Uniform Asymptotic Expansions of the Jacobi Functions and the Jacobi Polynomials

by

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QINGQING WANG

A thesis submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

MASTER OF SCIENCE

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Abstract

We examine the asymptotic behaviors of the Jacobi function

$$\varphi_\mu^{(\alpha,\beta)}(t) = {}_2F_1[\frac{1}{2}(\alpha+\beta+1-i\mu), \frac{1}{2}(\alpha+\beta+1+i\mu); \alpha+1; -\sinh^2 t],$$

and its zeros $t_{\mu,s}, s = 1, 2, ...,$ for large value of μ. We develop asymptotic expansions of $\varphi_\mu^{(\alpha,\beta)}(t)$ as $\mu \to \infty$, which are uniform with respect to t in $(0, \infty)$, and obtain asymptotic approximations for the zeros $t_{\mu,s}$ of $\varphi_\mu^{(\alpha,\beta)}(t)$, uniformly with respect to s. Bounds for the error terms in the expansions for both $\varphi_\mu^{(\alpha,\beta)}(t)$ and $t_{\mu,s}$ are given.

Similar results for the Jacobi polynomial are also obtained.

Our method involves the use of Olver's differential equations theory [11, Chap.12] and the Theorem stated in [6]. The Theorem is applied to find error bounds for the asymptotic approximations of zeros of $\varphi_\mu^{(\alpha,\beta)}(t)$ and $P_n^{(\alpha,\beta)}(\cos x)$.

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Chapter 1

Introduction

Let α, β and μ be real numbers with $\mu > 0$ and $\alpha \neq -1, -2, \ldots$. The Jacobi function is defined by

$$\varphi_\mu^{(\alpha,\beta)}(t) = {}_2F_1[\frac{1}{2}(\alpha+\beta+1-i\mu), \frac{1}{2}(\alpha+\beta+1+i\mu); \alpha+1; -\sinh^2 t] \qquad (1.1)$$

for $t > 0$, where ${}_2F_1(a, b; c; z)$ is the Gaussian hypergeometric function. This function plays an important role in the interactions of special functions and group theory, and for an excellent survey of this topic, we refer to Koornwinder [7].

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can also be expressed in terms of the hypergeometric function [16, p.63]

$$\frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(\alpha+n+1)} P_n^{(\alpha,\beta)}(x) = {}_2F_1[-n, n+\alpha+\beta+1; \alpha+1; \frac{1}{2}(1-x)], \qquad (1.2)$$

and this formula furnishes the extension of the polynomial $P_n^{(\alpha,\beta)}(x)$ to arbitrary values of the degree n. From (1.1) and (1.2), it is evident that

$$\varphi_\mu^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+1)\Gamma(\frac{1}{2}(i\mu-\alpha-\beta+1))}{\Gamma(\frac{1}{2}(i\mu+\alpha-\beta+1))} P^{(\alpha,\beta)}_{\frac{1}{2}(i\mu-\alpha-\beta-1)}(\cosh 2t), \qquad (1.3)$$

and for this reason, $\varphi_\mu^{(\alpha,\beta)}(t)$ is called the Jacobi function. This function is also related to the Legendre function [11, p.170, (12.05)]

$$P_n^{-m}(z) = 2^{-m}(z-1)^{m/2}(z+1)^{m/2}\ {}_2F_1(m-n, m+n+1;$$

$$m+1; \frac{1}{2} - \frac{1}{2}z), \tag{1.4}$$

and the relationship is

$$(\sinh t)^\alpha (\cosh t)^\alpha \varphi_{2\mu}^{(\alpha,\alpha)}(t) = \Gamma(\alpha+1) P^{-\alpha}_{-\frac{1}{2}+i\mu}(\cosh 2t). \tag{1.5}$$

Recently asymptotic approximations, complete with error bounds , have been obtained for the Jacobi polynomial $P_n^{(\alpha,\beta)}(\cos\theta)$ as $n \to \infty$, which are uniformly valid for $0 \le \theta \le \pi/2$, see [4] and [2]. Corresponding results for the Legendre function $P_n^{-m}(\cosh z)$ as $n \to +\infty$, which are uniformly valid for $0 < z < \infty$, can also be found in [11, p.466] and [14]. The purpose of this paper is to present similar results for the Jacobi function $\varphi_\mu^{(\alpha,\beta)}(t)$ as $\mu \to +\infty$, which is uniform with respect to t in $(0,\infty)$, and to use these results to obtain asymptotic approximations for the zeros $t_{\mu,k}$ of this function as $\mu \to \infty$, uniformly with respect to k. Asymptotic expansions of $\varphi_\mu^{(\alpha,\beta)}(t)$ have been given previously by Triméche [17] and Fitouhi & Hamza [3], using differential equation theory. Based on an integral representation of the Jacobi function, Schindler [13] (in the case when $\alpha = \beta$) and Stanton and Tomas [15] have derived asymptotic expansions of $\varphi_\mu^{(\alpha,\beta)}(t)$ as $t \to 0$, which is, in a sense, uniform with respect to μt. However, no error bounds were constructed for these expansions, and no study of the asymptotic behavior of the zeros $t_{\mu,k}$ was made in these investigations. In the present paper, we shall also use the differential equation

approach, and our work relies heavily on the results of Olver [11, Chap.12]. For completeness, we also include an asymptotic approximation of $\varphi_{\mu}^{(\alpha,\beta)}(t)$ as $t \to \infty$, which is uniform in $\mu \geq \delta > 0$, and a corresponding result for $t_{\mu,k}$ as $k \to \infty$. The problems studied in this paper have been suggested by R.A.Askey.

The leading term of the expansion of Jacobi polynomial, as $n \to \infty$, is provided by the well-known formula of "Hilb's type" given by Szegö [16, p.197]. Recently further results are reported in Frenzen & Wong [4] and Baratella & Gatteschi [2]. In [4], an expansion of $P_n^{(\alpha,\beta)}(\cos x)$ has been derived in terms of the Bessel functions $J_{\alpha+l}, l = 0, 1, 2, \ldots$. Consequently , a two-term representation for zeros of $P_n^{(\alpha,\beta)}(\cos x)$, with the O-term error, is obtained. Based on a different comparison equation, Baratella and Gatteschi [2] later derived a two-term approximation of zeros with a very sharp error bound. Unfortunatly, the parameters α and β in their result are restricted to a small interval of $[-1/2, 1/2]$. In Chapter 8 we shall establish the first and the second approximations of zeros of $P_n^{(\alpha,\beta)}(\cos x)$. In our results, error terms of the second approximations of zeros at least have the same order as that given by a general formula due to Olver [11, p.453], and α, β are real and $\alpha \geq 0$.

Chapter 2

Differential Equations

It is known that the Jacobi function $\varphi_\mu^{(\alpha,\beta)}(t)$ is the unique even C^∞-function on $\mathbf{R}$ which satisfies

$$v''(t) + [(2\alpha+1)\coth t + (2\beta+1)\tanh t]v'(t) + [\mu^2 + (\alpha+\beta+1)^2]v(t) = 0 \tag{2.1}$$

and $v(0) = 1$; see[7, p.2]. If we set

$$u(t) = (\sinh t)^{\alpha+\frac{1}{2}}(\cosh t)^{\beta+\frac{1}{2}}\varphi_\mu^{(\alpha,\beta)}(t), \tag{2.2}$$

then it is easily verified that

$$u''(t) + \{\mu^2 + \frac{\frac{1}{4}-\alpha^2}{\sinh^2 t} - \frac{\frac{1}{4}-\beta^2}{\cosh^2 t}\}u(t) = 0. \tag{2.3}$$

When $\alpha > -1/2$, we also have $u(0) = 0$.

To apply the asymptotic theory of Olver[11, pp.438-440], we shall restrict ourselves to the case $\alpha \geq 0$, and introduce the new variables

$$\begin{aligned} &(-\zeta)^{\frac{1}{2}} = t, && \zeta < 0, \\ &W(\zeta) = (-\zeta)^{\frac{1}{4}}u(t). \end{aligned} \tag{2.4}$$

The transformed equation is given by

$$\frac{d^2W}{d\zeta^2} = \{\frac{\mu^2}{4\zeta} + \frac{\alpha^2-1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta}\}W(\zeta), \quad \zeta < 0, \tag{2.5}$$

where

$$\psi(\zeta) = \frac{1}{4}\{\frac{\frac{1}{4}-\alpha^2}{\zeta} + [\frac{\frac{1}{4}-\alpha^2}{\sinh^2(-\zeta)^{\frac{1}{2}}} - \frac{\frac{1}{4}-\beta^2}{\cosh^2(-\zeta)^{\frac{1}{2}}}]\}. \tag{2.6}$$

Note that $\psi(\zeta)$ is analytic at $\zeta = 0$. Furthermore, if

$$\hat{\psi}(\zeta) \equiv \frac{\alpha^2-1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta}, \tag{2.7}$$

then

$$\lim_{\zeta \to 0^-} \zeta^2 \hat{\psi}(\zeta) = \frac{\alpha^2-1}{4}. \tag{2.8}$$

Chapter 3

Asymptotic Expansion

In [11, Chap.12,§§2-5], Olver has shown that asymptotic solutions of (2.5) are expressed in terms of the Bessel functions $J_\alpha(x)$ and $Y_\alpha(x)$. To state his result more precisely, we first recall the modulus function $M_\alpha(x)$ and the weight function $E_\alpha(x)$ defined by

$$M_\alpha(x) = \begin{cases} [2|Y_\alpha(x)J_\alpha(x)|]^{\frac{1}{2}} & (0 < x < X_\alpha) \\ [J_\alpha^2(x) + Y_\alpha^2(x)]^{\frac{1}{2}} & (x \geq X_\alpha), \end{cases} \tag{3.1}$$

$$E_\alpha(x) = \begin{cases} [-Y_\alpha(x)/J_\alpha(x)]^{\frac{1}{2}} & (0 < x < X_\alpha) \\ 1 & (x \geq X_\alpha), \end{cases} \tag{3.2}$$

where X_α denotes the smallest positive root of the equation $J_\alpha(x) + Y_\alpha(x) = 0$. We also need the following constants associated with these functions:

$$\begin{aligned} \lambda_2(\alpha) &= \sup\{\pi x M_\alpha^2(x)\}, \\ \lambda_3(\alpha) &= \sup\{\pi x |J_\alpha(x)| E_\alpha(x) M_\alpha(x)\}, \\ \lambda_4(\alpha) &= \sup\{\pi x |Y_\alpha(x)| E_\alpha^{-1}(x) M_\alpha(x)\}, \end{aligned} \tag{3.3}$$

each supremum being taken over the interval $(0, \infty)$.

Theorem 4.1 in [11, p.444] then states that for negative ζ, equation (2.5) has solutions $W_1(\mu,\zeta)$ and $W_2(\mu,\zeta)$ given by

$$W_1(\mu,\zeta) = (-\zeta)^{\frac{1}{2}} J_\alpha(\mu\sqrt{-\zeta}) \sum_{s=0}^{n} \frac{A_s(\zeta)}{\mu^{2s}} - \frac{(-\zeta)}{\mu} J_{\alpha+1}(\mu\sqrt{-\zeta}) \sum_{s=0}^{n-1} \frac{B_s(\zeta)}{\mu^{2s}} + \varepsilon_{2n+1,1}(\mu,\zeta), \tag{3.4}$$

$$W_2(\mu,\zeta) = (-\zeta)^{\frac{1}{2}} Y_\alpha(\mu\sqrt{-\zeta}) \sum_{s=0}^{n} \frac{A_s(\zeta)}{\mu^{2s}} - \frac{(-\zeta)}{\mu} Y_{\alpha+1}(\mu\sqrt{-\zeta}) \sum_{s=0}^{n-1} \frac{B_s(\zeta)}{\mu^{2s}} + \varepsilon_{2n+1,2}(\mu,\zeta), \tag{3.5}$$

where

$$|\varepsilon_{2n+1,1}(\mu,\zeta)| \le \lambda_3(\alpha)(-\zeta)^{\frac{1}{2}} E_\alpha^{-1}(\mu\sqrt{-\zeta}) M_\alpha(\mu\sqrt{-\zeta}) \times \exp\{\frac{\lambda_2(\alpha)}{\mu} \mathcal{V}_{\zeta,0}(\sqrt{-\zeta}B_0)\} \frac{\mathcal{V}_{\zeta,0}(\sqrt{-\zeta}B_n)}{\mu^{2n+1}}, \tag{3.6}$$

$$|\varepsilon_{2n+1,2}(\mu,\zeta)| \le \lambda_4(\alpha)(-\zeta)^{\frac{1}{2}} E_\alpha^{-1}(\mu\sqrt{-\zeta}) M_\alpha(\mu\sqrt{-\zeta}) \times \exp\{\frac{\lambda_2(\alpha)}{\mu} \mathcal{V}_{-\infty,\zeta}(\sqrt{-\zeta}B_0)\} \frac{\mathcal{V}_{-\infty,\zeta}(\sqrt{-\zeta}B_n)}{\mu^{2n+1}}, \tag{3.7}$$

$E_\alpha^{-1}(x) = 1/E_\alpha(x)$ and $\mathcal{V}_{a,b}(f)$ denotes the total variation of a function $f(x)$ on an interval (a,b). The coefficients $A_s(\zeta)$ and $B_s(\zeta)$ are analytic functions in a region

containing $\zeta = 0$, and are determined by the recursive formulas

$$B_s(\zeta) = -A_s'(\zeta) + \frac{1}{(-\zeta)^{\frac{1}{2}}} \int_\zeta^0 \{\psi(v)A_s(v) - (\alpha + \frac{1}{2})A_s'(v)\} \frac{dv}{(-v)^{\frac{1}{2}}}, \tag{3.8}$$

$$A_{s+1}(\zeta) = \alpha B_s(\zeta) - \zeta B_s'(\zeta) + \int \psi(\zeta) B_s(\zeta) d\zeta, \tag{3.9}$$

with $A_0(\zeta) = 1$.

