ON THREE THEOREMS CONCERNING THE RADICAL AND SEMI-SIMPLICITY

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by

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An Abstract of a Thesis

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The radical N of a finite dimensional algebra, or more generally of a ring R which satisfies the descending chain condition (d.c.c.), is usually defined as the union of all nilpotent left (right) ideals in R. Other definitions of the radical, namely those for arbitrary rings, narings, clusters, and alternative rings, are generalizations of N; and they coincide with this ideal when considered in a ring with d.c.c.

Fundamental to the structure theory of rings which satisfy the d.c.c., (i.e., the so-called classical theory), is that the radical N is a two-ideal; that R/N is semi-simple; and that a ring is semi-simple if and only if it is a finite direct sum of simple rings. For the more general definitions of the radical, a number of interesting analogues of these theorems have been proven. Taking them collectively reveals the following general information:

1. The fact that the radical is a two-ideal, is proved differently for different definitions of the radical. Each proof, however, depends mainly on the manner in which the radical is defined. On classifying the different definitions, it is found that four general methods suffice.

2. The analogues of the second theorem, that a ring, naring, cluster, etc., modulo its radical is semi-simple, are proved by one of two different methods. The most common of the two, used for all but two definitions of the radical, is based on the fact that the radical can always be expressed in terms of a specified set of elements. The other slighty more general method makes use of a suitable application of the second theorem on homomorphic mappings.

3. Lastly, analogues of the third theorem, (as well as the theorem itself), are proved by one of three general methods. First, there is the original or classical proof based on a number of results of the theory of idempotents. Secondly, there is a method based on an intersection characterization of the radical, and a fundamental theorem on subdirect sums; and thirdly, there is a method which results from a special and rather unusual definition of the radical.

A brief description of each particular method, illustrated by one or more specific examples, comprises the present dissertation.

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CHAPTER I

INTRODUCTION

The main object of this thesis is to become acquainted with the following general theorems taken from the theory of the radical and semi-simplicity:

Theorem 1. The radical N is a two-ideal.

Theorem 2. The difference ring (naring, cluster, alternative ring, non-associative algebra, alternative algebra, Jordan algebra) S/N is semi-simple, meaning that it is either a ring with zero radical or else in a particular case a finite direct sum of simple rings.*

Theorem 3. A ring (naring, cluster, etc.) S is semi-simple if and only if S is isomorphic to a subdirect sum of rings S_i, the S_i's suitably generalizing simple rings.

For the most part, these theorems were developed since 1942; and all of them have resulted from various extensions of the classical notion of the radical for algebras. The third is of particular interest as it expresses the existence of a number of semi-simple systems, whose structures suitably generalize the finite direct sum decomposition of classically semi-simple algebras.

For our particular purpose, we will study the different methods of proof which have been found expedient for these theorems. We have determined that there are

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* Throughout, unless otherwise stated, semi-simplicity will be regarded in the zero radical sense.

four general methods for theorem 1, two for theorem 2, and three for theorem 3. Each method is discussed briefly, and illustrated by one or more specific examples.

To familiarize the reader with the general theory of the radical and semi-simplicity, we begin with a brief synopsis of the various definitions of the radical, their relationships, and the explicit forms of the theorems above.

It is assumed that the reader is familiar with such concepts as group, ring, vector space, algebra, field, ideal, nil and nilpotent ideals, descending chain condition, direct sum, union, intersection, homomorphism, etc., all as one might find in B. L. Van der Waerden's "Modern Algebra."

With regard to notation, rings have been designated by R; algebras (which are assumed to have finite dimension unless otherwise stated) by A; rings or algebras by S; and residue class rings, i.e., difference rings by $S/I \equiv \{[s]; \text{ modulo } I\}$.

CHAPTER II

THE RADICAL AND STRUCTURE THEORY

Rings

For rings which satisfy the descending chain condition (d.d.c.), the three theorems of the introduction are a direct outgrowth of Wedderburn's contribution to the structure of algebras over a non-modular field [27].*

Following Albert's exposition of Wedderburn's results [1], it is well known that every nilpotent left, right, or two-ideal of an algebra A over an arbitrary field is contained in a maximal nilpotent two-ideal N_1 called its radical. Furthermore the difference algebra A/N_1 is semi-simple; and an algebra A is semi-simple if and only if it is isomorphic to a direct sum of simple algebras each of which is isomorphic to a total matrix algebra over a division algebra. Similarly, by defining the radical N_2 of a ring R with d.c.c. to be the union of all nilpotent left (right) ideals in R, it can be shown that N_2 is a two-ideal; R/N_2 is semi-simple; and a ring R is semi-simple if and only if R is isomorphic to a direct sum of simple rings each of which is isomorphic to a total matrix ring over a division ring.

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* The numbers in the square brackets refer to the bibliography given at the end.

The radical N_2 coincides with N_1; both are referred to as the classical radicals; and all other definitions of the radical are either generalizations or characterizations of one, or the other.

In the early 1940's, after the above so-called Wedderburn-Artin structure theorems (of R) were fully realized, Baer and Levitzki independently proposed definitions of the radical for arbitrary rings. It was hoped that these definitions would lead to further generalizations of Wedderburn's original results.

Baer [7], called an ideal N of a ring R a radical ideal if N is a two-ideal, a nil-ideal, and if R/N contains no non-zero nilpotent left (right) ideals. He defined the lower radical N_3 of R to be the intersection of all radical ideals, and the upper radical N_4 to be the union of all radical ideals.

Levitzki [18], called a left, right, or two-ideal I of R, a semi-nilpotent ideal if every subring generated by a finite number of its elements is nilpotent. He defined the radical N_5 to be the union of all semi-nilpotent two-ideals in R.

Although N_3, N_4, and N_5, coincide with N_2 when R satisfies the d.c.c., none of these definitions led to a structure for (arbitrary) semi-simple rings.

In 1945, Jacobson [15] finally overcame the difficulties presented by rings which are not restricted to

the d.c.c. On defining a completely new kind of radical he was able to provide an entirely satisfactory generalization of the Wedderburn-Artin theory. He abandoned the concepts of nil and semi-nilpotence in defining the radical, and used the notion of quasi-regularity introduced by Perlis [23]. The latter called an element $x \in R$ right quasi-regular if there exists a y such that $x + y + xy = 0$, and a right ideal quasi-regular if all of its elements are right quasi-regular. Jacobson defined the right (left) radical N_6 of R to be the union of all quasi-regular right (left) ideals of R.

He proved that N_6 the right radical coincides with the left radical; and that it is a two-ideal. He showed that the quotient R/N_6 is semi-simple, and that a ring R is semi-simple if and only if R is isomorphic to a subdirect sum of primitive rings* R_i each of which is isomorphic to a dense ring of linear transformations on a vector space over a division ring. This theorem is an immediate generalization of the classical result; for if R satisfies the d.c.c., the subdirect sum of primitive rings becomes a finite direct sum of simple rings.

This contribution created a great deal of interest, and in 1947 Brown and McCoy [10] contributed the G-radical.

They considered the two-ideal

$$G(a) \equiv \{ax - x + \Sigma(y_i a z_i - y_i z_i)\}$$

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* A ring R is called right primitive if it contains a maximal right ideal J such that the quotient ideal $(J:R) \equiv \{\alpha \in R; R\alpha \subseteq J\}$ is zero. A left primitive ring is defined similarly.

generated by the right ideal $\{ax - x\} \leq R$; and defined the G-radical N_7 to be the set of all $b \in R$ such that $a \in (b)$, the two-ideal generated by b, implies $a \in G(a)$. The element a is called G-regular.

