

Nonparametric Tests for Change in the Mean
with High Power for Changes Near the
Beginning or End of the Sample

by

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Abstract

Let n independent chronologically ordered observables either form a random sample with a finite positive variance, or be such that the first k^* observables have a common mean different from a common mean of the rest $n - k^*$ ones, $1 \leq k^* < n$, where the time k^* of the change in the mean is usually unknown. Through extensive Monte-Carlo simulations, we investigate finite-sample properties of nonparametric tests for such a change that are based on convergence in distribution of some trimmed suprema of an appropriately weighted tied-down partial sums process. By choosing suitable trimming sequences of a special form, we propose powerful tests for change in the mean with controlled type I errors. Moreover, these tests have a significantly higher power for changes that occur near the beginning or end of the observation period compared to other available tests in the literature that are based on untrimmed suprema and integrals of a weighted tied-down partial sums process. We also compare the performance of all the tests on real-world datasets.

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Dedication Page

To my family and friends, for all the good times and many laughs!

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Chapter 1

Introduction

Let $X_1, X_2, \dots, X_n$, $n > 1$ be non-degenerate, independent, chronologically ordered observables with finite means. We consider the null, no change in the mean, hypothesis

$$H_0 : X_1, \dots, X_n \text{ is a random sample on } X \text{ with } E(X) = \mu, \text{ Var}(X) = \sigma^2 < \infty \quad (1)$$

versus the at most one change (AMOC) in the mean alternative hypothesis

$$H_A : \text{there exists an unknown integer } k^*, 1 \leq k^* < n, \text{ such that} \quad (2)$$

$$E(X_1) = \dots = E(X_{k^*}) \neq E(X_{k^*+1}) = \dots = E(X_n).$$

To proceed with testing H_0 vs. H_A non-parametrically, one can compare the sample mean generated by the first $X_1, X_2, \dots, X_k$ with the sample mean of the remaining sample $X_{k+1}, \dots, X_n$, and see if there is a particular k for which the absolute value of this difference is large. Accordingly, let $S_k := \sum_{i=1}^k X_i$,

$1 \leq k \leq n$, then one can compare S_k/k with $S_n - S_k/(n - k)$ via the following standardized statistics:

$$V_n(k) := \left(\frac{k}{n} \left(1 - \frac{k}{n} \right) \right)^{-\frac{1}{2}} \left(\frac{S_k}{\sqrt{n}} - \frac{k}{n} \frac{S_n}{\sqrt{n}} \right), \quad 1 \leq k < n. \quad (3)$$

Further, one can reject H_0 in favor of H_A if $V_n := \max_{1 \leq k < n} |V_n(k)|/\hat{\sigma}_n$ is large,

where $\hat{\sigma}_n := \sqrt{\sum_{i=1}^n (X_i - S_n/n)^2/(n-1)}$. Then,

$$V_n = \frac{1}{\hat{\sigma}_n} \sup_{0 \leq t \leq 1} \frac{|Z_n(t)|}{\left(\frac{[(n+1)t]}{n} \left(1 - \frac{[(n+1)t]}{n} \right) \right)^{\frac{1}{2}}}, \quad (4)$$

where the tied-down partial sums process $\{Z_n(t), 0 \leq t \leq 1\}$ is given by

$$Z_n(t) = \begin{cases} \frac{S_{[(n+1)t]} - \frac{[(n+1)t]}{n} S_n}{\sqrt{n}}, & \text{for } 0 \leq t < 1, \\ 0, & \text{for } t = 1. \end{cases} \quad (5)$$

We note that V_n converges in distribution to ∞ under H_0 , as $n \rightarrow \infty$ (Csörgő and Horváth, 1997). Thus, in order to secure a non-degenerate limiting distribution of an eventual test statistic involving the maximum of (3) for testing H_0 versus H_A , there are various ways of standardizing the supremum of $|Z_n(t)|$. Some of these ways are discussed next.

Firstly, Martsynyuk (2016), introduces nonrandom positive real sequences $\{d_n\}_{n \geq 1}$ and $\{e_n\}_{n \geq 1}$, in order to trim the range over which the supremum of

$|Z_n(t)| / \left(\frac{[(n+1)t]}{n} \left(1 - \frac{[(n+1)t]}{n} \right) \right)^{1/2}$ is taken. Provided the nonrandom positive real sequences $\{d_n\}_{n \geq 1}$ and $\{e_n\}_{n \geq 1}$ satisfy the following conditions:

$$\begin{cases} d_n \left(1 + \frac{1}{e_n} \right) \leq 1, & \text{for all } n \geq 1, \\ \frac{d_n}{e_n} \leq \text{const.} < 1, & \text{for all } n \geq \text{some large } n_0, \\ d_n \rightarrow 0, \quad e_n \rightarrow 0, \quad nd_n \rightarrow \infty, & \text{as } n \rightarrow \infty, \end{cases} \quad (6)$$

then the following two convergences hold true (see Theorem 1.1 of Martsynyuk (2016)):

$$J_n^{[\frac{d_n}{e_n}, 1-d_n]} := \sqrt{d_n} \sup_{\frac{d_n}{e_n} \leq t \leq 1-d_n} \frac{|Z_n(t)|}{\hat{\sigma}_{[nt+1]} \frac{[(n+1)t]}{n} \left(1 - \frac{[(n+1)t]}{n} \right)} \xrightarrow{D} \sup_{0 \leq t \leq 1} |W(t)| \quad (7)$$

and

$$K_n^{[d_n, 1-\frac{d_n}{e_n}]} := \sqrt{d_n} \sup_{d_n \leq t \leq 1-\frac{d_n}{e_n}} \frac{|Z_n(t)|}{\hat{\sigma}_{[nt+1]} \frac{[(n+1)t]}{n} \left(1 - \frac{[(n+1)t]}{n} \right)} \xrightarrow{D} \sup_{0 \leq t \leq 1} |W(t)|, \quad (8)$$

where $\{W(t), 0 < t < 1\}$ denotes a standard Wiener process and

$$\hat{\sigma}_k^2 := \begin{cases} \frac{1}{n} \left(\sum_{i=1}^k \left(X_i - \frac{S_k}{k} \right)^2 + \sum_{i=k+1}^n \left(X_i - \frac{S_n - S_k}{n-k} \right)^2 \right), & \text{if } 1 \leq k < n, \\ \frac{1}{n} \sum_{i=1}^n \left(X_i - \frac{S_n}{n} \right)^2, & \text{if } k = n, \end{cases} \quad (9)$$

where the pooled variances $\{\hat{\sigma}_k^2, 1 \leq k \leq n\}$ constitute a weakly uniformly consistent sequence of estimators for σ^2 , i.e., as $n \rightarrow \infty$, $\max_{1 \leq k < n} |\hat{\sigma}_k^2 - \sigma^2| = o_P(1)$ (Lemma 2.1.1 of Csörgő and Horváth, 1997). Furthermore, as noted in Martsynyuk (2016), $\hat{\sigma}_{[nt+1]}$ in (7) and (8) can be replaced with the sample

standard deviation $\hat{\sigma}_n$. We note that the quantiles for $\sup_{0 \leq t \leq 1} |W(t)|$ are calculated in Table 1 of Csörgő and Horváth (1983). Using convergence in (7) or (8), one rejects H_A in favor of H_A when, respectively, $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ or $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ is large.

Another way of normalizing $|Z_n(t)|$ is with the use of $q(t)$ functions for reweighing $Z_n(t)$. Let Q be the class of positive functions $q(t) : (0, 1) \rightarrow (0, \infty)$ that are non-decreasing near 0 and non-increasing near 1, and put

$$I_0(q, c) = \int_0^1 \frac{e^{\frac{-cq^2(t)}{t(1-t)}}}{t(1-t)} dt, \quad c > 0. \quad (10)$$

Now, assuming that $X, X_1, X_2, \dots$ are i.i.d random variables with non-zero finite variance, and that $I_0(q, c) < \infty$ for $q(t)$ from Q and some $c > 0$, Theorem 2.1.1 of Csörgő and Horváth (1997) concludes: as $n \rightarrow \infty$,

$$T_n^q := \frac{1}{\hat{\sigma}_n} \sup_{0 < t < 1} \frac{|Z_n(t)|}{q(\frac{[(n+1)t]}{n})} = \frac{1}{\hat{\sigma}_n} \max_{1 \leq k < n} \frac{(\frac{k}{n}(1 - \frac{k}{n}))^{1/2}}{q(\frac{k}{n})} |V_n(k)| \xrightarrow{D} \sup_{0 < t < 1} \frac{|B(t)|}{q(t)}, \quad (11)$$

where $\{B(t), 0 < t < 1\}$ denotes a Brownian bridge.

Furthermore, following (ii) of Proposition 2.1.1 of Csörgő and Horváth (1997), assuming that $X, X_1, X_2, \dots$ are i.i.d random variables with a finite positive variance and that $q(t) \in Q$ satisfies

$$\int_0^1 \frac{t(1-t)}{q^2(t)} dt < \infty, \quad (12)$$

we have

$$L_n^q := \frac{1}{\hat{\sigma}_n^2} \int_0^1 \frac{Z_n^2(t)}{q\left(\frac{\lfloor (n+1)t \rfloor}{n}\right)} dt = \frac{1}{(n+1)\hat{\sigma}_n^2} \sum_{k=1}^{n-1} \frac{\frac{k}{n} \left(1 - \frac{k}{n}\right)}{q(k/n)} |V_n^2(k)| \xrightarrow{D} \int_0^1 \frac{B^2(t)}{q(t)} dt. \quad (13)$$

Convergence in (13) is an integral/average counterpart of (11).

Theorem 2.1.2 of Csörgő and Horváth (1997) presents a way of rescaling and centering V_n of (4) via a Darling-Erdős statistic, H_n , as follows. Assuming that $\{X, X_i\}_{i \geq 1}$ are i.i.d. with a positive variance and $E(X^2 \log \log(|X| + 1)) < \infty$, we have the following convergence: as $n \rightarrow \infty$, for $x \in \mathbf{R}$,

$$P\left(H_n := \frac{A(\log(n))}{\hat{\sigma}_n} \sup_{0 < t < 1} \frac{|Z_n(t)|}{\left(\frac{\lfloor (n+1)t \rfloor}{n} \left(1 - \frac{\lfloor (n+1)t \rfloor}{n}\right)\right)^{\frac{1}{2}}} - D(\log(n)) \leq x\right) \rightarrow e^{-2e^{-x}}, \quad (14)$$

where the numerical sequences $A(n) = \sqrt{2 \log(n)}$ and $D(n) = 2 \log(n) + 0.5 \log \log(n) - 0.5 \log(\pi)$.

A more general version of (7) and (8) is given in Theorem 1.2.7 in Horváth and Rice (2024) and reads: for $\kappa > \frac{1}{2}$,

$$C_n^{[t_1(n), 1-t_2(n)]} := \min(t_1(n), 1-t_2(n))^{\kappa - \frac{1}{2}} \frac{1}{\hat{\sigma}_n} \sup_{t_1 \leq t \leq t_2} \frac{|Z_n(t \frac{n+1}{n})|}{[t(1-t)]^\kappa} \rightarrow a(\kappa), \quad n \rightarrow \infty, \quad (15)$$

where for the trimming parameters,

$$\begin{aligned} t_1 &= t_1(n) < t_2 = t_2(n), \\ \min(t_1(n), 1 - t_2(n)) &\rightarrow 0, \\ n \min(t_1(n), 1 - t_2(n)) &\rightarrow \infty, \end{aligned} \tag{16}$$

as $n \rightarrow \infty$, and for the limit,

$$\begin{aligned} a(\kappa) &= \max(y_1^{\kappa-1/2} a_1(\kappa), y_2^{\kappa-1/2} a_2(\kappa)), \\ a_1(\kappa) &\stackrel{D}{=} a_2(\kappa) \stackrel{D}{=} \sup_{1 \leq t < \infty} \frac{|W(t)|}{t^k}, \\ y_1 &= \lim_{n \rightarrow \infty} \frac{\min(t_1(n), 1 - t_2(n))}{t_1(n)}, \quad \text{and} \quad y_2 = \lim_{n \rightarrow \infty} \frac{\min(t_1(n), 1 - t_2(n))}{1 - t_2(n)}. \end{aligned} \tag{17}$$

Making the substitutions $\kappa = 1$, $t_1(n) = d_n$, and $t_2(n) = 1 - \frac{d_n}{e_n}$, gives us a version of (8), while a version of (7) can be obtained by taking $k = 1$, $t_1(n) = \frac{d_n}{e_n}$, and $t_2(n) = 1 - d_n$. Furthermore, conditions $\min(t_1(n), 1 - t_2(n)) \rightarrow 0$ and $n \min(t_1(n), 1 - t_2(n)) \rightarrow \infty$, as $n \rightarrow \infty$, in (17) under the above substitutions are equivalent to $d_n \rightarrow 0$ and $nd_n \rightarrow \infty$, as $n \rightarrow \infty$, since $\min(d_n, 1 - \frac{d_n}{e_n}) = d_n$ and $\min(\frac{d_n}{e_n}, 1 - (1 - d_n)) = d_n$, provided d_n and e_n satisfy conditions (6).

Using (11), (13), (14), or (15), one can reject H_0 in favor of H_A if, respectively, T_n^q , L_n^q , H_n , or $C_n^{[t_1(n), 1-t_2(n)]}$ is large.

In Chapter 2, we will examine the finite-sample properties of (7)- and (8)-based tests, with the goal of finding powerful tests, regardless of where the change occurs in the sample, that have controlled type I errors, for a wide

variety of the underlying distributions of the data. In Chapter 3, we will conduct a similar search for an all-around balanced test, except this time with the (15)-based tests.

In Chapter 3, we will also compare the performance of our best (7)-, (8)-, and (15)-based tests with the (14)-based test that has extensively been studied by many, as well as with the all-around balanced tests proposed in Gallanosa and Martsynyuk (2026) from (11) and (13), namely with the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests with the weight functions

$$q_1(t) = \left(\log \frac{1}{t(1-t)} \right)^{-0.9} \quad \text{and} \quad q_2(t) = \left(\log \frac{1}{t(1-t)} \right)^{-4}, \quad (18)$$

which are nearly as powerful, regardless of the location of the change, as the overall most powerful sup-test in the literature. Moreover, compared to the latter tests, the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests have the type I errors that are closer to the asymptotic significance level across various distributions and are lower and converge faster for distributions that are more asymmetric, heavy-tailed, or both. Notably, we will demonstrate that our best $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ -, $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -, and $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests, with specially selected d_n and e_n in (11)/(13), and κ , $t_1(n)$, and $t_2(n)$ in (15), achieve the highest power for detecting the change near the beginning or end of the sample compared with the $T_n^{q_1}$ -, $L_n^{q_2}$ -, and H_n -based tests, and have similarly well-controlled type I errors as the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests.

The conclusions for the whole thesis are presented in Chapter 4, while Chapters 5–7 contain supplementary figures and tables for Chapters 2 and 3.

Contribution of Authors

Callum Lehingrat contributed to the development of the research, implementation of the numerical studies, formal analysis, and interpretation of the results. He also drafted and integrated the thesis manuscript and was responsible for maintaining coherence and clarity throughout the thesis. Dr. Yuliya V. Martynuk provided supervision, formulation of the research problems, conceptual and methodological guidance, overall direction throughout the research, and the refinement of the writing throughout the thesis.

Chapter 2

Powerful $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -Based Tests

2.1 Preliminaries

As it is not feasible to obtain the closed-form expressions for the finite-sample type I errors and powers for all the tests to be examined in this thesis. In order to estimate the type I errors and the powers for the tests based on the convergences of $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ in (7), $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ in (8), and H_n in (15), we used Monte-Carlo simulations with $m = 5000$ replications. We approximated the empirical type I errors for $n = 100, 300, \dots, 3700, 3900$, as well as for $n = 200$ and 1000, while for the empirical powers, we used $n = 100, 200, 500, 1000$, and 1500 using the procedure below:

- 1) For each replication $r = 1, 2, \dots, m$, generate $X_1^r, X_2^r, \dots, X_{k^*-1}^r$ from one of the probability density functions $f(x)$ as defined in Table 2.1. Then, introduce a shift/change $\Delta = v\sigma$, where $v = 0.5$ or 0.75 , and generate $X_{k^*}^r + \Delta, X_{k^*+1}^r + \Delta, X_{k^*+2}^r + \Delta, \dots, X_n^r + \Delta$, from the same probability density function. Note that σ is the theoretical standard deviation of the probability density function.
- 2) Compute the left-hand side statistic of (7), (8), or (15) for each $r = 1, 2, \dots, m$.
- 3) Compute the proportion of times out of the m replications when the left-hand side statistic of (7), (8), or (15) exceeds the $1 - \alpha$ critical value.

Type I errors were estimated in a similar fashion, except $X_1, \dots, X_n$ were generated all at once, and $\Delta = 0$.

For future reference, the probability density functions $f(x)$ examined in this thesis are listed in Table 2.1. These distributions were chosen such that the upcoming analyses include both symmetric and asymmetric distributions, as well as some heavy-tailed ones. Interestingly, the empirical performances of all the examined tests are similar on the Normal(0,1) and Exp(1), distributions, the t(3) and Frechet(2.5) distributions, and the Weibull(0.4,1) and Lognormal(0,1.5) distributions. As a result, the tables and figures relating to the Exp(1), Frechet(2.5), and Weibull(0.4,1) distributions will be delegated to

the appendices, and the main body of the thesis will include tables and figures for the Normal(0,1), t(3), and Lognormal(0,1.5) distributions.

Distribution	Probability Density Function
Normal(0,1)	$\frac{e^{-x^2/2}}{\sqrt{2\pi}}, x \in \mathbf{R}$
Exponential(1)	$e^{-x}, x > 0$
t(3)	$\frac{2(1+\frac{x^2}{3})^{-2}}{\pi\sqrt{3}}, x \geq 0$
Frechet(2.5)	$2.5x^{-3.5}e^{-x^{-2.5}}, x > 0$
Weibull(0.4,1)	$0.4x^{-0.6}e^{-x^{0.4}}, x > 0$
Lognormal(0,1.5)	$\frac{e^{-\ln^2(x)/4.5}}{1.5x\sqrt{2\pi}}, x > 0$

Table 2.1: Formulae of the 6 investigated distributions

2.2 Trends of $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests

Preliminary assessment of the finite-sample performance of the

$J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests

Preliminary investigations began with the hope of creating an all-around balanced test with controlled type I errors that converge quickly to the nominal level of $\alpha = 0.05$, as well as having high powers regardless of the value of k^* . As this investigation is similar in nature to the ones for the (11)- and (13)-based tests in Gallanosa and Martsynyuk (2026), we first examined their comparison

12 CHAPTER 2. POWERFUL $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - AND $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -BASED TESTS

study to get an idea of not only what to potentially expect with the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests, but also to observe how the $q(t)$ weight functions for their all-around balanced tests were selected. We found that while the best tests based on the convergences in (11) and (13) that were found in that paper were very powerful and had well-controlled type I errors, there were two potential areas for improvement. Firstly, the empirical type I errors on the skewed, asymmetric distributions exhibited a much slower convergence to the asymptotic level than those on the symmetric distributions. Secondly, the powers for a change on the tails, that is when $0.05 \leq \frac{k^*}{n} \leq 0.20$ and $0.80 \leq \frac{k^*}{n} \leq 0.95$, could potentially be improved upon. In general, those best tests were more sensitive to a change near the middle of the dataset, more specifically, when $0.25 \leq \frac{k^*}{n} \leq 0.75$.