We must now identify the function $(-\zeta)^{1/4}u(t)$ in (2.4) with one of the above two solutions or a linear combination of them. From (2.2) and (2.4), we have

$$(-\zeta)^{\frac{1}{4}}u(t) \sim (-\zeta)^{\frac{(\alpha+1)}{2}}, \tag{3.10}$$

as $\zeta \to 0^-$. Since $J_\alpha(x) \sim (x/2)^\alpha/\Gamma(\alpha+1)$ for x near 0, it follows from (3.4) and (3.6) that

$$W_1(\mu, \zeta) = \frac{\mu^\alpha}{2^\alpha \Gamma(\alpha+1)} (-\zeta)^{\frac{(\alpha+1)}{2}} \{\sum_{s=0}^{n} \frac{A_s(0)}{\mu^{2s}} + o(1)\}, \tag{3.11}$$

as $\zeta \to 0^-$. The content of the brackets reduces to $1 + o(1)$ if the arbitrary constants in the relation (3.9) are chosen so that

$$A_s(0) = 0, \qquad s = 1, 2, \ldots \quad . \tag{3.12}$$

Comparing (3.10) and (3.11) gives

$$(-\zeta)^{\frac{1}{4}}u(t) = \frac{2^\alpha \Gamma(\alpha+1)}{\mu^\alpha} W_1(\mu, \zeta);$$

consequently

$$(\sinh t)^{\alpha+\frac{1}{2}} (\cosh t)^{\beta+\frac{1}{2}} \varphi_\mu^{(\alpha,\beta)}(t)$$

$$= \frac{2^\alpha \Gamma(\alpha+1)}{\mu^\alpha} t^{-\frac{1}{2}} \{ t J_\alpha(\mu t) \sum_{s=0}^{n} \frac{A_s(-t^2)}{\mu^{2s}} \tag{3.13}$$
$$- \frac{t^2}{\mu} J_{\alpha+1}(\mu t) \sum_{s=0}^{n-1} \frac{B_s(-t^2)}{\mu^{2s}} + \varepsilon_{2n+1,1}(\mu, -t^2) \}$$

for $t > 0$. Note that for each n, expansion (3.13) gives (2n+1) that many terms, i.e., we always have an odd number of terms. By using the same argument as in [11, Chap.12], it is possible to derive an expansion which yields an even number of terms (see Appendix A), and the result is given by

$$(\sinh t)^{\alpha+\frac{1}{2}} (\cosh t)^{\beta+\frac{1}{2}} \varphi_\mu^{(\alpha,\beta)}(t)$$
$$= \frac{2^\alpha \Gamma(\alpha+1)}{\mu^\alpha} t^{-\frac{1}{2}} \{ t J_\alpha(\mu t) \sum_{s=0}^{n-1} \frac{A_s(-t^2)}{\mu^{2s}} \tag{3.14}$$
$$- \frac{t^2}{\mu} J_{\alpha+1}(\mu t) \sum_{s=0}^{n-1} \frac{B_s(-t^2)}{\mu^{2s}} + \varepsilon_{2n,1}(\mu, -t^2) \},$$

where

$$|\varepsilon_{2n,1}(\mu, -t^2)| \le \bar{\lambda}_3(\alpha)\, t\, E_\alpha^{-1}(\mu t) M_\alpha(\mu t)$$
$$\times \exp\{ \frac{\lambda_2(\alpha)}{\mu} \mathcal{V}_{0,t}(tB_0) \} \frac{\mathcal{V}_{0,t}(A_n)}{\mu^{2n}}, \tag{3.15}$$

and

$$\bar{\lambda}_3(\alpha) = \sup_{x \in (0,\infty)} \{ \pi x E_\alpha(x) M_\alpha(x) |J_{\alpha+1}(x)| \}. \tag{3.16}$$

The leading coefficients can be calculated explicitly. Since $A_0(\zeta) = 1$, it follows from (3.8) and (2.6) that

$$B_0(-t^2) = \frac{1}{2t} \int_0^t \{ (\frac{1}{4} - \alpha^2)(-s^{-2} + \operatorname{csch}^2 s) - (\frac{1}{4} - \beta^2) \operatorname{sech}^2 s \} ds;$$

accordingly

$$B_0(-t^2) = \frac{1}{2t}\{(\frac{1}{4} - \alpha^2)(t^{-1} - \coth t) - (\frac{1}{4} - \beta^2)\tanh t\}. \tag{3.17}$$

Multiplying (2.6) and (3.17) together gives

$$\begin{aligned}\psi(v)B_0(v) = \frac{1}{8}\{&-(\frac{1}{4} - \alpha^2)^2[v^{-2} - (-v)^{-\frac{3}{2}}\coth(-v)^{\frac{1}{2}} \\ &+v^{-1}\mathrm{csch}^2(-v)^{\frac{1}{2}} + (-v)^{-\frac{1}{2}}\coth(-v)^{\frac{1}{2}}\mathrm{csch}^2(-v)^{\frac{1}{2}}] \\ &+(\frac{1}{4} - \alpha^2)(\frac{1}{4} - \beta^2)[v^{-1}\mathrm{sech}^2(-v)^{\frac{1}{2}} + (-v)^{-\frac{3}{2}}\tanh(-v)^{\frac{1}{2}}] \\ &+(\frac{1}{4} - \beta^2)^2(-v)^{-\frac{1}{2}}\tanh(-v)^{\frac{1}{2}}\mathrm{sech}^2(-v)^{\frac{1}{2}}\}.\end{aligned}$$

Hence by (3.9),

$$\begin{aligned}A_1(-t^2) = \frac{1}{2}(\alpha + \frac{1}{2})\,t^{-1}&\{(\frac{1}{4} - \alpha^2)(t^{-1} - \coth t) - (\frac{1}{4} - \beta^2)\tanh t\} \\ &+\frac{1}{4}\{(\frac{1}{4} - \alpha^2)(t^{-2} - \mathrm{csch}^2 t) + (\frac{1}{4} - \beta^2)\mathrm{sech}^2 t\} \\ &+\frac{1}{4}\{(\frac{1}{4} - \alpha^2)^2(\frac{1}{2}t^{-2} - t^{-1}\coth t + \frac{1}{2}\mathrm{csch}^2 t) \\ &-(\frac{1}{4} - \alpha^2)(\frac{1}{4} - \beta^2)t^{-1}\tanh t - \frac{1}{2}(\frac{1}{4} - \beta^2)^2\mathrm{sech}^2 t\} + C,\end{aligned} \tag{3.18}$$

where

$$C = \frac{\alpha}{2}\{\frac{1}{3}(\frac{1}{4} - \alpha^2) + (\frac{1}{4} - \beta^2)\} + \frac{1}{8}\{(\frac{1}{4} - \alpha^2) + (\frac{1}{4} - \beta^2)\}^2 \tag{3.19}$$

is chosen so that $A_1(0) = 0$.

Chapter 4

Estimates for $\mathcal{V}_{0,t}(tB_0)$ and $\mathcal{V}_{0,t}(A_1)$

In order for the error bounds in (3.6) and (3.15) to be of any practical use, the total variations $\mathcal{V}_{0,t}(tB_n)$ and $\mathcal{V}_{0,t}(A_n)$ should be estimated. Here we illustrate only the cases involving B_0 and A_1. First we recall the property

$$\mathcal{V}_{a,b}(f) = \int_a^b |f'(t)|dt, \tag{4.1}$$

and observe the relation

$$\mathcal{V}_{\zeta,0}\{\sqrt{-\zeta}B_n(\zeta)\} = \mathcal{V}_{0,t}\{tB_n(-t^2)\}.$$

From (3.17), we have

$$\{tB_0(-t^2)\}' = \frac{1}{2}\{(\frac{1}{4} - \alpha^2)(-t^{-2} + \operatorname{csch}^2 t) - (\frac{1}{4} - \beta^2)\operatorname{sech}^2 t\}.$$

Since $-t^{-2} + \operatorname{csch}^2 t$ is always negative, it follows from (4.1) that

$$\mathcal{V}_{0,t}(tB_0) \leq \frac{1}{2}\{|\frac{1}{4} - \alpha^2|(\coth t - t^{-1}) + |\frac{1}{4} - \beta^2|\tanh t\}. \tag{4.2}$$

Consequently

$$\mathcal{V}_{0,t}(tB_0) \leq \mathcal{V}_{0,\infty}(tB_0) = \frac{1}{2}\{|\frac{1}{4} - \alpha^2| + |\frac{1}{4} - \beta^2|\}. \tag{4.3}$$

On the other hand, since both $t^{-1}\coth t - t^{-2}$ and $t^{-1}\tanh t$ are decreasing in the interval $0 < t < \infty$, we have $0 \le t^{-1}\coth t - t^{-2} \le 1/3$ and $0 \le t^{-1}\tanh t \le 1$. Hence (4.2) also gives

$$\mathcal{V}_{0,t}(tB_0) \le \frac{t}{6}\{|\frac{1}{4} - \alpha^2| + 3|\frac{1}{4} - \beta^2|\}. \tag{4.4}$$

Similarly, (3.18) gives

$$\begin{aligned}\{A_1(-t^2)\}' = &\frac{1}{2}(\frac{1}{2} + \alpha)(\frac{1}{4} - \alpha^2)T_1(t) \\ &+\frac{1}{4}\{2(\frac{1}{2} + \alpha)(\frac{1}{4} - \beta^2) + (\frac{1}{4} - \alpha^2)(\frac{1}{4} - \beta^2)\}T_2(t) \\ &+\frac{1}{4}(\frac{1}{4} - \alpha^2)T_3(t) - \frac{1}{4}(\frac{1}{4} - \alpha^2)^2T_4(t) \\ &+\frac{1}{4}\{2(\frac{1}{4} - \beta^2) - (\frac{1}{4} - \beta^2)^2\}T_5(t),\end{aligned}$$

where

$$T_1(t) = -2t^{-3} + t^{-2}\coth t + t^{-1}\mathrm{csch}^2 t,$$

$$T_2(t) = t^{-2}\tanh t - t^{-1}\sinh^2 t,$$

$$T_3(t) = -2t^{-3} + 2\coth t\mathrm{csch}^2 t,$$

$$T_4(t) = t^{-3} - t^{-2}\coth t - t^{-1}\mathrm{csch}^2 t + \coth t\mathrm{csch}^2 t,$$

and

$$T_5(t) = -\tanh t\,\mathrm{sech}^2 t.$$

It can be verified that for $t > 0$, we have

$$T_1(t) > 0, \qquad T_2(t) > 0, \qquad T_3(t) < 0,$$

$$T_4(t) < 0, \qquad T_5(t) < 0.$$

Hence

$$\begin{aligned}
&\int_0^t |T_1(t)|dt = t^{-2} - t^{-1}\coth t + \frac{1}{3} \le \frac{1}{3},\\
&\int_0^t |T_2(t)|dt = -t^{-1}\tanh t + 1 \le 1\\
&\int_0^t |T_3(t)|dt = -t^{-2} + \operatorname{csch}^2 t + \frac{1}{3} \le \frac{1}{3},\\
&\int_0^t |T_4(t)|dt = \frac{1}{2}t^{-2} - t^{-1}\coth t + \frac{1}{2}\operatorname{csch}^2 t + \frac{1}{2} \le \frac{1}{2}\\
&\int_0^t |T_5(t)|dt = -\frac{1}{2}\operatorname{sech}^2 t + \frac{1}{2} \le \frac{1}{2}.
\end{aligned} \tag{4.5}$$

From (4.1), it follows that

$$\begin{aligned}
\mathcal{V}_{0,t}(A_1) \le\ & \frac{1}{2}|\frac{1}{2}+\alpha||\frac{1}{4}-\alpha^2|\{t^{-2} - t^{-1}\coth t + \frac{1}{3}\}\\
&+\frac{1}{4}|2(\frac{1}{2}+\alpha)(\frac{1}{4}-\beta^2) + (\frac{1}{4}-\alpha^2)(\frac{1}{4}-\beta^2)|\{-t^{-1}\tanh t + 1\}\\
&+\frac{1}{4}|\frac{1}{4}-\alpha^2|\{-t^{-2} + \operatorname{csch}^2 t + \frac{1}{3}\}\\
&+\frac{1}{4}(\frac{1}{4}-\alpha^2)^2\{\frac{1}{2}t^{-2} - t^{-1}\coth t + \frac{1}{2}\operatorname{csch}^2 t + \frac{1}{2}\}\\
&+\frac{1}{4}|2(\frac{1}{4}-\beta^2) - (\frac{1}{4}-\beta^2)^2|\{-\frac{1}{2}\operatorname{sech}^2 t + \frac{1}{2}\},
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\mathcal{V}_{0,t}(A_1) &\le \mathcal{V}_{0,\infty}(A_1)\\
&= \frac{1}{6}(\frac{1}{2}+\alpha)|\frac{1}{4}-\alpha^2| + \frac{1}{12}|\frac{1}{4}-\alpha^2| + \frac{1}{8}(\frac{1}{4}-\alpha^2)^2\\
&\quad +\frac{1}{4}|2(\frac{1}{2}+\alpha)(\frac{1}{4}-\beta^2) + (\frac{1}{4}-\alpha^2)(\frac{1}{4}-\beta^2)|
\end{aligned} \tag{4.7}$$

$$+\frac{1}{8}|2(\frac{1}{4}-\beta^2)-(\frac{1}{4}-\beta^2)^2|.$$

It can also be shown that

$$\mathcal{V}_{0,t}(A_1) \leq t^2 \mathrm{V}(\alpha,\beta), \tag{4.8}$$

where

$$\begin{aligned}\mathrm{V}(\alpha,\beta) = {} & \frac{1}{90}(\frac{1}{2}+\alpha)|\frac{1}{4}-\alpha^2| + \frac{1}{6}(\frac{1}{2}+\alpha)|\frac{1}{4}-\beta^2| + \frac{1}{60}|\frac{1}{4}-\alpha^2| \\ & + \frac{1}{12}|\frac{1}{4}-\alpha^2||\frac{1}{4}-\beta^2| + \frac{1}{72}|\frac{1}{4}-\alpha^2|^2 \\ & + \frac{1}{8}\{2|\frac{1}{4}-\beta^2| - (\frac{1}{4}-\beta^2)^2\}.\end{aligned} \tag{4.9}$$

To do this, we return to (4.5) and put

$$X_i(t) = \frac{1}{t^2}\int_0^t |T_i(t)|dt, \qquad i = 1,...,5.$$

It can be verified that for each i, $X_i(t)$ is a nonnegative decreasing function in $(0,\infty)$, and that as $t \to 0^+$, we have

$$X_1(t) \to \frac{1}{45}, \qquad X_2(t) \to \frac{1}{3}, \qquad X_3(t) \to \frac{1}{15},$$

$$X_4(t) = X_1(t) + \frac{1}{2}X_3(t) \to \frac{1}{18}, \qquad X_5(t) \to \frac{1}{2}.$$

Thus it follows from (4.6) that

$$\frac{1}{t^2}\mathcal{V}_{0,t}(A_1) \leq \mathrm{V}(\alpha,\beta),$$

and (4.8) is proved.

Chapter 5

Zeros; First Approximation

The uniform expansions (3.13) and (3.14) can be used to determine the zeros of the Jacobi function $\varphi_\mu^{(\alpha,\beta)}(t)$, and the main tool in this regard is the following result stated in Hethcote [6].

Theorem A *In the interval* $[a-\rho, a+\rho]$, *suppose* $f(t) = g(t) + \epsilon(t)$, *where* $f(t)$ *is continuous,* $g(t)$ *is differentiable,* $g(a) = 0$,$m = min|g'(t)| > 0$, *and*

$$E = max|\epsilon(t)| < min\{|g(a-\rho)|, |g(a+\rho)|\}. \tag{5.1}$$

Then there exists a zero c *of* $f(t)$ *in the interval such that* $|c-a| \leq E/m$.

