Pure embeddings and pure-injectivity for topological modules

by

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Abstract

In this thesis, topological modules are examined from a model theoretic perspective. Specifically, an attempt is made to expand the classical concepts of pure-embedding and pure-injectivity for modules to an analogous concept for topological modules. Historical definitions of pure embeddings for topological modules are explored as well as one of my own. Ultimately, the classic model theoretic techniques are inadequate for capturing the concept of pure-injectivity for topological modules. Insightful reasons are provided as to why these standard techniques fail.

In addition, when examining different topologies on the direct sum of modules, a new approach was developed to construct the coproduct topology on the direct sum. This new approach uses different techniques from those already in the literature and this approach makes it less labour intensive to derive the coproduct topology's explicit form.

I want to express my sincerest gratitude to the following people. To Dr. Gàbor Lukàcs for providing an advanced copy of his book *Compact-like topological groups* which has been infinitely helpful and for his suggestion to consider T-filters as another approach to the coproduct topology on direct sums. To Dr. Peter Nicholas and Dr. Peter Loth for their e-mail correspondence and clarification. To Leslie Supnet for her love, commitment and unwavering positive attitude. To my family and friends for always being there with their unconditional support. To my peers and the faculty and staff at the University of Manitoba for their camaraderie and support.

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Contents

Introduction

In this thesis, I am interested in investigating *natural* concepts for pure-embeddings and pure-injectivity with respect to topological modules. In the case of both modules and topological modules, it is possible to approach them from two different perspectives, that is, from an algebraic perspective and from a model theoretic perspective. I will attempt to provide context by discussing the concepts in both the algebraic sense and the model theoretic sense, however, the emphasis will be put on the model theoretic perspective. There are close ties between model theory and algebra. For instance, for any left R-module M, the set defined by a pp-formula is a subgroup. Furthermore, it is possible to think of pp-formulas as generalized algebraic statements. This generalization is useful since it often provides additional information and insight.

In Chapter 1, all relevant background material is covered. It includes all of the rudimentary information required for the study of topological modules from a model theoretic perspective. Some of the standard and elementary proofs have been omitted for the sake of brevity, however, references are provided. A brief introduction is given to model theory, topology, topological groups and category theory. Furthermore, in Chapter 1.5, there is an introduction to the model theory of modules and more specifically, pure embeddings and pure-injectivity. These are the concepts that we will be trying to expand to topological modules in Chapter 4. In addition, positive primitive formulas are introduced. Positive primitive formulas are important in the study of model theory of modules. This concept is expanded to topological positive primitive formulas in Chapter 4.2.

In Chapter 2, the language used in topological structures is introduced, namely $\mathcal{L}_t$. Analogous concepts to those in classical model theory, introduced in Chapter 1.1, will be discussed for $\mathcal{L}_t$. Furthermore, the strengths and weaknesses of $\mathcal{L}_t$ are examined. This discussion will reveal which topological concepts can be captured in $\mathcal{L}_t$ and which concepts cannot. In addition, the language used to discuss topological modules will be introduced, namely, $\mathcal{L}_m$.

In Chapter 3, a coproduct is explored in the category of topological modules. The direct sum of modules is a coproduct in the category of modules and the direct sum plays an important role in the model theory of modules. Therefore, it is natural to consider different topologies on the direct sum in order to determine the role the direct sum will play in the model theory of topological modules. A topology is given on the direct sum which makes it a coproduct in the category of topological modules. Furthermore, an approach is given to determine this topology explicitly using an approach which is different from that in the literature.

In Chapter 4, different concepts for pure embeddings and pure-injectivity are explored. First, a concept of pure-injectivity in the category of locally compact abelian groups is considered, in order see if it is possible to expand this concept to the category of topological modules. In Chapter 4.2, a definition of topological positive primitive formulas is introduced. This concept is shown to be analogous to the concept of positive primitive formulas. Finally, different definitions of topological pure embeddings are given and it is shown why classical approaches for determining a natural concept for pure-injectivity fail in the study of model theory of topological modules.

Chapter 1

Background

This chapter provides a solid foundation for the study of topological modules in a model theoretic context.

1.1 Model Theory Preliminaries

In this section, I review basic model theoretic concepts. In model theory, we study mathematical structures using mathematical logic. These structures are studied by considering the structures definable by first-order formulas and by considering which structures satisfy specific first-order formulas. For further reference, consider [BM77], [Rot00] or [Mar02].

The definition of a first-order language and structure is presented here formally. At least three different kinds of logic will be presented throughout this thesis: ordinary first-order logic, two-sorted first-order logic and a weak second-order logic for topological structures. Only a detailed account of the first will be provided since the others are expansions/variations of this concept.

Definition 1.1. A first-order language $\mathcal{L}$ consists of:

(i) a set of distinct variables $\{v_i : i \in \mathbb{N}\}$;

(ii) the set of logical symbols $\{\neg, \wedge, \exists, =, (,)\}$;

(iii) a set of function symbols $\mathcal{F}$, and for each $f \in \mathcal{F}$, a positive integer n_f indicating that f represents a function of n_f variables;

(iv) a set of relation symbols $\mathcal{R}$, and for each $R \in \mathcal{R}$, a positive integer n_R indicating that R represents a n_R-ary relation;

(v) a set of constant symbols $\mathcal{C}$.

Note. Any or all of the sets $\mathcal{F}$, $\mathcal{R}$, and $\mathcal{C}$ may be empty.

The language for abelian groups is $(+, -, \mathbf{0})$ where $+$ is a binary function symbol, $-$ is a unary function symbol and $\mathbf{0}$ is a constant symbol.

Definition 1.2. The set of $\mathcal{L}$-*terms* is the smallest set $\mathcal{T}$ such that:

(i) $\forall c \in \mathcal{C}$, $c \in \mathcal{T}$;

(ii) $\forall i \in \mathbb{N}$, $v_i \in \mathcal{T}$;

(iii) $\forall f \in \mathcal{F}$, if $t_1, \ldots, t_{n_f} \in \mathcal{T}$ then $f(t_1, \ldots, t_{n_f}) \in \mathcal{T}$.

Some terms in the language of abelian groups are $\mathbf{0}$, $v_0 + v_1$ and $-(v_0 + v_1)$.

Definition 1.3. ϕ is an *atomic* $\mathcal{L}$-*formula* if ϕ is either:

(i) $t_1 = t_2$, where t_1 and t_2 are $\mathcal{L}$-terms or

(ii) $R(t_1, \ldots, t_{n_R})$, where $R \in \mathcal{R}$ and $t_1, \ldots, t_{n_R}$ are $\mathcal{L}$-terms.

For instance, $v_0+v_1 = v_1+v_2$ is an example of an atomic formula in the language of abelian groups. In fact, all atomic formulas in this language are of the form $t_1 = t_2$ since the language of abelian groups does not have any relation symbols.

Definition 1.4. The set of all $\mathcal{L}$-*formulas* is the smallest set $\mathcal{W}$ containing the atomic $\mathcal{L}$-formulas and such that:

(i) if $\phi \in \mathcal{W}$ then $\neg\phi \in \mathcal{W}$;

(ii) if $\phi, \psi \in \mathcal{W}$ then $(\phi \wedge \psi) \in \mathcal{W}$;

(iii) if ϕ is in $\mathcal{W}$ and v is a variable then $(\exists v)\phi \in \mathcal{W}$.

Note. We introduce the symbols $\vee$, $\rightarrow$, $\leftrightarrow$ and $\forall$ by defining them as follows:

$$\begin{aligned}
(\phi \vee \psi) &\quad \text{as} \quad \neg(\neg\phi \wedge \neg\psi);\\
(\forall v)\phi &\quad \text{as} \quad \neg(\exists v)\neg\phi;\\
(\phi \rightarrow \psi) &\quad \text{as} \quad (\neg\phi \vee \psi);\\
(\phi \leftrightarrow \psi) &\quad \text{as} \quad ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)).
\end{aligned}$$

If v is a variable in a formula ϕ and v occurs inside the scope of a $(\exists v)$ or a $(\forall v)$ quantifier then this occurrence of v is *bound* and we say ϕ has v as a *bound variable*. If v is a variable in a formula ϕ and v is not bound then v is *free* and we say ϕ has v as a *free variable*. It should be noted that ϕ can have v as a free and as a bound variable. For instance, consider the formula $v = v \wedge (\exists v)v = v$. A formula is called a *sentence* if there are no free variables. We use $\phi(v_1, \ldots, v_n)$ to denote an $\mathcal{L}$-formula ϕ whose free variables are among the distinct variables $v_1, \ldots, v_n$.

For many examples of sentences, see the descriptions of various theories following Definition 1.15.

Definition 1.5. An $\mathcal{L}$-*structure* $\mathcal{M}$ consists of the following:

(i) a non-empty set M, called the *universe* of $\mathcal{M}$;

(ii) for each $f \in \mathcal{F}$, a function $f^{\mathcal{M}} : M^{n_f} \to M$;

(iii) for each $R \in \mathcal{R}$, a relation $R^{\mathcal{M}} \subseteq M^{n_R}$;

(iv) for each $c \in \mathcal{C}$, an element $c^{\mathcal{M}} \in M$.

$f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ are called the *interpretations* of the symbols f, R, c in $\mathcal{M}$.

Definition 1.6. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, t be an $\mathcal{L}$-term whose variables are among the variables $v_1, \ldots, v_n$ and $\overline{a} = (a_1, \ldots, a_n) \in M^n$. The *interpretation* of t in $\mathcal{M}$ at $\overline{a}$, denoted $t^{\mathcal{M}}(\overline{a})$, is defined inductively as follows:

(i) if t is a constant symbol c then

$$t^{\mathcal{M}}(\overline{a}) = c^{\mathcal{M}};$$

(ii) if t is the variable v_i then

$$t^{\mathcal{M}}(\overline{a}) = a_i;$$

(iii) if t is $f(t_1, \ldots, t_{n_f})$, where $f \in \mathcal{F}$ and $t_1, \ldots, t_{n_f}$ are $\mathcal{L}$-terms, then

$$t^{\mathcal{M}}(\overline{a}) = f^{\mathcal{M}}({t_1}^{\mathcal{M}}(\overline{a}), \ldots, {t_{n_f}}^{\mathcal{M}}(\overline{a})).$$

Definition 1.7. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\phi(v_1, \ldots, v_n)$ be an $\mathcal{L}$-formula and $\overline{a} = (a_1, \ldots, a_n) \in M^n$. We define $\mathcal{M} \models \phi[\overline{a}]$ inductively as follows:

(i) if ϕ is $t_1 = t_2$ then

$$\mathcal{M} \models \phi[\overline{a}] \text{ if } {t_1}^{\mathcal{M}}(\overline{a}) = {t_2}^{\mathcal{M}}(\overline{a});$$

(ii) if ϕ is $R(t_1, \ldots, t_{n_R})$ then

$$\mathcal{M} \models \phi[\overline{a}] \text{ if } ({t_1}^{\mathcal{M}}(\overline{a}), \ldots, {t_{n_R}}^{\mathcal{M}}(\overline{a})) \in R^{\mathcal{M}};$$

(iii) if ϕ is $\neg\psi$ then

$$\mathcal{M} \models \phi[\overline{a}] \text{ if it is not the case that } \mathcal{M} \models \psi[\overline{a}];$$

(iv) if ϕ is $(\psi \wedge \theta)$ then

$$\mathcal{M} \models \phi[\overline{a}] \text{ if } \mathcal{M} \models \psi[\overline{a}] \text{ and } \mathcal{M} \models \theta[\overline{a}];$$

(v) (a) if ϕ is $(\exists v_i)\psi(\overline{v}, v_i)$, where $i > n$, then

$$\mathcal{M} \models \phi[\overline{a}] \text{ if there is a } b \in M \text{ such that } \mathcal{M} \models \psi[\overline{a}, b];$$

(b) if ϕ is $(\exists v_i)\psi(\overline{v})$, where $i \leq n$, then

$$\mathcal{M} \models \phi[\overline{a}] \text{ if there is a } b \in M \text{ such that}$$

$$\mathcal{M} \models \psi[a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n].$$

If $\mathcal{M} \models \phi[\overline{a}]$ then we say that $\mathcal{M}$ *satisfies* ϕ at $\overline{a}$ and that ϕ is *true* at $\overline{a}$ in $\mathcal{M}$. If it is not that case that $\mathcal{M} \models \phi[\overline{a}]$ then we write $\mathcal{M} \not\models \phi[\overline{a}]$.

For example, let $\mathcal{M} = (\mathbb{Z}, +, -, \mathbf{0})$, $\phi(v) = (\exists w)(v + w = \mathbf{0} \wedge w + w = \mathbf{0})$ and $\psi(v) = (\exists w)(w + w = v)$. So, $\mathcal{M} \models \phi[a]$ if and only if $a = 0$ and $\mathcal{M} \models \psi[a]$ if and only if a is even.

Definition 1.8. Let $\phi(v_1, \ldots, v_n)$ be an $\mathcal{L}$-formula. We let $\phi[M]$ denote the set of $\overline{a} \in M^n$ such that $\mathcal{M} \models \phi[\overline{a}]$. That is,

$$\phi[M] = \{\overline{a} \in M^n : \mathcal{M} \models \phi[\overline{a}]\}.$$

If $k \leq n$ and $\overline{b} \in M^{n-k}$, then we let

$$\phi[M, \overline{b}] = \left\{\overline{a} \in M^k : \mathcal{M} \models \phi[\overline{a}, \overline{b}]\right\}.$$

$\phi[M, \overline{b}]$ is referred to as a "definable set of M". For convenience, the "k" is often suppressed from the M^k. If Φ is a set of formulas in the same free variables, then we let

$$\Phi[M] = \bigcap \{\phi[M] : \phi \in \Phi\}.$$

The concept of a definable set is one of the key concepts in model theory since it can be used to classify and analyze structures. In addition, there is often a correlation between the complexity of the definable sets and the complexity of the structure.

It is possible to generalize the concept of language and structure to be many-sorted, however, we will only expand to the two-sorted case. A *two-sorted language* is a language where all the constants and variables are partitioned into two sorts. Informally, there are two different kinds, and each variable and constant is either of one sort or the other, but not both. Furthermore, the relation symbols and function symbols are type dependent. A *two-sorted structure*, $\mathcal{M}$, is a set M that is partitioned into two disjoint sets, non-empty sets M_1 and M_2. Now,

(i) for each $R \in \mathcal{R}$, $R^{\mathcal{M}} \subseteq M_{s_1} \times \ldots \times M_{s_{n_R}}$ where each $s_i \in \{1,2\}$;

(ii) for each $f \in \mathcal{F}$, $f^{\mathcal{M}} : M_{s_1} \times \cdots \times M_{s_{n_f}} \to M_{s_0}$ where each $s_i \in \{1,2\}$;

(iii) for each $c \in C$, $c^{\mathcal{M}} \in M_{s_1}$ where $s_1 \in \{1,2\}$.

Note that each s_i is determined by the sort of the underlying language. In addition, in a two-sorted language, Definitions 1.2-1.4, must be modified to respect the sort of each variable and constant symbol.

It should be remarked that a two-sorted structure for a two-sorted language is essentially identical to a one-sorted structure in a richer language. Furthermore, there are standard techniques for obtaining this first order language. This means that all of the standard theorems of ordinary first-order model theory apply in the two-sorted context.

Definition 1.9. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures. An *$\mathcal{L}$-homomorphism* $\eta : \mathcal{M} \to \mathcal{N}$ is a map from M to N such that:

(i) for all $f \in \mathcal{F}$ and for all $(a_1, \ldots, a_{n_f}) \in M^{n_f}$,

$$\eta(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \ldots, \eta(a_{n_f}));$$

(ii) for all $R \in \mathcal{R}$ and for all $(a_1, \ldots, a_{n_R}) \in M^{n_R}$,

$$(a_1, \ldots, a_{n_R}) \in R^{\mathcal{M}} \Rightarrow (\eta(a_1), \ldots, \eta(a_{n_R})) \in R^{\mathcal{N}};$$

(iii) for all $c \in \mathcal{C}$,

$$\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}.$$

If (ii) is $\Leftrightarrow$ instead of $\Rightarrow$ then η is called a *strong $\mathcal{L}$-homomorphism.* If η is a one-to-one strong $\mathcal{L}$-homomorphism then η is called an *$\mathcal{L}$-embedding.* Furthermore, if $M \subseteq N$ and if the inclusion map is a strong $\mathcal{L}$-homomorphism then $\mathcal{N}$ is called an *extension* of $\mathcal{M}$ and $\mathcal{M}$ is called a *substructure* of $\mathcal{N}$, denoted $\mathcal{M} \leq \mathcal{N}$. If η is a one-to-one and onto strong $\mathcal{L}$-homomorphism then η is called an *$\mathcal{L}$-isomorphism* and we say $\mathcal{M}$ is isomorphic to $\mathcal{N}$, denoted $\mathcal{M} \cong \mathcal{N}$.

Note. Let η be an $\mathcal{L}$-homomorphism from $\mathcal{M}$ to $\mathcal{N}$ and $\overline{a} = (a_1, \ldots, a_n) \in M^n$. For convenience, we let

$$\eta(\overline{a}) = (\eta(a_1), \ldots, \eta(a_n)).$$

Lemma 1.10. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures. If $\eta : \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-homomorphism and $\overline{a} = (a_1, \ldots, a_n) \in M^n$ then for all $\mathcal{L}$-terms t whose variables are among $v_1, \ldots, v_n$,*

$$\eta(t^{\mathcal{M}}(\overline{a})) = t^{\mathcal{N}}(\eta(\overline{a})).$$

Proof. This is easily proved by an induction on the complexity of $\mathcal{L}$-terms. □

Proposition 1.11. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures such that $\mathcal{M} \leq \mathcal{N}$. If*

$\phi(v_1, \ldots, v_n)$ is a quantifier-free $\mathcal{L}$-formula and $\overline{a} = (a_1, \ldots, a_n) \in M^n$ then

$$\mathcal{M} \models \phi[\overline{a}] \Leftrightarrow \mathcal{N} \models \phi[\overline{a}].$$

Proof. This is easily proved by an induction on the complexity of quantifier-free $\mathcal{L}$-formulas. □

Note. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures such that $\mathcal{M} \leq \mathcal{N}$. Clearly, if ψ is obtained from a quantifier-free formula by some universal quantifications then

$$\mathcal{N} \models \psi[\overline{a}] \Rightarrow \mathcal{M} \models \psi[\overline{a}]$$

and if ψ is obtained from a quantifier-free formula by some existential quantifications then

$$\mathcal{M} \models \psi[\overline{a}] \Rightarrow \mathcal{N} \models \psi[\overline{a}].$$

Definition 1.12. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $j : \mathcal{M} \to \mathcal{N}$ is an *elementary embedding* if for all n, for all $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and for all $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$,

$$\mathcal{M} \models \phi[a_1, \ldots, a_n] \Leftrightarrow \mathcal{N} \models \phi[j(a_1), \ldots, j(a_n)].$$

If $\mathcal{M} \leq \mathcal{N}$ then $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ or $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$, denoted $\mathcal{M} \prec \mathcal{N}$, if the inclusion map is an elementary embedding.

Note. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures such that $\mathcal{M} \prec \mathcal{N}$. It is clear that if $\phi(v_1, \ldots v_n)$ is an $\mathcal{L}$-formula then $\phi[M] = M^n \cap \phi[N]$.

Definition 1.13. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, denoted $\mathcal{M} \equiv \mathcal{N}$, if for all $\mathcal{L}$-sentences ϕ:

$$\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi.$$

Clearly if $\mathcal{M} \prec \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.

Theorem 1.14. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures. If $\eta : \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-isomorphism then $\mathcal{M} \equiv \mathcal{N}$.*

Proof. This is easily proved by an induction on the complexity of $\mathcal{L}$-formulas. □

Definition 1.15. Let $\mathcal{L}$ be a language. An $\mathcal{L}$-*theory* T is a set of $\mathcal{L}$-sentences. An $\mathcal{L}$-structure $\mathcal{M}$ is a *model* for T, denoted $\mathcal{M} \models T$, if for all $\phi \in T$, $\mathcal{M} \models \phi$. A theory is *satisfiable* if it has a model. If T is an $\mathcal{L}$-theory and ϕ is an $\mathcal{L}$-sentence such that $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$ then ϕ is a *logical consequence* of T, denoted $T \models \phi$. The *full theory* of $\mathcal{M}$, denoted $\mathsf{Th}(\mathcal{M})$, is the set of all $\mathcal{L}$-sentences ϕ such that $\mathcal{M} \models \phi$. If $\mathcal{C}$ is a class of $\mathcal{L}$-structures then the *theory* of $\mathcal{C}$, denoted $\mathsf{Th}(\mathcal{C})$, is the set of all $\mathcal{L}$-sentences ϕ such that $\mathcal{M} \models \phi$ for every $\mathcal{M} \in \mathcal{C}$. Finally, for any theory T we let $\mathsf{Mod}(T)$ denote the set all $\mathcal{L}$-structures that satisfy the theory T.

Theory (Theory of abelian groups). Let $\mathcal{L} = \{+, -, 0\}$, where $+$ is a binary function,$-$ is a unary function and 0 is a constant symbol. The theory of abelian groups is:

(i) $(\forall x)(0 + x = x)$;

(ii) $(\forall x)(\forall y)(\forall z)(x + (y + z) = (x + y) + z)$;

(iii) $(\forall x)(x + (-x) = 0)$;

(iv) $(\forall x)(\forall y)(x + y = y + x)$.

Theory (Theory of unital left R-modules). Let R be a ring with multiplicative identity 1. Let $\mathcal{L}_R = \{+, -, 0\} \cup \{\mathbf{r} : r \in R\}$ where $+$ is a binary function, 0 is a constant and for each $r \in R$, $\mathbf{r}$ is a unary function symbol. The theory of left R-modules is:

(i) The theory of abelian groups;

(ii) for each $r \in R$, $(\forall x)(\forall y)(\mathbf{r}(x + y) = \mathbf{r}(x) + \mathbf{r}(y))$;

(iii) for each $r, s, t \in R$ such that $rs = t$, $(\forall x)(\mathbf{r}(\mathbf{s}(x)) = \mathbf{t}(x))$;

(iv) $r, s, t \in R$ such that $r + s = t$, $(\forall x)(\mathbf{r}(x) + \mathbf{s}(x) = \mathbf{t}(x))$;

(v) $(\forall x)(\mathbf{1}(x) = x)$.

In $\mathcal{L}_R$, a unary function symbol is added for each element of the ring R. By incorporating the ring into the language, the theory of rings, an inherently complex first-order theory, is avoided. Moreover, this avoids building a two-sorted structure.

Theory (Theory of unital right R-modules). The theory of right R-modules is the theory of left R-modules with (iii) replaced by

(iii$'$) for each $r, s, t \in R$ such that $rs = t$, $(\forall x)(\mathbf{s}(\mathbf{r}(x)) = \mathbf{t}(x))$.

Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. Informally, a *proof* of ϕ from T is a finite sequence of $\mathcal{L}$-formulas $\phi_1, \ldots, \phi_n$ such that $\phi = \phi_n$ and for each $i \in \{1, \ldots, n\}$ either ϕ_i is an instance of an axiom of first-order logic, ϕ_i is in T or ϕ_i follows from

$\phi_1, \ldots, \phi_{i-1}$ by one of the standard rules of inference. If there is a proof of ϕ from T, we write $T \vdash \phi$.

Definition 1.16. An $\mathcal{L}$-theory T is *inconsistent* if there exists an $\mathcal{L}$-sentence ϕ such that $T \vdash \phi \wedge \neg\phi$. T is *consistent* if it is not inconsistent. A consistent $\mathcal{L}$-theory T is *complete* if for any $\mathcal{L}$-sentences ϕ either $T \vdash \phi$ or $T \vdash \neg\phi$.

Theorem 1.17 (Gödel's Completeness Theorem)**.** *Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence, then*

$$T \vdash \phi \Leftrightarrow T \models \phi.$$

Gödel's Completeness Theorem is one of the fundamental theorems in mathematical logic. It establishes a correspondence between semantic truth and provability in first-order logic.

Corollary 1.18. *Let T be an $\mathcal{L}$-theory. T is consistent if and only if T is satisfiable.*

Theorem 1.19 (Compactness Theorem)**.** *Let T be an $\mathcal{L}$-theory. T is satisfiable if and only if T is finitely satisfiable, that is, every finite subset of T is satisfiable.*

Proof. ($\Rightarrow$) Clearly, if T is satisfiable then every subset of T is satisfiable.

($\Leftarrow$) Suppose that T is not satisfiable. Since T is not satisfiable, T is inconsistent. So, let σ be a proof of a contradiction from T. Since σ is finite, only a finite number of assumptions from T are used in the proof. Thus, there is a finite $T_0 \subseteq T$ such that σ is a proof of a contradiction from T_0. So, T_0 is a finite unsatisfiable subset of T.

□

In other words, a set of first-order sentences has a model if and only if every finite subset of it has a model.

Corollary 1.20. *A set of $\mathcal{L}$-sentences that has arbitrarily large finite models, has an infinite model.*

Proof. Let Φ be a set of $\mathcal{L}$-sentences that has arbitrarily large finite models. Let

$$I_n = (\exists x_1) \cdots (\exists x_n) \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$$

and

$$\Phi^* = \Phi \cup \{I_1, I_2, I_3, \ldots\}.$$

Every finite subset Φ' of Φ^* can only contain finitely many I_k. Since I_k expresses "there is a model of size k" and Φ has a model $\mathcal{M}$ of size k such that $\mathcal{M} \models \Phi$, all finite subsets of Φ^* are satisfiable. Therefore, by the Compactness Theorem 1.19, Φ^* is satisfiable. However, Φ^* has only infinite models. Therefore $\Phi \subset \Phi^*$ must also have an infinite model. $\square$

Corollary 1.21 (Löwenheim-Skolem Theorem). *A countable set of $\mathcal{L}$-sentences that has an infinite model, has a countably infinite model.*

The compactness theorem and the Löwenheim-Skolem Theorem are both used in Lindström's theorem to characterize first-order logic. Lindström's theorem basically states that first-order logic is the *strongest* logic having both the compactness property and the Löwenheim-Skolem property.

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. Let $\mathcal{L}_A = \mathcal{L} \cup \{c_a : a \in A\}$ where for each $a \in A$, c_a is a new distinct constant symbol. Clearly, $\mathcal{M}$ can be viewed as an $\mathcal{L}_A$-structure by interpreting c_a as a. We let $\mathsf{Th}_A(\mathcal{M})$ denote the set of all $\mathcal{L}_A$-sentences that are true in $\mathcal{M}$.

Definition 1.22. Let p be a set of $\mathcal{L}_A$-formulas with free variables among $v_1, \ldots, v_n$. p is an *n-type* over A if $p \cup \mathsf{Th}_A(\mathcal{M})$ is satisfiable. p is a *complete n-type* over A if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas ϕ with free variables among $v_1, \ldots, v_n$. Incomplete types are called *partial types.* We let $S_n^{\mathcal{M}}(A)$ denote the set of all complete n-types over A.

For example, consider $\mathcal{M} = (\mathbb{Q}, <)$ and let $A = \mathbb{N}$ and let

$$p(v) = \{1 < v, 2 < v, 3 < v, \ldots\}.$$

If Φ is a finite subset of $p(v) \cup \mathsf{Th}_A(\mathcal{M})$, then Φ is satisfiable by choosing v to be a sufficiently large element of $\mathbb{Q}$. By the Compactness Theorem, $p(v) \cup \mathsf{Th}_A(\mathcal{M})$ is satisfiable. So, $p(v)$ is a 1-type. Furthermore, it is not realized in $(\mathbb{Q}, <)$ since any element realizing it is larger than every natural number.

Now, consider

$$q(v) = \{\phi(v) \in \mathcal{L}_A : \mathcal{M} \models \phi[1/2]\}.$$

For any $\phi(v) \in \mathcal{L}_A$, either $\mathcal{M} \models \phi[1/2]$ or $\mathcal{M} \models \neg\phi[1/2]$. Thus $q(v)$ is a complete 1-type.

Definition 1.23. Let κ be an infinite cardinal and let T be a complete theory with infinite models in a countable language $\mathcal{L}$. We say that $\mathcal{M} \models T$ is *κ-saturated* if, for all $A \subseteq M$ and all n, if $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$, then p is realized in $\mathcal{M}$. $\mathcal{M}$ is *saturated* if it is $|M|$-saturated.

Note. In general, the existence of saturated models depends on higher axioms of set theory. However, given certain set-theoretic assumptions, saturated models (usually with *very large* cadinality) exist for arbitrary theories.

For example, if $\mathcal{M}$ is an infinite dimensional vector space over any field then $\mathcal{M}$ is $dim(M)$-saturated.

Definition 1.24. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\overline{a} \in M^n$ and $B \subseteq M$. The *type* of $\overline{a}$ over B is

$$\begin{aligned} & \mathsf{tp}^{\mathcal{M}}(\overline{a}/B) \\ = \; & \{\phi(\overline{v}, \overline{b}) : \mathcal{M} \models \phi[\overline{a}, \overline{b}] \text{ where } \phi(v_1, \ldots, v_k) \text{ is a } \mathcal{L}\text{-formula and } \overline{b} \in B^{k-n}\}. \end{aligned}$$

So, $\mathsf{tp}^{\mathcal{M}}(\overline{a}/B)$ is a set of $\mathcal{L}_B$-formulas and is a complete type over B (in the sense of Definition 1.22). Also, we denote $\mathsf{tp}^{\mathcal{M}}(\overline{a}/\varnothing)$ by $\mathsf{tp}^{\mathcal{M}}(\overline{a})$ and refer to this as the type of $\overline{a}$ in $\mathcal{M}$.

Corollary 1.25. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures such that $\mathcal{M} \prec \mathcal{N}$. Then for all $\overline{a} \in M$, $\mathsf{tp}^{\mathcal{M}}(\overline{a}) = \mathsf{tp}^{\mathcal{N}}(\overline{a})$ and for all $A \subseteq M$, $\mathsf{tp}^{\mathcal{M}}(\overline{a}/A) = \mathsf{tp}^{\mathcal{N}}(\overline{a}/A)$.*

Proof. This follows directly from the Definition 1.12 and 1.24. □

1.2 Topological Preliminaries

In this section, I present some basic definitions and theorems from general topology needed in later sections. Since many of these results are well known, some of the proofs have been omitted. For further reference, please consult [Wil70] or [Eng89].

Definition 1.26. A *topological space* (X, τ) is a set X together with τ, a collection of subsets of X that includes $\varnothing$ and X, and which is closed under finite intersections and arbitrary unions. The sets contained in τ are called the *open sets* of X. A subset of X is a *closed set* of X if it is the complement of an open set of X. If $\tau = \{X, \varnothing\}$ then (X, τ) is called the *indiscrete* or *trivial topology* and if τ is the set of all subsets of X then (X, τ) is called the *discrete topology.* If (X, τ) is a topological space and $A \subseteq X$, an element $a \in A$ is an *interior point* of A if there is an open set U with $a \in U \subseteq A$. Furthermore, a *neighbourhood* of x in X is a subset of X containing x as an interior point.

Clearly, the set of closed subsets of X is closed under finite unions and arbitrary intersections.

Proposition 1.27. *Let (X, τ) be a topological space. If a set U contains a neighbourhood of each of its points then U is an open set.*

Definition 1.28. Let (X, τ) be a topological space. The *closure* of a subset A of X, denoted $cl_{(X,\tau)}(A)$ (or simply $cl_X(A)$ if the topology is clear from the context), is the smallest closed subset C of X such that $A \subseteq C$.

Definition 1.29. Let (X, τ) be a topological space.

(i) (X, τ) is a *T_0-space* if given any two distinct points $x, y \in X$, there exists an open set containing exactly one of them.

(ii) (X, τ) is a *T_1-space* if for any pair of distinct points $x, y \in X$, there exists open sets U and V such that $x \in U$, $y \notin U$, $y \in V$ and $x \notin V$.

(iii) (X, τ) is a *Hausdorff space* if for any pair of distinct points $x, y \in X$, there exists disjoint open sets U and V such that $x \in U$ and $y \in V$.

(iv) (X, τ) is a *regular space* if for each closed set F and each point $x \notin F$, there exists disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

(v) (X, τ) is a *normal space* if for any pair of disjoint closed sets E and F, there exists neigbourhoods A of E and B of F that are disjoint.

Note. Clearly, every Hausdorff space is a T_1-space and every T_1-space is a T_0-space. Hausdorff spaces are also called T_2-spaces. A regular T_1-space is called a T_3-space. Moreover, a T_3-space is Hausdorff and a normal T_1-space is regular.

Proposition 1.30. *(X, τ) is a T_1-space if and only if for each $x \in X$, $\{x\}$ is a closed set.*

Definition 1.31. If (X, τ) is a topological space, a *base* for τ is a collection $\mathfrak{B} \subseteq \tau$ such that every open set can be written as a union of elements of $\mathfrak{B}$. In other words, if $U \in \tau$ then there is $\mathfrak{U} \subseteq \mathfrak{B}$ such that

$$U = \bigcup \mathfrak{U}.$$

Theorem 1.32. *$\mathfrak{B} \subseteq \wp(X)$ is a base for some topology τ on X if and only if*

(i) $X = \bigcup \mathfrak{B}$ and

(ii) whenever $B_1, B_2 \in \mathfrak{B}$ with $x \in B_1 \cap B_2$, there is some $B_x \in \mathfrak{B}$ with

$$x \in B_x \subseteq B_1 \cap B_2.$$

Proof. $(\Rightarrow)$ Suppose that $\mathfrak{B}$ is a base for a topology τ on X.

(i) Since $X \in \tau$, $X = \bigcup \mathfrak{U}$ for some $\mathfrak{U} \subseteq \mathfrak{B}$. Therefore, $X = \bigcup \mathfrak{B}$.

(ii) Let $B_1, B_2 \in \mathfrak{B}$. Then $B_1 \cap B_2 \in \tau$ since $B_1, B_2 \in \tau$. Thus, $B_1 \cap B_2 = \bigcup \mathfrak{U}$ where $\mathfrak{U} \subseteq \mathfrak{B}$. Therefore, for each $x \in B_1 \cap B_2$ there exists a $B_x \in \mathfrak{B}$ such that $x \in B_x \subseteq B_1 \cap B_2$.

($\Leftarrow$) Suppose that (i) and (ii) hold and let τ be the collection of all unions of subcollections of $\mathfrak{B}$. We want to show that τ is a topology. Now, $\varnothing = \bigcup \varnothing$ and $\varnothing \subseteq \mathfrak{B}$. Also, $X = \bigcup \mathfrak{B}$ by (i). Therefore, $\varnothing \in \tau$ and $X \in \tau$. Clearly, the union of elements of τ is also in τ. Finally, take any $U_1, U_2 \in \tau$. Then, $U_1 = \bigcup \mathfrak{U}_1$ and $U_2 = \bigcup \mathfrak{U}_2$, where $\mathfrak{U}_1, \mathfrak{U}_2 \subseteq \mathfrak{B}$. By the distributive law,

$$U_1 \cap U_2 = \bigcup \mathfrak{U}_1 \cap \bigcup \mathfrak{U}_2 = \bigcup \{B_1 \cap B_2 : B_1 \in \mathfrak{U}_1, B_2 \in \mathfrak{U}_2\}.$$

By (ii), for every $x \in B_1 \cap B_2$, there is a $B_x \in \mathfrak{B}$ such that $x \in B_x \subseteq B_1 \cap B_2$. Therefore,

$$B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B_x.$$

So, by Definition 1.26, τ is a topology over X.

□

Note. Clearly, if $\mathfrak{B}$ is a base for a topology on X, then it is a base for only one topology. This topology is called the topology on X *generated by* $\mathfrak{B}$. Clearly, if $\mathcal{S}$ is any collection of subsets of X such that $\bigcup \mathcal{S} = X$, and if $\mathfrak{B}$ is the set of all intersections of finite subfamilies of $\mathcal{S}$, then $\mathfrak{B}$ is a basis for a topology on X.

Definition 1.33. If (X, τ) is a topological space, a *subbase* for τ is a collection $\mathcal{S} \subseteq \tau$ such that the collection of all finite intersections of elements from $\mathcal{S}$ form a base for τ.

Definition 1.34. If (X, τ) is a topological space and $A \subseteq X$, the collection $\sigma = \{U \cap A : U \in \tau\}$ is a topology on A called the *subspace topology.* Furthermore, (A, σ) is called a *subspace* of (X, τ).

Definition 1.35. Let (X, τ) and $Y, \sigma)$ be topological spaces and let $f : X \to Y$. Then f is *open* if for every open set U of X, $f[U]$ is an open set of Y and f is *closed* if for every closed set F of X, $f[F]$ is a closed set of Y.

Definition 1.36. Let (X, τ) and (Y, σ) be topological spaces and let $f : X \to Y$. Then f is *continuous at the point* $x \in X$ if for every neighbourhood V of $f(x)$ in Y, there is a neighbourhood U of x in X such that $f[U] \subseteq V$. Furthermore, f is *continuous* on X if f is continuous at each $x \in X$.

Equivalently, if (X, τ) and (Y, σ) are topological spaces and $f : X \to Y$, then f is continuous if and only if for each $V \in \sigma$, $f^{-1}[V] \in \tau$.

Theorem 1.37.

(i) The composition of continuous maps is continuous.

(ii) If X is equipped with the discrete topology then every mapping $f : X \to Y$ is continuous.

(iii) If Y is equipped with the indiscrete topology then every mapping $f : X \to Y$ is continuous.

