of the removal and go to the Criterion 2.

Criterion 2. Linear Ordering using the Mode.

In each class of T.F.Ns formed as a result of Criterion 1, look for modes. If all the modes are different, we can rank the triangular fuzzy numbers within the class according to the size of mode.

Fuzzy numbers having equal values of the mode within a class form a subclass. If modal values of some of the triangular fuzzy numbers in a class happen to be same then form sub-classes and go to Criterion 3.

Criterion 3. Linear Ordering using the Divergence

Divergence around a Mode in a Sub-class is given by $a_3 - a_1$.

In each sub-class consider the divergence around the mode and obtain sub-sub-classes. Rank the fuzzy numbers in each sub-class according to the size of the divergence.

1.4 Organization of the Thesis

Chapter 1 provides an introduction to the concepts and problems considered in the thesis. In Chapter 2, we review the literature of the related work done by other researchers. The distribution inventory management problem with fuzzy information (data) is solved using DRP approach in Chapter 3. Chapter 4 presents the inventory problem with variable demand rate under fuzzy information (data) with a finite planning horizon. Chapter 5 extends the variable demand lot-sizing inventory problem with fuzzy information (data) by incorporating backorders. Finally, the conclusion and the discussion on the contributions made in the thesis, along with some recommendations for further research, are given in Chapter 6.

CHAPTER 2

LITERATURE SURVEY

This chapter provides a survey of the literature dealing with inventory lot sizing problems, distribution inventory management problems and other concepts considered in this thesis. The purpose of this chapter is to review the developments, and to identify the status of existing literature in these areas.

2.1 Review of Literature on Inventory Lot Size Problem

2.1.1 Classification of Literature

We use the following classification, given by Zoller and Robrade (1988), to discuss the existing literature on lot-sizing problems.

1. Optimizing techniques
2. Stop rules (heuristics) and
3. Heuristic algorithms.

1. Optimizing Techniques

The optimization techniques include EOQ and Wagner-Whitin algorithm

- **Economic Order Quantity (EOQ)**

Harris (1915) is usually cited as the first to study economic lot size models that assume deterministic demand. He considered a model that assumes demand occur continuously over time. The basic formula to compute economic order lot size is as follows:

$$EOQ = \sqrt{\frac{2SR}{kC}}$$

where,

S = Fixed cost for the replenishment of an order,

R = demand rate of the item (normally annual usage rate),

C = cost of one unit,

k = cost of one dollar of item tied up in inventory for a unit of time.

Note that S and k should have same unit time basis (i.e. if annual demand is considered, then k must be considered for one year, not one month).

- **Wagner-Whitin algorithm**

EOQ model gives the optimal solution only under the assumption of a steady demand. But, when the demand rate varies from period to period, the EOQ formula no longer assures a minimum cost solution. To deal with variable demand, Wagner-Whitin (1958) presented a simple and most elegant dynamic economic lot size model. This model was based on some important theorems established in their paper. These theorems were themselves based upon the assumption that initial inventory is zero ($I_0 = 0$). Before stating their algorithm, we shall briefly state these theorems.

Theorem 1. There always exists an optimal policy such that

$$I_t \cdot X_t = 0 \quad \text{for } t = 1, 2, \dots, N;$$

Where I_t is the inventory entering a period t , X_t is the amount produced in period t and N is the length of planning horizon. This means that replenishment can be made only when the inventory level becomes zero, i.e. having positive inventory and producing at the same time never leads to optimality.

Theorem 2. There exists an optimal policy such that for all t

$$X_t = 0 \quad \text{or} \quad X_t = \sum_{j=t}^k d_j \quad \text{for some } k, \quad t \leq k \leq N$$

where X_t is the amount produced in period t and d_j is the demand in period j . This means that for any given period, production is either zero or is sum of subsequent demands for some number of periods in the future.

Theorem 3. There exists an optimal policy such that if demand d_{t^*} in a period t^* is satisfied by some amount $X_{t^{**}}$ produced in period t^{**} , $t^{**} < t^*$, then d_t , ($t = t^{**} + 1, \dots, t^* - 1$) is also satisfied by $X_{t^{**}}$.

Theorem 4. Given that $I_t = 0$ for period t , it is optimal to consider periods 1 through $(t - 1)$ by themselves.

Planning Horizon Theorem

The planning horizon theorem states in part that if it is optimal to incur a setup cost in period t^* when periods 1 through t^* are considered by themselves, then we may let $X_{t^*} > 0$ in the N period model without foregoing optimality. By theorems 1 and 4 it follows further that we adopt an optimal program for periods 1 through $t^* - 1$ considered separately.

The Algorithm

According to Wagner and Whitin (1958), the algorithm at period t^* , $t^* = 1, 2, \dots, N$, may be generally stated as:

1. Consider the policies of ordering at period t^{**} , $t^{**} = 1, 2, \dots, t^*$ and filling demands d_t , $t = t^{**}, t^{**} + 1, \dots, t^*$, by this order.

2. Determine the total cost of these t^* different policies by adding the ordering and holding costs associated with placing an order at period t^{**} , and the cost of acting optimally for periods 1 through $t^{**} - 1$ considered separately. The latter cost is computed previously in computations for periods $t = 1, 2, \dots, t^* - 1$.
3. From these t^* alternatives, select the minimum cost policy for periods 1 through t^* considered independently.
4. Proceed to period $t^* + 1$ (or stop if $t^* = N$).

It may be observed here that the literature has tended to either ignore or, at least, minimize the significance of the contribution of the Wagner-Whitin solution to this class of the problem due to its complicated nature and enormous computational efforts required. Frequently, excuses such as “the high computational burden and the near impossibility of explaining it to the average MRP user” or “the complexity of the procedure inhibits its understanding by the layman, and acts as an obstacle to its adoption process” were used to justify other approximate (not optimal) alternatives to the Wagner-Whitin procedure (Fordyce and Webster, 1984).

2. Stop Rules

Stop rules increase the cycle length t until some transformation of the controllable cost is reached. Controllable cost $C(t)$, is normally the sum of ordering and holding cost and given as

$$C(t) = R + H \cdot \sum_{h=1}^t (h-1) \cdot d_h$$

where d_h is the demand quantity in period h , H is the holding cost per period per unit, and R is the fixed cost of each replenishment.

- **Least Unit Cost Rule (LUC)**

This is probably the earliest heuristic. LUC chooses the least unit cost (setup + inventory carrying cost per period) over successive periods by adding the total carrying costs to the setup costs and finding the period for which the per unit cost is smallest. A shortcoming in LUC is that it may perform well on one set of data and poorly on another set of data.

- **The Silver and Meal Rule (SMR)**

This is perhaps the most famous heuristic method (Silver and Meal, 1973). Silver-Meal rule divides the total cost by the number of periods included in the lot. It computes the cost per period $P(t)$ as follows:

$$P(t) = C(t)/t,$$

and stops as soon as

$$P(t+1) > P(t).$$

- **Groff's Rule**

Groff (1979) introduced a policy under which the demand for a period is added to the lot if the marginal savings in ordering cost are greater than the marginal increase in carrying cost. In mathematical terms,

$$\text{Marginal savings in ordering cost} = (S/t) - (S / t+1) = S / (t. (t+1))$$

$$\text{Marginal increase in holding cost} = (1/2). H.d_{t+1}$$

Groff's rule adds the demand for the period to the lot if $S/(t. (t+1)) > (1/2).H. d_{t+1}$

and stops as soon as

$$(1/2).H. d_{t+1} = S / (t. (t+1))$$

- **Incremental Order Quantity (IOQ)**

Boe and Yilmax (1983) suggested that cycle length be increased so long as the incremental carrying costs $H.t.d_{t+1}$ does not exceed S and it stops as soon as

$$H.t.d_{t+1} = S$$

- **Period Order Quantity (POQ)**

Period Order Quantity is an EOQ based technique. It divides the EOQ quantity by the average demand during one period to obtain the number of periods whose requirements are to be covered by the lot size (rounded to the nearest positive integer).

$$T_{POQ} = EOQ / (\text{Average demand during one period})$$

If D is the average demand for one period, then

$$EOQ = \sqrt{\frac{2.S.D}{k.C}}$$

$$T_{POQ} = \sqrt{\frac{2.S}{k.C.D}}$$

Thus in POQ method, the time between orders remains fixed, but lot size changes. This approach does not minimize ordering and carrying costs, but it frequently is less costly than ordering each period or arbitrarily selecting a fixed order period.

3. **Heuristic Algorithms**

In the previous section we discussed some rules that were basically single pass stop rules. The stop rules terminate when some transformation of controllable cost is reached,

while algorithms seek to improve the decision by looking further ahead and comparing alternative solutions. In contrast to the stop rules, they are sensitive, in their computing time requirements, to the numerical structure of the demand rate.

- **IOQ Algorithm**

Trux (1972) proposed to use the IOQ rule to find a safe maximum and then examines if the corresponding lot can be split into two lots. Gaither (1983) determined two subsequent lengths and examined if shifting a demand from first lot to second lot is more profitable or not.

- **Part Period Algorithm**

To improve its performance under conditions of erratic demand, several authors have suggested modification of PPR.

PPA - FB: DeMatteis (1968) suggested that the cycle length determined by the PPR should be subjected to a forward or backward scan to determine if the periods of large demand exist.

PPA - BM: Blackburn and Millen (1980) proposed that the cycle length determined by PPR could be increased if a closer balance of ordering and carrying costs can be maintained.

PPA - MG: Karni (1981) combined, through an iterative procedure, that pairs of lots into a new single order which promises maximum gain in terms of net cost reduction.

- **Silver Meal Algorithm**

Silver and Meal (1973) observed that cost per period is not necessarily convex and may hence have many local minima, however, SMR identifies only the first minima.

Blackburn and Millen (1980) suggested that the absolute minima should be found by exhaustive enumeration of $C(t)$ over the entire planning horizon.

2.1.2 Other Approaches

Eppen et. al. (1969) evolved theorems that decrease the computational effort required to find optimal policies and established the existence of planning horizons. Zangwill (1969) extended the concepts of concave cost network to analyze the backorder version of Wagner-Whitin model. Montgomery et al.(1973)solved continuous review and periodic review inventory models that considered a mixture of back orders and lost sales.. Friedman and Hoch (1978) presented a model to find the lot size for perishable items by assuming fixed-plus-linear production functions and linear age-independent inventory cost functions. The issue of perishability was discussed extensively in Nahmias (1982). In his paper, Nahmias distinguished between fixed and variable lifetime inventories. Billington, McClain and Thomas (1983) presented a mathematical model that considers set up times and proposes a compression technique for the size of the product structure , but it does not effectively provide a solution method. In another work, they (1986) studied the multi-stage problem with constrained capacity in only one work center. Kim and Park (1985) considered a continuous review system with constant lead-time where a fraction of the unfilled demand was back ordered and the back order cost was assumed proportional to the length of time the back order existed

Fordyce and Webster (1984) presented the Wagner-Whitin algorithm in a simple and straightforward computational style in a tabular form, without using any mathematical notation or formulas. In 1985, they demonstrated the ability of Wagner-Whitin

Algorithm to be modified to situations in which unit cost price is not constant over the planning horizon and included quantity discounts. Carrying on with their research, Fordyce and Webster (1989) presented a back order version of the Wagner - Whittin algorithm.

James R. Evans (1984) developed a microcomputer program to implement the Wagner-Whittin algorithm. Prentis and Khumawala (1989) developed two heuristics based on branch and bound method to solve closed loop MRP lot sizing problems. Padmanabhan and Vrat (1990) developed an inventory model with a mixture of back orders and lost sales such that the backlogged demand rate was dependent upon the negative inventory level during the stock out period. Shtub (1990) presented a model of cellular production system and a heuristic lot sizing procedure that is based on tradeoff between setup cost and inventory carrying cost for MRP systems. McKnew et. al. (1991) presented a zero one linear formulation of the multilevel lot-sizing problem for MRP systems without capacity constraints. Roll and Karni (1991) present a model for multistage lot - sizing problems with unit lead-time for all components. The heuristic finds the lot-size by shifting production amounts between periods. Bretthauer et. al. (1994) formulated a resource constrained production and inventory management model as a nonlinear integer program.

Padmanabhan and Vrat (1995) presented inventory models for deteriorating items with stock-dependent selling rates and derived the profit functions with and without backlogging and complete backlogging cases. Arreola-Risa and DeCroix (1998) studied a stochastic-demand inventory system where the product supply is randomly disrupted for periods of random duration. They considered the stochastic- demand inventory system will become a mixture of back orders and lost sales during demand shortage. DeCroix

and Arreola-Risa (1998) explored the potential benefits of ordering economic incentives to back order as a strategy for inventory management when the system involves an unreliable supply. Martel (1998) formulated the problem with holding cost as a function of purchasing price. Moncer and Ben-Daya (1999) developed stochastic inventory models as continuous and periodic review models with mixture of backorders, lost sales and the base stock model. Ouyang and Chuang (1999) investigated an inventory model with a mixture of back orders and lost sales in which the back order rate was a random variable and the quantity was discounted on the inventory model. Grewal (1999) applied integer linear programming approach to solve lot-sizing problem with back orders allowed under both crisp and fuzzy environments. He then extended his model to incorporate quantity discounts. Hsu (2000) presented a lot size model for perishable inventory where stock deterioration rates depend on both the stocks' ages and their period of production. Pan and Hsiao (2001) presented inventory models with back order discount and variable lead-time to ensure that customers were willing to wait for the back orders. Chiu, Chen and Weng (2003) proposed a near optimal forward dynamic programming algorithm to solve the deterministic time-varying demand lot-sizing problem in which learning and forgetting in setups and production are considered simultaneously. Suerie and Stadlter (2003) provided a mixed integer-programming model for the capacitated lot-sizing problem with linked lot sizes.

None of the above approaches except Grewal (1999) consider the said problems under fuzzy environment. However, Grewal (1999) uses a complicated linear integer programming approach to solve the lot sizing problem with fuzzy data. In the present thesis, we extend further the simple technique developed by Fordyce and Webster (1984, 1989), to solve the lot size dynamic problem with finite horizon under fuzzy data.

2.2 Review of Literature on Distribution Inventory Management Problems

Some major sources of this literature include Production and Inventory Management Journal, Journal of Operation Research, Management Science Journal, European Journal of Operations Research (EJOR) and Decision Sciences etc. According to Clark and Scarf et. al (1960), there is a possibility that the distribution of stock in a pure distribution system can become imbalanced. That is, if one computes the ideal stock level for each location based on knowing only the total amount of stock in the system, then it could be the case that some retailers have stock in excess of the ideal levels for those locations. Since it is not possible to make shipments that reduce the stock level at a retailer, the ideal levels are not attainable. They extend their algorithm for serial systems to pure distribution systems by assuming that the probability of the system becoming unbalanced is negligible.

Veinott (1965) investigates conditions that ensure the optimality of base-stock policies. Schwarz (1973) derives a one-warehouse N-retailers inventory model. Goyal (1977) proposes a joint economic lot size model to minimize the total relevant costs for both the vendor and the single buyer. Graves (1979) shows that the Joint Replenishment problem is closely related to the One-warehouse Multi-Retailer System problem. Deuermeier and Schwarz (1981) construct and test an approximate model for predicting fill rates and expected backorders in two-echelon distribution system by assuming continuous review policy at all locations. In their model, the re-order point at each location, including the warehouse, is based only on the stock level at that location. Eppen and Schrage (1981) study centralized systems with fixed -length order cycles and derive a Newsboy-like

formula for determining an approximately optimal quantity of system stock. The order policy is referred to as (m, y) policy. Erkip, Hausman and Nahmias (1984) extend the model to allow demand to be correlated over time and between locations. Peter (1988) extends Eppen and Schrage's model to allow the warehouse to hold stock and proposes a policy to make allocations to the retailers in every period of the cycle. This policy is called a "ship-up-to- S " policy: the warehouse makes shipments to restore the inventory position of each retailer to some predetermined value, S , in every period for which the warehouse has sufficient stock. Blackburn and Millan (1982) introduce the concept of echelon stock in the multi-echelon system. Roundy (1985) considers a policy where each retailer orders at an integer or reciprocal of an integer multiple of the warehouse order interval for one warehouse multi-retailer system. This type of policy is called an integer-ratio policy. De Bodt and Graves (1985) proposes a nested continuous review policy for serial systems based on echelon stock. Caplin (1985) studies an economy of multiple retailers, within which each retailer is faced with the unit-sized demand at a time and assumed to employ a continuous (R, Q) type policy. He shows that each retailer's inventory position is independent of other's and uniformly distributed over $[R_i + 1, R_i + Q_i]$. Banerjee (1986) derives a joint economic lot size model for a single vendor, single buyer system where the vendor has a finite production rate. Goyal (1988) extends Banerjee's model by relaxing the lot-for-lot production assumption. Hill (1989) derives a central warehouse, multi-retailer model with shortage by using simulation.

Federgruen and Zipkin (1984) consider periodic review, multiple-echelon systems with interdependent demands under the assumption that the probability of imbalance is small. Zipkin (1984) develops a measure of stock imbalance in a multi-retailer system and formulate a dynamic programming to determine stocking policies when the problem of

stock imbalance can't be ignored. He also demonstrates analytically that the risk pooling benefit of warehouse decreases as the correlation in demand among the retailer increases. Jonsson and Silver (1984a,b) consider a single reallocation opportunity occurring at the beginning of the last period in the cycle by assuming costless transshipments are allowed. Erkip (1984a, b) predetermines the reallocation time by formulating an approximating dynamic program. Jackson and Muckstadt (1984a, b) consider a single, predetermined reallocation time and derive both exact and approximate optimality conditions that don't ignore the possibility of imbalance at the time of reallocation. Brown (1984) conducts numerical experiments with the exact two retailer version of Jackson and Muckstadt model.

Diaby and Martel (1993) formulate a mixed integer linear programming model and develop a Lagrangian relaxation-based procedure to determine the optimal purchasing and shipping quantities over a finite planning horizon for multi-echelon physical distribution systems with deterministic, time-varying demands. Lu and Posner (1994) introduce two heuristic procedures for one-warehouse multi-retailer system. Hariharan and Zipkin (1995) consider a continuous review model where customers place orders L units of time in advance of their requirements. They show that demand lead time L directly offsets the supply lead time and, as of consequence, base stock and (s, S) policies are optimal for zero and positive ordering costs, respectively. Atkins and Sun (1995) derives 98%-effective lot sizing for series inventory systems with backlogging. Ha and Kim (1997) integrates two-echelon inventory model using geometric programming. Yang and Wee (2000) extends an integrated vendor-buyer inventory system of deteriorating item. Gupta et al. (2000) considers the trade-off involved between inventory depletion and production costs under demand uncertainty. Gallego and Ozer (2001) examine a

periodic-review inventory model by allowing backorders and show that base stock and (s, S) policies are optimal for zero and positive ordering costs respectively under advance demand information. Sobel and Zang (2001) study a finite-horizon periodic review model where, in addition to spot demands, there are known commitments in every period. They assume that the commitments must be honored without delay, but allow spot demands to be backordered, and show that a modified (s, S) policy is optimal. Yang and Wee (2002a) analyzes an integrated single vendor multi-buyer inventory system of a deteriorating item. Chen, Feng and Simchi-Levi (2002) consider the periodic review two-echelon inventory system consisting of a supplier at the upper echelon and multiple retailers facing interrelated demands at the lower echelon, and show that under certain condition, inventory positions at each location are stationary, uniformly distributed and independent of the inventory positions at other locations. Ozer (2003) proposes a close-to-optimal solution based on the solution of a lower bound problem for a centralized system with one warehouse serving multiple retailers under advance demand information. He also provides an explicit solution for the system-wide inventory position. Martel (2003) develops rolling planning horizon policies to manage material flows in multi echelon supply-distribution networks with relatively general stochastic demand processes and procurement, transportation, inventory and shortage cost structures. Yao and Chiou (2003) considers an integrated supply chain model to minimize the vendor's total annual cost subject to the maximum cost that buyer may be prepared to incur.

It is important to point out here that none of the above work conducted by various researchers incorporates the fuzzy element present in most of the problems. In the present thesis we consider the distribution inventory management problem under fuzzy information.

CHAPTER 3

A SIMPLIFIED APPROACH TO LOT SIZING INVENTORY PROBLEM UNDER FUZZY INFORMATION WITH VARIABLE DEMAND RATE AND NO BACKORDERS ALLOWED

In the present chapter, we consider a variable demand rate inventory problem with no backorders allowed under fuzzy environment with a finite-planning horizon. We extend the Wagner –Whitin algorithm to find the lot size for a given product P, assuming that the demand for a given product P is represented not by a crisp number but by a triangular fuzzy number, and similarly, the acquisition cost and the carrying cost for a given product P are represented by other triangular fuzzy numbers. The approach can be extended further when the demand for a given product P is represented by a trapezoidal fuzzy number, and the acquisition cost and the carrying cost for a given product P are represented by other trapezoidal fuzzy numbers.

3.1 Introduction

A product (which could be a raw material, a purchased part or a semi finished product) is produced or purchased in lot quantities and placed in stock. As the stock is depleted by demands for the product, more of the product must be produced or purchased. The object of production planning is to minimize the cost of this cycle of filling and depleting the stock. To achieve this object, determination of optimal lot size is very important.

Most of the times the production planner works with the forecasted demand and limited budget. So, the demand is rarely known exactly as the forecasts do not always turn out to be crisp accurate. Thus, in practice there is always a component of fuzziness in available

data. Under condition of fuzzy demand and fuzzy costs, the models developed for the crisp problem may not deliver optimal solution. In present chapter, we create a model, by taking advantage of the fuzzy set theory developed by Zadeh and Bellman (1965), to deal with fuzziness in demand.

In 1984, Fordyce and Webster presented a simplified version of the Wagner-Whitin algorithm to determine the lot size under crisp environment. The present chapter is an extension of their work with fuzzy information (data). This chapter also demonstrates the simplicity of the technique when computations are made in several simple steps.

3.2 Lot Size Problem under Crisp Environment

3.2.1 Assumptions

For this model, the following assumptions are made:

1. The demand varies from one time period to another and is assumed to be known.
2. The units needed to satisfy demand during a particular period must be purchased during a previous period or at the beginning of the specific period during which they are needed.
3. Acquisition costs (setup costs of production run or ordering or follow-up costs for purchased parts) are fixed relative to the quantity acquired, but may vary from one time period to another.
4. Holding cost in a particular time period represents the cost of inventorying one unit of product from that period until the next period. Units carried forward more than one time period would be charged the accumulated holding cost of all periods from acquisition period through the period prior to the actual period of demand or use.

5. Holding cost may vary from one time period to another.
6. The replenishment lead-time is known with certainty so that delivery can be timed to occur accordingly.

3.2.2 Notation

D_j = number of units demanded during period j

H_i = holding cost (in \$) per unit during period i

C_{ij} = cumulative holding cost per unit during period j for the units needed during period j but purchased during period i , $i \leq j$

A_j = acquisition cost, irrespective of the number of units acquired, during period j .

N = the number of periods during the planning horizon

T_{ij} = cumulative cost (acquisition cost + carrying cost) during period j for the units needed during period j , but purchased during period i

M_i = the minimum prior period cost for period i

F_{ij} = decision cost (cumulative cost + the minimum prior period cost) during period j for the units needed during period j , but purchased during period i

L_i = lot size during period i

3.2.3 General Formulation

$$C_{ij} = \sum_{k=i}^{j-1} H_k \quad \text{for } j > i \quad (3.1)$$

$$C_{ij} = 0 \quad \text{for } j = i$$

$$T_{ij} = A_i + \sum_{k=i}^j (C_{ik} * D_k) \quad (3.2)$$

$$M_i = \text{minimum of } F_{(i-1)j} \quad (3.3)$$

$$\text{where } j = 1, 2, \dots, (i-1)$$

$$F_{ij} = T_{ij} + M_i \quad (3.4)$$

3.3 Lot Size with Fuzzy Information (Data)

3.3.1 Assumptions

1. The demand varies from one time period to another and is available in the form of triangular fuzzy numbers.
2. The units needed to satisfy demand during a particular period must be purchased during a previous period or at the beginning of the specific period during which they are needed.
3. Acquisition cost (setup cost of production run or ordering or follow-up cost for purchased parts) is available in the form of a triangular fuzzy number, and unchanged relative to the quantity acquired.
4. Holding cost in a particular time period, assumed to be fuzzy, represents the cost of inventorying one unit of product from that period until the next period. Units carried forward more than one time period would be charged the accumulated holding cost of all periods from acquisition period through the period prior to the actual period of demand or use.
5. Holding cost may vary from one time period to another.
6. The replenishment lead-time is known with certainty so that delivery can be timed to occur accordingly.

3.3.2 General Formulation

We assume that each of H_k in equation (3.1), A_i and D_k in equation (3.2) for $m = 1, 2, \dots, n$; $i=1, 2, \dots, n$; $k=1, 2, \dots, n$ is a T.F.N. of the type.

$$H_k = (H_{k1}, H_{k2}, H_{k3}) ; A_i = (A_{i1}, A_{i2}, A_{i3}) ; D_k = (D_{k1}, D_{k2}, D_{k3})$$

The values of H_k , A_i and D_k can be obtained from the experts who share the same information but different opinion.

If we set $H_k = (H_{k1}, H_{k2}, H_{k3})$ in equation (3.1), then C_{ik} is a triangular fuzzy number given by

$$C_{ik} = (C_{ik1}, C_{ik2}, C_{ik3}) \quad (3.5)$$

In (3.2), if we set $A_i = (A_{i1}, A_{i2}, A_{i3})$; $D_k = (D_{k1}, D_{k2}, D_{k3})$ and $C_{ik} = (C_{ik1}, C_{ik2}, C_{ik3})$ then

(i) each term in the second part $\sum_{k=i}^j (C_{ik} * D_k)$, in the R.H.S of (3.2), is obtained by

multiplying two T.F.N.'s .

(ii) each term obtained in (i) is itself a fuzzy number but not necessarily a T.F.N. and

(iii) T_{ij} is obtained by adding the results of each individual multiplication obtained in (i) to A_i .

Thus, we have

$$T_{ij} = A_i + \sum_{k=i}^j (C_{ik} * D_k) ; i = 1, 2, \dots, n ; \text{ and } j=1, 2, \dots, n$$

In view of (ii), it is important to point out here that each of the values For T_{ij} is a fuzzy number, not necessarily triangular.

Thus, T_{ij} is a fuzzy number given by

$$T_{ij} = (T_{ij1}, T_{ij2}, T_{ij3}) \quad (3.6)$$

Each fuzzy number, T_{ij} in equation (3.6) and its membership function are determined on the line of Kaufman and Gupta (1985, 1988) by using the interval of confidence for A_i , D_k and C_{ik} respectively, at α – level

$$A_i^\alpha = [(A_{i2} - A_{i1})\alpha + A_{i1}, (A_{i2} - A_{i3})\alpha + A_{i3}] \quad \forall \alpha \in [0,1] \quad (3.7)$$

$$D_k^\alpha = [(D_{k2} - D_{k1})\alpha + D_{k1}, (D_{k2} - D_{k3})\alpha + D_{k3}] \quad \forall \alpha \in [0,1] \quad (3.8)$$

$$C_{ik}^\alpha = [(C_{ik2} - C_{ik1})\alpha + C_{ik1}, (C_{ik2} - C_{ik3})\alpha + C_{ik3}] \quad \forall \alpha \in [0,1] \quad (3.9)$$

For $i = 1, 2, \dots, n$; $k = 1, 2, \dots, n$; We multiply the intervals in (3.8) and (3.9), and add the result to the interval in (3.7).

This yields the following interval of confidence.

Thus, we get

$$\left\{ \begin{aligned} & \sum_{k=i}^j [(D_{k2} - D_{k1}) * (C_{ik2} - C_{ik1})] \alpha^2 + \\ & \sum_{k=i}^j [(D_{k2} - D_{k1}) * C_{ik1} + (C_{ik2} - C_{ik1}) * D_{k1} + (A_{i2} - A_{i1})] \alpha + A_{i1} + \sum_{k=i}^j (D_{k1} * C_{ik1}), \\ & \sum_{k=i}^j [(D_{k2} - D_{k3}) * (C_{ik2} - C_{ik3})] \alpha^2 + \\ & \sum_{k=i}^j [(D_{k2} - D_{k3}) * C_{ik3} + (C_{ik2} - C_{ik3}) * D_{k3} + (A_{i2} - A_{i3})] \alpha + A_{i3} + \sum_{k=i}^j (D_{k3} * C_{ik3}) \end{aligned} \right\}$$

In this interval of confidence

1. Setting $\alpha = 0$, we get the end points T_{ij1} and T_{ij3}

$$T_{ij1} = [A_{i1} + \sum_{k=i}^j (D_{k1} * C_{ik1})], \text{ and } T_{ij3} = [A_{i3} + \sum_{k=i}^j (D_{k3} * C_{ik3})]$$

of the fuzzy number T_{ij}

2. Setting $\alpha = 1$ gives the interior point T_{ij2} of T_{ij}

$$T_{ij2} = [A_{i2} + \sum_{k=i}^j (D_{k2} * C_{ik2})]$$

The membership function is obtained from the interval of confidence by setting separately, each of the quadratic function, equal to x and solving each of those equations for α . Thus,

$$\begin{aligned} & \sum_{k=i}^j [(D_{k2} - D_{k1}) * (C_{ik2} - C_{ik1})] \alpha^2 + \\ & \sum_{k=i}^j [(D_{k2} - D_{k1}) * C_{ik1} + (C_{ik2} - C_{ik1}) * D_{k1} + (A_{i2} - A_{i1})] \alpha + A_{i1} + \\ & \sum_{k=i}^j (D_{k1} * C_{ik1}) = x \end{aligned} \quad (3.10)$$

solving (3.10) gives membership function between T_{ij1} and T_{ij2} satisfying $0 \leq \alpha \leq 1$.

Next, we set

$$\begin{aligned} & \sum_{k=i}^j [(D_{k2} - D_{k3}) * (C_{ik2} - C_{ik3})] \alpha^2 + \\ & \sum_{k=i}^j [(D_{k2} - D_{k3}) * C_{ik3} + (C_{ik2} - C_{ik3}) * D_{k3} + (A_{i2} - A_{i3})] \alpha + A_{i3} + \\ & \sum_{k=i}^j (D_{k3} * C_{ik3}) = x \end{aligned} \quad (3.11)$$

Solving (3.11) gives membership function between T_{ij2} and T_{ij3} satisfying $0 \leq \alpha \leq 1$.

Similarly, We determine each fuzzy number, F_{ij} in equation (3.4) and its membership function by using the interval of confidence for T_{ij} and M_j respectively, at α – level.

$$F_{ij} = (F_{ij1}, F_{ij2}, F_{ij3})$$

STEP 2. Obtaining Second Tableau

Obtain the cumulative cost T_{ij} associated with each time period by using (3.2) and prepare the second tableau in the form of a truncated upper triangular matrix having n rows and n columns.

Second Tableau

| Time → ↓ | | 1 | 2 | 3 | ... | m | ... | n-1 | n |
|----------------|--|----------|----------|----------|-----|----------|-----|-------------|----------|
| Row 1 | | T_{11} | T_{12} | T_{13} | ... | T_{1m} | ... | $T_{1,n-1}$ | T_{1n} |
| Row 2 | | | T_{22} | T_{23} | ... | T_{2m} | ... | $T_{2,n-1}$ | T_{2n} |
| | | ... | ... | ... | ... | ... | ... | ... | ... |
| Row m | | | | | | T_{mm} | ... | $T_{m,n-1}$ | T_{mn} |
| | | ... | ... | ... | ... | ... | ... | ... | ... |
| Row n | | | | | | | | | T_{nn} |

STEP 3. Obtaining Optimal Tableau

1. Obtain Row 1 of the optimal tableau by duplicating Row 1 of the second tableau.

Each subsequent row is obtained as follows:

2. To obtain Row 2 of the optimal tableau:
 - (a) Examine Column 1 of the optimal tableau and in the column identify the element with smallest value s_1 .
 - (b) Obtain Row 2 of the optimal tableau by adding s_1 to each element of Row 2 of the second tableau.

3. To obtain Row 3 of the optimal tableau:
 - (a) Examine Column 2 of the optimal tableau and in the column identify the element with smallest value s_2 .
 - (b) Obtain Row 3 of the optimal tableau by adding s_2 to each element of Row 3 of the second tableau.

In general therefore, we compute Row k , where $k = 1, 2, \dots, n$; of the optimal tableau as follows:

- (a) Examine Column $(k-1)$ of the optimal tableau and in the column identify the element with smallest value $s_{(k-1)}$.
- (b) Obtain Row k , where $k = 1, 2, \dots, n$; of the optimal tableau by adding $s_{(k-1)}$ to each element of Row k of the second tableau.

STEP 4. Identifying the optimal solution

1. (i) Examine the column n of the optimal tableau and identify the minimum value element s_n^* in it. Then,
 $s_n^* = g_n(1)$ is the minimum decision cost incurred from time 1 until time n , i.e. during the planning horizon.
- (ii) Suppose the position of s_n^* is in Row u , $u < n$, of the optimal tableau.

Then time u is the acquisition time immediately before time n .

2. (i) Examine the column u of the optimal tableau and identify the minimum value element s_u^* in it. Then,
 $s_u^* = g_u(1)$ is the minimum decision cost incurred from time 1 until time $u < n$.

- (ii) Suppose the position of s_u^* is in Row q , $q < u$, of the optimal tableau.

Then time q is the acquisition time immediately before time u .

Stopping Rule

We continue in the same manner until, in the optimal tableau, we identify a minimum value element v_b^* (say), in Column b , ($b < q$), such that Row 1 is the row in which v_b^* lies and $v_b^* = g_b(1)$. Once the minimum value element lies in Row 1, the optimal overall policy has been reached and the process terminates.

The optimal policy is to acquire units at time 1, time b , . . . , time q , time u , and the time n with the overall net optimal cost v_n^* .

3.3.4 Numerical Example with Fuzzy Information

We illustrate our method through the following numerical example (represented in Table 3.1 and Table 3.2). Demand, acquisition cost, and carrying cost are represented by triangular fuzzy numbers. We assume that acquisition cost remain constant throughout the planning horizon.

Table 3.1. Demand, Acquisition cost and Carrying cost (Week 1 to Week 3)

| Period | 1 | 2 | 3 |
|------------------|----------------|----------------|----------------|
| Demand | (17, 20, 25) | (45, 50, 52) | (9, 10, 12) |
| Acquisition Cost | (98, 100, 105) | (98, 100, 105) | (98, 100, 105) |
| Carrying Cost | (0.85, 1, 1.2) | (0.9, 1, 1.15) | (0.9, 1, 1.2) |

Table 3.2. Demand, Acquisition cost and Carrying cost (Week 4 to Week 6)

| Period | 4 | 5 | 6 |
|------------------|----------------|----------------|----------------|
| Demand | (8, 10, 12) | (48, 50, 53) | (18, 20, 25) |
| Acquisition Cost | (98, 100, 105) | (98, 100, 105) | (98, 100, 105) |
| Carrying Cost | (.95, 1, 1.1) | (.8, 1, 1.25) | (.85, 1, 1.3) |

3.3.5 Intervals of Confidence

3.3.5.1 Intervals of Confidence for Demand

$$d_1^\alpha = [3\alpha + 17, -5\alpha + 25]$$

$$d_2^\alpha = [5\alpha + 45, -2\alpha + 52]$$

$$d_3^\alpha = [1\alpha + 9, -2\alpha + 12]$$

$$d_4^\alpha = [2\alpha + 8, -2\alpha + 12]$$

$$d_5^\alpha = [2\alpha + 48, -3\alpha + 53]$$

$$d_6^\alpha = [2\alpha + 18, -5\alpha + 25]$$

3.3.5.2 Intervals of Confidence for Acquisition Cost

$$A_1^\alpha = [2\alpha + 98, -5\alpha + 105]$$

$$A_2^\alpha = [2\alpha + 98, -5\alpha + 105]$$

$$A_3^\alpha = [2\alpha + 98, -5\alpha + 105]$$

$$A_4^\alpha = [2\alpha + 98, -5\alpha + 105]$$

$$A_5^\alpha = [2\alpha + 98, -5\alpha + 105]$$

$$A_6^\alpha = [2\alpha + 98, -5\alpha + 105]$$

3.3.5.3 Intervals of Confidence for Carrying Cost

$$H_1^\alpha = [0.15\alpha + 0.85, -0.2\alpha + 1.2] \quad H_2^\alpha = [0.10\alpha + 0.90, -0.15\alpha + 1.15]$$

$$H_3^\alpha = [0.10\alpha + 0.90, -0.3\alpha + 1.2] \quad H_4^\alpha = [0.05\alpha + 0.95, -0.10\alpha + 1.10]$$

$$H_5^\alpha = [0.20\alpha + 0.80, -0.25\alpha + 1.25] \quad H_6^\alpha = [0.15\alpha + 0.85, -0.30\alpha + 1.30]$$

3.3.6 Calculating the Cumulative Carrying Cost per unit during Period j for the units purchased during Period 1 (C_{1j})

Obtaining Initial Tableau.

Using (3.1), we have

$$C_{11}^{\alpha} = 0$$

$$C_{12}^{\alpha} = \sum_{k=1}^1 H_k^{\alpha} = H_1^{\alpha} = [0.15\alpha + 0.85, -0.2\alpha + 1.2]$$

$$C_{13}^{\alpha} = \sum_{k=1}^2 H_k^{\alpha} = H_1^{\alpha} + H_2^{\alpha}$$

$$C_{13}^{\alpha} = [0.15\alpha + 0.85, -0.2\alpha + 1.2] + [0.10\alpha + 0.90, -0.15\alpha + 1.15]$$

$$C_{13}^{\alpha} = [0.25\alpha + 1.75, -0.35\alpha + 2.35]$$

We now set

$$0.25\alpha + 1.75 = x \text{ and } -0.35\alpha + 2.35 = x$$

This yields,

$$0.25\alpha + 1.75 - x = 0 \tag{3.12}$$

$$\text{and } -0.35\alpha + 2.35 - x = 0 \tag{3.13}$$

In (3.12) setting $\alpha = 0$ we get $x = 1.75$

In (3.13) setting $\alpha = 0$ we get $x = 2.35$

Setting $\alpha = 1$ in either we get $x = 2$

Therefore $C_{13} = (1.75, 2, 2.35)$

Similarly, we calculate C_{14} , C_{15} and C_{16}

$$C_{14} = (2.65, 3, 3.55); C_{15} = (3.6, 4, 4.65); C_{16} = (4.4, 5, 5.9)$$

Now, we determine membership function for C_{13}

Solving $0.25\alpha + 1.75 - x = 0$ for α we obtain

$$\alpha = \frac{x-1.75}{0.25} \quad \text{for } 1.75 \leq x \leq 2$$

and solving $-0.35\alpha + 2.35 - x = 0$ we obtain

$$\alpha = \frac{x-2.35}{-0.35} \quad \text{for } 2 \leq x \leq 2.35$$

Thus, the membership function for $C_{13} = (1.75, 2, 2.35)$ is

$$\mu_{C_{13}}(x) = \begin{cases} 0 & x \leq 1.75 \\ \frac{x-1.75}{0.25} & 1.75 \leq x \leq 2 \\ \frac{x-2.35}{-0.35} & 2 \leq x \leq 2.35 \\ 0 & x \geq 2.35 \end{cases}$$

Similarly, we obtain the fuzzy carrying costs for rest of C_{ij} 's (shown in Table 3.3).

Table 3.3. Cumulative Carrying Costs during Period j (Initial Tableau)

| Period | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|-----------|----------------|-----------------|-----------------|-----------------|-----------------|
| 1 | (0, 0, 0) | (0.85, 1, 1.2) | (1.75, 2, 2.35) | (2.65, 3, 3.55) | (3.6, 4, 4.65) | (4.4, 5, 5.9) |
| 2 | | (0, 0, 0) | (0.9, 1, 1.15) | (1.8, 2, 2.35) | (2.75, 3, 3.45) | (3.55, 4, 4.7) |
| 3 | | | (0, 0, 0) | (0.9, 1, 1.2) | (1.85, 2, 2.3) | (2.65, 3, 3.55) |
| 4 | | | | (0, 0, 0) | (0.95, 1, 1.1) | (1.75, 2, 2.35) |
| 5 | | | | | (0, 0, 0) | (0.8, 1, 1.25) |
| 6 | | | | | | (0, 0, 0) |

3.3.7 Calculating the Cumulative Cost during Period j for the units purchased during Period 1 (T_{1j})

Obtaining Second Tableau.

Using (3.2), we have

$$\begin{aligned} T_{11}^{\alpha} &= A_1^{\alpha} \quad (\because C_{11} = 0) \\ T_{11}^{\alpha} &= [2\alpha + 98, -5\alpha + 105] \end{aligned} \quad (3.14)$$

$$T_{12} = A_1 + \sum_{k=1}^2 (C_{1k} * D_k)$$

$$T_{12}^{\alpha} = A_1^{\alpha} + (C_{11}^{\alpha} * D_1^{\alpha}) + (C_{12}^{\alpha} * D_2^{\alpha})$$

$$\begin{aligned} T_{12}^{\alpha} &= [2\alpha + 98, -5\alpha + 105] + 0 + \\ &\quad \{[0.15\alpha + 0.85, -0.2\alpha + 1.2] * [5\alpha + 45, -2\alpha + 52]\} \\ T_{12}^{\alpha} &= [.75\alpha^2 + 13\alpha + 136.25, .40\alpha^2 - 17.8\alpha + 167.4] \end{aligned}$$

We now set

$$.75\alpha^2 + 13\alpha + 136.25 = x \text{ and } .40\alpha^2 - 17.8\alpha + 167.4 = x$$

This yields,

$$.75\alpha^2 + 13\alpha + 136.25 - x = 0 \quad (3.15)$$

$$\text{and } .40\alpha^2 - 17.8\alpha + 167.4 - x = 0 \quad (3.16)$$

In (3.15) setting $\alpha = 0$ we get $x = 136.25$

In (3.16) setting $\alpha = 0$ we get $x = 167.4$

Setting $\alpha = 1$ in either we get $x = 150$

Therefore $T_{12} = (136.25, 150, 167.40)$

Similarly, we calculate T_{13} , T_{14} , T_{15} and T_{16}

Now, we determine membership function for T_{12}

Solving quadratic equation (3.15) $.75\alpha^2 + 13\alpha + 136.25 - x = 0$ for α we obtain

$$\alpha = \frac{-13 + \sqrt{169 - 3(136.25 - x)}}{1.5} \quad \text{for } 136.25 \leq x \leq 150$$

And solving quadratic equation (3.16) $.40\alpha^2 - 17.8\alpha + 167.4 - x = 0$ for α we obtain

$$\alpha = \frac{17.8 - \sqrt{316.84 - 1.6(167.4 - x)}}{.80} \quad \text{for } 150 \leq x \leq 167.40$$

Thus, the membership function for $T_{12} = (136.25, 150, 167.40)$ is

$$\mu_{T_{12}}(x) = \begin{cases} 0 & x \leq 136.25 \\ \frac{-13 + \sqrt{169 - 3(136.25 - x)}}{1.5} & 136.25 \leq x \leq 150 \\ \frac{17.8 - \sqrt{316.84 - 1.6(167.4 - x)}}{0.80} & 150 \leq x \leq 167.4 \\ 0 & x \geq 167.40 \end{cases}$$

Similarly, we obtain the fuzzy cumulative costs, along with their membership functions, for rest of T_{ij} 's. Fuzzy cumulative costs are shown in Table 3.5.

In view of T_{12} , it is important to point out here that in Table 3.5, T_{ij} 's are parabolic fuzzy numbers. However, we may approximate a parabolic fuzzy number by a T.F.N. if by doing so the % age error obtained is less than 3%.