A more general approach to this definition, of interest itself, is given as follows: Assume there is defined an arbitrary mapping $a \longrightarrow F(a)$ of R into the set of two-ideals F(R) of R such that if $a \longrightarrow \bar{a}$ is any homomorphism of R onto a ring $\bar{R}$, then $F(\bar{a}) = \overline{F(a)}$. Call an element a F-regular if it is in F(a), and an ideal F-regular if every element in the ideal is F-regular. Finally, define the F-radical N_8 of R to be the set of all $b \in R$ such that (b), the two-ideal generated by b, is F-regular.

Since $G(\bar{a}) = \overline{G(a)}$, the G-radical <u>is</u> clearly a special case of the F-radical. In addition $N_7 \geq N_6$, since an element $b \in R$ is in the Jacobson radical if and only if $a \in \{ax - x\}$ for every $a \in \{bi + bz\}$. If R satisfies the d.c.c., Brown and McCoy have shown that $N_7 = N_6$ - which then coincide with N_2.

It should be noted, however, that the F-radical (itself) will not in general reduce to the classical radical. Obviously, there is no assurance that an arbitrary F-mapping will define a generalization of N_2. The latter occurs only if the F-mapping is specially chosen.

In spite of this fact, Brown and McCoy discovered that R/N_8 has zero F-radical; and also that a ring has zero

F-radical if and only if it is isomorphic to a subdirect sum of subdirectly irreducible rings with zero F-radical. In the case of the G-radical, it was proved that a ring R is semi-simple with respect to N_7 if and only if R is a subdirect sum of simple rings with unity element.

An equivalent definition of the G-radical was later proposed by Andrunakievitch [5]. Andrunakievitch considered the operation

$$a \circ b = a + b - ab$$

which we call quasi-multiplication. He defined a co-united ideal C of R to be a set satisfying:

1) that $a \in C$ and $r \in R$ implies $a \circ r \in C$;

2) for any $a_1, a_2, a_3 \in C$, $a_1 - a_2 + a_3 \in C$.

We note that this set is not an ideal in the ordinary sense, in fact it is not even closed under addition.

For the definition of the radical, Andrunakievitch considered the co-united ideal $R \circ a \circ R$ defined by

$$\left\{ \sum \lambda_i (x_i \circ a \circ y_i) \right\},$$

where $\sum \lambda_i = 1$. He called an element $b \in R$ a common radical element if $R \circ b \circ R = R$; and a two-ideal a common radical ideal if each of its elements is common radical. He defined the radical N_9 (which we call the Andrunakievitch radical) to be the union of all common radical ideals in R.

The notion of common radicality coincides with G-regularity, because an element $a \in R$ is common radical if and only if a is G-regular. Hence the Andrunakievitch radical, being $\{ b \in R$; every $a \in (b)$ is a common radical element$\}$, coincides with the G-radical. Evidently this is another approach to the theory of Brown and McCoy.

Among these definitions of the radical, there is one more due to McCoy [21].

McCoy calls a set M of a ring R an m-system if a, $b \in M$ implies that $axb \in M$ for some $x \in R$. He defined the radical N(I) of an ideal $I \leq R$ to be the set of all elements $x \in R$ with the property that every m-system which contains x contains an element of I. The radical N_{10} of R is defined to be the radical of the zero ideal. The set N_{10} is referred to as the p-radical.

It is shown that N_{10} is a nil ideal which contains every nilpotent right ideal of R; and in the presence of d.c.c. it coincides with the classical radical.

McCoy also obtained a form of the third theorem. He proved that a ring R is semi-simple with respect to N_{10} if and only if R is isomorphic to a subdirect sum of prime rings. It is interesting to note that a prime ring has zero p-radical, and that a prime ring with minimal left (right) ideals is a primitive ring.

Non-Associative Systems

Albert was the first to propose a definition of the radical for a non-associative system. In 1942, see [2], he showed that if a non-associative algebra A is homomorphic to a direct sum of non-zero simple algebras, then there exists a minimal two-ideal $N_{11} \subseteq A$, such that N_{11} is the intersection of all two-ideals $B \subseteq A$ with the property that A/B is a direct sum of non-zero simple algebras. Since the proof of the existence of N_{11} established that A/N_{11} is a finite direct sum of non-zero simple algebras, Albert defined a semi-simple algebra to be one which is a finite direct sum of simple algebras. He called N_{11} the radical of A.

Much later, Jenner [16] proposed a similar theory for non-associative rings. He considered the transformation ring T(R),* (as did Albert for A), and defined R to be semi-simple if T(R) satisfies the d.c.c.; R contains no absolute divisors of zero; and if R is a finite direct sum of simple rings.

For his definition of the radical, Jenner assumed that R is isomorphic to a semi-simple algebra R/H (where $H \subseteq R$) and that T(R) satisfies the d.c.c. He defined the radical N_{12} of R to be the complete reciprocal image of a

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* T(R) is formed by taking all finite sums and products of the right and left multiplications of R.

set M_j under the natural homomorphism

$$R \longrightarrow R/R^N,$$

where: (1) the set N is the classical radical of T(R); (2) R^N is the set of all images of R under N; and (3) M_j, a two-ideal contained in R/R^N, is the set of all $[w] \in R/R^N$ such that both $[w][d]$ and $[d][w]$ are in M_{j-1} for all $[d] \in R/R^N$. The set $M_1 \equiv \{[x] \in R/R^N;\ [x][a] = [a][x] = [0]$ for all $[a] \in R/R^N\}$.

It is shown that R/N_{12} is isomorphic to a direct sum of simple narings, where N_{12} is minimal in R having this property. As such, N_{12} is justifiably called the radical of R.

The set N_{11} coincide with N_1 if R is associative; and similarly N_{12} coincides with N_2 if R is associative and satisfies the d.c.c.

Between the Albert and Jenner contributions, Smiley [26] furthered the development of the structure of non-associative systems when he discovered that the notion of the F-radical of Brown and McCoy applies directly to narings.

For his definition of the radical Smiley considered the F-mapping $a \longrightarrow F(a)$ where

$$F(a) = F_1(a) \equiv \{ax - x + ya - y\}.$$

He defined the F_1-radical N_{13} to be the set of all $b \in R$, such that if $a \in (b)$ then $a \in F_1(a)$.

With this particular choice of $F_1(a)$ - regularity, there is no difficulty regarding the lack of associativity. It is easy to prove, as did Smiley, that N_{13} is a two-ideal; that R/N_{13} is semi-simple; and finally that a naring R is semi-simple with respect to N_{13} if and only if R is isomorphic to a subdirect sum of simple narings with unity element.

By examining the relationships between the $F_1(a)$-radical and those defined by Albert and Zorn, (the latter which we discuss below), Smiley found that if R is a non-associative algebra with unity element, then N_{13} coincides with N_{11}; and if R is hypercomplex alternative the radical N_{13} coincides with the Zorn radical N_{14}.

Alternative Systems

M. Zorn [28] was able to define a radical for an alternative ring, when he imposed the restriction that it be hypercomplex.

A hypercomplex alternative ring is one in which there are chain conditions on modules paralleling the d.c.c. on ideals in ordinary rings.

To define the radical, Zorn considered elements $a_{ik} \in R_{ik}$ such that $\{a_{ik} \cdot r_{ik}\}$ is nilpotent for each $r_{ik} \in R_{ik}$. The set R_{ik} is a component of R decomposed as a direct sum of special "Peirce" decomposition, and the elements a_{ik} are called singular.

Zorn thus assumed that there exists at least one finite set of idempotents $e_1, e_2, \cdots e_n$, with $e_i \cdot e_k = 0$ for $i \neq k$, such that

$$R = \sum_{i,k=1}^{n} R_{ik},$$

where $R_{ik} \equiv \{ x_{ik} \in R;\ e_j x_{ik} = \delta_{ji} x_{ik},$ and $x_{ik} e_j = \delta_{kj} x_{ik} \}$. He defined the radical N_{14} to be the set of all $\sum_{i,k=1}^{n} n_{ik}$ such that $n_{ik} \cdot R_{ik}$ is singular.