Initially, for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests, we investigated d_n 's of the following families with $a \geq 0$ and $b \geq 0$, with the goal of finding constants a and b such that the conditions in (6) were satisfied:

$$\frac{n^a(\log n)^{-b}}{n+1}, \quad \frac{n^a(\log n)^b}{n+1}, \quad \frac{n^a}{n+1}, \quad \frac{(\log n)^a}{n+1}, \quad \frac{(\log(\log n))^a}{n+1}. \quad (19)$$

We also wanted to increase the rate of convergence of the type I errors, as well as hopefully boost the empirical powers for a change near the middle. Notably, all potential families must converge to 0 more slowly than $\frac{1}{n+1}$, as per (6).

After a long investigation (described below), the 5 candidates from each

family that yield the most promising finite-sample performance of the tests, all with $e_n = d_n^{1-0.6}$, were as follows:

$$d_{n_1} = \frac{n^{0.985}(\log n)^{-1.45}}{n+1}, d_{n_2} = \frac{n^{0.42}(\log n)^{0.2}}{n+1}, d_{n_3} = \frac{n^{0.5}}{n+1}, d_{n_4} = \frac{(\log n)^{1.6}}{n+1},$$

$$\text{and } d_{n_5} = \frac{(\log(\log(n)))^{5.5}}{n+1}. \quad (20)$$

In Remark 1, we compare the tests with these 5 candidates with each other and determine the best d_n out of these 5. However, the process of coming up with these 5 combinations of d_n and e_n was incredibly arduous. Whilst we investigated hundreds (potentially thousands) of combinations of d_n and e_n to arrive at our recommendation for this chapter, a summary of what each stage of the investigation looked like is described below.

Firstly, we investigated the five families of d_n as in (19) while keeping $e_n = d_n^{1-0.99}$. Taking close to the full supremum in (7) and (8) yielded empirical type I errors that exhibited extremely slow convergence to the nominal level. While decreasing the values of d_n did slightly reduce the empirical type I errors on the Normal and Exponential distributions, it did very little with respect to reducing the empirical type I errors on the Student, Frechet, Weibull, and Lognormal distributions. Despite these high type I errors, empirical powers remained low. It appeared that the family of d_n had very little impact on these overarching trends. For the interested reader, we showcase some plots to highlight what the initial struggle looked like in Appendix C1. Notably, not

only was the convergence of the type I errors slow for the asymmetric, heavy-tailed distributions, the type I errors were often far above the nominal level for the symmetric distributions as well. Furthermore, while the power for a shift at $(\frac{k^*}{n} = 0.05, 0.10, 0.15, 0.85, 0.90, 0.95)$ was sufficiently high, the power curves often exhibited an M-shape, with the empirical powers at $\frac{k^*}{n} = 0.20, 0.80$ often higher than those for $\frac{k^*}{n} = 0.50$. Increasing the power in the middle proved to be unfruitful, as this only further inflated the type I errors on every distribution. *What this first investigation taught us was that d_n appears to primarily influence the convergence of the empirical type I errors on the Normal and Exponential distributions, but has a much smaller impact on the asymmetric, heavy-tailed distributions.*

After exhausting the five families of d_n as in (19) with a few results, we began to take $e_n = d_n^{1-\alpha}$ for various $0 < \alpha < 1$ with the hope of decreasing the empirical type I errors across all distributions, but in particular on the Weibull and Lognormal distributions. As we were hoping to closely match the empirical type I errors of the best (11) and (13)-based tests in Gallanosa and Martsynyuk (2026), if not out-perform them with respect to the empirical type I errors on the asymmetric, heavy-tailed distributions. Taking a slightly asymmetric supremum over $[d_n, 1 - d_n/e_n]$ or $[d_n/e_n, 1 - d_n]$ through decreasing the value of α in $e_n = d_n^{1-\alpha}$ resulted in the empirical type I errors converging more quickly to the nominal level, however the power curves began to exhibit some asymmetries. Keeping in mind that eventually we would have to compare our

tests to those from Gallanosa and Martsynyuk (2026), we thus chose an e_n that resulted in the empirical type I errors being as close as possible to the best tests found in that paper, especially on the Weibull and Lognormal distributions. An in-depth investigation is done in the next section demonstrating the trends associated with differing e_n 's, however we found that $e_n = d_n^{1-0.6}$ resulted in the type I errors on the Weibull and Lognormal distributions were a close match to those of the best tests in Gallanosa and Martsynyuk (2026). The primary downside of using this e_n was that the resulting power curves became extremely asymmetric. However, this turned out to not be as big of a problem as we had initially thought, mainly due to how tests based on convergences (7) and (8) behave with respect to one another. These trends will be investigated in Section 2.3 of this chapter. *What we learned during this period of the investigation will be covered more in depth in Remark 2, where we show that e_n is a major mechanism for controlling the empirical type I errors of the (7) and (8)-based tests, especially on the Frechet, Weibull, and Lognormal distributions.*

Afterwards, we returned to investigating the five families of d_n as in (19) while taking a slightly asymmetric supremum in (8), with the primary goal of controlling the type I errors on the Normal and Exponential distributions. After closely matching the type I errors between our tests and the $q(t)$ tests, we were able to observe slight differences in the performances of the five families of d_n as in (19). We observed the three-way trade-off between the convergence of the type I errors, the power for a change near the tails of the dataset, and the

power for a change near the middle of the dataset. In particular, all families of d_n as in (19) were producing relatively high empirical powers for changes on the left tail, whilst the type I errors and the power for a change near the middle was converging quite slowly. This was noticeably different from the three-way trade-off existing with the tests based on convergences (11) and (13) described in Gallanosa and Martsynyuk (2026), whereby the empirical powers for a change near the middle of the dataset were quite high, but the empirical powers on the tails were quite low.

Remark 1. *For the interested reader, below is a quick comparison of the (8)-based tests with the best candidates listed in (20) from each of the five families of d_n in (19), where these candidates were chosen such that the type I errors were less than 8% for $n = 100$ on the $\text{Normal}(0,1)$ distribution. This is to demonstrate our logic behind our recommendations for d_n and e_n . The d_{n_1} through d_{n_5} of (20) were selected through investigating the finite-sample performance of the tests for many different choices of parameters a and b in the d_n families in (19).*

From the upcoming plots, we observe in Figure 2.1 that the type I errors of all 5 d_n 's are converging in a relatively similar manner, especially on the heavy-tailed, asymmetric distributions. In Figure 2.2, we observe for the sample sizes of $n = 100$ and 200 , that the empirical powers are mainly a reflection of the empirical type I errors. Whilst d_{n_4} attains the highest empirical powers, it also has the highest empirical type I errors. However, in Figures 2.2-2.3, we observe

for $n = 500, 1000, 1500$ that d_{n_1} is giving the highest empirical powers out of all the d_{n_i} across every distribution. We also note that in general, the power for a change on the left tail of a dataset is approximately within 20-25% to the power for a change near the middle of a dataset for $n = 500$, and for $n \geq 1000$ the powers on the left tail are within 10-15% of those for a change near the middle. It is by looking at these higher sample sizes where we can rank the candidates d_{n_1} through d_{n_5} . Namely, (at least for $n = 100, 200, 500, 1000, 1500$), $d_{n_4} < d_{n_2} < d_{n_3} < d_{n_5} < d_{n_1}$.

For the remainder of the paper, we denote $d_{n_{ch1}}$ interchangeably with d_{n_1} .

Thus for Chapter 2, we will recommend d_{n_1} along with $e_n = d_n^{1-0.6}$, and will take this recommendation for d_n and e_n into our comparisons in Section 2.4.

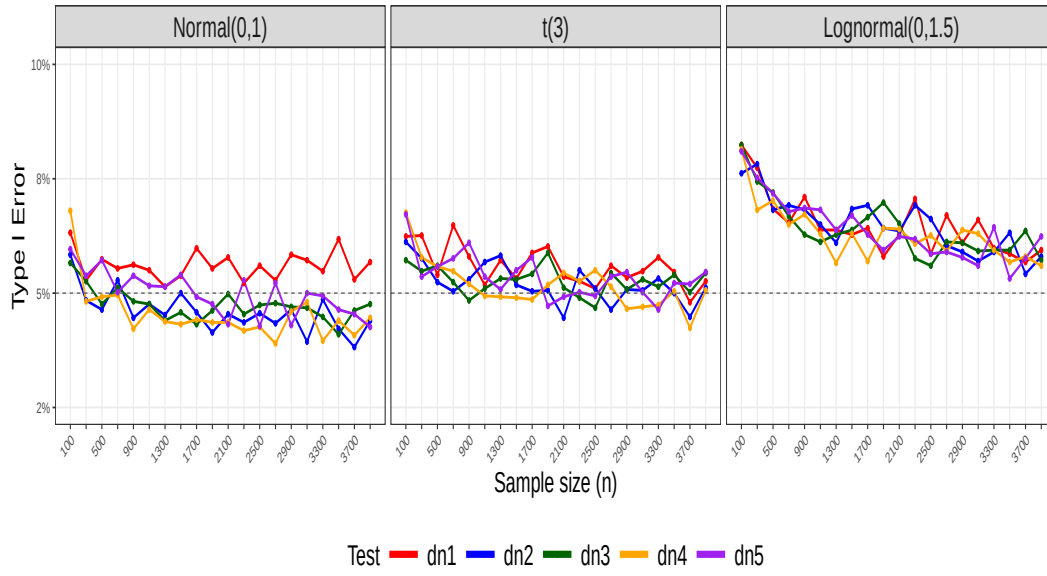


Figure 2.1: Empirical type I errors of the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}$ of (20), $i = \overline{1, 5}$, and $e_n = d_n^{1-0.6}$ on Normal(0,1), t(3), and Lognormal(0,1.5).

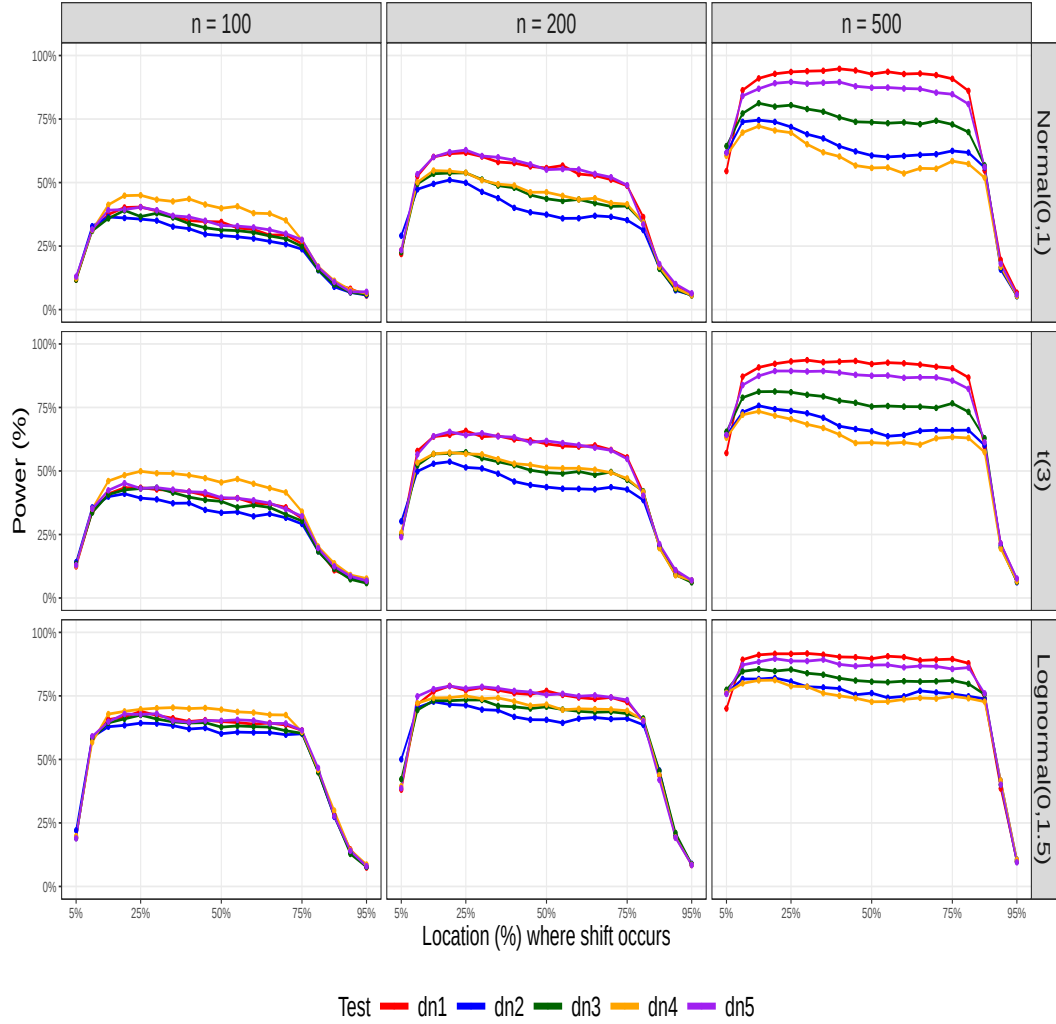


Figure 2.2: Empirical powers for $n = 100, 200, 500$ of the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}$ of (20), $i = \overline{1, 5}$, and $e_n = d_n^{1-0.6}$ on Normal(0,1), $t(3)$, and Lognormal(0,1.5).

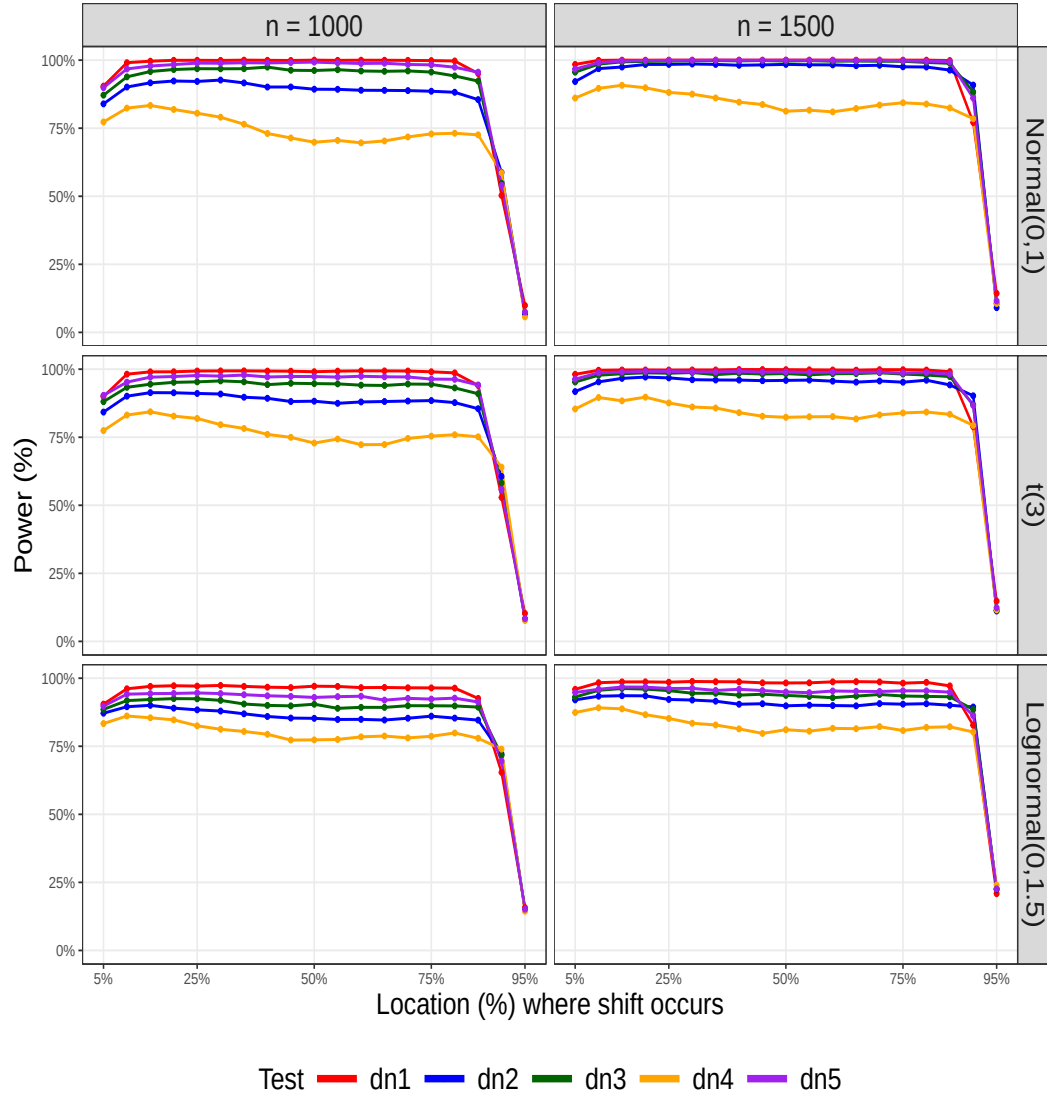


Figure 2.3: Empirical powers for $n = 1000, 1500$ of the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}$ of (20), $i = \overline{1, 5}$, and $e_n = d_n^{1-0.6}$ on Normal(0,1), $t(3)$, and Lognormal(0,1.5).

Remark 2. *With Figures 2.4-2.6 below, we demonstrate on the example of the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test that the parameter e_n is a major mechanism for controlling empirical type I errors.*

Recall that the test statistic $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ involves taking the supremum over the interval $[d_n, 1 - \frac{d_n}{e_n}]$. The magnitude of e_n solely serves the purpose of cutting off more of the right hand side of the supremum in (8), thus increasing the asymmetry of the test.

The effect of the choice of e_n can be observed in Figures 2.4-2.6. In particular, we observe a significant trade-off between the convergence of the type I errors and the power on the right-hand tail.

From the upcoming Figures 2.4-2.6, we observe a clear trade-off between the convergence of the empirical type I errors and the convergence of the empirical powers on the right tail. From Figure 2.4, we can observe that there is not much benefit in decreasing α in $e_n = d_n^{1-\alpha}$ to any value less than 0.6, since the type I error convergence is not further sped up on the asymmetric distributions. In addition, the convergence of the power on the right tail is significantly slowed through this decrease in α . Interestingly, taking the most symmetric supremum possible, which is done by setting $e_n = d_n^{1-0.9999\dots}$ leads to an incredibly slow type I convergence on both the symmetric and asymmetric distributions.