Table 3.4. Error Analysis for T_{12}

| α | Curvature | | T.F.N. | | Error | | % Error | |
|----------|-----------|---------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 136.25 | 167.4 | 136.25 | 167.4 | 0 | 0 | 0 | 0 |
| 0.1 | 137.5575 | 165.624 | 137.625 | 165.66 | 0.0675 | 0.036 | 0.04907 | 0.021736 |
| 0.2 | 138.88 | 163.856 | 139 | 163.92 | 0.12 | 0.064 | 0.086406 | 0.039059 |
| 0.3 | 140.2175 | 162.096 | 140.375 | 162.18 | 0.1575 | 0.084 | 0.112325 | 0.051821 |
| 0.4 | 141.57 | 160.344 | 141.75 | 160.44 | 0.18 | 0.096 | 0.127146 | 0.059871 |
| 0.5 | 142.9375 | 158.6 | 143.125 | 158.7 | 0.1875 | 0.1 | 0.131176 | 0.063052 |
| 0.6 | 144.32 | 156.864 | 144.5 | 156.96 | 0.18 | 0.096 | 0.124723 | 0.0612 |
| 0.7 | 145.7175 | 155.136 | 145.875 | 155.22 | 0.1575 | 0.084 | 0.108086 | 0.054146 |
| 0.8 | 147.13 | 153.416 | 147.25 | 153.48 | 0.12 | 0.064 | 0.081561 | 0.041717 |
| 0.9 | 148.5575 | 151.704 | 148.625 | 151.74 | 0.0675 | 0.036 | 0.045437 | 0.02373 |
| 1 | 150 | 150 | 150 | 150 | 0 | 0 | 0 | 0 |

As we see from Table 3.4, the % error is less than 3%, therefore, we may assume that T_{12} is a T.F.N.. The error analyses for all T_{ij} 's is shown in Appendix 1. Thus, from Table 3.4 and Appendix 1 we observe all T_{ij} 's can be approximated by triangular fuzzy numbers.

Table 3.5. Cumulative costs during Period j (Second Tableau)

| Period | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|----------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| 1 | (98, 100, 105) | (136.25, 150, 167.40) | (152, 170, 195.60) | (173.20, 200, 238.20) | (346, 400, 484.65) | (425.20, 500, 632.15) |
| 2 | | (98, 100, 105) | (106.10, 110, 118.80) | (120.50, 130, 147) | (252.50, 280, 329.85) | (316.40, 360, 447.35) |
| 3 | | | (98, 100, 105) | (105.20, 110, 119.40) | (194, 210, 241.30) | (241.70, 270, 330.05) |
| 4 | | | | (98, 100, 105) | (143.60, 150, 163.30) | (175.10, 190, 222.05) |
| 5 | | | | | (98, 100, 105) | (112.40, 120, 136.25) |
| 6 | | | | | | (98, 100, 105) |

3.3.8 Calculating the Minimum Prior Period Cost for Period i and the Decision Cost during Period j for the units purchased in Period i

Obtaining the Optimal Tableau.

As the minimum prior period cost for Period 1 is zero, therefore,

$$M_1 = 0$$

Thus, we start **Table 3.7** by adding zero to each entry in Row 1 of the **Table 3.5** to form Row 1 of **Table 3.7**. Now that Row 1 of **Table 3.7** is complete, we observe that the minimum cost of Period 1 is (98, 100, 105), as the (98, 100, 105) is the only entry in Column 1. Therefore, the acquisition cost M_2 for Period 1 is

$$M_2 = (98, 100, 105).$$

Since no backorder is allowed as the units must be purchased in Period 1, therefore, $M_2 = (98, 100, 105)$, the acquisition cost of Period 1, by necessity has to be incurred.

Row 2 of **Table 4.7** is completed by adding this cost to each entry in Row 2 of **Table 4.5**.

Thus, assuming that all the T_{ij} 's are T. F. N.'s, we have

$$F_{22} = M_2 + T_{22}$$

Therefore $F_{22} = (98, 100, 105) + (98, 100, 105) = (196, 200, 210)$

Similarly, we calculate the rest of F_{2j} 's.

Now, we determine the minimum prior period cost M_3 for Period 3.

Since, we want to keep the cost as low as possible, therefore, we identify the minimum of two entries in Column 2 of **Table 3.7**. For this purpose we calculate the associated ordinary number (A.O.N.) for each entry (Kaufman and Gupta, 1985).

For example, for $F_{12} = (136.25, 150, 167.40)$ and $F_{22} = (196, 200, 210)$,

$$\text{A.O.N. for } F_{12} = \frac{136.25 + 2 * 150 + 167.40}{4} = 150.91$$

$$\text{A.O.N. for } F_{22} = \frac{196 + 2 * 200 + 210}{4} = 210.5.$$

In a similar fashion we calculate the A. O. N. for each of the F_{ij} .

Table 3.6. Associated ordinary numbers for F_{ij} 's

| Period | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|--------|----------|----------|----------|----------|----------|
| 1 | 100.75 | 150.9125 | 171.9 | 202.85 | 407.6625 | 514.3375 |
| 2 | | 201.5 | 211.975 | 232.625 | 386.3375 | 471.6875 |
| 3 | | | 251.6625 | 262.0625 | 364.7375 | 428.85 |
| 4 | | | | 272.65 | 323.625 | 366.1875 |
| 5 | | | | | 303.6 | 325.0125 |
| 6 | | | | | | 404.35 |

From Table 3.6 we observe that $F_{12} < F_{22}$

Therefore, $M_3 = F_{12} = (136.25, 150, 167.40)$

and, $F_{33} = M_3 + T_{33}$

$$= (234.25, 250, 272.40)$$

Similarlry, we calculate rest of F_{ij} 's with their values as shown in Table 3.7.

Table 3.7. Decision costs during Period j (Optimal Tableau)

| Period | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|----------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| 1 | (98, 100, 105) | (136.25, 150, 167.40) | (152, 170, 195.60) | (173.20, 200, 238.20) | (346, 400, 484.65) | (425.20, 500, 632.15) |
| 2 | | (196, 200, 210) | (204.10, 210, 223.80) | (218.50, 230, 252) | (350.50, 380, 434.85) | (414.40, 460, 552.35) |
| 3 | | | (234.25, 250, 272.40) | (241.45, 260, 286.80) | (330.25, 360, 408.70) | (377.95, 420, 497.45) |
| 4 | | | | (250, 270, 300.60) | (295.60, 320, 358.90) | (327.10, 360, 417.65) |
| 5 | | | | | (271.20, 300, 343.20) | (285.60, 320, 374.45) |
| 6 | | | | | | (369.20, 400, 448.20) |

3.3.9 Calculating the Lot Size during Period j

For this purpose, we start from the last period of Table 3.7, Period 6. The minimum decision cost in Period 6 is $F_{56} = (285.60, 320, 374.45)$, calculated by ordering the fuzzy entries in Column 6, and it is obtained by acquiring Period 6's demand in Period 5.

Thus, lot size in Period 5 is given by

$$L_5^\alpha = D_5^\alpha + D_6^\alpha$$

$$L_5^\alpha = [2\alpha + 48, -3\alpha + 53] + [2\alpha + 18, -5\alpha + 25]$$

$$L_5^\alpha = [4\alpha + 66, -8\alpha + 78]$$

We now set

$$4\alpha + 66 = x \text{ and } -8\alpha + 78 = x$$

This yields,

$$4\alpha + 66 - x = 0 \tag{3.21}$$

$$\text{and } -8\alpha + 78 - x = 0 \tag{3.22}$$

In (3.21) setting $\alpha = 0$ we get $x = 66$

In (3.22) setting $\alpha = 0$ we get $x = 78$

Setting $\alpha = 1$ in either we get $x = 70$

Therefore $L_5 = (66, 70, 78)$

Now, we move to Period 4. The minimum decision cost for Period 4 is $F_{14} = (173.20, 200, 238.20)$, and it is obtained by acquiring Period 4 demand in Period 1. So, Period 1 is an acquisition period to satisfy the demand in Period 1, 2, 3 and 4.

Lot size in Period 1 is given by

$$L_1^\alpha = D_1^\alpha + D_2^\alpha + D_3^\alpha + D_4^\alpha$$

$$L_1^\alpha = [11\alpha + 79, -11\alpha + 101]$$

This yields,

Therefore, $L_1 = (79, 90, 101)$

Thus, we have lot size for each period as in Table 3.8.

Table 3.8. Lot Size during Period j

| Period | 1 | 2 | 3 | 4 | 5 | 6 |
|----------|-------------|-------------|-------------|-------------|------------|-------------|
| Lot Size | (79,90,101) | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) | (66,70,78) | (0, 0, 0) |

3.3.10 Interpretation of the Results

In Table 3.8, we calculated the lot size required to satisfy the demand during six weeks. The lot size in Week 1, $L_1 = (79, 90, 101)$, is used to satisfy the demand from Week 1 through Week 4. The lot size in Week 5, $L_5 = (66, 70, 78)$, is used to satisfy the demand in Week 5 and Week 6.

Fuzzy set theory permits the partial belonging of an element to a fuzzy set characterized by a membership function that takes values in the interval $[0, 1]$. Thus, fuzzy approach yields a relatively “more satisfactory and flexible solution” within a pre-specified intervals, whereas a conventional crisp set theory only permits an element either to belong (membership grade 1) or not to belong (membership grade 0) to the set.

CHAPTER 4

A SIMPLIFIED APPROACH TO LOT SIZING INVENTORY PROBLEM WITH VARIABLE DEMAND RATE AND BACKORDERS ALLOWED WITH FUZZY INFORMATION

In the present chapter, we modify Chapter 3 further by considering a variable demand rate inventory problem with backorders allowed, under fuzzy environment with a finite-planning horizon. We develop a technique based on dynamic programming to find the lot size for a given product P , assuming that the demand, the acquisition cost, the carrying cost and the back-ordering cost for a given product P are represented by triangular fuzzy numbers.

4.1 Introduction

In 1989, Webster and Francis considered a back-order version of the Wagner-Whitin algorithm to determine the lot size under crisp environment. The present chapter is an extension of their work with fuzzy information (data). This back-order (BO) model assumes that the demand in a given period can be met in future at some additional costs. Under certain circumstances this could be more economical than supplying the required units in the period of demand. The additional cost may be a function of the numbers of units, the time on BO, the number of orders on BO, the number of customers awaiting backordered units, or a variety of other variables. The BO costs in the model presented here assumes BO costs are expressed as "dollars per unit per period of time".

4.2 Lot Size Problem under Crisp Environment

4.2.1 Assumptions

For this model, the following assumptions are made (Webster and Francis (1989)).

1. The demand varies from one time period to another and is assumed to be known.
2. The units needed to satisfy demand during a particular period can be acquired at any time including backorders.
3. Acquisition costs (setup costs of production run or ordering or follow-up costs for purchased parts) are fixed relative to the quantity acquired, but may vary from one time period to another.
4. Holding cost in a particular time period represents the cost of inventorying one unit of product from that period until the next period. Units carried forward more than one time period would be charged the accumulated holding cost of all periods from acquisition period through the period prior to the actual period of demand or use.
5. Holding cost may vary from one time period to another.
6. Backorder cost represents the cost of backordering one unit of product from the period in which it was needed to the next time period. Units backordered more than one time period would be charged the accumulated backordering cost of all the periods from the time period of demand till the time period of acquisition.
7. Backorder cost may vary from one time period to another.
8. The replenishment lead-time is known with certainty so that delivery can be timed to occur accordingly.
9. The product is treated entirely independently of the other products i.e., benefits from joint replenishment do not exist or are ignored.

4.2.2 Notation

D_j = number of units demanded during period j

H_i = holding cost (in \$) per unit during period i

C_{ij} = cumulative holding cost per unit during period j for the units needed during period j but purchased during period i , $i \leq j$

b_i = backordering cost (in \$) per unit during period i

B_{ij} = cumulative backordering cost per unit during period j for the units needed during period j but purchased during period i , $i \geq j$

A_j = acquisition cost, irrespective of the number of units acquired, during period j .

n = the number of periods during the planning horizon

T_{ijk} = cumulative cost (acquisition cost + carrying cost + backorder cost) during period j for the units needed during period j , but purchased during period i for k periods of backorder

M_{ik} = the minimum prior period cost for row i of k periods of backorder section

F_{ijk} = decision cost (cumulative cost + the minimum prior period cost) during period j for the units needed during period j , but purchased during period i for k periods of backorder

L_i = lot size during period i

4.2.3 General Formulation

Using the notation given in Section (4.2.2) we have

$$C_{ij} = \sum_{m=i}^{j-1} H_m \quad \text{for } j > i \quad (4.1)$$

$$C_{ij} = 0 \quad \text{for } j \leq i$$

$$B_{ij} = \sum_{m=j}^{i-1} b_m \quad \text{for } j < i \quad (4.2)$$

$$B_{ij} = 0 \quad \text{for } j \geq i$$

For $k = 0$ i.e. no backorders allowed

$$T_{ij0} = A_i + \sum_{m=i}^j (C_{im} * D_m) \quad (4.3)$$

For $k > 0$ i.e. backorders allowed, and $i \leq j$

$$T_{ijk} = A_i + \sum_{m=i}^j (C_{im} * D_m) + \sum_{m=i-k}^i (B_{im} * D_m) \quad (4.4)$$

For $k > 0$ i.e. backorders allowed; $i > j$ and $i - j \leq k$

$$T_{ijk} = \sum_{m=i-k}^i (B_{im} * D_m) \quad (4.5)$$

For $k = 0$ i.e. no backorders allowed

$$M_{i0} = \text{minimum of } F_{m(i-1)k} \quad (4.6)$$

where $m = 2, 3, \dots, i$; $k = 1, 2, \dots, (i-1)$.

For $k > 0$ i.e. backorders allowed

$$M_{ik} = \text{minimum of } F_{m(i-1)0} \quad (4.7)$$

where $m = 1, 2, \dots, (i-1)$.

$$F_{ij0} = T_{ij0} + M_{i0} \quad (4.8)$$

$$F_{ijk} = T_{ijk} + M_{(i-k)k} \quad (4.9)$$

4.3 Lot Size with Fuzzy Information (Data)

4.3.1 Assumptions

1. The demand varies from one time period to another and is assumed to be known triangular fuzzy numbers.
2. The units needed to satisfy demand during a particular period can be acquired at any time including backorders.
3. Acquisition cost (setup cost of production run or ordering or follow-up cost for purchased parts) is available in the form of a triangular fuzzy number, and unchanged relative to the quantity acquired.
4. Holding cost in a particular time period, assumed to be available in the form of a triangular fuzzy number, represents the cost of inventorying one unit of product from that period until the next period. Units carried forward more than one time period would be charged the accumulated holding cost of all periods from acquisition period through the period prior to the actual period of demand or use.
5. Holding cost may vary from one time period to another.
6. Backorder cost, during a particular time period, is available in the form of a triangular fuzzy number, and represents the cost of backordering one unit of product from the period in which it was needed to the next time period. Units backordered more than one time period would be charged the accumulated backordering cost of all the periods from the time period of demand till the time period of acquisition.
7. Backorder cost may vary from one time period to another.

8. The replenishment lead-time is known with certainty so that delivery can be timed to occur accordingly.
9. The product is treated entirely independently of the other products i.e., benefits from joint replenishment do not exist or are ignored.

4.3.2 Computational Technique

Along the lines of Francis and Webster (1989), we now give a simple and straightforward computational technique to the discrete lot-size problem with backorders allowed. It will be explained through the progressive development of three tables and four steps, in which costs associated with the demand of a particular time period (columns) are identified as a function of that period during which the units are acquired. The first tableau contains the cumulative carrying costs per unit in the upper right triangular portion and cumulative backordering costs per unit in the lower left triangular portion, obtained as explained in Step 1 below. The second tableau shows the cumulative costs including the backordering costs, obtained as explained in Step 2 below. There are $(n+1)$ sections of this table; Section 1 shows the cumulative costs for no BO; Section 2 shows the cumulative costs for one period of BO; Section q contains the cumulative costs for $(q-1)$ periods of BO. The optimal tableau is obtained from the second table, as explained in Step 3, and also has $(n+1)$ sections. In Step 4, we identify the optimal solution from the optimal table.

The four steps of our algorithm are as follows:

STEP 1. Obtaining First Tableau

Obtain the cumulative carrying cost C_{ij} and the cumulative back ordering cost B_{ij} associated with each time period by using (4.1) and (4.2) respectively and prepare the following table in the form of a triangular matrix having n rows and n columns.

Initial Tableau

| Time → ↓ | | 1 | 2 | 3 | ... | m | ... | n-1 | n |
|-----------------------|--|------------|------------|------------|------------|------------|------------|-------------|------------|
| Row 1 | | 0 | C_{12} | C_{13} | ... | C_{1m} | ... | $C_{1,n-1}$ | C_{1n} |
| Row 2 | | B_{21} | 0 | C_{23} | ... | C_{2m} | ... | $C_{2,n-1}$ | C_{2n} |
| Row 3 | | B_{31} | B_{32} | 0 | ... | C_{3m} | ... | $C_{3,n-1}$ | C_{3n} |
| | | ... | ... | ... | ... | ... | ... | ... | ... |
| Row m | | B_{m1} | B_{m2} | B_{m3} | ... | 0 | ... | $C_{m,n-1}$ | C_{mn} |
| | | ... | ... | ... | ... | ... | ... | ... | ... |
| Row n | | B_{n1} | B_{n2} | B_{n3} | ... | B_{nm} | ... | $B_{n,n-1}$ | 0 |

STEP 2. Obtaining Second Tableau**1. Obtaining Section 1**

Obtain the cumulative cost T_{ij0} associated with each time period by using (4.3) and prepare Section 1 in the form of a truncated upper triangular matrix having n rows and n columns.

2. Obtaining Section 2

Obtain the cumulative cost T_{ij1} associated with each time period by using (4.4) and (4.5) prepare Section 2 in the form of a truncated upper triangular matrix having n rows and n columns.

3. Obtaining Section 3

Obtain the cumulative cost T_{ij2} associated with each time period by using (4.4) and (4.5) prepare Section 3 in the form of a truncated upper triangular matrix having $(n-1)$ rows and n columns.

In general therefore, we compute Section q , where $q = 2, 3, \dots, n+1$; of the second tableau as follows:

It shows the cumulative costs for $(q - 1)$ periods of back order. To obtain the cumulative cost $T_{ij(q-1)}$ associated with each time period we use (4.4) and (4.5), and prepare the section in the form of a truncated upper triangular matrix having $(n - q + 2)$ rows and n columns.

STEP 3. Obtaining Optimal Tableau

1. Obtaining Row 1 of each section

Obtain Row 1 of each section of the optimal tableau by duplicating Row 1 of the corresponding section of the second tableau.

2. Obtaining Row 2 of each section

To obtain Row 2 of Section 1

- I. Examine Column 1 of Section 2, Section 3, \dots , Section $n+1$ of the optimal tableau and in the column identify the element with smallest value s_1 .
- II. Obtain Row 2 of the first section of the optimal tableau by adding s_1 to each element of Row 2 of Section 1 of the second tableau.

To obtain Row 2 of Section 2

- III. Examine Column 1 of Section 1 of the optimal tableau and in the column identify the element with smallest value v_1 .
- IV. Obtain Row 2 of Section 2 of the optimal tableau by adding v_1 to each element of Row 2 of Section 2 of the second tableau.

To obtain Row 2 of Section 3

V. Examine Column 1 of Section 1 of the optimal tableau and in the column identify the element with smallest value v_1 .

VI. Obtain Row 2 of Section 3 of the optimal tableau by adding v_1 to each element of Row 2 of the third section of the second tableau.

In general therefore, we compute Row 2 of Section q , where $q = 2, 3, \dots, n+1$; of the optimal tableau as follows:

I. Examine Column 1 of Section 1 of the optimal tableau and in the column identify the element with smallest value v_1 .

II. Obtain Row 2 of Section q of the optimal tableau by adding v_1 to each element of Row 2 of Section q of the second tableau.

In general therefore, we compute Row p of Section q , where $p = 1, 2, \dots, n$; $q = 1, 2, \dots, n+1$; of the optimal tableau as follows:

To obtain Row p of Section 1

I. Examine Column $(p-1)$ of Section 2, Section 3, \dots , Section $(n+1)$ of the optimal tableau and in the column identify the element with smallest value $s_{(p-1)}$.

II. Obtain Row p of Section 1 of the optimal tableau by adding $s_{(p-1)}$ to each element of Row p of Section 1 of the second tableau.

To obtain Row p of Section q where $q = 2, 3, \dots, n+1$

III. Examine Column $(p-1)$ of Section 1 of the optimal tableau and in the column identify the element with smallest value $v_{(p-1)}$.

IV. Obtain Row p of Section q of the optimal tableau by adding $v_{(p-1)}$ to each element of Row p of Section q of the second tableau.

STEP 4. Identifying the optimal solution

- (i) Examine the column n of each section of the optimal tableau and identify the minimum value element s_n^* in it.
- (ii) Suppose $s_n^* = F_{mnk}$. Then Period m is the acquisition period for demand Period n .

If $m \geq n$, then F_{mnk} is obtained by acquiring Period n , Period $(n-1)$, . . ., Period $(n-k+1)$'s demand during Period m , then

- a) Ignore all rows that have acquisition Period n , Period $(n-1)$, . . ., Period $(n-k+1)$ of each section.
- b) Examine the column $(n-k)$ of each section of the optimal tableau and identify the minimum value element $s_{(n-k)}^*$ in it.
- c) Suppose $s_{(n-k)}^* = F_{h(n-k)g}$, then Period h is an acquisition period for units required for Period $(n-k)$.
- d) Ignore all rows that have acquisition Period $(n-k)$ of each section.
- e) Examine the column h of each section of the optimal tableau and identify the minimum value element s_h^* in it.
- f) Suppose $s_h^* = F_{hhg}$, then Period h is an acquisition period for units required for Period h .

If $m < n$, then F_{mnk} is obtained by acquiring Period n 's demand during Period m , then

- a) Ignore all rows that have acquisition Period n of each section.

- b) Examine the column m of each section of the optimal tableau and identify the minimum value element s_m^* in it.
- c) Suppose $s_m^* = F_{mng}$, then Period m is an acquisition period for units required for Period m .

Stopping Rule

We continue in the optimal tableau in the same manner until we reach at the first period of the planning horizon. At this stage we examine Column 1, and observe that the optimal overall policy has been reached and the process terminates.

4.3.3 Numerical Example Under Fuzzy Environment

We illustrate our method through the following numerical example (represented in Table 4.1 and Table 4.2). Demand, acquisition cost, and carrying cost are represented by triangular fuzzy numbers. We assume that acquisition cost remain constant throughout the planning horizon.

Table 4.1. Demand, Acquisition cost, Carrying and BO cost (Week 1 to Week 3)

| Period | 1 | 2 | 3 |
|-------------------|-----------------|------------------|------------------|
| Demand | (17, 20, 25) | (45, 50, 52) | (9, 10, 12) |
| Acquisition Cost | (98, 100, 105) | (98, 100, 105) | (98, 100, 105) |
| Carrying Cost | (0.85, 1, 1.2) | (0.9, 1, 1.15) | (0.9, 1, 1.2) |
| Backordering Cost | (0.2, 0.5, 0.7) | (0.25, 0.5, 0.6) | (0.35, 0.5, 0.7) |

Table 4.2. Demand, Acquisition cost , Carrying and BO cost (Week 4 to Week 6)

| Period | 4 | 5 | 6 |
|-------------------|------------------|-------------------|-------------------|
| Demand | (8, 10, 12) | (48, 50, 53) | (18, 20, 25) |
| Acquisition Cost | (98, 100, 105) | (98, 100, 105) | (98, 100, 105) |
| Carrying Cost | (.95, 1, 1.1) | (.8, 1, 1.25) | (.85, 1, 1.3) |
| Backordering Cost | (0.3, 0.5, 0.65) | (0.42, 0.5, 0.55) | (0.25, 0.5, 0.65) |

4.3.4 Intervals of Confidence

4.3.4.1 Intervals of Confidence of Demand

$$d_1^\alpha = [3\alpha + 17, -5\alpha + 25] \quad d_2^\alpha = [5\alpha + 45, -2\alpha + 52]$$

$$d_3^\alpha = [1\alpha + 9, -2\alpha + 12] \quad d_4^\alpha = [2\alpha + 8, -2\alpha + 12]$$

$$d_5^\alpha = [2\alpha + 48, -3\alpha + 53] \quad d_6^\alpha = [2\alpha + 18, -5\alpha + 25]$$

4.3.4.2 Intervals of Confidence of Acquisition Cost

$$A_1^\alpha = [2\alpha + 98, -5\alpha + 105] \quad A_2^\alpha = [2\alpha + 98, -5\alpha + 105]$$

$$A_3^\alpha = [2\alpha + 98, -5\alpha + 105] \quad A_4^\alpha = [2\alpha + 98, -5\alpha + 105]$$

$$A_5^\alpha = [2\alpha + 98, -5\alpha + 105] \quad A_6^\alpha = [2\alpha + 98, -5\alpha + 105]$$

4.3.4.3 Intervals of Confidence of Carrying Cost

$$H_1^\alpha = [0.15\alpha + 0.85, -0.2\alpha + 1.2] \quad H_2^\alpha = [0.10\alpha + 0.90, -0.15\alpha + 1.15]$$

$$H_3^\alpha = [0.10\alpha + 0.90, -0.3\alpha + 1.2] \quad H_4^\alpha = [0.05\alpha + 0.95, -0.10\alpha + 1.10]$$

$$H_5^\alpha = [0.20\alpha + 0.80, -0.25\alpha + 1.25] \quad H_6^\alpha = [0.15\alpha + 0.85, -0.30\alpha + 1.30]$$

4.3.4.4 Intervals of Confidence of Backordering Cost

$$b_1^\alpha = [0.3\alpha + 0.2, -0.2\alpha + 0.7] \quad b_2^\alpha = [0.25\alpha + 0.25, -0.1\alpha + 0.6]$$

$$b_3^\alpha = [0.15\alpha + 0.35, -0.2\alpha + 0.7] \quad b_4^\alpha = [0.02\alpha + 0.3, -0.15\alpha + 0.65]$$

$$b_5^\alpha = [0.08\alpha + 0.42, -0.05\alpha + 0.55] \quad b_6^\alpha = [0.25\alpha + 0.25, -0.15\alpha + 0.65]$$

4.3.5 Calculating the Cumulative Carrying Cost per unit and the Cumulative Backordering Cost per unit during Period j for the units purchased during Period i

Obtaining the First Tableau

Using (4.1), we have

$$C_{11}^\alpha = 0$$

$$C_{12}^\alpha = \sum_{k=1}^1 H_k^\alpha = H_1^\alpha = [0.15\alpha + 0.85, -0.2\alpha + 1.2]$$

$$C_{13}^\alpha = \sum_{k=1}^2 H_k^\alpha = H_1^\alpha + H_2^\alpha$$

$$C_{13}^\alpha = [0.15\alpha + 0.85, -0.2\alpha + 1.2] + [0.10\alpha + 0.90, -0.15\alpha + 1.15]$$

$$C_{13}^\alpha = [0.25\alpha + 1.75, -0.35\alpha + 2.35]$$

We now set

$$0.25\alpha + 1.75 = x \quad \text{and} \quad -0.35\alpha + 2.35 = x$$

This yields,

$$0.25\alpha + 1.75 - x = 0 \tag{4.10}$$

$$\text{and} \quad -0.35\alpha + 2.35 - x = 0 \tag{4.11}$$

In (4.10) setting $\alpha = 0$ we get $x = 1.75$

In (4.11) setting $\alpha = 0$ we get $x = 2.35$

Setting $\alpha = 1$ in either we get $x = 2$

Therefore $C_{13} = (1.75, 2, 2.35)$

Similarly, we calculate C_{14} , C_{15} and C_{16}

$$C_{14} = (2.65, 3, 3.55); C_{15} = (3.6, 4, 4.65); C_{16} = (4.4, 5, 5.9)$$

Membership function for $C_{13} = (1.75, 2, 2.35)$ is

$$\mu_{C_{13}}(x) = \begin{cases} 0 & x \leq 1.75 \\ \frac{x - 1.75}{0.25} & 1.75 \leq x \leq 2 \\ \frac{x - 2.35}{-0.35} & 2 \leq x \leq 2.35 \\ 0 & x \geq 2.35 \end{cases}$$

Again, we obtain

$$B_{31} = (0.45, 1, 1.30)$$

and its membership function $\mu_{B_{31}}(x)$ as

$$\mu_{B_{31}}(x) = \begin{cases} 0 & x \leq 0.45 \\ \frac{x - 0.45}{0.55} & 0.45 \leq x \leq 1 \\ \frac{x - 1.30}{-0.30} & 1 \leq x \leq 1.30 \\ 0 & x \geq 1.30 \end{cases}$$

We obtain in a similar manner the fuzzy carrying costs for rest of C_{ij} 's, and fuzzy backordering costs for rest of B_{ij} 's. We thus obtain Table 4.3.

Table 4.3. Cumulative Carrying Costs and Cumulative Backordering Costs during Period j
Demand Period

| Acquisition Period | 1 | 2 | 3 | 4 | 5 | 6 |
|--------------------|------------------|-------------------|------------------|-------------------|-------------------|-------------------|
| 1 | (0, 0, 0) | (0.85, 1, 1.2) | (1.75, 2, 2.35) | (2.65, 3, 3.55) | (3.6, 4, 4.65) | (4.4, 5, 5.9) |
| 2 | (0.2, 0.5, 0.7) | (0, 0, 0) | (0.9, 1, 1.15) | (1.8, 2, 2.35) | (2.75, 3, 3.45) | (3.55, 4, 4.7) |
| 3 | (0.45, 1, 1.3) | (0.25, 0.5, 0.6) | (0, 0, 0) | (0.9, 1, 1.2) | (1.85, 2, 2.3) | (2.65, 3, 3.55) |
| 4 | (0.8, 1.5, 2) | (0.6, 1, 1.3) | (0.35, 0.5, 0.7) | (0, 0, 0) | (0.95, 1, 1.1) | (1.75, 2, 2.35) |
| 5 | (1.1, 2, 2.65) | (0.9, 1.5, 1.95) | (0.65, 1, 1.35) | (0.3, 0.5, 0.65) | (0, 0, 0) | (0.8, 1, 1.25) |
| 6 | (1.52, 2.5, 3.2) | (1.32, 2, 2.5) | (1.07, 1.5, 1.9) | (0.72, 1, 1.2) | (0.42, 0.5, 0.55) | (0, 0, 0) |
| 7 | (1.77, 3, 3.85) | (1.57, 2.5, 3.15) | (1.32, 2, 2.55) | (0.97, 1.5, 1.85) | (0.67, 1, 1.2) | (0.25, 0.5, 0.65) |

4.3.6 Calculating the Cumulative cost including Back order cost during Period j for the units purchased during Period i for k periods of back orders (T_{ijk})

Obtaining the Second Tableau

1. No Back orders allowed.

Using (4.3), we have

$$T_{110}^{\alpha} = A_1^{\alpha} \quad (\because C_{11} = 0)$$

$$T_{110}^{\alpha} = [2\alpha + 98, -5\alpha + 105] \quad (4.12)$$

$$T_{120} = A_1 + \sum_{k=1}^2 (C_{1k} * D_k)$$

$$T_{120}^{\alpha} = A_1^{\alpha} + (C_{11}^{\alpha} * D_1^{\alpha}) + (C_{12}^{\alpha} * D_2^{\alpha})$$

$$T_{120}^{\alpha} = [2\alpha + 98, -5\alpha + 105] + 0 +$$

$$\{[0.15\alpha + 0.85, -0.2\alpha + 1.2] * [5\alpha + 45, -2\alpha + 52]\}$$

$$T_{120}^{\alpha} = [.75\alpha^2 + 13\alpha + 136.25, .40\alpha^2 - 17.8\alpha + 167.4]$$

We now set

$$.75\alpha^2 + 13\alpha + 136.25 = x \text{ and } .40\alpha^2 - 17.8\alpha + 167.4 = x$$

This yields,

$$.75\alpha^2 + 13\alpha + 136.25 - x = 0 \quad (4.13)$$

$$\text{and } .40\alpha^2 - 17.8\alpha + 167.4 - x = 0 \quad (4.14)$$

In (4.13) setting $\alpha = 0$ we get $x = 136.25$

In (4.14) setting $\alpha = 0$ we get $x = 167.4$

Setting $\alpha = 1$ in either we get $x = 150$

Therefore $T_{120} = (136.25, 150, 167.40)$

Now, we determine membership function for T_{120}

Solving quadratic equation (4.13) $.75\alpha^2 + 13\alpha + 136.25 - x = 0$ for α we obtain

$$\alpha = \frac{-13 + \sqrt{169 - 3(136.25 - x)}}{1.5} \quad \text{for } 136.25 \leq x \leq 150$$

And solving quadratic equation (4.14) $.40\alpha^2 - 17.8\alpha + 167.4 - x = 0$ for α we obtain

$$\alpha = \frac{17.8 - \sqrt{316.84 - 1.6(167.4 - x)}}{.80} \quad \text{for } 150 \leq x \leq 167.40$$

Thus, the membership function for $T_{120} = (136.25, 150, 167.40)$ is

$$\mu_{T_{120}}(x) = \begin{cases} 0 & x \leq 136.25 \\ \frac{-13 + \sqrt{169 - 3(136.25 - x)}}{1.5} & 136.25 \leq x \leq 150 \\ \frac{17.8 - \sqrt{316.84 - 1.6(167.4 - x)}}{0.80} & 150 \leq x \leq 167.4 \\ 0 & x \geq 167.40 \end{cases}$$

Similarly, we obtain the fuzzy cumulative costs for rest of T_{ij0} 's.

In view of T_{120} , it is important to point out here that in Table 4.6, T_{ij0} 's are parabolic fuzzy numbers. However, we may approximate a parabolic fuzzy number by a T.F.N. if by doing so the % age error obtained is less than 3%.

Table 4.4. Error Analysis for T_{120}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|---------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 136.25 | 167.4 | 136.25 | 167.4 | 0 | 0 | 0 | 0 |
| 0.1 | 137.5575 | 165.624 | 137.625 | 165.66 | 0.0675 | 0.036 | 0.04907 | 0.021736 |
| 0.2 | 138.88 | 163.856 | 139 | 163.92 | 0.12 | 0.064 | 0.086406 | 0.039059 |
| 0.3 | 140.2175 | 162.096 | 140.375 | 162.18 | 0.1575 | 0.084 | 0.112325 | 0.051821 |
| 0.4 | 141.57 | 160.344 | 141.75 | 160.44 | 0.18 | 0.096 | 0.127146 | 0.059871 |
| 0.5 | 142.9375 | 158.6 | 143.125 | 158.7 | 0.1875 | 0.1 | 0.131176 | 0.063052 |
| 0.6 | 144.32 | 156.864 | 144.5 | 156.96 | 0.18 | 0.096 | 0.124723 | 0.0612 |
| 0.7 | 145.7175 | 155.136 | 145.875 | 155.22 | 0.1575 | 0.084 | 0.108086 | 0.054146 |
| 0.8 | 147.13 | 153.416 | 147.25 | 153.48 | 0.12 | 0.064 | 0.081561 | 0.041717 |
| 0.9 | 148.5575 | 151.704 | 148.625 | 151.74 | 0.0675 | 0.036 | 0.045437 | 0.02373 |
| 1 | 150 | 150 | 150 | 150 | 0 | 0 | 0 | 0 |

As we see from Table 4.4, the % error is less than 3%, therefore, we may assume that T_{120} is a T.F.N.. The error analyses for all T_{ij0} 's is shown in Appendix 2. Thus, from Table 4.4 and Appendix 2 we observe all T_{ij0} 's can be approximated by triangular fuzzy numbers.

2. One period of back orders.

Now, for $k = 1, i > j$, using (4.5), we have

$$T_{211}^{\alpha} = \sum_{m=1}^1 B_{2m}^{\alpha} * D_m^{\alpha}$$

Now, for $m = 1$,

$$\begin{aligned} T_{211}^{\alpha} &= (B_{21}^{\alpha} * D_1^{\alpha}) \\ &= [0.3\alpha + 0.2, -0.2\alpha + 0.7] * [3\alpha + 17, -5\alpha + 25] \\ &= [0.9\alpha^2 + 5.7\alpha + 3.4, 1\alpha^2 - 8.5\alpha + 17.5] . \end{aligned}$$

Setting

$$0.9\alpha^2 + 5.7\alpha + 3.4 = x \text{ and } 1\alpha^2 - 8.5\alpha + 17.5 = x$$

yields,

$$0.9\alpha^2 + 5.7\alpha + 3.4 - x = 0 \quad (4.15)$$

$$\text{and } 1\alpha^2 - 8.5\alpha + 17.5 - x = 0 \quad (4.16)$$

In (4.15) setting $\alpha = 0$ we get $x = 3.4$

In (4.16) setting $\alpha = 0$ we get $x = 17.5$

Setting $\alpha = 1$ in either we get $x = 10$

Therefore, the parabolic fuzzy number

$$T_{211} = (3.4, 10, 17.5)$$

with membership function $\mu_{T_{211}}$

$$\mu_{T_{211}}(x) = \begin{cases} 0 & x \leq 3.4 \\ \frac{-5.7 + \sqrt{32.49 - 3.6(3.4 - x)}}{1.8} & 3.4 \leq x \leq 10 \\ \frac{8.5 - \sqrt{72.25 - 2(17.55 - x)}}{2} & 10 \leq x \leq 17.5 \\ 0 & x \geq 17.55 \end{cases}$$

Now, for $k = 1, i < j$, using (4.4), we have

$$T_{231}^{\alpha} = A_2 + \sum_{m=2}^3 (C_{2m} * D_m) + \sum_{m=1}^2 (B_{2m} * D_m)$$

$$T_{231}^{\alpha} = [2\alpha + 98, -5\alpha + 105] + [0.1\alpha + 0.9, -.15\alpha + 1.15] * [1\alpha + 9, -2\alpha + 12] \\ + [0.75\alpha^2 + 5\alpha + 4.25, .50\alpha^2 - 5.5\alpha + 15] \\ [0.9\alpha^2 + 5.7\alpha + 3.4, 1\alpha^2 - 8.5\alpha + 17.5]$$

$$T_{231}^{\alpha} = [1\alpha^2 + 9.5\alpha + 109.50, .1.3\alpha^2 - 17.6\alpha + 136.3]$$

Setting $1\alpha^2 + 9.5\alpha + 109.50 = x$ and $1.3\alpha^2 - 17.6\alpha + 136.3 = x$

yields, $1\alpha^2 + 9.5\alpha + 109.50 - x = 0$ (4.17)

and $1.3\alpha^2 - 17.6\alpha + 136.3 - x = 0$ (4.18)

In (4.17) setting $\alpha = 0$ we get $x = 109.50$

In (4.18) setting $\alpha = 0$ we get $x = 136.3$

Setting $\alpha = 1$ in either we get $x = 120$

Therefore, $T_{231} = (109.5, 120, 136.3)$

Similarly, we obtain the fuzzy cumulative costs for rest of T_{ijk} 's for $k=1, 2, \dots, n$

In view of T_{231} , it is important to point out here that in Table 4.6, T_{ijk} 's are parabolic fuzzy numbers. However, we may approximate a parabolic fuzzy number by a T.F.N. if by doing so the % age error obtained is less than 3%.

Table 4.5. Error Analysis for T_{231}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 109.50 | 136.30 | 109.5 | 136.3 | 0 | 0 | 0 | 0 |
| 0.1 | 110.46 | 134.55 | 110.55 | 134.67 | 0.09 | 0.117 | 0.081477 | 0.086955 |
| 0.2 | 111.44 | 132.83 | 111.6 | 133.04 | 0.16 | 0.208 | 0.143575 | 0.156589 |
| 0.3 | 112.44 | 131.14 | 112.65 | 131.41 | 0.21 | 0.273 | 0.186766 | 0.208179 |
| 0.4 | 113.46 | 129.47 | 113.7 | 129.78 | 0.24 | 0.312 | 0.211528 | 0.240986 |
| 0.5 | 114.50 | 127.83 | 114.75 | 128.15 | 0.25 | 0.325 | 0.218341 | 0.254254 |
| 0.6 | 115.56 | 126.21 | 115.8 | 126.52 | 0.24 | 0.312 | 0.207684 | 0.247211 |
| 0.7 | 116.64 | 124.62 | 116.85 | 124.89 | 0.21 | 0.273 | 0.180041 | 0.219071 |
| 0.8 | 117.74 | 123.05 | 117.9 | 123.26 | 0.16 | 0.208 | 0.135893 | 0.169034 |
| 0.9 | 118.86 | 121.51 | 118.95 | 121.63 | 0.09 | 0.117 | 0.075719 | 0.096286 |
| 1 | 120.00 | 120.00 | 120 | 120 | 0 | 0 | 0 | 0 |

As we see from Table 4.5, the % error is less than 3%, therefore, we may assume that T_{231} is a T.F.N.. The error analyses for all T_{ijk} 's is shown in Appendix 2. Thus, from Table 4.5 and Appendix 2 we observe all T_{ijk} 's can be approximated by triangular fuzzy numbers, therefore, in the analysis conducted in Section 4.3.6, use them as T. F. N.'s.

Values of all T_{ijk} 's are shown in Table 4.6.

Table 4.6. Cumulative Costs including Back-order Cost (Second Tableau)
Demand Period

| Acquisition Period | | 1 | 2 | 3 | 4 | 5 | 6 |
|-------------------------------------|-------|-------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| Section 1: No back orders | | | | | | | |
| 1 | Row 1 | (98, 100, 105) | (136.25, 150, 167.40) | (152, 170, 195.60) | (173.20, 200, 238.20) | (346, 400, 484.65) | (425.20, 500, 632.15) |
| 2 | Row 2 | | (101.4, 110, 122.5) | (109.5, 120, 136.3) | (123.90, 140, 164.5) | (255.9, 290, 347.35) | (319.80, 370, 464.85) |
| 3 | Row 3 | | | (116.9, 145, 168.7) | (124.1, 155, 183.1) | (212.9, 255, 305) | (206.6, 315, 393.8) |
| 4 | Row 4 | | | | (141.75, 185, 231) | (187.35, 235, 289.3) | (218.85, 275, 348.05) |
| 5 | Row 5 | | | | | (207.65, 225, 251.5) | (222.05, 245, 282.75) |
| 6 | Row 6 | | | | | | (233.42, 255, 284.85) |
| Section 2: One period of back order | | | | | | | |
| 2 | Row 1 | (3.40, 10, 17.50) | (101.40, 110, 122.50) | (109.5, 120, 136.30) | (123.90, 140, 164.5) | (255.9, 290, 347.35) | (319.8, 370, 464.85) |
| 3 | Row 2 | | (11.25, 25.00, 31.20) | (109.25, 125, 136.20) | (116.45, 135, 150.6) | (205.25, 235, 272.50) | (252.95, 295, 361.25) |
| 4 | Row 3 | | | (3.15, 5.00, 8.40) | (101.15, 105, 113.40) | (146.75, 155, 171.70) | (178.25, 195, 230.45) |
| 5 | Row 4 | | | | (2.40, 5.00, 7.80) | (100.40, 105, 112.80) | (114.80, 125, 144.05) |
| 6 | Row 5 | | | | | (20.16, 25, 29.15) | (118.16, 125, 134.15) |
| 7 | Row 6 | | | | | | (4.50, 10, 16.25) |

(Continued on next page)

Table 4.6. Cumulative Costs including Back-order Cost (Second Tableau)
Demand Period

| Acquisition Period | | 1 | 2 | 3 | 4 | 5 | 6 |
|--|-------|---|-----------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| Section 3: Two periods of back order | | | | | | | |
| 3 | Row 1 | | (18.90, 45.00, 63.70) | (116.90, 145.00, 168.70) | (124.10, 155.00, 183.10) | (212.90, 255.00, 305.00) | (260.60, 315.00, 393.75) |
| 4 | Row 2 | | | (30.15, 55.00, 76.00) | (128.15, 155.00, 181.00) | (173.75, 205.00, 239.30) | (205.25, 245.00, 298.05) |
| 5 | Row 3 | | | | (8.25, 15.00, 24.00) | (106.25, 115.00, 129.00) | (120.65, 135.00, 160.25) |
| 6 | Row 4 | | | | | (25.92, 35.00, 43.55) | (123.92, 135.00, 148.55) |
| 7 | Row 5 | | | | | | (36.66, 60.00, 79.85) |
| Section 4: Three periods of back order | | | | | | | |
| 4 | Row 1 | | | (43.75, 85.00, 126.00) | (141.75, 185.00, 231.00) | (187.35, 235.00, 289.30) | (218.85, 275.00, 348.05) |
| 5 | Row 2 | | | | (48.75, 90.00, 125.40) | (146.75, 190.00, 230.40) | (161.15, 210.00, 261.65) |
| 6 | Row 3 | | | | | (35.55, 50.00, 66.35) | (133.55, 150.00, 171.35) |
| 7 | Row 4 | | | | | | (44.42, 75.00, 102.05) |
| Section 5: Four periods of back order | | | | | | | |
| 5 | Row 1 | | | | (67.45, 130.00, 191.65) | (165.45, 230.00, 296.65) | (179.85, 250.00, 327.90) |
| 6 | Row 2 | | | | | (94.95, 150.00, 196.35) | (192.95, 250.00, 301.35) |
| 7 | Row 3 | | | | | | (56.30, 95.00, 132.65) |

(Continued on next page)

Table 4.6. Cumulative Costs including Back-order Cost (Second Tableau)
Demand Period

| Acquisition Period | | 1 | 2 | 3 | 4 | 5 | 6 |
|---------------------------------------|-------|---|---|---|---|------------------------|------------------------|
| Section 6: Five periods of back order | | | | | | | |
| 6 | Row 1 | | | | | (120.79,200.00,276.35) | (218.79,300.00,381.35) |
| 7 | Row 2 | | | | | | (126.95, 220, 296.45) |
| Section 7: Six periods of back order | | | | | | | |
| 7 | Row 1 | | | | | | (157.04, 280, 392.7) |

4.3.7 Obtaining the Optimal Tableau

Since, $M_{1k} = 0$, where $k = 1, 2, \dots, 6$,

we start **Table 4.8** by adding zero to each entry in Row 1 of the each section of the **Table 4.6** to form Row 1 of the corresponding section of **Table 4.8**.

