APPLICATION OF THE REACTION CONCEPT TO THE PROBLEM OF DETERMINING MUTUAL IMPEDANCE BETWEEN A PAIR OF COUPLED DIPOLE ANTENNAS

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ABSTRACT

The problem of defining impedance parameters for cylindrical dipole antennas is discussed. A review of the one-dimensional formulation of the antenna model is outlined. The elements of the reaction concept are presented and then specialized to the case of coupled cylindrical dipoles. A method of improving the trial approximations by the reaction method is discussed. Application of the reaction method then leads to an approximate expression for mutual impedance. A computer program is then used to calculate numerical results from this formula. Finally, graphs of computed numerical impedances are presented.

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CHAPTER I

INTRODUCTION

The concept of reaction was introduced by Rumsey[1*] as a fundamental observable to simplify the formulation of boundary value problems in electromagnetic theory. Beginning with the idea that classical analyses of electromagnetic problems are based on the theory of fields which satisfy Maxwell's equations, Rumsey suggests that, from the point of view of an experimenter, the postulate of fields may be questioned on the grounds that any experiment designed to measure these fields necessarily consists of measuring the effects of the fields over a small but finite region. The postulate is therefore incompatible with the process of performing the observation. Rumsey then introduced a physical observable which he termed "reaction" and gave it the symbol $\langle a,b \rangle$.

The reaction $\langle a,b \rangle$ is a scalar, and gives a measure of the coupling between two sources "a" and "b". While the field quantities are implicitly included in the formulation, the reaction method does not attempt to measure (or compute) these fields at a point, but rather includes the information carried by the field quantities as an integrated effect over the measuring or observing device.

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* The numeral denotes reference number as listed in bibliography

Approximate solutions to many problems in electromagnetic theory may be obtained by means of the reaction technique. Consider for example, the problem of determining the mutual impedance between a pair of coupled dipole antennas. This cannot be solved directly, since the current distributions on the antennas are not known. However, if assumed current distributions are used, application of the reaction concept leads directly to an approximate solution which is stationary (in the sense of the Calculus of Variations) with respect to small variations of the assumed current distributions about the true current distributions. That is, instead of attempting to solve the actual, but more difficult problem, the reaction method seeks to replace the correct (but unknown) current distributions with approximate distributions which are then adjusted so that their reactions with certain "test" sources are correct. In essence, the procedure is to make the approximate sources "look" the same as the correct sources according to the physical tests which are inherent in the problem.

The purpose of this investigation was to formulate by means of the reaction method an approximation to the mutual impedance between a pair of coupled dipole antennas. A computer program for the calculation of numerical results is derived from this formulation, and graphs of computed mutual impedances are presented.

In Chapter II, a brief summary of the circuit aspects of a dipole antenna is presented. The end effects and the gap problem are discussed, and the one-dimensional formulation reviewed.

Chapter III outlines the elements of the reaction concept method.

In Chapter IV, the reaction concept is applied to the problem of a pair of coupled dipole antennas, and the mutual impedance approximation derived.

Curves of mutual impedance are presented in Chapter V, along with a discussion of the results obtained.

CHAPTER II

DEVELOPMENT OF THE CIRCUIT MODEL

The problem of determining the impedance parameters of an antenna system is essentially a problem of attempting to find a solution to a set of three dimensional vector wave equations that satisfies the specified boundary conditions. No general method is available to handle this. Instead, the usual approach (developed by E. Hallén[7]) is to replace the three dimensional problem by a quasi-one dimensional problem and attempt to solve the latter.

The situation is further complicated by the fact that any pratical antenna is fed from a transmission line. This aspect of the problem must be carefully examined in order to gain an understanding of the operational significance of the defined impedances. This is the so called gap problem in antenna theory.

A discussion of this gap problem and an outline of the one dimensional formulation is presented in this chapter. The one dimensional model discussed is that to which the reaction method is to be applied in Chapter IV.

I THE GAP PROBLEM AND THE END EFFECT

Before considering the problem of coupled antennas, it is well to examine the single, cylindrical dipole antenna shown in figure 2 - 1. The antenna is center driven from a transmission line with a conductor separation $b = 2\delta$. If an attempt is made to define an input impedance Z_o for this antenna, it is found that transmission line effects cannot be ignored when b has a nonzero value. That is, with nonzero separation, an impedance cannot be defined that is a property of the dipole antenna alone. However, King[7] shows that if the separation is made sufficiently small (i.e., the following inequality is satisfied, namely, $\beta b \ll 1$, where β is the phase constant), the coupling between charges on the antenna and those in adjacent parts of the line is reduced sufficiently that an impedance Z_δ can be defined as

$$Z_\delta = \frac{V_\delta}{I_\delta}$$

As δ is made to approach zero, Z_δ approaches Z_o, and the impedance so defined is a property of the antenna structure alone, independent of the circuit to which it is connected.

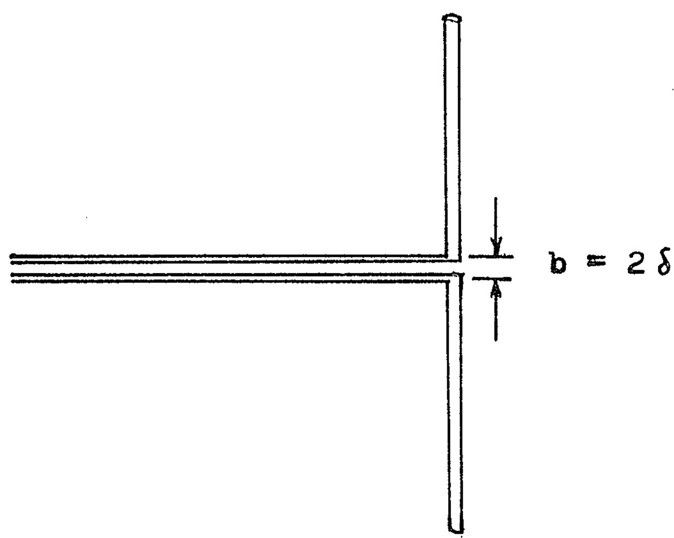

Figure 2-1. Center driven dipole antenna.

The result is a hypothetical antenna which extends unbroken from $z = -\ell$ to $z = +\ell$. In effect, the condition is equivalent to replacing the scalar potential difference V_δ across terminals that are separated by a finite distance by a discontinuity in scalar potential across terminals that are separated by a vanishingly small distance. This hypothetical driving source is termed a slice or belt generator.

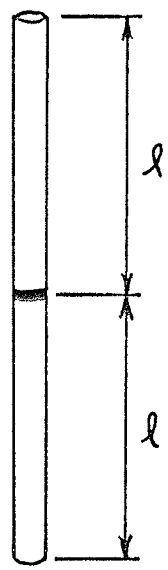

Figure 2-2. Cylindrical Antenna Driven by a Slice Generator.

Correlation of Theory and Experiment

Experiments[7] have shown that if the apparent impedance of an antenna terminating a transmission line is measured repeatedly as the spacing of the conductors is decreased progressively, and the values so obtained are extrapolated to zero line spacing, the values at zero spacing may be identified with those calculated from the configuration of figure 2-2 with $\delta = 0$. In this manner, an operational significance is given to the properties of antennas driven by slice generators.

Current Source Representation

In a preceeding section, the antenna driving mechanism was pictured as a hypothetical slice generator feeding current to the antenna conductors. However, as far as application of the reaction method is concerned, it is more convenient to use the current source representation of Harrington[8]. In this representation, the feeding mechanism is viewed as a short column of impressed current I_o existing across the gap as shown in figure 2-3. As Tai[4] points out, this representation of the gap problem leads to the same solution as the slice generator representation.

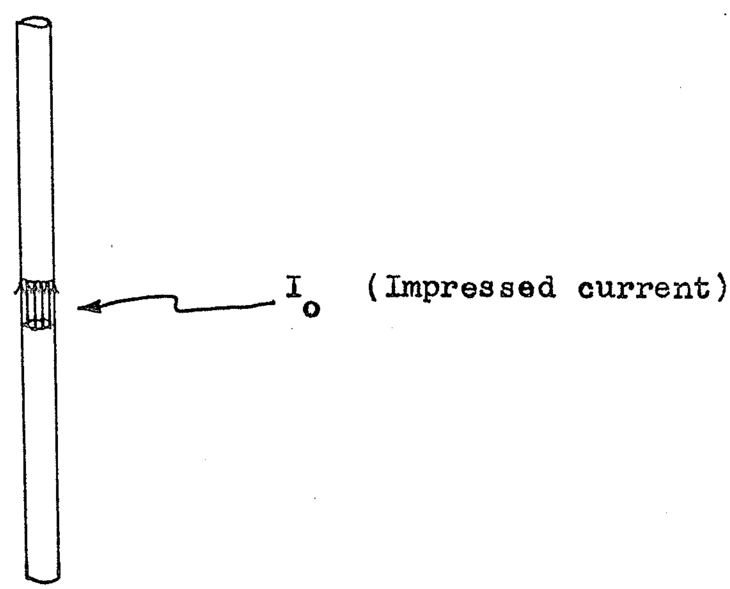

Figure 2-3. Antenna driven from a current source.

Coupled Antennas

If a pair of dipole antennas is considered, the transmission line effects noted previously are further complicated by the fact that coupling occurs not only between an antenna and its transmission line, but also between it and the transmission line of the adjacent antenna. In order to arrive at impedance parameters that are properties of the antenna configuration alone (independent of the external circuit), the same technique of using slice generators may be employed. The fundamental circuit of figure 2-4 then results. Impedance

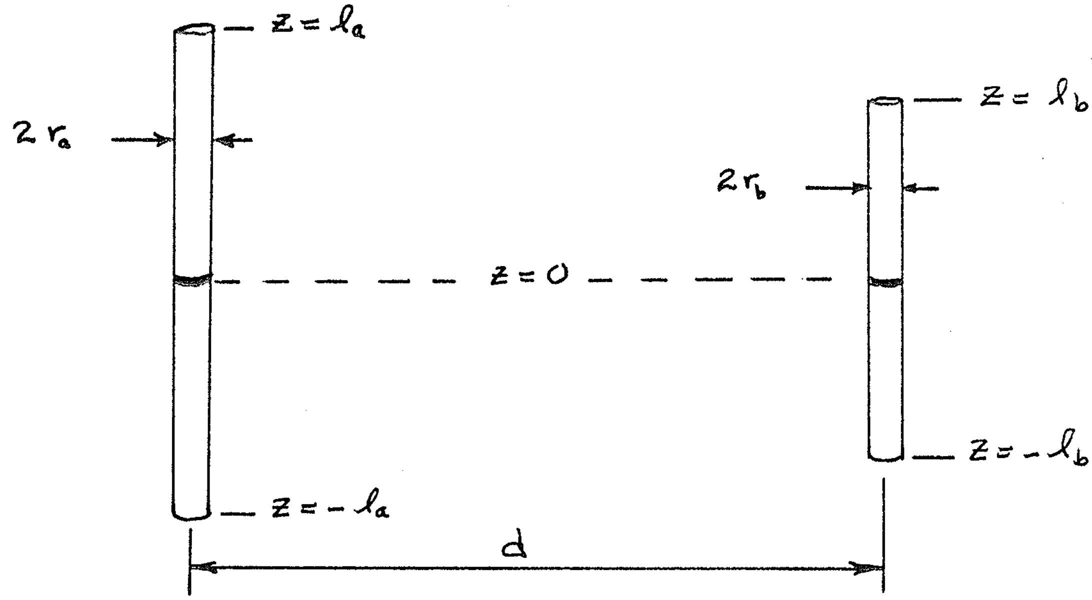

Figure 2-4. Coupled Dipole Antennas.

parameters for this case are governed only by the physical geometry of the system and the frequency. If all dimensions are expressed in wavelengths, the impedance parameters are determined only by the quantities r_a, ℓ_a, r_b, ℓ_b and d. Operational significance is again obtained by comparing the computed impedances with measured values which have been determined by extrapolating the measured results to zero gap spacing.

End Effect

In the one dimensional formulation to follow, the condition that the current is zero at the ends of the dipoles is imposed. While this is true for an antenna with hemispherical ends, it is not true when the antenna is composed of a solid cylindrical conductor with flat ends or a tube with open ends. However, for the latter two cases, King states that the effective half length of the antenna exceeds the physical half length by an amount that is difficult to determine accurately, but that is of the order of magnitude of the radius.

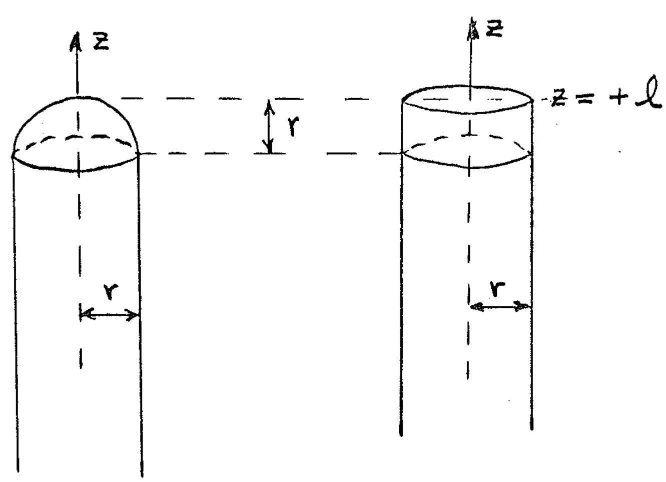

Figure 2-5. Antennas with Hemispherical and Plane Ends.

The model illustrated in figure 2-5 is a physically unavailable cylinder of radius r and half-length ℓ that has no chargeable surfaces beyond the edges at $z = \pm \ell$.

II ONE DIMENSIONAL FORMULATION

It was mentioned in a previous section that the three dimensional coupled antenna problem was not amenable to analysis and therefore, a one dimensional formulation would be presented instead. In essence, this involves replacing the volume distribution of current density in the antenna conductors by an axially distributed filamentary current along the center of the dipole. Rigorous justification of this procedure may be found in the literature[6].

Vector Potential Formulation

The electric field intensity $\mathbb{E}$ may be related to the charges and currents in a system through the potential functions Φ and $\mathbb{A}$. The defining relationships are

$$\mathbb{E} = -\nabla\Phi - \frac{\partial \mathbb{A}}{\partial t}$$

and

$$\nabla \cdot \mathbb{A} = -\mu\epsilon \frac{\partial \Phi}{\partial t}$$

When the time variations are harmonic, these reduce to

$$\mathbb{E} = -\nabla\Phi - j\omega\mathbb{A} \qquad \text{.......... } 2 - 1$$

$$\nabla\cdot\mathbb{A} = -j\omega\mu\epsilon\Phi \qquad \text{.......... } 2 - 2$$

Solving 2 - 2 for Φ and substituting into 2 - 1 yields an expression for $\mathbb{E}$ entirely in terms of $\mathbb{A}$. It is

$$\mathbb{E} = -j\frac{\omega}{\beta^2}\nabla(\nabla\cdot\mathbb{A}) - j\omega\mathbb{A} \qquad \text{.......... } 2 - 3$$

where $\beta = \omega\sqrt{\mu\epsilon}$ .... 2 - 4

The vector potential $\mathbb{A}$ is related to the currents in the system through the integral[8]

$$\mathbb{A} = \mu\iiint_{Volume}\frac{\mathbb{J}e^{-j\beta r}}{4\pi r}\,d\tau \qquad \text{.......... } 2 - 5$$

where r is the distance from the point at which $\mathbb{A}$ is being determined to the element of integration. $\mathbb{J}$ is the volume distribution of current density.

It may be seen from equation 2 - 5 that $\mathbb{A}$ is a vector in the same direction as $\mathbb{J}$. Thus, if the current distribution is entirely z-directed, $\mathbb{A}$ will have only a z-component as well. In the one dimensional formulation, only the z-component of $\mathbb{E}$ is of interest. Solving equation 2 - 3 for the z-component of $\mathbb{E}$ yields

$$E_z = -j\omega\left(A_z + \frac{1}{\beta^2}\frac{\partial^2 A_z}{\partial z^2}\right) \qquad \text{..........} \; 2 - 6$$

Consider the single dipole antenna shown in figure 2-6.

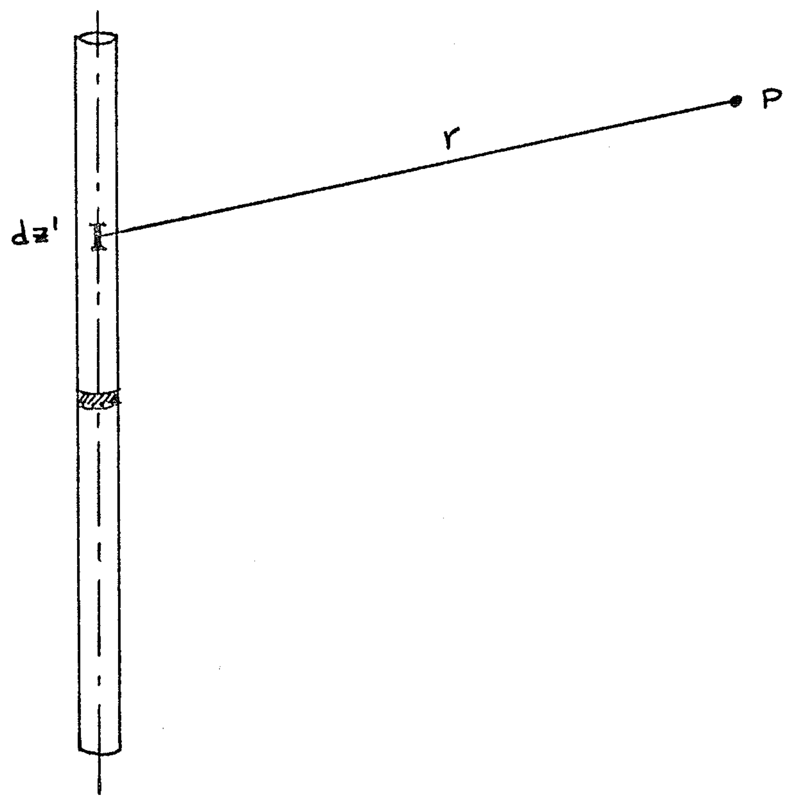

Figure 2-6. Single Dipole Antenna.

In the quasi-one dimensional formulation, the volume distribution of current density $\mathbb{J}$ is replaced by a filamentary z-directed current I(z'), and A_z calculated according to the integral

$$A_z = \frac{\mu}{4\pi} \int_{-\ell}^{\ell} I(z') \frac{e^{-j\beta r}}{r} dz' \qquad \text{.......... } 2 - 7$$

Note that the primed variable refers to the axis of the antenna.

Combining equations 2 - 6 and 2 - 7 yields an expression for E_z in terms of the current $I(z')$. It is

$$E_z = -j30\beta \int_{-\ell}^{\ell} I(z')\left(1 + \frac{1}{\beta^2}\frac{\partial^2}{\partial z^2}\right)\frac{e^{-j\beta r}}{r} dz' \qquad \text{.......... } 2 - 8$$

Consider the two antenna system of figure 2-7.

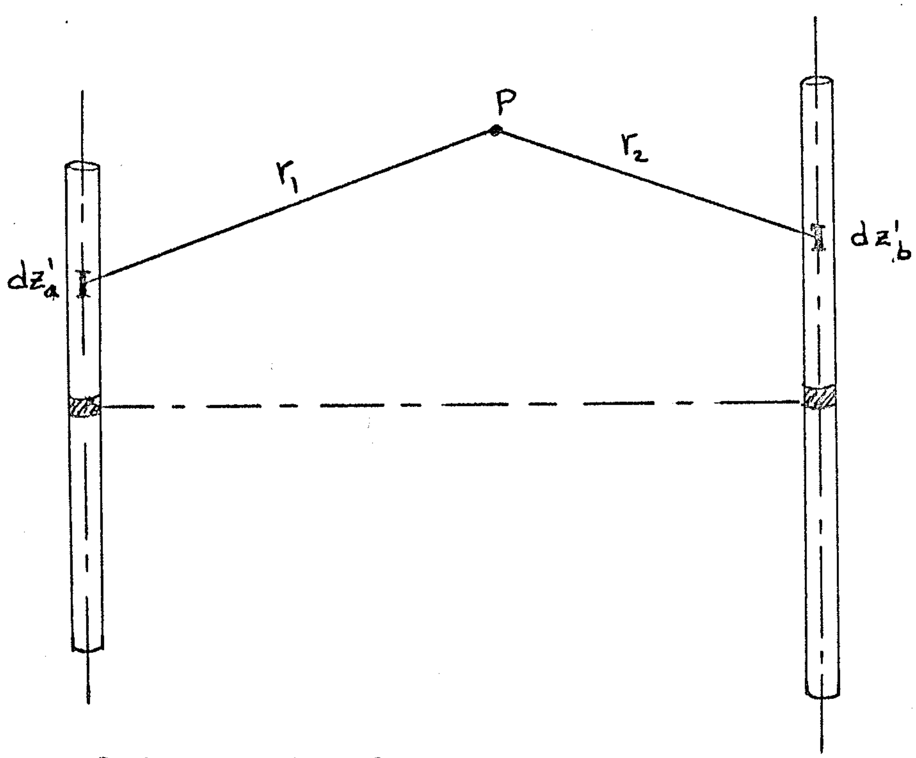

Figure 2-7. Geometry for two antennas.

The vector potential at any point P outside the antenna surfaces is obtained by summing the contributions due to each antenna. Thus

$$A_z = \frac{\mu}{4\pi}\left\{ \int_{-l_a}^{l_a} I_a(z_a')\frac{e^{-j\beta r_1}}{r_1}\,dz_a' + \int_{-l_b}^{l_b} I_b(z_b')\frac{e^{-j\beta r_2}}{r_2}\,dz_b' \right\} \qquad \text{..........} \; 2-9$$

If the true current distributions $I_a(z_a')$ and $I_b(z_b')$ were known, it would theoretically be possible to perform the integration and obtain A_z. Application of equation 2 - 6 would then yield E_z. In principle, it would thus be possible to solve the problem directly.

In view of the fact that the current distributions $I_a(z_a')$ and $I_b(z_b')$ are not known, and in fact can not be solved for directly, some sort of approximating procedure must be used. Iterative type solutions for the current distribution and the impedance parameters have been presented by King and Harrison[2] and by Tai[3]. Both these analyses were limited to the case of identical antennas. The results obtained were quite good for thin antennas.

CHAPTER III

THE REACTION CONCEPT

An outline of the reaction concept is presented in this chapter. The definition and properties of the reaction are considered and the impedance properties of a two port network are expressed in terms of the various reactions involved. A procedure for improving the trial approximations used in calculating the reactions is discussed.

Reciprocity Theorems

Consider two sets of AC sources $\mathbb{J}_a$ and $\mathbb{J}_b$ of the same frequency and existing in the same linear medium. Denote the fields produced by the "a" source acting alone as $\mathbb{E}_a$ and $\mathbb{H}_a$ and those produced by the "b" source acting alone as $\mathbb{E}_b$ and $\mathbb{H}_b$. These two sets of quantities may be related in a single equation known as a reciprocity theorem. Two forms of pure field reciprocity theorems are considered below.

Carson[9] has presented a pure field reciprocity theorem in the form of volume integrals involving electric current density and electric field intensity. It is

$$\iiint_{V_a} (\mathbb{E}_b \cdot \mathbb{J}_a)\, d\tau = \iiint_{V_b} (\mathbb{E}_a \cdot \mathbb{J}_b)\, d\tau \qquad \text{.......... } 3-1$$

where volume V_a includes antenna "a" and V_b includes antenna "b".

