SECOND LAYER CONDITIONS
IN
NOETHERIAN RINGS

BY

(PAUL) CHONG-HYUN KIM

A Thesis
Submitted to the Faculty of Graduate Studies
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Department of Mathematics and Astronomy
University of Manitoba
Winnipeg, Manitoba

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Abstract

The hierarchy of the various versions of the second layer condition, transferability of the second layer condition to extension rings, and relativization of it with respect to a torsion theory are investigated.

We show that the strong second layer condition is indeed stronger than the ordinary second layer condition by introducing an intermediate condition, called the right restricted strong second layer condition, and its equivalent versions in Chapter 2.

The second layer condition is passed to certain extension rings. Letzter proved that if R is a noetherian ring satisfying the second layer condition and S is an overring of R such that ${}_RS$ and S_R are finitely generated, then S satisfies the second layer condition. In Chapter 3, we extend this result to a noetherian ring satisfying just the right second layer condition. The proof makes use of finite annihilation of tame modules, thus dispensing with a symmetric dimension function implicitly required in Letzter's proof.

It is not known if the polynomial ring $R[x]$ satisfies the right second layer condition when R is a right noetherian ring with right second layer condition. We show that $R[x]$ satisfies the right second layer condition when R is a noetherian very strongly right AR-separated ring. Also, we observe that if $\mathrm{E}(R[x]/P)_{R[x]}$ is $(P \cap R)$-tame for each prime P of $R[x]$, then $R[x]$ satisfies the right second layer condition.

Let σ be a torsion radical on Mod-R, where R is a right σ-noetherian ring. In Chapter 4, we show that R is right fully σ-bounded iff $\mathrm{Spec}_{\sigma}(R)$ satisfies the right restricted strong second layer condition and R has local bijective Gabriel correspondence.

Prior to this, we establish that the local bijective Gabriel correspondence with respect to σ is equivalent to σ-torsionfree modules being tame. We also investigate localization of R with respect to a perfect torsion radical σ and finite annihilation of $\mathrm{E}(R/P)$, where P is a minimal prime ideal of R.

Contents

CHAPTER 1

Preliminaries

All rings considered are associative with identity element 1, and all right R-modules are unitary where R is a ring. Adjectives like noetherian, bounded, regular and so on describe two sided–(right and left)–properties.

If X is a nonempty subset of a right R-module M, then the *right annihilator* of X in R is the set

$$r_R(X) = \{r \in R \mid xr = 0 \text{ for all } x \in X\}.$$

This is a right ideal of R. If N is a submodule of M, then $r_R(N)$ is an ideal of R. If there is no danger of ambiguity, the subscript R will be dropped.

If X is a nonempty subset of R, then the *left annihilator* of X in M is the set

$$l_M(X) = \{m \in M \mid mx = 0 \text{ for all } x \in X\}.$$

This is not usually a submodule of M unless X is an ideal of R.

This chapter puts together known results and definitions for later use. However, in some cases, a proof is given that differs from that in the cited source.

1.1. Goldie Rings

Let R be a commutative ring. A proper ideal P of R is called *prime* if, for any two given elements $a, b \in R$ such that $ab \in P$, either $a \in P$ or $b \in P$. Note that the quotient ring R/P is an integral domain if P is a prime ideal of R. Note also that R/P is a field if P is a maximal ideal of R.

However, the latter is not the case when R is noncommutative, the ring of 2×2 matrices over a field being perhaps the simplest example. Thus, the definition of a prime ideal in a noncommutative ring has been modified. In 1928, W. Krull defined a proper ideal P of a ring R to be *prime* if, for any ideals $I, J \subseteq R$ with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. Ideals that are prime in the earlier (commutative) sense are called *completely prime*. A ring R is called a *prime ring* if 0 is a prime ideal.

An ideal S is called *semiprime* if S is an intersection of prime ideals. In particular, $N = \bigcap\{P \mid P \text{ is prime in } R\}$ is semiprime and is called the *prime radical* of R. The prime radical is always a nil ideal, and it is nilpotent if the ring is noetherian on one side. A ring R is called a *semiprime ring* if 0 is a semiprime ideal. An ideal P is called *right primitive* if $P = r(M)$ for some simple right R-module M. Right primitive ideals are prime ideals. The *Jacobson radical* of the ring R is defined as

$$\begin{aligned} J(R) &= \bigcap\{M \mid M \text{ is a maximal right ideal in } R\} \\ &= \bigcap\{P \mid P \text{ is a right primitive ideal of } R\}. \end{aligned}$$

If R is a ring, then $c \in R$ is called *right regular* if $r(c) = 0$. It is called *left regular* if $l(c) = 0$. Let I be an ideal of R. Then

$$\mathcal{C}'_R(I) = \{c \in R \mid c + I \text{ is right regular in } R/I\},$$

and similarly

$$'\mathcal{C}'_R(I) = \{c \in R \mid c + I \text{ is left regular in } R/I\}.$$

Now, $\mathcal{C}_R(0) = \mathcal{C}'(0) \cap {}'\mathcal{C}(0)$ is the set of regular elements of R. If there is no danger of ambiguity, then the subscript is usually dropped.

A ring R is a *right order* in a ring Q if

1. every regular element of R is invertible in Q, and
2. every element $q \in Q$ can be written as $q = ac^{-1}$ for some $a \in R$ and $c \in \mathcal{C}_R(0)$.

A nonempty subset $0 \notin C$ of R is called *multiplicative* if $1 \in C$ and C is closed under multiplication. A multiplicative set C of R is called *right Ore* if $cR \cap rC \neq \emptyset$ for every $c \in C$ and $r \in R$. The set C is called *right reversible* if whenever $cr = 0$ for some $r \in R, c \in C$, there exists $d \in C$ such that $rd = 0$. If C is right Ore and right reversible, it is called a *right denominator set.* In this case, the set $T = \{r \in R \mid rc = 0 \text{ for some } c \in C\}$ is a two sided ideal and the set $\overline{C} = \{c + T \mid c \in C\}$ is a right Ore set consisting of regular elements of $\overline{R} = R/T$. Thus, a ring of right quotients can be constructed for $\overline{R}$, that is, a ring Q containing $\overline{R}$ in which each $\overline{c} \in \overline{C}$ is invertible and whose elements are of the form $\overline{r}\,\overline{c}^{-1}$ where $\overline{r} \in \overline{R}, \overline{c} \in \overline{C}$. We say that R can be *localized with respect to* C. In particular, if P is a prime ideal of R and $\mathcal{C}(P)$ is a right denominator set, then we also say that R can be localized at P.

Let R be a commutative ring and let P be a prime ideal of R. Then R can be localized at P since $\mathcal{C}(P) = \{r \in R \mid r \notin P\}$ is trivially a right denominator set in R. However, if R is not commutative, then $\mathcal{C}(P)$ need not be a right denominator set. The structure of right denominator sets is in part limited by the existence of *linked* prime ideals. In fact, if C is a right Ore set in a right noetherian ring R such that $C \subseteq \mathcal{C}_R(P)$, then $C \subseteq \mathcal{C}_R(Q)$ for any prime ideal Q linked to P [29, Lemma 12.17]. This represents the main stumbling block for P to be right localizable.

Let M be a module and N be a submodule of M. Then N is called an *essential submodule* of M if $N \cap L \neq 0$ for every nonzero submodule L of M. This is denoted by $N \subseteq_e M$. A module M is called *uniform* if every nonzero submodule of M is essential in M.

A module M is of *finite (uniform) rank* if there are finitely many uniform submodules $U_1, \dots, U_n$ of M such that $U_1 \oplus \cdots \oplus U_n \subseteq_e M$. In this case, we say that $\operatorname{rank}(M) = n$.

A ring R is called *right Goldie* if R has ACC (ascending chain condition) on right annihilators and R_R has finite rank. The key result for verifying the right Ore condition for $\mathcal{C}(0)$ in a semiprime right Goldie ring is the following.

PROPOSITION 1.1 (Goldie). *Let R be a semiprime right Goldie ring. Then a right ideal E is essential in R_R iff E contains a regular element of R.*

This leads to a fundamental result about noncommutative localization.

THEOREM 1.2 (Goldie's Theorem). [**23**, Theorem A] [**22**, Theorem 13] [**24**, Theorems 4.1, 4.2] *A ring R is a right order in a semisimple artinian ring Q iff R is a semiprime right Goldie ring.*

When R is a semiprime right Goldie ring, then an additive rank function on the right R-modules can be defined by factoring out the torsion part of the module. Goldie named this function the *reduced rank* [**26**].

DEFINITION. Let R be a semiprime right Goldie ring and let M be a right R-module. Then the *reduced rank* or *torsionfree rank* $\rho_R(M)$ of M is defined to be rank$(M/Z(M))$, where

$$Z(M) = \{m \in M \mid mc = 0 \text{ for some } c \in \mathcal{C}(0)\}$$

is the torsion submodule of M.

If R is a ring with nilpotent prime radical N such that R/N is right Goldie, then the reduced rank $\rho_R(M)$ of M is defined to be

$$\sum_{i=0}^{n-1} \rho_{R/N}(MN^i/MN^{i+1}),$$

where n is the index of nilpotency of N.

The reduced rank is additive in the sense that $\rho(M) = \rho(L) + \rho(M/L)$ whenever $L \subseteq M$ [**56**, Lemma 1.2 (ii)], while the (uniform) rank is not.

LEMMA 1.3. [**56**, Proposition 4.1.3(i)] *Let R be a ring with nilpotent prime radical N where R/N is right Goldie, and let M be a right R-module. Then $\rho_R(M) = 0$ iff for every $m \in M$ there exists $c \in \mathcal{C}_R(N)$ such that* $mc = 0$.

1.2. Bounded Rings

In a commutative ring, every right (or left) ideal is trivially an ideal. If an arbitrary right noetherian ring R contains "enough" ideals, then it shares a very useful property with commutative rings, namely that for any finitely generated right R-module M, there exist finitely many elements $m_1, \ldots, m_n \in M$ such that $r(M) = r(m_1, \ldots, m_n)$. In this case, M is called *finitely annihilated.*

The following definition formalizes the notion of a ring R having "enough" ideals.

DEFINITION. A ring R is said to be *right bounded* if every essential right ideal E of R contains a nonzero ideal I of R. R is said to be *right fully bounded* if R/P is right bounded for every prime ideal P in R. If R is right noetherian and right fully bounded, then R is called *right FBN.*

A module M is called *faithful* if $r(M) = 0$, and it is called *fully faithful* if $r(N) = 0$ for every nonzero submodule N of M. A module M is called *prime* if M is fully faithful as an $R/r(M)$-module. In this case, $r(M)$ turns out to be a prime ideal of R, and M is called P-prime if $P = r(M)$.

Let M be a right R-module. A prime ideal P of R is called an *annihilator prime* of M if $P = r(N)$ for some submodule N of M. A submodule M_1 of M is called an *affiliated submodule* of M if $M_1 = l_M(P)$ for some ideal P, which is maximal among the annihilators of nonzero submodules of M. A series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

is called an *affiliated series* of M if M_i/M_{i-1} is an affiliated submodule of M/M_{i-1}, for $i = 1, \ldots, n$. Let $P_i = r_R(M_i/M_{i-1})$, for $i = 1, \ldots, n$. Then $\{P_1, \ldots, P_n\}$ is called the set of *affiliated primes* of M corresponding to the affiliated series $M_0 \subset \cdots \subset M_n$.

Let R be a right noetherian ring, and let U be a uniform right R-module. Then there exists a maximum member P in the set of the right annihilators of nonzero submodules of U, which turns out to be a prime ideal. It is called the *assassinator* or the *associated prime* of U, and is denoted by $\mathrm{Ass}(U) = P$. If M is an arbitrary right R-module, then $\mathrm{Ass}(M)$ is defined as the set of associated primes of uniform submodules of M.

Set $V = l_U(P)$ where U is a uniform module, and observe that V is a P-prime submodule of U. By [**29**, Proposition 6.11], the torsion submodule

$$Z(V) = \{v \in V \mid vc = 0 \text{ for some } c \in \mathcal{C}_R(P)\}$$

is either 0 or V. If $Z(V) = 0$, i.e., if V is a torsionfree right R/P-module, then U is called *P-tame.* If $Z(V) = V$, then V is a torsion right R/P-module and in this case U is called *P-wild.*

A module M is called *tame* if all its uniform submodules are tame. A module M is called *wild* if all its uniform submodules are wild.

DEFINITION. Let $\mathcal{X}$ be a nonempty set of prime ideals of a right noetherian ring R. A right R-module M is called *$\mathcal{X}$-tame* if it is tame and $\operatorname{Ass}(M) \subseteq \mathcal{X}$.

Jategaonkar observed in [**39**, p. 23] that right FBN rings have no wild modules.

LEMMA 1.4. *Let R be a right noetherian ring. Then R is a right FBN ring iff all right R-modules are tame.*

PROOF. $(\Rightarrow)$: Let U be a uniform right R-module with $\operatorname{Ass}(U) = P$. Let $V = l_U(P)$. Now V is either torsion or torsionfree as a right R/P-module, so assume that it is torsion. If $0 \neq v \in V$, then $r(v)/P \subseteq_e R/P$. Thus by right boundedness of R/P, there exists an ideal I such that $0 \neq I/P \subseteq r(v)/P$. Then $0 = vI = vRI$, which leads to the contradiction $P \subset I \subseteq r(vR) = P$. Thus U is tame.

$(\Leftarrow)$: Without loss of generality, let R be a prime right noetherian ring. If, for any $I \subseteq_e R$, $r(R/I) \neq 0$, then $0 \subset r(R/I) \subseteq I$ shows that R is right bounded. So assume that $r(R/I) = 0$. Set $\mathcal{E} = \{E \subseteq_e R \mid r(R/E) = 0\}$.

Since $I \in \mathcal{E} \neq \emptyset$, there exists a maximal element E in $\mathcal{E}$. We claim that R/E is a uniform and fully faithful right R-module. If E_1, E_2 are right ideals strictly including E and $E = E_1 \cap E_2$, then

$$R/E = R/(E_1 \cap E_2) \hookrightarrow R/E_1 \oplus R/E_2.$$

Hence either $r(R/E_1) = 0$ or $r(R/E_2) = 0$, either of which gives a contradiction to the maximality of E. This shows that R/E is uniform.

Next we show that R/E is fully faithful. Let F/E be a nonzero submodule of R/E. Then $r(R/F)r(F/E) \subseteq r(R/E) = 0$ implies that $r(R/F) = 0$ or $r(F/E) = 0$. The former again contradicts the choice of E. Thus R/E is fully faithful as a right R-module. Hence the claim is proved.

It now follows that $\mathrm{Ass}(R/E) = 0$. So R/E is torsionfree as a right R-module as all right R-modules are tame. But R/E is at the same time torsion as a right R-module since $E \subseteq_e R$. Thus $R/E = 0$, so $r(R/E) = R$, which contradicts $r(R/E) = 0$. Hence there exists a nonzero ideal J such that $J \subseteq I$, proving that R is right bounded. □

THEOREM 1.5. [**17**, Théorème II 8] *If R is a right FBN ring, then any finitely generated right R-module M is finitely annihilated.*

PROOF. A module M can be embedded into a direct sum of $\mathrm{rank}(M)$ many uniform modules M/M_i by [**18**, Lemma 7.9]. If each M/M_i is finitely annihilated, then M is finitely annihilated. So without loss of generality, assume that M is a uniform module. Also assume that $r(M) = 0$, as the fully bounded property is inherited by factor rings. Let $\mathrm{Ass}(M) = P$. We will use noetherian induction on M to show that M is finitely annihilated. So suppose that M is not finitely annihilated, but M/N is finitely annihilated for every $0 \neq N \subseteq M$.

Now, $M/l_M(P)$ is finitely annihilated by the induction hypothesis since $l_M(P) \neq 0$. So there exist $m_1, \dots, m_k \in M$ such that

$$r(m_1 + l_M(P), \dots, m_k + l_M(P)) = r(M/l_M(P)).$$

Let $r(m_1, \dots, m_k) = I$. Since

$$r(m_1, \dots, m_k) \subseteq r(m_1 + l_M(P), \dots, m_k + l_M(P)) = r(M/l_M(P)),$$

we have $MIP = 0$. Since $r(M) = 0$, the right ideal I can be regarded as an R/P-module. If $I = 0$, then M is annihilated by $r(m_1, \dots, m_k)$. Otherwise, there exists $x_1 \in M$ such that $x_1 I \neq 0$. Set $I_0 = I$ and $I_1 = I_0 \cap r(x_1)$. Observe that $I_0 \supset I_1$. If $I_1 = 0$, then again M is finitely annihilated. If $I_1 \neq 0$, then there exists $x_2 \in M$ such that $x_2 I_1 \neq 0$. Set $I_2 = I_1 \cap r(x_2)$. Continuing in this fashion, setting $I_{i+1} = I_i \cap r(x_{i+1})$, we get

$$0 \neq I_i/I_{i+1} \cong x_{i+1} I_i \subseteq x_{i+1} I \subseteq l_M(P)$$

which is torsionfree as an R/P-module. Note that $\rho_{R/P}(I_i/I_{i+1}) = 1$ for all i. Since I_0 is a finitely generated and hence a noetherian right R/P-module, $\rho_{R/P}(I_0) < \infty$. It now follows from the additivity of the reduced rank that

$$I_n = I_{n+1} = \cdots$$

for some $n \geq 0$. This contradicts the fact that $I_n \supset I_{n+1} \supset \cdots$. Thus, M is finitely annihilated. □

We can also prove the converse of Theorem 1.5. If U is a uniform module with $\mathrm{Ass}(U) = P$, then set $V = l_U(P)$. Since V is a P-prime module, vR is also P-prime uniform for every $0 \neq v \in V$. However, $r(vR) = r(v_1, \ldots, v_n)$ for some $v_i \in vR$ as every finitely generated module is finitely annihilated. Then $R/P = R/r(vR) \hookrightarrow \bigoplus_{i=1}^{n} v_iR \subseteq (vR)^n$, the direct sum of n copies of vR, and thus vR is tame. Since $vR \subseteq U$, the uniform module U is tame. By Lemma 1.4, the ring R is right fully bounded.

1.3. Torsion Theory

As torsion theory is investigated extensively in [6], [61], [27] and [71] among others, only the essential definitions and results will be quoted for later use.

As indicated in Section 1.1, the attempt to localize a noncommutative ring at a prime ideal is hindered by linked prime ideals. A more abstract approach to localization, developed by Gabriel [21], uses quotient categories, that are derived from torsion radicals. The term "torsion radical" is used by Maranda [54], and is sometimes referred to as a *hereditary torsion theory* [71], [19],[48]. It has been also called an *idempotent kernel functor* [27].

Definition. Let R be a ring, and let Mod-R be the category of right R-modules. A covariant functor σ from Mod-R to Mod-R is called a *preradical* if $\sigma(M) \subseteq M$ for all right R-modules M and σf is the restriction of the homomorphism $f : M \to N$ to $\sigma(M)$ where M, N are right R-modules.

Note that $\sigma f(\sigma(M)) = f(\sigma(M)) \subseteq \sigma(N)$.

Let τ be another preradical. Then the preradicals $\sigma : \tau$ and $\sigma\tau$ are defined by

$$(\sigma : \tau)(M)/\sigma(M) = \tau(M/\sigma(M)),\ (\sigma\tau)(M) = \sigma(\tau(M)).$$

If $(\sigma : \sigma) = \sigma$, then σ is called a *radical*. If $\sigma(N) = N \cap \sigma(M)$ for any submodule N of M, then the preradical σ is called a *torsion preradical*. Finally, a radical that is a torsion preradical is called *torsion radical*.

Lemma 1.6. (i) *Let M be a right R-module. Then*

$$\mathrm{rad}_M(K) = \bigcap\{\ker f \mid f : K \to M\},\ K \in \text{Mod-R}$$

defines a radical.

(ii) *Let E be an injective right R-module. Then*

$$\mathrm{rad}_E(K) = \bigcap\{\ker f \mid f : K \to E\},\ K \in \text{Mod-R}$$

defines a torsion radical.

Proof. (i). It is clear that $\mathrm{rad}_M(K)$ is a submodule of K. For any homomorphism $g : K \to L$, we must show that $g(\mathrm{rad}_M(K)) \subseteq \mathrm{rad}_M(L)$.

Let $h : L \to M$. Then $h \circ g : K \to M$. So $(h \circ g)(\operatorname{rad}_M(K)) = 0$ and $g(\operatorname{rad}_M(K)) \subseteq \operatorname{rad}_M(L)$. Thus rad_M is a preradical.

Note that each homomorphism $f : K \to M$ induces a homomorphism $\bar{f} : K/\operatorname{rad}_M(K) \to M$ with $\ker \bar{f} = \ker f/\operatorname{rad}_M(K)$. Similarly each homomorphism $g : K/\operatorname{rad}_M(K) \to M$ induces a homomorphism $\bar{g} : K \to M$ with $\ker g = \ker \bar{g}/\operatorname{rad}_M(K)$. So

$$\operatorname{rad}_M(K/\operatorname{rad}_M(K)) = \bigcap\{\ker g \mid g : K/\operatorname{rad}_M(K) \to M\} = 0.$$

(ii). If $f : N \to E$ is a homomorphism and $N \subseteq K$, then f can be extended to a homomorphism $\bar{f} : K \to E$. We observe that $\ker f = \ker \bar{f} \cap N$. Also if $g : K \to E$ is a homomorphism, then the restriction $g|_N : N \to E$ is a homomorphism from N and $\ker g|_N = \ker g \cap N$. Thus $\operatorname{rad}_E(N) = \operatorname{rad}_E(K) \cap N$. □

Let M be a right R-module and let σ be a torsion radical. Then $\sigma(M)$ is called the *σ-torsion submodule* of M. If $\sigma(M) = 0$, then M is called *σ-torsionfree*. If $\sigma(M) = M$, then M is called *σ-torsion*.

A submodule N of M is called *σ-dense* in M if $\sigma(M/N) = M/N$. A submodule N is called *σ-closed* in M if $\sigma(M/N) = 0$. For $N \subseteq M$, the submodule

$$\overline{N}^{\sigma} = \bigcap\{N' \mid N \subseteq N' \subseteq M,\ \sigma(M/N') = 0\}$$

is called the *σ-closure* of N in M. Note that $\overline{N}^{\sigma}/N = \sigma(M/N)$ and $\overline{N}^{\sigma}$ is σ-closed in M.

Let $\mathcal{F}$ be a filter on the set of right ideals of a ring R. Then $\mathcal{F}$ is called a *topologizing filter* if $r(x+I) \in \mathcal{F}$ for any $I \in \mathcal{F}$ and $x \in R$. A topologizing filter $\mathcal{G}$ is called a *Gabriel filter* if whenever I is a right ideal of R and there exists $J \in \mathcal{G}$ such that $r(j+I) \in \mathcal{G}$ for all $j \in J$, then $I \in \mathcal{G}$.

If σ is a torsion radical, then $\{I \subseteq R_R \mid \sigma(R/I) = R/I\}$ defines a Gabriel filter. If $\mathcal{G}$ is a Gabriel filter on R, then for each right R-module M, we define the submodule $\sigma(M)$ to be $\{m \in M \mid r(m) \in \mathcal{G}\}$. The functor σ is a torsion radical [71].

Let σ be a torsion radical. Then the class of σ-torsion modules is closed under direct sums, extensions, quotient modules and submodules. The class of σ-torsionfree modules is closed under direct products, essential extensions and submodules.

LEMMA 1.7. *Let σ be a torsion radical on* Mod-R *and let M be a right R-module. Then*

(i) $\overline{N_1 \cap N_2}^{\sigma} = \overline{N_1}^{\sigma} \cap \overline{N_2}^{\sigma}$ *for any* $N_1, N_2 \subseteq M$.

(ii) $\overline{F/E}^{\sigma} = \overline{F}^{\sigma}/E$ *for any* $E \subseteq F \subseteq M$.

PROOF. (i). Since $M/(\overline{N_1}^{\sigma} \cap \overline{N_2}^{\sigma}) \hookrightarrow M/\overline{N_1}^{\sigma} \oplus M/\overline{N_2}^{\sigma}$, we have $\sigma(M/(\overline{N_1}^{\sigma} \cap \overline{N_2}^{\sigma})) = 0$. Thus, $\overline{N_1 \cap N_2}^{\sigma} \subseteq \overline{N_1}^{\sigma} \cap \overline{N_2}^{\sigma}$.

Also $(\overline{N_1}^{\sigma} \cap \overline{N_2}^{\sigma})/(N_1 \cap N_2)$ is σ-torsion as submodules and quotient modules of a σ-torsion module are σ-torsion. Thus $\overline{N_1 \cap N_2}^{\sigma} = \overline{N_1}^{\sigma} \cap \overline{N_2}^{\sigma}$.

(ii). Since $F \subseteq \overline{F}^{\sigma}$ and $\sigma(M/\overline{F}^{\sigma}) = 0$, it follows that $\overline{F/E}^{\sigma} \subseteq \overline{F}^{\sigma}/E$. On the other hand,

$$\sigma\left(\frac{\overline{F}^{\sigma}/E}{F/E}\right) \cong \sigma\left(\frac{\overline{F}^{\sigma}}{F}\right) = \frac{\overline{F}^{\sigma}}{F}$$

implies that $\overline{F}^{\sigma}/E \subseteq \overline{F/E}^{\sigma}$. □

A module M is called *σ-injective* if, for any module L and any submodule $K \subseteq L$ with $\sigma(L/K) = L/K$, every R-module homomorphism $f : K \to M$ can be extended to a homomorphism $\bar{f} : L \to M$.

Let σ be a torsion radical, and let M be a σ-torsionfree right R-module. Then the *quotient module M_σ* of M with respect to σ is the submodule of the injective hull $\mathrm{E}(M)$ defined by $M_\sigma/M = \sigma(\mathrm{E}(M)/M)$. Note that M_σ contains M by definition. The quotient module M_σ is σ-torsionfree since $\mathrm{E}(M)$ is σ-torsionfree. Next we observe that M_σ is σ-injective. If K is a submodule of L such that L/K is σ-torsion and $f : K \to M_\sigma$ is a homomorphism, then f can be extended to a homomorphism $\bar{f} : L \to \mathrm{E}(M)$. We claim that $\bar{f}(L) \subseteq M_\sigma$. Since L/K is σ-torsion, $r(x + K)$ is σ-dense in R for every $x \in L$. Then $\bar{f}(x)r(x + K) \subseteq \bar{f}(K) = f(K) \subseteq M_\sigma$. However, $\mathrm{E}(M)/M_\sigma$ is σ-torsionfree, yielding $\bar{f}(x) \in M_\sigma$. Therefore, M_σ is σ-injective.

One can thus define a functor Q_σ from Mod-R to Mod-R$/\sigma$, where Mod-R$/\sigma$ is the category of right R-modules that are σ-torsionfree and σ-closed in their injective hulls, by

$$\frac{Q_\sigma(M)}{M/\sigma(M)} = \sigma\left(\frac{\mathrm{E}(M/\sigma(M))}{M/\sigma(M)}\right).$$

Sometimes $Q_\sigma(M)$ is written as M_σ for simplicity, even when M is not σ-torsionfree.

If σ is not a radical, then $M/\sigma(M)$ is not necessarily σ-torsionfree. If σ is not a torsion radical, then $\sigma(M) = 0$ need not imply that $\sigma(\mathrm{E}(M)) = 0$. The σ-injectivity of $Q_\sigma(M)$ is obtained from the following proposition.

PROPOSITION 1.8. [27, Proposition 3.3] *If $0 \to F \to E \to L \to 0$ is an exact sequence where E is σ-injective and L is σ-torsionfree, then F is also σ-injective.*

The quotient functor Q_σ is left exact, but not right exact [27, Section 7 Examples 2, 3]. If $Q_\sigma(M) \cong M \otimes Q_\sigma(R)$ for any right R-module M, then Q_σ is right exact and commutes with direct sums. In this case, σ is called a *perfect torsion radical.* A perfect torsion radical can be characterized as follows.

PROPOSITION 1.9. [6, Theorem 3.1] [27, Theorem 4.3] *The following are equivalent for a torsion radical σ.*

(i) *The inclusion $V_\sigma : \text{Mod-R}/\sigma \to \text{Mod-R}_\sigma$ is an equivalence.*

(ii) *$V_\sigma Q_\sigma$ is naturally isomorphic to $(\cdot) \otimes_R R_\sigma$.*

(iii) *The inclusion $U_\sigma : \text{Mod-R}/\sigma \to \text{Mod-R}$ is exact and preserves direct sums.*

(iv) *Every right R_σ-module is σ-torsionfree as an R-module.*

(v) *$DR_\sigma = R_\sigma$ for all σ-dense right ideals D in R.*

(vi) *R_σ is a flat left R-module and is an epimorphic image of R, and the canonical homomorphism $D \otimes R_\sigma \to R$ is an isomorphism for every σ-dense D.*

(vii) *Every $Q_\sigma(R)$-module is σ-torsionfree and σ-injective.*

(viii) $Q_\sigma(M) \cong M \otimes Q_\sigma(R)$.

Let σ be a torsion radical. Then there exists a 1-1 correspondence between subobjects of $Q_\sigma(M)$ in the quotient category Mod-R $/\sigma$ and σ-closed submodules of M. Note that $R_\sigma = Q_\sigma(R)$ has a natural ring structure arising from that of R. If σ is perfect, then there is a 1-1 correspondence between the right ideals of R_σ and σ-closed right ideals of R.

1.4. Dimensions

Let S be a nonempty set. A relation $\leq$ on S is called a *partial order* if (i) it is reflexive, i.e., $s \leq s$ for all $s \in S$, (ii) anti-symmetric, i.e., if $s \leq t$ and $t \leq s$ then $s = t$, and (iii) transitive, i.e., if $s \leq t$ and $t \leq r$ then $s \leq r$. The pair $(S, \leq)$ is called a *poset*.

For any right R-module M there is a dimension that measures how far M deviates from being an artinian module. Gabriel and Rentschler defined it in terms deviations of posets of submodules of M and called it Krull dimension. Krause extended the definition to include infinite ordinals.

DEFINITION. Let $(P, \leq)$ be a poset. Define $b/a = \{x \in P \mid a \leq x \leq b\}$ for any $a \leq b \in P$.

If $(P, \leq)$ is trivial, then $\operatorname{dev}(P)$, the *deviation* of P, is set to be -1. If P is not trivial and satisfies DCC (descending chain condition), then $\operatorname{dev}(P) = 0$. For an arbitrary ordinal α, $\operatorname{dev}(P)$ is defined to be α if

(i) $\operatorname{dev}(P) \neq \beta$ for all ordinals $\beta < \alpha$.

(ii) If $p_1 \geq p_2 \geq \cdots$ is any descending chain, then $\operatorname{dev}(p_i/p_{i+1}) < \alpha$ for all but finitely many i.

DEFINITION. Let M be a right R-module. The *Krull dimension* of M, denoted by $|M|$, is defined as $\operatorname{dev}(\mathcal{M})$ where $\mathcal{M}$ is the poset of submodules of M, partially ordered by inclusion.

For a ring R regarded as a right R-module, $|R_R|$ is called the right Krull dimension.

PROPOSITION 1.10. [30, Lemma 1.1, Proposition 1.3, Corollary 5.9, Corollary 7,5], [21]

(i) *If M is a noetherian module, then $|M|$ exists.*

(ii) *If $|M|$ exists, then $|M| = \max\{|N|, |M/N|\}$ for any submodule N of M. Note that $|N|$, $|M/N|$ exist whenever $|M|$ does.*

(iii) *If R is a ring with right Krull dimension, then*

$$\begin{aligned}|R_R| &= |(R/N)_R| = |(R/N)_{R/N}| \\ &= \max\{|(R/P)_{R/P}| \mid P \text{ is a minimal prime ideal in } R\}\end{aligned}$$

where N is the prime radical of R.