For $k = 0$, Calculating the minimum prior period cost for Row 2 of Section 1.

Using (4.6), $M_{20} = \text{minimum of } F_{m1k}$,

where $m = 2, 3, \dots, 7; k = 1, 2, \dots, 6$.

Again, $M_{20} = F_{211} = (3.4, 10, 17.5)$

and $F_{220} = M_{20} + T_{220}$

$= (101.4, 110, 122.5)$

Similarly, we calculate the rest of the F_{ij0} 's.

Calculating the minimum prior period cost for Row 2 for $k > 0$.

Using (4.7), $M_{2k} = F_{110}$ and $F_{ijk} = T_{ijk} + M_{(i-k)k}$

Thus, $F_{321} = (109.25, 125, 136.20)$

Similarly, we calculate the rest of the F_{ij1} 's.

Determining the minimum prior period cost for Row 3 for $k > 0$.

$M_{30} = \text{minimum of } F_{m2k}$ where $m = 2, 3$ and $k = 1, 2$.

$M_{30} = \text{minimum of } [F_{221}, F_{321}, F_{222}, F_{322}]$

To calculate M_{30} , we order these fuzzy entries.

$F_{221} = (101.40, 110, 122.50)$

$F_{321} = (109.25, 125, 136.20)$

$F_{222} = 0$

$F_{322} = (18.90, 45, 63.70)$

$$\text{A.O.N. for } F_{221} = \frac{101.4 + 2*110 + 122.50}{4} = 110.97$$

$$\text{A.O.N. for } F_{321} = \frac{109.25 + 2*125 + 136.20}{4} = 123.86$$

$$\text{A.O.N. for } F_{322} = \frac{18.90 + 2*45 + 63.70}{4} = 43.15$$

Therefore, $M_{30} = F_{322} = (18.90, 45, 63.70)$,

and $F_{330} = (116.90, 145, 168.70)$

Determining the minimum prior period cost for Row 3 for $k > 0$.

$M_{31} = \text{minimum of } F_{m20}; m = 1, 2$

$M_{31} = \text{minimum of } [F_{120}, F_{220}]$

$F_{120} = (136.25, 150, 167.40)$

$F_{220} = (101.4, 110, 122.50)$

$$\text{A.O.N. for } F_{120} = \frac{136.25 + 2*150 + 167.40}{4} = 150.91$$

$$\text{A.O.N. for } F_{220} = \frac{101.4 + 2*110 + 122.50}{4} = 110.98$$

$\therefore F_{220} < F_{120}$

(A.O.N. for each of F_{ijk} 's are shown in Table 4.7.)

Therefore, $M_{31} = F_{220} = (101.4, 110, 122.50)$

and $F_{431} = (104.55, 115, 130.9)$

Similarly, we calculate the rest of the F_{ijk} 's. Values of all F_{ijk} 's are shown in Table 4.8.

Table 4.7. Associated ordinary numbers for F_{ijk} 's

| Demand Period | | | | | | |
|--|--------|----------|----------|----------|----------|----------|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| Section 1: No back orders | | | | | | |
| Row 1 | 100.75 | 150.9125 | 171.9 | 202.85 | 407.6625 | 514.3375 |
| Row 2 | | 110.975 | 121.45 | 142.1 | 295.8125 | 381.1625 |
| Row 3 | | | 143.9 | 154.3 | 256.975 | 321.0875 |
| Row 4 | | | | 185.6875 | 236.6625 | 279.225 |
| Row 5 | | | | | 227.2875 | 248.7 |
| Row 6 | | | | | | 257.0675 |
| Section 2: One Period of back order | | | | | | |
| Row 1 | 10.225 | 110.975 | 121.45 | 142.1 | 295.8125 | 381.1625 |
| Row 2 | | 123.8625 | 224.6125 | 235.0125 | 337.6875 | 401.8 |
| Row 3 | | | 116.3625 | 217.1125 | 268.0875 | 310.65 |
| Row 4 | | | | 126.5 | 227.25 | 248.6625 |
| Row 5 | | | | | 166.9275 | 267.6775 |
| Row 6 | | | | | | 237.475 |
| Section 3: Two Periods of back order | | | | | | |
| Row 1 | | 43.15 | 143.9 | 154.3 | 256.975 | 321.0875 |
| Row 2 | | | 154.7875 | 255.5375 | 306.5125 | 349.075 |
| Row 3 | | | | 126.5375 | 227.2875 | 248.7 |
| Row 4 | | | | | 156.3175 | 257.0675 |
| Row 5 | | | | | | 201.2275 |
| Section 4: Three Periods of back order | | | | | | |
| Row 1 | | | 84.9375 | 185.6875 | 236.6625 | 279.225 |
| Row 2 | | | | 189.2875 | 290.0375 | 311.45 |
| Row 3 | | | | | 161.45 | 262.2 |
| Row 4 | | | | | | 195.5675 |
| Section 5: Four Periods of back order | | | | | | |
| Row 1 | | | | 129.775 | 230.525 | 251.9375 |
| Row 2 | | | | | 248.575 | 349.325 |
| Row 3 | | | | | | 205.7125 |
| Section 6: Five Periods of back order | | | | | | |
| Row 1 | | | | | 199.285 | 300.035 |
| Row 2 | | | | | | 316.6 |
| Section 7: Six Periods of back order | | | | | | |
| Row 1 | | | | | | 277.435 |

Table 4.8. Decision Costs including Back-order Costs (Optimal Tableau)
Demand Period

| Acquisition Period | | 1 | 2 | 3 | 4 | 5 | 6 |
|-------------------------------------|-------|-----------------|-----------------------|-----------------------|-----------------------|-----------------------|--------------------------|
| Section 1: No back orders | | | | | | | |
| 1 | Row 1 | (98, 100, 105) | (136.25, 150, 167.40) | (152, 170, 195.60) | (173.20, 200, 238.20) | (346, 400, 484.65) | (425.20, 500, 632.15) |
| 2 | Row 2 | | (101.4, 110, 122.5) | (109.5, 120, 136.3) | (123.90, 140, 164.5) | (255.9, 290, 347.35) | (319.80, 370, 464.85) |
| 3 | Row 3 | | | (116.9, 145, 168.7) | (124.1, 155, 183.1) | (212.9, 255, 305) | (206.6, 315, 393.8) |
| 4 | Row 4 | | | | (141.75, 185, 231) | (187.35, 235, 289.3) | (218.85, 275, 348.05) |
| 5 | Row 5 | | | | | (207.65, 225, 251.5) | (222.05, 245, 282.75) |
| 6 | Row 6 | | | | | | (233.42, 255, 284.85) |
| Section 2: One period of back order | | | | | | | |
| 2 | Row 1 | (3.4, 10, 17.5) | (101.40, 110, 122.5) | (109.5, 120, 136.30) | (123.9, 140, 164.5) | (255.9, 290, 347.35) | (319.8, 370, 464.85) |
| 3 | Row 2 | | (109.25, 125, 136.20) | (207.25, 225, 241.20) | (214.45, 235, 255.60) | (303.25, 335, 377.50) | (350.95, 395.00, 466.25) |
| 4 | Row 3 | | | (104.55, 115, 130.9) | (202.55, 215, 235.9) | (248.15, 265, 294.2) | (279.65, 305, 352.95) |
| 5 | Row 4 | | | | (111.9, 125, 144.1) | (209.9, 225, 249.1) | (224.3, 245, 280.35) |
| 6 | Row 5 | | | | | (144.06, 165, 193.65) | (242.06, 265, 298.65) |
| 7 | Row 6 | | | | | | (212.15, 235, 267.75) |

(Continued on next page)

Table 4.8. Decision Costs including Back-order Costs (Optimal Tableau)
Demand Period

| Acquisition Period | 1 | 2 | 3 | 4 | 5 | 6 | | |
|--|-------|---|---------------------|------------------------|--------------------------|--------------------------|--------------------------|-----------------------|
| Section 3: Two periods of back order | | | | | | | | |
| 3 | Row 1 | | (18.90,45.00,63.70) | (116.90,145.00,168.70) | (124.10,155.00, 183.10) | (212.90, 255.00, 305.00) | (260.60, 315.00, 393.75) | |
| 4 | Row 2 | | | (128.15,155.00,181.00) | (226.15, 255.00, 286.00) | (271.75, 305.00, 344.30) | (303.25, 345.00, 403.05) | |
| 5 | Row 3 | | | | | (109.65,125.00,146.5) | (207.65, 225, 251.5) | (222.05, 245, 282.75) |
| 6 | Row 4 | | | | | (135.42,155,179.85) | | (233.42, 255, 284.85) |
| 7 | Row 5 | | | | | | | (160.56, 200, 244.35) |
| Section 4: Three periods of back order | | | | | | | | |
| 4 | Row 1 | | | (43.75,85.00,126.00) | (141.75,185.00, 231.00) | (187.35, 235.00, 289.30) | (218.85, 275.00, 348.05) | |
| 5 | Row 2 | | | | (146.75,190.00, 230.40) | (244.75, 290.00, 335.40) | (259.15, 310.00, 366.65) | |
| 6 | Row 3 | | | | | | (136.95, 160, 188.85) | (234.95, 260, 293.85) |
| 7 | Row 4 | | | | | | | |
| Section 5: Four periods of back order | | | | | | | | |
| 5 | Row 1 | | | | (67.45,130.00,191.65) | (165.45, 230.00, 296.65) | (179.85, 250.00, 327.90) | |
| 6 | Row 2 | | | | | (192.95, 250.00, 301.35) | (290.95, 350.00, 406.35) | |
| 7 | Row 3 | | | | | | | (157.7, 205, 255.15) |

(Continued on next page)

Table 4.8. Decision Costs including Back-order Costs (Optimal Tableau)
Demand Period

| Acquisition Period | | 1 | 2 | 3 | 4 | 5 | 6 |
|---------------------------------------|-------|---|---|---|---|--------------------------|--------------------------|
| Section 6: Five periods of back order | | | | | | | |
| 6 | Row 1 | | | | | (120.79, 200.00, 276.35) | (218.79, 300.00, 381.35) |
| 7 | Row 2 | | | | | | (224.95, 320, 401.45) |
| Section 7: Six periods of back order | | | | | | | |
| 7 | Row 1 | | | | | | (157.04, 280, 392.7) |

4.3.8 Calculating the Lot Size during Period I

We start from the last period of the optimal tableau (Table 4.8), Period 6. The minimum decision cost in period 6 is $F_{763} = (153.92, 195, 238.35)$, calculated by ordering the fuzzy entries in Column 6. Because it is obtained by acquiring Period 4, Period 5 and Period 6's demand in Period 7, it implies that the unit required for demand period 4, 5 and 6 are backordered. This rules out Section 7, Section 6, Section 5, Section 4, Row 2 to Row 5 of Section 3 and Row 3 to Row 6 of Section 2, and thus requires that the least cost alternative for Period 3 be selected from Section 1, Row 1 & Row 2 of Section 2, and Row 1 of Section 3. This least cost is $(109.5, 120, 136.30)$. While it occurs in two places i.e. Row 2 of Section 1 (F_{230}) and Row 1 of Section 2 (F_{231}), there is only one distinct strategy. F_{230} & F_{231} , both imply the acquisition of Period 3's demand in Period 2. The least cost for Period 2 is $(101.40, 110, 122.5)$, it also occurs in two places, Row 2 of Section 1 (F_{220}) and Row 1 of Section 2 (F_{221}). As we observe from the notation, the acquisition period in both cases is Period 2. Now we move to Period 1. The minimum decision cost for Period 1 is $F_{211} = (3.4, 10, 17.5)$, indicating that units required in Period 1 were backorder and must also be produced in Period 2. So, Period 2 is an acquisition period for 1, 2, and 3.

Thus, lot size in Period 2 is given by

$$\begin{aligned} L_2 &= D_1 + D_2 + D_3 \\ &= (71, 80, 89) \end{aligned}$$

Table 4.9. Optimal Schedule (with backorder allowed in the last period of the planning horizon)

| Period | Demand | Action |
|--------|--------------|---------------------------|
| 1 | (17, 20, 25) | BO (1period) |
| 2 | (45, 50, 52) | Produce (71, 80, 89) |
| 3 | (9, 10, 12) | From Inventory (1 period) |
| 4 | (8, 10, 12) | BO (3 periods) |
| 5 | (48, 50, 53) | BO (2 periods) |
| 6 | (18, 20, 25) | BO (1 period) |

No Back Orders allowed in Final Period

It may be unacceptable to end a planning horizon with units on backorder. In that event, it is only necessary to ignore those alternatives in Period 6 based on this, i.e., last row of each section from Section 2 to Section 7. The resulting least cost alternative is indicated by the cost of $F_{561} = (224.3, 245, 280.35)$, and is obtained by acquiring Period 6's demand in Period 5. The least cost for Period 4 is $F_{541} = (111.9, 125, 144.1)$, indicating that units required for Period 4 were on backordered and must also be produced in Period 5. So, Period 5 is an acquisition period to satisfy the demand in Period 4, 5 and 6. This also rules out Section 7, Section 6, Section 5, Section 4, Row 2 to Row 5 of Section 3 and Row 3 to Row 6 of Section 2, and thus requires that the least cost alternative for Period 3 be selected from Section 1, Row 1 & Row 2 of Section 2, and Row 1 of Section 3. This least cost is $(109.5, 120, 136.30)$. While it occurs in two places i.e. Row 2 of Section 1 (F_{230}) and Row 1 of Section 2 (F_{231}), there is only one distinct strategy. F_{230} & F_{231} , both implies the

acquisition of Period 3's demand in Period 2. The least cost for Period 2 is (101.40, 110, 122.5), and it also occurs in two places, Row 2 of Section 1 (F_{220}) and Row 1 of Section 2 (F_{221}). As we see from the notation, the acquisition period in both cases is Period 2. Now we move to Period 1. The minimum decision cost for Period 1 is $F_{211} = (3.4, 10, 17.5)$, indicating that units required in Period 1 were backorder and must also be produced in Period 2. So, Period 2 is an acquisition period to satisfy the demand in Period 1, 2 and 3.

Thus, lot size during Period 2, $L_2 = (71, 80, 89)$ (as calculated above)

Lot size during Period 5 is given by

$$\begin{aligned} L_5 &= D_4 + D_5 + D_6 \\ &= (74, 80, 90) \end{aligned}$$

Table 4.10. Optimal Schedule (with no backorder allowed in the last period of planning horizon)

| Period | Demand | Action |
|--------|--------------|---------------------------|
| 1 | (17, 20, 25) | BO (1 period) |
| 2 | (45, 50, 52) | Produce (71, 80, 89) |
| 3 | (9, 10, 12) | From Inventory (1 period) |
| 4 | (8, 10, 12) | BO (1 period) |
| 5 | (48, 50, 53) | Produce (74, 80, 90) |
| 6 | (18, 20, 25) | From Inventory (1 period) |

4.3.9 Interpretation of the Results

Table 4.9 and Table 4.10 show two different optimal strategies. In Table 4.9, we calculate the optimal strategy, by allowing backorder in the last period of planning horizon, to

satisfy the demand during six weeks. This schedule indicate that we acquire the units required for Period 1, Period 2 and Period 3 during Period 3, and backorder Period 4's demand for 3 periods, Period 5's demand for 2 periods and Period 6' demand for 1 period. The total cost for this strategy is (153.92.195, 238.35).

In Table 4.10, we calculate the optimal strategy by not allowing backorder in the last period of planning horizon to satisfy the demand during six weeks. According to this schedule, the units required for Period 1, Period 2 and Period 3 must be acquired during Period 3, and the units required for Period 4, Period 5 and Period 6 must be acquired during Period 5. The total cost for this strategy is (224.3, 245, 280.35). This indicates that there is a 27% increase in cost if we don't allow backorder in the lat period of planning horizon.

CHAPTER 5

DISTRIBUTION INVENTORY MANAGEMENT PROBLEM UNDER FUZZY INFORMATION (DATA)

In the present chapter, we consider a two echelon distribution problem in which there are one central warehouse and multiple branch warehouses. The projected orders from branch warehouses along with direct sales constitute the demand on the central warehouse. We formulate the problem under two different scenarios; one not allowing backorder, and other allowing backorder. We first characterize the solution to no-backordering version of the problem in which the demand, the expected inventory on hand, the expected receipt of an order and expected release of an order of a given product P at a given branch warehouse k during a given time period j are represented by a special type of triangular fuzzy numbers. We extend the crisp distribution requirements planning (DRP) approach to find the demand of a product P on a central warehouse during period j with fuzzy data. We then proceed with the backordering version of the problem in which the demand, the expected inventory on hand, the expected receipt of an order and expected release of an order of a given product P at a given branch warehouse k during a given time period j are represented by triangular fuzzy numbers. We extend the crisp DRP approach but with backorders allowed to find the demand of product P on a central warehouse during period j .

5.1 Introduction

According to Fogarty et. al (1991), the DRP is the time-phased replenishment needs of branch warehouses summed by period. These requirements are based on the difference

between customer demand and the on-hand and in-transit inventory. In a branch warehouse environment the DRP provides a solid link between distribution and manufacturing by providing a record of the quantity and timings of likely orders.

The problem with most distribution systems is the assumption that the inventory will be available in the central facility. So they focus on how best to allocate that inventory to the different distribution centers. Shortages at the central supply facility can be frequent, and these shortages are one of the most serious problems in managing a distribution network. The fundamental problem, therefore, is how to get a system integrated with manufacturing that will not only distribute the product in the best way, but also will make sure that the product is available for distribution. DRP is the way to integrate distribution and manufacturing (Martin 1983). However, both Fogarty et al. (1991) and Martin (1983) deal with only crisp data. In the present chapter, using DRP approach, we deal with the problem of distribution inventory management with fuzzy data.

5.2 DRP with Variable Demand Rate and No Backorders allowed under Fuzzy Information (Data)

We assume that j represents the number of periods and k represents the number of branch warehouses which get their supplies from the Central warehouse, where $j = 1, 2, \dots, m$; and $k = 1, 2, \dots, p$.

5.2.1 Assumptions

For the fuzzy model, we make the following assumptions:

1. The demand at branch warehouses varies from one time period to another and is available in the form of special triangular fuzzy numbers.
2. The units needed to satisfy demand during a particular period must be purchased during a previous period or at the beginning of a particular period during which they are needed.
3. The economic shipping quantity for each warehouse is available in the form of special triangular fuzzy number.
4. The beginning inventory is available in the form of special triangular fuzzy number and in-transit inventory is zero.
5. The lead-time is known with certainty so that delivery can be timed accordingly.
6. Warehouses are not allowed to redistribute their stock by lateral transshipment.

5.2.2 Notation

d_{kj} = Demand of product P at branch warehouse k during period j

I_{kj} = Expected inventory on hand of product P at branch warehouse k during period j

R_{kj} = Expected receipt of an order of product P at branch warehouse k during period j

O_{kj} = Expected release of an order of product P at branch warehouse k during period j

D_{cj} = Total demand of product P on central warehouse during period j

I_{cj} = Expected Inventory on hand of product P at central warehouse during period j

S_j = Direct sales of product P from central warehouse during period j

R_{cj} = Expected receipt of an order of product P at central warehouse during period j

O_{cj} = Expected release of an order of product P at central warehouse during period j

5.2.3 General Formulation

The formulas to compute I_{kj} 's and D_{cj} 's under crisp environment are given below:

$$I_{kj} = I_{kr} + R_{kj} - d_{kj} \quad (5.1)$$

$$\text{where} \quad r = j - 1$$

$$k = 1, 2, \dots, p$$

$$j = 1, 2, \dots, m$$

$$D_{cj} = \sum_{k=1}^p O_{kj} + S_j \quad (5.2)$$

We now assume that each of I_{kr} , R_{kj} and d_{kj} in equation (5.1); O_{kj} and S_j in equation (5.2) for $k = 1, 2, \dots, p$; $j = 1, 2, \dots, m$ is a special triangular fuzzy number as given below.

$$I_{kr} = [I_{kr1}, I_{kr2}, (I_{kr3} + \delta)] \quad \text{where } \delta \rightarrow 0$$

$$R_{kj} = [R_{kj1}, R_{kj2}, (R_{kj3} + \delta)] \quad \text{where } \delta \rightarrow 0$$

$$d_{kj} = [d_{kj1}, d_{kj2}, (d_{kj3} + \delta)] \quad \text{where } \delta \rightarrow 0$$

$$O_{kj} = [O_{kj1}, O_{kj2}, (O_{kj3} + \delta)] \quad \text{where } \delta \rightarrow 0$$

$$S_j = [S_{j1}, S_{j2}, (S_{j3} + \delta)] \quad \text{where } \delta \rightarrow 0$$

The values of I_{kj} , R_{kj} , d_{kj} , O_{kj} and S_j can be obtained from the experts who share the same information but different opinion.

If we set $I_{kr} = [I_{kr1}, I_{kr2}, (I_{kr3} + \delta)]$; $R_{kj} = [R_{kj1}, R_{kj2}, (R_{kj3} + \delta)]$; $d_{kj} = [d_{kj1}, d_{kj2}, (d_{kj3} + \delta)]$ equation (1), then I_{kj} is a special triangular fuzzy number (S.T.F.N.) given by

$$I_{kj} = [I_{kj1}, I_{kj2}, (I_{kj3} + \delta)] \quad \text{where } \delta \rightarrow 0 \quad (5.3)$$

Similarly, if we set $O_{kj} = [O_{kj1}, O_{kj2}, (O_{kj3} + \delta)]$; $S_j = [S_{j1}, S_{j2}, (S_{j3} + \delta)]$ in equation (5.2), then D_j is a special triangular fuzzy number (S.T.F.N.) given by

$$D_{ej} = [D_{ej1}, D_{ej2}, (D_{ej3} + \delta)] \quad \text{where } \delta \rightarrow 0 \quad (5.4)$$

We determine each S.T.F.N., I_{kj} in equation (5.3) and its membership function on the line of Kaufman and Gupta (1985, 1988) by using the interval of confidence for I_{kr} , R_{kj} and d_{kj} respectively, at α - level as follows.

$$I_{kr}^\alpha = \{(I_{kr2} - I_{kr1})\alpha + I_{kr1}, [I_{kr2} - (I_{kr3} + \delta)]\alpha + (I_{kr3} + \delta)\} \quad \forall \alpha \in [0,1] \quad (5.5)$$

$$R_{kj}^\alpha = \{(R_{kj2} - R_{kj1})\alpha + R_{kj1}, [R_{kj2} - (R_{kj3} + \delta)]\alpha + (R_{kj3} + \delta)\} \quad \forall \alpha \in [0,1] \quad (5.6)$$

$$d_{kj}^\alpha = \{(d_{kj2} - d_{kj1})\alpha + d_{kj1}, [d_{kj2} - (d_{kj3} + \delta)]\alpha + (d_{kj3} + \delta)\} \quad \forall \alpha \in [0,1] \quad (5.7)$$

For $r = 0, 1, \dots, m-1$; $k = 1, 2, \dots, p$; $j = 1, 2, \dots, m$, (5.5), (5.6) and (5.7) yield

$$\begin{aligned} I_{kr}^\alpha + R_{kj}^\alpha - d_{kj}^\alpha &= \{[(I_{kr2} - I_{kr1}) + (R_{kj2} - R_{kj1}) - d_{kj2} + (d_{kj3} + \delta)]\alpha + I_{kr1} + R_{kj1} \\ &\quad - (d_{kj3} + \delta), [I_{kr2} - (I_{kr3} + \delta) + R_{kj2} - (R_{kj3} + \delta) - (d_{kj2} - d_{kj1})]\alpha \\ &\quad + (I_{kr3} + \delta) + (R_{kj3} + \delta) - d_{kj1}\} \end{aligned}$$

In interval of confidence $\{[(I_{kr2} - I_{kr1}) + (R_{kj2} - R_{kj1}) - d_{kj2} + (d_{kj3} + \delta)]\alpha + I_{kr1} + R_{kj1} - (d_{kj3} + \delta), [I_{kr2} - (I_{kr3} + \delta) + R_{kj2} - (R_{kj3} + \delta) - (d_{kj2} - d_{kj1})]\alpha + (I_{kr3} + \delta) + (R_{kj3} + \delta) - d_{kj1}\}$

1. Setting $\alpha = 0$, we get the end points I_{kj1} and I_{kj3} of the fuzzy number I_{kj}

$$I_{kj1} = [I_{kr1} + R_{kj1} - (d_{kj3} + \delta)] \quad \text{where } \delta \rightarrow 0$$

$$I_{kj3} = [(I_{kr3} + \delta) + (R_{kj3} + \delta) - d_{kj1}] \quad \text{where } \delta \rightarrow 0$$

2. Setting $\alpha = 1$ gives the interior point I_{kj2} of I_{kj}

$$I_{kj2} = [I_{kr2} + R_{kj2} - d_{kj2}, I_{kr2} + R_{kj2} - d_{kj2}]$$

The corresponding membership functions from the above interval of confidence are obtained by setting

$$[(I_{kr2} - I_{kr1}) + (R_{kj2} - R_{kj1}) - d_{kj2} + (d_{kj3} + \delta)]\alpha + I_{kr1} + R_{kj1} - (d_{kj3} + \delta) = x \quad (5.8)$$

Next, we set

$$[I_{kr2} - (I_{kr3} + \delta) + R_{kj2} - (R_{kj3} + \delta) - (d_{kj2} - d_{kj1})]\alpha + (I_{kr3} + \delta) + (R_{kj3} + \delta) - d_{kj1} = x \quad (5.9)$$

Solving (5.8) and (5.9) give membership functions, respectively, between I_{kj1} and I_{kj2} , and I_{kj2} and I_{kj3} satisfying $0 \leq \alpha \leq 1$.

Similarly, we determine each S.T.F.N., D_{cj} in equation (5.4) and its membership function by using the interval of confidence for O_{kj} and S_j respectively, at α - level

$$D_{cj1} = (O_{kj1} + S_{j1})$$

$$D_{cj2} = (O_{kj2} + S_{j2})$$

$$D_{cj3} = [(O_{kj3} + \delta) + (S_{j3} + \delta)] \quad \text{where } \delta \rightarrow 0$$

5.3.3 Computational Technique

Along the lines of Fogarty et al. (1989), in this section we give steps for a simple and straightforward computational technique to solve the distribution inventory management problem without backorders under fuzzy information. Later, in Section 3.4.3, we modify the technique to incorporate backorders.

Step 1. Obtaining the Expected Order Release Period at Branch Warehouse k

- (i) Starting with the first period, using (5.1) calculate the expected on hand inventory in each period, up to and including the period in which the left end point of the expected on hand inventory is less than zero.

- (ii) Suppose the left end point (I_{kn1}) of the expected on hand inventory (I_{kn}) is less than zero in Period n , schedule an expected receipt in Period n and recalculate the expected on hand inventory I_{kn} by using (5.1).
- (iii) Calculate the expected order release period by subtracting the lead-time from the expected receipt period.
- (iv) Go to Period $(n+1)$ and repeat Step 1 (i) – (iii).

Continue the process to the end of the planning horizon.

Step 2. Obtaining Total Demand on the Central Warehouse

Calculate the total demand on the central warehouse during each period of the planning horizon by using (5.2).

5.3.4 Numerical Example without allowing Backorders with Fuzzy Information (Data)

In this section, to illustrate the method, we consider a numerical example with three branch warehouses WH_1 , WH_2 , WH_3 , one central warehouse CWH , and eight time periods M_1 , M_2 , M_3 , M_4 , M_5 , M_6 , M_7 and M_8 . The projected warehouse orders and direct sales constitute the demand on the CWH . The lead-time, order quantity, beginning inventory on hand and forecast demand for each warehouse are given in terms of the special triangular fuzzy numbers as follows:

Table 5.3.1. Lead-time, order quantity and beginning inventory, where $\delta \rightarrow 0$

| | Lead Time (Periods) | Order quantity | Beginning Inventory |
|-----------------|------------------------|-------------------------------|---------------------|
| WH ₁ | 2 | (475, 500, 500 + δ) | (204, 204, 204) |
| WH ₂ | 3 | (150, 200, 200 + δ) | (100, 100, 100) |
| WH ₃ | 2 | (550, 700, 700 + δ) | (600, 600, 600) |
| CWH | 3 | (900, 1000, 1000 + δ) | (950, 950, 950) |

Table 5.3.2. Demand in product units (Period 1 to Period 3), where $\delta \rightarrow 0$

| | M ₁ | M ₂ | M ₃ |
|-----------------|----------------------------|----------------------------|----------------------------|
| WH ₁ | (50, 80, 80 + δ) | (70, 80, 80 + δ) | (68, 80, 80 + δ) |
| WH ₂ | (15, 30, 30 + δ) | (10, 30, 30 + δ) | (18, 30, 30 + δ) |
| WH ₃ | (70, 120, 120 + δ) | (90, 120, 120 + δ) | (95, 120, 120 + δ) |
| Direct Supply | (35, 50, 50 + δ) | (30, 50, 50 + δ) | (40, 50, 50 + δ) |

Table 5.3.3. Demand in product units (Period 4 to Period 6), where $\delta \rightarrow 0$

| | M ₄ | M ₅ | M ₆ |
|-----------------|-----------------------------|-----------------------------|-----------------------------|
| WH ₁ | (60, 70, 70 + δ) | (62, 80, 80 + δ) | (70, 90, 90 + δ) |
| WH ₂ | (85, 100, 100 + δ) | (75, 100, 100 + δ) | (78, 100, 100 + δ) |
| WH ₃ | (150, 200, 200 + δ) | (100, 150, 150 + δ) | (120, 150, 150 + δ) |
| Direct Supply | (37, 50, 50 + δ) | (40, 50, 50 + δ) | (30, 50, 50 + δ) |

Table 5.3.4. Demand in product units (Period 7 to Period 8), where $\delta \rightarrow 0$

| | M_7 | M_8 |
|-----------------|----------------------------|-----------------------------|
| WH ₁ | (70, 90, 90 + δ) | (60, 90, 90 + δ) |
| WH ₂ | (75, 100, 100 + δ) | (90, 100, 100 + δ) |
| WH ₃ | (75, 100, 100 + δ) | (110, 150, 150 + δ) |
| Direct Supply | (35, 50, 50 + δ) | (25, 50, 50 + δ) |

5.3.5 Intervals of Confidence

Using Tables 5.3.1–5.3.4, below, we compute the intervals of confidence for demand, for direct sales, and for beginning inventory so as to calculate the total demand on the CWH.

5.3.5.1 Intervals of Confidence for Demand

$$d_{11}^{\alpha} = [30\alpha + 50, (0 - \delta)\alpha + (80 + \delta)]$$

$$d_{12}^{\alpha} = [10\alpha + 70, (0 - \delta)\alpha + (80 + \delta)]$$

$$d_{13}^{\alpha} = [12\alpha + 68, (0 - \delta)\alpha + (80 + \delta)]$$

$$d_{14}^{\alpha} = [10\alpha + 60, (0 - \delta)\alpha + (70 + \delta)]$$

$$d_{15}^{\alpha} = [18\alpha + 80, (0 - \delta)\alpha + (80 + \delta)]$$

$$d_{16}^{\alpha} = [20\alpha + 70, (0 - \delta)\alpha + (90 + \delta)]$$

$$d_{17}^{\alpha} = [25\alpha + 65, (0 - \delta)\alpha + (90 + \delta)]$$

$$d_{18}^{\alpha} = [20\alpha + 70, (0 - \delta)\alpha + (90 + \delta)]$$

$$d_{21}^{\alpha} = [15\alpha + 15, (0 - \delta)\alpha + (30 + \delta)]$$

$$d_{22}^{\alpha} = [20\alpha + 10, (0 - \delta)\alpha + (30 + \delta)]$$

$$d_{23}^{\alpha} = [12\alpha + 18, (0-\delta)\alpha + (30+\delta)]$$

$$d_{24}^{\alpha} = [10\alpha + 10, (0-\delta)\alpha + (20+\delta)]$$

$$d_{25}^{\alpha} = [10\alpha + 20, (0-\delta)\alpha + (30+\delta)]$$

$$d_{26}^{\alpha} = [15\alpha + 20, (0-\delta)\alpha + (35+\delta)]$$

$$d_{27}^{\alpha} = [20\alpha + 15, (0-\delta)\alpha + (35+\delta)]$$

$$d_{28}^{\alpha} = [15\alpha + 20, (0-\delta)\alpha + (35+\delta)]$$

$$d_{31}^{\alpha} = [50\alpha + 70, (0-\delta)\alpha + (120+\delta)]$$

$$d_{32}^{\alpha} = [30\alpha + 90, (0-\delta)\alpha + (120+\delta)]$$

$$d_{33}^{\alpha} = [25\alpha + 95, (0-\delta)\alpha + (120+\delta)]$$

$$d_{34}^{\alpha} = [25\alpha + 75, (0-\delta)\alpha + (100+\delta)]$$

$$d_{35}^{\alpha} = [40\alpha + 80, (0-\delta)\alpha + (120+\delta)]$$

$$d_{36}^{\alpha} = [40\alpha + 100, (0-\delta)\alpha + (140+\delta)]$$

$$d_{37}^{\alpha} = [30\alpha + 110, (0-\delta)\alpha + (140+\delta)]$$

$$d_{38}^{\alpha} = [50\alpha + 90, (0-\delta)\alpha + (140+\delta)]$$

5.3.5.2 Intervals of Confidence for Direct Sales

$$S_1^{\alpha} = [15\alpha + 35, (0-\delta)\alpha + (50+\delta)]$$

$$S_2^{\alpha} = [20\alpha + 30, (0-\delta)\alpha + (50+\delta)]$$

$$S_3^{\alpha} = [10\alpha + 40, (0-\delta)\alpha + (50+\delta)]$$

$$S_4^{\alpha} = [13\alpha + 37, (0-\delta)\alpha + (50+\delta)]$$

$$S_5^\alpha = [10\alpha + 40, (0-\delta)\alpha + (50+\delta)]$$

$$S_6^\alpha = [20\alpha + 30, (0-\delta)\alpha + (50+\delta)]$$

$$S_7^\alpha = [15\alpha + 35, (0-\delta)\alpha + (50+\delta)]$$

$$S_8^\alpha = [25\alpha + 25, (0-\delta)\alpha + (50+\delta)]$$

5.3.5.3 Intervals of Confidence for Beginning Inventory

$$I_{10}^\alpha = [0\alpha + 204, 0\alpha + 204] \quad I_{20}^\alpha = [0\alpha + 100, 0\alpha + 100]$$

$$I_{30}^\alpha = [0\alpha + 600, 0\alpha + 600] \quad I_{c0}^\alpha = [0\alpha + 950, 0\alpha + 950]$$

5.3.6 Obtaining the Expected Order Release Period at Warehouse 1

Using equation (5.1), we have

$$I_{11}^\alpha = I_{10}^\alpha - d_{11}^\alpha + R_{11}^\alpha$$

$$I_{11}^\alpha = I_{10}^\alpha + (-d_{11}^\alpha) + R_{11}^\alpha$$

$$I_{11}^\alpha = [0\alpha + 204, 0\alpha + 204] + [(\delta)\alpha - (80+\delta), -30\alpha - 50] + 0$$

$$I_{11}^\alpha = [\delta\alpha + 124 - \delta, -30\alpha + 154]$$

We now set $\delta\alpha + 124 - \delta = x$ and $-30\alpha + 154 = x$

This yields,

$$\delta\alpha + [(124 - \delta) - x] = 0 \tag{5.10}$$

$$\text{and} \quad -30\alpha + (154 - x) = 0 \tag{5.11}$$

In (5.10) and (5.11) setting $\alpha = 0$, we get $x = (124 - \delta)$ and $x = 154$, respectively.

Setting $\alpha = 1$ in either (5.10) or (5.11), we get $x = 124$.

Therefore $I_{11} = (124 - \delta, 124, 154)$

Similarly, we obtain $I_{12} = (44 - 2\delta, 44, 84)$

Continuing in the same manner, we find that I_{131} would be negative unless an order was received. Therefore, an expected receipt R_{13} is planned for Period 3, and

$$R_{13} = (475, 500, 500 + \delta)$$

such that $R_{13}^\alpha = [25\alpha + 475, (0 - \delta)\alpha + (500 + \delta)]$.

When the two-period lead-time is applied, we obtain that the expected order be planned for release in Period 1, and

$$O_{11} = (475, 500, 500 + \delta), \quad \text{and} \quad O_{11}^\alpha = [25\alpha + 475, (0 - \delta)\alpha + (500 + \delta)]$$

To determine membership function for I_{13} , we have

$$I_{13}^\alpha = I_{12}^\alpha + R_{13}^\alpha - d_{13}^\alpha = [(25 + 3\delta)\alpha + (439 - 3\delta), (-52 - \delta)\alpha + (516 + \delta)]$$

Setting $(25 + 3\delta)\alpha + (439 - 3\delta) = x$ and $(-52 - \delta)\alpha + (516 + \delta) = x$,

$$\text{yields,} \quad (25 + 3\delta)\alpha + (439 - 3\delta - x) = 0 \quad (5.12)$$

$$\text{and} \quad (-52 - \delta)\alpha + (516 + \delta - x) = 0 \quad (5.13)$$

In (5.12), (5.13) setting $\alpha = 0$, we get $x = 439 - 3\delta$ and $x = 516 + \delta$ respectively.

Setting $\alpha = 1$ in either we get $x = 464$

Solving $(25 + 3\delta)\alpha + (439 - 3\delta - x) = 0$ for α we obtain

$$\alpha = \frac{x - 439 + 3\delta}{25 + 3\delta} \quad \text{for } 439 - 3\delta \leq x \leq 464$$

Similarly, from (5.13) we obtain

$$\alpha = \frac{x - 516 - \delta}{-52 - \delta} \quad \text{for } 464 \leq x \leq (516 + \delta)$$

Thus, the membership function for $I_{13} = (439 - 3\delta, 464, 516 + \delta)$ is

$$\mu_{I_{13}}(x) = \begin{cases} 0 & x \leq 439 - 3\delta \\ \frac{x - 439 + 3\delta}{25 + 3\delta} & 439 - 3\delta \leq x \leq 464 \\ \frac{x - 516 - \delta}{-52 - \delta} & 464 \leq x \leq 516 + \delta \\ 0 & x \geq 516 + \delta \end{cases}$$

Similarly, we obtain the expected on hand inventories and expected order releases for rest of I_{kj} 's, and O_{ak} 's respectively, and are in Tables 5.3.5–5.3.10.

Table 5.3.5. Distribution Requirements Planning for Warehouse 1 (Period 1 to Period 4)

| | | M_1 | M_2 | M_3 | M_4 |
|----------------------------|-------------------|----------------------------|--------------------------|--------------------------------------|--------------------------------------|
| Forecasted Demand | | $(50, 80, 80 + \delta)$ | $(70, 80, 80 + \delta)$ | $(68, 80, 80 + \delta)$ | $(60, 70, 70 + \delta)$ |
| Expected Order Receipt | | | | $(475, 500, 500 + \delta)$ | |
| Expected Inventory on hand | $(204, 204, 204)$ | $(124 - \delta, 124, 154)$ | $(44 - 2\delta, 44, 84)$ | $(439 - 3\delta, 464, 516 + \delta)$ | $(369 - 4\delta, 394, 456 + \delta)$ |
| Expected Order Release | | $(475, 500, 500 + \delta)$ | | | |

Table 5.3.6. Distribution Requirements Planning for Warehouse 1 (Period 5 to Period 8)

| | | M_5 | M_6 | M_7 | M_8 |
|----------------------------|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|------------------------------------|
| Forecasted Demand | | $(62, 80, 80 + \delta)$ | $(70, 90, 90 + \delta)$ | $(65, 90, 90 + \delta)$ | $(70, 90, 90 + \delta)$ |
| Expected Order Receipt | | | | | |
| Expected Inventory on hand | $(369 - 4\delta, 394, 456 + \delta)$ | $(289 - 5\delta, 314, 394 + \delta)$ | $(199 - 6\delta, 224, 324 + \delta)$ | $(109 - 7\delta, 134, 259 + \delta)$ | $(19 - 8\delta, 44, 189 + \delta)$ |
| Expected Order Release | | | | | |

Table 5.3.7. Distribution Requirements Planning for Warehouse 2 (Period 1 to Period 4)

| | | M_1 | M_2 | M_3 | M_4 |
|----------------------------|-------------------|----------------------------|--------------------------|--------------------------|--------------------------------------|
| Forecasted Demand | | $(15, 30, 30 + \delta)$ | $(10, 30, 30 + \delta)$ | $(18, 30, 30 + \delta)$ | $(10, 20, 20 + \delta)$ |
| Expected Order Receipt | | | | | $(150, 200, 200 + \delta)$ |
| Expected Inventory on hand | $(100, 100, 100)$ | $(70 - \delta, 70, 85)$ | $(40 - 2\delta, 40, 75)$ | $(10 - 3\delta, 10, 57)$ | $(140 - 4\delta, 190, 247 + \delta)$ |
| Expected Order Release | | $(150, 200, 200 + \delta)$ | | | |

Table 5.3.8. Distribution Requirements Planning for Warehouse 2 (Period 5 to Period 8)

| | | M_5 | M_6 | M_7 | M_8 |
|----------------------------|--------------------------------------|--------------------------------------|-------------------------------------|------------------------------------|-----------------------------------|
| Forecasted Demand | | $(20, 30, 30 + \delta)$ | $(20, 35, 35 + \delta)$ | $(15, 35, 35 + \delta)$ | $(20, 35, 35 + \delta)$ |
| Expected Receipt | | | | | |
| Expected Inventory on hand | $(140 - 4\delta, 190, 247 + \delta)$ | $(110 - 5\delta, 160, 227 + \delta)$ | $(75 - 6\delta, 125, 207 + \delta)$ | $(40 - 7\delta, 90, 192 + \delta)$ | $(5 - 8\delta, 55, 172 + \delta)$ |
| Expected Order Release | | | | | |

Table 5.3.9. Distribution Requirements Planning for Warehouse 3 (Period 1 to Period 4)

| | | M_1 | M_2 | M_3 | M_4 |
|----------------------------|-------------------|----------------------------|-----------------------------|-----------------------------|-----------------------------|
| Forecasted Demand | | $(70, 120, 120 + \delta)$ | $(90, 120, 120 + \delta)$ | $(95, 120, 120 + \delta)$ | $(75, 100, 100 + \delta)$ |
| Expected Order Receipt | | | | | |
| Expected Inventory on hand | $(600, 600, 600)$ | $(480 - \delta, 480, 530)$ | $(360 - 2\delta, 360, 440)$ | $(240 - 3\delta, 240, 345)$ | $(140 - 4\delta, 140, 270)$ |
| Expected Order Release | | | | | $(550, 700, 700 + \delta)$ |

Table 5.3.10. Distribution Requirements Planning for Warehouse 3 (Period 5 to Period 8)

| | | M_5 | M_6 | M_7 | M_8 |
|----------------------------|-----------------------------|---------------------------|--------------------------------------|--------------------------------------|--------------------------------------|
| Forecasted Demand | | $(80, 120, 120 + \delta)$ | $(100, 140, 140 + \delta)$ | $(110, 140, 140 + \delta)$ | $(90, 140, 140 + \delta)$ |
| Expected Order Receipt | | | $(550, 700, 700 + \delta)$ | | |
| Expected Inventory on hand | $(140 - 4\delta, 140, 270)$ | $(20 - 5\delta, 20, 190)$ | $(430 - 6\delta, 580, 790 + \delta)$ | $(290 - 7\delta, 440, 680 + \delta)$ | $(150 - 8\delta, 300, 590 + \delta)$ |
| Expected Order Release | | | | | |

5.3.7 Calculating the Total Demand on CWH during Period 1 (D_1)

Using (5.2)

$$D_{cl}^{\alpha} = \sum_{k=1}^3 O_{kl}^{\alpha} + S_1^{\alpha}$$

$$D_{cl}^{\alpha} = [O_{11}^{\alpha} + O_{21}^{\alpha} + O_{31}^{\alpha} + S_1^{\alpha}]$$

$$D_{cl}^{\alpha} = [25\alpha + 475, (0-\delta)\alpha + (500+\delta)] + [50\alpha + 150, (0-\delta)\alpha + (200+\delta)] + [15\alpha + 35, (0-\delta)\alpha + (50+\delta)]$$

$$D_{cl}^{\alpha} = [90\alpha + 660, (0-3\delta)\alpha + (750+3\delta)]$$

We now set

$$90\alpha + 660 = x \text{ and } (0-3\delta)\alpha + (750+3\delta) = x$$

This yields,

$$90\alpha + 660 - x = 0 \quad (5.14)$$

$$(0-3\delta)\alpha + (750+3\delta) - x = 0 \quad (5.15)$$

In (5.14) setting $\alpha = 0$ we get $x = 660$

In (5.15) setting $\alpha = 0$ we get $x = (750+3\delta)$

Setting $\alpha = 1$ in either we get $x = 750$

Therefore $D_{cl} = (660, 750, 750+3\delta)$

Now, the membership function is obtained as follows.