A second pure field reciprocity theorem involving electric- and magnetic-field intensities was derived by Lorentz[9] in the form of the surface integral expression below.

$$\iint_{s_a} (\mathbb{E}_a \times \mathbb{H}_b - \mathbb{E}_b \times \mathbb{H}_a) \cdot d\mathbb{S}$$

$$= \iint_{s_b} (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b) \cdot d\mathbb{S} \qquad \ldots\ldots\ldots\ldots 3 - 2$$

Surface s_a encloses antenna "a" and surface s_b encloses antenna "b".

The steps leading to equations 3 - 1 and 3 - 2 are outlined in Appendix A.

Definition of Reaction

Rumsey has given the name "reaction" to the integrals appearing in equations 3 - 1 and 3 - 2. By definition, the reaction of field "a" on source "b" is

$$\langle a,b \rangle = \iiint_{V_b} (\mathbb{E}_a \cdot \mathbb{J}_b)\, d\tau \qquad \ldots\ldots\ldots\ldots 3 - 3$$

In this notation, the reciprocity theorem becomes

$$\langle b, a\rangle = \langle a, b\rangle \qquad \text{.......... } 3 - 4$$

In view of the equivalence of the two reciprocity equations, an alternate statement of reaction is

$$\langle a, b\rangle = \iint_{S_b} (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b) \cdot d\,\mathbb{S} \qquad \text{.......... } 3 - 5$$

Useful Identities

Let "c" represent a third source of the same frequency as "a" and "b", and existing in the same linear medium. Making use of the linearity of the field equations, the following useful identity is obtained

$$\langle a, (b + c)\rangle = \langle a, b\rangle + \langle a, c\rangle \qquad \text{.......... } 3 - 6$$

Another useful identity is

$$\langle Aa, b\rangle = A\langle a, b\rangle = \langle a, Ab\rangle \qquad \text{.......... } 3 - 7$$

where A is a scalar quantity.

A Reciprocity Theorem of the Mixed Type

Kouyoumjian[10] has developed an expression for the voltage induced in one antenna by another in terms of the reaction. The physical situation is depicted in figures 3-1 and 3-2.

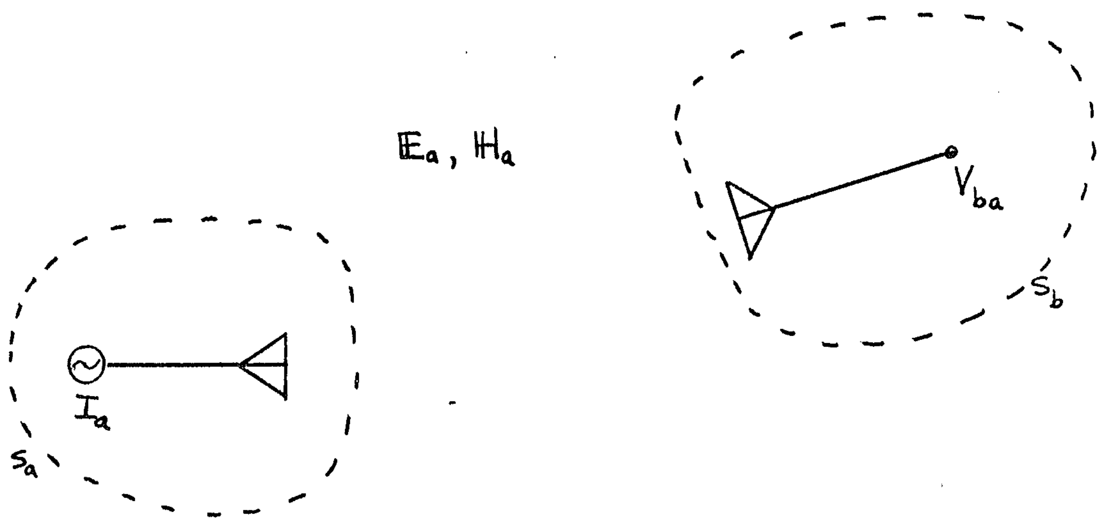

Figure 3-1. First situation: Antenna "a" transmits and antenna "b" receives.

As shown in figure 3-1, antenna "a" is driven by a current source I_a at its terminals, and antenna "b" is open circuited. Voltage V_{ba} is the open circuit voltage at the terminals of antenna "b".

In the second situation shown in figure 3-2, antenna "b" is driven by a current source I_b, and voltage V_{ab} is the open circuit voltage induced at the terminals of antenna "a".

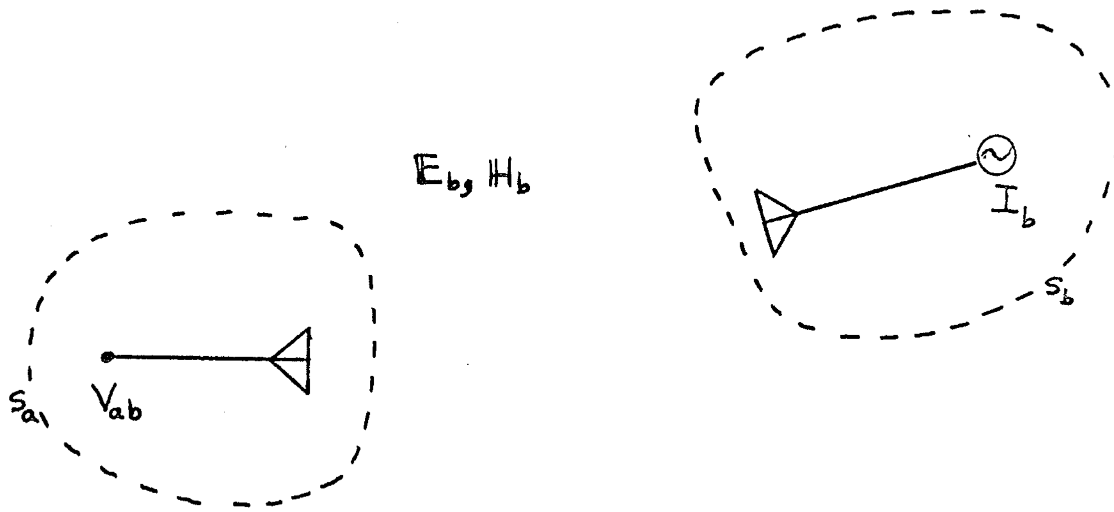

Figure 3-2. Second situation: Antenna "b" transmits and antenna "a" receives.

In terms of the above defined quantities, Kouyoumjian derived the following expressions

$$\langle a,b \rangle = -V_{ba} I_b \qquad \text{.......... } 3-8$$

$$\langle b,a \rangle = -V_{ab} I_a \qquad \text{.......... } 3-9$$

Equations 3 - 8 and 3 - 9 relate field quantities to terminal (circuit) quantities, and thus, they are reciprocity theorems of the mixed (field-circuit) type.

Impedance in Terms of Reactions

To relate the above reciprocity theorems to the usual circuit theory representation of a two port network, let the two antennas be represented (insofar as their terminal behaviour is concerned) by the following matrix equation

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} Z_{aa} & Z_{ab} \\ Z_{ba} & Z_{bb} \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix} \qquad \text{..........} \quad 3 - 10$$

Let the partial response V_{ij} be the voltage at port "i" due to source I_j at port "j". Each current source sees the other port open-circuited; hence

$$Z_{ij} = \frac{V_{ij}}{I_j} \qquad \text{..........} \quad 3 - 11$$

In terms of the circuit reactions, $\langle j,i \rangle = - V_{ij}\, I_j$; thus

$$Z_{ij} = - \frac{\langle j,i \rangle}{I_i\, I_j} \qquad \text{..........} \quad 3 - 12$$

Equation 3 - 10 may now be expanded as

$$\begin{aligned} I_a\, V_a &= -\langle a,a \rangle - \langle a,b \rangle \\ I_b\, V_b &= -\langle b,a \rangle - \langle b,b \rangle \end{aligned} \qquad \text{..........} \quad 3 - 13$$

These equations are to be applied to the analysis of a pair of coupled dipole antennas in Chapter IV.

A Modified Reaction Integral

Neither equation 3 - 3 nor equation 3 - 5 is suitable for evaluating the reaction $\langle a,b \rangle$, since the total fields $\mathbb{E}_a$ and $\mathbb{E}_b$ vanish on the perfectly conducting surface of the antenna. Richmond[11] however, presents an alternate equation which is useful for computing $\langle a,b \rangle$. By resolving the fields into incident and scattered components, he obtains the following expression

$$\langle a,b \rangle = - \iint_{S_b} (\mathbb{E}_a^i \times \mathbb{H}_b) \cdot d\mathbb{S} \qquad \text{.......... } 3 - 14$$

where $\mathbb{E}_a^i$ is the incident electric field intensity. Note that this no longer vanishes on the antenna surface, since the incident, rather than the total field is used.

The surface current $\mathbb{J}_{s_b}$ on the metal can be introduced in place of $\hat{n} \times \mathbb{H}_b$ to obtain the following result

$$\langle a,b \rangle = \iint_{S_b} (\mathbb{E}_a^i \cdot \mathbb{J}_{s_b}) dS \qquad \text{.......... } 3 - 15$$

Combining equations 3 - 12 and 3 - 15 gives

$$Z_{ab} = -\frac{1}{I_a I_b} \iint_{S_b} (\mathbb{E}_a^i \cdot \mathbb{J}_{S_b})\, dS \qquad \text{.......... } 3 - 16$$

If the true expressions for $\mathbb{E}_a^i$ and $\mathbb{J}_{S_b}$ were known, they could be substituted into equation 3 - 16, and (at least in principle), the integration performed to obtain the mutual impedance directly. However, such is not the case, and some sort of approximating procedure must be employed.

Constraints

In many cases, evaluation of $\langle a,b \rangle$ by application of any of the defining integrals is impossible because the true fields and sources are unknown. However, it is often possible to determine approximations to the desired reactions by assuming trial fields (or sources) to approximate the true fields (or sources). To be specific, suppose an approximation to the reaction $\langle a,b \rangle$ is desired. Let the correct value of $\langle a,b \rangle$ be denoted by $\langle c_a, c_b \rangle$. (The "c" stands for correct). If it were possible to adjust the approximation $\langle a,b \rangle$ such that

$$\langle a,b \rangle = \langle c_a, c_b \rangle \qquad \text{.......... } 3 - 17$$

then the impedance calculated according to equation 3 - 16 would be correct. Obviously, this is too much to expect: indeed equation 3 - 17 cannot be enforced because it is not known how to calculate $\langle c_a, c_b \rangle$. It is possible however, to utilize the reactions $\langle c_a, b \rangle$ and $\langle a, c_b \rangle$ in the following manner. Let the approximation $\langle a, b \rangle$ be constrained according to

$$\langle a,b \rangle = \langle c_a, b \rangle = \langle a, c_b \rangle \qquad \text{..........} \; 3 - 18$$

(This is a restriced case of the more general constraints imposed by equations 3 - 19 and 3 - 20). Enforcing equation 3 - 18 makes "b" look the same to "a" as it looks to the correct value "c_a": simultaneously, "a" is made to look the same to "b" as it looks to "c_b".

The reaction $\langle a,b \rangle$ constrained according to equation 3 - 18 is stationary for small variations of "a" and "b" about the correct values "c_a" and "c_b". This is shown in Appendix B.

It was mentioned previously that equation 3 - 18 is to be regarded as a special case of a more general restriction. This will now be considered in more detail. Equation 3 - 18 suggests that the reaction between some arbitrary test source "x" and the approximation "a" is to be made equal to the reaction between this test source and the correct "a", namely "c_a". Expressed mathematically, the condition is

$$\langle x,a\rangle = \langle x,c_a\rangle \qquad \text{.......... } 3 - 19$$

If in the process of enforcing 3 - 19, every available* test source "x" is used, and the approximation "a" adjusted such that 3 - 19 holds for all "x", then "a" and "c" are indistinquishable from the point of view of any measurements that can be made using the test sources available.

Similarly for the "b" approximation, enforce the condition

$$\langle y,b\rangle = \langle y,c_b\rangle \qquad \text{.......... } 3 - 20$$

where "y" is any arbitrary test source available to check against "c_b".

Available Test Sources

In the coupled dipole antenna problem, the only available test sources are "a" and "b". Letting "x"

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* A test source is considered to be available if its reaction with the correct source can be calculated.

take on these values in equation 3 - 19 yields

$$\langle a,a\rangle = \langle a,c_a\rangle$$
$$\langle b,a\rangle = \langle b,c_a\rangle \qquad \ldots\ldots\ldots\ldots 3 - 21$$

These same test sources ("a" and "b") are available as the "y" test source. Using them in equation 3 - 20 gives two more relationships

$$\langle a,b\rangle = \langle a,c_b\rangle$$
$$\langle b,b\rangle = \langle b,c_b\rangle \qquad \ldots\ldots\ldots\ldots 3 - 22$$

Equations 3 - 21 and 3 - 22 are used in Chapter IV to adjust the approximations for the coupled antenna problem.

Trial Distribution of Linear Combinations

The above procedure implies that some means must be available whereby the approximation can be adjusted at each step in the process. This may be done by including adjustable constants (variational parameters) in the definition of the trial distributions, and choosing those parameters which best suit the conditions of the problem.

A simple means to include these parameters is to express the trial distribution as a linear combination of functions of the form

$$a = U\,u + V\,v$$
$$b = M\,m + N\,n \qquad \ldots\ldots\ldots\ldots\ 3 - 23$$

where the adjustable parameters U, V, M and N are to be determined. (Note: If a higher order approximation is desired, additional parameters and functions would be included). The functions u, v, m and n are to be chosen such that the expressions for "a" and "b" satisfy the boundary conditions of the problem for any choice of U, V, M and N.

CHAPTER IV

APPLICATION OF THE REACTION CONCEPT TO THE COUPLED ANTENNA PROBLEM

As mentioned in Chapter II, a direct solution for the impedance parameters can not be obtained since the actual current distributions on the antennas are unknown. Thus, some sort of approximating procedure is required.

The approximating procedure to be applied here is based on the reaction concept of V. H. Rumsey. It yields the same set of equations as obtained by Levis and Tai[5] using a variational technique.* Formulation of the problem in terms of the reaction concept follows.

Antenna Impedance Equations in Terms of Reactions

The antenna configuration to be considered is shown in figure 4-1. From the terminals, the antenna system may be considered as a two port network. The two port equations are

$$V_a = Z_{aa}I_a(0) + Z_{ab}I_b(0)$$
$$V_b = Z_{ba}I_a(0) + Z_{bb}I_b(0) \qquad \text{.......... } 4 - 1$$

The parameters Z_{aa}, Z_{ab} and Z_{bb} characterize the impedance behaviour of the antennas. Note that $Z_{ab} = Z_{ba}$.

* Levis and Tai made no numerical computations to complete the problem (reference 20).

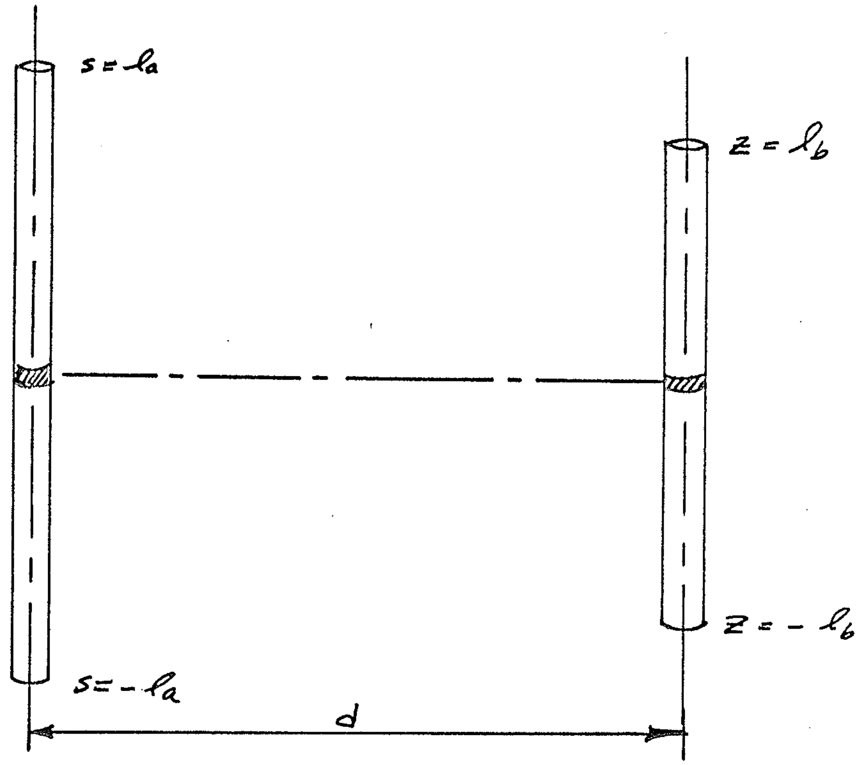


Figure 4-1. Two Coupled Antennas.

In terms of the reactions developed in Chapter III, equation 4 - 1 may be written as

$$I_a^2(0)Z_{aa} + I_a(0)I_b(0)Z_{ab} = -\langle a,a\rangle - \langle a,b\rangle$$
$$I_a(0)I_b(0)Z_{ba} + I_b^2(0)Z_{bb} = -\langle b,a\rangle - \langle b,b\rangle$$

.... 4 - 2

The set of equations 4 - 2 are stationary with respect to small variations of the assumed current distributions about the true current distributions (see Appendix B). Thus, if trial currents are assumed which differ from the true currents by errors of the first order, the calculated impedances will differ from the true impedances by errors of the second order.

Choice of Trial Functions

The choice of trial distributions is to be made such that the boundary conditions of the one dimensional model are satisfied. Also, since it is desirable to use trial distributions which closely approximate the true current distributions, the choice can be influenced by certain experimental or theoretical results which suggest what form the approximation should take. For example, the current distribution along an open-ended dissipative transmission line (with small attenuation) is of the form

$$I(z) = I_0 \left[\sin\beta(\ell - z) - j\alpha(\ell - z)\cos\beta(\ell - z)\right]$$

Based on this, Tai[4] suggests a trial current of the form

$$I(z) = A\sin\beta(\ell - z) + B\beta(\ell - z)\cos\beta(\ell - z)$$

where A and B are the adjustable parameters to be determined.

Approximation Using Trial Functions

For convenience, the axial direction on antenna "a" will be denoted by "s", and that on antenna "b" by "z". The primes on the variables will now be dropped. In the event that any confusion should arise, the primes can be reinserted at that point in the discussion.

The assumed trial currents to be used are

$$I_a(s) = U \sin\beta(\ell_a - |s|) + V\beta(\ell_a - |s|) \cos\beta(\ell_a - |s|)$$

$$I_b(z) = M \sin\beta(\ell_b - |z|) + N\beta(\ell_b - |z|) \cos\beta(\ell_b - |z|)$$

.... 4 - 3

Define the following shorthand notation for the trial functions

$$u(s) = \sin\beta(\ell_a - |s|)$$
$$v(s) = \beta(\ell_a - |s|) \cos\beta(\ell_a - |s|)$$
$$m(z) = \sin\beta(\ell_b - |z|)$$
$$n(z) = \beta(\ell_b - |z|) \cos\beta(\ell_b - |z|)$$

.... 4 - 4

At the feed points

$$u(0) = u_o$$
$$v(0) = v_o$$
$$m(0) = m_o$$
$$n(0) = n_o$$

.... 4 - 5

Using 4 - 4, the currents 4 - 3 may be expressed as

$$I_a(s) = U\,u(s) + V\,v(s)$$
$$I_b(z) = M\,m(z) + N\,n(z)$$

.... 4 - 6

At the terminals

$$I_a(0) = U\,u_o + V\,v_o$$
$$I_b(0) = M\,m_o + N\,n_o \qquad \text{.......... } 4 - 7$$

According to the reaction concept, all trial distributions should appear the same to the trial fields as to the true fields; hence, enforce the following conditions

$$\langle a,a\rangle = \langle a,c_a\rangle$$
$$\langle b,b\rangle = \langle b,c_b\rangle$$
$$\langle a,b\rangle = \langle a,c_b\rangle$$
$$\langle b,a\rangle = \langle b,c_a\rangle \qquad \text{.......... } 4 - 8$$

Consider the reaction $\langle a,b\rangle$. Since $a = U\,u + V\,v$, this may be expanded as

$$\langle a,b\rangle = U\,\langle u,b\rangle + V\,\langle v,b\rangle \qquad \text{.......... } 4 - 9$$

Similarly

$$\langle a,c_b\rangle = U\,\langle u,c_b\rangle + V\,\langle v,c_b\rangle \qquad \text{.......... } 4 - 10$$

To force equality of these reactions, impose the conditions

$$\langle u,b\rangle = \langle u,c_b\rangle$$
$$\langle v,b\rangle = \langle v,c_b\rangle \qquad \text{.......... } 4 - 11$$

Recall that the reaction between any two sources "w" and "x" is by definition

$$\langle w,x\rangle = \iiint_{Volume} (\mathbb{E}_w \cdot \mathbb{J}_x)\, d\tau$$

where $\mathbb{E}_w$ is the electric field due to the source $\mathbb{J}_w$. Note that the integrand has a value only where the current distribution $\mathbb{J}_x$ exists, namely the volume containing the "x" antenna. Thus, the integration need be carried out only throughout the volume containing the "x" antenna.

If the antenna is assumed to be a perfect conductor, $\hat{n} \times \mathbb{E}_w^c = 0$ on the antenna surface, except at the feed. (Note: Symbol "c" stands for correct). Thus

$$\langle c_w, x\rangle = -\, V_{xw}\, I_x \qquad \text{.......... } 4 - 12$$

for any "x", where V_{xw} is the voltage across the "x" antenna terminals due to the field $\mathbb{E}_w^c$, and I_x is the component of current at the feedpoint associated with $\mathbb{J}_x$.