If σ is a torsion radical, then the Krull dimension can be relativized with respect to σ [**37**].

DEFINITION. Define $\mathcal{K}_\sigma = \{N \subseteq M \mid \sigma(M/N) = 0\}$. The *relative σ-Krull dimension* of M, denoted by $|M|_\sigma$, is defined to be $\operatorname{dev}(\mathcal{K}_\sigma)$.

In 1928, Krull [**47**] defined for a commutative ring R the *classical Krull dimension* of R as the supremum of lengths of chains of prime ideals. The definition has been extended to noncommutative rings and to include infinite ordinals. Recall that Spec(R) is the set of prime ideals of R.

DEFINITION. Let R be a ring and let α be an ordinal. Define $X_{-1} = \emptyset$, and assume that X_β has been defined for all $\beta < \alpha$. Set

$$X_\alpha = \{P \in \operatorname{Spec}(R) \mid Q \supset P \text{ with } Q \in \operatorname{Spec}(R) \Rightarrow Q \in X_\beta \text{ for some } \beta < \alpha\}.$$

If $\operatorname{Spec}(R) = X_\gamma$ for some ordinal γ, then R is said to have *classical Krull dimension.* If α is the least such ordinal, we write $\operatorname{cl.K.dim}(R) = \alpha$.

Not all rings have classical Krull dimension. However, if the ring has right Krull dimension, then cl.K.dim(R) exists, and $\operatorname{cl.K.dim}(R) \leq |R_R|$. For a right FBN ring R, it is known that $\operatorname{cl.K.dim}(R) = |R_R|$ [44, Theorem 2.4].

PROPOSITION 1.11. [44, Proposition 1.2, Lemma 1.3] *Let R be a ring with classical Krull dimension.*

(i) *If I is an ideal of R, then* $\operatorname{cl.K.dim}(R/I) \leq \operatorname{cl.K.dim}(R)$.

(ii) *If* $P \subset Q$ *are primes in* R*, then* $\text{cl.K.dim}(R/Q) < \text{cl.K.dim}(R/P)$. *Hence,* R *has* ACC *on prime ideals.*

(iii) *If* R *is a right noetherian ring, then there is a minimal prime ideal* P *such that* $\text{cl.K.dim}(R/P) = \text{cl.K.dim}(R)$.

1.5. Finite Annihilation

A ring R is right artinian iff every right R-module M is finitely annihilated. If R is a commutative ring, then every finitely generated right R-module is finitely annihilated. If R is a right fully bounded right noetherian (right FBN) ring, then every finitely generated right R-module is finitely annihilated [**17**, Théorème II 8]. If R is a right noetherian ring satisfying the right restricted strong second layer condition, then every finitely generated tame right R-module is finitely annihilated [**67**, Proposition 1.2].

Definition. A right R-module M is called a Δ-*module* if R has DCC for right annihilators of subsets of M.

Lemma 1.12. [**8**, Proposition 1.4] [**20**] *A module* M *is a* Δ-*module iff any nonempty subset* S *of* M *is finitely annihilated.*

Proof. $(\Leftarrow)$: Let $r(S_1) \supseteq r(S_2) \supseteq \cdots$ be a descending chain where each S_i is a subset of M. Without loss of generality, we may assume that $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$.

Let $S = \bigcup S_i$. Then $r(S) = r(x_1, \ldots, x_m)$ for some $x_i \in S$. There exists some n such that $\{x_1, \ldots, x_m\} \subseteq S_n$, so

$$r(S) = r(x_1, \ldots, x_m) \supseteq r(S_k) \supseteq r(S) \text{ for all } k \geq n.$$

$(\Rightarrow)$: Let S be a nonempty subset of M. Assume that S is not finitely annihilated. Then there exists an infinite sequence $s_1, s_2, \ldots$ in S such that $r(s_1) \supset r(s_1, s_2) \supset \cdots$ and $r(s_1, \ldots, s_i) \supset r(S)$ for all i. This obviously contradicts the DCC on right annihilators on M. □

LEMMA 1.13. *Let $M, M_1, M_2, \ldots, M_n$ be Δ-modules. Then*

(i) *Any $0 \neq N \subseteq M$ is also a Δ-module.*

(ii) *For any nonzero ideal $I \subseteq R$, $M/l_M(I)$ is a Δ-module.*

(iii) *$M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is a Δ-module.*

(iv) *If I is an ideal such that $I \subseteq r(M)$, then M_R is a Δ-module iff $M_{R/I}$ is a Δ-module.*

PROOF. (i), (iii) and (iv) follow from the definition of a Δ-module.

(ii). Suppose that there exists a sequence $x_1, x_2, \cdots \in M$ such that

$$r(x_1 + l_M(I)) \supseteq r(x_1 + l_M(I), x_2 + l_M(I)) \supseteq \cdots .$$

However, $r(x_1,\ldots,x_n) = r(x_1,\ldots,x_n,x_{n+1}) = \ldots$ for some n, as M is a Δ-module. We claim that

$$\begin{aligned} &r(x_1 + l_M(I),\ldots,x_n + l_M(I)) \\ &= r(x_1 + l_M(I),\ldots,x_n + l_M(I),x_{n+1} + l_M(I)) \\ &= \ldots. \end{aligned}$$

Let $J = r(x_1 + l_M(I),\ldots,x_n + l_M(I))$. So $x_iJ \subseteq l_M(I)$ for all $i \leq n$. It follows that $x_iJI = 0$ for $i \leq n$, and hence $x_kJI = 0$ for all k. Thus

$$J \subseteq r(x_1 + l_M(I),\ldots,x_k + l_M(I)) \subseteq J$$

for all $k \geq n$. This proves the claim. □

LEMMA 1.14. [10, Lemma 1.3]

(i) *Let M, N be right R-modules such that (a) M can be embedded into a direct sum of copies of N and (b) N is a homomorphic image of a direct sum of copies of M. Then M is finitely annihilated iff N is.*

(ii) *Let M, N be right R-modules such that M can be embedded into a direct product of copies of N. If every submodule of N is finitely annihilated, then the same holds for M.*

(iii) *If every fully invariant submodule N of M is finitely annihilated, then every submodule of M is finitely annihilated.*

(iv) *If $N_1, \ldots, N_n$ are right R-modules such that every submodule of each N_i is finitely annihilated, then every submodule of $\bigoplus_{i=1}^{n} N_i$ is finitely annihilated.*

PROOF. (i). First we claim that $r(M) = r(N)$. Since $M \hookrightarrow \bigoplus N$, $r(M) \supseteq r(\bigoplus N) = r(N)$. Let $f \colon \bigoplus M \to N$ be the surjective homomorphism. Then $r(N) = r(f(\oplus M)) \supseteq r(\oplus M) = r(M)$. Therefore, $r(M) = r(N)$ proves the claim.

Suppose that $r(M) = r(m_1, \ldots, m_k)$ for some $m_i \in M$. Without loss of generality, each $m_i = n_{i,1} \oplus \cdots \oplus n_{i,l_i}$ where $n_{i,j} \in N$, so

$$r(N) = r(M) = r(m_1, \ldots, m_k) = r(n_{1,1}, \ldots, n_{k,l_k}) \supseteq r(N).$$

Hence N is finitely annihilated.

Suppose that $r(N) = r(n_1, \ldots, n_k)$ for some $n_1, \ldots, n_k \in N$. Since each $n_i = f(m_{i,1} \oplus \cdots \oplus m_{i,l_i})$ where $m_{i,j} \in M$, it follows that

$$\begin{aligned} r(M) = r(N) &= r(n_1, \ldots, n_k) \\ &= r(f(\oplus m_{1,j}), \ldots, f(\oplus m_{k,j})) \\ &\supseteq r(m_{1,1}, \ldots, m_{k,l_k}) \supseteq r(M). \end{aligned}$$

Hence M is annihilated by $r(m_{1,1}, \ldots, m_{k,l_k})$.

(ii). Let M' be a submodule of M and let

$$N' = \sum \{f(M') \mid f : M' \to N\}.$$

Observe that M' can be embedded in a direct product of copies of N', as M is embedded into $\prod N$. Then $r(M') \subseteq r(N') = r(\prod N') \subseteq r(M')$. Since N' is finitely annihilated, there exist $n_1, \ldots, n_k \in N'$ such that $r(N') = r(n_1, \ldots, n_k)$. Now each $n_i = \sum_{j=1}^{r_i} f_{ij}(m_{ij})$ for some $f_{ij} : M' \to N$ and $m_{ij} \in M'$. We have

$$r(M') \subseteq r(m_{11}, \ldots, m_{kr_k}) \subseteq r(n_1, \ldots, n_k) = r(N') = r(M').$$

Hence M' is finitely annihilated.

(iii). Let M' be a submodule of M and let

$$N = \sum \{f(M') \mid f \in \mathrm{End}_R(M)\}.$$

Then since N is a fully invariant submodule of M, by hypothesis N is finitely annihilated. Note that $M' = i(M') \hookrightarrow N$. Define $\phi : \bigoplus_{f \in \mathrm{End}(M)} M' \to N$ by $\phi(m_1 + \cdots + m_k) = f_1(m_1) + \cdots + f_k(m_k)$ where $f_i \in \mathrm{End}(M)$, so ϕ is surjective. By (i), the result follows.

(iv). Observe that the submodules invariant under endomorphisms of $\bigoplus_{i=1}^n N_i$ are of the form $\bigoplus_{i=1}^n N_i'$ where $N_i' \subseteq N_i$. Now apply (iii). ☐

LEMMA 1.15. *Let R be a right FBN ring. If M is a finitely generated P-primary uniform right R-module such that $r_R(M) = Q \in \mathrm{Spec}(R)$, then $Q = P$.*

PROOF. Suppose $Q \subset P$. Let $I = r_R(M/l_M(P))$, so $MIP = 0$. Thus $IP \subseteq Q$, so either $I \subseteq Q$ or $P \subseteq Q$. The latter is ruled out by $Q \subset P$.

For the former, $I = Q$ since $r(M) = Q$. Then $M/l_M(P)$ is a right R/Q-module, finitely generated by, say $m_1 + l_M(P), \ldots, m_k + l_M(P)$. Note that $l_M(P) \subseteq_e M$, so $r_{R/Q}(m_i + l_M(P)) \subseteq_e R/Q$ for all i. Since R/Q is right bounded, there exists a nonzero ideal $K/Q \subseteq \bigcap_{i=1}^{k} r_{R/Q}(m_i + l_M(P))$. Now $K \subseteq r_R(M/l_M(P)) = I = Q$ yields the contradiction. □

Note that Lemma 1.15 shows that all right FBN rings satisfy the right strong second layer condition [See Chapter 2]. Moreover, Theorem 1.5 can now be strengthened. This was given as an exercise in [29, Exercise 8E].

PROPOSITION 1.16. *Let R be a right FBN ring and let M be a finitely generated right R-module. Then M is a Δ-module.*

PROOF. By Lemma 1.13, without loss of generality, we may assume that M is a uniform module since $M \hookrightarrow_e M/M_1 \oplus M/M_2 \oplus \cdots \oplus M/M_n$ where each M/M_i is uniform and $n = \text{rank}(M)$ [18, Lemma 7.9].

Let $P = \text{Ass}(M)$ and set $l_{R/r_R(S)}(P) = L/r_R(S)$. In view of Lemma 1.12, it suffices to show that any nonempty subset S of M is finitely annihilated. We claim that

$$L/r_R(S) \subseteq_e R/r_R(S).$$

Let $U/r(S)$ be a prime uniform submodule of $R/r(S)$ such that $r(U/r(S)) = \text{Ass}(U/r(S)) = Q$. Clearly, $r(SU) = Q$. As SU is a P-tame uniform module, $Q = P$ by Lemma 1.15. This gives $U/r(S) \subseteq L/r_R(S)$, proving the claim.

Suppose that S is not finitely annihilated. Then there exists $s_1 \in S$ such that $r(s_1) \supset r(S)$ and $r(s_1) \cap L \supset r(S)$. So there exists $s_2 \in S$ such that

$s_2(r(s_1) \cap L) \neq 0$ and $r(s_1, s_2) \supset r(S)$. Continuing inductively, we get an infinite sequence $s_1, s_2, \cdots \in S$ such that

$$r(s_1, \ldots, s_n) \cap L \supset r(s_1, \ldots, s_n, s_{n+1}) \cap L \text{ for all } n.$$

Now

$$0 \neq \frac{r(s_1, \ldots, s_n) \cap L}{r(s_1, \ldots, s_n, s_{n+1}) \cap L} \hookrightarrow s_{n+1}L,$$

each of which is a torsionfree right R/P-module. Note that $L/r(S)$ has finite reduced rank over R/P, say k. However,

$$\begin{aligned} k &= \rho_{R/P}(\frac{L}{r(S)}) \\ &= \rho_{R/P}(\frac{L}{L \cap r(s_1)}) + \rho_{R/P}(\frac{L \cap r(s_1)}{L \cap r(s_1, s_2)}) \\ &+ \cdots + \rho_{R/P}(\frac{L \cap r(s_1, \ldots, s_k)}{L \cap r(s_1, \ldots, s_k, s_{k+1})}) > k. \end{aligned}$$

This is a contradiction. □

Let M be a right R-module. It was noted in [7, p. 1778] that rad_M-closed right ideals of R are right annihilators of subsets of M.

LEMMA 1.17. *Let $\sigma = \mathrm{rad}_M$ for some right R-module M. A right ideal I is σ-closed in R iff it is the right annihilator of some subset of M.*

PROOF. $(\Rightarrow)$: If I is σ-closed, then

$$\sigma(R/I) = \mathrm{rad}_M(R/I) = \bigcap\{\ker f \mid f : R/I \to M\} = 0.$$

Let $S = \{f(1 + I) \mid f : R/I \to M\}$. We claim that $I = r(S)$. Since $f(1 + I)I = f(I/I) = 0$, it follows that $I \subseteq r(S)$.

If $r \in r(S)$, then $f(1+I)r = f(r+I) = 0$ for all f, so $r + I \in \ker f$ for all f. Therefore, $r + I \in \mathrm{rad}_M(R/I) = 0$. Hence $I = r(S)$.

$(\Leftarrow)$: If $I = r(S)$ for some subset S of M, then

$$R/I = R/r(S) = R/\bigcap_{s\in S} r(s) \hookrightarrow \prod_{s\in S} sR \hookrightarrow M^S.$$

Thus, $\sigma(R/I) = 0$ by definition. □

Thus, if R is right rad_M-artinian, i.e., if R has DCC on rad_M-closed right ideals, then M is finitely annihilated, in fact, a Δ-module. Moreover, every rad_M-torsionfree right R-module is finitely annihilated.

There is another way of defining torsion radicals in terms of multiplicative sets, which is related to rad_E where E is an injective module. When C is a multiplicative subset of R,

$$\rho_C = \{M \in \text{Mod-R} \mid \text{ for any } m \in M \text{ there is some } c \in C \text{ with } mc = 0\}$$

defines a torsion class. We define

$$\rho_C(M) = \{m \in M \mid \text{ for any } r \in R, mrc = 0 \text{ for some } c \in C\}.$$

A partial order can be defined for the set of preradicals on Mod-R. If σ, ρ are preradicals, then we say that $\sigma \leq \rho$ if $\sigma(M) \subseteq \rho(M)$ for all right R-modules M. If $\sigma = \mathrm{rad}_{\mathrm{E}(M)}$ where $\mathrm{E}(M)$ is the injective hull of M, then σ is the largest torsion radical for which M is torsionfree.

LEMMA 1.18. *Let σ be a torsion radical. Then $\sigma \leq \mathrm{rad}_M$ iff $\sigma(M) = 0$.*

PROOF. $(\Rightarrow)$: If $\sigma \leq \mathrm{rad}_M$, then

$$\sigma(M) \subseteq \mathrm{rad}_M(M) = \bigcap\{\ker f \mid f : M \to M\} = 0.$$

$(\Leftarrow)$: Let N be a module. If $\sigma(M) = 0$, then $f(\sigma(N)) \subseteq \sigma(M) = 0$ for any $f : N \to M$. Thus $\sigma(N) \subseteq \bigcap \ker f = \mathrm{rad}_M(N)$. □

LEMMA 1.19. [9, Proposition 1] *If every essential right ideal I of R contains a regular element of R, then $\rho_{C(0)} = \mathrm{rad}_{\mathrm{E}(R)}$.*

PROOF. Set $\sigma = \mathrm{rad}_{\mathrm{E}(R)}$. Since $\rho_{C(0)}(R) = 0$, we have $\rho_{C(0)} \leq \mathrm{rad}_{\mathrm{E}(R)}$ by Lemma 1.18.

Let D be a σ-dense right ideal in R. We claim that D is essential in R. If not, there exists a non-zero right ideal $I \subseteq R$ such that $I \cap D = 0$. Then $I = I/(I \cap D) \cong (I + D)/D \hookrightarrow R/D$, hence I is σ-torsion. But the inclusion map $\phi : I \to R$ is not a zero map, while $\mathrm{Hom}(I, R) = 0$ as I is σ-torsion. This contradiction shows that any σ-dense right ideal D of R is essential in R. The right ideal D now contains a regular element $c \in R$.

If $\rho_{C(0)} < \sigma$, then there exists a module M such that

$$N_1 = \rho_{C(0)}(M) \subset \sigma(M) = N_2. \text{ So}$$

$$\rho_{C(0)}(N_2/N_1) \subseteq \rho_{C(0)}(M/N_1) = 0 \subsetneq \sigma(N_2/N_1) = N_2/N_1 \neq 0.$$

Hence, without loss of generality, assume that $\rho_{C(0)}(M) = 0 \subset \sigma(M) = M$. For any $0 \neq m \in M$, the cyclic submodule $mR \cong R/r(m)$ is σ-torsion, so

$r(m)$ contains an element $c \in \mathcal{C}(0)$, whence $mc = 0$, contradicting the fact that M is $\rho_{\mathcal{C}(0)}$-torsionfree. Thus $\rho_{\mathcal{C}(0)} = \sigma$. $\square$

Since every essential right ideal of a semiprime right Goldie ring R contains a regular element of R, in view of Lemma 1.19, $\rho_{\mathcal{C}(0)} = \mathrm{rad}_{\mathrm{E}(R)}$ for any semiprime right Goldie ring R.

CHAPTER 2

Second Layer Conditions

2.1. Introduction

One of the ways to study the structure of a ring R is to embed it into a ring S with a richer structure, provided information can be passed down from S to R. The ring of integers furnishes the most elementary example, where the overring is the field of rationals in which every nonzero integer has an inverse. This idea of inverting nonzero elements is widely used for commutative integral domains, producing fields of fractions.

More generally, if C is a multiplicatively closed subset of a commutative ring R, then a ring S can be constructed so that every element s of S is a "fraction" r/c or rc^{-1} for some r in R and c in C. In this case, we denote S by RC^{-1} and call it the quotient ring of R with respect to C. In particular, if $C = \mathcal{C}(P) = R \setminus P$ for some prime ideal P, then RC^{-1} is a local ring with the unique maximal ideal PRC^{-1}. Thus, the term localization is also used for constructing a quotient ring and we say that R is localized at P or at C.

However, for noncommutative rings, localizing is not as simple as in the commutative case, and the question of localizability was first raised by Van der Waerden around 1930 [74]. An attempt to form a quotient ring, if possible, inevitably calls for a decision as to whether the right or left quotient ring is to be built.

Suppose that R is a noncommutative integral domain. If R is to have a right quotient ring D, where nonzero elements of R are invertible in D and every element of D is written as rc^{-1} for some $r, c \in R$, then R has to satisfy some obvious conditions: If c is a nonzero element and r is an element of R, then the product $c^{-1}r$ also need be an element of D. In other words, there must exist some $t, d \in R$ such that $c^{-1}r = td^{-1}$. Multiplying on the appropriate side by c and d respectively, we get a common multiple $rd = ct$. Therefore, the ring R is required to satisfy the common multiple condition $cR \cap rR \neq 0$ for any given nonzero c and r in R.

Ore realized in 1931 that this condition was not only necessary, but also sufficient when he constructed quotient rings from noetherian domains [**62**]. The condition that $cR \cap rC \neq \emptyset$ for any $c \in C$ and any $r \in R$, where C is a multiplicatively closed subset of R, is nowadays called the right Ore condition. Together with another condition, called right reversibility (that is automatically satisfied if R is right noetherian), it allows the construction of a right ring of fractions of R. The set C is called right reversible if whenever $cr = 0$ for $c \in C, r \in R$ then $rc' = 0$ for some $c' \in C$.

However, in 1936, Malcev [**53**] produced an example of a domain that did not have a division ring of quotients. Since no large classes of rings could be found at the time that satisfied Ore's condition, interest waned for some years until Tamari [**72**] showed in 1951 that universal enveloping algebras of finite dimensional Lie algebras satisfied the condition on both sides and thus had right and left quotient division rings.

The real impetus for localization came from Goldie who observed, in 1957, that Ore's noetherian domain had one particularly interesting characteristic. He discovered a dichotomy in the structure of domains: A domain R has either an infinite direct sum of right ideals or any two nonzero right ideals have a nonzero intersection. The latter results in a localizable domain, which is now called an Ore domain. This observation led to the concept of rank, which measures how many independent uniform submodules a module contains. It became a valuable tool in the study of noncommutative rings [22]. For example, the quotient ring of a prime ring with finite rank n and maximum condition on right annihilators is a ring of $n \times n$ matrices over a division ring. This fact paved the way for the study of quotient rings of noncommutative rings to progress rapidly.

During the years following Goldie's fundamental results, considerable research was conducted on which rings satisfy the Ore condition and Ore's method has emerged as the natural and correct way to form a quotient ring. However, often cases are encountered where a right Ore set C does not necessarily consist of regular elements of the ring R , and whence there may not be an embedding. In this case, it is necessary first to factor out the torsion part $t_C(R) = \{r \in R \mid rc = 0 \text{ for some } c \in C\}$, and then the torsionfree factor ring $R/t_C(R)$ is embedded into its quotient ring so that regular elements of $R/t_{\mathcal{C}}(R)$ are invertible in S. The ring S is still called a quotient ring of R with respect to $\mathcal{C}$.

In 1966, Small developed a criterion that tells when a noetherian ring has an artinian quotient ring using Ore's method [68, Theorems 2.10, 2.11, 2.12]. Thus this method has proved to be powerful and elegant for localizing noncommutative rings, provided it can be applied to them at all. However, its major drawback is that Ore's condition fails quite unexpectedly even when applied to some "nice" noetherian rings. While it is tempting to expect $\mathcal{C}(P)$ to be right Ore for a prime ideal P when trying to localize R at P, this is often not the case.

For a simple example, consider $U(\mathrm{g})$, the universal enveloping algebra of the 2-dimensional solvable Lie algebra g over an algebraically closed field k of characteristic 0 with k-basis $\{x, y\}$ such that $[x, y] = x$. In 1976, Müller [57] showed that $U(\mathrm{g})$ cannot be localized at some prime ideals, although it can be at the zero prime ideal.

It was Jategaonkar who observed the difficulty with Ore's method in localizing $U(\mathrm{g})$ and other classes of noetherian rings such as group rings of polycyclic-by-finite groups, and initiated an investigation as to why it failed. While he was comparing the structure of artinian rings with that of fully bounded noetherian rings, he recognized that prime ideals are closely related to injective hulls of certain modules, and bimodules. Moreover, as the quotient ring $R\mathcal{C}^{-1}$ of R with respect to $\mathcal{C}$ satisfies the condition

$RC^{-1}/R = t_C(\mathrm{E}(R)/R)$ if R is C-torsionfree, the necessity for studying the "layered structure" of certain modules presents itself.

He suggested a new point of view on localization by looking into the structure of prime factor rings and layers of their indecomposable injective modules. Noticing that localization is closely connected to a certain short series of submodules and a relation between prime ideals arising as annihilators of layers of submodules, he defined the second layer condition. In the process, it turns out that if C is a right Ore set contained in $\mathcal{C}(P)$ for some prime P then C has to be in every $\mathcal{C}(Q)$ where Q is linked to P. Hence linked primes were the stumbling block for $\mathcal{C}(P)$ to be right Ore.

The idea of the second layer condition for a prime ideal was first alluded to in his 1974 paper [**36**] on fully bounded noetherian rings. He observed that a certain "undesirable case" never arose in the second layer $E/l_E(P)$ of an indecomposable injective module E where P is the prime ideal that is maximal among annihilators of nonzero submodules of E. The second layer condition took on a provisional shape in 1982 when it was introduced as $(*)$ in [**39**], and then, in 1986, evolved into the present definition in terms of second layers of tame modules in [**40**].

The second layer condition, as it turns out, has a close relation with Ore's method. If a nonempty set of prime ideals is classically localizable, then it has to satisfy the second layer condition. Conversely if it satisfies the second layer condition along with some other properties, then it is (classically) localizable. Hence, Ore's method benefits from a thorough understanding of the second layer condition. Furthermore, the second layer condition can be stated in terms of finite annihilation of certain modules. Thus it is a weakened version of an artinian ring property that every module is finitely annihilated. Since an artinian ring is its own quotient ring, rings equipped with the second layer condition can be expected to have quotient rings with a richer structure. In some sense, a ring with second layer condition can be viewed as a poor man's artinian ring.

Currently well-understood noetherian rings such as polynomial identity rings, group rings of polycyclic-by-finite groups and enveloping algebras of finite dimensional solvable Lie algebras all satisfy the second layer condition. These rings are known to be "accessible, pleasant and useful" [**40**, p. 219].

However, the awkwardness in defining the second layer condition in terms of layers has created ad-hoc approach to applications and contributed to a proliferation of definitions [**39**], [**29**], [**67**], [**52**].

The purpose of this chapter is to present the existing definitions and to investigate the relations between them. In the next chapter it is also shown that the second layer condition is inherited by certain extension rings, namely, finite extensions and some centralizing extensions.

2.2. Links

Let R be a right noetherian ring. A semiprime ideal I is said to be *right localizable* if $C(I)$ is a right Ore set in R. We can check the localizability of I by using a uniform module and its layers. If M is a finitely generated uniform module with a submodule L that is isomorphic to a right ideal of R/I, and such that M/L is $\mathcal{C}(I)$-torsion, then I is right localizable iff $MI = 0$ [**34**], [**35**].

If I is a prime ideal, then it is difficult to verify that I is right localizable since, in most cases, $MI = 0$ is a rare occurrence; cf., Proposition 2.2. Rather it happens frequently that I is linked to some other prime ideal that is an associated prime ideal of M/L.

DEFINITION. Let P and Q be prime ideals of a right noetherian ring R. We say that Q is *linked* to P, denoting this by $Q \rightsquigarrow P$, if there exists an ideal A such that $QP \subseteq A \subset Q \cap P$ and $(Q \cap P)/A$ is torsionfree as a right R/P-module and fully faithful as a left R/Q-module.

The set

$$\Omega^r(P) = \{Q \in \mathrm{Spec}(R) \mid Q = P_n \rightsquigarrow \cdots \rightsquigarrow P_1 = P \text{ for some primes } P_i\} \cup \{P\}$$

is called the *right clique* of P.

DEFINITION. Let $\mathcal{X}$ be a nonempty subset of Spec(R). Then $\mathcal{X}$ is said to be *right link closed* if $Q \rightsquigarrow P$ for some $P \in \mathcal{X}$ implies that $Q \in \mathcal{X}$.

The name link was introduced by Müller [**57**]. It is easy to see from the definition of link that if 0 is a prime ideal then it is neither linked to nor from any other prime ideal. Also a minimal prime ideal of a semiprime noetherian ring cannot be linked to nor from any other prime ideal [**39**, Lemma 3.3]. In a commutative noetherian ring, if P and Q are linked prime ideals, then $P = Q$, so cliques consist of only one element. There are more possibilities for the cardinality of cliques in noncommutative rings, they may even be infinite as shown in the next example. However, Stafford showed that cliques are at most countable [**70**, Corollaries 3.10, 3.13].

EXAMPLE 2.1. [**57**, Lemma 12] Let k be an algebraically closed field of characteristic 0. Let g be the Lie algebra with k-basis $\{x, y\}$ such that $[y, x] = x$, and let $S = U(\mathrm{g})$ be its enveloping algebra, which can be viewed as the skew polynomial ring $S = k[x][y; x\frac{d}{dx}]$. The primes of S are $0, xS$ and $P_a = xS + (y - a)S$ where $a \in k$, and they are completely prime. The right clique $\Omega^r(P_a)$ of P_a is the countably infinite set $\{P_a, P_{a+1}, \dots\}$.

We now show that $U(\mathrm{g})$ cannot be localized at the prime ideal P_a. Note that $y - a - 1 \notin P_a$, and so $y - a - 1 \in \mathcal{C}(P_a)$. Suppose that $\mathcal{C}(P_a)$ is a right Ore set, so $(y-a-1)S \cap x\mathcal{C}(P_a) \neq \emptyset$. There exist $s \in S$ and $d \in \mathcal{C}(P_a)$ so that $(y - a - 1)s = xd \in xS$. Since xS is completely prime and $(y - a - 1) \notin xS$, we get $s \in xS$, so $s = xs_1$ for some $s_1 \in S$. Then

$$xd = (y - a - 1)s = (y - a - 1)xs_1$$

$$= (yx - ax - x)s_1 = (xy + x - ax - x)s_1 = x(y - a)s_1,$$

so it follows that $(y - a)s_1 = d \in \mathcal{C}(P_a) \cap P_a = \emptyset$, an obvious contradiction. Hence, $\mathcal{C}(P_a)$ is not right Ore. □

Let P be a prime ideal. It turns out that any right Ore set contained in $\mathcal{C}(P)$ has to be in $\bigcap\{\mathcal{C}(Q) \mid Q \rightsquigarrow P\}$. Thus the trouble with the attempt to localize R at P stems from the primes linked to P.

PROPOSITION 2.2. [**40**, Theorem 5.4.5] *Let P be a prime ideal of a right noetherian ring R. If C is a right Ore set in R contained in $\mathcal{C}(P)$, then C is contained in $\mathcal{C}(Q)$ for every $Q \rightsquigarrow P$.*

PROOF. Let $(Q \cap P)/A$ be a linking bimodule of Q and P. In other words, $(Q \cap P)/A$ is torsionfree as a right R/P-module and fully faithful as a left R/Q-module. By replacing C with $\overline{C} = \{c + A \mid c \in C\}$ and R with R/A, we may assume that $A = 0$ since $\overline{C}$ is a right Ore set in R/A. It suffices to show that every $c \in C$ is right regular modulo Q by [**29**, Lemma 5.7]. If $cr \in Q$, then $(cr)x = 0$ for all $x \in Q \cap P$. Since C is right reversible by [**25**, Proposition 2], there exists $d \in C \subseteq \mathcal{C}(P)$ such that $(rx)d = 0$. It follows that $rx = 0$, as $rx \in Q \cap P$ and $Q \cap P$ is a torsionfree right R/P-module. Thus $r \in l(Q \cap P) = Q$, proving that $C \subseteq \mathcal{C}'(Q) = \{c \in R \mid cr = 0 \Rightarrow r = 0\}$. □

2.3. The Second Layer Condition

Let P be a prime ideal of a ring R. Jategaonkar observed in 1974 that when R is a right fully bounded right noetherian ring, the second layer $\mathrm{E}(R/P)/l_{\mathrm{E}(R/P)}(P)$ of the R-injective hull $\mathrm{E}(R/P)$ of R/P was never wild. This observation led to the next lemma that subsequently became the cornerstone in the study of second layer conditions.