Proof. This follows directly from the definitions. □

Note. Let $A \subseteq X$ and let (A, σ), (X, τ) be topological spaces. If (A, σ) is a subspace of (X, τ) then the inclusion map is continuous.

Theorem 1.38. *If (X, τ) and (Y, σ) are topological spaces and $f : X \to Y$ is continuous then:*

(i) If $\mathfrak{B}$ is a base for σ then $\{f^{-1}[B] : B \in \mathfrak{B}\}$ is a base for τ.

(ii) If $\mathcal{S}$ is a subbase for σ then $\{f^{-1}[S] : S \in \mathcal{S}\}$ is a subbase for τ.

Proof. (i) Let $\mathfrak{B}$ be a base for σ. Then $Y = \bigcup \mathfrak{B}$. So,

$$X = f^{-1}[Y] = f^{-1}\left[\bigcup \mathfrak{B}\right] = \bigcup_{B \in \mathfrak{B}} f^{-1}[B].$$

Now, consider $f^{-1}[B_1]$ and $f^{-1}[B_2]$ where $B_1, B_2 \in \mathfrak{B}$. Suppose

$$x \in f^{-1}[B_1] \cap f^{-1}[B_2] = f^{-1}[B_1 \cap B_2].$$

So, $f(x) \in B_1 \cap B_2$. Therefore, there exists $B_3 \in \mathfrak{B}$ with $f(x) \in B_3 \subseteq B_1 \cap B_2$. So,

$$x \in f^{-1}[B_3] \subseteq f^{-1}[B_1 \cap B_2] = f^{-1}[B_1] \cap f^{-1}[B_2].$$

This shows that $\{f^{-1}[B] : B \in \mathfrak{B}\}$ is a base for τ.

(ii) This follows directly from (i) and Definition 1.33 since f^{-1} preserves intersections.

□

Theorem 1.39. *If (X, τ) and (Y, σ) are topological spaces, $\mathfrak{B}$ is a base for σ and $\mathcal{S}$ is a subbase for σ then:*

(i) $f : X \to Y$ is continuous if and only if for every $B \in \mathcal{B}$, $f^{-1}[B] \in \tau$.

(ii) $f : X \to Y$ *is continuous if and only if for every* $S \in \mathcal{S}$, $f^{-1}[S] \in \tau$.

Proof. (i) ($\Rightarrow$) Vacuously.

($\Leftarrow$) Suppose that for every $B \in \mathcal{B}$, $f^{-1}[B] \in \tau$. Let U be an open set of Y. So, $U = \bigcup \mathfrak{U}$ where $\mathfrak{U} \subseteq \mathfrak{B}$. Now,

$$f^{-1}[U] = f^{-1}\left[\bigcup \mathfrak{U}\right] = \bigcup_{V \in \mathfrak{U}} f^{-1}[V].$$

Since $\mathfrak{U} \subseteq \mathfrak{B}$, $f^{-1}[V] \in \tau$ for each $V \in \mathfrak{U}$. So, $\bigcup_{V \in \mathfrak{U}} f^{-1}[V] \in \tau$ since the union of open sets is an open set. Therefore, f is continuous.

(ii) ($\Rightarrow$) Vacuously.

($\Leftarrow$) Let $\mathfrak{B}$ be the base generated by $\mathcal{S}$. Let $B \in \mathfrak{B}$. So, $B = \bigcap_{i \in I} S_i$ where I is finite and $S_i \in \mathcal{S}$ for each $i \in I$. So,

$$f^{-1}[B] = f^{-1}\left[\bigcap_{i \in I} S_i\right] = \bigcap_{i \in I} f^{-1}[S_i]$$

is open since the finite intersection of open sets is open. Therefore by (i), f is continuous.

□

Definition 1.40. Let (X, τ) and (Y, σ) be topological spaces. Then a bijection $f : X \to Y$ is a *homeomorphism* if both f and f^{-1} are continuous.

Note. Homeomorphisms are both open and closed.

Definition 1.41. Let (X, τ) and (Y, σ) be topological spaces. A mapping $f : X \to Y$ is a *topological embedding* if f is a homeomorphism from X onto

$f[X]$.

Lemma 1.42. *Let* (X, τ) *and* (Y, σ) *be topological spaces. If there exists a topological embedding* $f : X \to Y$ *then, up to homeomorphism,* (X, τ) *is a subspace of* (Y, σ).

Definition 1.43. Let (X, τ) be a topological space. A subset D of X is *dense* in X if $D \cap U \neq \varnothing$ for every non-empty $U \in \tau$.

Definition 1.44. A *cover* of a topological space (X, τ) is a collection $\mathfrak{A}$ of subsets of X whose union is all of X. A *subcover* of a cover $\mathfrak{A}$ is a subcollection of $\mathfrak{A}$ that is a cover. An open cover of (X, τ) is a cover consisting of open sets. (X, τ) is *compact* if every open cover of X has a finite subcover. $Y \subseteq X$ is a *compact subset* of X if Y is compact with respect to the subspace topology.

Definition 1.45. A topological space (X, τ) is *compactly generated* if it satisfies the following condition: A subspace A is closed in X if and only if $A \cap K$ is closed in K for all compact subspaces $K \subseteq X$.

Note. Every compact Hausdorff space is normal.

Theorem 1.46. *Every closed subset of a compact space is compact.*

Proof. Let (X, τ) be a compact space and let F be a closed set of X. Let $\mathfrak{U}$ be any open cover of F. Then for each $U \in \mathfrak{U}$ we can find an open set V_U of X such that $V_U \cap F = U$. Now $\{X \setminus F\} \cup \{V_U : U \in \mathfrak{U}\}$ is an open cover of X and therefore has a finite subcover by compactness. This means that the intersections with F of this finite subcover form a finite subcover of F from $\mathfrak{U}$. Therefore, F is compact. □

Theorem 1.47. *The continuous image of a compact space is compact.*

Proof. Suppose that X is compact and that f is a continuous map of X onto Y. If $\mathfrak{U}$ is an open cover of Y, then $\{f^{-1}[U] : U \in \mathfrak{U}\}$ is an open cover of X. So by compactness, a finite subcover exists, say $\{f^{-1}[U_1], \ldots, f^{-1}[U_m]\}$. Now, since f is onto, $\{U_1, \ldots, U_n\}$ is an open cover of Y. Therefore, Y is compact. $\square$

Lemma 1.48 (Alexander's Subbase Lemma). *Let (X, τ) be a topological space and $\mathcal{S}$ be a subbase for τ. Then X is compact if and only if every cover of X by elements from $\mathcal{S}$ has a finite subcover.*

Definition 1.49. A topology τ_1 over a set X is *coarser* than a topology τ_2 over the same set if $\tau_1 \subseteq \tau_2$. Alternatively, we say τ_2 is *finer* than τ_1.

Theorem 1.50. *The set of all topologies on X forms a complete lattice with respect to $\subseteq$.*

Proof. Let $\mathcal{T}$ be the set of all topologies on X partially ordered by inclusion. Since arbitrary meets can be expressed in terms of arbitrary joins, it is sufficient to show the existence of arbitrary meets in $\mathcal{T}$. Furthermore, it is clear that the intersection of any non-empty family of topologies on X is also a topology on X. Let $A \subseteq \mathcal{T}$. If $A = \varnothing$ then $\bigwedge A$ is the discrete topology. If $A \neq \varnothing$ then $\bigwedge A = \bigcap A$. $\square$

Theorem 1.51. *The identity map, $id_X : (X, \tau_1) \to (X, \tau_2)$, is continuous if and only if τ_1 is finer than τ_2. Furthermore, the identity map is open if and only if τ_1 is coarser than τ_2.*

Proof. (i) ($\Rightarrow$) Let τ_1 be finer than τ_2 and let $U \in \tau_2$. Therefore, $U \in \tau_1$ since $\tau_2 \subseteq \tau_1$. Now, ${id_X}^{-1}[U] = U$ so id_X is continuous.

($\Leftarrow$) Suppose that id_X is continuous and let $U \in \tau_2$. So, ${id_X}^{-1}[U] = U \in \tau_1$ since id_X is continuous. Therefore, τ_1 is finer than τ_2.

(ii) ($\Rightarrow$) Let τ_1 be coarser than τ_2 and let $U \in \tau_1$. Therefore, $U \in \tau_2$ since $\tau_1 \subseteq \tau_2$. So, $id_X[U] = U$ so id_X is open.

($\Leftarrow$) Suppose that id_X is open and let $U \in \tau_1$. So, $id_X[U] = U \in \tau_2$ since id_X is continuous. Therefore, τ_1 is coarser than τ_2.

$\square$

Definition 1.52. Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces. The *Tychonoff topology* on $\prod_{i\in I} X_i$, denoted τ_{prod}, is the coarsest topology making each projection map $\pi_j : \prod_{i\in I} X_i \to X_j$ continuous.

Theorem 1.53. *Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces. The Tychonoff topology on $\prod_{i\in I} X_i$ is generated by all sets of the form $\prod_{i\in I} U_i$, where*

(i) for each $i \in I$, $U_i \in \tau_i$ and

(ii) for only finitely many $i \in I$, $U_i \neq X_i$.

Proof. Let $\mathfrak{B}$ be the collection of sets of the form $\prod_{i\in I} U_i$ where for each $i \in I$, $U_i \in \tau_i$ and for only finitely many $i \in I$, $U_i \neq X_i$. It suffices to show that elements of $\mathfrak{B}$ are open with respect to the Tychonoff topology and that $\mathfrak{B}$ is a basis for a topology on $\prod_{i\in I} X_i$.

(i) Since, by assumption, each π_j is continuous, the inverse image of an open set V of X_j must be open with respect to the Tychonoff topology. Now, ${\pi_j}^{-1}[V] = \prod_{i\in I} U_i$ where $U_j = V$ and $U_i = X_i$ for $i \neq j$. So, these sets must be open in the Tychonoff topology. Now, any element of $\mathfrak{B}$ is the finite intersection of sets of this form. So, $\mathfrak{B} \subseteq \tau_{\mathsf{prod}}$.

(ii) We must show $\mathfrak{B}$ is a basis for a topology on $\prod_{i\in I} X_i$. Clearly, $\prod_{i\in I} X_i \in \mathfrak{B}$ and $\varnothing \in \mathfrak{B}$. Now, let $\prod_{i\in I} U_i \in \mathfrak{B}$ and $\prod_{i\in I} V_i \in \mathfrak{B}$ be such that $z \in \prod_{i\in I} U_i \cap \prod_{i\in I} V_i$. Then, $z \in \prod_{i\in I}(U_i \cap V_i) = \prod_{i\in I} U_i \cap \prod_{i\in I} V_i$ and $\prod_{i\in I}(U_i \cap V_i)$ is in $\mathfrak{B}$. Therefore, $\mathfrak{B}$ is a basis for a topology on $\prod_{i\in I} X_i$.

□

Note. Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces. From the previous proof, it is clear that τ_{prod} is the topology on $\prod_{i\in I} X_i$ which has for a subbase the collection

$$\left\{\pi_i{}^{-1}[U_i] : i \in I, U_i \in \tau_i\right\}.$$

Theorem 1.54 (Tychonoff). *Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces such that $\prod_{i\in I} X_i \neq \varnothing$. Then $\left(\prod_{i\in I} X_i, \tau_{\mathsf{prod}}\right)$ is compact if and only if (X_i, τ_i) is compact for each $i \in I$.*

Theorem 1.55. *Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces and and let $\prod_{i\in I} X_i$ be equipped with the Tychonoff topology. Then for every topological space (Y, σ), $f : Y \to \prod_{i\in I} X_i$ is continuous if and only if $\pi_i \circ f$ is continuous for each $i \in I$.*

Proof. ($\Rightarrow$) If f is continuous, then $\pi_i \circ f$ is continuous since π_i is continuous and the composition of two continuous functions is continuous.

($\Leftarrow$) Suppose that each $f_i = \pi_i \circ f$ is continuous for each $i \in I$. The sets of the form $\pi_i{}^{-1}[U_i]$, where $i \in I$ and $U_i \in \tau_i$ form a subbase for τ_{prod}. Also,

$$f^{-1}[\pi_i{}^{-1}[U_i]] = f_i{}^{-1}[U_i]$$

where $f_i^{-1}[U_i] \in \sigma$ for each $i \in I$ since f_i is continuous. Therefore, f is continuous.

□

Definition 1.56. Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces. The *box topology* on $\prod_{i\in I} X_i$, denoted τ_{box}, is generated by taking as a base sets of the form $\prod_{i\in I} U_i$, where $U_i \in \tau_i$ for each $i \in I$.

Note. Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces. It is clear that the box topology over $\prod_{i\in I} X_i$ is finer than the Tychonoff topology over $\prod_{i\in I} X_i$. Furthermore, if I is infinite and each τ_i is non-trivial, then the box topology over $\prod_{i\in I} X_i$ is strictly finer than the Tychonoff topology over $\prod_{i\in I} X_i$.

Definition 1.57. Let (X, τ) and (Y, σ) be topological spaces. Let $\mathsf{C}(X, Y)$ be the set of all the continuous maps from X to Y. The *compact-open topology* on $\mathsf{C}(X, Y)$, denote τ_{co}, is the topology having for a subbase the sets

$$(K, U) = \{f \in \mathsf{C}(X, Y) : f[K] \subseteq U\}$$

where K is compact in X and U is open in Y.

Theorem 1.58. *Let (X, τ) and (Y, σ) be topological spaces where τ is the discrete topology. The compact-open topology on $\mathsf{C}(X, Y)$ is the topology induced by the Tychonoff topology on the set of all maps from X to Y.*

Proof. If τ is the discrete topology then every function from X to Y is continuous, therefore $\mathsf{C}(X, Y)$ is the set of all maps from X to Y. Furthermore, $K \subseteq X$ is compact if and only if K is finite.

$$\begin{aligned}(K,U) &= \{f \in \mathsf{C}(X,Y) : f[K] \in U\}\\ &= \prod_{x\in X} U_x\end{aligned}$$

where

$$U_x = \begin{cases} Y & \text{if } x \notin K \\ U & \text{if } x \in K \end{cases}.$$

Therefore, $\tau_{\mathsf{co}} = \tau_{\mathsf{prod}}$ on the set of all maps from X to Y. □

Theorem 1.59. *Let (X,τ) and (Y,σ) be topological spaces. If (Y,σ) is a T_0-space, a T_1-space, a Hausdorff space, or a regular space then so is $(\mathsf{C}(X,Y),\tau_{\mathsf{co}})$.*

Proof. (i) Suppose that (Y,σ) is a T_0-space. Let $f,g \in \mathsf{C}(X,Y)$ such that $f \neq g$. So, there is an $x \in X$ such that $f(x) \neq g(x)$. Without loss of generality, there exists an open set U of Y such that $f(x) \in U$ and $g(x) \notin U$ since (Y,σ) is a T_0-space. Clearly, $f \in (\{x\},U)$, $g \notin (\{x\},U)$ and $(\{x\},U)$ is an open set of τ_{co}. Therefore, $(\mathsf{C}(X,Y),\tau_{\mathsf{co}})$ is a T_0-space.

(ii) Suppose that (Y,σ) is a T_1-space. Let $f,g \in \mathsf{C}(X,Y)$ such that $f \neq g$. So, there is an $x \in X$ such that $f(x) \neq g(x)$. So, there exists $U,V \in \sigma$ such that $f(x) \in U$, $g(x) \notin U$, $g(x) \in V$ and $f(x) \notin V$. Clearly, $f \in (\{x\},U)$, $g \notin (\{x\},U)$, $g \in (\{x\},V)$, $f \notin (\{x\},V)$, and $(\{x\},U)$ and $(\{x\},V)$ are open sets of τ_{co}. Therefore, $(\mathsf{C}(X,Y),\tau_{\mathsf{co}})$ is a T_1-space.

(iii) Suppose that (Y,σ) is a Hausdorff space. Let $f,g \in \mathsf{C}(X,Y)$ such that $f \neq g$. So, there is an $x \in X$ such that $f(x) \neq g(x)$. So, there exists $U,V \in \sigma$

such that $f(x) \in U$, $g(x) \in V$ and $U \cap V = \varnothing$. Clearly, $f \in (\{x\}, U)$, $g \in (\{x\}, V)$, and $(\{x\}, U)$ and $(\{x\}, V)$ are disjoint open sets of τ_{co}. Therefore, $(\mathsf{C}(X,Y), \tau_{\mathsf{co}})$ is a Hausdorff space.

(iv) Suppose that (Y, σ) is a regular space. Let F be a closed set of $(\mathsf{C}(X,Y), \tau_{\mathsf{co}})$. τ_{co} is the topology having for a subbase sets of the form (K, U) where K is compact in X and U is open in Y. Therefore, without loss of generality, $F = \mathsf{C}(X,Y) \setminus (K, U)$ for some compact set K of X and some open set U of Y. Let $f \notin F$ and $F' = Y \setminus U$. Since $f \notin F$, there exists a $x \in K$ such that $f(x) \in U$. So, $f(x) \notin F'$. Since (Y, σ) is a regular space, there exists $U', V' \in \sigma$ such that $f(x) \in U'$ and $F' \subseteq V'$. So, $f \in (\{x\}, U')$ and $F \subseteq (K, F') \subseteq (K, V')$. Moreover, $(\{x\}, U')$ and (K, V) are open sets in τ_{co}. Therefore, $(\mathsf{C}(X,Y), \tau_{\mathsf{co}})$ is a regular space.

□

1.3 Topological Abelian Groups

In this section, I present some basic concepts and theorems about topological abelian groups. For further reference, please see [Hig74] and [Pon66]. I have also had the benefit of receiving an advance copy of Gàbor Lukàcs' forthcoming book [Luk] about topological abelian groups. I have expanded many of these results to cover topological modules.

Definition 1.60. A *topological abelian group* is a structure $\langle\langle G; +, -, 0\rangle, \tau\rangle$ such that $\langle G; +, -, 0\rangle$ is an abelian group, τ is a topology on G and

(i) $+ : G \times G \to G$ is a continuous map where $G \times G$ is equipped with the product topology;

(ii) $- : G \to G$ is a continuous map.

Let M be a left R-module. The above definition can be expanded to topological modules by adding the additional condition that

(iii) $\mathbf{r} : M \to M$, defined by $m \mapsto r \cdot m$, is a continuous map for each $r \in R$.

Note. When discussing topological modules it is standard to assume that R has a topology. In this case, the definition requires that the scalar multiplication $\cdot : R \times M \to M$ is continuous, in addition to $+ : M \times M \to M$ and $- : M \to M$ being continuous. In contrast, in our case, R will simply be a ring and we will assume that for each $r \in R$, $\mathbf{r} : M \to M$, defined by $m \mapsto r \cdot m$ is continuous as a unary function on M. This is equivalent to considering only rings with the discrete topology.

It is clear that any subgroup of a topological group is a topological group with respect to the subspace topology and that every submodule of a topological module is a topological module with respect to the subspace topology. For the following theorems dealing primarily with topological groups analogous results for topological left R-modules follow immediately unless otherwise noted.

Let (G, τ) be a topological group and let g be a fixed element of G. The constant map $x \mapsto g$ and the identity map are continuous maps from G to G. This implies that $x \mapsto (g, x)$ is a continuous map from G to $G \times G$. Since addition is continuous and the composition of continuous functions is continuous, we get a continuous map

$t_g : x \mapsto g + x$ from G to G, called the *translation* by g. This map has a continuous inverse, namely t_{-g}, so t_g is a homeomorphism.

Notation. If $A, B \subseteq G$ and $g \in G$ where (G, τ) is a topological abelian group, then we let

$$A + g = \{a + g : a \in A\} = g + A;$$

$$A + B = \bigcup_{b \in B} (A + b) = \bigcup_{a \in A} (a + B) = B + A;$$

$$-A = \{-a : a \in A\};$$

$$A - B = A + (-B).$$

If $A \subseteq M$ where (M, τ) is a topological left R-module and $r \in R$, then we let

$$rA = \{ra : a \in A\};$$

$$r^{-1}A = \{x \in M : rx \in A\}.$$

Note. In general, $A - A \neq \{0\}$! Furthermore, if A is an open set then, $r^{-1}A$ is an open set since $r : M \to M$ is continuous. Similar, if A is a closed set then, $r^{-1}A$ is a closed set. On the other hand, if A is open or closed nothing can be determined about rA since $r : M \to M$ need not be an open or closed map.

Proposition 1.61. *Let (G, τ) be a topological abelian group, $A, B \subseteq G$ and $g \in G$.*

(i) G is a homogeneous space, that is, given $a, b \in G$ there is a homeomorphism $G \to G$ sending a to b. Intuitively, this means that G looks the same topologically at every point.

(ii) A is an open set $\Rightarrow A + g$ is an open set.

(iii) A is a closed set $\Rightarrow A+g$ is a closed set.

(iv) A is an open set $\Rightarrow A+B$ is an open set.

(v) A is a closed set and B is finite $\Rightarrow A+B$ is a closed set.

Proof. (i) t_{b-a} is a homeomorphism sending a to b.

(ii) t_g is a homeomorphism and therefore open.

(iii) t_g is a homeomorphism and therefore closed.

(iv) $A+B=\bigcup_{b\in B}(A+b)$ and a union of open sets is an open set.

(v) $A+B$ is the union of a finite number of closed sets and is therefore a closed set.

$\square$

Note. Proposition 1.61(i) implies that the topology on a topological abelian group (G,τ) is determined by the set of open neighbourhoods of 0. That is, $U\in\tau$ if and only if for every g in G there is $W_g\in\tau$, $0\in W_g$ such that $g+W_g\subseteq U$. We let $\mathcal{N}(G)$ denote the collection of open neighbourhoods of 0.

Corollary 1.62. *Let (G_1,τ_1) and (G_2,τ_2) be topological groups and $f:G_1\to G_2$ be a homomorphism. Then*

(i) f is continuous if and only if f is continuous at the identity.

(ii) f is open onto its image if and only if for each $U\in\mathcal{N}(G_1)$, $f[U]=V\cap f[G_1]$ for some $V\in\tau_2$.

Proof. This is a direct consequence of Proposition 1.61(i). $\square$

Corollary 1.63. *Let (G,τ) be a topological abelian group. Let $U \in \mathcal{N}(G)$. The following statements hold:*

(i) If $a \in U$ there exists $W \in \mathcal{N}(G)$ such that $W + a \subseteq U$.

(ii) There exists $W \in \mathcal{N}(G)$ such that $W - W \subseteq U$.

(iii) $cl_G(U) \subseteq U - U$.

Proof. (i) Clearly, $U - a \in \mathcal{N}(G)$ and $(U - a) + a \subseteq U$.

(ii) The map $f : G \times G \to G$ given by $(a,b) \mapsto a - b$ is continuous. Thus, $f^{-1}[U]$ is an open set, contains $(0,0)$, and therefore contains a set of the form $A \times B$ where $A, B \in \mathcal{N}(G)$. Let $W = A \cap B$. So, $W \in \mathcal{N}(G)$. Furthermore, $W \times W \subseteq f^{-1}[U]$, that is, $W - W \subseteq U$.

(iii) Let $a \in cl_G(U)$. Since $U + a$ is an open neighbourhood of a, $(U + a) \cap U \neq \varnothing$. Therefore, there are $b, c \in U$ such that $a + b = c$. Therefore, $a = c - b \in U - U$.

□

Proposition 1.64. *Let (G,τ) be a topological abelian group. The following are equivalent:*

(i) (G,τ) is a Hausdorff space.

(ii) $\{0\}$ is a closed subgroup of G.

(iii) The intersection of all the open neighbourhoods of 0 is $\{0\}$.

Proof. (i) $\Rightarrow$ (ii): Let (G,τ) be a Hausdorff space. So, for all $g \in G$, $\{g\}$ is a closed set. Therefore, in particular, $\{0\}$ is a closed set. Furthermore, $\{0\}$ is a subgroup of G. Therefore, $\{0\}$ is a closed subgroup of G.

(ii) $\Rightarrow$ (iii): Let $g \in G$ be such that $g \neq 0$. Now, $\{g\}$ is a closed set since G is a homogeneous space. Since $\{g\}$ is closed set and $0 \notin \{g\}$, there exists $U \in \mathcal{N}(G)$ such that $U \cap \{g\} = \varnothing$. Hence, g is not in the intersection of all the open neighbourhoods of 0. Since g was arbitrary, the intersection of all the open neighbourhoods of 0 is $\{0\}$.

(iii) $\Rightarrow$ (i): Let $a, b \in G$ such that $a \neq b$. Since $a - b \neq 0$, there exists $U \in \mathcal{N}(G)$ which does not contain $a - b$. Also, there exists $W \in \mathcal{N}(G)$ such that $W - W \subseteq U$. So, $a - b \notin W - W$. Therefore, $(W + (a - b)) \cap W = \varnothing$ and $(W + a) \cap (W + b) = \varnothing$. Since W is open, $W + a$ and $W + b$ are open neighbourhoods of a and b respectively. Therefore, G is Hausdorff. $\square$

Corollary 1.65. *Every topological abelian group that is a T_0-space is a Hausdorff space.*

Proof. Let (G, τ) be a T_0 topological abelian group. Let $a \in G$. Since G is T_0, there is an open set U such that U contains either a or 0 but not both. By homogeneity, without loss of generality, U contains 0 and not a. So, the intersection of all the open neighbourhoods of 0 is $\{0\}$. Therefore by Proposition 1.64, (G, τ) is Hausdorff. $\square$

Theorem 1.66. *Let (G, τ) be a topological group and let H be a subgroup of G. If H is open, then H is closed.*

Proof. Suppose H is an open subgroup of G. So, H is an open neighbourhood of 0. Therefore, by Proposition 1.63(iii), $cl_G(H) \subseteq H - H$. But, H is an open subgroup, so $H - H = H$. Therefore, H is a closed. $\square$

Theorem 1.67. *Let G be an abelian group and $\mathfrak{B}_0$ a collection of subsets of G containing* 0 *such that* *(i)* $-B = B$ *for every* $B \in \mathfrak{B}_0$*;*

(ii) for every $B \in \mathfrak{B}_0$*, there exists* $B' \in \mathfrak{B}_0$ *such that* $B' + B' \subseteq B$*;*

(iii) for every $B \in \mathfrak{B}_0$ *and* $x \in B$*, there exists* $B' \in \mathfrak{B}_0$ *such that* $B' + x \subseteq B$*;*

(iv) for every $B_1, B_2 \in \mathfrak{B}_0$*, there exists* $B_3 \in \mathfrak{B}_0$ *such that* $B_3 \subseteq B_1 \cap B_2$*.*

Then the family

$$\mathfrak{B} = \{B + g : B \in \mathfrak{B}_0, g \in G\}$$

is a base for a topology τ *on* G *such that* (G, τ) *is a topological group. Moreover, if* $\bigcap \mathfrak{B}_0 = \{0\}$ *then the topology generated by* $\mathfrak{B}_0$ *is Hausdorff.*

Proof. Let

$$\mathfrak{B}' = \{B_1 \cap \cdots \cap B_k : B_i \in \mathfrak{B}_0, k \in \mathbb{N}\}.$$

That is, $\mathfrak{B}'$ is the family of all finite intersections of the members of $\mathfrak{B}_0$. Now, set

$$\tau = \left\{U \subseteq G : \text{for all } x \in U \text{ there exists } B' \in \mathfrak{B}' \text{ such that } B' + x \subseteq O\right\}.$$

With some work it can be shown that τ is a topology on G and (G, τ) is a topological group. By (iii), each member of $\mathfrak{B}_0$ belongs to τ. Let $U \in \tau$ and $x \in U$. Since τ is a group topology on G, there exist $B_1, \ldots B_k \in \mathfrak{B}_0$ such that

$$(B_1 \cap \cdots \cap B_k) + x \subseteq \mathfrak{B}_0.$$

By (iv) there is a $B' \in \mathfrak{B}_0$ such that $B' \subseteq B_1 \cap \cdots \cap B_k$, and thus $B' + x \subseteq U$ and $B' + x \in \mathfrak{B}$. Therefore, $\mathfrak{B}$ is a base for τ. $\square$

In order for $\mathfrak{B}_0$ to generate a module topology on a left R-module we require the additional condition that

(v) for every $r \in R$ and $B \in \mathfrak{B}_0$, there is a $B' \in \mathfrak{B}_0$ such that $rB' \subseteq B$.

1.3.1 Filters and Convergence

In this section, we cover two closely related topics, namely filters and convergence. The concept of a filter will be used and expanded upon in Chapter 3.

Definition 1.68. Let X be a set. $\mathcal{F}$ is a *filter* on X if $\mathcal{F}$ is a non-empty collection of subsets of X such that

(i) $\varnothing \notin \mathcal{F}$;

(ii) if $F_1, \ldots, F_n \in \mathcal{F}$ then $F_1 \cap \cdots \cap F_n \in \mathcal{F}$;

(iii) if $F \in \mathcal{F}$ and $F \subseteq F'$ then $F' \in \mathcal{F}$.

Definition 1.69. Let X be a set. $\mathfrak{B}$ is a *filter-base* on X if $\mathfrak{B}$ is a non-empty collection of subsets of X such that

(i) $\varnothing \notin \mathfrak{B}$;

(ii) for every $B_1, B_2 \in \mathfrak{B}$, there exists $B \in \mathfrak{B}$ such that $B \subseteq B_1 \cap B_2$.

Lemma 1.70. *Let X be a set.*

(i) Let (X, τ) be a topological space and $x \in X$. If $\mathcal{F}_x$ is the collection of all neighbourhoods of x then $\mathcal{F}_x$ is a filter on X.

(ii) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on X then $\mathcal{F}_1 \cap \mathcal{F}_2$ is a filter on X.

(ii) If $\mathfrak{B}$ is a filter-base on X then

$$\mathcal{F} = \{A : \text{there exists } B \in \mathfrak{B} \text{ such that } B \subseteq A\}$$

is a filter on X.

Proof. These follow directly from the definitions. □

Definition 1.71. Let (X, τ) be a topological space. Let $x \in X$, $\mathcal{F}$ be a filter on X and $\mathfrak{B}$ be a filter-base on X. Then $\mathcal{F}$ *converges* to x, denoted by $\mathcal{F} \xrightarrow{\tau} x$, if for every open neighbourhood U of x, there is $F \in \mathcal{F}$ such that $F \subseteq U$ (or equivalently, $U \in \mathcal{F}$). Similarly, $\mathfrak{B}$ *converges* to x, denoted by $\mathfrak{B} \xrightarrow{\tau} x$, if for every neighbourhood U of x, there is a $B \in \mathfrak{B}$ such that $B \subseteq U$. If $\mathcal{F}$ ($\mathfrak{B}$) converges to x then x is a *limit* of $\mathcal{F}$ ($\mathfrak{B}$), denoted $\lim \mathcal{F}$ ($\lim \mathfrak{B}$).

Let (X, τ) be a topological space and $x \in X$. Clearly, $\mathcal{F}_x$, the collection of all neighbourhoods of x, converges to x. Also, if $\mathfrak{B}$ is a a filter-base on X and $\mathcal{F}$ is the filter generated by $\mathfrak{B}$, then $\mathfrak{B} \xrightarrow{\tau} x$ if and only if $\mathcal{F} \xrightarrow{\tau} x$.

Proposition 1.72. *Let (X, τ) be a topological space. (X, τ) is Hausdorff if and only if every filter-base on X converges to at most one limit.*

Proof. Suppose that (X, τ) is Hausdorff. Let $\mathfrak{B}$ be a filter-base with more than one limit, that is, assume $\mathfrak{B} \xrightarrow{\tau} x$ and $\mathfrak{B} \xrightarrow{\tau} y$ where $x \neq y$. Since (X, τ) is Hausdorff there exists open neighbourhoods U of x and V of y such that $U \cap V = \varnothing$. However, by convergence there exists $B_1, B_2 \in \mathfrak{B}$ such that $B_1 \subseteq U$ and $B_2 \subseteq V$. Since $\mathfrak{B}$ is a filter base, $B_1 \cap B_2$ contains an element of $\mathfrak{B}$. This is a contradiction since $B_1 \cap B_2 \subseteq U \cap V = \varnothing$, so $\mathfrak{B}$ has at most one limit. Suppose that (X, τ) is not

Hausdorff. Then there exists $x \neq y$ such that for every open neighbourhood U of x and V of y, $U \cap V \neq \varnothing$. Let $\mathfrak{B}$ be the set of open neighbourhoods of x and y. Clearly, $\mathfrak{B}$ is a filter-base on X that converges to x and to y. $\square$

Proposition 1.73. *Let (X, τ) be a topological space and let $Y \subseteq X$. The following are equivalent:*

(i) There exists a filter-base $\mathfrak{B}$ whose elements are all contained in Y such that $\mathfrak{B} \xrightarrow{\tau} x$.

(ii) There exists a filter $\mathcal{F}$ such that $Y \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\tau} x$.

(iii) $x \in cl_X(Y)$.

Proof.

(i)⇒(ii) Suppose that $\mathfrak{B}$ is a filter-base whose elements are all contained in Y such that $\mathfrak{B} \xrightarrow{\tau} x$. Then the filter associated with $\mathfrak{B}$ satisfies the properties of (ii).

(ii)⇒(iii) Suppose that $\mathcal{F}$ is a filter such that $Y \in \mathcal{F}$ and $\mathcal{F} \xrightarrow{\tau} x$. There exists some $U \in \tau$ such that $cl_X(Y) = X \setminus U$. Suppose that $x \notin cl_X(Y)$, so $x \in U$. Therefore, $U \in \mathcal{F}$. However, $cl_X(Y) \in \mathcal{F}$ since $Y \subseteq cl_X(Y)$ and $Y \in \mathcal{F}$. Therefore, $\varnothing = U \cap cl_X(Y) \in \mathcal{F}$. This is a contradiction, therefore, $x \in cl_X(Y)$.

(iii)⇒(i) Suppose $x \in cl_X(Y)$. Define $\mathfrak{B} = \{U \cap Y : U \text{ is a neighbourhood of } x\}$. It is easy to confirm that $\mathfrak{B}$ is a filter-base with the desired properties.

$\square$

Corollary 1.74. *Let (X, τ) be a topological space and $A \subseteq X$. Then A is closed if and only if A contains all limits of all convergent filters $\mathcal{F}$ with $A \in \mathcal{F}$.*

Proof. This follows immediately from Proposition 1.73. $\square$

Proposition 1.75. *Let (M, τ) be a topological left R-module.*

(i) If $\mathcal{F}$ is a filter on M than so are $-\mathcal{F}$ and $r\mathcal{F}$.

(ii) If $\mathcal{F} \xrightarrow{\tau} x$, then $-\mathcal{F} \xrightarrow{\tau} -x$ and $r\mathcal{F} \xrightarrow{\tau} rx$.

Proof. (i) follows directly from the definition. (ii) Let $\mathcal{F} \xrightarrow{\tau} x$. Let U be an open neighbourhood of $-x$ and let V be an open neighbourhood of rx. So, $-U$ and $r^{-1}V$ are open neighbourhoods of x. Since $\mathcal{F} \xrightarrow{\tau} x$, there are $F_1, F_2 \in \mathcal{F}$ such that $F_1 \subseteq -U$ and $F_2 \subseteq r^{-1}V$. Therefore, $-F_1 \subseteq U$, $rF_2 \subseteq V$. Furthermore, $-F_1 \in -\mathcal{F}$, $rF_2 \in r\mathcal{F}$. So, $-\mathcal{F} \xrightarrow{\tau} -x$ and $r\mathcal{F} \xrightarrow{\tau} rx$. $\square$

1.3.2 Cauchy Filters and Convergence

In this section, we cover two closely related topics, namely Cauchy filters and completion. The concept of the completion of a topological module will be used in Chapter 4.

Definition 1.76. Let (M, τ) be a topological left R-module and τ be Hausdorff. A *Cauchy filter* on M is a filter $\mathcal{F}$ such that for every $U \in \mathcal{N}(M)$, there is $F \in \mathcal{F}$ satisfying $F - F \subseteq U$. A Hausdorff R-module (M, τ) is *complete* if every Cauchy filter on M converges.

Lemma 1.77. *Let (M, τ) be a topological left R-module and $r \in R$.*

(i) Every convergent filter is a Cauchy filter.

(ii) If $\mathcal{F}$ is a Cauchy filter on M, then so is $-\mathcal{F}$.

(iii) If $\mathcal{F}$ is a Cauchy filter on M and A is a submodule of M, then the restriction $\mathcal{F}_A = \{F \subseteq A : F \in \mathcal{F}\}$ is a Cauchy filter on A in the subspace topology.

Proof. (i) Suppose that $\mathcal{F} \xrightarrow{\tau} x$ where $x \in G$ and let $U \in \mathcal{N}(G)$. By Corollary 1.63, there exists $W \in \mathcal{N}(G)$ such that $W - W \subseteq U$. Furthermore, $W + x$ is open and $x \in W + x$, so there exists $F \in \mathcal{F}$ such that $F \subseteq W + x$

$$F - F \subseteq (W + x) - (W + x) = W - W \subseteq U.$$

So, $\mathcal{F}$ is a Cauchy filter. (ii) and (iii) follow directly from the definitions. $\square$

Theorem 1.78. *Let (N, τ) be a complete module. A submodule M of N is complete in the subspace topology if and only if M is closed in N.*

Proof. Suppose that M is complete in the subspace topology, and let $x \in cl_N(M)$. By Proposition 1.73, there is a filter $\mathcal{F}$ on N such that $\mathcal{F} \xrightarrow{\tau} x$ and $M \in \mathcal{F}$. So, by Lemma 1.77, $\mathcal{F}$ is Cauchy on N and the restriction $\mathcal{F}_M = \{F \subseteq M : F \in \mathcal{F}\}$ a Cauchy filter on M in the subspace topology. Since M is complete, there is $y \in M$ such that $\mathcal{F}_M \xrightarrow{\tau} y$. Hence, $\mathcal{F} \xrightarrow{\tau} y$. So, by Proposition 1.72, $x = y$. Therefore, $x \in M$. Hence, M is closed.