He showed that the radical N_{14} is the set of all nilpotent elements of R, and also that a hypercomplex alternative ring which is semi-simple with respect to N_{14} is a direct sum of simple rings.

For the definition of the radical of ordinary alternative rings, Smiley [25] considered an extension of Jacobson's radical. Having shown that a naring is semi-simple (re $F_1(a)$) if and only if it is isomorphic to a sub-direct sum of simple narings with unity element, the $F_1(a)$-radical naturally applies to alternative rings. He was, however, interested in defining something smaller. Thus, instead of considering $F_1(a) \equiv \{ax - x + ya - y\}$ he considered $Q(a) \equiv \{ax - x\}$.

In other words, Smiley considered the operation $x \circ y = x + y + xy$ which he called the quasi product of x with y. He defined the right radical N_r of an alternative ring R to be the set of all $b \in R$ such that $(b)_r$, the right ideal

generated by b, has the property that each $a \in (b)_r$ yields $a \circ z = 0$ for some $z \in R$. Equivalently this means that each $a \in (b)_r$ is contained in $\{az - z\} = Q(a)$. A left radical N_l was defined similarly and N_{15} the radical of R is taken either as N_r or N_l.

Smiley was able to show that $N_r = N_l$, and that if R is hypercomplex alternative then N_{15} coincides with Zorn's radical. This is to be expected since N_{15} is contained in the $F_l(a)$-radical which, as we stated, also coincides with Zorn's radical if the naring R is hypercomplex alternative. He was, however, unable to show that N_{15} led to a generalization of the Wedderburn-Artin structure theorem for semi-simple rings.

For alternative algebras with unity element there are three definitions of the radical proposed by Dubish and Perlis [11]. The first of these radicals, which is called the N-radical N_{16} of A is defined to be the totality of elements $n \in A$ such that $g + n$ is regular for every regular $g \in A$. The Q-radical N_{17} is defined to be the sum of all quasi-regular right ideals, and the S-radical (similar to Albert's N_{11}) is taken to be the intersection of all two-ideals B such that A/B is a direct sum of simple alternative algebras.

It is shown that the N-radical coincides with the S and Q-radical; and the proof of the existence of the S-radical provides a structure for semi-simple alternative

algebras in a manner similar to Albert's result for non-associative algebras.

In the case that A is associative the above radicals coincide with the classical radical N_1. Actually it was shown by Perlis [23] that N_1 the classical radical coincides with the set of all elements $r \in A$ such that $x+\alpha r$ is quasi-regular for every quasi-regular $x \in A$ and α in the base field. Furthermore, if A contains a unity element then N_1 coincides with the set of all $n \in A$ such that $g+n$ is regular for every regular $g \in A$.

Jordan Algebras

The structure of Jordan algebras is due almost entirely to Albert. For the definition of the radical [3], he considered the derived series of a Jordan algebra A which is the sequence $A = A^{(0)} \supset A^{(1)} \supset A^{(2)} \supset \cdots \supset A^{(n)}$. By definition $A^{(k+1)}$ is the square of $A^{(k)}$, and it is either zero or a subalgebra of A. The elements of $A^{(k)}$ are finite sums of quantities of the form $aS_1 \cdots S_k$, where each S_i is a right multiplication of the transformation algebra T(A), and $a \in A$. Albert called an algebra A solvable if $A^{(k)} = 0$ for some interger k. He established that there exists a maximal solvable two-ideal, the radical N_{17}, and proved that a Jordan algebra modulo this two-ideal is a direct sum of simple Jordan algebras each of which is a Jordan algebra of linear transformations with the exception of algebras of degree 3 and 27 over their centers.

Clusters

An extension of the Jacobson radical provides the basis for the theory of the radical and semi-simplicity of clusters.* Brown [9], in 1951, showed that a cluster modulo its right (or left) Jacobson radical is semi-simple; and that a cluster is semi-simple if and only if it is isomorphic to a subdirect sum of right (or left) primitive clusters each of which suitably generalizes a simple ring.

The phrase 'or left', in the brackets, denotes the fact that in a cluster the right and left Jacobson radicals are not necessarily equal. Hence Jacobson's theory is not entirely generalized, and the present meaning of semi-simplicity must be restricted to either the right, or left radical is zero.

In passing we note that the theorem on semi-simple clusters implies that an alternative ring is semi-simple with respect to the Jacobson radical if and only if it is isomorphic to a subdirect sum of primitive alternative rings.** This follows, for in an alternative ring the right and left Jacobson radicals coincide; and if a semi-simple cluster is commutative with respect to addition and obeys the alternative law its subdirect sum decomposition becomes a subdirect sum of primitive alternative rings.

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* A cluster is an additive group, closed under a multiplication which is left and right distributive with respect to addition.

** We remarked earlier that Smiley lacked this result in his extension of the Jacobson radical to alternative rings.

A Characterization of the Radical

An important characterization of the radical, which we have neglected so far, is that N of S coincides with the intersection of all left, right, or two-ideals I of S such that these ideals have a well defined property or else S/I is a type of ring suitably generalizing a simple ring.

In the case of the classical radical we can prove that N_2 is the intersection of all maximal left (right) ideals [14]. For the Jacobson radical, there is the theorem that N_6 coincides with the intersection of all two-ideals $M \leq R$ such that R/M is primitive; and if R has a unity element, then N_6 is the intersection of all maximal right (left) ideals. Both the G and the Andrunakievitch radicals, and more generally the F-radical, afford similar characterization. In the case of the p-radical, we have that N_{10} is the intersection of all prime ideals in R.

Type Definitions of the Radical

From the preceding material, it is seen that the radical is defined in one of the following forms:

type 1. a maximal or minimal two-ideal which fulfills one or more particular requirements,

type 2. a set of elements each having a specified property,

type 3. the infinite union of certain left, right or two-ideals.

Definitions of the first type are, by necessity, restricted (1) to finite dimensional algebras (where maximality is assured by the finite dimensionality); and (2) to theories of the kind proposed by Albert and Jenner for non-associative systems. All of the other definitions of the radical are fully classified by types 2 and 3. Both types suitably generalize the classical radical N_1, which itself can be defined either setwise or elementwise.

Actually we can summarize this section by saying that all of the usual definitions of the radical are either of type 2 or 3; where if desirable, all definitions of the third type can be characterized by a particular definition of the second type or vice versa.

CHAPTER III

THEOREM 1. THE RADICAL IS A TWO-IDEAL

The proof of this theorem is established by one of four general methods depending on the manner in which the radical is defined. Referring to the definitions classified in the previous section, we have concluded the following:

If the radical is defined as type 1, then in order to prove the theorem it is shown that such a <u>two-ideal exists</u>.

If the radical is defined in the manner of type 2, the theorem is proved either directly, or else one shows that N is the intersection of certain two-ideals, from which the theorem follows as a corollary.

For the radical defined in the third form, (consider first the radical as a union of one sided ideals), then the theorem is proved either by:

1) characterizing the radical elementwise and applying one of the two methods just described, or

2) showing that the union of any two of the <u>left</u> (right) ideals is again a <u>left</u> (right) ideal of the same kind, and also that it is a <u>right</u> (left) ideal.

The latter is justified since an infinite union is defined to be the set of all <u>finite</u> sums.

If on the other hand the radical is defined as the union of certain two-ideals, then the theorem is proved by

showing that the union of any two is again a two-ideal of the same kind. As in the case of the one-sided ideals, this is sufficient because an infinite union of two-ideals is also the set of all finite sums.

As an example of the proof for type 1, we consider the theorem for the classical radical of an algebra. It is required that one prove the existence of a maximal nilpotent two-ideal. The actual statement of the theorem is contained in the following:

> Every nilpotent left, right, or two-ideal is contained in a unique maximal nilpotent two-ideal N_1 of A.