Lemma 2.3 (Jategaonkar's Main Lemma). [39, Main Lemma 2.2] *Let R be a right noetherian ring and let P be a prime ideal. Let M be a P-tame module and set $r(M) = A$ and $L = l_M(P)$. Assume that $r(N) = A$ for any submodule $N \not\subseteq L$, and that M/L is Q-prime for some prime ideal Q. Then one of the following mutually exclusive cases occurs:*

(i) *$Q = A \subset P$. In this case, M/L is a torsion R/Q-module.*

(ii) *$Q \rightsquigarrow P$ via A.*

Proof. Since $0 \subset L \subset M$ is an affiliated series with corresponding affiliated primes P, Q, we see that $MQP = 0$, so $QP \subseteq A$. On the other hand, $A \subseteq P = \mathrm{Ass}(M)$ and $A \subseteq Q = r(M/L)$ give $A \subseteq Q \cap P$. Thus $QP \subseteq A \subseteq Q \cap P$. This yields 2 cases: (i) $A = Q \cap P$, and (ii) $A \subset Q \cap P$.

(i) If $A = Q \cap P$, then $MPQ \subseteq M(Q \cap P) = 0$. If $MP \subseteq L$, then $P \subseteq r(M/L) = Q$ and thus $A = P$. But this contradicts $L = l_M(P) \subset M$. So we must have $MP \not\subseteq L$, yielding $Q \subseteq r(MP) = A$ by the hypothesis imposed on M. Hence $A = Q \subseteq P$. However, $Q = P$ is ruled out because $L \subset M$. Consequently, $A = Q \subset P$. Also, in this case, since M is an essential extension of the R/Q-module L, M/L is torsion as an R/Q-module.

(ii) If $A \subset Q \cap P$, then we must show that $(Q \cap P)/A$ is torsionfree on the right and fully faithful on the left.

Let B/A be the right torsion submodule of $(Q \cap P)/A$. If $B/A \neq 0$, then $MB \neq 0$ and so $L \cap MB \neq 0$. Any nonzero element x in this intersection can be written as $x = m_1b_1 + \cdots + m_kb_k$ for some $m_i \in M$ and $b_i \in B$.

Since B/A is torsion, there exists $c \in \mathcal{C}(P)$ such that $b_i c = 0$ for all i. This means that $xc = 0$, which is impossible since x belongs to the torsionfree R/P-module L. Thus $(Q \cap P)/A$ is torsionfree on the right.

For the left side, let B/A be a nonzero left R/Q-submodule of $(Q \cap P)/A$ and set $J = l_R(B/A)$. Note that J is an ideal containing Q. Since $JB \subseteq A$, we get $MJB = 0$. If $MJ \not\subseteq L$, then $A = r(MJ) \supseteq B$. This contradicts the fact that $A \subset B$. Therefore, $MJ \subseteq L$, whence $J \subseteq r(M/L) = Q$. This shows that $J = Q$, so $(Q \cap P)/A$ is fully faithful on the left. □

Note that in Jategaonkar's Main Lemma a P-tame module M with $r(N) = r(M)$, whenever $N \not\subseteq L$, can be constructed by choosing a submodule M' such that $L \subset M' \subseteq M$ and $r(M)$ is maximal among $r(N)$ where $L \subset N \subseteq M$ from R being right noetherian.

The case (i) in the above lemma is called *undesirable* since the second layer M/L is a torsion R/Q-module (wild). In case (ii), if $(Q \cap P)/A$ is torsionfree as a left R/Q-module, then M/L is tame [40, Lemma 6.1.1]. This is the case when R is noetherian. Thus the case (ii) is *desirable*. In a right noetherian ring without the second layer condition, we can only produce a Q-tame submodule of M/L, but cannot guarantee that M/L is tame.

If M is just P-primary in Jategaonkar's Main Lemma, then the conclusion is similar to the tame case except that in the desirable case, we can only show that the linking bimodule $(Q\cap P)/A$ is fully faithful on both sides [**40**, Lemma 6.1.2].

We are now in position to display a relation between links for prime ideals and the structure of tame modules using the Main Lemma. The links can be also characterized in terms of layers of tame modules.

THEOREM 2.4. [**28**, Theorem 1.3] *Let R be a right noetherian ring. A prime ideal Q is linked to a prime ideal P iff there exists a finitely generated P-tame uniform module M such that the second layer $M/l_M(P)$ of M contains a Q-tame submodule.*

PROOF. ($\Rightarrow$): Let Q be linked to P via A. Since Q/A is linked to P/A via 0, assume that A is 0. So $Q\cap P$ is a torsionfree right R/P-module. Since $Q/(Q\cap P)\cong(Q+P)/P\subseteq R/P$, it follows that $Q/(Q\cap P)$ is torsionfree as well. Combining the above results, we obtain that Q is a torsionfree right R/P-module. Moreover, we claim that $Q_R\subseteq_e R_R$. Let I be a nonzero right ideal of R with $I\cap Q=0$. Then $IQ\subseteq I\cap Q=0$, so $I\subseteq l_R(Q)\subseteq l_R(Q\cap P)=Q$. Hence $I=I\cap Q=0$, contradicting $I\neq 0$.

Thus R_R is a P-tame module. Since R_R is noetherian, it has finite rank, say k. Thus

$$R\hookrightarrow_e \mathrm{E}(R)=\bigoplus_{i=1}^{k}E_i$$

where each E_i is an indecomposable injective P-tame module. Note that $Q \subseteq l_R(P) \subseteq l_R(Q \cap P) = Q$. Hence

$$\begin{aligned}\frac{R}{Q} = \frac{R}{l_R(P)} &= \frac{R}{R \cap l_{\oplus_{i=1}^k E_i}(P)} \\ &\cong \frac{R + l_{\oplus_{i=1}^k E_i}(P)}{l_{\oplus_{i=1}^k E_i}(P)} \hookrightarrow \frac{\bigoplus_{i=1}^k E_i}{l_{\oplus_{i=1}^k E_i}(P)} \cong \bigoplus_{i=1}^k \frac{E_i}{l_{E_i}(P)}.\end{aligned}$$

If U/Q is a uniform submodule of R/Q, then $U/Q \hookrightarrow E_i/l_{E_i}(P)$ for some i. Let $N/l_{E_i}(P)$ be the submodule of $E_i/l_{E_i}(P)$ isomorphic to U/Q. Note that N is uniform since E_i is uniform, and also note that $N/l_{E_i}(P)$ is a Q-prime module. Now choose $m \in N$, but $m \notin l_{E_i}(P)$. Then $mR/l_{mR}(P) \cong (mR + l_{E_i}(P))/l_{E_i}(P)$ is a Q-tame module. Set $M = mR$. Then M is the required module, that is, it is finitely generated P-tame uniform with $M/l_M(P)$ being Q-tame.

($\Leftarrow$): Without loss of generality, assume that $M/l_M(P)$ is a Q-tame module and $0 \subset l_M(P) \subset M$ forms an affiliated series with affiliated primes P, Q satisfying the hypotheses of the Main Lemma. The undesirable case $\tau(M) = Q \subset P$ cannot occur, since it would imply that $M/l_M(P)$ is a torsion right R/Q-module, contradicting the fact that $M/l_M(P)$ is tame, i.e., torsionfree over R/Q. Thus the other case, $Q \rightsquigarrow P$, holds. □

We can now establish a connection between the second layer condition and Jategaonkar's Main Lemma. For noetherian rings, the second layer condition for a prime ideal is sometimes defined in terms of the Main Lemma.

DEFINITION. Let R be a right noetherian ring and P be a prime ideal. The prime ideal P is said to satisfy the ***right affiliated second layer condition*** if the undesirable case in the Main Lemma never arises.

It is said to satisfy the ***right affiliated strong second layer condition*** if the undesirable case never arises for P-primary modules.

Let R be a right noetherian ring. Let M be a P-primary right R-module with an affiliated series $0 \subset l_M(P) \subset M$ and P, Q as affiliated primes such that $r(M)$ is maximal among annihilators of submodules properly containing L. If P satisfies the right affiliated strong second layer condition, then $r(M) = Q \subset P$ never occurs. The right affiliated strong second layer condition is formally weaker than the right strong second layer condition defined below.

When Jategaonkar introduced the second layer condition, he first called it $(*)$, which is a weakened version of the strong second layer condition. The standard definition is given in terms of the second layer of an injective module.

DEFINITION. Let R be a right noetherian ring. We say that a prime ideal P of R satisfies the *right second layer condition* if the second layer $\mathrm{E}(R/P)/l_{\mathrm{E}(R/P)}(P)$ of the injective module $\mathrm{E}(R/P)_R$ is tame.

It is said to satisfy the *right strong second layer condition* if, for every prime ideal $Q \subset P$, every finitely generated P/Q-primary right R/Q-module M is unfaithful as an R/Q-module.

If $\mathcal{X}$ is a nonempty set of prime ideals, then we say that $\mathcal{X}$ satisfies the *right (strong) second layer condition* if every $P \in \mathcal{X}$ does so. We also say that the ring R satisfies the *right (strong) second layer condition* if Spec(R) does so.

REMARK. If M is a uniform P-tame module and if P satisfies the right second layer condition, then $M/l_M(P)$ is tame since M has a nonzero submodule isomorphic to a uniform right ideal of R/P.

Let M be a finitely generated P-tame module. Then the second layer condition on R yields that $M/l_M(P)$ is tame and $Q \rightsquigarrow P$ for every associated prime Q of $M/l_M(P)$. Therefore, an affiliated series of M with corresponding affiliated primes $P_n \rightsquigarrow P_{n-1} \rightsquigarrow \cdots \rightsquigarrow P_1 = P$ can be constructed. Accordingly M is annihilated by a product of primes from $\Omega^r(P)$.

LEMMA 2.5. [40, Lemma 7.1.2] *Let R be a right noetherian ring and let $\mathcal{X}$ be a nonempty set of prime ideals such that $\mathcal{X}$ is right link closed and satisfies the right second layer condition. Then every finitely generated $\mathcal{X}$-tame module M is annihilated by a finite product of primes from $\mathcal{X}$.*

PROOF. Since M is finitely generated, it is a noetherian module. The proof is carried out by noetherian induction. So suppose that every $\mathcal{X}$-tame homomorphic image M/L, where $0 \neq L \subseteq M$, is annihilated by a product of finitely many primes in $\mathcal{X}$. We discuss two cases: (i) M is not uniform, (ii) M is uniform.

(i) If M is not uniform, then there exist nonzero submodules N_1, N_2 such that $N_1 \cap N_2 = 0$. Let M_1 be a submodule of M maximal with respect to properties $N_1 \subseteq M_1$ and $M_1 \cap N_2 = 0$, so that $M_1 \oplus N_2 \subseteq_e M$ and

$$N_2 \cong \frac{M_1 + N_2}{M_1} \subseteq_e \frac{M}{M_1}.$$

Now M_2 can be constructed similarly so that $M_1 \oplus M_2 \subseteq_e M$ and

$$M_1 \cong \frac{M_1 + M_2}{M_2} \subseteq_e \frac{M}{M_2}.$$

Then by the induction hypothesis, M/M_1 is annihilated by a product of primes $P_1 \cdots P_n$ where $P_i \in \mathcal{X}$, since M/M_1 is also $\mathcal{X}$-tame, being an essential extension of the $\mathcal{X}$-tame module N_2. Similarly, M/M_2 is annihilated by $Q_1 \cdots Q_m$ where $Q_j \in \mathcal{X}$. Note that

$$M = \frac{M}{M_1 \cap M_2} \hookrightarrow \frac{M}{M_1} \oplus \frac{M}{M_2}.$$

Thus, $M(P_1 \cdots P_n)(Q_1 \cdots Q_m) \subseteq M_1 \cap M_2 = 0$. Hence, M is annihilated by a finite product of primes from $\mathcal{X}$.

(ii) If M is uniform, then set $\text{Ass}(M) = P \in \mathcal{X}$. If $M = l_M(P)$, then the assertion is trivial. So assume that $l_M(P) \subset M$. Since $M/l_M(P)$ is noetherian, it has finite rank, say k. Hence there exist submodules $M_1, \dots, M_k$ such that

$$\frac{M}{l_M(P)} \hookrightarrow_e \frac{M}{M_1} \oplus \cdots \oplus \frac{M}{M_k}$$

where $M_1 \cap \cdots \cap M_k = l_M(P)$ and each M/M_i is uniform [18, Lemma 7.9]. Then each M/M_i is Q_i-tame where $Q_i = \text{Ass}(M/M_i) \in \text{Ass}(M/l_M(P))$ since

P satisfies the right second layer condition. Note also that Q_i satisfies the right second layer condition since $Q_i \rightsquigarrow P$ by Theorem 2.4 implies $Q_i \in \mathcal{X}$. By the induction hypothesis, each M/M_i is annihilated by a finite product of primes and so $M/l_M(P)$ is annihilated by a finite product of primes $P_1 \cdots P_n$ from $\mathcal{X}$. It now follows that $M(P_1 \cdots P_n)P = 0$. Hence, M is annihilated by a finite product of primes from $\mathcal{X}$. ■

REMARK. As a consequence, if a right R-module M is finitely generated P-tame and $\Omega^r(P)$ satisfies the right second layer condition, then M is, in fact, annihilated by a finite product of prime ideals from $\Omega^r(P)$.

2.4. Rings with the Second Layer Condition

We have already seen that right FBN rings satisfy the right (strong) second layer condition [Lemma 1.15]. Many noetherian rings arising "naturally" are known to satisfy the second layer condition. These rings have a separation characteristic that an ideal between two comparable prime ideals has the Artin-Rees property.

DEFINITION. Let I be an ideal of a ring R. It is said to satisfy the *right AR-property* if for any right ideal J there exists a positive integer n such that $J \cap I^n \subseteq JI$.

The ring R is said to be *right AR-separated* if, for any two prime ideals $Q \subset P$, there exists an ideal I such that $Q \subset I \subseteq P$ and I/Q has the right AR-property in R/Q.

The right AR-property for I can be formulated in terms of modules. Namely, if M is a finitely generated right R-module with $L \subseteq_e M$ such that $LI = 0$, then $MI^n = 0$ for some n [**56**, Theorem 4.2.2].

LEMMA 2.6. [**39**, Proposition 4.1] *If R is a right noetherian and right AR-separated ring, then R satisfies the right strong second layer condition.*

PROOF. Let $Q \subset P$ be prime ideals and let M be a finitely generated P/Q-primary right R/Q-module. Suppose that M is faithful over R/Q. By AR-separatedness, there exists an ideal I such that $Q \subset I \subseteq P$ and I/Q has the right AR-property. Since M is P/Q-primary, $l_M(P) \subseteq_e M$. Now $l_M(P)I \subseteq l_M(P)P = 0$, and so $MI^n = 0$ for some n by the right AR-property of I/Q. Thus $I^n \subseteq r(M) = Q$ yields that $I \subseteq Q$, which contradicts $Q \subset I$. Hence, P satisfies the right strong second layer condition.

□

The following corollary is a consequence of the fact that all of the rings listed there are AR-separated.

COROLLARY 2.7. [**11**], [**13**], [**39**]

(i) *Group rings of polycyclic-by-finite groups over commutative noetherian rings satisfy the right strong second layer condition.*

(ii) *The enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ over a field k satisfies the right strong second layer condition iff $\mathfrak{g}$ is solvable or k has positive characteristic.*

PROPOSITION 2.8. *If R is a right noetherian ring with classical Krull dimension zero, then R satisfies the right strong second layer condition.*

PROOF. Let P be a prime ideal of R. Since every prime ideal is maximal, there is no prime ideal properly contained in any given prime ideal P. Therefore the strong second layer condition is trivially satisfied by P. $\square$

If R, S are noetherian rings and ${}_S B_R$ is a noetherian bimodule, then in general it is not known if the left Krull dimension $|{}_S B|$ is equal to the right Krull dimension $|B_R|$. However, for $|B_R| = 0$, equality was proved by Lenagan [50]. For fully bounded rings, the symmetry of Krull dimensions was shown in [36]. Concerning the classical Krull dimension, the symmetry $\text{cl.K.dim}(S) = \text{cl.K.dim}(R)$ is obtained for rings satisfying the second layer condition by applying the next lemma twice.

LEMMA 2.9. [38, Theorem H] *Let R and S be prime noetherian rings satisfying the right second layer condition. If ${}_S B_R$ is a noetherian bimodule that is torsionfree on each side, then* $\text{cl.K.dim}(S) \geq \text{cl.K.dim}(R)$.

PROOF. Induction on $\alpha = \text{cl.K.dim}(R)$ is used. The cases $\alpha = -1, 0$ hold trivially. Let $\alpha > 0$ and assume that the lemma is true for any ring R with $\text{cl.K.dim}(R) = \beta < \alpha$. If $\text{cl.K.dim}(R) = \alpha$, then for any $\beta < \alpha$ there exists a nonzero prime ideal P in R such that $\text{cl.K.dim}(R/P) = \beta$. Since R satisfies the right second layer condition, there exists a subbimodule factor C/D of B such that C/D is torsionfree as a right R/P-module and torsionfree as a left $S/l_S(C/D)$-module by [**39**, Lemma 1.2]. Then $Q = l_S(C/D) \neq 0$ by [**29**, Proposition 7.4] since $r(C/D) = P \neq 0$. Moreover, $Q \in \mathrm{Spec}(S)$. By the induction hypothesis,

$$\mathrm{cl.K.dim}(R/P) = \beta \leq \mathrm{cl.K.dim}(S/Q) < \mathrm{cl.K.dim}(S).$$

This gives the desired result. □

REMARK. Jategaonkar called a noetherian bimodule ${}_SB_R$ a *right cell* if $r_R(B) = P \in \mathrm{Spec}(R)$, B is torsionfree as a right R/P-module and B/C is unfaithful as a right R/P-module for every nonzero subbimodule C of B. Hence the noetherian bimodule B and the factor C/D in the above proof can be regarded as right cells.

Note that every nonzero noetherian bimodule contains a cell.

Over an FBN ring R it was shown that the Krull dimension of a finitely generated module M is the same as that of any essential submodule [**36**, Theorem 3.5]. We can extend this result to noetherian rings with the second layer condition if for every prime factor ring the classical Krull dimension coincides with the Krull dimension.

PROPOSITION 2.10. *Let R be a noetherian ring satisfying the second layer condition. If, for every prime factor ring R/P, the classical Krull dimension of R/P is equal to the right Krull dimension, then $|M| = |N|$ for every finitely generated tame right R-module M with $N \subseteq_e M$.*

PROOF. Assume first that M is a uniform module. We claim that $|M| = |R/\operatorname{Ass}(M)|$. Set $\operatorname{Ass}(M) = P, L = l_M(P)$. We proceed by noetherian induction, assuming the claim to be true for all proper factor modules that are tame and uniform. If $L = M$, then $r(M) = P$, so $|M| \leq |R/P|$. Now equality holds since M contains a submodule that is isomorphic to a uniform right ideal of R/P.

Assume therefore that $L \subset M$. Now M/L is tame since

$$M/L \hookrightarrow \mathrm{E}(R/P)/l_{\mathrm{E}(R/P)}(P)$$

and P satisfies the right second layer condition. Also, M/L can be essentially embedded in a direct sum of uniform modules $M/M_1, \ldots, M/M_k$, each of which is tame and its associated prime $Q_i = \operatorname{Ass}(M/M_i)$ is linked to P by Theorem 2.4. By the inductive hypothesis, $|M/M_i| = |R/Q_i|$ for $i = 1, \ldots, k$. Since $|M/L| \leq \max\{|M/M_i| \mid i = 1, \ldots, k\} \leq |M/L|$, it follows that $|M/L| = |R/Q_i|$ for some i. Now

$$|R/Q_i| = \text{cl.K.dim}(R/Q_i) = \text{cl.K.dim}(R/P) = |R/P|,$$

where the first and last equalities hold by hypothesis, and the second one holds by Lemma 2.9. Thus

$$|M| = \max\{|M/L|, |L|\} = \max\{|R/Q_i|, |R/P|\} = |R/P|,$$

proving the claim. The assertion follows from this since $\operatorname{Ass}(N) = \operatorname{Ass}(M)$ for $N \subseteq_e M$.

For the general case, assume that M is not uniform. Then M can be embedded essentially into a finite direct sum of uniform modules $M/M_1 \oplus \cdots \oplus M/M_k$ where $M_1 \cap \cdots \cap M_k = 0$ and $k = \operatorname{rank}(M)$. For any given essential submodule N of M, note that $N \cap (M/M_i) \neq 0$ for each i, and $|N \cap (M/M_i)| = |M/M_i|$ by the uniform case. Thus,

$$|N| \geq |(N \cap (M/M_1)) \oplus \cdots \oplus (N \cap (M/M_k))|$$

$$= \max\{|N \cap (M/M_i)|\} = \max\{|M/M_i|\} \geq |M|.$$

Therefore, $|M| = |N|$ for every $N \subseteq_e M$. □

While there are abundant examples of noetherian rings satisfying the second layer condition, there are also noetherian rings that fail to satisfy the second layer condition.

EXAMPLE 2.11. Let k be a field of characteristic 0. Let $S = k[x][y; d/dx]$ be the first Weyl algebra $A_1(k)$ over k. The right ideal xS is maximal in S and has $R = k + xS$ as its idealizer in S. Thus R is a noetherian integral domain having just $0, xS, R$ as its ideals.

Let $E = \mathrm{E}(R/xS)_R$. Since $xS_R \subseteq_e R_R$, there is an R-module monomorphism $\phi : R \to xS$ given by $\phi(r) = xr$, for all $r \in R$. Also $R \to R/xS \to E$ produces a nonzero homomorphism, which can be extended to a nonzero homomorphism $\alpha : xS \to E$. So $\alpha(xS) \cap R/xS \neq 0$ and hence $r_R(E) \subset xS$. (Otherwise, $r_R(E) = xS$ and so $\alpha(xS) = \alpha((xS)^2) = \alpha(xS)xS \subseteq ExS = 0$

leading to a contradiction.) Thus $L = l_E(xS) \subset E$. Now there exists a submodule M of E such that M satisfies the hypotheses of the Main Lemma, with an affiliated series $0 \subset U \subset M$ and xS, Q as its corresponding affiliated primes.

Assume that $Q = xS$. Then $0 = MxSxS = MxS$, contradicting $U = l_M(xS) \subset M$. Thus $Q = 0 \subset xS$ follows, producing the undesirable case, hence xS does not satisfy the right (affiliated) second layer condition. □

Note that in this example $|R_R| = 1$, while any ring R with $|R_R| = 0$ satisfies the right second layer condition. The ring R in the above example has very few ideals. Such rings generally do not satisfy the strong second layer condition as the following result shows.

PROPOSITION 2.12. [14, Lemma 8] *Let R be a noetherian ring such that there exists a prime factor ring T where the number of its proper ideals is finite, but greater than one. Then R does not satisfy the right strong second layer condition.*

PROOF. Suppose that R satisfies the right strong second layer condition. Then so does T. Let P be a nonzero prime ideal of T, and set

$$I(P) = \bigcap \{Q \mid Q \in \Omega^r(P)\}.$$

Note that there are only finitely many primes in $\Omega^r(P)$. So $I(P) \neq 0$ since $I(P) = 0$ would imply $Q = 0$ for some $Q \in \Omega^r(P)$, which is impossible since 0 is not linked to any prime. Define $I = \bigcap_{n \geq 1} I(P)^n$. Then again $I \neq 0$. Let

$E = \mathrm{E}(T/P)_T$. We claim that $EI = 0$. It suffices to show that each finitely generated submodule M of E is annihilated by I. By Lemma 2.5, there exist $P_1, \ldots, P_n \in \Omega^r(P)$ such that $MP_n \cdots P_1 = 0$. Thus $MI(P)^n = 0$, and so $MI = 0$ proving the claim.

However, note that over a prime noetherian ring, any injective module E is faithful. For if $J = r(E) \neq 0$, then J contains a regular element c of T. Let e be a nonzero element in E. Define a module homomorphism $\phi : cT \to E$ by $\phi(ct) = et$ for for every $ct \in cT$. Then ϕ is well-defined and can be extended to $\Phi : T \to E$ since E is injective. But then

$$\Phi(c) = \Phi(1c) = \Phi(1)c \subseteq Ec = 0,$$

contradicting the fact that $\phi(c) = \Phi(c) = e \neq 0$. This contradiction shows that R does not satisfy the right strong second layer condition. ☐

So far, all known noetherian rings that do not satisfy the right strong second layer condition are of the type described in Proposition 2.12.

2.5. The Hierarchy of Second Layer Conditions

When R is a noetherian ring, the right strong second layer condition on a prime ideal P of R implies the right second layer condition on P by the noetherian version of Jategaonkar's Main Lemma [**40**, Corollary 6.1.4]. In order to establish the same implication for right noetherian rings, we introduce an intermediate condition and call it the restricted strong second layer condition. This notion was originally labeled $\{*\}_r$ in [**39**] and later (†)

in **[29]**. Also, some authors use this as the second layer condition since it coincides with the usual definition when all of the prime ideals are involved **[67]**.

DEFINITION. Let R be a right noetherian ring. A prime ideal P is said to satisfy the ***right restricted strong second layer condition*** if no finitely generated P-tame module M is annihilated by a prime ideal strictly contained in P.

Thus, if P satisfies the right restricted strong second layer condition and M is a finitely generated P-tame module with $r(M) = Q \in \operatorname{Spec}(R)$, then it follows that $Q = P$. Note the formal resemblance between the above definition and that of the strong second layer condition where P-primary modules are employed instead of P-tame modules. One of the advantages of working with the restricted strong second layer condition is that it can be characterized in terms of finite annihilation of finitely generated modules, without any reference to second layers.

THEOREM 2.13. *Let R be a right noetherian ring and let $\mathcal{X}$ be a nonempty subset of* $\operatorname{Spec}(R)$. *Then the following statements are equivalent.*

(i) *$\mathcal{X}$ satisfies the right restricted strong second layer condition.*

(ii) *If M is a finitely generated $\mathcal{X}$-tame module, then M^I is also $\mathcal{X}$-tame for any nonempty index set I.*

(iii) *Any finitely generated $\mathcal{X}$-tame module M is a Δ-module.*

(iv) *For any prime ideal Q and any essential right ideal E/Q of R/Q such that R/E is $\mathcal{X}$-tame, $r_R(R/E) \supset Q$.*

(v) *Let σ be a torsion radical on right R-modules, such that each $P \in \mathcal{X}$ is σ-closed in R, and let M be a finitely generated $\mathcal{X}$-tame module. Then $|M|_\sigma = |R/r(M)|_\sigma$.*

PROOF. (i) ⇒ (ii): First assume that M is a finitely generated uniform right R-module with $P = \operatorname{Ass}(M) \in \mathcal{X}$. Let $Q \in \operatorname{Ass}(M^I)$. So $Q = r(nR)$ for some $n = (n_i)_{i \in I}$ with each $n_i \in M$. Note that

$$Q = r(nR) = \bigcap_{i \in I} r(n_i R) = r(\sum_{i \in I} n_i R).$$

Now $N = \sum_{i \in I} n_i R$ is a submodule of the noetherian P-tame module M, hence finitely generated P-tame. Thus $Q \subseteq P$, and $Q = P$ follows from this since P satisfies the right restricted strong second layer condition. Consequently, $\operatorname{Ass}(M^I) = P$.

Next, we show that M^I is tame. Set $L = l_{M^I}(P)$. Suppose that L is not a torsionfree right R/P-module. Then there exist $0 \neq (m_i)_{i \in I} \in L$ and $c \in \mathcal{C}_R(P)$ such that $m_i c = 0$ for all $i \in I$. Since $(m_i) \neq 0$, we have $m_i \neq 0$ for some i. Now $m_i RP = m_i P = 0$ implies that $m_i R$ is a torsionfree right R/P-module, which contradicts $m_i c = 0$. Thus M^I is P-tame, proving the uniform case.

If M is a finitely generated $\mathcal{X}$-tame right R-module, then M can be embedded essentially into a finite direct sum of uniform modules $M_1, \ldots, M_n$

with each $\mathrm{Ass}(M_j) = P_j \in \mathcal{X}$ [18, Lemma 7.9]. Note that

$$M^I \hookrightarrow (\bigoplus_{j=1}^{n} M_j)^I.$$

Since the latter is obviously isomorphic to $\bigoplus_{j=1}^{n}(M_j{}^I)$, the uniform case yields that M^I is $\mathcal{X}$-tame.

(ii) $\Rightarrow$ (iii): Since $M \hookrightarrow_e \bigoplus_{i=1}^{n} M_i$ where each M_i is a uniform module and $n = \mathrm{rank}(M)$, and since a finite direct sum of Δ-modules is a Δ-module by Lemma 1.13, we may assume that M is just a uniform module with $P = \mathrm{Ass}(M) \in \mathcal{X}$.

In order to show (iii), we use Lemma 1.12. Let S be a nonempty subset of M. Suppose that S is not finitely annihilated. Note that

$$R/r(S) \hookrightarrow \prod_{s \in S} sR \hookrightarrow M^S,$$

and thus $R/r(S)$ is P-tame by (ii). Set $L/r(S) = l_{R/r(S)}(P)$. Now, there exists $s_1 \in S$ such that $r(s_1) \supset r(S)$ and thus $r(s_1) \cap L \supset r(S)$ since $R/r(S)$ is P-tame (primary). This, in turn, yields $s_2 \in S$ such that $s_2[r(s_1) \cap L] \neq 0$ and $r(s_1, s_2) \supset r(S)$. Continuing inductively, we get $s_1, s_2, \cdots \in S$ such that

$$r(s_1, \ldots, s_n) \cap L \supset r(s_1, \ldots, s_n, s_{n+1}) \cap L$$

and $r(s_1, \ldots, s_n) \supset r(S)$ for all n.

Observe that

$$\frac{L \cap r(s_1, \ldots, s_n)}{L \cap r(s_1, \ldots, s_n, s_{n+1})} \cong s_{n+1}(L \cap r(s_1, \ldots, s_n)) \neq 0$$

is torsionfree as an R/P-module, so it has a positive reduced rank. Since R is a right noetherian ring, $\rho_{R/P}(l_{R/r(S)}(P))$ is finite, say k. Then

$$\begin{aligned}
k &= \rho_{R/P}(l_{R/r(S)}(P)) = \rho(L/r(S)) \\
&= \rho\left(\frac{L}{L \cap r(s_1)}\right) + \sum_{i=1}^{k} \rho\left(\frac{L \cap r(s_1, \ldots, s_i)}{L \cap r(s_1, \ldots, s_i, s_{i+1})}\right) \\
&> k,
\end{aligned}$$

which is a contradiction.

(iii) $\Rightarrow$ (iv): Let $(E/Q)_{R/Q} \subseteq_e (R/Q)_{R/Q}$ with R/E being $\mathcal{X}$-tame. By (iii), $r(R/E) = r(x_1 + E, x_2 + E, \ldots, x_k + E)$ for finitely many elements $x_i + E \in R/E$. Note that each $r(x_i + E)/Q$ is essential in R/Q since $E/Q \subseteq_e R/Q$, so

$$r(x_1 + E, \ldots, x_k + E)/Q = \bigcap_{i=1}^{k} r(x_i + E)/Q \subseteq_e R/Q.$$

In other words, $r(R/E) \supset Q$.

(iv) $\Rightarrow$ (i): Let M be a finitely generated P-tame right R-module. Assume that $r(M) = Q \in \operatorname{Spec}(R)$. Without loss of generality, set $M = mR$. Now $r(m)/Q$ cannot be essential in R/Q, since this would imply that there is an ideal J such that $Q \subset J \subseteq r(m)$, and $mRJ = mJ = 0$ leads to a contradiction. Thus there exists some nonzero right ideal $I \supset Q$ such that $r(m) \cap I = Q$. Consequently,

$$I/Q = I/(r(m) \cap I) \cong (I + r(m))/r(m) \hookrightarrow R/r(m) \cong M.$$

Hence, $Q = \operatorname{Ass}(I/Q) = \operatorname{Ass}(R/r(m)) = P$.