Suppose that M is closed in N and let $\mathcal{F}$ be a Cauchy filter on M. Let $\mathcal{F}'$ be the filter on G generated by $\mathcal{F}$. If $U \in \mathcal{N}(N)$, then $U \cap M \in \mathcal{N}(M)$ and so there is an $F \in \mathcal{F}$ such that $F - F \subseteq U \cap M$. In particular, $F \in \mathcal{F}'$ and $F - F \subseteq U$. So, $\mathcal{F}'$ is a Cauchy filter on N. Since N is complete, there is $x \in N$ such that $\mathcal{F}' \xrightarrow{\tau} x$. So by Proposition 1.73, $x \in M$ since M is closed and $\mathcal{F} \xrightarrow{\tau\restriction M} x$. So, M is complete. $\square$

For instance, $(\mathbb{R}, Euc)$ is complete, but $(\mathbb{Q}, Euc)$ is not complete.

Lemma 1.79. *Let (M_1, τ_1) and (M_2, τ_2) be topological modules. Let $f : M_1 \to M_2$ be a continuous homomorphism. For every Cauchy filter $\mathcal{F}$ in M_1, $f[\mathcal{F}]$ is a Cauchy filter in M_2.*

Proof. Let $U \in \mathcal{N}(M_2)$. Since f is continuous $f^{-1}[U] \in \mathcal{N}(M_1)$. There is an $F \in \mathcal{F}$ such that $F - F \subseteq f^{-1}[U]$. Therefore, $f[F] - f[F] \subseteq U$. □

Corollary 1.80. *Let (M, τ) be a topological left R-module and $r \in R$. If $\mathcal{F}$ is a Cauchy filter on M, then so is $r\mathcal{F}$.*

Proof. Follows directly from Lemma 1.79 since $r : M \to M$ is continuous. □

Theorem 1.81. *Let (M, τ) be a topological module. Let D be a dense submodule of M, and let L be a complete module. Every continuous homomorphism $f : D \to L$ extends uniquely to a continuous homomorphism $\widetilde{f} : M \to L$. That is, the following diagram can be completed commutatively:*

$$\begin{tikzcd} D \arrow[r, hook] \arrow[d, "f"'] & M \arrow[dl, dashed, "\exists! \widetilde{f}"] \\ L & \end{tikzcd}$$

Proof. I will only provide the construction of $\widetilde{f}$. It is then straightforward to show that$\widetilde{f}$ is well-defined, a continuous homomorphism and unique. Let $m \in M$. Since D is dense in M, there is a filter $\mathcal{F}$ on M such that $\mathcal{F} \xrightarrow{\tau} m$ and $D \in \mathcal{F}$ by Proposition 1.73. Since $\mathcal{F}$ is a Cauchy filter in M so is its restriction

$$\mathcal{F}_D = \{F \subseteq D : F \in \mathcal{F}\}.$$

By Lemma 1.79, $f[\mathcal{F}_D]$ is a Cauchy filter on L. Since L is complete, $\lim f[\mathcal{F}_D]$ exists.

So, let

$$\widetilde{f}(g) = \lim f[\mathcal{F}_D].$$

$\square$

Definition 1.82. A *completion* of a topological module (M, τ) is a complete topological module $(\widetilde{M}, \widetilde{\tau})$ that contains M as a dense submodule.

For instance, $(\mathbb{R}, Euc)$ is the completion of $(\mathbb{Q}, Euc)$.

Theorem 1.83 (Raĭkov-Completion Theorem. See [Luk]). *Every topological module admits a completion, which is unique up to topological isomorphism.*

1.4 Category Theory Preliminaries

In this section, I present some basic definitions and theorems from category theory that will be used in later sections. For further reference, please consult [ML98] or [HS73].

Definition 1.84. A *category* $\mathcal{C}$ consists of:

(i) a class $\mathsf{ob}(\mathcal{C})$ of *objects*;

(ii) for each pair of objects X and Y, a set of *morphisms* from X to Y, denoted $\mathsf{Hom}\,(X, Y)$;

(iii) for any three objects X, Y and Z a binary operation,

$$\circ : \mathsf{Hom}\,(Y, Z) \times \mathsf{Hom}\,(X, Y) \to \mathsf{Hom}\,(X, Z)$$

such that if $f \in \mathsf{Hom}\,(W,X)$, $g \in \mathsf{Hom}\,(X,Y)$ and $h \in \mathsf{Hom}\,(Y,Z)$ then

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

(iv) for every object X, a morphism $1_X \in \mathsf{Hom}\,(X,X)$, called the *identity morphism* for X, such that for every morphism $f \in \mathsf{Hom}\,(X,Y)$ and every morphism $g \in \mathsf{Hom}\,(Z,X)$ we have

$$1_X \circ g = g$$

and

$$f \circ 1_X = f.$$

The categories we will be most interested in are:

(i) The *category of topological spaces*, denoted **Top**, whose class of objects is the class of all topological spaces and for any two topological spaces (X,τ) and (Y,τ), $\mathsf{Hom}\,((X,\tau),(Y,\tau))$ is the set of all continuous functions from (X,τ) to (Y,τ).

(ii) The *category of Hausdorff spaces*, denoted $\mathbf{Top_2}$, whose class of objects is the class of all Hausdorff topological spaces and for any two topological spaces (X,τ) and (Y,τ), $\mathsf{Hom}\,((X,\tau),(Y,\tau))$ is the set of all continuous functions from (X,τ) to (Y,τ).

(iii) The *category of abelian groups*, denoted **Ab**, whose class of objects is the class of all abelian groups and for any two groups G and H, $\mathsf{Hom}\,(G,H)$ is the set of all homomorphisms from G to H.

(iv) The *category of topological abelian groups*, denoted **TopAb**, whose class of objects is the class of all topological abelian groups and for any two topological abelian groups (G, τ) and (H, τ), $\mathsf{Hom}((G, \tau), (H, \tau))$ is the set of all continuous homomorphisms from (G, τ) to (H, τ).

(v) The *category of left R-modules*, denoted **RMod**, whose class of objects is the class of all left R-modules and for any two modules M and N, $\mathsf{Hom}(M, N)$ is the set of all module homomorphisms from M to N.

(vi) The category of all topological left R-modules where R is a ring with the discrete topology, denoted **TopRMod**, whose class of objects is the class of all topological left R-modules and for any two topological left R-modules (M, τ) and (N, σ), $\mathsf{Hom}((M, \tau), (N, \tau))$ is the set of all continuous module homomorphisms from (M, τ) to (N, σ).

Definition 1.85. Let $\mathcal{C}$ be a category. A morphism $f \in \mathsf{Hom}(X, Y)$ is an *epimorphism* if for all morphisms $h, k \in \mathsf{Hom}(Y, Z)$ such that

$$h \circ f = k \circ f$$

we have $h = k$.

For **Ab**, **Top** and **RMod** the epimorphisms are precisely the morphisms which are surjective on the underlying sets. However, in $\mathbf{Top}_2$ the epimorphisms are precisely the continuous functions with dense images.

Definition 1.86. Let $\mathcal{C}$ be a category. A morphism $f \in \mathsf{Hom}(X, Y)$ is a *monomorphism* if for all morphisms $h, k \in \mathsf{Hom}(Z, X)$ such that

$$f \circ h = f \circ k$$

we have $h = k$.

For **Top**, **Top**$_2$, **Ab**, **TopAb**, **RMod** and **TopRMod** the monomorphisms are precisely the morphisms which are injective on the underlying sets.

Definition 1.87. Let $\mathcal{C}$ be a category. A morphism $f \in \mathsf{Hom}(X, Y)$ is an *isomorphism* if there exist $g, h \in \mathsf{Hom}(Y, X)$ such that

$$f \circ g = 1_Y$$

and

$$h \circ f = 1_X.$$

In any category, every identity is an isomorphism. A morphism in **Ab** is an isomorphism if and only if it a group isomorphism. A morphism in **Top** is an isomorphism if and only if it is a homeomorphism. It is important to note that every morphism that is an isomorphism is a monomorphism and an epimorphism, but not every morphism that is both a monomorphsim and epimorphism is an isomorphism. For instance, consider **Top**$_2$. The inclusion map $\mathbb{Q}$ to $\mathbb{R}$ is a non-surjective epimorphism.

Definition 1.88. Let $\mathcal{C}$ be a category and let $\{X_i : i \in I\}$ be an indexed family of objects in $\mathcal{C}$. Then a *$\mathcal{C}$-product* of the set $\{X_i\}_{i \in I}$ is an object X together with a collection of morphisms $\pi_i \in \mathsf{Hom}\,(X, X_i)$ such that for any object Y and any collection of morphisms $f_i \in \mathsf{Hom}\,(Y, X_i)$, there exists a unique morphism $f \in \mathsf{Hom}\,(Y, X)$ such that $f_i = \pi_i \circ f$. That is, for each $i \in I$ the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_i"'] & X \arrow[d, "\pi_i"] \\ & X_i \end{tikzcd}$$

Example 1.89. If $\{(X_i, \tau_i)\}_{i\in I}$ is a family of topological spaces then $\left(\prod_{i\in I} X_i, \tau_{\mathsf{prod}}\right)$ together with the set of morphisms $\{\pi_i\}_{i\in I}$ is a **Top**-product.

Definition 1.90. Let $\mathcal{C}$ be a category and let $\{X_i : i \in I\}$ be an indexed family of objects in $\mathcal{C}$. Then a *$\mathcal{C}$-coproduct* of the set $\{X_i\}_{i\in I}$ is an object X together with a collection of morphisms $\varepsilon_i \in \mathsf{Hom}\,(X_i, X)$ such that for any object Y and any collection of morphisms $f_i \in \mathsf{Hom}\,(X_i, Y)$, there exists a unique morphism $f \in \mathsf{Hom}\,(X, Y)$ such that $f_i = f \circ \varepsilon_i$. That is, for each $i \in I$ the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, dashed, "\exists! f"] & Y \\ X_i \arrow[u, "\varepsilon_i"] \arrow[ur, "f_i"'] & \end{tikzcd}$$

Example 1.91. If $\{(M_i, \tau_i)\}_{i\in I}$ is a family of left R-modules then $\bigoplus_{i\in I} M_i$ is a coproduct in the category of left R-modules. See Definition 1.97 and the theorems that follow.

Since both the definition of product and the definition of coproduct assert the existence of a unique morphism it is trivial to show that the product and the co-product of a family, when they exist, are unique up to isomorphism.

Definition 1.92. Let $\mathcal{C}$ be a category. An object I is *$\mathcal{C}$-injective* provided that for each monomorphism $f \in \mathsf{Hom}(Y, X)$ and each morphism $g \in \mathsf{Hom}(Y, I)$, there exists a morphism $h \in \mathsf{Hom}(X, I)$ such that $g = h \circ f$. That is, the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[dl, dashed, "\exists h"] \\ I & \end{tikzcd}$$

Example 1.93. Any vector space is injective in the category of vector spaces over a fixed field. An abelian group is injective in the category of abelian groups if and only if it is divisible.

Proposition 1.94. *A topological space is categorically* **Top**-*injective if and only if it is indiscrete.*

Proof. $\Leftarrow$ Suppose that (I, τ) is a topological space where I is non-empty and τ is the indiscrete topology. Let f be a continuous one-to-one function from (Y, τ_1) to (X, τ_2). Let g be a continuous map from (Y, τ_1) to (I, τ). Let $i \in I$. Let

$$h(x) = \begin{cases} g \circ f^{-1}(x) & \text{if } x \in f[Y] \\ i & \text{if } x \notin f[Y] \end{cases}.$$

Now, h is a continuous map from (X, τ_2) to (I, τ) since τ is the indiscrete topology on I and every mapping into the indiscrete topology is continuous.

Furthermore, $g = h \circ f$. Therefore, (I, τ) is categorically **Top**-injective.

$\Rightarrow$ Now, suppose that (I, τ) is injective. The identity map from (I, τ_1) to (I, τ_2) is continuous and one-to-one if τ_1 is the discrete topology and τ_2 is the indiscrete topology. Now, the identity map from (I, τ_1) to (I, τ) is continuous, since every mapping from the discrete topology is continuous. Therefore, the identity map from (I, τ_2) to (I, τ) is continuous. Hence, τ must be the indiscrete topology.

□

1.4.1 Category of Left R-Modules

In this section, we will assume that R is a fixed ring with unity.

Notation. If M and N are left R-modules then $\mathsf{Hom}\,(M, N)$ is the set of R-module homomorphisms from M to N. We will assume that $\{M_i\}_{i \in I}$ is a family of left R-modules and that H is an arbitrary left R-module.

Proposition 1.95. *Let M and N be left R-modules.*

(i) Let $f, g \in \mathsf{Hom}_R\,(M, N)$. Define $f + g$ as follows:

$$(f + g)(a) = f(a) + g(a)$$

for all $a \in M$. Then $\mathsf{Hom}_R\,(M, N)$ forms an abelian group.

(ii) In addition, suppose R is commutative. Let $f, g \in \mathsf{Hom}_R\,(M, N)$. Define $f+g$ as in (i) and for each $r \in R$ define rf

$$rf(a) = f(ra)$$

for all $a \in M$. Then $\mathsf{Hom}_R(M, N)$ forms a left R-module.

In general, if R is not commutative then $\mathsf{Hom}_R(M, N)$ will not be a left R-module.

Notation. If $\alpha : M_2 \to M_1$ is a group homomorphism then it is possible to assign to a homomorphism $\phi : M_1 \to N$, the homomorphism $\phi \circ \alpha : M_2 \to N$ creating $\alpha^* = \mathsf{Hom}_R(\alpha, N) : \mathsf{Hom}_R(M_1, N) \to \mathsf{Hom}_R(M_2, N)$ a group homomorphism. Similarly, if $\beta : N_1 \to N_2$ is a group homomorphism then it is possible to assign to a homomorphism $\phi : M \to N_1$, the homomorphism $\beta \circ \phi : M \to N_2$ creating $\beta^* = \mathsf{Hom}_R(M, \beta) : \mathsf{Hom}_R(M, N_1) \to \mathsf{Hom}_R(M, N_2)$ a group homomorphism.

Now, let $\mathsf{Hom}_R(M, -)$ be the category where objects are groups of the form $\mathsf{Hom}_R(M, A)$ where A is any left R-module and the morphisms are group homomorphisms of the form $\mathsf{Hom}_R(M, \beta) : \mathsf{Hom}_R(M, A) \to \mathsf{Hom}_R(M, B)$ where $\beta : A \to B$ is a group homomorphism. Let $\mathsf{Hom}_R(-, N)$ be the category where objects are groups of the form $\mathsf{Hom}_R(A, N)$ where A is any left R-module and the morphisms are group homomorphisms of the form $\mathsf{Hom}(\alpha, N) : \mathsf{Hom}_R(A, N) \to \mathsf{Hom}_R(B, N)$ where $\alpha : B \to A$ is a group homomorphism.

Proposition 1.96. *Let S and T be rings, M be a R-S bimodule and N be a R-T bimodule. Let $f \in \mathsf{Hom}_R(M, N)$. For each $s \in S$ and $t \in T$ define fs and tf as follows:*

$$fs(a) = f(as)$$

and

$$tf(a) = (f(a))t$$

for all $a \in M$. Then $\mathsf{Hom}_R(M, N)$ forms a T-S bimodule.

Definition 1.97. Let $\{M_i\}_{i\in I}$ be a family of left R-modules. The *direct sum* of $\{M_i\}_{i\in I}$, denoted $\underline{M} = \bigoplus_{i\in I} M_i$, is the set of all sequences $(a_i)_{i\in I}$ where $a_i \in M_i$ for all $i \in I$ and $a_i = 0$ for all but finitely many $i \in I$. The *direct product* of $\{M_i\}_{i\in I}$, denoted $\overline{M} = \prod_{i\in I} M_i$, is the set of all sequences $(a_i)_{i\in I}$ where $a_i \in M_i$ for all $i \in I$. Now, let $a_i, b_i \in M_i$ for each $i \in I$ and $r \in R$ then we define $(a_i)_{i\in I} + (b_i)_{i\in I}$ and $r((a_i)_{i\in I})$ as follows:

$$(a_i)_{i\in I} + (b_i)_{i\in I} = (a_i + b_i)_{i\in I}$$

and

$$r((a_i)_{i\in I}) = (r(a_i))_{i\in I}.$$

Clearly, $\underline{M}$ and $\overline{M}$ are both left R-modules. Furthermore, $\underline{M}$ is a submodule of $\overline{M}$. Finally, we let $\pi_i : \overline{M} \to M_i$ and $\varepsilon_i : M_i \to \underline{M}$ be the canonical projection and injection, respectively. That is, for $(a_j)_{j\in I} \in \overline{M}$ and $b \in M_i$, $\pi_i((a_j)_{j\in I}) = a_i$ and $\varepsilon_i(b) = (c_j)_{j\in J}$ where $c_j = 0$ for $j \neq i$ and $c_i = b$.

So, $\pi_i \circ \varepsilon_i = id_{M_i}$ and $\pi_i \circ \varepsilon_k = 0$ for $i \neq k$. Also, for $a \in \underline{M}$, the support of a, $\mathsf{supp}\,(a) = \{k \in I : \pi_k(a) \neq 0\}$, is finite, so $\sum_{i\in I} \varepsilon_i \circ \pi_i(a)$ is well defined. Thus we have a morphism $\sum_{i\in I} \varepsilon_i \circ \pi_i : \underline{M} \to \underline{M}$ where, clearly, $\sum_{i\in I} \varepsilon_i \circ \pi_i = id_{\underline{M}}$. Clearly, $\prod_{i\in I} \pi_i = (\pi_i)_{i\in I} = id_{\overline{M}}$.

Proposition 1.98. *For every family of module homomorphisms $f_i \in \mathsf{Hom}_R\,(M_i, H)$ there is a unique $f \in \mathsf{Hom}_R\,(\underline{M}, H)$ such that $f_i = f \circ \varepsilon_i$ for each $i \in I$. Hence, $(\underline{M}, \{\varepsilon_i\}_{i\in I})$ is the* **RMod**-*coproduct.*

Proof. Let $a \in \underline{M}$. Define f by

$$f(a) = \sum_{i \in I} f_i \circ \pi_i(a).$$

Now, f is well-defined since $\mathsf{supp}\,(a)$ is finite for each $a \in \underline{M}$. It can be easily verified that f is the unique homomorphism such that $f_i = f \circ \varepsilon_i$ for all $i \in I$. $\square$

Proposition 1.99. *For every family of module homomorphisms $f_i \in \mathsf{Hom}_R\,(H, M_i)$ there is a unique $f \in \mathsf{Hom}_R\,(H, \overline{M})$ such that $f_i = \pi_i \circ f$ for each $i \in I$. Hence, $(\overline{M}, \{\pi_i\}_{i \in I})$ is the* **RMod***-product.*

Proof. Let $a \in H$. Define f by

$$f(a) = (f_i(a))_{i \in I}.$$

It can be easily verified that f is the unique homomorphism such that $f_i = \pi_i \circ f$ for all $i \in I$. $\square$

Proposition 1.100. $\alpha : \mathsf{Hom}_R\left(H, \prod_{i \in I} M_i\right) \to \prod_{i \in I} \mathsf{Hom}_R\,(H, M_i)$ *defined by*

$$\alpha(f) = (\pi_i \circ f)_{i \in I}$$

is a group isomorphism.

Proof. $\prod_{i \in I} \mathsf{Hom}_R\,(H, M_i)$ is the product of $\{\mathsf{Hom}_R\,(H, M_i)\}_{i \in I}$ in $\mathsf{Hom}_R\,(H, -)$ with projection maps

$$p_i : \prod_{i \in I} \mathsf{Hom}_R\,(H, M_i) \to \mathsf{Hom}_R\,(H, M_i)\,.$$

Clearly,

$$\mathsf{Hom}_R(H, \pi_i) = p_i \circ \alpha.$$

Since $\prod_{i \in I} \mathsf{Hom}_R(H, M_i)$ is the product, α is a homomorphism.

Since $\overline{M}$ is the product of $\{M_i\}_{i \in I}$ there exists a unique morphism g such that $f_i = \pi_i \circ g$, so α is a bijection. $\square$

Proposition 1.101. *$\beta : \mathsf{Hom}_R(\underline{M}, H) \to \prod_{i \in I} \mathsf{Hom}_R(M_i, H)$ defined by*

$$\beta(f) = (f \circ \varepsilon_i)_{i \in I}$$

is a group isomorphism.

Proof. $\prod_{i \in I} \mathsf{Hom}_R(M_i, H)$ is the product of $\{\mathsf{Hom}_R(M_i, H)\}_{i \in I}$ in $\mathsf{Hom}_R(-, H)$ with projection maps

$$p_i : \prod_{i \in I} \mathsf{Hom}_R(M_i, H) \to \mathsf{Hom}_R(M_i, H).$$

Clearly,

$$\mathsf{Hom}_R(\varepsilon_i, H) = p_i \circ \beta.$$

Since $\prod_{i \in I} \mathsf{Hom}_R(M_i, H)$ is the product, β is a homomorphism.

Since $\underline{M}$ is the coproduct of $\{M_i\}_{i \in I}$ there exists a unique morphism g such that $f_i = g \circ \varepsilon_i$, so β is a bijection. $\square$

Proposition 1.102. *Define a map $\gamma : \bigoplus_{i \in I} \mathsf{Hom}_R(M_i, H) \to \mathsf{Hom}_R\left(\overline{M}, H\right)$ by*

$$\gamma((f_i)_{i \in I}) = \sum_{i \in I} f_i \circ \pi_i.$$

Then γ is a monomorphism and its image consists of all module homomorphisms from $\overline{M}$ to H that vanish on all but a finite number of the submodules M_i.

Proof. $\bigoplus_{i\in I} \mathsf{Hom}_R(M_i, H)$ is the coproduct of $\{\mathsf{Hom}_R(M_i, H)\}_{i\in I}$ in $\mathsf{Hom}_R(-, H)$ with injection maps

$$e_i : \mathsf{Hom}_R(M_i, H) \to \bigoplus_{i\in I} \mathsf{Hom}_R(M_i, H).$$

Clearly,

$$\mathsf{Hom}_R(\pi_i, H) = \gamma \circ e_i.$$

Since $\bigoplus_{i\in I} \mathsf{Hom}_R(M_i, H)$ is the coproduct, γ is a homomorphism.

Let $(f_i)_{i\in I} \in \bigoplus_{i\in I} \mathsf{Hom}_R(M_i, H)$ and $(g_i)_{i\in I} \in \bigoplus_{i\in I} \mathsf{Hom}_R(M_i, H)$. Since $\underline{M}$ is the coproduct, the families $f_i : M_i \to H$ and $g_i : M_i \to H$ of R-module homomorphisms gives rise to unique maps $f : \underline{M} \to H$ and $g : \underline{M} \to H$ such that $f_i = f \circ \varepsilon_i$ and $g_i = g \circ \varepsilon_i$. So,

$$\begin{aligned}
&\gamma((f_i)_{i\in I}) = \gamma((g_i)_{i\in I}) \\
\Rightarrow\ & \sum_{i\in I} f_i \circ \pi_i = \sum_{i\in I} g_i \circ \pi_i \\
\Rightarrow\ & \sum_{i\in I} f \circ \varepsilon_i \circ \pi_i = \sum_{i\in I} g \circ \varepsilon_i \circ \pi_i \\
\Rightarrow\ & f \circ \sum_{i\in I} \varepsilon_i \circ \pi_i = g \circ \sum_{i\in I} \varepsilon_i \circ \pi_i \\
\Rightarrow\ & f \circ id_{\underline{M}} = g \circ id_{\underline{M}} \\
\Rightarrow\ & f = g.
\end{aligned}$$

So, γ is a monomorphism.

Now, suppose that $f \in \mathsf{Hom}_R\left(\overline{M}, H\right)$ vanishes on all but a finite number of the submodules M_i. Consider, $f_i = f \circ \varepsilon_i$. So, $f_i \in \mathsf{Hom}_R\left(M_i, H\right)$ and $f_i = 0$ for all but a finite number of the submodules M_i. Therefore, $(f_i)_{i \in I} \in \bigoplus_{i \in I} \mathsf{Hom}_R\left(M_i, H\right)$. Since $\gamma((f_i)_{i\in I}) = f$, f is in the image of γ. Conversely, suppose that $f \in \mathsf{Hom}_R\left(\overline{M}, H\right)$ is in the image of γ. So, for some $(g_i)_{i\in I} \in \bigoplus_{i\in I} \mathsf{Hom}_R\left(M_i, H\right)$,

$$f = \gamma((g_i)_{i\in I}) = \sum_{i\in I} g_i \circ \pi_i.$$

But $g_i = 0$ for all but finitely many $i \in I$. Therefore, f must vanish on all but a finite number of the submodules M_i. Hence, γ has the desired properties. □

Analogous results to Theorems 1.100 - 1.102 for the category of topological modules will be discussed in Chapter 3.

1.5 Pure-injective Modules

In this section, I will introduce some basic concepts of pure-injective modules from a model theoretic perspective. For further reference, [Pre88] provides an in depth study of pure-injective modules from a model theoretic perspective and [Wis91] provides an in depth study of pure-injective modules from an algebraic perspective. Many of the proofs in this section can be found in [Pre88] and have not been duplicated here. In Chapter 4, I investigate whether or not techniques analogous to the ones used by Prest in [Pre88] can be extended to topological modules.

Notation. In this section we will assume that R is an arbitrary fixed ring with multiplicative identity 1 and it will be assumed that all modules are unital left R-modules. Recall that $\mathcal{L}_R$ is the language of unital left R-modules. Suppose that M

is a unital left R-module, $a \in M$ and $r \in R$. For convenience, we will often write $\mathbf{r}(a)$ as $\mathbf{r}a$. This convention allows us to write equations using matrix notation. Recall that if M and N are left R-modules and $\phi \in \mathsf{Hom}_R(M, N)$, then the *kernel* of ϕ is

$$ker(\phi) = \{a \in M : \phi(a) = 0\}.$$

Definition 1.103. Let A, B and C be left R-modules. A sequence of homomorphisms

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$

is *exact* at B if $im(\phi) = ker(\psi)$. A sequence of homomorphisms

$$0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$$

is *short-exact* if it is exact at A, B, and C.

Proposition 1.104. *Let A, B and C be left R-modules.*

(i) $0 \to A \xrightarrow{\phi} B$ is exact at A if and only if ϕ is injective.

(ii) $A \xrightarrow{\psi} B \to 0$ is exact at B if and only if ψ is surjective.

Proof. (i) Let ϕ be injective. $ker(\phi) = \{0\}$ since $\phi(0) = 0$ and if $\phi(a) = 0$ then $a = 0$ since ϕ is injective. Therefore, $0 \to A \xrightarrow{\phi} B$ is exact at A.

(ii) Let ψ be surjective. $im(\psi) = B$ since ψ is surjective. Therefore, $A \xrightarrow{\psi} B \to 0$ is exact at B.

□

Definition 1.105. Let A, B and C be left R-modules. A short-exact sequence of homomorphisms

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

splits if $im(f)$ is a direct summand of B.

Theorem/Definition 1.106. *A left R-module M is injective if it satisfies one of the following equivalent conditions:*

(i) If M is a submodule of N then there exists a submodule A of N such that $M \oplus A \equiv N$.

(ii) If A is a submodule of B and $f : A \to M$ is a homomorphism then there exists a homomorphism $g : B \to M$ such that $\forall x \in A,\ g(x) = f(x)$.

(iii) If $f : A \to B$ is a monomorphism and $g : A \to M$ is an arbitrary homomorphism then there exists a homomorphism $h : B \to M$ such that $h \circ f = g$. That is, the following diagram can be completed commutatively:

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "\exists h"] \\ M & \end{tikzcd}$$

(iv) Every exact sequence $0 \to M \to A \to B \to 0$ splits.

Clearly, a left R-module is categorically **RMod**-injective if and only if it is an injective R-module.

Definition 1.107. In any language $\mathcal{L}$, a *positive primitive* formula, or simply a pp-formula, is an existential quantification of a conjunction of atomic formulas. In particular, for modules, an $\mathcal{L}_R$-formula ϕ is positive primitive, if it is equivalent to a formula of the following form:

$$(\exists w_1)\dots(\exists w_l)\bigwedge_{i=1}^{m}\left(\sum_{j=1}^{n}\mathbf{r_{ij}}(v_j)+\sum_{k=1}^{l}\mathbf{s_{ik}}(w_k)=0\right)$$

where $r_{ij}, s_{ik} \in R$ for $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq l$.

Note. It is convenient to express the above as a matrix:

$$(\exists w_1)\dots(\exists w_l)\begin{pmatrix} \mathbf{r_{11}} & \cdots & \mathbf{r_{1n}} & \mathbf{s_{11}} & \cdots & \mathbf{s_{1l}} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathbf{r_{m1}} & \cdots & \mathbf{r_{mn}} & \mathbf{s_{m1}} & \cdots & \mathbf{s_{1l}} \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_n \\ w_1 \\ \vdots \\ w_l \end{pmatrix}=0$$

or as

$$(\exists\overline{w})(\mathbf{R}(\overline{v})=-\mathbf{S}(\overline{w}))$$

where $r_{ij}, s_{ik} \in R$ for $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq l$ and where $\mathbf{R}$ and $\mathbf{S}$ are the the obvious matrices.

Example 1.108 (Examples of pp-formulas). Let $r \in R$. Consider the following formulas:

(i) $\mathbf{r}(v) = 0$. Clearly, $\mathcal{M} \models \mathbf{r}(a) = 0$ if and only if r annihilates a.

(ii) $(\exists w)(\mathbf{1}(v) = \mathbf{r}(w))$. Clearly, $\mathcal{M} \models (\exists w)\mathbf{1}(a) = \mathbf{r}(w)$ if and only if a is divisible by r in M; in other words, there exists $b \in M$ such that $a = rb$.

Algebraically, a pp-formula expresses the solvability of a system of linear equations.

Lemma 1.109 (Linearity of pp-formulas). *Suppose that $\phi(v_1, \ldots, v_n)$ is a pp-formula. Then for any R-module M and any $\overline{a}, \overline{b} \in M^n$ we have*

(i) $\mathcal{M} \models \phi[\overline{0}]$;

(ii) $\mathcal{M} \models \phi[\overline{a}]$ *and* $\mathcal{M} \models \phi[\overline{b}] \Rightarrow \mathcal{M} \models \phi[\overline{a} - \overline{b}]$;

(iii) if $c \in Z(R)$ *then* $\mathcal{M} \models \phi[\overline{a}] \Rightarrow \mathcal{M} \models \phi[c\overline{a}]$ *(where $Z(R)$ is the center of R).*

Proof. (i) $\phi[\overline{0}]$ is equivalent to

$$(\exists \overline{w})(\mathbf{R}(\overline{0}) + \mathbf{S}(\overline{w})) = 0$$

or

$$\mathbf{S}(\overline{0}) = \overline{0}.$$

Now, $\overline{0} \in M^n$ and $\mathbf{S}(\overline{0}) = \overline{0}$. Therefore, $\mathcal{M} \models \phi[\overline{0}]$.

(ii) Suppose $\mathcal{M} \models \phi[\overline{a}]$ and $\mathcal{M} \models \phi[\overline{b}]$. Now, $\phi[\overline{a}]$ is equivalent to

$$(\exists \overline{w})(\mathbf{R}(\overline{a}) = -\mathbf{S}(\overline{w})).$$

Since $\mathcal{M} \models \phi[\overline{a}]$, there exists $\overline{c} \in M^l$ such that $\mathbf{R}(\overline{a}) = -\mathbf{S}(\overline{c})$. Similarly, $\phi[\overline{b}]$ is equivalent to $(\exists \overline{w})(\mathbf{R}(\overline{b}) = -\mathbf{S}(\overline{w}))$ and since $\mathcal{M} \models \phi[\overline{b}]$, there exists $\overline{d} \in M^l$

such that $\mathbf{R}(\overline{b}) = -\mathbf{S}(\overline{c})$. Now, $\overline{c} - \overline{d} \in M^l$ and

$$\begin{aligned} &\mathbf{R}(\overline{a} - \overline{b}) \\ &= \mathbf{R}(\overline{a}) - \mathbf{R}(\overline{b}) \\ &= -\mathbf{S}(\overline{c}) + \mathbf{S}(\overline{d}) \\ &= -(\mathbf{S}(\overline{c}) - \mathbf{S}(\overline{d})) \\ &= -\mathbf{S}(\overline{c} - \overline{d}). \end{aligned}$$

Therefore, $\mathcal{M} \models \phi[\overline{a} - \overline{b}]$.

(iii) Suppose $\mathcal{M} \models \phi[\overline{a}]$. Now, $\phi[\overline{a}]$ is equivalent to

$$(\exists \overline{w})(\mathbf{R}(\overline{a}) = -\mathbf{S}(\overline{w})).$$

Since $\mathcal{M} \models \phi[\overline{a}]$, there exists $\overline{d} \in M^l$ such that $\mathbf{R}(\overline{a}) = -\mathbf{S}(\overline{d})$. This means, $\mathbf{c}(\mathbf{R}(\overline{a})) = \mathbf{c}(-\mathbf{S}(\overline{d}))$. Since $c \in Z(R)$, we have $\mathbf{R}(\mathbf{c}(\overline{a})) = -\mathbf{S}(\mathbf{c}(\overline{d}))$. Now, since $c\overline{d} \in M^l$ we have $\mathcal{M} \models \phi[c\overline{a}]$.

□

Corollary 1.110 ([Pre88], Corollary 2.2-2.3, p. 16-18). *Let $\phi(v_1, \ldots, v_n)$ be a pp-formula, $\Phi(v_1, \ldots, v_n)$ be a set of pp-formulas and M be any R-module.*

(i) $\phi[M]$ is a subgroup of M^n stable under endomorphisms of the module.

(ii) $\Phi[M]$ is a subgroup of M^n stable under endomorphisms of the module.

(iii) Let $k \leq n$ and $\overline{b} = (b_1, \ldots, b_k) \in M^k$. Then $\phi[M, \overline{b}]$ is either empty or is a coset of the subgroup $\phi[M, \overline{0}]$ of M^{n-k}.

(iv) *Let $k \leq n$ and $\overline{b} = (b_1, \ldots, b_k) \in M^k$. Then $\Phi[M, \overline{b}]$ is either empty or is a coset of the subgroup $\Phi[M, \overline{0}]$ of M^{n-k}.*

Proof. (i) By Lemma 1.109, it suffices to check that $\phi[M]$ is stable under endomorphisms of the module. Let $f \in End(M)$, $\overline{a} = (a_1, \ldots, a_n) \in \phi[M]$ and $\phi(v_1, \ldots, v_n)$ be of the form:

$$(\exists w_1) \ldots (\exists w_l) \bigwedge_{i=1}^{m} \left(\sum_{j=1}^{n} \mathbf{r_{ij}} v_j + \sum_{k=1}^{l} \mathbf{s_{ik}} w_k = 0 \right)$$

where $r_{ij}, s_{ik} \in R$ for $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq l$. Since $\mathcal{M} \models \phi[\overline{a}]$ there exists $\overline{b} = (b_1, \ldots, b_l) \in M^l$ such that

$$\bigwedge_{i=1}^{m} \left(\sum_{j=1}^{n} \mathbf{r_{ij}} a_j + \sum_{k=1}^{l} \mathbf{s_{ik}} b_k = 0 \right).$$

Applying f we get

$$\bigwedge_{i=1}^{m} \left(\sum_{j=1}^{n} \mathbf{r_{ij}} f(a_j) + \sum_{k=1}^{l} \mathbf{s_{ik}} f(b_k) = 0 \right).$$

Therefore, $f(\overline{a}) \in \phi[M]$.

(ii) $\Phi[M] = \bigcap \{\phi[M] : \phi \in \Phi\}$. So the result follows from (i).

(iii) Let $\phi(M, \overline{b})$ be non-empty and $\overline{a}, \overline{c} \in \phi[M, \overline{b}]$. Therefore, $\mathcal{M} \models \phi[\overline{a}, \overline{b}]$ and $\mathcal{M} \models \phi[\overline{c}, \overline{b}]$. So, Lemma 1.106 implies that $\mathcal{M} \models \phi[\overline{a} - \overline{c}, \overline{0}]$. So, $\overline{a} - \overline{c} \in \phi[M, \overline{0}]$. Conversely, if $\overline{d} \in \phi(M, \overline{0})$ then $\mathcal{M} \models \phi[\overline{d}, \overline{0}]$. Therefore, Lemma 1.106 implies $\mathcal{M} \models \phi[\overline{d} + \overline{c}, \overline{b}]$. So, $\overline{c} + \overline{d} \in \phi[M, \overline{b}]$. Hence, $\phi[M, \overline{b}]$ is a coset of $\phi[M, \overline{0}]$.

(iv) Similar to (iii).