The existence of N_1 is established by considering N, the nilpotent two-ideal in A having the largest dimension as a vector space, and applying two lemmas:

Lemma 1. If B_1 and B_2 are nilpotent left, right or two-ideals then (B_1, B_2) is a nilpotent left, right, or two-ideal.

Lemma 2. If B is a nilpotent left, right ideal then BA is a two-ideal and (B, BA) is a nilpotent two-ideal.

Theorem 1.1. Every nilpotent left, right or two-ideal is contained in a unique maximal nilpotent two-ideal N_1 of A.

Proof: Let N be a nilpotent two-ideal of A, which, as a vector space, has maximal dimension. Consider any other nilpotent two-ideal I of A. By lemma 1, (N, I) is a nilpotent two-ideal, and since N has maximal dimension, $N \geqslant (N, I)$ and so $N \geqslant I$.

Now, let L be any nilpotent left ideal of A. The set $L \subseteq (L, LA)$; and since (L, LA) is a nilpotent two-ideal by lemma 2, $(L, LA) \subseteq N$. This implies $L \subseteq N$.

Similarly N contains all nilpotent right ideals. On defining N to be N_1, the proof is complete.

As an example of the theorem for a definition of the second type, we first consider the case when the theorem is proved directly. We will show that the G-radical is a two-ideal, i.e., that N_7 is a module, and that N_7R and $RN_7 \subseteq N_7$. The two latter properties are easily shown, but the module property is more difficult. We require

Lemma 3. If a-c is G-regular in R, then a is G-regular if $c \in G(a)$.

Proof: Since $a-c \in G(a-c)$ by assumption, there exist elements $r, x_i, y_i \in R$ such that

$$a-c = (a-c)r-r + \sum (x_i [a-c] y_i - x_i y_i).$$

Transposing and rearranging, we have

$$a = [ar-r + \sum (x_i a y_i - x_i y_i)] + [c-cr + \sum x_i c y_i].$$

Clearly, the quantity in the first bracket is in G(a). But $c-cr + \sum x_i c y_i$ is also in G(a).

By hypothesis $c \in G(a)$, and therefore there exist elements $z, s_i, t_i \in R$ such that

$$c = az-z + \sum (s_i a t_i - s_i t_i).$$

As a result of this statement,

$$-cr = -[a.zr-zr + \sum (s_i a.t_i r - s.t_i r)],$$
$$= -[am-m + \sum (s_i a u_i - s u_i)].$$

Adding c,-cr, and $\sum x_i cy_i$ we see that $c-cr+\sum x_i cy_i$ is of the form $ar'-r'+\sum (x_i'ay_i'-x_i'y_i')$, for some $r',x_i',y_i' \in R$. As such, $c-cr+\sum x_i cy_i$ is in G(a), and therefore $a \in G(a)$.

Before proving the theorem, we remark that a nilpotent element is G-regular. This fact follows directly from the lemma above.

Theorem 1.2. The G-radical N_7 of R is a two-ideal.

Proof: Let z be an element of the G-radical and consider $(tz)^*$, the two-ideal generated by tz, for any $t \in R$. Since (z) is G-regular, and since $(tz) \leqslant (z)$, (tz) is contained in N_7. Hence $RN_7 \leqslant N_7$, and similarly $N_7R \leqslant N_7$.

The proof of the module property is shown by proving that if $z, w \in N_7$, and $a \in (z-w)$, then $a \in G(a)$.

Hence let z, w, be elements of N_7, and assume that $a \in (z-w)$.

Since $(z-w) \leqslant (z)-(w)$, the element a is of the form u-v, where $u \in (z)$ and $v \in (w)$. Also since (z) is G-regular, there exist elements $r, x_i, y_i \in R$ such that

$$u = ur-r+\sum (x_i uy_i - x_i y_i).$$

As a result, the element

$$\begin{aligned} a-[ar-r+\textstyle\sum (x_i ay_i-x_i y_i)] &= (u-v)-[(u-v)r-r+\textstyle\sum (x_i (u-v)y_i-x_i y_i)] \\ &= u-[ur-r+\textstyle\sum (x_i uy_i-x_i y_i)]+(-v+vr+\textstyle\sum x_i vy_i) \\ &= -v+vr+\textstyle\sum x_i vy_i. \end{aligned}$$

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* The set $(tz) = \{ x(tz)y+u(tz)+(tz)v+tzi \}$ where i is an integer.

Since c is in the G-regular two-ideal (w), the right hand side of this expression is G-regular. Consequently, the left hand side $a-[ar-r]+\sum(x_i a y_i - x_i y_i)$ is also G-regular; and the lemma just proved implies that $a \in G(a)$. This completes the proof.

We now illustrate a proof of theorem 1, again re type 2, but which follows from showing that the radical N is the intersection of certain two-ideals in R. It is interesting to note, that in the first part of this characterization one must always make use of Zorn's lemma. The manner in which it is applied is common to all forms.

For our purpose we have chosen to show that the Jacobson cluster radical N_{18} is the intersection of all two-ideals $M' = \{x \in R;\ (x) \leq M\}$, where M is a modular maximal right ideal in (the cluster) R.

In the course of the proof, we will require:

Lemma 4. If $a, b \in R$, then $Q(a+b) = \{(a-b)x-x\}$ is contained in $Q(a)+(b)_r$; the latter being $\{bj+br+bi\}$ where $r \in R$, j and i are integers.

Without fully restating the intersection property, we prove:

Lemma 5. The radical $N_{18} = \bigcap M'$.

Proof: Let a be an element of R such that $a \notin N_{18}$. Then there exists an element $b \in (a)$ such that $b \notin Q(b)$. Now take

all right ideals M_i, (where i runs over an index set of arbitrary cardinal number), which contain Q(b), but not b. Order them by inclusion, and consider the maximal right ideal M of this set which exists as a result of Zorn's lemma. The set M is modular; for by construction $Q(b) = \{bx-x\}$ is in M. Furthermore M is maximal in R. This follows because there is no other M_k such that

$$Q(b) \leqslant M_1 \leqslant M_2 \leqslant \cdots \leqslant M \leqslant M_k;$$

and because there exists no other proper right ideal K such that $M \subset K \subset R$.

Considering now the ideal $M' \equiv \{x \in R;\ (x) \leqslant M\}$ and the element $b \in (a)$, it is seen that since $b \notin M$, $b \notin M'$. This implies that $a \notin M'$; because assuming the contrary would result in (a) being in M' which implies that $b \in M$, leading to a contradiction.

Since $a \notin M'$, $a \notin \cap M'$. Thus we have shown that $a \notin N_{18}$, implies that $a \notin \cap M'$. This means of course that $\cap M' \leqslant N_{18}$ of R.

To show that $N_{18} \leqslant \cap M'$, let a be an element such that $a \notin \cap M'$. From this assumption, it follows that there is at least one M', such that $a \notin M'$. Also there is an element $b \in (a)$ such that $b \notin M$. (M is the modular maximal right ideal associated with M').

Since M is maximal, $R = (b)_r + M$. Also since M is modular, there exists an $e \in R$ such that $ex-x \in M$ for every $x \in R$. It follows that $e = c+m$ for some $c \in (b)_r$ and $m \in M$.

Now suppose that $a \in N_{18}$; and consider the element $c \in (b)_r$. Since $(b)_r \subseteq (a)$, $c \in (a)$; and so $c \in Q(c)$. But $c = e-m$, hence c is also in $Q(e-m)$.

By lemma 4, $Q(e-m) \subseteq Q(e) + (m)_r$, and in this case $(m)_r \subseteq M$. The set $Q(e)$ is also in M since $ex-x \in M$ for every $x \in R$. (The set $Q(e) = \{ex-x\}$ by definition). Hence $Q(e-m) \subseteq M$, which implies that $c \in M$. But $c = e-m$ and hence $e-m \in M$. This, and the fact that $m \in M$, implies that $e \in M$ which is a contradiction.