(iii) $\Rightarrow$ (v): Let M be a finitely generated $\mathcal{X}$-tame module. Then there exist $m_1, m_2, \ldots, m_k \in M$ such that

$$R/r(M) = R/r(m_1, m_2, \ldots, m_k) \hookrightarrow \bigoplus_{i=1}^{k} m_i R \hookrightarrow M^k.$$

Thus $|R/r(M)|_\sigma \leq |M^k|_\sigma = |M|_\sigma$. On the other hand, $|M|_\sigma \leq |R/r(M)|_\sigma$ holds in general. So the equality is achieved.

(v) $\Rightarrow$ (iv): Let Q be a prime ideal of R, and let E/Q be an essential right ideal of R/Q such that R/E is $\mathcal{X}$-tame. Suppose $r(R/E) = Q$. Observe that

$$R/E \hookrightarrow_e \mathrm{E}(R/E) \hookrightarrow \bigoplus_{i=1}^{n} \mathrm{E}(R/P_i),$$

where $P_i \in \mathcal{X}$ and $n = \mathrm{rank}(R/E)$. Note that R/E is σ-torsionfree. Since we assume that $r(R/E) = Q$, we get $\sigma(R/Q) = 0$, as $R/Q \hookrightarrow \prod(R/E)$. Now $|R/E|_\sigma < |R/Q|_\sigma$, as E/Q contains a regular element of R/Q. However, by (v),

$$|R/E|_\sigma = |R/r(R/E)|_\sigma = |R/Q|_\sigma$$

since R/E is a finitely generated $\mathcal{X}$-tame module. This is a contradiction.

□

In the above theorem, the right restricted strong second layer condition for $\mathcal{X}$ is essential for M^I to be $\mathcal{X}$-tame. If $\mathcal{X}$ does not satisfy the right restricted strong second layer condition, then M^I may just be tame, but is not guaranteed to be Ass(M)-tame as the following example indicates.

EXAMPLE 2.14. Let $A_1 = k[x][y; d/dx]$ where k is a field of characteristic 0. If $R = k + xA_1$, then R is a noetherian domain and $0, P = xA_1$ and R are the only ideals of R [**56**, Example 1.3.10].

The right R-module $\mathrm{E}(R/P)_R$ contains a finitely generated submodule M with $r(M) = 0$ by [**56**, Example 4.3.15]. Observe that

$$\{0\} = \mathrm{Ass}(R_R) \subseteq \mathrm{Ass}(M^M) \text{ as } R \hookrightarrow \prod_{m \in M} mR \hookrightarrow M^M.$$

Note that M^M is tame by Example 4.3 and Proposition 4.1 in Chapter 4. Thus $0 \in \mathrm{Ass}(M^M)$, yet $\mathrm{Ass}(M) = P$. This happens because P does not satisfy the right restricted strong second layer condition as pointed out in Example 2.11. $\square$

By (ii) in the preceding theorem, the restricted strong second layer condition for $\mathcal{X}$ is equivalent to the $\mathcal{X}$-tameness of direct powers of finitely generated $\mathcal{X}$-tame modules. The following lists several other equivalent conditions.

LEMMA 2.15. *Let R be a right noetherian ring, and let $\mathcal{X}$ be a nonempty subset of* $\mathrm{Spec}(R)$. *Then the following are equivalent.*

(i) *Direct powers of $\mathcal{X}$-tame right R-modules are $\mathcal{X}$-tame.*

(ii) *Direct products of $\mathcal{X}$-tame right R-modules are $\mathcal{X}$-tame.*

(iii) *Direct products of finitely generated $\mathcal{X}$-tame right R-modules are $\mathcal{X}$-tame.*

(iv) *Direct products of cyclic $\mathcal{X}$-tame right R-modules are $\mathcal{X}$-tame.*

PROOF. (i) $\Rightarrow$ (ii): Let $\{M_i\}_{i\in I}$ be a family of $\mathcal{X}$-tame modules and let $M = \prod_{i\in I} M_i$. Note that $M' = \bigoplus_{i\in I} M_i$ is also $\mathcal{X}$-tame. Then $M \hookrightarrow (M')^I$, and $(M')^I$ is tame by (i). Therefore, $M = \prod M_i$ is $\mathcal{X}$-tame.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i): Let M be an $\mathcal{X}$-tame module. Note that if each mR is tame for $m = (m_i) \in M^I$ where $m_i \in M$, then M^I is tame. Now $\prod_{i\in I} m_iR$ is $\mathcal{X}$-tame by (iv). Since $mR \hookrightarrow \prod_{i\in I} m_iR$, it follows that mR is $\mathcal{X}$-tame. ☐

When Jategaonkar introduced the right second layer condition, he labeled it $\{*\}_r$. A prime ideal P in a right noetherian ring R is said to satisfy $\{*\}_r$ if, for any given prime $Q \subset P$, every finitely generated essential extension of an unfaithful P/Q-tame right R/Q-module is unfaithful over R/Q. This is equivalent to the restricted strong second layer condition.

PROPOSITION 2.16. *Let P be a prime ideal in a right noetherian ring R. Then P satisfies $\{*\}_r$ iff it satisfies the right restricted strong second layer condition.*

PROOF. ($\Rightarrow$) : Let M be a finitely generated P-tame right R-module with $r(M) = Q \in \mathrm{Spec}(R)$. Suppose $Q \subset P$, then $l_M(P) \subset M$. In fact, $l_M(P)$ is essential in M since M is P-primary. Hence, $\{*\}_r$ implies that M is unfaithful over R/Q, which is an obvious contradiction. Thus $Q = P$, and so P satisfies the right restricted strong second layer condition.

($\Leftarrow$) : If P does not satisfy $\{*\}_r$, then for some prime $Q \subset P$ there exists a finitely generated faithful R/Q-module M with $L \subseteq_e M$ where L is a

P-tame unfaithful R/Q-module. Note that M is also P-tame since $L \subseteq_e M$. So M is a finitely generated P-tame module with $r(M) = Q \subset P$. This contradicts the right restricted strong second layer condition of P. $\square$

DEFINITION. Let R be a right noetherian ring and $\mathcal{X}$ be a nonempty subset of $\operatorname{Spec}(R)$. The ring R is said to be ***right $\mathcal{X}$-tame bounded*** if, for every $E \subseteq_e R$ such that R/E is $\mathcal{X}$-tame, $r_R(R/E) \neq 0$. The ring R is said to be ***right fully $\mathcal{X}$-tame bounded*** if every prime factor ring R/P is right $\mathcal{X}$-tame bounded.

By Theorem 2.13, the right restricted strong second layer condition for R is thus equivalent to R being right fully $\operatorname{Spec}(R)$-tame bounded. Having taken this into account, we can deduce the right second layer condition from the right restricted strong second layer condition.

PROPOSITION 2.17. *Let R be a right noetherian ring and let $\mathcal{X}$ be a nonempty subset of* $\operatorname{Spec}(R)$. *If $\mathcal{X}$ satisfies the right restricted strong second layer condition, then it satisfies the right second layer condition.*

PROOF. Let $P \in \mathcal{X}$ satisfy the right restricted strong second layer condition. Let $\overline{U} = (U + l_{E(R/P)}(P))/l_{E(R/P)}(P)$ be a Q-prime uniform submodule of $E(R/P)_R/l_{E(R/P)_R}(P)$. Without loss of generality, assume that U is a finitely generated submodule of $E(R/P)$. By Theorem 2.13, U is finitely annihilated. Moreover, $(U + l_{E(R/P)}(P))/l_{E(R/P)}(P) \cong U/l_U(P)$ is finitely annihilated by Lemma 1.13. Thus

$$R/Q = R/r(U/l_U(P)) = R/(\bigcap_{i=1}^{n} r(u_i + l_U(P)) \hookrightarrow \bigoplus_{i=1}^{n}(u_iR + l_U(P))/l_U(P)$$

for some $u_1, \ldots, u_n \in U$, so $(U + l_{\mathrm{E}(R/P)}(P))/l_{\mathrm{E}(R/P)}(P)$ is tame. Therefore, the second layer of $\mathrm{E}(R/P)$ is tame, proving that P satisfies the right second layer condition. Since $P \in \mathcal{X}$ was chosen arbitrarily, $\mathcal{X}$ satisfies this condition. ☐

If $\Omega^r(\mathcal{X}) = \bigcup_{P\in\mathcal{X}} \Omega^r(P) = \mathcal{X}$, then $\mathcal{X}$ is right link closed. Note that Spec(R) is trivially right link closed. For the converse of Proposition 2.17, the link closure of $\mathcal{X}$ is necessary. Shapiro noted the equivalence of the right restricted strong second layer condition and the right second layer condition for right link closed subsets of Spec(R) [67, p. 1790]. However, it should be noted that, for a set of prime ideals that is not link closed, the right restricted strong second layer condition is strictly stronger than the right second layer condition (See Section 2.6).

PROPOSITION 2.18. *Let R be a right noetherian ring and let $\mathcal{X}$ be a nonempty right link closed subset of* Spec(R). *If $\mathcal{X}$ satisfies the right second layer condition, then it satisfies the right restricted strong second layer condition.*

PROOF. Let $P \in \mathcal{X}$. Suppose that M is a finitely generated P-tame right R-module with $P \supset r(M) = Q \in$ Spec(R). Define

$$\overline{\mathcal{X}} = \{T/Q \mid T \supset Q \text{ and } T \in \mathcal{X}\}.$$

Then $P/Q \in \overline{\mathcal{X}}$ and $\overline{\mathcal{X}}$ is still right link closed, since $T'/Q \rightsquigarrow T/Q$, where $T/Q \in \overline{\mathcal{X}}$, implies that there exists an ideal I such that

$$(T'T)/Q \subseteq I/Q \subset (T' \cap T)/Q$$

with $${}_{(R/Q)/(T'/Q)}\left(\frac{T'/Q \cap T/Q}{I/Q}\right)_{(R/Q)/(T/Q)} \cong {}_{R/T'}\left(\frac{T' \cap T}{I}\right)_{R/T}$$

torsionfree on the right and fully faithful on the left. This means $T' \rightsquigarrow T$, thus $T' \in \mathcal{X}$ and $T'/Q \in \overline{\mathcal{X}}$. Passing to the factor ring R/Q, we may assume that $Q = 0$, so R is a prime right noetherian ring. Since M is P-tame, it is $\overline{\mathcal{X}}$-tame. Thus there exist, by Lemma 2.5, primes $P_1, \ldots, P_n \in \overline{\mathcal{X}}$ such that $MP_n \cdots P_1 = 0$. So $P_i = 0$ for some i. This contradicts the fact that $P_i \neq 0$ as $P_i \in \overline{\mathcal{X}}$ for all $i = 1, \ldots, n$. Thus $Q = r(M) = P$ follows, giving the right restricted strong second layer condition for P. ☐

COROLLARY 2.19. *Let R be a right noetherian ring satisfying the right second layer condition. Then R satisfies the right restricted strong second layer condition.*

PROOF. Since Spec(R) is trivially right link closed, R satisfies the right restricted strong second layer condition by Proposition 2.18. ☐

COROLLARY 2.20. *Let P be a prime ideal in a right noetherian ring R. Consider the following statements:*

(i) *P satisfies the right strong second layer condition.*

(ii) *P satisfies the right restricted strong second layer condition.*

(iii) *P satisfies the right second layer condition.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

PROOF. (i) $\Rightarrow$ (ii): By definition.

(ii) $\Rightarrow$ (iii): By Proposition 2.17. □

COROLLARY 2.21. *Let P be a minimal prime ideal in a right noetherian ring R. Then P satisfies the right second layer condition.*

PROOF. By definition any minimal prime ideal P satisfies the right strong second layer condition, hence Corollary 2.20 gives the result. □

Corollary 2.21 was also proved by Boyle and Kosler [12].

For noetherian rings, Goodearl and Warfield [29] and Byun [15] define the right (strong) second layer condition of a prime ideal P in terms of an affiliated series of a finitely generated (P-primary) P-tame module M. The right second layer condition and the right affiliated second layer condition coincide on noetherian rings while the former implies the latter in general.

PROPOSITION 2.22. *Let R be a right noetherian ring and let P be a prime ideal in R. If P satisfies the right second layer condition, then it satisfies the right affiliated second layer condition.*

PROOF. Suppose that P does not satisfy the right affiliated second layer condition. Then there exists a P-tame module M satisfying the hypotheses of Jategaonkar's Main Lemma and resulting in the undesirable case. Set

$L = l_M(P)$ and $r(M) = Q \subset P$. There exist uniform submodules U_i of L such that

$$\bigoplus_{i \in I} U_i \subseteq_e L \subseteq_e M.$$

Since L is a torsionfree right R/P-module, each $U_i \hookrightarrow \mathrm{E}(R/P)_R$ and so $\bigoplus_{i\in I} U_i \hookrightarrow \bigoplus_{i\in I} \mathrm{E}(R/P)$. As $\bigoplus_{i\in I} \mathrm{E}(R/P)$ is also an injective module by [**63**, Theorem 1] [**5**, Theorem 1.1], $L \hookrightarrow \bigoplus_{i\in I} \mathrm{E}(R/P)$ and moreover, $M \hookrightarrow \bigoplus_{i\in I} \mathrm{E}(R/P)$. Hence,

$$\begin{aligned} M/l_M(P) &= M/(M \cap l_{\oplus \mathrm{E}(R/P)}(P)) \\ &\cong (M + l_{\oplus \mathrm{E}(R/P)}(P))/l_{\oplus \mathrm{E}(R/P)}(P) \hookrightarrow \bigoplus_{i\in I} (\mathrm{E}(R/P)/l_{\mathrm{E}(R/P)}(P)), \end{aligned}$$

which is tame as direct sums of tame modules are tame. This contradicts M/L being torsion as an R/Q-module. □

If R is noetherian, then the converse of Proposition 2.22 holds.

PROPOSITION 2.23. *Let R be a noetherian ring and let P be a prime ideal satisfying the right affiliated second layer condition. Then P satisfies the right second layer condition.*

PROOF. Let $\overline{U} = (U + l_{\mathrm{E}(R/P)}(P))/l_{\mathrm{E}(R/P)}(P)$ be a Q-prime uniform submodule of $\mathrm{E}(R/P)/l_{\mathrm{E}(R/P)}(P)$. By normalizing U to a suitable submodule, we can assume that U satisfies the hypotheses of Jategaonkar's Main Lemma. Then $Q \rightsquigarrow P$ via $r(U)$, and $(Q \cap P)/r(U)$ is torsionfree on the right and faithful as a left R/Q-module. Now it follows that it is also torsionfree as a left R/Q-module since every noetherian bimodule is tame. Thus

$U/l_U(P)$ is Q-tame by the noetherian version of the Main Lemma. This shows that the second layer of $\mathrm{E}(R/P)$ is tame. □

It is obvious from Corollary 2.20 and Proposition 2.22 that a prime ideal P satisfying the right restricted strong second layer condition also satisfies the right affiliated second layer condition. The latter says that the undesirable case does not arise for P-tame modules satisfying the hypotheses of Jategaonkar's Main Lemma. The following result characterizes the right restricted strong second layer condition in terms of the undesirable case not arising for a wider class of P-tame modules.

PROPOSITION 2.24. *Let R be a right noetherian ring and let P be a prime ideal of R. Then the following are equivalent.*

(i) *P satisfies the right restricted strong second layer condition.*

(ii) *There does not exist a finitely generated P-tame right R-module M such that $r(M/l_M(P)) = Q \in \mathrm{Spec}(R), Q \subset P$ and $MQ = 0$.*

PROOF. (i)⇒(ii): Suppose that there is a finitely generated P-tame module M with $MQ = 0$, $Q \subset P$, and $r(M/l_M(P)) = Q$. Then

$$Q \subseteq r(M) \subseteq r(M/l_M(P)) = Q,$$

giving that $r(M) = Q$. But then $Q = P$ by the right restricted strong second layer condition on P. This is a contradiction.

(ii)⇒(i): Let M be a finitely generated P-tame right R-module with $r(M) = Q \in \mathrm{Spec}(R)$. Suppose $Q \subset P$. Note that $Mr(M/l_M(P))P = 0$, so $r(M/l_M(P))P \subseteq Q$ and hence $r(M/l_M(P)) \subseteq Q$ since $P \supset Q$. Now it follows from $Q \subseteq r(M/l_M(P))$ that $r(M/l_M(P)) = Q$. This contradicts (ii). ☐

Next, we present conditions that are equivalent to the strong second layer condition.

PROPOSITION 2.25. *Let R be a right noetherian ring and let $\mathcal{X}$ be a nonempty subset of* Spec(R). *Then the following are equivalent.*

(i) *$\mathcal{X}$ satisfies the right strong second layer condition.*

(ii) *If $P \in \mathcal{X}$, then M^I is P-primary for any finitely generated P-primary right R-module M and any nonempty index set I.*

(iii) $\mathrm{Ass}(M) = \mathrm{Ass}(M^I)$ *for any finitely generated right R-module M with* $\mathrm{Ass}(M) \subseteq \mathcal{X}$.

(iv) *If M is a finitely generated right R-module with* $\mathrm{Ass}(M) \subseteq \mathcal{X}$, *then* $\{P \mid P$ *is an annihilator prime of* $M\} = \mathrm{Ass}(M)$.

PROOF. (i) ⇒ (ii): Let M be a finitely generated P-primary right R-module with $P \in \mathcal{X}$. If $Q \in \mathrm{Ass}(M^I)$, then there exists a Q-prime submodule $N \subseteq M^I$. Let $0 \neq n = (n_i)_{i \in I} \in N$, so

$$Q = r(nR) = r((n_i)R) = \bigcap_{i\in I} r(n_iR) = r(\sum_{i\in I} n_iR) = r(\sum_{i=1}^{k} n_iR)$$

since $\sum_{i\in I} n_iR \subseteq M$ is noetherian. As $\sum_{i=1}^{k} n_iR$ is a finitely generated P-primary module, we obtain $Q = P$.

(ii) $\Rightarrow$ (iii): Observe that $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(M^I)$ as $M \hookrightarrow M^I$. Since any finitely generated right R-module M can be essentially embedded into a finite direct sum of uniform modules, without loss of generality, we assume that M is P-primary. So by (ii), $\operatorname{Ass}(M^I) \subseteq \operatorname{Ass}(M)$.

(iii) $\Rightarrow$ (iv): Let $P = r(N)$ for some submodule N of M. Since

$$R/P = R/r(N) \hookrightarrow \prod_{n \in N} nR \hookrightarrow M^N,$$

we get $P = \operatorname{Ass}(R/P) \subseteq \operatorname{Ass}(M^N) = \operatorname{Ass}(M)$ by (iii).

(iv) $\Rightarrow$ (i): Let M be a finitely generated P-primary R/Q-module where $Q \subset P$. If $r(M) = Q$, then $Q = P = \operatorname{Ass}(M)$ by (iv), contradicting $Q \subset P$. ☐

Even though the right affiliated strong second layer condition of a single prime ideal does not imply the right strong second layer condition(see Section 2.6), the two definitions are equivalent when applied to the whole ring.

PROPOSITION 2.26. *Let R be a right noetherian ring. Then the following are equivalent.*

(i) *R satisfies the right strong second layer condition.*

(ii) *R satisfies the right affiliated strong second layer condition.*

PROOF. (i) $\Rightarrow$ (ii): Let P satisfy the right strong second layer condition. If P did not satisfy the right affiliated strong second layer condition, then the undesirable case of the Main Lemma would arise, that is, there would be a finitely generated P-primary right R-module M with an affiliated series $0 \subset l_M(P) \subset M$ such that $M/l_M(P)$ is a Q-prime uniform module and $r(M) = Q \subset P$. Since P satisfies the right strong second layer condition, this is impossible.

(ii) ⇒ (i): Let Q be a prime ideal such that $Q \subset P$, and let M be a finitely generated P-primary faithful right R/Q-module. Without loss of generality, we may assume that M is uniform. Set

$$\mathcal{M} = \{N \subset M \mid r(M/N) = Q, M/N \text{ is } P'\text{-primary uniform where } P' \supset Q\}.$$

Then $0 \in \mathcal{M} \neq \emptyset$. Since M is noetherian, a maximal element N can be chosen in $\mathcal{M}$. Now replace M by M/N and set $\text{Ass}(M) = P'$. Note that $r(M/l_M(P'))P' \subseteq r(M) = Q$ implies that $r(M/l_M(P')) = Q$. Set $L = l_M(P')$. There exist finitely many submodules $L_1, \ldots, L_n$ such that $\bigcap_{i=1}^n L_i = L$, $L \subseteq L_i \subseteq M$ and M/L_i is uniform for all $i = 1, \ldots, n$. Thus $M/L \hookrightarrow_e \oplus_{i=1}^n M/L_i$ by [18, Lemma 7.9]. Hence

$$\bigcap_{i=1}^n r(M/L_i) = r(\bigoplus_{i=1}^n M/L_i) \subseteq r(M/L) = Q,$$

so $r(M/L_i) \subseteq Q$ for some i. This means that $r(M/L_i) = Q$, since $Q = r(M/L) \subseteq r(M/L_i)$ in any case. Set $\text{Ass}(M/L_i) = P_i'$. Note that $P_i' \supseteq Q$. By the assumption on M, we get $P_i' = Q$. Hence

$$Q = P_i' = \text{Ass}(M/L_i) \subseteq \text{Ass}(M/L).$$

Now choose a submodule M' of M such that M'/L is Q-prime uniform and $r(M')$ is maximal among annihilators of submodules of M properly containing L. Then

$$Q = r(M) \subseteq r(M') \subseteq r(M'/L) = Q.$$

Note that M' is P-primary, and that $0 \subset L \subset M'$ is an affiliated series with $r(M') = Q \subset P'$, which is not allowed by the right affiliated strong second layer condition of P'. □

We now summarize the relations among the various second layer conditions.

THEOREM 2.27. *Let R be a right noetherian ring and let P be a prime ideal of R. Consider the following statements:*

(sslc) *P satisfies the right strong second layer condition.*

(rsslc) *P satisfies the right restricted strong second layer condition.*

(slc) *P satisfies the right second layer condition.*

(asslc) *P satisfies the right affiliated strong second layer condition.*

(aslc) *P satisfies the right affiliated second layer condition.*

Then

(i) *(sslc) $\Rightarrow$ (rsslc) $\Rightarrow$ (slc) $\Rightarrow$ (aslc).*

(ii) *(aslc) $\Rightarrow$ (slc) if R is noetherian.*

(iii) *(sslc) $\Rightarrow$ (asslc) $\Rightarrow$ (aslc).*

(iv) *(slc) $\not\Rightarrow$ (rsslc), in general, even if R is noetherian.*

PROOF. (i). By Corollary 2.20 and Proposition 2.22.

(ii). By Proposition 2.23.

(iii). By definition.

(iv) is the topic of the next section. ☐

If R is a right noetherian ring and $\mathcal{X}$ is a right link closed nonempty subset of Spec(R), then the right restricted strong second layer condition on $\mathcal{X}$ is equivalent to the right second layer condition on $\mathcal{X}$. If $\mathcal{X}$ is the whole Spec(R), then the right strong second layer condition on R is equivalent to the right affiliated strong second layer condition on R and the right restricted strong second layer condition on R is equivalent to the right second layer condition on R.

The following diagram summarizes the relation between various definitions of the second layer condtions given in this chapter.

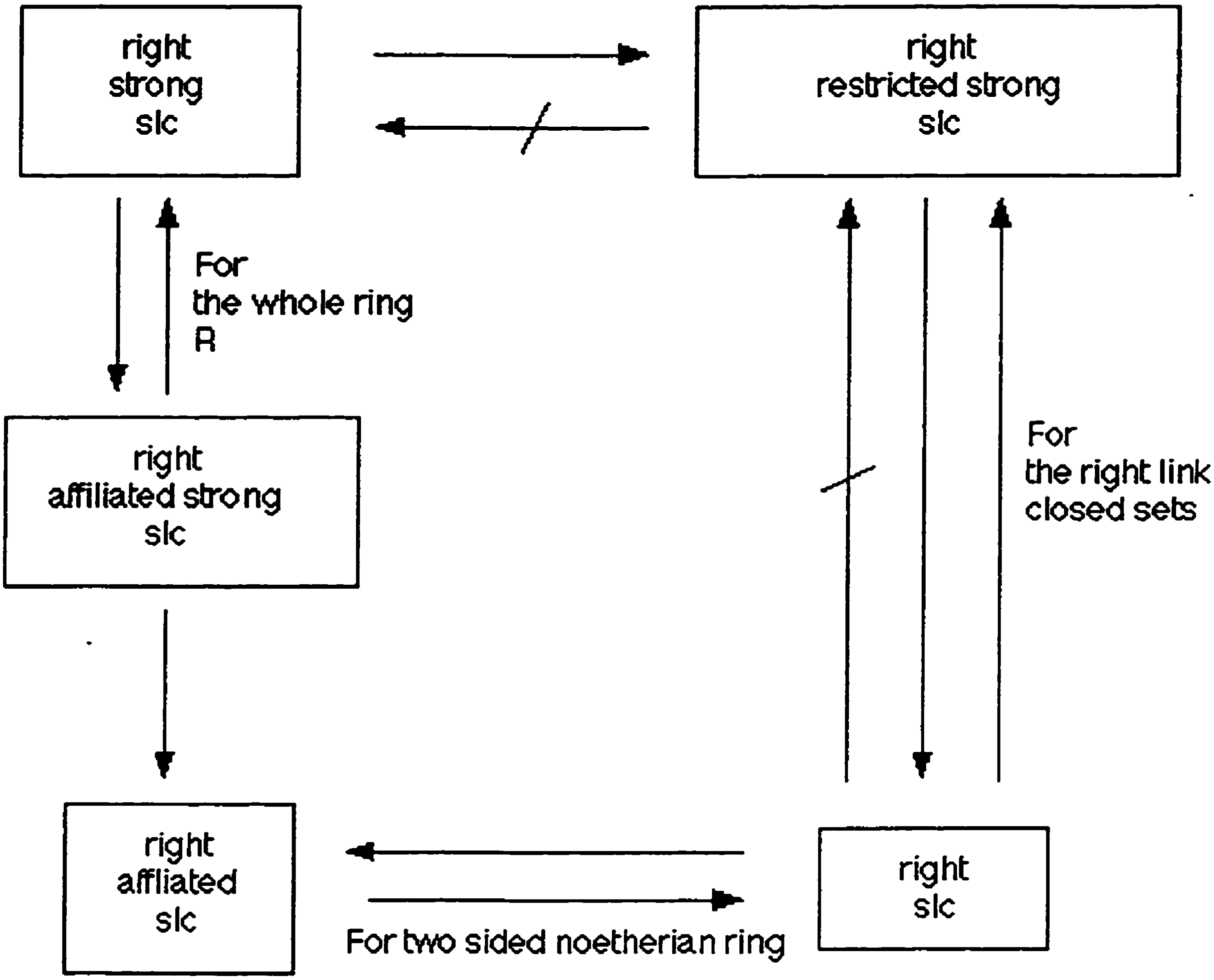

FIGURE 1. R is right noetherian.

2.6. Examples

In this section, we present a prime ideal, in a noetherian ring, which does not satisfy the right restricted strong second layer condition, but satisfies the right affiliated second layer condition. Note that this prime ideal also satisfies the right second layer condition since the ring is two-sided noetherian. Certainly, it cannot satisfy the right strong second layer condition.

EXAMPLE 2.28. [**29**, Exercise 11M] Let $R = k+\theta S$ where $S = A_1(k) = k[x][\theta; d/dx]$ and k is a field of characteristic zero. Set

$$T = \begin{pmatrix} R & R \\ \theta S & R \end{pmatrix}, Q = \begin{pmatrix} \theta S & R \\ \theta S & R \end{pmatrix}, P = \begin{pmatrix} R & R \\ \theta S & \theta S \end{pmatrix} \text{ and } A = \begin{pmatrix} R & R \\ S & S \end{pmatrix}.$$

(a) We claim that (i) T is a prime noetherian ring, and (ii) P and Q are prime ideals in T.

PROOF. (i) Note that R is the idealizer of θS in S and θS is a maximal right ideal in S. Since S is a right noetherian ring, R_R and θS_R are noetherian by [**56**, Theorem 1.1.12]. Hence T is a right noetherian ring by [**56**, Proposition 1.1.7]. In order to show that T is a left noetherian ring, observe that R is also the idealizer of the maximal left ideal $S\theta$ of S. So ${}_RR$ is noetherian. Hence R is noetherian and so ${}_R(\theta S)$ is noetherian, as it is an ideal of R.

To prove that T is a prime ring, assume that

$$\begin{pmatrix} x_1 & x_2 \\ \theta s & x_3 \end{pmatrix} \begin{pmatrix} r_1 & r_2 \\ \theta t & r_3 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ \theta u & y_3 \end{pmatrix} = 0$$

for all $\begin{pmatrix} r_1 & r_2 \\ \theta t & r_3 \end{pmatrix} \in T$, and assume that $\begin{pmatrix} x_1 & x_2 \\ \theta s & x_3 \end{pmatrix} \neq 0$. By choosing various specific matrices $\begin{pmatrix} r_1 & r_2 \\ \theta t & r_3 \end{pmatrix}$, we can show that $\begin{pmatrix} y_1 & y_2 \\ \theta u & y_3 \end{pmatrix} = 0$. For example, if $x_1 \neq 0$ then choose $r_1 = 1, r_2 = r_3 = \theta t = 0$. So

$$0 = \begin{pmatrix} x_1 & x_2 \\ \theta s & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ \theta u & y_3 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ \theta s & 0 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ \theta u & y_3 \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 \\ \theta s y_1 & \theta s y_2 \end{pmatrix},$$

yielding $y_1 = y_2 = 0$ since S is a domain. Now

$$0 = \begin{pmatrix} x_1 & x_2 \\ \theta s & x_3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \theta u & y_3 \end{pmatrix} = \begin{pmatrix} x_1 \theta u & x_1 y_3 \\ \theta s \theta u & \theta s y_3 \end{pmatrix},$$

so $\theta u = y_3 = 0$ follows. The case $x_1 = 0$ and the subcases arising from it are treated in a similar manner. Thus, T is a prime ring.

(ii) $T/P = \begin{pmatrix} R & R \\ \theta S & R \end{pmatrix} \Big/ \begin{pmatrix} R & R \\ \theta S & \theta S \end{pmatrix} \cong \begin{pmatrix} 0 & 0 \\ 0 & R/\theta S \end{pmatrix} \cong \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \cong k,$

hence P is a prime ideal in T.

A similar argument shows that Q is also a prime ideal. □

(b) We claim that (i) A_T is a submodule of $M_2(S)$, $(T/P)_T \subseteq_e (A/P)_T$ and (ii) $r_T(A/P) = 0$. Hence we conclude that the prime P in T does not satisfy the right restricted strong second layer condition.

PROOF. (i) First, we show that A is a T-submodule of $M_2(S)$.

$$AT = \begin{pmatrix} R & R \\ S & S \end{pmatrix} \begin{pmatrix} R & R \\ \theta S & R \end{pmatrix} = \begin{pmatrix} R^2 + R\theta S & R^2 + R^2 \\ SR + S\theta S & SR + SR \end{pmatrix} = \begin{pmatrix} R & R \\ S & S \end{pmatrix} = A.$$

So A is a T-submodule of $M_2(S)$.