□

Definition 1.111. Let $\mathcal{M}$ be an $\mathcal{R}$-module. Let $\bar{c} \in M$ and $A \subseteq M$. The *pp-type* of $\bar{c}$ over A in $\mathcal{M}$ is

$$\begin{aligned} & pp^{\mathcal{M}}(\bar{c}/A) \\ = \ & \{\phi(\bar{v},\bar{a}) : \mathcal{M} \models \phi[\bar{c},\bar{a}] \text{ where } \phi \text{ is a pp formula and } \bar{a} \in A\}. \end{aligned}$$

Definition 1.112. If p is a complete type then the *pp-part* of p is

$$p^+ = \{\phi : \phi \text{ is pp, } \phi \in p\}.$$

Furthermore,

$$p^- = \{\neg\phi : \phi \text{ is pp, } \neg\phi \in p\}.$$

Theorem 1.113 ([Pre88], p. 27)**.** *Let $f : \mathcal{M} \to \mathcal{N}$ be any $\mathcal{L}$-homomorphism and let $\bar{a} \in M$. Then*

$$pp^{\mathcal{M}}(\bar{a}) \subseteq pp^{\mathcal{N}}(f(\bar{a})).$$

In particular, if $\mathcal{M}$ is a submodule of $\mathcal{N}$ and $\bar{a} \in M$ then $pp^{\mathcal{M}}(\bar{a}) \subseteq pp^{\mathcal{N}}(\bar{a})$.

Definition 1.114. Let A be a subgroup of B. A is *pure* in B if $nA = nB \cap A$ for every positive integer n.

Proposition 1.115. *Let A be a subgroup of B and let $a \in A$. If A is pure in B then a is divisible by n in A if and only if a is divisible by n in B.*

Proof. This follows directly from the definition. □

Note that, nB is defined by the pp-formula $(\exists w)(nw = v)$.

Definition 1.116. Let A, B and C be abelian groups. A short exact sequence

$$0 \to A \xrightarrow{\alpha} B \to C \to 0$$

is *pure* if $\alpha[A]$ is a pure subgroup of B.

Definition 1.117. A *torsion group* is a group consisting of elements of finite order. A group is *torsion-free* if the only element of finite order is the identity.

Theorem 1.118. *Let G be an abelian group and H a subgroup of G.*

(i) If H is a direct summand of G then H is a pure subgroup.

(ii) If H is a divisible subgroup of G then H is a pure subgroup.

(iii) If the G/H is torsion-free, then H is a pure subgroup.

(iv) If H is a torsion group then H is a pure subgroup.

Purity for abelian groups is extended to R-modules as follows:

Definition 1.119. Let $\mathcal{M}$ be a submodule of $\mathcal{N}$. The embedding is *pure*, denoted $\mathcal{M} \prec_1^+ \mathcal{N}$, if $\forall \overline{a} \in M$,

$$pp^{\mathcal{M}}(\overline{a}) = pp^{\mathcal{N}}(\overline{a}).$$

That is,

$$\mathcal{M} \prec_1^+ \mathcal{N} \Leftrightarrow \text{ for every pp-formula } \phi \text{ and } \overline{a} \in M, \mathcal{M} \models \phi[\overline{a}] \Leftrightarrow \mathcal{N} \models \phi[\overline{a}].$$

Furthermore, $\mathcal{M} \prec_1^+ \mathcal{N}$ implies that for all pp-formulas $\phi(v_1, \ldots, v_n)$,

$$\phi[M] = \phi[N] \cap M^n.$$

Equivalently, $\mathcal{M} \prec_1^+ \mathcal{N}$ if and only if every finite system of linear equations with coefficients in R, parameters in M and a solution in N already has a solution in M.

Algebraically, $\mathbb{Z}$-modules and abelian groups are equivalent. Model theoretically, there is no distinction either since the language for $\mathbb{Z}$-modules is definably equivalent to the language for abelian groups since multiplication by n can be defined as an iterated sum. Purity for $\mathbb{Z}$-modules is equivalent to the usual purity for abelian groups.

Proposition 1.120 ([Pre88], Corollary 2.$\mathbb{Z}$2, p.47). *Let M and N be $\mathbb{Z}$-modules such that M is a submodule of N. Then*

$$M \prec_1^+ N \Leftrightarrow rN \cap M = rM$$

for all $r \in \mathbb{Z}$.

Proof. Suppose that $M \prec_1^+ N$ and $r \in \mathbb{Z}$ and that $m = rn \in rN \cap M$. Then $\mathcal{N} \models (\exists w)(m = rw)$ and since M is pure in N, $\mathcal{M} \models (\exists w)(m = rw)$. Therefore, $m \in rM$. So, $M \cap rN \subseteq rM$. Trivially, $rM \subseteq M \cap rN$. So, $M \cap rN = rM$ for all $r \in \mathbb{Z}$.

Suppose $M \cap rN = rM$ for all $r \in \mathbb{Z}$. Now, let $\phi(v_1, \ldots, v_n)$ be a pp-formula. So, by [Pre88] (Theorem 2.$\mathbb{Z}$1, p.46), $\phi(\overline{v})$ is equivalent, in every module, to a conjunction of formulas of the form $r'|t(\overline{v})$ or the form $t'(\overline{v}) = 0$ for suitable elements of $r' \in \mathbb{Z}$ and terms t, t'. Since atomic formulas of the form $t'(\overline{v}) = 0$ are preserved in both directions, we only need to consider formulas of the form $r'|t(\overline{v})$. So, without loss of generality, let $\phi(\overline{v})$ be $r'|t(\overline{v})$. Now, let $\overline{a} = (a_1, \ldots a_n) \in M^n$ and $N \models r'|t(\overline{a})$. So, $t(\overline{a}) \in r'M$ since $M \cap r'N \subseteq r'M$. Therefore, $M \models r'|t(\overline{a})$. Thus, M is pure in

N. □

Definition 1.121. $\mathcal{M}$ is *pp-compact* if whenever Φ is a set of pp-formulas with parameters from M which is finitely satisfied in $\mathcal{M}$ then Φ is satisfied in $\mathcal{M}$.

Theorem/Definition 1.122. *A left R-module M is* pure-injective *if one of the following equivalent conditions hold:*

(i) $\mathcal{M}$ *is* pp-compact.

(ii) $\mathcal{M}$ *is injective over pure embeddings, that is, any diagram as follows can be completed commutatively as shown:*

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "\prec_1^+"] \arrow[d, "f"'] & \mathcal{B} \arrow[dl, dashed, "\exists \overline{f}"] \\ \mathcal{M} & \end{tikzcd}$$

Proof.

(i) $\Rightarrow$ (ii) Suppose that $\mathcal{M}$ is pp-compact. Let $\overline{b}$ be an enumeration of B. Let $\overline{v}_b$ be new distinct variables indexed by the elements of $\overline{b}$. Consider

$$\Phi(\overline{v}_b) = \left\{\phi(\overline{v}_b, \overline{a}) : \phi \text{ is pp and } B \models \phi[\overline{b}, \overline{a}] \text{ where } \overline{a} \in A\right\}.$$

Now, let

$$\Phi^{'}(\overline{v}_b) = \{\phi(\overline{v}_b, f(\overline{a})) : \phi \in \Phi\}.$$

We want to show that every finite subset of Φ is satisfied in $\mathcal{M}$. In order to do this, it is sufficient to choose a single formula in $\Phi^{'}$ since Φ is closed under finite conjunctions.

Let $\psi \in \Phi$.

$$
\begin{aligned}
\mathcal{B} \models \psi[\overline{b}, \overline{a}] & \\
\Rightarrow \quad & \mathcal{B} \models (\exists \overline{v})\psi[\overline{v}, \overline{a}] \\
\Rightarrow \quad & \mathcal{A} \models (\exists \overline{v})\psi[\overline{v}, \overline{a}] \\
\text{since} \quad & \mathcal{A} \prec_1^+ \mathcal{B} \text{ and } \overline{a} \in A \\
\Rightarrow \quad & \mathcal{M} \models (\exists \overline{v})\psi[\overline{v}, f(\overline{a})] \\
\Rightarrow \quad & \mathcal{M} \models \phi[\overline{c}, f(\overline{a})] \text{ where } \overline{c} \in M.
\end{aligned}
$$

So, Φ' is finitely satisfied in $\mathcal{M}$. Therefore, Φ' is satisfied in $\mathcal{M}$, that is, there is $\overline{a} \in M$, indexed by $\overline{b}$, such that $\mathcal{M} \models \Phi'[\overline{a}]$. Define $\overline{f}$ to be the map which carries $\overline{b}$ to $\overline{a}$ component-wise. Clearly, $\overline{f}$ extends f. Now, consider the pp-formula $v_1 + \mathbf{r}v_2 = v_3$. So,

$$
\begin{aligned}
\mathcal{B} \models \mathbf{b_1} + \mathbf{rb_2} = \mathbf{b_3} & \\
\Rightarrow \quad & \overline{v}_{b_1} + \mathbf{r}\overline{v}_{b_2} = \overline{v}_{b_3} \in \Phi \\
\Rightarrow \quad & \overline{f}(b_1) + r\overline{f}(b_2) = \overline{f}(b_3).
\end{aligned}
$$

Therefore, $\overline{f}$ is a homomorphism.

(ii) $\Rightarrow$ (i) Suppose that $\mathcal{M}$ is injective over pure embeddings and that Φ is a collection of pp-formulas that is finitely satisfiable in $\mathcal{M}$. Therefore, Φ is consistent, so there exists an elementary extension of $\mathcal{M}$ that realizes Φ, say $\mathcal{N}$. Elementary extensions are stronger than pure embeddings. Consider the following diagram:

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook, "\prec"] \arrow[d, "1_M"'] & \mathcal{N} \arrow[dl, "f"] \\ \mathcal{M} & \end{tikzcd}$$

By pure-injectivity the identity lifts. Clearly, if $\bar{b}$ is a solution of Φ in N, then $f(\bar{b})$ is a solution of Φ in M. So, Φ is satisfied in $\mathcal{M}$. So, $\mathcal{M}$ is pp-compact.

□

It is clear that injectives are pure injective.

Definition 1.123. A left R-module M is equationally compact if whenever Σ is a system of linear equations over M which is finitely satisfiable in M (that is, every finite subset of Σ has a solution in M), then Σ has a solution in M.

Definition 1.124. A *compatible topology*, τ, on M is one that makes (M, τ) a topological module.

Theorem 1.125 (W. Taylor. See [CH06]). *Let M be a left R-module. If M can be equipped with a compatible, compact, Hausdorff topology then M is equationally compact.*

Theorem 1.126 ([Pre88], Theorem 2.8, p. 28-29). *A left R-module M is* pure-injective *if one of the following equivalent conditions hold:*

(i) every system of equations over M which is finitely satisfied in M actually has a solution in M.

(ii) every partial pp-type in one variable over M which is finitely satisfied in $\mathcal{M}$ is actually realized in $\mathcal{M}$.

(iii) if $\mathcal{M}$ is purely embedded in $\mathcal{N}$ then this embedding is split. In other words, $N = M \bigoplus M'$ for some M'.

(iv) if $\overline{a}$ is in $\mathcal{A}$, $\overline{b}$ is in M and $pp^{\mathcal{A}}(\overline{a}) \subseteq pp^{\mathcal{M}}(\overline{b})$ then there is a $\mathcal{L}$-homomorphism $f : \mathcal{A} \to \mathcal{M}$ with $f(\overline{a}) = \overline{b}$.

Corollary 1.127. *If a left R-module M has a compatible, compact Hausdorff topology it is pp-compact.*

Proof. Suppose M has a compatible, compact Hausdorff topology. So, by 1.125, M is equationally compact. Now, by (i) of Theorem 1.126, equationally compact modules are pure-injective. Therefore, M is pp-compact. □

Theorem 1.128. *Any direct product of pure-injectives is pure-injective.*

Proof. Let $\{M_i\}_{i \in I}$ be a family left R-module that are pure-injective. Let $f : C \to B$ be a pure-embedding and $g : C \to \overline{M}$ be a module homomorphism. Since M_i is pure-injective, there exists $h_i : C \to M_i$ such that $h_i \circ f = \pi_i \circ g = g_i$. Now, let $h = (h_i)_{i \in I} : C \to \overline{M}$. So,

$$h \circ f = (h_i)_{i \in I} \circ f = (h_i \circ f)_{i \in I} = (g_i)_{i \in I} = g.$$

So, $\overline{M}$ is pure-injective. □

Theorem 1.128 is inherently a categorical result and therefore can be extended to any category with a "reasonable" concept of injectivity and a "reasonable" concept of products.

Theorem 1.129 (G. Sabbagh. See [Pre88], Corollary 2.24, p. 40). *For any set $\{M_i\}_{i\in I}$ of modules,*

$$\bigoplus_{i\in I} M_i \equiv \prod_{i\in I} M_i.$$

Theorem 1.130 (E. Fisher. See [Pre88], Corollary 2.26, p. 40). *If M is elementarily equivalent to N then any pure embedding of M into N is an elementary embedding.*

So, an immediate corollary is

Corollary 1.131 ([Pre88], Corollary 2.28, p. 41). *For any set $\{M_i\}_{i\in I}$ of modules,*

$$\bigoplus_{i\in I} M_i \prec \prod_{i\in I} M_i.$$

Note. We will see in Chapter 4 that Corollary 1.131 cannot hold for topological modules in general.

Chapter 2

Topological Model Theory

2.1 Preliminaries

Topological model theory is the study of topology from a model theoretic perspective. In Garavaglia's 1978 paper the *Model Theory of Topological Structures* [Gar78], Garavaglia provides a brief history of topological model theory. It appears that the first results in topological model theory were discovered independently by three different groups of mathematicians. In 1974, at the University of Wisconsin, two Ph. D. students, McKee and Sgro, wrote their Ph. D. dissertations on topological model theory [McK74] [Sgr74]. In 1975, Garavaglia announced a series of abstracts in the AMS Notices [Gar75] of results obtained while working on his Ph.D. thesis [Gar76] and later published in his 1978 paper *Model Theory of Topological Structures* [Gar78]. Finally, in 1976, Ziegler published *A language for topological structures which satisfies a Lindström theorem* [Zie76]. In 1980, Ziegler and Flum published the first monograph on topological model theory [FZ80]. Many of the basic definitions and results from [FZ80] will be summarized or expanded on in this chapter. In addition, some of the results from Kucera's 1986 paper *Stability theory for topological logic, with applications to topological modules* [Kuc86] will be summarized.

In this section, the formal language used to study topological structures, namely $\mathcal{L}_t$, will be introduced. The language $\mathcal{L}_t$ is interesting since many topological concepts can be expressed in $\mathcal{L}_t$ and $\mathcal{L}_t$ satisfies a compactness theorem, a completeness theorem and a Löwenheim-Skolem theorem.

Definition 2.1. The language $\mathcal{L}_2$ is the two-sorted language obtained by adding to $\mathcal{L}$ a set of new distinct variables $\{V_i : i \in \mathbb{N}\}$ and a binary symbol $\in$ with sorting $v_i \in V_j$. Variables of the first sort, namely $\{v_i : i \in \mathbb{N}\}$, are called individual variables and variables of the second sort, namely $\{V_i : i \in \mathbb{N}\}$, are called set variables.

Terms and formulas of $\mathcal{L}_2$ are defined as in Definitions 1.2-1.4. A structure for $\mathcal{L}_2$ consists of an $\mathcal{L}$-structure $\mathcal{M}$, a non-empty set τ, and a binary relation $\in$ on $M \times \tau$. The intended interpretation for an element of the second sort is as a set of elements of the first sort. First-order languages of this type have been used to study logics which are inherently second-order. The study of topology using model theoretic techniques uses this type of language since topology involves genuinely higher-order concepts. The quantifiers $(\exists X)\phi$ and $(\forall X)\phi$ are intended to be interpreted as "there exists a subset A of the universe such that $\phi[A]$ holds" and "for every subset A of the universe $\phi[A]$ holds" respectively. $\langle \mathcal{M}, \tau \rangle$ is called a *weak structure* for $\mathcal{L}_2$ if $\varnothing \neq \tau \subseteq \wp(M)$. Weak structures are interpreted in $\mathcal{L}_2$ by viewing $\in$ as set membership.

Definition 2.2. Let $\langle \mathcal{M}, \tau \rangle$ be a weak structure for $\mathcal{L}_2$, $\phi(v_1, \ldots, v_n, V_1, \ldots, V_m)$ an $\mathcal{L}_2$-formula, $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and $\overline{A} = (A_1, \ldots, A_m)$ where $A_i \in \tau$. We define

$\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}]$ for $\mathcal{L}_2$-formulas by extending the definition of $\models$ for $\mathcal{L}$-formulas inductively as follows:

(i) if ϕ is $t \in V_i$ then

$$\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}] \text{ if } t^{\mathcal{M}}(\overline{a}) \in A_i;$$

(ii) (a) if ϕ is $(\exists V_i)\psi(\overline{v}, \overline{V}, V_i)$, where $i > m$, then

$$\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}] \text{ if there is a } B \in \tau \text{ such that } \langle \mathcal{M}, \tau \rangle \models \psi[\overline{a}, \overline{A}, B];$$

(b) if ϕ is $(\exists V_i)\psi(\overline{v}, \overline{V})$, where $i \leq m$, then

$$\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}] \text{ if there is a } B \in \tau \text{ such that}$$

$$\langle \mathcal{M}, \tau \rangle \models_2 \psi[\overline{a}, A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_m].$$

If $\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}]$ then we say that $\langle \mathcal{M}, \tau \rangle$ *satisfies* ϕ at $(\overline{a}, \overline{A})$ and that ϕ is *true* at $(\overline{a}, \overline{A})$ in $\langle \mathcal{M}, \tau \rangle$.

Definition 2.3.

$$\mathbf{ext} = (\forall X)(\forall Y)(X = Y \leftrightarrow (\forall z)(z \in X \leftrightarrow z \in Y)).$$

Any model $\langle \mathcal{M}, \mathcal{S}, \epsilon \rangle$ that satisfies **ext** can be interpreted as a weak structure $\langle \mathcal{M}, \tau \rangle$ by replacing S with τ where τ is defined as

$$\{\{a \in M : \langle M, S, \epsilon \rangle \models a \in s\} : s \in S\}$$

and ϵ with $\in$. Clearly $\langle \mathcal{M}, \mathcal{S}, \epsilon \rangle$ and $\langle \mathcal{M}, \tau, \in \rangle$ are isomorphic. For the remainder of this work, all of the structures will be models of **ext** and will therefore, without loss of generality, be weak structures.

Definition 2.4. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. $\langle \mathcal{M}, \tau \rangle$ is called a *topological structure* if τ is a topology on M.

In particular, a topological structure is a weak structure.

Let $\langle \mathcal{M}, \tau \rangle$ be a weak structure for $\mathcal{L}_2$, $\phi(v_1, \ldots, v_n, V_1, \ldots, V_m)$ an $\mathcal{L}_2$-formula and $\overline{A} = (A_1, \ldots, A_m)$ where $A_i \in \tau$. We let $\phi(M, \overline{A})$ denote the set of $\overline{a} \in M^n$ such that $\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}]$. That is,

$$\phi(M, \overline{A}) = \left\{ \overline{a} \in M^n : \langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}] \right\}.$$

Definition 2.5. Two weak structures for $\mathcal{L}_2$, $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$, are $\mathcal{L}_2$-*elementarily equivalent*, $\langle \mathcal{M}, \tau \rangle \equiv_2 \langle \mathcal{N}, \sigma \rangle$, if for all $\mathcal{L}_2$-sentences ϕ:

$$\langle \mathcal{M}, \tau \rangle \models_2 \phi \Leftrightarrow \langle \mathcal{N}, \sigma \rangle \models_2 \phi.$$

Definition 2.6.

$$\begin{aligned} \mathbf{bas} = & (\forall x)(\exists X)(x \in X) \wedge \\ & (\forall x)(\forall X)(\forall Y)[(x \in X \wedge x \in Y) \rightarrow \\ & (\exists Z)[x \in Z \wedge (\forall z)[z \in Z \rightarrow (z \in X \wedge z \in Y)]]]. \end{aligned}$$

Clearly,

$$\langle \mathcal{M}, \tau \rangle \models_2 \mathbf{bas} \Leftrightarrow \tau \text{ is a base for a topology on } M.$$

If $\langle \mathcal{M}, \tau \rangle$ is a topological structure then it is a weak model of **bas**. If $\langle \mathcal{M}, \tau \rangle$ is a weak model for **bas**, we let $\widetilde{\tau}$ denote the topology on M generated by τ as a basis. So,

$$\langle \mathcal{M}, \tau \rangle \models_2 \mathbf{bas} \Leftrightarrow \widetilde{\tau} \text{ is a topology on } M.$$

Let $\langle \mathcal{M}, \tau \rangle$ be a weak structure for $\mathcal{L}_2$. There does not exist a set of axioms Σ in $\mathcal{L}_2$ such that $\langle \mathcal{M}, \tau \rangle \models_2 \Sigma \Leftrightarrow \tau$ is a topology on M, since this is inherently a second-order concept over τ. However, **bas** is an axiom such that $\langle \mathcal{M}, \tau \rangle \models_2 \mathbf{bas} \Leftrightarrow \tau$ is a basis for a topology on M. This observation suggests that concepts of topological model theory should be chosen to be invariant with respect to the choice of the basis.

Definition 2.7. Let $\phi(v_1, \ldots, v_n, V_1, \ldots, V_m)$ an $\mathcal{L}_2$-formula. ϕ is *invariant for topologies* if for all weak models of **bas** $\langle \mathcal{M}, \tau \rangle$, $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and $\overline{A} = (A_1, \ldots, A_m)$ where $A_i \in \tau$ we have

$$\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}] \Leftrightarrow \langle \mathcal{M}, \widetilde{\tau} \rangle \models_2 \phi[\overline{a}, \overline{A}].$$

It is clear that ϕ is invariant for topologies if and only if for all topological structures $\langle \mathcal{M}, \tau \rangle$ and any base σ for τ

$$\langle \mathcal{M}, \tau \rangle \models_2 \phi \Leftrightarrow \langle \mathcal{M}, \sigma \rangle \models_2 \phi.$$

Definition 2.8. Let $\langle \mathcal{M}, \tau \rangle$ be a topological structure. $\langle \mathcal{M}, \tau \rangle$ is $(\kappa$-$)$*saturated* if τ has a basis σ such that $\langle \mathcal{M}, \sigma \rangle$ is (κ-)saturated as a two-sorted structure.

Theorem 2.9 ([FZ80], Lemma 1.18, p. 89). *Let $\langle \mathcal{M}, \tau \rangle$ be an $\aleph_1$-saturated topo-*

logical structure. Then τ is closed under countable intersections.

Proof. Let σ be a basis for τ such that $\langle \mathcal{M}, \sigma \rangle$ is $\aleph_1$-saturated as a two-sorted structure. Suppose that for $i \in \aleph_0$, $U_i \in \tau$ and $a \in \bigcap_{i \in \aleph_0} U_i$. Choose neighbourhoods $A_i \in \sigma$ with $a \in A_i$ and $A_i \subseteq U_i$.

Consider

$$\phi = \{a \in X\} \cup \{\forall y(y \in X \rightarrow y \in A_i : i \in \aleph_0\}.$$

Clearly, ϕ is finitely satisfiable in $\langle (M, a, A_0, A_1, \ldots), \sigma \rangle$. Hence, there is an $A \in \sigma$ with $a \in A$ and $A \subseteq A_i \subseteq U_i$ for $i = 0, 1, 2, \ldots$. So, $\bigcap_{i \in \aleph_0} U_i$ is open. $\square$

Corollary 2.10. *Let τ be a Hausdorff topology on M such that $\langle \mathcal{M}, \tau \rangle$ is $\aleph_1$-saturated as a topological structure. If $\langle \mathcal{M}', \tau' \rangle$ is a countable subspace of $\langle \mathcal{M}, \tau \rangle$ then τ' is the discrete topology on M'.*

Proof. Let τ be a Hausdorff topology on M such that $\langle \mathcal{M}, \tau \rangle$ is $\aleph_1$-saturated as a two-sorted structure, and let $\langle \mathcal{M}', \tau' \rangle$ be a countable subspace of $\langle \mathcal{M}, \tau \rangle$. Since $\langle \mathcal{M}, \tau \rangle$ is $\aleph_1$-saturated, by Theorem 2.9 τ is closed under countable intersections. Since $\langle \mathcal{M}', \tau' \rangle$ is a countable subspace of $\langle \mathcal{M}, \tau \rangle$, τ' is a Hausdorff topology on M' and is closed under countable intersections of open sets. Now, for each $x \in M'$, $\{x\}$ is closed since τ' is a Hausdorff topology on M'. τ' is closed under countable intersections of open sets, so it is closed under countable unions of closed sets. Since M' is countable and closed under countable unions of closed sets, for every $x \in M'$, $M' \setminus \{x\}$ is closed. Therefore, for every $x \in M'$, $\{x\}$ is open in τ'. So, τ' is the discrete topology on M'. $\square$

Note. This corollary implies that substructures of topological structures will not, in general, be subspaces.

Theorem 2.11 ([FZ80], Lemma 2.2, p. 114). *An $\aleph_1$-saturated topological group has a basis at 0 consisting of subgroups.*

Proof. Let σ be a basis for τ such that $\langle \mathcal{M}, \sigma\rangle$ is $\aleph_1$-saturated as a two-sorted structure. Now, τ is closed under countable intersections by theorem 2.9. Since M is a topological group, for every $U \in \sigma$ with $0 \in U$, we obtain a sequence $U_0 \supseteq U_1 \supseteq \cdots$ such that $U_i \in \sigma$, $0 \in U_i$ and $U_{i+1} - U_{i+1} \subseteq U_i \subseteq U$. So, $\bigcap_{i\in\omega} U_i$ is a subgroup of A and $\bigcap_{i\in\omega} U_i \in \tau$. □

Although this result is strong, it also demonstrates that it will be difficult to exhibit explicit examples of saturated topological groups. However, using quantifier elimination results, it is clear that the indiscrete topology and the discrete topology are κ-saturated if and only if the underlying group is κ-saturated.

Definition 2.12. A topological group $\langle M, \tau\rangle$ is *locally pure* if for every $n \geq 0$,

$$(\forall X)_0(\exists Y)_0(\forall x)(\exists y)(nx \in Y \to (y \in X \wedge ny = nx))$$

holds in $\langle M, \tau\rangle$.

Theorem 2.13 ([FZ80], Lemma 3.4, p.115). *An $\aleph_1$-saturated locally pure group has a basis at 0 consisting of pure subgroups.*

Example 2.14. Consider the Euclidean topology on the additive group of rationals, and an $\aleph_1$-saturated $\mathcal{L}_2$-elementary extension $\langle \mathbb{Q}^{(\kappa)}, \sigma\rangle$ of $\langle \mathbb{Q}, Euc\rangle$. The abelian groups elementarily equivalent to the rationals are just the $\mathbb{Q}$ vector spaces. Any linearly ordered group is locally pure (since $nx \in (-nb, nb)$ implies $x \in (-b, b)$), therefore, by Theorem 2.13, $\widetilde{\sigma}$ has a basis σ_0 at 0 of open, pure, subgroups. So,

$\langle \mathbb{Q}^{(\kappa)}, \sigma_0 \rangle$ has a countable $\mathcal{L}_2$-elementary substructure $\langle \mathbb{Q}^{(\aleph_0)}, \sigma' \rangle$, where again σ' is a basis at 0 for a topology consisting of open, pure subgroups. Notice that $\mathbb{Q}^n, n < \aleph_0$, cannot have a non-discrete Hausdorff topology with a basis at 0 of open pure subgroups; the pure subgroups are just the $\mathbb{Q}$-subspaces. However, $\mathbb{Q}^{(\aleph_0)}$ does, by letting σ' consist of all subspaces with finite co-dimension.

Definition 2.15. Let ϕ be an $\mathcal{L}_2$-formula and let t be an $\mathcal{L}$-term. ϕ is in *negation normal form* if ϕ is written using only the logical connectives $\neg$, $\vee$, $\wedge$, $\forall$ and $\exists$ and all the negation signs occur only in front of atomic formulas. ϕ is *positive* in X if each free occurrence of X in the negation normal form of ϕ is of the form $t \in X$ and ϕ is *negative* in X if each free occurrence of X in the negation normal form of ϕ is of the form $\neg(t \in X)$.

Note. If ψ is any $\mathcal{L}_2$-formula, it is logically equivalent to a formula ϕ in negation normal form. Furthermore, if ϕ and ϕ' are both negation normal forms of ψ, then ϕ is negative (positive) in X if and only if ϕ' is negative (positive) in X, so negative and positive are well-defined for $\mathcal{L}_2$-formulas. It is also possible that an $\mathcal{L}_2$-formula may not be positive nor negative in X.

Lemma 2.16. *Let $\phi(v_1, \ldots, v_n, V_1, \ldots, V_m, V_{m+1})$ be an $\mathcal{L}_2$-formula, $\langle \mathcal{M}, \tau \rangle$ a weak structure, $\overline{a} = (a_1, \ldots, a_n) \in M^n$, $\overline{A} = (A_1, \ldots, A_m)$ where $A_i \in \tau$ for $1 \leq i \leq m+1$. Assume $\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}, A_{m+1}]$. Then*

1. *If ϕ is positive in V_{m+1}, then $\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}, W]$ for any $W \in \tau$ such that $A_{m+1} \subseteq W \subseteq M$.*

2. *If ϕ is negative in V_{m+1}, then $\langle \mathcal{M}, \tau \rangle \models_2 \phi[\overline{a}, \overline{A}, W]$ for any $W \in \tau$ such that $W \subseteq A_{m+1}$.*

Proof. This is easily proved by an induction on the complexity of $\mathcal{L}_2$-formulas. $\square$

Definition 2.17. The set of all $\mathcal{L}_t$-*formulas* is the smallest set $\mathcal{W}_t$ containing the atomic $\mathcal{L}_2$-formulas and such that:

(i) if $\phi \in \mathcal{W}_t$ then $\neg\phi \in \mathcal{W}_t$;

(ii) if $\phi, \psi \in \mathcal{W}_t$ then $(\phi \wedge \psi) \in \mathcal{W}_t$;

(iii) if ϕ is in $\mathcal{W}_t$ then $(\exists v_i)\phi \in \mathcal{W}_t$;

(iv) if t is an $\mathcal{L}$-term, ϕ is in $\mathcal{W}_t$ and ϕ is positive in V_i then

$$(\forall V_i)(t \in V_i \rightarrow \phi) \in \mathcal{W}_t;$$

(v) if t is an $\mathcal{L}$-term, ϕ is in $\mathcal{W}_t$ and ϕ is negative in V_i then

$$(\exists V_i)(t \in V_i \wedge \phi) \in \mathcal{W}_t.$$

Notation. If t is an $\mathcal{L}$-term, ϕ is in $\mathcal{W}_t$ and ϕ is positive in V_i then we abbreviate $(\forall V_i)(t \in V_i \rightarrow \phi)$ as $(\forall V_i)_t\phi$. Similarly, if t is an $\mathcal{L}$-term, ϕ is in $\mathcal{W}_t$ and ϕ is negative in V_i then we abbreviate $(\exists V_i)(t \in V_i \wedge \phi)$ as $(\exists V_i)_t\phi$. The intended interpretation of $(\forall V_i)_t\phi$ is "for every open neighbourhood U of t, $\phi[U]$ holds" and the intended interpretation of $(\exists V_i)_t\phi$ is "there is an open neighbourhood U of t such that $\phi[U]$ holds". In addition, for an $\mathcal{L}_t$-formula ϕ we let $\phi(v_1, \ldots, v_n, V_1{}^+, \ldots, V_l{}^+, V_{l+1}{}^-, \ldots, V_m{}^-)$ indicate that ϕ is positive in $V_1, \ldots, V_l$ and negative in $V_{l+1}, \ldots, V_m$.

Definition 2.18. Let $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ be weak models of **bas**. We relativize $\models$, $\equiv$ and $\cong$ to $\mathcal{L}_t$ as follows:

$\models_t$: for all $\mathcal{L}_t$-formulas ϕ if $\langle \mathcal{M}, \tau \rangle \models_2 \phi$ then $\langle \mathcal{M}, \tau \rangle \models_t \phi$;

$\equiv_t$: if for all $\mathcal{L}_t$-sentences ϕ,

$$\langle \mathcal{M}, \tau \rangle \models_t \phi \Leftrightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi$$

then $\langle \mathcal{M}, \tau \rangle \equiv_t \langle \mathcal{N}, \sigma \rangle$;

$\cong_t$: if η is an $\mathcal{L}$-isomorphism from $\mathcal{M}$ to $\mathcal{N}$ and η is a homeomorphism from $\langle M, \widetilde{\tau} \rangle$ to $\langle N, \widetilde{\sigma} \rangle$ then $\langle \mathcal{M}, \tau \rangle \cong_t \langle \mathcal{N}, \sigma \rangle$.

Theorem 2.19 ([FZ80], Lemma 2.3, p. 6). *$\mathcal{L}_t$-formulas are invariant for topologies.*

Proof. This is proved by an induction on the complexity of $\mathcal{L}_t$-formulas.
Let $\langle \mathcal{M}, \tau \rangle$ be a weak model of **bas**.
We must show if $\phi(v_1, \ldots, v_n, V_1{}^+, \ldots, V_l{}^+, V_{l+1}{}^-, \ldots, V_m{}^-) \in \mathcal{L}_t$,
$\overline{a} = (a_1, \ldots, a_n) \in M^n$, $\overline{A} = (A_1, \ldots, A_m)$ where $A_i \in \tau$, then

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}] \Leftrightarrow \langle \mathcal{M}, \widetilde{\tau} \rangle \models_t \phi[\overline{a}, \overline{A}]. \tag{2.1}$$

This proof is routine and it is provided, in part, in [FZ80] on page 6. Since the cases dealing with set quantifiers illustrates weak-second-order techniques I will provide part of the proof here. The part provided in [FZ80], is the case $\phi = (\exists X)_t \psi$, therefore, I provide the case $\phi = (\forall X)_t \psi$ where (2.1) holds for ψ. First, set $a_0 = t^{\mathcal{M}}[\overline{a}]$.

$\Rightarrow$ Assume $\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}]$. Let $V' \in \widetilde{\tau}$ such that $a_0 \in V'$. So, there is a $V \in \tau$ such that $a_0 \in V \subseteq V'$ since τ is a basis for $\widetilde{\tau}$. Since $\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}]$,

$\langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{a}, \overline{A}, V]$. By the induction hypothesis, $\langle \mathcal{M}, \widetilde{\tau} \rangle \models_t \psi[\overline{a}, \overline{A}, V]$. Since $\phi \in \mathcal{L}_t$, ψ is positive in X. So, by Lemma 2.16, $\langle \mathcal{M}, \widetilde{\tau} \rangle \models_t \psi[\overline{a}, \overline{A}, V']$ since $V \subseteq V'$. Since V' was arbitrary, $\langle \mathcal{M}, \widetilde{\tau} \rangle \models_t \phi[\overline{a}, \overline{A}]$.

$\Leftarrow$ Assume $\langle \mathcal{M}, \widetilde{\tau} \rangle \models_t \phi[\overline{a}, \overline{A}]$. Therefore, for all $V \in \widetilde{\tau}$ such that $a_0 \in V$ we have $\langle \mathcal{M}, \widetilde{\tau} \rangle \models_t \psi[\overline{a}, \overline{A}, V]$. In particular, this holds for all $V \in \tau$, where $a_0 \in V$ since $\tau \subseteq \widetilde{\tau}$. Therefore, by the induction hypothesis $\langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{a}, \overline{A}, V]$. Therefore, $\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}]$.

□

Theorem 2.20 ([Gar78], [McK74]). *An $\mathcal{L}_2$-sentence is invariant for topologies if and only if it is equivalent in $\mathcal{L}_2$, for topological structures, to an $\mathcal{L}_t$-sentence.*

Since $\mathcal{L}_t$ is an ordinary first-order logic, we obtain the following two results.

Theorem 2.21 ([Gar78], Compactness Theorem for $\mathcal{L}_t$). *Let T be an $\mathcal{L}_t$-theory. T is satisfiable if and only if T is finitely satisfiable, that is, every finite subset of T is satisfiable.*

Corollary 2.22. *A set of $\mathcal{L}_t$-sentences that has arbitrary large finite models, has an infinite model.*

Theorem 2.23 ([FZ80], Löwenheim-Skolem Theorem for $\mathcal{L}_t$). *A countable set of $\mathcal{L}_t$-sentences that has a topological model has a countable model $\langle \mathcal{M}, \tau \rangle$, that is, $\mathcal{M}$ is a countable $\mathcal{L}$-structure and τ is a topology on M with a countable basis.*

Theorem 2.24 ([Gar78], [FZ80]). *Let $f \in \mathcal{L}$ be an n-ary function symbol and let $R \in \mathcal{L}$ be an n-ary relation symbol. The following properties can be expressed in $\mathcal{L}_t$:*

(i) base, denoted **bas**.

(ii) f is continuous, denoted $\mathbf{cont}_f$.

(iii) f is open, denoted $\mathbf{open}_f$.

(iv) R is open, denoted $\mathbf{open}_R$.

(v) R is closed, denoted $\mathbf{closed}_R$.

(vi) T_0-space.

(vii) T_1-space.

(viii) Hausdorff space, denoted **haus**.

(ix) regular space, denoted **reg**.

(x) discrete topology, denoted **disc**.

(xi) trivial topology, denoted **triv**.