The result 4 - 12 applied to the reactions in equations 4 - 11 yields

$$\langle b,u\rangle = \langle c_b,u\rangle = -\, V_{ab}\, u_o$$
$$\langle b,v\rangle = \langle c_b,v\rangle = -\, V_{ab}\, v_o$$

The remaining reactions in 4 - 8 may be treated in a similar manner. The results are presented below.

$$\langle a,u\rangle = -\ V_{aa}\ u_o \qquad \ldots (a)$$
$$\langle a,v\rangle = -\ V_{aa}\ v_o \qquad \ldots (b)$$
$$\langle a,m\rangle = -\ V_{ba}\ m_o \qquad \ldots (c)$$
$$\langle a,n\rangle = -\ V_{ba}\ n_o \qquad \ldots (d)$$
$$\langle b,u\rangle = -\ V_{ab}\ u_o \qquad \ldots (e)$$
$$\langle b,v\rangle = -\ V_{ab}\ v_o \qquad \ldots (f)$$
$$\langle b,m\rangle = -\ V_{bb}\ m_o \qquad \ldots (g)$$
$$\langle b,n\rangle = -\ V_{bb}\ n_o \qquad \ldots (h)$$

$$\ldots\ldots\ldots\ldots 4 - 13$$

If (a) and (e) are added, the result is

$$\langle a,u\rangle + \langle b,u\rangle = -\ u_o(V_{aa} + V_{ab}) = -\ u_o\ V_a$$

where V_a is the total voltage across the terminals of the "a" antenna. Since $a = U\ u + V\ v$ and $b = M\ m + N\ n$, the left hand side of this equation may be expanded according to the identities of Chapter III. Thus

$$U\langle u,u\rangle + V\langle u,v\rangle + M\langle u,m\rangle + N\langle u,n\rangle = -\ u_o\ V_a$$

Combining (b) and (f), (c) and (g) and (d) and (h) yields three more equations. In matrix form, the resulting four equations are

$$\begin{bmatrix} \langle u,u\rangle & \langle u,v\rangle & \langle u,m\rangle & \langle u,n\rangle \\ \langle u,v\rangle & \langle v,v\rangle & \langle m,v\rangle & \langle n,v\rangle \\ \langle u,m\rangle & \langle v,m\rangle & \langle m,m\rangle & \langle m,n\rangle \\ \langle u,n\rangle & \langle v,n\rangle & \langle m,n\rangle & \langle n,n\rangle \end{bmatrix} \begin{bmatrix} U \\ V \\ M \\ N \end{bmatrix} = \begin{bmatrix} -u_o V_a \\ -v_o V_a \\ -m_o V_b \\ -n_o V_b \end{bmatrix} \qquad \text{.......... } 4-14$$

Equation 4 - 14 may be written in a more compact form by defining $[\mathcal{R}]$ as the square matrix of reactions, $[\rho]$ as the column matrix of parameters, and $[\mathcal{V}]$ as the column matrix appearing on the right hand side of 4 - 14. In this notation, 4 - 14 becomes

$$[\mathcal{R}][\rho]=[\mathcal{V}] \qquad \text{.......... } 4-15$$

Denote the inverse of the reaction matrix by $[\Upsilon]$ Solving for the parameter matrix yields the result

$$[\rho]=[\Upsilon][\mathcal{V}] \qquad \text{.......... } 4-16$$

In expanded form, equation 4 - 16 is

$$\begin{bmatrix} U \\ V \\ M \\ N \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \begin{bmatrix} -u_o V_a \\ -v_o V_a \\ -m_o V_b \\ -n_o V_b \end{bmatrix} \qquad \text{.......... } 4-17$$

All entries in the $[T]$ matrix are determined when $[R]$ is inverted. Both the $[R]$ and the $[T]$ matrices are symmetric.

Using the base currents defined as 4 - 7 in the two port equations 4 - 1 yields

$$V_a = Z_{aa}(Uu_o + Vv_o) + Z_{ab}(Mm_o + Nn_o)$$
$$V_b = Z_{ba}(Uu_o + Vv_o) + Z_{bb}(Mm_o + Nn_o)$$

.... 4 - 18

If the parameters U, V, M and N from 4 - 17 are substituted into 4 - 18, the following result is obtained

$$V_a \Big\{ Z_{aa}(u_o^2 t_{11} + 2u_o v_o t_{12} + v_o^2 t_{22}) + Z_{ab}(u_o m_o t_{13} + m_o v_o t_{23} + u_o n_o t_{14} + n_o v_o t_{24}) + 1 \Big\}$$
$$+ V_b \Big\{ Z_{aa}(u_o m_o t_{13} + m_o v_o t_{23} + u_o n_o t_{14} + n_o v_o t_{24}) + Z_{ab}(m_o^2 t_{33} + 2m_o n_o t_{34} + n_o^2 t_{44}) \Big\} = 0$$

This can be interpreted as a set which must hold for any arbitrary set of driving voltages V_a and V_b. This can be the case only if the coefficients of V_a and V_b vanish individually. Setting these coefficients equal to zero allows us to solve for Z_{ab}. The final result is

$$Z_{ab} = \frac{u_o m_o t_{13} + u_o n_o t_{14} + v_o m_o t_{23} + v_o n_o t_{24}}{\Delta} \qquad \text{.......... } 4 - 19$$

where Δ is the determinant

$$\Delta = \begin{vmatrix} d_1 & d_2 \\ d_3 & d_4 \end{vmatrix}$$

with

$$d_1 = u_o^2 t_{11} + 2\, u_o v_o t_{12} + v_o^2 t_{22}$$

$$d_2 = u_o m_o t_{13} + u_o n_o t_{14} + m_o v_o t_{23} + n_o v_o t_{24}$$

$$d_3 = d_2$$

$$d_4 = m_o^2 t_{33} + 2\, m_o n_o t_{34} + n_o^2 t_{44}$$

$$\text{.......... } 4 - 20$$

Equation 4 - 19 with auxiliary relations 4 - 20 forms the basis of the computer solution for Z_{ab}.

All entries in the $[\mathcal{R}]$ matrix are reactions between the various trial functions, and may be integrated directly as shown in Appendix C. Although the integrations are long and involved, all reduce to closed form, with the final results expressed in terms of sine and cosine integrals. When all entries have been computed, the resulting matrix is inverted to determine the $[\Upsilon]$ matrix. With all t_{ij} known, equation 4 - 19 yields the numerical values for Z_{ab} for that particular case under consideration. The computer program for performing the computations is shown in Appendix D, and the numerical results are presented in graphical form in Chapter V.

CHAPTER V

RESULTS AND CONCLUSIONS

Mutual impedances between pairs of coupled dipoles were calculated on the IBM System 360 Computer using the program shown in Appendix D. The results are presented in graphical form on the following pages.

To specify the length to diameter ratio, the commonly used parameter Ω is employed. It is defined as

$$\Omega = 2 \ln\left(\frac{2\ell}{a}\right)$$

where ℓ = half-length of dipole

a = radius of dipole

Comparison with King's Results

Since a limited amount of experimental data was available, most of the results were checked by comparison with King's [7] data. King's data is known to be good for short, thin antennas. It can be seen from the graphs that the results check very nicely with King's for the conditions stated. Note however, for thicker, longer antennas, the results calculated according to the reaction method deviate somewhat from King's data.

Comparison with Experiment

Two sets of experimental data are presented by King.

Figure 5-2 shows the results for identical, half-wave dipoles of moderate thickness ($\Omega = 9.3$). For this case, it may be seen that both King's and the newly computed results agree quite well with experiment, with the results calculated by the reaction method agreeing somewhat better than King's.

Experimental results for identical, full-wave length dipoles are shown as figure 5-3. Although King compared his calculated results with the data shown, this comparison is not really valid, since the calculated results are for the case $\Omega = 10.0$ and $\beta\ell = 3.157$, whereas, the experimental results are for the case $\Omega = 10.7$ and $\beta\ell = \pi$. Data computed from the reaction formulation is shown for both cases for comparison. As may be seen, neither King's results nor the reaction method results agree very well with experiment.

The failure of both methods at antiresonance can be attributed to the inadequacy of the current representation for this particular length. At this length, the terminal current is very small, but the current at other points on the antenna can be very large. Figure 5-1 (a) illustrates the general nature of the problem. Two possible distributions are shown, each having the same terminal current, but quite different values at other points on the dipole. Since the field quantities depend on the current over the whole length of the antenna, quite different results will be obtained in each case. This problem is not unique to the full-wavelength case, but is less serious for other lengths (see figure 5-1(b)).

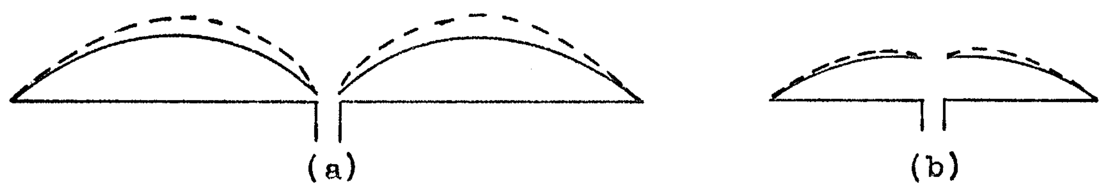


Figure 5-1. Possible approximate current distributions.

An improvement in the reaction method solution could likely be obtained by adding additional terms to the current approximation. An analytic solution to closed form for this case would represent a formidable task. This suggests that any further attempts at improving the solution should be made using the techniques of numerical integration.

Conclusions

The mutual impedance expression derived in Chapter IV represents a useful approximation for determining mutual impedance between coupled dipole antennas of moderate length and thickness. With a six second solution time per answer, the program in Appendix D is useful to quickly obtain approximate numerical data for coupled dipoles. While the results appear to be better than King's, no conclusion can be drawn in this regard until further experimental data is available.

Figure 5-2. Mutual impedance curves for coupled dipoles.

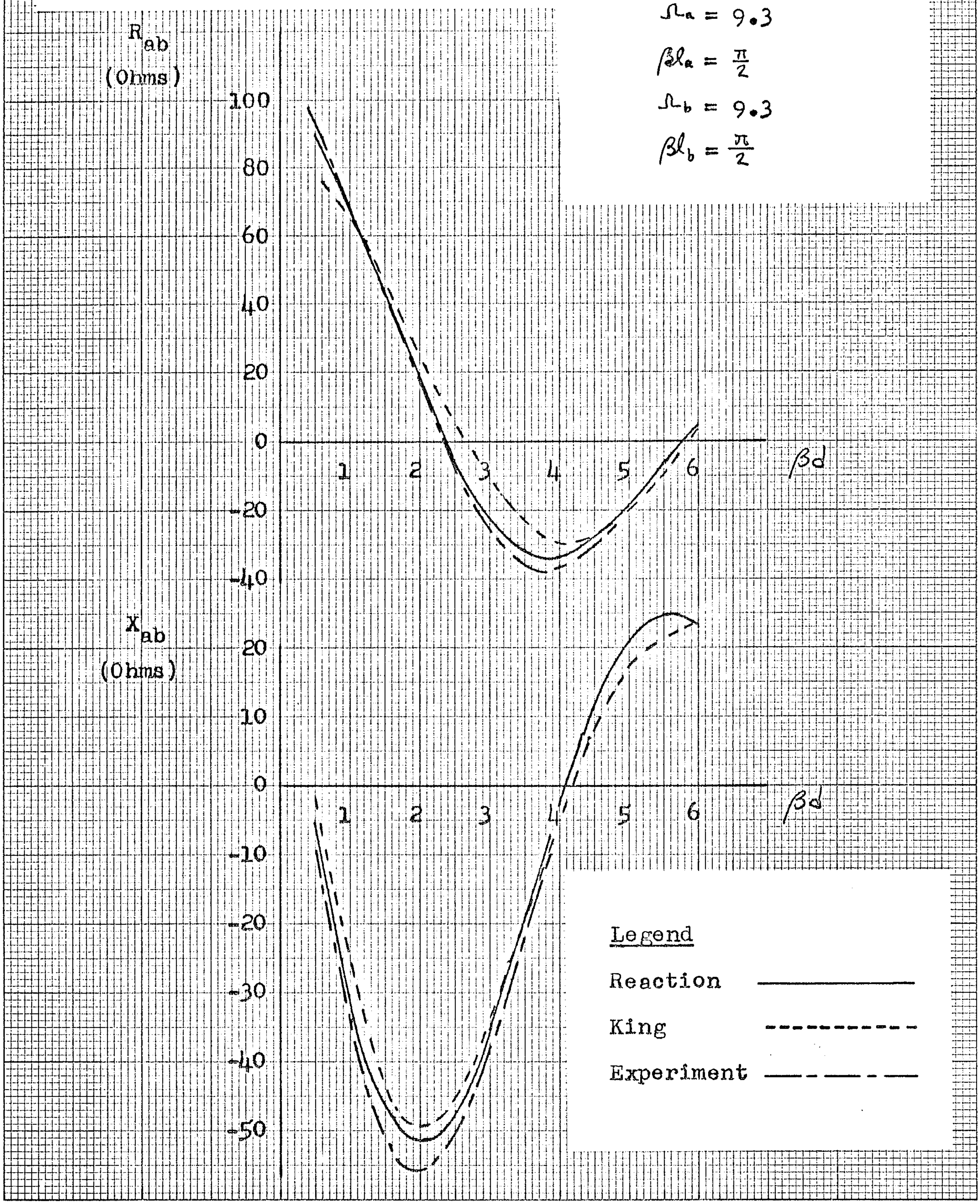


Figure 5-3. Mutual impedance curves for coupled dipoles.

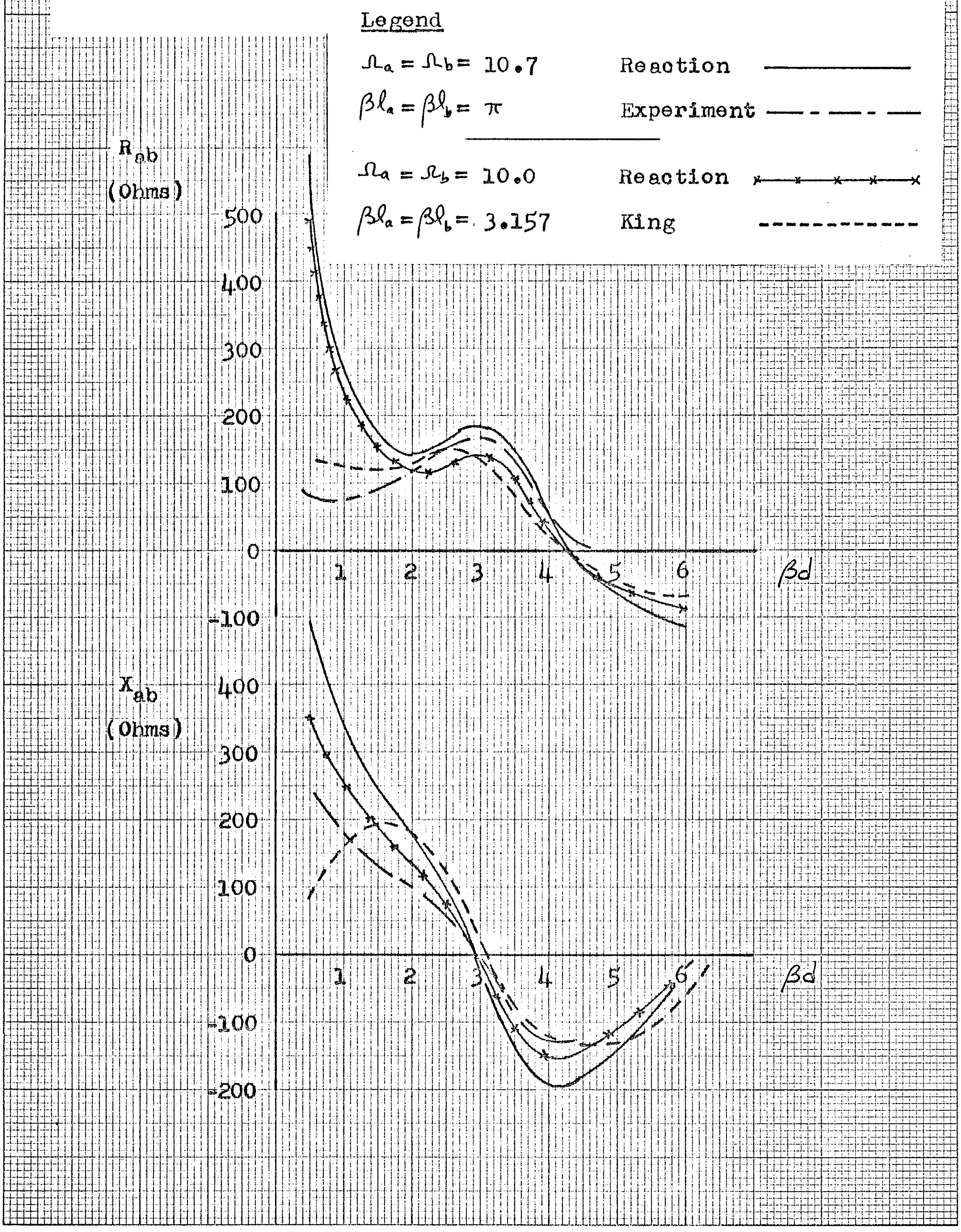


Figure 5-4. Mutual impedance curves for coupled dipoles.

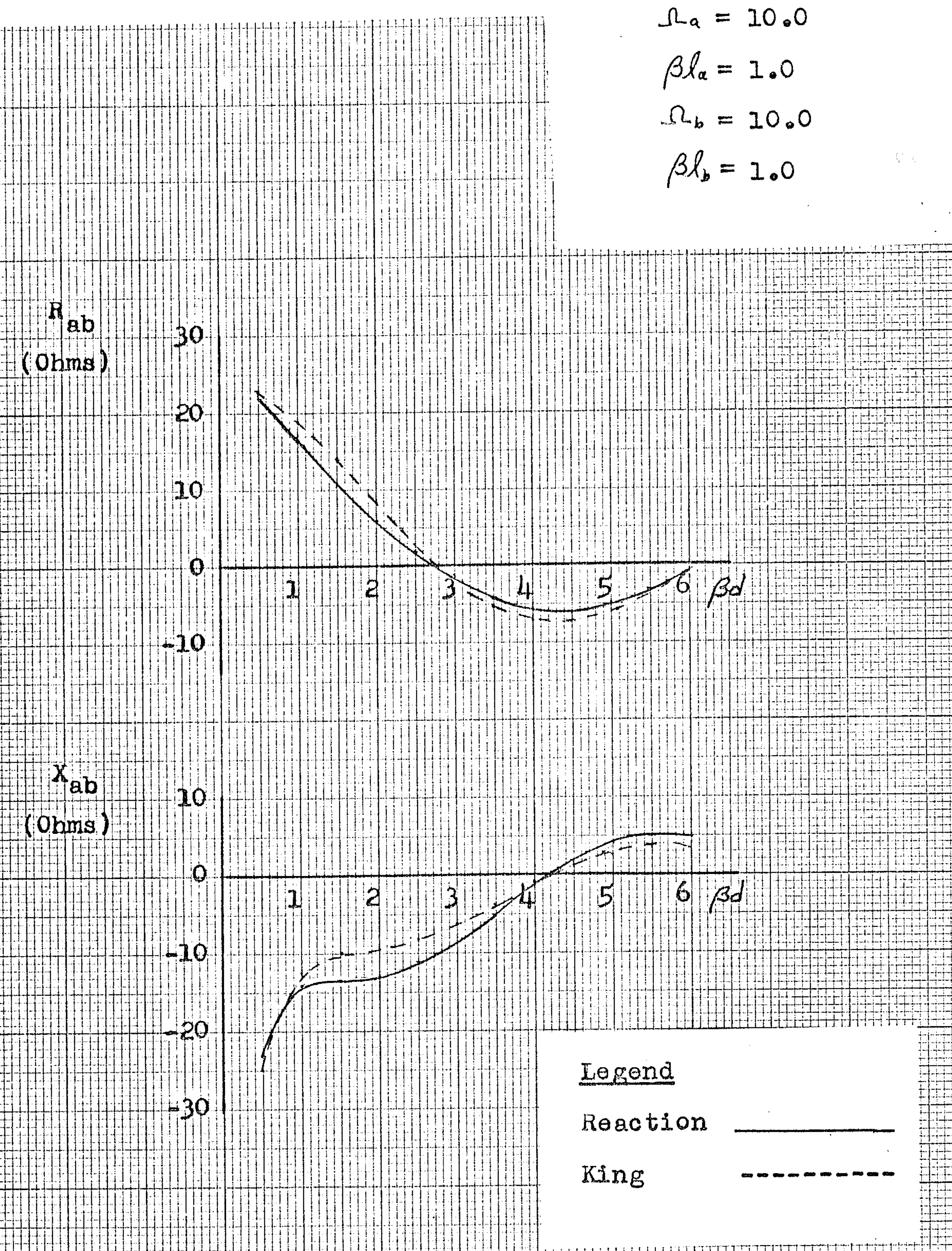


Figure 5-5. Mutual impedance curves for coupled dipoles.

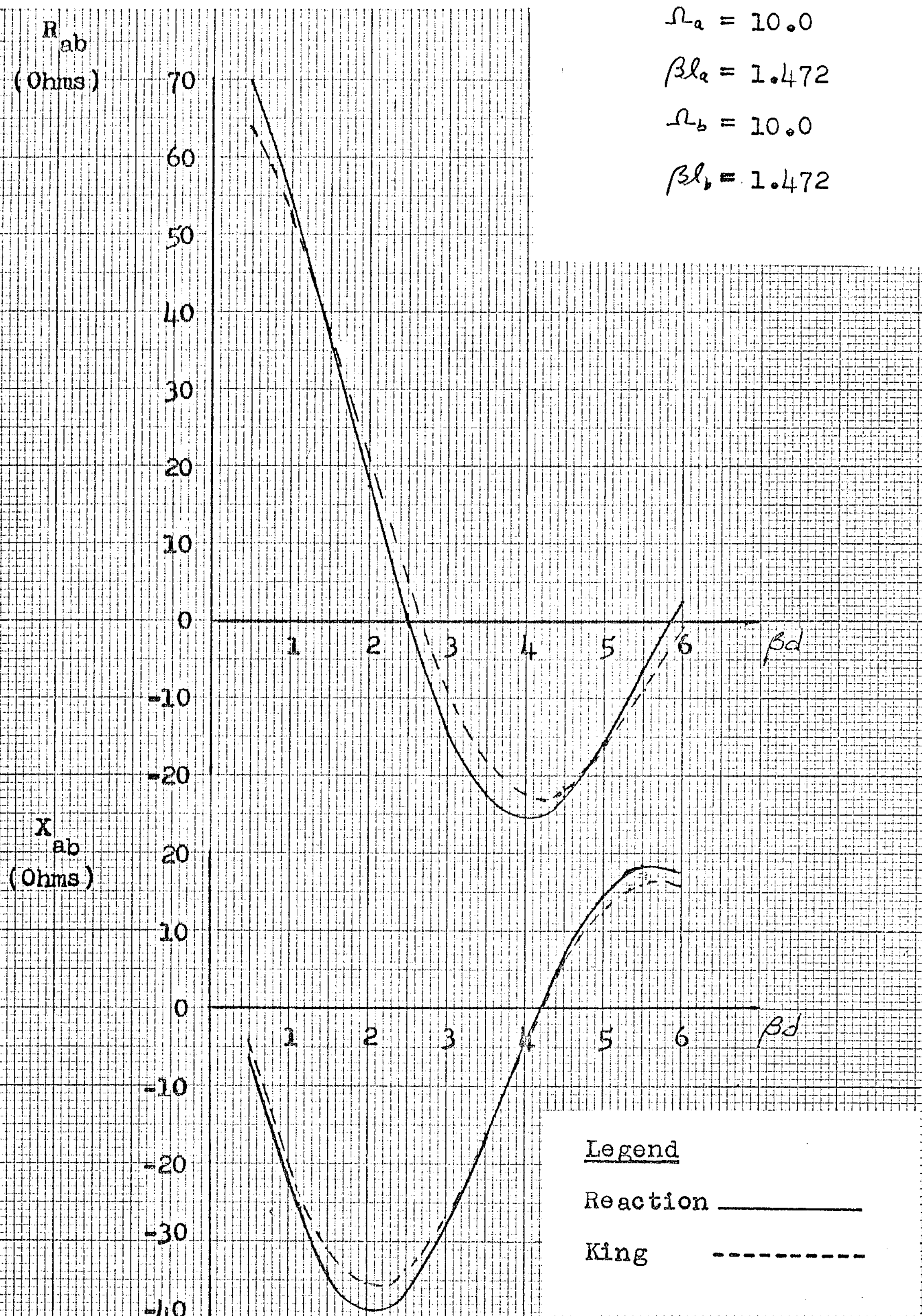


Figure 5-6. Mutual impedance curves for coupled dipoles.