In order to show that $T/P \subseteq_e A/P$, let $0 \neq a \in A/P$. Then

$$a = \begin{pmatrix} 0 & 0 \\ f(x) & g(x) \end{pmatrix} + \begin{pmatrix} R & R \\ \theta S & \theta S \end{pmatrix} \quad \text{for some } f(x),\ g(x) \in k[x],$$

since any $s \in S$ can be written as $s = p(x) + \theta s'$, where $p(x) \in k[x]$ and $s' \in S$. It is required that some nonzero multiple of a is in T/P, that is, we have to find $\begin{pmatrix} r_1 & r_2 \\ \theta s & r_3 \end{pmatrix}$ such that

$$\begin{aligned} 0 \neq & \left[\begin{pmatrix} 0 & 0 \\ f(x) & g(x) \end{pmatrix} + \begin{pmatrix} R & R \\ \theta S & \theta S \end{pmatrix}\right] \begin{pmatrix} r_1 & r_2 \\ \theta s & r_3 \end{pmatrix} \\ = & \begin{pmatrix} 0 & 0 \\ f(x)r_1 + g(x)\theta s & f(x)r_2 + g(x)r_3 \end{pmatrix} + \begin{pmatrix} R & R \\ \theta S & \theta S \end{pmatrix} \\ \in & \begin{pmatrix} R & R \\ \theta S & R \end{pmatrix} \Big/ \begin{pmatrix} R & R \\ \theta S & \theta S \end{pmatrix}. \end{aligned}$$

So we need $f(x)r_1 + g(x)\theta s \in \theta S$ and $f(x)r_2 + g(x)r_3 \in R \setminus \theta S$. First note that $f(x)\theta^n = \sum_{i=0}^{n} \binom{n}{i} \theta^i f^{(n-i)}(x)$. If $f(x) \neq 0$, then take $r_1 = \theta^{\deg f+1}$ and $s = 0, r_2 = \theta^{\deg f}, r_3 = 0$. If $g(x) \neq 0$, then take $r_1 = 0, s = \theta^{\deg g}, r_2 = 0$ and $r_3 = \theta^{\deg g}$. These will yield a nonzero multiple of a in T/P.

(ii) In order to show that $r_T(A/P) = 0$, let $\begin{pmatrix} r_1 & r_2 \\ \theta s & r_3 \end{pmatrix} \in r_T(A/P)$. Then

$$\begin{pmatrix} R & R \\ S & S \end{pmatrix} \begin{pmatrix} r_1 & r_2 \\ \theta s & r_3 \end{pmatrix} = \begin{pmatrix} Rr_1 + R\theta s & Rr_2 + Rr_3 \\ Sr_1 + S\theta s & Sr_2 + Sr_3 \end{pmatrix} \subseteq \begin{pmatrix} R & R \\ \theta S & \theta S \end{pmatrix}.$$

Hence, it follows that $Sr_1 + S\theta s \in \theta S$ and $Sr_2 + Sr_3 \in \theta S$. Thus $r_1, r_2, r_3, \theta s \in r_S(S/\theta S)$. Since S is simple, $r_S(S/\theta S) = 0$.

Since $r_T(A/P) = 0 \in \operatorname{Spec}(T)$ and $(A/P)_T$ is P-tame finitely generated, P does not satisfy the right restricted strong second layer condition. □

(c) Let $E = \mathrm{E}((R/\theta S)_R)$ and make the row $E' = (E, E)$ into a right $M_2(R)$-module in the obvious way. We claim that E' is an injective right $M_2(R)$-module.

PROOF. Let I be a right ideal of $M_2(R)$ and let $f : I \to E'$ be a right $M_2(R)$-module homomorphism. Set $I_1 = \left\{ \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} \middle| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in I \right\}$. Then I_1 is a right R-submodule of $M_2(R)$. Define I_2 similarly. Note that the entries in I_1 are the same as those in I_2 since columns of I can be switched by a simple column operation.

Define $f_1 : I_1 \to E$ by $f_1 \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} = \pi_1[f(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})]$, where π_1 is the projection of E' onto $(E, 0)$, so that f_1 is an R-module homomorphism. Likewise for $f_2 : I_2 \to E$. Since E_R is an injective module, f_1 can be extended to an R-module homomorphism $\overline{f_1} : \begin{pmatrix} R & 0 \\ R & 0 \end{pmatrix} \to E$. We define $\overline{f_2} : \begin{pmatrix} 0 & R \\ 0 & R \end{pmatrix} \to E$ by $\overline{f_2} \begin{pmatrix} 0 & r_{12} \\ 0 & r_{22} \end{pmatrix} = \overline{f_1} \begin{pmatrix} r_{12} & 0 \\ r_{22} & 0 \end{pmatrix}$, and claim that $\overline{f_2}$ is an extension of f_2. Observe that

$$(f_1 \begin{pmatrix} a_{12} & 0 \\ a_{22} & 0 \end{pmatrix}, 0) = f[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \left[(f_1 \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix}, f_2 \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \left[f_2 \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}, f_1 \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= (f_2 \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}, 0).$$

Therefore, $f_1 \begin{pmatrix} a_{12} & 0 \\ a_{22} & 0 \end{pmatrix} = f_2 \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}$ for any $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in I$ proving the claim.

Define $F : M_2(R) \to (E, E)$ by

$$F \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = (\overline{f_1} \begin{pmatrix} r_{11} & 0 \\ r_{21} & 0 \end{pmatrix}, \overline{f_2} \begin{pmatrix} 0 & r_{12} \\ 0 & r_{22} \end{pmatrix}).$$

We claim that

(i) F is an extension of f.

(ii) F is a right $M_2(R)$-module homomorphism.

(i) follows from the definition of F.

(ii) Let $r = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in M_2(R)$ and $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in M_2(R)$. It is to be shown that $F(xr) = F(x)r$. Since

$$xr = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} x_{11}r_{11} + x_{12}r_{21} & x_{11}r_{12} + x_{12}r_{22} \\ x_{21}r_{11} + x_{22}r_{21} & x_{21}r_{12} + x_{22}r_{22} \end{pmatrix},$$

we get

$$F(xr) = \left[\overline{f_1} \begin{pmatrix} x_{11}r_{11} + x_{12}r_{21} & 0 \\ x_{21}r_{11} + x_{22}r_{21} & 0 \end{pmatrix}, \overline{f_2} \begin{pmatrix} 0 & x_{11}r_{12} + x_{12}r_{22} \\ 0 & x_{21}r_{12} + x_{22}r_{22} \end{pmatrix} \right].$$

Also,

$$
\begin{aligned}
F(x)r &= \left[F\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right]\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \\
&= \left(\overline{f_1}\begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix} \quad \overline{f_2}\begin{pmatrix} 0 & x_{12} \\ 0 & x_{22} \end{pmatrix}\right)\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \\
&= \left[\overline{f_1}\begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix} r_{11} + \overline{f_2}\begin{pmatrix} 0 & x_{12} \\ 0 & x_{22} \end{pmatrix} r_{21}, \overline{f_1}\begin{pmatrix} x_{11} & 0 \\ x_{21} & 0 \end{pmatrix} r_{12} + \overline{f_2}\begin{pmatrix} 0 & x_{12} \\ 0 & x_{22} \end{pmatrix} r_{22}\right] \\
&= \left[\overline{f_1}\begin{pmatrix} x_{11}r_{11} & 0 \\ x_{21}r_{11} & 0 \end{pmatrix} + \overline{f_2}\begin{pmatrix} 0 & x_{12}r_{21} \\ 0 & x_{22}r_{21} \end{pmatrix}, \overline{f_1}\begin{pmatrix} x_{11}r_{12} & 0 \\ x_{21}r_{12} & 0 \end{pmatrix} + \overline{f_2}\begin{pmatrix} 0 & x_{12}r_{22} \\ 0 & x_{22}r_{22} \end{pmatrix}\right].
\end{aligned}
$$

Since $\overline{f_2}$ was defined as $\overline{f_2}\begin{pmatrix} 0 & r_{12} \\ 0 & r_{22} \end{pmatrix} = \overline{f_1}\begin{pmatrix} r_{12} & 0 \\ r_{22} & 0 \end{pmatrix}$, it follows that F is an $M_2(R)$-module homomorphism. □

(d) We claim that $M_2(R)$ is projective as a left T-module, and conclude that E' is also injective as a right T-module.

PROOF. Note that

$$
\begin{pmatrix} R & R \\ \theta S & R \end{pmatrix}\begin{pmatrix} R & R \\ R & R \end{pmatrix} = \begin{pmatrix} RR+RR & RR+RR \\ \theta SR+RR & \theta SR+RR \end{pmatrix} \subseteq \begin{pmatrix} R & R \\ R & R \end{pmatrix},
$$

so $M_2(R)$ is a left T-module. Write

$$
M_2(R) = \begin{pmatrix} R & R \\ R & R \end{pmatrix} = {}_T\begin{pmatrix} R & 0 \\ R & 0 \end{pmatrix} \oplus {}_T\begin{pmatrix} 0 & R \\ 0 & R \end{pmatrix}
$$

and note that

$$ {}_T\begin{pmatrix} R & 0 \\ R & 0 \end{pmatrix} \cong {}_T\begin{pmatrix} 0 & R \\ 0 & R \end{pmatrix}. $$

Thus, ${}_TM_2(R)$ is a projective module since a summand of a free module is projective and a direct sum of projective modules is projective by [**66**, Proposition 2.8.3, Lemma 2.8.4] where

$$ T = {}_T\begin{pmatrix} R & 0 \\ \theta S & 0 \end{pmatrix} \oplus {}_T\begin{pmatrix} 0 & R \\ 0 & R \end{pmatrix}. $$

By claim (c) above, E' is injective as a right $M_2(R)$-module, which as a left T-module is projective. Therefore E'_T is injective by Lemma 2.29 below. ☐

(e) (i) We claim that the row $B = (0, R/\theta S)$ is a right T-submodule of E', and that $B \cong (T/P)_T$.

PROOF. Define $\phi : (T/P)_T \to B$ by

$$ \phi\left(\begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix} + \begin{pmatrix} R & R \\ \theta S & \theta S \end{pmatrix}\right) = (0, r + \theta S). $$

Then ϕ is not only well defined, but also a T-isomorphism. ☐

(ii) If M is any T-submodule of E' such that $M \supset B$, we claim that M contains the row $C = (R/\theta S, R/\theta S)$.

PROOF. Since $(0, R/\theta S) \subset M \subseteq (\mathrm{E}(R/\theta S), \mathrm{E}(R/\theta S))$, the elements of M are of the form $m = (m_1, m_2)$ where $m_i \in \mathrm{E}(R/\theta S)$. Suppose that all the elements of M are of the form $(0, m_2)$. Since

$$(0, m_2)\begin{pmatrix} r_1 & r_2 \\ \theta s & r_3 \end{pmatrix} = (m_2\theta s, m_2 r_3) \in M,$$

we get $m_2\theta s = 0$ for all $s \in S$. So $m_2 \in l_{\mathrm{E}(R/\theta S)_R}(\theta S) = R/\theta S$, since $\mathrm{E}(R/\theta S)_R$ is a uniform right R-module with socle $R/\theta S$. Thus $M = B$, contradicting the hypothesis $B \subset M$. Therefore, there exists at least one $(m_1, m_2) \in M$ such that $m_1 \neq 0$. Since $R/\theta S \subseteq_e \mathrm{E}(R/\theta S)$, we have $0 \neq m_1 r_1 \in R/\theta S$ for some $r_1 \in R$. As $m_1 r_1$ generates the simple module $R/\theta S$, it follows that $B \subset (R/\theta S, R/\theta S) \subseteq M$. □

(iii) Now, if M has an affiliated series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$, then we claim that P, Q must be its first two affiliated primes.

Proof. First, we show that $B_T \subseteq_e E'_T$. It is to be shown that for any $0 \neq (e_1, e_2) \in E'$ there exists $\begin{pmatrix} r_1 & r_2 \\ \theta s & r_3 \end{pmatrix} \in T$ such that

$$0 \neq (e_1, e_2)\begin{pmatrix} r_1 & r_2 \\ \theta s & r_3 \end{pmatrix} = (e_1 r_1 + e_2\theta s, e_1 r_2 + e_2 r_3) \in (0, R/\theta S)$$

.

If $0 \neq e_1 \in \mathrm{E}(R/\theta S)$, then $0 \neq e_1 r \in R/\theta S$ for some $r \in R$ since $R/\theta S \subseteq_e \mathrm{E}(R/\theta S)$. Setting $r_1 = 0, s = 0, r_2 = r$ and $r_3 = 0$ produces the required element of T. If $e_2 \neq 0$, then $0 \neq e_2 r \in R/\theta S$ for some $r \in R$, so set $r_1 = 0, s = 0, r_2 = 0, r_3 = r$ in this case. It follows that E' is P-primary, in fact, P-tame since P is a maximal right ideal of T. Now the first layer of M is $M_1 = l_M(P) = l_{E'}(P) \cap M = (0, R/\theta S) \cap M = B$.

Since $C = (R/\theta S, R/\theta S) \subseteq M'$ for any M' with $B \subset M' \subseteq E'$, the simple module C/B is essential in M_2/M_1 and $Q = r_T(C/B) = \mathrm{Ass}(M_2/M_1)$. Thus, M_2/M_1 is Q-primary. Since $T/Q \cong R/\theta S \cong k$, M_2/M_1 is Q-tame as well. □

(f) We claim that P satisfies the right second layer condition, but not the right strong second layer condition.

Proof. First, we show that P satisfies the right affiliated strong second layer condition (and hence the right affiliated second layer condition and consequently also the right second layer condition by Theorem 2.27). For this, we have to show that there does not exist a finitely generated uniform right T-module M with an affiliated series $0 \subset U \subset M$ and corresponding prime ideals P, Q' such that M/U is uniform, $Q' \subset P$ and $MQ' = 0$. Assume that there is such a module. Since $T/P \cong k$, the uniform module U can be identified with the module B above. Since $B \subseteq E' = \mathrm{E}(B)_T$, the module M can be considered as a submodule of E' containing B. By the above, the first two affiliated prime ideals of any affiliated series of M are thus P and Q. Since they are P and Q' by assumption, $Q = Q'$. Thus $Q = Q' \subset P$, which gives a contradiction.

Note that by (b), P does not satisfy the right restricted strong second layer condition, so it cannot satisfy the right strong second layer condition by Theorem 2.27. □

The above example also shows that for a single prime the right affiliated strong second layer condition does not imply the right strong second layer condition.

The following lemma is slightly more general than [**16**, Ch. 2, Proposition 6.2a], but the proof is essentially the same since every projective module is flat.

LEMMA 2.29. *Let R, S be rings such that ${}_RS$ is a flat left R-module, and let E_S be an injective module. Then E is injective as a right R-module.*

PROOF. Let $N \subseteq M$ be right R-modules and let $f : N \to E$ be a right R-module homomorphism. By a well-known result of homological algebra (see [**66**, Proposition 2.10.9]),

$$s' : \mathrm{Hom}_R(N, \mathrm{Hom}_S(S, E)) \to \mathrm{Hom}_S(N \otimes_R S, E)$$

defined by $[s'(\phi)](n \otimes s) = [\phi(n)](s)$ is an isomorphism. Also $\mathrm{Hom}_S(S, E) \cong E$. Hence f corresponds to $s'(f) \in \mathrm{Hom}(N \otimes S, E)$, so that $s'(f)(n \otimes 1) = f(n)$.

Since S is a flat left R-module, $N \otimes_R S \subseteq M \otimes_R S$. Thus $s'(f)$ can be extended to an S-module homomorphism $s''(f) : M \otimes S \to E$. Now $s''(f)$

gives an R-module homomorphism $\overline{f} : M \to E$ via $\overline{f}(m) = s''(f)(m \otimes 1)$, which is an extension of f since $\overline{f}(n) = s''(f)(n \otimes 1) = s'(f)(n \otimes 1) = f(n)$ for all $n \in N$. $\square$

CHAPTER 3

Second Layer Conditions in Extension Rings

3.1. Introduction

In recent years, a good deal of research has been concerned with the structure of extensions of rings that satisfy the second layer condition. Investigations in this direction started with Brown and Jategaonkar who showed independently that a group ring RG satisfies the strong second layer condition whenever R is a commutative noetherian ring and G is a polycyclic-by-finite group [13], [39]. Recall that a commutative noetherian ring satisfies the strong second layer condition. Bell then weakened the hypotheses on R to include artinian rings, noetherian P. I. rings and principal ideal rings, and showed that a strongly G-graded ring S with base ring R satisfies the second layer condition [11].

In this chapter, we consider some of the simplest cases: finite extensions and polynomial extensions. Let R and S be rings such that S contains R as a subring. The ring S is said to be a right module finite extension of R if S is finitely generated as a right R-module. We begin with module finite extensions of a class of rings satisfying the second layer condition, in particular, artinian rings and FBN rings as they have well-understood structures. The former have the property that every module is finitely annihilated, while the latter are characterized by the Gabriel (H)-condition that every finitely generated module is finitely annihilated. It is easy to see that a module finite extension of an artinian ring is also artinian. In 1990, Letzter showed that a module finite extension of an FBN ring is also an FBN ring [51]. We give a new proof of this in terms of finite annihilation.

By following the pattern set by artinian rings and FBN rings in terms of finite annihilation, we continue with a more general class of rings: noetherian rings satisfying the second layer condition. As it has been established in Chapter 2, these rings can also be characterized by a version of finite annihilation. Here, only the finitely generated tame modules are finitely annihilated. We show, by finite annihilation of modules, that a module finite extension ring of a noetherian ring satisfying the right second layer condition also satisfies the right second layer condition. This is a generalization of an earlier result by Letzter who showed that a module finite extension of a noetherian ring satisfying the two-sided second layer condition satisfies the same.

We also observe that tameness of modules over the extension ring can be transferred to the base ring and vice versa. That is, if S is a module finite extension of a ring R, then an S-module M is tame as an S-module iff it is tame as an R-module.

However, when infinite extensions of noetherian rings are considered, there are still a fair number of open questions. For example, it is still not known if the polynomial ring $R[x]$ satisfies the second layer condition when the base ring R does [11, p. 109]. So far, it is known that $R[x]$ satisfies the strong second layer condition when R is an FBN ring. With the help from a technical lemma of Byun [15] and a recent result established by Kosler [43], we give a sufficient condition for a prime ideal P in $R[x]$ to satisfy the second layer condition.

3.2. Finite Extensions

The observation that all finitely generated tame modules are finitely annihilated if the ring satisfies the second layer condition is convenient and practical, particularly when dealing with extension rings. It originated from prime Goldie rings where the torsionfree modules, though not necessarily finitely generated, are finitely annihilated.

LEMMA 3.1. *Let R be a prime right Goldie ring. Then every torsionfree right R-module M is a Δ-module.*

PROOF. Suppose that there is a subset S of M with an infinite descending chain $r(s_1) \supset r(s_1, s_2) \supset \cdots$, where $s_i \in S$. Set $I_0 = R$ and $I_n = r(s_1, \ldots, s_n)$ for each $n \geq 1$. Note that $0 \neq I_n/I_{n+1} \cong s_{n+1}I_n \subseteq M$. Since M is torsionfree, $\rho(I_n/I_{n+1}) \geq 1$. Note also that $\rho(R)$ is finite, say k, as R is prime right Goldie. However, by the additivity of the reduced rank,

$$k = \rho(R) = \sum_{i=0}^{k} \rho(I_i/I_{i+1}) \geq k + 1 > k.$$

This is an obvious contradiction. □

DEFINITION. Let R, S be rings such that R is a subring of S. The ring S is said to be a *right module finite extension* of R if S is finitely generated as a right R-module.

Similarly, we define *left module finite extensions*. We say that S is a *module finite extension* of R if it is right and left module finite over R.

When S is a module finite extension of R, some properties of right S-modules are inherited by right R-modules, and vice versa. For example, tameness of a module is preserved in both directions. In fact, we can even show that going down can be achieved without S being finitely generated as an R-module. First, we observe that every Δ-module is tame.

LEMMA 3.2. *Let R be a right noetherian ring and let M be a right R-module such that every submodule is finitely annihilated. Then M is tame.*

PROOF. Let U be a prime uniform submodule of M. Set $P = \mathrm{Ass}(U)$. Since U is finitely annihilated, there exist some $u_1, \ldots, u_k \in U$ such that $r(U) = r(u_1, \ldots, u_k)$. So,

$$R/P = R/r(u_1, \ldots, u_k) \hookrightarrow U^k.$$

If V/P is a uniform right ideal of R/P, then $V/P \hookrightarrow U$, and so U is tame. Therefore, M is tame. □

PROPOSITION 3.3 (Down). *Let R be a right noetherian ring, and let $S \supseteq R$ be a right noetherian ring satisfying the right second layer condition.*

Then every finitely generated tame right S-module M is tame as a right R-module.

PROOF. Note that M is a Δ-module as an S-module by Proposition 2.18 and Theorem 2.13, and it obviously remains a Δ-module when considered as an R-module. Hence M is tame as a right R-module by Lemma 3.2. $\square$

In order to set up the converse to the previous proposition, we require a lemma that describes a relation between regular elements of subrings and overrings. It is a variation of Joseph-Small [41, Corollary 3.7] and is basically due to Warfield [75, Corollary 2] where the two-sided second layer condition was used.

LEMMA 3.4. *Let R be a noetherian ring satisfying the right second layer condition and let S be a module finite extension ring of R. Then the set $\mathcal{C}_{R/(P\cap R)}(N/(P\cap R))$ is contained in $\mathcal{C}_{S/P}(0)$ for every prime ideal P of S, where $N/(P\cap R)$ denotes the prime radical of $R/(P\cap R)$.*

PROOF. Note that the hypotheses of the lemma hold for $R/(P\cap R)\cong (R+P)/P$ and its extension ring S/P. So we may assume that S is a prime noetherian ring. Since ${}_RS_S$ is a noetherian bimodule, there exists a left affiliated series

$$0 = S_0 \subset S_1 \subset \cdots \subset S_n = S$$

with affiliated prime ideals $Q_1, \ldots, Q_n \in \operatorname{Spec}(R)$. We claim that each factor ${}_{R/Q_i}(S_i/S_{i-1})_S$ is torsionfree on both sides. First, we show this for the left hand side. Suppose that the torsion submodule $Z({}_{R/Q_i}S_i/S_{i-1}) = W/S_{i-1}$ is nonzero. Since W/S_{i-1} is finitely generated as a right S-module, say by $w_1 + S_{i-1}, \ldots, w_k + S_{i-1}$, we get

$$l_{R/Q_i}(W/S_{i-1}) = l_{R/Q_i}(w_1 + S_{i-1}, \ldots, w_k + S_{i-1}).$$

Set $I/Q_i = l_{R/Q_i}(W/S_{i-1})$. Then

$$R/I \cong (R/Q_i)/(I/Q_i) \hookrightarrow (W/S_{i-1})^k.$$

Thus, I/Q_i cannot be 0, for if it were then the torsionfree module R/Q_i would be a submodule of the torsion module $(W/S_{i-1})^k$. However, $I/Q_i \neq 0$ contradicts the fact that the affiliated prime ideal Q_i, by definition, is maximal among left annihilators of nonzero left R-submodules of S/S_{i-1}. Thus S_i/S_{i-1} is torsionfree on the left.

For the right hand side, observe first that $S/S_0 = S$ is torsionfree. Suppose that S/S_i is not torsionfree for some $i > 0$, but that S/S_{i-1} is torsionfree. Let $V/S_i = Z(S/S_i)_S \neq 0$ be the torsion submodule of S/S_i. Since V/S_i is finitely generated on the left by, say $v_1 + S_i, \ldots, v_m + S_i$, we have $r_S(V/S_i) = r_S(v_1 + S_i, \ldots, v_m + S_i)$. Note that $r_S(v_j + S_i) \subseteq_e S$ since $v_j + S_i \in Z(S/S_i)_S$. By Goldie's Theorem, there exists $c \in \mathcal{C}_S(O)$ such that $c \in r_S(V/S_i)$. Observe that ${}_RV \cong {}_RVc$, as c is a regular element of S. So it follows from

$$Q_1Q_2 \cdots Q_iVc \subseteq Q_1Q_2 \cdots Q_iS_i = 0$$

that $Q_1 \cdots Q_i V = 0$. Consequently, $V \subseteq l_S(Q_1 \cdots Q_i) = S_i \subset V$, a contradiction. Thus S/S_i and hence S_{i+1}/S_i is torsionfree for all i.

If every left affiliated prime ideal Q_i is minimal, then the desired result $\mathcal{C}_R(N) \subseteq \mathcal{C}_S(0)$ follows. To show this, let $c \in \mathcal{C}_R(N)$. It suffices to show that c is right regular in S, by [**29**, Lemma 5.7]. If $cs = 0$ for some $0 \neq s \in S$, then let i be the smallest integer such that $s \in S_i$. Then $s + S_{i-1} \neq 0$ and $c(s + S_{i-1}) = 0$. Since $\mathcal{C}_R(N) \subseteq \mathcal{C}_R(Q_i)$ by [18, Theorem 1.25], $c \in \mathcal{C}_R(Q_i)$. This contradicts the fact that S_i/S_{i-1} is torsionfree as a left R/Q_i-module. Thus, c is regular in S.

Now it remains to show that every Q_i is a minimal prime. First observe that $\operatorname{minspec}(R) \subseteq \{Q_1, \ldots, Q_n\}$ since $Q_1 \ldots Q_n \subseteq l_R(S) = 0$. For the other inclusion, form a right affiliated series

$$0 = T_0 \subset \cdots \subset T_m = S_i/S_{i-1}$$

of ${}_{R/Q_i}(S_i/S_{i-1})_R$. Let $P_1, \ldots, P_m \in \operatorname{Spec}(R)$ be its affiliated primes. Since $(S_i/S_{i-1})_S$ is torsionfree, $r_R(S_i/S_{i-1}) = 0$ and so $P_m \cdots P_1 \subseteq 0$. Hence

$$\operatorname{minspec}(R) \subseteq \{P_1, \ldots, P_m\}.$$

Thus $\text{cl.K.dim}(R) = \text{cl.K.dim}(R/P_j)$ for some j. On the other hand,

$$\text{cl.K.dim}(R/Q_i) \geq \text{cl.K.dim}(R/P_j)$$

by Lemma 2.9, since R satisfies the right second layer condition. Thus

$$\text{cl.K.dim}(R/Q_i) = \text{cl.K.dim}(R),$$

showing that Q_i is a minimal prime for all $i = 1, \dots, n$, and so the set of minimal primes $\text{minspec}(R) = \{Q_1, \dots, Q_n\}$. $\square$

Since $\mathcal{C}_{R/(P\cap R)}(N/(P\cap R)) \subseteq \mathcal{C}_{S/P}(0)$, the set $\mathcal{C}_{R/(P\cap R)}(N/(P\cap R))$ of $R/(P\cap R)$ is also regular in $R/(P\cap R)$, and thus the hypotheses of Small's theorem are satisfied. In other words, $R/(P\cap R)$ is a right order in a right artinian ring.

PROPOSITION 3.5 (Up). *Let R be a noetherian ring satisfying the right second layer condition and let S be a module finite extension ring of R. Then a finitely generated right S-module M is tame whenever it is tame as a right R-module.*

PROOF. Suppose that M_S is tame as a right R-module. Let U be a prime uniform S-submodule of M with $P = r_S(U) = \text{Ass}(U_S)$. Since U is finitely generated and P is a prime ideal, we may assume that $U = uS$ for some $u \in U$. By passing S to S/P, we may also assume that $P = 0$ since $R/P\cap R$ and S/P satisfy the hypotheses of the proposition. Hence S is a prime noetherian ring. Thus U is either torsionfree or torsion as a right S-module. Suppose it is torsion. Then there exists $c \in \mathcal{C}_S(0)$ such that $uc = 0$. Note that $uS \cong S/r_S(u)$ and $S_R \cong cS_R \subseteq r_S(u)$. By the additivity of the reduced rank,

$$\rho_R(uS) = \rho_R(S/r_S(u)) \leq \rho_R(S/cS) = 0$$

as $\rho_R(S)$ is finite. Thus uS_R is $\mathcal{C}_R(N)$-torsion by Lemma 1.3. Since uS is finitely generated as a right R-module (S is right module finite over R) and since R satisfies the right second layer condition, $r_R(uS) = r_R(us_1, \dots, us_n)$ for some $us_1, \dots, us_n \in uS$. Set $I = r_R(uS)$. Combining $R/I \hookrightarrow (uS_R)^n$ together with uS_R being $\mathcal{C}_R(N)$-torsion, we obtain that R/I is $\mathcal{C}_R(N)$-torsion, and so $I \cap \mathcal{C}_R(N) \neq \emptyset$. Furthermore, by Lemma 3.4, $I \cap \mathcal{C}_S(0) \neq \emptyset$. This means that ${}_S(S/SI)$ is torsion, and hence there exists a nonzero ideal J in S such that $J \subseteq l_S(S/SI)$ by [**29**, Lemma 7.3]. But then

$$uSJ = uJS \subseteq uSI = 0,$$

contradicting that $r_S(uS) = 0$. Therefore, U has to be torsionfree, and M_S is tame. □

If every finitely generated right R-module is finitely annihilated, then R is said to satisfy the *Gabriel H-condition.* Characterizing a right FBN ring in terms of finite annihilation is useful as we can see in the following new proof that a finite extension ring of an FBN ring is also an FBN ring.

PROPOSITION 3.6. [**51**, Proposition 4.9] *Let S be a right module finite extension ring of a right FBN ring R. Then S is also right FBN.*

PROOF. Since R is right noetherian, S_R is noetherian and therefore S_S is noetherian. Note that R/I is right FBN for any ideal I of R. Now $R/(P \cap S) \hookrightarrow S/P$ for any prime ideal P of S, so assume without loss of generality that S is a prime right noetherian ring.

Suppose $E_S \subseteq_e S_S$ with $r_S(S/E) = 0$. Since S is a prime right Goldie ring, Goldie's Theorem provides an element $c \in \mathcal{C}_S(0) \cap E$. Now $(cS)_S \cong S_S$ as S-modules, so $(cS)_R \cong S_R$ as R-modules. Since $\rho_R(S_R)$ is finite, it follows that $\rho_R(S/cS) = 0$ by the additivity of reduced rank. Note that $\rho_R(S/E) \leq \rho_R(S/cS) = 0$.

By Lemma 1.3, $(S/E)_R$ is $\mathcal{C}_R(N)$-torsion. Now, by Theorem 1.5,

$$r_R(S/E) = r_R(s_1 + E, \ldots, s_n + E).$$

Since $r_R(S/E) = R \cap r_S(S/E) = R \cap 0 = 0$, it follows that

$$R = R/r_R(S/E) \hookrightarrow \bigoplus_{i=1}^{n} (s_i R + E)/E \hookrightarrow (S/E)^n,$$

so $\rho(R_R) = 0$. This is impossible, so $r_S(S/E) > 0$, proving that S is right bounded. $\square$

By using a similar argument, we can now prove the main result of this section that the second layer condition is carried over to a module finite extension ring. First, we record Bell's assertion on G-graded rings where G is a finite group, which served as precept for Letzter's finite extension. We also record Letzter's Theorem for comparison.

DEFINITION. Let G be a monoid and let R be a ring. A ring S is called a *strongly G-graded* ring if $S = \oplus_{g \in G} Rg$ where Rg is an additive subgroup of S such that $Rg \cdot Rh = Rgh$ for any $g, h \in G$. We call $R = R1$ the *base ring* of S.