Therefore $a \notin N_{18}$; for we have proved that $a \notin \cap M'$ implies that $a \notin N_{18}$. This proves the latter part, i.e., that $N_{18} \subseteq \cap M'$.

Theorem 1.3. The radical N_{18} is a two-ideal.

Since the intersection of two-ideals is a two-ideal, the proof follows as a corollary of lemma 5.

Lastly, to illustrate a proof of the theorem for the radical defined as the union of certain two-ideals (type 3); we show that N_9, the union of all common radical ideals, is a common radical ideal.

In accord with an earlier remark,i.e., that an infinite union is the set of all finite sums, it is sufficient to prove:

Lemma 6. The sum of two common radical ideals is a common radical ideal.

Proof: Let J_1 and J_2 be two common radical ideals and consider the element $a = a_1 + a_2$, where $a_i \in J_i$, $(i = 1, 2)$.

Since J_1 is a common radical ideal there exist a finite number of elements $x_i, y_i \in R$ such that

$$\sum \lambda_i \ (x_i \circ a_1 \circ y_i) = 0,$$

where $\sum \lambda_i = 1$. It follows that

$$\begin{aligned}
\sum \lambda_i (x_i \circ a \circ y_i) &= \sum \lambda_i (x_i \circ [a_1 + a_2] \circ y_i) \\
&= \sum \lambda_i (x_i \circ a_1 + x_i \circ a_2 - x_i) \circ y_i \\
&= \sum \lambda_i (x_i \circ a_1 \circ y_i + x_i \circ a_2 \circ y_i - x_i \circ y_i) \\
&= \sum \lambda_i (x_i \circ a_1 \circ y_i) \\
&\qquad + \sum \lambda_i (x_i \circ a_2 \circ y_i - x_i \circ y_i) \\
&= \sum \lambda_i (x_i \circ a_2 \circ y_i - x_i \circ y_i).
\end{aligned}$$

Define $x_i \circ a_2 \circ y_i - x_i \circ y_i$ to be the element a_{2i}. This gives that $\sum \lambda_i (x_i \circ a \circ y_i) = \sum \lambda_i a_{2i}$, where $a_{2i} \in J_2$. Clearly

$$\begin{aligned}
a_{2i} &= (x_i + a_2 - x_i a_2) \circ y_i - x_i y_i \\
&= x_i + a_2 - x_i a_2 + y_i - x_i y_i - a_2 y_i + x_i a_2 y_i - x_i - y_i + x_i y_i \\
&= a_2 - x_i a_2 - a_2 y_i + x_i a_2 y_i.
\end{aligned}$$

Hence $\sum \lambda_i (x_i \circ a \circ y_i) \in J_2$, and since J_2 is a common radical ideal, there exist a finite number of elements $s_j, t_j \in R$ such that

$$\sum \nu_j \left\{ s_j \circ \left[\sum \lambda_i (x_i \circ a \circ y_i) \right] \circ t_j \right\} = 0,$$

where $\sum \nu_j = 1$. We now make use of the fact that $a \circ (b \circ c) = (a \circ b) = c$, that $\sum \lambda_i a_j = a_j$, and consequently that $\sum \nu_j \left\{ a_j \circ \left[\sum \lambda_i b_i \right] \right\}$ $= \sum \nu_j \sum \lambda_i (a_j \circ b_i)$. Thus, by virtue of the summation and the properties of quasi-multiplication:

$$\sum \omega_j \{ s_j \circ [\sum \lambda_i (x_i \circ a \circ y_i)] \circ t_j \}$$

$$= \sum \omega_j \sum \lambda_i \{ s_j \circ (x_i \circ a \circ y_i) \circ t_j \},$$

where $s_j \circ (x_i \circ a \circ y_i) \circ t_j$ can be multiplied in whatever manner is desirable. Hence

$$\sum \omega_j \{ s_j \circ [\sum \lambda_i (x_i \circ a \circ y_i)] \circ t_j \}$$

$$= \sum \sum \omega_j \lambda_i (s_j \circ x_i) \circ a \circ (y_i \circ t_j) = 0,$$

where $\sum \sum \omega_j \lambda_i = 1$. This implies that a is a common radical element *; and since $a = a_1 + a_2$ for arbitrary $a_i \in J_i$, $J_1 + J_2$ is a common radical ideal.

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* Indeed, having $\sum \sum \omega_j \lambda_i (s_j \circ x_i) \circ a \circ (y_i \circ t_i) = 0$, implies $\{ \sum \sum \omega_j \lambda_i (s_j \circ x_i) \circ a \circ (y_i \circ t_j) \} \equiv R \circ a \circ R = R$.

CHAPTER IV

THEOREM 2. S/N IS SEMI-SIMPLE

The two definitions of semi-simplicity, i.e., the usual one as well as the Albert and Jenner definition, resolve the proofs and the immediate results of this theorem into two different classes.

For the usual definition: that a ring or algebra S is semi-simple if its radical N is the zero ideal, two appropriate methods of proof suffice. It is of particular interest, that unlike our observations regarding the proofs of theorem 1, the fact that there are three types of definitions of the radical is of no consequence. Here, all proofs employ the property that N can always be characterized by a specified set of elements (type 2); provided it is not already defined as such.

Using either of the two methods, it is shown that b, the coset representative of any non-zero element $[b]$ in the radical $\bar{N}$ of S/N, is in the radical N of S. This implies that $[b] = [0] = \bar{N}$, which proves the theorem.

For all definitions of the radical N, except the F and p-radical the above is proved directly. One requires only the properties associated with the radical N of S (or those associated with $\bar{N}$); and it is found immediately that b, the representative of $[b] \in \bar{N}$, (by virtue of definition or characterization of N (or $\bar{N}$)), is in N of S.

For the F-radical, defined only on the basis of an arbitrary F-mapping, a slightly more general method is employed. The proof that $[b] \neq [0] \in \bar{N}$ implies $b \in N$ of S, is based on a suitable application of the second theorem on homomorphic mappings. First, as we will see, it is kept in mind that the radical N_8 of R is the intersection of all ideals $M \leqslant R$ such that R/M is subdirectly irreducible with zero F-radical. Then, the second theorem is applied to one of the difference rings R/M with the result that $R/M \cong (R/N_8)/(M/N_8)$; and that $\bar{N}_8$ is the intersection of all M/N_8 of R/N_8. By the property of M, the coset representative b, of any non-zero $[b] \in \bar{N}_8$, is in $\cap M \leq R$. Since $\cap M = N_8$, $b \in N_8$ which implies that $[b] = [0]$.

It follows that the above could be used with respect to all F-type radicals; in fact it can be applied to the proof for all radicals N which are intersections of two-ideals. In practice, however, it is used only for the F-radical itself (i.e., by necessity), and with respect to the p-radical where it is more expedient.

Regarding the exceptional definition of semi-simplicity, i.e., that a ring S is semi-simple if it is a direct sum of simple rings, we get not only a different type of proof but also a different type of theorem. In a sense, it is a form of theorem 3. Hence we leave it until the next chapter, where we discuss this theorem in general.

To illustrate some examples of the direct method, we prove the theorem for the classical and Jacobson radical (where we must first characterize in terms of a definition of type 2); and also for the G-radical of Brown and McCoy.

We begin by proving:

Theorem 2.1. A/N_1 is semi-simple.

Proof: Let $[b]$, be a non-zero element in the radical $\bar{N}_1$ of A/N_1. Since $\bar{N}_1$ coincides with the set of all properly nilpotent elements of A/N_1, there exist integers α and β such that $[b]^{\alpha} = [b^{\alpha}] = [0]$, and $([b][y])^{\beta} = [by]^{\beta} = [by^{\beta}] = [0]$ for any $[y] \in A/N_1$.