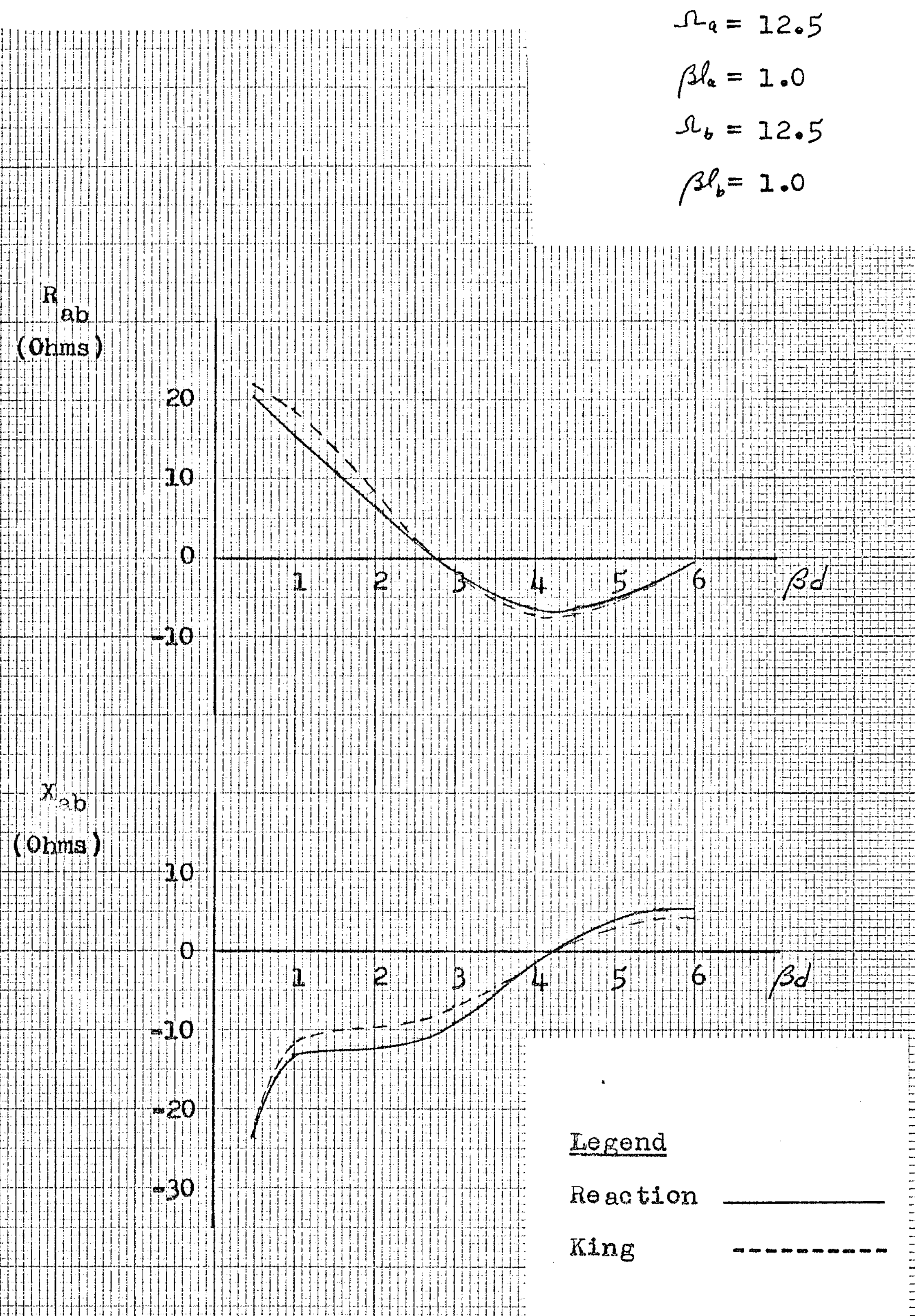


Figure 5-7. Mutual impedance curves for coupled dipoles.

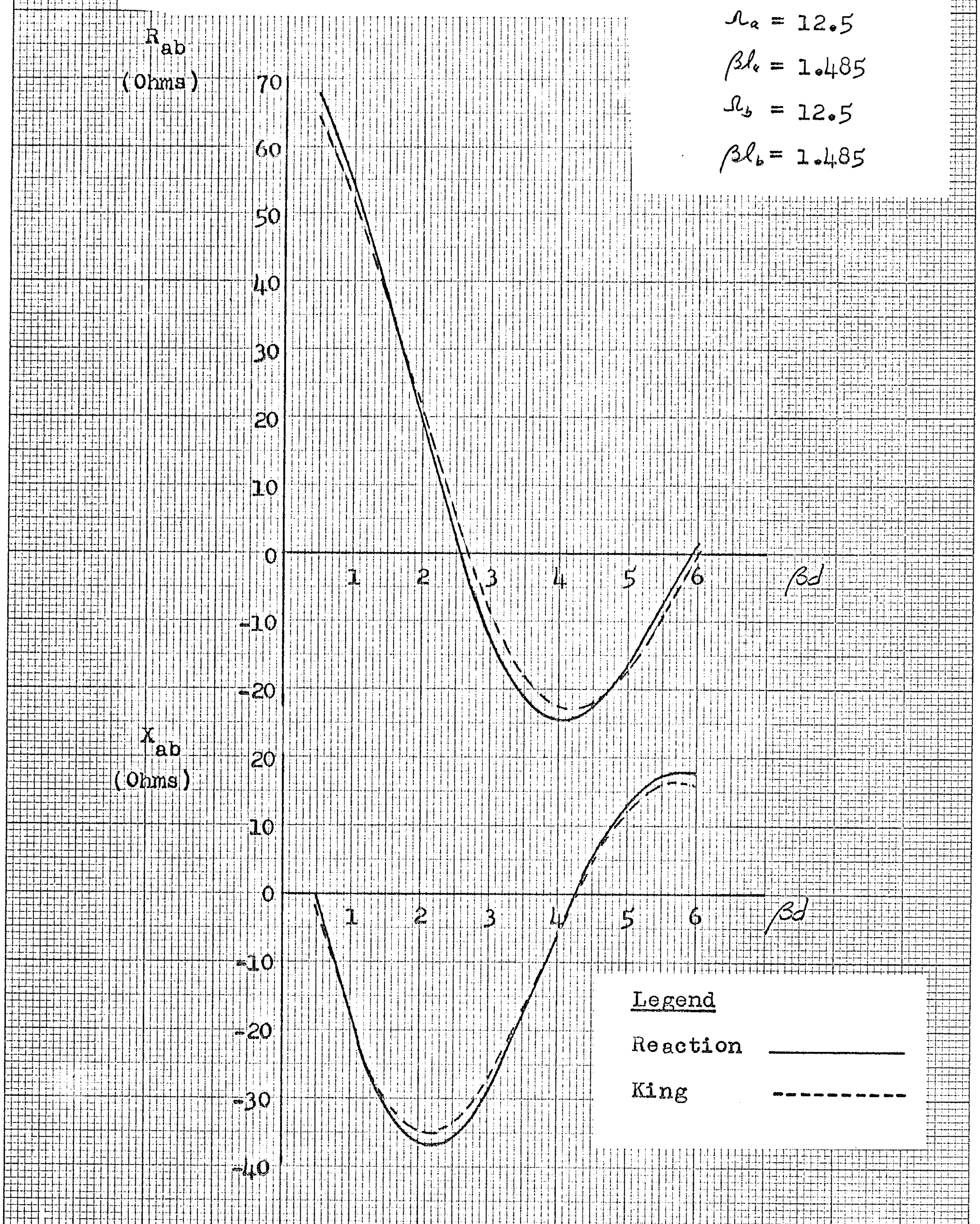


Figure 5-8. Mutual impedance curves for coupled dipoles.

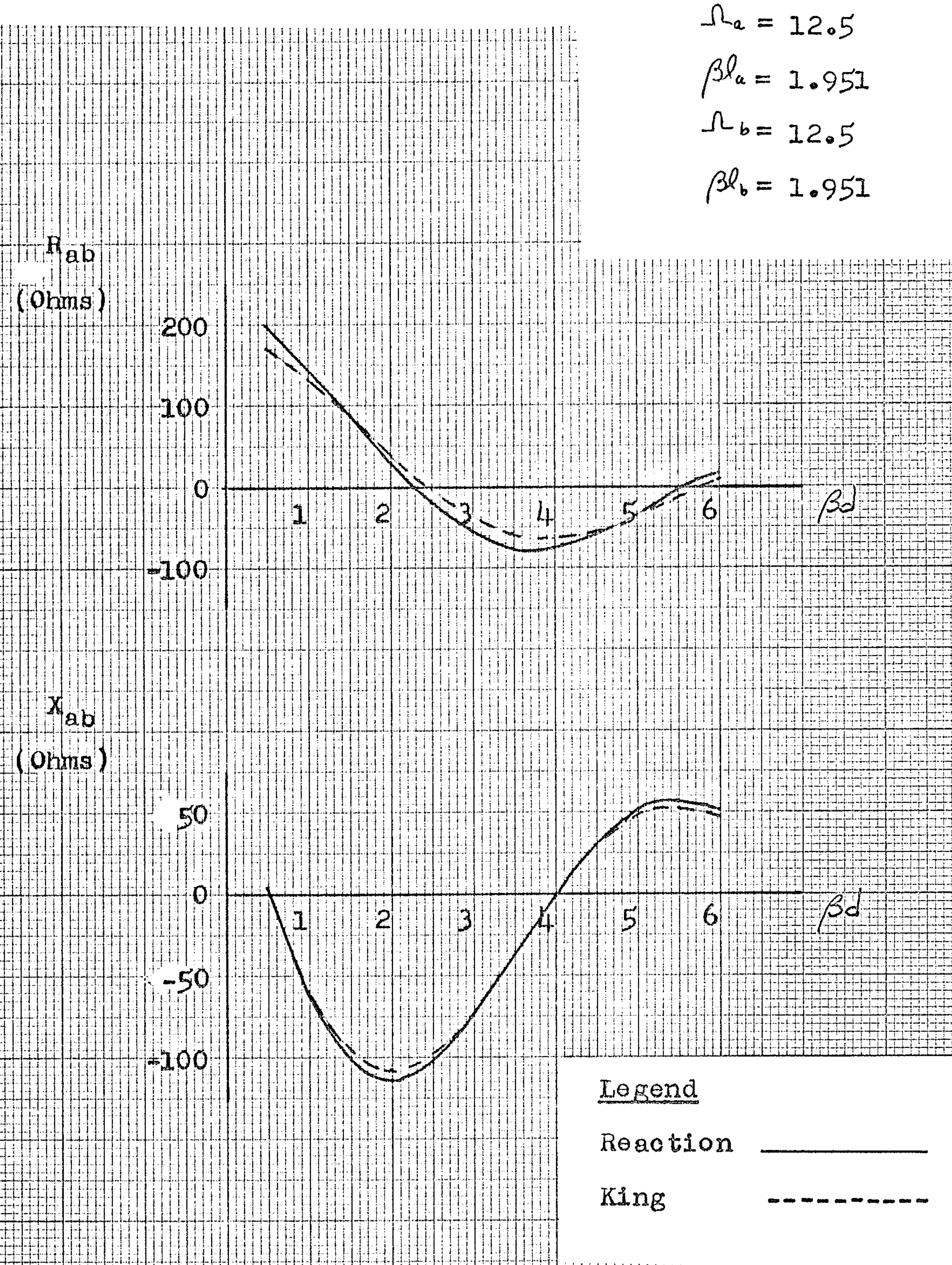


Figure 5-9. Mutual impedance curves for coupled dipoles.

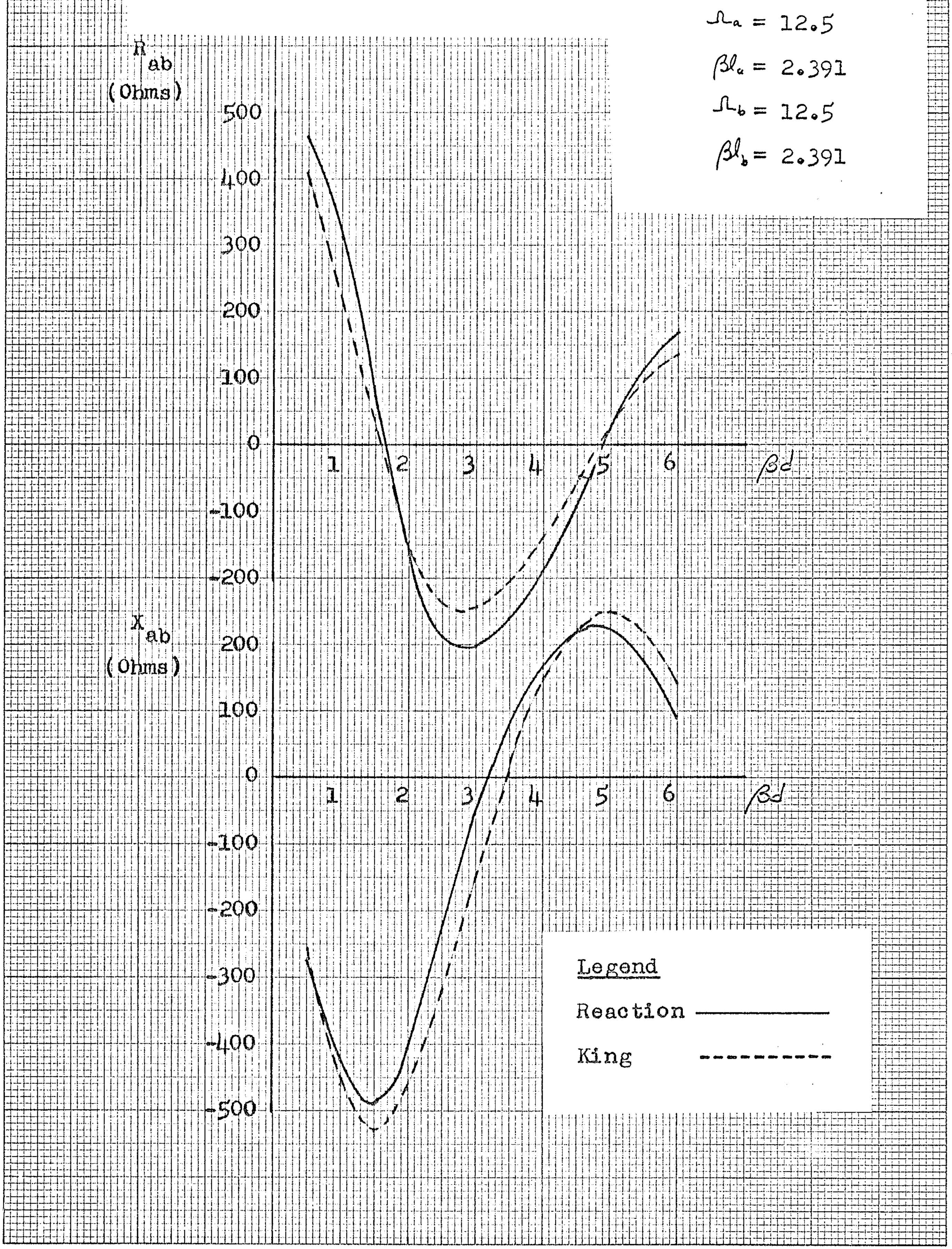


Figure 5-10. Mutual impedance curves for coupled dipoles.

$\Omega_a = 15.0$

$\beta l_a = 1.0$

$\Omega_b = 15.0$

$\beta l_b = 1.0$

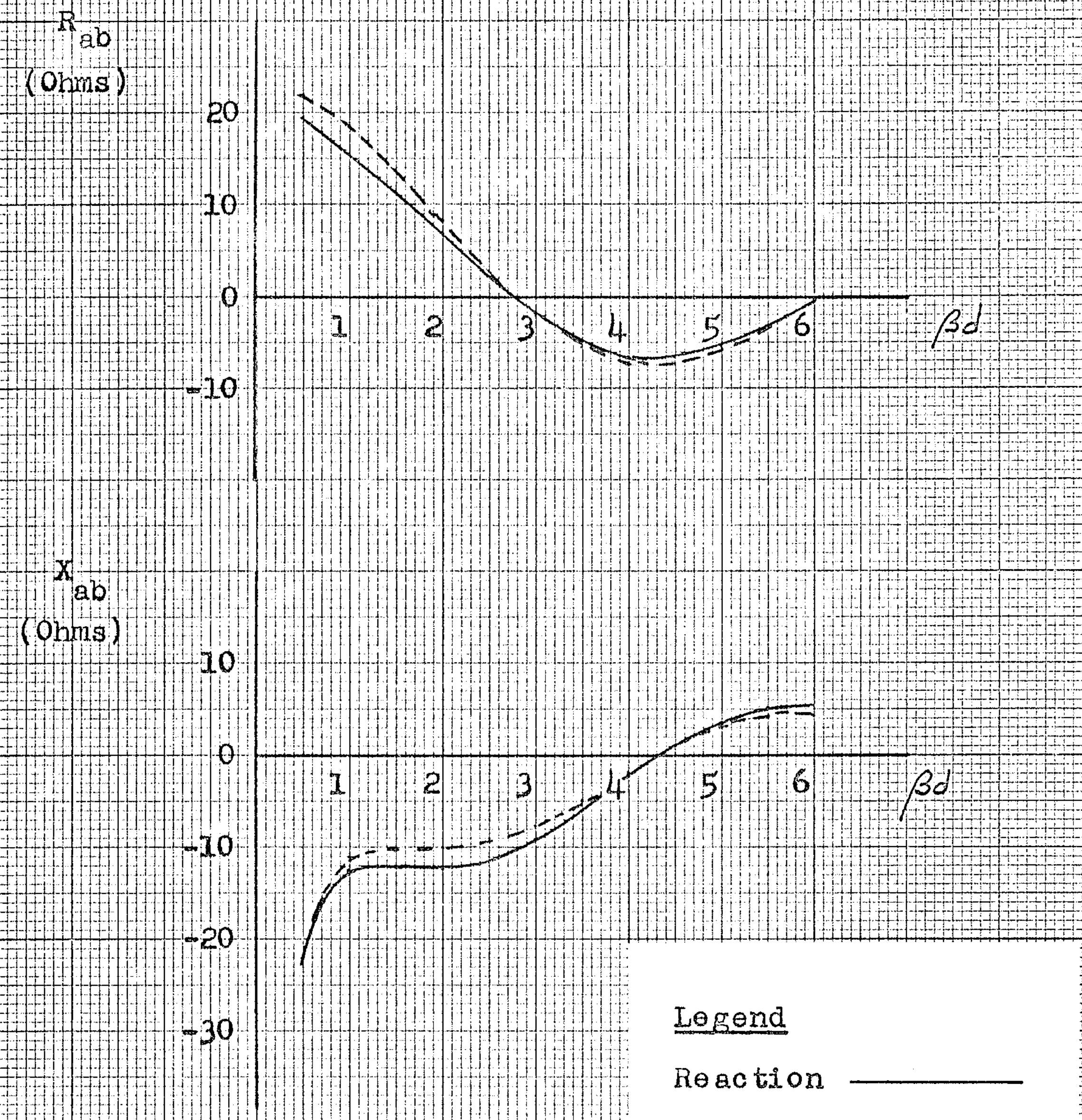


Figure 5-11. Mutual impedance curves for coupled dipoles.

$\Omega_a = 15.0$

$\beta l_a = 1.970$

$\Omega_b = 15.0$

$\beta l_b = 1.970$

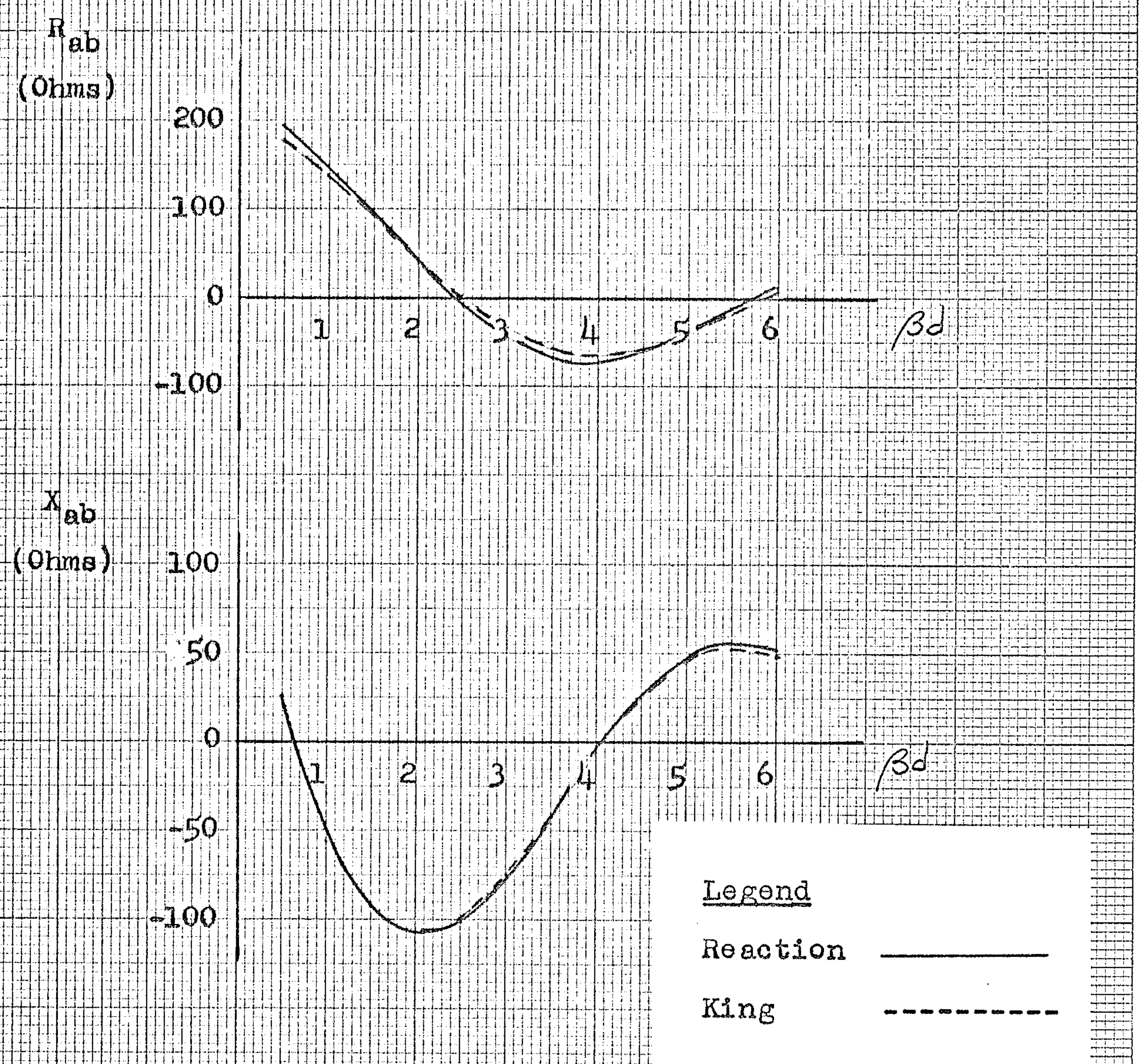


Figure 5-12. Mutual impedance curves for coupled dipoles.