In (3.13) we now take $n = 0$. This gives

$$(\sinh t)^{\alpha+\frac{1}{2}}(\cosh t)^{\beta+\frac{1}{2}}\varphi_\mu^{(\alpha,\beta)}(t)$$
$$= \frac{2^\alpha \Gamma(\alpha+1)}{\mu^\alpha} t^{\frac{1}{2}}\{J_\alpha(\mu t) + t^{-1}\varepsilon_{1,1}(\mu, -t^2)\}. \tag{5.2}$$

In terms of the phase function $\theta_\alpha(x)$ defined by

$$\theta_\alpha(x) = -\frac{1}{4}\pi, \qquad 0 < x \leq X_\alpha,$$

$$\theta_\alpha(x) = \tan^{-1}\{Y_\alpha(x)/J_\alpha(x)\}, \qquad x \geq X_\alpha, \tag{5.3}$$

where the branch of the inverse tangent is chosen to make $\theta_\alpha(x)$ continuous, the Bessel function can be expressed as

$$J_\alpha(x) = E_\alpha^{-1}(x) M_\alpha(x) \cos\theta_\alpha(x). \tag{5.4}$$

It is known that

$$\theta'_\alpha(x) = \frac{2}{\pi x M_\alpha^2(x)}, \qquad x > X_\alpha, \tag{5.5}$$

and

$$\theta_\alpha(j_{\alpha,s}) = (s - \frac{1}{2})\pi, \qquad \theta_\alpha(y_{\alpha,s}) = (s-1)\pi, \tag{5.6}$$

where $j_{\alpha,s}$ and $y_{\alpha,s}$ are the sth positive zero of $J_\alpha(x)$ and $Y_\alpha(x)$; see [11, p.437]. To apply the above theorem to (5.2), we take

$$f(t) = \frac{\mu^\alpha}{2^\alpha \Gamma(\alpha+1)} t^{-\frac{1}{2}} (\sinh t)^{\alpha+\frac{1}{2}} (\cosh t)^{\beta+\frac{1}{2}} \times E_\alpha(\mu t) M_\alpha^{-1}(\mu t) \varphi_\mu^{(\alpha,\beta)}(t), \tag{5.7}$$

$$g(t) = \cos\theta_\alpha(\mu t), \tag{5.8}$$

$$\varepsilon(t) = t^{-1} E_\alpha(\mu t) M_\alpha^{-1}(\mu t) \varepsilon_{1,1}(\mu, -t^2). \tag{5.9}$$

From (3.6) with $(-\zeta)^{1/2} = t$ (see(2.4)), it follows that

$$|\varepsilon(t)| \leq \lambda_3(\alpha) \exp\{\frac{\lambda_2(\alpha)}{\mu} \mathcal{V}_{0,t}[tB_0(-t^2)]\} \frac{\mathcal{V}_{0,t}[tB_0(-t^2)]}{\mu}, \tag{5.10}$$

which coupled with (4.3) yields

$$|\varepsilon(t)| \leq \frac{\lambda_3(\alpha)(A+B)}{2\mu} \exp\{\frac{\lambda_2(\alpha)(A+B)}{2\mu}\}, \tag{5.11}$$

where

$$A = |\frac{1}{4} - \alpha^2|, \qquad B = |\frac{1}{4} - \beta^2|. \tag{5.12}$$

In view of (4.4), the estimate (5.10) also gives

$$|\varepsilon(t)| \leq \lambda_3(\alpha) \exp\{\frac{\lambda_2(\alpha)(A+B)}{2\mu}\} \frac{(A+3B)}{6} \frac{t}{\mu}. \tag{5.13}$$

Differentiation and use of (5.5) give

$$g'(t) = -\frac{2\mu \sin\theta_\alpha(\mu t)}{\pi \mu t M_\alpha^2(\mu t)}, \qquad \mu t > X_\alpha. \tag{5.14}$$

Since $\lambda_2(\alpha) \geq \pi x M_\alpha^2(x)$ by (3.3), we have

$$|g'(t)| \geq \frac{2\mu |\sin\theta_\alpha(\mu t)|}{\lambda_2(\alpha)}, \qquad \mu t \geq X_\alpha. \tag{5.15}$$

According to (5.6), the sth zero of $g(t)$ is $a_s = j_{\alpha,s}/\mu$ and $\sin\theta_\alpha(j_{\alpha,s}) = \pm 1$. We now choose $\rho_s = \mu^{-3/2}$ with μ being sufficiently large so that

$$|\sin\theta_\alpha(\mu t)| \geq |\sin\theta_\alpha(j_{\alpha,s} \pm \mu^{-\frac{1}{2}})| \geq \frac{1}{2} \tag{5.16}$$

for $a_s - \rho_s \leq t \leq a_s + \rho_s$. (Note that by (5.5), $\theta_\alpha(x)$ is an increasing function for $x > X_\alpha$.) Coupling (5.15) and (5.16) gives

$$m = \min |g'(t)| \geq \frac{\mu}{\lambda_2(\alpha)} > 0, \tag{5.17}$$

the minimum being taken over the interval $[a_s - \rho_s, a_s + \rho_s]$. To verify that condition (5.1) holds, we observe from (5.11) that

$$E = \max |\varepsilon(t)| \leq \frac{e\lambda_3(\alpha)(A+B)}{2\mu}, \tag{5.18}$$

if we choose

$$\mu \geq \frac{1}{2}\lambda_2(\alpha)(A+B). \tag{5.19}$$

From (5.13), we also have

$$E = \max |\varepsilon(t)| \leq \frac{e\lambda_3(\alpha)(A+3B)}{6\mu}(\frac{j_{\alpha,s}}{\mu} + \mu^{-\frac{3}{2}}),$$

which of course implies

$$E \leq 0.5473\lambda_3(\alpha)(A+3B)\frac{j_{\alpha,s}}{\mu^2}, \tag{5.20}$$

if (5.19) holds and $\mu \geq 4$. Here we have made use of the fact that $j_{\alpha,s} \geq j_{0,1} = 2.40482$ for $\alpha \geq 0$ and $s \geq 1$.

On the other hand, since $g(a_s) = 0$ by (5.6), we have

$$g(a_s \pm \rho_s) = \pm g'(\xi_\pm)\rho_s,$$

where $a_s < \xi_+ < a_s + \rho_s$ and $a_s - \rho_s < \xi_- < a_s$. From (5.15) and (5.16) it follows that

$$|g(a_s \pm \rho_s)| \geq \frac{1}{\lambda_2(\alpha)}\mu^{-\frac{1}{2}}. \tag{5.21}$$

Thus, in view of (5.18), condition (5.1) holds if

$$\mu^{\frac{1}{2}} \geq \frac{e}{2}\lambda_2(\alpha)\lambda_3(\alpha)(A+B). \tag{5.22}$$

By Theorem A, (5.17) and (5.22), the Jacobi function $\varphi_\mu^{(\alpha,\beta)}(t)$ has a zero $t_{\mu,s}$ satisfying

$$|t_{\mu,s} - \frac{j_{\alpha,s}}{\mu}| \leq \frac{e\lambda_2(\alpha)\lambda_3(\alpha)(A+B)}{2\mu^2}, \tag{5.23}$$

if (5.19) and the second inequality in (5.16) hold. On the other hand, from (5.17) and (5.20), we also have

$$|t_{\mu,s} - \frac{j_{\alpha,s}}{\mu}| \leq 0.5473\ \lambda_2(\alpha)\lambda_3(\alpha)(A+3B)\frac{j_{\alpha,s}}{\mu^3} \tag{5.24}$$

if, in addition, (5.22) holds.

Note that since X_α is a zero of $J_\alpha(x) + Y_\alpha(x)$, from (3.3) we have $\lambda_3(\alpha) \geq \pi X_\alpha M_\alpha^2(X_\alpha)/\sqrt{2}$. Furthermore, since $xM_\alpha^2(x) \geq 2/\pi$ if $\alpha \geq 1/2$ (see [18, pp.446 and 447]), it follows that $\lambda_3(\alpha) \geq \sqrt{2}$. From the integral representation of $J_\alpha^2(x) + Y_\alpha^2(x)$ in [18, p.444], it is readily seen that $M_\alpha^2(x)$ is an increasing function of α. If $0 \leq \alpha \leq 1/2$, then $xM_\alpha^2(x)$ is also an increasing function of x; see [18, p.446]. Since X_α is increasing in α (see [18, p.508]), $X_\alpha M_\alpha^2(X_\alpha) \geq X_\alpha M_0^2(X_\alpha) \geq X_0 M_0^2(X_0)$. Using $X_0 = 0.23$ (see [11, p.438]), the last quantity is easily computed to be greater or equal to 0.93. Consequently, in both cases $0 \leq \alpha \leq 1/2$ and $\alpha \geq 1/2$, we have $\lambda_3(\alpha) \geq e^{-1}$. Thus, condition (5.19) is included in condition (5.22) for $\mu \geq 1$.

The condition in the second inequality of (5.16) can also be made more explicit. In view of (5.6), the Mean Value Theorem gives

$$\sin\theta_\alpha(j_{\alpha,s} + \mu^{-\frac{1}{2}}) = \pm 1 + \cos\theta_\alpha(j_{\alpha,s} + \xi)\ \theta'_\alpha(j_{\alpha,s} + \xi)\ \mu^{-\frac{1}{2}} \tag{5.25}$$

for some $0 < \xi < \mu^{-\frac{1}{2}}$. Since $j_{\alpha,s} + \xi > X_\alpha$ and $xM_\alpha^2(x) \geq 2/\pi$ if $\alpha \geq 1/2$, we have, from (5.5), $|\theta'_\alpha(j_{\alpha,s} + \xi)| \leq 1$ if $\alpha \geq 1/2$. For $0 \leq \alpha \leq 1/2$, we can repeat an earlier

argument (in the previous paragraph) to show that $xM_\alpha^2(x) \geq X_0 M_\alpha^2(X_0) \geq 0.93$ for all $x \geq X_\alpha$. Thus from (5.5) we have $|\theta'_\alpha(j_{\alpha,s} + \xi)| \leq 0.685$ if $0 \leq \alpha \leq 1/2$. In both cases, we conclude

$$|\sin\theta_\alpha(j_{\alpha,s} + \mu^{-\frac{1}{2}})| \geq 1 - \mu^{-\frac{1}{2}}. \tag{5.26}$$

Therefore, (5.16) holds if $\mu^{1/2} \geq 2$. Summarizing the above results gives the following theorem

Theorem 1 *If* $\mu^{\frac{1}{2}} \geq max\{2, e\lambda_2(\alpha)\lambda_3(\alpha)(A+B)/2\}$ *then the zeros* $t_{\mu,s}$ *of the Jacobi function* $\varphi_\mu^{(\alpha,\beta)}(t)$ *satisfy the uniform asymptotic estimates (5.23) and (5.24).*

Note that both results (5.23) and (5.24) are needed, since the quantity $j_{\alpha,s}/\mu$ may tend to zero or infinity.

Chapter 6

Zeros; Second Approximation

To show that the error estimates in (5.23) and (5.24) are of the correct order, we recall a result given in [11, p.453, Ex.7.2], which states that the sth negative zero of the solution $W_1(\mu,\zeta)$ in (3.4) are given by

$$\zeta = -\eta_s + \frac{2\eta_s B_0(-\eta_s)}{\mu^2} + \eta_s^{\frac{1}{2}} O(\mu^{-3}), \qquad \mu \to \infty, \tag{6.1}$$

uniformly with respect to unbounded s, where $\eta_s = j_{\alpha,s}^2/\mu^2$. Since $t = (-\zeta)^{1/2}$ by (2.4), it is readily seen that

$$t_{\mu,s} = \gamma_s - \frac{\gamma_s B_0(-\gamma_s^2)}{\mu^2} + \gamma_s B_0^2(-\gamma_s^2) O(\mu^{-4})$$
$$+ B_0(-\gamma_s^2) O(\mu^{-5}) + O(\mu^{-3}), \qquad \mu \to \infty, \tag{6.2}$$

uniformly with respect to s, where $\gamma_s = j_{\alpha,s}/\mu$. By the argument used for (4.4), it is easily shown that

$$|B_0(-t^2)| \le \frac{1}{6}(A + 3B). \tag{6.3}$$

Hence (6.2) gives

$$t_{\mu,s} = \gamma_s - \frac{\gamma_s B_0(-\gamma_s^2)}{\mu^2} + \gamma_s O(\mu^{-4}) + O(\mu^{-3}). \tag{6.4}$$

Using (3.17), (4.2) and (4.3), it can also be shown that

$$|tB_0(-t^2)| \leq \frac{1}{2}(A+B), \qquad t>0. \tag{6.5}$$

In view of (6.3) and (6.5), it is now clear from the second term in (6.4) that the error estimates for the one-term approximation (5.23)- (5.24) are of the right asymptotic order.

In some recent investigations of the monotonicity of the Lebesgue constants for the Legendre series [12] and of the relative extrema of the Jacobi polynomial $P_n^{(0,-1)}(x)$ [19], we have found necessary to use the numerical bounds for the error term associated with the two-term asymptotic expansion of the zeros of the Jacobi polynomials. For this reason, we thought it is appropriate here to derive similar results for the two-term approximation (6.4), i.e., to find the implied constants hidden behind the O-symbols in (6.4). To do this, we first take $n=1$ in (3.14) and obtain

$$(\sinh t)^{\alpha+\frac{1}{2}}(\cosh t)^{\beta+\frac{1}{2}}\varphi_\mu^{(\alpha,\beta)}(t) = \frac{2^\alpha\Gamma(\alpha+1)}{\mu^\alpha}t^{\frac{1}{2}}\{J_\alpha(\mu t) - \frac{tB_0(-t^2)}{\mu}J_{\alpha+1}(\mu t) + t^{-1}\varepsilon_{2,1}(\mu,-t^2)\}. \tag{6.6}$$

Making use of the recurrence relation

$$J_{\alpha+1}(x) = -J'_\alpha(x) + \frac{\alpha}{x}J_\alpha(x)$$

and Taylor's theorem, we can rewrite (6.6) as

$$(\sinh t)^{\alpha+\frac{1}{2}}(\cosh t)^{\beta+\frac{1}{2}}\varphi_\mu^{(\alpha,\beta)}(t) = \frac{2^\alpha\Gamma(\alpha+1)}{\mu^\alpha}t^{\frac{1}{2}}\{J_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu}) - \frac{t^2B_0^2(-t^2)}{2\mu^2}J''_\alpha(\xi)$$

$$-\frac{\alpha B_0(-t^2)}{\mu^2}J_\alpha(\mu t) + t^{-1}\varepsilon_{2,1}(\mu, -t^2)\}, \qquad (6.7)$$

where $|\xi - \mu t| \leq |tB_0(-t^2)|/\mu$. Let b_s denote the roots of the equation

$$J_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu}) = 0,$$

that is,

$$\mu b_s + \frac{b_s B_0(-b_s^2)}{\mu} = j_{\alpha,s}, \qquad s = 1, 2, \ldots \ . \qquad (6.8)$$

We shall first prove the following result.

Lemma *Let $\gamma_s = j_{\alpha,s}/\mu$ and put*

$$A = |\frac{1}{4} - \alpha^2|, \qquad B = |\frac{1}{4} - \beta^2|. \qquad (6.9)$$

Then for $\mu \geq max\{5, 5(A+3B)^2\}$ and $s = 1, 2, \ldots$, we have

$$b_s = \gamma_s - \frac{\gamma_s B_0(-\gamma_s^2)}{\mu^2} + \sigma_s(\mu), \qquad (6.10)$$

where

$$|\sigma_s(\mu)| \leq 0.0280(A+3B)^2\frac{\gamma_s}{\mu^4}. \qquad (6.11)$$

Proof. Equation (6.8) can be written as

$$b_s(1 + \frac{B_0(-b_s^2)}{\mu^2}) = \gamma_s,$$

from which it follows that

$$b_s = \gamma_s + \sigma_s^*(\mu), \qquad (6.12)$$

where

$$\sigma_s^*(\mu) = -\frac{\gamma_s}{\mu^2}\,\frac{B_0(-b_s^2)}{1+\frac{1}{\mu^2}B_0(-b_s^2)}. \tag{6.13}$$

If $\mu \geq 5$ then we have

$$1 - \frac{1}{6\sqrt{5}\mu^{3/2}} \geq \frac{149}{150}. \tag{6.14}$$

On the other hand, if $\mu \geq 5(A+3B)^2$ then we also have

$$\frac{1}{6\sqrt{5}\mu^{3/2}} \geq \frac{A+3B}{6\mu^2}. \tag{6.15}$$

Coupling (6.3) and the last two inequalities, we obtain

$$|1 + \frac{B_0(-b_s^2)}{\mu^2}| \geq 1 - \frac{A+3B}{6\mu^2} \geq \frac{149}{150}. \tag{6.16}$$

Inserting (6.3) and (6.16) in (6.13) gives

$$|\sigma_s^*(\mu)| \leq 0.1678(A+3B)\frac{\gamma_s}{\mu^2}. \tag{6.17}$$

Now we rewrite (6.8) in the form

$$b_s = \gamma_s - \frac{b_s B_0(-b_s^2)}{\mu^2},$$

and replace b_s on the right-hand side by (6.12). This yields, after an application of Taylor's theorem,

$$b_s = \gamma_s - \frac{\gamma_s B_0(-\gamma_s^2)}{\mu^2} + \sigma_s(\mu),$$

where

$$\sigma_s(\mu) = -\frac{1}{\mu^2}\sigma_s^*(\mu)\frac{d}{dt}[tB_0(-t^2)]|_{t=\eta}, \tag{6.18}$$

and $|\eta - \gamma_s| < |\sigma_s^*(\mu)|$. The derivative of $tB_0(-t^2)$ is given in Chapt.4; see the equation preceding (4.2). Since $0 \leq \operatorname{sech}^2 t \leq 1$ and $-1/3 \leq \operatorname{csch}^2 t - t^{-2} \leq 0$ as indicated in (4.5), we have

$$|\frac{d}{dt}[tB_0(-t^2)]| \leq \frac{1}{6}(A + 3B) \tag{6.19}$$

for all $t > 0$. The error bound in (6.11) now follows immediately from (6.17), (6.18) and (6.19), and the lemma is proved. □

We shall next use Theorem A to obtain estimates for the difference between b_s and $t_{\mu,s}$, the sth zero of the Jacobi function. In particular, we shall show that as $\mu \to \infty$, we have

$$t_{\mu,s} = b_s + \gamma_s^2 O(\mu^{-\frac{5}{2}}), \tag{6.20}$$

and

$$t_{\mu,s} = b_s + O(\mu^{-3}), \tag{6.21}$$

both uniformly with respect to unbounded s; cf. (6.4). Clearly, (6.20) is preferable if s is fixed, whereas (6.21) is a stronger result if $\gamma_s = j_{\alpha,s}/\mu$ is large.