Proof. (i)

$$\begin{aligned}\mathbf{bas} = &(\forall x)(\exists X)_x(x = x) \wedge (\forall X)_x(\forall Y)_x(\exists Z)_x \\ &(\forall z)(z \in Z \rightarrow (z \in X \wedge z \in Y)).\end{aligned}$$

(ii)

$$\mathbf{cont}_f = (\forall x_1)\cdots(\forall x_n)(\forall Y)_{f(x_1,\ldots,x_n)}(\exists X_1)_{x_1}\cdots(\exists X_n)_{x_n}$$
$$(\forall y_1)\cdots(\forall y_n)[y_1 \in X_1 \wedge \cdots \wedge y_n \in X_n \rightarrow f(y_1,\ldots,y_n) \in Y].$$

(iii)

$$\mathbf{open}_f = (\forall x_1)\cdots(\forall x_n)(\forall X_1)_{x_1}\cdots(\forall X_n)_{x_n}(\exists Y)_{f(x_1,\ldots,x_n)}.$$
$$(\forall y_1)\cdots(\forall y_n)[f(y_1,\ldots,y_n) \in Y \rightarrow y_1 \in X_1 \wedge \cdots \wedge y_n \in X_n].$$

(iv)

$$\mathbf{open}_R = (\forall x_1)\cdots(\forall x_n)(R(x_1,\ldots,x_n) \rightarrow$$
$$(\exists X_1)_{x_1}\cdots(\exists X_n)_{x_n}(\forall y_1)\cdots(\forall y_n)$$
$$((y_1 \in X_1 \wedge \cdots \wedge y_n \in X_n) \rightarrow R(y_1,\ldots y_n))).$$

(v)

$$\mathbf{closed}_R = \mathbf{open}_{\neg R}.$$

(vi)

$$(\forall x)(\forall y)(x = y \vee (\exists X)_x \neg(y \in X) \vee (\exists Y)_y \neg(x \in Y)).$$

(vii)

$$(\forall x)(\forall y)(x = y \vee (\exists X)_x (\exists Y)_y (\neg(y \in X) \wedge \neg(x \in Y))).$$

(viii)

$$\begin{aligned}\textbf{haus} =& (\forall x)(\forall y)(x = y \vee (\exists X)_x (\exists Y)_y \\ & (\forall z)\neg(z \in X \vee z \in Y).\end{aligned}$$

(ix)

$$\begin{aligned}\textbf{reg} =& (\forall x)(\forall X)_x (\exists Y)_x \\ & (\forall y)(y \in X \vee (\exists W)_y (\forall z)(\neg(z \in W) \vee \neg(z \in Y))).\end{aligned}$$

(x)

$$\textbf{disc} = (\forall x)(\exists X)_x (\forall y)(y \in X \rightarrow y = x).$$

(xi)

$$\textbf{triv} = (\forall x)(\forall X)_x (\forall y)(y \in X).$$

□

Note. It is worth emphasizing that **ext** is not an $\mathcal{L}_t$-formula, but **bas** is an $\mathcal{L}_t$-formula.

Theory (Theory of Topological Abelian Groups). Let $\mathcal{L} = \{+, -, 0\}$, where $+$ is a binary function, $-$ is a unary function and 0 is a constant symbol. The theory of topological abelian groups is:

The theory of abelian groups;

$\mathbf{cont}_+$;

$\mathbf{cont}_-$.

Theory (Theory of Topological Unital Left R-Modules). Let R be a ring with multiplicative identity 1. Let $\mathcal{L} = \{+, -, 0\} \cup \{\mathbf{r} : r \in R\}$, where $+$ is a binary function, 0 is a constant and $\mathbf{r}$ is a unary function symbol for each $r \in R$. The theory of topological left R-modules is:

The theory of topological abelian groups;

The theory of unital left R-modules;

$\forall r \in R,\ \mathbf{cont}_{\mathbf{r}}$.

Example 2.25 ([FZ80], Corollary 3.5, 3.6, Exercise 3.7, p. 8). In [FZ80], Flum and Ziegler show that the class of normal spaces and the class of connected spaces are not axiomatizable by a sentence of $\mathcal{L}_t$. The explanation that the class of compact spaces are not axiomatizable by sentences of $\mathcal{L}_t$ is left as an excercise. I have also included that the class of locally compact spaces is not axiomatizable in $\mathcal{L}_t$. These are all consequences of the Löwenheim-Skolem Theorem and the Compactness Theorem for $\mathcal{L}_t$. For instance,

(i) each countable regular space is normal, however, there exists uncountable regular spaces that are not normal;

(ii) each connected and ordered topological field is isomorphic to the field of real numbers, which is uncountable;

(iii) there exist arbitrarily large finite, discrete, compact spaces, however, an infinite, discrete space is not compact;

(iv) each countable, Hausdorff, locally compact abelian group is discrete, however, there exist uncountable, Hausdorff, locally compact abelian groups that are not discrete. Locally compact abelian groups are topological abelian groups where 0 has a compact neighbourhood. LCA groups will be discussed in detail in Chapter 4.

Another language we wish to consider in Chapter 4 is a language that is useful for the study of topological structures where the topology is determined by the set of neighbourhoods of some fixed point, namely, $\mathcal{L}_m$. $\mathcal{L}_m$ is the language associated with monotone structures and, in particular, topological groups and modules.

Definition 2.26. Given a set A and a non-empty set ν of subsets of A, ν is a *monotone system* if $B \in \nu$ and $B \subseteq C \subseteq A$ imply that $C \in \nu$.

Given a set A and a non-empty set ν of subsets of A, we let

$$\hat{\nu} = \{C : B \subseteq C \subseteq A \text{ for some } B \in \nu\}$$

denote the least monotone system containing ν.

Definition 2.27. $\langle \mathcal{M}, \nu \rangle$ is called a *monotone $\mathcal{L}$-structure* if $\mathcal{M}$ is an $\mathcal{L}$-structure and ν is a monotone system.

Definition 2.28. An L_2 sentence is called *invariant for monotone structures*, if for any $\mathcal{L}$-structure, $\langle \mathcal{M}, \nu \rangle$, and any L_2 formula ϕ we have

$$\langle \mathcal{M}, \nu \rangle \models \phi \Leftrightarrow \langle \mathcal{M}, \hat{\nu} \rangle \models \phi.$$

Similarly to the treatment of $\mathcal{L}_t$, when restricting to monotone structures and L_2-sentences that are invariant for monotone structures, the compactness theorem and the Löwenheim-Skolem Theorem hold.

Definition 2.29. The set of all *$\mathcal{L}_m$-formulas* is the smallest set $\mathcal{W}_m$ containing the atomic $\mathcal{L}_2$-formulas and such that:

(i) if $\phi \in \mathcal{W}_m$ then $\neg\phi \in \mathcal{W}_m$;

(ii) if $\phi, \psi \in \mathcal{W}_m$ then $(\phi \wedge \psi) \in \mathcal{W}_m$;

(iii) if ϕ is in $\mathcal{W}_m$ then $(\exists v_i)\phi \in \mathcal{W}_m$;

(iv) if t is an $\mathcal{L}$-term, ϕ is in $\mathcal{W}_m$ and ϕ is positive in V_i then

$$(\forall V_i)\phi \in \mathcal{W}_m;$$

(v) if t is an $\mathcal{L}$-term, ϕ is in $\mathcal{W}_m$ and ϕ is negative in V_i then

$$(\exists V_i)\phi \in \mathcal{W}_m.$$

Proposition 2.30 ([FZ80], Lemma 8.5, p. 53). *Every L_m-sentence is invariant for monotone structures.*

Theorem 2.31 ([FZ80], Theorem 8.6, p. 53). *Each $\mathcal{L}_2$-sentence that is invariant for monotone structures is equivalent to an $\mathcal{L}_m$-sentence.*

Theorem 2.32 ([FZ80], Theorem 8.7, p. 53). *Let Φ be a set of $\mathcal{L}_m$-sentences. If we are restricted to the class of monotone structures that are models of Φ then there is no logic stronger than $\mathcal{L}_m$ that still satisfies the compactness theorem and the Löwenheim-Skolem Theorem.*

Let $\langle \mathcal{M}, \tau \rangle$ be a topological left R-module and let ν_τ be the monotone system on M consisting of the neighbourhoods of 0 in M. For any $\phi \in \mathcal{L}_t$ there is a $\phi' \in \mathcal{L}_m$ and for any $\psi \in \mathcal{L}_m$ there is a $\phi' \in \mathcal{L}_t$ such that

$$(\mathcal{M}, \tau) \models_t \phi \Leftrightarrow (\mathcal{M}, \nu_\tau) \models_m \phi'$$

and

$$(\mathcal{M}, \tau) \models_t \psi \Leftrightarrow (\mathcal{M}, \nu_\tau) \models_m \psi'.$$

Theorem 2.33 ([FZ80], Corollary 2.9, p. 117). *The $\mathcal{L}_m$-theory of the group of rationals with the Euclidean topology is axiomatized by "torsion-free, $\neq \{0\}$, divisible, Hausdorff and locally pure".*

Example 2.34. So, in fact, $(\mathbb{Q}, Euc)$ is $\mathcal{L}_m$-equivalent to $(\mathbb{R}, Euc)$ and $\mathbb{Q}^{(\aleph_0)}$ with the topology consisting of all $\mathbb{Q}$-subspaces of finite co-dimension. Now, as abelian groups, $\mathbb{R} \cong \mathbb{Q}^{(c)}$. In contrast, $(\mathbb{R}, Euc)$ is not a product of two of its subgroups. Furthermore, topologically there are many topologies on $\mathbb{Q}^{(c)}$ that can be considered.

Example 2.35. Consider any algebraic embedding ε of $\mathbb{Q}$ into $\mathbb{Q}^{(\aleph_0)}$. Let $\mathbb{Q}^{(\aleph_0)}$ be equipped with the topology consisting of subspaces of finite co-dimension, σ. Let $q \in \mathbb{Q}$. There exists K such that $\varepsilon[\mathbb{Q}] + K = \mathbb{Q}^{(\aleph_0)}$. K has co-dimension 1 so it is open. So, $\varepsilon(q) + K$ is open. However, $(\varepsilon(q) + K) \cap \varepsilon[\mathbb{Q}] = \{q\}$. Therefore, $\varepsilon[\mathbb{Q}]$ is discrete. So, every finite dimensional subspace is discrete. However, every infinite dimensional subspace carries the topology consisting of subpaces of finite co-dimension. So, if K is an infinite dimensional subspace, then

$$(K, \sigma \restriction K) \cong_t (\mathbb{Q}^{(\aleph_0)}, \sigma).$$

Proposition 2.36. $(\mathbb{Q}^{(\aleph_0)}, \sigma)$ *can be written as the product of any finite number of copies of itself if* σ *is the topology consisting of subspaces of finite co-dimension.*

Proof. Consider $(\mathbb{Q}^{(\aleph_0)}, \sigma) \times (\mathbb{Q}^{(\aleph_0)}, \sigma)$. A basic open set in the product topology is $H \times K$ where H and K are both subspaces of finite co-dimension. However, $H \times K$ is a subspace of $\mathbb{Q}^{(\aleph_0)} \times \mathbb{Q}^{(\aleph_0)}$ of finite co-dimension. So the product topology is the topology consisting of subspaces of finite co-dimension. Furthermore,

$$\mathbb{Q}^{(\aleph_0)} \times \mathbb{Q}^{(\aleph_0)} \cong \mathbb{Q}^{(\aleph_0)}.$$

$\square$

Therefore, $(\mathbb{Q}^{(\aleph_0)}, \sigma)$ is not direct sum/direct product indecomposable.

Finally, we introduced the concept of a partial homeomorphism which was first introduced by Flum and Ziegler in [FZ80]. This concept is used in order to replicate a standard model theoretic technique, namely the "Back and Forth" argument, in $\mathcal{L}_t$. This technique will be used in Chapter 4.

Definition 2.37. $p = (p^0, p^1, p^2)$ is a *partial homeomorphism* from $\langle \mathcal{M}, \tau \rangle$ to $\langle \mathcal{N}, \sigma \rangle$ if:

(i) p^0 is a partial isomorphism from $\mathcal{M}$ to $\mathcal{N}$, that is, a one-to-one mapping with $dom(p^0) \subseteq M$ and $rg(p^0) \subseteq N$ and for each $R \in \mathcal{L}$, $f \in \mathcal{L}$, $\overline{a}, b \in dom(p^0)$:

(a)
$$\mathcal{M} \models R(\overline{a}) \Leftrightarrow \mathcal{N} \models R(p^0(\overline{a}));$$

(b)
$$\mathcal{M} \models f(\overline{a}) = b \Leftrightarrow \mathcal{N} \models f(p^0(\overline{a})) = p^0(b);$$

(ii) p^1 and p^2 are relations $p^1, p^2 \subseteq \tau \times \sigma$ satisfying:

(a) if $\langle U, V \rangle \in p^1$, $a \in dom(p^0)$ and $a \in U$ then $p^0(a) \in V$;

(b) if $\langle U, V \rangle \in p^2$, $b \in rg(p^0)$, say $p^0(a) = b$, and $b \in V$ then $a \in U$.

Notation. If p and q are partial homeomorphisms from from $\langle \mathcal{M}, \tau \rangle$ to $\langle \mathcal{N}, \sigma \rangle$ we write $p \subseteq q$ if $p^0 \subseteq q^0$, $p^1 \subseteq q^1$ and $p^2 \subseteq q^2$.

Definition 2.38. Let $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ be topological modules. $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ are *partially homeomorphic*, denoted $\langle \mathcal{M}, \tau \rangle \simeq_t^p \langle \mathcal{N}, \sigma \rangle$ if there exists a set of partial homeomorphisms I from $\langle \mathcal{M}, \tau \rangle$ to $\langle \mathcal{N}, \sigma \rangle$ with the following back and forth properties:

forth$_1$: For $p \in I$ and $a \in M$ there is a $q \in I$ with $p \subseteq q$ and $a \in dom(q^0)$.

forth$_2$: For $p \in I$, $a \in dom(p^0)$ and $U \in \tau$ with $a \in U$ there are $q \in I$ and $V \in \sigma$ such that $p \subseteq q$, $p^0(a) \in V$ and Uq^2V.

back$_1$: For $p \in I$ and $b \in N$ there is a $q \in I$ with $p \subseteq q$ and $b \in rg(q^0)$.

back$_2$: For $p \in I$, $b \in rg(p^0)$, say $p^0(a) = b$, and $V \in \sigma$ with $b \in V$ there are $q \in I$ and $U \in \tau$ such that $p \subseteq q$, $a \in U$ and Uq^1V.

Lemma 2.39 ([FZ80], Lemma 4.6, p. 17). *Partially homeomorphic structures are $\mathcal{L}_t$-equivalent.*

Lemma 2.40 ([FZ80], Lemma 4.7, p. 17). *Suppose that $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ are $\aleph_0$-saturated. Then*

$$\langle \mathcal{M}, \tau \rangle \equiv_t \langle \mathcal{N}, \sigma \rangle \Rightarrow \langle \mathcal{M}, \tau \rangle \simeq_t^p \langle \mathcal{N}, \sigma \rangle.$$

Theorem 2.41 (Ehrenfeucht-Fraïssé Theorem, [FZ80], Theorem 4.13, p. 21). *Let $\mathcal{L}$ be finite. For any two topological structures $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$,*

$$\langle \mathcal{M}, \tau \rangle \equiv_t \langle \mathcal{N}, \sigma \rangle \Leftrightarrow \langle \mathcal{M}, \tau \rangle \simeq_t^p \langle \mathcal{N}, \sigma \rangle.$$

Chapter 3

Coproduct for Topological Modules

3.1 Topologies on the direct sum

A significant property of the direct sum is that it can be endowed with a topology that makes it the coproduct in the category of topological modules. In [Hig77], Higgins explicitly described the coproduct topology on the direct sum for a special case. In [Nic02], Nickolas expands upon the results of Higgins and provides a detailed study and analysis of the coproduct topology. In addition, Nickolas provided relationships between the coproduct topology and other natural topologies on the direct sum. Many of the properties of the coproduct topology that are discussed here came from an exercise found in [DPS90].

Notation. We will assume that R is a fixed ring with unity. If (M, τ) and (N, σ) are topological left R-modules then $\mathsf{CHom}_R(M, N)$ is the set of continuous module homomorphisms between (M, τ) and (N, σ). We regard $\mathsf{CHom}_R(M, N)$ as a subspace of $(C(M, N), \tau_{\mathsf{co}})$ where τ_{co} is the compact-open topology. Furthermore, we will also assume that $(M_i, \tau_i)_{i\in I}$ is a family of topological left R-modules, (H, τ) is an arbitrary topological left R-module and $\underline{M} = \bigoplus_{i\in I} M_i$ is the direct sum, and $\overline{M} = \prod_{i\in I} M_i$ is the direct product in the category of modules.

Theorem 3.1. *$\mathsf{CHom}_R(M, N)$ is a topological abelian group.*

Proof. We first show that $\mathsf{CHom}_R(M, N)$ is a subgroup of $\mathsf{Hom}_R(M, N)$. Let $f, g \in \mathsf{CHom}_R(M, N)$. Now, since N is a topological module $- : N \to N$ is continuous. So, $-f \in \mathsf{CHom}_R(M, N)$ since the composition of continuous functions is continuous. Now, $(f, g) : M \times M \to N \times N$ is continuous since f and g are continuous. Furthermore, $f + g = + \circ (f, g)$ is continuous since the composition of continuous functions is continuous. Therefore, $\mathsf{CHom}_R(M, N)$ is a subgroup of $\mathsf{Hom}_R(M, N)$.

Now, we want to show that $\mathsf{CHom}_R(M, N)$ is a topological abelian group. Consider $- : \mathsf{CHom}_R(M, N) \to \mathsf{CHom}_R(M, N)$. Let (K, U) be a basic open set of $\mathsf{CHom}_R(M, N)$. Now,

$$\begin{aligned} & -f \in (K, U) \\ & \Leftrightarrow \ (-f)[K] \subseteq U \\ & \Leftrightarrow \ f[K] \subseteq -U \\ & \Leftrightarrow \ f \in (K, -U). \end{aligned}$$

However, $-U$ is open since $-$ is continuous on N. Therefore, $-$ is continuous on $\mathsf{CHom}_R(M, N)$.

Consider $+ : \mathsf{CHom}_R(M, N) \times \mathsf{CHom}_R(M, N) \to \mathsf{CHom}_R(M, N)$.
Let $f, g \in \mathsf{CHom}_R(M, N)$. Let (K, U) be a basic open neighbourhood of $f + g$. Let

$x_0 \in K$.

$$\begin{aligned} & f+g \in (K,U) \\ \Leftrightarrow \quad & (f+g)[K] \subseteq U \\ \Rightarrow \quad & (f+g)(x_0) \in U \\ \Leftrightarrow \quad & f(x_0)+g(x_0) \in U. \end{aligned}$$

$+$ is continuous with respect to N, since N is a topological module. Therefore, there exists open neighbourhoods of $f(x_0)$ and $g(x_0)$, say W and V respectively, such that $W+V \subseteq U$. So,

$$+[(\{x_0\},W),(\{x_0\},V)] \subseteq (K,U).$$

Therefore, $+$ is continuous. $\square$

Theorem 3.2. *Let S and T be rings with unity. If (M,τ) is a topological R-S bimodule and (N,σ) is a topological R-T bimodule then $\mathsf{CHom}_R(M,N)$ is a topological T-S bimodule.*

Proof. Let $x \in M$ and $f \in \mathsf{CHom}_R(M,N)$. For every $s \in S$ and $t \in T$,

$$fs(x) = f(s(x)) = f(xs) \text{ and } tf(x) = t(f(x)) = (f(x))t.$$

Now, $\mathsf{Hom}_R(M,N)$ is a T-S bimodule. It is clear that $\mathsf{CHom}_R(M,N)$ is a subbimodule of $\mathsf{Hom}_R(M,N)$. We want to show that $\mathsf{CHom}_R(M,N)$ is a topological T-S bimodule.

By Theorem 3.1, $\mathsf{CHom}_R(M,N)$ is a topological abelian group.

Consider $s \in S$. Let (K,U) be a basic open set of $\mathsf{CHom}_R(M,N)$.

$$\begin{aligned} & fs \in (K,U) \\ \Leftrightarrow \quad & f[s[K]] \subseteq U \\ \Leftrightarrow \quad & f[s[K]] \subseteq U \\ \Leftrightarrow \quad & f \in (s[K],U). \end{aligned}$$

Since s is continuous on M, $s[K]$ is compact. So, s is continuous on $\mathsf{CHom}_R(M,N)$.

Consider $t \in T$. Let (K,U) be a basic open set of $\mathsf{CHom}_R(M,N)$.

$$\begin{aligned} & tf \in (K,U) \\ \Leftrightarrow \quad & t[f[K]] \subseteq U \\ \Leftrightarrow \quad & f[K] \subseteq t^{-1}[U] \\ \Leftrightarrow \quad & f \in (K,t^{-1}[U]). \end{aligned}$$

However, $t^{-1}[U]$ is open in N, since t is continuous on N. So, t is continuous on $\mathsf{CHom}_R(M,N)$.

Hence, $\mathsf{CHom}_R(M,N)$ is a topological T-S bimodule. □

Notation. Let (M,σ) and (N,τ) be topological R-modules. Let $\mathsf{CHom}_R(M,-)$ be the category where objects are topological groups of the form $\mathsf{CHom}_R(M,A)$ where (A,τ_1) is any topological left R-module and the morphisms are continuous group homomorphisms of the form

$$\mathsf{CHom}_R(M,\beta) : \mathsf{CHom}_R(M,A) \to \mathsf{CHom}_R(M,B)$$

where $\beta : (A,\tau_1) \to (B,\tau_2)$ is a continuous group homomorphism.

Let $\mathsf{CHom}_R(-,N)$ be the category where objects are topological groups of the form $\mathsf{CHom}_R(A,N)$ where (A,τ_1) is any topological left R-module and the morphisms are continuous group homomorphisms of the form

$$\mathsf{CHom}_R(\alpha,N) : \mathsf{CHom}_R(A,N) \to \mathsf{CHom}_R(B,N)$$

where $\alpha : (B,\tau_2) \to (A,\tau_1)$ is a continuous group homomorphism.

Theorem 3.3. *The set of topologies that make M a topological module forms a complete lattice with respect to $\subseteq$.*

Proof. Let $\mathcal{T} = \{\mathcal{T}_i\}_{i\in I}$ be the set of all topologies that make M a topological module partially ordered by inclusion. Let $A \subseteq \mathcal{T}$. Since arbitrary meets can be expressed in terms of arbitrary joins, it is sufficient to show the existence of arbitrary joins. If $A = \varnothing$, then $\bigvee A$ is the indiscrete topology. If $A \neq \varnothing$ then let τ be the topology generated by $\bigcup A$ as a subbase. We want to show that $-$, r and $+$ are continuous with respect to τ. It suffices to test continuity on the subbase.

(i) It suffices to check that $(-)^{-1}[V]$ is open for every V in the subbase for τ. For such a V, $V \in \mathcal{T}_i$ for some $i \in I$. Since $-$ is continuous with respect to $\mathcal{T}_i$, $(-)^{-1}[V] \in \mathcal{T}_i \subseteq \tau$. Therefore, $-$ is continuous with respect to τ.

(ii) It suffices to check that $(r)^{-1}[V]$ is open for every V in the subbase for τ. For such a V, $V \in \mathcal{T}_i$ for some $i \in I$. Since r is continuous with respect to $\mathcal{T}_i$, $(r)^{-1}[V] \in \mathcal{T}_i \subseteq \tau$. Therefore, r is continuous with respect to τ.

(iii) It suffices to check that $(+)^{-1}[V]$ is open for every V in the subbase for τ. For such a V, $V \in \mathcal{T}_i$ for some $i \in I$. Since $+$ is continuous with respect to $\mathcal{T}_i$, $(+)^{-1}[V]$ is open with respect to the product topology on $(M, \mathcal{T}_i) \times (M, \mathcal{T}_i)$. So, $(+)^{-1}[V]$ is open with respect to the product topology on $(M, \tau) \times (M, \tau)$. Hence, $+$ is continuous with respect to τ.

So, $\tau = \bigvee A$. This is sufficient to show that $\mathcal{T}$ forms a complete lattice with respect to $\subseteq$. □

Notice that the finest topology is the discrete topology and the coarsest topology is the indiscrete topology. It is also worth noting that the meet of a family of module topologies is not simply the intersection of these topologies since this intersection does not form a module topology (even though it does form a topology).

In general, it is not the case that the set of Hausdorff topologies that make M a topological module forms a complete lattice with respect to $\subseteq$ since the indiscrete topology is not Hausdorff. Furthermore, the intersection of two Hausdorff topologies need not be Hausdorff. Consider, $\mathbb{Z}$ with the p-adic topology and $\mathbb{Z}$ with the q-adic topology where $p \neq q$. Both of these topologies are Hausdorff, however, the intersection is the indiscrete topology which is not Hausdorff.

Definition 3.4. The *coproduct topology*, denoted τ_{coprod}, is the supremum of all the left R-module topologies on $\underline{M}$ such that for all $i \in I$, ε_i is continuous.

Proposition 3.5. *τ_{coprod} is finer than the restriction of the box topology, τ_{box}, to $\underline{M}$.*

Proof. Clearly, for each $i \in I$, $\varepsilon_i : (M_i, \tau_i) \to (\underline{M}, \tau_{\mathsf{box}})$ is an embedding. $\square$

Proposition 3.6. *For every topological left R-module (H, τ) and every family $f_i : (M_i, \tau_i) \to (H, \tau)$ of module homomorphisms, the module homomorphism $f : (\underline{M}, \tau_{\mathsf{coprod}}) \to (H, \tau)$ defined by $f = \sum_{i \in I} f_i \circ \pi_i$ is continuous if and only if each f_i is continuous.*

Proof. ($\Rightarrow$) If f is continuous with respect to τ_{coprod} then $f_i = f \circ \varepsilon_i$ is continuous since ε_i is continuous.

($\Leftarrow$) For each $i \in I$ and for each open neighbourhood $V \in \tau$,

$$\varepsilon_i^{-1}(f^{-1}[V]) = f_i^{-1}[V] \in \tau_i,$$

due to the continuity of f_i. Now, $f^{-1}[\tau]$ is coarser than τ_{coprod} since it is a topology on $\underline{M}$ that makes each ε_i continuous. Therefore f is continuous.

$\square$

Proposition 3.7. *Let $(\underline{M}, \tau_1)$ be a topological module such that each ε_i is continuous. If for every left R-module (H, τ) and for every family of continuous module homomorphisms $f_i : (M_i, \tau_i) \to (H, \tau)$ the homomorphism $f = \sum_{i \in I} f_i \circ \pi_i$ is continuous, then $\tau_1 = \tau_{\mathsf{coprod}}$.*

Proof. In particular, for $(H, \tau) = (\underline{M}, \tau_{\mathsf{coprod}})$, and for the family of continuous maps $\varepsilon_i : (M_i, \tau_i) \to (\underline{M}, \tau_{\mathsf{coprod}})$ we have $id_{\underline{M}} = \sum_{i \in I} \varepsilon_i \circ \pi_i$ is continuous. Therefore, $\tau_{\mathsf{coprod}} \subseteq \tau_1$. Furthermore, $\tau_1 \subseteq \tau_{\mathsf{coprod}}$ since τ_{coprod} is the finest topology such that all ε_i are continuous. Therefore, $\tau_1 = \tau_{\mathsf{coprod}}$. $\square$

Corollary 3.8. *If (M_i, τ_i) is a family of topological left R-modules, then $(\underline{M}, \tau_{\mathsf{coprod}})$ is the coproduct in the category of topological left R-modules.*

Theorem 3.9 ([Nic02], Theorem 3.3, p. 413). *If I is countable, then τ_{coprod} coincides with τ_{box}.*

Proof. By Proposition 3.7, it is enough to show that for an arbitrary left R-module (H, τ), if $f_i : (M_i, \tau_i) \to (H, \tau)$ is continuous for each $i \in \aleph_0$ then $\sum_{i \in \aleph_0} f_i \circ \pi_i$ is continuous with respect to the box topology on $\bigoplus\limits_{i \in \aleph_0} M_i$.

Let $U \in \mathcal{N}(H)$ and let

$$U \supseteq U_0 \supseteq \cdots \supseteq U_n \supseteq \cdots$$

be open neighbourhoods of 0 in H such that $U_{n+1} + U_{n+1} \subseteq U_n$ for each $n \in \aleph_0$ and $U_0 + U_0 \subseteq U$. So,

$$\sum_{k=1}^{n} U_k \subseteq U$$

for each $n \in \aleph_0$.

This means that $W = \prod\limits_{i \in \aleph_0} {f_i}^{-1}[U_i] \cap \underline{M}$ is an open neigbourhood of 0 with respect to the box topology. Take $w \in W$. So, $\pi_i(w) = 0$ for all but finitely many $i \in I$, say, $\pi_i(w) \neq 0$ for $i_1, \ldots, i_N$. So,

$$(\sum_{i \in \omega} f_i \circ \pi_i)(w) \in \sum_{j=1}^{N} U_{i_j} \subseteq U.$$

Therefore, $\sum\limits_{i \in \omega} f_i \circ \pi_i$ is continuous with respect to the box topology. Therefore, $\tau_{\mathsf{box}} = \tau_{\mathsf{coprod}}$. ∎

However, the countability assumption in the preceding is essential, as shown by Higgins in [Hig77].

In any topological left R-module, if V is an open neighbourhood of 0, define

$$(1/2)V = \{x : x \in V \text{ and } 2x \in V\}$$

and define $(1/2^n)V$ inductively as $(1/2)((1/2^{n-1})V)$.

For $x \in V$, (x/V) is the smallest $1/2^n$ such that $x \in (1/2^n)(V)$. If $x \notin (1/2^n)(V)$ for all $n \geq 0$ then $(x/V) = 0$.

Theorem 3.10 ([Hig77], Corollary, p. 157). *If I is uncountable, each (M_i, τ_i) is non-discrete, Hausdorff and for every $i \in I$ there is an open neighbourhood U_i of 0 in M_i such that every neighbourhood of 0 contains a set $(1/2^n)U_i$ for some $n \in \aleph_0$ then τ_{coprod} is strictly finer than τ_{box}.*

In particular, $(\mathbb{Q}, Euc)$ is non-discrete, Hausdorff and there is an open neighbourhood U of 0, namely $(-1, 1)$, such that every neighbourhood of 0 contains a set $(1/2^n)U$ for some $n \in \aleph_0$. So, if I is uncountable then the coproduct topology on $\mathbb{Q}^{(I)}$ is strictly finer than the box topology on $\mathbb{Q}^{(I)}$.

Let (M_i, τ_i) be topological left R-modules. Let $V_i \in \mathcal{N}(M_i)$ for each $i \in I$. Define

$$U(\{V_i\}_{i\in I}) = \left\{ m \in \underline{M} : \pi_i(m) \in V_i \text{ for each } i \in I, \text{ and } \sum_{i\in I} (\pi_i(m)/V_i) < 1 \right\}$$

and

$$\mathbb{U} = \left\{ U(\{V_i\}_{i\in I}) : V_i \in \mathcal{N}(M_i) \text{ for each } i \in I \right\}.$$

In [Kap48], Kaplan showed that $\mathbb{U}$ forms a fundamental system of open neighourhoods of 0 for a group topology on $\underline{M}$, called the *asterisk topology* and denoted τ_*.

There is the obvious relationship between the asterisk topology, the box topology and the coproduct topology, namely,

$$\tau_{\mathsf{box}} \subseteq \tau_* \subseteq \tau_{\mathsf{coprod}}.$$

Theorem 3.11 ([Nic02], Theorem 3.11, p. 418). *τ_{box} is equal to τ_* if and only if for all but countably many indices i the following holds: for every open neighbourhood U of M_i, there exists an open neighbourhood V of 0 in M_i such that*

$$V \subseteq \bigcap_{n=1}^{\infty}(1/2^n)U.$$

Theorem 3.11 provides us with a condition for determining when the coproduct topology is strictly finer than the box topology, namely, when the condition stated fails.

Proposition 3.12. *$(\underline{M}, \tau_{\mathsf{coprod}})$ is a Hausdorff space if and only if each (M_i, τ_i) is a Hausdorff space.*

Proof. ($\Rightarrow$) Suppose $\tau_{\mathsf{co-prod}}$ is Hausdorff. Let $i \in I$ and consider $x, y \in M_i$ such that $x \neq y$. If $x \neq y$ then $\varepsilon_i(x) \neq \varepsilon_i(y)$. So, there exists $U_{\varepsilon_i(x)}, U_{\varepsilon_i(y)} \in \tau_{\mathsf{coprod}}$ such that

$$U_{\varepsilon_i(x)} \cap U_{\varepsilon_i(y)} = \varnothing.$$

Clearly, ${\varepsilon_i}^{-1}(U_{\varepsilon_i(x)})$ and ${\varepsilon_i}^{-1}(U_{\varepsilon_i(y)})$ are disjoint, τ_i-open neighbourhoods of x and y.

Therefore, each τ_i is Hausdorff.

($\Leftarrow$) Suppose each τ_i is Hausdorff. Let $x, y \in \underline{M}$ such that $x \neq y$. So, there exists an i such that $\pi_i(x) \neq \pi_i(y)$. Since M_i is Hausdorff there exists $U_{\pi_i(x)}, U_{\pi_i(y)} \in \tau_i$ such that

$$U_{\pi_i(x)} \cap U_{\pi_i(y)} = \varnothing.$$

Since τ_{coprod} is finer than τ_{prod}, each π_i is continuous. Therefore, ${\pi_i}^{-1}[U_{\pi_i(x)}]$ and ${\pi_i}^{-1}[U_{\pi_i(y)}]$ are disjoint open neighbourhoods of x and y. Therefore, τ_{coprod} is Hausdorff.

□

Proposition 3.13. *$(\underline{M}, \tau_{\mathsf{coprod}})$ is discrete if and only if each (M_i, τ_i) is discrete.*

Proof. Obvious. □

Proposition 3.14. *$\underline{M}$ is dense in $(\overline{M}, \tau_{\mathsf{prod}})$.*

Proof. Let U be a non-empty member of the standard basis of $\overline{M}$. So U is of the form $\prod_{i \in I} U_i$ where U_i is open in M_i for each $i \in I$ and $U_i = M_i$ for all but finitely many $i \in I$. Let $\alpha(i) = 0$ for all $i \in I$ where $U_i = M_i$ and let $\alpha(i) = a_i$ for some $a_i \in U_i$ for each $i \notin I$. Therefore, $\alpha \in \underline{M} \cap \prod_{i \in I} U_i$. So, $\underline{M}$ is dense in $(\overline{M}, \tau_{\mathsf{prod}})$. □

The following results are obtained from exercises on pages 66-69 of [DPS90].

Theorem 3.15 (Compare with Theorem 1.100)**.**

$$\alpha : \mathsf{Hom}_R\left(H, \overline{M}\right) \to \prod_{i \in I} \mathsf{Hom}_R\left(H, M_i\right)$$

defined by

$$\alpha(f) = (\pi_i \circ f)_{i \in I}$$

gives a topological isomorphism

$$\alpha' : \mathsf{CHom}_R\left(H, \overline{M}\right) \to \prod_{i \in I} \mathsf{CHom}_R\left(H, M_i\right)$$

when the products are equipped with the product topology.

Proof.

(i) Let $f \in \mathsf{CHom}_R\left(H, \overline{M}\right)$. Then

$$\pi_i \circ f \in \mathsf{CHom}_R\left(H, M_i\right)$$

since $\pi_i \in \mathsf{CHom}_R\left(\overline{M}, M_i\right)$. In Theorem 1.100 it was shown that

$$\alpha : \mathsf{Hom}_R\left(H, \overline{M}\right) \to \prod_{i \in I} \mathsf{Hom}_R\left(H, M_i\right)$$

is a group isomorphism. Therefore α' is one-to-one. Now, let

$$(f_i)_{i \in I} \in \prod_{i \in I} \mathsf{CHom}_R\left(H, M_i\right).$$

Then

$$\prod_{i \in I} f_i \in \mathsf{CHom}_R\left(H, \overline{M}\right),$$

since $(\overline{M}, \tau_{prod})$ is the product topology. Therefore, α' is onto.

$\prod_{i \in I} \mathsf{CHom}_R\left(H, M_i\right)$ is the product of $\{\mathsf{CHom}_R\left(H, M_i\right)\}_{i \in I}$ in $\mathsf{CHom}_R\left(H, -\right)$

with projections maps

$$p_i : \prod_{i \in I} \mathsf{CHom}_R(H, M_i) \to \mathsf{CHom}_R(H, M_i).$$

Clearly,

$$\mathsf{CHom}_R(H, \pi_i) = p_i \circ \alpha'.$$

Since $(\overline{M}, \tau_{\mathsf{prod}})$ is the product, α' is a topological homomorphism.

We must show that α' is a open. Without loss of generality, let (K, U) be a basic open set for $\mathsf{CHom}_R(H, \overline{M})$, that is, K is compact in H and U is a member of the standard basis for $\overline{M}$. So, $U = \prod_{i \in I} U_i$ where U_i is open in M_i and $U_i = M_i$ for all but finitely many $i \in I$. Now, a trivial calculation shows $(f_i)_{i \in I} \in \alpha'[(K, U)]$ if and only if $(f_i)_{i \in I} \in \prod_{i \in I}(K, U_i)$. Now, $(K, U_i) = \mathsf{CHom}_R(H, M_i)$ for all but finitely many $i \in I$ since $U_i = M_i$ for all but finitely many $i \in I$. Therefore $\alpha'[(K, U)]$ is open in $\prod_{i \in I} \mathsf{CHom}_R(H, M_i)$. Therefore, α' is open. □

Proposition 3.16 (Compare with Theorem 1.101).

$$\beta : \mathsf{Hom}_R(\underline{M}, H) \to \prod_{i \in I} \mathsf{Hom}_R(M_i, H)$$

defined by

$$\beta(f) = (f \circ \varepsilon_i)_{i \in I}$$

restricts to a continuous isomorphism

$$\beta' : \mathsf{CHom}_R(\underline{M}, H) \to \prod_{i \in I} \mathsf{CHom}_R(M_i, H)$$

when $\underline{M}$ is equipped with the coproduct topology and $\prod_{i\in I} \mathsf{Hom}_R(M_i, H)$ *is equipped the product topology.*

Proof. Let $f \in \mathsf{CHom}_R(\underline{M}, H)$. Then

$$f \circ \varepsilon_i \in \mathsf{CHom}_R(M_i, H)$$

since $\varepsilon_i \in \mathsf{CHom}_R(M_i, \underline{M})$. Therefore, $\beta'(f) \in \prod_{i\in I} \mathsf{CHom}_R(M_i, H)$. In Theorem 1.101 it was shown that

$$\beta : \mathsf{Hom}_R(\underline{M}, H) \to \prod_{i\in I} \mathsf{Hom}_R(M_i, H)$$

is a group isomorphism. Therefore β' is one-to-one.