Solving $90\alpha + 660 - x = 0$ for α we obtain

$$\alpha = \frac{x-660}{90} \quad 660 \leq x \leq 750$$

Solving $(0-3\delta)\alpha + (750+3\delta) - x = 0$ for α we obtain

$$\alpha = \frac{x - 750 - 3\delta}{-3\delta} \quad 750 \leq x \leq (750 + 3\delta)$$

Thus, the membership function for $D_{c1} = (660, 750, 750 + 3\delta)$ is

$$\mu_{D_{c1}}(x) = \begin{cases} 0 & x \leq 660 \\ \frac{x - 660}{90} & 660 \leq x \leq 750 \\ \frac{x - 750 - 3\delta}{-3\delta} & 750 \leq x \leq (750 + 3\delta) \\ 0 & x \geq (750 + 3\delta) \end{cases}$$

Similarly, we obtain the fuzzy demands along with their respective membership functions for rest of D_{cj} 's.

As in 5.3.6 for different periods, we obtain expected on hand inventory and expected order release period (see Tables 5.3.11 and 5.3.12) along with the corresponding membership functions.

Table 5.3.11. Distribution Requirements Planning for Central Warehouse (Period 1 to Period 4)

| | | M_1 | M_2 | M_3 | M_4 |
|----------------------------|-------------------|------------------------------|-----------------------------|-----------------------------|--------------------------------------|
| Forecasted Demand | | $(660, 750, 750 + 3\delta)$ | $(30, 50, 50 + \delta)$ | $(40, 50, 50 + \delta)$ | $(587, 750, 750 + 2\delta)$ |
| Expected Receipt | | | | | $(900, 1000, 1000 + \delta)$ |
| Expected Inventory on hand | $(950, 950, 950)$ | $(200 - 3\delta, 200, 290)$ | $(150 - 4\delta, 150, 260)$ | $(100 - 5\delta, 100, 220)$ | $(250 - 7\delta, 350, 633 + \delta)$ |
| Expected Order Release | | $(900, 1000, 1000 + \delta)$ | | | |

Table 5.3.12. Distribution Requirements Planning for Central Warehouse (Period 5 to Period 8)

| | | M_5 | M_6 | M_7 | M_8 |
|----------------------------|--------------------------------------|--------------------------------------|--------------------------------------|---------------------------------------|--------------------------------------|
| Forecasted Demand | | $(40, 50, 50 + \delta)$ | $(30, 50, 50 + \delta)$ | $(35, 50, 50 + \delta)$ | $(25, 50, 50 + \delta)$ |
| Expected Receipt | | | | | |
| Expected Inventory on hand | $(250 - 7\delta, 350, 633 + \delta)$ | $(200 - 8\delta, 300, 593 + \delta)$ | $(150 - 9\delta, 250, 563 + \delta)$ | $(100 - 10\delta, 200, 528 + \delta)$ | $(50 - 11\delta, 150, 503 + \delta)$ |
| Expected Order Release | | | | | |

5.3.8 Interpretation of the Results

From Tables 5.3.11 and 5.3.12, we calculate the demand on CWH during 8 Periods by assuming $\delta = 0.005$. The demand during Period 1, Period 2, Period 3, Period 4, Period 5, Period 6, Period 7 and Period 8 is (660, 750, 750.015), (30, 50, 50.005), (40, 50, 50.005), (587, 750, 750.010), (40, 50, 50.005), (30, 50, 50.005), (35, 50, 50.005) and (25, 50, 50.005) respectively. This yields the replenishment strategy to satisfy the demand from Period 1 through Period 8. For example, the beginning inventory satisfies the demand from Period 1 through Period 3, and to satisfy the demand from Period 4 through Period 8, we must schedule an expected receipt during Period 4. Also, we observe that when the three-week lead-time is applied the expected order release is planned during Period 1.

In Appendix 3, we calculate the various values of x , by assuming $\delta = 0.005$, when α lies between 0 and 1, and plot the membership function graphs for different values of x . For example, the demand during Period 1 is (660, 750, 750.015). As we observe from the membership function graph D_{c1} , when $660 \leq x \leq 750$ the membership function increases monotonically to the left and goes to its maximum value of 1 (level of one's believe about the belongingness of x to A , or level of truth of x belonging to A , or degree of compatibility of x to A) at an interior point $x = 750$, and when $750 \leq x \leq 750.015$ the membership function decreases monotonically to the right and goes to 0 at a right end point $x = 750.015$, starting from 1 at $x = 750$. Similarly we have the membership function graph for D_{c2} to D_{c8} along with the values of x .

5.4 Distribution Requirements Planning with Variable Demand Rate and Backorders allowed under Fuzzy Information (Data)

We now consider the option of backordering.

5.4.1 Assumptions

For this model, the following assumptions, in addition to the ones in Section 5.2.1, are made :

1. The demand at branch warehouses varies from one time period to another and is available in the form of triangular fuzzy numbers.
2. The units needed to satisfy demand in a particular period can be acquired at any time including the backorders.
3. The economic shipping quantity for each warehouse is available in the form of triangular fuzzy number.
4. The beginning inventory is available in the form of triangular fuzzy numbers and in transit inventory is zero.

5.4.2 General Formulation

We assume that each of I_{kr} , R_{kj} and d_{kj} in equation (5.1); O_{kj} and S_j in equation (5.2) for $k = 1, 2, \dots, p$; $j = 1, 2, \dots, m$ is a triangular fuzzy number of the type.

$$I_{kr} = [I_{kr1}, I_{kr2}, I_{kr3}]$$

$$R_{kj} = [R_{kj1}, R_{kj2}, R_{kj3}]$$

$$d_{kj} = [d_{kj1}, d_{kj2}, d_{kj3}]$$

$$O_{kj} = [O_{kj1}, O_{kj2}, O_{kj3}]$$

$$S_j = [S_{j1}, S_{j2}, S_{j3}]$$

The values of I_{kr} , R_{kj} , d_{kj} , O_{kj} and S_j can, for example, be obtained from the experts who share the same information but different opinion.

If we set $I_{kr} = [I_{kr1}, I_{kr2}, I_{kr3}]$; $R_{kj} = [R_{kj1}, R_{kj2}, R_{kj3}]$; $d_{kj} = [d_{kj1}, d_{kj2}, d_{kj3}]$ equation (5.1), then I_{kj} is a triangular fuzzy number (T.F.N) given by

$$I_{kj} = [I_{kj1}, I_{kj2}, I_{kj3}] \quad (5.16)$$

Similarly, if we set $O_{kj} = [O_{kj1}, O_{kj2}, O_{kj3}]$; $S_j = [S_{j1}, S_{j2}, S_{j3}]$ in equation (5.2), then D_j is a triangular fuzzy number (T.F.N) given by

$$D_{cj} = [D_{cj1}, D_{cj2}, D_{cj3}] \quad (5.17)$$

Each T.F.N, I_{kj} in equation (5.16) and its membership function are determined on the line of Kaufman and Gupta (1985, 1988) by using the interval of confidence for I_{kr} , R_{kj} and d_{kj} respectively, at α - level

$$I_{kr}^\alpha = [(I_{kr2} - I_{kr1})\alpha + I_{kr1}, (I_{kr2} - I_{kr3})\alpha + I_{kr3}] \quad \forall \alpha \in [0,1] \quad (5.18)$$

$$R_{kj}^\alpha = [(R_{kj2} - R_{kj1})\alpha + R_{kj1}, (R_{kj2} - R_{kj3})\alpha + R_{kj3}] \quad \forall \alpha \in [0,1] \quad (5.19)$$

$$d_{kj}^\alpha = [(d_{kj2} - d_{kj1})\alpha + d_{kj1}, (d_{kj2} - d_{kj3})\alpha + d_{kj3}] \quad \forall \alpha \in [0,1] \quad (5.20)$$

for $r = 0, 1, \dots, m-1$; $k = 1, 2, \dots, p$; $j = 1, 2, \dots, m$.

(5.18), (5.19) and (5.20), along with $r = 0, 1, \dots, m-1$; $k = 1, 2, \dots, p$; $j = 1, 2, \dots, m$, yield the following interval of confidence.

$$[(I_{kr2} - I_{kr1} + R_{kj2} - R_{kj1} - d_{kj2} + d_{kj3})\alpha + (I_{kr1} + R_{kj1} - d_{kj3}), (I_{kr2} - I_{kr3} + R_{kj2} - R_{kj3} - d_{kj2} - d_{kj1})\alpha + I_{kr3} + R_{kj3} - d_{kj1}].$$

In this interval of confidence

1. Setting $\alpha = 0$, we get the end points I_{kj1} and I_{kj3}

$$I_{kj1} = (I_{kr1} + R_{kj1} - d_{kj3}), \text{ and } I_{kj3} = (I_{kr3} + R_{kj3} - d_{kj1})$$

of the fuzzy number I_{kj} .

1. Setting $\alpha = 1$ gives the interior point I_{kj2} of I_{kj}

$$I_{kj2} = (I_{kr2} + R_{kj2} - d_{kj2}, I_{kr2} + R_{kj2} - d_{kj2}).$$

The membership function is obtained from the interval of confidence by setting separately, each of the function, equal to x and solving each of those equations for α . Thus,

$$(I_{kr2} - I_{kr1} + R_{kj2} - R_{kj1} - d_{kj2} + d_{kj3})\alpha + (I_{kr1} + R_{kj1} - d_{kj3}) = x \quad (5.21)$$

Next, we set

$$(I_{kr2} - I_{kr3} + R_{kj2} - R_{kj3} - d_{kj2} - d_{kj1})\alpha + (I_{kr3} + R_{kj3} - d_{kj1}) = x \quad (5.22)$$

Using (5.21) and (5.22), we obtain the membership function for I_{kj} .

Similarly, we determine each T.F.N. D_{ej} (see (5.17)) along with its membership function by using the interval of confidence for O_{kj} and S_j respectively, at α - level, where

$$D_{ej1} = (O_{kj1} + S_{j1})$$

$$D_{ej2} = (O_{kj2} + S_{j2})$$

$$D_{ej3} = (O_{kj3} + S_{j3}).$$

5.4.3 Computational Technique

Below, we now modify the computational technique given in 5.3.3 for the distribution requirements planning problem by incorporating backorders.

Step 1. Obtaining the Expected Order Release Period at Branch Warehouse k

- (i) Starting with the first period, using (5.1) calculate the expected on hand inventory in each period, up to and including the period in which the interior point of the expected on hand inventory is less than zero.
- (ii) Suppose the interior point (I_{kn2}) of the expected on hand inventory (I_{kn}) is less than zero in Period n , schedule an expected receipt in Period n and recalculate the expected on hand inventory I_{kn} by using (5.1).

- (iii) Calculate the expected order release period by subtracting the lead-time from the expected receipt period.
- (iv) Go to Period (n+1) and repeat Step 1 (i) – (iii).

Continue the process to the end of the planning horizon.

Step 2. Obtaining Total Demand on the Central Warehouse

Calculate the total demand on the central warehouse during each period of the planning horizon by using equation (5.2).

5.4.4 Numerical Example with Backorders allowed with Fuzzy Information (Data)

We illustrate our method through the following numerical example in which we have three branch warehouses WH_1 , WH_2 , WH_3 , one CWH, and eight time periods M_1 , M_2 , M_3 , M_4 , M_5 , M_6 , M_7 and M_8 . The projected orders from the branch warehouses and direct sales from the CWH constitute the demand on the CWH. In Tables 5.4.1–5.4.4 below, we represent the lead-time, order quantity, beginning inventory on hand and forecast demand for each warehouse in the form of triangular fuzzy numbers.

Table 5.4.1. Lead-time, order quantity and beginning inventory

| | Lead Time (Periods) | Order quantity | Beginning Inventory |
|---------------|------------------------|-------------------|---------------------|
| WH_1 | 2 | (300, 500, 600) | (204, 204, 204) |
| WH_2 | 3 | (150, 200, 300) | (100, 100, 100) |
| WH_3 | 2 | (600, 700, 850) | (600, 600, 600) |
| Direct Supply | 3 | (950, 1000, 1100) | (950, 950, 950) |

Table 5.4.2. Demand in product units (Period 1 to Period 3)

| | M ₁ | M ₂ | M ₃ |
|-----------------|----------------|-----------------|----------------|
| WH ₁ | (65, 80, 97) | (72, 80, 95) | (68, 80, 100) |
| WH ₂ | (15, 30, 40) | (10, 30, 40) | (18, 30, 45) |
| WH ₃ | (70, 120, 135) | (100, 120, 145) | (90, 120, 145) |
| Direct Supply | (40, 50, 65) | (30, 50, 60) | (42, 50, 62) |

Table 5.4.3. Demand in product units (Period 4 to Period 6)

| | M ₄ | M ₅ | M ₆ |
|-----------------|----------------|----------------|-----------------|
| WH ₁ | (60, 70, 90) | (71, 80, 99) | (70, 90, 109) |
| WH ₂ | (10, 20, 35) | (20, 30, 45) | (20, 35, 55) |
| WH ₃ | (75, 100, 120) | (80, 120, 140) | (100, 140, 160) |
| Direct Supply | (37, 50, 65) | (40, 50, 70) | (30, 50, 60) |

Table 5.4.4 Demand in product units (Period 7 to Period 8)

| | M ₇ | M ₈ |
|-----------------|-----------------|-----------------|
| WH ₁ | (65, 90, 98) | (60, 90, 100) |
| WH ₂ | (15, 35, 45) | (20, 35, 50) |
| WH ₃ | (110, 140, 170) | (100, 140, 185) |
| Direct Supply | (35, 50, 60) | (40, 50, 70) |

5.4.5 Intervals of Confidence

Using Tables 5.4.1-5.4.4 below, we compute the intervals of confidence for demand, for direct sales, and for beginning inventory so as to calculate the total demand on the CWH.

5.4.5.1 Intervals of Confidence for Demand

| | |
|---|---|
| $d_{11}^{\alpha} = [15\alpha + 65, -17\alpha + 97]$ | $d_{12}^{\alpha} = [8\alpha + 72, -15\alpha + 95]$ |
| $d_{13}^{\alpha} = [12\alpha + 68, -20\alpha + 100]$ | $d_{14}^{\alpha} = [10\alpha + 60, -20\alpha + 90]$ |
| $d_{15}^{\alpha} = [9\alpha + 80, -19\alpha + 99]$ | $d_{16}^{\alpha} = [20\alpha + 70, -19\alpha + 109]$ |
| $d_{17}^{\alpha} = [25\alpha + 65, -8\alpha + 98]$ | $d_{18}^{\alpha} = [30\alpha + 60, -10\alpha + 100]$ |
| $d_{21}^{\alpha} = [15\alpha + 15, -10\alpha + 40]$ | $d_{22}^{\alpha} = [20\alpha + 10, -10\alpha + 40]$ |
| $d_{23}^{\alpha} = [12\alpha + 18, -15\alpha + 45]$ | $d_{24}^{\alpha} = [10\alpha + 10, -15\alpha + 35]$ |
| $d_{25}^{\alpha} = [10\alpha + 20, -15\alpha + 45]$ | $d_{26}^{\alpha} = [15\alpha + 20, -20\alpha + 55]$ |
| $d_{27}^{\alpha} = [20\alpha + 15, -10\alpha + 45]$ | $d_{28}^{\alpha} = [15\alpha + 20, -15\alpha + 50]$ |
| $d_{31}^{\alpha} = [50\alpha + 70, -15\alpha + 135]$ | $d_{32}^{\alpha} = [20\alpha + 100, -25\alpha + 145]$ |
| $d_{33}^{\alpha} = [30\alpha + 95, -25\alpha + 145]$ | $d_{34}^{\alpha} = [25\alpha + 75, -20\alpha + 120]$ |
| $d_{35}^{\alpha} = [40\alpha + 80, -20\alpha + 140]$ | $d_{36}^{\alpha} = [40\alpha + 100, -20\alpha + 160]$ |
| $d_{37}^{\alpha} = [30\alpha + 110, -30\alpha + 170]$ | $d_{38}^{\alpha} = [40\alpha + 100, -45\alpha + 185]$ |

5.4.5.2 Intervals of Confidence of Direct Sales

| | |
|--|--|
| $S_1^{\alpha} = [10\alpha + 40, -15\alpha + 65]$ | $S_2^{\alpha} = [15\alpha + 35, -10\alpha + 60]$ |
| $S_3^{\alpha} = [8\alpha + 42, -12\alpha + 62]$ | $S_4^{\alpha} = [13\alpha + 37, -15\alpha + 65]$ |
| $S_5^{\alpha} = [10\alpha + 40, -20\alpha + 70]$ | $S_6^{\alpha} = [20\alpha + 30, -10\alpha + 60]$ |
| $S_7^{\alpha} = [15\alpha + 35, -10\alpha + 60]$ | $S_8^{\alpha} = [10\alpha + 40, -20\alpha + 70]$ |

5.4.5.3 Intervals of Confidence of Beginning Inventory

$$I_{10}^{\alpha} = [0\alpha + 204, 0\alpha + 204] \quad I_{20}^{\alpha} = [0\alpha + 100, 0\alpha + 100]$$

$$I_{30}^{\alpha} = [0\alpha + 600, 0\alpha + 600] \quad I_{c0}^{\alpha} = [0\alpha + 950, 0\alpha + 950]$$

5.4.6 Obtaining the Expected Order Release Period at Warehouse 2

$$I_{21}^{\alpha} = I_{20}^{\alpha} - d_{21}^{\alpha} + R_{21}^{\alpha}$$

$$I_{21}^{\alpha} = I_{20}^{\alpha} + (-d_{21}^{\alpha}) + R_{21}^{\alpha}$$

$$I_{21}^{\alpha} = [0\alpha + 100, 0\alpha + 100] + [10\alpha - 40, -15\alpha - 15] + 0$$

$$I_{21}^{\alpha} = [10\alpha + 60, -15\alpha + 85]$$

We now set

$$10\alpha + 60 = x \text{ and } -15\alpha + 85 = x$$

This yields,

$$10\alpha + 60 - x = 0 \quad (5.23)$$

$$\text{and} \quad -15\alpha + 85 - x = 0 \quad (5.24)$$

In (5.23) setting $\alpha = 0$ we get $x = 60$

In (5.24) setting $\alpha = 0$ we get $x = 85$

Setting $\alpha = 1$ in either we get $x = 70$

Therefore $I_{21} = (60, 70, 85)$

Similarly, we calculate I_{22} and I_{23}

$$I_{22} = (20, 40, 75), \quad I_{23} = (-25, 10, 57),$$

Similarly, $I_{24} = (-35, -10, 25)$,

We observe that I_{242} is negative. Therefore, an order should be received in Period 4.

And from Table 5.4.1, $R_{24} = (150, 200, 300)$.

When the three-Period lead-time is applied, the expected order is planned for release in Period 1. Therefore,

$$O_{21} = (150, 200, 300)$$

$$O_{21}^{\alpha} = (50\alpha + 150, -100\alpha + 300)$$

Along the same lines as for I_{21} ,

$$I_{24} = (90, 190, 347)$$

with its membership function given by

$$\mu_{I_{24}}(x) = \begin{cases} 0 & x \leq 90 \\ \frac{x - 90}{100} & 90 \leq x \leq 190 \\ \frac{x - 347}{-157} & 190 \leq x \leq 347 \\ 0 & x \geq 347 \end{cases}$$

Similarly, we obtain the expected on hand inventories and expected order releases for rest of I_{kj} 's, and O_{ak} 's respectively, and are in Tables 5.4.5–5.4.10.

Table 5.4.5. Distribution Requirements Planning for WH₁ (Period 1 to Period 4)

| | | M ₁ | M ₂ | M ₃ | M ₄ |
|----------------------------|-----------------|-----------------|----------------|-----------------|-----------------|
| Forecasted Demand | | (65, 80, 97) | (72, 80, 95) | (68, 80, 100) | (60, 70, 90) |
| Expected Order Receipt | | | | (300, 500, 600) | |
| Expected Inventory on hand | (204, 204, 204) | (107, 124, 139) | (12, 44, 67) | (212, 464, 599) | (122, 394, 539) |
| Expected Order Release | | (300, 500, 600) | | | |

Table 5.4.6. Distribution Requirements Planning for WH₁ (Period 5 to Period 8)

| | | M5 | M6 | M7 | M8 |
|----------------------------|-----------------|----------------|-----------------|------------------|-----------------|
| Forecasted Demand | | (71, 80, 90) | (70, 90, 109) | (65, 90, 98) | (60, 90, 100) |
| Expected Order Receipt | | | | | |
| Expected Inventory on hand | (122, 394, 539) | (23, 314, 468) | (-86, 224, 398) | (-184, 134, 333) | (-284, 44, 273) |
| Expected Order Release | | | | | |

Table 5.4.7. Distribution Requirements Planning for WH₂ (Period 1 to Period 4)

| | | M ₁ | M ₂ | M ₃ | M ₄ |
|----------------------------|-----------------|-----------------|----------------|----------------|-----------------|
| Forecasted Demand | | (15, 30, 40) | (10, 30, 40) | (18, 30, 45) | (10, 20, 35) |
| Expected Order Receipt | | | | | (150, 200, 300) |
| Expected Inventory on hand | (100, 100, 100) | (60, 70, 85) | (20, 40, 75) | (-25, 10, 57) | (90, 190, 347) |
| Expected Order Release | | (150, 200, 300) | | | |

Table 5.4.8. Distribution Requirements Planning for WH₂ (Period 5 to Period 8)

| | | M ₅ | M ₆ | M ₇ | M ₈ |
|----------------------------|----------------|----------------|-----------------|----------------|-----------------|
| Forecasted Demand | | (20, 30, 45) | (20, 35, 55) | (15, 35, 45) | (20, 35, 50) |
| Expected Order Receipt | | | | | |
| Expected Inventory on hand | (90, 190, 347) | (45, 160, 327) | (-10, 125, 307) | (-55, 90, 292) | (-105, 55, 272) |
| Expected Order Release | | | | | |

Table 5.4.9. Distribution Requirements Planning for WH₃ (Period 1 to Period 4)

| | | M ₁ | M ₂ | M ₃ | M ₄ |
|----------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Forecasted Demand | | (70, 120, 135) | (90, 120, 145) | (95, 120, 145) | (75, 100, 120) |
| Expected Order Receipt | | | | | |
| Expected Inventory on hand | (600, 600, 600) | (465, 480, 530) | (320, 360, 430) | (175, 240, 340) | (55, 140, 265) |
| Expected Order Release | | | | | (600, 700, 850) |

Table 5.4.10. Distribution Requirements Planning for WH₃ (Period 5 to Period 8)

| | | M ₅ | M ₆ | M ₇ | M ₈ |
|----------------------------|----------------|-----------------|-----------------|-----------------|-----------------|
| Forecasted Demand | | (80, 120, 140) | (100, 140, 160) | (110, 140, 170) | (100, 140, 185) |
| Expected Order Receipt | | | (600, 700, 850) | | |
| Expected Inventory on hand | (55, 140, 265) | (-85, 20, 185) | (355, 580, 935) | (185, 440, 825) | (0, 300, 725) |
| Expected Order Release | | | | | |

5.4.7 Calculating the Total Demand on CWH during Period 1 (D_{c1})

$$D_{c1}^{\alpha} = \sum_{k=1}^3 O_{k1}^{\alpha} + S_1^{\alpha}$$

$$D_{c1}^{\alpha} = [O_{11}^{\alpha} + O_{21}^{\alpha} + O_{31}^{\alpha} + S_1^{\alpha}]$$

$$D_{c1}^{\alpha} = (200\alpha + 300, -100\alpha + 600) + (50\alpha + 150, -100\alpha + 300) \\ + (10\alpha + 40, -15\alpha + 65)$$

$$D_{c1}^{\alpha} = [260\alpha + 490, -215\alpha + 965]$$

We now set $260\alpha + 490 = x$ and $-215\alpha + 965 = x$

This yields, $260\alpha + 490 - x = 0$, and $-215\alpha + 965 - x = 0$

Therefore $D_{c1} = (490, 750, 965)$

With membership function

$$\mu_{D_{c1}}(x) = \begin{cases} 0 & x \leq 490 \\ \frac{x - 490}{260} & 490 \leq x \leq 750 \\ \frac{x - 965}{-215} & 750 \leq x \leq 965 \\ 0 & x \geq 965 \end{cases}$$

Similarly, we obtain the fuzzy demands for rest of D_{cj} 's.

Furthermore, the expected on hand inventories and expected order releases for rest of I_{cj} 's, and O_{cj} 's respectively and are as in Tables 5.4.11–5.4.12.

Table 5.4.11. Distribution Requirements Planning for CWH (Period 1 to Period 4)

| | | M ₁ | M ₂ | M ₃ | M ₄ |
|----------------------------|-----------------|-------------------|-----------------|------------------|-------------------|
| Forecasted Demand | | (490, 750, 965) | (35, 50, 60) | (42, 50, 62) | (637, 750, 915) |
| Expected Order Receipt | | | | | (950, 1000, 1100) |
| Expected Inventory on hand | (950, 950, 950) | (-15, 200, 460) | (-75, 150, 425) | (-137, 100, 383) | (-102, 350, 846) |
| Expected Order Release | | (950, 1000, 1100) | | | |

Table 5.4.12. Distribution Requirements Planning for CWH (Period 5 to Period 8)

| | M4 | M5 | M6 | M7 | M8 |
|----------------------------|------------------|------------------|------------------|------------------|------------------|
| Forecasted Demand | | (40, 50, 70) | (30, 50, 60) | (35, 50, 60) | (40, 50, 70) |
| Expected Order Receipt | | | | | |
| Expected Inventory on hand | (-102, 350, 846) | (-172, 300, 806) | (-232, 250, 776) | (-292, 200, 741) | (-362, 150, 701) |
| Expected Order Release | | | | | |

5.4.8 Interpretation of the Results

From Tables 5.4.11 and 5.4.12, we calculate the demand on CWH during 8 Periods. The demand during Period 1, Period 2, Period 3, Period 4, Period 5, Period 6, Period 7 and Period 8 is (490, 750, 965), (35, 50, 60), (42, 50, 62), (637, 750, 915), (40, 50, 70), (30, 50, 60), (35, 50, 60) and (40, 50, 70) respectively. This also gives us the replenishment strategy to satisfy the demand from Period 1 through Period 8. The beginning inventory will satisfy the demand from Period 1 through Period 3. To satisfy demand from Period 4 through Period 8, we must schedule an expected receipt during Period 4. When the three-week lead-time is applied the expected order release is planned during Period 1.

CHAPTER 6

CONCLUSION, CONTRIBUTION AND RECOMMENDATIONS

In the present chapter, we state the contributions and conclusions of this dissertation. Finally, we give some recommendations for further research on the problems considered in this dissertation.

6.1 Conclusion and Contribution

In the present dissertation, two important problems in the field of industrial engineering i.e. lot size inventory control problem (addressed by Wagner and Whitin, 1958) and distribution inventory management problem have been revisited under fuzzy environment. Using fuzzy algebra approach we deal with inventory lot sizing and distribution inventory management problem when the data known is imprecise. We believe this is more realistic approach to tackle these problems as most of the times data is forecasted, and forecasts rarely-if-ever turn out to be crisply correct. Therefore, the models based on precise knowledge of demand have little practical application.

Main contributions in this thesis are included in Chapters 3–5. Using fuzzy data, Chapter 3 and 4 deals with lot-sizing problem without back-ordering and with back ordering, respectively. Chapter 5 modifies, under fuzzy data, the distribution requirement planning (DRP) approach for a two-level distribution inventory management problem consisting of one central warehouse and a finite number of branch warehouses. We develop the technique for both without and with back-ordering situations. In Chapter 3, we discuss the distribution inventory management problem under fuzzy information (data). In all the

problems the data available is assumed to be available in the form of triangular fuzzy numbers. Throughout the thesis, we have used Excel for the purpose of numerical calculations.

Though most real productions systems still exceed the limits of the proposed models, yet their potential impact on both practical and theoretical decision making is apparent.

6.2 Recommendations for Future Research

It is believed that a number of extensions are possible to both the lot sizing inventory problem and distribution requirement planning problem. Results of Chapters 3 and 4 can be extended to the case when the estimates for various parameters involved are provided in the form of Tr.F.N.'s or general fuzzy numbers. The results of Chapter 5 can also be extended to the case when the data available is in the form of trapezoidal fuzzy numbers (Tr.F.N.'s) or general fuzzy numbers. Furthermore, the problem may be further extended from two-level to multi-level distribution inventory management systems and/or when the lead time available is fuzzy.

REFERENCES

- Arreola-Risa, A., DeCroix, G. A.,** "Inventory Management Under Random Supply Disruptions and Partial Backorder", *Naval Research Logistics* 45, 1998, pp. 687 –703.
- Atkins, D., Sun, D.,** "98%-Effective Lot Sizing for Series Inventory Systems with Backlogging", *Operation Research* 43 (2), 1995, pp. 335-345.
- Banerjee, A.,** "A joint Economic Lot Size model for Purchaser and Vendor", *Decision Sciences* 17, 1986, pp. 292-311.
- Blackburn, G. G., Millen, R.,** "Improved Heuristics for Multi-Stage Requirements Planning Systems", *Management Sci.*, 1982, Vol. 28, pp. 44-56.
- Blackburn, J. D., Millen, R. A.,** "Heuristic Lot Sizing Performance in a Rolling Schedule Environment", *Decision Science*, Vol. 11, 1980, pp. 691-701.
- Boe, W. J., Yilmax, C.,** "The Incremental Order Quantity", *Production and Inventory Management*, Vol. 24, Number 2, Second Quarter, 1983, pp. 94-100.
- Bookbinder, J. H., Heath, Donald B.,** "Replenishment Analysis in Distribution Requirements Planning", *Decision Sciences* 19, 1988, pp. 477-489.
- Bretthauer Kurt, Shetty Bala, Syam Siddhartha, White Susan,** "A Model for Resource Constrained Production and Inventory Management", *Decision Sciences*, Vol. 25, Number 4, July/August 1994.
- Brown, K. A.,** "A Two-Period, Two-Echelon, Two-Retailer Inventory Stocking Problem", M.S. thesis, School of Operations Research and Industrial Engineering, Cornell University, 1984.
- Caplin, A. S.,** "The Variability of Aggregate Demand with (S,s) Inventory Policies", *Econometrica* 53 (6), pp. 1395-1409.
- Chen, F. Y., Feng, Y., Simchi-Levi, D.,** "Uniform Distribution of Inventory Positions in Two-Echelon Periodic Review Systems with Batch-Ordering Policies and Interdependent Demands", *European Journal Of Operation Research* 140, 2002, pp. 648-654.
- Chiu, H. N., Chen, H. M., Weng, L. C.,** "Deterministic Time-Varying Demand Lot-Sizing Models with Learning and Forgetting in setups and Production", *Production and Operations Management*, Vol.12, No. 1, 2003, pp. 120 - 127.
- Clark, A. J., Scarf, H.,** "Optimal Policies for a Multi-Echelon Inventory Problem", *Management Sci.*, Vol. 6, Number 4, 1960, pp. 475-490.

De Bodt, M.A., Graves, S. C., "Continuous-Review Policies for a Multi-Echelon Inventory Problem with Stochastic Demand", *Management Sci.*, Vol. 31, pp. 1286-1299.

DeCroix, G. A., Arreola-Risa, A., "On Offering Economic Incentives to Backorder", *IIE Transactions* 30, 1998, pp. 715-721.

DeMatteis, J. J., "An Economic Lot Sizing Technique.I: The Part Period Algorithm", *IBM System Journal*, Vol. 7, Number 1, 1968, pp. 30-38.

Deuermeyer, B. L., Schwarz, L. B., "A Model for the Analysis of System Service Level in warehouse-Retailer Distribution Systems: The Identical Retailer Case", *Multi-Level Production/Inventory Control Systems: Theory and Practice*, L. B. Schwarz, North-Holland, Amsterdam, 1981. pp. 163-193.

Diaby, M., Martel, A., "Dynamic Lot Sizing for Multi-Echelon Distribution Systems with Purchasing and Transportation Price Discounts", *Operation Research* 41 (1), 1993, pp. 48-58.

Eppen, G., Schrage, L., "Centralized Ordering Policies in a Multi-Warehouse System with Lead Times and Random Demand", *Multi-Level Production/Inventory Control Systems: Theory and Practice*, L. B. Schwarz, North-Holland, Amsterdam, 1981. pp. 51-67.

Eppen, G. D., Gould, F. J., Pashigan, B. D., "Extensions of the Planning Horizon Theorem in Dynamic Lot Size Model", *Management Science*, Vol. 15, Number 5, Jan. 1969, pp. 268-277.

Erkip, N. K., "A Restricted Class of Allocation Policies in a Two-Echelon Inventory System", Technical Report No. 628, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, 1984a.

Erkip, N. K., "Approximate Policies in Multi-Echelon Inventory Systems", Ph.D. dissertation, Department of Industrial Engineering and Management, Stanford University, June 1984b.

Erkip, N. K., Hausman, W., Nahmias, S., "Optimal and Near Optimal Policies in Multi-Echelon Inventory Systems with Correlated Demands", presented at the TIMS XXVI International Meeting, Copenhagen, Denmark, June 17-21, 1984.

Federgruen, A., Zipkin, P., "Approximations of Dynamic Multi location Production and Inventory Problems", *Management Sci.*, 1984, Vol. 30, pp. 69-84.

Fogarty, D. W., Blackstone, J. H., and Hoffmann, T. R., *Production and Inventory Management*, South-Western Company, 1991, Cincinnati, Ohio, USA.

Fogarty, D. W., Stonebraker, P. W., and Hoffmann, T. R., Production and Operations Management, South-Western Publishing Company, 1989, Cincinnati, Ohio, USA

Fordyce, James M., Webster, Francis M., "Non Constant Unit Cost Price with the Wagner-Whitin Algorithm", Production and Inventory Management, First Quarter, 1985, pp. 70-80.

Fordyce, James M., Webster, Francis M., "The Wagner Whitin Algorithm Made Simple", Production and Inventory Management, Second Quarter, 1984, pp. 21-27.

Francis, M., Webster, Jr., "A Back-Order Version of the Wagner Whitin Discrete-Demand EOQ Algorithm by Dynamic Programming", Production and Inventory Management, Forth Quarter, 1989.

Gaither, N. G., "An Improved Lot Sizing Model for MRP Systems", Production and Inventory Management, Vol. 24, Number 3, Third Quarter, 1983, pp. 10-20.

Gill, A., "Analysis of Lot Size Inventory And Equipment Replacement Problems", M.Sc. Thesis, Department of Mechanical and Industrial Engineering, University of Manitoba, 1992.

Gorham, T., "Dynamic Order Quantities", Production and Inventory Management, Vol. 9, Number 1, 1968, pp. 75-81.

Goyal, S. K. , "A joint Economic Lot Size model for Purchaser and Vendor", Decision Sciences 19, 1988, pp. 236-241.

Goyal, S. K., "Integrated Inventory Model For A Single Supplier-Single Customer Problem", International Journal of Production Research 15, 1977, pp. 107 –111.

Graves, S. C., "On the Deterministic Demand Multi-Product Single-Machine Lot Scheduling Problem", Management Sci., Vol. 25, Number 2, 1979, pp. 276-280.

Grewal, H., "Lot Sizing Problem Under Fuzzy Environment", M.Sc. Thesis, Department of Mechanical and Industrial Engineering, University of Manitoba, 1999.

Groff, G., "A Lot sizing Rule for Time Phased component Demand", Production and Inventory Management, Vol. 20, Number 1, 1979, pp. 47-53.
Gallego, G., Ozer, O., "Integrating Replenishment Decisions with Advance Demand Information", Management Sci., Vol. 47, Number 10, 2001, pp. 1344-1360..

Gupta, A., Maranas, C. D., McDonald, C. H., "Mid-term Supply Chain Planning under Demand Uncertainty: Customer Demand Satisfaction and Inventory Management", Computers and Chemical Engineering 24, 2000, pp. 2613-2625.

Ha, D., Kim, S. L., "Implementation of JIT Purchasing: An Integrated Approach", *Production Planning and Control* 8 (2), 1997, pp. 152-157.

Hariharan, R., Zipkin, P., "Customer Order Information, Lead Times and Inventories", *Management Sci.*, Vol. 41, 1995, pp. 1599-1607.

Harris, F. W., *Operations and Cost, Factory Management Series*, A.W. Shaw Co., Chicago, Chapter 2, 1915.

Heskett, James L., "Logistics-Essential to Strategy", *Harvard Business Review*, Nov-Dec 1977, pp. 85-96.

Hill, R. M., "Allocating Warehouse Stock in a Retail Chain", *Journal of the Operation Society* 40 (11), 1989, pp. 983-991.

Hsu, V. N., "Dynamic Economic Lot Size Model with Perishable Inventory", *Management Sci.*, Vol.46, Number 8, 2000, pp. 1159 - 1169.

Jackson, Peter L., "Stock Allocation in a Two-Echelon Distribution System OR What To Do Until Your Ship Comes In", *Management Sci.*, 1988, Vol. 34, Number 7, pp. 880-895.

Jackson, P. L. and Muckstadt, J. A., "A Two-Period, Two-Echelon Inventory Stocking and Allocation Problem", Technical Report No. 616, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, 1984a.

Jackson, P. L. and Muckstadt, J. A., "Risk Pooling in a Two-Period, Two-Echelon Inventory Stocking and Allocation Problem", Technical Report No. 634, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, 1984b.

Jonsson, H. Silver, E. A., "Analysis of an Extreme Case of a Push Control System", Faculty of Management, The university of Calgary, Calgary, Alberta, 1984a.

Jonsson, H. Silver, E. A., "Stock Allocation from a Central Warehouse in a Two-Location Push type Inventory Control System", WP, Faculty of Management, The university of Calgary, Calgary, Alberta, 1984b.

Karni, R., " Maximum Part Period Gain (MPG)- A Lot Sizing Procedure for Unconstrained and Constrained Requirements Planning Systems", *Production and Inventory Management*, Vol.22, Number 2, Second Quarter, 1981, pp. 91-98.

Kaufmann, A. and Gupta, M. M. (1985), *Introduction to Fuzzy Arithmetic Theory and Applications*, Von Nostrand Reinhold Company, New York. USA.

Kaufmann, A. and Gupta, M. M. (1988), *Mathematical Fuzzy Models in Engineering and Management Science*, North Holland, New York, USA.

Kim, D. H., Park, K. S., "(Q, r) Inventory Model with a Mixture of lost Sales and Time-Weighted Backorders", Journal of the Operational Research Society 36, 1985, pp. 231 – 238.

Lu, L., Posner, M., "Approximation Procedures for the One-Warehouse Multi-Retailer System", Management Sci., Vol. 40, Number 10, 1994, pp. 1305-1316.

Martel, A., "Policies for Multi-Echelon Supply: DRP Systems with Probabilistic Time-Varying Demands 1", INFOR, 2003, Vol. 41, Iss.1.

Martel, A., Gascon A., "Dynamic lot-sizing with price changes and price dependent holding costs", European Journal of Operational Research, Vol. 111, 1998, pp. 114-128.

Martin, Andre J., DRP Distribution Resource Planning, Oliver Wight Limited Publications, Inc., 1983.

McKnew, Mark A., Saydam Cem, Coleman, B. Jay, "An Efficient Zero-One Formulation of the Multilevel Lot Sizing Problem", Decision Sciences, Vol. 22, Number 2, Spring 1991.

Mendoza, A. G., "An Economic Lot sizing Technique. II: Mathematical Analysis of the Part Period Algorithm", IBM System Journal, Vol. 7, Number 1, 1968, pp. 39- 46.

Moncer H., and Ben-Daya M., "Some stochastic inventory models with deterministic variable lead time", European Journal of Operational Research, Vol. 113, 1999, pp. 42- 51.

Montgomery, D. C., Bazaraa, M. S., Keswani, A. K., "Inventory Models with a Mixture of Backorders and Lost sales", Naval Research Logistic Quarterly 20, 1973, pp. 255 –263.

Nahmias, S., "Perishable Inventory Theory: A Review", Management Sci., Vol. 30, 1982, pp. 680 - 708.

Ouyang, L. Y., Chuang, B. R., "(Q, R, L) Inventory Model Involving Quantity Discounts and a Stochastic Backorder Rate", Production Planning and Control 10, 1999, pp. 426 –433.

Ozer, O., "Replenishment Strategies for Distribution Systems Under Advance Demand Information", Management Sci., 2003, Vol. 49, Number 3, pp. 255-272.

Padmanabhan, G., Vrat, P., "Inventory Model with a Mixture of Back Orders and Lost Sales", International Journal of Systems Sciences 21, 1990, pp. 1721 –1726.

Padmanabhan, G., Vrat, P., "An EOQ Model for Perishable Items Under Stock Dependent Selling Rate", *European Journal of Operational Research* 86, 1995, pp. 281 – 292.

Pan, C. H., Hsiao, Y. C., "Inventory Models with Backorder Discounts and Variable Lead-Time", *International Journal of Systems Science* 32, 2001, pp. 925 –929.

Porter, R. W., "Centralized Inventory Management in the Multilevel Distribution Network ", *APICS 22nd Annual International Conference Proceedings* (1979), pp. 81-82.

Prentis, Eric L., Khumawala, Basheer M., "Efficient Heuristics for MRP Lot Sizing with Variable Production/Purchasing Costs", *Decision Sciences*, Vol. 20, Number 3, Summer 1989.

Roundy, R., "98%-Effective Integer-Ratio Lot-Sizing for One-Warehouse Multi-Retailer Systems", *Management Sci.*, Vol. 31, Number 11, 1985, pp. 1416-1430.

Schwarz, L. B., "A Simple Continuous Review Deterministic One-warehouse N-retailer inventory problem", *Management Sci.*, Vol. 19, Number 5, 1973, pp. 555-566.

Shtub, A., "Lot sizing in MRP/GT systems", *Production Planning and Control*, Vol. 1, Number 1, Jan-March 1990, pp. 40-44. **Sobel, M. J., Zhang, R. Q.,** "Inventory Policies for Systems with Stochastic and Deterministic Demand", *Operation Research*, 2001, Vol.45, pp. 931-939.

Silver, E. A., Meal, H. C., "A heuristic for Selecting Lot Sizing Quantities for the case of Deterministic Time Varying Demand Rate and Discrete Opportunities for Replenishment", *Production and Inventory Management*, Vol. 14, Number 2, 1973, pp. 64-74.

Trux, W., "Einkauf und Lagerdisposition mit Datenverarbeitung.", 2. Aufl., Munchen: Moderne Industrie, 1972.

Veinott, A., "Optimal Policy for a Multi-Product, Dynamic, Non-Stationary Inventory Problem", *Management Sci.* 12, 1965, pp. 206-222.

Vollmann, T. E., Berry, W. L., Whybark, D. C., *Manufacturing Planning And Control Systems*, Third Edition, Irwin, 1992, Homewood, Boston, USA.