To prove (6.20) and (6.21), we first estimate μt and $\mu t + tB_0(-t^2)/\mu$ for t in the interval $[b_s - \rho, b_s + \rho]$ with $\rho = \mu^{-5/2}$. Let $\Delta_s = t - b_s$. Then by the Mean Value Theorem

$$\mu t + \frac{t}{\mu}B_0(-t^2) = \mu b_s + \frac{b_s}{\mu}B_0(-b_s^2)$$
$$+\{\mu + \frac{1}{\mu}[tB_0(-t^2)]'_{t=\zeta}\}\Delta_s,$$

where $|\zeta - b_s| < |\Delta_s| \leq \rho$. From (6.8) and (6.19), it follows that

$$|\mu t + \frac{t}{\mu} B_0(-t^2) - j_{\alpha,s}| \leq \mu^{-\frac{3}{2}}(1 + \frac{A+3B}{6\mu^2}).$$

If $\mu \geq 5$ and $\mu \geq 5(A+3B)^2$, then by (6.15)

$$|\mu t + \frac{t}{\mu} B_0(-t^2) - j_{\alpha,s}| \leq 0.0901.$$

Consequently, we have

$$\begin{aligned} \mu t + \frac{t}{\mu} B_0(-t^2) &\geq j_{\alpha,s} - 0.0901. \\ &> y_{\alpha,1} > X_\alpha, \qquad \alpha \geq 0, \end{aligned} \tag{6.22}$$

for all $t \in [b_s - \rho, b_s + \rho]$ and $\rho = \mu^{-5/2}$, where X_α is the first positive zero of $J_\alpha(x) + Y_\alpha(x)$ (see (3.1) and (3.2)). The second inequality in (6.22) follows from the fact that $j_{\alpha,1} - y_{\alpha,1} > 0.9444$, which can be proved by using a lower bound for $j_{\alpha,1}$ and an upper bound for $y_{\alpha,1}$ (see Appendix B). Since $t = b_s + \Delta_s$, by (6.8) μt can be written as

$$\begin{aligned} \mu t &= \mu b_s + \mu \Delta_s + \frac{b_s}{\mu} B_0(-b_s^2) - \frac{b_s}{\mu} B_0(-b_s^2) \\ &= j_{\alpha,s} + \mu \Delta_s - \frac{b_s}{\mu} B_0(-b_s^2). \end{aligned} \tag{6.23}$$

Using the bound in (6.5), we have

$$|\mu t - j_{\alpha,s}| \leq \mu^{-\frac{3}{2}} + \frac{A+B}{2\mu}.$$

By (6.15), this implies

$$\mu t \geq j_{\alpha,s} - 0.1895 \geq y_{\alpha,1} > X_\alpha \tag{6.24}$$

for $\mu \geq 5$ and $\mu \geq 5(A+3B)^2$. Under the same conditions, since

$$\begin{aligned} |b_s| &\leq \frac{j_{\alpha,s}}{\mu}\{1+0.1678\frac{(A+3B)}{\mu^2}\} \\ &\leq \frac{j_{\alpha,s}}{\mu}(1+\frac{0.1678}{25}) \end{aligned} \tag{6.25}$$

by (6.12), (6.17) and (6.15), we also have from (6.23) and (6.3)

$$\begin{aligned} |\mu t - j_{\alpha,s}| &\leq \{j_{0,1}^{-1} + 1.00672\ \frac{(A+3B)}{6\sqrt{\mu}}\} j_{\alpha,s}\mu^{-\frac{3}{2}} \\ &\leq 0.4909\ j_{\alpha,s}\mu^{-\frac{3}{2}}. \end{aligned} \tag{6.26}$$

Here we have used again (6.15) and the fact that $j_{\alpha,1}$ is an increasing function of α and $j_{0,1} = 2.40482$; see [11, p.246] and [1, p.409].

We now return to the asymptotic approximation (6.7). For $t \in [b_s - \rho, b_s + \rho]$ and $\rho = \mu^{-5/2}$, we let $x = \mu t + tB_0(-t^2)/\mu$. Since $x > X_\alpha$ by (6.22), it follows from (3.1), (3.2) and (5.3) that

$$J_\alpha(x) = M_\alpha(x)\cos\theta_\alpha(x),$$

$$M_\alpha(x) = \{J_\alpha^2(x) + Y_\alpha^2(x)\}^{\frac{1}{2}}$$

and

$$\theta_\alpha(x) = \tan^{-1}\{Y_\alpha(x)/J_\alpha(x)\}.$$

Take

$$\begin{aligned} f(t) = \frac{\mu^\alpha}{2^\alpha\Gamma(\alpha+1)} t^{-\frac{1}{2}}(\sinh t)^{\alpha+\frac{1}{2}}(\cosh t)^{\beta+\frac{1}{2}} \\ \times\varphi_\mu^{(\alpha,\beta)}(t) M_\alpha^{-1}(\mu t + \frac{tB_0(-t^2)}{\mu}), \end{aligned} \tag{6.27}$$

$$g(t) = \cos\theta_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu}), \tag{6.28}$$

and

$$\varepsilon(t) = M_\alpha^{-1}(\mu t + \frac{tB_0(-t^2)}{\mu})\{-\frac{t^2B_0^2(-t^2)}{2\mu^2}J_\alpha''(\xi) - \frac{\alpha B_0(-t^2)}{\mu^2}J_\alpha(\mu t) + t^{-1}\varepsilon_{2,1}(\mu, -t^2)\}, \tag{6.29}$$

so that (6.6) becomes $f(t) = g(t) + \varepsilon(t)$. In what follows, we shall derive two pairs of upper bounds for the error

$$E = \max\{|\varepsilon(t)| : \quad b_s - \rho \le t \le b_s + \rho\}, \tag{6.30}$$

one pair for the uniform asymptotic approximation (6.20) and the other pair for (6.21). The bounds for E will be obtained by estimating each individual term in (6.29) separately. First we show that

$$M_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu}) \ge 0.9899\{\frac{2}{\pi(j_{\alpha,s} + 0.0901)}\}^{\frac{1}{2}} \tag{6.31}$$

for $t \in [b_s - \rho, b_s + \rho]$ and $\rho = \mu^{-5/2}$. Since $M_\alpha(x)$ is a decreasing function of x (see [18, p.446]), we have from the inequality preceding (6.22)

$$M_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu}) \ge M_\alpha(j_{\alpha,s} + 0.0901)$$

for t in the specified interval. If $0 \le \alpha \le 1/2$ then we can use the asymptotic approximation [18, p.449]

$$J_\alpha^2(x) + Y_\alpha^2(x) = \frac{2}{\pi x}\{1 + \frac{4\alpha^2 - 1}{8x^2} + \delta_3(x)\}$$

to obtain

$$J_\alpha^2(x) + Y_\alpha^2(x) \geq \frac{2}{\pi x}\{1 + \frac{4\alpha^2 - 1}{8x^2}\}$$
$$\geq \frac{2}{\pi x}(1 - \frac{1}{8x^2}), \tag{6.32}$$

since $\delta_3(x)$ is positive in this case. Recalling that $j_{\alpha,s} \geq j_{0,1} = 2.40482$, the last two inequalities together give

$$M_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu}) \geq 0.9899\{\frac{2}{\pi(j_{\alpha,s} + 0.0901)}\}^{\frac{1}{2}}.$$

If $\alpha \geq 1/2$ then by the inequality $M_\alpha^2(x) \geq 2/(\pi x)$ for $x \geq \alpha \geq 1/2$,

$$M_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu}) \geq \{\frac{2}{\pi(j_{\alpha,s} + 0.0901)}\}^{\frac{1}{2}}.$$

Thus in either case we have (6.31).

Next we estimate the term

$$\frac{t^2 B_0^2(-t^2)}{2\mu^2} J_\alpha''(\xi)$$

in (6.29), where $|\xi - \mu t| \leq |tB_0(-t^2)|/\mu$. From the differential equation

$$J_\alpha''(x) = -\frac{1}{x}J_\alpha'(x) - \frac{x^2 - \alpha^2}{x^2}J_\alpha(x),$$

and the recurrence relation

$$J_\alpha'(x) = -J_{\alpha+1}(x) + \frac{\alpha}{x}J_\alpha(x), \tag{6.33}$$

we have

$$|J''_\alpha(\xi)| \le \frac{1}{\xi}|J_{\alpha+1}(\xi)| + (1 + \frac{|\alpha - \alpha^2|}{\xi^2})|J_\alpha(\xi)|. \tag{6.34}$$

For real α and positive x, it is known [11, p.270] that

$$|H^{(1)}_\alpha(x)| = |H^{(2)}_\alpha(x)| \le \sqrt{\frac{2}{\pi x}} \exp\{|\alpha^2 - \frac{1}{4}|\frac{1}{x}\}. \tag{6.35}$$

Hence it follows from the identity $J_\alpha(x) = \{H^{(1)}_\alpha(x) + H^{(2)}_\alpha(x)\}/2$ that

$$|J_\alpha(x)| \le e\sqrt{\frac{2}{\pi x}} \tag{6.36}$$

if $x \ge |\alpha^2 - 1/4|$. As a consequence, we obtain

$$|J''_\alpha(\xi)| \le e\,(\frac{1}{\xi} + 1 + \frac{|\alpha - \alpha^2|}{\xi^2})\sqrt{\frac{2}{\pi\xi}} \tag{6.37}$$

if $\xi \ge (\alpha + 1)^2 - 1/4$. By (6.5) and the inequality following (6.23), we have

$$|\xi - j_{\alpha,s}| \le \mu^{-\frac{3}{2}} + \frac{A + B}{\mu}.$$

If $\mu \ge 5$ and $\mu \ge 5(A+3B)^2$ then it can be shown as in (6.15) that $|\xi - j_{\alpha,s}| \le 0.2895$ and

$$\begin{aligned} \xi &\ge j_{\alpha,s} - 0.2895 \\ &\ge j_{0,1} - 0.2895 = 2.1153. \end{aligned} \tag{6.38}$$

Since the condition $\xi \geq (\alpha+1)^2 - 1/4$ implies $\xi \geq |\alpha - \alpha^2|$, coupling (6.37) and (6.38) gives

$$|J''_\alpha(\xi)| \leq 2e\{\frac{2}{\pi(j_{\alpha,s} - 0.2895)}\}^{\frac{1}{2}}, \tag{6.39}$$

provided that $j_{\alpha,s} \geq 0.2895 + (\alpha+1)^2 - 1/4$. If we use the bound $|J_\alpha(x)| \leq 1$ for $\alpha \geq 0$ and $x \geq 0$, then we have from (6.34)

$$|J''_\alpha(\xi)| \leq (\frac{1}{\xi} + 1 + \frac{|\alpha - \alpha^2|}{\xi^2}). \tag{6.40}$$

Since $j_{\alpha,s} \geq j_{\alpha,1} \geq \sqrt{\alpha(\alpha+2)}$ (see [18, p.486,(5)]), it follows from (6.38) that $\xi \geq \alpha$, which in turn implies $\xi^2 \geq |\alpha - \alpha^2|$. Hence, by (6.38) and (6.40),

$$|J''_\alpha(\xi)| \leq 2.4728 \tag{6.41}$$

with no restriction on $j_{\alpha,s}$; cf. (6.39). Coupling (6.5) and (6.39), we obtain

$$|\frac{t^2 B_0^2(-t^2)}{2\mu^2} J''_\alpha(\xi)| \leq 0.6796(A+B)^2\{\frac{2}{\pi(j_{\alpha,s} - 0.2895)}\}^{\frac{1}{2}} \frac{1}{\mu^2}, \tag{6.42}$$

whereas a combination of (6.3), (6.26) and (6.39) gives

$$|\frac{t^2 B_0^2(-t^2)}{2\mu^2} J''_\alpha(\xi)| \leq 0.0823(A+3B)^2\{\frac{2}{\pi(j_{\alpha,s} - 0.2895)}\}^{\frac{1}{2}} \frac{j_{\alpha,s}^2}{\mu^4}. \tag{6.43}$$

Both (6.42) and (6.43) hold for $t \in [b_s - \rho, b_s + \rho]$, $\rho = \mu^{-5/2}$ and $j_{\alpha,s} \geq 0.0395 + (\alpha+1)^2$. If we use (6.40) instead of (6.39), then we have the corresponding results

$$|\frac{t^2 B_0^2(-t^2)}{2\mu^2} J''_\alpha(\xi)| \leq 0.3091(A+B)^2 \frac{1}{\mu^2}, \tag{6.44}$$

and

$$|\frac{t^2 B_0^2(-t^2)}{2\mu^2} J''_\alpha(\xi)| \leq 0.0375(A+3B)^2 \frac{j_{\alpha,s}^2}{\mu^4}. \tag{6.45}$$

Using similar arguments, one can construct estimates for the term $\alpha B_0(-t^2)$ $J_\alpha(\mu t)/\mu^2$ in (6.29). Indeed, from (6.23), we can write $\mu t = j_{\alpha,s} + \delta_s$ with

$$\delta_s = \mu \Delta_s - \frac{b_s B_0(-b_s^2)}{\mu}.$$

By the inequality preceding (6.24), we have

$$|\delta_s| \leq \mu^{-\frac{3}{2}} + \frac{A+B}{2\mu} \leq 0.1895 \tag{6.46}$$

for $\mu \geq 5$ and $\mu \geq 5(A+3B)^2$. For the second inequality in (6.46), see (6.15). On the other hand, by (6.26) we also have

$$|\delta_s| \leq 0.4909 j_{\alpha,s} \mu^{-\frac{3}{2}}. \tag{6.47}$$

An application of the Mean Value Theorem and the recurrence relation (6.33) gives

$$\begin{aligned} J_\alpha(\mu t) &= \delta_s J'_\alpha(\zeta_s) \\ &= \delta_s \{-J_{\alpha+1}(\zeta_s) + \frac{\alpha}{\zeta_s} J_\alpha(\zeta_s)\}, \end{aligned} \tag{6.48}$$

where $|\zeta_s - j_{\alpha,s}| < |\delta_s|$. In view of (6.46), and (6.24), ζ_s satisfies

$$\zeta_s \geq j_{\alpha,s} - 0.1895 \geq y_{\alpha,1} > \alpha;$$

see also [1, p.370, (9.5.2)]. Hence it follows from (6.3), (6.48), (6.36) and (6.47) that

$$|\frac{\alpha B_0(-t^2)}{\mu^2} J_\alpha(\mu t)| \leq 0.4449\ \alpha(A+3B)\{\frac{2}{\pi(j_{\alpha,s}-0.1895)}\}^{\frac{1}{2}} \frac{j_{\alpha,s}}{\mu^{7/2}}, \tag{6.49}$$

or from (6.46) that

$$|\frac{\alpha B_0(-t^2)}{\mu^2} J_\alpha(\mu t)| \leq 0.1718\ \alpha(A+3B)\{\frac{2}{\pi(j_{\alpha,s}-0.1895)}\}^{\frac{1}{2}} \frac{1}{\mu^2}, \tag{6.50}$$

if $j_{\alpha,s} \geq (\alpha+1)^2 - 0.0605$. If instead of (6.36) we use the bound $|J_\alpha(x)| \leq 1$ for all $x \geq 0$ and $\alpha \geq 0$, then (6.49) and (6.50) are replaced by

$$|\frac{\alpha B_0(-t^2)}{\mu^2} J_\alpha(\mu t)| \leq 0.1637\ \alpha(A+3B)\frac{j_{\alpha,s}}{\mu^{7/2}}, \tag{6.51}$$

and

$$|\frac{\alpha B_0(-t^2)}{\mu^2} J_\alpha(\mu t)| \leq 0.0632\ \alpha(A+3B)\frac{1}{\mu^2}. \tag{6.52}$$