$\Omega_a = 20.0$

$\beta l_a = 1.0$

$\Omega_b = 20.0$

$\beta l_b = 1.0$

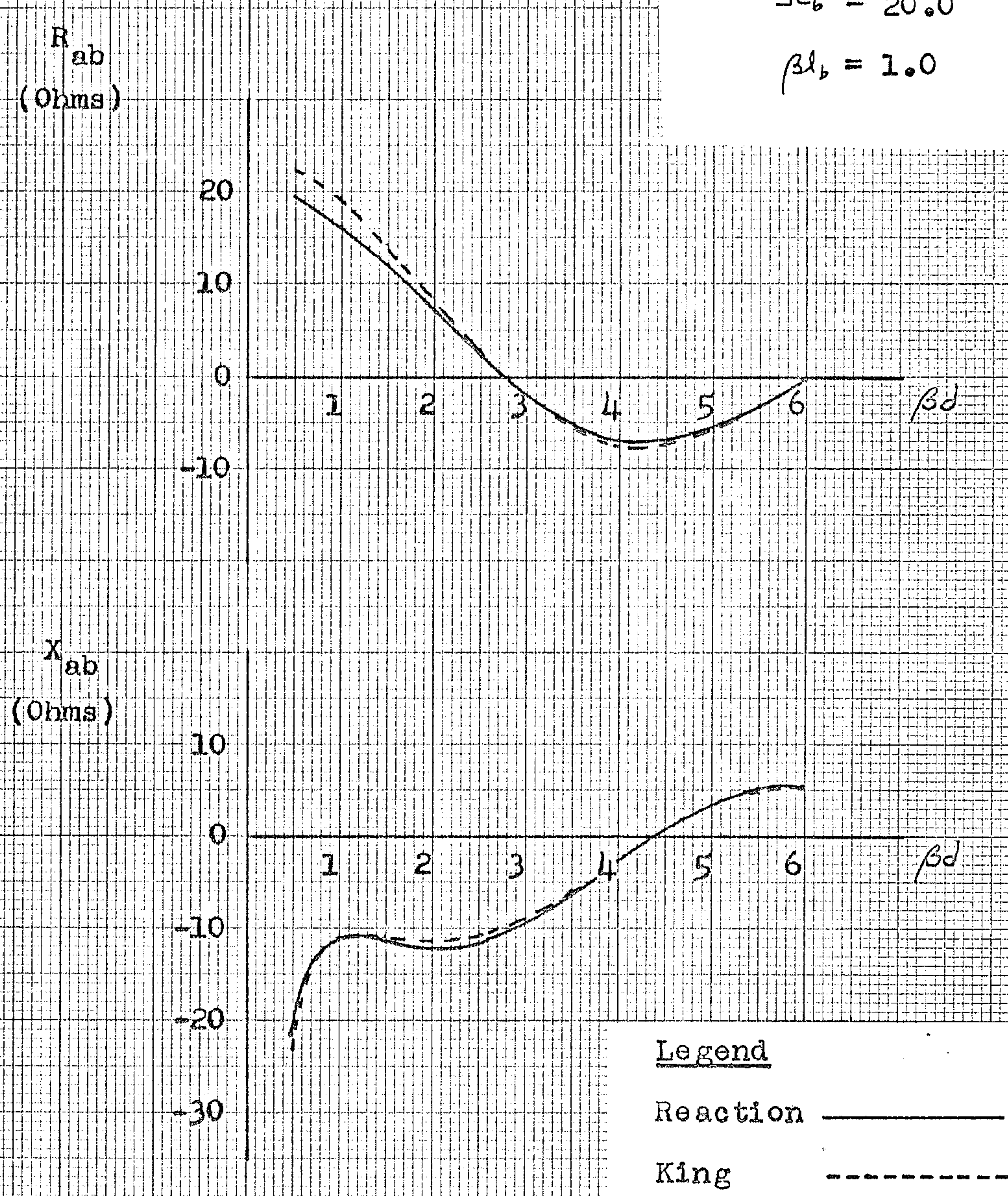


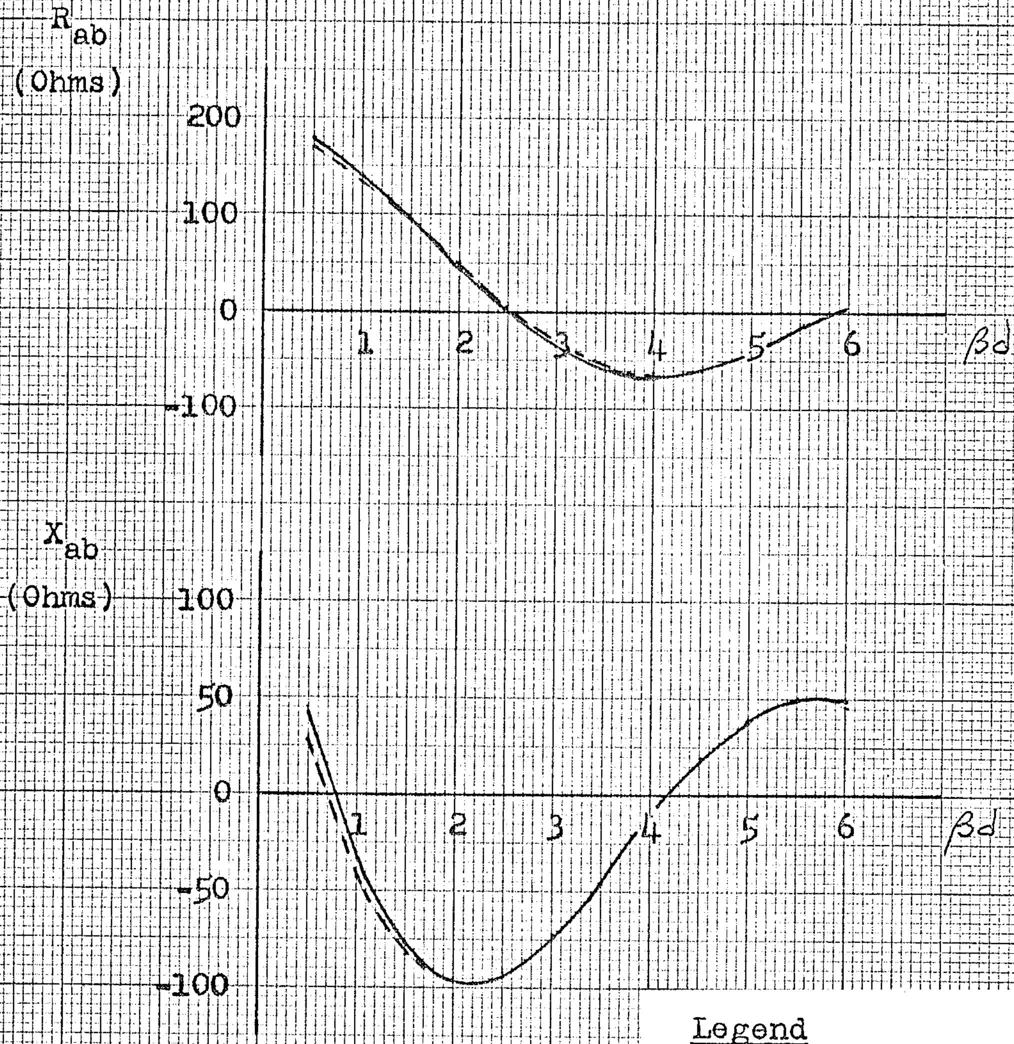
Figure 5-13. Mutual impedance curves for coupled dipoles.

$\Omega_a = 20.0$

$\beta l_a = 1.973$

$\Omega_b = 20.0$

$\beta l_b = 1.973$

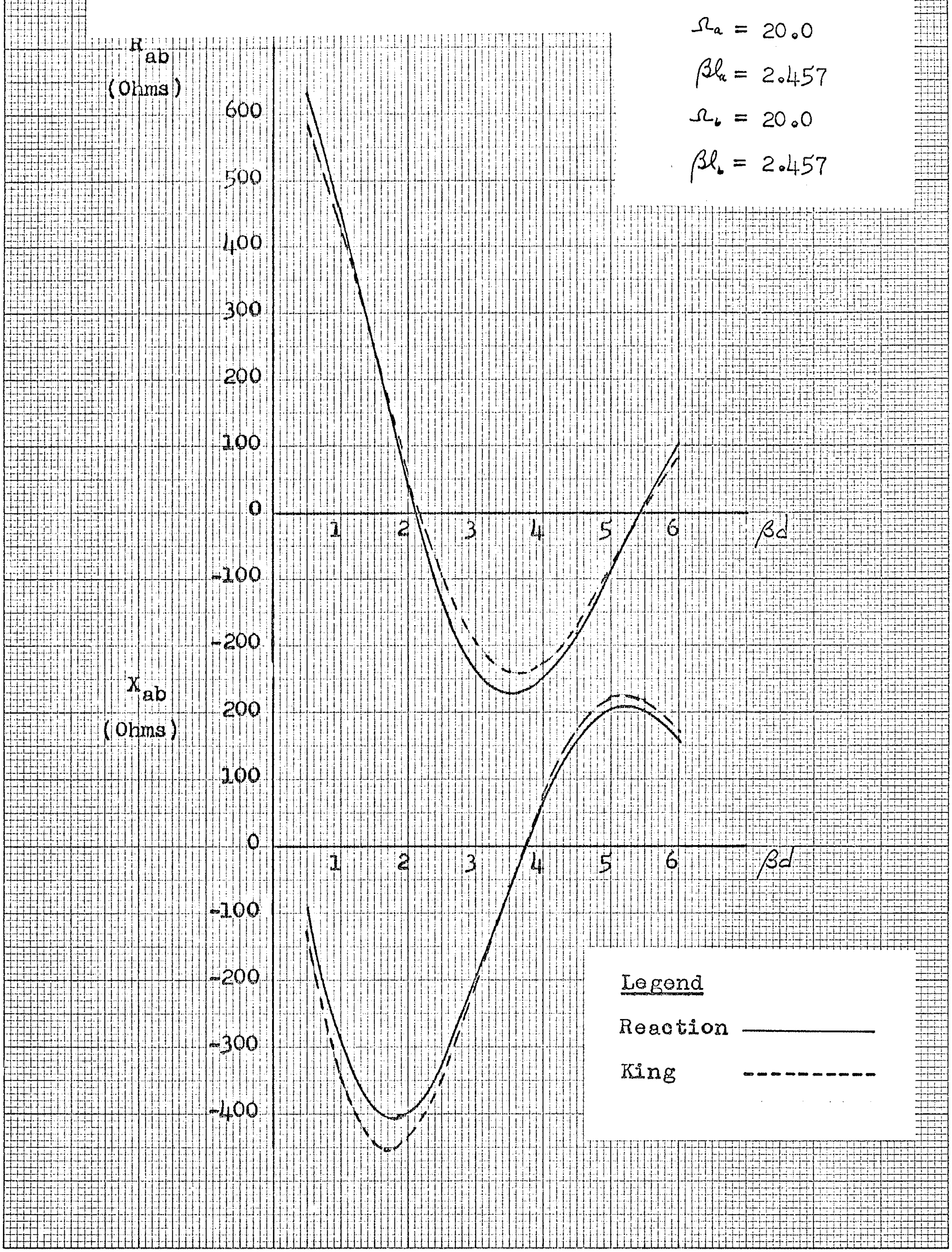


Legend

Reaction ————

King - - - - - -

Figure 5-14. Mutual impedance curves for coupled dipoles.



APPENDIXES

APPENDIX A

RECIPROCITY AND THE REACTION CONCEPT

In this appendix, various forms of the reciprocity theorem are considered and related to the reaction theorem. The development is essentially that of Richmond[11,12] and Harrington[8].

Reciprocity

In most equations, all quantities involved are understood to relate to a common situation. A reciprocity theorem on the other hand, brings together quantities from two different situations into a single equation.

Consider for example, the situation depicted by figures 3-1 and 3-2. The reciprocity theorems of equation 3 - 1 and 3 - 2 relate the quantities $\mathbb{E}_a$, $\mathbb{H}_a$ and $\mathbb{J}_a$ when antenna "a" is transmitting and antenna "b" is receiving to the quantities $\mathbb{E}_b$, $\mathbb{H}_b$ and $\mathbb{J}_b$ when antenna "b" is transmitting and antenna "a" is receiving. Furthermore, the field quantities may be related to the circuit quantities by means of the "mixed" reciprocity theorems of equations 3 - 8 and 3 - 9.

Lorentz Reciprocity Theorem

For the following discussion, it is convenient to divide space into regions as shown in figure A-1.

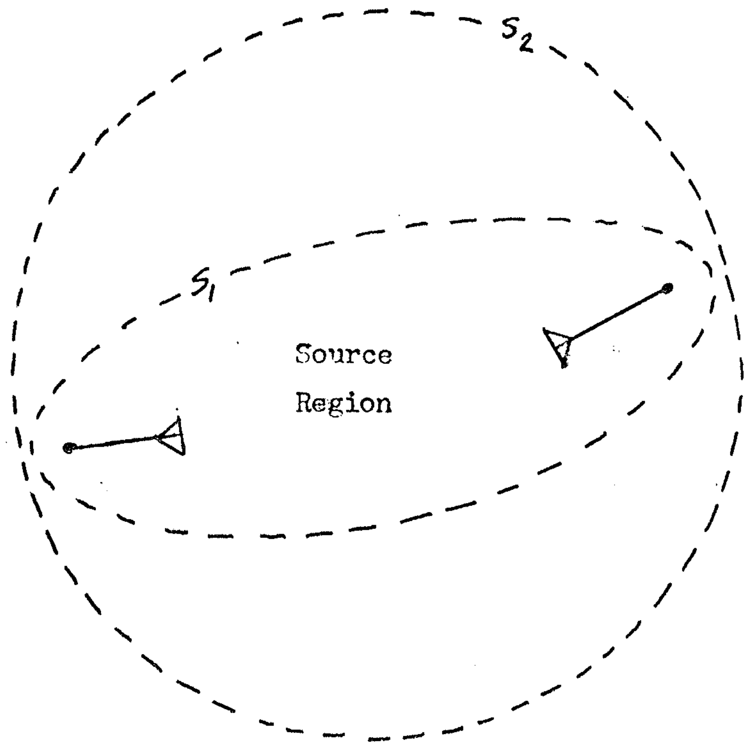


Figure A-1. Division of space into regions.

Surface s_1 encloses all sources, and s_2 is chosen as the surface of a large sphere.

Let antenna "a" transmit into the enviroment which is free space everywhere outside the regions bounded by s_a and s_b as shown in figure A-2. The region bounded by s_b is occupied by the open circuited receiving antenna.

Denote the fields produced by antenna "a" in this enviroment as $\mathbb{E}_a$ and $\mathbb{H}_a$. External to the source

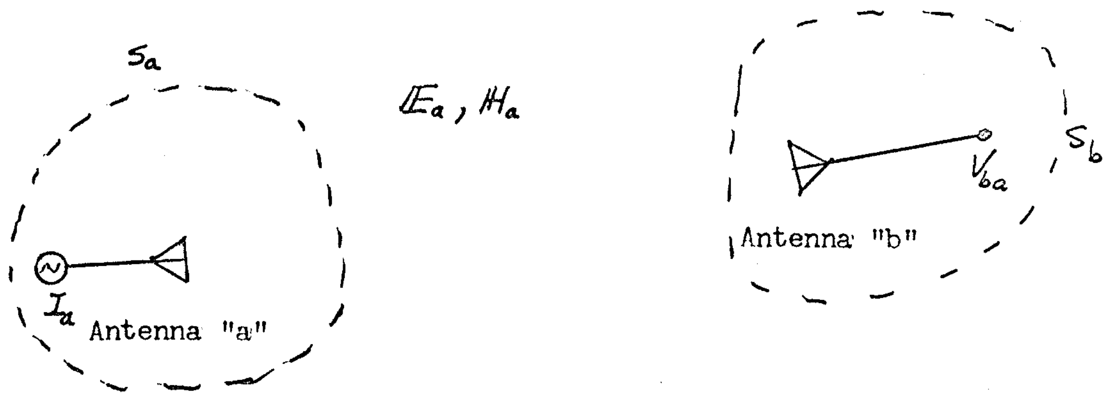


Figure A-2. Transmission between two antennas.

region bounded by s_1, the field equations are

$$-\nabla \times \mathbb{E}_a = \hat{z}\, \mathbb{H}_a$$

$$\nabla \times \mathbb{H}_a = \hat{y}\, \mathbb{E}_a$$

.... A - 1

where the free space parameters $\hat{z}$ and $\hat{y}$ are given by

$$\hat{z} = j\omega\mu$$

$$\hat{y} = j\omega\epsilon$$

.... A - 2

when the fields vary harmonically with time.

Now let antenna "b" radiate in the enviroment which is free space everywhere outside s_a and s_b.

Within s_a is located the open circuited receiving antenna. External to the source region, the "b" fields are related by

$$-\nabla \times \mathbb{E}_b = \hat{z}\, \mathbb{H}_b$$

$$\nabla \times \mathbb{H}_b = \hat{y}\, \mathbb{E}_b$$

.... A - 3

Scalarly multiplying the second of equations A - 1 by $\mathbb{E}_b$ and the first of A - 3 by $\mathbb{H}_a$ and adding yields

$$-\nabla \cdot (\mathbb{E}_b \times \mathbb{H}_a) = \hat{y}\, \mathbb{E}_a \cdot \mathbb{E}_b + \hat{z}\, \mathbb{H}_a \cdot \mathbb{H}_b$$

.... A - 4

where the left hand side has been simplified by the identity

$$\nabla \cdot (\mathbb{A} \times \mathbb{B}) = \mathbb{B} \cdot (\nabla \times \mathbb{A}) - \mathbb{A} \cdot (\nabla \times \mathbb{B})$$

.... A - 5

A similar development in terms of the other two of equations A - 1 and A - 3 yields

$$-\nabla \cdot (\mathbb{E}_a \times \mathbb{H}_b) = \hat{y}\, \mathbb{E}_a \cdot \mathbb{E}_b + \hat{z}\, \mathbb{H}_a \cdot \mathbb{H}_b$$

.... A - 6

Subtracting A - 6 from A - 4 results in

$$\nabla \cdot (\mathbb{E}_a \times \mathbb{H}_b - \mathbb{E}_b \times \mathbb{H}_a) = 0$$

.... A - 7

Equation A - 7 is valid in the source free region bounded by s_1 and s_2. By introducing suitable cuts, the region of interest can be made simply connected. Then, applying the divergence theorem and integrating throughout this source free region yields

$$\iint_{s_1+s_2} (\mathbb{E}_a \times \mathbb{H}_b - \mathbb{E}_b \times \mathbb{H}_a) \cdot d\mathbb{S} = 0$$

.... A - 8

In view of the radiation conditions, the contribution to this integral vanishes over the surface s_2 when the radius of the large sphere is allowed to increase to infinity. Equation A - 8 then reduces to

$$\iint_{s_1} (\mathbb{E}_a \times \mathbb{H}_b - \mathbb{E}_b \times \mathbb{H}_a) \cdot d\mathbb{S} = 0$$

.... A - 9

This is the generalized Lorentz reciprocity theorem as presented by Richmond.

The Reaction Integral

Since equation A - 7 is valid at all points outside of s_a and s_b, equation A - 9 can be reduced to

$$\iint_{S_a} (\mathbb{E}_a \times \mathbb{H}_b - \mathbb{E}_b \times \mathbb{H}_a) \cdot d\mathbb{S} + \iint_{S_b} (\mathbb{E}_a \times \mathbb{H}_b - \mathbb{E}_b \times \mathbb{H}_a) \cdot d\mathbb{S} = 0$$

Finally

$$\iint_{S_a} (\mathbb{E}_a \times \mathbb{H}_b - \mathbb{E}_b \times \mathbb{H}_a) \cdot d\mathbb{S} = \iint_{S_b} (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b) \cdot d\mathbb{S} \qquad \text{.......... A - 10}$$

Rumsey has given the name "reaction" to each of the integrals appearing in equation A - 10. Thus, by definition

$$\langle a, b \rangle = \iint_{S_b} (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b) \cdot d\mathbb{S} \qquad \text{.......... A - 11}$$

In this notation, the reciprocity theorem becomes

$$\langle b, a \rangle = \langle a, b \rangle \qquad \text{.......... A - 12}$$

Sources of the Fields

The enviroment under consideration is free space everywhere outside of the two small subregions bounded by s_a and s_b respectively. The constants of free space are denoted by μ and ϵ , and the characteristics of the antenna materials are described by σ_a , μ_a and ϵ_a for antenna "a", and by σ_b , μ_b and ϵ_b for antenna "b".

Let antenna "a" transmit into the enviroment which is free space everywhere outside of s_a, except for the region bounded by s_b which encloses the open circuited receiving antenna. Denote the fields in this enviroment as $\mathbb{E}_a$ and $\mathbb{H}_a$. The primary source of the fields is the source current $\mathbb{J}_a^i$ within s_a. Scattering from the antenna structures can be considered to have as its sources the conduction currents within s_a and s_b.

Within s_a, the field equations are

$$-\nabla \times \mathbb{E}_a = \hat{z}_a \mathbb{H}_a$$

$$\nabla \times \mathbb{H}_a = \hat{y}_a \mathbb{E}_a + \mathbb{J}_a^i \qquad \text{.......... A - 13}$$

where

$$\hat{z}_a = j\omega\mu_a$$

$$\hat{y}_a = (\sigma_a + j\omega\epsilon_a)$$

Consider the second of equations A - 13. Adding and subtracting $\hat{y}\,\mathbb{E}_a$ yields

$$\nabla \times \mathbb{H}_a = \hat{y}\,\mathbb{E}_a + (\hat{y}_a - \hat{y})\mathbb{E}_a + \mathbb{J}_a^i$$

$$= \hat{y}\,\mathbb{E}_a + \mathbb{J}_a$$

where $$\mathbb{J}_a = (\hat{y}_a - \hat{y})\mathbb{E}_a + \mathbb{J}_a^i$$

Similarly

$$-\nabla \times \mathbb{E}_a = \hat{z}\,\mathbb{H}_a + \mathbb{M}_a$$

where $$\mathbb{M}_a = (\hat{z}_a - \hat{z})\,\mathbb{H}_a$$

Since $\epsilon_a = \epsilon$ and $\mu_a = \mu$ for non-magnetic conductors, the effective "current" densities reduce to

$$\mathbb{J}_a = \sigma_a \mathbb{E}_a + \mathbb{J}_a^i \qquad \text{.......... A - 14}$$

$$\mathbb{M}_a = 0$$

Thus

$$\left.\begin{aligned} -\nabla \times \mathbb{E}_a &= \hat{z}\,\mathbb{H}_a \\ \nabla \times \mathbb{H}_a &= \hat{y}\,\mathbb{E}_a + \mathbb{J}_a \end{aligned}\right\} \text{ within } s_a$$

and

$$\left.\begin{aligned} -\nabla \times \mathbb{E}_a &= \hat{z}\,\mathbb{H}_a \\ \nabla \times \mathbb{H}_a &= \hat{y}\,\mathbb{E}_a \end{aligned}\right\} \text{ in free space}$$

$$\text{.......... A - 15}$$

It must be noted however, that the total fields $\mathbb{E}_a$ and $\mathbb{H}_a$ cannot be determined from equations A - 15 alone, since the contribution from the induced currents within s_b have not yet been accounted for. This will now be considered. Within s_b, the field equations are

$$-\nabla \times \mathbb{E}_a = \hat{z}_b \mathbb{H}_a$$

$$\nabla \times \mathbb{H}_a = \hat{y}_b \mathbb{E}_a$$

Noting that $\mu_b = \mu$ and $\epsilon_b = \epsilon$, these can be written as

$$-\nabla \times \mathbb{E}_a = \hat{z} \mathbb{H}_a$$

$$\nabla \times \mathbb{H}_a = \hat{y} \mathbb{E}_a + \mathbb{J}_a^s$$

.... A - 16

where

$$\mathbb{J}_a^s = (\hat{y}_b - \hat{y}) \mathbb{E}_a = \sigma_b \mathbb{E}_a$$ .... A - 17

Insofar as determining the "a" fields is concerned, the effective currents given by equations A - 14 and A - 17 can be thought of as source currents* in an otherwise homogeneous region.