Wagner, H. M., "A Postscript to Dynamic Problems in the Theory of the Firm", *Naval Research Logistics Quarterly*, Vol. 7, 1960, pp. 7-12.

Wagner, H. M., and Whitin T. M., "Dynamic Version of Lot Size Model", *Management Science*, 5, 1958, pp. 89-96.

Yang, P. C., Wee, H. M., "Economic Ordering Policy of Deteriorated Items for Vendor and Buyer: An Integrated Approach", *Production Planning and Control* 11 (5), 2000, pp. 474-480.

Yang, P. C., Wee, H. M., "A Single-Vendor and Multiple-Buyers Production-Inventory Policy for a Deteriorating Item", *European Journal of Operation Research* 143, 2002a, pp. 570-581.

Yao, M. J., Chiou, C. C., "On A Replenishment Coordination Model in an Integrated Supply Chain with One Vendor and Multiple Buyers", *European Journal Of Operation Research* , 2003.

Zadeh, L. A., "Fuzzy Sets", *Information and Control*, 8, 1965, pp. 338-353.

Zangwill, W. I., "A Backlogging Model and Multi-Echelon Model of a Dynamic Economic Lot Size Production System- A Network Approach", *Management Science*, Vol. 15, Number 9, May 1969, pp. 506-527.

Zimmermann, H. J., *Fuzzy Set theory and its application*, Kluwer-Nijhoff Publishing, Second Edition, 1991.

Zipkin, P., "On the Imbalance of Inventories in Multi-Echelon Systems", *Math. Oper. Res.*, 1984, Vol. 9.

Zoller, K., Robrade A., "Dynamic Lot sizing Techniques: Survey and Comparison", *Journal of Operations Management*, Vol. 7, Number 4, December 1988, pp. 125-148.

APPENDIX 1

Error Analysis for T_{ij}

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Error analysis corresponding to T_{12}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|---------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 136.25 | 167.4 | 136.25 | 167.4 | 0 | 0 | 0 | 0 |
| 0.1 | 137.5575 | 165.624 | 137.625 | 165.66 | 0.0675 | 0.036 | 0.04907 | 0.021736 |
| 0.2 | 138.88 | 163.856 | 139 | 163.92 | 0.12 | 0.064 | 0.086406 | 0.039059 |
| 0.3 | 140.2175 | 162.096 | 140.375 | 162.18 | 0.1575 | 0.084 | 0.112325 | 0.051821 |
| 0.4 | 141.57 | 160.344 | 141.75 | 160.44 | 0.18 | 0.096 | 0.127146 | 0.059871 |
| 0.5 | 142.9375 | 158.6 | 143.125 | 158.7 | 0.1875 | 0.1 | 0.131176 | 0.063052 |
| 0.6 | 144.32 | 156.864 | 144.5 | 156.96 | 0.18 | 0.096 | 0.124723 | 0.0612 |
| 0.7 | 145.7175 | 155.136 | 145.875 | 155.22 | 0.1575 | 0.084 | 0.108086 | 0.054146 |
| 0.8 | 147.13 | 153.416 | 147.25 | 153.48 | 0.12 | 0.064 | 0.081561 | 0.041717 |
| 0.9 | 148.5575 | 151.704 | 148.625 | 151.74 | 0.0675 | 0.036 | 0.045437 | 0.02373 |
| 1 | 150 | 150 | 150 | 150 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{13}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|-------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 152.00 | 195.60 | 152 | 195.6 | 0 | 0 | 0 | 0 |
| 0.1 | 153.71 | 192.94 | 153.8 | 193.04 | 0.09 | 0.099 | 0.058552 | 0.051311 |
| 0.2 | 155.44 | 190.30 | 155.6 | 190.48 | 0.16 | 0.176 | 0.102934 | 0.092484 |
| 0.3 | 157.19 | 187.69 | 157.4 | 187.92 | 0.21 | 0.231 | 0.133596 | 0.123076 |
| 0.4 | 158.96 | 185.10 | 159.2 | 185.36 | 0.24 | 0.264 | 0.150981 | 0.142629 |
| 0.5 | 160.75 | 182.53 | 161 | 182.8 | 0.25 | 0.275 | 0.155521 | 0.150664 |
| 0.6 | 162.56 | 179.98 | 162.8 | 180.24 | 0.24 | 0.264 | 0.147638 | 0.146686 |
| 0.7 | 164.39 | 177.45 | 164.6 | 177.68 | 0.21 | 0.231 | 0.127745 | 0.130178 |
| 0.8 | 166.24 | 174.94 | 166.4 | 175.12 | 0.16 | 0.176 | 0.096246 | 0.100604 |
| 0.9 | 168.11 | 172.46 | 168.2 | 172.56 | 0.09 | 0.099 | 0.053536 | 0.057404 |
| 1 | 170.00 | 170.00 | 170 | 170 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{14}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 173.20 | 238.20 | 173.2 | 238.2 | 0 | 0 | 0 | 0 |
| 0.1 | 175.73 | 234.18 | 175.88 | 234.38 | 0.153 | 0.198 | 0.087067 | 0.08455 |
| 0.2 | 178.29 | 230.21 | 178.56 | 230.56 | 0.272 | 0.352 | 0.152562 | 0.152905 |
| 0.3 | 180.88 | 226.28 | 181.24 | 226.74 | 0.357 | 0.462 | 0.197365 | 0.204174 |
| 0.4 | 183.51 | 222.39 | 183.92 | 222.92 | 0.408 | 0.528 | 0.222329 | 0.237419 |
| 0.5 | 186.18 | 218.55 | 186.6 | 219.1 | 0.425 | 0.55 | 0.22828 | 0.251659 |
| 0.6 | 188.87 | 214.75 | 189.28 | 215.28 | 0.408 | 0.528 | 0.216019 | 0.245865 |
| 0.7 | 191.60 | 211.00 | 191.96 | 211.46 | 0.357 | 0.462 | 0.186323 | 0.218959 |
| 0.8 | 194.37 | 207.29 | 194.64 | 207.64 | 0.272 | 0.352 | 0.139941 | 0.169812 |
| 0.9 | 197.17 | 203.62 | 197.32 | 203.82 | 0.153 | 0.198 | 0.077599 | 0.097239 |
| 1 | 200.00 | 200.00 | 200 | 200 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{15}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|-------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 346.00 | 484.65 | 346 | 484.65 | 0 | 0 | 0 | 0 |
| 0.1 | 351.18 | 475.81 | 351.4 | 476.185 | 0.225 | 0.3735 | 0.064071 | 0.078497 |
| 0.2 | 356.40 | 467.06 | 356.8 | 467.72 | 0.4 | 0.664 | 0.112233 | 0.142167 |
| 0.3 | 361.68 | 458.38 | 362.2 | 459.255 | 0.525 | 0.8715 | 0.145158 | 0.190125 |
| 0.4 | 367.00 | 449.79 | 367.6 | 450.79 | 0.6 | 0.996 | 0.163488 | 0.221435 |
| 0.5 | 372.38 | 441.29 | 373 | 442.325 | 0.625 | 1.0375 | 0.167842 | 0.235107 |
| 0.6 | 377.80 | 432.86 | 378.4 | 433.86 | 0.6 | 0.996 | 0.158814 | 0.230095 |
| 0.7 | 383.28 | 424.52 | 383.8 | 425.395 | 0.525 | 0.8715 | 0.136977 | 0.205289 |
| 0.8 | 388.80 | 416.27 | 389.2 | 416.93 | 0.4 | 0.664 | 0.102881 | 0.159513 |
| 0.9 | 394.38 | 408.09 | 394.6 | 408.465 | 0.225 | 0.3735 | 0.057052 | 0.091524 |
| 1 | 400.00 | 400.00 | 400 | 400 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{16}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 425.20 | 632.15 | 425.2 | 632.15 | 0 | 0 | 0 | 0 |
| 0.1 | 432.35 | 618.16 | 432.68 | 618.935 | 0.333 | 0.7785 | 0.077021 | 0.125939 |
| 0.2 | 439.57 | 604.34 | 440.16 | 605.72 | 0.592 | 1.384 | 0.134678 | 0.229012 |
| 0.3 | 446.86 | 590.69 | 447.64 | 592.505 | 0.777 | 1.8165 | 0.173879 | 0.307522 |
| 0.4 | 454.23 | 577.21 | 455.12 | 579.29 | 0.888 | 2.076 | 0.195495 | 0.359659 |
| 0.5 | 461.68 | 563.91 | 462.6 | 566.075 | 0.925 | 2.1625 | 0.200357 | 0.383481 |
| 0.6 | 469.19 | 550.78 | 470.08 | 552.86 | 0.888 | 2.076 | 0.189262 | 0.376917 |
| 0.7 | 476.78 | 537.83 | 477.56 | 539.645 | 0.777 | 1.8165 | 0.162967 | 0.337747 |
| 0.8 | 484.45 | 525.05 | 485.04 | 526.43 | 0.592 | 1.384 | 0.122201 | 0.263596 |
| 0.9 | 492.19 | 512.44 | 492.52 | 513.215 | 0.333 | 0.7785 | 0.067657 | 0.151921 |
| 1 | 500.00 | 500.00 | 500 | 500 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{23}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 106.10 | 118.80 | 106.1 | 118.8 | 0 | 0 | 0 | 0 |
| 0.1 | 106.48 | 117.89 | 106.49 | 117.92 | 0.009 | 0.027 | 0.008452 | 0.022902 |
| 0.2 | 106.86 | 116.99 | 106.88 | 117.04 | 0.016 | 0.048 | 0.014972 | 0.041028 |
| 0.3 | 107.25 | 116.10 | 107.27 | 116.16 | 0.021 | 0.063 | 0.019581 | 0.054265 |
| 0.4 | 107.64 | 115.21 | 107.66 | 115.28 | 0.024 | 0.072 | 0.022297 | 0.062496 |
| 0.5 | 108.03 | 114.33 | 108.05 | 114.4 | 0.025 | 0.075 | 0.023143 | 0.065602 |
| 0.6 | 108.42 | 113.45 | 108.44 | 113.52 | 0.024 | 0.072 | 0.022137 | 0.063465 |
| 0.7 | 108.81 | 112.58 | 108.83 | 112.64 | 0.021 | 0.063 | 0.0193 | 0.055962 |
| 0.8 | 109.20 | 111.71 | 109.22 | 111.76 | 0.016 | 0.048 | 0.014651 | 0.042968 |
| 0.9 | 109.60 | 110.85 | 109.61 | 110.88 | 0.009 | 0.027 | 0.008212 | 0.024357 |
| 1 | 110.00 | 110.00 | 110 | 110 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{24}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|-------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 120.50 | 147.00 | 120.5 | 147 | 0 | 0 | 0 | 0 |
| 0.1 | 121.41 | 145.21 | 121.45 | 145.3 | 0.045 | 0.09 | 0.037066 | 0.061979 |
| 0.2 | 122.32 | 143.44 | 122.4 | 143.6 | 0.08 | 0.16 | 0.065402 | 0.111545 |
| 0.3 | 123.25 | 141.69 | 123.35 | 141.9 | 0.105 | 0.21 | 0.085196 | 0.148211 |
| 0.4 | 124.18 | 139.96 | 124.3 | 140.2 | 0.12 | 0.24 | 0.096634 | 0.171478 |
| 0.5 | 125.13 | 138.25 | 125.25 | 138.5 | 0.125 | 0.25 | 0.0999 | 0.180832 |
| 0.6 | 126.08 | 136.56 | 126.2 | 136.8 | 0.12 | 0.24 | 0.095178 | 0.175747 |
| 0.7 | 127.05 | 134.89 | 127.15 | 135.1 | 0.105 | 0.21 | 0.082648 | 0.155682 |
| 0.8 | 128.02 | 133.24 | 128.1 | 133.4 | 0.08 | 0.16 | 0.06249 | 0.120084 |
| 0.9 | 129.01 | 131.61 | 129.05 | 131.7 | 0.045 | 0.09 | 0.034882 | 0.068384 |
| 1 | 130.00 | 130.00 | 130 | 130 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{25}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 252.50 | 329.85 | 252.5 | 329.85 | 0 | 0 | 0 | 0 |
| 0.1 | 255.16 | 324.65 | 255.25 | 324.865 | 0.09 | 0.2115 | 0.035272 | 0.065146 |
| 0.2 | 257.84 | 319.50 | 258 | 319.88 | 0.16 | 0.376 | 0.062054 | 0.117682 |
| 0.3 | 260.54 | 314.40 | 260.75 | 314.895 | 0.21 | 0.4935 | 0.080602 | 0.156965 |
| 0.4 | 263.26 | 309.35 | 263.5 | 309.91 | 0.24 | 0.564 | 0.091165 | 0.18232 |
| 0.5 | 266.00 | 304.34 | 266.25 | 304.925 | 0.25 | 0.5875 | 0.093985 | 0.193042 |
| 0.6 | 268.76 | 299.38 | 269 | 299.94 | 0.24 | 0.564 | 0.089299 | 0.188392 |
| 0.7 | 271.54 | 294.46 | 271.75 | 294.955 | 0.21 | 0.4935 | 0.077337 | 0.167594 |
| 0.8 | 274.34 | 289.59 | 274.5 | 289.97 | 0.16 | 0.376 | 0.058322 | 0.129837 |
| 0.9 | 277.16 | 284.77 | 277.25 | 284.985 | 0.09 | 0.2115 | 0.032472 | 0.07427 |
| 1 | 280.00 | 280.00 | 280 | 280 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{26}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 316.40 | 447.35 | 316.4 | 447.35 | 0 | 0 | 0 | 0 |
| 0.1 | 320.59 | 438.09 | 320.76 | 438.615 | 0.171 | 0.5265 | 0.053339 | 0.120181 |
| 0.2 | 324.82 | 428.94 | 325.12 | 429.88 | 0.304 | 0.936 | 0.093591 | 0.21821 |
| 0.3 | 329.08 | 419.92 | 329.48 | 421.145 | 0.399 | 1.2285 | 0.121247 | 0.292558 |
| 0.4 | 333.38 | 411.01 | 333.84 | 412.41 | 0.456 | 1.404 | 0.136779 | 0.341601 |
| 0.5 | 337.73 | 402.21 | 338.2 | 403.675 | 0.475 | 1.4625 | 0.140647 | 0.363614 |
| 0.6 | 342.10 | 393.54 | 342.56 | 394.94 | 0.456 | 1.404 | 0.133293 | 0.356765 |
| 0.7 | 346.52 | 384.98 | 346.92 | 386.205 | 0.399 | 1.2285 | 0.115145 | 0.31911 |
| 0.8 | 350.98 | 376.53 | 351.28 | 377.47 | 0.304 | 0.936 | 0.086616 | 0.248583 |
| 0.9 | 355.47 | 368.21 | 355.64 | 368.735 | 0.171 | 0.5265 | 0.048105 | 0.14299 |
| 1 | 360.00 | 360.00 | 360 | 360 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{34}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 105.20 | 119.40 | 105.2 | 119.4 | 0 | 0 | 0 | 0 |
| 0.1 | 105.66 | 118.42 | 105.68 | 118.46 | 0.018 | 0.036 | 0.017035 | 0.030399 |
| 0.2 | 106.13 | 117.46 | 106.16 | 117.52 | 0.032 | 0.064 | 0.030152 | 0.054488 |
| 0.3 | 106.60 | 116.50 | 106.64 | 116.58 | 0.042 | 0.084 | 0.0394 | 0.072105 |
| 0.4 | 107.07 | 115.54 | 107.12 | 115.64 | 0.048 | 0.096 | 0.04483 | 0.083085 |
| 0.5 | 107.55 | 114.60 | 107.6 | 114.7 | 0.05 | 0.1 | 0.04649 | 0.08726 |
| 0.6 | 108.03 | 113.66 | 108.08 | 113.76 | 0.048 | 0.096 | 0.044431 | 0.084459 |
| 0.7 | 108.52 | 112.74 | 108.56 | 112.82 | 0.042 | 0.084 | 0.038703 | 0.07451 |
| 0.8 | 109.01 | 111.82 | 109.04 | 111.88 | 0.032 | 0.064 | 0.029356 | 0.057237 |
| 0.9 | 109.50 | 110.90 | 109.52 | 110.94 | 0.018 | 0.036 | 0.016438 | 0.032461 |
| 1 | 110.00 | 110.00 | 110 | 110 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{35}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|-------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 194.00 | 241.30 | 194 | 241.3 | 0 | 0 | 0 | 0 |
| 0.1 | 195.56 | 238.05 | 195.6 | 238.17 | 0.045 | 0.117 | 0.023011 | 0.049149 |
| 0.2 | 197.12 | 234.83 | 197.2 | 235.04 | 0.08 | 0.208 | 0.040584 | 0.088574 |
| 0.3 | 198.70 | 231.64 | 198.8 | 231.91 | 0.105 | 0.273 | 0.052845 | 0.117857 |
| 0.4 | 200.28 | 228.47 | 200.4 | 228.78 | 0.12 | 0.312 | 0.059916 | 0.136562 |
| 0.5 | 201.88 | 225.33 | 202 | 225.65 | 0.125 | 0.325 | 0.06192 | 0.144236 |
| 0.6 | 203.48 | 222.21 | 203.6 | 222.52 | 0.12 | 0.312 | 0.058974 | 0.140409 |
| 0.7 | 205.10 | 219.12 | 205.2 | 219.39 | 0.105 | 0.273 | 0.051196 | 0.124591 |
| 0.8 | 206.72 | 216.05 | 206.8 | 216.26 | 0.08 | 0.208 | 0.0387 | 0.096273 |
| 0.9 | 208.36 | 213.01 | 208.4 | 213.13 | 0.045 | 0.117 | 0.021598 | 0.054926 |
| 1 | 210.00 | 210.00 | 210 | 210 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{36}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 241.70 | 330.05 | 241.7 | 330.05 | 0 | 0 | 0 | 0 |
| 0.1 | 244.42 | 323.68 | 244.53 | 324.045 | 0.108 | 0.3645 | 0.044186 | 0.112611 |
| 0.2 | 247.17 | 317.39 | 247.36 | 318.04 | 0.192 | 0.648 | 0.07768 | 0.204164 |
| 0.3 | 249.94 | 311.18 | 250.19 | 312.035 | 0.252 | 0.8505 | 0.100825 | 0.273311 |
| 0.4 | 252.73 | 305.06 | 253.02 | 306.03 | 0.288 | 0.972 | 0.113955 | 0.318628 |
| 0.5 | 255.55 | 299.01 | 255.85 | 300.025 | 0.3 | 1.0125 | 0.117394 | 0.338615 |
| 0.6 | 258.39 | 293.05 | 258.68 | 294.02 | 0.288 | 0.972 | 0.111459 | 0.331686 |
| 0.7 | 261.26 | 287.16 | 261.51 | 288.015 | 0.252 | 0.8505 | 0.096456 | 0.296172 |
| 0.8 | 264.15 | 281.36 | 264.34 | 282.01 | 0.192 | 0.648 | 0.072687 | 0.230308 |
| 0.9 | 267.06 | 275.64 | 267.17 | 276.005 | 0.108 | 0.3645 | 0.04044 | 0.132237 |
| 1 | 270.00 | 270.00 | 270 | 270 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{45}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 143.60 | 163.30 | 143.6 | 163.3 | 0 | 0 | 0 | 0 |
| 0.1 | 144.23 | 161.94 | 144.24 | 161.97 | 0.009 | 0.027 | 0.00624 | 0.016673 |
| 0.2 | 144.86 | 160.59 | 144.88 | 160.64 | 0.016 | 0.048 | 0.011045 | 0.029889 |
| 0.3 | 145.50 | 159.25 | 145.52 | 159.31 | 0.021 | 0.063 | 0.014433 | 0.039561 |
| 0.4 | 146.14 | 157.91 | 146.16 | 157.98 | 0.024 | 0.072 | 0.016423 | 0.045596 |
| 0.5 | 146.78 | 156.58 | 146.8 | 156.65 | 0.025 | 0.075 | 0.017033 | 0.0479 |
| 0.6 | 147.42 | 155.25 | 147.44 | 155.32 | 0.024 | 0.072 | 0.01628 | 0.046377 |
| 0.7 | 148.06 | 153.93 | 148.08 | 153.99 | 0.021 | 0.063 | 0.014184 | 0.040928 |
| 0.8 | 148.70 | 152.61 | 148.72 | 152.66 | 0.016 | 0.048 | 0.01076 | 0.031452 |
| 0.9 | 149.35 | 151.30 | 149.36 | 151.33 | 0.009 | 0.027 | 0.006026 | 0.017845 |
| 1 | 150.00 | 150.00 | 150 | 150 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{46}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 175.10 | 222.05 | 175.1 | 222.05 | 0 | 0 | 0 | 0 |
| 0.1 | 176.54 | 218.66 | 176.59 | 218.845 | 0.054 | 0.1845 | 0.030589 | 0.084377 |
| 0.2 | 177.98 | 215.31 | 178.08 | 215.64 | 0.096 | 0.328 | 0.053937 | 0.152337 |
| 0.3 | 179.44 | 212.00 | 179.57 | 212.435 | 0.126 | 0.4305 | 0.070217 | 0.203062 |
| 0.4 | 180.92 | 208.74 | 181.06 | 209.23 | 0.144 | 0.492 | 0.079595 | 0.235702 |
| 0.5 | 182.40 | 205.51 | 182.55 | 206.025 | 0.15 | 0.5125 | 0.082237 | 0.249377 |
| 0.6 | 183.90 | 202.33 | 184.04 | 202.82 | 0.144 | 0.492 | 0.078305 | 0.24317 |
| 0.7 | 185.40 | 199.18 | 185.53 | 199.615 | 0.126 | 0.4305 | 0.06796 | 0.216131 |
| 0.8 | 186.92 | 196.08 | 187.02 | 196.41 | 0.096 | 0.328 | 0.051358 | 0.167277 |
| 0.9 | 188.46 | 193.02 | 188.51 | 193.205 | 0.054 | 0.1845 | 0.028654 | 0.095586 |
| 1 | 190.00 | 190.00 | 190 | 190 | 0 | 0 | 0 | 0 |

Error analysis corresponding to T_{56}

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 112.40 | 136.25 | 112.4 | 136.25 | 0 | 0 | 0 | 0 |
| 0.1 | 113.12 | 134.51 | 113.16 | 134.625 | 0.036 | 0.1125 | 0.031823 | 0.083635 |
| 0.2 | 113.86 | 132.80 | 113.92 | 133 | 0.064 | 0.2 | 0.056211 | 0.150602 |
| 0.3 | 114.60 | 131.11 | 114.68 | 131.375 | 0.084 | 0.2625 | 0.073301 | 0.20021 |
| 0.4 | 115.34 | 129.45 | 115.44 | 129.75 | 0.096 | 0.3 | 0.083229 | 0.23175 |
| 0.5 | 116.10 | 127.81 | 116.2 | 128.125 | 0.1 | 0.3125 | 0.086133 | 0.244499 |
| 0.6 | 116.86 | 126.20 | 116.96 | 126.5 | 0.096 | 0.3 | 0.082147 | 0.237718 |
| 0.7 | 117.64 | 124.61 | 117.72 | 124.875 | 0.084 | 0.2625 | 0.071407 | 0.210653 |
| 0.8 | 118.42 | 123.05 | 118.48 | 123.25 | 0.064 | 0.2 | 0.054047 | 0.162536 |
| 0.9 | 119.20 | 121.51 | 119.24 | 121.625 | 0.036 | 0.1125 | 0.0302 | 0.092583 |
| 1 | 120.00 | 120.00 | 120 | 120 | 0 | 0 | 0 | 0 |

APPENDIX 2

Error Analysis for T_{ijk}

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Error Analysis

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Corresponding to T211

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|-------|-------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 3.40 | 17.50 | 3.4 | 17.5 | 0 | 0 | 0 | 0 |
| 0.1 | 3.98 | 16.66 | 4.06 | 16.75 | 0.081 | 0.09 | 2.035687 | 0.540216 |
| 0.2 | 4.58 | 15.84 | 4.72 | 16 | 0.144 | 0.16 | 3.146853 | 1.010101 |
| 0.3 | 5.19 | 15.04 | 5.38 | 15.25 | 0.189 | 0.21 | 3.640917 | 1.396277 |
| 0.4 | 5.82 | 14.26 | 6.04 | 14.5 | 0.216 | 0.24 | 3.708791 | 1.683029 |
| 0.5 | 6.48 | 13.50 | 6.7 | 13.75 | 0.225 | 0.25 | 3.474903 | 1.851852 |
| 0.6 | 7.14 | 12.76 | 7.36 | 13 | 0.216 | 0.24 | 3.023516 | 1.880878 |
| 0.7 | 7.83 | 12.04 | 8.02 | 12.25 | 0.189 | 0.21 | 2.413485 | 1.744186 |
| 0.8 | 8.54 | 11.34 | 8.68 | 11.5 | 0.144 | 0.16 | 1.686973 | 1.410935 |
| 0.9 | 9.26 | 10.66 | 9.34 | 10.75 | 0.081 | 0.09 | 0.874824 | 0.844278 |
| 1 | 10.00 | 10.00 | 10 | 10 | 0 | 0 | 0 | 0 |

Corresponding to T221

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 101.40 | 122.50 | 101.4 | 122.5 | 0 | 0 | 0 | 0 |
| 0.1 | 102.18 | 121.16 | 102.26 | 121.25 | 0.081 | 0.09 | 0.079273 | 0.074282 |
| 0.2 | 102.98 | 119.84 | 103.12 | 120 | 0.144 | 0.16 | 0.139838 | 0.133511 |
| 0.3 | 103.79 | 118.54 | 103.98 | 118.75 | 0.189 | 0.21 | 0.182097 | 0.177155 |
| 0.4 | 104.62 | 117.26 | 104.84 | 117.5 | 0.216 | 0.24 | 0.206454 | 0.204673 |
| 0.5 | 105.48 | 116.00 | 105.7 | 116.25 | 0.225 | 0.25 | 0.213321 | 0.215517 |
| 0.6 | 106.34 | 114.76 | 106.56 | 115 | 0.216 | 0.24 | 0.203114 | 0.209132 |
| 0.7 | 107.23 | 113.54 | 107.42 | 113.75 | 0.189 | 0.21 | 0.176255 | 0.184957 |
| 0.8 | 108.14 | 112.34 | 108.28 | 112.5 | 0.144 | 0.16 | 0.133166 | 0.142425 |
| 0.9 | 109.06 | 111.16 | 109.14 | 111.25 | 0.081 | 0.09 | 0.074272 | 0.080964 |
| 1 | 110.00 | 110.00 | 110 | 110 | 0 | 0 | 0 | 0 |

Corresponding to T231

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 109.50 | 136.30 | 109.5 | 136.3 | 0 | 0 | 0 | 0 |
| 0.1 | 110.46 | 134.55 | 110.55 | 134.67 | 0.09 | 0.117 | 0.081477 | 0.086955 |
| 0.2 | 111.44 | 132.83 | 111.6 | 133.04 | 0.16 | 0.208 | 0.143575 | 0.156589 |
| 0.3 | 112.44 | 131.14 | 112.65 | 131.41 | 0.21 | 0.273 | 0.186766 | 0.208179 |
| 0.4 | 113.46 | 129.47 | 113.7 | 129.78 | 0.24 | 0.312 | 0.211528 | 0.240986 |
| 0.5 | 114.50 | 127.83 | 114.75 | 128.15 | 0.25 | 0.325 | 0.218341 | 0.254254 |
| 0.6 | 115.56 | 126.21 | 115.8 | 126.52 | 0.24 | 0.312 | 0.207684 | 0.247211 |
| 0.7 | 116.64 | 124.62 | 116.85 | 124.89 | 0.21 | 0.273 | 0.180041 | 0.219071 |
| 0.8 | 117.74 | 123.05 | 117.9 | 123.26 | 0.16 | 0.208 | 0.135893 | 0.169034 |
| 0.9 | 118.86 | 121.51 | 118.95 | 121.63 | 0.09 | 0.117 | 0.075719 | 0.096286 |
| 1 | 120.00 | 120.00 | 120 | 120 | 0 | 0 | 0 | 0 |

Corresponding to T241

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 123.90 | 164.50 | 123.9 | 164.5 | 0 | 0 | 0 | 0 |
| 0.1 | 125.38 | 161.87 | 125.51 | 162.05 | 0.126 | 0.18 | 0.100491 | 0.1112 |
| 0.2 | 126.90 | 159.28 | 127.12 | 159.6 | 0.224 | 0.32 | 0.176523 | 0.200904 |
| 0.3 | 128.44 | 156.73 | 128.73 | 157.15 | 0.294 | 0.42 | 0.228908 | 0.267977 |
| 0.4 | 130.00 | 154.22 | 130.34 | 154.7 | 0.336 | 0.48 | 0.258454 | 0.311244 |
| 0.5 | 131.60 | 151.75 | 131.95 | 152.25 | 0.35 | 0.5 | 0.265957 | 0.329489 |
| 0.6 | 133.22 | 149.32 | 133.56 | 149.8 | 0.336 | 0.48 | 0.252207 | 0.321457 |
| 0.7 | 134.88 | 146.93 | 135.17 | 147.35 | 0.294 | 0.42 | 0.217978 | 0.28585 |
| 0.8 | 136.56 | 144.58 | 136.78 | 144.9 | 0.224 | 0.32 | 0.164035 | 0.221331 |
| 0.9 | 138.26 | 142.27 | 138.39 | 142.45 | 0.126 | 0.18 | 0.09113 | 0.12652 |
| 1 | 140.00 | 140.00 | 140 | 140 | 0 | 0 | 0 | 0 |

Corresponding to T251

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 255.90 | 347.35 | 255.9 | 347.35 | 0 | 0 | 0 | 0 |
| 0.1 | 259.14 | 341.31 | 259.31 | 341.615 | 0.171 | 0.3015 | 0.065988 | 0.088335 |
| 0.2 | 262.42 | 335.34 | 262.72 | 335.88 | 0.304 | 0.536 | 0.115847 | 0.159836 |
| 0.3 | 265.73 | 329.44 | 266.13 | 330.145 | 0.399 | 0.7035 | 0.150152 | 0.213543 |
| 0.4 | 269.08 | 323.61 | 269.54 | 324.41 | 0.456 | 0.804 | 0.169464 | 0.24845 |
| 0.5 | 272.48 | 317.84 | 272.95 | 318.675 | 0.475 | 0.8375 | 0.174328 | 0.263499 |
| 0.6 | 275.90 | 312.14 | 276.36 | 312.94 | 0.456 | 0.804 | 0.165275 | 0.25758 |
| 0.7 | 279.37 | 306.50 | 279.77 | 307.205 | 0.399 | 0.7035 | 0.142821 | 0.229526 |
| 0.8 | 282.88 | 300.93 | 283.18 | 301.47 | 0.304 | 0.536 | 0.107468 | 0.178112 |
| 0.9 | 286.42 | 295.43 | 286.59 | 295.735 | 0.171 | 0.3015 | 0.059703 | 0.102053 |
| 1 | 290.00 | 290.00 | 290 | 290 | 0 | 0 | 0 | 0 |

Corresponding to T261

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 319.80 | 464.85 | 319.8 | 464.85 | 0 | 0 | 0 | 0 |
| 0.1 | 324.57 | 454.75 | 324.82 | 455.365 | 0.252 | 0.6165 | 0.077642 | 0.135569 |
| 0.2 | 329.39 | 444.78 | 329.84 | 445.88 | 0.448 | 1.096 | 0.136008 | 0.246412 |
| 0.3 | 334.27 | 434.96 | 334.86 | 436.395 | 0.588 | 1.4385 | 0.175905 | 0.330723 |
| 0.4 | 339.21 | 425.27 | 339.88 | 426.91 | 0.672 | 1.644 | 0.198109 | 0.386582 |
| 0.5 | 344.20 | 415.71 | 344.9 | 417.425 | 0.7 | 1.7125 | 0.20337 | 0.411943 |
| 0.6 | 349.25 | 406.30 | 349.92 | 407.94 | 0.672 | 1.644 | 0.192413 | 0.404631 |
| 0.7 | 354.35 | 397.02 | 354.94 | 398.455 | 0.588 | 1.4385 | 0.165937 | 0.362328 |
| 0.8 | 359.51 | 387.87 | 359.96 | 388.97 | 0.448 | 1.096 | 0.124613 | 0.282566 |
| 0.9 | 364.73 | 378.87 | 364.98 | 379.485 | 0.252 | 0.6165 | 0.069093 | 0.162721 |
| 1 | 370.00 | 370.00 | 370 | 370 | 0 | 0 | 0 | 0 |

Corresponding to T321

145

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|--------|-------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 11.25 | 31.20 | 11.25 | 31.2 | 0 | 0 | 0 | 0 |
| 0.1 | 12.51 | 30.56 | 12.625 | 30.58 | 0.1125 | 0.018 | 0.899101 | 0.058897 |
| 0.2 | 13.80 | 29.93 | 14 | 29.96 | 0.2 | 0.032 | 1.449275 | 0.106923 |
| 0.3 | 15.11 | 29.30 | 15.375 | 29.34 | 0.2625 | 0.042 | 1.736973 | 0.143354 |
| 0.4 | 16.45 | 28.67 | 16.75 | 28.72 | 0.3 | 0.048 | 1.823708 | 0.167411 |
| 0.5 | 17.81 | 28.05 | 18.125 | 28.1 | 0.3125 | 0.05 | 1.754386 | 0.178253 |
| 0.6 | 19.20 | 27.43 | 19.5 | 27.48 | 0.3 | 0.048 | 1.5625 | 0.174978 |
| 0.7 | 20.61 | 26.82 | 20.875 | 26.86 | 0.2625 | 0.042 | 1.273499 | 0.156611 |
| 0.8 | 22.05 | 26.21 | 22.25 | 26.24 | 0.2 | 0.032 | 0.907029 | 0.1221 |
| 0.9 | 23.51 | 25.60 | 23.625 | 25.62 | 0.1125 | 0.018 | 0.478469 | 0.070307 |
| 1 | 25.00 | 25.00 | 25 | 25 | 0 | 0 | 0 | 0 |

Corresponding to T331

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 109.25 | 136.20 | 109.25 | 136.2 | 0 | 0 | 0 | 0 |
| 0.1 | 110.71 | 135.06 | 110.825 | 135.08 | 0.1125 | 0.018 | 0.101615 | 0.013327 |
| 0.2 | 112.20 | 133.93 | 112.4 | 133.96 | 0.2 | 0.032 | 0.178253 | 0.023893 |
| 0.3 | 113.71 | 132.80 | 113.975 | 132.84 | 0.2625 | 0.042 | 0.230845 | 0.031627 |
| 0.4 | 115.25 | 131.67 | 115.55 | 131.72 | 0.3 | 0.048 | 0.260304 | 0.036454 |
| 0.5 | 116.81 | 130.55 | 117.125 | 130.6 | 0.3125 | 0.05 | 0.267523 | 0.0383 |
| 0.6 | 118.40 | 129.43 | 118.7 | 129.48 | 0.3 | 0.048 | 0.253378 | 0.037085 |
| 0.7 | 120.01 | 128.32 | 120.275 | 128.36 | 0.2625 | 0.042 | 0.218727 | 0.032731 |
| 0.8 | 121.65 | 127.21 | 121.85 | 127.24 | 0.2 | 0.032 | 0.164406 | 0.025156 |
| 0.9 | 123.31 | 126.10 | 123.425 | 126.12 | 0.1125 | 0.018 | 0.091232 | 0.014274 |
| 1 | 125.00 | 125.00 | 125 | 125 | 0 | 0 | 0 | 0 |

Corresponding to T341

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 116.45 | 150.60 | 116.45 | 150.6 | 0 | 0 | 0 | 0 |
| 0.1 | 118.17 | 148.99 | 118.305 | 149.04 | 0.1305 | 0.054 | 0.11043 | 0.036245 |
| 0.2 | 119.93 | 147.38 | 120.16 | 147.48 | 0.232 | 0.096 | 0.193449 | 0.065136 |
| 0.3 | 121.71 | 145.79 | 122.015 | 145.92 | 0.3045 | 0.126 | 0.250184 | 0.086423 |
| 0.4 | 123.52 | 144.22 | 123.87 | 144.36 | 0.348 | 0.144 | 0.281731 | 0.09985 |
| 0.5 | 125.36 | 142.65 | 125.725 | 142.8 | 0.3625 | 0.15 | 0.289161 | 0.105152 |
| 0.6 | 127.23 | 141.10 | 127.58 | 141.24 | 0.348 | 0.144 | 0.273516 | 0.102058 |
| 0.7 | 129.13 | 139.55 | 129.435 | 139.68 | 0.3045 | 0.126 | 0.235808 | 0.090288 |
| 0.8 | 131.06 | 138.02 | 131.29 | 138.12 | 0.232 | 0.096 | 0.177021 | 0.069553 |
| 0.9 | 133.01 | 136.51 | 133.145 | 136.56 | 0.1305 | 0.054 | 0.09811 | 0.039559 |
| 1 | 135.00 | 135.00 | 135 | 135 | 0 | 0 | 0 | 0 |

Corresponding to T351

146

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 205.25 | 272.50 | 205.25 | 272.5 | 0 | 0 | 0 | 0 |
| 0.1 | 208.07 | 268.62 | 208.225 | 268.75 | 0.1575 | 0.135 | 0.075697 | 0.050258 |
| 0.2 | 210.92 | 264.76 | 211.2 | 265 | 0.28 | 0.24 | 0.132752 | 0.090648 |
| 0.3 | 213.81 | 260.94 | 214.175 | 261.25 | 0.3675 | 0.315 | 0.171884 | 0.12072 |
| 0.4 | 216.73 | 257.14 | 217.15 | 257.5 | 0.42 | 0.36 | 0.19379 | 0.140002 |
| 0.5 | 219.69 | 253.38 | 220.125 | 253.75 | 0.4375 | 0.375 | 0.199147 | 0.148002 |
| 0.6 | 222.68 | 249.64 | 223.1 | 250 | 0.42 | 0.36 | 0.188611 | 0.144208 |
| 0.7 | 225.71 | 245.94 | 226.075 | 246.25 | 0.3675 | 0.315 | 0.162821 | 0.128083 |
| 0.8 | 228.77 | 242.26 | 229.05 | 242.5 | 0.28 | 0.24 | 0.122394 | 0.099067 |
| 0.9 | 231.87 | 238.62 | 232.025 | 238.75 | 0.1575 | 0.135 | 0.067927 | 0.056576 |
| 1 | 235.00 | 235.00 | 235 | 235 | 0 | 0 | 0 | 0 |

Corresponding to T351

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 252.95 | 361.25 | 252.95 | 361.25 | 0 | 0 | 0 | 0 |
| 0.1 | 256.93 | 354.24 | 257.155 | 354.625 | 0.2205 | 0.3825 | 0.08582 | 0.107977 |
| 0.2 | 260.97 | 347.32 | 261.36 | 348 | 0.392 | 0.68 | 0.15021 | 0.195785 |
| 0.3 | 265.05 | 340.48 | 265.565 | 341.375 | 0.5145 | 0.8925 | 0.194114 | 0.262128 |
| 0.4 | 269.18 | 333.73 | 269.77 | 334.75 | 0.588 | 1.02 | 0.21844 | 0.305636 |
| 0.5 | 273.36 | 327.06 | 273.975 | 328.125 | 0.6125 | 1.0625 | 0.224061 | 0.324861 |
| 0.6 | 277.59 | 320.48 | 278.18 | 321.5 | 0.588 | 1.02 | 0.211822 | 0.318273 |
| 0.7 | 281.87 | 313.98 | 282.385 | 314.875 | 0.5145 | 0.8925 | 0.182531 | 0.284252 |
| 0.8 | 286.20 | 307.57 | 286.59 | 308.25 | 0.392 | 0.68 | 0.136968 | 0.221088 |
| 0.9 | 290.57 | 301.24 | 290.795 | 301.625 | 0.2205 | 0.3825 | 0.075884 | 0.126974 |
| 1 | 295.00 | 295.00 | 295 | 295 | 0 | 0 | 0 | 0 |

Corresponding to T431

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|-------|-------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 3.15 | 8.40 | 3.15 | 8.4 | 0 | 0 | 0 | 0 |
| 0.1 | 3.32 | 8.02 | 3.335 | 8.06 | 0.0135 | 0.036 | 0.406443 | 0.448654 |
| 0.2 | 3.50 | 7.66 | 3.52 | 7.72 | 0.024 | 0.064 | 0.686499 | 0.835946 |
| 0.3 | 3.67 | 7.30 | 3.705 | 7.38 | 0.0315 | 0.084 | 0.857493 | 1.151316 |
| 0.4 | 3.85 | 6.94 | 3.89 | 7.04 | 0.036 | 0.096 | 0.934094 | 1.382488 |
| 0.5 | 4.04 | 6.60 | 4.075 | 6.7 | 0.0375 | 0.1 | 0.928793 | 1.515152 |
| 0.6 | 4.22 | 6.26 | 4.26 | 6.36 | 0.036 | 0.096 | 0.852273 | 1.532567 |
| 0.7 | 4.41 | 5.94 | 4.445 | 6.02 | 0.0315 | 0.084 | 0.713719 | 1.415094 |
| 0.8 | 4.61 | 5.62 | 4.63 | 5.68 | 0.024 | 0.064 | 0.521059 | 1.139601 |
| 0.9 | 4.80 | 5.30 | 4.815 | 5.34 | 0.0135 | 0.036 | 0.281162 | 0.678733 |
| 1 | 5.00 | 5.00 | 5 | 5 | 0 | 0 | 0 | 0 |

Corresponding to T441

147

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 101.15 | 113.40 | 101.15 | 113.4 | 0 | 0 | 0 | 0 |
| 0.1 | 101.52 | 112.52 | 101.535 | 112.56 | 0.0135 | 0.036 | 0.013298 | 0.031993 |
| 0.2 | 101.90 | 111.66 | 101.92 | 111.72 | 0.024 | 0.064 | 0.023553 | 0.057319 |
| 0.3 | 102.27 | 110.80 | 102.305 | 110.88 | 0.0315 | 0.084 | 0.0308 | 0.075815 |
| 0.4 | 102.65 | 109.94 | 102.69 | 110.04 | 0.036 | 0.096 | 0.035069 | 0.087317 |
| 0.5 | 103.04 | 109.10 | 103.075 | 109.2 | 0.0375 | 0.1 | 0.036395 | 0.091659 |
| 0.6 | 103.42 | 108.26 | 103.46 | 108.36 | 0.036 | 0.096 | 0.034808 | 0.088672 |
| 0.7 | 103.81 | 107.44 | 103.845 | 107.52 | 0.0315 | 0.084 | 0.030343 | 0.078186 |
| 0.8 | 104.21 | 106.62 | 104.23 | 106.68 | 0.024 | 0.064 | 0.023031 | 0.060029 |
| 0.9 | 104.60 | 105.80 | 104.615 | 105.84 | 0.0135 | 0.036 | 0.012906 | 0.034025 |
| 1 | 105.00 | 105.00 | 105 | 105 | 0 | 0 | 0 | 0 |

Corresponding to T451

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 146.75 | 171.70 | 146.75 | 171.7 | 0 | 0 | 0 | 0 |
| 0.1 | 147.55 | 169.97 | 147.575 | 170.03 | 0.0225 | 0.063 | 0.015249 | 0.037066 |
| 0.2 | 148.36 | 168.25 | 148.4 | 168.36 | 0.04 | 0.112 | 0.026961 | 0.066568 |
| 0.3 | 149.17 | 166.54 | 149.225 | 166.69 | 0.0525 | 0.147 | 0.035194 | 0.088265 |
| 0.4 | 149.99 | 164.85 | 150.05 | 165.02 | 0.06 | 0.168 | 0.040003 | 0.10191 |
| 0.5 | 150.81 | 163.18 | 150.875 | 163.35 | 0.0625 | 0.175 | 0.041442 | 0.107247 |
| 0.6 | 151.64 | 161.51 | 151.7 | 161.68 | 0.06 | 0.168 | 0.039567 | 0.104017 |
| 0.7 | 152.47 | 159.86 | 152.525 | 160.01 | 0.0525 | 0.147 | 0.034432 | 0.091954 |
| 0.8 | 153.31 | 158.23 | 153.35 | 158.34 | 0.04 | 0.112 | 0.026091 | 0.070784 |
| 0.9 | 154.15 | 156.61 | 154.175 | 156.67 | 0.0225 | 0.063 | 0.014596 | 0.040228 |
| 1 | 155.00 | 155.00 | 155 | 155 | 0 | 0 | 0 | 0 |