From observing Figures 2.4-2.6, we observe that a value of $0.8 \leq \alpha \leq 0.9$ lends to a fairly balanced test in terms of type I errors and the convergence on

the right-hand tail of the power curve. However, for the sake of reducing the empirical type I errors as much as possible on the asymmetric distributions, we took $\alpha = 0.6$. With the reasoning that $\alpha = 0.5$ does not seem to provide much of an improvement in the empirical type I errors, but further increases the asymmetry of the test.

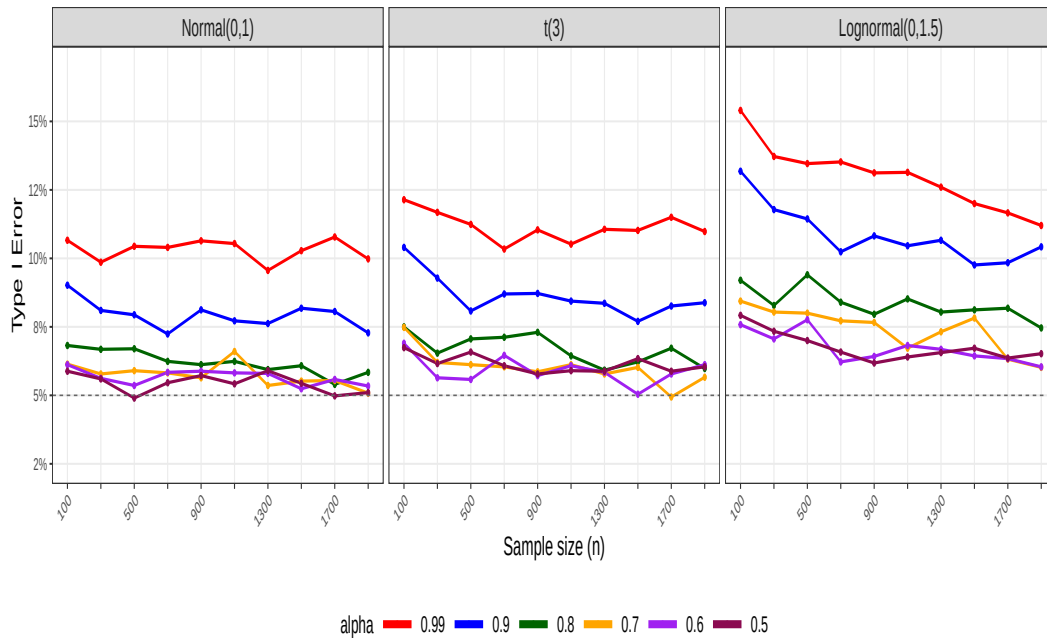


Figure 2.4: Empirical type I errors for the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-\alpha}$, $\alpha = 0.99, 0.9, 0.8, 0.7, 0.6, 0.5$ on Normal(0,1), t(3), and Lognormal(0,1.5).

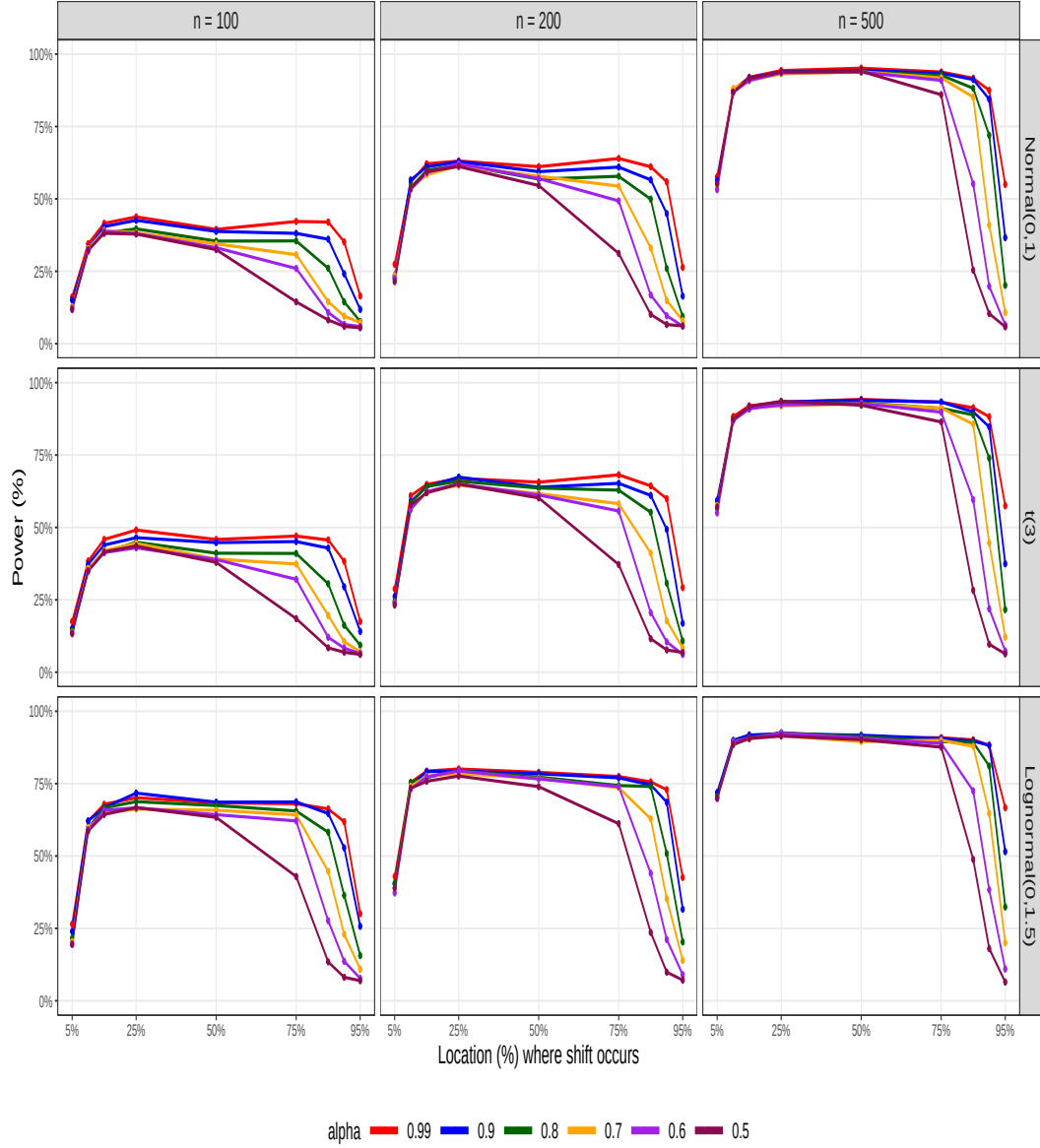


Figure 2.5: Empirical powers for $n = 100, 200, 500$ for the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-\alpha}$, $\alpha = 0.99, 0.9, 0.8, 0.7, 0.6, 0.5$ for Normal(0,1), t(3), and Lognormal(0,1.5).

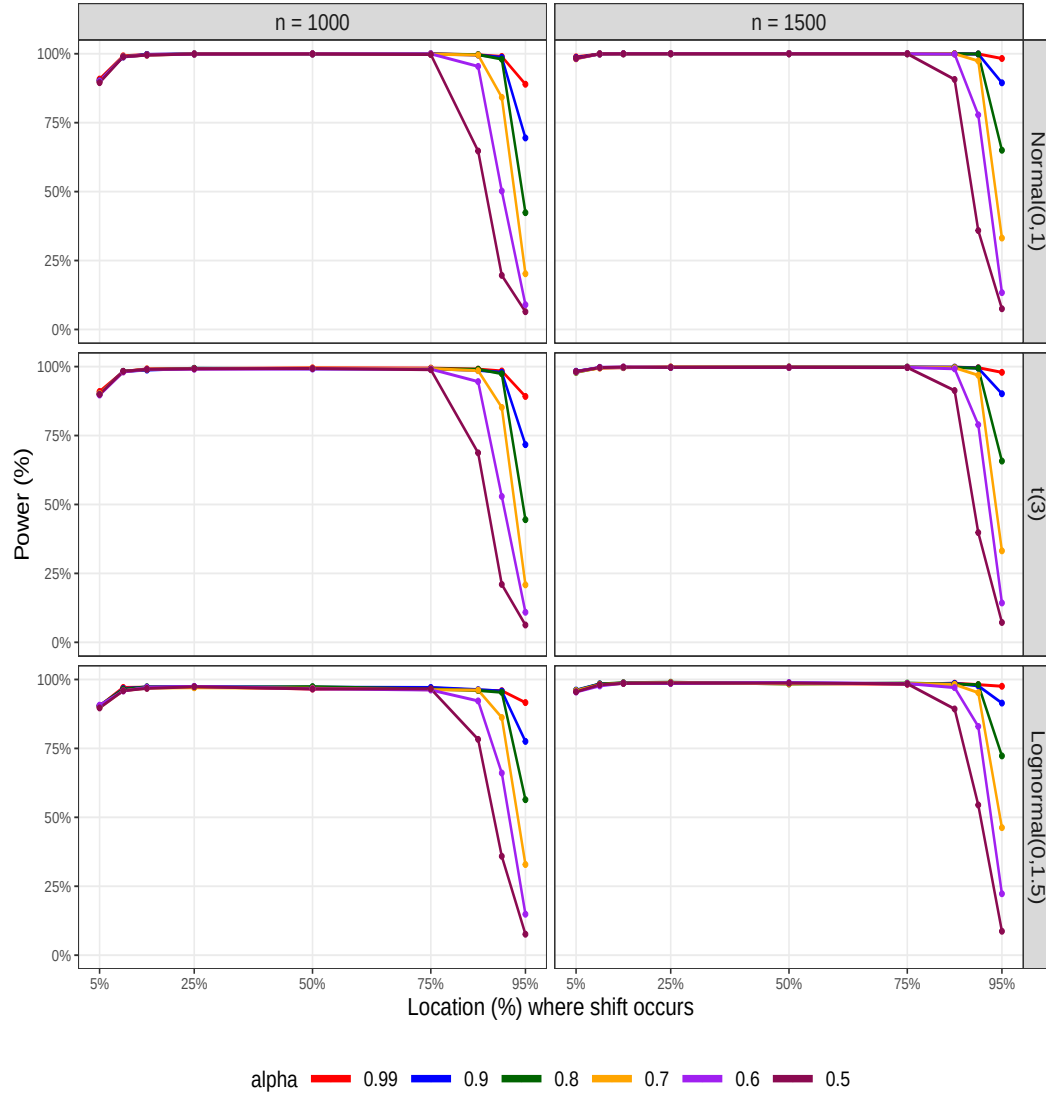


Figure 2.6: Empirical powers for $n = 1000, 1500$ for the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-\alpha}$, $\alpha = 0.99, 0.9, 0.8, 0.7, 0.6, 0.5$ for Normal(0,1), t(3), and Lognormal(0,1.5).

2.3 $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - vs. $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests

The numerical analyses done so far was mainly focused on the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test. In this section, we compare the latter test to the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ -based test from the convergence in (7).

From Figures 2.7, we observe that the empirical type I errors of both of these tests are extremely similar. However, the main difference is that the empirical powers, when viewed as functions of $\frac{k^*}{n}$, exhibit opposite asymmetries (see Figures 2.8 and 2.9). Therefore, if one, through some prior knowledge of the dataset, suspects there is a change on the left-hand tail of the sample, then using (8) over (7) will provide better results. Conversely, if one suspects a change is more likely to occur on the right hand side of the dataset, then using (7) will provide superior results over using (8).

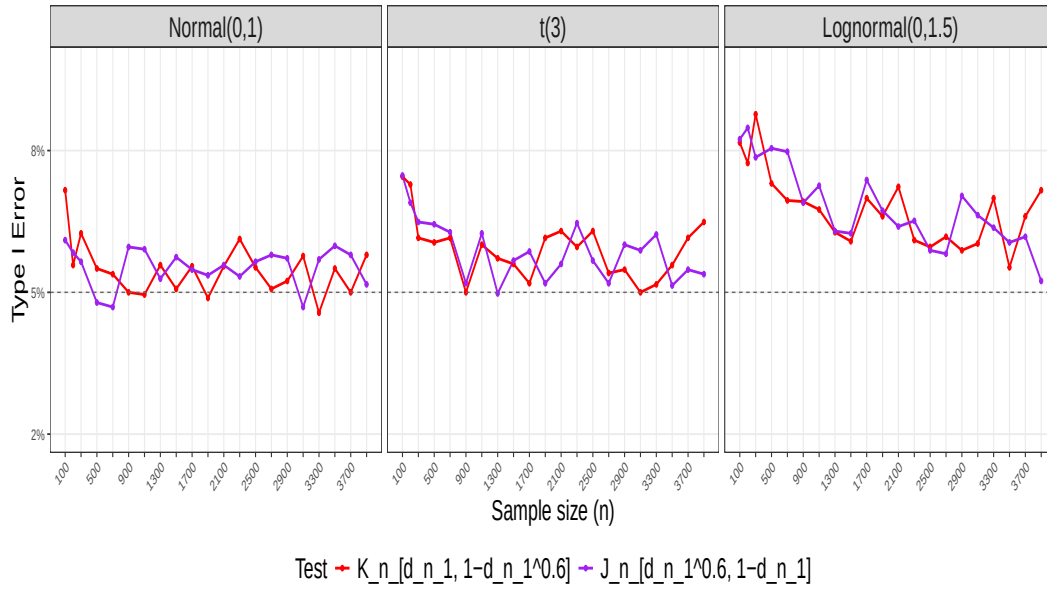


Figure 2.7: Empirical type I errors for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - vs. $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests on Normal(0,1), t(3), and Lognormal(0,1.5), using $d_n = d_{n_1}$ of (20) and $e_n = d_n^{1-0.6}$

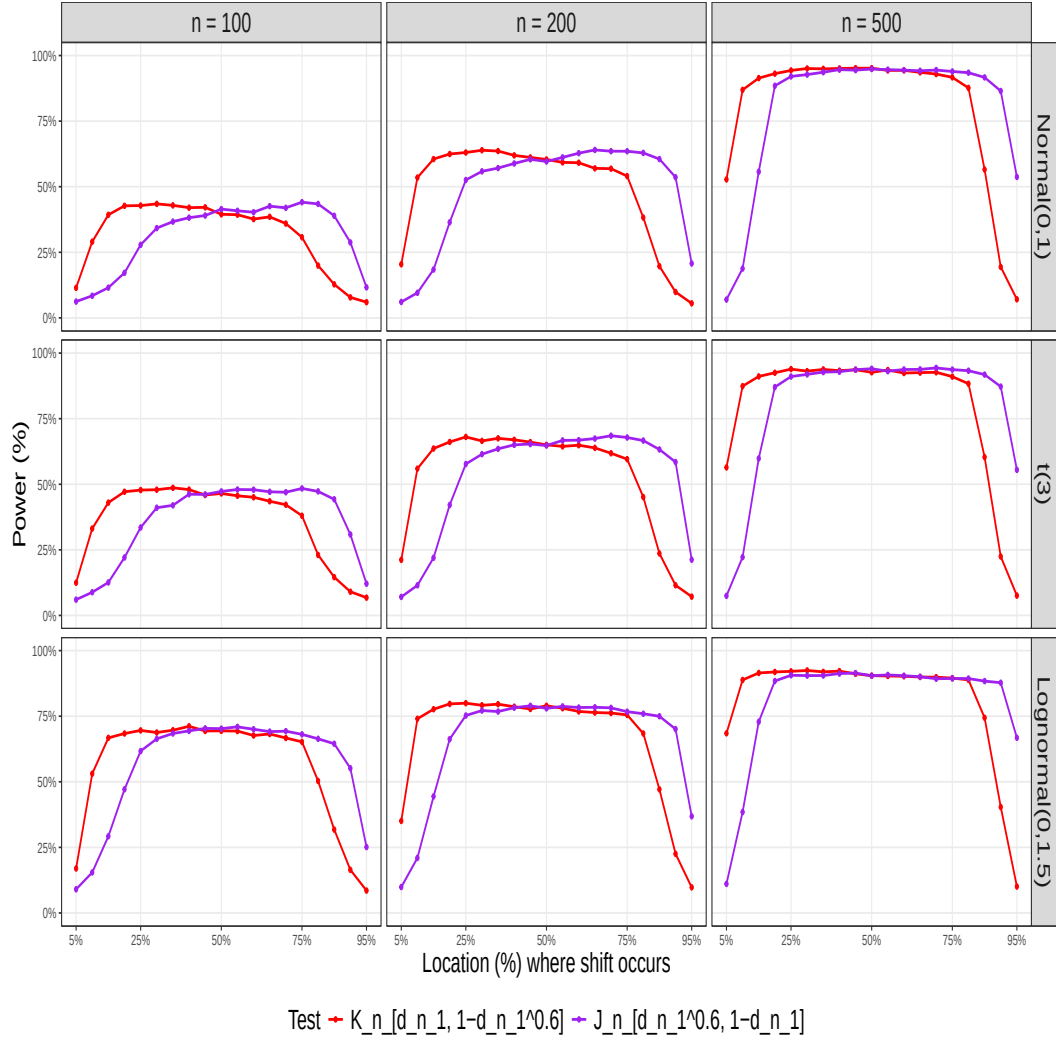


Figure 2.8: Empirical powers for $n = 100, 200, 500$ for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - vs. $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests on Normal(0,1), t(3), and Lognormal(0,1.5), using $d_n = d_{n1}$ of (20) and $e_n = d_n^{1-0.6}$