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* Note that since $\mathbb{J}_a$ and $\mathbb{J}_a^s$ are functions of $\mathbb{E}_a$, they are not known until $\mathbb{E}_a$ is obtained. On the other hand, $\mathbb{E}_a$ cannot be found unless $\mathbb{J}_a$ and $\mathbb{J}_a^s$ are known. Thus, one is led to an integral equation problem. Unless suitable approximations can be made, the problem is virtually an intractable one. Nonetheless, it is useful at this stage to treat the current densities $\mathbb{J}_a$ and $\mathbb{J}_a^s$ as if they were known.

Reaction Expressed as a Volume Integral

Maxwell's equations within the region bounded by s_b are

$$-\nabla \times \mathbb{E}_a = \hat{z}_b \mathbb{H}_a$$

$$\nabla \times \mathbb{H}_a = \hat{y}_b \mathbb{E}_a$$

$$-\nabla \times \mathbb{E}_b = \hat{z}_b \mathbb{H}_b$$

$$\nabla \times \mathbb{H}_b = \hat{y}_b \mathbb{E}_b + \mathbb{J}_b^i$$

.... A - 18

By application of the divergence theorem, the surface integral in equation A - 11 becomes a volume integral of

$$\nabla \cdot (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b)$$

Using the vector identity of equation A - 5 along with equations A - 18 yields

$$\nabla \cdot (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b) = \mathbb{E}_a \cdot \mathbb{J}_b^i$$

Thus

$$\iint_{s_b} (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b) \cdot d\mathbb{S} = \iiint_{v_b} (\mathbb{E}_a \cdot \mathbb{J}_b^{i})\, d\tau$$

This then results in the following alternate form of the reaction integral

$$\langle a,b \rangle = \iiint_{v_b} (\mathbb{E}_a \cdot \mathbb{J}_b^{i})\, d\tau \qquad \text{.......... A - 19}$$

Reaction in Terms of the Incident Fields

While equations A - 11 and A - 19 are expressions for the reaction ⟨a,b⟩ , they are difficult to apply since the total fields $\mathbb{E}_a$ and $\mathbb{H}_a$ involve contributions from both effective currents ($\mathbb{J}_a$ of equation A - 14 and $\mathbb{J}_a^s$ of equation A - 17). An alternate form derived by Richmond circumvents this difficulty. The development is outlined below.

The environmentunder consideration is as follows- antenna "a" enclosed within s_a and radiating in the presence of antenna "b" (open-circuited) within s_b. The total fields in this environment can be considered as the superposition of two components

(1) That due to the volume distribution of electric

current $\mathbb{J}_a$ radiating into free space.

(2) That due to the volume distribution of electric current $\mathbb{J}_a^s$ radiating into free space.

The field quantities are resolved into components according to

$$\mathbb{E}_a = \mathbb{E}_a^i + \mathbb{E}_a^s$$

$$\mathbb{H}_a = \mathbb{H}_a^i + \mathbb{H}_a^s$$

.... A - 20

The fields $\mathbb{E}_a^i$ and $\mathbb{H}_a^i$ are defined as those components of the total fields which are due to the currents within s_a alone. Similarly, $\mathbb{E}_a^s$ and $\mathbb{H}_a^s$ are those components of the total fields which are due to currents within s_b alone. That is, the source of the incident fields ($\mathbb{E}_a^i$ and $\mathbb{H}_a^i$) is the current distribution

$$\mathbb{J}_a = (\hat{y}_a - \hat{y})\mathbb{E}_a + \mathbb{J}_a^i \qquad \text{(within } s_a\text{)}$$

.... A - 21

and the source of the scattered fields ($\mathbb{E}_a^s$ and $\mathbb{H}_a^s$) is the current distribution

$$\mathbb{J}_a^s = (\hat{y}_b - \hat{y})\mathbb{E}_a \qquad \text{(within } s_b\text{)}$$

.... A - 22

Maxwell's equations for the incident components are

$$\left.\begin{aligned} -\nabla \times \mathbb{E}_a^i &= \hat{z}\, \mathbb{H}_a^i \\ \nabla \times \mathbb{H}_a^i &= \hat{y}\, \mathbb{E}_a^i + \mathbb{J}_a \end{aligned}\right\} \text{ within } s_a$$

$$\left.\begin{aligned} -\nabla \times \mathbb{E}_a^i &= \hat{z}\, \mathbb{H}_a^i \\ \nabla \times \mathbb{H}_a^i &= \hat{y}\, \mathbb{E}_a^i \end{aligned}\right\} \text{ in free space}$$

$$\left.\begin{aligned} -\nabla \times \mathbb{E}_a^i &= \hat{z}\, \mathbb{H}_a^i \\ \nabla \times \mathbb{H}_a^i &= \hat{y}\, \mathbb{E}_a^i \end{aligned}\right\} \text{ within } s_b$$

.... A - 23

Maxwell's equations for the scattered components are

$$\left.\begin{aligned} -\nabla \times \mathbb{E}_a^s &= \hat{z}\, \mathbb{H}_a^s \\ \nabla \times \mathbb{H}_a^s &= \hat{y}\, \mathbb{E}_a^s \end{aligned}\right\} \text{ within } s_a$$

$$\left.\begin{aligned} -\nabla \times \mathbb{E}_a^s &= \hat{z}\, \mathbb{H}_a^s \\ \nabla \times \mathbb{H}_a^s &= \hat{y}\, \mathbb{E}_a^s \end{aligned}\right\} \text{ in free space}$$

$$\left.\begin{aligned} -\nabla \times \mathbb{E}_a^s &= \hat{z}\,\mathbb{H}_a^s \\ \nabla \times \mathbb{H}_a^s &= \hat{y}\,\mathbb{E}_a^s + \mathbb{J}_a^s \end{aligned}\right\} \text{ within } s_b$$

.... A - 24

Now consider the reaction $\langle a,b \rangle$.

$$\langle a,b \rangle = \iint_{s_b} (\mathbb{E}_b \times \mathbb{H}_a - \mathbb{E}_a \times \mathbb{H}_b) \cdot d\,\mathbb{S}$$

$$= \iint_{s_b} (\mathbb{E}_b \times \mathbb{H}_a^i - \mathbb{E}_a^i \times \mathbb{H}_b) \cdot d\,\mathbb{S}$$

$$+ \iint_{s_b} (\mathbb{E}_b \times \mathbb{H}_a^s - \mathbb{E}_a^s \times \mathbb{H}_b) \cdot d\,\mathbb{S}$$

.... A - 25

It may now be shown with the aid of equations A - 24, A - 3, A - 5 and the divergence theorem that the last integral in A - 25 vanishes. Thus, the reaction $\langle a,b \rangle$ reduces to

$$\langle a,b \rangle = \iint_{s_b} (\mathbb{E}_b \times \mathbb{H}_a^i - \mathbb{E}_a^i \times \mathbb{H}_b) \cdot d\,\mathbb{S}$$

.... A - 26

Equation A - 26 may be converted to a volume integral by following the same steps used to convert A - 11 to A - 19. The final result is

$$\langle a, b \rangle = \iiint_{v_b} (\mathbb{E}_a^i \cdot \mathbb{J}_b)\, d\tau \qquad \text{.......... A - 27}$$

where $\mathbb{J}_b$ is the volume distribution of electric current given by

$$\mathbb{J}_b = (\hat{y}_b - \hat{y})\mathbb{E}_b + \mathbb{J}_b^i \qquad \text{.......... A - 28}$$

within s_b.

Approximations for Good Conductors

In practice, a cylindrical dipole will be made from material which is a good conductor. In this case, the currents will tend to concentrate as a thin layer near the surface, and quickly go to zero as one progresses inward from the surface.

From a mathematical point of view, it is convenient to approximate the behaviour of a practical conductor by that of a perfect conductor. In this case, the fields within the conductor are zero and the tangential component of the total electric field vanishes at the surface. The currents are then concentrated as a sheet of current over the surface. The conduction currents are no longer obtained from $\sigma \mathbb{E}$, but rather from $\hat{n} \times \mathbb{H}$. Thus, the current density $\mathbb{J}_b$ of equation A - 27 becomes, in the limit, the surface current density $\mathbb{J}_{b_s}$ given by

$$\mathbb{J}_{b_s} = \hat{n} \times \mathbb{H}_b \qquad \text{.......... A - 29}$$

The volume integration now becomes a surface integration over the surface where the current $\mathbb{J}_{b_s}$ exists. Equation A - 27 then reduces to

$$\langle a,b \rangle = \iint_{S_b} (\mathbb{E}_a^i \cdot \mathbb{J}_{b_s})\, dS \qquad \text{.......... A - 30}$$

when antenna "b" is composed of perfect conductors.

Now consider equation A - 19. The integrand vanishes everywhere except across the gap. Integrating over the cross-section of the gap yields the total driving current I_b. The current I_b is assumed to be constant across the length of the gap. Thus, equation A - 19 reduces to

$$\langle a,b \rangle = \int_{-\delta}^{\delta} E_{a_z}\, I_b\, dz$$

$$= I_b \int_{-\delta}^{\delta} E_{a_z}\, dz$$

Finally,

$$\langle a, b \rangle = - V_{ba} I_b \qquad \text{.......... A - 31}$$

where V_{ba} is the voltage at the terminals of antenna "b" due to the fields of antenna "a", and I_b is the driving current of antenna "b" when it transmits.

Approximating the Reactions

In all previous discussions, the current densities were treated as if they were known - in reality, they are not. However, since the medium is homogeneous except for two small islands of matter, it is possible to assume current distributions in these subregions, and from these, obtain approximate expressions for $\mathbb{E}$ and $\mathbb{H}$ elsewhere.

Consider for example, the situation when antenna "a" is transmitting. The fields produced by the current distributions within s_a were denoted by $\mathbb{E}_a^i$ and $\mathbb{H}_a^i$. Outside of s_a, the fields may be calculated by the vector potential method as outlined in Chapter II, using as the source, the assumed current distribution within s_a.

To simplify the formulation of the problem, the fields outside of s_a are often approximated by replacing the surface current density by a z-directed filamentary current concentrated on the axis of the dipole and calculating the fields produced by this. The vector potential is then found from application of equation 2 - 7, and the z-component of the electric field from equation 2 - 8.

Considering $\mathbb{J}_{b_s}$ to be z-directed, the dot product in equation A - 30 becomes simply the product of the z-component of $\mathbb{E}_a^i$ and the z-component of $\mathbb{J}_{b_s}$. Integrating around the circumference of antenna "b" yields the total current $I_b(z)$ at any point z. Equation A - 30 then reduces to

$$\langle a,b\rangle = \int_{-\ell_b}^{\ell_b} E_{a_z}^i \, I_b(z) \, dz$$

Substituting from equation 2 - 8 gives

$$\langle a,b\rangle = -j30\beta \int_{-\ell_b}^{\ell_b}\int_{-\ell_a}^{\ell_a} I_a(z')I_b(z) \, G(z',z) \, dz' \, dz \qquad \text{.......... A - 32}$$

where

$$G(z',z) = \left(1 + \frac{1}{\beta^2}\frac{\partial^2}{\partial z^2}\right)\frac{e^{-j\beta r}}{r} \qquad \text{.......... A - 33}$$

APPENDIX B

STATIONARY FORMULAS FOR IMPEDANCE

Stationary Integrals

Consider the integral shown as equation B - 1.

$$N = \int_a^b F(x,y,y')\, dx \qquad \text{.......... B - 1}$$

If the substitution

$$\bar{y} = y(x) + \alpha\eta(x) \qquad \text{.......... B - 2}$$

is made, the integral becomes a function of α. That is

$$N(\alpha) = \int_a^b F(x,\bar{y},\bar{y}')\, dx \qquad \text{.......... B - 3}$$

The integral $N(\alpha)$ is said to be stationary for small α if the following condition is satisfied

$$\left.\frac{dN(\alpha)}{d\alpha}\right|_{\alpha=0} = 0 \qquad \text{.......... B - 4}$$

Enforcing the condition of equation B - 4 leads to Euler's equation (which is a first necessary condition) for determining the extremal function $y(x)$ which minimizes or maximizes the integral.

In some cases, it is not possible to solve for y(x) by this procedure. However, if it can be shown that the integral N is stationary, then an approximate solution may be obtained by assuming a trial function $\bar{y}(x)$, and evaluating the integral B - 3 with this choice. This of course does not yield the correct value N(0) which would be obtained if the true y(x) were used, but for small α , the error will be small. This is shown pictorially in figure B - 1.

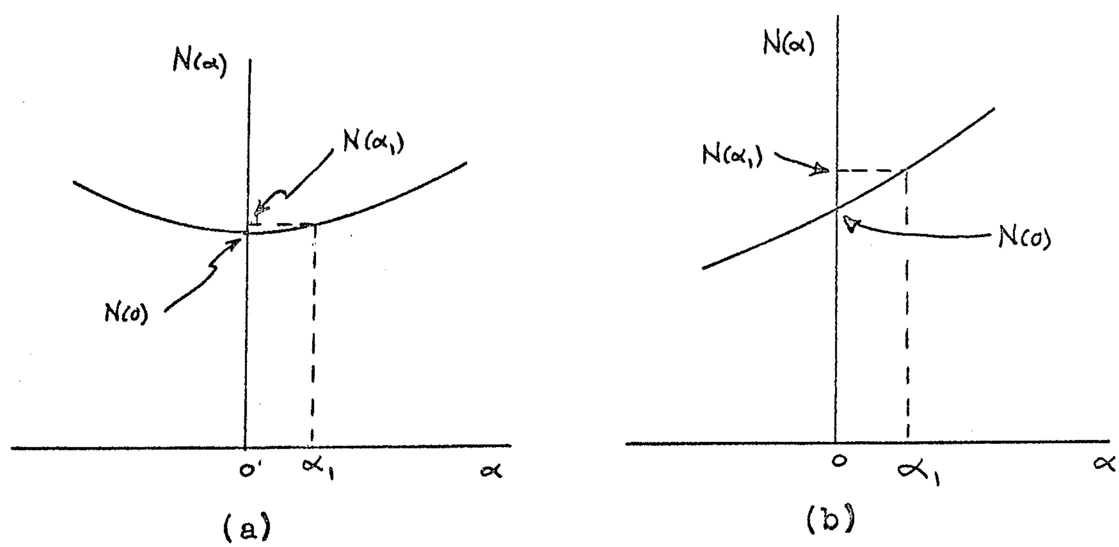

Figure B - 1. N(α) versus α for (a) stationary formula and (b) non-stationary formula.

For a given error $\delta y = \alpha \eta(x)$ in the approximation, the corresponding error in N(α) is, in general, much smaller for a stationary integral than for a non-stationary integral.

The Ritz Procedure

The Ritz method is a procedure for obtaining approximate solutions of problems which have a stationary character. The procedure consists essentially in assuming that the desired extremal can be approximated by a linear combination of "n" suitably chosen functions of the form

$$y \approx c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x) \qquad \text{.......... B - 5}$$

where the "c's" are constants to be determined. Usually the functions $\phi_k(x)$ are chosen in such a manner that this expression satisfies the specified boundary conditions for any choice of the "c's". In physical problems, the general nature of the solution is often known, and a set of ϕ's is chosen in such a way that some linear combination of them may be expected to satisfactorally approximate the solution.

Stationary Character of the Reactions

If two variational parameters α_a and α_b are involved, the condition for a stationary integral is

$$\left.\frac{\partial N(\alpha_a, \alpha_b)}{\partial \alpha_a}\right|_{\alpha_a = \alpha_b = 0} = \left.\frac{\partial N(\alpha_a, \alpha_b)}{\partial \alpha_b}\right|_{\alpha_a = \alpha_b = 0} = 0 \qquad \text{.......... B - 6}$$

Applying this idea, it is easily shown that the reactions of Chapter III are stationary. For example, consider $\langle a,b\rangle$. It is desired to show that the reaction $\langle a,b\rangle$ is stationary for small variations of "a" and "b" about the correct values "c_a" and "c_b". To show this, let

$$a = c_a + \alpha_a e_a$$

$$b = c_b + \alpha_b e_b$$

Expanding $\langle a,b\rangle$ yields

$$\langle a,b\rangle = \langle c_a,c_b\rangle + \alpha_a\langle e_a,c_b\rangle + \alpha_b\langle c_a,e_b\rangle + \alpha_a\alpha_b\langle e_a,e_b\rangle$$

The constraints

$$\langle a,b\rangle = \langle c_a,b\rangle = \langle a,c_b\rangle$$

leads to two more relations

$$\langle a,b\rangle = \langle c_a,c_b\rangle + \alpha_b\langle c_a,e_b\rangle$$

$$\langle a,b\rangle = \langle c_a,c_b\rangle + \alpha_a\langle e_a,c_b\rangle$$

Substituting these into the above equations yields

$$\langle a,b\rangle = \langle c_a,c_b\rangle - \alpha_a\alpha_b\langle e_a,e_b\rangle \qquad \text{.......... B - 7}$$

Since $\langle c_a,c_b\rangle$ is not dependent upon either α_a or α_b , it is easily seen that B - 6 is satisfied for $\langle a, b \rangle$, thus proving its stationary character.

Stationary Formulas for Impedance

With the aid of equation A - 32, the set of simultaneous equations 4 - 2 may be written as

$$I_a^2(0)\; Z_{aa} + 2\, I_a(0)\; I_b(0)\; Z_{ab} + I_b^2(0)\, Z_{bb} = \iint I_a\; \bar{G}_{aa}\; I_a + 2 \iint I_a\; \bar{G}_{ab}\; I_b + \iint I_b\; \bar{G}_{bb}\; I_b$$

.... B - 8

where the specific arguments of the integrals have been omitted for convenience. That is, the following short hand notation has been used

$$\iint I_m\; \bar{G}_{mn}\; I_n = -\, j30\beta \int_{-l_m}^{l_m} \int_{-l_n}^{l_n} I_m(z)\; G_{mn}(z, z')\; I_n(z')\; dz'dz$$

Let the currents I_a and I_b be varied about their true values. Now, as shown by Levis and Tai[5], the set of equations obtained from B - 8 by means of the variational calculus shows that

$$\delta Z_{aa} = \delta Z_{ab} = \delta Z_{bb} = 0 \qquad \text{.......... B - 9}$$

Thus, the simultaneous equations 4 - 2 represent a set for which the impedances are stationary with respect to small variations of the assumed current distributions about the true current distributions.

APPENDIX C

INTEGRATING THE REACTIONS

Because of the reciprocal nature of the reactions, the matrix $[\mathcal{R}]$ is symmetrical about the principal diagonal, and consequently only ten of the sixteen reactions are distinct. Furthermore, all ten of these reactions may be obtained from four general integrals. The four general integrals represent the reactions $\langle u,m\rangle$, $\langle u,n\rangle$, $\langle v,m\rangle$ and $\langle v,n\rangle$. By replacing ℓ_b by ℓ_a in these integrals, one obtains the three reactions $\langle u,u\rangle$, $\langle u,v\rangle$ and $\langle v,v\rangle$. Alternatively, if ℓ_a is replaced by ℓ_b in the above general integrals, the reactions $\langle m,m\rangle$, $\langle m,n\rangle$ and $\langle n,n\rangle$ are obtained. These ten reactions completely specify the reaction matrix.

Integration of the reaction integrals is long and involved, and only the main steps are outlined below.

I THE REACTION $\langle u,m\rangle$

The reaction $\langle u,m\rangle$ may be expressed as

$$\langle u,m\rangle = -j\,30\beta \int_{-\ell_a}^{\ell_a}\int_{-\ell_b}^{\ell_b} I_u(s)\, I_m(z)\, G(s,z)\; ds\, dz \qquad \text{.......... C - 1}$$

where $$G(s,z) = \left(1 + \frac{1}{\beta^2}\frac{\partial^2}{\partial z^2}\right)\frac{e^{-j\beta r}}{r}$$

$$r = \sqrt{(z - s)^2 + d^2}$$

$$\qquad\qquad\text{.......... C - 2}$$

Inserting the expressions for $I_u(s)$ and $I_m(z)$ yields

$$\langle u,m\rangle = -j\,30\beta\int_{-l_a}^{l_a}\sin\beta(l_a - |s|)\left[\int_{-l_b}^{l_b}\sin\beta(l_b - |z|)\,G(s,z)\,dz\right] ds$$

Define the inner integral as A. Thus

$$A = \int_{-l_b}^{l_b}\sin\beta(l_b - |z|)\,G(s,z)\,dz \qquad\text{.......... C - 3}$$

$$= \int_{-l_b}^{0}\sin\beta(l_b + z)\;G(s,z)\;dz + \int_{0}^{l_b}\sin\beta(l_b - z)\;G(s,z)\;dz$$

$$= A_1 + A_2$$

Consider A_1.

$$A_1 = \int_{-l_b}^{0}\sin\beta(l_b + z)\;\frac{e^{-j\beta r}}{r}\;dz$$

$$+ \frac{1}{\beta^2}\int_{-l_b}^{0}\sin\beta(l_b + z)\,\frac{\partial^2}{\partial z^2}\left(\frac{e^{-j\beta r}}{r}\right)\;dz$$

$$= A_1' + \frac{1}{\beta^2}A_1''$$

$$A_1'' = \int_{-\ell_b}^{0} \sin\beta(\ell_b + z)\frac{\partial^2}{\partial z^2}\left(\frac{e^{-j\beta r}}{r}\right) dz$$

Integrating by parts twice yields

$$A_1'' = \sin\beta\ell_b \frac{\partial}{\partial z}\left(\frac{e^{-j\beta r}}{r}\right)\Bigg|_{z=0} - \beta \cos\beta(\ell_b + z)\,\frac{e^{-j\beta r}}{r}\Bigg|_{z=-\ell_b}^{z=0}$$

$$-\beta^2 \int_{-\ell_b}^{0} \sin\beta(\ell_b + z)\,\frac{e^{-j\beta r}}{r}\, dz$$

When this expression is divided by β^2, it is seen that the last term is equal to $-A_1'$. After cancellation, A_1 reduces to

$$A_1 = \frac{1}{\beta^2} \sin\beta\ell_b \frac{\partial}{\partial z}\left(\frac{e^{-j\beta r}}{r}\right)\Bigg|_{z=0}$$

$$-\frac{1}{\beta}\cos\beta(\ell_b + z)\,\frac{e^{-j\beta r}}{r}\Bigg|_{z=-\ell_b}^{z=0}$$

A similar development yields the following expression for A_2

$$A_2 = -\frac{1}{\beta^2}\sin\beta\ell_b \frac{\partial}{\partial z}\left(\frac{e^{-j\beta r}}{r}\right)\Bigg|_{z=0} + \frac{1}{\beta}\cos\beta(\ell_b - z)\frac{e^{-j\beta r}}{r}\Bigg|_{z=0}^{z=\ell_b}$$

Recall that $A = A_1 + A_2$. Thus,

$$A = \frac{1}{\beta}\left\{\cos\beta(\ell_b - z)\frac{e^{-j\beta r}}{r}\Bigg|_{z=0}^{z=\ell_b} - \cos\beta(\ell_b + z)\frac{e^{-j\beta r}}{r}\Bigg|_{z=-\ell_b}^{z=0}\right\}$$

$$= \frac{1}{\beta}\left\{\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} - 2\cos\beta\ell_b\frac{e^{-j\beta R_o}}{R_o}\right\} \qquad \text{.......... C - 4}$$

where

$$R_o = \sqrt{s^2 + d^2}$$

$$R_1 = \sqrt{(\ell_b - s)^2 + d^2}$$

$$R_2 = \sqrt{(\ell_b + s)^2 + d^2}$$

Substituting this into the expression for $\langle u,m\rangle$ gives

$$\langle u,m\rangle = -j\,30\int_{-\ell_a}^{\ell_a}\sin\beta(\ell_a - |s|)\left[\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} - 2\cos\beta\ell_b\frac{e^{-j\beta R_o}}{R_o}\right]ds$$

Finally

$$\langle u,m\rangle = -j\,60\int_0^{\ell_a} \sin\beta(\ell_a - s)\left[\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} - 2\cos\beta\ell_b\,\frac{e^{-j\beta R_0}}{R_0}\right] ds$$

.... C - 5

The remaining integration required to reduce equation C - 5 to closed form will be postponed until section V.