It follows that b^{α}, the coset representative of $[b^{\alpha}]$, is in N_1 of A. Furthermore, since N_1 coincides with the set of all properly nilpotent elements of A, there exists an integer γ such that $(b^{\alpha})^{\gamma} = 0$. Hence b is nilpotent.

Similarly it can be shown that the element by is nilpotent, and so b is properly nilpotent. This required that $b \in N_1$ of A, which implies that $[b] = [0]$. From this it follows that N_1 is the zero ideal in A/N_1.

The proof of this theorem shows clearly that $[b] \neq [0]$ in $\bar{N}$ of S/N leads directly to $b \in N$. The next theorem, with respect to the Jacobson radical, shows this fact even more simply.

Theorem 2.2. The difference ring R/N_6 is semi-simple.

Proof: Let $\bar{N}_6$ be the Jacobson radical of R/N_6, choose any non-zero coset $[b] \in \bar{N}_6$, and consider an element $[a]$ in the right ideal $\{[bi + bz]\}$ generated by $[b]$.

Since $[a]$ is right quasi-regular*, there exists a right quasi-inverse $[x]$ such that $[a] + [x] + [a][x] = [0]$. It follows that $a + x + ax \in N_6$ of R, and so $a + x + ax$ is right quasi-regular in R. Hence there is an element $y \in R$ such that $(a + x + ax) + y + (a + x + ax)y = 0$. Rearranging this expression, we obtain that $a + (x+y+xy) + a(x + y + xy) = 0$, which implies that a is right quasi-regular in R. Since a is the coset representative of an arbitrary element $[a] \in \{[bi + bz]\}$, b, the representative of $[b]$, is in the radical N_6. This requires that $[b] = [0] = \bar{N}_6 \leq R/N_6$.

Now turning to the proof of the theorem for the G-radical, we deviate slightly from the direct method by making a suitable application of lemma 3. This, however, merely eliminates such manipulations as were required to prove that $a + x + ax \in N_6$ implies that $b \in N_6$. We might mention that in the more recent proof of theorem 2 for the Jacobson cluster radical, Brown makes use of a similar type lemma, (lemma 4), in the same way as is done below.

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* As we stated earlier, $N_6 = \{b \in R$; for every $a \in \{bi + bz\}$, $a \in \{ax - x\}\}$.

Theorem 2.3. The difference ring R/N_7 is semi-simple.

Proof: Let $\bar{N}_7$ be the G-radical of R/N_7, and let $[b] \neq [0]$ be an element of $\bar{N}_7$. Consider also an element $a \in (b)$.

Since $R \longrightarrow R/N_7$ under the natural homomorphism, $b \longrightarrow [b]$. Consequently $(b) \longrightarrow ([b])$ and so $a \longrightarrow [a] \in ([b])$. The element $[a]$ is G-regular, for it is in the radical $\bar{N}_7$ of R/N_7. It follows that

$$[a] \in G([a]) = \left\{ [a][r] - [r] + \sum ([x_i][a][y_i] - [x_i][y_i]) \right\}$$

and hence

$$[a] = [a][r] - [r] + \sum ([x_i][a][y_i] - [x_i][y_i])$$

for certain cosets $[x_i]$, $[y_i]$, $[r] \in R/N_7$.

However, the element $[a][r] - [r] + \sum ([x_i][a][y_i] - [x_i][y_i])$ has other representatives than $a \in R$. In particular it has a representative c which is in G(a). Consequently $[a] = [c]$, and so $[a] - [c] = [0]$. This implies that $a - c \in N_7$ of R, and hence a-c is G-regular.

By lemma 3, $a \in G(a)$; and since we assumed that $a \in (b)$, $b \in N_7$ of R.

It follows that $[b] = [0]$, and so $\bar{N}_7 = ([0])$.

Next, we consider the proof of the theorem with respect to the F-radical. As we mentioned earlier, it is based on a suitable application of the second theorem an homomorphic mappings. In this particular case we apply the second theorem to one of the set of difference rings R/M,

where N_8 is the intersection of all M such that R/M is subdirectly irreducible with zero F-radical.

Theorem 2.4. The difference ring R/N_8 is semi-simple.*

Proof: Let $\bar{N}_8$ be the F-radical of R/N_8 and consider an element $[b] \neq [0] \in \bar{N}_8$. Let M be any ideal in R such that R/M is subdirectly irreducible with zero F-radical.

By the second theorem an homomorphic mappings, $R/M \cong (R/N_8)/(M/N_8)$, and so $(R/N_8)/(M/N_8)$ is a subdirectly irreducible ring with zero F-radical. From the intersection characterization of the F-radical it follows that $\bar{N}_8$ is the intersection of all M/N_8. Hence [b] is in every quotient M/N_8, which means that b is in every M of R. But N_8 of R coincides with the intersection of all M, thus $b \in N_8$. This implies that $[b] = [0]$, contradicting our original assumption that $\bar{N}_8 \neq ([0])$.

In concluding this section we illustrate a completely different kind of proof, i.e., that of theorem 2 with respect to the Levitzki radical N_5. The method employed is somewhat like the direct method, although one does not make any specific reference to an arbitrary element of the radical $\bar{N}_5$ of R/N_5.

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* It should be noted that the concept of semi-simplicity is not truly generalized when used with respect to a ring with zero F-radical. For conformity, however, we feel that the use of the term semi-simple is justified.

We prove directly:

Theorem 2.5. The difference ring R/N_5 has zero N_5-radical.

Proof: First we show that if $G \supseteq N_5$ is a semi-regular right ideal, then G/N_5 is semi-regular in R/N_5.*

Assume that $G \supseteq N_5$ of R is a semi-regular right ideal. By definition of G it follows that there exist a finite number of elements $r_1, r_2, \cdots r_n \in G$ such that the ring T, generated by these elements, is potent.

Consider the quotient G/N_5 and assume it is semi-nilpotent in R/N_5.** Consider also the set $\bar{T} \equiv (T + N_5)/N_5$ which is nilpotent. This follows, because

$$(T + N_5)/N_5 = \{[t + n]\} = \{[t] + [n]\},$$

(where $[n] = [0]$) and so $\bar{T} = \{[t]\}$; and also because $T \subseteq G$ which implies $\bar{T} \subseteq G/N_5$, (where G/N_5 is assumed to be semi-nilpotent).

Now let λ be the integer such that $\bar{T}^{\lambda} = [0]$, and denote the finite set of products $r_{i_1} \cdot r_{i_2} \cdots r_{i_\lambda}$ by $u_1, u_2, \cdots u_m$. Generate the ring $U = \{u_1, u_2, \cdots u_m\}$.

Since T is potent, $T^{\lambda} \neq 0$ for any λ; and since $T^{\sigma\lambda+1} \subseteq U^{\sigma} \cdot T$,*** $U^{\sigma} \neq 0$ for any integer σ.

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* A left, right or two-ideal which is not semi-nilpotent is called semi-regular.

** This is equivalent to assuming that the radical N_5 of R/N_5 is non-zero.

*** A simple calculation shows that T is not necessarily in U.

On the other hand, by the definition of U, $U \leq T^{\lambda}$; and since $T^{\lambda} \leq N_5$, (because $\overline{T}^{\lambda} = [0]$), $U \leq N_5$. But this is a contradiction, and so G/N_5 is semi-regular.

Lastly, since G ranges between R and N_5, the quotient G/N_5 ranges between $[0]$ and R/N_5. Therefore there are no semi-nilpotent ideals in R/N_5. The conclusion, is that the radical $\overline{N}_5$ of R/N_5 is zero.