PROPOSITION 3.7. [11, Proposition 7.5] *Let G be a finite group and R be a noetherian ring and let S be a strongly G-graded ring with base ring R. Then R satisfies the right second layer condition iff S does.*

THEOREM 3.8. [51, Theorem 4.2] *Let S be a module finite extension ring of a noetherian ring R. If R satisfies the second layer condition, then so does S. Moreover, if R satisfies the strong second layer condition, then so does S.*

THEOREM 3.9. *Let S be a module finite extension ring of a noetherian ring R. If R satisfies the right second layer condition, then so does S.*

PROOF. Suppose that S does not satisfy the right second layer condition. Then there exists a prime ideal P of S such that S/P is not right tame bounded by Theorem 2.13 and Corollary 2.20. By passing S to S/P, we can assume that the prime noetherian ring S is not right tame bounded. Set

$$\mathcal{E} = \{E_S \subseteq_e S_S \mid S/E \text{ is tame, but } r_S(S/E) = 0\}.$$

By the assumption on S, the set $\mathcal{E}$ is not empty. A maximal $E \in \mathcal{E}$ can be chosen as S_S is noetherian. We claim that S/E is a uniform right S-module. If it is not, then by [18, Lemma 7.9] there exist nonzero submodules $E_1/E, \dots, E_n/E$ of S/E, where $n = \text{rank}(S/E_S)$, such that $S/E \hookrightarrow_e \bigoplus_{i=1}^n S/E_i$, $\bigcap_{i=1}^n E_i = E$ and each S/E_i is uniform. Since essential extensions of tame modules are tame, each S/E_i is also tame. Moreover, $r_S(\bigoplus_{i=1}^n S/E_i) \subseteq r_S(S/E) = 0$ yields $r_S(S/E_i) = 0$ for some i. This E_i

contradicts the maximality of E. Thus S/E must be uniform, proving the claim.

Set $Q = \text{Ass}(S/E)_S$ and $L/E = l_{S/E}(Q)$. Note that $(L/E)_S \subseteq_e S/E$. Since $(S/E)_R$ is noetherian as well, there exists a right R-submodule K/E of S/E maximal with respect to the property that $K/E \cap L/E = 0$. Then

$$L/E \cong \frac{L/E \oplus K/E}{K/E} \subseteq_e \frac{S/E}{K/E} \cong S/K.$$

Note that $(L/E)_R$ is tame, since it is a Δ-module as a right S/Q-module by Lemma 3.1 and so a Δ-module as a right R-module. Thus S/K is tame as a right R-module as well.

On the other hand, the reduced rank $\rho_R(S/K)$ will be shown to be 0. In that case, the fact that S/K is $\mathcal{C}_R(N)$-torsion and is finitely annihilated by Theorem 2.13, gives $r_R(S/K) \cap \mathcal{C}_R(N) \neq \emptyset$, as $R/r_R(S/K) \hookrightarrow \bigoplus_{i=1}^m (S/K)_R$ for some m. Then by Lemma 3.4, $r_R(S/K) \cap \mathcal{C}_S(0) \neq \emptyset$. Set $I = r_R(S/K)$. It follows that ${}_S(S/SI)$ is torsion, and so by [29, Lemma 7.3] there exists a nonzero ideal J of S such that

$$J = JS \subseteq SI \subseteq K.$$

If $J \subseteq E$, then $0 \neq J \subseteq r_S(S/E) = 0$, which is an obvious contradiction. Otherwise, i.e., if $J \not\subseteq E$, then

$$\frac{L}{E} \cap \frac{J+E}{E} \subseteq \frac{L}{E} \cap \frac{K}{E} = 0.$$

But then $L/E \subseteq_e S/E$ gives a contradiction.

In order to complete the proof, we need to prove that $\rho_R(S/K) = 0$. Since $E_S \subseteq_e S_S$, there exists $c \in E \cap \mathcal{C}_S(0)$. Now observe that $S_R \cong cS_R$, and so $\rho_R(S/cS) = 0$ since $\rho_R(S)$ is finite. Thus,

$$\rho_R(S/K) \leq \rho_R(S/E) \leq \rho_R(S/cS)$$

produces the desired result.

Therefore, S satisfies the right second layer condition. $\square$

The above theorem is a generalization of [**51**, Theorem 4.2] in the sense that it does not require the full strength of two sided second layer condition on R. The proof given relies heavily on finite annihilation of tame modules, which turned out to be a useful characterization of the right second layer condition. The two sided second layer condition gives rise to a symmetric dimension function on bimodules, which is used implicitly in the proof of [**51**, Theorem 4.2] to show the existence of two sided artinian quotient ring of $R/(P \cap R)$, hence the minimal spectrum is right and left link closed. Thus, with the existence of a symmetric dimension function like Gelfand-Kirillov dimension [**46**, Corollary 5.4], finite extensions of noetherian algebras produce the same result for second layer conditions. However, due to the lack of suitable finite annihilation of modules when R satisfies the right strong second layer condition, our argument cannot be utilized in this case.

COROLLARY 3.10. *Let R be a ring with center Z that is a noetherian ring. If R is finitely generated as a right Z-module, then $R[x]$ satisfies the second layer condition.*

PROOF. Note that $Z[x]$ satisfies the second layer condition since it is a commutative noetherian ring. Let R, as a right Z-module, be generated by $r_1, \ldots, r_n \in R$. Then $R[x] = r_1 Z[x] + \cdots + r_n Z[x]$, so by Theorem 3.9 $R[x]$ satisfies the right second layer condition.

The left second layer condition follows by symmetry. □

Going back to the bimodule dimension defined in [39], we obtain equality for the classical Krull dimensions of R and S.

COROLLARY 3.11. *Let S be a module finite extension ring of a noetherian ring R satisfying the right second layer condition. Then*

$$\text{cl.K.dim}(R) = \text{cl.K.dim}(S).$$

PROOF. For the noetherian bimodule ${}_SS_R$, the right second layer condition of R gives that $\mu({}_SS_R) = \mu({}_RR_R)$ where μ is the bimodule Krull dimension defined in terms of deviation on subbimodules [39, Theorem 1.5]. By definition, $\mu({}_SS_R) \geq \mu({}_SS_S)$. On the other hand, since S satisfies the right second layer condition by Theorem 3.9, cl.K.dim$(S) = \mu({}_SS_S)$ and cl.K.dim$(R) = \mu({}_RR_R)$ by [39, Theorem 1.7]. Furthermore, we have cl.K.dim$(S) \geq$ cl.K.dim(R) by Lemma 2.9. Putting these together, we get

$$\text{cl.K.dim}(S) \geq \text{cl.K.dim}(R) = \mu({}_SS_R) \geq \mu({}_SS_S) = \text{cl.K.dim}(S),$$

and thus cl.K.dim(S) = cl.K.dim(R). □

3.3. Centralizing Extensions

Even though it is not known in general [11, p. 109] if the second layer condition is satisfied by $R[x]$ for a noetherian ring R satisfying it, some special cases have an affirmative answer. For example, $R[x]$ satisfies the right strong second layer condition if R is an FBN ring. In the last five years we have seen some progress in centralizing and normalizing extensions, providing us with a better perspective on the structure of extension rings [43], [15].

Definition. Let R, S be rings such that R is a subring of S. The ring S is said to be a *centralizing extension* of R if there exists a subset C of S such that $S = RC = CR$ where $cr = rc$ for every $r \in R, c \in C$. It is called a *normalizing extension* of R if there exists a subset N of S such that $S = RN = NR$ where $nR = Rn$ for every n in N.

Lemma 3.12. *Let S be a centralizing extension of R. If P is a prime ideal of S, then $P \cap R$ is a prime ideal of R.*

Proof. Let C centralize R and let $S = RC = CR$. Suppose that I and J are ideals of R such that $IJ \subseteq P \cap R$. Observe that

$$IJ \cdot S = IJ(RC) = IC \cdot JC = IS \cdot JS.$$

On the other hand, $IJ \cdot S \subseteq (P \cap R)S \subseteq P$. Combining these, we get $IS \cdot JS \subseteq P$ and so either $IS \subseteq P$ or $JS \subseteq P$. In the first case,

$$I \subseteq (IS \cap R) \subseteq P \cap R.$$

Similarly for JS. Consequently, $P \cap R$ is a prime ideal of R. □

It should be noted that $P \cap R$ is only a semiprime ideal of R when S is a normalizing extension of R and right noetherian [**75**, Theorem 3].

We recall one elementary result: If R is a right noetherian ring, then the polynomial ring $R[x]$ is also right noetherian by the Hilbert Basis Theorem [**31**, Theorems I, III]. Also, note that if S is a simple artinian ring with center Z, then every ideal of $S[x]$ is principal generated by a central element [**29**, Proposition 15.1]. In this case, the center of $S[x]$ is just $Z[x]$. More generally, the center of the polynomial ring $S[x_1, \ldots, x_n]$ in n commuting indeterminates is $Z[x_1, \ldots, x_n]$, and every ideal of $S[x_1, \ldots, x_n]$ is generated by elements in $Z[x_1, \ldots, x_n]$ [**29**, Proposition 15.5].

The main results of this section rely on one technical lemma established by Byun [**15**]. We follow his definitions and lemmas to arrive at a more complete understanding of centralizing extensions.

DEFINITION. Let R be a right noetherian ring and let L, M and N be right R-modules. If $0 \to L \to M \to N \to 0$ is an exact sequence such that $0 \subset L \subset_e M$ is an affiliated series with affiliated primes P, Q, then it is called a *right second layer series.*

DEFINITION. Let k be a commutative ring. A k-algebra A is said to satisfy the condition (†) if, for any division k-algebra D, any right noetherian factor ring of $A \otimes_k D$ satisfies the right affiliated second layer condition. It is said to satisfy the condition (‡) if, for any division k-algebra D, any right noetherian factor of $A \otimes_k D$ satisfies the right affiliated strong second layer condition.

LEMMA 3.13. [15, Lemma A] *Let R and A be k-subalgebras of S where A centralizes R and $R \cup A$ generates S. Assume that R and S are right noetherian and that $Q \subset P$ are prime ideals of S with $Q \cap R = P \cap R$.*

(i) *If A satisfies the condition (†), then there does not exist a second layer series $0 \to L \to M \to N \to 0$ of right S-modules with affiliated primes P, Q such that $MQ = 0$ and $L_{S/P}$ is torsionfree.*

(ii) *If A satisfies the condition (‡), then there does not exist a second layer series $0 \to L \to M \to N \to 0$ of right S-modules with affiliated primes P, Q such that $MQ = 0$ and $M_{R/(Q \cap R)}$ is torsionfree.*

DEFINITION. Let R be a prime ring. Then R is called *sub-bounded* if for every nonzero prime ideal P of R there exist an element $c \in R$ and a nonzero ideal I such that $I \subseteq cR \cap Rc \subsetneq P$. A ring R is called *fully sub-bounded* if every prime factor ring R/P is sub-bounded.

An FBN ring is an example of a fully sub-bounded ring.

LEMMA 3.14. [15, Lemma B] *Let S be a centralizing extension of R and $Q \subset P$ be prime ideals of S with $Q \cap R \subset P \cap R$. If $R/(Q \cap R)$ is sub-bounded, then there does not exist a second layer series $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ over S with affiliated primes P, Q such that $MQ = 0$.*

While Byun's assertion is based on the affiliated series (second layer series) and hence is directly related to the affiliated second layer condition, his assertion–particularly [15, Lemma A]–can be generalized to the right restricted strong second layer condition. We first define a condition that is very similar to (†).

DEFINITION. Let k be a commutative ring. A k-algebra A is said to satisfy the condition $(\star)_r$ if for any k-division algebra D any right noetherian factor of $A \otimes_k D$ satisfies the right restricted strong second layer condition.

REMARK. (i) If $A \otimes_k D$ is right noetherian, then the condition $(\star)_r$ implies that $A \otimes_k D$ itself satisfies the right restricted strong second layer condition.

(ii) If A satisfies $(\star)_r$ then so does any factor A/B of A, since $A/B \otimes_k D \cong (A \otimes_k D)/(B \otimes_k D)$.

(iii) Since any simple artinian ring R can be regarded as $M_n(D)$ for some division ring D and since $A \otimes_k M_n(D) \cong M_n(A \otimes_k D)$, we may conclude that $A \otimes_k R$ satisfies the right restricted strong second layer condition whenever A satisfies $(\star)_r$.

The following lemma is a variation of [15, Lemma A], whose proof is almost identical to that of Byun's.

LEMMA 3.15. *Let R and A be k-subalgebras of S where A centralizes R and $R \cup A$ generates S. Assume that R and S are right noetherian and that $Q \subset P$ are prime ideals of S with $Q \cap R = P \cap R$. If A satisfies $(\star)_r$, then there does not exist a finitely generated P-tame right S-module M with* $r_S(M) = Q \in \operatorname{Spec}(S)$.

PROOF. Suppose there exists such a module M. Before arriving at a contradiction, we will reduce R to the case where R is a simple artinian algebra so that (iii) of the previous remark can be utilized.

First, we make R into a prime right Goldie ring. Define $\pi : S \to S/Q$ to be the canonical map. So $\pi(R) = R+Q/Q \cong R/Q\cap R$ and $\pi(A)$ generate the centralizing extension $\pi(S)$. By (ii) of the previous remark, $\pi(A)$ satisfies $(\star)_r$. So we may assume that $Q = 0$, whence S is a prime right noetherian ring and R is a prime right noetherian ring .

Let $C = \mathcal{C}_R(0)$. Then C is a right Ore (denominator) set in R. Note that C is also right Ore in S, since for any $s = r_1a_1 + \cdots + r_na_n \in S$ where $r_i \in R, a_i \in A$, and $c \in C$, there exist $d \in C$ and $r'_i \in R$ such that $cr'_i = r_id$ for all i. Set $s' = r'_1a_1 + \cdots + r'_na_n$ to get $cs' = sd \in cS \cap sC$. Consider the canonical map $\theta : S \to SC^{-1}$. It follows that SC^{-1} is a centralizing extension of RC^{-1} by $\theta(A)$. By (ii) of the remark, R now can be regarded as a simple artinian ring.

Observe also that $C \cap P = \emptyset$ since

$$C \cap P = C \cap P \cap R = C \cap \{0\}.$$

Hence, by [**29**, Lemma 9.21] S and S/P are C-torsionfree, and $C \subseteq \mathcal{C}_S(0)$ as well as $C \subseteq \mathcal{C}_S(P)$. Furthermore, $S \subseteq SC^{-1}$ may be assumed.

We now claim that M is C-torsionfree. Since $l_M(P)$ is torsionfree as an S/P-module, it is $\mathcal{C}_S(P)$-torsionfree and so C-torsionfree. Set $L = l_M(P)$. Since $L \subseteq_e M$, it follows that M is C-torsionfree, proving the claim. Thus we may assume that $M \subseteq MC^{-1}$.

Note that 0 and PC^{-1} are prime ideals of SC^{-1} by [**29**, Theorem 9.22], and IC^{-1} is an ideal of SC^{-1} for every ideal I of R by [**29**, Theorem 9.20].

Next we claim that MC^{-1} is a finitely generated PC^{-1}-tame right SC^{-1}-module with $r_{SC^{-1}}(MC^{-1}) = 0$. Since M is a finitely generated right S-module, MC^{-1} is also finitely generated as a right SC^{-1}-module. Let $I = r_{SC^{-1}}(MC^{-1})$. So $MIC^{-1} = MC^{-1}I = 0$. Note that $I \cap S$ is an ideal of S and $(I \cap S)C^{-1} = I$. So $M(I \cap S) = 0$, and hence $I \cap S = 0$. This proves that $r_{SC^{-1}}(MC^{-1}) = 0$.

If $T \in \text{Ass}(MC^{-1})$, then there exists a T-prime uniform submodule U in MC^{-1}. So $U \cap M$ is a uniform S-submodule of M and $(U \cap M)C^{-1} = U$. Moreover, $U \cap M$ is $(T \cap S)$-prime, where $T \cap S$ is a prime ideal of S. Hence $T \cap S \in \text{Ass}(M) = P$, giving $\text{Ass}(MC^{-1}) = PC^{-1}$. Note that $l_{MC^{-1}}(PC^{-1}) = LC^{-1}$. It now follows that $LC^{-1} \subseteq_e MC^{-1}$. We will show that LC^{-1} is SC^{-1}/PC^{-1}-torsionfree. Suppose that

$$\begin{aligned} 0 \neq xc^{-1} &\in t(LC^{-1}) \\ &= \{sd^{-1} \in LC^{-1} \mid r_{SC^{-1}}(sd^{-1})/PC^{-1} \subseteq_e SC^{-1}/PC^{-1}\}. \end{aligned}$$

Then $x = xc^{-1}c \in t(LC^{-1})$. So $r_{SC^{-1}}(x)/PC^{-1} \subseteq_e SC^{-1}/PC^{-1}$ yields that $r_S(x)/P \subseteq_e S/P$, for if there exists an ideal $0 \neq J/P \subseteq S/P$ such that $r_S(x) \cap J = P$, then

$$JC^{-1} \cap r_{SC^{-1}}(x) = JC^{-1} \cap r_S(x)C^{-1} = PC^{-1}.$$

This contradicts the fact that x belongs to $t(LC^{-1})$. Therefore, MC^{-1} is PC^{-1}-tame.

However, any right noetherian factor of $A \otimes R$, where R is simple artinian, satisfies the right restricted strong second layer condition, giving $QC^{-1} = PC^{-1}$ and thus $Q = P$. This contradicts the assumption that $Q \subset P$. □

If R is a right noetherian ring with center Z, then $R[x] \cong R \otimes_Z Z[x]$. Note that $Z[x]$ satisfies the condition $(\star)_r$, since, for any division Z-algebra D, every ideal of a homomorphic image of $Z[x] \otimes_Z D \cong D[x]$ is principal generated by a central element and thus has the right AR-property by [**56**, Proposition 4.2.6]. Hence every right noetherian factor ring of $D[x]$ satisfies the right strong second layer condition by Lemma 2.6, since every right AR-ring is right AR-separated. Hence it satisfies the right restricted strong second layer condition.

The polynomial ring $R[x]$ satisfies the right strong second layer condition if R is an FBN ring [**15**, Theorem C]. Recall that an ideal I of a ring R is called *polynormal (polycentral)* if it is generated by a finite set of elements $a_1, \ldots, a_n$ such that $a_j + \sum_{i=1}^{j-1} a_i R$ is normal (central) in $R/\sum_{i=1}^{j-1} a_i R$ for all $j = 1, \ldots, n$. If every ideal is polynormal (polycentral), then the ring R is called a *polynormal (polycentral) ring.* If R is a noetherian polynormal ring, then $R[x]$ satisfies the right second layer condition [**15**]. Note that a noetherian polycentral ring, being a polynormal ring, satisfies the right second layer condition, and that a noetherian polycentral ring is also an AR-ring [**56**, Theorem 4.2.7]. We now investigate extensions of noetherian AR-rings.

DEFINITION. An ideal I of a ring R is said to have the *very strong right AR-property* if the Rees ring $\mathcal{R}(I) = R + It + I^2t^2 + \cdots$ is a right noetherian ring. A ring R is called *very strong right AR* if every ideal has the very strong right AR-property.

LEMMA 3.16. [**29**, Lemma 11.12] *Let R be a right noetherian ring and I be an ideal of R. If I has the very strong right AR-property, then I has the right AR-property.*

We now show that the very strong AR-property of an ideal is inherited by polynomial extension rings. The proof of the following lemma is similar to that of [**11**, Lemma 7.1].

Lemma 3.17. *Let R be a right noetherian ring. If an ideal I of R has the very strong right AR-property, then so does $IR[x]$ in $R[x]$.*

Proof. It is to be shown that the Rees ring $\mathcal{R}(IR[x])$ of $IR[x]$ is right noetherian. We claim that $\mathcal{R}(IR[x]) \cong \mathcal{R}(I)[x]$. Note that

$$\begin{aligned}\mathcal{R}(IR[x]) &= R[x] + IR[x]t + (IR[x])^2t^2 + \cdots \\ &= R[x] + I[x]t + I^2[x]t^2 + \cdots ,\end{aligned}$$

so a typical element is of the form

$$(r_0 + r_1x + \cdots + r_{n_0}x^{n_0}) + (i_{1,0} + i_{1,1}x + \cdots + i_{1,n_1}x^{n_1})t + \cdots + (i_{m,0} + i_{m,1}x + \cdots + i_{m,n_m}x^{n_m})t^m$$

where $i_{h,j} \in I^h$. Rearranging this as

$$(r_0 + i_{1,0}t + \cdots + i_{m,0}t^m) + (r_1 + i_{1,1}t + \cdots + i_{m,1}t^m)x + \cdots + f_n(t)x^n \in \mathcal{R}(I)[x]$$

where $f_n(t)$ is the sum of the "coefficients" of x^n, we get a ring isomorphism, proving the claim. Thus, $\mathcal{R}(IR[x])$ is right noetherian, since $\mathcal{R}(I)[x]$ is right noetherian by the Hilbert Basis Theorem. □

Definition. A ring R is said to be *very strongly right AR-separated* if for any prime ideals $Q \subset P$ of R there exists an ideal I such that $Q \subset I \subseteq P$ and I/Q has the very strong right AR-property.

Any right noetherian very strong right AR-ring is very strongly right AR-separated. Now, we will show that $R[x]$ satisfies the second layer condition when R is very strongly AR-separated.

THEOREM 3.18. *Let R be a right noetherian ring that is very strongly right AR-separated. Then $R[x]$ satisfies the right restricted strong second layer condition.*

PROOF. Suppose that $R[x]$ does not satisfy the right restricted strong second layer condition. Then there exists $P\,\mathrm{Spec}(R[x])$ such that P does not satisfy the right restricted strong second layer condition. There exists a finitely generated P-tame right $R[x]$-module M with $r_{R[x]}(M) = Q \in \mathrm{Spec}(R[x])$ and $Q \subset P$.

Note that $Q \cap R$ and $P \cap R$ are prime ideals of R by Lemma 3.12. If $Q \cap R = P \cap R$, then we can employ Lemma 3.15 to show that such a module M cannot exist, since if Z is the center of R then $Z[x]$ satisfies the condition $(\star)_r$.

If $Q \cap R \subset P \cap R$, then by the hypothesis on R there exists an ideal I such that $Q \cap R \subset I \subseteq P \cap R$ and $I/(Q \cap R)$ has the very strong right AR-property. By passing to factor rings, we may assume that $Q = 0$. Then, by Lemma 3.17, $IR[x]$ has the very strong right AR-property in $R[x]$, and thus by Lemma 3.16 it has the right AR-property. Note that

$$IR[x] \subseteq (P \cap R)R[x] \subseteq P.$$

So $M(IR[x])^n = 0$, as $l_M(P)IR[x] \subseteq l_M(P)P = 0$ and $l_M(P)_{R[x]} \subseteq_e M_{R[x]}$. Accordingly, $(IR[x])^n \subseteq r(M) = 0$ and hence $IR[x] = 0$, leading to the contradiction $I = 0$. Therefore, P must satisfy the right restricted strong second layer condition, and so must $R[x]$. □

Recall that, if α is an endomorphism of a ring R, then an additive map $\delta : R \rightarrow R$ is called an α-derivation if $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. The skew polynomial ring $R[x; \alpha, \delta]$ is the polynomial extension ring of R with multiplication $xr = \alpha(r)x + \delta(r)$ where $r \in R$.

PROPOSITION 3.19. *Let D be a division ring, let α be an automorphism on D and let δ be an α-derivation of D. Then $D[x; \alpha, \delta]$ satisfies the strong second layer condition.*

PROOF. Note that D is trivially noetherian and $D[x; \alpha, \delta]$ is a principal right and left ideal ring by [**29**, Theorem 1.11]. So every ideal of $D[x; \alpha, \delta]$ is generated by a normal element by [**11**, Lemma 6.3], and thus has the AR-property by [**56**, Proposition 4.2.6]. Hence $D[x; \alpha, \delta]$ satisfies the strong second layer condition by Lemma 2.6. □

For a general right noetherian ring satisfying the right second layer condition, we show that the second layer condition of a prime ideal P of $R[x]$ depends on the tameness of $\mathrm{E}(R[x]/P)_{R[x]}$ as an R-module by using Lemma 3.15. This was first observed by Kosler [**43**, Corollary 3.5] for rings with two-sided second layer condition.

Theorem 3.20. *Let R be a right noetherian ring. If P is a prime ideal in $R[x]$ such that $\mathrm{E}(R[x]/P)_{R[x]}$ is $(P \cap R)$-tame as a right R-module and $P \cap R$ satisfies the right restricted strong second layer condition, then P satisfies the right restricted strong second layer condition.*

Proof. Let Z be the center of R. Then note that $R[x] \cong R \otimes Z[x]$ and that $Z[x]$ centralizes R [**33**, Ch. 4, Proposition 4.8]. By the remark following the definition of $(\star)_r$, we see that $Z[x]$ satisfies $(\star)_r$.

For convenience of notation, set $S = R[x]$. Suppose that P does not satisfy the right restricted strong second layer condition. Then there exists a finitely generated P-tame right S-module M with $r_S(M) = Q \in \mathrm{Spec}(S)$ where $Q \subset P$.

We claim that $Q \cap R = P \cap R$, which leads to a contradiction to Lemma 3.15. Since M is finitely generated as an S-module, there exists a finitely generated R-submodule N of M such that $M = NS$. Also, $N \hookrightarrow M \hookrightarrow \bigoplus^n \mathrm{E}(S/P)_S$ as R-modules, implying that N is $(P \cap R)$-tame as an R-module. Now if $r_R(N) = Q \cap R$, then $Q \cap R = P \cap R$ by the right restricted strong second layer condition for $P \cap R$. In this case, the claim is proved. So observe that $NQS = NSQ = MQ = 0$, and get that $N(Q \cap R) \subseteq NQS = 0$, whence $Q \cap R \subseteq r_R(N)$. Let $r_R(N) = I$, so $MIS = NSIS = NIS = 0$. Thus $IS \subseteq Q$, and consequently $I \subseteq Q \cap R$ shows that $r_R(N) = Q \cap R$. □

COROLLARY 3.21. *Let R be a right noetherian ring satisfying the right second layer condition. If $\mathrm{E}(R[x]/P)_{R[x]}$ is $(P\cap R)$-tame as a right R-module for every prime P of $R[x]$, then $R[x]$ satisfies the right second layer condition.*

PROOF. By Theorem 2.27 and Proposition 2.18, the right second layer condition and the right restricted strong second layer condition are equivalent. Thus Theorem 3.20 gives the desired result. □

A partial converse to the above theorem can be deduced from Proposition 3.3 [Down] in the previous section. If P satisfies the right second layer condition, then any finitely generated P-tame right $R[x]$-module M is a Δ-module and so it is also tame as a right R-module. Thus $\mathrm{E}(R[x]/P)_{R[x]}$ is tame as a right R-module. But, we cannot claim that it is $(P\cap R)$-tame, even though, by [**43**, Corollary 3.5],it is if R satisfies the second layer condition.

COROLLARY 3.22. *If R is a right artinian ring, then $R[x]$ satisfies the right restricted strong second layer condition.*

PROOF. Let P be a prime ideal of $R[x]$, and set $E = \mathrm{E}(R[x]/P)_{R[x]}$. It will be shown that E is $(P\cap R)$-tame so that P satisfies the right restricted strong second layer condition by Theorem 3.20. First, observe that E is tame as a right R-module since R is right artinian. Suppose that $Q \in \mathrm{Ass}(E)$ as a right R-module. Then there exists a right R-submodule $M \subseteq E$ such that $Q = r_R(M) = \mathrm{Ass}(M)$. Thus $MR[x]QR[x] = MQR[x] = 0$, and so

$QR[x] \subseteq P$, yielding $Q \subseteq P \cap R$. However, since in an artinian ring all prime ideals are maximal, $Q = P \cap R$. This proves that E is $(P \cap R)$-tame. □

REMARK. The above corollary can be also viewed as a corollary to Theorem 3.18, since any right artinian ring is (vacuously) very strongly AR-separated. Similarly, this holds for a simple right noetherian ring. Finally the same holds for a right noetherian PI-ring, since if $Q < P \in \mathrm{Spec}(R)$ then P/Q, being a nonzero ideal of a prime right Goldie ring R/Q, contains a central element of R/Q by [**65**, Theorem 1.6.27].

According to [**15**, Theorem C], $R[x]$ satisfies the right strong second layer condition when R is an FBN ring. Furthermore, any artinian ring is an FBN ring. Thus, in case R is an artinian ring, the conclusion of the above corollary can be strengthened to the strong second layer condition.

3.4. Second Layer conditions of Quotient Rings

We look at the transfer of second layer conditions to quotient rings. When C is a right Ore set in a right noetherian ring R satisfying the right second layer condition, then RC^{-1} satisfies the right second layer condition[**40**, Proposition 8.1.4]. This can be generalized to a torsion radical σ if I_σ is an ideal of R_σ for every ideal I of R. Recall that if R is σ-torsionfree, then $I_\sigma/I = \sigma(\mathrm{E}(I)/I)$ and I_σ is a right ideal of R_σ. First, we examine relations between $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R_\sigma)$, which are very similar to those of a ring and its quotient ring with respect to a right Ore set.

LEMMA 3.23. *Let R be a ring. Let σ be a perfect torsion radical on* Mod-R *such that R is σ-torsionfree. If I is a right ideal of R_σ, then*

(i) *$I \cap R$ is a right ideal of R.*

(ii) $(I \cap R)_\sigma = I$.

PROOF. (i). Since $\sigma(R) = 0$, $R \subseteq R_\sigma$ and hence $I \cap R$ is a right ideal of R.

(ii). Note that

$$\frac{I}{I \cap R} \cong \frac{I + R}{R} \subseteq \frac{R_\sigma}{R},$$

so $I/I \cap R$ is σ-torsion. Thus $I \subseteq (I \cap R)_\sigma$. Also note that

$$\frac{(I \cap R)_\sigma}{I} \cong \frac{\frac{(I \cap R)_\sigma}{I \cap R}}{\frac{I}{I \cap R}},$$

yielding that $(I \cap R)_\sigma/I$ is σ-torsion since $(I \cap R)_\sigma/(I \cap R)$ is σ-torsion. On the other hand, $(I \cap R)_\sigma/I$ is σ-torsionfree since σ is perfect. Hence, $I = (I \cap R)_\sigma$. □

LEMMA 3.24. *Let R be a ring. Let σ be a perfect torsion radical on* Mod-R *such that R is σ-torsionfree. If I is a right ideal of R, then $I_\sigma = IR_\sigma$.*

PROOF. Note that IR_σ is a right ideal of R_σ and hence σ-torsionfree as a right R_σ-module. Since $R \subseteq R_\sigma$, we have $I \subseteq IR_\sigma$. Observe that IR_σ/I is σ-torsion, and so $IR_\sigma \subseteq I_\sigma$. On the other hand, I_σ/IR_σ is σ-torsionfree, so $I_\sigma = IR_\sigma$. □

LEMMA 3.25. *Let R be a right noetherian ring, and let σ be a perfect torsion radical on* Mod-R *such that R is σ-torsionfree. If I_σ is an ideal of R_σ for every ideal I of R, then $P \cap R$ is a prime ideal of R for every prime P in R_σ.*

PROOF. Note that R can be regarded as a subring of R_σ since $\sigma(R) = 0$. Let I, J be ideals of R such that $IJ \subseteq P \cap R$. Then $(IJ)_\sigma \subseteq (P \cap R)_\sigma = P$, where the last equality is established in Lemma 3.23.

We claim that $(IJ)_\sigma = I_\sigma J_\sigma$. Since $I_\sigma = IR_\sigma$ by Lemma 3.24,

$$I_\sigma J_\sigma = IR_\sigma JR_\sigma = IJR_\sigma = (IJ)_\sigma.$$

Thus either $I_\sigma \subseteq P$ or $J_\sigma \subseteq P$, so $I \subseteq P \cap R$ or $J \subseteq P \cap R$. □

If M is a σ-torsionfree right R-module where σ is a torsion radical, then $\mathrm{E}(M)_R$ is an injective hull $\mathrm{E}(M_\sigma)_{R_\sigma}$ of M_σ as a right R_σ-module [71, Proposition IX 2.5]. Now the following lemma is specialized for prime ideals of R_σ.

LEMMA 3.26. *Let R be a right noetherian ring, and let σ a perfect torsion radical on* Mod-R *such that R is σ-torsionfree. Then*

$$\mathrm{E}(R_\sigma/P)_{R_\sigma} \cong \mathrm{E}(R/P \cap R)_R$$

as R-modules for every prime ideal P of R_σ.