II THE REACTION $\langle v,m\rangle$

The reaction $\langle v,m\rangle$ is given by

$$\langle v,m\rangle = -j\,30\beta\int_{-\ell_a}^{\ell_a} \beta(\ell_a - |s|)\cos\beta(\ell_a - |s|)\left[\int_{-\ell_b}^{\ell_b} \sin\beta(\ell_b - |z|)\, G(s,z)\, dz\right] ds$$

The inner integral is recognized as A in equation C - 3. The result C - 4 may be inserted directly to give

$$\langle v,m\rangle = -j\,60\int_0^{\ell_a} \beta(\ell_a - s)\cos\beta(\ell_a - s)\left[\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} - 2\cos\beta\ell_b\,\frac{e^{-j\beta R_0}}{R_0}\right] ds$$

.... C - 6

III THE REACTION <v,n>

$$\langle v,n\rangle = -j\,30\beta \int_{-\ell_a}^{\ell_a} \beta(\ell_a - |s|)\cos\beta(\ell_a - |s|)\,B(s)\,ds$$

where $$B(s) = \int_{-\ell_b}^{\ell_b} \beta(\ell_b - |z|)\cos\beta(\ell_b - |z|)\,G(s,z)\,dz \qquad \text{.......... C - 7}$$

Let $B(s) = B_1 + B_2$

$$B_1 = \int_{-\ell_b}^{0} \beta(\ell_b + z)\cos\beta(\ell_b + z)\,G(s,z)\,dz$$

$$B_2 = \int_{0}^{\ell_b} \beta(\ell_b - z)\cos\beta(\ell_b - z)\,G(s,z)\,dz$$

It is convenient to further subdivide these integrals. For example, B_1 may be expressed as

$$B_1 = B_1' + \frac{1}{\beta^2} B_1''$$

$$B_1' = \int_{-\ell_b}^{0} \beta(\ell_b + z)\cos\beta(\ell_b + z)\,\frac{e^{-j\beta r}}{r}\,dz$$

$$B_1'' = \int_{-\ell_b}^{0} \beta(\ell_b + z)\cos\beta(\ell_b + z)\,\frac{\partial^2}{\partial z^2}\left(\frac{e^{-j\beta r}}{r}\right)dz$$

Consider B_1''. Integrating by parts twice yields

$$B_1'' = \beta \ell_b \cos\beta\ell_b \frac{\partial}{\partial z}\left(\frac{e^{-j\beta r}}{r}\right)\Bigg|_{z=0} - \beta\cos\beta(\ell_b+z)\frac{e^{-j\beta r}}{r}\Bigg|_{z=-\ell_b}^{z=0}$$

$$- 2\beta^2 \int_{-\ell_b}^{0} \sin\beta(\ell_b+z)\frac{e^{-j\beta r}}{r}\,dz + \beta^2 \ell_b \sin\beta\ell_b \frac{e^{-j\beta r}}{r}\Bigg|_{z=0}$$

$$- \beta^2 \int_{-\ell_b}^{0} \beta(\ell_b+z)\cos\beta(\ell_b+z)\frac{e^{-j\beta r}}{r}\,dz$$

When this expression is divided by β^2, it is seen that the last term is equal to $-B_1'$. After cancellation, B_1 reduces to

$$B_1 = \frac{1}{\beta}\left\{ \ell_b \cos\beta\ell_b \frac{\partial}{\partial z}\left(\frac{e^{-j\beta r}}{r}\right)\Bigg|_{z=0} + \frac{e^{-j\beta R_2}}{R_2} \right.$$

$$+ \left(\beta\ell_b \sin\beta\ell_b - \cos\beta\ell_b\right)\frac{e^{-j\beta R_0}}{R_0}$$

$$\left. - 2\beta \int_{0}^{\ell_b} \sin\beta(\ell_b - z)\frac{e^{-j\beta \bar{r}}}{\bar{r}}\,dz \right\}$$

where R_o and R_2 are the same as defined in section I, and $\bar{r}$ is given by

$$\bar{r} = \sqrt{(z + s)^2 + d^2}$$

A similar development yields the following expression for B_2

$$B_2 = \frac{1}{\beta}\Bigg\{ - \ell_b \cos\beta\ell_b \frac{\partial}{\partial z}\left(\frac{e^{-j\beta r}}{r}\right)_{z=0} + \frac{e^{-j\beta R_1}}{R_1}$$

$$+ \left(\beta\ell_b \sin\beta\ell_b - \cos\beta\ell_b\right)\frac{e^{-j\beta R_o}}{R_o}$$

$$- 2\beta\int_0^{\ell_b} \sin\beta(\ell_b - z)\frac{e^{-j\beta r}}{r}\, dz \Bigg\}$$

Make the following definition

$$F(s,z) = \frac{e^{-j\beta r}}{r} + \frac{e^{-j\beta\bar{r}}}{\bar{r}}$$

With this definition, B_1 and B_2 may be added to obtain $B(s)$, which, after cancellation is

$$B = \frac{1}{\beta}\left\{ \frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} + 2\left(\beta l_b \sin\beta l_b - \cos\beta l_b\right)\frac{e^{-j\beta R_0}}{R_0} - 2\beta\int_0^{l_b} \sin\beta(l_b - z)\, F(s,z)\, dz \right\} \qquad \text{.......... C - 8}$$

This may now be substituted into the expression for $\langle v,n\rangle$. When this is done, it is seen that a group of terms combines to form $\langle v,m\rangle$. The final result for $\langle v,n\rangle$ is

$$\langle v,n\rangle = \langle v,m\rangle - j\,120\,\beta l_b \sin\beta l_b \int_0^{l_a} \beta(l_a - s)\cos\beta(l_a - s)\frac{e^{-j\beta R_0}}{R_0}\, ds + j\,120\beta\int_0^{l_a} \beta(l_a - s)\cos\beta(l_a - s)\left[\int_0^{l_b} \sin\beta(l_b - z)\, F(s,z)\, dz\right] ds \qquad \text{.......... C - 9}$$

The final evaluation of this expression is postponed to Section V.

IV THE REACTION ⟨u,n⟩

$$\langle u,n \rangle = -j\,30\,\beta \int_{-\ell_a}^{\ell_a} \sin\beta(\ell_a - |s|)\; B(s)\, ds$$

where
$$B(s) = \int_{-\ell_b}^{\ell_b} \beta(\ell_b - |z|) \cos\beta(\ell_b - |z|)\; G(s,z)\, dz \qquad \text{.......... C - 10}$$

This integral has already been shown to reduce to equation C - 8. If the result C - 8 is substituted into the above expression for ⟨u,n⟩ , the following is obtained

$$\langle u,n \rangle = \langle u,m \rangle - j\,120\,\beta\ell_b \sin\beta\ell_b \int_0^{\ell_a} \sin\beta(\ell_a - s)\,\frac{e^{-j\beta R_o}}{R_o}\, ds$$

$$+\, j\,120\,\beta \int_0^{\ell_a} \sin\beta(\ell_a - s)\left[\int_0^{\ell_b} \sin\beta(\ell_b - z)\, F(s,z)\, dz\right] ds \qquad \text{.......... C - 11}$$

Evaluation of this expression is considered in the next section.

V EVALUATION OF THE INTEGRALS

All reactions considered in the previous sections can be expressed in terms of the following six integrals

$$N_1 = \int_0^{l_a} \sin\beta(l_a - s)\left[\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2}\right] ds$$

.... C - 12

$$N_2 = \int_0^{l_a} \sin\beta(l_a - s)\ \frac{e^{-j\beta R_0}}{R_0}\ ds$$

.... C - 13

$$N_3 = \int_0^{l_a} \beta(l_a - s)\cos\beta(l_a - s)\left[\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2}\right] ds$$

.... C - 14

$$N_4 = \int_0^{l_a} \beta(l_a - s)\cos\beta(l_a - s)\frac{e^{-j\beta R_0}}{R_0}\ ds$$

.... C - 15

$$N_5 = \int_0^{l_a} \beta(l_a - s)\cos\beta(l_a - s)\left[\int_0^{l_b} \sin\beta(l_b - z)\ F(s,z)\, dz\right] ds$$

.... C - 16

$$N_6 = \int_0^{l_a} \sin\beta(l_a - s)\left[\int_0^{l_b} \sin\beta(l_b - z)\ F(s,z)\, dz\right] ds$$

.... C - 17

In terms of the above defined integrals, the reactions may be expressed as

$$\langle u,m\rangle = -j60(N_1 - 2\cos\beta\ell_b\, N_2)$$

$$\langle v,m\rangle = -j60(N_3 - 2\cos\beta\ell_b\, N_4)$$

$$\langle v,n\rangle = -j60(N_3 + 2(\beta\ell_b\sin\beta\ell_b - \cos\beta\ell_b)N_4 - 2\beta N_5)$$

$$\langle u,n\rangle = -j60(N_1 + 2(\beta\ell_b\sin\beta\ell_b - \cos\beta\ell_b)N_2 - 2\beta N_6)$$

.... C - 18

Function Subroutines

In order to systemize the evaluation of equations C - 12 to C - 17, the following function subroutines are defined

$$E(x_1,x_2) = \int_{x_1}^{x_2} \frac{e^{-j\beta(R-x)}}{R}\,dx$$

$$H(x_1,x_2,x_3) = \int_{x_1}^{x_2} (x+x_3)\frac{e^{-j\beta(R-x)}}{R}\,dx$$

$$W(x_1,x_2,x_3) = \int_{x_1}^{x_2} (x+x_3)^2\frac{e^{-j\beta(R-x)}}{R}\,dx$$

where $R = \sqrt{x^2 + d^2}$

Evaluation of $E(x_1, x_2)$ is straightforward. Define a change of variables as

$$z = \beta (R - x)$$

Then
$$\frac{dz}{dx} = \beta\left(\frac{dR}{dx} - 1\right) = \beta\left(\frac{x}{R} - 1\right)$$

$$= -\frac{\beta(R-x)}{R} = -\frac{z}{R}$$

Thus
$$\frac{dx}{R} = -\frac{dz}{z}$$

Substituting into the integral defining $E(x_1, x_2)$ yields

$$E(x_1, x_2) = -\int_{z_1}^{z_2} \frac{e^{-jz}}{z}\, dz = \int_{z_2}^{z_1} \frac{e^{-jz}}{z}\, dz$$

$$= \int_{z_2}^{z_1} \frac{\cos z}{z}\, dz - j\int_{z_2}^{z_1} \frac{\sin z}{z}\, dz$$

$$= \left[Ci(z_1) - Ci(z_2) \right] - j\left[Si(z_1) - Si(z_2) \right]$$

.... C - 19

where

$$z_1 = \beta\left(\sqrt{x_1^2 + d^2} \quad - x_1\right)$$

$$z_2 = \beta\left(\sqrt{x_2^2 + d^2} \quad - x_2\right)$$

In a similar manner, the expression for $H(x_1, x_2, x_3)$ is shown to be

$$H(x_1, x_2, x_3) = (x_3 + j\frac{\beta d^2}{2})E(x_1, x_2) + j\frac{1}{2\beta}\left(e^{-jz_2} - e^{-jz_1}\right) + \frac{\beta d^2}{2}\left(\frac{e^{-jz_2}}{z_2} - \frac{e^{-jz_1}}{z_1}\right)$$

.... C - 20

where z_1 and z_2 are the same as defined above.

The expression for $W(x_1, x_2, x_3)$ is

$$W(x_1, x_2, x_3) = \left(x_3^2 + \frac{d^2}{2} + \frac{\beta^2 d^4}{8}\right)E(x_1, x_2) + 2x_3 H(x_1, x_2, 0) - \frac{1}{4\beta^2}\left[(1 + jz_2)e^{-jz_2} - (1 + jz_1)e^{-jz_1}\right] + \frac{\beta^2 d^4}{8}\left[\left(\frac{e^{-jz_2}}{z_2^2} - \frac{e^{-jz_1}}{z_1^2}\right) - j\left(\frac{e^{-jz_2}}{z_2} - \frac{e^{-jz_1}}{z_1}\right)\right]$$

.... C - 21

Reduction of the Integrals to Closed Form

Consider N_1.

$$N_1 = \int_0^{\ell_a} \sin\beta(\ell_a - s)\left[\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2}\right] ds$$

But $$\sin\beta(\ell_a - s) = \frac{1}{2j}\left[e^{j\beta(\ell_a - s)} - e^{-j\beta(\ell_a - s)}\right]$$

Substituting this into the expression for N_1 and rearranging yields

$$N_1 = \frac{1}{2j}\Bigg\{ e^{j\beta(\ell_a - \ell_b)} \int_0^{\ell_a} \frac{e^{-j\beta(R_1 + s - \ell_b)}}{R_1} ds$$

$$- e^{-j\beta(\ell_a - \ell_b)} \int_0^{\ell_a} \frac{e^{-j\beta(R_1 - s + \ell_b)}}{R_1} ds + e^{j\beta(\ell_a + \ell_b)} \int_0^{\ell_a} \frac{e^{-j\beta(R_2 + s + \ell_b)}}{R_2} ds$$

$$- e^{-j\beta(\ell_a + \ell_b)} \int_0^{\ell_a} \frac{e^{-j\beta(R_2 - s - \ell_b)}}{R_2} ds \Bigg\}$$

Consider the first integral in this expression.

Define $x = \ell_b - s$

Then $dx = - ds$

$$R_1 = \sqrt{(\ell_b - s)^2 + d^2} = \sqrt{x^2 + d^2} = R$$

Substituting yields

$$\int_0^{\ell_a} \frac{e^{-j\beta(R_1 + s - \ell_b)}}{R_1} ds = - \int_{\ell_b}^{\ell_b - \ell_a} \frac{e^{-j\beta(R - x)}}{R} dx$$

$$= - E(\ell_b, \ell_b - \ell_a)$$

All other integrals in N_1 may be treated in a similar manner. The final result is

$$N_1 = 0.5j \Big\{ e^{j\beta(\ell_a - \ell_b)} E(\ell_b, \ell_b - \ell_a)$$

$$+ e^{-j\beta(\ell_a - \ell_b)} E(-\ell_b, \ell_a - \ell_b) + e^{j\beta(\ell_a + \ell_b)} E(-\ell_b, -\ell_a - \ell_b)$$

$$+ e^{-j\beta(\ell_a + \ell_b)} E(\ell_b, \ell_a + \ell_b) \Big\}$$

Since N_2, N_3 and N_4 may be reduced in exactly the same manner, only the results are presented below.

$$N_2 = j0.5\left\{e^{j\beta\ell_a}E(0,-\ell_a) + e^{-j\beta\ell_a}E(0,\ell_a)\right\}$$

$$\begin{aligned} N_3 = -0.5\beta\Big\{ & e^{j\beta(\ell_a-\ell_b)}H(\ell_b,\ell_b-\ell_a,\ell_a+\ell_b) \\ & + e^{j\beta(\ell_a+\ell_b)}H(-\ell_b,-\ell_a-\ell_b,\ell_a+\ell_b) \\ & + e^{-j\beta(\ell_a-\ell_b)}H(-\ell_b,\ell_a-\ell_b,\ell_b-\ell_a) \\ & + e^{-j\beta(\ell_a+\ell_b)}H(\ell_b,\ell_a+\ell_b,-\ell_a-\ell_b)\Big\} \end{aligned}$$

$$N_4 = -0.5\beta\left\{e^{j\beta\ell_a}H(0,-\ell_a,\ell_a) - e^{-j\beta\ell_a}H(0,\ell_a,-\ell_a)\right\}$$

The integral N_5 is

$$N_5 = \int_0^{\ell_a}\int_0^{\ell_b}\beta(\ell_a-s)\cos\beta(\ell_a-s)\sin\beta(\ell_b-z)F(s,z)\,dz\,ds$$

Now $\cos\beta(\ell_a - s)$ and $\sin\beta(\ell_b - z)$ may be expressed in exponential form. Making use of the definition of $F(s,z)$, the integral N_5 may be expanded as follows

$$N_5 = -j0.25\beta\Big\{ e^{j\beta(l_a+l_b)} \int_0^{l_a}\int_0^{l_b} \frac{(l_a-s)}{r} e^{-j\beta(r+s+z)} \, dz\, ds$$

$$+ e^{j\beta(l_a+l_b)} \int_0^{l_a}\int_0^{l_b} \frac{(l_a-s)}{\bar{r}} e^{-j\beta(\bar{r}+s+z)} \, dz\, ds$$

$$- e^{-j\beta(l_a+l_b)} \int_0^{l_a}\int_0^{l_b} \frac{(l_a-s)}{r} e^{-j\beta(r-s-z)} \, dz\, ds$$

$$- e^{-j\beta(l_a+l_b)} \int_0^{l_a}\int_0^{l_b} \frac{(l_a-s)}{\bar{r}} e^{-j\beta(\bar{r}-s-z)} \, dz\, ds$$

$$- e^{j\beta(l_a-l_b)} \int_0^{l_a}\int_0^{l_b} \frac{(l_a-s)}{r} e^{-j\beta(r+s-z)} \, dz\, ds$$

$$- e^{j\beta(l_a-l_b)} \int_0^{l_a}\int_0^{l_b} \frac{(l_a-s)}{\bar{r}} e^{-j\beta(\bar{r}+s-z)} \, dz\, ds$$

$$+ e^{-j\beta(l_a-l_b)} \int_0^{l_a}\int_0^{l_b} \frac{(l_a-s)}{r} e^{-j\beta(r-s+z)} \, dz\, ds$$

$$+ e^{-j\beta(l_a-l_b)} \int_0^{l_a}\int_0^{l_b} \frac{(l_a-s)}{\bar{r}} e^{-j\beta(\bar{r}-s+z)} \, dz\, ds \Big\}$$

At this point, it is convenient to define two functions as follows

$$Q(A,B) = \int_0^A \beta(A-s)\, e^{-j2\beta s} \left\{ \int_0^B \frac{e^{-j\beta(r+z-s)}}{r}\, dz \right\} ds$$

$$G(A,B) = \int_0^A \beta(A-s) \left\{ \int_0^B \frac{e^{-j\beta(r+z-s)}}{r}\, dz \right\} ds$$

By making appropriate changes of variable, all integrals in N_5 can be expressed in terms these two functions. The final result is

$$N_5 = -j0.25 \Big\{ e^{j\beta(l_a+l_b)} \left[Q(l_a, l_b) + G(-l_a, l_b) \right]$$

$$+ e^{-j\beta(l_a+l_b)} \left[Q(-l_a, -l_b) + G(l_a, -l_b) \right]$$

$$+ e^{j\beta(l_a-l_b)} \left[Q(l_a, -l_b) + G(-l_a, -l_b) \right]$$

$$+ e^{-j\beta(l_a-l_b)} \left[Q(-l_a, l_b) + G(l_a, l_b) \right] \Big\}$$

Evaluation of Q(A,B) and G(A,B) is postponed until N_6 has been considered.

After expansion, N_6 may be expressed as

$$N_6 = 0.25\Bigg\{ e^{j\beta(\ell_a-\ell_b)} \Bigg[\int_0^{\ell_a}\int_0^{\ell_b} \frac{e^{-j\beta(r+s-z)}}{r}\,dz\,ds$$

$$+ \int_0^{\ell_a}\int_0^{\ell_b} \frac{e^{-j\beta(\bar{r}+s-z)}}{\bar{r}}\,dz\,ds \Bigg]$$

$$+ e^{-j\beta(\ell_a-\ell_b)} \Bigg[\int_0^{\ell_a}\int_0^{\ell_b} \frac{e^{-j\beta(r-s+z)}}{r}\,dz\,ds + \int_0^{\ell_a}\int_0^{\ell_b} \frac{e^{-j\beta(\bar{r}-s+z)}}{\bar{r}}\,dz\,ds \Bigg]$$

$$- e^{j\beta(\ell_a+\ell_b)} \Bigg[\int_0^{\ell_a}\int_0^{\ell_b} \frac{e^{-j\beta(r+s+z)}}{r}\,dz\,ds + \int_0^{\ell_a}\int_0^{\ell_b} \frac{e^{-j\beta(\bar{r}+s+z)}}{\bar{r}}\,dz\,ds \Bigg]$$

$$- e^{-j\beta(\ell_a+\ell_b)} \Bigg[\int_0^{\ell_a}\int_0^{\ell_b} \frac{e^{-j\beta(r-s-z)}}{r}\,dz\,ds + \int_0^{\ell_a}\int_0^{\ell_b} \frac{e^{-j\beta(\bar{r}-s-z)}}{\bar{r}}\,dz\,ds \Bigg] \Bigg\}$$

Define

$$P(A,B) = \int_0^{A}\int_0^{B} \frac{e^{-j\beta(r-s+z)}}{r}\,dz\,ds$$

$$Q(A,B) = \int_0^{A} e^{-j2\beta s} \Bigg[\int_0^{B} \frac{e^{-j\beta(r-s+z)}}{r}\,dz \Bigg] ds$$

In this notation, N_6 becomes

$$N_6 = 0.25\Big\{ e^{j\beta(\ell_a-\ell_b)}\left[P(-\ell_a,-\ell_b) - T(\ell_a,-\ell_b)\right]$$

$$+ e^{-j\beta(\ell_a-\ell_b)}\left[P(\ell_a,\ell_b) - T(-\ell_a,\ell_b)\right]$$

$$+ e^{j\beta(\ell_a+\ell_b)}\left[P(-\ell_a,\ell_b) - T(\ell_a,\ell_b)\right]$$

$$+ e^{-j\beta(\ell_a+\ell_b)}\left[P(\ell_a,-\ell_b) - T(-\ell_a,-\ell_b)\right]\Big\}$$

Consider the function P(A,B).

$$P(A,B) = \int_0^A \left[\int_0^B \frac{e^{-j\beta(r-s+z)}}{r}\,dz\right] ds$$

Define $\quad w = \beta(r-s+z)$

$$\frac{dw}{dz} = \beta\left(\frac{dr}{dz}+1\right) = \beta\left(\frac{z-s}{r}+1\right)$$

$$= \beta\frac{(r-s+z)}{r}$$

$$= \frac{w}{r}$$

Thus $$\frac{dz}{r} = \frac{dw}{w}$$

Upon substitution, there results

$$P(A,B) = \int_0^A \left[\int_{w_0}^{w_B} \frac{e^{-jw}}{w} dw \right] ds$$

where $$w_0 = \beta(R_0 - s)$$

$$w_B = \beta(R_B - s + B)$$

$$R_0 = \sqrt{s^2 + d^2}$$

$$R_B = \sqrt{(s-B)^2 + d^2}$$

In order to integrate by parts, make the following definitions

$$u = \int_{w_0}^{w_B} \frac{e^{-jw}}{w} dw$$

$$\frac{dv}{ds} = 1$$

Then $$v = s$$

To determine $\frac{du}{ds}$, differentiate under the integral. The integrand is not a function of s, but the limits are. Therefore

$$\frac{du}{ds} = \frac{e^{-j W_B}}{W_B} \frac{dW_B}{ds} - \frac{e^{-j W_o}}{W_o} \frac{dW_o}{ds}$$

$$= \frac{e^{-j\beta(R_o - s)}}{R_o} - \frac{e^{-j\beta(R_B - s + B)}}{R_B}$$

Substituting yields

$$P(A,B) = \left[s \int_{W_o}^{W_B} \frac{e^{-jw}}{w} dw \right] \Bigg|_{s=0}^{s=A} - \int_0^A \frac{s e^{-j\beta(R_o - s)}}{R_o} ds$$

$$+ \int_0^A \frac{s e^{-j\beta(R_B - s + B)}}{R_B} ds$$

Finally

$$P(A,B) = A\, E(AB, A) - H(0, A, 0) + H(-B, AB, B)$$

where $AB = A - B$

A similar development yields the following formulae.

$$T(A,B) = \frac{j}{2\beta}\Big\{ e^{-j2\beta A}\, E(AB,A) - E(-B,0) + E(0,-A) - e^{-j2\beta B}\, E(B,-AB)\Big\}$$

$$Q(A,B) = \frac{1}{4\beta}\Big\{ e^{-j2\beta A}\, E(A,AB) + (1 - j2\beta A)\, E(-B,0) + (1 - j2\beta A)\left[e^{-j2\beta B}\, E(B,-AB) - E(0,-A)\right] + j2\beta\left[H(0,-A,0) - e^{-j2\beta B}\, H(B,-AB,-B)\right]\Big\}$$

$$G(A,B) = \beta\Big\{\frac{A}{2}^2\, E(AB,A) - A\left[H(0,A,0) + H(-B,AB,B)\right] + 0.5\left[W(0,A,0) - W(-B,AB,B)\right]\Big\}$$

APPENDIX D

THE COMPUTER PROGRAM

The program was coded in FORTRAN for operation on the IBM System 360 computer. Although lengthy, the program is rather simple, with very little branching. Liberal use was made of function subprograms to simplify the programming. All arithmetic is performed in single precision, with the exception of the matrix inversion routine which is done in double precision. Evaluation of the sine integrals and cosine integrals is performed by means of a series expansion for arguments less than one, and by a Hasting's approximation for arguments greater than one. Total running time per impedance calculation is about six seconds.