Finally we come to the estimation of the term $t^{-1}\varepsilon_{2,1}(\mu,-t^2)$ in (6.29). Since $\mu t > X_\alpha$ by (6.24), we have from (3.15) and (3.2)

$$|t^{-1}\varepsilon_{2,1}(\mu,-t^2)| \leq \bar{\lambda}_3(\alpha) M_\alpha(\mu t) \exp\{\frac{\lambda_2(\alpha)}{\mu}\mathcal{V}_{0,t}(tB_0)\}\frac{\mathcal{V}_{0,t}(A_1)}{\mu^2}.$$

In view of (4.3), this reduces to

$$|t^{-1}\varepsilon_{2,1}(\mu,-t^2)| \leq e\bar{\lambda}_3(\alpha) M_\alpha(\mu t)\frac{\mathcal{V}_{0,t}(A_1)}{\mu^2}$$

if $\mu \geq \lambda_2(\alpha)(A+B)/2$. Recall that $\mathcal{V}_{0,t}(A_1) \leq \mathcal{V}_{0,\infty}(A_1)$ by (4.7) and that $\mathcal{V}_{0,t}(A_1) \leq t^2 V(\alpha,\beta)$ by (4.8); for values of $\mathcal{V}_{0,\infty}(A_1)$ and $V(\alpha,\beta)$, see (4.7) and (4.9). From (3.1), (6.24) and (6.35), we also have

$$M_\alpha(\mu t) = |H_\alpha^{(1)}(\mu t)| \leq e\{\frac{2}{\pi(j_{\alpha,s} - 0.1895)}\}^{\frac{1}{2}} \tag{6.53}$$

if $j_{\alpha,s} \geq \alpha^2 - 0.0605$. Thus a combination of the above estimates gives

$$|t^{-1}\varepsilon_{2,1}(\mu, -t^2)| \leq e^2 \bar{\lambda}_3(\alpha)\{\frac{2}{\pi(j_{\alpha,s} - 0.1895)}\}^{\frac{1}{2}} \frac{\mathcal{V}_{0,\infty}(A_1)}{\mu^2} \tag{6.54}$$

if $j_{\alpha,s} \geq \alpha^2 - 0.0605$. Since

$$t \leq (1 + 0.4909\mu^{-\frac{3}{2}})\frac{j_{\alpha,s}}{\mu} \leq 1.0440\frac{j_{\alpha,s}}{\mu}, \qquad \mu \geq 5,$$

by (6.26), the same estimation also gives

$$|t^{-1}\varepsilon_{2,1}(\mu, -t^2)| \leq 8.0536\ \bar{\lambda}_3(\alpha) V(\alpha,\beta)\{\frac{2}{\pi(j_{\alpha,s} - 0.1895)}\}^{\frac{1}{2}} \frac{j_{\alpha,s}^2}{\mu^4} \tag{6.55}$$

if $j_{\alpha,s} \geq \alpha^2 - 0.0605$. If instead of (6.53) we use the bound $M_\alpha(\mu t) \leq M_\alpha(y_{\alpha,1}) = |J_\alpha(y_{\alpha,1})| \leq 1$ (see (6.24)), then (6.54) and (6.55) are replaced by

$$|t^{-1}\varepsilon_{2,1}(\mu, -t^2)| \leq e\bar{\lambda}_3(\alpha)\frac{\mathcal{V}_{0,\infty}(A_1)}{\mu^2} \tag{6.56}$$

and

$$|t^{-1}\varepsilon_{2,1}(\mu, -t^2)| \leq 2.9628\ \bar{\lambda}_3(\alpha) V(\alpha,\beta)\frac{j_{\alpha,s}^2}{\mu^4}, \tag{6.57}$$

with no restriction on $j_{\alpha,s}$.

We are now ready to provide an estimate for the error $E = \max|\varepsilon(t)|$ defined in (6.30). First, we replace each term in (6.29) by its appropriate lower or upper bound given in (6.31), (6.42), (6.50), and (6.54). Next, we take out the largest of the factors of the form $\{2/(\pi(j_{\alpha,s} - \delta))\}^{1/2}$, where δ is equal to either 0.2895 or 0.1895. Then we observe that

$$\frac{j_{\alpha,s} + 0.0901}{j_{\alpha,s} - 0.2895} \leq \frac{1 + \frac{0.0901}{j_{0,1}}}{1 - \frac{0.2895}{j_{0,1}}} \leq 1.1795$$

for all $\alpha \geq 0$ and $s = 1, 2, \ldots$. The final result is

$$E \leq 1.0972 C_1(\alpha, \beta) \frac{1}{\mu^2}, \qquad \text{if } j_{\alpha,s} \geq (\alpha+1)^2 + 0.0395, \tag{6.58}$$

where

$$C_1(\alpha, \beta) = 0.6796\ (A+B)^2 + 0.1718\ \alpha(A+3B)$$
$$+e^2 \bar{\lambda}_3(\alpha) \mathcal{V}_{0,\infty}(A_1). \tag{6.59}$$

If we use the alternative set of estimates given in (6.43), (6.49), and (6.55), then the result is

$$E \leq 1.0972 C_2(\alpha, \beta) \frac{j_{\alpha,s}^2}{\mu^{7/2}}, \qquad \text{if } j_{\alpha,s} \geq (\alpha+1)^2 + 0.0395, \tag{6.60}$$

where

$$C_2(\alpha, \beta) = 0.0369\ (A+3B)^2 + 0.1851\ \alpha(A+3B)$$
$$+3.6017\ \bar{\lambda}_3(\alpha) V(\alpha, \beta). \tag{6.61}$$

Here we have made use of the assumptions $\mu \geq 5$ and $j_{\alpha,s} \geq j_{0,1} \geq 2.40482$. Bounds for E without restriction on $j_{\alpha,s}$ can be obtained from (6.44), (6.52) and (6.56), or

(6.45), (6.51) and (6.57). The respective results are

$$E \leq 0.4745 C_3(\alpha,\beta)\frac{j_{\alpha,s}^{1/2}}{\mu^2}, \tag{6.62}$$

where

$$C_3(\alpha,\beta) = 0.8403\ (A+B)^2 + 0.1718\ \alpha(A+3B)$$
$$+e^2\bar{\lambda}_3(\alpha)\mathcal{V}_{0,\infty}(A_1), \tag{6.63}$$

and

$$E \leq 0.4745 C_4(\alpha,\beta)\frac{j_{\alpha,s}^{5/2}}{\mu^{7/2}}, \tag{6.64}$$

where

$$C_4(\alpha,\beta) = 0.0456\ (A+3B)^2 + 0.1851\ \alpha(A+3B)$$
$$+3.6018\ \bar{\lambda}_3(\alpha)V(\alpha,\beta). \tag{6.65}$$

The estimates (6.58) and (6.62) will be used together to give (6.21), whereas the estimates (6.60) and (6.64) will be coupled together to yield (6.20).

To apply Theorem A, we also need an estimate for

$$m = \min\{|g'(t)| : \quad b_s - \rho \leq t \leq b_s + \rho\}. \tag{6.66}$$

From (6.28), we have

$$g'(t) = -\sin\theta_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu})\ \theta'_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu})\{\mu + \frac{[tB_0(-t^2)]'}{\mu}\}.$$

Using (6.19) and (6.15), it is easily seen that

$$|\mu + \frac{[tB_0(-t^2)]'}{\mu}| \geq \mu(1 - \frac{A+3B}{6\mu^2}) \geq 0.9933\ \mu.$$

Since $\mu t + tB_0(-t^2)/\mu > X_\alpha$ by (6.22), it follows from (5.5) and (3.3) that

$$\theta'_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu}) \geq \frac{2}{\lambda_2(\alpha)}.$$

In what follows, we shall show that

$$|\sin\theta_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu})| \geq 0.9077. \tag{6.67}$$

A combination of these estimates gives

$$m \geq \frac{1.8032}{\lambda_2(\alpha)}\mu. \tag{6.68}$$

To prove (6.67), we first note that

$$\sin\theta_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu})|_{t=b_s} = \sin\theta_\alpha(j_{\alpha,s}) = \pm 1,$$

on account of (6.8) and (5.6). By the Mean Value Theorem, we have

$$\sin\theta_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu})$$

$$= \pm 1 + \cos\theta_\alpha(\eta)\,\theta'_\alpha(\eta)\{\mu + \frac{1}{\mu}[tB_0(-t^2)]'_{t=\zeta}\}\Delta_s,$$

where $\Delta_s = t - b_s$, $|\zeta - b_s| \leq |\Delta_s|$ and $\eta \equiv \mu\zeta + \zeta B_0(-\zeta^2)/\mu$. Recall that

$$\mu t + \frac{tB_0(-t^2)}{\mu} = j_{\alpha,s} + \{\mu + \frac{1}{\mu}[tB_0(-t^2)]'_{t=\zeta}\}\Delta_s,$$

and

$$|\mu t + \frac{tB_0(-t^2)}{\mu} - j_{\alpha,s}| \leq 0.0901; \tag{6.69}$$

see the two equations following (6.21). Hence it follows that

$$|\sin\theta_\alpha(\mu t + \frac{tB_0(-t^2)}{\mu})| \geq 1 - 0.0901\ \theta'_\alpha(\eta). \tag{6.70}$$

Since $\eta \geq j_{\alpha,s} - 0.0901 > X_\alpha$ by (6.69) (see (6.22)),

$$\theta'_\alpha(\eta) = \frac{2}{\pi\eta M^2_\alpha(\eta)}.$$

Using (6.32), we have $\theta'_\alpha(\eta) \leq 1.0239$. The result in (6.67) is now obtained by inserting this in (6.70).

Since $g(b_s \pm \rho) = g(b_s) \pm \rho g'(\xi_\pm) = \pm\rho g'(\xi_\pm)$ for some ξ_- in $(b_s - \rho, b_s)$ and ξ_+ in $(b_s, b_s + \rho)$,we have by (6.68)

$$\min\{|g(b_s - \rho)|, |g(b_s + \rho)|\} \geq \frac{1.8032}{\lambda_2(\alpha)}\mu^{-\frac{3}{2}}. \tag{6.71}$$

Thus, as long as μ is chosen sufficiently large so that the bounds for E given in (6.58) and (6.62) are less then $1.8032/\lambda_2(\alpha)\mu^{3/2}$, Theorem A applies. From [5, p.34], we also have the estimate

$$|j_{\alpha,s} - (s + \frac{1}{2}\alpha - \frac{1}{4})\pi| \leq \frac{0.188|4\alpha^2 - 1|}{s - 0.314 + \frac{1}{2}\alpha} \leq 0.1363, \tag{6.72}$$

for $s \geq 0.314 - \alpha/2 + 1.38|4\alpha^2 - 1|$. Let s_α be the smallest integer such that

$$s_\alpha \geq 0.7039 + 0.1367\alpha + 1.38|4\alpha^2 - 1|. \tag{6.73}$$

Then it is easily seen from (6.72) that for all $s \geq s_\alpha$,

$$\begin{aligned} j_{\alpha,s} \geq j_{\alpha,s_\alpha} &\geq (\alpha + 1)^2 + 0.0395 \\ &> (\alpha + 1)^2 - 0.0605. \end{aligned} \tag{6.74}$$

Summarizing the above results, we obtain the following theorems

Theorem 2 *Let $\alpha \geq 0$ and $\mu \geq 5$. Suppose that μ also satisfies* (i) $\mu \geq 5(A + 3B)^2$ *and* (ii) $\mu \geq \lambda_2(\alpha)(A + B)/2$. *Let $t_{\mu,s}$ denote the positive zeros of the*

Jacobi function, and b_s denote the roots of equation (6.8). Furthermore, let s_α be defined as in (6.73). Then for all $s \geq s_\alpha$ and $\mu^{1/2} \geq 0.6085\ \lambda_2(\alpha)C_1(\alpha,\beta)$, we have

$$|t_{\mu,s} - b_s| \leq 0.6085\ \lambda_2(\alpha)C_1(\alpha,\beta)\mu^{-3}, \tag{6.75}$$

and

$$|t_{\mu,s} - b_s| \leq 0.6085\ \lambda_2(\alpha)C_2(\alpha,\beta)\frac{j_{\alpha,s}^2}{\mu^{9/2}}, \tag{6.76}$$

where the constants $C_1(\alpha,\beta)$ and $C_2(\alpha,\beta)$ are given explicitly in (6.59) and (6.61), respectively.

Note that to apply Theorem A, we need only one upper bound to make E less than $min\{|g(b_s - \rho)|, |g(b_s + \rho)|\}$. Thus in the above theorem we have used only the bound in (6.58) to ensure this condition, but have used both bounds (6.58) and (6.60) to derive the approximations (6.75) and (6.76).

Theorem 3 *Under the same conditions as in Theorem 2, we have for all $s \geq 1$*

$$|t_{\mu,s} - b_s| \leq 0.2632\ \lambda_2(\alpha)C_3(\alpha,\beta)\frac{j_{\alpha,s}^{1/2}}{\mu^3}, \tag{6.77}$$

if $\mu^{1/2} \geq 0.2632\ \lambda_2(\alpha)C_3(\alpha,\beta)j_{\alpha,s}^{1/2}$, and

$$|t_{\mu,s} - b_s| \leq 0.2632\ \lambda_2(\alpha)C_4(\alpha,\beta)\frac{j_{\alpha,s}^{5/2}}{\mu^{9/2}}, \tag{6.78}$$

if $\mu^2 \geq 0.2632\ \lambda_2(\alpha)C_4(\alpha,\beta)j_{\alpha,s}^{5/2}$, where the constants $C_3(\alpha,\beta)$ and $C_4(\alpha,\beta)$ are given in (6.63) and (6.65), respectively.

Note that the constants $C_1(\alpha,\beta)$ and $C_3(\alpha,\beta)$ are almost identical, and that the same is true for the constants $C_2(\alpha,\beta)$ and $C_4(\alpha,\beta)$. The order estimate in (6.20)

now follows from (6.76) and (6.78), whereas the order estimate in (6.21) follows from (6.75) and (6.77).