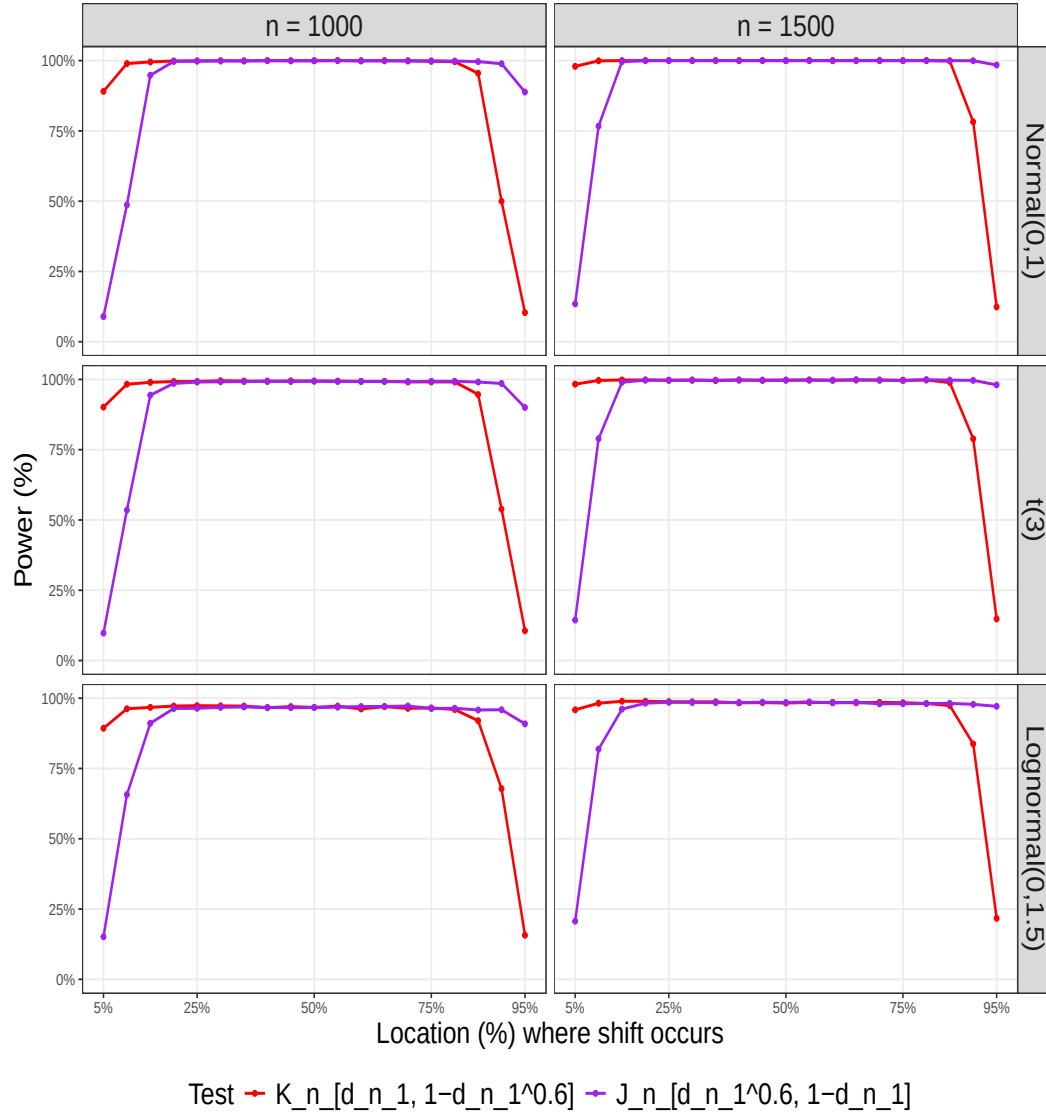


Figure 2.9: Empirical powers for $n = 1000, 1500$ for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - vs. $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ - based tests on Normal(0,1), $t(3)$, and Lognormal(0,1.5), using $d_n = d_{n_1}$ of (20) and $e_n = d_n^{1-0.6}$

Remark 3. *As mentioned in Chapter 1, we have the choice to use the sample standard deviation $\hat{\sigma}_n$ in place of the pooled standard deviation $\hat{\sigma}_{[nt+1]}$ in (7) and (8).*

In general, using $\hat{\sigma}_{[nt+1]}$ over $\hat{\sigma}_n$ for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests is to be recommended (see Figure 2.10). The empirical type I errors are extremely similar across all distributions. However, the empirical powers are boosted by roughly 10-15% for $n = 100$, and roughly 5-10% for $n = 200$ by the use of $\hat{\sigma}_{[nt+1]}$.

We observe in Figure 2.11 that for $n = 100$, the use of $\hat{\sigma}_{[nt+1]}$ improves the power for a change from approximately $0.2 \leq \frac{k^}{n} \leq 0.8$ by around 5-10%. For $n = 200, 500$ the empirical powers are either approximately equal or higher from the use of $\hat{\sigma}_{[nt+1]}$ with the (8)-based test. In Figure 2.12, we observe for $n = 1000, 1500$ that using $\hat{\sigma}_{[nt+1]}$ or $\hat{\sigma}_n$ for the (8)-based test results in almost completely identical empirical powers.*

For Figures 2.10-2.12 below, we used the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test, however using the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ -based test would have also shown these same trends.

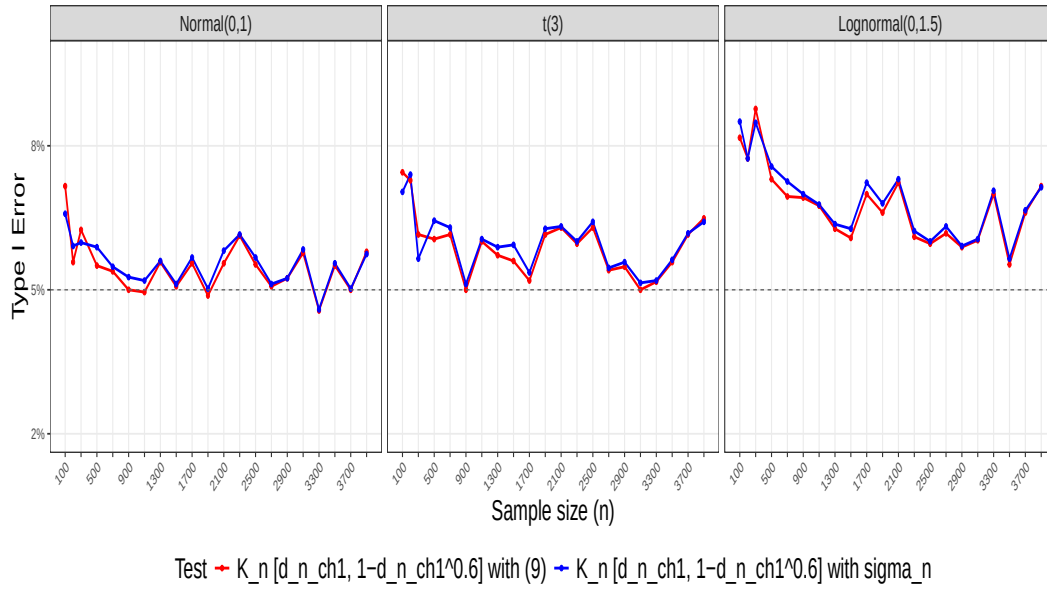
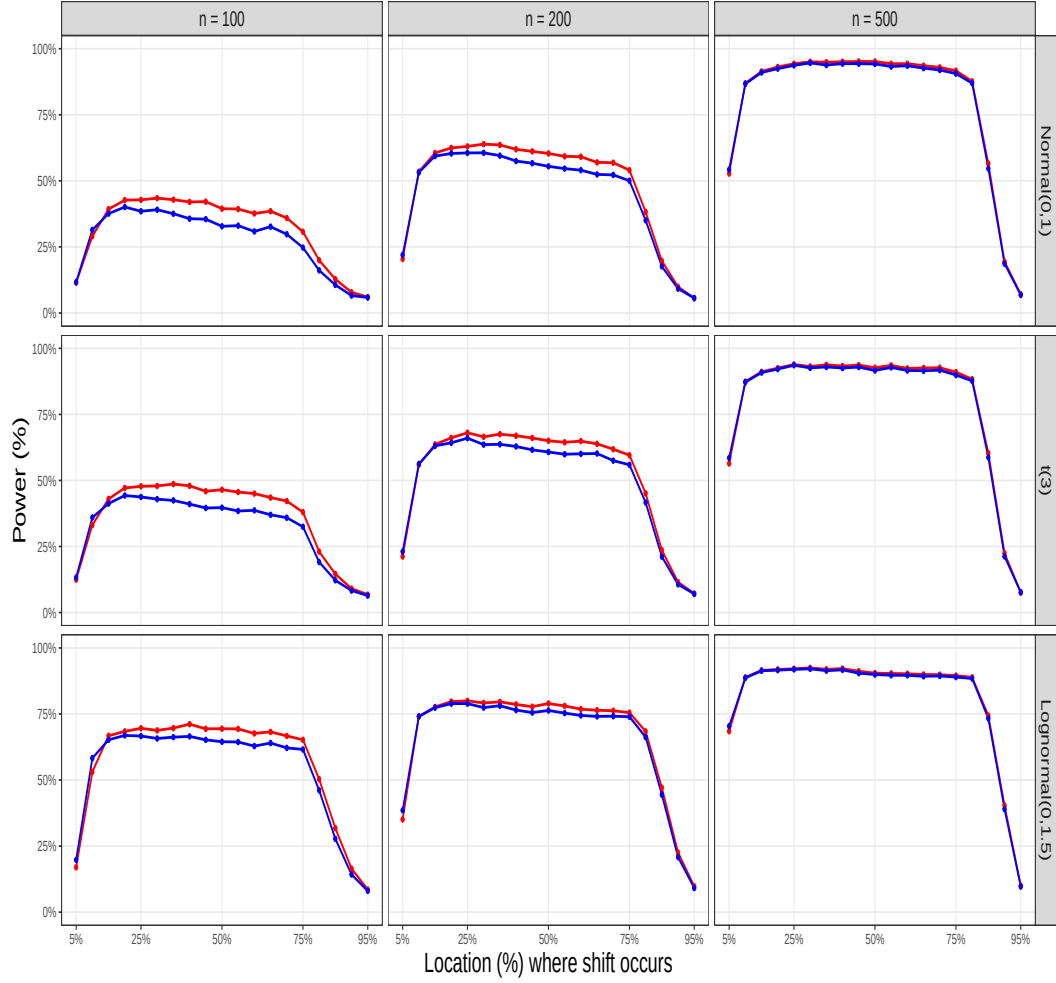
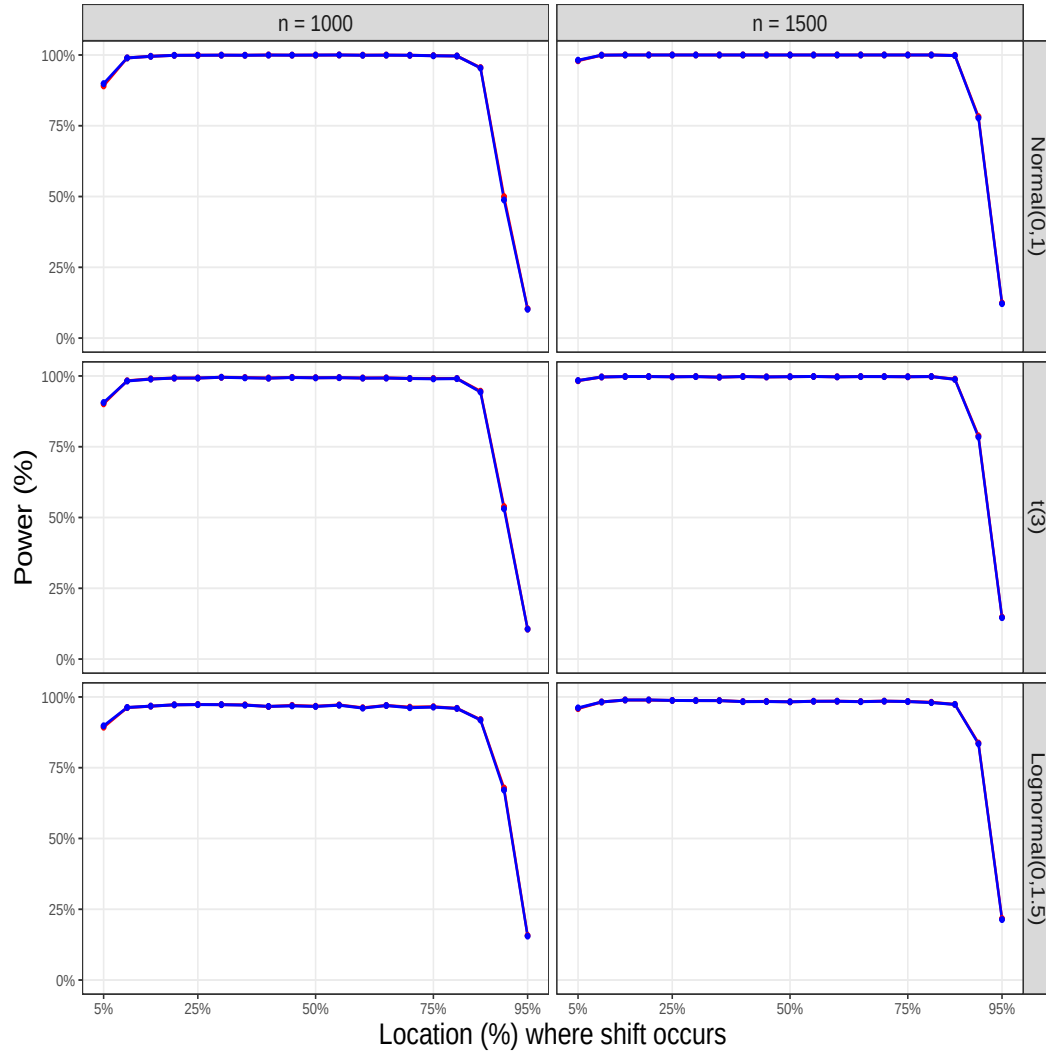


Figure 2.10: Empirical type I errors using $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9) for the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-0.6}$ on Normal(0,1), t(3), and Lognormal(0,1.5).



Test $\rightarrow K_n[d_n, 1-d_n]$ with (9) $\rightarrow K_n[d_n, 1-d_n]$ with σ_n

Figure 2.11: Empirical powers for $n = 100, 200, 500$ using $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9) for the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-0.6}$ on Normal(0,1), t(3), and Lognormal(0,1.5).



Test —•— $K_n[d_n, 1-d_n/e_n]$ with σ_n —•— $K_n[d_n, 1-d_n/e_n]$ with $\sigma_{[nt+1]}$

Figure 2.12: Empirical powers for $n = 1000, 1500$ using $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9) for the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-0.6}$ on Normal(0,1), t(3), and Lognormal(0,1.5).

2.4 Comparison with the T_n^{q1} -, L_n^{q2} -, and H_n -based tests

We now compare the empirical performance of our best $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests, with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_n^{1-0.6}$, with the best all-around balanced tests based on the convergences in (11) and (13), namely with the T_n^{q1} - and L_n^{q2} -based from Gallanosa and Martsynyuk (2026), as well as with the H_n -based test based on (14).

Note that the empirical power comparisons in the main body of the thesis are for a shift of 0.5σ . We included empirical power comparisons for a shift of 0.75σ of all tests, including the upcoming ones from Chapter 3, in Section 6.6. During some preliminary investigations, we found that the sensitivity of all tests was similarly low for a shift of $\Delta < 0.5\sigma$, and similarly high for a shift of $\Delta > 0.75\sigma$.

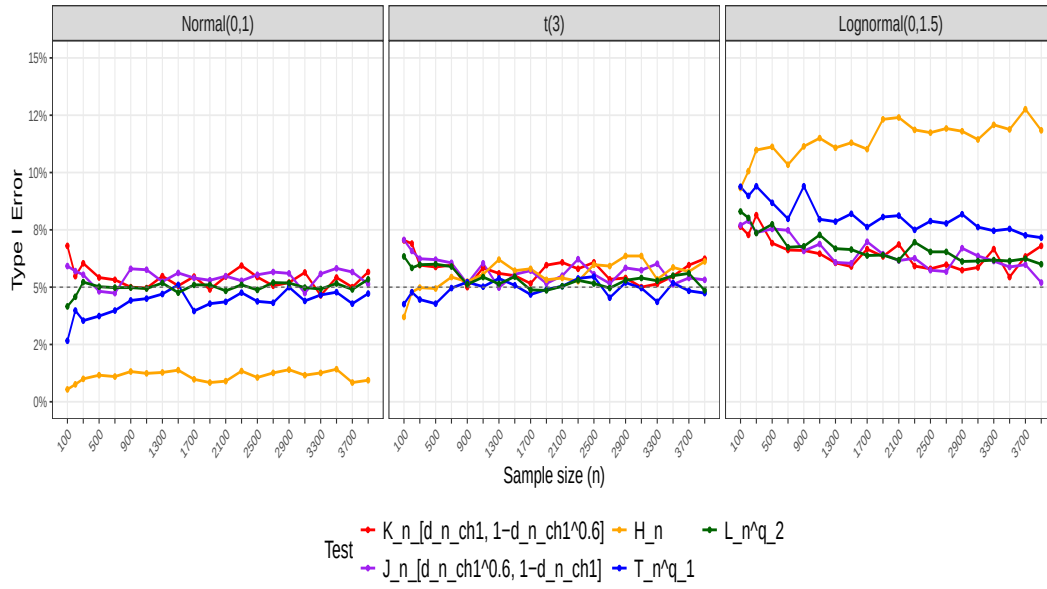


Figure 2.13: Empirical type I errors for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_1} = d_{n_{ch1}}$ of (20) and $e_n = d_n^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$ -, and H_n -based tests on Normal(0,1), t(3), and Lognormal(0,1.5).

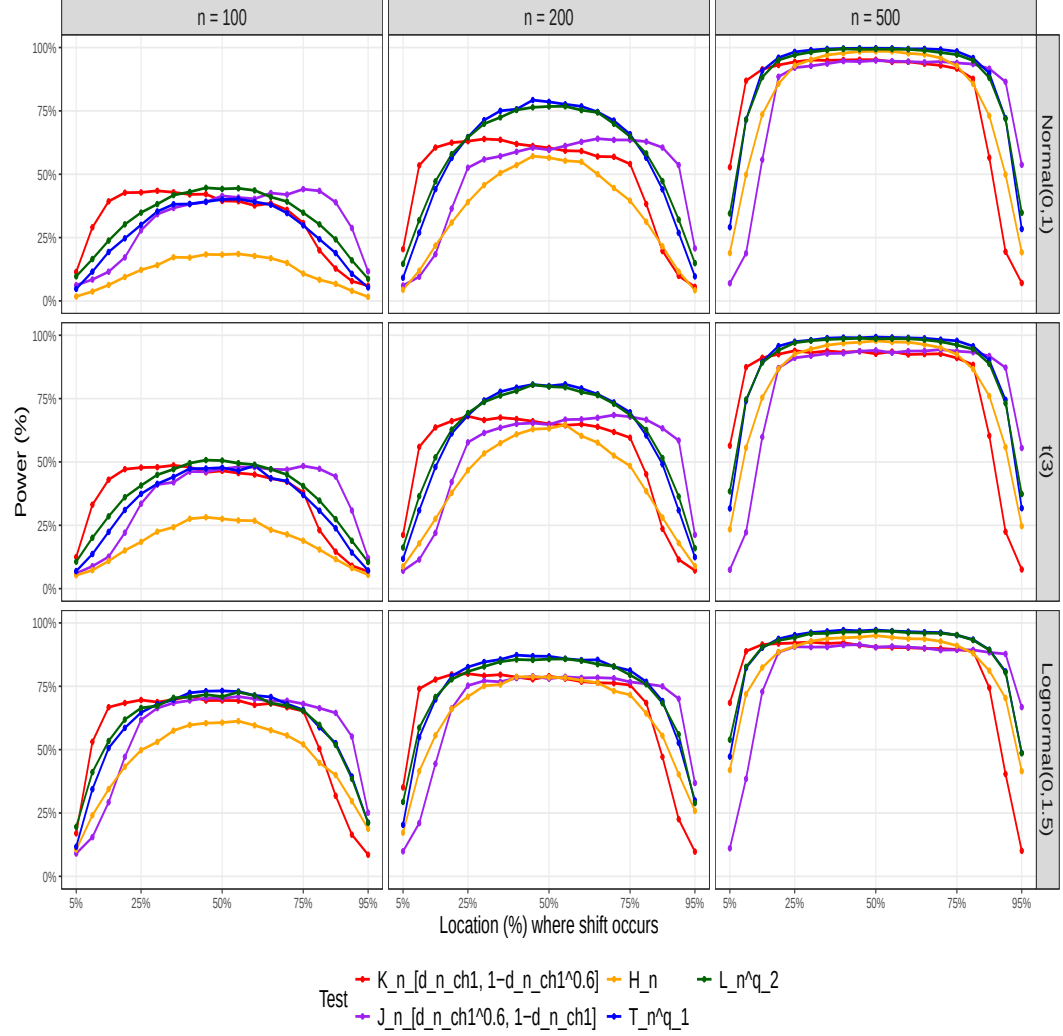


Figure 2.14: Empirical powers for a shift of 0.5σ and $n = 100, 200, 500$ for the $J_n^{[d_n, 1-d_n]}$ - and $K_n^{[d_n, 1-d_n]}$ -based tests with $d_n = d_{n_1} = d_{n_{ch1}}$ of (20) and $e_n = d_n^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$ -, and H_n -based tests on Normal(0,1), t(3), and Lognormal(0,1.5).