Corresponding to T461

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 178.25 | 230.45 | 178.25 | 230.45 | 0 | 0 | 0 | 0 |
| 0.1 | 179.86 | 226.68 | 179.925 | 226.905 | 0.0675 | 0.2205 | 0.03753 | 0.097272 |
| 0.2 | 181.48 | 222.97 | 181.6 | 223.36 | 0.12 | 0.392 | 0.066123 | 0.17581 |
| 0.3 | 183.12 | 219.30 | 183.275 | 219.815 | 0.1575 | 0.5145 | 0.08601 | 0.23461 |
| 0.4 | 184.77 | 215.68 | 184.95 | 216.27 | 0.18 | 0.588 | 0.097418 | 0.272624 |
| 0.5 | 186.44 | 212.11 | 186.625 | 212.725 | 0.1875 | 0.6125 | 0.10057 | 0.288762 |
| 0.6 | 188.12 | 208.59 | 188.3 | 209.18 | 0.18 | 0.588 | 0.095684 | 0.28189 |
| 0.7 | 189.82 | 205.12 | 189.975 | 205.635 | 0.1575 | 0.5145 | 0.082974 | 0.250828 |
| 0.8 | 191.53 | 201.70 | 191.65 | 202.09 | 0.12 | 0.392 | 0.062653 | 0.19435 |
| 0.9 | 193.26 | 198.32 | 193.325 | 198.545 | 0.0675 | 0.2205 | 0.034927 | 0.111181 |
| 1 | 195.00 | 195.00 | 195 | 195 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|-------|-------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 2.40 | 7.80 | 2.4 | 7.8 | 0 | 0 | 0 | 0 |
| 0.1 | 2.62 | 7.49 | 2.66 | 7.52 | 0.036 | 0.027 | 1.371951 | 0.360336 |
| 0.2 | 2.86 | 7.19 | 2.92 | 7.24 | 0.064 | 0.048 | 2.240896 | 0.667408 |
| 0.3 | 3.10 | 6.90 | 3.18 | 6.96 | 0.084 | 0.063 | 2.713178 | 0.913441 |
| 0.4 | 3.34 | 6.61 | 3.44 | 6.68 | 0.096 | 0.072 | 2.870813 | 1.089588 |
| 0.5 | 3.60 | 6.33 | 3.7 | 6.4 | 0.1 | 0.075 | 2.777778 | 1.185771 |
| 0.6 | 3.86 | 6.05 | 3.96 | 6.12 | 0.096 | 0.072 | 2.484472 | 1.190476 |
| 0.7 | 4.14 | 5.78 | 4.22 | 5.84 | 0.084 | 0.063 | 2.030948 | 1.090531 |
| 0.8 | 4.42 | 5.51 | 4.48 | 5.56 | 0.064 | 0.048 | 1.449275 | 0.870827 |
| 0.9 | 4.70 | 5.25 | 4.74 | 5.28 | 0.036 | 0.027 | 0.765306 | 0.513992 |
| 1 | 5.00 | 5.00 | 5 | 5 | 0 | 0 | 0 | 0 |

Corresponding to T551

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 100.40 | 112.80 | 100.4 | 112.8 | 0 | 0 | 0 | 0 |
| 0.1 | 100.82 | 111.99 | 100.86 | 112.02 | 0.036 | 0.027 | 0.035706 | 0.024109 |
| 0.2 | 101.26 | 111.19 | 101.32 | 111.24 | 0.064 | 0.048 | 0.063206 | 0.043169 |
| 0.3 | 101.70 | 110.40 | 101.78 | 110.46 | 0.084 | 0.063 | 0.082599 | 0.057067 |
| 0.4 | 102.14 | 109.61 | 102.24 | 109.68 | 0.096 | 0.072 | 0.093985 | 0.065689 |
| 0.5 | 102.60 | 108.83 | 102.7 | 108.9 | 0.1 | 0.075 | 0.097466 | 0.068918 |
| 0.6 | 103.06 | 108.05 | 103.16 | 108.12 | 0.096 | 0.072 | 0.093146 | 0.066637 |
| 0.7 | 103.54 | 107.28 | 103.62 | 107.34 | 0.084 | 0.063 | 0.081131 | 0.058726 |
| 0.8 | 104.02 | 106.51 | 104.08 | 106.56 | 0.064 | 0.048 | 0.061529 | 0.045065 |
| 0.9 | 104.50 | 105.75 | 104.54 | 105.78 | 0.036 | 0.027 | 0.034448 | 0.025531 |
| 1 | 105.00 | 105.00 | 105 | 105 | 0 | 0 | 0 | 0 |

Corresponding to T561

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 114.80 | 144.05 | 114.8 | 144.05 | 0 | 0 | 0 | 0 |
| 0.1 | 115.75 | 142.01 | 115.82 | 142.145 | 0.072 | 0.1395 | 0.062204 | 0.098236 |
| 0.2 | 116.71 | 139.99 | 116.84 | 140.24 | 0.128 | 0.248 | 0.109672 | 0.177153 |
| 0.3 | 117.69 | 138.01 | 117.86 | 138.335 | 0.168 | 0.3255 | 0.142745 | 0.235853 |
| 0.4 | 118.69 | 136.06 | 118.88 | 136.43 | 0.192 | 0.372 | 0.161769 | 0.273413 |
| 0.5 | 119.70 | 134.14 | 119.9 | 134.525 | 0.2 | 0.3875 | 0.167084 | 0.288883 |
| 0.6 | 120.73 | 132.25 | 120.92 | 132.62 | 0.192 | 0.372 | 0.159035 | 0.28129 |
| 0.7 | 121.77 | 130.39 | 121.94 | 130.715 | 0.168 | 0.3255 | 0.137963 | 0.249637 |
| 0.8 | 122.83 | 128.56 | 122.96 | 128.81 | 0.128 | 0.248 | 0.104207 | 0.192903 |
| 0.9 | 123.91 | 126.77 | 123.98 | 126.905 | 0.072 | 0.1395 | 0.058108 | 0.110046 |
| 1 | 125.00 | 125.00 | 125 | 125 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|--------|--------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 20.16 | 29.15 | 20.16 | 29.15 | 0 | 0 | 0 | 0 |
| 0.1 | 20.63 | 28.72 | 20.644 | 28.735 | 0.0144 | 0.0135 | 0.069803 | 0.047003 |
| 0.2 | 21.10 | 28.30 | 21.128 | 28.32 | 0.0256 | 0.024 | 0.121313 | 0.084818 |
| 0.3 | 21.58 | 27.87 | 21.612 | 27.905 | 0.0336 | 0.0315 | 0.155711 | 0.113011 |
| 0.4 | 22.06 | 27.45 | 22.096 | 27.49 | 0.0384 | 0.036 | 0.17409 | 0.131128 |
| 0.5 | 22.54 | 27.04 | 22.58 | 27.075 | 0.04 | 0.0375 | 0.177462 | 0.138696 |
| 0.6 | 23.03 | 26.62 | 23.064 | 26.66 | 0.0384 | 0.036 | 0.166771 | 0.135216 |
| 0.7 | 23.51 | 26.21 | 23.548 | 26.245 | 0.0336 | 0.0315 | 0.142891 | 0.120167 |
| 0.8 | 24.01 | 25.81 | 24.032 | 25.83 | 0.0256 | 0.024 | 0.106638 | 0.093002 |
| 0.9 | 24.50 | 25.40 | 24.516 | 25.415 | 0.0144 | 0.0135 | 0.058772 | 0.053146 |
| 1 | 25.00 | 25.00 | 25 | 25 | 0 | 0 | 0 | 0 |

Corresponding to T661

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 118.16 | 134.15 | 118.16 | 134.15 | 0 | 0 | 0 | 0 |
| 0.1 | 118.83 | 133.22 | 118.844 | 133.235 | 0.0144 | 0.0135 | 0.012118 | 0.010133 |
| 0.2 | 119.50 | 132.30 | 119.528 | 132.32 | 0.0256 | 0.024 | 0.021422 | 0.018141 |
| 0.3 | 120.18 | 131.37 | 120.212 | 131.405 | 0.0336 | 0.0315 | 0.027958 | 0.023977 |
| 0.4 | 120.86 | 130.45 | 120.896 | 130.49 | 0.0384 | 0.036 | 0.031773 | 0.027596 |
| 0.5 | 121.54 | 129.54 | 121.58 | 129.575 | 0.04 | 0.0375 | 0.032911 | 0.028949 |
| 0.6 | 122.23 | 128.62 | 122.264 | 128.66 | 0.0384 | 0.036 | 0.031417 | 0.027989 |
| 0.7 | 122.91 | 127.71 | 122.948 | 127.745 | 0.0336 | 0.0315 | 0.027336 | 0.024665 |
| 0.8 | 123.61 | 126.81 | 123.632 | 126.83 | 0.0256 | 0.024 | 0.020711 | 0.018927 |
| 0.9 | 124.30 | 125.90 | 124.316 | 125.915 | 0.0144 | 0.0135 | 0.011585 | 0.010723 |
| 1 | 125.00 | 125.00 | 125 | 125 | 0 | 0 | 0 | 0 |

Corresponding to T761

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|-------|--------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 4.50 | 16.25 | 4.5 | 16.25 | 0 | 0 | 0 | 0 |
| 0.1 | 5.01 | 15.56 | 5.05 | 15.625 | 0.045 | 0.0675 | 0.899101 | 0.433874 |
| 0.2 | 5.52 | 14.88 | 5.6 | 15 | 0.08 | 0.12 | 1.449275 | 0.806452 |
| 0.3 | 6.05 | 14.22 | 6.15 | 14.375 | 0.105 | 0.1575 | 1.736973 | 1.10779 |
| 0.4 | 6.58 | 13.57 | 6.7 | 13.75 | 0.12 | 0.18 | 1.823708 | 1.326455 |
| 0.5 | 7.13 | 12.94 | 7.25 | 13.125 | 0.125 | 0.1875 | 1.754386 | 1.449275 |
| 0.6 | 7.68 | 12.32 | 7.8 | 12.5 | 0.12 | 0.18 | 1.5625 | 1.461039 |
| 0.7 | 8.25 | 11.72 | 8.35 | 11.875 | 0.105 | 0.1575 | 1.273499 | 1.344143 |
| 0.8 | 8.82 | 11.13 | 8.9 | 11.25 | 0.08 | 0.12 | 0.907029 | 1.078167 |
| 0.9 | 9.41 | 10.56 | 9.45 | 10.625 | 0.045 | 0.0675 | 0.478469 | 0.639356 |
| 1 | 10.00 | 10.00 | 10 | 10 | 0 | 0 | 0 | 0 |

Corresponding to T322

150

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|-------|-------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 18.90 | 63.70 | 18.9 | 63.7 | 0 | 0 | 0 | 0 |
| 0.1 | 21.25 | 61.68 | 21.51 | 61.83 | 0.261 | 0.153 | 1.228293 | 0.248067 |
| 0.2 | 23.66 | 59.69 | 24.12 | 59.96 | 0.464 | 0.272 | 1.961447 | 0.455703 |
| 0.3 | 26.12 | 57.73 | 26.73 | 58.09 | 0.609 | 0.357 | 2.331457 | 0.618364 |
| 0.4 | 28.64 | 55.81 | 29.34 | 56.22 | 0.696 | 0.408 | 2.429828 | 0.731026 |
| 0.5 | 31.23 | 53.93 | 31.95 | 54.35 | 0.725 | 0.425 | 2.321857 | 0.788132 |
| 0.6 | 33.86 | 52.07 | 34.56 | 52.48 | 0.696 | 0.408 | 2.05528 | 0.78353 |
| 0.7 | 36.56 | 50.25 | 37.17 | 50.61 | 0.609 | 0.357 | 1.665709 | 0.710405 |
| 0.8 | 39.32 | 48.47 | 39.78 | 48.74 | 0.464 | 0.272 | 1.180181 | 0.561195 |
| 0.9 | 42.13 | 46.72 | 42.39 | 46.87 | 0.261 | 0.153 | 0.619526 | 0.327504 |
| 1 | 45.00 | 45.00 | 45 | 45 | 0 | 0 | 0 | 0 |

Corresponding to T332

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 116.90 | 168.70 | 116.9 | 168.7 | 0 | 0 | 0 | 0 |
| 0.1 | 119.45 | 166.18 | 119.71 | 166.33 | 0.261 | 0.153 | 0.218503 | 0.092071 |
| 0.2 | 122.06 | 163.69 | 122.52 | 163.96 | 0.464 | 0.272 | 0.380153 | 0.16617 |
| 0.3 | 124.72 | 161.23 | 125.33 | 161.59 | 0.609 | 0.357 | 0.48829 | 0.221419 |
| 0.4 | 127.44 | 158.81 | 128.14 | 159.22 | 0.696 | 0.408 | 0.546122 | 0.256908 |
| 0.5 | 130.23 | 156.43 | 130.95 | 156.85 | 0.725 | 0.425 | 0.556729 | 0.271696 |
| 0.6 | 133.06 | 154.07 | 133.76 | 154.48 | 0.696 | 0.408 | 0.523057 | 0.264811 |
| 0.7 | 135.96 | 151.75 | 136.57 | 152.11 | 0.609 | 0.357 | 0.447923 | 0.235251 |
| 0.8 | 138.92 | 149.47 | 139.38 | 149.74 | 0.464 | 0.272 | 0.334015 | 0.181979 |
| 0.9 | 141.93 | 147.22 | 142.19 | 147.37 | 0.261 | 0.153 | 0.183895 | 0.103928 |
| 1 | 145.00 | 145.00 | 145 | 145 | 0 | 0 | 0 | 0 |

Corresponding to T342

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 124.10 | 183.10 | 124.1 | 183.1 | 0 | 0 | 0 | 0 |
| 0.1 | 126.91 | 180.10 | 127.19 | 180.29 | 0.279 | 0.189 | 0.219839 | 0.104941 |
| 0.2 | 129.78 | 177.14 | 130.28 | 177.48 | 0.496 | 0.336 | 0.382173 | 0.189676 |
| 0.3 | 132.72 | 174.23 | 133.37 | 174.67 | 0.651 | 0.441 | 0.49051 | 0.253115 |
| 0.4 | 135.72 | 171.36 | 136.46 | 171.86 | 0.744 | 0.504 | 0.548204 | 0.294125 |
| 0.5 | 138.78 | 168.53 | 139.55 | 169.05 | 0.775 | 0.525 | 0.558458 | 0.311526 |
| 0.6 | 141.90 | 165.74 | 142.64 | 166.24 | 0.744 | 0.504 | 0.524328 | 0.304098 |
| 0.7 | 145.08 | 162.99 | 145.73 | 163.43 | 0.651 | 0.441 | 0.448721 | 0.27057 |
| 0.8 | 148.32 | 160.28 | 148.82 | 160.62 | 0.496 | 0.336 | 0.334403 | 0.209628 |
| 0.9 | 151.63 | 157.62 | 151.91 | 157.81 | 0.279 | 0.189 | 0.183999 | 0.119908 |
| 1 | 155.00 | 155.00 | 155 | 155 | 0 | 0 | 0 | 0 |

Corresponding to T352

151

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|-------|-------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 212.90 | 305.00 | 212.9 | 305 | 0 | 0 | 0 | 0 |
| 0.1 | 216.80 | 299.73 | 217.11 | 300 | 0.306 | 0.27 | 0.141141 | 0.090081 |
| 0.2 | 220.78 | 294.52 | 221.32 | 295 | 0.544 | 0.48 | 0.246404 | 0.162977 |
| 0.3 | 224.82 | 289.37 | 225.53 | 290 | 0.714 | 0.63 | 0.317593 | 0.217714 |
| 0.4 | 228.92 | 284.28 | 229.74 | 285 | 0.816 | 0.72 | 0.35645 | 0.253271 |
| 0.5 | 233.10 | 279.25 | 233.95 | 280 | 0.85 | 0.75 | 0.36465 | 0.268577 |
| 0.6 | 237.34 | 274.28 | 238.16 | 275 | 0.816 | 0.72 | 0.343805 | 0.262505 |
| 0.7 | 241.66 | 269.37 | 242.37 | 270 | 0.714 | 0.63 | 0.295461 | 0.233879 |
| 0.8 | 246.04 | 264.52 | 246.58 | 265 | 0.544 | 0.48 | 0.221106 | 0.181461 |
| 0.9 | 250.48 | 259.73 | 250.79 | 260 | 0.306 | 0.27 | 0.122163 | 0.103954 |
| 1 | 255.00 | 255.00 | 255 | 255 | 0 | 0 | 0 | 0 |

Corresponding to T362

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 260.60 | 393.75 | 260.6 | 393.75 | 0 | 0 | 0 | 0 |
| 0.1 | 265.67 | 385.36 | 266.04 | 385.875 | 0.369 | 0.5175 | 0.138894 | 0.134291 |
| 0.2 | 270.82 | 377.08 | 271.48 | 378 | 0.656 | 0.92 | 0.242224 | 0.24398 |
| 0.3 | 276.06 | 368.92 | 276.92 | 370.125 | 0.861 | 1.2075 | 0.31189 | 0.327309 |
| 0.4 | 281.38 | 360.87 | 282.36 | 362.25 | 0.984 | 1.38 | 0.34971 | 0.382409 |
| 0.5 | 286.78 | 352.94 | 287.8 | 354.375 | 1.025 | 1.4375 | 0.357423 | 0.407296 |
| 0.6 | 292.26 | 345.12 | 293.24 | 346.5 | 0.984 | 1.38 | 0.336691 | 0.399861 |
| 0.7 | 297.82 | 337.42 | 298.68 | 338.625 | 0.861 | 1.2075 | 0.289102 | 0.357865 |
| 0.8 | 303.46 | 329.83 | 304.12 | 330.75 | 0.656 | 0.92 | 0.216171 | 0.278932 |
| 0.9 | 309.19 | 322.36 | 309.56 | 322.875 | 0.369 | 0.5175 | 0.119344 | 0.160536 |
| 1 | 315.00 | 315.00 | 315 | 315 | 0 | 0 | 0 | 0 |

Corresponding to T432

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|--------|-------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 30.15 | 76.00 | 30.15 | 76 | 0 | 0 | 0 | 0 |
| 0.1 | 32.44 | 73.81 | 32.635 | 73.9 | 0.1935 | 0.09 | 0.596458 | 0.121935 |
| 0.2 | 34.78 | 71.64 | 35.12 | 71.8 | 0.344 | 0.16 | 0.989188 | 0.223339 |
| 0.3 | 37.15 | 69.49 | 37.605 | 69.7 | 0.4515 | 0.21 | 1.215229 | 0.302202 |
| 0.4 | 39.57 | 67.36 | 40.09 | 67.6 | 0.516 | 0.24 | 1.303886 | 0.356295 |
| 0.5 | 42.04 | 65.25 | 42.575 | 65.5 | 0.5375 | 0.25 | 1.27862 | 0.383142 |
| 0.6 | 44.54 | 63.16 | 45.06 | 63.4 | 0.516 | 0.24 | 1.158405 | 0.379987 |
| 0.7 | 47.09 | 61.09 | 47.545 | 61.3 | 0.4515 | 0.21 | 0.958731 | 0.343755 |
| 0.8 | 49.69 | 59.04 | 50.03 | 59.2 | 0.344 | 0.16 | 0.692348 | 0.271003 |
| 0.9 | 52.32 | 57.01 | 52.515 | 57.1 | 0.1935 | 0.09 | 0.369829 | 0.157867 |
| 1 | 55.00 | 55.00 | 55 | 55 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|-------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 128.15 | 181.00 | 128.15 | 181 | 0 | 0 | 0 | 0 |
| 0.1 | 130.64 | 178.31 | 130.835 | 178.4 | 0.1935 | 0.09 | 0.148115 | 0.050474 |
| 0.2 | 133.18 | 175.64 | 133.52 | 175.8 | 0.344 | 0.16 | 0.258305 | 0.091095 |
| 0.3 | 135.75 | 172.99 | 136.205 | 173.2 | 0.4515 | 0.21 | 0.332588 | 0.121394 |
| 0.4 | 138.37 | 170.36 | 138.89 | 170.6 | 0.516 | 0.24 | 0.372902 | 0.140878 |
| 0.5 | 141.04 | 167.75 | 141.575 | 168 | 0.5375 | 0.25 | 0.381104 | 0.149031 |
| 0.6 | 143.74 | 165.16 | 144.26 | 165.4 | 0.516 | 0.24 | 0.358972 | 0.145314 |
| 0.7 | 146.49 | 162.59 | 146.945 | 162.8 | 0.4515 | 0.21 | 0.308205 | 0.129159 |
| 0.8 | 149.29 | 160.04 | 149.63 | 160.2 | 0.344 | 0.16 | 0.23043 | 0.099975 |
| 0.9 | 152.12 | 157.51 | 152.315 | 157.6 | 0.1935 | 0.09 | 0.127201 | 0.057139 |
| 1 | 155.00 | 155.00 | 155 | 155 | 0 | 0 | 0 | 0 |

Corresponding to T452

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 173.75 | 239.30 | 173.75 | 239.3 | 0 | 0 | 0 | 0 |
| 0.1 | 176.67 | 235.75 | 176.875 | 235.87 | 0.2025 | 0.117 | 0.114619 | 0.049628 |
| 0.2 | 179.64 | 232.23 | 180 | 232.44 | 0.36 | 0.208 | 0.200401 | 0.089566 |
| 0.3 | 182.65 | 228.74 | 183.125 | 229.01 | 0.4725 | 0.273 | 0.258688 | 0.119351 |
| 0.4 | 185.71 | 225.27 | 186.25 | 225.58 | 0.54 | 0.312 | 0.290776 | 0.138502 |
| 0.5 | 188.81 | 221.83 | 189.375 | 222.15 | 0.5625 | 0.325 | 0.297915 | 0.146512 |
| 0.6 | 191.96 | 218.41 | 192.5 | 218.72 | 0.54 | 0.312 | 0.281309 | 0.142852 |
| 0.7 | 195.15 | 215.02 | 195.625 | 215.29 | 0.4725 | 0.273 | 0.242118 | 0.126967 |
| 0.8 | 198.39 | 211.65 | 198.75 | 211.86 | 0.36 | 0.208 | 0.181461 | 0.098275 |
| 0.9 | 201.67 | 208.31 | 201.875 | 208.43 | 0.2025 | 0.117 | 0.10041 | 0.056165 |
| 1 | 205.00 | 205.00 | 205 | 205 | 0 | 0 | 0 | 0 |

Corresponding to T462

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 205.25 | 298.05 | 205.25 | 298.05 | 0 | 0 | 0 | 0 |
| 0.1 | 208.98 | 292.47 | 209.225 | 292.745 | 0.2475 | 0.2745 | 0.118434 | 0.093856 |
| 0.2 | 212.76 | 286.95 | 213.2 | 287.44 | 0.44 | 0.488 | 0.206806 | 0.170063 |
| 0.3 | 216.60 | 281.49 | 217.175 | 282.135 | 0.5775 | 0.6405 | 0.266624 | 0.227536 |
| 0.4 | 220.49 | 276.10 | 221.15 | 276.83 | 0.66 | 0.732 | 0.299333 | 0.265123 |
| 0.5 | 224.44 | 270.76 | 225.125 | 271.525 | 0.6875 | 0.7625 | 0.306321 | 0.281612 |
| 0.6 | 228.44 | 265.49 | 229.1 | 266.22 | 0.66 | 0.732 | 0.288916 | 0.275719 |
| 0.7 | 232.50 | 260.27 | 233.075 | 260.915 | 0.5775 | 0.6405 | 0.24839 | 0.246086 |
| 0.8 | 236.61 | 255.12 | 237.05 | 255.61 | 0.44 | 0.488 | 0.18596 | 0.191281 |
| 0.9 | 240.78 | 250.03 | 241.025 | 250.305 | 0.2475 | 0.2745 | 0.102792 | 0.109787 |
| 1 | 245.00 | 245.00 | 245 | 245 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|--------|-------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 8.25 | 24.00 | 8.25 | 24 | 0 | 0 | 0 | 0 |
| 0.1 | 8.86 | 23.01 | 8.925 | 23.1 | 0.0675 | 0.09 | 0.762066 | 0.391134 |
| 0.2 | 9.48 | 22.04 | 9.6 | 22.2 | 0.12 | 0.16 | 1.265823 | 0.725953 |
| 0.3 | 10.12 | 21.09 | 10.275 | 21.3 | 0.1575 | 0.21 | 1.556709 | 0.995733 |
| 0.4 | 10.77 | 20.16 | 10.95 | 20.4 | 0.18 | 0.24 | 1.671309 | 1.190476 |
| 0.5 | 11.44 | 19.25 | 11.625 | 19.5 | 0.1875 | 0.25 | 1.639344 | 1.298701 |
| 0.6 | 12.12 | 18.36 | 12.3 | 18.6 | 0.18 | 0.24 | 1.485149 | 1.30719 |
| 0.7 | 12.82 | 17.49 | 12.975 | 17.7 | 0.1575 | 0.21 | 1.228789 | 1.200686 |
| 0.8 | 13.53 | 16.64 | 13.65 | 16.8 | 0.12 | 0.16 | 0.886918 | 0.961538 |
| 0.9 | 14.26 | 15.81 | 14.325 | 15.9 | 0.0675 | 0.09 | 0.473435 | 0.56926 |
| 1 | 15.00 | 15.00 | 15 | 15 | 0 | 0 | 0 | 0 |

Corresponding to T552

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|-------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 106.25 | 129.00 | 106.25 | 129 | 0 | 0 | 0 | 0 |
| 0.1 | 107.06 | 127.51 | 107.125 | 127.6 | 0.0675 | 0.09 | 0.06305 | 0.070583 |
| 0.2 | 107.88 | 126.04 | 108 | 126.2 | 0.12 | 0.16 | 0.111235 | 0.126944 |
| 0.3 | 108.72 | 124.59 | 108.875 | 124.8 | 0.1575 | 0.21 | 0.144871 | 0.168553 |
| 0.4 | 109.57 | 123.16 | 109.75 | 123.4 | 0.18 | 0.24 | 0.164279 | 0.194868 |
| 0.5 | 110.44 | 121.75 | 110.625 | 122 | 0.1875 | 0.25 | 0.169779 | 0.205339 |
| 0.6 | 111.32 | 120.36 | 111.5 | 120.6 | 0.18 | 0.24 | 0.161696 | 0.199402 |
| 0.7 | 112.22 | 118.99 | 112.375 | 119.2 | 0.1575 | 0.21 | 0.140352 | 0.176485 |
| 0.8 | 113.13 | 117.64 | 113.25 | 117.8 | 0.12 | 0.16 | 0.106073 | 0.136008 |
| 0.9 | 114.06 | 116.31 | 114.125 | 116.4 | 0.0675 | 0.09 | 0.059181 | 0.077379 |
| 1 | 115.00 | 115.00 | 115 | 115 | 0 | 0 | 0 | 0 |

Corresponding to T562

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 120.65 | 160.25 | 120.65 | 160.25 | 0 | 0 | 0 | 0 |
| 0.1 | 121.98 | 157.52 | 122.085 | 157.725 | 0.1035 | 0.2025 | 0.084849 | 0.128553 |
| 0.2 | 123.34 | 154.84 | 123.52 | 155.2 | 0.184 | 0.36 | 0.149186 | 0.232498 |
| 0.3 | 124.71 | 152.20 | 124.955 | 152.675 | 0.2415 | 0.4725 | 0.193644 | 0.310442 |
| 0.4 | 126.11 | 149.61 | 126.39 | 150.15 | 0.276 | 0.54 | 0.21885 | 0.360938 |
| 0.5 | 127.54 | 147.06 | 127.825 | 147.625 | 0.2875 | 0.5625 | 0.225424 | 0.38249 |
| 0.6 | 128.98 | 144.56 | 129.26 | 145.1 | 0.276 | 0.54 | 0.21398 | 0.373547 |
| 0.7 | 130.45 | 142.10 | 130.695 | 142.575 | 0.2415 | 0.4725 | 0.185123 | 0.332506 |
| 0.8 | 131.95 | 139.69 | 132.13 | 140.05 | 0.184 | 0.36 | 0.139451 | 0.257714 |
| 0.9 | 133.46 | 137.32 | 133.565 | 137.525 | 0.1035 | 0.2025 | 0.07755 | 0.147463 |
| 1 | 135.00 | 135.00 | 135 | 135 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|--------|--------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 25.92 | 43.55 | 25.92 | 43.55 | 0 | 0 | 0 | 0 |
| 0.1 | 26.76 | 42.65 | 26.828 | 42.695 | 0.0648 | 0.0495 | 0.242124 | 0.116073 |
| 0.2 | 27.62 | 41.75 | 27.736 | 41.84 | 0.1152 | 0.088 | 0.417077 | 0.210768 |
| 0.3 | 28.49 | 40.87 | 28.644 | 40.985 | 0.1512 | 0.1155 | 0.53066 | 0.282607 |
| 0.4 | 29.38 | 40.00 | 29.552 | 40.13 | 0.1728 | 0.132 | 0.588171 | 0.330017 |
| 0.5 | 30.28 | 39.14 | 30.46 | 39.275 | 0.18 | 0.1375 | 0.594452 | 0.351325 |
| 0.6 | 31.20 | 38.29 | 31.368 | 38.42 | 0.1728 | 0.132 | 0.553931 | 0.344756 |
| 0.7 | 32.12 | 37.45 | 32.276 | 37.565 | 0.1512 | 0.1155 | 0.470664 | 0.308415 |
| 0.8 | 33.07 | 36.62 | 33.184 | 36.71 | 0.1152 | 0.088 | 0.348365 | 0.240293 |
| 0.9 | 34.03 | 35.81 | 34.092 | 35.855 | 0.0648 | 0.0495 | 0.190436 | 0.138247 |
| 1 | 35.00 | 35.00 | 35 | 35 | 0 | 0 | 0 | 0 |

Corresponding to T662

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 123.92 | 148.55 | 123.92 | 148.55 | 0 | 0 | 0 | 0 |
| 0.1 | 124.96 | 147.15 | 125.028 | 147.195 | 0.0648 | 0.0495 | 0.051855 | 0.03364 |
| 0.2 | 126.02 | 145.75 | 126.136 | 145.84 | 0.1152 | 0.088 | 0.091413 | 0.060377 |
| 0.3 | 127.09 | 144.37 | 127.244 | 144.485 | 0.1512 | 0.1155 | 0.118968 | 0.080003 |
| 0.4 | 128.18 | 143.00 | 128.352 | 143.13 | 0.1728 | 0.132 | 0.134811 | 0.092309 |
| 0.5 | 129.28 | 141.64 | 129.46 | 141.775 | 0.18 | 0.1375 | 0.139233 | 0.097079 |
| 0.6 | 130.40 | 140.29 | 130.568 | 140.42 | 0.1728 | 0.132 | 0.13252 | 0.094092 |
| 0.7 | 131.52 | 138.95 | 131.676 | 139.065 | 0.1512 | 0.1155 | 0.114959 | 0.083124 |
| 0.8 | 132.67 | 137.62 | 132.784 | 137.71 | 0.1152 | 0.088 | 0.086833 | 0.063943 |
| 0.9 | 133.83 | 136.31 | 133.892 | 136.355 | 0.0648 | 0.0495 | 0.048421 | 0.036315 |
| 1 | 135.00 | 135.00 | 135 | 135 | 0 | 0 | 0 | 0 |

Corresponding to T762

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|--------|--------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 36.66 | 79.85 | 36.66 | 79.85 | 0 | 0 | 0 | 0 |
| 0.1 | 38.89 | 77.74 | 38.994 | 77.865 | 0.1044 | 0.1215 | 0.268452 | 0.156283 |
| 0.2 | 41.14 | 75.66 | 41.328 | 75.88 | 0.1856 | 0.216 | 0.451116 | 0.285473 |
| 0.3 | 43.42 | 73.61 | 43.662 | 73.895 | 0.2436 | 0.2835 | 0.561052 | 0.38513 |
| 0.4 | 45.72 | 71.59 | 45.996 | 71.91 | 0.2784 | 0.324 | 0.608956 | 0.452602 |
| 0.5 | 48.04 | 69.59 | 48.33 | 69.925 | 0.29 | 0.3375 | 0.603664 | 0.485001 |
| 0.6 | 50.39 | 67.62 | 50.664 | 67.94 | 0.2784 | 0.324 | 0.552539 | 0.479177 |
| 0.7 | 52.75 | 65.67 | 52.998 | 65.955 | 0.2436 | 0.2835 | 0.461762 | 0.431694 |
| 0.8 | 55.15 | 63.75 | 55.332 | 63.97 | 0.1856 | 0.216 | 0.336559 | 0.338802 |
| 0.9 | 57.56 | 61.86 | 57.666 | 61.985 | 0.1044 | 0.1215 | 0.181371 | 0.1964 |
| 1 | 60.00 | 60.00 | 60 | 60 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|-------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 43.75 | 126.00 | 43.75 | 126 | 0 | 0 | 0 | 0 |
| 0.1 | 47.49 | 121.59 | 47.875 | 121.9 | 0.3825 | 0.315 | 0.80539 | 0.259078 |
| 0.2 | 51.32 | 117.24 | 52 | 117.8 | 0.68 | 0.56 | 1.325019 | 0.477653 |
| 0.3 | 55.23 | 112.97 | 56.125 | 113.7 | 0.8925 | 0.735 | 1.615896 | 0.650644 |
| 0.4 | 59.23 | 108.76 | 60.25 | 109.6 | 1.02 | 0.84 | 1.7221 | 0.772343 |
| 0.5 | 63.31 | 104.63 | 64.375 | 105.5 | 1.0625 | 0.875 | 1.678184 | 0.83632 |
| 0.6 | 67.48 | 100.56 | 68.5 | 101.4 | 1.02 | 0.84 | 1.511559 | 0.835322 |
| 0.7 | 71.73 | 96.57 | 72.625 | 97.3 | 0.8925 | 0.735 | 1.244206 | 0.761145 |
| 0.8 | 76.07 | 92.64 | 76.75 | 93.2 | 0.68 | 0.56 | 0.893914 | 0.604491 |
| 0.9 | 80.49 | 88.79 | 80.875 | 89.1 | 0.3825 | 0.315 | 0.4752 | 0.35479 |
| 1 | 85.00 | 85.00 | 85 | 85 | 0 | 0 | 0 | 0 |

Corresponding to T443

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|-------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 141.75 | 231.00 | 141.75 | 231 | 0 | 0 | 0 | 0 |
| 0.1 | 145.69 | 226.09 | 146.075 | 226.4 | 0.3825 | 0.315 | 0.262539 | 0.139328 |
| 0.2 | 149.72 | 221.24 | 150.4 | 221.8 | 0.68 | 0.56 | 0.454181 | 0.253119 |
| 0.3 | 153.83 | 216.47 | 154.725 | 217.2 | 0.8925 | 0.735 | 0.580176 | 0.339547 |
| 0.4 | 158.03 | 211.76 | 159.05 | 212.6 | 1.02 | 0.84 | 0.645447 | 0.396675 |
| 0.5 | 162.31 | 207.13 | 163.375 | 208 | 1.0625 | 0.875 | 0.654601 | 0.42245 |
| 0.6 | 166.68 | 202.56 | 167.7 | 203.4 | 1.02 | 0.84 | 0.611951 | 0.414692 |
| 0.7 | 171.13 | 198.07 | 172.025 | 198.8 | 0.8925 | 0.735 | 0.521526 | 0.37109 |
| 0.8 | 175.67 | 193.64 | 176.35 | 194.2 | 0.68 | 0.56 | 0.387089 | 0.289196 |
| 0.9 | 180.29 | 189.29 | 180.675 | 189.6 | 0.3825 | 0.315 | 0.212155 | 0.166416 |
| 1 | 185.00 | 185.00 | 185 | 185 | 0 | 0 | 0 | 0 |

Corresponding to T453

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 187.35 | 289.30 | 187.35 | 289.3 | 0 | 0 | 0 | 0 |
| 0.1 | 191.72 | 283.53 | 192.115 | 283.87 | 0.3915 | 0.342 | 0.2042 | 0.120623 |
| 0.2 | 196.18 | 277.83 | 196.88 | 278.44 | 0.696 | 0.608 | 0.354769 | 0.218837 |
| 0.3 | 200.73 | 272.21 | 201.645 | 273.01 | 0.9135 | 0.798 | 0.455086 | 0.293154 |
| 0.4 | 205.37 | 266.67 | 206.41 | 267.58 | 1.044 | 0.912 | 0.508361 | 0.341998 |
| 0.5 | 210.09 | 261.20 | 211.175 | 262.15 | 1.0875 | 0.95 | 0.517641 | 0.363706 |
| 0.6 | 214.90 | 255.81 | 215.94 | 256.72 | 1.044 | 0.912 | 0.485816 | 0.356517 |
| 0.7 | 219.79 | 250.49 | 220.705 | 251.29 | 0.9135 | 0.798 | 0.415621 | 0.318573 |
| 0.8 | 224.77 | 245.25 | 225.47 | 245.86 | 0.696 | 0.608 | 0.309644 | 0.247908 |
| 0.9 | 229.84 | 240.09 | 230.235 | 240.43 | 0.3915 | 0.342 | 0.170333 | 0.142448 |
| 1 | 235.00 | 235.00 | 235 | 235 | 0 | 0 | 0 | 0 |

Corresponding to T463

156

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 218.85 | 348.05 | 218.85 | 348.05 | 0 | 0 | 0 | 0 |
| 0.1 | 224.03 | 340.25 | 224.465 | 340.745 | 0.4365 | 0.4995 | 0.194841 | 0.146806 |
| 0.2 | 229.30 | 332.55 | 230.08 | 333.44 | 0.776 | 0.888 | 0.338415 | 0.267026 |
| 0.3 | 234.68 | 324.97 | 235.695 | 326.135 | 1.0185 | 1.1655 | 0.434002 | 0.358649 |
| 0.4 | 240.15 | 317.50 | 241.31 | 318.83 | 1.164 | 1.332 | 0.484705 | 0.41953 |
| 0.5 | 245.71 | 310.14 | 246.925 | 311.525 | 1.2125 | 1.3875 | 0.493463 | 0.447382 |
| 0.6 | 251.38 | 302.89 | 252.54 | 304.22 | 1.164 | 1.332 | 0.463051 | 0.439767 |
| 0.7 | 257.14 | 295.75 | 258.155 | 296.915 | 1.0185 | 1.1655 | 0.396093 | 0.394084 |
| 0.8 | 262.99 | 288.72 | 263.77 | 289.61 | 0.776 | 0.888 | 0.295064 | 0.307562 |
| 0.9 | 268.95 | 281.81 | 269.385 | 282.305 | 0.4365 | 0.4995 | 0.162299 | 0.17725 |
| 1 | 275.00 | 275.00 | 275 | 275 | 0 | 0 | 0 | 0 |

Corresponding to T543

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 48.75 | 125.40 | 48.75 | 125.4 | 0 | 0 | 0 | 0 |
| 0.1 | 52.54 | 121.69 | 52.875 | 121.86 | 0.3375 | 0.171 | 0.642398 | 0.140522 |
| 0.2 | 56.40 | 118.02 | 57 | 118.32 | 0.6 | 0.304 | 1.06383 | 0.257592 |
| 0.3 | 60.34 | 114.38 | 61.125 | 114.78 | 0.7875 | 0.399 | 1.305158 | 0.348834 |
| 0.4 | 64.35 | 110.78 | 65.25 | 111.24 | 0.9 | 0.456 | 1.398601 | 0.411612 |
| 0.5 | 68.44 | 107.23 | 69.375 | 107.7 | 0.9375 | 0.475 | 1.369863 | 0.442994 |
| 0.6 | 72.60 | 103.70 | 73.5 | 104.16 | 0.9 | 0.456 | 1.239669 | 0.439713 |
| 0.7 | 76.84 | 100.22 | 77.625 | 100.62 | 0.7875 | 0.399 | 1.02489 | 0.39812 |
| 0.8 | 81.15 | 96.78 | 81.75 | 97.08 | 0.6 | 0.304 | 0.739372 | 0.314127 |
| 0.9 | 85.54 | 93.37 | 85.875 | 93.54 | 0.3375 | 0.171 | 0.394564 | 0.183144 |
| 1 | 90.00 | 90.00 | 90 | 90 | 0 | 0 | 0 | 0 |

Corresponding to T553

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 146.75 | 230.40 | 146.75 | 230.4 | 0 | 0 | 0 | 0 |
| 0.1 | 150.74 | 226.19 | 151.075 | 226.36 | 0.3375 | 0.171 | 0.223899 | 0.0756 |
| 0.2 | 154.80 | 222.02 | 155.4 | 222.32 | 0.6 | 0.304 | 0.387597 | 0.136927 |
| 0.3 | 158.94 | 217.88 | 159.725 | 218.28 | 0.7875 | 0.399 | 0.495478 | 0.183127 |
| 0.4 | 163.15 | 213.78 | 164.05 | 214.24 | 0.9 | 0.456 | 0.55164 | 0.213299 |
| 0.5 | 167.44 | 209.73 | 168.375 | 210.2 | 0.9375 | 0.475 | 0.55991 | 0.226487 |
| 0.6 | 171.80 | 205.70 | 172.7 | 206.16 | 0.9 | 0.456 | 0.523865 | 0.221678 |
| 0.7 | 176.24 | 201.72 | 177.025 | 202.12 | 0.7875 | 0.399 | 0.44684 | 0.197798 |
| 0.8 | 180.75 | 197.78 | 181.35 | 198.08 | 0.6 | 0.304 | 0.33195 | 0.153709 |
| 0.9 | 185.34 | 193.87 | 185.675 | 194.04 | 0.3375 | 0.171 | 0.1821 | 0.088204 |
| 1 | 190.00 | 190.00 | 190 | 190 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 161.15 | 261.65 | 161.15 | 261.65 | 0 | 0 | 0 | 0 |
| 0.1 | 165.66 | 256.20 | 166.035 | 256.485 | 0.3735 | 0.2835 | 0.22546 | 0.110655 |
| 0.2 | 170.26 | 250.82 | 170.92 | 251.32 | 0.664 | 0.504 | 0.390001 | 0.200944 |
| 0.3 | 174.93 | 245.49 | 175.805 | 246.155 | 0.8715 | 0.6615 | 0.498189 | 0.269457 |
| 0.4 | 179.69 | 240.23 | 180.69 | 240.99 | 0.996 | 0.756 | 0.554276 | 0.314693 |
| 0.5 | 184.54 | 235.04 | 185.575 | 235.825 | 1.0375 | 0.7875 | 0.562216 | 0.335053 |
| 0.6 | 189.46 | 229.90 | 190.46 | 230.66 | 0.996 | 0.756 | 0.525694 | 0.328833 |
| 0.7 | 194.47 | 224.83 | 195.345 | 225.495 | 0.8715 | 0.6615 | 0.448133 | 0.294218 |
| 0.8 | 199.57 | 219.83 | 200.23 | 220.33 | 0.664 | 0.504 | 0.332722 | 0.229272 |
| 0.9 | 204.74 | 214.88 | 205.115 | 215.165 | 0.3735 | 0.2835 | 0.182425 | 0.131933 |
| 1 | 210.00 | 210.00 | 210 | 210 | 0 | 0 | 0 | 0 |

Corresponding to T653

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|-------|--------|--------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 35.55 | 66.35 | 35.55 | 66.35 | 0 | 0 | 0 | 0 |
| 0.1 | 36.89 | 64.59 | 36.995 | 64.715 | 0.1035 | 0.1215 | 0.280552 | 0.188099 |
| 0.2 | 38.26 | 62.86 | 38.44 | 63.08 | 0.184 | 0.216 | 0.48097 | 0.343599 |
| 0.3 | 39.64 | 61.16 | 39.885 | 61.445 | 0.2415 | 0.2835 | 0.609179 | 0.463527 |
| 0.4 | 41.05 | 59.49 | 41.33 | 59.81 | 0.276 | 0.324 | 0.672285 | 0.544666 |
| 0.5 | 42.49 | 57.84 | 42.775 | 58.175 | 0.2875 | 0.3375 | 0.67667 | 0.583531 |
| 0.6 | 43.94 | 56.22 | 44.22 | 56.54 | 0.276 | 0.324 | 0.628072 | 0.576348 |
| 0.7 | 45.42 | 54.62 | 45.665 | 54.905 | 0.2415 | 0.2835 | 0.531663 | 0.519026 |
| 0.8 | 46.93 | 53.05 | 47.11 | 53.27 | 0.184 | 0.216 | 0.392107 | 0.407132 |
| 0.9 | 48.45 | 51.51 | 48.555 | 51.635 | 0.1035 | 0.1215 | 0.213616 | 0.235861 |
| 1 | 50.00 | 50.00 | 50 | 50 | 0 | 0 | 0 | 0 |