PROOF. Since σ is perfect, $(\frac{R}{P \cap R})_\sigma \cong R_\sigma/P$. Also

$$\left(\frac{R}{P \cap R}\right)_\sigma \subseteq \mathrm{E}\left(\frac{R}{P \cap R}\right)_R$$

by the definition of the quotient module since $R/P\cap R$ is σ-torsionfree.

Consider

$$\begin{tikzcd}
(\frac{R}{P\cap R})_\sigma \arrow[r, "f"] \arrow[d, "g"'] & \mathrm{E}((\frac{R}{P\cap R})_\sigma)_{R_\sigma} \\
\mathrm{E}(\frac{R}{P\cap R})_R &
\end{tikzcd}$$

where f, g are R-module embeddings. Since $\mathrm{E}(R/(P\cap R))_R$ is injective as an R-module and since $(\frac{R}{P\cap R})_\sigma \subseteq_e \mathrm{E}((\frac{R}{P\cap R})_\sigma)_{R_\sigma}$ as R-modules, g can be extended to a monomorphism

$$G:\mathrm{E}((\frac{R}{P\cap R})_\sigma)_{R_\sigma}\to \mathrm{E}(\frac{R}{P\cap R})_R.$$

Note that $\mathrm{E}(R/(P\cap R))_R$ is σ-torsionfree and σ-injective, and so it can be regarded as an R_σ-module. This makes G an isomorphism, proving the assertion. □

PROPOSITION 3.27. *Let R be a right noetherian ring and let σ be a perfect torsion radical such that R is σ-torsionfree and I_σ is an ideal of R_σ for every ideal I of R. If R satisfies the right second layer condition, then R_σ satisfies the right second layer condition.*

PROOF. First, note that R_σ is right noetherian since R is right noetherian and σ is perfect. Let P be a prime ideal of R_σ. Let M be a finitely generated P-tame right R_σ-module with $r_{R_\sigma}(M)=Q\in\mathrm{Spec}(R_\sigma)$. Suppose $Q\subset P$. Since M is finitely generated as an R_σ-module, there exists a finitely generated right R-module N such that $M=NR_\sigma$.

We claim that M is $P \cap R$-tame. Without loss of generality, assume that M is a uniform right R_σ-module by [18, Lemma 7.9]. So $M \hookrightarrow \mathrm{E}(R_\sigma/P)_{R_\sigma}$. By Lemma 3.2b, $\mathrm{E}(R_\sigma/P)_{R_\sigma} \cong \mathrm{E}(R/P \cap R)_R$ as R-modules. Since

$$r_R(M) = r_{R_\sigma}(M) \cap R = Q \cap R,$$

we deduce that $r_R(N) = Q \cap R$. By the right second layer condition on $P \cap R$, it follows that $Q \cap R = P \cap R$. But this cannot happen as $Q \subset P$. $\square$

CHAPTER 4

Torsion Theoretic Second Layer Conditions

4.1. Introduction

Let R be a ring and let σ be a torsion radical on the category Mod-R of right R-modules. Quite often statements regarding R-modules can be relativized with respect to σ with minor modifications. For example, a ring S is shown to be right σ-noetherian iff direct sums of σ-torsionfree injective right S-modules are injective. This bears a close parallel to a well-known result by Papp [**63**] and Bass [**5**] that a ring R is right noetherian iff direct sums of injective R-modules are injective. Also Jategaonkar relativized the Krull dimension with respect to σ resulting in a dimension function with similar properties [**37**].

However, occasionally a blind relativization bogs down so that no equivalent formulation may be possible. This happens to be the case with FBN rings. An attempt to extend Krause's 1972 result, that FBN rings are characterized by the 1-1 correspondence between the set of prime ideals and the set of isomorphism classes of indecomposable injective modules, to σ-bounded rings ends short of achieving the equivalence. In reformulating hypotheses relative to σ, we define a ring R to be right σ-bounded if every σ-closed essential right ideal contains a nonzero ideal of R. We produce a counter-example to show that a 1-1 correspondence between the set of σ-torsionfree indecomposable injective modules and the set of σ-closed prime ideals need not imply that the ring is fully σ-bounded. The 1-1 correspondence mentioned above is called a local bijective Gabriel correspondence.

In this chapter, we investigate sufficient and necessary conditions for a right σ-noetherian ring R to be fully σ-bounded and learn that the second layer condition plays an important role here, too. In doing so, we find that, with an additional condition, called ideal invariance, on the structure of ideals, the equivalence between fully σ-bounded rings and rings with local bijective Gabriel correspondence can be achieved. In the absence of the ideal invariance, we conclude that the local bijective Gabriel correspondence and the second layer condition are not only sufficient, but also necessary for a right σ-noetherian ring to be fully σ-bounded.

Afterwards we extend the classical Krull dimension defined by Kosler [**42**] and explore its properties relative to σ.

Motivation for this chapter was provided by [**1**] where the equivalence between fully σ-boundedness and the local bijective Gabriel correspondence was claimed to hold. Unfortunately, the proof contained an error, which was propagated in [**3**]. Additional material on σ-relative FBN rings can be found in [**49**].

4.2. Gabriel Correspondences

Let R be a ring. There is a surjective mapping ϕ from the set of isomorphism classes $[E]$ of indecomposable injectives to the set of prime ideals of R, given by $\phi([E]) = \text{Ass}(E)$. If R is a commutative noetherian ring, then Matlis [**55**, Proposition 3.1] showed, in 1958, that this mapping is, in fact, a 1-1 correspondence. Later in 1972, Krause [**44**, Theorem 3.5] showed that this 1-1 correspondence characterizes right FBN rings. Specifically, he showed that the following statements are equivalent:

(i) R is a right FBN ring.

(ii) There is a 1-1 correspondence between the set of isomorphism classes of indecomposable injective right R-modules and the set of prime ideals of R.

(iii) For any indecomposable injective right R-module E with $\text{Ass}(E) = P$ there exists a P-prime cyclic submodule eR such that $|eR| = |R/P|$.

If R has property (ii), then R is said to have *bijective Gabriel correspondence*. In this section, we try to give a similar formulation of right FBN rings relative to a torsion radical σ in terms of Gabriel correspondence.

DEFINITION. Let R be a ring and let σ be a torsion radical on Mod-R. We define $\text{Spec}_\sigma(R)$ to be the set $\{P \in \text{Spec}(R) \mid \sigma(R/P) = 0\}$ of σ-closed prime ideals.

Note that if R is a right noetherian ring, then for any given prime ideal P of R, either $\sigma(R/P) = 0$ or $\sigma(R/P) = R/P$ [**40**, Proposition 5.4.2]. In other words, P is either σ-closed or σ-dense in R.

The ring R is said to have *local bijective Gabriel correspondence* with respect to σ if the map $[E] \to \mathrm{Ass}(E)$ is a 1-1 correspondence between the set of isomorphism classes of σ-torsionfree indecomposable injective modules and $\mathrm{Spec}_\sigma(R)$.

Let σ be a torsion radical. A ring R is called *right σ-noetherian* if R has ACC on σ-closed right ideals. It is called *right σ-artinian* if it has DCC on σ-closed right ideals. A ring R is right σ-noetherian iff every σ-torsionfree injective right R-module is a direct sum of σ-torsionfree indecomposable injective submodules or, equivalently, if a direct sum of σ-torsionfree injective right R-modules is again injective [**73**, Theorem 1.2].

A right R-module M is called *σ-noetherian* if M has ACC on σ-closed submodules. It is called *σ-artinian* if it has DCC on σ-closed submodules.

If R is right σ-noetherian and M is a finitely generated right R-module, then M is σ-noetherian. Note that, if M is a σ-noetherian module, then M contains a submodule N such that $\sigma(M/N) = M/N$ and N is finitely generated. In this case, M is called *σ-finitely generated.* The class of right σ-noetherian modules is closed under submodules and factor modules.

A σ-noetherian module M need not contain a uniform submodule. However, if M is a σ-torsionfree σ-noetherian module, then the existence of a uniform submodule is assured. This can be seen as follows. Assume that M is not uniform, so there exist nonzero submodules N_1 and N_2 with $N_1 \cap N_2 = 0$. Since M is torsionfree, $\overline{N_1^\sigma} \cap \overline{N_2^\sigma} = 0$. Since M is σ-noetherian and not uniform, there exists a σ-closed submodule K that is maximal with respect to the property that $K \cap L = 0$ for some nonzero submodule L, which is easily seen to be uniform, due to the maximality of K. Accordingly, in order to ensure the existence of uniform submodules, we will restrict the discussion to σ-torsionfree modules in what follows. In particular, we will consider tame modules only when they are σ-torsionfree. If R is right σ-noetherian and P is a σ-closed prime ideal, then R/P turns out to be a prime right Goldie ring [**60**, Lemma 4.1], [**7**, Proposition 2]. Moreover, for any given nonzero right ideal I of a prime σ-torsionfree right σ-noetherian ring R, we can see that $I^{\mathrm{rank}(R_R)}$ contains a finitely generated essential free submodule [**29**, Corollary 6.26].

Recall that when U is a uniform right R-module, then $\mathrm{Ass}(U)$ can be defined as $\bigcup\{r(V) \mid 0 \neq V \subseteq U\} = r(U')$ for some $U' \subseteq U$. Note that $\mathrm{Ass}(U)$ is a prime ideal.

Now we look at several conditions equivalent to the local bijective Gabriel correspondence with respect to a torsion radical σ.

PROPOSITION 4.1. *Let σ be a torsion radical on* Mod-R *and let R be a right σ-noetherian ring. Then the following are equivalent.*

(i) *R has local bijective Gabriel correspondence with respect to σ.*

(ii) *Direct products of* $\mathrm{Spec}_\sigma(R)$*-tame right* R*-modules are tame and* σ *is cogenerated by* $\bigoplus_{P\in\mathrm{Spec}_\sigma(R)} \mathrm{E}(R/P)$.

(iii) *Any nonzero* σ*-torsionfree right* R*-module* M *is tame.*

(iv) *For every* σ*-torsionfree indecomposable injective right* R*-module* E *with the associated prime* $\mathrm{Ass}(E) = P$*, there exists a* P*-prime cyclic submodule* eR *of* E *such that* $|eR|_\sigma = |R/P|_\sigma$.

PROOF. (i) ⇒ (ii): Let $\{M_i \mid i \in I\}$ be a set of $\mathrm{Spec}_\sigma(R)$-tame modules. Since each M_i is σ-torsionfree and since products of σ-torsionfree modules are σ-torsionfree, $\prod_{i\in I} M_i$ is σ-torsionfree. Now, let U be a uniform submodule of $\prod_{i\in I} M_i$. Then $\mathrm{E}(U) \cong E_P$ by (i), where $\mathrm{Ass}(U) = P \in \mathrm{Spec}_\sigma(R)$ and E_P is the indecomposable injective summand of $\mathrm{E}(R/P)$. Thus U is $\mathrm{Spec}_\sigma(R)$-tame, and so $\prod_{i\in I} M_i$ is tame proving that the direct product is tame.

Next, we show that σ is cogenerated by $\bigoplus_{P\in\mathrm{Spec}_\sigma(R)} \mathrm{E}(R/P)$. Since any torsion radical is cogenerated by an injective (σ-torsionfree) module [**40**, p. 10], let $E = \bigoplus_{i\in I} E_i$, where each E_i is indecomposable injective, be a cogenerator for σ. Since each E_i is σ-torsionfree, by (i) $E_i \cong E_{P_i}$ with $P_i \in \mathrm{Spec}_\sigma(R)$. Note that $\bigoplus_{P\in\mathrm{Spec}_\sigma(R)} \mathrm{E}(R/P)$ is injective. Also note that $\bigoplus_{P\in\mathrm{Spec}_\sigma(R)} E_P$ is σ-torsionfree. Thus $\bigoplus_{P\in\mathrm{Spec}_\sigma(R)} \mathrm{E}(R/P)$ is a cogenerator of σ.

(ii) ⇒ (iii): Let M be a nonzero σ-torsionfree right R-module. Then

$$M \hookrightarrow \prod \left(\bigoplus_{P\in\mathrm{Spec}_\sigma(R)} E(R/P) \right) \subseteq \prod \prod_{P\in\mathrm{Spec}_\sigma(R)} \mathrm{E}(R/P),$$

as M is σ-torsionfree. By (ii), $\prod(\prod_{P\in \mathrm{Spec}_\sigma(R)} \mathrm{E}(R/P))$ is tame. So M is tame.

(iii) $\Rightarrow$ (iv): Let E be an indecomposable injective σ-torsionfree right R-module with $\mathrm{Ass}(E) = P$. Set $F = l_E(P)$, so F is a P-prime submodule. Choose $0 \neq e \in F$, so eR is tame since E is tame by (iii). Hence it follows that $r(e)/P$ is not an essential right ideal of R/P, and so there exists a right ideal $I \supset P$ such that $I \cap r(e) = P$. Note that

$$\begin{aligned}
|R/P|_\sigma &= |I/P|_\sigma \\
&= |I/(I \cap r(e))|_\sigma = |(I + r(e))/r(e)|_\sigma \\
&\leq |R/r(e)|_\sigma = |eR|_\sigma \leq |R/P|_\sigma.
\end{aligned}$$

This shows that $|eR|_\sigma = |R/P|_\sigma$.

(iv) $\Rightarrow$ (i): Let $[E] \to \mathrm{Ass}(E)$ be the map f from the isomorphism classes $[E]$ of indecomposable injective σ-torsionfree right R-modules to $\mathrm{Spec}_\sigma(R)$. Since f is surjective, it suffices to show that it is injective. Suppose that $f([E]) = f([E'])$, so $\mathrm{Ass}(E) = \mathrm{Ass}(E') = P$. By (iv), there exist $e \in E$ and $e' \in E'$ such that eR and $e'R$ are P-prime modules with $|eR|_\sigma = |R/P|_\sigma = |e'R|_\sigma$. Note that $r(e)/P$ and $r(e')/P$ are not essential in R/P. Thus $eR \cong R/r(e)$ contains a submodule which is isomorphic to a uniform right ideal of R/P by [**29**, Lemma 6.17]. The same is true for $e'R$. This means that $E \cong \mathrm{E}(eR) \cong E_P \cong \mathrm{E}(e'R) \cong E'$. □

DEFINITION. Let R be a ring and σ be a torsion radical on Mod-R. The ring R is said to be *right σ-bounded* if every σ-closed essential right ideal I of R contains a nonzero ideal J of R. It is said to be *right fully σ-bounded* if every prime factor ring R/P is right σ-bounded.

LEMMA 4.2. *Let σ be a torsion radical on* Mod-R *and let R be a right σ-noetherian ring. If R is right fully σ-bounded, then R has local bijective Gabriel correspondence with respect to σ.*

PROOF. Let E be an indecomposable injective σ-torsionfree module with $\mathrm{Ass}(E) = P$. Let $e \in l_E(P)$ and note that $r(e)$ is σ-closed. Since $P = r(eR) = r(R/r(e))$ is the largest two-sided ideal contained in $r(e)$, the fully σ-boundedness of R implies that $r(e)/P$ is not essential in R/P. Consequently, $|eR|_\sigma = |R/r(e)|_\sigma = |R/P|_\sigma$, so (iv) of Proposition 4.1 holds. This proves the assertion. □

In view of the equivalence of (i), (ii), and (iii) at the beginning of this section and taking into account Proposition 4.1 as well as the previous lemma, one is tempted to conclude that a σ-noetherian ring with local bijective Gabriel correspondence with respect to σ is right fully σ-bounded. However, this fails even for a noetherian ring as indicated by the following example.

EXAMPLE 4.3. Let $A_1 = k[x][y; d/dx]$ be the first Weyl algebra over a field k of characteristic 0. Set $R = k + xA_1$. Then R is the idealizer of the maximal right ideal xA_1, and it is a noetherian domain. It is easy to see that $0, P = xA_1$ and R are the only ideals of R.

Let σ be the torsion radical cogenerated by $\mathrm{E}(R)_R \oplus \mathrm{E}(R/P)_R$. It will be shown that R has local bijective Gabriel correspondence, but that it is not right fully σ-bounded.

First, let E be a σ-torsionfree indecomposable injective right R-module with $\mathrm{Ass}(E) = P$. Let eR be a cyclic P-prime submodule of E. Then $r_R(e) = P$ since $r_R(e) \supseteq P$ and P is a maximal right ideal of R. Hence, $eR \cong R/P \hookrightarrow \mathrm{E}(R/P)_R$, and so $E \cong E_P = \mathrm{E}(R/P)_R$.

Next, let E be a σ-torsionfree indecomposable injective right R-module with $\mathrm{Ass}(E) = 0$. If $E \not\cong \mathrm{E}(R)_R$, then $E \cong \mathrm{E}(R/A)_R$ for some nonzero right ideal A of R. Since R is a domain and $0 \neq A \subseteq R$, we get $|R/A| < |R|_R$. Also note that $|R_R| = |A_1| = 1$ by [**56**, Proposition 6.5.2 and 6.6.8]. Therefore, $|R/A| = 0$, i.e., R/A is an artinian and uniform right R-module. So R/A may be assumed to be a simple σ-torsionfree.

Now,

$$\begin{aligned} 0 &\neq \mathrm{Hom}(R/A, \mathrm{E}(R)_R \oplus \mathrm{E}(R/P)_R) \\ &\cong \mathrm{Hom}(R/A, \mathrm{E}(R)_R) \oplus \mathrm{Hom}(R/A, \mathrm{E}(R/P)_R). \end{aligned}$$

Note that $\mathrm{Hom}(R/A, \mathrm{E}(R)_R) = 0$, for otherwise R would contain a nonzero minimal right ideal, so $\mathrm{soc}(R_R) \neq 0$ and R would be artinian by [**29**, Corollary 6.16].

Thus, $\mathrm{Hom}(R/A, \mathrm{E}(R/P)_R) \neq 0$. In other words, $\mathrm{E}(R/P)_R$ contains a submodule isomorphic to R/A, so that

$$\mathrm{Ass}(R/A) = \mathrm{Ass}(\mathrm{E}(R/P)) = P.$$

This contradicts $\mathrm{Ass}(E) = 0$. Therefore, $E \cong \mathrm{E}(R)_R$.

Consequently, R has local bijective Gabriel correspondence with respect to σ.

We proceed to show that the prime ring R is not right bounded. Note that $xP \subseteq_e R$ since R is a domain. We claim that xP is σ-closed in R.

As $xP = x^2A_1 \subset xA_1 = P$ and $P \not\subseteq r(R/xP)$, so $r(R/xP) = 0$. Note that $P/xP = xA_1/x^2A_1 \cong A_1/xA_1 = A_1/P$ as right A_1-modules, hence as right R-modules. Also $(A_1/P)_R$ is seen to be uniform as in the proof of [**56**, Theorem 1.1.12 (ii)], so $R/P \subseteq_e A_1/P \cong P/xP$. Therefore, $P/xP \hookrightarrow \mathrm{E}(R/P)_R$, and so P/xP is σ-torsionfree. Since R/P is σ-torsionfree and extensions of σ-torsionfree modules by σ-torsionfree modules are σ-torsionfree, R/xP is thus σ-torsionfree. Since $r(R/xP) = 0$, it follows that xP does not contain a nonzero ideal. □

The question arises what additional condition has to be imposed on a torsion radical σ in order to assure that for a right σ-noetherian ring R the local bijective Gabriel correspondence with respect to σ is equivalent to right fully σ-boundedness. As is shown below, one such condition is the ideal invariance of σ.

DEFINITION. Let R be a ring with right Krull dimension. An ideal I of R is said to be ***right ideal invariant*** with respect to the Krull dimension if $|M \otimes_R I| \leq |M|$ for any finitely generated right R-module M, or equivalently $|I/TI| \leq |R/T|$ for every right ideal $T \subseteq R$. The ring R is called ***right ideal invariant*** if every ideal is right ideal invariant.

An ideal I is said to be ***right weakly ideal invariant*** if $|M \otimes_R I| < |R/I|$ for every finitely generated right R-module M with $|M| < |R/I|$.

Let R be a ring and let σ be a torsion radical. Then σ is said to be *(right) ideal invariant* if $\sigma(I/DI) = I/DI$ or equivalently $|I/DI|_\sigma = -1$, for any ideal I and any σ-dense right ideal D in R.

The concept of ideal invariance was first introduced by Stafford [**69**]. It is a sort of generalization of the AR-property and has been applied to study polycyclic group rings and enveloping algebras. The definition with respect to a torsion theory was given by Robson [**64**]. One practical use of ideal invariance is furnished by the following: If R is a Krull homogeneous ring with Krull dimension and the prime radical N is right weakly ideal invariant, then R is a right order in a right artinian ring [**45**, Theorem 8], [**58**, Corollary 11].

We establish an elementary property of ideal invariance and proceed to show that ideal invariance closes the gap between the local bijective Gabriel correspondence and the right fully σ-boundedness of a right σ-noetherian ring.

LEMMA 4.4. [3, Proposition 2.5] *Let σ be an ideal invariant torsion radical. Let N be a σ-dense submodule of a σ-torsionfree right R-module M. Then $r_R(N) = r_R(M)$.*

PROOF. Certainly, $r_R(N) \supseteq r_R(M)$.

In order to establish equality, set $I = r_R(N)$. Since N is σ-dense in M, for any given $m \in M$ there exists a σ-dense right ideal $D_m \subseteq R$ such that $mD_m \subseteq N$. Thus $mD_mI = 0$. Note that $\sigma(I/D_mI) = I/D_mI$ by the ideal invariance of σ, hence for any $x \in I$ there exists a σ-dense right ideal $D_x \subseteq R$ such that $xD_x \subseteq D_mI$. Thus, $mxD_x \subseteq mD_mI = 0$. However, M is σ-torsionfree, so $mx = 0$ for every $m \in M$ and every $x \in I$. Consequently, $r_R(M) = I = r_R(N)$. □

The following result was proved in [3, Theorem 2.9] as a direct consequence of [1, Theorem 17]. However, [1] contains an error, which was pointed out in Example 4.3. However, the result as stated still holds, and we now give a new proof.

PROPOSITION 4.5. *Let R be a right σ-noetherian ring and let σ be ideal invariant. Then the following are equivalent.*

(i) *R is right fully σ-bounded.*

(ii) *R has local bijective Gabriel correspondence with respect to σ.*

PROOF. (i) $\Rightarrow$ (ii) follows from Lemma 4.2.

(ii) $\Rightarrow$ (i): Let P be a prime ideal of R. Suppose that $E/P \subseteq_e R/P$ and that $(R/P)/(E/P) \cong R/E$ is σ-torsionfree. We proceed to show that E/P contains a nonzero ideal of R/P. If P is not σ-closed, then $E \supseteq \overline{P}^{\sigma} \supset P$. Since $\overline{P}^{\sigma}$ is easily seen to be an ideal, the desired result follows. Therefore, assume that P is σ-closed. Define

$$\mathcal{E} = \{E/P \mid E/P \subseteq_e R/P,\ R/E \text{ is } \sigma\text{-torsionfree and } r(R/E) = P\},$$

and assume that $\mathcal{E} \neq \emptyset$. Since R/P is also right σ-noetherian, there exists a maximal $E/P \in \mathcal{E}$. We claim that R/E is uniform and fully faithful as an R/P-module. Suppose R/E is not uniform. Then there exist $0 \neq I_1/E,\ I_2/E \subseteq R/E$ such that $I_1 \cap I_2 = E$. Since $\sigma(R/E) = 0$, the σ-closure of E $\overline{E}^{\sigma} = E$. Note that

$$\overline{E} = \overline{I_1 \cap I_2}^{\sigma} = \overline{I_1}^{\sigma} \cap \overline{I_2}^{\sigma} = E \text{ by Lemma 1.7.}$$

$$\text{So } R/E = R/(\overline{I_1}^{\sigma} \cap \overline{I_2}^{\sigma}) \hookrightarrow R/\overline{I_1}^{\sigma} \oplus R/\overline{I_2}^{\sigma},$$

and we get either $r(R/\overline{I_1}^{\sigma}) = P$ or $r(R/\overline{I_2}^{\sigma}) = P$. Since $E \subset I_i \subseteq \overline{I_i}^{\sigma}$ for $i = 1, 2$, the maximality of E is contradicted. This shows that R/E is uniform.

Now, in order to show that R/E is fully faithful, let $0 \neq F/E \subseteq R/E$. Note that $r(\overline{F}^{\sigma}/E) = r(\overline{F/E}^{\sigma}) = r(F/E)$ by Lemma 1.7 and Lemma 4.4, as F/E is σ-torsionfree. Since

$$r(R/\overline{F}^{\sigma})r(\overline{F}^{\sigma}/E) \subseteq r(R/E) = P,$$

we get either $r(R/\overline{F}^{\sigma}) = P$ or $r(\overline{F}^{\sigma}/E) = P$. The former leads to a contradiction to the maximality of E in $\mathcal{E}$, as $E \subset F \subseteq \overline{F}^{\sigma}$. So $r(F/E) = P$, and hence R/E is fully faithful. This proves the claim.

Consequently, R/E is P-tame by the local bijective Gabriel correspondence. But $E/P \subseteq_e R/P$ implies that $r(x+E)/P \subseteq_e R/P$ for any $0 \neq x+P$. This contradicts that R/E is P-tame. □

While the ideal invariance of σ is sufficient for the right fully σ-boundedness of a right σ-noetherian ring with local bijective Gabriel correspondence, it seems that only a weaker condition is necessary. Specifically, this condition is the right restricted strong second layer condition for $\text{Spec}_{\sigma}(R)$. This is established in Theorem 4.8 below. We first show that if σ is ideal invariant and R is right fully σ-bounded, then $\text{Spec}_{\sigma}(R)$ satisfies the right restricted strong second layer condition, in fact, it satisfies the following.

DEFINITION. Let R be a right σ-noetherian ring and let P be a σ-closed prime ideal of R. Then P is said to satisfy the *right σ-restricted strong second layer condition* if whenever M is a σ-noetherian P-tame right R-module with $r(M) = Q \in \text{Spec}(R)$ then $Q = P$.

Since every finitely generated right R-module is σ-noetherian, the right σ-restricted strong second layer condition is formally stronger than the right restricted strong second layer condition.

PROPOSITION 4.6. *Let σ be an ideal invariant torsion radical. If R is a right σ-noetherian right fully σ-bounded ring, then* $\mathrm{Spec}_\sigma(R)$ *satisfies the right σ-restricted strong second layer condition.*

PROOF. Let $P \in \mathrm{Spec}_\sigma(R)$. Suppose that M is a σ-noetherian P-tame module with $r(M) = Q \in \mathrm{Spec}(R)$. Note that $Q \in \mathrm{Spec}_\sigma(R)$. There exists a finitely generated σ-dense submodule N such that $N = n_1R + \cdots + n_kR$ and $r(M) = r(N)$ by Lemma 4.4. Since $r(N) = \bigcap_{i=1}^{k} r(n_iR) = Q$, there exists some i such that $Q = r(n_iR)$. If $r(n_i)/Q \subseteq_e R/Q$, then there exists an ideal I such that $Q \subset I \subseteq r(n_i)$ by right σ-boundedness of R/Q, since $R/r(n_i)$ is σ-torsionfree. But this leads to the contradiction $Q \subset I \subseteq r(n_iR) = Q$. Hence, $r(n_i)/Q$ is not essential in R/Q and thus there exists a right ideal J such that $Q \subset J$ and $J \cap r(n_i) = Q$. Observe that

$$J/Q = J/(J \cap r(n_i)) \cong (J + r(n_i))/r(n_i) \hookrightarrow R/r(n_i) \cong n_iR.$$

Thus $Q = \mathrm{Ass}(J/Q) = \mathrm{Ass}(n_iR) = P$, proving that P satisfies the right σ-restricted strong second layer condition. □

Let R be a right σ-noetherian ring. The right σ-restricted strong second layer condition on R can be characterized in several ways, analogous to the characterization of the right restricted strong second layer condition in Theorem 2.13.

PROPOSITION 4.7. *Let R be a right σ-noetherian ring. If $P \in \mathrm{Spec}_\sigma(R)$, then the following are equivalent.*

(i) *P satisfies the right σ-restricted strong second layer condition.*

(ii) *M^I is P-tame for every P-tame σ-noetherian module M for any nonempty index set I.*

(iii) *Every P-tame σ-noetherian module M is finitely annihilated.*

PROOF. (i)⇒(ii): Since M is σ-torsionfree, M^I is σ-torsionfree. Let $Q \in \operatorname{Ass}(M^I)$. So $Q = r(U)$ for some uniform submodule of M^I. Without loss of generality, we can assume that U is Q-prime and that $Q = r(mR)$ for some $m = (m_i)_{i\in I} \in U$, where each $m_i \in M$. Then

$$Q = r((m_i)R) = \bigcap_{i\in I} r(m_iR) = r(\sum_{i\in I} m_iR).$$

Since $\sum_{i\in I} m_iR$ is a submodule of the σ-noetherian module M, it is also σ-noetherian. It is also P-tame. Now, the right σ-restricted strong second layer condition on P forces $Q = P$. Thus we have shown that M^I is P-primary.

Next, we show that M^I is tame. Set $L = l_{M^I}(P)$. If $r(x)/P \subseteq_e R/P$ for some $0 \neq x = (x_i)_{i\in I} \in L$, then $r(x_i)/P \subseteq_e R/P$ for all x_i, contradicting the fact that M is P-tame.

(ii)⇒(iii): Since $R/r(M) \hookrightarrow M^M$, we can assume without loss of generality that $r(M) = 0$. By (ii) and $R \hookrightarrow M^M$, obtain that R is P-tame. Since R is σ-torsionfree and right σ-noetherian, $l_R(P) \subseteq_e R_R$.

Suppose that M is not finitely annihilated. Take $m_1 \in M$, so $r(m_1) \neq 0$, and hence $r(m_1) \cap l_R(P) \neq 0$. There exists an element $m_2 \in M$ such that $m_2(r(m_1) \cap l_R(P)) \neq 0$, in other words, $r(m_1) \cap l_R(P) \supset r(m_1, m_2) \cap l_R(P)$.

Continuing inductively, we get a sequence $m_1, m_2, \cdots \in M$ such that

$$0 \neq \frac{l_R(P) \cap r(m_1, \ldots, m_k)}{l_R(P) \cap r(m_1, \ldots, m_k, m_{k+1})} \cong m_{k+1}(l_R(P) \cap r(m_1, \ldots, m_k)),$$

and so each factor is R/P-torsionfree since M is P-tame. Since R/P is a prime right Goldie ring, the reduced rank with respect to R/P can be defined. Since R is σ-noetherian, σ-torsionfree and since $l_R(P)$ is torsionfree as a right R/P-module, it follows that $\rho_{R/P}(l_R(P)) < \infty$ [7, Lemma (1)]. Set $\rho_{R/P}(l_R(P)) = n$. On the other hand,

$$\begin{aligned}\rho_{R/P}(l_R(P)) \geq \rho_{R/P}&\left(\frac{l_R(P)}{l_R(P) \cap r(m_1)}\right) \\ &+ \sum_{i=1}^{n} \rho_{R/P}\left(\frac{l_R(P) \cap r(m_1, \ldots, m_i)}{l_R(P) \cap l_R(P) \cap r(m_1, \ldots, m_{i+1})}\right) > n.\end{aligned}$$

This is a contradiction. Thus, M is finitely annihilated.

(iii)⇒(i): Let M be a σ-noetherian P-tame right R-module with $r(M) = Q \in \text{Spec}(R)$. Since $Q = r(M) = r(m_1, \ldots, m_n)$ for some $m_1, \ldots, m_n \in M$,

$$R/Q \hookrightarrow \bigoplus_{i=1}^{n} m_i R \subseteq M^n.$$

Thus $Q = \text{Ass}(R/Q) = \text{Ass}(M) = P$. □

We proceed to establish the main theorem of this section that the second layer condition closes the gap between the right fully σ-boundedness and the local bijective Gabriel correspondence with respect to σ.