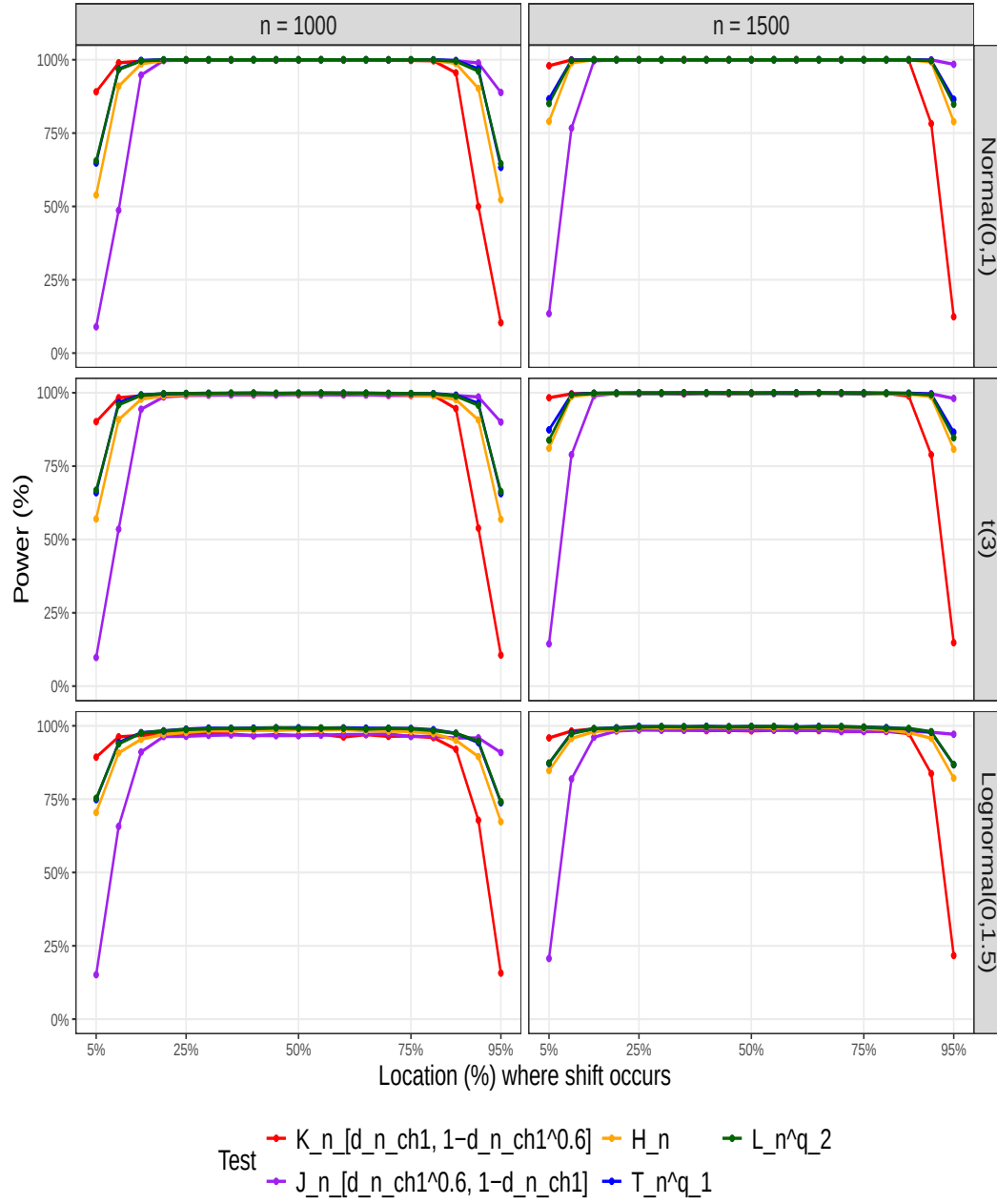


Figure 2.15: Empirical powers for a shift of 0.5σ and $n = 1000, 1500$ for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_1} = d_{n_{ch1}}$ of (20) and $e_n = d_n^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$ -, and H_n -based tests on Normal(0,1), t(3), and Lognormal(0,1.5).

Figure 2.13 showcases that the empirical type I errors of the our best all-around $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests, with our $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_n^{1-0.6}$, exhibit a similar convergence to the nominal level when compared with the empirical type I errors of the $L_n^{q_2}$ - or (13)-based test, notably on the asymmetric, heavy-tailed Lognormal(0,1.5) distribution. For $n = 100, 200$ the type I errors are around 0.5-1% closer to the nominal level for the (13)-based test than for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests.

Comparing the empirical type I error performances of the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with those of the $T_n^{q_1}$ - or (11)-based test, we observe that the performance on the heavy-tailed, asymmetric Lognormal(0,1.5) distributions is superior for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests by around 1-2%. The empirical type I errors are similar on the symmetric and moderately skewed distributions, with the main difference being that the type I errors of the (7)- and (8)-based tests are converging from above the nominal level, whereas those of the (11)-based test are converging from below.

Finally, the (14)- or the Darling-Erdős-statistic-based test exhibits increasing empirical type I errors on the heavy-tailed, asymmetric Lognormal(0,1.5) distribution, and empirical type I errors of around 1% on the Normal(0,1) distribution. On the $t(3)$ and the Exp(1) distributions, the empirical type I errors converge slightly faster to the nominal level and at least for $n =$

100, 200, 500 are around 0.5-1% closer to the nominal level than those of the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests.

When examining the empirical powers in Figure 2.14, we note that the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests exhibit higher empirical powers than the $T_n^{q_1}$ and $L_n^{q_2}$ -based tests for $0 \leq \frac{k^*}{n} \leq 0.3$ for $n = 100$, for $0 \leq \frac{k^*}{n} \leq 0.2$ for $n = 200$ and $0 \leq \frac{k^*}{n} \leq 0.1$ for $n = 500$. We note that the empirical powers of the H_n -based test are lower than those of the $T_n^{q_1}$ and $L_n^{q_2}$ -based tests everywhere. Thus, also due to the higher empirical type I errors of the H_n -based test, it is logical from now on to only compare the performance of the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests to those of the $T_n^{q_1}$ and $L_n^{q_2}$ -based tests.

We observe that for $n = 100, 200$, the empirical powers of the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests are quite behind those of the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests on the symmetric, light-tailed distributions by as far as 20% for a change at $\frac{k^*}{n} = 0.5$. The empirical powers are closer on these distributions for $n = 100$, however this is accompanied by inflated type I errors for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests. On the other hand, the type I errors of the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests are much closer to the nominal level of 0.05. For $n = 500$, the power for a change at $\frac{k^*}{n} = 0.5$ is much closer among the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ -, $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -, $T_n^{q_1}$ -, and $L_n^{q_2}$ -based tests. The powers of the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based

tests are only behind the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests by around 5-10% in the worst cases. In exchange, we note that the empirical powers at $\frac{k^*}{n} = 0.05, 0.1$ are significantly higher for the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test than for the $T_n^{q_1}$ - and $L_n^{q_2}$ -based. At $\frac{k^*}{n} = 0.05$ these powers are approximately 25-30% higher, and at $\frac{k^*}{n} = 0.10$ they are anywhere between 5-20% higher. Replacing the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with the $J_n^{[\frac{d_n}{e_n}, 1 - d_n]}$ -based test yields the exact same differences except now for $\frac{k^*}{n} = 0.95, 0.90$ instead of $\frac{k^*}{n} = 0.05, 0.10$.

Figure 2.15 showcases that the empirical powers have converged to 1 for all the tests, however at $\frac{k^*}{n} = 0.05$, the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test is anywhere from approximately 15-20% higher than the $T_n^{q_1}$ - and $L_n^{q_2}$ -based for $n = 1000$, and roughly 10-15% higher than the $T_n^{q_1}$ - and $L_n^{q_2}$ -based for $n = 1500$.

While the asymmetry of the power curves of the $J_n^{[\frac{d_n}{e_n}, 1 - d_n]}$ - and $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based tests may present an issue, using the $J_n^{[\frac{d_n}{e_n}, 1 - d_n]}$ -based test instead of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test will invert these asymmetries. Whether to use the $J_n^{[\frac{d_n}{e_n}, 1 - d_n]}$ - or $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test requires some prior knowledge on where one suspects a change is more likely to occur. If one expects the change is occurring towards the end of the dataset, then using the $J_n^{[\frac{d_n}{e_n}, 1 - d_n]}$ -based test will yield better results. On the other hand, if one suspects a change is likely to occur near the beginning of the dataset, then the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test in place is to

be recommended. One should keep in mind that the power asymmetries begin to take effect at roughly 80% for $n = 100, 200$, roughly 85% for $n = 500, 1000$, and roughly 90% for $n = 1500$.

2.5 Conclusions of Chapter 2

Tests based on convergences (7) and (8) provide an interesting alternative to the current AMOC in the mean tests in the literature, with the potential of achieving extremely high powers for the change on the tails of the sample at the expense of the symmetry of the power curves when viewed as functions of the location of the change (k^*/n). *In order to recommend these tests, one should have some prior general knowledge about which side of the dataset the change may occur.* If one has such knowledge, then (7) and (8)-based tests with $d_n = d_{n_1} = d_{n_{ch1}}$ of (20) and $e_n = d_n^{1-0.6}$ are to be recommended if $n \geq 500$, or for any n if the data is asymmetric and heavy-tailed. Otherwise, the $L_n^{q_2}$ -based test from the convergence (13), with the weight function $q_2(t)$ of (18), is a more all-around balanced test and is to be recommended over the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests, especially if one has no prior knowledge about the dataset in practice.

Chapter 3

Powerful $C_n^{[t_1(n), 1-t_2(n)]}$ -Based Tests

3.1 Preliminaries

While in the previous chapter we found the powerful $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests, the performance of these tests for the shifts at $0.25 \leq \frac{k^*}{n} \leq 0.75$ for $n = 100, 200$ was not fully satisfactory to us. However, to the best of our knowledge, we had exhausted all the tools available to us to improve the empirical performance of these two tests. As outlined in the Introduction, we were able to relate the convergence in (15) to the convergences in (7) and (8). The primary advantage this provided was the ability to alter a new parameter κ . We hoped that there would be some combinations of κ , d_n , and e_n that would yield the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with a superior finite-sample performance

compared to the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests from Chapter 2. With this in mind, we therefore investigate the finite-sample performance of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests in this chapter.

While the limiting distribution of the (7)- and (8)-based tests was previously tabulated, the limiting distribution of (15) is more complex, depends on κ , d_n , and e_n , and had to be tabulated by us through the following steps.

- 1) For each replication $r = 1, 2, \dots, m$, generate i.i.d. r.v.s $X_1^r, X_2^r, \dots, X_n^r$ from the Normal(0,1) distribution, with $m = n = 30,000$.
- 2) For each r , compute the left-hand side statistic of (15) for a combination of d_n , e_n , and κ we are interested in.
- 3) Order the m test statistics from lowest to highest, then estimate the $(1 - \alpha)$ quantile of the limiting distribution of (15) with the $(1 - \alpha)$ quantile of the m ordered test statistics.
- 4) Increase n (thus increasing m) until the estimated quantiles do not appear to be changing. These will be the corresponding $(1 - \alpha)$ quantile estimates.

Values for the estimated 0.95 quantiles for the $C_n^{[t_1(n), 1-t_2(n)]}$ -based test with different values of κ , and with $d_n = d_{n_{ch1}} = d_{n_1}$ of (20) and $e_n = d_n^{1-0.6}$, which we will use initially, are reported in Table 3.1. Again, for each new combination of d_n , e_n and κ , we need to re-tabulate the quantiles of the limiting distribution of the convergence in (15).

Table 3.1: The 0.95 quantile of $a(\kappa)$ of (15) for different values of κ , with $d_n = d_{n_{ch1}} = d_{n_1}$ of (20) and $e_n = d_n^{1-0.6}$

κ	0.95 quantile of $a(\kappa)$
1.2	2.228935
1.1	2.249073
1.0	2.275527
0.9	2.314825
0.8	2.388221
0.75	2.443131
0.7	2.519460
0.6	2.753893
0.51	3.074950

3.2 Trends of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests

Preliminary investigations of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests were done starting with $\kappa = 1.2$ and then decreasing the values of κ by 0.1 as far as we could go such that the condition of $\kappa > 0.5$ was satisfied. The power functions for $\kappa > 1.2$ were extremely low and asymmetric regardless of the values of d_n and e_n made in the substitution for $t_1(n)$ and $t_2(n)$, and thus were eliminated from the comparison. It is also important to note that each value of κ will also have a corresponding combination of d_n and e_n that will optimize its performance. Therefore, in this comparison, the general trends in the three-way trade-off among the power for a change on the tails, the power for a change in the middle, and the convergence of the type I errors were the primary focus.

Since (15) reduces to (7) or (8) when $\kappa = 1$, one would expect $d_n = d_{n_{ch1}} = d_{n_1}$ of (20) and $e_n = d_n^{1-0.6}$ to perform well for values of κ relatively close to 1.

However, the goal of this section is just to observe some general trends, and not to obtain optimal $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests.

From Figure 3.1, we note that the empirical type I error convergences are extremely similar for the different values of κ , with the only notable difference being $\kappa = 0.6, 0.51$ where the type I errors are in general much lower.

In terms of the power functions, we note that from Figures 3.2 and 3.3, the power for a change in the middle is gradually improved as κ decreases. However, in turn the power for a change on the extreme tails becomes slightly weaker. It is difficult to make a determination on which value of κ is the most superior to the others. However, since our goal from the beginning was to create a powerful test for changes on the tails, it appears that choosing κ in the 0.7-0.8 range will allow us to maintain similar empirical powers on the tails whilst in addition boosting the power functions for a change near the middle of the dataset. It is worth mentioning that further investigation was done with $\kappa = 0.6$ as well as $\kappa = 0.51$, however the convergence of the empirical powers on the tails was slow regardless of our choice of d_n and e_n .

In preliminary investigations, we also experimented with some different e_n 's. However, as the primary reason behind choosing $e_n = d_n^{1-0.6}$ as oppose to $e_n = d_n^{1-0.99}$ was to accelerate the convergence of the empirical type I errors, the trends we observed regardless of our value of κ were identical to the trends observed Remark 2 of Chapter 2. Thus, an in depth comparison of different e_n 's is not included in this chapter. Interested readers may again look to the

Chapter 3 Appendix for the trend analysis/comparisons below for Exp(1), Frechet(2.5), and Weibull(0.4,1) observables.

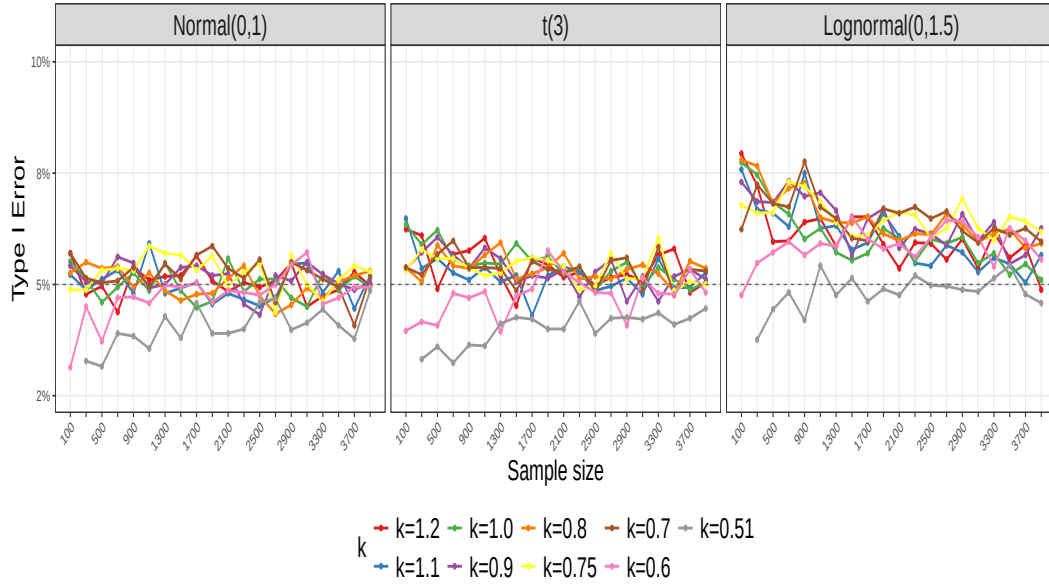


Figure 3.1: Empirical type I errors of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_1} = d_{n_{ch1}}$ of (20), $t_2(n) = 1 - d_{n_{ch1}}/e_n$, $e_n = d_{n_{ch1}}^{1-0.6}$ and $\kappa = 1.2, 1.1, 1.0, 0.9, 0.8, 0.75, 0.7, 0.6, 0.51$ across Normal(0,1), t(3), and Lognormal(0,1.5).