Let $(f_i)_{i\in I} \in \prod_{i\in I} \mathsf{CHom}_R(M_i, H)$. By Proposition 3.6

$$f = \sum_{i\in I} f_i \circ \pi_i \in \mathsf{CHom}_R(\underline{M}, H)$$

and clearly $\beta'(f) = (f_i)_{i\in I}$. Therefore, β' is onto.

$\prod_{i\in I} \mathsf{CHom}_R(M_i, H)$ is the product of $\{\mathsf{CHom}_R(M_i, H)\}_{i\in I}$ in $\mathsf{CHom}_R(-, H)$ with projection maps

$$p_i : \prod_{i\in I} \mathsf{CHom}_R(M_i, H) \to \mathsf{CHom}_R(M_i, H).$$

Clearly,

$$\mathsf{CHom}_R(\varepsilon_i, H) = p_i \circ \beta'.$$

Since $\prod_{i\in I} \mathsf{CHom}_R(M_i, H)$ is the product, β' is a continuous homomorphism. $\square$

Note that it is not claimed that β' is a homeomorphism.

Proposition 3.17 (Compare with Theorem 1.102).

$$\gamma : \bigoplus_{i\in I} \mathsf{Hom}_R(M_i, H) \to \mathsf{Hom}_R\left(\overline{M}, H\right)$$

defined by

$$\gamma((f_i)_{i\in I}) = \sum_{i\in I} f_i \circ \pi_i$$

restricts to a continuous monomorphism

$$\gamma' : \bigoplus_{i\in I} \mathsf{CHom}_R(M_i, H) \to \mathsf{CHom}_R\left(\overline{M}, H\right)$$

when $\overline{M}$ is equipped the product topology and $\bigoplus_{i\in I} \mathsf{CHom}_R(M_i, H)$ is equipped with the coproduct topology.

Proof. Let

$$(f_i)_{i\in I} \in \bigoplus_{i\in I} \mathsf{CHom}_R(M_i, H).$$

So, $f_i = 0$ for all but finitely many $i \in I$. Therefore, $\sum_{i\in I} f_i \circ \pi_i$ is well defined. Furthermore, since the composition of continuous functions is continuous and a finite sum of continuous maps is continuous,

$$\sum_{i\in I} f_i \circ \pi_i \in \mathsf{CHom}_R\left(\overline{M}, H\right).$$

In Theorem 1.102, it was shown that γ is a group monomorphism, so γ' is a monomorphism.

$\bigoplus_{i\in I} \mathsf{CHom}_R(M_i, H)$ is the coproduct of $\{\mathsf{CHom}_R(M_i, H)\}_{i\in I}$ in $\mathsf{CHom}_R(-, H)$ with injection maps

$$e_i : \mathsf{CHom}_R(M_i, H) \to \bigoplus_{i\in I} \mathsf{CHom}_R(M_i, H).$$

Clearly,

$$\mathsf{CHom}_R(\pi_i, H) = \gamma' \circ e_i.$$

Since $\bigoplus_{i\in I} \mathsf{CHom}_R(M_i, H)$ is the coproduct, γ' is a continuous homomorphism. □

Definition 3.18. Let H be a left R-module. H has the *no small submodule property* if there exists a neighbourhood of 0 containing no non-trivial submodules of H.

For example, $(\mathbb{Q}, Euc)$ has the no small submodule property and $(\overline{\mathbb{Z}_{(p)}}, \tau_{\mathsf{p-adic}})$ does not have the no small submodule property.

Proposition 3.19. *The monomorphism γ' is an isomorphism whenever H has the no small submodule property.*

Proof. Let $f \in \mathsf{CHom}_R(\overline{M}, H)$ and let U be a neighbourhood of 0 in H such that U contains no nontrivial submodule of H. Since f is continuous, there exists an open neighbourhood V of 0 in $\overline{M}$ such that $f[V] \subseteq U$. Without loss of generality, there is a finite set $F \subseteq I$ such that $V = \prod_{i\in I} V_i$ where V_i is open in M_i for each $i \in I$ and for $i \notin F$, $V_i = M_i$. So,

$$f\left[\prod_{i\in I} V_i\right] \subseteq U.$$

Therefore,

$$\prod_{i\in I\setminus F} f[M_i] \times \prod_{i\in F} \{0\}$$

is a submodule of H and

$$\prod_{i\in I\setminus F} f[M_i] \times \prod_{i\in F} \{0\} \subseteq U.$$

Since U contains no nontrivial submodule of H, this submodule must be $\{0\}$. So, f vanishes on all but finitely many submodules of M_i. So, $f \in \mathsf{Im}\left(\gamma'\right)$ by Theorem 1.102. □

3.2 T-Filters and coproduct of topological modules

In this section, the concept of T-filters introduced by Zelenyuk and Protasov in [ZP90] is used in order to provide a description of the coproduct for topological abelian groups. Many of the early results and notation can be found in Zelenyuk and Protasov's comprehensive book titled *Topologies on Groups Determined by Sequences* [PZ99]. I expand on the work of Zelenyuk and Protasov by considering topological modules; that is, in addition to topological abelian groups, I also consider a continuous scalar multiplication. A description of the coproduct for topological abelian groups was obtained by Higgins in [Hig77], and was expanded upon by Nicholas in [Nic02]. In this section, I obtain an alternative description of the coproduct for topological left R-modules and we obtain some of the previously known results using only the concept of T-filters.

Definition 3.20. A filter $\mathcal{F}$ on a left R-module M is called a *T-filter* if there exists a Hausdorff module topology τ on M in which $\mathcal{F} \xrightarrow{\tau} 0$. Given any T-filter on a left R-module M, we let $M_{\mathcal{F}}$ be the topological left R-module (M, τ) such that τ is the finest module topology in which $\mathcal{F} \xrightarrow{\tau} 0$.

Let (M, τ) be a topological left R-module and $\mathcal{F}_0$ be the collection of all neighbourhoods of 0. Then $\mathcal{F}_0$ is clearly a T-filter and $(M, \tau) = M_{\mathcal{F}_0}$. This means that every module topology on M is determined by a T-filter. Moreover, consider any filter $\mathcal{F}$, such that $\mathcal{F}_0 \subseteq \mathcal{F}$. It is clear that $\mathcal{F}$ is a T-filter and that τ is contained in the topology of the topological module $M_{\mathcal{F}}$.

Notation. Let M be a left R-module. Let $\mathcal{F}$ be a family of non-empty subsets of M. Now, for every $F \in \mathcal{F}$, let $F^* = F \cup (-F) \cup \{0\}$ and for every $\overline{F} = (F_n)_{n\in\omega} \in \mathcal{F}^\omega$, let $\overline{F}^* = (F_n^*)_{n\in\omega}$. For every $\overline{F} = (F_n)_{n\in\omega} \in \mathcal{F}^\omega$, let

(i) $\sum(\overline{F}) = \bigcup\limits_{n\in\omega}(F_0 + \cdots + F_n)$;

(ii) $\sum(\mathcal{F}) = \left\{\sum(\overline{F}^*) : \overline{F} \in \mathcal{F}^\omega\right\}$.

Furthermore, let $\overline{F}' = (F_n')_{n\in\omega} \in \mathcal{F}^\omega$ and let

$$\overline{F} \cap \overline{F}' = (F_0 \cap F_0', F_1 \cap F_1', \ldots) \in \mathcal{F}^\omega;$$

$$\bigcap \overline{F} = (F_0, F_0 \cap F_1, F_0 \cap F_1 \cap F_2, \ldots) \in \mathcal{F}^\omega.$$

Let M be a left R-module and let $\mathcal{F}$ be a filter. Let $\overline{F}, \overline{F'} \in \mathcal{F}^\omega$ be such that $\overline{F} \subseteq \overline{F'}$, that is, $F_i \subseteq F_i'$ for each $i \in \omega$. Then $\sum(\overline{F}) \subseteq \sum(\overline{F'})$ and $\bigcap \overline{F} \subseteq \overline{F}$.

Lemma 3.21 ([PZ99], Lemma 2.1.1). *Let M be a left R-module. For every filter $\mathcal{F}$ on M the following statements hold:*

(i) For every $A, B \in \sum(\mathcal{F})$*, there exists* $C \in \sum(\mathcal{F})$ *such that* $C \subseteq A \cap B$*.*

(ii) $\sum(\mathcal{F})$ *is a filter-base on* M*.*

(iii) $0 \in A$ *for every* $A \in \sum(\mathcal{F})$*.*

(iv) $A = -A$ *for every* $A \in \sum(\mathcal{F})$*.*

(v) For every $C \in \sum(\mathcal{F})$*, there exists* $A, B \in \sum(\mathcal{F})$ *such that* $A + B \subseteq C$*.*

Proof. (i) Let $\overline{A}, \overline{B} \in \mathcal{F}^\omega$ such that $A = \sum(\overline{A}^*)$ and $B = \sum(\overline{B}^*)$. Let $C_n = A_n \cap B_n$ and let $C = \sum(\overline{C}^*)$. Since $\mathcal{F}$ is a filter $C_n \in \mathcal{F}$, so $C \in \sum(\mathcal{F})$ and $C \subseteq A \cap B$. (ii) $\sum(\mathcal{F})$ is non-empty since $\mathcal{F}$ is non-empty. So, by (i), $\sum(\mathcal{F})$ is a filter-base on M. (iii) and (iv) follow directly from the definitions. (v) Let $\overline{C} \in \mathcal{F}^\omega$ such that $C = \sum(\overline{C}^*)$. Set $A = \sum((C^*_{2n})_{n\in\omega})$ and $B = \sum((C^*_{2n+1})_{n\in\omega})$. Then $A, B \in \sum(\mathcal{F})$ and $A + B \subseteq C$. ☐

Lemma 3.22 ([PZ99], Lemma 2.1.2). *Let* (M, τ) *be a topological left* R*-module and* $\mathcal{F}$ *a filter on* M *that converges to zero in* τ*. Then for every neighbourhood* U *of zero there exists* $V \in \sum(\mathcal{F})$ *such that* $V \subseteq U$*.*

Proof. Construct a set $\{V_n\}_{n\in\omega}$ of neighbourhoods of 0 such that $V_i \in \mathcal{F}$, $V_0 + V_0 \subseteq U$, $V_{n+1} + V_{n+1} \subseteq V_n$ and $V_n = V_n{}^*$ for all $n \in \omega$. Put $V = \sum((V_n)_{n\in\omega})$. Then $V \subseteq U$ and $V \in \sum(\mathcal{F})$. ☐

Notation. Let M be a left R-module, let $\mathfrak{A}$ be a collection of non-empty subsets of M and define

$$R^{-1}\mathfrak{A} = \bigcup_{r\in R} \bigcup_{A\in\mathfrak{A}} r^{-1}A.$$

Recall, $r^{-1}A = \{m \in M : rm \in A\}$.

Theorem 3.23 ([PZ99], Theorem 2.1.3). *A filter on a left R-module M is a T-filter if and only if $\bigcap \sum(\mathcal{F}) = \{0\}$. For every T-filter $\mathcal{F}$, $\sum(\mathcal{F})$ forms a base of neighbourhoods of* 0 *for a Hausdorff group topology and $R^{-1}\sum(\mathcal{F})$ forms a base of neighbourhoods at* 0 *for the topological R-module $M_{\mathcal{F}}$.*

Proof. Let $\mathcal{F}$ be a T-filter and τ a Hausdorff module topology such that $\mathcal{F} \xrightarrow{\tau} 0$. Since τ is Hausdorff, by Lemma 3.22 $\bigcap \sum(\mathcal{F}) = \{0\}$.

Now suppose that $\bigcap \sum(\mathcal{F}) = \{0\}$. By Lemma 3.21 and 1.67, $\sum(\mathcal{F})$ is a base of neigbourhoods of 0 for some Hausdorff group topology σ on M.

Now, in order for $R^{-1}\sum \mathcal{F}$ to be a module topology on M, for every $r \in R$, and for every $B \in R^{-1}\sum(\mathcal{F})$, there must exist $A \in R^{-1}\sum(\mathcal{F})$ such that $rA \subseteq B$. So, let $r \in R$ and $B \in R^{-1}\sum(\mathcal{F})$. So, $B = s^{-1}\sum(\overline{B}^*)$ for some $s \in R$ and $\sum(\overline{B}^*) \in \sum(\mathcal{F})$. Let $A = r^{-1}B$. So, $A = (sr)^{-1}\sum(\overline{B}^*)$. Therefore, $A \in R^{-1}\sum(\mathcal{F})$ and $rA \subseteq B$. So, $R^{-1}\sum \mathcal{F}$ forms a base of neighbourhoods at 0 for some Hausdorff module topology σ on M. By Lemma 3.22, σ is a topology on the left R-module $M_{\mathcal{F}}$. □

Now that I have introduced the concept of T-filters for topological left R-modules, I am going to use it to obtain an alternative description of the coproduct topology. This description of the coproduct topology is explicit and it requires less work to obtain than the previous descriptions.

Notation. Let $\{M_i\}_{i \in I}$ be a family of left R-modules and $\underline{M} = \bigoplus_{i \in I} M_i$. Clearly, for every $i \in i$, $\varepsilon_i[\mathcal{N}(M_i)]$ is a filter-base on $\underline{M}$. By Theorem 1.70, there exists a filter on $\underline{M}$ generated $\varepsilon_i[\mathcal{N}(M_i)]$. We will let $\mathfrak{F}_i$ denote the filter on $\underline{M}$ generated by $\varepsilon_i[\mathcal{N}(M_i)]$.

Proposition 3.24. *Let $\{(M_i, \tau_i)\}_{i\in I}$ be a family of topological left R-modules and τ be a topology on $\underline{M}$. $\mathfrak{F}_i \xrightarrow{\tau} 0$ if and only if ε_i is continuous.*

Proof. Suppose ε_i is continuous and let $V \in \mathcal{N}(\underline{M})$. Therefore, $\varepsilon_i^{-1}[V] \in \tau_i$. So,

$$\varepsilon_i[\varepsilon_i^{-1}[V]] \in \mathfrak{F}_i$$

and

$$\varepsilon_i[\varepsilon_i^{-1}[V]] \subseteq V.$$

Therefore, $\mathfrak{F}_i \xrightarrow{\tau} 0$.

Now, suppose $\mathfrak{F}_i \xrightarrow{\tau} 0$ and $V \in \mathcal{N}(\underline{M})$. So, there exists $V' \in \varepsilon_i[\mathcal{N}(M_i)]$ such that $V' \subseteq V$. Therefore, $V' = \varepsilon_i[U]$ for some $U \in \tau_i$ and $\varepsilon_i[U] \subseteq V$. So, ε_i is continuous. □

Theorem 3.25. *Let $\{(M_i, \tau_i)\}_{i\in I}$ be a family of topological left R-modules and let τ be a topology on $\underline{M}$. $\bigcap_{i\in I} \mathfrak{F}_i \xrightarrow{\tau} 0$ if and only if ε_i is continuous for every $i \in I$.*

Proof. $\bigcap_{i\in I} \mathfrak{F}_i$ is a filter on $\underline{M}$ by Theorem 1.70 (i).

Suppose ε_i is continuous for each $i \in I$ and let $V \in \mathcal{N}(\underline{M})$. By Proposition 3.24, $\mathfrak{F}_i \xrightarrow{\tau} 0$ for each $i \in I$. So, for each $i \in I$ there is an $F_i \in \mathfrak{F}_i$ such that $F_i \subseteq V$. Then $F = \bigcup_{j\in I} F_j \in \mathfrak{F}_i$ for each $i \in I$. Therefore, $F \in \bigcap_{i\in I} \mathfrak{F}_i$. Moreover, $F \subseteq V$. Therefore, $\bigcap_{i\in I} \mathfrak{F}_i \xrightarrow{\tau} 0$.

Now, suppose $\bigcap_{i\in I} \mathfrak{F}_i \xrightarrow{\tau} 0$. Clearly, $\mathfrak{F}_i \xrightarrow{\tau} 0$ for each $i \in I$. Therefore, by Proposition 3.24, ε_i is continuous for each $i \in I$. □

Let $\{(M_i, \tau_i)\}_{i\in I}$ be a family of topological left R-modules. $(\underline{M}, \tau_{\mathsf{prod}} \restriction \underline{M})$ is a module topology that makes each ε_i continuous. So $\bigcap_{i\in I}\mathfrak{F}_i$ is a T-filter on $\underline{M}$. Therefore, there exists a finest topology on $\underline{M}$ in the set of topologies under which $\bigcap_{i\in I}\mathfrak{F}_i$ converges to 0.

Corollary 3.26. *The finest topology on $\underline{M}$ under which $\bigcap_{i\in I}\mathfrak{F}_i$ converges to 0 is the coproduct topology.*

Proof. This follows directly from Theorem 3.25. $\square$

Hence, the coproduct topology on the coproduct of a family of topological abelian groups is given by $\sum(\bigcap_{i\in I}\mathfrak{F}_i)$, and the coproduct topology on the coproduct of a family of topological left R-modules is given by $R^{-1}\sum(\bigcap_{i\in I}\mathfrak{F}_i)$. Using T-filters it is possible to obtain an explicit description of a basis for the coproduct topology on the coproduct of a family of topological abelian groups. This description was originally obtained by Chasco and Domínguez through a different approach.

Let $\mathfrak{F} = \bigcap_{i\in I}\mathfrak{F}_i$ where $\mathfrak{F}_i$ is the filter on $\underline{M}$ generated by $\varepsilon_i[\mathcal{N}(M_i)]$, that is,

$$\mathfrak{F}_i = \{F : \text{ there exists } B \in \varepsilon_i[\mathcal{N}(M_i)] \text{ such that } B \subseteq F\}.$$

So, $F \in \mathfrak{F}_i$ if and only if there exists $0 \in U_i \in \tau_i$ such that $F \supseteq \varepsilon_i[U_i]$. Therefore, $F \in \mathfrak{F}$ if and only if for all $i \in I$, there exists $0 \in U_i \in \tau_i$ such that $F \supseteq \varepsilon_i[U_i]$. In particular, if $0 \in U_i \in \tau_i$ for all $i \in I$, then

$$\bigcup_{i\in I}\varepsilon_i[U_i] \in \mathfrak{F}$$

and the collection of all sets of this form is a base for the filter $\mathfrak{F}$.

Now let $(U_i^n)_{i\in I}^{n\in\omega}$ be any family with $0 \in U_i^n \in \tau_i$ for all $i \in I$ and $n \in \omega$. Let

$$F_n = \bigcup_{i\in I} \varepsilon_i[U_i^n].$$

So, $F_n \in \mathfrak{F}$ and

$$\sum(\overline{F}) = \bigcup_{N\in\omega}\sum_{n=1}^{N} F_n = \bigcup_{N\in\omega}\sum_{n=1}^{N}\bigcup_{i\in I}\varepsilon_i[U_i^n].$$

So, by simply relabeling

$$\bigcup_{N\in\omega}\sum_{n=1}^{N}\bigcup_{i\in I}\varepsilon_i[U_i^n] = \bigcup_{N\in\omega}\ \bigcup_{(i_1,\ldots,i_N)\in I^N}\ \sum_{n=1}^{N}\varepsilon_{i_n}[U_{i_n}^n]$$

where $0 \in U_i^n \in \tau_i$.

Therefore, by 3.23, I get the result by Chasco and Domínguez:

Proposition 3.27 ([CD03], Proposition 5). *Let $\{(G_i,\tau_i)\}_{i\in I}$ be a family of topological abelian groups and*

$$((U_i^n))_{n\in\omega})_{i\in I} \in \prod_{i\in I}\mathcal{N}(G_i)^\omega.$$

Define

$$\mathbb{U}(((U_i^n)_{n\in\omega})_{i\in I}) = \bigcup_{N\in\omega}\ \bigcup_{(i_1,\ldots,i_N)\in I^N}\ \sum_{n=1}^{N}\varepsilon_{i_n}(U_{i_n}^n).$$

Then the family

$$\left\{\mathbb{U}(((U_i^n)_{n\in\omega})_{i\in I}) : ((U_i^n)_{n\in\omega})_{i\in I} \in \prod_{i\in I}\mathcal{N}(M_i)^\omega\right\}$$

is a neighbourhood basis at 0 for $(\underline{G}, \tau_{\mathsf{coprod}})$.

It is possible to extend this result to topological R-modules. Let $\mathfrak{F} = \bigcap_{i\in I} R^{-1}\mathfrak{F}_i$.

So, $F \in \mathfrak{F}_i$ if and only if there exists $0 \in U_i \in \tau_i$ such that $F \supseteq \varepsilon_i[U_i]$. Therefore, $F \in \mathfrak{F}$ if and only if for all $i \in I$, there exists $r \in R$ and $0 \in U_i \in \tau_i$ such that $F \supseteq r^{-1}\varepsilon_i[U_i]$. In particular, if $0 \in U_i \in \tau_i$ for all $i \in I$, then

$$\bigcup_{r \in R} \bigcup_{i \in I} r^{-1}\varepsilon_i[U_i] \in \mathfrak{F}$$

and the collection of all sets of this form is a base for the filter $\mathfrak{F}$.

Now let $(U_i^n)_{i \in I}^{n \in \omega}$ be any family with $0 \in U_i^n \in \tau_i$ for all $i \in I$ and $n \in \omega$. Let

$$F_n = \bigcup_{r \in R} \bigcup_{i \in I} r^{-1}\varepsilon_i[U_i^n].$$

So, $F_n \in \mathfrak{F}$ and

$$\sum(\overline{F}) = \bigcup_{N \in \omega} \sum_{n=1}^{N} F_n = \bigcup_{N \in \omega} \sum_{n=1}^{N} \bigcup_{r \in R} \bigcup_{i \in I} r^{-1}\varepsilon_i[U_i^n].$$

So, by simply relabeling

$$\bigcup_{N \in \omega} \sum_{n=1}^{N} \bigcup_{r \in R} \bigcup_{i \in I} r^{-1}\varepsilon_i[U_i^n] = \bigcup_{r \in R} \bigcup_{N \in \omega} \bigcup_{(i_1, \dots, i_N) \in I^N} \sum_{n=1}^{N} r^{-1}\varepsilon_{i_n}[U_{i_n}^n]$$

where $0 \in U_i^n \in \tau_i$.

Therefore, by 3.23, I get the result:

Proposition 3.28. *Let $\{(M_i, \tau_i)\}_{i \in I}$ be a family of topological R-modules and*

$$((U_i^n))_{n \in \omega})_{i \in I} \in \prod_{i \in I} \mathcal{N}(M_i)^{\omega}.$$

Define

$$\mathbb{U}(((U_i^n)_{n\in\omega})_{i\in I}) = \bigcup_{r\in R}\bigcup_{N\in\omega}\bigcup_{(i_1,\ldots,i_N)\in I^N}\sum_{n=1}^{N} r^{-1}\varepsilon_{i_n}(U_{i_n}^n).$$

Then the family

$$\left\{\mathbb{U}(((U_i^n)_{n\in\omega})_{i\in I}) : ((U_i^n)_{n\in\omega})_{i\in I} \in \prod_{i\in I}\mathcal{N}(M_i)^\omega\right\}$$

is a neighbourhood basis at 0 *for* $(\underline{M}, \tau_{\mathsf{coprod}})$.

Chapter 4

Pure-injectivity for Topological Modules

4.1 Topologically Pure Extensions in Locally Compact Abelian Groups

In Loth's paper, *On t-pure and almost pure exact sequences of LCA groups* [Lot06], a definition of pure injectivity was provided for locally compact topological abelian (LCA) groups. In this section, Loth's results from [Lot01] and [Lot06] are given in order to provide a historical background and a point of contrast to our investigation of pure injectivity. Additional information about LCA groups can be found in Armacost's book, *The Structure of Locally Compact Abelian Groups* [Arm81].

Definition 4.1. Let (G, τ) be a topological abelian group. (G, τ) is *locally compact* if 0 has a compact neighborhood.

Let $\mathfrak{L}$ denote the class of Hausdorff locally compact abelian (LCA) groups with continuous homomorphisms.

Example 4.2. (i) The groups $\mathbb{Z}_n$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_{p^\infty}$ with the discrete topology. In fact, all countable LCA groups are discrete [see [Arm81], p. 7].

(ii) $\overline{\mathbb{Z}_p}$ with the p-adic topology is a compact Hausdorff group and hence a LCA group.

(iii) The additive group $\mathbb{R}$ with the Euclidean topology.

(iv) The circle group $\mathbb{T}$ is the multiplicative group of all complex numbers z such that $|z| = 1$, with the usual Euclidean topology. $\mathbb{T}$ may also be realized as the additive group $\mathbb{R}/\mathbb{Z}$.

From this example, it is already clear that we will require a broader context for a generalization of model theoretic concepts. In model theory, the Löwenheim-Skolem Theorem states that every countable theory which is satisfiable, is satisfiable in a countable structure. Since all countable LCA groups are discrete and discreteness is expressible in $\mathcal{L}_t$, we are restricted to discrete infinite models. However, from a topological perspective, we do not want to be restricted only to the discrete topology since this topology does not provide us with any real topological insight. Moreover, it was shown in Chapter 2 that local compactness is not axiomatizable in $\mathcal{L}_t$ and, hence, LCA groups are not axiomatizable in $\mathcal{L}_t$.

Definition 4.3. Let (X, τ) and (Y, σ) be topological spaces. A continuous map $f : X \to Y$ is *proper* if it is open onto its image.

Note. Let (G, τ) and (H, σ) be a topological groups. If H is a subspace of G then the inclusion map is proper. If G and H are LCA and $f : G \to H$ is proper then the image of f must be closed in H since H is locally compact.

Proposition 4.4. *Let (X, τ_1), (Y, τ_2) and (Z, τ_3) be topological spaces.*

(i) Let $f : X \to Y$ be proper. Then for all $W \subseteq Y$, if $f^{-1}[W]$ is open in X, then there exists a V open in Y such that $W \cap f[X] = V \cap f[X]$.

(ii) Let $f : X \to Y$ be proper. If f is one-to-one and onto then f is a homeomorphism.

(iii) If $f : X \to Y$ is one-to-one and continuous, then it is proper if and only if it is an embedding.

(iv) Let $f : X \to Y$ and $g : Y \to Z$ be proper. If f is surjective or g is injective then $g \circ f$ is proper.

Proof. (i) Let $f : X \to Y$ be proper and let $W \subseteq Y$ such that $f^{-1}[W]$ is open in X. Since f is proper, there exists V in Y such that $f[f^{-1}[W]] = V \cap f[X]$. Also, $W \cap f[X] = f[f^{-1}[W]]$. Therefore, $W \cap f[X] = V \cap f[X]$. (ii), (iii), (iv) follow directly from the definitions. □

Note. It is important to note that if f and g are proper $g \circ f$ may not be proper. Consider $\mathbb{Z}$ with the discrete topology, $\mathbb{R}$ with the Euclidean topology and $\mathbb{T}$ as a subspace of $\mathbb{C}$ with the usual topology. Now, let $f : \mathbb{Z} \to \mathbb{R}$ be defined by $f(x) = \sqrt{2}x$ and $g : \mathbb{R} \to \mathbb{T}$ be defined by $g(x) = e^{ix} = \cos x + i \sin x$. Clearly, f and g are proper. Now, $g \circ f[\mathbb{Z}]$ is dense in $\mathbb{T}$. Since $\mathbb{T}$ is not discrete, $g \circ f$ cannot be open onto its image. So, $g \circ f$ is not proper.

Definition 4.5. An exact sequence

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} G_n$$

in $\mathfrak{L}$ is *proper exact* if each morphism α_i is proper.

Definition 4.6. A proper short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

in $\mathfrak{L}$ is called an *extension* of C by A.

Note. This is often referred to as an extension of A by C, however, I am following Loth's convention in [Lot06].

Definition 4.7. A proper exact sequence $0 \to A \xrightarrow{\alpha} B \to C \to 0$ in $\mathfrak{L}$ is *t-pure* if $\alpha[A]$ is a topologically pure subgroup of B, that is, if $cl_B(n\alpha[A]) = cl_B(nB) \cap \alpha[A]$ for every positive integer n.

In particular, by setting $n = 1$, we see that $\alpha[A]$ is closed in B. Also, this definition of topological purity has some consequences with respect to topological positive primitive formulas which will be introduced in Section 4.2.

Definition 4.8. A proper exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{L}$ is *topologically pure* if

$$0 \to cl_A(nA) \to cl_B(nB) \to cl_C(nC) \to 0$$

is proper exact for all positive integers n.

Since this sequence is proper, the image of A is locally compact and is therefore closed in B. Therefore, it is evident that a topologically pure exact sequence is t-pure. In addition, Loth provides an example of a t-pure exact sequence that is not topologically pure, namely, Example 3.5 on page 197 in [Lot01].

Definition 4.9. Let $G \in \mathfrak{L}$. A homomorphism $f : G \to \mathbb{T}$ is called a *character* of G. The *dual group* of G, denoted $\hat{G}$, is the collection of all continuous characters of G, namely, $\hat{G} = \mathsf{CHom}\,(G, \mathbb{T})$. Furthermore, if $G_1, G_2 \in \mathfrak{L}$ and $f \in \mathsf{CHom}\,(G_1, G_2)$, we define $f^* : \hat{G}_2 \to \hat{G}_1$ by $f^*(f_2) = f_2 \circ f$ for all $f_2 \in \hat{G}_2$.

Using standard techniques it can be shown that f^* is a continuous homomorphism.

Proposition 4.10 (See [Arm81], p. 8). *Let $G \in \mathfrak{L}$. If G is compact then $\hat{G}$ is discrete and if G is discrete then $\hat{G}$ is compact.*

The next theorem allows us to identify G with $\hat{\hat{G}}$ and f with $(f^*)^*$.

Theorem 4.11 (Pontryagin Duality Theorem). *For any $G \in \mathfrak{L}$ and $x \in G$ define $\phi_x(f) = f(x)$ for all $f \in \hat{G}$. Then $\phi_x \in \mathsf{CHom}_R\left(\hat{G}, \mathbb{T}\right) = \hat{\hat{G}}$. The map $\Phi : G \to \hat{\hat{G}}$ defined by $\Phi(x) = \phi_x$ for all $x \in G$ is a topological isomorphism from G onto $\hat{\hat{G}}$.*

Consider $\mathbb{Z}$ with the discrete topology, $\mathbb{R}$ with the discrete topology and $\mathbb{T}$ as a subspace of $\mathbb{C}$ with the usual topology. Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by the inclusion map, $f(x) = x$ and $g : \mathbb{R} \to \mathbb{T}$ be defined by $g(x) = e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x$. It is clear that f and g are continuous homomorphisms and that the sequence $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{T} \to 0$ is exact. From an algebraic perspective, the fact that this sequence is exact seems counter-intuitive. That is, since $\mathbb{Z}$ and $\mathbb{R}$ both have the discrete topology, $\mathbb{T}$ should also have the discrete topology. However, the category LCA is not an abelian category, so the topology on $\mathbb{T}$ is not uniquely determined by exactness. Now, $\hat{\mathbb{R}}$ is compact by 4.10 so the image of g^* cannot be closed in $\hat{\mathbb{R}}$ since it is countably infinite. So, the image of g^* cannot equal the kernel of f^*. Therefore, the dual sequence $0 \to \hat{\mathbb{T}} \xrightarrow{g^*} \hat{\mathbb{R}} \xrightarrow{f^*} \hat{\mathbb{Z}} \to 0$ cannot be exact. This fact and the next theorem provides us with motivation for using proper maps.

Proposition 4.12 (See [Arm81], Proposition 9.14, p.122). *Let*

$$0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 0$$

be a proper exact sequence in $\mathfrak{L}$. The the dual sequence

$$0 \to \hat{G}_3 \xrightarrow{g^*} \hat{G}_2 \xrightarrow{f^*} \hat{G}_1 \to 0$$

is also a proper exact sequence in $\mathfrak{L}$.

Definition 4.13. An element $x \in G$ is said to be *compact* if x lies in some compact subgroup of G. We write $b(G)$ for the set of all compact elements of G.

Proposition 4.14. *Let G be a LCA group.*

(i) The identity subgroup is both pure and topologically pure.

(ii) $b(G)$ is both pure and topologically pure.

Proof. (i) is trivial. (ii) $b(G)$ is pure since $G/b(G)$ is torsion-free. That is, $nb(G) = nG \cap b(G)$ for all $n \in \omega$. Now, $cl_G(nb(G)) = nb(G)$. So, the result follows. $\square$

Example 4.15 (See [Lot01], Example 2.4). An exact sequence that is pure need not be topologically pure and an exact sequence that is topologically pure need not be pure. Let p be prime and n a positive integer and let H be a densely divisible LCA group (that is, it possesses a dense divisible subgroup), such that $H/p^nH \neq 0$. It is non-trivial to find such a group H, but such a group exists and was first given in [Kha95]. Furthermore, there is a non-splitting extension

$$0 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{Z}(p^n) \longrightarrow 0$$

in $\mathfrak{L}$. This extension is topologically pure but is not pure. In addition, the dual sequence

$$0 \longrightarrow \mathbb{Z}(p^n) \longrightarrow \hat{G} \longrightarrow \hat{H} \longrightarrow 0$$

is pure since $\hat{H}$ is torsion-free, but not topologically pure.

Definition 4.16. Let G be a LCA group. G is *t-pure injective* in $\mathfrak{L}$ if for every t-pure exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$ and continuous homomorphism $f : A \longrightarrow G$ there is a continuous homomorphism $\overline{f} : B \longrightarrow G$ such that $\overline{f} \circ \alpha = f$.

Definition 4.17. Let G be a LCA group. G is a *topological torsion group* if

$$G = \left\{ x \in G : \lim_{n \to \infty} n!x = 0 \right\}.$$

If $G \in \mathfrak{L}$ is a torsion group algebraically then each element has finite order, so for each $x \in G$ and for a sufficiently large n, $n!x = 0$. So, $\lim_{n\to\infty} n!x = 0$ for each $x \in G$. Now, in order to capture this concept in the topological setting, we only require a weaker condition, namely, that a group is torsion in the limit.

Theorem 4.18 (Theorem 2.7, [Lot06]). *G is t-pure injective in $\mathfrak{L}$ implies $G \cong \mathbb{R}^n \oplus \mathbb{T}^m \oplus G'$ where n is a non-negative integer and m is cardinal and G' is a topological torsion group.*

Definition 4.19. Let $G \in \mathfrak{L}$. A closed subgroup H of G is *t-pure* in G if whenever K is a closed subgroup of G such that $K \supseteq H$ and K/H is compactly generated, then H splits from K.

Theorem 4.20 (Khan, See [Arm81], p. 99). *Let $G \in \mathfrak{L}$ and let H be a closed subgroup of G.*

(i) If H is a topological direct summand of G then H is t-pure in G.

(ii) If H is t-pure in G then H is pure in G.

It is worth noting that (i) of Theorem 4.20 is a characteristic of ordinary algebraic purity.

In [Lot06] on page 800, Loth states that a t-pure exact sequence need not be pure. This result contradicts Khan's original 1973 result given in Theorem 4.20. This contradiction occurs since Khan and Loth are unfortunately using different definitions of "t-pure". Loth continues to conduct research on t-pure sequences and on LCA groups in general. Given the complexity of this subject matter, one can hope that a survey paper will be written in the near future clearly connecting the various definitions given for topological sequences and purity to their analogous algebraic counterparts.

4.2 Topological Positive Primitive Formulas

In this section, we present Garavaglia's definition for topological positive primitive formulas (tppfs) in the language L_t first introduced in [Gar76]. In addition, some properties of tppfs are presented.

Notation. We will assume that R is a fixed ring with unity and that $\mathcal{L}$ is a fixed language of abstract left R-modules. The theory of topological left R-modules, T_{mod}, is the set of all $\mathcal{L}_t$ sentences such that for each weak structure $\langle \mathcal{M}, \sigma \rangle$ of $\mathcal{L}$, $\langle \mathcal{M}, \sigma \rangle \models_t T_{mod}$ if and only if $\langle \mathcal{M}, \widetilde{\sigma} \rangle$ is a topological module. Since every topological abelian group is homogeneous, we may restrict set quantifiers to neighbourhoods of 0, namely, $(\exists X)_0$ and $(\forall X)_0$. For topological modules, an atomic formula is, without loss of generality, either of the form $t = 0$ or $t \in V$, where t is some term in the language of modules.