CHAPTER V

THEOREM 3. A RING S IS SEMI SIMPLE IF AND ONLY IF S IS ISOMORPHIC TO A SUBDIRECT SUM OF RINGS S_i, THE S_i's SUITABLY GENERALIZING SIMPLE RINGS

By allowing this theorem to include the two exceptional forms of theorem 2, i.e., those for difference rings which are semi-simple if they are finite direct sums, we find three general methods of proof which have been developed.

The first to be illustrated depends on certain results of the theory of idempotents, and we refer to it as the original or classical method. It is used with respect to the theorem for both forms of the classical radical, as well as for N_{14} and N_{17}. In general, the idempotents of a semi-simple ring S are used to decompose S into a finite direct sum of simple rings. Finiteness on the dimension, or minimum condition, is required in S.

For arbitrary rings, narings, and clusters, said to be semi-simple with respect to an F-type radical, a newer and more powerful method is applied. The latter simply makes use of a fundamental theorem on subdirect sums, and the fact that the radical N is the intersection of certain two-ideals.

The fundamental theorem, which is of considerable interest itself, states the following:

A ring (naring or cluster) R^* is isomorphic to a subdirect sum of rings R_i if and only if, for each i, R contains a two-ideal L_i such that $R/L_i \cong R_i$ and the intersection of the L_i's is zero.

To illustrate its present application, consider a ring R with zero F-type radical. From the intersection property, (in terms of <u>two-ideals</u>), the radical N of R coincides with the intersection of certain ideals M_i of R, such that $\cap M_i = 0$. By taking the difference R/M_i, which suitably generalizes a simple ring, and associating this quotient with R/L_i above, one immediately provides the proof for a particular form of the theorem in question.

For the converse, it is shown that a subdirect sum of rings R_i, each with zero radical, is isomorphic to a ring R with zero radical. The proof of this statement follows readily because: (1) From the fundamental theorem on subdirect sums, a ring R which is isomorphic to a subdirect sum of rings R_i contains ideals B_i such that $R/B_i \cong R_i$, and $\cap B_i = 0$. (2) Under the natural homomorphism of $R \longrightarrow \bar{R}$, the F-type radical N of R (under consideration) maps into the F-type radical N_i of R_i for each i. Since $R/B_i \cong R_i$, $N \subseteq B_i$; and since $\cap B_i = 0$, $\cap B_i = N = 0$ in R.

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* Throughout this particular discussion, we imply ring, naring, and cluster R, when we write ring R.

Since F-regularity includes Q-regularity (i.e., quasi-regularity), G-regularity and F_1-regularity, this particular method provides a form of theorem 3 for all of the remaining ring radicals, except the p-radical. Actually, this method is applicable to every radical N which is the intersection of certain two-ideals, and which under a given homomorphism, $R \longrightarrow \bar{R}$, maps onto $\bar{N}$ of $\bar{R}$.

Lastly, we remark on the third method, i.e., the one by which Albert and Jenner provide a structure for non-associative systems. Recalling what we stated earlier: both authors define a ring or algebra to be semi-simple if it is a finite direct sum of simple rings or algebras. They show, under certain hypothesis, that there exists a minimal ideal $N \leq S$ such that S/N is semi-simple in their particular sense.*

In essence, the above results by virtue of certain existing relationships between properties of a naring (algebra) S and its associative transformation ring (algebra) T(S). Having found the desired relationships, one is led not only to the conditions under which it is possible to construct a radical, but to the construction itself.

To illustrate the classical method, we prove

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* Note that in this way, Jenner actually provides a finite decomposition for the naring R/N_{12}, instead of the usual subdirect sum.

Theorem 3.1. A ring R which satisfies the d.c.c., and which is semi-simple with respect to the classical radical N_2, is a direct sum of simple rings.

Proof: Let C be the center of R, that is let $C = \{x \in R;$ $xy = yx$ for every $y \in R\}$. Consider the set of all idempotents in C. Denote these by $e_{\alpha 1}, e_{\alpha 2}, \cdots e_{\alpha n}$, and assume that $e_1, e_2, \cdots e_n$, are all idempotents in C such that Re_i is simple for each e_i.

Consider all two-ideals of the form

$$R(1 - \sum_{i=1}^{n} e_i).$$

Because R satisfies the d.c.c., among these is a minimal ideal. Let this minimal ideal be R(1-e) where $e = \sum_{i=1}^{n} e_i$. Since $R = R1 = R(1-e+e)$, it follows that $R = R(1-e) + Re$, from which $R = R(1-e) + Re_1 + Re_2 + \cdots + Re_n$. Each Re_i in the last expression is simple. Furthermore $Re_1 + Re_2 + \cdots + Re_n$ is a direct sum. To show this, let $B = Re_1 + Re_2 + \cdots + Re_n$, and consider $0 \in B$ which is of the form $0 = r_1e_1 + r_2e_2 + \cdots + r_ne_n$ for certain $r_i \in R$. Since $Re_i \cdot Re_j = 0$,* $Re_i \cdot e_j = 0$, which implies that $re_i \cdot e_j = 0$ for all $r \in R$. Hence $0 \cdot e_1 = r_1e_1e_1 + r_2e_2e_1 + \cdots + r_ne_ne_1$ i.e., $0 \cdot e_1 = r_1e_1 + 0 + \cdots + 0$, and so $r_1e_1 = 0$.

Similarly $r_ie_i = 0$, and hence B is a direct sum.

Consequently, rewriting R, it is now of the form

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* If $i \neq j$, the set Re_i is distinct from Re_j since both are simple ideals. Furthermore $Re_i \cdot Re_j$ is contained in R_i and R_j, which implies that if $Re_i \cdot Re_j \neq 0$, then $Re_i = Re_j$. This is a contradiction. Hence $Re_i \cdot Re_j = 0$.

$$R = R(1-e) \dotplus Re_1 \dotplus Re_2 \dotplus \cdots \dotplus Re_n$$

1. If $R(1-e) = 0$, then the theorem is proved.

2. If $R(1-e) \neq 0$, then it contains a non-zero simple-ideal because R satisfies the d.c.c. Call this ideal Re_{n+1}, and let $e' = e+e_{n+1}$. It follows that

$$\begin{aligned} R(1-e) &= Re_{n+1} + R(1-e-e_{n+1}) \\ &= Re_{n+1} + R(1-e'). \end{aligned}$$

This implies that $R(1-e') \subsetneq R(1-e)$ which is, however, a contradiction since $R(1-e)$ is a minimal ideal of that form. Hence $R(1-e)$ is always zero, and

$$R = Re_1 \dotplus Re_2 \dotplus \cdots \dotplus Re_n.$$

For an example of the second method which we described, we prove theorem 3 for a ring which is semi-simple with respect to the F-radical itself.

Keeping in mind the two pertinent results on subdirect sums, and also that the F-radical of a ring R is the intersection of all two-ideals $M \subsetneq R$ such that R/M is subdirectly irreducible with zero F-radical, we prove:

Theorem 3.2. A ring R has zero F-radical if and only if R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i with zero F-radical.

Proof: Assume that R is a ring with zero F-radical. Then there exist ideals $M_i \subsetneq R$ such that R/M_i is subdirectly

irreducible with zero F-radical and $\bigcap M_i = 0$. Consider all R/M_i and apply the fundamental theorem on subdirect sums. It follows that R is a subdirect sum of subdirectly irreducible rings R/M_i each with zero F-radical.

Conversely, assume that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i with zero F-radical. The second result, previously described, implies R is a ring with zero F-radical.

Finally, as an example of the Albert and Jenner method, we prove (the Jenner theorem): if T(R) of a naring R satisfies the d.c.c. and there exists an ideal $H \leq R$ such that R/H is semi-simple,* then there exists a unique set $M \subset R$ (the radical N_{12}) such that R/M is semi-simple. We proceed by presenting a series of lemmas which will ultimately lead us to the proof of this theorem.