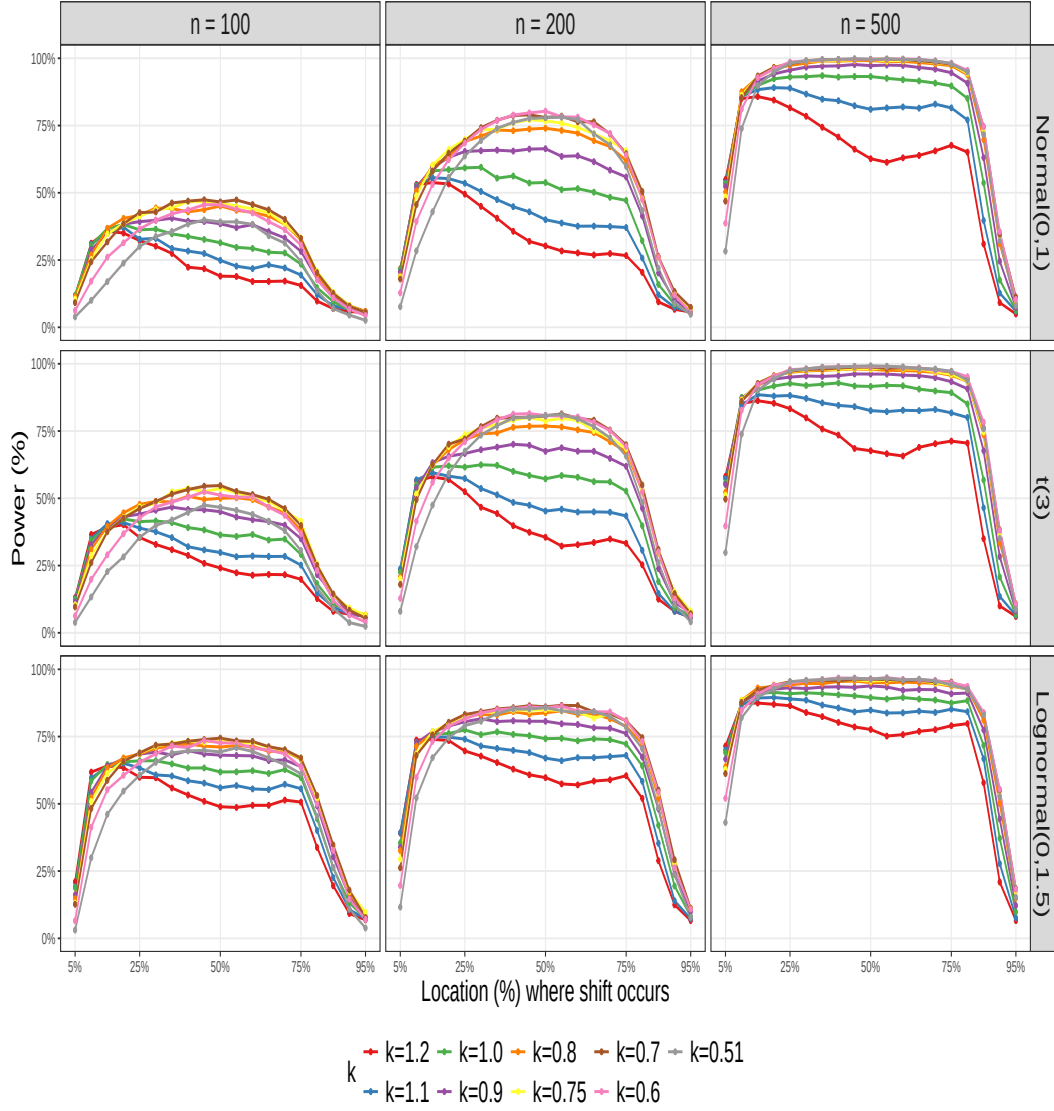


Figure 3.2: Empirical powers for $n = 100, 200, 500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_1}$ of (20), $t_2(n) = 1 - d_{n_1}/e_n$, $e_n = d_{n_1}^{1-0.6}$, and $\kappa = 1.2, 1.1, 1.0, 0.9, 0.8, 0.75, 0.7, 0.6, 0.51$ across Normal(0,1), $t(3)$, and Lognormal(0,1.5).

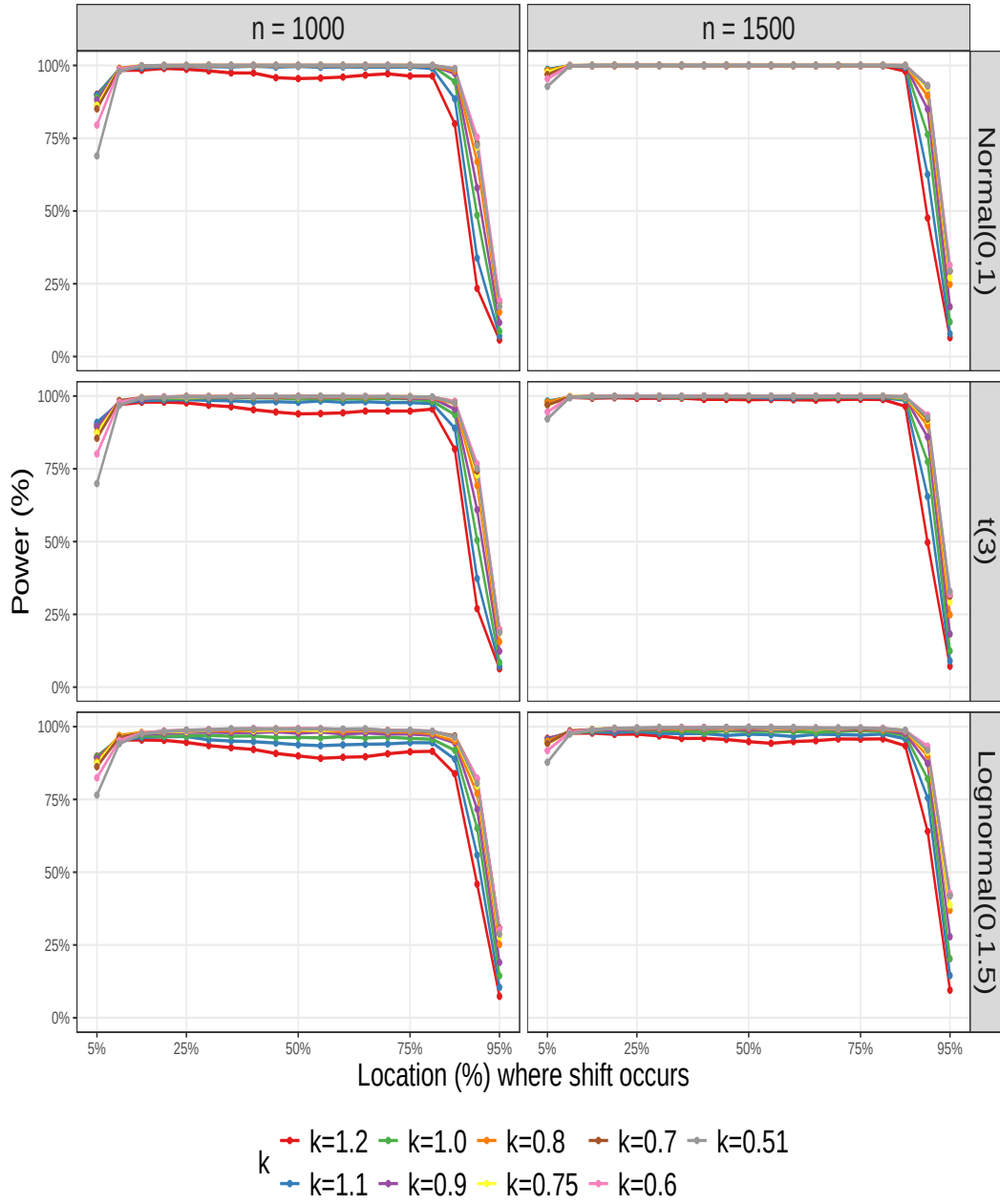


Figure 3.3: Empirical powers for $n = 1000, 1500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_1}$ of (20), $t_2(n) = 1 - d_{n_1}/e_n$, $e_n = d_{n_1}^{1-0.6}$, and $\kappa = 1.2, 1.1, 1.0, 0.9, 0.8, 0.75, 0.7, 0.6, 0.51$ across Normal(0,1), $t(3)$, and Lognormal(0,1.5).

To demonstrate that a potentially suboptimal choice of d_n is not solely to blame for the trends we see above, we picked $\kappa = 1.2$ and $\kappa = 0.75$ and investigated the performance of $d_{n_i}, i = \overline{1, 5}$. Note that each time we altered d_n and κ , we again estimated the quantiles of the limiting distribution with $m = 30,000$ repetitions.

	$\kappa = 1.2$	$\kappa = 0.75$
d_{n_1}	2.228935	2.443131
d_{n_2}	2.150753	2.346122
d_{n_3}	2.153696	2.345553
d_{n_4}	2.147252	2.336131
d_{n_5}	2.161320	2.347442

Table 3.2: The 0.95 quantile of $a(\kappa)$ of (15) for $\kappa = 1.2$ and $\kappa = 0.75$, along with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - \frac{d_{n_i}}{e_{n_i}}$, and $e_{n_i} = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, using $m = n = 30,000$.

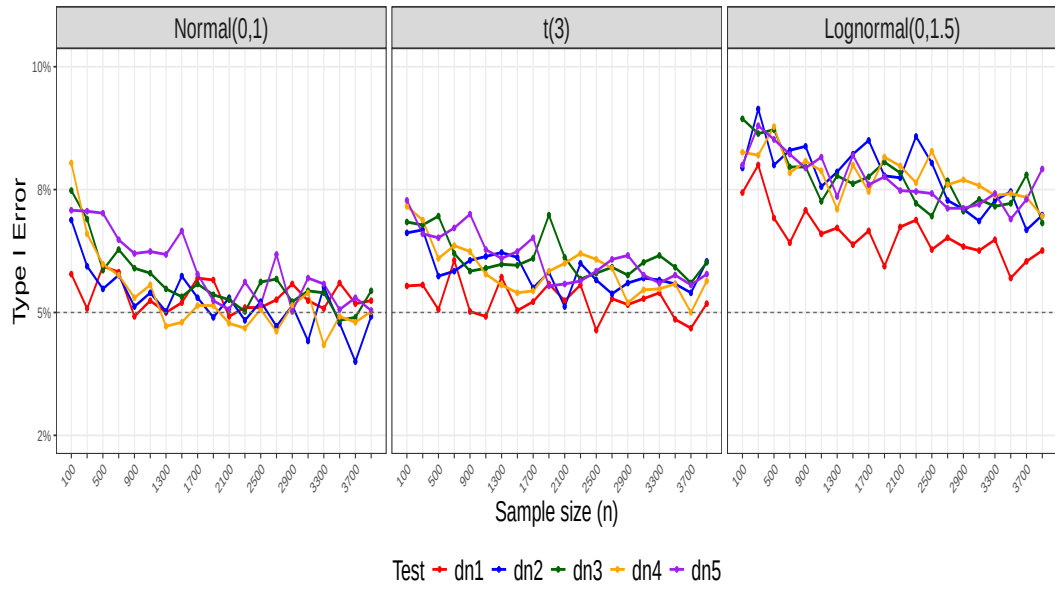


Figure 3.4: Empirical type I errors of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_{n_i}$, and $e_{n_i} = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 0.75$, across Normal(0,1), t(3), and Lognormal(0,1.5).

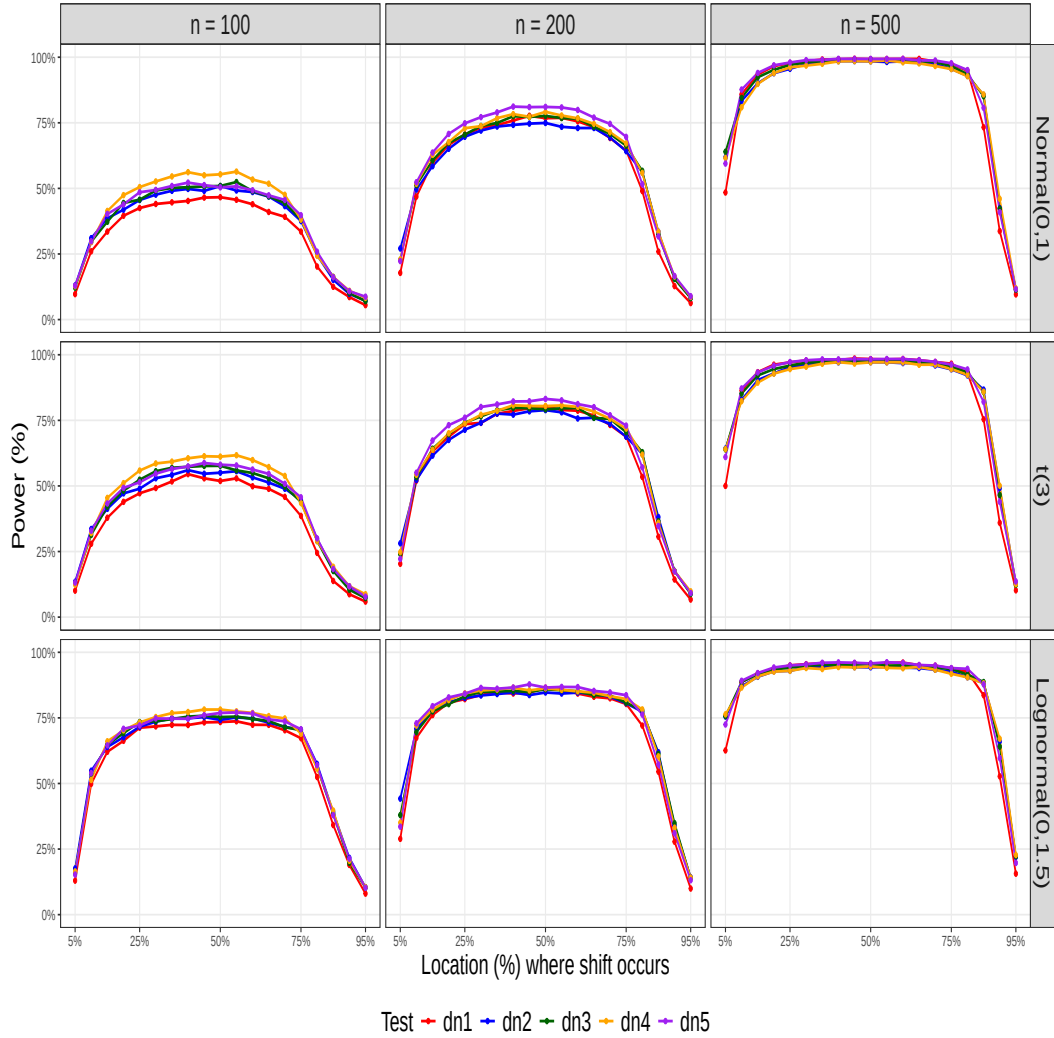


Figure 3.5: Empirical powers for $n = 100, 200, 500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_{n_i}$, and $e_{n_i} = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 0.75$, across Normal(0,1), $t(3)$, and Lognormal(0,1.5).

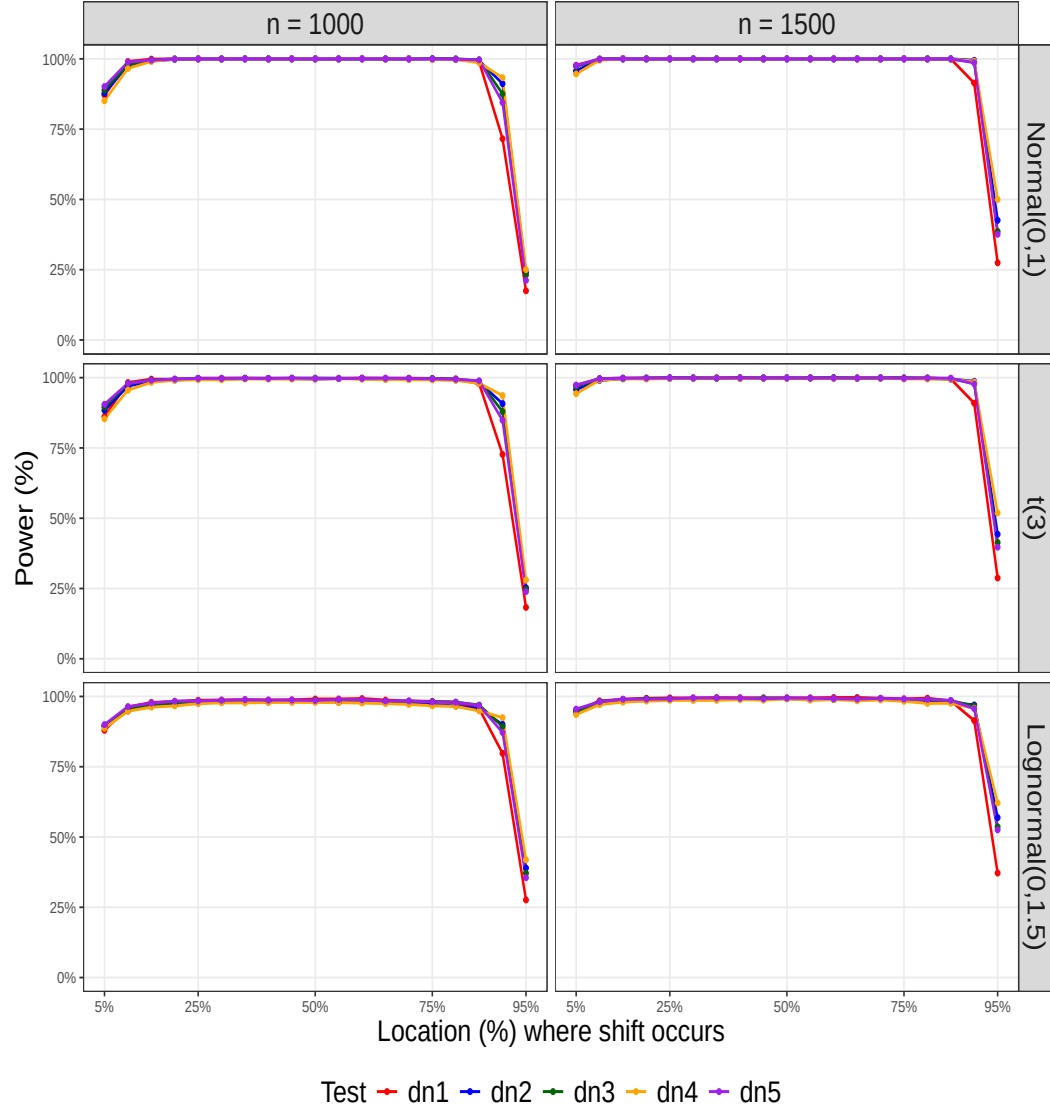


Figure 3.6: Empirical powers for $n = 1000, 1500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_{n_i}$, and $e_{n_i} = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 0.75$, across Normal(0,1), t(3), and Lognormal(0,1.5).

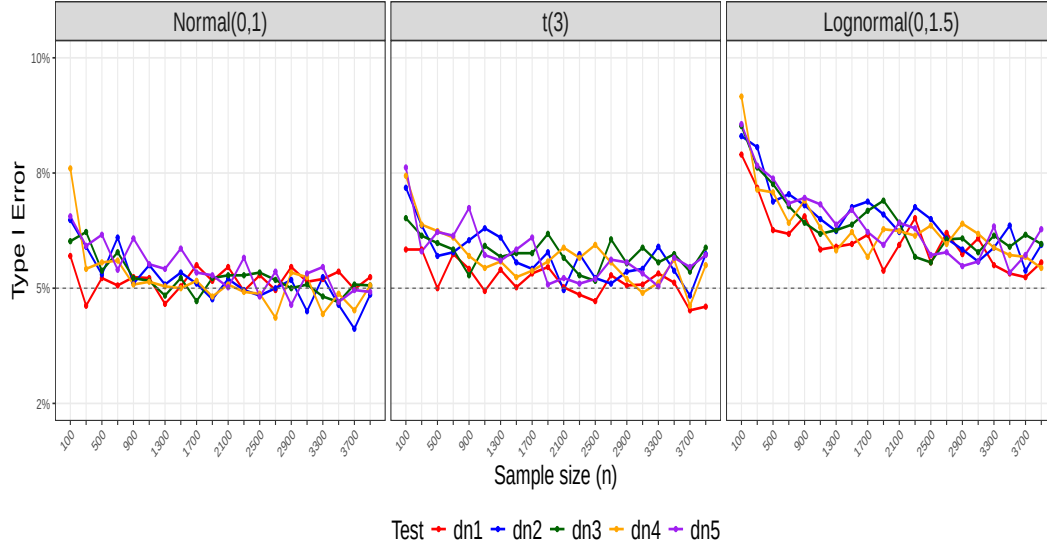


Figure 3.7: Empirical type I errors of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_{n_i}$, and $e_{n_i} = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 1.2$, across Normal(0,1), t(3), and Lognormal(0,1.5).