Chapter 7

Large-t Behavior

Turning our attention to the asymptotic behavior of $\varphi_\mu^{(\alpha,\beta)}(t)$ as $t \to \infty$, we first recall the following Liouville-Green approximation. Let $f(t)$ be a real, positive, twice continuously differentiable function on a finite or infinite interval (a_1, a_2), and let $g(t)$ be a continuous real or complex function on (a_1, a_2). Theorem 2.2 in [11, p.196] states that the differential equation

$$\frac{d^2u}{dt^2} + \{f(t) - g(t)\}u = 0 \tag{7.1}$$

has two linearly independent, twice continuously differentiable solutions

$$u_1(t) = f^{-1/4}(t)\ \exp\{i\int f^{1/2}(t)dt\}\{1 + \varepsilon_1(t)\}, \tag{7.2}$$

$$u_2(t) = f^{-1/4}(t)\ \exp\{-i\int f^{1/2}(t)dt\}\{1 + \varepsilon_2(t)\}, \tag{7.3}$$

such that

$$|\varepsilon_j(t)| \le \exp\{\mathcal{V}_{a,t}(F)\} - 1, \qquad j = 1, 2, \tag{7.4}$$

where a is an arbitrary point in the closure of (a_1, a_2), and

$$F(t) = \int\{\frac{1}{f^{1/4}}\frac{d^2}{dt^2}(\frac{1}{f^{1/4}}) - \frac{g}{f^{1/2}}\}dt. \tag{7.5}$$

To apply this result to (2.3), we take $f(t) = \mu^2$ and

$$g(t) = -\frac{\frac{1}{4} - \alpha^2}{\sinh^2 t} + \frac{\frac{1}{4} - \beta^2}{\cosh^2 t},$$

and get the two linearly independent solution

$$u_1(t) = \mu^{-1/2}\, e^{i\mu t}\{1 + \varepsilon_1(t)\}, \tag{7.6}$$

$$u_2(t) = \mu^{-1/2}\, e^{-i\mu t}\{1 + \varepsilon_2(t)\}, \tag{7.7}$$

with $|\varepsilon_j(t)| \le \exp\{\mathcal{V}_{t,\infty}(F)\} - 1$. By (7.5),

$$F(t) = \frac{1}{\mu}\int \{\frac{\frac{1}{4} - \alpha^2}{\sinh^2 t} - \frac{\frac{1}{4} - \beta^2}{\cosh^2 t}\} dt,$$

from which it follows that

$$\begin{aligned}\int_t^\infty |F'(t)|dt &\le \frac{1}{\mu}\{A\int_t^\infty \operatorname{csch}^2 t dt + B\int_t^\infty \operatorname{sech}^2 t dt\} \\ &\le \frac{1}{\mu}\{A(\coth t - 1) + B(1 - \tanh t)\},\end{aligned}$$

where A and B are as given in (6.9). Simple estimation gives

$$\mathcal{V}_{t,\infty}(F) \le \frac{1}{\mu}\{4Ae^{-2t} + 2Be^{-2t}\}$$

for $t \ge 1/2$. Thus the error terms in (7.6) and (7.7) satisfies

$$|\varepsilon_j(t)| \le \exp\{\frac{1}{\mu}(4A + 2B)e^{-2t}\} - 1, \quad j = 1, 2, \tag{7.8}$$

for $t \geq 1/2$. Since $e^x - 1 \leq xe^x$, we have

$$|\varepsilon_j(t)| \leq \frac{e}{\mu}(4A + 2B)e^{-2t}, \quad j = 1, 2, \tag{7.9}$$

if $t \geq \ln[(4A + 2B)/\mu]/2$. The general solution $\tilde{u}(t)$ of (2.3) is a linear combination of the two solutions $u_1(t)$ and $u_2(t)$ given in (7.6) and (7.7). Hence

$$\tilde{u}(t) = \frac{1}{\mu^{1/2}}\{c_1 e^{i\mu t} + c_2 e^{-i\mu t} + \tilde{\varepsilon}(t)\}, \tag{7.10}$$

where c_1 and c_2 are arbitrary constants and $\tilde{\varepsilon}(t)$ satisfies

$$|\tilde{\varepsilon}(t)| \leq (|c_1| + |c_2|)\{\exp[\frac{1}{\mu}(4A + 2B)e^{-2t}] - 1\} \tag{7.11}$$

for $t \geq 1/2$. Note that $\tilde{\varepsilon}(t) \to 0$, as $\mu \to \infty$. Consequently,

$$\tilde{u}(t) = \frac{1}{\mu^{1/2}}\{c_1 e^{i\mu t} + c_2 e^{-i\mu t} + O(\frac{1}{\mu})\}, \tag{7.12}$$

as $\mu \to \infty$. Let $u(t)$ be the function defined by (2.2), i.e.,

$$u(t) = (\sinh t)^{\alpha+\frac{1}{2}}(\cosh t)^{\beta+\frac{1}{2}}\varphi_\mu^{(\alpha,\beta)}(t). \tag{7.13}$$

For fixed $t > 0$, we have from (5.2) and (3.6)

$$u(t) = \frac{2^\alpha \Gamma(\alpha + 1)}{\mu^\alpha} t^{\frac{1}{2}}\{J_\alpha(\mu t) + O(\frac{1}{\mu})\},$$

as $\mu \to \infty$; see also (5.11). The asymptotic behavior of the Bessel function then gives

$$u(t) = \frac{2^\alpha \Gamma(\alpha + 1)}{\mu^\alpha} t^{\frac{1}{2}}\{\sqrt{\frac{2}{\pi\mu t}}\cos(\mu t - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi) + O(\frac{1}{\mu})\}, \tag{7.14}$$

as $\mu \to \infty$. Comparing (7.10) and (7.12), we conclude that $u(t) = \tilde{u}(t)$ if we choose

$$c_1 = \sqrt{\frac{1}{2\pi}}\,\frac{2^\alpha \Gamma(\alpha + 1)}{\mu^\alpha} e^{-i(\frac{1}{2}\alpha + \frac{1}{4})\pi},$$

$$c_2 = \sqrt{\frac{1}{2\pi}}\,\frac{2^\alpha\Gamma(\alpha+1)}{\mu^\alpha}e^{i(\frac{1}{2}\alpha+\frac{1}{4})\pi},$$

and consequently we obtain from (7.6) and (7.7)

$$u(t) = \frac{2^\alpha\Gamma(\alpha+1)}{\mu^\alpha}\sqrt{\frac{2}{\pi\mu}}\{\cos(\mu t - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi) + \varepsilon^*(t)\}, \tag{7.15}$$

where

$$|\varepsilon^*(t)| \le \exp\{\frac{1}{\mu}(4A+2B)e^{-2t}\} - 1. \tag{7.16}$$

Note that the error term $\varepsilon^*(t)$ is exponentially small. Indeed, we have

$$\varepsilon^*(t) = O(e^{-2t}), \tag{7.17}$$

as $t \to \infty$.

To derive an asymptotic formula for the zeros $t_{\mu,s}$ for large values of s, we will make use of the following corollary of Theorem A again stated in [5]

Theorem B *In the interval* $[n\pi - \psi - \rho, n\pi - \psi + \rho]$, *where* $\rho < \pi/2$, *suppose* $f(t) = \sin(t+\psi) + \varepsilon(t)$, $f(t)$ *is continuous and* $E = \max|\varepsilon(t)| < \sin\rho$. *Then there exists a zero* c *of* $f(t)$ *in the interval such that* $|c - (n\pi - \psi)| \le E\rho csc\rho$.

To apply this result, we rewrite (7.15) in the form

$$u(t) = \frac{2^\alpha\Gamma(\alpha+1)}{\mu^\alpha}\sqrt{\frac{2}{\pi\mu}}\{\sin(\mu t - \frac{1}{2}\alpha\pi + \frac{1}{4}\pi) + \varepsilon^*(t)\}, \tag{7.18}$$

and take $\psi = -\alpha\pi/2 + \pi/4$ and $\rho = \pi/4$. For t satisfying $s\pi + \alpha\pi/2 - \pi/2 \le \mu t \le$

$s\pi + \alpha\pi/2$, we have from (7.16)

$$|\varepsilon^*(t)| \leq \exp\{\frac{1}{\mu}(4A + 2B)e^{\frac{-2(s\pi+\frac{1}{2}\alpha\pi-\frac{1}{2}\pi)}{\mu}}\} - 1. \tag{7.19}$$

Note that if $A = B = 0$ then $\varepsilon^*(t) = 0$; cf [7, (2.11)]. Since $2 > 1/\ln(1 + 1/\sqrt{2})$, to have the right-hand side of (7.19) bounded by $\sin(\pi/4) = 1/\sqrt{2}$, we require

$$(s + \frac{1}{2}\alpha - \frac{1}{2})\pi \geq \frac{\mu}{2}\ln\{\frac{2}{\mu}(4A + 2B)\}. \tag{7.20}$$

Furthermore, since $e^x - 1 \leq xe^x$, coupling (7.19) and (7.20) gives

$$E = \max|\varepsilon^*(t)| \leq 1.7072\,(4A + 2B)e^{\frac{-2(s\pi+\frac{1}{2}\alpha\pi-\frac{1}{2}\pi)}{\mu}}\frac{1}{\mu}. \tag{7.21}$$

From Theorem B, it follows that

$$|t_{\mu,s} - (s + \frac{1}{2}\alpha - \frac{1}{4})\frac{\pi}{\mu}| \leq 1.8963\,(4A + 2B)e^{\frac{-2(s\pi+\frac{1}{2}\alpha\pi-\frac{1}{2}\pi)}{\mu}}\frac{1}{\mu^2}, \tag{7.22}$$

for all integer s satisfying (7.20). In particular, we have

$$t_{\mu,s} = (s + \frac{1}{2}\alpha - \frac{1}{4})\frac{\pi}{\mu} + O(e^{-\frac{2\pi s}{\mu}}), \tag{7.23}$$

as $s \to \infty$.

Chapter 8

Zeros of Jacobi Polynomial

In this chapter, we shall consider the approximations of zeros of Jacobi polynomial $P_n^{(\alpha,\beta)}(\cos x)$, as $n \to \infty$, by using Olver's differential equations theory [11, Chap.12] and Theorem A.

It is well known that the differential equation

$$\frac{d^2w}{dx^2} + \{\mu^2 + \frac{\frac{1}{4} - \alpha^2}{4\sin^2\frac{x}{2}} + \frac{\frac{1}{4} - \beta^2}{4\cos^2\frac{x}{2}}\}w(x) = 0, \tag{8.1}$$

$$\mu = n + \frac{\alpha + \beta + 1}{2},$$

is satisfied by

$$w(x) = (\sin\frac{x}{2})^{\alpha+\frac{1}{2}}(\cos\frac{x}{2})^{\beta+\frac{1}{2}}P_n^{(\alpha,\beta)}(\cos x), \tag{8.2}$$

for $0 < x < \pi$. Observing the similarity between (8.1) and (2.3), we can apply the method used in previous chapters to determine zeros of $P_n^{(\alpha,\beta)}(\cos x)$. We still assume that α and β are real numbers with $\alpha \geq 0$. Without loss of generality, we may also restrict our attention to $0 < x \leq \pi/2$, since

$$P_n^{(\alpha,\beta)}(\cos x) = (-1)^n P_n^{(\beta,\alpha)}(-\cos x).$$

By introducing new variables $(-\zeta)^{1/2} = x$ and $W(\zeta) = (2x)^{1/2}w(x)$, $\zeta < 0$, we have

$$W''(\zeta) = \{\frac{\mu^2}{4\zeta} + \frac{\alpha^2 - 1}{4\zeta^2} + \frac{\psi(\zeta)}{\zeta}\}W(\zeta), \tag{8.3}$$

in which

$$\psi(\zeta) = \frac{1 - 4\alpha^2}{16\zeta} + \frac{1}{4}\{\frac{\frac{1}{4} - \alpha^2}{4\sin^2\frac{x}{2}} + \frac{\frac{1}{4} - \beta^2}{4\cos^2\frac{x}{2}}\}. \tag{8.4}$$

Applying Theorem 4.1 [11, p.444], we obtain an asymptotic expansion for Jacobi polynomial (see [2])

$$\begin{aligned}(\sin\frac{x}{2})^{\alpha+\frac{1}{2}}(\cos\frac{x}{2})^{\beta+\frac{1}{2}}P_n^{(\alpha,\beta)}(\cos x) \\ = 2^{-\frac{1}{2}}\frac{\Gamma(n+\alpha+1)}{n!\mu^\alpha}x^{-\frac{1}{2}}\{xJ_\alpha(\mu x)\sum_{s=0}^{p}\frac{A_s(-x^2)}{\mu^{2s}} \\ -\frac{x^2}{\mu}J_{\alpha+1}(\mu x)\sum_{s=0}^{p-1}\frac{B_s(-x^2)}{\mu^{2s}} + \varepsilon_{2p+1,1}(\mu, -x^2)\},\end{aligned} \tag{8.5}$$

where

$$\begin{aligned}|\varepsilon_{2p+1,1}(\mu, -x^2)| \le \lambda_3(\alpha)xE_\alpha^{-1}(\mu x)M_\alpha(\mu x) \\ \times \exp\{\frac{\lambda_2(\alpha)}{\mu}\mathcal{V}_{0,x}(xB_0)\}\frac{\mathcal{V}_{0,x}(xB_p)}{\mu^{2p+1}}.\end{aligned} \tag{8.6}$$

And we can also derive an expansion which yields an even terms as following

$$\begin{aligned}(\sin\frac{x}{2})^{\alpha+\frac{1}{2}}(\cos\frac{x}{2})^{\beta+\frac{1}{2}}P_n^{(\alpha,\beta)}(\cos x) \\ = 2^{-\frac{1}{2}}\frac{\Gamma(n+\alpha+1)}{n!\mu^\alpha}x^{-\frac{1}{2}}\{xJ_\alpha(\mu x)\sum_{s=0}^{p-1}\frac{A_s(-x^2)}{\mu^{2s}}\end{aligned} \tag{8.7}$$

$$-\frac{x^2}{\mu}J_{\alpha+1}(\mu x)\sum_{s=0}^{p-1}\frac{B_s(-x^2)}{\mu^{2s}}+\varepsilon_{2p,1}(\mu,-x^2)\},$$

where

$$|\varepsilon_{2p,1}(\mu,-x^2)|\le\bar{\lambda}_3(\alpha)xE_\alpha^{-1}(\mu x)M_\alpha(\mu x)$$
$$\times\exp\{\frac{\lambda_2(\alpha)}{\mu}\mathcal{V}_{0,x}(xB_0)\}\frac{\mathcal{V}_{0,x}(A_p)}{\mu^{2p}} \tag{8.8}$$

(see Appendix A). The coefficients $A_s(-x^2)$ and $B_s(-x^2)$ are defined by the recursive formulas (3.8) and (3.9), with $\psi(\zeta)$ given in (8.4) and $A_s(0)=0$ $(s=1,2,...)$. The explicit determinations of the first coefficients given by

$$xB_0(-x^2)=-\frac{1}{4}\{(\frac{1}{4}-\alpha^2)(\cot\frac{x}{2}-\frac{2}{x})-(\frac{1}{4}-\beta^2)\tan\frac{x}{2}\}, \tag{8.9}$$

and

$$\begin{aligned}A_1(-x^2)=&-\frac{1}{4}(\frac{1}{2}+\alpha)x^{-1}\{(\frac{1}{4}-\alpha^2)(\cot\frac{x}{2}-\frac{2}{x})-(\frac{1}{4}-\beta^2)\tan\frac{x}{2}\}\\&+\frac{1}{16}\{(\frac{1}{4}-\alpha^2)(\frac{4}{x^2}-\csc^2\frac{x}{2})-(\frac{1}{4}-\beta^2)\sec^2\frac{x}{2}\}\\&-\frac{1}{8}\{(\frac{1}{4}-\alpha^2)^2(x^{-2}-x^{-1}\cot\frac{x}{2}+\frac{1}{4}\csc^2\frac{x}{2})\\&+(\frac{1}{4}-\alpha^2)(\frac{1}{4}-\beta^2)x^{-1}\tan\frac{x}{2}+\frac{1}{4}(\frac{1}{4}-\beta^2)^2\sec^2\frac{x}{2}\}+C_1,\end{aligned} \tag{8.10}$$

where

$$C_1=\frac{1}{32}\{(\frac{1}{4}-\alpha^2)+(\frac{1}{4}-\beta^2)\}^2-\frac{1}{8}\alpha\{\frac{1}{3}(\frac{1}{4}-\alpha^2)+(\frac{1}{4}-\beta^2)\}.$$

In order to find error bounds for the asymptotic approximations of zeros of $P_n^{(\alpha,\beta)}(\cos x)$, the total variations $\mathcal{V}_{0,x}\{xB_0(-x^2)\}$ and $\mathcal{V}_{0,x}\{A_1(-x^2)\}$ should be estimated. Carrying out analysis similar to Chapter 4, and observing that $x\le\pi/2$,

we derive

$$\mathcal{V}_{0,x}\{xB_0(-x^2)\} \le \frac{x}{2\pi}(0.2733A + B), \tag{8.11}$$

$$\le \frac{1}{4}(0.2733A + B), \tag{8.12}$$

$$\mathcal{V}_{0,x}\{A_1(-x^2)\} \le \frac{x^2}{4}\,V(\alpha,\beta) \le \mathcal{V}_{0,\frac{\pi}{2}}\{A_1(-x^2)\},$$

where

$$V(\alpha,\beta) = [0.0030(\frac{1}{2}+\alpha) + 0.0047]\,A + 0.0038\,A^2 \tag{8.13}$$

$$+0.0277(\frac{3}{2}+\alpha)(\frac{1}{2}+\alpha)\,B + 0.1014|(\frac{1}{4}-\beta^2) + \frac{1}{2}\,B^2|,$$

$$\mathcal{V}_{0,\frac{\pi}{2}}\{A_1(-x^2)\} = [0.0019(\frac{1}{2}+\alpha) + 0.0029]\,A + 0.0024\,A^2 \tag{8.14}$$

$$+0.0171(\frac{3}{2}+\alpha)(\frac{1}{2}+\alpha)\,B + 0.0625|(\frac{1}{4}-\beta^2) + \frac{1}{2}\,B^2|,$$

and

$$A = |\frac{1}{4} - \alpha^2|, \qquad B = |\frac{1}{4} - \beta^2|.$$

The first approximation for zeros $x_{\mu,s}$ of $P_n^{(\alpha,\beta)}(\cos x)$, complete with error bound, can be stated by following theorem:

Theorem 4 *If* $\mu^{1/2} \ge \max\{2, e\,\lambda_2(\alpha)\,\lambda_3(\alpha)(0.2733A + B)/4\}$, *then the zeros* $x_{\mu,s}$ *of the Jacobi polynomial* $P_n^{(\alpha,\beta)}(\cos x)$ *satisfy the uniform asymptotic estimates*

$$|x_{\mu,s} - \frac{j_{\alpha,s}}{\mu}| \le 0.6796\,\lambda_2(\alpha)\,\lambda_3(\alpha)(0.2733A + B)\frac{1}{\mu^2}, \tag{8.15}$$

$$|x_{\mu,s} - \frac{j_{\alpha,s}}{\mu}| \le 0.5227\,\lambda_2(\alpha)\,\lambda_3(\alpha)(0.2733A + B)\frac{j_{\alpha,s}}{\mu^3}. \tag{8.16}$$

Obviously, the second approximation is more accurate for small s and large μ.