If $\overline{A} = (A_1, \ldots, A_m)$ where $A_i \in \tau$, we let $\phi[M, \overline{A}]$ denote the set of $\overline{a} \in M^n$ such that $\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}]$. That is,

$$\phi[M, \overline{A}] = \left\{ \overline{a} \in M^n : \langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}] \right\}.$$

If $k \leq n$ and $\overline{b} \in M^{n-k}$, then we let

$$\phi[M, \overline{b}, \overline{A}] = \left\{ \overline{a} \in M^k : \langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{b}, \overline{A}] \right\}.$$

Furthermore, if Φ is a set of formulas, then we let

$$\Phi[M, \overline{A}] = \bigcap \left\{ \phi[M, \overline{A}] : \phi \in \Phi \right\}.$$

Definition 4.21. A *topological positive primitive formula*, tppf, is a formula of $\mathcal{L}_t$ of the form $Q_1 \cdots Q_n\phi$ where ϕ is a conjunction of atomic formulas of $\mathcal{L}_t$ and each Q_i is of the form $(\exists v)$ or $(\forall V)_0$ for some variables v and V.

Notation. Suppose that $\langle \mathcal{M}, \tau\rangle$ and $\langle \mathcal{N}, \sigma\rangle$ are topological modules such that $f : M \to N$ is a continuous homomorphism. For notational convenience, if $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and $\overline{A} = (A_1, \ldots, A_m)$ where each $A_i \in \sigma$, we let

$$f(\overline{a}) = (f(a_1), \ldots, f(a_n))$$

and

$$f^{-1}[\overline{A}] = (f^{-1}[A_1], \ldots, f^{-1}[A_m]).$$

Observe that since f is a continuous homomorphism each $f^{-1}[A_i] \in \tau$. Similarly, if $M \subseteq N$ then we let $\overline{A} \cap M = (A_1 \cap M, \ldots, A_m \cap M)$. Observe that if $\tau = \sigma \restriction M$ then each $A_i \cap M \in \tau$.

Note. Any free set variables of a tppf occur only positively.

In [Kuc86], Kucera expanded on the results of Garavaglia to show the following:

Corollary 4.22 ([Kuc86], Corollary 3.7). *Every formula of $\mathcal{L}_t$ with only individual variables free is equivalent in topological R-modules to a Boolean combination of tppfs with only individual variables free.*

This result is powerful since it gives us a model theory of topological modules which is analogous to the standard model of modules. However, this result only holds for formulas of $\mathcal{L}_t$ with only individual variables free, that is, it does not give any indication to as to what happens to formulas with free set variables.

Theorem 4.23 ([Kuc86], Theorem 3.13). *Let T be a complete $\mathcal{L}_t$-theory of Hausdorff topological modules with a topologically compact model. Then every tppf with only individual variables free is equivalent in T to a tppf with no set quantifiers, that is, to a pp-formula.*

This is once again a powerful result since an immediate consequence is that if $\langle \mathcal{M}, \tau \rangle$ is compact, it has the same definable sets as $\mathcal{M}$. However, once again, this result only holds for formulas of $\mathcal{L}_t$ with only individual variables free, that is, it does not give any indication to as to what happens to formulas with free set variables.

Lemma 4.24. *Suppose that $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ are topological modules such that $f : M \to N$ is a continuous homomorphism. Then for all tppf $\phi(\overline{v}, \overline{V})$, for all $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and for all $\overline{A} = (A_1, \ldots, A_m)$ where each $A_i \in \sigma$,*

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, f^{-1}[\overline{A}]] \Rightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi[f(\overline{a}), \overline{A}].$$

Proof. This is proved by induction on the complexity of tppfs. Let $\phi(\overline{v}, \overline{V})$ be a tppf. Let $\overline{a} = (a_1, \ldots, a_n) \in M$ and let $\overline{A} = (A_1, \ldots, A_n)$ where each $A_i \in \sigma$. The cases "$t = 0$", conjunction and existential quantification follow as they would in ordinary modules. There are only two cases involving set variables to consider.

(a) Suppose $\phi(\overline{v}, \overline{V})$ is $t \in V_i$. Then

$$\begin{aligned}
&\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, f^{-1}[\overline{A}]] \\
&\Rightarrow\ t^{\mathcal{M}}(\overline{a}) \in f^{-1}[A_i] \\
&\Rightarrow\ f(t^{\mathcal{M}}(\overline{a})) \in A_i \\
&\Rightarrow\ t^{\mathcal{N}}(f(\overline{a})) \in A_i \\
&\Rightarrow\ \langle \mathcal{N}, \sigma \rangle \models_t \phi[f(\overline{a}), \overline{A}].
\end{aligned}$$

(b) Suppose ϕ is $(\forall W)_0 \psi(\overline{v}, \overline{V}, W)$. Then

$$\begin{aligned}
&\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, f^{-1}[\overline{A}]] \\
&\Rightarrow\ \langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{a}, f^{-1}[\overline{A}], C] \\
&\quad\ \text{for all } C \text{ with } 0 \in C \in \tau \\
&\Rightarrow\ \langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{a}, f^{-1}[\overline{A}], f^{-1}[B]] \\
&\quad\ \text{for all } B \text{ with } 0 \in B \in \sigma, \\
&\quad\ (\text{since } f \text{ is continuous,} \\
&\quad\ f^{-1}[B] \text{ is open in } \tau) \\
&\Rightarrow\ \langle \mathcal{N}, \sigma \rangle \models_t \phi[f(\overline{a}), \overline{A}, B] \\
&\quad\ \text{for all } B \text{ with } 0 \in B \in \sigma \\
&\quad\ (\text{by the induction hypothesis}) \\
&\Rightarrow\ \langle \mathcal{N}, \sigma \rangle \models_t \phi[f(\overline{a}), \overline{A}].
\end{aligned}$$

Therefore, the result follows by induction on the complexity of tppfs. □

Corollary 4.25. *If $\langle \mathcal{M}, \tau \rangle$ is a subspace of $\langle \mathcal{N}, \sigma \rangle$ then for all tppf $\phi(\overline{v}, \overline{V})$, all $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and all $\overline{A} = (A_1, \ldots, A_m)$ where each $A_i \in \tau$,*

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A} \cap M] \Rightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi[\overline{a}, \overline{A}].$$

Proof. This follows directly from Lemma 4.24. □

Lemma 4.26. *Let $\langle \mathcal{M}, \tau \rangle$ be a topological module. Then for all tppf $\phi(\overline{v}, \overline{V})$, all $\overline{a} = (a_1, \ldots, a_n) \in M^n$, all $\overline{A} = (A_1, \ldots, A_m)$ where each $A_i \in \tau$ and all $\overline{B} = (B_1, \ldots B_m)$ where $B_i \in \tau$ and $A_i \subseteq B_i$,*

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}] \Rightarrow \langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{B}].$$

Proof. The cases "$t = 0$", conjunction and existential quantification follow as they would in ordinary modules. There are only two cases involving set variables to consider.

(a) Suppose $\phi(\overline{v}, \overline{V})$ is $t \in V_i$. Then

$$\begin{aligned} &\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}] \\ &\Rightarrow t^{\mathcal{M}}(\overline{a}) \in A_i \\ &\Rightarrow t^{\mathcal{M}}(\overline{a}) \in B_i \\ &\Rightarrow \langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{B}]. \end{aligned}$$

(b) Suppose ϕ is $(\forall W)_0\psi(\overline{v},\overline{V},W)$. Then

$$\begin{aligned}
&\langle\mathcal{M},\tau\rangle \models_t \phi[\overline{a},\overline{A}] \\
&\Rightarrow \langle\mathcal{M},\tau\rangle \models_t \psi[\overline{a},\overline{A},A] \\
&\quad \text{for all } A\in\tau \\
&\Rightarrow \langle\mathcal{M},\tau\rangle \models_t \psi[f(\overline{a}),\overline{B},A] \\
&\quad \text{for all } A\in\tau \\
&\quad \text{(by the induction hypothesis)} \\
&\Rightarrow \langle\mathcal{M},\tau\rangle \models_t \phi[\overline{a},\overline{B}].
\end{aligned}$$

Therefore, the result follows by induction on the complexity of tppfs. □

For the purposes of the following argument (Corollary 4.27), if $\phi(\overline{v},\overline{V})$ is a tppf define $\phi^-(\overline{u})$ as follows: since ϕ is a tppf, it is the quantification of a conjunction of atomic formulas. From this conjunction, we delete each component of the form $t(\overline{v})\in V_i$, where V_i is a free set variable.

Corollary 4.27. *Let $\phi(\overline{v},\overline{V})$ be a tppf. Then ϕ is satisfiable in $\langle\mathcal{M},\tau\rangle$ if and only if*

$$\langle\mathcal{M},\tau\rangle \models_t \phi[\overline{a},\overline{M}]$$

for some $\overline{a}\in M^n$. Hence, ϕ is satisfiable in $\langle\mathcal{M},\tau\rangle$ if and only if the formula ϕ^- is satisfiable in $\langle\mathcal{M},\tau\rangle$.

Proof. Suppose that $\phi(\overline{v},\overline{V})$ is satisfiable in $\langle\mathcal{M},\tau\rangle$. This implies that there exists

$\overline{a} = (a_1, \dots, a_n) \in M^n$ and $\overline{A} = (A_1, \dots, A_m)$ where $A_i \in \tau$ such that

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}].$$

So,

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{M}]$$

since $A_i \subseteq M$. Moreover, $t(\overline{v}) \in M$ is vacuously true. The converse is vacuous.

Now, suppose that ϕ^- is satisfiable in $\langle \mathcal{M}, \tau \rangle$. This implies that there exists $\overline{a} = (a_1, \dots, a_n) \in M^n$ such that

$$\langle \mathcal{M}, \tau \rangle \models_t \phi^-[\overline{a}].$$

Since $t(\overline{v}) \in M$ is vacuously true, it follows that

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{M}].$$

Therefore, ϕ is satisfiable in $\langle \mathcal{M}, \tau \rangle$. $\square$

In terms of satisfiability, if ϕ is a tppf, without loss of generality it is possible to consider ϕ^-. Furthermore, let $\phi(\overline{v})$ be a a tppf with only individual variables free. Let $\phi_0(\overline{v}, \overline{w}, \overline{V})$ be obtained from ϕ by removing all of the outermost quantifiers. Let $\underline{\phi}(\overline{v})$ be $(\exists \overline{v})(\forall \overline{Y})_0 \phi_0$ and let $\overline{\phi}(\overline{v})$ be $(\forall \overline{Y})_0 (\exists \overline{y}) \phi_0$. Clearly,

$$\models_t (\forall x)[(\underline{\phi} \to \phi) \wedge (\phi \to \overline{\phi})].$$

Lemma 4.28 ([Gar78], Lemma 1, p. 28). *Let $\langle \mathcal{M}, \tau \rangle$ be a topological module and* *suppose that τ has a basis at 0 consisting of subgroups of $\mathcal{M}$. Let $A_1, \ldots, A_m \subseteq \tau$ be open subgroups of $\mathcal{M}$ and let $\phi(v_1, \ldots, v_n, V_1, \ldots, V_m)$ be a tppf. Then*

(i) $\langle \mathcal{M}, \tau \rangle \models \phi[\overline{0}, \overline{A}]$;

(ii) $\langle \mathcal{M}, \tau \rangle \models \phi[\overline{a}, \overline{A}]$ *and* $\mathcal{M} \models \phi[\overline{b}, \overline{A}] \Rightarrow \langle \mathcal{M}, \tau \rangle \models \phi[\overline{a} - \overline{b}, \overline{A}]$;

(iii) $\phi[M, A_1, \ldots, A_m]$ *is a subgroup of* M^n.

Proof. Let τ_0 be a basis at 0 for τ consisting of subgroups of M. Suppose that $\phi(v_1, \ldots, v_n, V_1, \ldots, V_m)$ is a tppf, $\overline{a}, \overline{b} \in M^n$ and that $A_1 \ldots A_m \subseteq \tau$ are open subgroups of M. For (i) and (ii) we proceed by induction on the complexity of tppfs. The cases "$t = 0$", conjunction and existential quantification follow as they would in ordinary modules. There are only two cases involving set variables to consider.

(i) (a) Suppose ϕ is $t \in V_i$. $\phi[0]$ is equivalent to $(\exists \overline{w})(\mathbf{R}\overline{0} + \mathbf{S}\overline{w}) \in V_i$ or $(\exists \overline{w})(\mathbf{S}\overline{w}) \in V_i$. Now, $\overline{0} \in M^n$ and $0 = \mathbf{S}\overline{0} \in V_i$ since V_i is a subgroup of M. So, $\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{0}, \overline{A}]$.

(b) Suppose ϕ is $(\forall W)_0 \psi(\overline{v}, \overline{V}, W)$. By the induction hypothesis,

$$\langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{0}, \overline{A}, B]$$

for all $B \in \tau_0$. Since τ_0 is a basis at 0 for τ we have

$$\langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{0}, \overline{A}, C]$$

for all $C \in \tau$ with $0 \in C$. So, $\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{0}, \overline{A}]$.

Therefore, the result follows by induction on the complexity of tppfs.

(ii) (a) Suppose ϕ is $t \in V_i$. Now, if $t^{\mathcal{M}}(\overline{a}) \in A_i$ and $t^{\mathcal{M}}(\overline{b}) \in A_i$ then

$$t^{\mathcal{M}}(\overline{a} - \overline{b}) = t^{\mathcal{M}}(\overline{a}) - t^{\mathcal{M}}(\overline{b}) \in A_i$$

since A_i is a subgroup of M. So, $\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a} - \overline{b}, \overline{A}]$.

(b) Suppose ϕ is $(\forall W)_0 \psi(\overline{v}, \overline{V}, W)$, $\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, \overline{A}]$ and $\mathcal{M} \models_t \phi[\overline{b}, \overline{A}]$. So, for every $B \in \tau$, $\langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{a}, \overline{A}, B]$ and $\langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{b}, \overline{A}, B]$. By the induction hypothesis, $\langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{a} - \overline{b}, \overline{A}, C]$ for every $C \in \tau_0$. Since τ_0 is a basis at 0 for τ we have $\langle \mathcal{M}, \tau \rangle \models_t \psi[\overline{a} - \overline{b}, \overline{A}, B]$ for all $0 \in B \in \tau$. Therefore, $\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a} - \overline{b}, \overline{A}]$.

So, the result follows by induction on the complexity of tppfs.

(iii) This follows directly from (i) and (ii).

□

Corollary 4.29. *Let $\phi(\overline{v})$ be a tppf with no set variables free. Then in any topological module $\langle M, \tau \rangle$, $\phi[M]$ is a subgroup of M.*

Proof. This follows directly from Lemma 4.28, Theorem 2.11 and the fact that every module has an $\aleph_1$-saturated elementary extension. □

Although Corollary 4.29 provides a powerful algebraic result, it does not necessarily provide us with any topological insight.

(i) $t(\overline{v}) = 0$ defines a closed subgroup;

(ii) $t(\overline{v}) \in V$ defines an open subgroup and therefore defines a closed subgroup;

(iii) $(\forall V)_0\phi(\overline{v}, V)$ defines an arbitrary intersection of subgroups, and the arbitrary intersection of closed subgroups is a closed subgroup, but;

(iv) $(\exists v)\phi(v, \overline{V})$ defines a projection of a subgroup, and projection maps are not closed.

Therefore, it is not possible to draw any conclusions about the topological properties of the definable subgroups of M.

4.3 Definitions of Embedding for Topological Modules

Different definitions of elementary embeddings have been proposed to relativize $\prec$ to the language L_t. In 1980, in [FZ80], Flum and Ziegler introduced a concept of topological substructure and extension. Furthermore, they characterized $\mathcal{L}_t$-sentences which were preserved under these extensions and substructures. Kucera [Kuc86] introduced a definition for the relation $\prec_t$ in 1986. Independently, Majewski [Maj87] provided a different definition for the relation $\prec_t$. In this section, we will consider both of these definitions in addition to introducing some alternative definitions. In adition, a concept of pure embedding will be introduced.

First we will consider the concept of an embedding in the two-sorted sense. Suppose that $\langle\mathcal{M}, \tau\rangle$ and $\langle\mathcal{N}, \sigma\rangle$ are topological modules and $f : \langle\mathcal{M}, \tau\rangle \xrightarrow{\prec_2} \langle\mathcal{N}, \sigma\rangle$. Let $m \in M$ and $U \in \tau$. Then $m \in U \Leftrightarrow f(m) \in f(U)$. Now, if $M \subseteq N$ then

$$m \in U \Leftrightarrow m \in f(U).$$

Thus, if f is the inclusion map (or if $M \subseteq N$), then $U = f(U) \cap M$. Therefore, $\tau \subseteq \sigma \restriction M$. That is, τ is coarser than the subspace topology induced by σ. In fact, it need not be the subspace topology. For instance, suppose that $\langle \mathcal{M}, \tau \rangle \prec_2 \langle \mathcal{N}, \sigma \rangle$, M is countable and σ is Hausdorff and not discrete. Then τ will not be discrete. However, Corollary 2.10 states that if σ is a Hausdorff topology on N such that $\langle \mathcal{N}, \tau \rangle$ is $\aleph_1$-saturated as a topological structure and if $\langle \mathcal{M}, \tau' \rangle$ is a countable subspace of $\langle \mathcal{N}, \sigma \rangle$ then τ' is the discrete topology on M. Therefore, τ will be strictly coarser than $\sigma \restriction M$. This implies that using a two-sorted elementary embedding is not the best approach to studying the first-order relationships between topological structures.

Next we consider the definition of topological purity given for LCA groups in Chapter 4.1. That is, A is a topologically pure subgroup of B if

$$cl_B(nA) = cl_B(nB) \cap A$$

for every positive integer n. In any topological structure, if a set W is definable by the $\mathcal{L}_t$ formula $\phi(v)$ then $cl(W)$ is also a definable set. Consider the formula,

$$(\forall V)_v((\exists u)\phi(u) \wedge u \in V).$$

In fact, if our structure is a topological abelian group, and if a set W is definable by a tppf $\phi(v)$ then $cl(W)$ is also definable by a tppf. Consider the formula,

$$(\forall V)_0((\exists u)\phi(u) \wedge (u - v) \in V).$$

So, topological purity in this sense only requires that these closure formulas be respected. Topological purity does not require that all tppfs be respected nor does it require all ordinary pp-formulas to respected. Model theoretically, we would expect the definition of purity for topological modules to be analogous to case of purity for ordinary modules. Furthermore, purity for topological modules should imply purity for ordinary modules.

Definition 4.30 (Compare with Definition 1.119). Suppose that $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ are topological modules such that $M \subseteq N$. We relativize $\prec_2$ as follows:

[Kuc86] $\prec_t^1$: if for all $\phi(\overline{v}) \in \mathcal{L}_t$ with only individual variables free, and for all $\overline{a} = (a_1, \ldots, a_n) \in M^n$,

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}] \Leftrightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi[\overline{a}]$$

then $\langle \mathcal{M}, \tau \rangle \prec_t^1 \langle \mathcal{N}, \sigma \rangle$.

[Maj87] $\prec_t^2$: if τ is a subspace of σ, and if for all $\phi(\overline{v}) \in \mathcal{L}_t$ with only individual variables free, and for all $\overline{a} = (a_1, \ldots, a_n) \in M^n$,

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}] \Leftrightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi[\overline{a}]$$

then $\langle \mathcal{M}, \tau \rangle \prec_t^2 \langle \mathcal{N}, \sigma \rangle$.

$\prec_t^3$: if $f : \langle \mathcal{M}, \tau \rangle \to \langle \mathcal{N}, \sigma \rangle$ is proper, and if for all $\phi(\overline{v}, \overline{V}) \in \mathcal{L}_t$, for all $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and for all $\overline{A} = (A_1, \ldots, A_m)$ where each $A_i \in \sigma$,

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, f^{-1}[\overline{A}]] \Leftrightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi[f(\overline{a}), \overline{A}]$$

then $\langle \mathcal{M}, \tau \rangle \prec_t^3 \langle \mathcal{N}, \sigma \rangle$.

Clearly, all three versions imply that $\mathcal{M} \prec \mathcal{N}$.

Lemma 4.31. *Suppose that $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ are topological modules.*

(i) If $\langle \mathcal{M}, \tau \rangle \prec_t^3 \langle \mathcal{N}, \sigma \rangle$ then $\langle \mathcal{M}, \tau \rangle \prec_t^2 \langle \mathcal{N}, \sigma \rangle$ and $\langle \mathcal{M}, \tau \rangle \prec_t^1 \langle \mathcal{N}, \sigma \rangle$.

(ii) If $\langle \mathcal{M}, \tau \rangle \prec_t^2 \langle \mathcal{N}, \sigma \rangle$ then $\langle \mathcal{M}, \tau \rangle \prec_t^1 \langle \mathcal{N}, \sigma \rangle$.

Proof. This follows directly from the definitions. □

Lemma 4.32. *Suppose that $\langle \mathcal{A}, \tau \rangle$, $\langle \mathcal{B}, \sigma \rangle$ and $\langle \mathcal{C}, \gamma \rangle$ are topological modules.*

(i) For $* = 1, 2$ or 3 if $\langle \mathcal{A}, \tau \rangle \prec_t^* \langle \mathcal{B}, \sigma \rangle$ then $\langle \mathcal{A}, \tau \rangle \equiv_t \langle \mathcal{B}, \sigma \rangle$.

(ii) For $* = 1, 2$ or 3, $\langle \mathcal{A}, \tau \rangle \prec_t^* \langle \mathcal{A}, \tau \rangle$.

(iii) For $* = 1, 2$ or 3, if $\langle \mathcal{A}, \tau \rangle \prec_t^* \langle \mathcal{B}, \sigma \rangle$ and $\langle \mathcal{B}, \tau \rangle \prec_t^* \langle \mathcal{C}, \gamma \rangle$ then $\langle \mathcal{A}, \tau \rangle \prec_t^* \langle \mathcal{C}, \gamma \rangle$.

(iv) If $\langle \mathcal{A}, \tau \rangle \prec_t^1 \langle \mathcal{C}, \gamma \rangle$, $\langle \mathcal{B}, \sigma \rangle \prec_t^1 \langle \mathcal{C}, \gamma \rangle$ and $A \subseteq B$ then $\langle \mathcal{A}, \tau \rangle \prec_t^1 \langle \mathcal{B}, \sigma \rangle$.

(v) For $* = 2$ or 3, if $\langle \mathcal{A}, \tau \rangle \prec_t^* \langle \mathcal{C}, \gamma \rangle$, $\langle \mathcal{B}, \sigma \rangle \prec_t^* \langle \mathcal{C}, \gamma \rangle$, $A \subseteq B$ and τ is a subspace of σ then $\langle \mathcal{A}, \tau \rangle \prec_t^* \langle \mathcal{B}, \sigma \rangle$.

Proof. This follows directly from the definitions. □

Example 4.33. Recall the theory of $(\mathbb{Q}, Euc)$ explored in Theorem 2.33, Example 2.34 and Example 2.35. In [FZ80], it is claimed in Exercise 2.10 on page 117 that with respect to this theory, every $\mathcal{L}_m$-formula $\phi(v_1, \ldots, v_n, V_1, \ldots, V_m)$ is equivalent in $\mathcal{L}_m$ to a quantifier free formula $\psi(v_1, \ldots, v_n, V_1, \ldots, V_m)$ in the sense that for all models $\langle \mathcal{M}, \nu \rangle$ that are "torsion-free, $\neq \{0\}$, divisible, Hausdorff and locally pure" and where ν is a monotone system, for any $\overline{a} = (a_1, \ldots, a_n) \in M^n$, and any divisible subgroups $\overline{A} = (A_1, \ldots, A_m)$ where $A_i \in \nu$ and $M \supsetneq A_1 \supsetneq \cdots \supsetneq A_m$, we have

$$\langle \mathcal{M}, \nu \rangle \models_m (\phi \leftrightarrow \psi)[\overline{a}, \overline{A}].$$

By this quantifier elimination result it is easy to see that $(\mathbb{Q}, Euc) \prec_t^1 (\mathbb{R}, Euc)$ and $\mathbb{Q} \prec_t^2 (\mathbb{R}, Euc)$ since $\mathbb{Q} \prec_1^+ \mathbb{R}$ and $\prec_t^1$, $\prec_t^2$ do not deal with free set variables.

At this point it becomes relevant to discuss the context we will be working in. Considering $f : M \to N$ to be only a module homomorphism (as in $\prec_t^1$) without any topological restrictions is topologically uninteresting. Without free set parameters (as for $\prec_t^2$), it appears we would not be able to impose any topological conditions by means of formulas and types. However, by Lemma 4.28, if $\langle M, \tau \rangle$ is a topological module and $\phi(v_1, \ldots, v_n)$ is a tppf with only individual variables free, then $\phi[M]$ is a subgroup of M^n. This is analogous to Corollary 1.110 from Chapter 1.5. It is also natural to impose the additional condition that M is a subspace of N. Furthermore, if we allow for free set variables then Corollary 4.29 breaks down in general since tppfs with set parameters need not define subgroups. It may be worthwhile to let our set variables be open subgroups since this once again allows the tppfs to define subgroups, however, we begin to lose some of the most basic examples, for instance, the rationals or the reals with the standard Euclidean topology.

Definition 4.34. Suppose that $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ are topological modules and $f : M \to N$ is an embedding. We relativize $\prec_1^+$ as follows: if for all tppf, $\phi(\overline{v}, \overline{V})$, for all $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and for all $\overline{A} = (A_1, \ldots, A_m)$ where each $A_i \in \sigma$,

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, f^{-1}[\overline{A}]] \Leftrightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi[f(\overline{a}), \overline{A}]$$

then $\langle \mathcal{M}, \tau \rangle \prec_t^+ \langle \mathcal{N}, \sigma \rangle$.

Hence if $\langle \mathcal{M}, \tau \rangle$ is a subspace of $\langle \mathcal{N}, \sigma \rangle$ then $\langle \mathcal{N}, \sigma \rangle \prec_t^+ \langle \mathcal{N}, \tau \rangle$ if and only if for all tppfs $\phi(v_1, \ldots, v_n, V_1, \ldots, V_m)$, we have

$$\phi[M, \overline{A} \cap M] = \phi[N, \overline{A}] \cap M^n$$

where each $\overline{A} = (A_1, \ldots A_m)$ and each $A_i \in \sigma$.

It is worth emphasizing that an embedding $f : M \rightarrow N$ is a topological embedding, that is, f is a homeomorphism from M onto $f[M]$. This definition seems ideal since it is analogous to the definition for ordinary modules; if A is open in σ then $f^{-1}[A]$ is open in τ since embeddings are continuous and Lemma 4.24 holds if f is continuous. That is, if $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ are topological modules, $f : M \rightarrow N$ is continuous, $\phi(v_1, \ldots, v_n, V_1, \ldots, V_m)$ is a tppf then for all $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and for all $\overline{A} = (A_1, \ldots, A_m)$ where each $A_i \in \sigma$,

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, f^{-1}[\overline{A}]] \Rightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi[f(\overline{a}), \overline{A}].$$

This is analogous to pp-formulas from Chapter 1.5. However, there is one immediate drawback to this definition.

Theorem 4.35. *Let M be a subspace of N and $\langle \mathcal{M}, \tau \cap M \rangle \prec_t^+ \langle \mathcal{N}, \tau \rangle$. Then M is dense in N.*

Proof. Consider $\phi = (\exists v)(v \in V)$. Let U be a non-empty open set in N. So,

$$\langle \mathcal{N}, \tau \rangle \vDash_t \phi[U].$$

So,

$$\langle \mathcal{M}, \tau \cap M \rangle \vDash_t \phi[U \cap M].$$

So, $U \cap M$ is a non-empty open set in M. Therefore, M is dense in N. □

This implies that if we allow free set parameters our concept of purity is probably too strong since one of the main properties of purity for ordinary modules is that if $\mathcal{M}$ is a direct summand of $\mathcal{N}$ then $\mathcal{M}$ is pure in $\mathcal{N}$. However, for topological modules a direct summand will almost never be pure since direct summands are closed and hence not dense. It is possible to circumvent this problem if we restrict the open sets considered.

Theorem 4.36. *Suppose that $\langle M_1, \tau_1 \rangle$ and $\langle M_2, \tau_2 \rangle$ are topological modules. Let $a_1 \in M_1$, $a_2 \in M_2$ and suppose that $A_1 \times A_2$ is a basic open neighbourhood of* 0 *in τ_{prod}. Let $\phi(v, V)$ be a tppf then*

$$\begin{aligned} &\langle \mathcal{M}_1 \times \mathcal{M}_2, \tau_1 \times \tau_2 \rangle \vDash_t \phi[(a_1, a_2), A_1 \times A_2] \\ \Leftrightarrow \quad &\langle \mathcal{M}_1, \tau_1 \rangle \vDash_t \phi[a_1, A_1] \text{ and } \langle \mathcal{M}_2, \tau_2 \rangle \vDash_t \phi[a_2, A_2]. \end{aligned}$$

Proof. Let $\phi(v, V)$ be a tppf. Let $a_1 \in M_1$, $a_2 \in M_2$ and suppose that $A_1 \times A_2$ is a basic open neighbourhood of 0 in τ_{prod}.

This is proved by induction on the complexity of tppfs. The cases "$t = 0$" and conjunction follow trivially, so the only cases to consider are $t \in V_j$, $(\forall W)_0 \psi(v, V, W)$

and $(\exists w)\psi(v, w, V)$.

(i) Suppose $\phi(v, V)$ is $t \in V_j$. Then

$$\begin{aligned}
&\langle \mathcal{M}_1 \times \mathcal{M}_2, \tau_1 \times \tau_2 \rangle \models_t \phi[(a_1, a_2), A_1 \times A_2] \\
&\quad \Leftrightarrow \quad t^{\mathcal{M}_1 \times \mathcal{M}_2}((a_1, a_2)) \in A_1 \times A_2 \\
&\quad \Leftrightarrow \quad t^{\mathcal{M}_1}(a_1) \in A_1 \text{ and } t^{\mathcal{M}_2}(a_2) \in A_2 \\
&\quad \Leftrightarrow \quad \langle \mathcal{M}_1, \tau_1 \rangle \models_t \phi[a_1, A_1] \text{ and } \langle \mathcal{M}_2, \tau_2 \rangle \models_t \phi[a_2, A_2].
\end{aligned}$$

(ii) Suppose ϕ is $(\forall W)_0 \psi(v, V, W)$. If $A_1 \times A_2$ is a basic open neighbourhood of 0 in τ_{prod} then $0 \in A_1 \in \tau_1$ and $0 \in A_2 \in \tau_2$. Furthermore, if $0 \in A_1 \in \tau_1$ and $0 \in A_2 \in \tau_2$, then $A_1 \times A_2$ is a basic open neighbourhood of 0 in τ_{prod}.

$$\begin{aligned}
&\langle \mathcal{M}_1 \times \mathcal{M}_2, \tau_1 \times \tau_2 \rangle \models_t \phi[(a_1, a_2), A_1 \times A_2] \\
&\quad \Leftrightarrow \quad \langle \mathcal{M}_1 \times \mathcal{M}_2, \tau_1 \times \tau_2 \rangle \models_t \psi[(a_1, a_2), A_1 \times A_2, B_1 \times B_2] \\
&\qquad\quad \text{for all } B_1 \times B_2 \text{ in the basis of 0 of } \tau_{\mathsf{prod}} \\
&\quad \Leftrightarrow \quad \langle \mathcal{M}_1, \tau_1 \rangle \models_t \psi[a_1, A_1, B_1] \text{ and } \langle \mathcal{M}_2, \tau_2 \rangle \models_t \psi[a_2, A_2, B_2] \\
&\qquad\quad \text{for all } 0 \in B_1 \in \tau_1 \text{ and for all } 0 \in B_2 \in \tau_2 \\
&\qquad\quad \text{(by the induction hypothesis)} \\
&\quad \Leftrightarrow \quad \langle \mathcal{M}_1, \tau_1 \rangle \models_t \phi[a_1, A_1] \text{ and } \langle \mathcal{M}_2, \tau_2 \rangle \models_t \phi[a_2, A_2].
\end{aligned}$$

(iii) Suppose ϕ is $(\exists w)\psi(v, w, V)$.

$$\begin{aligned}
&\langle \mathcal{M}_1 \times \mathcal{M}_2, \tau_1 \times \tau_2 \rangle \models_t \phi[(a_1, a_2), A_1 \times A_2] \\
&\quad \Leftrightarrow \quad \langle \mathcal{M}_1 \times \mathcal{M}_2, \tau_1 \times \tau_2 \rangle \models_t \phi[(a_1, a_2), b, A_1 \times A_2] \\
&\qquad\quad \text{for some } b \in M_1 \times M_2 \\
&\quad \Leftrightarrow \quad \langle \mathcal{M}_1 \times \mathcal{M}_2, \tau_1 \times \tau_2 \rangle \models_t \phi[(a_1, a_2), (b_1, b_2), A_1 \times A_2] \\
&\qquad\quad \text{for some } b_1 \in M_1 \text{ and } b_2 \in M_2 \\
&\quad \Leftrightarrow \quad \langle \mathcal{M}_1, \tau_1 \rangle \models_t \psi[a_1, b_1, A_1] \text{ and } \langle \mathcal{M}_2, \tau_2 \rangle \models_t \psi[a_2, b_2, A_2] \\
&\qquad\quad \text{for some } b_1 \in M_1 \text{ and } b_2 \in M_2 \\
&\qquad\quad \text{(by the induction hypothesis)} \\
&\quad \Leftrightarrow \quad \langle \mathcal{M}_1, \tau_1 \rangle \models_t \phi[a_1, A_1] \text{ and } \langle \mathcal{M}_2, \tau_2 \rangle \models_t \phi[a_2, A_2].
\end{aligned}$$

Therefore, the result follows by induction on the complexity of tppfs. $\square$

Corollary 4.37. *Suppose that $\langle M_1, \tau_1 \rangle$ and $\langle M_2, \tau_2 \rangle$ are topological modules. Let $\overline{a} = (a_1, \ldots, a_n) \in M_1$, $\overline{b} = (b_1, \ldots, b_n) \in M_2$ and suppose that $A_1 \times B_1, \ldots, A_m \times B_m$ are basic open neighbourhoods of 0 in τ_{prod}. Let $\phi(\overline{v}, \overline{V})$ be a tppf then*

$$\begin{aligned}
&\langle \mathcal{M}_1 \times \mathcal{M}_2, \tau_1 \times \tau_2 \rangle \models_t \phi[(a_1, b_1), \ldots, (a_n, b_n), A_1 \times B_1, \ldots, A_m \times B_m] \\
&\quad \Leftrightarrow \quad \langle \mathcal{M}_1, \tau_1 \rangle \models_t \phi[\bar{a}, A_1, \ldots, A_m] \text{ and } \langle \mathcal{M}_2, \tau_2 \rangle \models_t \phi[\bar{b}, B_1, \ldots, B_m].
\end{aligned}$$

Proof. This is a trivial generalization of Theorem 4.36. $\square$

Theorem 4.38. *Suppose that $\langle M_i, \tau_i \rangle$ and $\langle N_i, \sigma_i \rangle$ are $\aleph_0$-saturated for each $i \in I$.*

If $\langle M_i, \tau_i\rangle \equiv_t \langle N_i, \sigma_i\rangle$ *for each* $i \in I$ *then*

$$\langle \underline{\mathcal{M}}, \tau_{\mathsf{coprod}}\rangle \equiv_t \langle \underline{\mathcal{N}}, \sigma_{\mathsf{coprod}}\rangle.$$

Proof. By Lemma 2.40, since $\langle M_i, \tau_i\rangle$ and $\langle N_i, \sigma_i\rangle$ are $\aleph_0$-saturated for each $i \in I$,

$$\langle M_i, \tau_i\rangle \equiv_t \langle N_i, \sigma_i\rangle \Rightarrow \langle M_i, \tau_i\rangle \simeq_t^p \langle N_i, \sigma_i\rangle.$$

Therefore, there is a set of partial homeomorphisms S_i from $\langle M_i, \tau_i\rangle$ to $\langle N_i, \sigma_i\rangle$ for each $i \in I$ satisfying forth$_1$, forth$_2$, back$_1$ and back$_2$. We let $p_i = (p_i^0, p_i^1, p_i^2)$ denote a typical element of S_i. I use $\prod\limits_{i\in I} S_i$ to define a set T of partial homeomorphisms from $\langle \underline{\mathcal{M}}, \tau_{\mathsf{coprod}}\rangle$ to $\langle \underline{\mathcal{N}}, \sigma_{\mathsf{coprod}}\rangle$.

Define $\mathfrak{f}$ as follows: for $(p_i)_{i\in I} \in \prod_{i\in I} S_i$ let $\mathfrak{f}((p_i)_{i\in I}) = p = (p^0, p^1, p^2)$ where

$$\begin{aligned} p^0 &= \langle \varepsilon_i^N \circ p_i^0 \circ \pi_i^M\rangle_{i\in I}; \\ p^k &= \left\{ (\prod_{i\in I} U_i, \prod_{i\in I} V_i) : (U_i, V_i) \in p_i^k \text{ for } i \in I \right\} \end{aligned}$$

for $k = 1, 2$. Clearly, the range of p^0 is a subset of $\underline{N}$ and p^0 is one-to-one. In addition, notice that if $U_i \in \tau_i$ and $V_i \in \sigma_i$ then $\prod\limits_{i\in I} U_i \in \tau_{\mathsf{coprod}}$ and $\prod_{i\in I} V_i \in \sigma_{\mathsf{coprod}}$ since the coproduct topology is finer than the box topology.