THEOREM 4.8. *Let R be a right σ-noetherian ring. Then the following are equivalent.*

(i) *R is right fully σ-bounded.*

(ii) (a) *R has local bijective Gabriel correspondence with respect to σ.*

(b) $\mathrm{Spec}_\sigma(R)$ *satisfies the right restricted strong second layer condition.*

Proof. (i)⇒(ii): Part (a) follows from Lemma 4.2. In order to prove (b), let $P \in \mathrm{Spec}_\sigma(R)$. Let M be a finitely generated P-tame module with $r(M) = Q \in \mathrm{Spec}(R)$. Recall that M is σ-torsionfree, and hence $Q \in \mathrm{Spec}_\sigma(R)$ as $R/Q = R/r(M) \hookrightarrow M^M$. Without loss of generality, we may assume that $Q = r(mR)$ where $M = mR$. If $r(m)/Q \subseteq_e R/Q$, then there exists an ideal I such that $Q \subset I \subseteq r(m)$ by the right σ-boundedness of R/Q. So $mRI \subseteq mI = 0$, whence $I \subseteq r(mR) = Q$, a contradiction. So $r(m)/Q$ is not essential in R/Q, hence there exists a nonzero right ideal J/Q such that $J/Q \cap r(m)/Q = 0$. Then $Q = P$ follows from

$$I/Q = I/(r(m) \cap I) \cong (I + r(m))/r(m) \hookrightarrow mR.$$

Therefore, P satisfies the right restricted strong second layer condition.

(ii)⇒(i): Let P be a prime ideal and let $E/P \subseteq_e R/P$ such that R/E is σ-torsionfree. We show that $r(R/E) \supset P$. If P is not σ-closed, then $E \supseteq \overline{P}^\sigma \supsetneq P$. Since $\overline{P}^\sigma$ is easily seen to be an ideal, the assertion follows in this case.

Thus assume that P is σ-closed. Since R/E is a right σ-noetherian and σ-torsionfree module, R/E has finite rank [**3**, p. 852]. Hence there exist finitely many σ-closed right ideals $E_i \supset E$ such that $E = \bigcap_{i=1}^n E_i$, R/E_i is uniform for each i and $R/E \hookrightarrow \bigoplus_{i=1}^{n} R/E_i$. Since $r(R/E) = \bigcap_{i=1}^{n} r(R/E_i)$, the assertion will follow if we can show that $r(R/E_i) \supset P$ for each i. Thus we may without loss of generality assume that R/E is uniform. Assume that $r(R/E) = P$ and let $\mathrm{Ass}(R/E) = Q$. Note that $Q \in \mathrm{Spec}_\sigma(R)$ and $P \subseteq Q$. Since R is assumed to have local bijective Gabriel correspondence with respect to σ, $\mathrm{E}(R/E) \cong E_Q$. Since R/E is a finitely generated Q-tame right R-module with $P = r(R/E) \subseteq Q$, the right restricted strong second layer condition for $\mathrm{Spec}_\sigma(R)$ forces $Q = P$, so R/E is P-tame. However, since R/P is a right Goldie prime ring, E contains an element $c \in \mathcal{C}(P)$, so R/E, being a homomorphic image of R/cR, is $\mathcal{C}(P)$-torsion. This contradiction shows that $r(R/E) \supset P$. $\square$

REMARK. It would be interesting to know whether condition (b) in the preceding theorem can be replaced by the right σ-restricted strong second layer condition. In view of Proposition 4.7, this would allow one to characterize right fully σ-bounded rings as those right σ-noetherian rings for which each σ-torsionfree σ-noetherian right module is finitely annihilated. If σ is ideal invariant, then σ-restricted strong second layer condition can be used for (b). At this time we can only prove the following.

PROPOSITION 4.9. *Let R be a right σ-noetherian ring. Then the following are equivalent.*

(i) (a) *R has local bijective Gabriel correspondence with respect to σ.*

(b) $\mathrm{Spec}_\sigma(R)$ *satisfies the right σ-restricted strong second layer condition.*

(ii) *Every σ-torsionfree σ-noetherian right R-module M is finitely annihilated.*

PROOF. (i)⇒(ii): Note that M embeds in a finite direct sum of uniform σ-torsionfree σ-noetherian modules N_i with

$$\mathrm{Ass}(N_i) = P_i \in \mathrm{Ass}(M) \subseteq \mathrm{Spec}_\sigma(R).$$

By (i)(a), $N_i \subseteq \mathrm{E}(N_i) \cong E_{P_i}$, so each N_i is P_i-tame. By Proposition 4.7, each submodule of each N_i is finitely annihilated, so M is finitely annihilated by Lemma 1.14.

(ii)⇒(i): Let P be a prime ideal, and let $E/P \subseteq_e R/P$ such that $\sigma(R/E) = 0$. Then $r(R/E) = \bigcap_{i+1}^{n} r(x_i + E)$, so $r(R/E) \supset P$ since $r(x_i + E)/P \subseteq_e R/P$ for each i. Thus R is right fully σ-bounded, hence (i)(a) holds holds by Theorem 4.8. If $P \in \mathrm{Spec}_\sigma(R)$ and M is a P-tame σ-noetherian right R-module, then M is σ-torsionfree and thus finitely annihilated by hypothesis. Thus P satisfies the right σ-restricted strong second layer condition by Proposition 4.7. ☐

4.3. Relative Krull Dimension

For a right FBN ring R, it has been shown that the classical Krull dimension of R is equal to the Krull dimension of R_R in the sense of Gabriel and Rentschler [**44**, Theorem 2.4]. This has been generalized to right σ-noetherian right fully σ-bounded rings in [49] where relative classical Krull dimension with respect to σ is defined in terms of σ-closed prime ideals.

DEFINITION. Let R be a ring and let σ be a torsion radical on Mod-R. Set $\mathrm{Spec}_\sigma^{-1}(R) = \emptyset$, and for an ordinal $\alpha > -1$ define

$$\mathrm{Spec}_\sigma^\alpha(R) = \{P \in \mathrm{Spec}_\sigma(R) \mid P \subset Q \in \mathrm{Spec}_\sigma(R) \Rightarrow Q \in \cup_{\beta<\alpha} \mathrm{Spec}_\sigma^\beta(R)\}.$$

The *classical σ-Krull dimension* of R is the smallest ordinal α such that $\mathrm{Spec}_\sigma(R) = \mathrm{Spec}_\sigma^\alpha(R)$ and is denoted by $\mathrm{cl.K}_\sigma.\mathrm{dim}(R)$.

If R is a right σ-noetherian ring, then $\mathrm{cl.K}_\sigma.\mathrm{dim}(R)$ exists, cf., [**29**, Proposition 12.1].

In this section we show that rings with local bijective Gabriel correspondence with respect to a torsion radical exhibit the same behavior as right FBN rings with respect to the equality of the relative Krull dimension and the relative classical Krull dimension.

DEFINITION. Let R be a ring and let σ be a torsion radical. For a right R-module M, define $\mathcal{K}_\sigma(M) = \{N \subseteq M \mid \sigma(M/N) = 0\}$. The *relative σ-Krull dimension* of M is defined as $\mathrm{dev}(\mathcal{K}_\sigma(M))$ and denoted by $|M|_\sigma$.

Set $\Gamma_\sigma = \{N \subseteq M \mid M/N$ is $\mathrm{Spec}_\sigma(R)$-tame $\}$. Then $\gamma_\sigma(M)$ is defined as $\mathrm{dev}(\Gamma_\sigma(M))$.

If M is σ-torsion, then $\gamma_\sigma(M) = -1$. If a right R-module M has Krull dimension, then $\gamma_\sigma(M)$ is defined. The relative dimension γ_σ is an extension of Kosler's ***classical Krull dimension*** Cl.dim [**42**], which arises as γ_σ for the special case $\mathrm{Spec}(R) = \mathrm{Spec}_\sigma(R)$. It was proved that, for a right noetherian ring R satisfying the right second layer condition,

$$\mathrm{Cl.dim}(R_R) = \mathrm{Cl.K.dim}(R)$$

[**42**, Theorem 2.9]. This equality can also be established for right noetherian rings with local bijective Gabriel correspondence, which do not necessarily satisfy the right second layer condition [Example 4.3].

PROPOSITION 4.10. *Let R be a right σ-noetherian ring and let M be a σ-noetherian right R-module. Then $\gamma_\sigma(M) \leq |M|_\sigma$.*

PROOF. Since M is σ-noetherian, $|M|_\sigma$ exists. Since $\mathrm{Spec}_\sigma(R)$-tame modules are σ-torsionfree, $\Gamma_\sigma(M) \subseteq \mathcal{K}_\sigma(M)$. Thus $\gamma_\sigma(M) \leq |M|_\sigma$. □

COROLLARY 4.11. *Let R be a right σ-noetherian ring with local bijective Gabriel correspondence with respect to σ. Then $\gamma_\sigma(M) = |M|_\sigma$ for any σ-noetherian module M.*

PROOF. By Proposition 4.1, any σ-torsionfree module is $\mathrm{Spec}_\sigma(R)$-tame. □

LEMMA 4.12. *Let R be a σ-torsionfree prime right σ-noetherian ring. Let E be an essential right ideal of R. Then*

(i) *If R/E is $\mathrm{Spec}_\sigma(R)$-tame, then $\gamma_\sigma(R/E) < \gamma_\sigma(R_R)$.*

(ii) $|R/E|_\sigma < |R_R|_\sigma$.

PROOF. (i). Define $E_0 = R$ and $E_1 = E$. Suppose E_i has been defined for $i = 0, 1, \dots, k$ where $E_i \subseteq_e R$, $E_{i-1} \supset E_i$, $\gamma_\sigma(E_{i-1}/E_i) \geq \gamma_\sigma(R/E)$ and R/E_i is $\mathrm{Spec}_\sigma(R)$-tame for all $i \leq k$.

Choose a regular element c of R in E_k. This is possible since R is a prime right Goldie ring and $E_k \subseteq_e R$. We define E_{k+1} to be a submodule of E_k that is maximal with respect to $E_{k+1}/cE_k \cap cR/cE_k = 0$. This can be chosen by Zorn's lemma. Then $E_{k+1}/cE_k \oplus cR/cE_k \subseteq_e E_k/cE_k$ and

$$cR/cE_k \cong \frac{E_{k+1}/cE_k \oplus cR/cE_k}{E_{k+1}/cE_k} \subseteq_e E_k/E_{k+1}.$$

Observe that $R/E_k \cong cR/cE_k$ as c is a regular element. Thus E_k/E_{k+1} is $\mathrm{Spec}_\sigma(R)$-tame, and hence R/E_{k+1} is $\mathrm{Spec}_\sigma(R)$-tame, as an extension of a tame module by a tame module is tame.

Note that $c^2R \subseteq cE_k \subseteq E_{k+1}$. Since c^2 is also regular, E_{k+1} contains a regular element, hence $E_{k+1} \subseteq_e R$. We also have

$$\gamma_\sigma(E_k/E_{k+1}) \geq \gamma_\sigma(R/E_k) \geq \gamma_\sigma(R/E)$$

since $R/E_k \hookrightarrow E_k/E_{k+1}$. Thus E_{k+1} has been constructed satisfying the required conditions.

Considering the infinite decreasing sequence $R = E_0 \supset E_1 \supset E_2 \supset \cdots$, we see that R/E_i is $\mathrm{Spec}_\sigma(R)$-tame and that $\gamma_\sigma(E_i/E_{i+1}) \geq \gamma_\sigma(R/E)$ for all i. Thus $\gamma_\sigma(R) > \gamma_\sigma(R/E)$.

(ii). Let c be a regular element of R in E. Then for the decreasing sequence $R \supset cR \supset c^2R \supset \cdots$, we have that $|c^iR/c^{i+1}R|_\sigma < |R|_\sigma$ for all but finitely many i. Since $R/cR \cong c^iR/c^{i+1}R$, it now follows that $|R/E|_\sigma \leq |R/cR|_\sigma < |R|_\sigma$. □

LEMMA 4.13. *Let R be a right σ-noetherian ring. Then*

$$\text{cl.K}_\sigma.\text{dim}(R) \leq \gamma_\sigma(R_R) \leq |R_R|_\sigma.$$

PROOF. The second inequality was obtained in Proposition 4.10.

The first inequality is proved by induction on $\alpha = \gamma_\sigma(R)$. If $\alpha = -1$, then for any prime ideal P of R, R/P cannot be $\text{Spec}_\sigma(R)$-tame, that is, $P \notin \text{Spec}_\sigma(R)$. Thus, $\text{Spec}_\sigma(R) = \emptyset$, so $\text{cl.K}_\sigma.\text{dim}(R) = -1$.

Assume that the inequality holds for all ordinals $\beta < \alpha$. It has to be shown that $\text{cl.K}_\sigma.\text{dim}(R) \leq \alpha$ when $\alpha = \gamma_\sigma(R)$. We show that if $P \in \text{Spec}(R)$, then $P \in \text{Spec}^\alpha_\sigma(R)$. Let $Q \supset P$ be a prime ideal in $\text{Spec}_\sigma(R)$. Since $Q/P \subseteq_e R/P$, we get $\beta = \gamma_\sigma(R/Q) < \gamma_\sigma(R/P) \leq \gamma_\sigma(R) = \alpha$ by Lemma 4.12. By the inductive hypothesis, $\text{cl.K}_\sigma.\text{dim}(R/Q) \leq \gamma_\sigma(R/Q)$. Therefore $\text{Spec}^\beta_\sigma(R/Q) = \text{Spec}_\sigma(R/Q)$, and hence $Q \in \text{Spec}^\beta_\sigma(R)$. Thus, $P \in \text{Spec}^\alpha_\sigma(R)$, so $\text{Spec}_\sigma(R) = \text{Spec}^\alpha_\sigma(R)$. Hence $\text{cl.K}_\sigma.\text{dim}(R) \leq \gamma_\sigma(R)$. □

In order to establish equality between dimensions, a lemma is required.

LEMMA 4.14. *Let R be a right σ-noetherian ring with local bijective Gabriel correspondence. If M is a σ-noetherian right R-module that is not σ-torsion, then $|M|_\sigma = |R/P|_\sigma$ for some prime ideal of R.*

PROOF. Since $|M|_\sigma = |M/\sigma(M)|_\sigma$, assume that M is σ-torsionfree. Suppose that the assertion is not true. Consider

$$\mathcal{M} = \{N < M \mid \sigma(M/N) = 0, |M/N|_\sigma \neq |R/P|_\sigma \text{ for any } P \in \operatorname{Spec}_\sigma(R)\}.$$

Since $0 \in \mathcal{M}$, there exists a maximal $N \in \mathcal{M}$ such that for any $N' \supset N$ with $\sigma(M/N') = 0$, $|M/N'|_\sigma = |R/P|_\sigma$ for some prime P. Replace M by M/N and choose a prime uniform submodule $U \subseteq M$ where $\operatorname{Ass}(U) = P$. Note that $P \in \operatorname{Spec}_\sigma(R)$ and U is tame by Proposition 4.1. If $\overline{U}^\sigma \subset M$, then $|M/\overline{U}^\sigma|_\sigma = |R/Q|_\sigma$ for some prime Q by the maximality of N. Also $|\overline{U}^\sigma|_\sigma = |U|_\sigma = |R/P|_\sigma$, where the second equality holds by U being tame. Hence

$$\begin{aligned} |M|_\sigma &= \max\{|U|_\sigma, |M/\overline{U}^\sigma|_\sigma\} \\ &= \max\{|R/P|_\sigma, |R/Q|_\sigma\}. \end{aligned}$$

If $\overline{U}^\sigma = M$, then $|M|_\sigma = |U|_\sigma = |R/P|_\sigma$ gives the result. ∎

PROPOSITION 4.15. *Let R be a right σ-noetherian ring having local bijective Gabriel correspondence with respect to σ. Then*

$$\text{cl.K}_\sigma.\text{dim}(R) = \gamma_\sigma(R_R) = |R_R|_\sigma.$$

PROOF. By Corollary 4.11 and Lemma 4.13, it suffices to prove that $|R_R|_\sigma \leq \text{cl.K}_\sigma.\text{dim}(R)$. This is shown by induction on $\alpha = |R_R|_\sigma$. If $\alpha = -1$, then for any prime ideal P, the prime ring R/P is σ- torsion. Thus, $\operatorname{Spec}_\sigma(R) = \emptyset$, and so $\text{cl.K}_\sigma.\text{dim}(R) = -1$. Assume that conclusion holds for all ordinals $\beta < \alpha$. Let $|R_R|_\sigma = \alpha$. By Lemma 4.14, there exists a prime ideal $P \in \mathrm{Spec}_\sigma(R)$ such that $|R|_\sigma = |R/P|_\sigma$. If $|R/P|_\sigma \leq \mathrm{cl.K}_\sigma.\mathrm{dim}(R/P)$, then the assertion follows. Hence, assume that R is prime σ-torsionfree and right σ-noetherian. In order to establish the inequality $|R_R|_\sigma \leq \mathrm{cl.K}_\sigma.\mathrm{dim}(R)$, it has to be shown that, given any infinite descending chain $R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of σ-closed right ideals, $|I_i/I_{i+1}|_\sigma < \mathrm{cl.K}_\sigma.\mathrm{dim}(R)$ for all but finitely many i. Since $|R_R|_\sigma = \alpha$, $|I_i/I_{i+1}|_\sigma < \alpha$ for almost all i.

Let M be a σ-torsionfree subfactor of R with $|M|_\sigma = \beta < \alpha$. We proceed to show that $\beta < \mathrm{cl.K}_\sigma.\mathrm{dim}(R)$. Define

$$\mathcal{M} = \{N \subseteq M \mid \sigma(M/N) = 0 \text{ and } |M/N|_\sigma = \beta\}.$$

Since M is also σ-noetherian and $0 \in \mathcal{M} \neq \emptyset$, there exists a maximal $N \in \mathcal{M}$. We claim that M/N is β-critical with respect to σ. If N'/N is a nonzero submodule of M/N such that $(M/N)/(N'/N) \cong M/N'$ is σ-torsionfree, then $|M/N'|_\sigma < \beta$ by the choice of N. Since M/N is σ-torsionfree, M/N is uniform. Let $\mathrm{Ass}(M/N) = Q$. Since R has local bijective Gabriel correspondence with respect to σ, it follows from $Q \in \mathrm{Spec}_\sigma(R)$ that M/N is Q-tame by Proposition 4.1. Let nR be a Q-prime cyclic submodule of M/N. Since M/N is β-critical, $|M/N|_\sigma = |nR|_\sigma$. Since $r(n)/Q$ is not an essential right ideal of R/Q, and since $|R/Q|_\sigma = |A/Q|_\sigma$ for every right ideal $A \supset Q$, it follows that

$$|R/Q|_\sigma = |nR|_\sigma = |M/N|_\sigma = \beta < \alpha.$$

By the inductive hypothesis, $\mathrm{cl.K}_\sigma.\mathrm{dim}(R/Q) = \beta$. Now $Q \neq 0$ since otherwise $\alpha = |R_R|_\sigma = \beta$. Thus

$$|M|_\sigma = \beta = \mathrm{cl.K}_\sigma.\mathrm{dim}(R/Q) < \mathrm{cl.K}_\sigma.\mathrm{dim}(R).$$

□

4.4. Δ-modules under Torsion Theory

If R is a right noetherian ring, then all minimal prime ideals satisfy the right second layer condition. Hence any finitely generated submodule of $\mathrm{E}(R/P)_R$ is a Δ-module if P is a minimal prime ideal. In fact, $\mathrm{E}(R/P)$ is a Δ-module if P is a minimal prime ideal of a right noetherian ring R [**60**, Corollaire 4.4]. We establish this for a ring R that is a right order in a right artinian ring. The ring R need not be a right noetherian ring. Recall the following characterization.

PROPOSITION 4.16. [**56**, Theorem 4.14] *A ring R with prime radical N is a right order in a right artinian ring iff*

1. *R/N is right Goldie and N is nilpotent.*
2. *$\rho(R)$ is finite.*
3. $\mathcal{C}_R(0) = \mathcal{C}_R(N)$.

PROPOSITION 4.17. *If R is a right order in a right artinian ring S, then $\mathrm{E}(R/P)_R$ is a Δ-module for every minimal prime ideal P of R.*

PROOF. Let N be the prime radical of R, and let $\sigma = \mathrm{rad}_{\mathrm{E}(R/N)}$ be the torsion radical cogenerated by $\mathrm{E}(R/N)$. Since R is a right order in a right artinian ring, R/N is right Goldie and $\mathcal{C}(0) = \mathcal{C}(N)$ by Proposition 4.16. Hence $\rho_{\mathcal{C}(N)} = \sigma$ by Lemma 1.19, $R_\sigma = R\mathcal{C}(N)^{-1}$ is a right artinian ring and σ is a perfect torsion radical as $\mathcal{C}(0)$ is a right denominator set. Since $\mathrm{E}(R/P) \hookrightarrow \mathrm{E}(R/N)$ for any minimal prime ideal P, we obtain that $\mathrm{E}(R/P)$ is σ-torsionfree. Recall that R_σ is right artinian iff R has DCC on σ-closed right ideals in R, as σ is perfect. If X is a subset of $\mathrm{E}(R/P)$, then

$$R/r(X) \hookrightarrow \prod_{x \in X} xR \hookrightarrow (\mathrm{E}(R/P))^X.$$

So $r(X)$ is a σ-closed right ideal in R. This shows that an arbitrary nonempty subset of $\mathrm{E}(R/P)$ is finitely annihilated. □

COROLLARY 4.18. *If R is a right order in a right artinian ring S, then any σ-torsion free module M is a Δ-module where $\sigma = \mathrm{rad}_{\mathrm{E}(R/N)}$.*

Therefore, any R_σ-module is a Δ-module as an R-module.

PROOF. Since σ is a perfect torsion radical as seen in the proof of Proposition 4.17, any R_σ-module M is a σ-torsionfree R-module and so $r_R(X)$ is σ-closed for any nonempty subset X of M. □

Proposition 4.17 can be generalized to an arbitrary semiprime ideal I of a ring R with torsion radical σ cogenerated by $E(R/I)$ if R_σ is right artinian.

Recall that a right R-module C is called a *cogenerator* in Mod-R if every right R-module can be embedded in a product of copies of C.

LEMMA 4.19. [2, Proposition 18.15] *Let E be an injective R-module. Then the following are equivalent.*

(i) *E is a cogenerator.*

(ii) $\mathrm{Hom}(T, E) \neq 0$ *for all simple right R-modules T.*

(iii) *E cogenerates all simple right R-modules, i.e., every simple module embeds in a direct product of copies of E.*

PROPOSITION 4.20. *Let I be a semiprime ideal of a ring R such that R/I is right Goldie, and let $\sigma = \mathrm{rad}_{\mathrm{E}(R/I)}$ be the torsion radical cogenerated by $\mathrm{E}(R/I)$. Note that $\sigma(R) = r_R(\mathrm{E}(R/I))$, and set $K = \sigma(R)$. If I_σ is the Jacobson radical $J(R_\sigma)$ of R_σ and R_σ is a right artinian ring, then $\mathrm{E}(R/P)_{R/K}$ is a Δ-module, for every minimal prime ideal P/K of R/K.*

PROOF. Note that, by Lemma 1.19, $\rho_{\mathcal{C}(I)} = \sigma$ as R/I is a semiprime right Goldie ring. Also $(R/K)_\sigma = R_\sigma$ by definition of R_σ as $K = \sigma(R)$. Since R_σ is right artinian and $I_\sigma = J(R_\sigma)$, we see that R_σ/I_σ is a semisimple artinian ring. If M is a simple right R_σ-module, then for any $0 \neq m \in M$,

$$M \cong m(R_\sigma/I_\sigma) \cong (R_\sigma/I_\sigma)/r_{R_\sigma/I_\sigma}(m) \cong L/I_\sigma$$

where L/I_σ is a minimal right ideal of R_σ/I_σ. Thus any simple R_σ-module M embeds in R_σ/I_σ, hence $\mathrm{E}(R/I)$ is a cogenerator of Mod-R_σ since $R_\sigma/I_\sigma \hookrightarrow$

$(R/I)_\sigma \hookrightarrow \mathrm{E}(R/I)$. Therefore any R_σ-module is σ-torsionfree as an R-module, so σ is a perfect torsion radical.

Now, we claim that $\mathcal{C}_R(I)$ is a right Ore set in R. For any $c \in \mathcal{C}(I)$, if R/cR is $\mathcal{C}(I)$-torsion, then the right Ore condition is satisfied. Since $(R/I)/(cR+I/I)$ is σ-torsion and σ is perfect, the short exact sequence

$$0 \to (cR+I) \to R \to (R/cR+I) \to 0$$

produces the short exact sequence

$$0 \to (cR+I)_\sigma \to R_\sigma \to (R/cR+I)_\sigma \to 0.$$

Thus $(cR+I)_\sigma = (cR)_\sigma + I_\sigma = R_\sigma$, as $\sigma(R/cR+I) = R/cR+I$, and consequently,

$$(R_\sigma/(cR)_\sigma)J(R_\sigma) = ((cR)_\sigma + J(R_\sigma))/(cR)_\sigma = R_\sigma/(cR)_\sigma.$$

Since R_σ is assumed to be right artinian and so right noetherian, it follows from Nakayama's Lemma [**59**, II], [**32**, Theorem 10], [**4**, Theorem 1], that $(cR)_\sigma = R_\sigma$. Again it follows that $(R/cR)_\sigma = R_\sigma/(cR)_\sigma = 0$, thus R/cR is σ-torsion, whence $\mathcal{C}(I)$-torsion. This proves that $\mathcal{C}(I)$ is right Ore in R.

Moreover, $C = \{c+K | c \in \mathcal{C}(I)\}$ becomes a right Ore set in R/K. It is also a right denominator set in R/K since R_σ is right artinian and $R/K \hookrightarrow R_\sigma$.

We proceed to show that $(R/K)_\sigma = R_\sigma = (R/K)C^{-1}$. First, we claim that $r_{(R/K)_\sigma}(c+K) = 0$ for any $c \in \mathcal{C}(I)$. For, assume that $0 \neq x \in r_{(R/K)_\sigma}(c+K)$, so $(c+K)x = 0$. Now, for any $x \in (R/K)_\sigma$ there exists

$d \in \mathcal{C}(I)$ such that $xd \in R/K$. Let $xd = r + K$ for some $r \in R$, so $cr \in K$. By the right reversibility of C, we get $rc' \in K = \sigma(R)$ for some $c' \in \mathcal{C}(I)$, so $rc'c'' = 0$ for some $c'' \in \mathcal{C}(I)$. Thus $r \in K$, and x is $\mathcal{C}(I)$-torsion, contradicting the fact that R_σ is σ-torsionfree.

It now follows that any element $c \in C$ becomes invertible in R_σ. Consider the diagram

$$\begin{tikzcd}
R/K \arrow[r, "f"] \arrow[d, "\phi"'] & (R/K)C^{-1} \\
(R/K)_\sigma = R_\sigma, &
\end{tikzcd}$$

where $\phi : R/K \hookrightarrow (R/K)_\sigma$ and $f : R/K \to (R/K)C^{-1}$ denote the canonical maps. Note that $\ker f = 0$ and that $(R/K)C^{-1}/(R/K)$ is σ-torsion. Since R_σ is σ-injective, there exists an extension map $\Phi : (R/K)C^{-1} \to (R/K)_\sigma$ such that $\phi = \Phi f$.

We claim that Φ is a ring isomorphism. If $(r + K)(c + K)^{-1} \in \ker \Phi$, then $\Phi(r + K) = 0$, so $\phi(r + K) = 0$. But ϕ is an embedding, so $r + K = 0$, thus $\ker \Phi = 0$. The surjectivity follows from the definition of quotient rings with respect to σ.

Now R/K is a right order in a right artinian ring R_σ. Therefore by Proposition 4.17, $\mathrm{E}(P/K)_{R/K}$ is a Δ-module for any $P/K \in \mathrm{minspec}(R/K)$.

□

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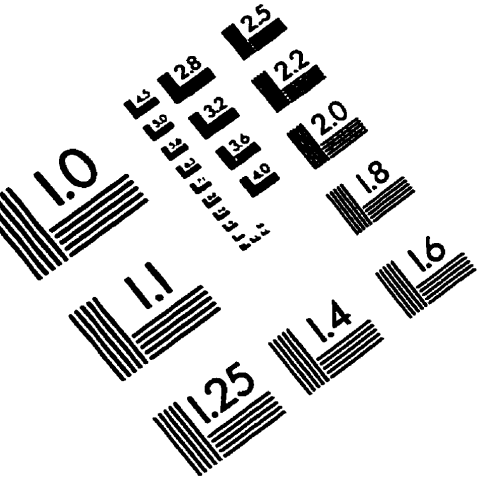
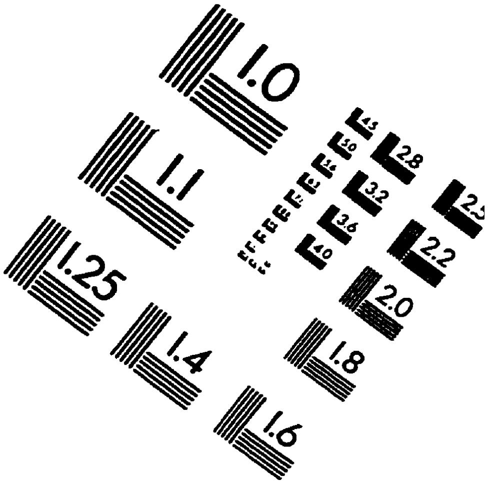
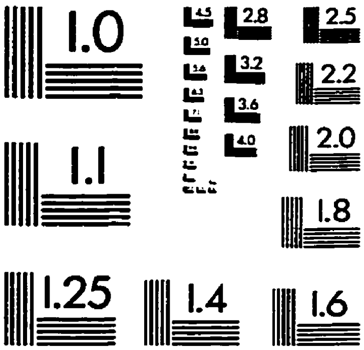
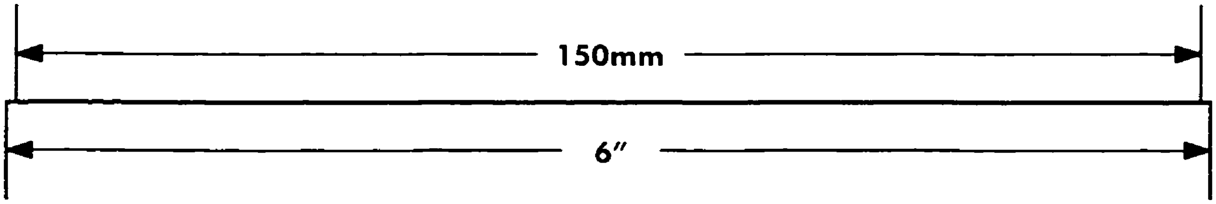
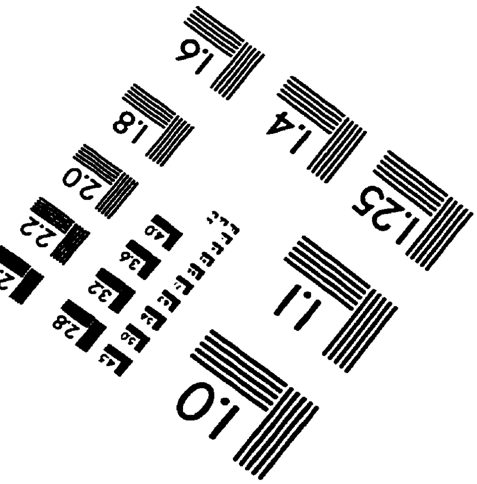
IMAGE EVALUATION TEST TARGET (QA-3)











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