A complete listing of the program follows.

```
/JOB
/FTC
C       PROGRAM FOR THE CALCULATION OF THE MUTUAL IMPEDANCE BETWEEN
C       A PAIR OF DIPOLES
C
    1 FORMAT (2F11.7, 2F11.8, F9.5, F10.4, I3)
    2 FORMAT (1H0, 11HBETA * A = F6.3, 5X, 10HOMEGA A = F6.2)
    3 FORMAT (1H , 11HBETA * B = F6.3, 5X, 10HOMEGA B = F6.2)
    4 FORMAT (1H , 11HBETA * D = F7.2)
    5 FORMAT (1H , 6HRAB = F20.8, 5X, 6HXAB = F20.8)
C
      DOUBLE PRECISION R(9,9)
      COMMON BETA, D, DD
C
C
   10 READ (1,1) HA, HB, RA, RB, DI, FREQ, N
C
      I = 0
C
      A = HA
      B = HA
      DD = RA
      GO TO 25
C
   15 B = HB
      DD = DI
      GO TO 25
C
   20 A = HB
      DD = RB
C
   25 I = I + 1
      BETA = FREQ * (2.0959601E-04)
      D = BETA * DD
      AK = A - B
      AM = A + B
      AP = BETA * A
      SAL = SIN(BETA * AK)
      CAL = COS(BETA * AK)
      SAN = SIN(BETA * AM)
      CAN = COS(BETA * AM)
      SAP = SIN(AP)
      CAP = COS(AP)
      AQ = BETA * B
      SAQ = SIN(AQ)
      CAQ = COS(AQ)
      AV = AQ * CAQ
      AY = AP * CAP
      AZ = 2.0 * CAQ
      AW = 2.0 * AQ * SAQ
      AX = 2.0 * BETA
      EA = E(B,-AK)
      EB = EE(B,-AK)
      EC = E(-B,-AM)
      ED = EE(-B,-AM)
      EU = E(-B,AK)
      EG = EE(-B,AK)
      EH = E(B,AM)
      EK = EE(B,AM)
      EL = E(0.,A)
```

```
      EM  =  EE(0.,A)
      EN  =  E(0.,-A)
      EP  =  EE(0.,-A)
      A1  =  0.5  *  (SAL*(EU-EA)  -  CAL*(EB+EG)  -  SAN*(EC-EH)  -  CAN*(ED+EK))
      B1  =  0.5  *  (SAL*(EG-EB)  +  CAL*(EA+EU)  -  SAN*(ED-EK)  +  CAN*(EC+EH))
      A2  =  0.5  *  (SAP*(EL-EN)  -  CAP*(EP+EM))
      B2  =  0.5  *  (SAP*(EM-EP)  +  CAP*(EN+FL))
      BE  =  0.5  *  BETA
      FA  =  Y(B,-AK,AK)
      FB  =  YY(B,-AK,AK)
      FC  =  Y(-B,-AM,AM)
      FD  =  YY(-B,-AM,AM)
      FE  =  Y(-B,AK,-AK)
      FG  =  YY(-B,AK,-AK)
      FH  =  Y(B,AM,-AM)
      FK  =  YY(B,AM,-AM)
      FL  =  Y(0.,A,-A)
      FM  =  YY(0.,A,-A)
      FN  =  Y(0.,-A,A)
      FP  =  YY(0.,-A,A)
      A3  =  BE  *  (SAL*(FB-FG)-CAL*(FA+FE)-CAN*(FC+FH)+SAN*(FD-FK))
      B3  =  BE  *  (SAL*(FE-FA)-CAL*(FB+FG)-CAN*(FD+FK)+SAN*(FH-FC))
      A4  =  BE  *  (SAP*(FP-FM)-CAP*(FN+FL))
      B4  =  BE  *  (SAP*(FL-FN)-CAP*(FP+FM))
      QA  =  Q(A,B)  +  G(-A,B)
      QB  =  QQ(A,B)  +  GG(-A,B)
      QC  =  -Q(-A,-B)  -  G(A,-B)
      QD  =  -QQ(-A,-B)  -  GG(A,-B)
      QE  =  -Q(A,-B)  -  G(-A,-B)
      QU  =  -QQ(A,-B)  -  GG(-A,-B)
      QG  =  Q(-A,B)  +  G(A,B)
      QH  =  QQ(-A,B)  +  GG(A,B)
      A5  =  .25  *  (CAL*(QH-QU)  -  SAL*(QE+QG)  +  CAN*(QB-QD)  +  SAN*(QA+QC))
      B5  =  .25  *  (CAL*(QE-QG)  -  SAL*(QU+QH)  -  CAN*(QA-QC)  +  SAN*(QB+QD))
      PA  =  P(-A,-B)  -  T(A,-B)
      PB  =  PP(-A,-B)  -  TT(A,-B)
      PC  =  P(A,B)  -  T(-A,B)
      PD  =  PP(A,B)  -  TT(-A,B)
      PE  =  T(A,B)  -  P(-A,B)
      PU  =  TT(A,B)  -  PP(-A,B)
      PG  =  T(-A,-B)  -  P(A,-B)
      PH  =  TT(-A,-B)  -  PP(A,-B)
      A6  =  .25  *  (CAL*(PA+PC)  -  SAL*(PB-PD)  -  CAN*(PE+PG)  -  SAN*(PH-PU))
      B6  =  .25  *  (CAL*(PB+PD)  +  SAL*(PA-PC)  -  CAN*(PU+PH)  -  SAN*(PE-PG))
      UM  =  60.  *  (B1  -  AZ  *  B2)
      UA  =  -  60.  *  (A1  -  AZ  *  A2)
      VM  =  60.  *  (B3  -  AZ  *  A4)
      VA  =  -  60.  *  (A3  -  AZ  *  A4)
      VN  =  60.  *  (B3  +  (AW  -  AZ)  *  B4  -  AX  *  B5)
      VB  =  -  60.  *  (A3  +  (AW  -  AZ)  *  A4  -  AX  *  A5)
      UN  =  60.  *  (  B1  +  (AW  -  AZ)  *  B2  -  AX  *  B6)
      UB  =  -  60.  *  (A1  +  (AW  -  AZ)  *  A2  -  AX  *  A6)
C
      IF  (I  -  2)  30,  35,  40
C
   30 R(1,1)  =  UM
      R(5,1)  =  UA
      R(1,2)  =  UN
      R(5,2)  =  UB
      R(2,2)  =  VN
```

```
      R(6,2) = VB
      GO TO 15
C
   35 R(1,3) = UM
      R(5,3) = UA
      R(1,4) = UN
      R(5,4) = UB
      R(2,3) = VM
      R(6,3) = VA
      R(2,4) = VN
      R(6,4) = VB
      GO TO 20
C
   40 R(3,3) = UM
      R(7,3) = UA
      R(3,4) = UN
      R(7,4) = UB
      R(4,4) = VN
      R(8,4) = VB
C
      DO 45 I = 1, 4
      DO 45 J = I, 4
      L = I + 4
      M = J + 4
      R(J,I) = R(I,J)
   45 R(M,I) = R(L,J)
C
      DO 50 I = 1, 4
      DO 50 J = 1, 4
      L = I + 4
      M = J + 4
      R(I,M) = - R(L,J)
   50 R(L,M) = R(I,J)
C
C     BEGIN INVERSION OF REACTION MATRIX
C
      DO 55 K = 2, 9
   55 R(K,9) = 0.0
      R(1,9) = 1.0
      DO 70 M = 1, 8
      DO 60 J = 1, 8
   60 R(9,J) = R(1, J+1)/R(1,1)
      DO 65 I = 2, 8
      L = I - 1
      DO 65 J = 1, 8
   65 R(L,J) = R(I, J+1) - R(I,1) * R(9,J)
      DO 70 K = 1, 8
   70 R(8,K) = R(9,K)
C
      A = SAP * (SAQ*R(1,3)+AV*R(1,4)) + AY*(SAQ*R(2,3)+AV*R(2,4))
      B = SAP*(SAQ*R(5,3)+AV*R(5,4)) + AY*(SAQ*R(6,3)+AV*R(6,4))
      D1P = SAP*(SAP*R(1,1)+2.0*AY*R(1,2))+AY*AY*R(2,2)
      D1PP = SAP*(SAP*R(5,1)+2.0*AY*R(5,2))+AY*AY*R(6,2)
      D2P = SAQ*(SAP*R(3,1)+AY*R(3,2))+AV*(SAP*R(4,1)+AY*R(4,2))
      D2PP = SAQ*(SAP*R(7,1)+AY*R(7,2))+AV*(SAP*R(8,1)+AY*R(8,2))
      D3P = A
      D3PP = B
      D4P = SAQ*(SAQ*R(3,3)+2.0*AV*R(3,4))+AV*AV*R(4,4)
      D4PP = SAQ*(SAQ*R(7,3)+2.0*AV*R(7,4))+AV*AV*R(8,4)
      C = D1P*D4P - D1PP*D4PP - D2P*D3P + D2PP*D3PP
```

```
      D = D1PP*D4P + D1P*D4PP - D2P*D3PP - D2PP*D3P
      RAB = (A*C + B*D)/(C*C + D*D)
      XAB = (B*C - A*D)/(C*C + D*D)
      A = BETA * HA
      B = 2.0 * ALOG (2.0 * HA/RA)
      WRITE (3,2) A, B
      B = 2.0 * ALOG (2.0 * HB/RB)
      WRITE (3,3) AQ, B
      B = BETA * DI
      WRITE (3,4) B
      WRITE (3,5) RAB, XAB
C
      IF (N) 75, 10, 75
   75 CALL EXIT
      END
/FTC
      FUNCTION Z(X)
      COMMON BETA, D, DD
      IF (X) 100, 105, 110
  100 Z = BETA * (SQRT (X*X + DD*DD) - X)
      RETURN
  105 Z = BETA * DD
      RETURN
  110 IF (DD/X - 0.09) 115, 115, 100
  115 H = DD/X
      Z = 0.5 * BETA * X * (H*H - 0.25*H*H*H*H)
      RETURN
      END
/FTC
      FUNCTION E(X1,X2)
      E = CI(Z(X1)) - CI(Z(X2))
      RETURN
      END
/FTC
      FUNCTION EE(X1,X2)
      EE = SI(Z(X2)) - SI(Z(X1))
      RETURN
      END
/FTC
      FUNCTION Y(X1, X2, X3)
      COMMON BETA, D, DD
      Y = 0.5*DD*((SIN (Z(X2))-SIN (Z(X1)))/D+D*(COS (Z(X2))/Z(X2)-
     1  COS (Z(X1))/Z(X1))+D*EE(X1,X2)) + X3*E(X1,X2)
      RETURN
      END
/FTC
      FUNCTION YY(X1, X2, X3)
      COMMON BETA, D, DD
      YY=0.5*DD*((COS (Z(X2))-COS (Z(X1)))/D-D*(SIN (Z(X2))/Z(X2)-
     1  SIN (Z(X1))/Z(X1))-D*E(X1,X2)) + X3*EE(X1,X2)
      RETURN
      END
/FTC
      FUNCTION BB(X1)
      COMMON BETA, D, DD
      BB = 1.0/(D*D) - D*D/(2.0*Z(X1)*Z(X1))
      RETURN
      END
/FTC
      FUNCTION AA(X1)
```

```
      COMMON BETA, D, DD
      AA = Z(X1)/(D*D) + D*D/(2.0*Z(X1))
      RETURN
      END
/FTC
      FUNCTION W(X1, X2, X3)
      COMMON BETA, D, DD
      W = 0.25*DD*DD*(BB(X1)*COS (Z(X1))+AA(X1)*SIN (Z(X1))
     1  -BB(X2)*COS (Z(X2))-AA(X2)*SIN (Z(X2))-(2.0+0.5*D*D)*E(X1,X2))
     2   + X3*X3*E(X1,X2) + 2.0*X3*Y(X1,X2,0)
      RETURN
      END
/FTC
      FUNCTION WW(X1,X2,X3)
      COMMON BETA, D, DD
      WW = 0.25*DD*DD*(AA(X1)*COS (Z(X1))-BB(X1)*SIN (Z(X1))
     1   -AA(X2)*COS (Z(X2))+BB(X2)*SIN (Z(X2))-(2.0+0.5*D*D)*EE(X1,X2))
     2   + X3*X3*EE(X1,X2) + 2.0*X3*YY(X1,X2,0)
      RETURN
      END
/FTC
      FUNCTION G(A,B)
      COMMON BETA, D, DD
      AB = A - B
      G = BETA*(A*(0.5*A*E(AB,A)-Y(0.,A,0.)+Y(-B,AB,B))
     1   + 0.5*(W(0.,A,0.)-W(-B,AB,B)))
      RETURN
      END
/FTC
      FUNCTION GG(A,B)
      COMMON BETA, D, DD
      AB = A - B
      GG = BETA*(A*(0.5*A*EE(AB,A)-YY(0.,A,0.)+YY(-B,AB,B))
     1   + 0.5*(WW(0.,A,0.)-WW(-B,AB,B)))
      RETURN
      END
      FUNCTION Q(A,B)
      COMMON BETA, D, DD
      AP = 2.0 * BETA
      AB = A - B
      AN = AP * A
      AM = AP * B
      X1 = E(B,-AB)+AP*(A*EE(B,-AB)+YY(B,-AB,-B))
      X2 = EE(B,-AB)-AP*(A*E(B,-AB)+Y(B,-AB,-B))
      Q = 0.25*(COS (AN)*E(A,AB)+SIN (AN)*EE(A,AB) + COS (AM)*X1
     1   +SIN (AM)*X2+E(-B,0.)+AN*(EE(-B,0.)-EE(0.,-A))-E(0.,-A)
     2   -AP*YY(0.,-A,0.))/BETA
      RETURN
      END
/FTC
      FUNCTION QQ(A,B)
      COMMON BETA, D, DD
      AP = 2.0 * BETA
      AB = A - B
      AN = AP * A
      AM = AP * B
      X1 = E(B,-AB)+AP*(A*EE(B,-AB)+YY(B,-AB,-B))
      X2 = EE(B,-AB)-AP*(A*E(B,-AB)+Y(B,-AB,-B))
      QQ = 0.25*(COS (AN)*EE(A,AB)-SIN (AN)*E(A,AB)+COS (AM)*X2
     1   -SIN (AM)*X1-EE(0.,-A)+EE(-B,0.)+AN*(E(0.,-A)-E(-B,0.))
```

```
     2    + AP*Y(0.,-A,0.))/BETA
      RETURN
      END
/FTC
      FUNCTION P(A,B)
      AB = A - B
      P = A*E(AB,A) - Y(0.,A,0.) + Y(-B,AB,B)
      RETURN
      END
/FTC
      FUNCTION PP(A,B)
      AB = A - B
      PP = A*EE(AB,A) - YY(0.,A,0.) + YY(-B,AB,B)
      RETURN
      END
/FTC
      FUNCTION T(A,B)
      COMMON BETA, D, DD
      X1 = 2.0 * BETA * A
      X2 = 2.0 * BETA * B
      AB = A - B
      T = (SIN (X1)*E(AB,A) - COS (X1)*EE(AB,A) + EE(-B,0.) - EE(0.,-A)
     1  + COS (X2)*EE(B,-AB) - SIN (X2)*E(B,-AB))/(2.0*BETA)
      RETURN
      END
/FTC
      FUNCTION TT(A,B)
      COMMON BETA, D, DD
      X1 = 2.0 * BETA * A
      X2 = 2.0 * BETA * B
      AB = A - B
      TT = (SIN (X1)*EE(AB,A) + COS (X1)*E(AB,A) - E(-B,0.) + E(0.,-A)
     1  - COS (X2)*E(B,-AB) - SIN (X2)*EE(B,-AB))/(2.0*BETA)
      RETURN
      END
/FTC
      FUNCTION PPP(X)
      X2 = X*X
      AN=38.102495/X+X*(335.67732+X2*(265.18703+X2*(38.027264+X2)))
      DB = 157.10542+X2*(570.23628+X2*(322.62491+X2*(40.021433+X2)))
      PPP = AN/DB
      RETURN
      END
/FTC
      FUNCTION QQQ(X)
      X2 = X*X
      AN= 21.821899/X2+352.0185+X2*(302.75787+X2*(42.242855+X2))
      DB = 449.69033+X2*(1114.9789+X2*(432.48598+X2*(48.196927+X2)))
      QQQ = AN/DB
      RETURN
      END
/FTC
      FUNCTION SI(X)
      X2 = X*X
      XB = X
      DB = X
      AN = 3.0
      IF(X-1.0) 515, 515, 525
  515 DB = -X2*DB/(AN*AN*(AN-1.0))
      XB = XB + DB
```

```
      DB = AN*DB
      AN = AN + 2.0
      IF(ABS (DB) - 4.0E-09) 520, 520, 515
  520 SI = XB
      RETURN
  525 SI = 1.5707963 - PPP(X)*COS (X) - QQQ(X)*SIN (X)
      RETURN
      END
/FTC
      FUNCTION CI(X)
      X2 = X * X
      DB = -X2/2.0
      XB = DB/2.0
      AN = 3.0
      IF(X-1.) 500, 500, 510
  500 DB = -X2*DB/(AN*(AN+1.0)**2)
      XB = XB + DB
      DB = (AN + 1.0) * DB
      AN = AN + 2.0
      IF(ABS (DB) - 4.0E-09) 505, 505, 500
  505 CI = XB + 0.57721566 + ALOG(X)
      RETURN
  510 CI = PPP(X)*SIN (X) - QQQ(X)*COS (X)
      RETURN
      END
/DATA
```

BIBLIOGRAPHY

1. Rumsey, V. H., "Reaction Concept in Electromagnetic Theory," Physical Review, Vol. 94, June 1954.

2. King, R. and C. W. Harrison, Jr., "Mutual and Self-Impedance for Coupled Antennas," Journal of Applied Physics, Vol. 15, June 1944.

3. Tai, C. T., "Coupled Antennas," Proceedings of the I.R.E., April, 1948.

4. Tai, C. T., "A Variational Solution to the Problem Cylindrical Antennas," Tech. Report no. 12, Aircraft Radiation Systems Laboratory, Stanford Research Institute, Stanford, California, 1950

5. Levis, C. A. and C. T. Tai, "A Method of Analyzing Coupled Antennas of Unequal Sizes," IRE Transactions on Antennas and Propogation, April, 1956.

6. King, R. and C. W. Harrison, Jr., "The Distribution of Current along a Symmetrical Center-Driven Antenna," Proceedings of the IRE, 1943.

7. King, R. W. P., "The Theory of Linear Antennas," Harvard University Press, 1956.

8. Harrington, R. F. , "Time-Harmonic Electromagnetic Fields," McGraw-Hill Book Co. Inc., 1961.

9. Carson, J. R., "Reciprocal Theorems in Radio Communication," Proceedings of the IRE, Vol. 17, June 1929.

10. Kouyoumjian, R. G., "The Calculation of Echo Areas of Perfectly Conducting Objects by the Variational Method," Antenna Lab., Ohio State Univ. Res. Foundation, Columbus, Rept. No. 444-13, 1953.

11. Richmond, J. H., "A Reaction Theorem and Its Application to Antenna Impedance Calculations," IRE Transactions on Antennas and Propagation, November 1961.

12. Richmond, J. H., "On the Theory of Scattering by Dielectric and Metal Objects," Ohio State Univ. Res. Foundation, Columbus, Ohio, Antenna Lab. Rept. 786-3, April, 1958.

13. Collin, R. E. , "Field Theory of Guided Waves," McGraw-Hill Book Co. Inc., 1960

14. Fano, R. M., Lan Jen Chu, and R. B. Adler, "Electromagnetic Energy Transmission and Radiation," John Wiley & Sons Inc., 1960.

15. Schelkunoff, S. A., "Electromagnetic Waves," D. Van Nostrand Co., Inc., 1943.

16. Ramo, S., and J. R. Whinnery, "Fields and Waves in Modern Radio," John Wiley & Sons, Inc., 1953.

17. Jordan, E. C., "Electromagnetic Waves and Radiating Systems," Prentice-Hall, Inc., 1950.

18. Pipes, L. A., "Applied Mathematics for Engineers and Physicists," McGraw-Hill Book Co. Inc., 1958.

19. Hildebrand, F. B., "Methods of Applied Mathematics," Prentice-Hall, Inc., 1952.

20. Tai, C. T., Private communication with Dr. Tai dated August 29, 1966.