Now, define

$$T = \mathfrak{f}\left[\prod_{i\in I} S_i\right].$$

It will now be shown that T is a set of partial homeorphisms from $\langle \underline{\mathcal{M}}, \tau_{\mathsf{coprod}}\rangle$ to $\langle \underline{\mathcal{N}}, \sigma_{\mathsf{coprod}}\rangle$

forth$_1$: Let $p = (p^0, p^1, p^2) \in T$, say $p^0 = \langle \varepsilon_i^N \circ p_i^0 \circ \pi_i^M \rangle_{i \in I}$ and $a \in \underline{M}$. So, $a_i = \pi_i^M(a) \in M_i$ for each $i \in I$. There is a q_i in S_i with $p_i \subseteq q_i$ and $a_i \in dom(q_i^0)$ for each $i \in I$. Let $q = \mathfrak{f}((q_i)_{i \in I})$. Clearly, $p \subseteq q$ and $a \in dom(q^0)$.

forth$_2$: Let $p = (p^0, p^1, p^2) \in T$, say $p^0 = \langle \varepsilon_i^N \circ p_i^0 \circ \pi_i^M \rangle_{i \in I}$, $a \in dom(p_0)$ and $U \in \tau_{\mathsf{coprod}}$ with $a \in U$. So, $a_i = \pi_i^M(a) \in M_i$, $a_i \in (\varepsilon_i^M)^{-1}[U] = U_i$ and $a_i \in dom(p_i^0)$ for each $i \in I$. Therefore, for each $i \in I$ there is a $q_i \in S_i$ and $V_i \in \sigma_i$ such that $p_i \subseteq q_i$, $p_i^0(a_i) \in V_i$ and $(U_i, V_i) \in q_i^2$. Let $q = \mathfrak{f}((q_i)_{i \in I})$. Clearly, $p \subseteq q$. Consider, $V = \prod_{i \in I} V_i \in \sigma_{\mathsf{coprod}}$. Now, $p^0(a) = (p_i^0(a_i))_{i \in I} \in V$ since $p_i^0(a_i) \in V_i$. Furthermore, $(U, V) \in q^2$, since $(U_i, V_i) \in q_i^2$ for each for $i \in I$.

back$_1$: Let $p = (p^0, p^1, p^2) \in T$, say $p^0 = \langle \varepsilon_i^N \circ p_i^0 \circ \pi_i^M \rangle_{i \in I}$ and $b \in rg(p^0)$, say $p^0(a) = b$. So, for each $i \in I$, $b_i = \pi_i^N(b) \in N_i$ and $b_i \in rg(p_i^0)$, since $p_i^0(\pi_i(a)) = b_i$. There is a $q_i \in S_i$ with $p_i \subseteq q_i$ and $b_i \in rg(q_i^0)$, say $b_i = q_i^0(c_i)$ for each $i \in I$. Let $q = \mathfrak{f}((q_i)_{i \in I})$. Clearly, $p \subseteq q$ and $b \in rg(q^0)$.

back$_2$: Let $p = (p^0, p^1, p^2) \in T$, say $p^0 = \langle \varepsilon_i^N \circ p_i^0 \circ \pi_i^M \rangle_{i \in I}$, $b \in rg(p^0)$, say $p^0(a) = b$, $V \in \sigma_{\mathsf{coprod}}$ and $b \in V$. So, for each $i \in I$, $b_i = \pi_i^N(b) \in N_i$ and $b_i \in rg(p_i^0)$, since $p_i^0(\pi_i(a)) = b_i$. Therefore, for each $i \in I$, there exists $q_i \in S_i$ and $U_i \in \tau_i$ with $p_i \subseteq q_i, a_i \in U_i$ and $(U_i, V_i) \in q_i^1$ where $V_i = (\varepsilon_i^N)^{-1}[V]$. Let $q = \mathfrak{f}((q_i)_{i \in I})$. Clearly, $p \subseteq q$. Consider, $U = \prod_i U_i \in \tau_{\mathsf{coprod}}$. Now, $a \in U$. Furthermore, $(U, V) \in q^1$, since $(U_i, V_i) \in q_i^1$ for each $i \in I$.

So, $\langle \underline{\mathcal{M}}, \tau_{\mathsf{coprod}} \rangle \simeq_t^p \langle \underline{\mathcal{N}}, \sigma_{\mathsf{coprod}} \rangle$. However, partially homeomorphic structures are L_t-equivalent by Lemma 2.39, so $\langle \underline{\mathcal{M}}, \tau_{\mathsf{co-prod}} \rangle \equiv_t \langle \underline{\mathcal{N}}, \sigma_{\mathsf{co-prod}} \rangle$. $\square$

Ideally, Theorem 4.38 would be true without the additional assumption that the topological modules be $\aleph_0$-saturated as two-sorted structures. In order to complete the theorem, it would be enough to show that if (M_i, τ_i) is a two-sorted elementary substructure of an (N_i, σ_i), where τ_i and σ_i are suitably chosen bases for the topologies and (N_i, σ_i) is $\aleph_0$-saturated, then $\langle \underline{\mathcal{M}}, \tau_{\mathsf{coprod}} \rangle \equiv_t \langle \underline{\mathcal{N}}, \sigma_{\mathsf{coprod}} \rangle$. Furthermore, by Theorem 2.41, if L is finite, then $\langle \underline{\mathcal{M}}, \tau_{\mathsf{coprod}} \rangle \equiv_t \langle \underline{\mathcal{N}}, \sigma_{\mathsf{coprod}} \rangle$ without the additional assumption that the topological modules be $\aleph_0$-saturated. In particular, this result holds for abelian groups.

Lemma 4.39. *Let $\langle \mathcal{M}_i, \tau_i \rangle$ be topological modules, $\overline{a} = (a_1, \ldots, a_n) \in \underline{M}$ and let $\overline{A} = (A_1, \ldots, A_n)$ where each A_i is a basic open neighbourhood of 0 in τ_{prod}. Let $c \in \overline{M}$. Now, for each $i \in I$, let $d_i = \pi_i(c_i)$ if $\pi_i(a_j) \neq 0$ for some $j = 1, \ldots, n$ or if $\pi_i[A_k] \neq M_i$ for some $k = 1, \ldots m$; otherwise, set $d_i = 0$. Let $d = (d_i)_{i \in I}$.*

$$\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \vDash_t \psi[\overline{a}, c, \overline{A}] \Rightarrow \langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \vDash_t \psi[\overline{a}, d, \overline{A}].$$

Proof. Let $\phi(\overline{v}, \overline{V})$ be a tppf. Let $\overline{a} = (a_1, \ldots, a_n) \in \underline{M}$ and let $\overline{A} = (A_1, \ldots, A_n)$ where each A_i are basic open neighbourhoods of 0 in τ_{prod}. Let $c \in \overline{M}$. Let d be constructed as above.

This is proved by induction on the complexity of tppfs. The case for conjunction, the case when ϕ is $(\exists w)\psi(\overline{v}, w, \overline{V})$ and the case when ϕ is $(\forall W)_0\psi(\overline{v}, \overline{V}, W)$ follow trivially. So, the only cases we have to consider are when $\phi(\overline{v}, \overline{V})$ is $t = 0$ and $t \in V_j$.

(i) Suppose $\phi(\overline{v}, \overline{V})$ is $t = 0$ and $\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \vDash_t \phi[\overline{a}, c, \overline{A}]$. So,

$$\mathbf{R}(a_1 \ldots a_n, c) + \mathbf{S}(b_1, \ldots, b_m) = 0$$

where $b_k \in \overline{M}$ for $k = 1 \ldots m$. Now, if $\pi_i(a_j) = 0$ for $j = 1, \ldots, n$, then set $b^{'}_{k_i} = 0$ for $k = 1 \ldots m$. Otherwise, set $b_{k_i} = \pi_i(b_k)$ for $k = 1 \ldots m$. So, let $b^{'}_k = (b_{k_i})_{i \in I}$ for $k = 1 \ldots m$.

$$\mathbf{R}(a_1 \ldots a_n, d) + \mathbf{S}(b^{'}_1, \ldots, b^{'}_m) = 0$$

where $b^{'}_k \in \overline{M}$ for $k = 1 \ldots m$. Therefore, $\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \models_t \phi[\overline{a}, d, \overline{A}]$.

(ii) Suppose $\phi(\overline{v}, \overline{V})$ is $t \in V_j$. Then

$$\begin{aligned}
&\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \models_t \phi[\overline{a}, c, \overline{A}] \\
&\Rightarrow \; t^{\overline{\mathcal{M}}}(\overline{a}, c) \in A_j \\
&\Rightarrow \; t^{\mathcal{M}_i}(\pi_i(a_1), \ldots, \pi_i(a_n), \pi_i(c)) \in \pi_i[A_j] \text{ for each } i \in I.
\end{aligned}$$

So, if $\pi_i(a_j) = 0$ for $i = 1, \ldots, n$ and $\pi_i[A_j] = M_i$ then

$$t^{\mathcal{M}_i}(\pi_i(\overline{a}), \pi_i(d)) = t^{\mathcal{M}_i}(0, 0) = 0 \in M_i.$$

Otherwise,

$$t^{\mathcal{M}_i}(\pi_i(\overline{a}), \pi_i(d)) = t^{\mathcal{M}_i}(\pi_i(a_1), \ldots, \pi_i(a_n), \pi_i(c)) \in \pi_i[A_j].$$

So, $t^{\mathcal{M}_i}(\pi_i(\overline{a}), \pi_i(d)) \in \pi_i[A_j]$ for each $i \in I$. Hence,

$$\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \models_t \phi[\overline{a}, d, \overline{A}].$$

Therefore, the result follows by induction on the complexity of tppfs. □

Here is a result analogous to Corollary 1.131 in Chapter 1.5.

Theorem 4.40. *Let $\langle \mathcal{M}_i, \tau_i \rangle$ be topological modules and $\overline{a} = (a_1, \ldots, a_n) \in \underline{M}$ and let $\overline{A} = (A_1, \ldots, A_n)$ where each A_i are basic open neighbourhoods of 0 in τ_{prod}. Then*

$$\langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction \underline{M} \rangle \vDash_t \phi[\overline{a}, \overline{A} \cap \underline{M}] \Leftrightarrow \langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \vDash_t \phi[\overline{a}, \overline{A}].$$

Proof. Let $\phi(\overline{v}, \overline{V})$ be a tppf. Let $\overline{a} = (a_1, \ldots, a_n) \in \underline{M}$ and let $\overline{A} = (A_1, \ldots, A_n)$ where each A_i are basic open neighbourhoods of 0 in τ_{prod}. We are considering $\underline{M}$ as a subspace of $\overline{M}$, therefore by Corollary 4.24,

$$\langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction \underline{M} \rangle \vDash_t \phi[\overline{a}, \overline{A} \cap \underline{M}] \Rightarrow \langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \vDash_t \phi[\overline{a}, \overline{A}].$$

So, we need to show that

$$\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \vDash_t \phi[\overline{a}, \overline{A}] \Rightarrow \langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction M \rangle \vDash_t \phi[\overline{a}, \overline{A} \cap \underline{M}].$$

This is proved by induction on the complexity of tppfs. The cases "$t = 0$" and conjunction follow directly from $\underline{M} \prec_1^+ \overline{M}$. There are only three cases involving set variables to consider.

(i) Suppose $\phi(\overline{v}, \overline{V})$ is $t \in V_j$. Then

$$\begin{aligned}
&\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \models_t \phi[\overline{a}, \overline{A}] \\
&\quad \Rightarrow \; t^{\overline{\mathcal{M}}}(\overline{a}) \in A_j \\
&\quad \Rightarrow \; t^{\underline{\mathcal{M}}}(\overline{a}) \in A_j \\
&\qquad\quad (\text{since } (t^{\overline{\mathcal{M}}}(\overline{a}) = t^{\underline{\mathcal{M}}}(\overline{a})) \\
&\quad \Rightarrow \; t^{\underline{\mathcal{M}}}(\overline{a}) \in A_j \cap \underline{M} \\
&\quad \Rightarrow \; \langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction \underline{M} \rangle \models_t \phi[\overline{a}, \overline{A} \cap \underline{M}].
\end{aligned}$$

(ii) Suppose ϕ is $(\forall W)_0 \psi(\overline{v}, \overline{V}, W)$. Since $\tau_{\mathsf{prod}} \restriction \underline{M}$ is a subspace of τ_{prod} on $\overline{M}$, for every $U \in \tau_{\mathsf{prod}}$, $U \cap \underline{M} \in \tau_{\mathsf{prod}} \restriction \underline{M}$. Furthermore, for every $U \in \tau_{\mathsf{prod}} \restriction \underline{M}$, there exists $V \in \tau$ such that $U = V \cap \underline{M}$.

$$\begin{aligned}
&\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \models_t \phi[\overline{a}, \overline{A}] \\
&\quad \Rightarrow \; \langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \models_t \psi[\overline{a}, \overline{A}, A] \\
&\qquad\quad \text{for all } 0 \in A \in \tau_{\mathsf{prod}} \\
&\quad \Rightarrow \; \langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction \underline{M} \rangle \models_t \psi[\overline{a}, \overline{A} \cap \underline{M}, A \cap \underline{M}] \\
&\qquad\quad \text{for all } 0 \in A \in \tau_{\mathsf{prod}} \\
&\qquad\quad (\text{by the induction hypothesis}) \\
&\quad \Rightarrow \; \langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction \underline{M} \rangle \models_t \psi[\overline{a}, \overline{A} \cap \underline{M}, B] \\
&\qquad\quad \text{for all } 0 \in B \in \tau \restriction \underline{M} \\
&\quad \Rightarrow \; \langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction \underline{M} \rangle \models_t \phi[\overline{a}, \overline{A} \cap \underline{M}].
\end{aligned}$$

(iii) Suppose ϕ is $(\exists w)\psi(\overline{v}, w, \overline{V})$. Let $\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \models_t \phi[\overline{a}, \overline{A}]$. So, there exists $c \in \overline{M}$ such that $\langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction \underline{M} \rangle \models_t \psi[\overline{a}, d, \overline{A} \cap M]$. Now, for each $i \in I$, let $d_i = \pi_i(c_i)$ if $\pi_i(a_j) \neq 0$ for some $j = 1, \ldots, n$ or if $\pi_i[A_k] \neq M_i$ for some $k = 1, \ldots m$; otherwise, set $d_i = 0$. Let $d = (d_i)_{i \in I}$. Therefore, $\langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle \models_t \psi[\overline{a}, d, \overline{A}]$ by Lemma 4.39. Notice that $d \in \underline{M}$ since $d_i \neq 0$ for at most finitely many $i \in I$.

$$\langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction \underline{M} \rangle \models_t \psi[\overline{a}, d, \overline{A} \cap M]$$

(by the induction hypothesis)

$$\Rightarrow \quad \langle \underline{\mathcal{M}}, \tau_{\mathsf{prod}} \restriction \underline{M} \rangle \models_t \phi[\overline{a}, \overline{A} \cap M].$$

Therefore, the result follows by induction on the complexity of tppfs. □

Ideally, we would like to be able to prove the more general form of Theorem 4.40, namely:

Conjecture 4.41. *Let $\langle \mathcal{N}, \tau \rangle$ be a topological module such that M is a dense submodule of N and $M \prec_1^+ N$. Then*

$$\langle \mathcal{M}, \tau \restriction M \rangle \prec_t^+ \langle \mathcal{N}, \tau \rangle.$$

The difficulty in proving this lies only with existential quantification of individual variables and this part of the proof in Theorem 4.40 relies heavily on the relationship between $\overline{M}$ and $\underline{M}$.

Note. In general, we will not be able to obtain

$$\langle \underline{\mathcal{M}}, \tau_{\mathsf{coprod}} \rangle \prec_t^+ \langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle$$

for any version of $\prec_t^+$ since the coproduct topology is finer than the subspace topology induced by τ_{prod}. For instance, the product of an infinite number of discrete modules each with more than one element is not discrete under the product topology. However, the direct sum of an an infinite number of discrete modules each with more than one element is discrete under the coproduct topology since the coproduct topology is finer than the box topology. This will be one of the points in which our theory differs from that of ordinary module theory since in ordinary module theory, for any set $\{M_i\}_{i\in I}$ of modules,

$$\bigoplus_{i\in I} M_i \prec_1^+ \prod_{i\in I} M_i.$$

In fact,

$$\bigoplus_{i\in I} M_i \prec \prod_{i\in I} M_i.$$

Flum and Ziegler were able to prove the following, however neglected to include the $\aleph_0$-saturated assumption. It is clearly required in their proof unless $\mathcal{L}$ is finite.

Theorem 4.42 ([FZ80], Theorem 6.1, p. 32). *Suppose that $\langle M_i, \tau_i\rangle$ and $\langle N_i, \sigma_i\rangle$ are $\aleph_0$-saturated for each $i \in I$. If $\langle \mathcal{M}_i, \tau_i\rangle \equiv_t \langle \mathcal{N}_i, \sigma_i\rangle$ for each $i \in I$ then*

$$\prod_{i\in I}\langle \mathcal{M}_i, \tau_i\rangle \equiv_t \prod_{i\in I}\langle \mathcal{N}_i, \sigma_i\rangle.$$

Definition 4.43. Suppose that $\langle \mathcal{M}, \tau\rangle$ is a topological module. $\langle \mathcal{M}, \tau\rangle$ is *tpp-compact* if for some basis τ_0 of τ, if Φ is a set of tppfs which is finitely satisfiable in $\langle \mathcal{M}, \tau_0\rangle$ then Φ is satisfiable in $\langle \mathcal{M}, \tau\rangle$.

Definition 4.44. $\langle \mathcal{M}, \tau\rangle$ is injective over topological pure embeddings if any diagram of proper maps can be completed commutatively as shown:

$$\begin{tikzcd}
\langle \mathcal{A}, \sigma \restriction A \rangle \arrow[r, hook, "\prec_t^+"] \arrow[d, "f"'] & \langle \mathcal{B}, \sigma \rangle \arrow[dl, dashed, "\exists \overline{f}"] \\
\langle \mathcal{M}, \tau \rangle &
\end{tikzcd}$$

An interesting and useful model-theoretic concept of purity and pure-injectivity would ideally satisfy the following:

Conjecture 4.45. *$\langle \mathcal{M}, \tau \rangle$ is tpp-compact if and only if it is injective over topological pure embeddings.*

The following arguments show the standard techniques for ordinary modules and, in addition, explains why these techniques must break down for topological modules. Suppose that $\langle \mathcal{M}, \tau \rangle$ is tpp-compact, $\langle \mathcal{A}, \sigma \restriction A \rangle \prec_t^+ \langle \mathcal{B}, \sigma \rangle$ and $f : (A, \sigma) \to (M, \tau)$ is an embedding. Now, we want to construct a proper map $\overline{f} : \langle \mathcal{B}, \sigma \rangle \to \langle \mathcal{M}, \tau \rangle$. Using standard techniques it is possible to construct $\overline{f}$ so that it is open onto its image: Let $\overline{b}$ be an enumeration of B and let $\overline{U}$ be an enumeration of σ. Let $\overline{v_b}$ and $\overline{V_U}$ correspond to $\overline{b}$ and $\overline{\tau}$ respectively.

Consider

$$\Phi(\overline{v_b}, \overline{V_U}) = \left\{ \phi(\overline{v_b}, \overline{a}, \overline{V_U}) : \phi \text{ is tppf and } B \models \phi[\overline{b}, \overline{a}, \overline{U}] \text{ where } \overline{a} \in A \right\}.$$

Now, let

$$\Phi'(\overline{v_b}, \overline{V_U}) = \left\{ \phi(\overline{v_b}, f(\overline{a}), \overline{V_U}) : \phi \in \Phi \right\}.$$

We want to show that every finite subset of Φ is satisfied in $\langle \mathcal{M}, \tau \rangle$. In order to do this, it is sufficient to consider a single formula in Φ' since Φ is closed under finite conjunctions.

Let $\psi \in \Phi$.

$$
\begin{aligned}
&\langle \mathcal{B}, \sigma\rangle \models \psi[\overline{b}, \overline{a}, \overline{U}] \\
&\quad \Rightarrow \quad \langle \mathcal{B}, \sigma\rangle \models (\exists \overline{v})\psi[\overline{v}, \overline{a}, \overline{U}] \\
&\quad \Rightarrow \quad \langle \mathcal{A}, \sigma \restriction A\rangle \models (\exists \overline{v})\psi[\overline{v}, \overline{a}, \overline{U} \cap A] \\
&\quad \text{since} \quad \langle \mathcal{A}, \sigma \cap A\rangle \prec^{+} \langle \mathcal{B}, \sigma\rangle \\
&\qquad\qquad \overline{U} \cap A = f^{-1}[V] \text{ for some } V \in \tau \\
&\quad \Rightarrow \quad \langle \mathcal{M}, \tau\rangle \models (\exists \overline{v})\psi[\overline{v}, f(\overline{a}), \overline{V}] \\
&\quad \Rightarrow \quad \langle \mathcal{M}, \tau\rangle \models \phi[\overline{c}, f(\overline{a}), \overline{V}] \text{ where } \overline{c} \in M.
\end{aligned}
$$

So, Φ' is finitely satisfied in $\langle \mathcal{M}, \tau\rangle$. Therefore, Φ' is satisfied in $\langle \mathcal{M}, \tau\rangle$, that is, there are $\overline{m} \in M$ and $\overline{U}' \in \tau$ such that $\langle \mathcal{M}, \tau\rangle \models \Phi[\overline{m}, \overline{U}']$. Define $\overline{f}$ to be the two-sorted map which carries $\overline{b}$ to $\overline{m}$ and $\overline{U}$ to $\overline{U}'$ component-wise. Now, consider the tpp-formula $v \in V$. So,

$$\langle \mathcal{B}, \sigma\rangle \models b_1 \in U_1 \Rightarrow \langle \mathcal{M}, \tau\rangle \models \overline{f}(b_1) \in \overline{f}(U_1).$$

Let $U \in \sigma$ and $b \in \overline{f}[U]$. So, there exists $a \in U$ such that $b = \overline{f}(a)$. Therefore, $b = \overline{f}(a) \in \overline{f}(U)$ where $\overline{f}(U) \in \tau$. Thus, $\overline{f}[U] \subseteq \overline{f}(U)$. So, $\overline{f}[U] = \overline{f}(U) \cap \overline{f}[B]$. Therefore, $\overline{f}$ is open onto its image.

There are three main model theoretic problems in proving that $\overline{f}$ is continuous.

(i) Let U be an open neighbourhood of τ. If $\overline{f}$ is continuous, then $\overline{f}^{-1}[U] \in \sigma$.

Now, $f^{-1}[U] \in \sigma \restriction A$ since f is continuous. Therefore, $f^{-1}[U] = U' \cap A$ for some $U' \in \sigma$. Therefore, $\overline{f}^{-1}[U] = U' \cap A$. However, there is nothing to guarantee that $U' \cap A$ is open in σ, unless $A \in \sigma$.

(ii) Consider $\widetilde{M}$, the completion of M. Since A is dense in B, by Theorem 1.67 there exists a unique map from B to $\widetilde{M}$, say $\widetilde{f}$, such that the following diagram commutes:

$$\begin{tikzcd} \langle A, \sigma \restriction A\rangle \arrow[r, hook, "\mu"] \arrow[d, "f"'] & \langle B, \sigma\rangle \arrow[dl, dashed, "\exists \overline{f}"'] \arrow[d, dashed, "\exists! \widetilde{f}"] \\ \langle M, \widetilde{\tau} \restriction M\rangle \arrow[r, hook, "\nu"] & \langle \widetilde{M}, \widetilde{\tau}\rangle \end{tikzcd}$$

That is, $f = \overline{f} \circ \mu$, $\widetilde{f} = \overline{f} \circ \nu$ and $\widetilde{f} \circ \mu = \nu \circ f$.

Now, from the proof of Theorem 1.67, it is possible to define $\widetilde{f}$ as follows: Let $b \in B$. Pick any filter $\mathcal{F}$ on B such that $\mathcal{F} \longrightarrow b$ and $A \in \mathcal{F}$. Since A is dense in B such a filter exists. Since $\mathcal{F}$ is a Cauchy filter in B so is its restriction

$$\mathcal{F}_A = \{F \subseteq A : F \in \mathcal{F}\}.$$

By Lemma 1.66, $f \circ \nu[\mathcal{F}_A]$ is a Cauchy filter on $\widetilde{M}$. Since $\widetilde{M}$ is complete, $\lim f[\mathcal{F}_A]$ exists. Let

$$\widetilde{f}(b) = \lim f[\mathcal{F}_A].$$

Since $\widetilde{f}$ and $f \circ \nu$ are continuous, $\widetilde{f}(\lim \mathcal{F}) = \lim \widetilde{f}[\mathcal{F}]$ and $f \circ \nu(\lim \mathcal{F}_A) = \lim f \circ \nu[\mathcal{F}_A]$. Furthermore, $\tilde{f}$ is unique and $\tilde{f}(x) = \overline{f}(x)$ for all $x \in M$.

The main hindrance to using this powerful result is that it is impossible to formalize "lim" in $\mathcal{L}_t$. By definition, x is the limit of $\mathcal{F}$, if for every open neighbourhood U of x, there exists $F \in \mathcal{F}$ such that $U \in F$; and "for every open neighbourhood U of x, $U \in F$" is beyond the expressive power of $\mathcal{L}_t$.

(iii) Finally, consider $\Phi = tp^+(\overline{B}, \overline{\sigma}/A)$. Then $\overline{f}(\Phi)$ is finitely satisfiable and therefore satisfiable in $\langle \mathcal{M}, \tau \rangle$. However, by Corollary 4.27, any set variables can be realized by the open set M. Therefore, this does not provided any insight into the construction of $\overline{f}$. Furthermore, there is no reason to assume $\overline{f}$ is continuous.

This implies there does not seem to be a *natural* way to force $\overline{f}$ to be continuous.

Now suppose that $\langle \mathcal{M}, \tau \rangle$ is injective over topological pure embeddings. It is clear that using the classical proof does not necessarily produce a continuous map. Using the classical proof, let Φ be a consistent system of equations over M. Then there exists an elementary extension $\mathcal{M}'$ of $\mathcal{M}$ such that Φ has a solution. Now, if $\mathcal{M} \prec \mathcal{M}'$ then $\mathcal{M} \prec_1^+ \mathcal{M}'$. So, consider the following diagram:

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook, "\prec"] \arrow[d, "f"'] & \mathcal{M}' \arrow[dl, dashed, "\overline{f}"] \\ \mathcal{M} & \end{tikzcd}$$

where $f = 1_M$. Since M is injective, there exists $\overline{f} : M' \to M$ such that $\overline{f} \restriction M = 1_M$. So, if there is a solution $\overline{b}$ of Φ in M' then $\overline{f}(\overline{b})$ is a solution to Φ in M. So, $\mathcal{M}$ is pp-compact.

For topological modules the only concept of elementary extensions we have is as two-sorted structures. However, it is unlikely that this sort of extension is a topological extension since it has already been shown that if $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ are topological modules such that $\langle \mathcal{M}, \tau \rangle \prec_2 \langle \mathcal{N}, \sigma \rangle$, then τ is coarser than the subspace topology induced by σ. Furthermore, it would be difficult to obtain a theory of elementary extension for topological modules without a concept of pure embedding since elementary extensions are stronger than pure embeddings.

These ideas seem to carry over for any *natural* definition of injective over pure-embeddings and tpp-compact.

4.4 Conclusion

In this thesis, the following results were obtained through my research:

(i) An alternative approach, using T-filters, to construct Chasco and Domínguez's explicit description of the coproduct topology on the direct sum of topological abelain groups. I have also expanded this result to topological left R-modules in Proposition 3.28.

(ii) Let $\langle \mathcal{M}, \tau \rangle$ be an $\aleph_1$-saturated topological structure. Using Flum and Ziegler's result that τ is closer under intersections, I obtained Corollary 2.10; that is, if τ is hausdorff and $\langle \mathcal{M}', \tau' \rangle$ is a countable subspace of $\langle \mathcal{M}, \tau \rangle$ then τ' is discrete.

(iii) Suppose that $\langle \mathcal{M}, \tau \rangle$ and $\langle \mathcal{N}, \sigma \rangle$ are topological modules such that $f : M \to N$ is a continuous homomorphism. In order to obtain a concept of pure embedding for topological modules that included set variables and that was analogous to the concept of pure embedding for modules, I first proved Lemma 4.24, namely, for all tppf $\phi(\overline{v}, \overline{V})$, for all $\overline{a} = (a_1, \ldots, a_n) \in M^n$ and for all

$\overline{A} = (A_1, \ldots, A_m)$ where each $A_i \in \sigma$,

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, f^{-1}[\overline{A}]] \Rightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi[f(\overline{a}), \overline{A}].$$

This became the foundation for my definition of pure embedding, namely, $\langle \mathcal{M}, \tau \rangle \prec_1^+ \langle \mathcal{N}, \sigma \rangle$ if

$$\langle \mathcal{M}, \tau \rangle \models_t \phi[\overline{a}, f^{-1}[\overline{A}]] \Leftrightarrow \langle \mathcal{N}, \sigma \rangle \models_t \phi[f(\overline{a}), \overline{A}].$$

Furthermore, I provided a comparison of my definition of pure embedding with previously proposed definitions in Lemma 4.31 and Lemma 4.32. One of the main restrictions of my definition of pure embedding was shown in Theorem 4.35; that is, if M be a subspace of N and $\langle \mathcal{M}, \tau \cap M \rangle \prec_t^+ \langle \mathcal{N}, \tau \rangle$ then M is dense in N. This is a restriction since for ordinary modules if $\mathcal{M}$ is a direct summand of $\mathcal{N}$ then $\mathcal{M}$ is pure in $\mathcal{N}$. However, for topological modules a direct summand will almost never be pure. I circumvented this problem by restricting the open sets considered and obtained Theorem 4.36, Corollary 4.37, Lemma 4.39 and Theorem 4.40. Furthermore, I provided reasons as to why, for any definition of purity, we will not be able to obtain that

$$\langle \underline{\mathcal{M}}, \tau_{\mathsf{coprod}} \rangle \prec_t^+ \langle \overline{\mathcal{M}}, \tau_{\mathsf{prod}} \rangle$$

despite the fact that for ordinary modules

$$\underline{M} \prec \overline{M}.$$

(iv) Let $\langle M_i, \tau_i \rangle$ and $\langle N_i, \sigma_i \rangle$ be $\aleph_0$-saturated for each $i \in I$. Using a Back and Forth argument and properties of the coproduct topology, I obtained Theorem 4.38 which states that if $\langle M_i, \tau_i \rangle \equiv_t \langle N_i, \sigma_i \rangle$ for each $i \in I$ then

$$\langle \underline{\mathcal{M}}, \tau_{\mathsf{coprod}} \rangle \equiv_t \langle \underline{\mathcal{N}}, \sigma_{\mathsf{coprod}} \rangle.$$

(v) Finally, I have shown that standard model theoretic techniques do not seem apply when looking for a rich concept of pure injectivity for topological modules. Moreover, I have shown this for any reasonable definition of pure embedding. This seems to imply that this is the wrong approach to take when studying topological modules. Furthermore, if there is a concept of pure injectivity for topological modules it will not be analogous to pure injectivity for modules. At this moment in time, a model theoretic approach seems to be the best approach since there does not appear to be any standard, simple or insightful examples in order to provide guidance with this pursuit. For topological modules, all we can claim is the existence of $\aleph_0$-saturated models and even in classical model theory of modules, coming up with particular $\aleph_0$-saturated models is usually difficult. The recent discovery of a *reasonable* definition of pure injectivity for LCA groups, namely t-pure injectivity, may be the closest concept to pure injectivity for topological modules that is obtainable. Furthermore, this concept seems analogous to the concept of pure injectivity for modules. However, LCA groups are not axiomatizable in $\mathcal{L}_t$ and countable LCA groups are discrete, which implies that standard model theoretic techniques do not apply anyways, unless we are restricted to the discrete topology.

Bibliography

[Arm81] David L. Armacost, *The structure of locally compact abelian groups*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 68, Marcel Dekker Inc., New York, 1981. MR MR637201 (83h:22010)

[BM77] J. L. Bell and M. Machover, *A course in mathematical logic*, North-Holland Publishing Co., Amsterdam, 1977. MR MR0472455 (57 #12155)

[CD03] M. J. Chasco and X. Domínguez, *Topologies on the direct sum of topological abelian groups*, Topology Appl. **133** (2003), no. 3, 209–223. MR MR2000499 (2004j:22003)

[CH06] J. P. Coleman and J. Hyndman, *Walter Taylor's contributions to universal algebra*, Algebra Universalis **55** (2006), no. 2-3, 113–117, Special issue dedicated to Walter Taylor. MR MR2280220

[DPS90] Dikran N. Dikranjan, Ivan R. Prodanov, and Luchezar N. Stoyanov, *Topological groups*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 130, Marcel Dekker Inc., New York, 1990, Characters, dualities and minimal group topologies. MR MR1015288 (91e:22001)

[Eng89] Ryszard Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author. MR MR1039321 (91c:54001)

[FZ80] Jörg Flum and Martin Ziegler, *Topological model theory*, Lecture Notes in Mathematics, vol. 769, Springer, Berlin, 1980. MR MR560706 (81j:03059)

[Gar75] S. Garavaglia, *Notices of the american mathematical society*, Abstracts 75T-E36 (June 1975); 75T-E61 (October, 1976); 75T-E79 (November, 1975) 1975.

[Gar76] ———, *Model theory of topological structures*, Ph.D. thesis, Yale University, 1976.

[Gar78] ———, *Model theory of topological structures*, Ann. Math. Logic **14** (1978), no. 1, 13–37. MR MR0491140 (58 #10406)

[Hig74] P. J. Higgins, *Introduction to topological groups*, Cambridge University Press, London, 1974, London Mathematical Society Lecture Note Series, No. 15. MR MR0360908 (50 #13355)

[Hig77] ———, *Coproducts of topological abelian groups*, J. Algebra **44** (1977), no. 1, 152–159. MR MR0457615 (56 #15820)

[HS73] Horst Herrlich and George E. Strecker, *Category theory: an introduction*, Allyn and Bacon Inc., Boston, Mass., 1973, Allyn and Bacon Series in Advanced Mathematics. MR MR0349791 (50 #2284)

[Kap48] Samuel Kaplan, *Extensions of the Pontrjagin duality. I. Infinite products*, Duke Math. J. **15** (1948), 649–658. MR MR0026999 (10,233c)

[Kha95] J. A. Khan, *The finite torsion subgroup of an LCA group need not split*, Period. Math. Hungar. **31** (1995), no. 1, 43–44. MR MR1349292 (96g:22007)

[Kuc86] T. G. Kucera, *Stability theory for topological logic, with applications to topological modules*, J. Symbolic Logic **51** (1986), no. 3, 755–769. MR MR853855 (88b:03057)

[Lot01] Peter Loth, *Topologically pure extensions*, Abelian groups, rings and modules (Perth, 2000), Contemp. Math., vol. 273, Amer. Math. Soc., Providence, RI, 2001, pp. 191–201. MR MR1817162 (2001m:22008)

[Lot06] ———, *On t-pure and almost pure exact sequences of LCA groups*, J. Group Theory **9** (2006), no. 6, 799–808. MR MR2272718 (2007j:22009)

[Luk] Gàbor Lukàcs, *Compact-like topological groups*, Research and Exposition in Mathematics 31, Heldermann Verlag, Lemgo, (to appear).

[Maj87] Mirosław Majewski, *Elementary extensions of topological models in L_t language*, Studia Logica **46** (1987), no. 3, 255–264. MR MR938719 (89h:03066)

[Mar02] David Marker, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002, An introduction. MR MR1924282 (2003e:03060)

[McK74] T. A. McKee, *Some applications of model theory to topology*, Ph.D. thesis, University of Wisconsin, 1974.

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR MR1712872 (2001j:18001)

[Nic02] Peter Nickolas, *Coproducts of abelian topological groups*, Topology Appl. **120** (2002), no. 3, 403–426. MR MR1897271 (2003c:22006)

[Pon66] L. S. Pontryagin, *Topological groups*, Translated from the second Russian edition by Arlen Brown, Gordon and Breach Science Publishers, Inc., New York, 1966. MR MR0201557 (34 #1439)

[Pre88] Mike Prest, *Model theory and modules*, London Mathematical Society Lecture Note Series, vol. 130, Cambridge University Press, Cambridge, 1988. MR MR933092 (89h:03061)

[PZ99] I. Protasov and E. Zelenyuk, *Topologies on groups determined by sequences*, Mathematical Studies Monograph Series, vol. 4, VNTL Publishers, Lviv, 1999. MR MR1718814 (2001i:22005)

[Rot00] Philipp Rothmaler, *Introduction to model theory*, Algebra, Logic and Applications, vol. 15, Gordon and Breach Science Publishers, Amsterdam, 2000, Prepared by Frank Reitmaier, Translated and revised from the 1995 German original by the author. MR MR1800596 (2001h:03002)

[Sgr74] J. Sgro, *Completeness theorems for topological models*, Ph.D. thesis, University of Wisconsin, 1974.

[Wil70] Stephen Willard, *General topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970. MR MR0264581 (41 #9173)

[Wis91] Robert Wisbauer, *Foundations of module and ring theory*, german ed., Algebra, Logic and Applications, vol. 3, Gordon and Breach Science Publishers, Philadelphia, PA, 1991, A handbook for study and research. MR MR1144522 (92i:16001)

[Zie76] Martin Ziegler, *A language for topological structures which satisfies a lindström theorem*, Bulletin of the American Mathematical Society **82** (1976), 568–570.

[ZP90] E. G. Zelenyuk and I. V. Protasov, *Topologies on abelian groups*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 5, 1090–1107. MR MR1086087 (93b:22001)

Index