The proof is similar to that of Theorem 1. Taking $p = 0$ in (8.5),we have

$$(\sin\frac{x}{2})^{\alpha+\frac{1}{2}}(\cos\frac{x}{2})^{\beta+\frac{1}{2}}P_n^{(\alpha,\beta)}(\cos x)$$

$$= \frac{2^{-\frac{1}{2}}\Gamma(n+\alpha+1)}{n!\mu^\alpha}x^{\frac{1}{2}}\{J_\alpha(\mu x) + x^{-1}\varepsilon_{1,1}(\mu, -x^2)\}. \tag{8.17}$$

Set

$$f(x) = \frac{2^{\frac{1}{2}}n!\mu^\alpha}{\Gamma(n+\alpha+1)}x^{-\frac{1}{2}}(\sin\frac{x}{2})^{\alpha+\frac{1}{2}}(\cos\frac{x}{2})^{\beta+\frac{1}{2}}$$
$$\times E_\alpha(\mu x)M_\alpha^{-1}(\mu x)P_n^{(\alpha,\beta)}(\cos x),$$

$$g(x) = \cos\theta_\alpha(\mu x),$$

$$\varepsilon(x) = x^{-1}E_\alpha(\mu x)M_\alpha^{-1}(\mu x)\varepsilon_{1,1}(\mu, -x^2),$$

and let $a_s = j_{\alpha,s}/\mu$, $\rho = \mu^{-3/2}$. Since for any $x \in [a_s - \rho, a_s + \rho]$, and $\mu \geq 2$, we have

$$\mu x = \mu a_s - \mu(x - a_s)$$

$$\geq j_{\alpha,s} - \mu^{-\frac{1}{2}} > y_{\alpha,1} > X_\alpha. \tag{8.18}$$

From (5.17) and (5.21)

$$m = \min|g'(x)| \geq \frac{\mu}{\lambda_2(\alpha)},$$

$$|g(a_s \pm \rho)| \geq \frac{1}{\lambda_2(\alpha)}\mu^{-\frac{1}{2}},$$

if $\mu^{1/2} \geq 2$. In view of (8.6) and (8.12), we have

$$E = \max|\varepsilon(x)| \leq \frac{e}{4}\lambda_3(\alpha)(0.2733A + B)\frac{1}{\mu},$$

if

$$\mu \geq \lambda_2(\alpha)(0.2733A + B)/4. \tag{8.19}$$

On the other hand, coupling (8.6) and (8.11), and observing from (8.18) that $x \leq j_{\alpha,s}/\mu + \mu^{-3/2}$, we also have

$$E \leq 0.5227\,\lambda_3(\alpha)(0.2733A + B)\frac{j_{\alpha,s}}{\mu^2}.$$

when $\mu^{1/2} \geq 2$ and (8.19) holds. If we now suppose that

$$\mu^{\frac{1}{2}} \geq \max\{2, \frac{e}{4}\lambda_2(\alpha)\lambda_3(\alpha)(0.2733A + B)\},$$

then conditions (5.1) and (8.19) are satisfied (see the arguments below (5.24)) , the inequalities (8.15) and (8.16) may be obtained.

Consider now the second approximation of zeros $x_{\mu,s}$ of $P_n^{(\alpha,\beta)}(\cos x)$. Let $p = 1$ in (8.7). By making use of the recurrence relation $J_{\alpha+1}(x) = -J'_\alpha(x) + \mu J_\alpha(x)/x$ and Taylor's theorem, we have

$$\begin{aligned}
&(\sin\frac{x}{2})^{\alpha+\frac{1}{2}}(\cos\frac{x}{2})^{\beta+\frac{1}{2}}P_n^{(\alpha,\beta)}(\cos x) \\
&\quad = \frac{2^{-\frac{1}{2}}\Gamma(n+\alpha+1)}{n!\mu^\alpha}x^{\frac{1}{2}}\{J_\alpha(\mu x + \frac{xB_0(-x^2)}{\mu}) \\
&\quad\quad -\frac{x^2B_0^2(-x^2)}{2\mu^2}J''_\alpha(xi) - \frac{\alpha B_0(-x^2)}{\mu^2}J_\alpha(\mu x) + x^{-1}\varepsilon_{2,1}(\mu, -x^2)\},
\end{aligned} \tag{8.20}$$

in which $|\xi - \mu x| \leq |xB_0(-x^2)/\mu|$. Again, we let b_s denote the roots of the equation

$$J_\alpha(\mu x + \frac{xB_0(-x^2)}{\mu}) = 0, \tag{8.21}$$

and note that

$$|B_0(-x^2)| \leq \frac{1}{2\pi}(0.2733A + B), \tag{8.22}$$

$$|(xB_0(-x^2))'| \leq \frac{1}{4}(0.1895A + B), \tag{8.23}$$

for $0 \leq x \leq \pi/2$. Then

$$b_s = \gamma_s - \frac{\gamma_s B_0(-\gamma_s^2)}{\mu^2} + \sigma_s(\mu), \tag{8.24}$$

where $\gamma_s = j_{\alpha,s}/\mu$, and

$$|\sigma_s(\mu)| \leq 0.0404(0.2733A + B)(0.1895A + B)\frac{j_{\alpha,s}}{\mu^5}, \tag{8.25}$$

provided that $\mu \geq 5$, and $\mu^{1/2} \geq (0.2733A + B)$.

With $x \in [b_s - \rho, b_s + \rho]$, $\rho = \mu^{-5/2}$, $\mu \geq 5$, and $\mu^{1/2} \geq (0.2733A + B)$, it can be verified that $\mu x + xB_0(-x^2)/\mu > y_{\alpha,1} > X_\alpha$ and $\mu x > y_{\alpha,1} > X_\alpha$. Furthermore, we have following estimates

$$|\mu x + \frac{xB_0(-x^2)}{\mu} - j_{\alpha,s}| \leq 0.0915,$$

$$|\mu x - j_{\alpha,s}| \leq 0.2013,$$

and

$$|\mu x - j_{\alpha,s}| \leq 0.5773\ j_{\alpha,s}\mu^{-\frac{3}{2}}.$$

Define

$$f(x) = \frac{2^{\frac{1}{2}}n!\mu^\alpha}{\Gamma(n+\alpha+1)}(\sin\frac{x}{2})^{\alpha+\frac{1}{2}}(\cos\frac{x}{2})^{\beta+\frac{1}{2}}x^{-\frac{1}{2}}$$
$$\times P_n^{(\alpha,\beta)}(\cos x)M_\alpha^{-1}(\mu x + \frac{xB_0(-x^2)}{\mu}),$$

$$g(x) = \cos\theta_\alpha(\mu x + \frac{xB_0(-x^2)}{\mu}),$$

$$\varepsilon(x) = M_\alpha^{-1}(\mu x + \frac{xB_0(-x^2)}{\mu})\{-\frac{x^2B_0^2(-x^2)}{2\mu^2}J''_\alpha(\xi)$$

$$-\frac{\alpha B_0(-x^2)}{\mu^2}J_\alpha(\mu x) + x^{-1}\varepsilon_{2,1}(\mu, -x^2)\}.$$

and assume that $\mu \geq 5$, $\mu^{1/2} \geq (0.2733A + B)$, $\mu \geq \lambda_2(\alpha)(0.2733A + B)/4$, then, in a similar manner to Chapter 6, we can obtain following estimates

$$E \leq 3.0005\, C_1(\alpha,\beta)\frac{1}{\mu^2}, \qquad \text{if} \quad j_{\alpha,s} \geq (\alpha+1)^2 + 0.0631, \tag{8.26}$$

$$C_1(\alpha,\beta) = 0.0625(0.2733A + B)^2 \tag{8.27}$$

$$+0.0641\ \alpha(0.2733A + B) + e\ \bar{\lambda}_3(\alpha)\mathcal{V}_{0,\frac{\pi}{2}}(A_1);$$

$$E \leq 3.0005\, C_2(\alpha,\beta)\frac{j_{\alpha,s}^{\frac{3}{2}}}{\mu^{\frac{7}{2}}}, \qquad \text{if} \quad j_{\alpha,s} \geq (\alpha+1)^2 + 0.0631, \tag{8.28}$$

$$C_2(\alpha,\beta) = 0.0363(0.2733A + B)^2 \tag{8.29}$$

$$+0.1186\ \alpha(0.2733A + B) + 0.3564\ e\ \bar{\lambda}_3(\alpha)V(\alpha,\beta);$$

$$E \leq 1.2901\, C_3(\alpha,\beta)\frac{j_{\alpha,s}^{\frac{1}{2}}}{\mu^2}, \tag{8.30}$$

$$C_3(\alpha,\beta) = 0.0775(0.2733A + B)^2 \tag{8.31}$$

$$+0.0641\ \alpha(0.2733A + B) + e\ \bar{\lambda}_3(\alpha)\mathcal{V}_{0,\frac{\pi}{2}}(A_1);$$

$$E \leq 1.2901\, C_4(\alpha,\beta)\frac{j_{\alpha,s}^{2}}{\mu^{\frac{7}{2}}}, \tag{8.32}$$

$$C_4(\alpha,\beta) = 0.0449(0.2733A + B)^2 \tag{8.33}$$

$$+0.1186\ \alpha(0.2733A + B) + 0.3564\ e\ \bar{\lambda}_3(\alpha)V(\alpha,\beta),$$

and

$$m \geq \frac{1.7720}{\lambda_2(\alpha)}\mu, \tag{8.34}$$

$$\{|g(b_s - \rho)|, |g(b_s + \rho)|\} \geq \frac{1.7720}{\lambda_2(\alpha)\mu^{-\frac{3}{2}}}, \tag{8.35}$$

where $E = \max|\varepsilon(x)|$, $m = \min|g'(x)|$ with $x \in [b_s - \rho, b_s + \rho]$, $V(\alpha, \beta)$ given in (8.13). Here we have used the fact that x is less than or equal to $\pi/2$, hence, it follows from (8.18) that $j_{\alpha,s}/\mu \leq x + \mu^{-3/2} \leq 1.6603$.

If we let s_α be the smallest integer, such that

$$s_\alpha \geq 1.38\,|4\alpha^2 - 1| + 0.1367\,\alpha + 0.7114, \tag{8.36}$$

it is easy to verify from (6.72) that for all $s \geq s_\alpha$

$$j_{\alpha,s} \geq j_{\alpha,s_\alpha} \geq (\alpha + 1)^2 + 0.0631.$$

Then, by using estimates (8.26), (8.28), (8.34), (8.35), and Theorem A, we obtain

Theorem 5 *Let $\alpha \geq 0$ and $\mu \geq 5$. Suppose that μ also satisfies* (i) $\mu \geq (0.2733A + B)^2$ *and* (ii) $\mu \geq \lambda_2(\alpha)(0.2733A + B)/4$. *Let $x_{\mu,s}$ denote the zeros of the Jacobi polynomial, $0 \leq x \leq \pi/2$, and b_s denote the roots of equation (8.21). Furthermore, let s_α be defined as in (8.36). Then for all $s \geq s_\alpha$ and $\mu^{1/2} \geq 1.6933\,\lambda_2(\alpha)C_1(\alpha, \beta)$, we have*

$$|x_{\mu,s} - b_s| \leq 1.6933\,\lambda_2(\alpha)C_1(\alpha, \beta)\mu^{-3}, \tag{8.37}$$

and

$$|x_{\mu,s} - b_s| \leq 1.6933\,\lambda_2(\alpha)C_2(\alpha,\beta)\frac{j_{\alpha,s}^{3/2}}{\mu^{9/2}}, \tag{8.38}$$

where $C_1(\alpha,\beta)$ *and* $C_2(\alpha,\beta)$ *are given in (8. 27) and (8.29),respectively.*

From (8.24) and (8.25), we can see that (8.37) gives the same order estimate as that in (6.4). While for small s and large μ, (8.38) is a better result. On the other hand, if we use (8.30) and (8.32) instead of (8.26) and (8.28) in Theorem 5, the results are given by

Theorem 6 *Under the same conditions as in Theorem 5, we have for all* $s \geq 1$

$$|x_{\mu,s} - b_s| \leq 0.7281\,\lambda_2(\alpha)C_3(\alpha,\beta)\frac{j_{\alpha,s}^{1/2}}{\mu^3}, \tag{8.39}$$

if $\mu^{1/2} \geq 0.7281\,\lambda_2(\alpha)C_3(\alpha,\beta)j_{\alpha,s}^{1/2}$ *and*

$$|x_{\mu,s} - b_s| \leq 0.7281\,\lambda_2(\alpha)C_4(\alpha,\beta)\frac{j_{\alpha,s}^{2}}{\mu^{9/2}}, \tag{8.39}$$

if $\mu^2 \geq 0.7281\,\lambda_2(\alpha)C_4(\alpha,\beta)j_{\alpha,s}^{2}$, *where the constants* $C_3(\alpha,\beta)$ *and* $C_4(\alpha,\beta)$ *are given in (8.31) and (8.33), respectively.*

Appendix A

It can be verified that equation (2.5) is formally satisfied by the expansion

$$W = |\zeta|^{\frac{1}{2}} J_\alpha(\mu|\zeta|^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{A_s(\zeta)}{\mu^{2s}} - \frac{|\zeta|}{\mu} J_{\alpha+1}(\mu|\zeta|^{\frac{1}{2}}) \sum_{s=0}^{\infty} \frac{B_s(\zeta)}{\mu^{2s}}, \qquad (A.1)$$

in which $A_s(\zeta)$ and $B_s(\zeta)$ denote the analytic coefficients determined by the recursive formulas (3.8) and (3.9) with $A_0(\zeta) = 1$. Instead of taking odd number of terms in (A.1) (see (3.5)), we assume

$$W_{2n,1}(\mu,\zeta) = (-\zeta)^{\frac{1}{2}} J_\alpha(\mu(-\zeta)^{\frac{1}{2}}) \sum_{s=0}^{n-1} \frac{A_s(\zeta)}{\mu^{2s}}$$
$$- \frac{(-\zeta)}{\mu} J_{\alpha+1}(\mu(-\zeta)^{\frac{1}{2}}) \sum_{s=0}^{n-1} \frac{B_s(\zeta)}{\mu^{2s}} + \varepsilon_{2n,1}(\mu,\zeta), \qquad (A.2)$$

then, it is found that

$$\varepsilon''_{2n,1}(\mu,\zeta) - \{\frac{\mu^2}{4\zeta} + \frac{\alpha^2-1}{4\zeta^2}\}\varepsilon_{2n,1}(\mu,\zeta)$$
$$= \frac{\psi(\zeta)}{\zeta}\varepsilon_{2n,1}(\mu,\zeta) + J_{\alpha+1}(\mu(-\zeta)^{\frac{1}{2}})\frac{A'_n(\zeta)}{\mu^{2n-1}}, \qquad (A.3)$$

primes denoting differentiations with respect to ζ.