The first of these, fundamental to the present theory states the following:

Lemma 7. If R is a naring without absolute divisors of zero and if T(R) satisfies the d.c.c., then R is a finite direct sum of simple rings $R_1 \dot{+} R_2 \dot{+} \cdots \dot{+} R_n$ if and only if $T(R) = T(R_1) \dot{+} T(R_2) \dot{+} \cdots \dot{+} T(R_n)$, such that the $T(R_i)$ are simple.

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* Semi-simplicity is now regarded in the Jenner sense.

The next three lemmas, which are required, are taken from the theory of rings and their linear transformation spaces.

Lemma 8. If R is a naring, then for any two-ideal $H \subsetneq R$, $S = \{s \in T(R);\ R^S \subseteq H\}$ is a two-ideal in T(R).

Proof: 1) $R^{T(R)S} \subsetneq H$, and

2) $R^{S \cdot T(R)} \subsetneq H$.

Both statements follow readily, since

$$R^{TS} \subseteq R^S \subsetneq H,^* \text{ and}$$
$$R^{ST} \subseteq H^T \subseteq H.$$

Lemma 9. Right and left multiplications by a fixed element in the naring R/H, define a unique residue class in T/S.

Proof: Denote the elements of T/S by $[A]$, $[B]$, $[C]$, $[D]$, and assume that $[U]$ and $[V]$ are right multiplications; consider $[x][U]$ and $[x][V]$ of R/H. If $[x][U] = [x][V]$ for every $[x] \in R/H$, then it is clear that $x^{(U-V)} \in H$. From the definition of S, this means that $U-V \in S$, and so $[U] = [V] \in T/S$. The same argument would suffice for left multiplications.

However, if $[x][U] \neq [x][V]$ for some $[x] \in R/H$, it follows that $[U] \neq [V] \in T/S$. The same applies for left multiplications.

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* Henceforth we abbreviate T(R) by T.

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Lemma 10. $T/S \cong T(R/H)$.

Proof: By lemma 9 each multiplication by a fixed element in R/H, determines a coset in T/S. Consequently each element of T(R/H) (which is simply the set of all representatives generated by right and left multiplications in R/H), is isomorphic to a unique element in T/S. Clearly this completes the proof.

At this point, we consider certain properties of a set M'_j which will be used directly in defining the radical N_{12}.

Lemma 11. The set $M'_{j+1} \equiv \{w \in R;\ wd,\ dw \in M'_j$ for all $d \in R\}$* consists of all $w \in R$ such that the right and left multiplications R_w, L_w, of T are left annihilators of

$$T^j = T \cdot T \cdot \cdots \cdot T,\ j \text{ times}.$$

Lemma 12. If T satisfies the d.c.c. then there exists an integer k such that $M'_k = M'_{k+1}$ (i.e., the sequence M'_i terminates producing a.c.c. on M'_i).

Lemma 13. If T satisfies the d.c.c., then for the terminating set M'_k (of the last lemma) the difference ring R/M'_k has no absolute divisors of zero. Furthermore M'_k is contained in every ideal H for which R/H has no absolute divisors of zero.

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* $M'_1 = \{x \in R;\ xa = ax = 0$ for all $a \in R\}$, plus 0.
$M'_2 = \{y \in R;\ yb, by \in M'_1$ for all $b \in R\}$.
$M'_j = \{z \in R;\ zc, cz \in M'_{j-1}$ for all $c \in R\}$.

Proof: Let the elements of R/M'_k be denoted by $[r]$, $[a]$, etc., and assume that $[r][a] = [0] \in R/M'_k$. This assumption implies that $ra \in M'_k$. Similarly assuming that $[a][r] = [0] \in R/M'_k$ results in a.r also being in M'_k. But by the definition of M'_i, these statements put $r \in M'_{k+1}$. However, $M'_{k+1} = M'_k$, and so r is also in M'_k. This result defines $[r]$ to be the zero coset in R/M'_k, which with the assumption that $[r][a] = [0] = [a][r]$, implies that there are no divisors of zero in R/M'_k. (i.e., if $[x][y] = [0] = [x][y] \in R/M'_k$ then either $[x]$ or $[y] = [0]$).

To prove the second part, consider any ideal $H \subsetneq R$, such that R/H has no absolute divisors of zero. Clearly M'_k, the set of all zero divisors, is in H; for all elements of H define the zero coset in R/H. Now by induction assume that $M'_i \subsetneq H$. To show that M'_{i+1} is also in H, consider any element $x \in M'_{i+1}$. The elements x.a and a.x are then in M'_i for all $a \in R$. But M'_i is in H, and so x.a and a.x are also in H. In R/H this means that $[x][a] = [0] = [a][x]$. But here there are no absolute divisors of zero, and so $[x] = [0]$. It follows that $x \in H$, and therefore $M'_{i+1} \subsetneq H$, which completes the proof.

Lemma 14. If T satisfies the d.c.c. and there exists an ideal $H \subsetneq R$ such that R/H is semi-simple (in Jenner's sense), then $R^N \subsetneq H$ where N is the classical radical of R.

Proof: Consider $T(R/H)$, and the set T/S, where

$$S \equiv \{s \in T(R);\ R^S \leq H\}.$$

By lemma 10, $T(R/H) \cong T/S$; but furthermore under the present assumptions T/S is (classically) semi-simple by lemma 7. As a result, N the classical radical of R is in S, and so $R^N \leq R^S \leq H$.

Now under the same assumptions, we present (as is done in Jenner's paper) the arguments leading to the formulation of theorem 3.

Consider the ideal M_k for the difference ring R/R^N. By lemma 12, M_k exists for R/R^N, since $T(R/R^N)$ satisfies the d.c.c. (The latter assertion results from the fact that R/H is assumed semi-simple and $R^N \leq H$).

Secondly, let M denote the complete reciprocal image of M_k under the natural homomorphism $R \longrightarrow R/R^N$. The set M is a two-ideal in R, since M_k is a two-ideal in R/R^N.

Examining the difference ring R/M, and the transformation ring $T(R/M)$, the following facts suffice:

1. By the second theorem on homomorphic mappings $R/M \cong (R/R^N)/M_k$.

2. R/M has no absolute divisors of zero since by lemma 13 $(R/R^N)/M_k$ is devoid of these elements.

3. By lemma 10, $T(R/M) \cong T/S$ where $S \equiv \{s \in T;\ R^S \leq M\}$. (Henceforth S will be thought to be only in this form).

Lastly, consider the classical radical N, the set S, and T/S. Since $R^N \subseteq R$ defines the zero coset in R/R^N and since this coset is in M_k of R/R^N, the set of all pre-images of M_k must contain R^N. Hence both R^S, and R^N are in M. But by lemma 9, $T(R/M) \cong T/S$, and therefore $N \subseteq S$. From this, the final inference is that T/S is classically semi-simple.

Combining these last statements with statement 3 it follows that R/M has the classically semi-simple transformation ring T/S. The latter is a finite direct sum of simple rings $T(R_i)$.

All these results together with lemma 7 imply:

Theorem 3.3. If T(R) of a naring R satisfies the d.c.c. and there exists an ideal $H \subsetneq R$ such that R/H is semi-simple, then there exists a set $M \subsetneq R$ (the radical N_{12}) such that R/M is semi-simple.

Summary

1. The assumption that T(R) satisfies the d.c.c. allows one to talk about the classical radical N - from which it is ultimately shown that $T(R/M) \cong$ direct sum of simple transformation rings.

2. The assumption that R/H is semi-simple gives rise to the existence of M_k from which (after defining M) it is established that

1) R/M has no divisors of zero;
2) T(R/M) satisfies the d.c.c. (since T(R) does); and
3) that R/M is a direct sum of simple rings.

Note: M is actually the minimal ideal having this property. Note further that theorem 3.3. also proves theorem 1 for N_{12}.

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