Corresponding to T663

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 133.55 | 171.35 | 133.55 | 171.35 | 0 | 0 | 0 | 0 |
| 0.1 | 135.09 | 169.09 | 135.195 | 169.215 | 0.1035 | 0.1215 | 0.076615 | 0.071854 |
| 0.2 | 136.66 | 166.86 | 136.84 | 167.08 | 0.184 | 0.216 | 0.134645 | 0.129447 |
| 0.3 | 138.24 | 164.66 | 138.485 | 164.945 | 0.2415 | 0.2835 | 0.174692 | 0.172171 |
| 0.4 | 139.85 | 162.49 | 140.13 | 162.81 | 0.276 | 0.324 | 0.197349 | 0.199402 |
| 0.5 | 141.49 | 160.34 | 141.775 | 160.675 | 0.2875 | 0.3375 | 0.203198 | 0.210493 |
| 0.6 | 143.14 | 158.22 | 143.42 | 158.54 | 0.276 | 0.324 | 0.192813 | 0.204783 |
| 0.7 | 144.82 | 156.12 | 145.065 | 156.405 | 0.2415 | 0.2835 | 0.166755 | 0.181589 |
| 0.8 | 146.53 | 154.05 | 146.71 | 154.27 | 0.184 | 0.216 | 0.125575 | 0.140211 |
| 0.9 | 148.25 | 152.01 | 148.355 | 152.135 | 0.1035 | 0.1215 | 0.069814 | 0.079927 |
| 1 | 150.00 | 150.00 | 150 | 150 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|--------|--------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 44.42 | 102.05 | 44.42 | 102.05 | 0 | 0 | 0 | 0 |
| 0.1 | 47.28 | 99.16 | 47.478 | 99.345 | 0.1998 | 0.1845 | 0.422605 | 0.186062 |
| 0.2 | 50.18 | 96.31 | 50.536 | 96.64 | 0.3552 | 0.328 | 0.70784 | 0.34056 |
| 0.3 | 53.13 | 93.50 | 53.594 | 93.935 | 0.4662 | 0.4305 | 0.877507 | 0.460406 |
| 0.4 | 56.12 | 90.74 | 56.652 | 91.23 | 0.5328 | 0.492 | 0.949408 | 0.54222 |
| 0.5 | 59.16 | 88.01 | 59.71 | 88.525 | 0.555 | 0.5125 | 0.938213 | 0.582304 |
| 0.6 | 62.24 | 85.33 | 62.768 | 85.82 | 0.5328 | 0.492 | 0.856107 | 0.576599 |
| 0.7 | 65.36 | 82.68 | 65.826 | 83.115 | 0.4662 | 0.4305 | 0.713282 | 0.520654 |
| 0.8 | 68.53 | 80.08 | 68.884 | 80.41 | 0.3552 | 0.328 | 0.518322 | 0.40958 |
| 0.9 | 71.74 | 77.52 | 71.942 | 77.705 | 0.1998 | 0.1845 | 0.278497 | 0.238002 |
| 1 | 75.00 | 75.00 | 75 | 75 | 0 | 0 | 0 | 0 |

Corresponding to T544

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 67.45 | 191.65 | 67.45 | 191.65 | 0 | 0 | 0 | 0 |
| 0.1 | 73.12 | 185.02 | 73.705 | 185.485 | 0.5805 | 0.4635 | 0.793852 | 0.250511 |
| 0.2 | 78.93 | 178.50 | 79.96 | 179.32 | 1.032 | 0.824 | 1.307521 | 0.461635 |
| 0.3 | 84.86 | 172.07 | 86.215 | 173.155 | 1.3545 | 1.0815 | 1.596149 | 0.62851 |
| 0.4 | 90.92 | 165.75 | 92.47 | 166.99 | 1.548 | 1.236 | 1.702558 | 0.745683 |
| 0.5 | 97.11 | 159.54 | 98.725 | 160.825 | 1.6125 | 1.2875 | 1.660445 | 0.80702 |
| 0.6 | 103.43 | 153.42 | 104.98 | 154.66 | 1.548 | 1.236 | 1.496635 | 0.805611 |
| 0.7 | 109.88 | 147.41 | 111.235 | 148.495 | 1.3545 | 1.0815 | 1.232703 | 0.733651 |
| 0.8 | 116.46 | 141.51 | 117.49 | 142.33 | 1.032 | 0.824 | 0.886156 | 0.582307 |
| 0.9 | 123.16 | 135.70 | 123.745 | 136.165 | 0.5805 | 0.4635 | 0.471321 | 0.341558 |
| 1 | 130.00 | 130.00 | 130 | 130 | 0 | 0 | 0 | 0 |

Corresponding to T554

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 165.45 | 296.65 | 165.45 | 296.65 | 0 | 0 | 0 | 0 |
| 0.1 | 171.32 | 289.52 | 171.905 | 289.985 | 0.5805 | 0.4635 | 0.338831 | 0.160092 |
| 0.2 | 177.33 | 282.50 | 178.36 | 283.32 | 1.032 | 0.824 | 0.581972 | 0.291686 |
| 0.3 | 183.46 | 275.57 | 184.815 | 276.655 | 1.3545 | 1.0815 | 0.738306 | 0.392454 |
| 0.4 | 189.72 | 268.75 | 191.27 | 269.99 | 1.548 | 1.236 | 0.815931 | 0.4599 |
| 0.5 | 196.11 | 262.04 | 197.725 | 263.325 | 1.6125 | 1.2875 | 0.822232 | 0.491342 |
| 0.6 | 202.63 | 255.42 | 204.18 | 256.66 | 1.548 | 1.236 | 0.763946 | 0.483901 |
| 0.7 | 209.28 | 248.91 | 210.635 | 249.995 | 1.3545 | 1.0815 | 0.647217 | 0.434488 |
| 0.8 | 216.06 | 242.51 | 217.09 | 243.33 | 1.032 | 0.824 | 0.47765 | 0.339785 |
| 0.9 | 222.96 | 236.20 | 223.545 | 236.665 | 0.5805 | 0.4635 | 0.260355 | 0.196231 |
| 1 | 230.00 | 230.00 | 230 | 230 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 179.85 | 327.90 | 179.85 | 327.9 | 0 | 0 | 0 | 0 |
| 0.1 | 186.25 | 319.53 | 186.865 | 320.11 | 0.6165 | 0.576 | 0.331009 | 0.180263 |
| 0.2 | 192.78 | 311.30 | 193.88 | 312.32 | 1.096 | 1.024 | 0.568512 | 0.328947 |
| 0.3 | 199.46 | 303.19 | 200.895 | 304.53 | 1.4385 | 1.344 | 0.72121 | 0.443292 |
| 0.4 | 206.27 | 295.20 | 207.91 | 296.74 | 1.644 | 1.536 | 0.797029 | 0.520318 |
| 0.5 | 213.21 | 287.35 | 214.925 | 288.95 | 1.7125 | 1.6 | 0.803189 | 0.556812 |
| 0.6 | 220.30 | 279.62 | 221.94 | 281.16 | 1.644 | 1.536 | 0.746269 | 0.549309 |
| 0.7 | 227.52 | 272.03 | 228.955 | 273.37 | 1.4385 | 1.344 | 0.632262 | 0.49407 |
| 0.8 | 234.87 | 264.56 | 235.97 | 265.58 | 1.096 | 1.024 | 0.466633 | 0.387064 |
| 0.9 | 242.37 | 257.21 | 242.985 | 257.79 | 0.6165 | 0.576 | 0.254365 | 0.223938 |
| 1 | 250.00 | 250.00 | 250 | 250 | 0 | 0 | 0 | 0 |

Corresponding to T654

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 94.95 | 196.35 | 94.95 | 196.35 | 0 | 0 | 0 | 0 |
| 0.1 | 100.05 | 191.50 | 100.455 | 191.715 | 0.4095 | 0.2115 | 0.409314 | 0.110442 |
| 0.2 | 105.23 | 186.70 | 105.96 | 187.08 | 0.728 | 0.376 | 0.691805 | 0.201388 |
| 0.3 | 110.51 | 181.95 | 111.465 | 182.445 | 0.9555 | 0.4935 | 0.864632 | 0.271226 |
| 0.4 | 115.88 | 177.25 | 116.97 | 177.81 | 1.092 | 0.564 | 0.94237 | 0.318202 |
| 0.5 | 121.34 | 172.59 | 122.475 | 173.175 | 1.1375 | 0.5875 | 0.937468 | 0.340407 |
| 0.6 | 126.89 | 167.98 | 127.98 | 168.54 | 1.092 | 0.564 | 0.860601 | 0.335762 |
| 0.7 | 132.53 | 163.41 | 133.485 | 163.905 | 0.9555 | 0.4935 | 0.720972 | 0.301998 |
| 0.8 | 138.26 | 158.89 | 138.99 | 159.27 | 0.728 | 0.376 | 0.526537 | 0.236636 |
| 0.9 | 144.09 | 154.42 | 144.495 | 154.635 | 0.4095 | 0.2115 | 0.284206 | 0.136961 |
| 1 | 150.00 | 150.00 | 150 | 150 | 0 | 0 | 0 | 0 |

Corresponding to T664

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 192.95 | 301.35 | 192.95 | 301.35 | 0 | 0 | 0 | 0 |
| 0.1 | 198.25 | 296.00 | 198.655 | 296.215 | 0.4095 | 0.2115 | 0.206562 | 0.071452 |
| 0.2 | 203.63 | 290.70 | 204.36 | 291.08 | 0.728 | 0.376 | 0.357508 | 0.129341 |
| 0.3 | 209.11 | 285.45 | 210.065 | 285.945 | 0.9555 | 0.4935 | 0.456938 | 0.172884 |
| 0.4 | 214.68 | 280.25 | 215.77 | 280.81 | 1.092 | 0.564 | 0.508669 | 0.201252 |
| 0.5 | 220.34 | 275.09 | 221.475 | 275.675 | 1.1375 | 0.5875 | 0.516253 | 0.213568 |
| 0.6 | 226.09 | 269.98 | 227.18 | 270.54 | 1.092 | 0.564 | 0.482998 | 0.208907 |
| 0.7 | 231.93 | 264.91 | 232.885 | 265.405 | 0.9555 | 0.4935 | 0.411979 | 0.186289 |
| 0.8 | 237.86 | 259.89 | 238.59 | 260.27 | 0.728 | 0.376 | 0.30606 | 0.144674 |
| 0.9 | 243.89 | 254.92 | 244.295 | 255.135 | 0.4095 | 0.2115 | 0.167907 | 0.082966 |
| 1 | 250.00 | 250.00 | 250 | 250 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|-------|---------|-------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 56.30 | 132.65 | 56.3 | 132.65 | 0 | 0 | 0 | 0 |
| 0.1 | 59.91 | 128.60 | 60.17 | 128.885 | 0.261 | 0.2835 | 0.435661 | 0.220448 |
| 0.2 | 63.58 | 124.62 | 64.04 | 125.12 | 0.464 | 0.504 | 0.729835 | 0.404442 |
| 0.3 | 67.30 | 120.69 | 67.91 | 121.355 | 0.609 | 0.6615 | 0.90489 | 0.548083 |
| 0.4 | 71.08 | 116.83 | 71.78 | 117.59 | 0.696 | 0.756 | 0.979123 | 0.647072 |
| 0.5 | 74.93 | 113.04 | 75.65 | 113.825 | 0.725 | 0.7875 | 0.967634 | 0.696671 |
| 0.6 | 78.82 | 109.30 | 79.52 | 110.06 | 0.696 | 0.756 | 0.88298 | 0.691649 |
| 0.7 | 82.78 | 105.63 | 83.39 | 106.295 | 0.609 | 0.6615 | 0.735676 | 0.626222 |
| 0.8 | 86.80 | 102.03 | 87.26 | 102.53 | 0.464 | 0.504 | 0.534587 | 0.493992 |
| 0.9 | 90.87 | 98.48 | 91.13 | 98.765 | 0.261 | 0.2835 | 0.287227 | 0.287871 |
| 1 | 95.00 | 95.00 | 95 | 95 | 0 | 0 | 0 | 0 |

Corresponding to T655

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 120.79 | 276.35 | 120.79 | 276.35 | 0 | 0 | 0 | 0 |
| 0.1 | 128.04 | 268.19 | 128.711 | 268.715 | 0.6741 | 0.5265 | 0.526489 | 0.196317 |
| 0.2 | 135.43 | 260.14 | 136.632 | 261.08 | 1.1984 | 0.936 | 0.884862 | 0.359801 |
| 0.3 | 142.98 | 252.22 | 144.553 | 253.445 | 1.5729 | 1.2285 | 1.100083 | 0.487082 |
| 0.4 | 150.68 | 244.41 | 152.474 | 245.81 | 1.7976 | 1.404 | 1.19302 | 0.574454 |
| 0.5 | 158.52 | 236.71 | 160.395 | 238.175 | 1.8725 | 1.4625 | 1.18122 | 0.617838 |
| 0.6 | 166.52 | 229.14 | 168.316 | 230.54 | 1.7976 | 1.404 | 1.07952 | 0.612737 |
| 0.7 | 174.66 | 221.68 | 176.237 | 222.905 | 1.5729 | 1.2285 | 0.900529 | 0.554186 |
| 0.8 | 182.96 | 214.33 | 184.158 | 215.27 | 1.1984 | 0.936 | 0.655008 | 0.436702 |
| 0.9 | 191.40 | 207.11 | 192.079 | 207.635 | 0.6741 | 0.5265 | 0.352185 | 0.254215 |
| 1 | 200.00 | 200.00 | 200 | 200 | 0 | 0 | 0 | 0 |

Corresponding to T665

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 218.79 | 381.35 | 218.79 | 381.35 | 0 | 0 | 0 | 0 |
| 0.1 | 226.24 | 372.69 | 226.911 | 373.215 | 0.6741 | 0.5265 | 0.297962 | 0.141271 |
| 0.2 | 233.83 | 364.14 | 235.032 | 365.08 | 1.1984 | 0.936 | 0.512501 | 0.257041 |
| 0.3 | 241.58 | 355.72 | 243.153 | 356.945 | 1.5729 | 1.2285 | 0.651088 | 0.345359 |
| 0.4 | 249.48 | 347.41 | 251.274 | 348.81 | 1.7976 | 1.404 | 0.720549 | 0.404138 |
| 0.5 | 257.52 | 339.21 | 259.395 | 340.675 | 1.8725 | 1.4625 | 0.727121 | 0.431146 |
| 0.6 | 265.72 | 331.14 | 267.516 | 332.54 | 1.7976 | 1.404 | 0.676506 | 0.423995 |
| 0.7 | 274.06 | 323.18 | 275.637 | 324.405 | 1.5729 | 1.2285 | 0.573917 | 0.380133 |
| 0.8 | 282.56 | 315.33 | 283.758 | 316.27 | 1.1984 | 0.936 | 0.424123 | 0.296828 |
| 0.9 | 291.20 | 307.61 | 291.879 | 308.135 | 0.6741 | 0.5265 | 0.231486 | 0.171159 |
| 1 | 300.00 | 300.00 | 300 | 300 | 0 | 0 | 0 | 0 |

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|---------|--------|--------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 126.95 | 296.45 | 126.95 | 296.45 | 0 | 0 | 0 | 0 |
| 0.1 | 135.58 | 288.40 | 136.255 | 288.805 | 0.6795 | 0.4005 | 0.501197 | 0.138867 |
| 0.2 | 144.35 | 280.45 | 145.56 | 281.16 | 1.208 | 0.712 | 0.836843 | 0.25388 |
| 0.3 | 153.28 | 272.58 | 154.865 | 273.515 | 1.5855 | 0.9345 | 1.034385 | 0.342835 |
| 0.4 | 162.36 | 264.80 | 164.17 | 265.87 | 1.812 | 1.068 | 1.116052 | 0.40332 |
| 0.5 | 171.59 | 257.11 | 173.475 | 258.225 | 1.8875 | 1.1125 | 1.100022 | 0.43269 |
| 0.6 | 180.97 | 249.51 | 182.78 | 250.58 | 1.812 | 1.068 | 1.001282 | 0.428036 |
| 0.7 | 190.50 | 242.00 | 192.085 | 242.935 | 1.5855 | 0.9345 | 0.832286 | 0.386156 |
| 0.8 | 200.18 | 234.58 | 201.39 | 235.29 | 1.208 | 0.712 | 0.603451 | 0.303524 |
| 0.9 | 210.02 | 227.24 | 210.695 | 227.645 | 0.6795 | 0.4005 | 0.323548 | 0.176242 |
| 1 | 220.00 | 220.00 | 220 | 220 | 0 | 0 | 0 | 0 |

Corresponding to T766

| α | Curvature | | T.F.N | | Error | | % Error | |
|----------|-----------|--------|---------|--------|--------|-------|----------|----------|
| | Left | Right | Left | Right | Left | Right | Left | Right |
| 0 | 157.04 | 392.70 | 157.04 | 392.7 | 0 | 0 | 0 | 0 |
| 0.1 | 168.32 | 380.65 | 169.336 | 381.43 | 1.0116 | 0.783 | 0.600982 | 0.205702 |
| 0.2 | 179.83 | 368.77 | 181.632 | 370.16 | 1.7984 | 1.392 | 1.000036 | 0.377473 |
| 0.3 | 191.57 | 357.06 | 193.928 | 358.89 | 2.3604 | 1.827 | 1.23215 | 0.511674 |
| 0.4 | 203.53 | 345.53 | 206.224 | 347.62 | 2.6976 | 2.088 | 1.32543 | 0.604286 |
| 0.5 | 215.71 | 334.18 | 218.52 | 336.35 | 2.81 | 2.175 | 1.302675 | 0.650857 |
| 0.6 | 228.12 | 322.99 | 230.816 | 325.08 | 2.6976 | 2.088 | 1.182544 | 0.646456 |
| 0.7 | 240.75 | 311.98 | 243.112 | 313.81 | 2.3604 | 1.827 | 0.98043 | 0.585609 |
| 0.8 | 253.61 | 301.15 | 255.408 | 302.54 | 1.7984 | 1.392 | 0.709121 | 0.462231 |
| 0.9 | 266.69 | 290.49 | 267.704 | 291.27 | 1.0116 | 0.783 | 0.379313 | 0.269547 |
| 1 | 280.00 | 280.00 | 280 | 280 | 0 | 0 | 0 | 0 |

APPENDIX 3

DRP with Variable Demand Rate and No Backorders

allowed under Fuzzy Information

(Membership Function Tables and Graphs)

(PAGE 162- 191)

Interval of Confidence for $0 \leq \alpha \leq 1$

| Corresponding to I_{11} | | | |
|---------------------------|------------------------------|----------|------------------------------|
| α | $X = 0.005 \alpha + 123.995$ | α | $X = -30.00 \alpha + 154.00$ |
| 0 | 123.995 | 1 | 124.00 |
| 0.1 | 123.996 | 0.9 | 127.00 |
| 0.2 | 123.996 | 0.8 | 130.00 |
| 0.3 | 123.997 | 0.7 | 133.00 |
| 0.4 | 123.997 | 0.6 | 136.00 |
| 0.5 | 123.998 | 0.5 | 139.00 |
| 0.6 | 123.998 | 0.4 | 142.00 |
| 0.7 | 123.999 | 0.3 | 145.00 |
| 0.8 | 123.999 | 0.2 | 148.00 |
| 0.9 | 124.000 | 0.1 | 151.00 |
| 1 | 124.000 | 0.0 | 154.00 |

| Corresponding to I_{12} | | | |
|---------------------------|-----------------------------|----------|-----------------------------|
| α | $X = 0.010 \alpha + 43.990$ | α | $X = -40.00 \alpha + 84.00$ |
| 0 | 43.990 | 1.0 | 44.00 |
| 0.1 | 43.991 | 0.9 | 48.00 |
| 0.2 | 43.992 | 0.8 | 52.00 |
| 0.3 | 43.993 | 0.7 | 56.00 |
| 0.4 | 43.994 | 0.6 | 60.00 |
| 0.5 | 43.995 | 0.5 | 64.00 |
| 0.6 | 43.996 | 0.4 | 68.00 |
| 0.7 | 43.997 | 0.3 | 72.00 |
| 0.8 | 43.998 | 0.2 | 76.00 |
| 0.9 | 43.999 | 0.1 | 80.00 |
| 1 | 44.000 | 0.0 | 84.00 |

| Corresponding to I_{13} | | | |
|---------------------------|------------------------------|----------|---------------------------|
| α | $X = 25.02 \alpha + 438.985$ | α | $X = -52 \alpha + 516.01$ |
| 0 | 438.985 | 1.0 | 464.000 |
| 0.1 | 441.487 | 0.9 | 469.201 |
| 0.2 | 443.988 | 0.8 | 474.401 |
| 0.3 | 446.490 | 0.7 | 479.602 |
| 0.4 | 448.991 | 0.6 | 484.802 |
| 0.5 | 451.493 | 0.5 | 490.003 |
| 0.6 | 453.994 | 0.4 | 495.203 |
| 0.7 | 456.496 | 0.3 | 500.404 |
| 0.8 | 458.997 | 0.2 | 505.604 |
| 0.9 | 461.499 | 0.1 | 510.805 |
| 1 | 464.000 | 0.0 | 516.005 |

| Corresponding to I_{14} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 25.020 \alpha + 368.980$ | α | $X = -62.005 \alpha + 456.01$ |
| 0 | 368.980 | 1.0 | 394.000 |
| 0.1 | 371.482 | 0.9 | 400.201 |
| 0.2 | 373.984 | 0.8 | 406.401 |
| 0.3 | 376.486 | 0.7 | 412.602 |
| 0.4 | 378.988 | 0.6 | 418.802 |
| 0.5 | 381.490 | 0.5 | 425.003 |
| 0.6 | 383.992 | 0.4 | 431.203 |
| 0.7 | 386.494 | 0.3 | 437.404 |
| 0.8 | 388.996 | 0.2 | 443.604 |
| 0.9 | 391.498 | 0.1 | 449.805 |
| 1 | 394.000 | 0.0 | 456.005 |

| Corresponding to I_{15} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 25.025 \alpha + 288.975$ | α | $X = -80.005 \alpha + 394.01$ |
| 0 | 288.975 | 1.0 | 314.000 |
| 0.1 | 291.478 | 0.9 | 322.001 |
| 0.2 | 293.980 | 0.8 | 330.001 |
| 0.3 | 296.483 | 0.7 | 338.002 |
| 0.4 | 298.985 | 0.6 | 346.002 |
| 0.5 | 301.488 | 0.5 | 354.003 |
| 0.6 | 303.990 | 0.4 | 362.003 |
| 0.7 | 306.493 | 0.3 | 370.004 |
| 0.8 | 308.995 | 0.2 | 378.004 |
| 0.9 | 311.498 | 0.1 | 386.005 |
| 1 | 314.000 | 0.0 | 394.005 |

| Corresponding to I_{16} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 25.030 \alpha + 198.970$ | α | $X = -100.01 \alpha + 324.01$ |
| 0 | 198.970 | 1.0 | 224.000 |
| 0.1 | 201.473 | 0.9 | 234.001 |
| 0.2 | 203.976 | 0.8 | 244.001 |
| 0.3 | 206.479 | 0.7 | 254.002 |
| 0.4 | 208.982 | 0.6 | 264.002 |
| 0.5 | 211.485 | 0.5 | 274.003 |
| 0.6 | 213.988 | 0.4 | 284.003 |
| 0.7 | 216.491 | 0.3 | 294.004 |
| 0.8 | 218.994 | 0.2 | 304.004 |
| 0.9 | 221.497 | 0.1 | 314.005 |
| 1 | 224.000 | 0.0 | 324.005 |

| Corresponding to I_{17} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 25.035 \alpha + 108.965$ | α | $X = -125.01 \alpha + 259.01$ |
| 0 | 108.965 | 1.0 | 134.000 |
| 0.1 | 111.469 | 0.9 | 146.501 |
| 0.2 | 113.972 | 0.8 | 159.001 |
| 0.3 | 116.476 | 0.7 | 171.502 |
| 0.4 | 118.979 | 0.6 | 184.002 |
| 0.5 | 121.483 | 0.5 | 196.503 |
| 0.6 | 123.986 | 0.4 | 209.003 |
| 0.7 | 126.490 | 0.3 | 221.504 |
| 0.8 | 128.993 | 0.2 | 234.004 |
| 0.9 | 131.497 | 0.1 | 246.505 |
| 1 | 134.000 | 0.0 | 259.005 |

| Corresponding to I_{18} | | | |
|---------------------------|------------------------------|----------|-------------------------------|
| α | $X = 25.040 \alpha + 18.960$ | α | $X = -145.01 \alpha + 189.01$ |
| 0 | 18.960 | 1 | 44.000 |
| 0.1 | 21.464 | 0.9 | 58.501 |
| 0.2 | 23.968 | 0.8 | 73.001 |
| 0.3 | 26.472 | 0.7 | 87.502 |
| 0.4 | 28.976 | 0.6 | 102.002 |
| 0.5 | 31.480 | 0.5 | 116.503 |
| 0.6 | 33.984 | 0.4 | 131.003 |
| 0.7 | 36.488 | 0.3 | 145.504 |
| 0.8 | 38.992 | 0.2 | 160.004 |
| 0.9 | 41.496 | 0.1 | 174.505 |
| 1 | 44.000 | 0.0 | 189.005 |

| Corresponding to I_{21} | | | |
|---------------------------|-----------------------------|----------|------------------------------|
| α | $X = 0.005 \alpha + 69.995$ | α | $X = -15.000 \alpha + 85.00$ |
| 0 | 69.995 | 1 | 70.000 |
| 0.1 | 69.996 | 0.9 | 71.500 |
| 0.2 | 69.996 | 0.8 | 73.000 |
| 0.3 | 69.997 | 0.7 | 74.500 |
| 0.4 | 69.997 | 0.6 | 76.000 |
| 0.5 | 69.998 | 0.5 | 77.500 |
| 0.6 | 69.998 | 0.4 | 79.000 |
| 0.7 | 69.999 | 0.3 | 80.500 |
| 0.8 | 69.999 | 0.2 | 82.000 |
| 0.9 | 70.000 | 0.1 | 83.500 |
| 1 | 70.000 | 0.0 | 85.000 |

| Corresponding to I_{22} | | | |
|---------------------------|-----------------------------|----------|------------------------------|
| α | $X = 0.010 \alpha + 39.990$ | α | $X = -35.000 \alpha + 75.00$ |
| 0 | 39.990 | 1 | 40.000 |
| 0.1 | 39.991 | 0.9 | 43.500 |
| 0.2 | 39.992 | 0.8 | 47.000 |
| 0.3 | 39.993 | 0.7 | 50.500 |
| 0.4 | 39.994 | 0.6 | 54.000 |
| 0.5 | 39.995 | 0.5 | 57.500 |
| 0.6 | 39.996 | 0.4 | 61.000 |
| 0.7 | 39.997 | 0.3 | 64.500 |
| 0.8 | 39.998 | 0.2 | 68.000 |
| 0.9 | 39.999 | 0.1 | 71.500 |
| 1 | 40.000 | 0.0 | 75.000 |

| Corresponding to I_{23} | | | |
|---------------------------|----------------------------|----------|------------------------------|
| α | $X = 0.015 \alpha + 9.985$ | α | $X = -47.000 \alpha + 57.00$ |
| 0 | 9.985 | 1 | 10.000 |
| 0.1 | 9.987 | 0.9 | 14.700 |
| 0.2 | 9.988 | 0.8 | 19.400 |
| 0.3 | 9.990 | 0.7 | 24.100 |
| 0.4 | 9.991 | 0.6 | 28.800 |
| 0.5 | 9.993 | 0.5 | 33.500 |
| 0.6 | 9.994 | 0.4 | 38.200 |
| 0.7 | 9.996 | 0.3 | 42.900 |
| 0.8 | 9.997 | 0.2 | 47.600 |
| 0.9 | 9.999 | 0.1 | 52.300 |
| 1 | 10.000 | 0.0 | 57.000 |

| Corresponding to I_{24} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 50.020 \alpha + 139.980$ | α | $X = -57.005 \alpha + 247.01$ |
| 0 | 139.980 | 1 | 190.000 |
| 0.1 | 144.982 | 0.9 | 195.701 |
| 0.2 | 149.984 | 0.8 | 201.401 |
| 0.3 | 154.986 | 0.7 | 207.102 |
| 0.4 | 159.988 | 0.6 | 212.802 |
| 0.5 | 164.990 | 0.5 | 218.503 |
| 0.6 | 169.992 | 0.4 | 224.203 |
| 0.7 | 174.994 | 0.3 | 229.904 |
| 0.8 | 179.996 | 0.2 | 235.604 |
| 0.9 | 184.998 | 0.1 | 241.305 |
| 1 | 190.000 | 0.0 | 247.005 |

| Corresponding to I_{25} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 50.025 \alpha + 109.975$ | α | $X = -67.005 \alpha + 227.01$ |
| 0 | 109.975 | 1 | 160.000 |
| 0.1 | 114.978 | 0.9 | 166.701 |
| 0.2 | 119.980 | 0.8 | 173.401 |
| 0.3 | 124.983 | 0.7 | 180.102 |
| 0.4 | 129.985 | 0.6 | 186.802 |
| 0.5 | 134.988 | 0.5 | 193.503 |
| 0.6 | 139.990 | 0.4 | 200.203 |
| 0.7 | 144.993 | 0.3 | 206.904 |
| 0.8 | 149.995 | 0.2 | 213.604 |
| 0.9 | 154.998 | 0.1 | 220.305 |
| 1 | 160.000 | 0.0 | 227.005 |

| Corresponding to I_{26} | | | |
|---------------------------|------------------------------|----------|-------------------------------|
| α | $X = 50.030 \alpha + 74.970$ | α | $X = -82.005 \alpha + 207.01$ |
| 0 | 74.970 | 1 | 125.000 |
| 0.1 | 79.973 | 0.9 | 133.201 |
| 0.2 | 84.976 | 0.8 | 141.401 |
| 0.3 | 89.979 | 0.7 | 149.602 |
| 0.4 | 94.982 | 0.6 | 157.802 |
| 0.5 | 99.985 | 0.5 | 166.003 |
| 0.6 | 104.988 | 0.4 | 174.203 |
| 0.7 | 109.991 | 0.3 | 182.404 |
| 0.8 | 114.994 | 0.2 | 190.604 |
| 0.9 | 119.997 | 0.1 | 198.805 |
| 1 | 125.000 | 0.0 | 207.005 |

| Corresponding to I_{27} | | | |
|---------------------------|------------------------------|----------|-------------------------------|
| α | $X = 50.035 \alpha + 39.965$ | α | $X = -102.01 \alpha + 192.01$ |
| 0 | 39.965 | 1 | 90.000 |
| 0.1 | 44.969 | 0.9 | 100.201 |
| 0.2 | 49.972 | 0.8 | 110.401 |
| 0.3 | 54.976 | 0.7 | 120.602 |
| 0.4 | 59.979 | 0.6 | 130.802 |
| 0.5 | 64.983 | 0.5 | 141.003 |
| 0.6 | 69.986 | 0.4 | 151.203 |
| 0.7 | 74.990 | 0.3 | 161.404 |
| 0.8 | 79.993 | 0.2 | 171.604 |
| 0.9 | 84.997 | 0.1 | 181.805 |
| 1 | 90.000 | 0.0 | 192.005 |

| Corresponding to I_{28} | | | |
|---------------------------|-----------------------------|----------|-------------------------------|
| α | $X = 50.040 \alpha + 4.960$ | α | $X = -117.01 \alpha + 172.01$ |
| 0 | 4.960 | 1 | 55.000 |
| 0.1 | 9.964 | 0.9 | 66.701 |
| 0.2 | 14.968 | 0.8 | 78.401 |
| 0.3 | 19.972 | 0.7 | 90.102 |
| 0.4 | 24.976 | 0.6 | 101.802 |
| 0.5 | 29.980 | 0.5 | 113.503 |
| 0.6 | 34.984 | 0.4 | 125.203 |
| 0.7 | 39.988 | 0.3 | 136.904 |
| 0.8 | 44.992 | 0.2 | 148.604 |
| 0.9 | 49.996 | 0.1 | 160.305 |
| 1 | 55.000 | 0.0 | 172.005 |

| Corresponding to I_{31} | | | |
|---------------------------|------------------------------|----------|------------------------------|
| α | $X = 0.005 \alpha + 479.995$ | α | $X = -50.00 \alpha + 530.00$ |
| 0 | 479.995 | 1 | 480.000 |
| 0.1 | 479.996 | 0.9 | 485.000 |
| 0.2 | 479.996 | 0.8 | 490.000 |
| 0.3 | 479.997 | 0.7 | 495.000 |
| 0.4 | 479.997 | 0.6 | 500.000 |
| 0.5 | 479.998 | 0.5 | 505.000 |
| 0.6 | 479.998 | 0.4 | 510.000 |
| 0.7 | 479.999 | 0.3 | 515.000 |
| 0.8 | 479.999 | 0.2 | 520.000 |
| 0.9 | 480.000 | 0.1 | 525.000 |
| 1 | 480.000 | 0.0 | 530.000 |

| Corresponding to I_{32} | | | |
|---------------------------|------------------------------|----------|-------------------------------|
| α | $X = 0.010 \alpha + 359.990$ | α | $X = -80.000 \alpha + 440.00$ |
| 0 | 359.990 | 1 | 360.000 |
| 0.1 | 359.991 | 0.9 | 368.000 |
| 0.2 | 359.992 | 0.8 | 376.000 |
| 0.3 | 359.993 | 0.7 | 384.000 |
| 0.4 | 359.994 | 0.6 | 392.000 |
| 0.5 | 359.995 | 0.5 | 400.000 |
| 0.6 | 359.996 | 0.4 | 408.000 |
| 0.7 | 359.997 | 0.3 | 416.000 |
| 0.8 | 359.998 | 0.2 | 424.000 |
| 0.9 | 359.999 | 0.1 | 432.000 |
| 1 | 360.000 | 0.0 | 440.000 |

| Corresponding to I_{33} | | | |
|---------------------------|------------------------------|----------|-------------------------------|
| α | $X = 0.015 \alpha + 239.985$ | α | $X = -105.00 \alpha + 345.00$ |
| 0 | 239.985 | 1 | 240.000 |
| 0.1 | 239.987 | 0.9 | 250.500 |
| 0.2 | 239.988 | 0.8 | 261.000 |
| 0.3 | 239.990 | 0.7 | 271.500 |
| 0.4 | 239.991 | 0.6 | 282.000 |
| 0.5 | 239.993 | 0.5 | 292.500 |
| 0.6 | 239.994 | 0.4 | 303.000 |
| 0.7 | 239.996 | 0.3 | 313.500 |
| 0.8 | 239.997 | 0.2 | 324.000 |
| 0.9 | 239.999 | 0.1 | 334.500 |
| 1 | 240.000 | 0.0 | 345.000 |

| Corresponding to I_{34} | | | |
|---------------------------|------------------------------|----------|-------------------------------|
| α | $X = 0.020 \alpha + 139.980$ | α | $X = -130.00 \alpha + 270.00$ |
| 0 | 139.980 | 1 | 140.000 |
| 0.1 | 139.982 | 0.9 | 153.000 |
| 0.2 | 139.984 | 0.8 | 166.000 |
| 0.3 | 139.986 | 0.7 | 179.000 |
| 0.4 | 139.988 | 0.6 | 192.000 |
| 0.5 | 139.990 | 0.5 | 205.000 |
| 0.6 | 139.992 | 0.4 | 218.000 |
| 0.7 | 139.994 | 0.3 | 231.000 |
| 0.8 | 139.996 | 0.2 | 244.000 |
| 0.9 | 139.998 | 0.1 | 257.000 |
| 1 | 140.000 | 0.0 | 270.000 |

| Corresponding to I_{35} | | | |
|---------------------------|-----------------------------|----------|-------------------------------|
| α | $X = 0.025 \alpha + 19.975$ | α | $X = -170.00 \alpha + 190.00$ |
| 0 | 19.975 | 1 | 20.000 |
| 0.1 | 19.978 | 0.9 | 37.000 |
| 0.2 | 19.980 | 0.8 | 54.000 |
| 0.3 | 19.983 | 0.7 | 71.000 |
| 0.4 | 19.985 | 0.6 | 88.000 |
| 0.5 | 19.988 | 0.5 | 105.000 |
| 0.6 | 19.990 | 0.4 | 122.000 |
| 0.7 | 19.993 | 0.3 | 139.000 |
| 0.8 | 19.995 | 0.2 | 156.000 |
| 0.9 | 19.998 | 0.1 | 173.000 |
| 1 | 20.000 | 0.0 | 190.000 |

| Corresponding to I_{36} | | | |
|---------------------------|-------------------------------|----------|----------------------------|
| α | $X = 150.03 \alpha + 429.970$ | α | $X = -210 \alpha + 790.01$ |
| 0 | 429.970 | 1 | 580.000 |
| 0.1 | 444.973 | 0.9 | 601.001 |
| 0.2 | 459.976 | 0.8 | 622.001 |
| 0.3 | 474.979 | 0.7 | 643.002 |
| 0.4 | 489.982 | 0.6 | 664.002 |
| 0.5 | 504.985 | 0.5 | 685.003 |
| 0.6 | 519.988 | 0.4 | 706.003 |
| 0.7 | 534.991 | 0.3 | 727.004 |
| 0.8 | 549.994 | 0.2 | 748.004 |
| 0.9 | 564.997 | 0.1 | 769.005 |
| 1 | 580.000 | 0.0 | 790.005 |

| Corresponding to I_{37} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 150.04 \alpha + 289.965$ | α | $X = -240.01 \alpha + 680.01$ |
| 0 | 289.965 | 1 | 440.000 |
| 0.1 | 304.969 | 0.9 | 464.001 |
| 0.2 | 319.972 | 0.8 | 488.001 |
| 0.3 | 334.976 | 0.7 | 512.002 |
| 0.4 | 349.979 | 0.6 | 536.002 |
| 0.5 | 364.983 | 0.5 | 560.003 |
| 0.6 | 379.986 | 0.4 | 584.003 |
| 0.7 | 394.990 | 0.3 | 608.004 |
| 0.8 | 409.993 | 0.2 | 632.004 |
| 0.9 | 424.997 | 0.1 | 656.005 |
| 1 | 440.000 | 0.0 | 680.005 |

| Corresponding to I_{38} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 150.04 \alpha + 149.960$ | α | $X = -290.01 \alpha + 590.01$ |
| 0 | 149.960 | 1 | 300.000 |
| 0.1 | 164.964 | 0.9 | 329.001 |
| 0.2 | 179.968 | 0.8 | 358.001 |
| 0.3 | 194.972 | 0.7 | 387.002 |
| 0.4 | 209.976 | 0.6 | 416.002 |
| 0.5 | 224.980 | 0.5 | 445.003 |
| 0.6 | 239.984 | 0.4 | 474.003 |
| 0.7 | 254.988 | 0.3 | 503.004 |
| 0.8 | 269.992 | 0.2 | 532.004 |
| 0.9 | 284.996 | 0.1 | 561.005 |
| 1 | 300.000 | 0.0 | 590.005 |

| Corresponding to O11 & R ₁₃ | | | |
|--|-------------------------------|----------|------------------------------|
| α | $X = 25.000 \alpha + 475.000$ | α | $X = -0.005 \alpha + 500.01$ |
| 0 | 475.000 | 1 | 500.000 |
| 0.1 | 477.500 | 0.9 | 500.001 |
| 0.2 | 480.000 | 0.8 | 500.001 |
| 0.3 | 482.500 | 0.7 | 500.002 |
| 0.4 | 485.000 | 0.6 | 500.002 |
| 0.5 | 487.500 | 0.5 | 500.003 |
| 0.6 | 490.000 | 0.4 | 500.003 |
| 0.7 | 492.500 | 0.3 | 500.004 |
| 0.8 | 495.000 | 0.2 | 500.004 |
| 0.9 | 497.500 | 0.1 | 500.005 |
| 1 | 500.000 | 0.0 | 500.005 |

| Corresponding to O21 & R ₂₄ | | | |
|--|-------------------------------|----------|------------------------------|
| α | $X = 50.000 \alpha + 150.000$ | α | $X = -0.005 \alpha + 200.01$ |
| 0 | 150.000 | 1 | 200.000 |
| 0.1 | 155.000 | 0.9 | 200.001 |
| 0.2 | 160.000 | 0.8 | 200.001 |
| 0.3 | 165.000 | 0.7 | 200.002 |
| 0.4 | 170.000 | 0.6 | 200.002 |
| 0.5 | 175.000 | 0.5 | 200.003 |
| 0.6 | 180.000 | 0.4 | 200.003 |
| 0.7 | 185.000 | 0.3 | 200.004 |
| 0.8 | 190.000 | 0.2 | 200.004 |
| 0.9 | 195.000 | 0.1 | 200.005 |
| 1 | 200.000 | 0.0 | 200.005 |

| Corresponding to O34 & R ₃₆ | | | |
|--|-------------------------------|----------|------------------------------|
| α | $X = 150.00 \alpha + 550.000$ | α | $X = -0.001 \alpha + 700.00$ |
| 0 | 550.000 | 1 | 700.000 |
| 0.1 | 565.000 | 0.9 | 700.000 |
| 0.2 | 580.000 | 0.8 | 700.000 |
| 0.3 | 595.000 | 0.7 | 700.000 |
| 0.4 | 610.000 | 0.6 | 700.000 |
| 0.5 | 625.000 | 0.5 | 700.001 |
| 0.6 | 640.000 | 0.4 | 700.001 |
| 0.7 | 655.000 | 0.3 | 700.001 |
| 0.8 | 670.000 | 0.2 | 700.001 |
| 0.9 | 685.000 | 0.1 | 700.001 |
| 1 | 700.000 | 0.0 | 700.001 |

| Corresponding to D_{c1} | | | |
|---------------------------|-------------------------------|----------|------------------------------|
| α | $X = 90.000 \alpha + 660.000$ | α | $X = -0.015 \alpha + 750.02$ |
| 0 | 660.000 | 1 | 750.000 |
| 0.1 | 669.000 | 0.9 | 750.002 |
| 0.2 | 678.000 | 0.8 | 750.003 |
| 0.3 | 687.000 | 0.7 | 750.005 |
| 0.4 | 696.000 | 0.6 | 750.006 |
| 0.5 | 705.000 | 0.5 | 750.008 |
| 0.6 | 714.000 | 0.4 | 750.009 |
| 0.7 | 723.000 | 0.3 | 750.011 |
| 0.8 | 732.000 | 0.2 | 750.012 |
| 0.9 | 741.000 | 0.1 | 750.014 |
| 1 | 750.000 | 0.0 | 750.015 |

| Corresponding to D_{c2} | | | |
|---------------------------|------------------------------|----------|-----------------------------|
| α | $X = 20.000 \alpha + 30.000$ | α | $X = -0.005 \alpha + 50.01$ |
| 0 | 30.000 | 1.0 | 50.000 |
| 0.1 | 32.000 | 0.9 | 50.001 |
| 0.2 | 34.000 | 0.8 | 50.001 |
| 0.3 | 36.000 | 0.7 | 50.002 |
| 0.4 | 38.000 | 0.6 | 50.002 |
| 0.5 | 40.000 | 0.5 | 50.003 |
| 0.6 | 42.000 | 0.4 | 50.003 |
| 0.7 | 44.000 | 0.3 | 50.004 |
| 0.8 | 46.000 | 0.2 | 50.004 |
| 0.9 | 48.000 | 0.1 | 50.005 |
| 1 | 50.000 | 0.0 | 50.005 |

| Corresponding to D_{c3} | | | |
|---------------------------|------------------------------|----------|-----------------------------|
| α | $X = 10.000 \alpha + 40.000$ | α | $X = -0.005 \alpha + 50.01$ |
| 0 | 40.000 | 1.0 | 50.000 |
| 0.1 | 41.000 | 0.9 | 50.001 |
| 0.2 | 42.000 | 0.8 | 50.001 |
| 0.3 | 43.000 | 0.7 | 50.002 |
| 0.4 | 44.000 | 0.6 | 50.002 |
| 0.5 | 45.000 | 0.5 | 50.003 |
| 0.6 | 46.000 | 0.4 | 50.003 |
| 0.7 | 47.000 | 0.3 | 50.004 |
| 0.8 | 48.000 | 0.2 | 50.004 |
| 0.9 | 49.000 | 0.1 | 50.005 |
| 1 | 50.000 | 0.0 | 50.005 |