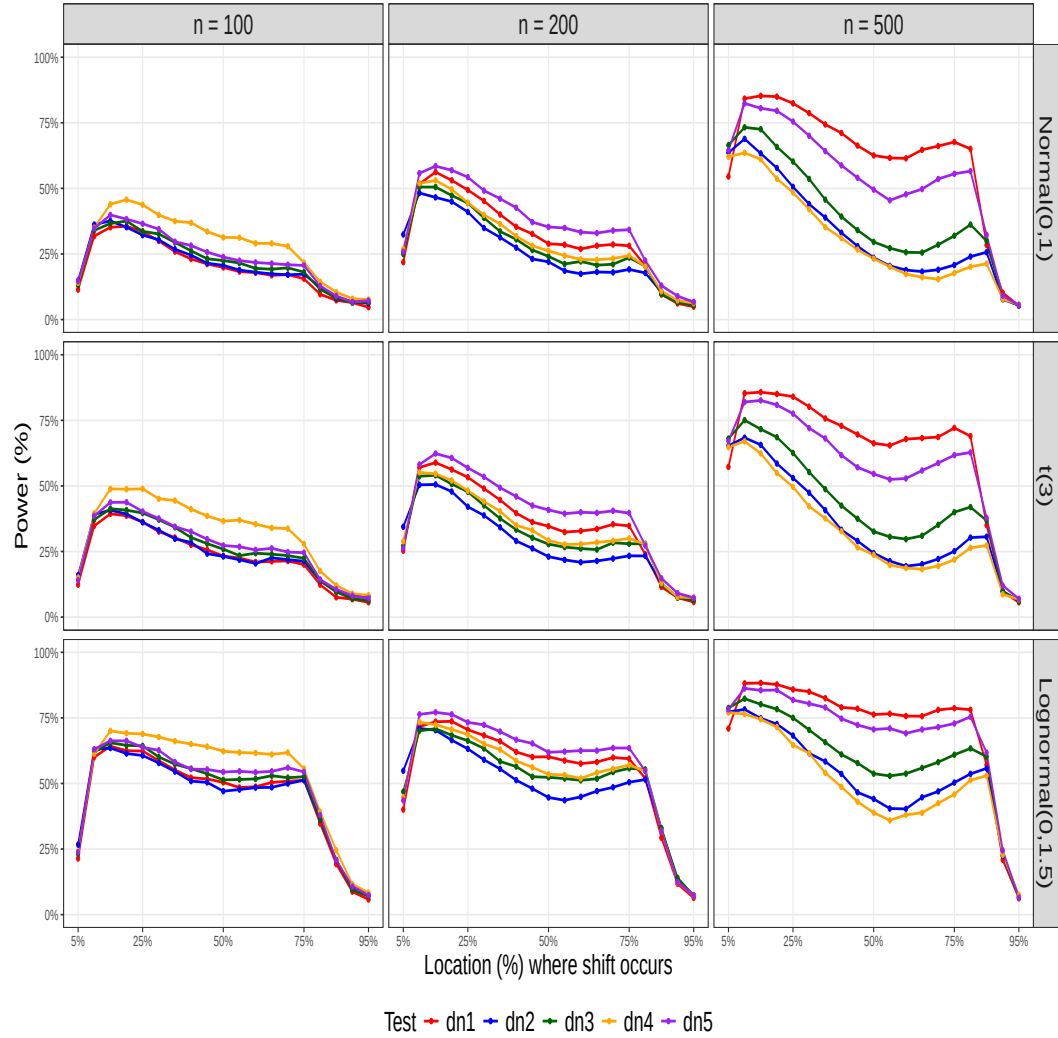


Figure 3.8: Empirical powers for $n = 100, 200, 500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_{n_i}$, and $e_{n_i} = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 1.2$, across Normal(0,1), $t(3)$, and Lognormal(0,1.5).

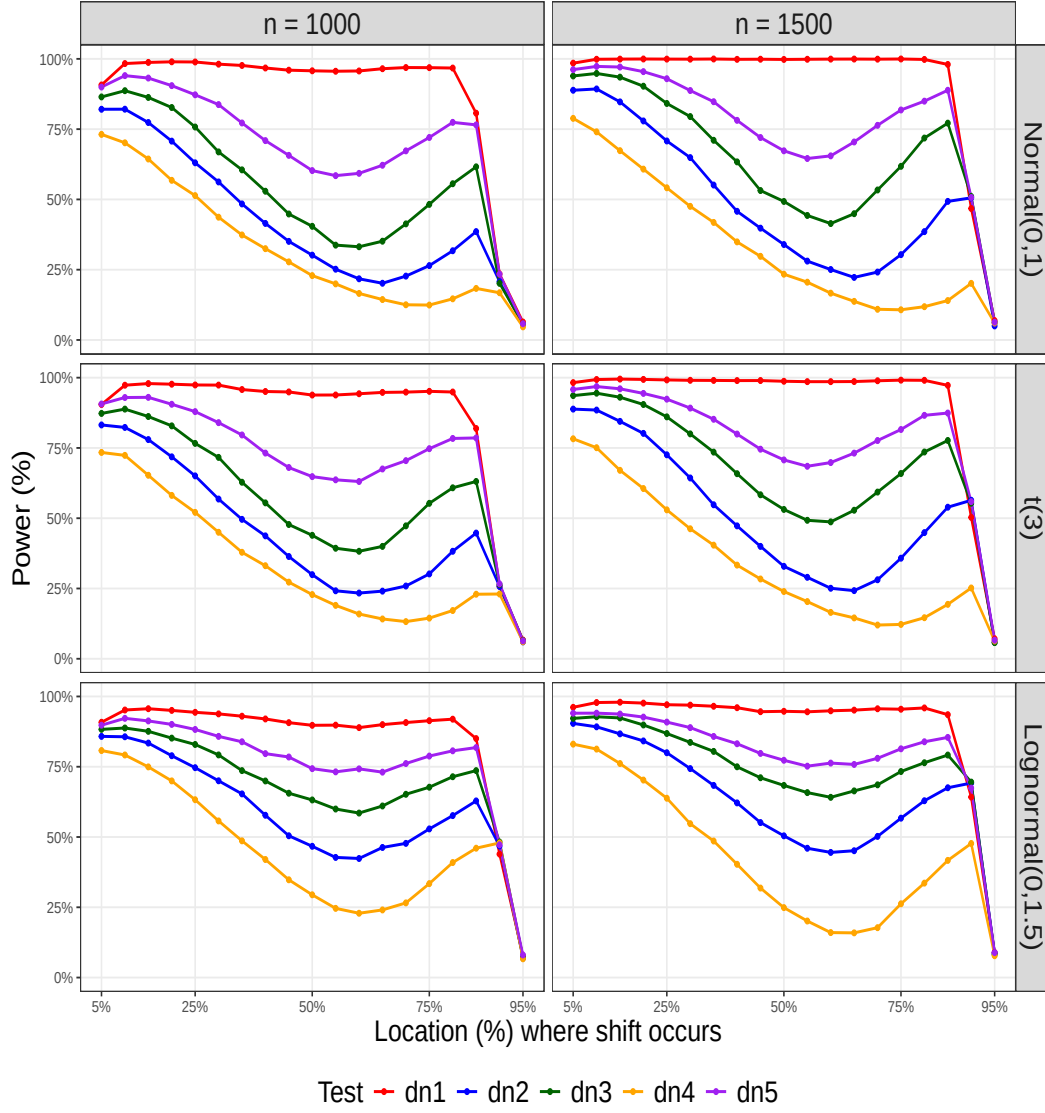


Figure 3.9: Empirical powers for $n = 1000, 1500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_{n_i}$, and $e_{n_i} = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 1.2$, across Normal(0,1), t(3), and Lognormal(0,1.5).

We note the similar trends of the power functions regardless of the choice of d_{n_i} , this is especially clear for $\kappa = 1.2$ as shown in Figures 3.8 and 3.9. As a result, one can be fairly confident that the trends we see from the various κ 's are not solely attributable to a sub-optimal choice of d_n made at the beginning of this section. Based on these analyses, we made the observation that d_{n_1} through d_{n_5} appeared to be exhibiting similar trends in terms of the magnitudes of the empirical type I errors as well as the shape of the empirical power curves when viewed as functions of the location of change. As a result, we decided to improve upon $d_n = d_{n_1}$ for the upcoming comparisons. While certainly d'_n s of the families in (19) could also be altered with different a and b to achieve similar results to our suggestion of

$$d_n = d_{n_{ch2}} = d_{n_6} = \frac{n^{0.9}(\log(n))^{-1.2}}{n+1}, \quad e_n = d_{n_6}^{1-0.6}, \quad \text{and} \quad \kappa = 0.75, \quad (3.1)$$

the latter combination performs sufficiently well compared to the other related tests, as will be seen from Section 3.3.

Remark 5 *In principle, in (15), we have the option of replacing $\hat{\sigma}_n$ with $\hat{\sigma}_{[nt+1]}$ of (9). Below is a quick comparison of the respective (15)-based tests with $t_1(n) = d_{n_6}$ and $t_2(n) = 1 - d_{n_6}/e_n$, with d_{n_6} , e_n , and κ as in (3.1).*

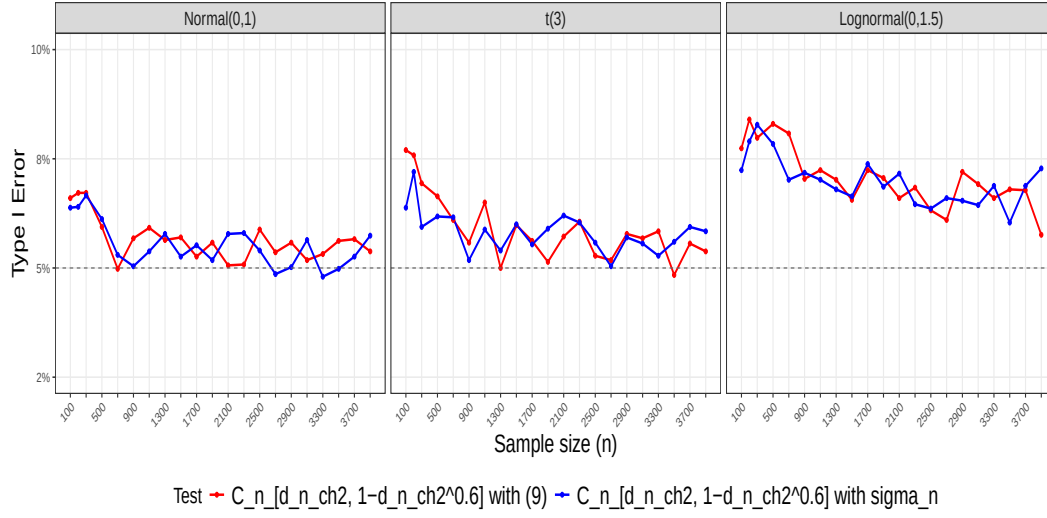


Figure 3.10: Empirical type I errors of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9), with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, with d_{n_6} , e_n , and κ as in (3.1), on Normal(0,1), t(3), and Lognormal(0,1.5).

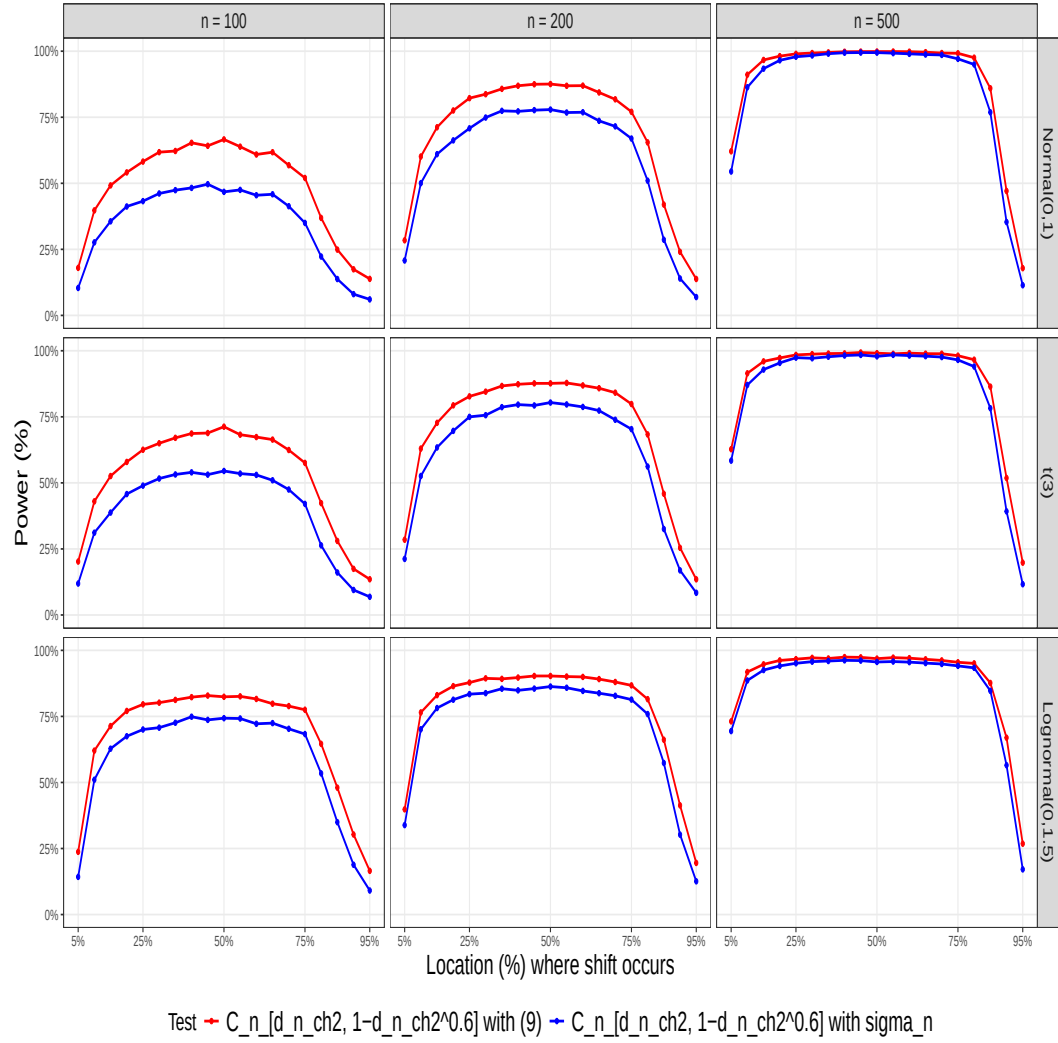


Figure 3.11: Empirical powers for $n = 100, 200, 500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9), with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, with d_{n_6} , e_n , and κ as in (3.1), on Normal(0,1), $t(3)$, and Lognormal(0,1.5).

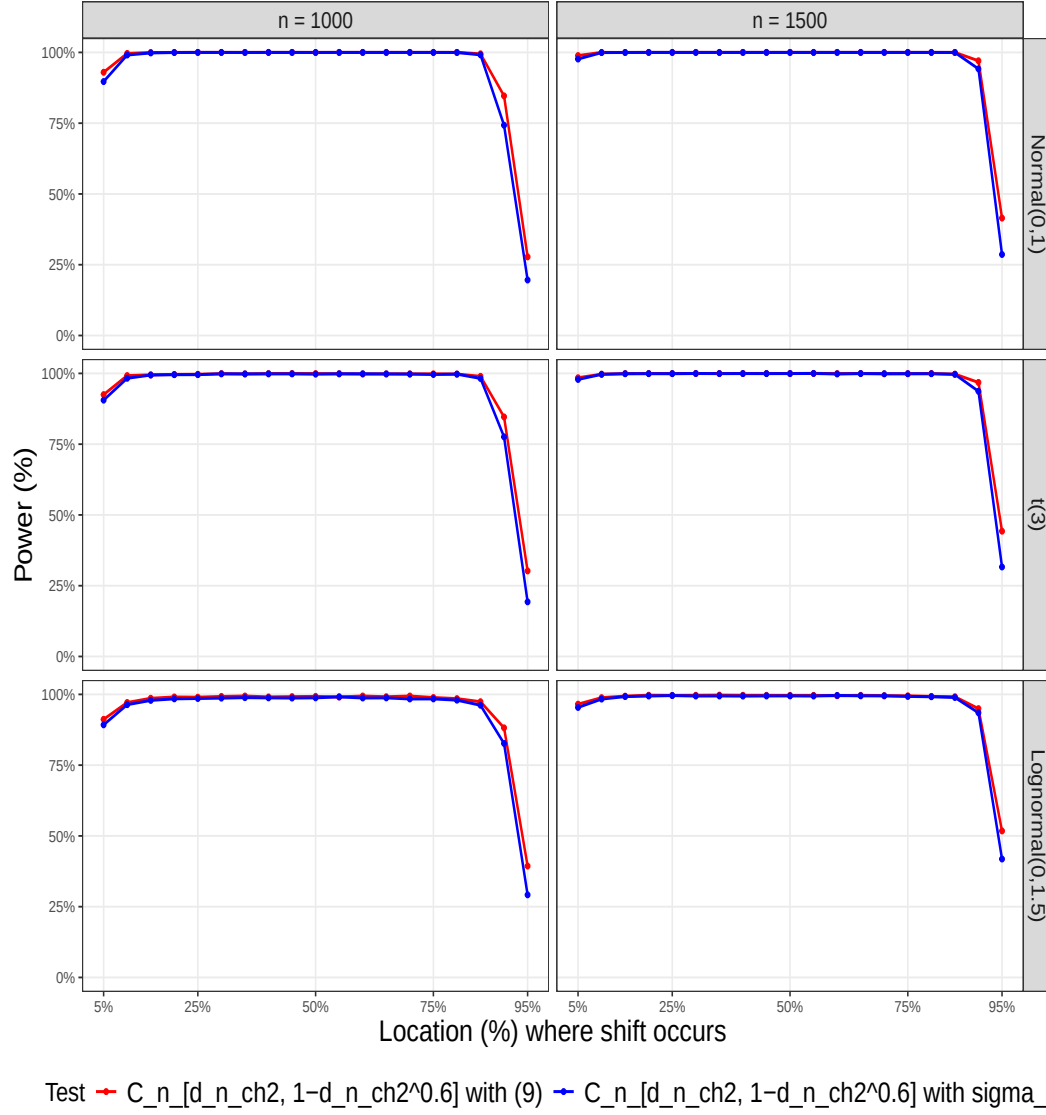


Figure 3.12: Empirical powers for $n = 1000, 1500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9), with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, with d_{n_6} , e_n , and κ as in (3.1), on Normal(0,1), t(3), and Lognormal(0,1.5).