The complementary functions associated with the left-hand side of this equation are $(-\zeta)^{\frac{1}{2}}J_\alpha(\mu(-\zeta)^{\frac{1}{2}})$ and $(-\zeta)^{\frac{1}{2}}Y_\alpha(\mu(-\zeta)^{\frac{1}{2}})$, with Wronskian $-1/\pi$. Therefore, by using the method of variation of parameters, we have

$$\varepsilon_{2n,1}(\mu,\zeta) = \int_\zeta^0 \pi(\zeta t)^{\frac{1}{2}}\{J_\alpha(\mu(-\zeta)^{\frac{1}{2}})Y_\alpha(\mu(-t)^{\frac{1}{2}}) - Y_\alpha(\mu(-\zeta)^{\frac{1}{2}})J_\alpha(\mu(-t)^{\frac{1}{2}})\}$$

$$\times\{\frac{\psi(t)}{t}\varepsilon_{2n,1}(\mu,t) + \frac{1}{\mu^{2n-1}}J_{\alpha+1}(\mu(-t)^{\frac{1}{2}})\frac{d}{dt}(A_n(t))\}dt. \qquad (A.4)$$

Let ζ be replaced by $-\tilde{\zeta}$, (A.4) can be rewritten in the form

$$\varepsilon_{2n,1}(\mu,-\tilde{\zeta}) = \int_0^{\tilde{\zeta}} K(\tilde{\zeta},v)\{\frac{\psi(-v)}{v^{1/2}}\varepsilon_{2n,1}(\mu,-v)$$

$$+\frac{1}{\mu^{2n-1}}v^{\frac{1}{2}}J_{\alpha+1}(\mu v^{\frac{1}{2}})\frac{d}{dv}(A_n(-v))\}dv, \qquad (A.5)$$

where

$$K(\tilde{\zeta},v) = -\pi\tilde{\zeta}^{\frac{1}{2}}\{J_\alpha(\mu\tilde{\zeta}^{\frac{1}{2}})Y_\alpha(\mu v^{\frac{1}{2}}) - Y_\alpha(\mu\tilde{\zeta}^{\frac{1}{2}})J_\alpha(\mu v^{\frac{1}{2}})\}. \qquad (A.6)$$

Since the Bessel functions can be expressed as

$$J_\alpha(x) = E_\alpha^{-1}(x)M_\alpha(x)\cos\theta_\alpha(x),$$

$$Y_\alpha(x) = E_\alpha(x)M_\alpha(x)\sin\theta_\alpha(x),$$

and $E_\alpha(x)$ is a nonincreasing function of α, for $\alpha \geq 0$ and $x > 0$ (see [11, pp.437,443]), we derive

$$|K(\tilde{\zeta},v)| \leq \pi\tilde{\zeta}^{\frac{1}{2}}E_\alpha^{-1}(\mu\tilde{\zeta}^{\frac{1}{2}})E_\alpha(\mu v^{\frac{1}{2}})M_\alpha(\mu\tilde{\zeta}^{\frac{1}{2}})M_\alpha(\mu v^{\frac{1}{2}}),$$

where $0 < v \leq \tilde{\zeta}$. By applying Theorem 10.1 in [11, p.219] with

$$h(\tilde{\zeta}) = \varepsilon_{2n,1}(\mu,-\tilde{\zeta}),$$

$$\phi(v) = \frac{1}{\mu^{2n}}\frac{d}{dv}(A_n(-v)),$$

$$J(v) = \mu v^{\frac{1}{2}} J_{\alpha+1}(\mu v^{\frac{1}{2}}),$$

$$\psi_0(v) = \frac{\psi(-v)}{v^{1/2}},$$

$$\psi_1(v) \equiv 0,$$

and subsequently replacing $-\tilde{\zeta}$ by ζ, we obtain the inequality

$$|\varepsilon_{2n,1}(\mu,\zeta)| \leq \bar{\lambda}_3(\alpha)(-\zeta)^{\frac{1}{2}} E_\alpha^{-1}(\mu(-\zeta)^{\frac{1}{2}}) M_\alpha(\mu(-\zeta)^{\frac{1}{2}})$$

$$\times \exp\{\frac{\lambda_2(\alpha)}{\mu} \mathcal{V}_{\zeta,0}[(-\zeta)^{\frac{1}{2}} B_0(\zeta)]\} \frac{\mathcal{V}_{\zeta,0}[A_n(\zeta)]}{\mu^{2n}}, \qquad (A.7)$$

where

$$\bar{\lambda}_3(\alpha) = \sup_{x\in(0,\infty)} \{\pi x E_\alpha(x) M_\alpha(x) |J_{\alpha+1}(x)|\}. \qquad (A.8)$$

Appendix B

Hethcote in his dissertation [5] has shown that for $\alpha \geq 1$, the first positive zero $j_{\alpha,1}$ of $J_\alpha(x)$ satisfies

$$j_{\alpha,1} = \alpha + 1.85576\,\alpha^{\frac{1}{3}} + c\,\alpha^{-\frac{1}{3}}, \qquad (B.1)$$

where $0.500 \leq c \leq 1.357$. Similarly, we can prove that if $y_{\alpha,1}$ is the first positive zero of $Y_\alpha(x)$, then

$$y_{\alpha,1} = \alpha + 0.9316\,\alpha^{\frac{1}{3}} + b\,\alpha^{-\frac{1}{3}}, \qquad (B.2)$$

where $0.0267 \leq b \leq 0.4888$ for $\alpha \geq 2.5$.

To prove (B.2), let us consider the first approximation for the Bessel function of the second kind:

$$Y_\alpha(\alpha z) = -\frac{1}{\alpha^{\frac{1}{3}}}\left(\frac{4\zeta}{1-z^2}\right)^{\frac{1}{4}}\{B_i(\alpha^{\frac{2}{3}}\zeta) + \epsilon_{1,2}\}, \qquad (B.3)$$

where

$$|\epsilon_{1,2}| \leq \frac{2\lambda_2 e^{\frac{n_0}{\alpha}}}{\alpha} E(\alpha^{\frac{2}{3}}\zeta) M(\alpha^{\frac{2}{3}}\zeta) \mathcal{V}_{-\infty,\zeta}(|\zeta|^{\frac{1}{2}} B_0), \qquad (B.4)$$

for real positive α and z [10, p.209]. $B_i(x)$ is the second kind of Airy function, ζ is defined by the transformation

$$\zeta\left(\frac{d\zeta}{dz}\right)^2 = \frac{1-z^2}{z^2}, \qquad \zeta(1) = 0. \qquad (B.5)$$

And

$$E(x) = \begin{cases} \exp(\frac{2}{3}x^{\frac{3}{2}}), & x > 0, \\ 1, & x \leq 0, \end{cases}$$

$$M(x) = \{E^2(x)A_i^2(x) + E^{-2}(x)B_i^2(x)\}^{\frac{1}{2}},$$

$$\lambda_1 = \max_{(-\infty,\infty)}\{\pi|x|^{\frac{1}{2}}M^2(x)\} \doteq 1.430,$$

$$\lambda_2 = \max_{(-\infty,\infty)}\{\pi|x^{\frac{1}{2}}B_i(x)|E^{-1}(x)M(x)\} \doteq 1.315,$$

$$\mathcal{V}_{-\infty,\infty}(|\zeta|^{\frac{1}{2}}B_0(\zeta)) \doteq 0.1051,$$

$$n_0 = 2\lambda_1\mathcal{V}_{-\infty,\infty}(|\zeta|^{\frac{1}{2}}B_0(\zeta)) \doteq 0.3006;$$

see Olver [8, p.9; 9, p.751; 10, p.207].

In order to apply Theorem A, let

$$f(\alpha^{\frac{2}{3}}\zeta) = -\alpha^{\frac{1}{3}}(\frac{1-z^2}{4\zeta})^{\frac{1}{4}}Y_\alpha(\alpha z), \tag{B.6}$$

$$g(\alpha^{\frac{2}{3}}\zeta) = B_i(\alpha^{\frac{2}{3}}\zeta), \tag{B.7}$$

$$\epsilon(\alpha^{\frac{2}{3}}\zeta) = \epsilon_{1,2}(\alpha^{\frac{2}{3}}\zeta), \tag{B.8}$$

with $\alpha^{2/3}\zeta$ as the independent variable. Since the only zeros of $B_i(x)$ occur on the negative real axis, let $b_1 = -1.1737$ denote the first zero of $B_i(x)$ with the smallest absolute value. If we take $\rho = 0.20$, then $\alpha^{2/3}\zeta \in [-1.3737, -0.9734]$, and $\alpha^{2/3}\zeta < 0$ implies

$$E(\alpha^{\frac{2}{3}}\zeta) = 1,$$

$$M(\alpha^{\frac{2}{3}}\zeta) = \{A_i^2(\alpha^{\frac{2}{3}}\zeta) + B_i^2(\alpha^{\frac{2}{3}}\zeta)\}^{\frac{1}{2}}.$$

From tables of Airy function [1, pp.476-477], $M(\alpha^{2/3}\zeta) \leq 0.5523$,

$$|\epsilon_{1,2}| \leq \frac{0.1722}{\alpha}, \qquad \text{if} \quad \alpha \geq 2.5, \tag{B.9}$$

$$\min\{|B_i(b_1-\rho)|, |B_i(b_1+\rho)|\}$$

$$\geq \min\{|B_i(-1.3)|, |B_i(-0.98)|\} \geq 0.0757, \tag{B.10}$$

and

$$\min_{x\in[b_1-\rho, b_1+\rho]} |B_i'(x)| \geq |B_i'(-1.4)| \geq 0.5816. \tag{B.11}$$

Comparing (B.9) to (B.10) for $\alpha \geq 2.5$ and by the Theorem A, we have

$$\alpha^{\frac{2}{3}}\zeta_1 = b_1 + \delta, \tag{B.12}$$

where

$$|\delta| \leq \frac{0.2961}{\alpha}, \qquad \alpha \geq 2.5. \tag{B.13}$$

From definition (B.5) and the Mean Value Theorem,

$$\frac{2}{3}(-\zeta)^{\frac{3}{2}} = (z^2-1)^{\frac{1}{2}} - \sec^{-1} z, \qquad z \geq 1, \tag{B.14}$$

$$z'(\zeta) = -\frac{z(-\zeta)^{1/2}}{(z^2-1)^{1/2}}, \tag{B.15}$$

and

$$z(\zeta) = 1 - 2^{-\frac{1}{3}}\zeta + \frac{z''(\theta)}{2!}\zeta^2, \qquad \zeta < \theta < 0. \tag{B.16}$$

Note that $z(\zeta)$ is a positive decreasing function of ζ, $y_{\alpha,1} = \alpha\, z(\zeta_1)$, by substituting (B.12) into (B.16), we obtain

$$y_{\alpha,1} = \alpha - 2^{-\frac{1}{3}}(b_1+\delta)\,\alpha + \frac{z''(\theta_1)}{2!}(b_1+\delta)^2\alpha^{-\frac{1}{3}} \tag{B.17}$$

$$= \alpha + 0.9316\,\alpha^{\frac{1}{3}} + b\,\alpha^{-\frac{1}{3}}, \qquad \alpha \geq 2.5,$$

where $\zeta_1 < \theta_1 < 0$, and

$$b = -2^{-\frac{1}{3}}\delta\,\alpha^{\frac{2}{3}} + \frac{z''(\theta_1)}{2!}(b_1+\delta)^2. \qquad (B.18)$$

It is easy to calculate that $|2^{-1/3}\delta\,\alpha^{2/3}| \le 0.1732$, $\alpha \ge 2.5$. Observing the second term of (B.18), since

$$z''(\zeta) = \frac{z[(z^2-1)^{3/2} - 2(-\zeta)^{3/2}]}{2(-\zeta)^{1/2}(z^2-1)^2},$$

and $z''(\zeta)$ is a nondecreasing function for $\zeta \le 0$, $z''(0) = 0.378$; see [5, p.85]. It can be solved from (B.12) and (B.14) that

$$\zeta_1 = (b_1+\delta)\alpha^{-\frac{2}{3}} \ge -0.7015, \qquad \alpha \ge 2.5,$$

$$z(-0.7015) \doteq 1.6487.$$

Therefore,

$$0.3591 = z''(-0.7015) \le z''(\theta_1) \le z''(0) = 0.378,$$

and

$$0.1999 \le \frac{z''(\theta_1)}{2!}(b_1+\delta)^2 \le 0.3156, \qquad \alpha \ge 2.5.$$

Hence, we arrive at the result:

$$y_{\alpha,1} = \alpha + 0.9316\,\alpha^{\frac{1}{3}} + b\,\alpha^{-\frac{1}{3}},$$

where $0.0267 \le b \le 0.4888$ for $\alpha \ge 2.5$.

In consequence of (B.1) and (B.2), we have

$$j_{\alpha,1} - y_{\alpha,1} \ge 0.9241\,\alpha^{\frac{1}{3}} + 0.0112\,\alpha^{-\frac{1}{3}} \ge 1.2624, \qquad (B.19)$$

for $\alpha \geq 2.5$. When $0 \leq \alpha \leq 2.5$, since $j_{\alpha,1}$ and $y_{\alpha,1}$ are increasing functions of α [11, pp.246, 249], from tables of the Bessel function [1; 18], it follows that

$$j_{\alpha,1} - y_{\alpha,1} \geq j_{0,1} - y_{\frac{1}{3},1} = 1.0518, \qquad 0 \leq \alpha \leq \frac{1}{3};$$

$$j_{\alpha,1} - y_{\alpha,1} \geq j_{\frac{1}{3},1} - y_{\frac{1}{2},1} = 1.3318, \qquad \frac{1}{3} \leq \alpha \leq \frac{1}{2};$$

$$j_{\alpha,1} - y_{\alpha,1} \geq j_{\frac{1}{2},1} - y_{1,1} = 0.9444, \qquad \frac{1}{2} \leq \alpha \leq 1;$$

$$j_{\alpha,1} - y_{\alpha,1} \geq j_{1,1} - y_{\frac{3}{2},1} = 1.0333, \qquad 1 \leq \alpha \leq \frac{3}{2};$$

$$j_{\alpha,1} - y_{\alpha,1} \geq j_{\frac{3}{2},1} - y_{2,1} = 1.1091, \qquad \frac{3}{2} \leq \alpha \leq 2;$$

$$j_{\alpha,1} - y_{\alpha,1} \geq j_{2,1} - y_{\frac{5}{2},1} = 1.1760, \qquad 2 \leq \alpha \leq \frac{5}{2}.$$

Comparing all differences, we conclude that for $\alpha \geq 0$

$$j_{\alpha,1} - y_{\alpha,1} \geq 0.9444. \tag{B.20}$$

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