| Corresponding to D_{c4} | | | |
|---------------------------|-------------------------------|----------|------------------------------|
| α | $X = 163.00 \alpha + 587.000$ | α | $X = -0.010 \alpha + 750.01$ |
| 0 | 587.000 | 1.0 | 750.000 |
| 0.1 | 603.300 | 0.9 | 750.001 |
| 0.2 | 619.600 | 0.8 | 750.002 |
| 0.3 | 635.900 | 0.7 | 750.003 |
| 0.4 | 652.200 | 0.6 | 750.004 |
| 0.5 | 668.500 | 0.5 | 750.005 |
| 0.6 | 684.800 | 0.4 | 750.006 |
| 0.7 | 701.100 | 0.3 | 750.007 |
| 0.8 | 717.400 | 0.2 | 750.008 |
| 0.9 | 733.700 | 0.1 | 750.009 |
| 1 | 750.000 | 0.0 | 750.010 |

| Corresponding to D_{c5} | | | |
|---------------------------|------------------------------|----------|-----------------------------|
| α | $X = 10.000 \alpha + 40.000$ | α | $X = -0.005 \alpha + 50.01$ |
| 0 | 40.000 | 1.0 | 50.000 |
| 0.1 | 41.000 | 0.9 | 50.001 |
| 0.2 | 42.000 | 0.8 | 50.001 |
| 0.3 | 43.000 | 0.7 | 50.002 |
| 0.4 | 44.000 | 0.6 | 50.002 |
| 0.5 | 45.000 | 0.5 | 50.003 |
| 0.6 | 46.000 | 0.4 | 50.003 |
| 0.7 | 47.000 | 0.3 | 50.004 |
| 0.8 | 48.000 | 0.2 | 50.004 |
| 0.9 | 49.000 | 0.1 | 50.005 |
| 1 | 50.000 | 0.0 | 50.005 |

| Corresponding to D_{c6} | | | |
|---------------------------|------------------------------|----------|-----------------------------|
| α | $X = 20.000 \alpha + 30.000$ | α | $X = -0.005 \alpha + 50.01$ |
| 0 | 30.000 | 1.0 | 50.000 |
| 0.1 | 32.000 | 0.9 | 50.001 |
| 0.2 | 34.000 | 0.8 | 50.001 |
| 0.3 | 36.000 | 0.7 | 50.002 |
| 0.4 | 38.000 | 0.6 | 50.002 |
| 0.5 | 40.000 | 0.5 | 50.003 |
| 0.6 | 42.000 | 0.4 | 50.003 |
| 0.7 | 44.000 | 0.3 | 50.004 |
| 0.8 | 46.000 | 0.2 | 50.004 |
| 0.9 | 48.000 | 0.1 | 50.005 |
| 1 | 50.000 | 0.0 | 50.005 |

| Corresponding to D_{c7} | | | |
|---------------------------|------------------------------|----------|-----------------------------|
| α | $X = 15.000 \alpha + 35.000$ | α | $X = -0.005 \alpha + 50.01$ |
| 0 | 35.000 | 1.0 | 50.000 |
| 0.1 | 36.500 | 0.9 | 50.001 |
| 0.2 | 38.000 | 0.8 | 50.001 |
| 0.3 | 39.500 | 0.7 | 50.002 |
| 0.4 | 41.000 | 0.6 | 50.002 |
| 0.5 | 42.500 | 0.5 | 50.003 |
| 0.6 | 44.000 | 0.4 | 50.003 |
| 0.7 | 45.500 | 0.3 | 50.004 |
| 0.8 | 47.000 | 0.2 | 50.004 |
| 0.9 | 48.500 | 0.1 | 50.005 |
| 1 | 50.000 | 0.0 | 50.005 |

| Corresponding to D_{c8} | | | |
|---------------------------|------------------------------|----------|-----------------------------|
| α | $X = 25.000 \alpha + 25.000$ | α | $X = -0.005 \alpha + 50.01$ |
| 0 | 25.000 | 1.0 | 50.000 |
| 0.1 | 27.500 | 0.9 | 50.001 |
| 0.2 | 30.000 | 0.8 | 50.001 |
| 0.3 | 32.500 | 0.7 | 50.002 |
| 0.4 | 35.000 | 0.6 | 50.002 |
| 0.5 | 37.500 | 0.5 | 50.003 |
| 0.6 | 40.000 | 0.4 | 50.003 |
| 0.7 | 42.500 | 0.3 | 50.004 |
| 0.8 | 45.000 | 0.2 | 50.004 |
| 0.9 | 47.500 | 0.1 | 50.005 |
| 1 | 50.000 | 0.0 | 50.005 |

| Corresponding to I_{c1} | | | |
|---------------------------|------------------------------|----------|------------------------------|
| α | $X = 0.015 \alpha + 199.985$ | α | $X = -90.00 \alpha + 290.00$ |
| 0 | 199.985 | 1 | 200.000 |
| 0.1 | 199.987 | 0.9 | 209.000 |
| 0.2 | 199.988 | 0.8 | 218.000 |
| 0.3 | 199.990 | 0.7 | 227.000 |
| 0.4 | 199.991 | 0.6 | 236.000 |
| 0.5 | 199.993 | 0.5 | 245.000 |
| 0.6 | 199.994 | 0.4 | 254.000 |
| 0.7 | 199.996 | 0.3 | 263.000 |
| 0.8 | 199.997 | 0.2 | 272.000 |
| 0.9 | 199.999 | 0.1 | 281.000 |
| 1 | 200.000 | 0.0 | 290.000 |

| Corresponding to I_{c2} | | | |
|---------------------------|------------------------------|----------|------------------------------|
| α | $X = 0.020 \alpha + 149.980$ | α | $X = -110.0 \alpha + 260.00$ |
| 0 | 149.980 | 1 | 150.000 |
| 0.1 | 149.982 | 0.9 | 161.000 |
| 0.2 | 149.984 | 0.8 | 172.000 |
| 0.3 | 149.986 | 0.7 | 183.000 |
| 0.4 | 149.988 | 0.6 | 194.000 |
| 0.5 | 149.990 | 0.5 | 205.000 |
| 0.6 | 149.992 | 0.4 | 216.000 |
| 0.7 | 149.994 | 0.3 | 227.000 |
| 0.8 | 149.996 | 0.2 | 238.000 |
| 0.9 | 149.998 | 0.1 | 249.000 |
| 1 | 150.000 | 0.0 | 260.000 |

| Corresponding to I_{c3} | | | |
|---------------------------|-----------------------------|----------|------------------------------|
| α | $X = 0.025 \alpha + 99.975$ | α | $X = -120.0 \alpha + 220.00$ |
| 0 | 99.975 | 1 | 100.000 |
| 0.1 | 99.978 | 0.9 | 112.000 |
| 0.2 | 99.980 | 0.8 | 124.000 |
| 0.3 | 99.983 | 0.7 | 136.000 |
| 0.4 | 99.985 | 0.6 | 148.000 |
| 0.5 | 99.988 | 0.5 | 160.000 |
| 0.6 | 99.990 | 0.4 | 172.000 |
| 0.7 | 99.993 | 0.3 | 184.000 |
| 0.8 | 99.995 | 0.2 | 196.000 |
| 0.9 | 99.998 | 0.1 | 208.000 |
| 1 | 100.000 | 0.0 | 220.000 |

| Corresponding to I_{c4} | | | |
|---------------------------|-------------------------------|----------|------------------------------|
| α | $X = 100.04 \alpha + 249.965$ | α | $X = -283.0 \alpha + 633.01$ |
| 0 | 249.965 | 1 | 350.000 |
| 0.1 | 259.969 | 0.9 | 378.301 |
| 0.2 | 269.972 | 0.8 | 406.601 |
| 0.3 | 279.976 | 0.7 | 434.902 |
| 0.4 | 289.979 | 0.6 | 463.202 |
| 0.5 | 299.983 | 0.5 | 491.503 |
| 0.6 | 309.986 | 0.4 | 519.803 |
| 0.7 | 319.990 | 0.3 | 548.104 |
| 0.8 | 329.993 | 0.2 | 576.404 |
| 0.9 | 339.997 | 0.1 | 604.705 |
| 1 | 350.000 | 0.0 | 633.005 |

| Corresponding to I_{c5} | | | |
|---------------------------|-------------------------------|----------|------------------------------|
| α | $X = 100.04 \alpha + 199.960$ | α | $X = -293.0 \alpha + 593.01$ |
| 0 | 199.960 | 1 | 300.000 |
| 0.1 | 209.964 | 0.9 | 329.301 |
| 0.2 | 219.968 | 0.8 | 358.601 |
| 0.3 | 229.972 | 0.7 | 387.902 |
| 0.4 | 239.976 | 0.6 | 417.202 |
| 0.5 | 249.980 | 0.5 | 446.503 |
| 0.6 | 259.984 | 0.4 | 475.803 |
| 0.7 | 269.988 | 0.3 | 505.104 |
| 0.8 | 279.992 | 0.2 | 534.404 |
| 0.9 | 289.996 | 0.1 | 563.705 |
| 1 | 300.000 | 0.0 | 593.005 |

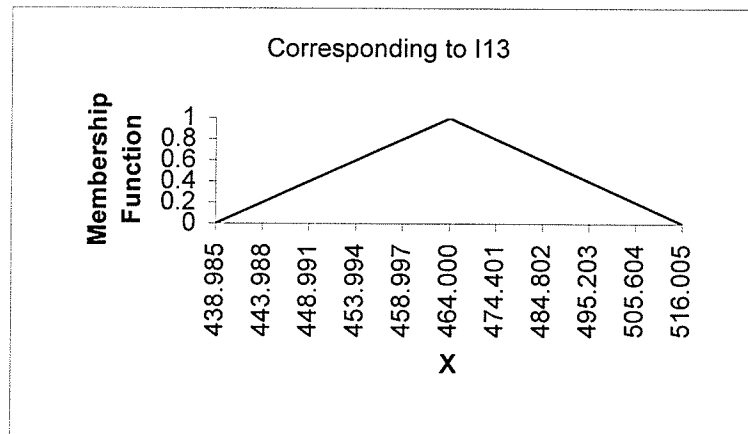
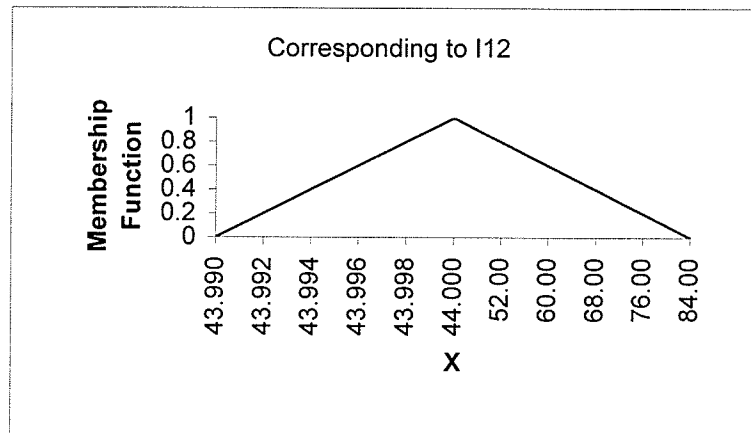
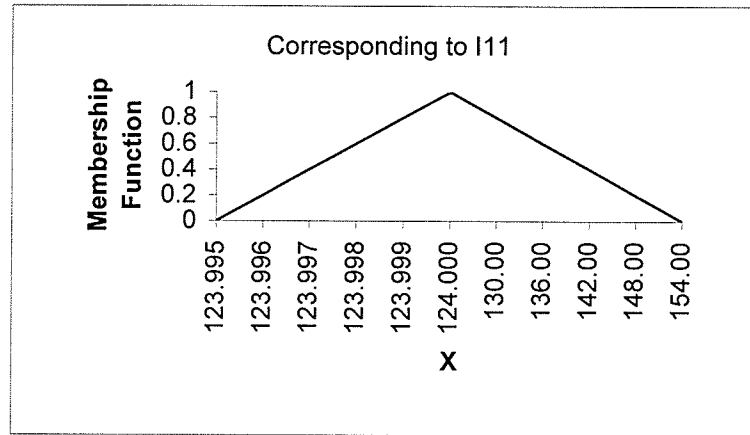
| Corresponding to I_{c6} | | | |
|---------------------------|-------------------------------|----------|------------------------------|
| α | $X = 100.05 \alpha + 149.955$ | α | $X = -313.0 \alpha + 563.01$ |
| 0 | 149.955 | 1 | 250.000 |
| 0.1 | 159.960 | 0.9 | 281.301 |
| 0.2 | 169.964 | 0.8 | 312.601 |
| 0.3 | 179.969 | 0.7 | 343.902 |
| 0.4 | 189.973 | 0.6 | 375.202 |
| 0.5 | 199.978 | 0.5 | 406.503 |
| 0.6 | 209.982 | 0.4 | 437.803 |
| 0.7 | 219.987 | 0.3 | 469.104 |
| 0.8 | 229.991 | 0.2 | 500.404 |
| 0.9 | 239.996 | 0.1 | 531.705 |
| 1 | 250.000 | 0.0 | 563.005 |

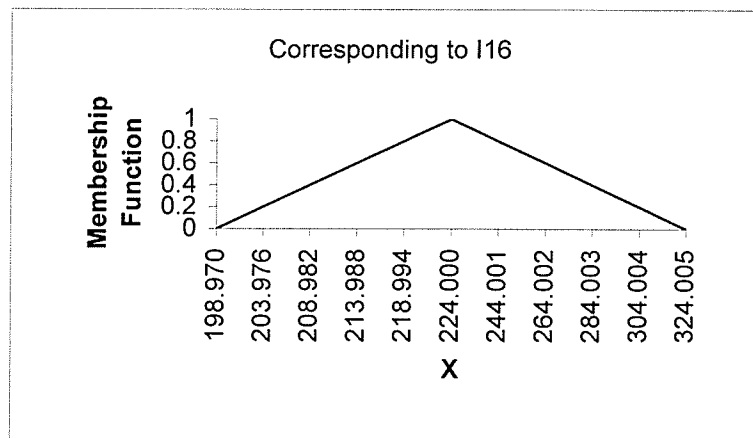
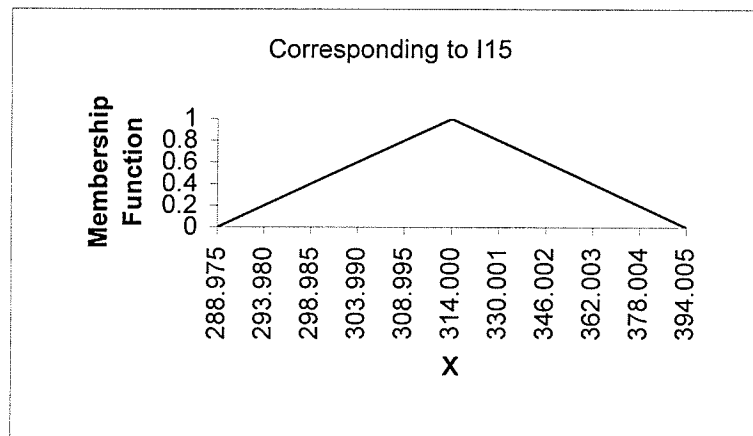
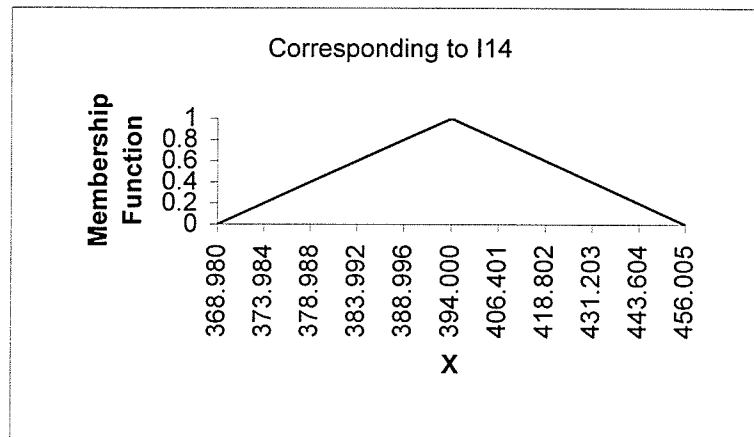
| Corresponding to I_{c7} | | | |
|---------------------------|------------------------------|----------|------------------------------|
| α | $X = 100.05 \alpha + 99.950$ | α | $X = -328.0 \alpha + 528.01$ |
| 0 | 99.950 | 1 | 200.000 |
| 0.1 | 109.955 | 0.9 | 232.801 |
| 0.2 | 119.960 | 0.8 | 265.601 |
| 0.3 | 129.965 | 0.7 | 298.402 |
| 0.4 | 139.970 | 0.6 | 331.202 |
| 0.5 | 149.975 | 0.5 | 364.003 |
| 0.6 | 159.980 | 0.4 | 396.803 |
| 0.7 | 169.985 | 0.3 | 429.604 |
| 0.8 | 179.990 | 0.2 | 462.404 |
| 0.9 | 189.995 | 0.1 | 495.205 |
| 1 | 200.000 | 0.0 | 528.005 |

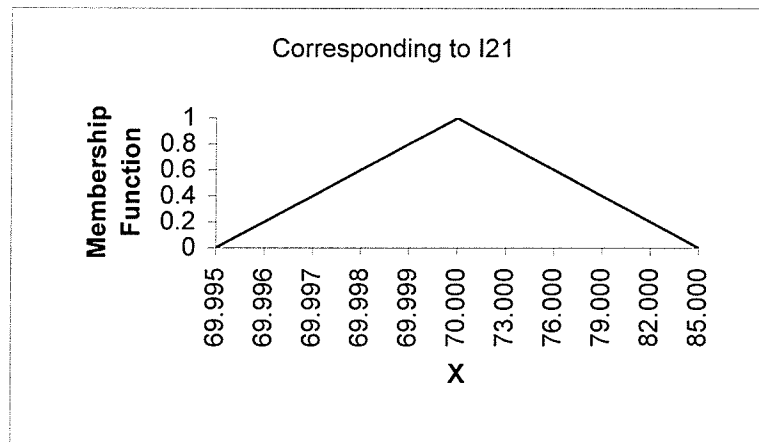
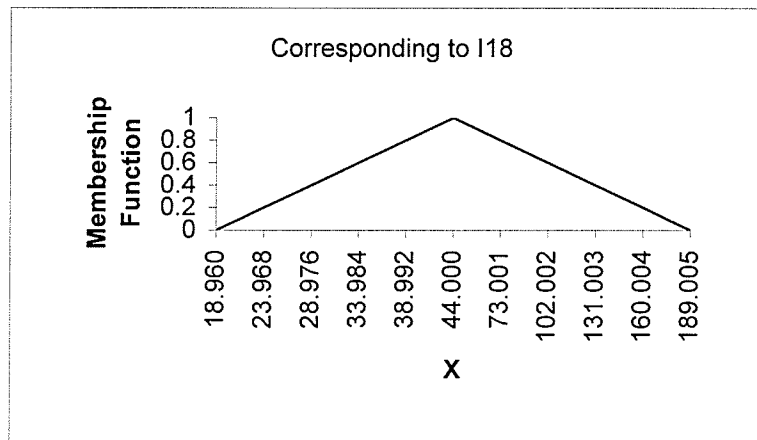
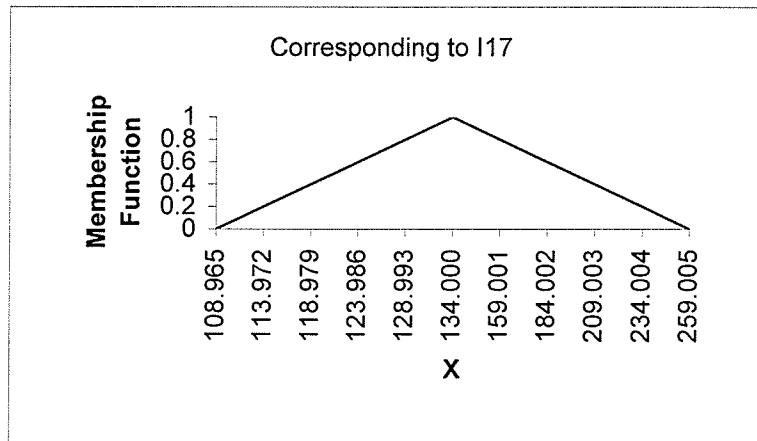
| Corresponding to I_{c8} | | | |
|---------------------------|------------------------------|----------|------------------------------|
| α | $X = 100.06 \alpha + 49.945$ | α | $X = -353.0 \alpha + 503.01$ |
| 0 | 49.945 | 1 | 150.000 |
| 0.1 | 59.951 | 0.9 | 185.301 |
| 0.2 | 69.956 | 0.8 | 220.601 |
| 0.3 | 79.962 | 0.7 | 255.902 |
| 0.4 | 89.967 | 0.6 | 291.202 |
| 0.5 | 99.973 | 0.5 | 326.503 |
| 0.6 | 109.978 | 0.4 | 361.803 |
| 0.7 | 119.984 | 0.3 | 397.104 |
| 0.8 | 129.989 | 0.2 | 432.404 |
| 0.9 | 139.995 | 0.1 | 467.705 |
| 1 | 150.000 | 0.0 | 503.005 |

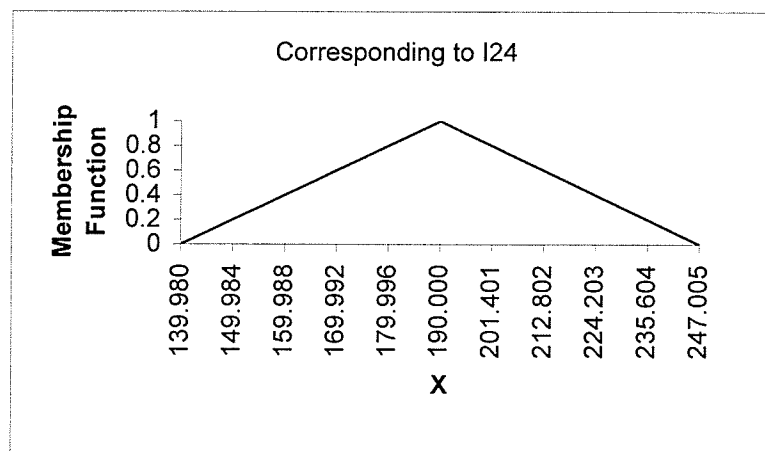
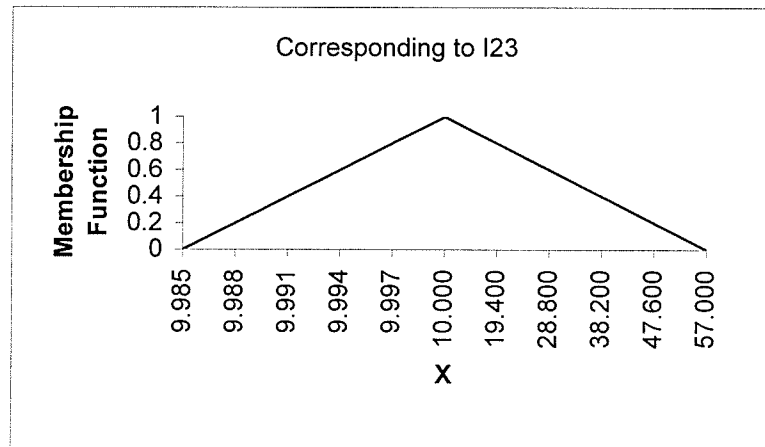
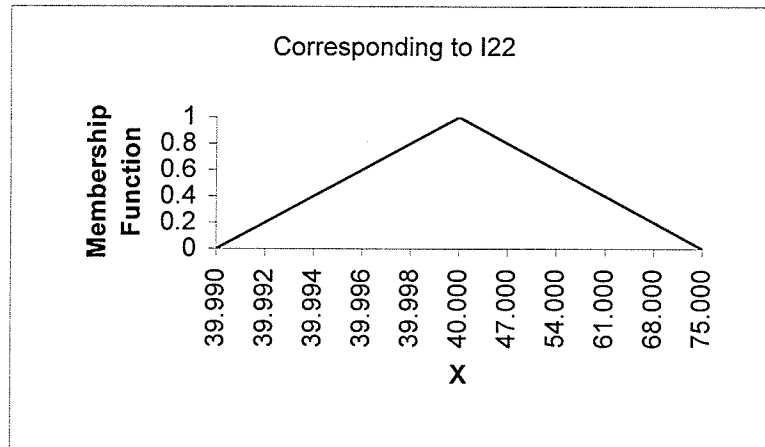
| Corresponding to R_{c4} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 100.00 \alpha + 900.000$ | α | $X = -0.005 \alpha + 1000.01$ |
| 0 | 900.000 | 1 | 1000.000 |
| 0.1 | 910.000 | 0.9 | 1000.001 |
| 0.2 | 920.000 | 0.8 | 1000.001 |
| 0.3 | 930.000 | 0.7 | 1000.002 |
| 0.4 | 940.000 | 0.6 | 1000.002 |
| 0.5 | 950.000 | 0.5 | 1000.003 |
| 0.6 | 960.000 | 0.4 | 1000.003 |
| 0.7 | 970.000 | 0.3 | 1000.004 |
| 0.8 | 980.000 | 0.2 | 1000.004 |
| 0.9 | 990.000 | 0.1 | 1000.005 |
| 1 | 1000.000 | 0.0 | 1000.005 |

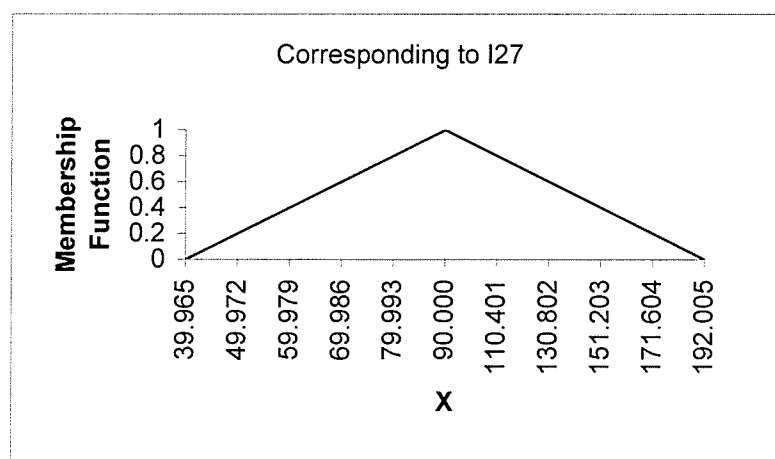
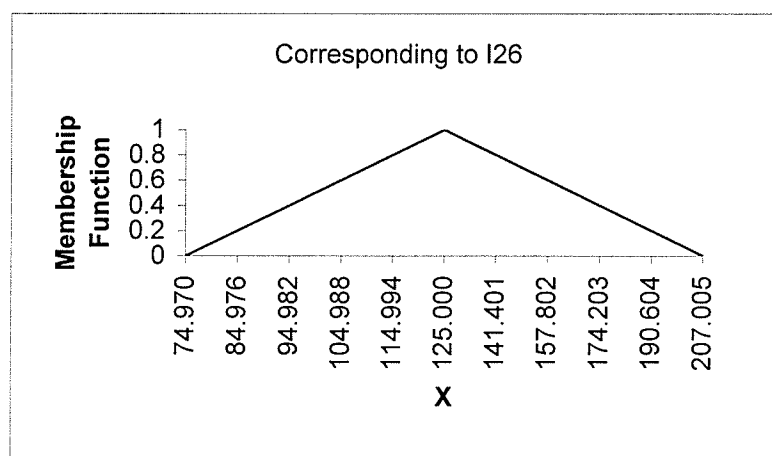
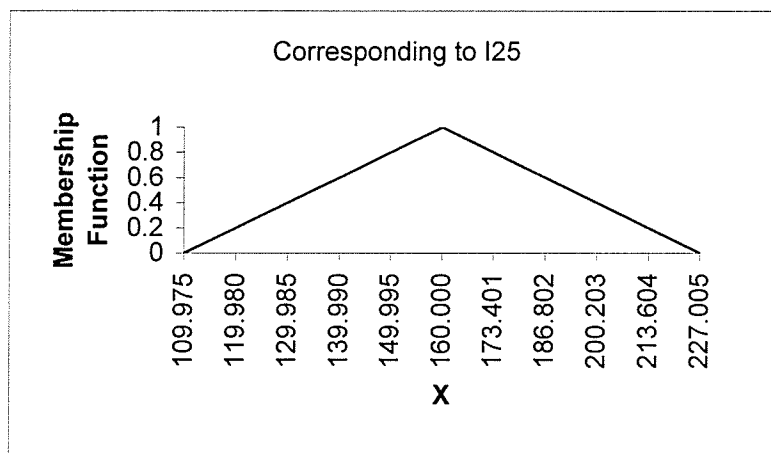
| Corresponding to O_{c1} | | | |
|---------------------------|-------------------------------|----------|-------------------------------|
| α | $X = 100.00 \alpha + 900.000$ | α | $X = -0.005 \alpha + 1000.01$ |
| 0 | 900.000 | 1 | 1000.000 |
| 0.1 | 910.000 | 0.9 | 1000.001 |
| 0.2 | 920.000 | 0.8 | 1000.001 |
| 0.3 | 930.000 | 0.7 | 1000.002 |
| 0.4 | 940.000 | 0.6 | 1000.002 |
| 0.5 | 950.000 | 0.5 | 1000.003 |
| 0.6 | 960.000 | 0.4 | 1000.003 |
| 0.7 | 970.000 | 0.3 | 1000.004 |
| 0.8 | 980.000 | 0.2 | 1000.004 |
| 0.9 | 990.000 | 0.1 | 1000.005 |
| 1 | 1000.000 | 0.0 | 1000.005 |

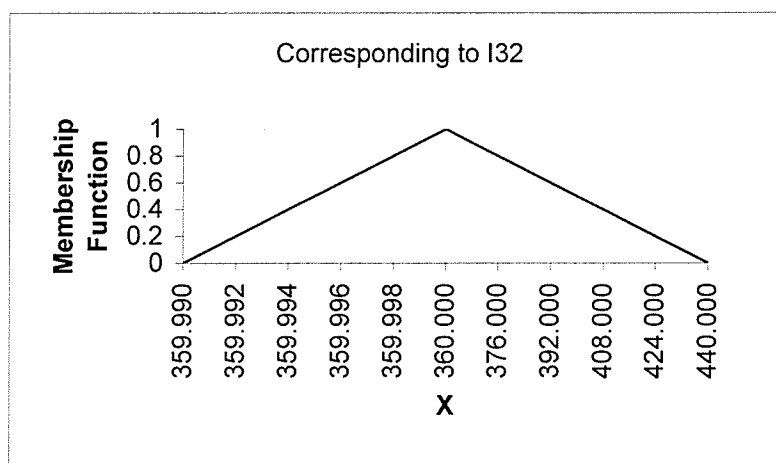
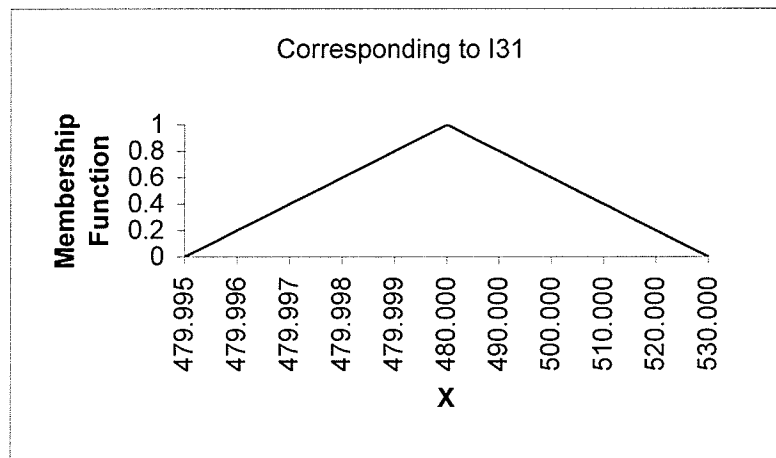
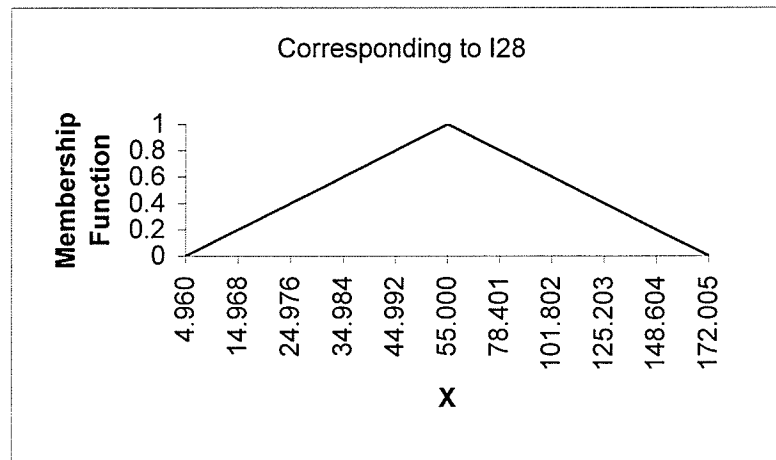
Graph of Membership Functions for I_{kj}'s

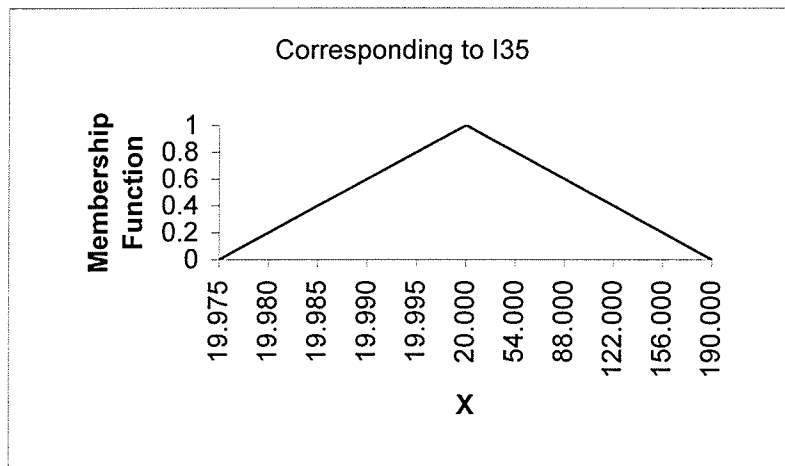
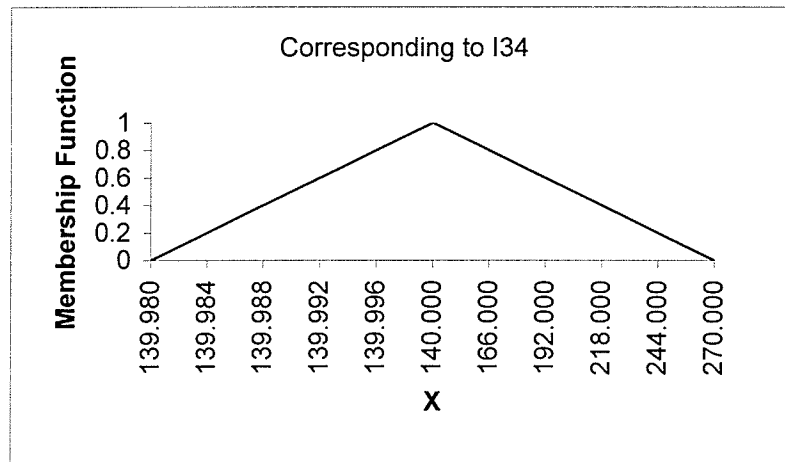
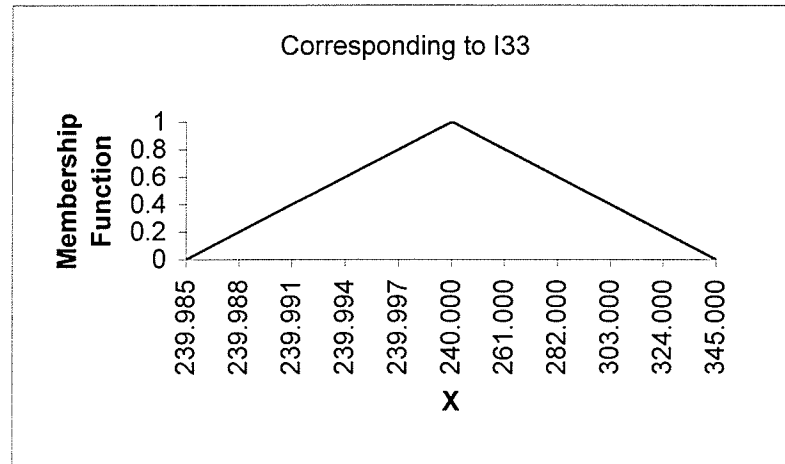


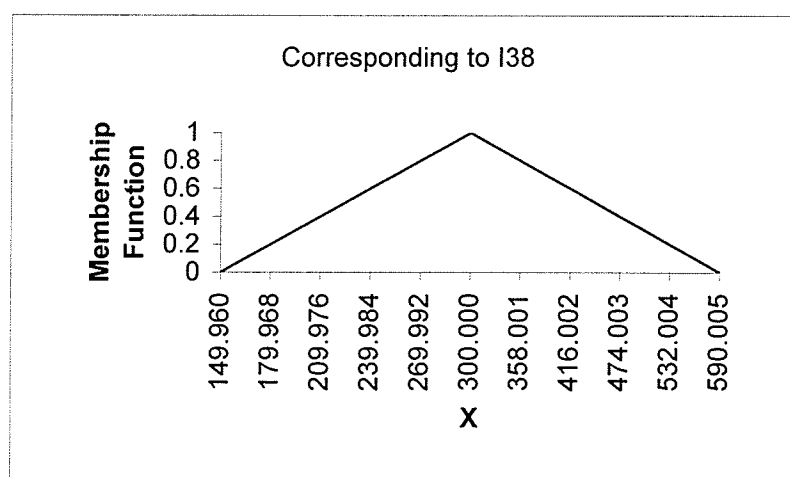
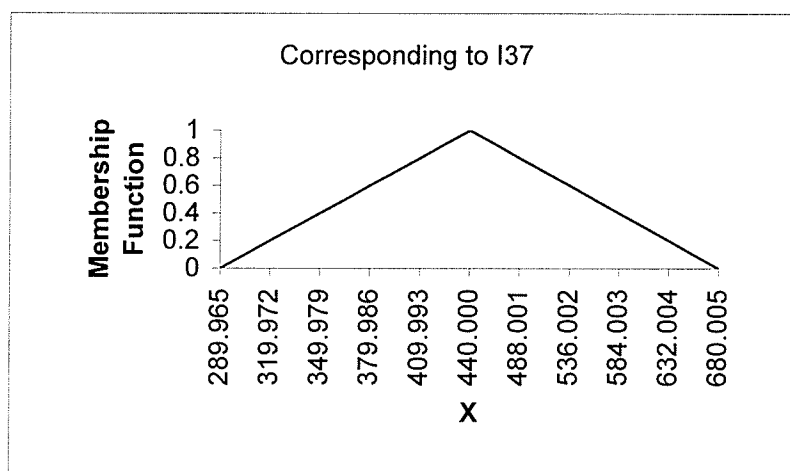
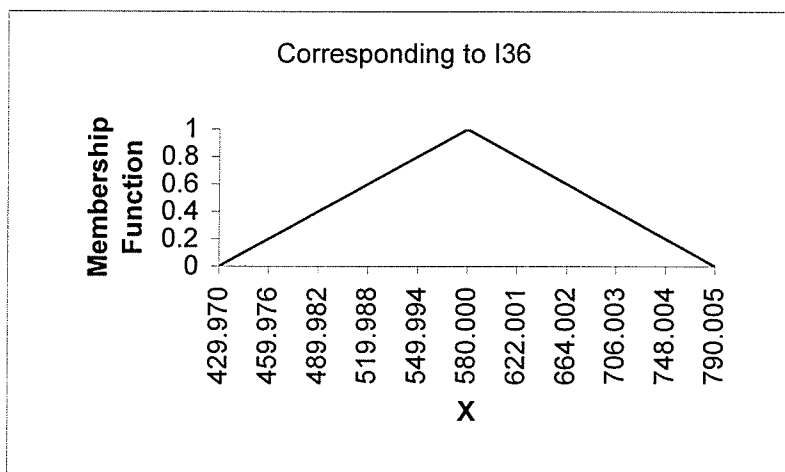


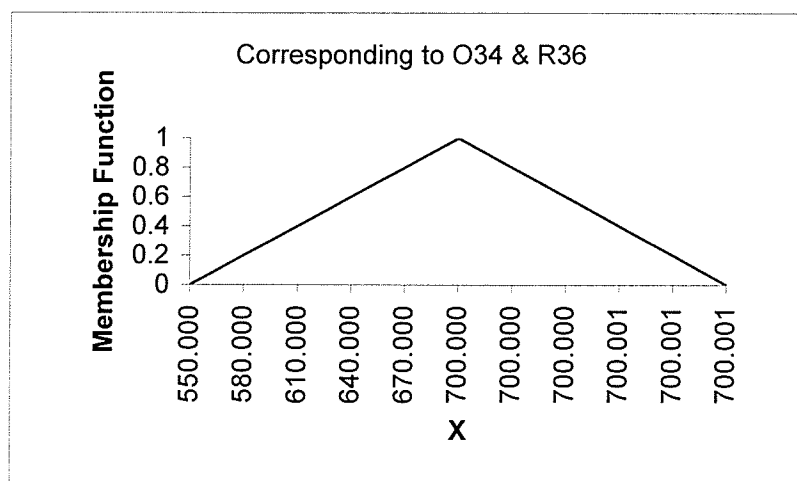
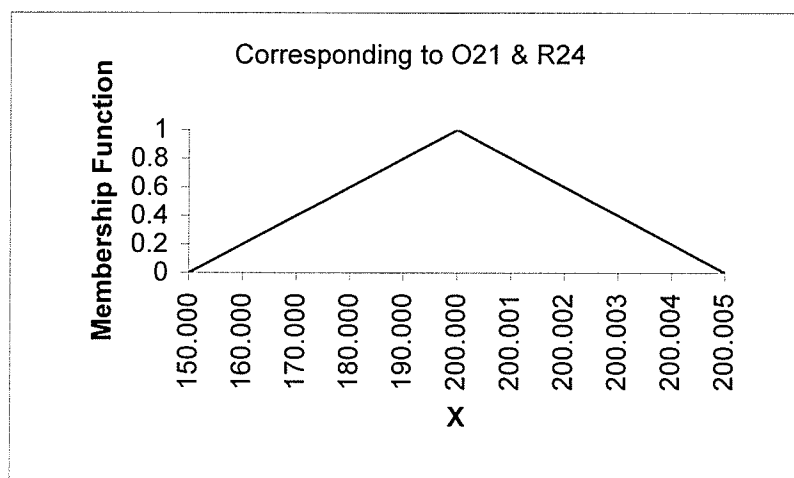
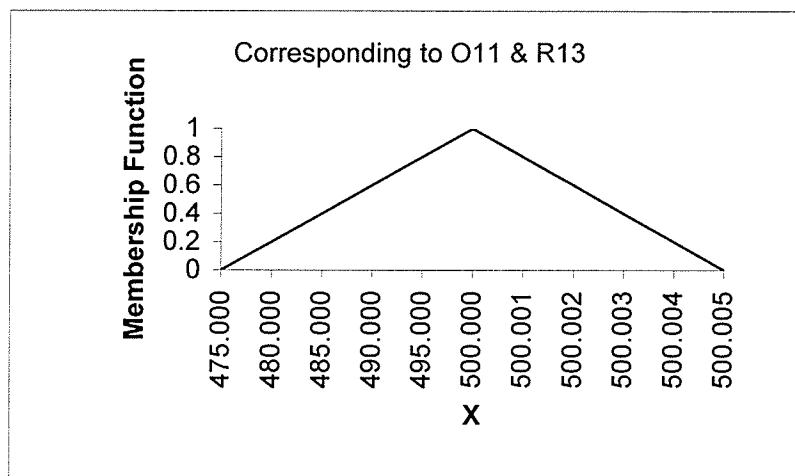












Graph of Membership Functions for Dcj's