We observe from Figure 3.10 that the convergences in the type I errors are relatively similar for both tests. However, in Figures 3.11-3.12, we observe a difference in the empirical powers primarily for $n = 100, 200$. However, this can be attributed to the slight inflation of the empirical type I errors for these sample sizes. As a result, the comparisons done in the next section will be done using $\hat{\sigma}_n$ as opposed to $\hat{\sigma}_{[nt+1]}$, as the type I errors are slightly closer to the nominal level for smaller n .

3.3 Comparison with tests from (7), (8), (11), (13), (15)

Finally, we compare the $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}/e_n]}$ - and $C_n^{[d_{n_{ch2}}/e_n, 1-d_{n_{ch2}}]}$ - tests, with $d_{n_{ch2}}$, e_n , and κ as in (3.1), with the best tests obtained in Chapter 2, and with the powerful $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests found in Gallanosa and Martsynyuk (2026) (see (11), (13), and (18)). Other κ 's close to 0.75 will provide similar results to what is shown below. Since the differences for the powers for a change near the middle of a dataset are much closer in between the tests, one can refer to the Tables 3.3-3.5 below for the exact differences in the power functions for $\frac{k^*}{n} = 0.05, 0.15, 0.25, 0.5, 0.75, 0.85, 0.95$ and the change size $\Delta = 0.5\sigma$. The only test we are not including in the upcoming comparisons is the H_n -based test, as it outperformed by the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests in terms of the convergence of the empirical type I errors as well as the empirical powers, which was

demonstrated clearly in the previous chapter.

For these comparisons, we used $m = n = 100,000$ replications in order to empirically estimate the 0.95 quantile of the limiting distribution of (15) with $\kappa = 0.75$, $t_1(n) = d_{n_{ch2}}$, and $t_2(n) = 1 - \frac{d_{n_{ch2}}}{e_n}$ as in (3.1), since such m and n had been used to get the critical values for the $T_n^{q_1}$ and $L_n^{q_2}$. This estimated quantile, 2.406535, is a bit different from the one listed in Table 3.1, 2.443131, where we used $m = n = 30,000$.

Table 3.3: Empirical powers for a change at k^*/n on Normal(0,1), with $\Delta = 0.5\sigma$

n	Test statistic	$\frac{k^*}{n} = 0.05$	0.15	0.25	0.50	0.75	0.85	0.95
100	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.1146	0.3930	0.4282	0.3948	0.3074	0.1278	0.0600
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0622	0.1154	0.2786	0.4146	0.4410	0.3892	0.1164
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.1038	0.3560	0.4324	0.4676	0.3500	0.1376	0.0606
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0674	0.1360	0.3466	0.4714	0.4302	0.3512	0.1030
	$T_n^{q_1}$	0.0478	0.1932	0.3004	0.4004	0.2982	0.1888	0.0534
	$L_n^{q_2}$	0.0970	0.2382	0.3486	0.4422	0.3482	0.2428	0.0874
200	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.2044	0.6054	0.6304	0.6038	0.5404	0.1970	0.0554
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0608	0.1844	0.5256	0.5962	0.6352	0.6054	0.2074
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.2078	0.6102	0.7080	0.7790	0.6694	0.2864	0.0694
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0730	0.2862	0.6576	0.7794	0.7202	0.6018	0.2046
	$T_n^{q_1}$	0.0916	0.4412	0.6466	0.7856	0.6580	0.4402	0.0968
	$L_n^{q_2}$	0.1464	0.4722	0.6452	0.7674	0.6484	0.4726	0.1492
500	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.5278	0.9140	0.9432	0.9518	0.9168	0.5654	0.0708
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0698	0.5572	0.9204	0.9486	0.9392	0.9166	0.5374
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.5446	0.9338	0.9790	0.9946	0.9710	0.7688	0.1142
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1004	0.7586	0.9748	0.9944	0.9756	0.9360	0.5496
	$T_n^{q_1}$	0.2910	0.9090	0.9828	0.9972	0.9848	0.8988	0.2838
	$L_n^{q_2}$	0.3452	0.8830	0.9706	0.9936	0.9720	0.8804	0.3480
1000	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.8908	0.9954	0.9992	0.9996	0.9976	0.9554	0.1034
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0896	0.9478	0.9982	0.9996	0.9992	0.9966	0.8884
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.8970	0.9982	1.0000	0.9998	1.0000	0.9914	0.1960
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1930	0.9918	1.0000	0.9998	1.0000	0.9980	0.9052
	$T_n^{q_1}$	0.6480	0.9978	1.0000	1.0000	1.0000	0.9974	0.6330
	$L_n^{q_2}$	0.6556	0.9960	1.0000	1.0000	1.0000	0.9948	0.6456
1500	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9796	1.0000	1.0000	1.0000	1.0000	0.9984	0.1238
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1346	0.9962	1.0000	1.0000	1.0000	1.0000	0.9844
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9766	0.9998	1.0000	1.0000	1.0000	0.9998	0.2862
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.2964	0.9998	1.0000	1.0000	1.0000	1.0000	0.9820
	$T_n^{q_1}$	0.8668	1.0000	1.0000	1.0000	1.0000	1.0000	0.8652
	$L_n^{q_2}$	0.8508	1.0000	1.0000	1.0000	1.0000	1.0000	0.8484

Table 3.4: Empirical powers for a change at k^*/n on $t(3)$, with $\Delta = 0.5\sigma$

n	Test statistic	$\frac{k^*}{n} = 0.05$	0.15	0.25	0.50	0.75	0.85	0.95
100	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.1248	0.4300	0.4778	0.4650	0.3800	0.1458	0.0678
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0602	0.1258	0.3352	0.4726	0.4840	0.4426	0.1208
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.1190	0.3876	0.4896	0.5450	0.4204	0.1608	0.0688
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0668	0.1576	0.4130	0.5456	0.4944	0.4050	0.1146
	$T_n^{q_1}$	0.0692	0.2240	0.3742	0.4774	0.3696	0.2382	0.0718
	$L_n^{q_2}$	0.1064	0.2852	0.4070	0.5054	0.4050	0.2742	0.1050
200	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.2120	0.6362	0.6804	0.6502	0.5956	0.2362	0.0716
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0706	0.2198	0.5772	0.6476	0.6782	0.6326	0.2124
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.2122	0.6340	0.7498	0.8040	0.7034	0.3254	0.0840
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0844	0.3266	0.7118	0.8050	0.7406	0.6382	0.2174
	$T_n^{q_1}$	0.1180	0.4800	0.6832	0.7998	0.6958	0.4910	0.1242
	$L_n^{q_2}$	0.1622	0.5182	0.6930	0.7972	0.6852	0.5154	0.1594
500	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.5642	0.9110	0.9388	0.9268	0.9102	0.6034	0.0756
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0746	0.5984	0.9106	0.9402	0.9370	0.9180	0.5550
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.5842	0.9288	0.9740	0.9788	0.9654	0.7826	0.1164
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1202	0.7838	0.9680	0.9770	0.9706	0.9300	0.5736
	$T_n^{q_1}$	0.3164	0.8958	0.9744	0.9928	0.9780	0.9016	0.3174
	$L_n^{q_2}$	0.3840	0.8920	0.9704	0.9858	0.9616	0.8870	0.3732
1000	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9016	0.9896	0.9928	0.9938	0.9910	0.9462	0.1058
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0976	0.9440	0.9906	0.9944	0.9932	0.9906	0.9002
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9056	0.9942	0.9956	0.9974	0.9962	0.9822	0.1930
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1950	0.9822	0.9960	0.9972	0.9968	0.9926	0.8912
	$T_n^{q_1}$	0.6594	0.9930	0.9974	0.9984	0.9972	0.9918	0.6562
	$L_n^{q_2}$	0.6682	0.9912	0.9978	0.9990	0.9976	0.9888	0.6636
1500	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9832	0.9980	0.9972	0.9974	0.9972	0.9888	0.1480
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1440	0.9898	0.9966	0.9974	0.9958	0.9974	0.9808
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9786	0.9988	0.9990	0.9994	0.9990	0.9966	0.3162
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.3168	0.9984	0.9988	0.9994	0.9990	0.9966	0.9738
	$T_n^{q_1}$	0.8736	0.9984	0.9994	0.9996	0.9998	0.9988	0.8658
	$L_n^{q_2}$	0.8384	0.9982	0.9998	0.9988	0.9988	0.9976	0.8466

Table 3.5: Empirical powers for a change at k^*/n on Lognormal(0,1.5), with $\Delta = 0.5\sigma$

n	Test statistic	$\frac{k^*}{n} = 0.05$	0.15	0.25	0.50	0.75	0.85	0.95
100	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.1696	0.6674	0.6958	0.6942	0.6522	0.3178	0.0854
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0902	0.2924	0.6172	0.7026	0.6808	0.6450	0.2512
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.1426	0.6278	0.7004	0.7432	0.6832	0.3496	0.0910
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0860	0.3368	0.6528	0.7500	0.6976	0.6176	0.2336
	$T_n^{q_1}$	0.1168	0.5062	0.6464	0.7318	0.6570	0.5256	0.2120
	$L_n^{q_2}$	0.1954	0.5342	0.6646	0.7092	0.6534	0.5186	0.2110
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.3508	0.7766	0.7996	0.7896	0.7550	0.4712	0.0976
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0988	0.4442	0.7526	0.7800	0.7672	0.7498	0.3684
200	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.3386	0.7816	0.8342	0.8630	0.8136	0.5736	0.1258
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1070	0.5332	0.8066	0.8576	0.8088	0.7488	0.3738
	$T_n^{q_1}$	0.2032	0.6978	0.8258	0.8682	0.8124	0.6918	0.2996
	$L_n^{q_2}$	0.2936	0.7078	0.8082	0.8572	0.7944	0.6832	0.2890
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.6848	0.9148	0.9210	0.9046	0.8952	0.7440	0.1008
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1108	0.7288	0.9060	0.9042	0.8938	0.8838	0.6678
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.6944	0.9254	0.9510	0.9560	0.9412	0.8468	0.1710
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1516	0.8218	0.9408	0.9532	0.9376	0.9076	0.6840
500	$T_n^{q_1}$	0.4726	0.9022	0.9520	0.9720	0.9516	0.8942	0.4870
	$L_n^{q_2}$	0.5392	0.9050	0.9424	0.9680	0.9524	0.8932	0.4858
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.8932	0.9674	0.9734	0.9672	0.9654	0.9200	0.1570
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1514	0.9108	0.9642	0.9670	0.9636	0.9582	0.9088
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.8924	0.9788	0.9854	0.9878	0.9840	0.9616	0.2918
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.2724	0.9524	0.9822	0.9870	0.9822	0.9690	0.9082
	$T_n^{q_1}$	0.7484	0.9766	0.9888	0.9934	0.9910	0.9744	0.7378
	$L_n^{q_2}$	0.7530	0.9752	0.9870	0.9912	0.9898	0.9744	0.7412
1000	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9588	0.9892	0.9876	0.9828	0.9840	0.9738	0.2170
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.2068	0.9612	0.9856	0.9848	0.9804	0.9818	0.9712
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9538	0.9922	0.9958	0.9946	0.9928	0.9890	0.4184
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.3990	0.9848	0.9932	0.9940	0.9920	0.9888	0.9590
	$T_n^{q_1}$	0.8718	0.9898	0.9978	0.9976	0.9956	0.9890	0.8672
	$L_n^{q_2}$	0.8730	0.9906	0.9958	0.9978	0.9956	0.9912	0.8678
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9588	0.9892	0.9876	0.9828	0.9840	0.9738	0.2170
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.2068	0.9612	0.9856	0.9848	0.9804	0.9818	0.9712
1500	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9538	0.9922	0.9958	0.9946	0.9928	0.9890	0.4184
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.3990	0.9848	0.9932	0.9940	0.9920	0.9888	0.9590
	$T_n^{q_1}$	0.8718	0.9898	0.9978	0.9976	0.9956	0.9890	0.8672
	$L_n^{q_2}$	0.8730	0.9906	0.9958	0.9978	0.9956	0.9912	0.8678
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9588	0.9892	0.9876	0.9828	0.9840	0.9738	0.2170
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.2068	0.9612	0.9856	0.9848	0.9804	0.9818	0.9712
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9538	0.9922	0.9958	0.9946	0.9928	0.9890	0.4184
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.3990	0.9848	0.9932	0.9940	0.9920	0.9888	0.9590

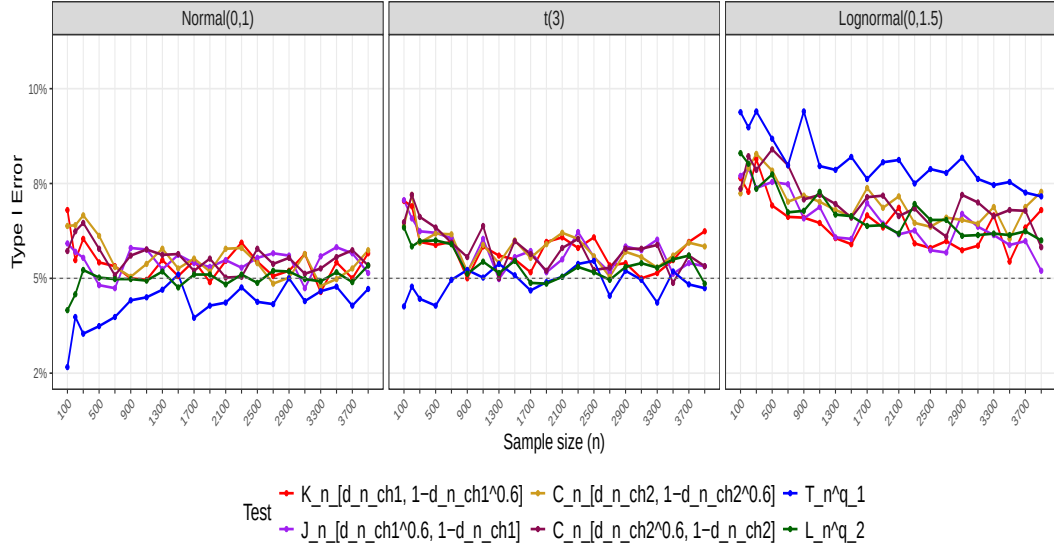


Figure 3.13: Empirical type I errors for the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ -, $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -, $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -, $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -, $T_n^{q_1}$ -, and $L_n^{q_2}$ -based tests on Normal(0,1), t(3), and Lognormal(0,1.5).

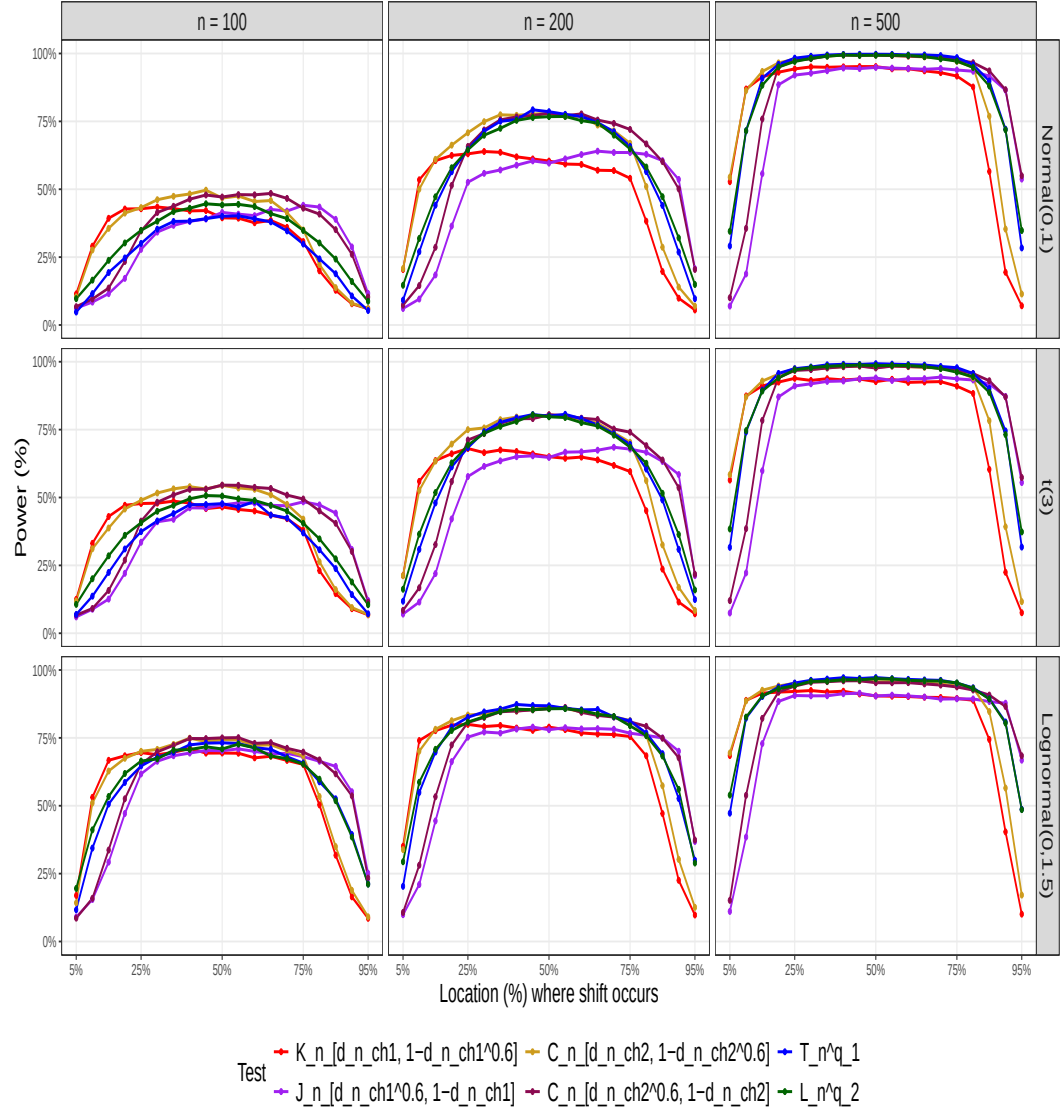


Figure 3.14: Empirical powers for the shift of 0.5σ and $n = 100, 200, 500$ for the $J_n^{[d_{n_{ch1}^{0.6}}, 1-d_{n_{ch1}}]}$, $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$, $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$, $C_n^{[d_{n_{ch2}^{0.6}}, 1-d_{n_{ch2}}]}$, $T_n^{q_1}$, and $L_n^{q_2}$ -based tests on Normal(0,1), t(3), and Lognormal(0,1.5).

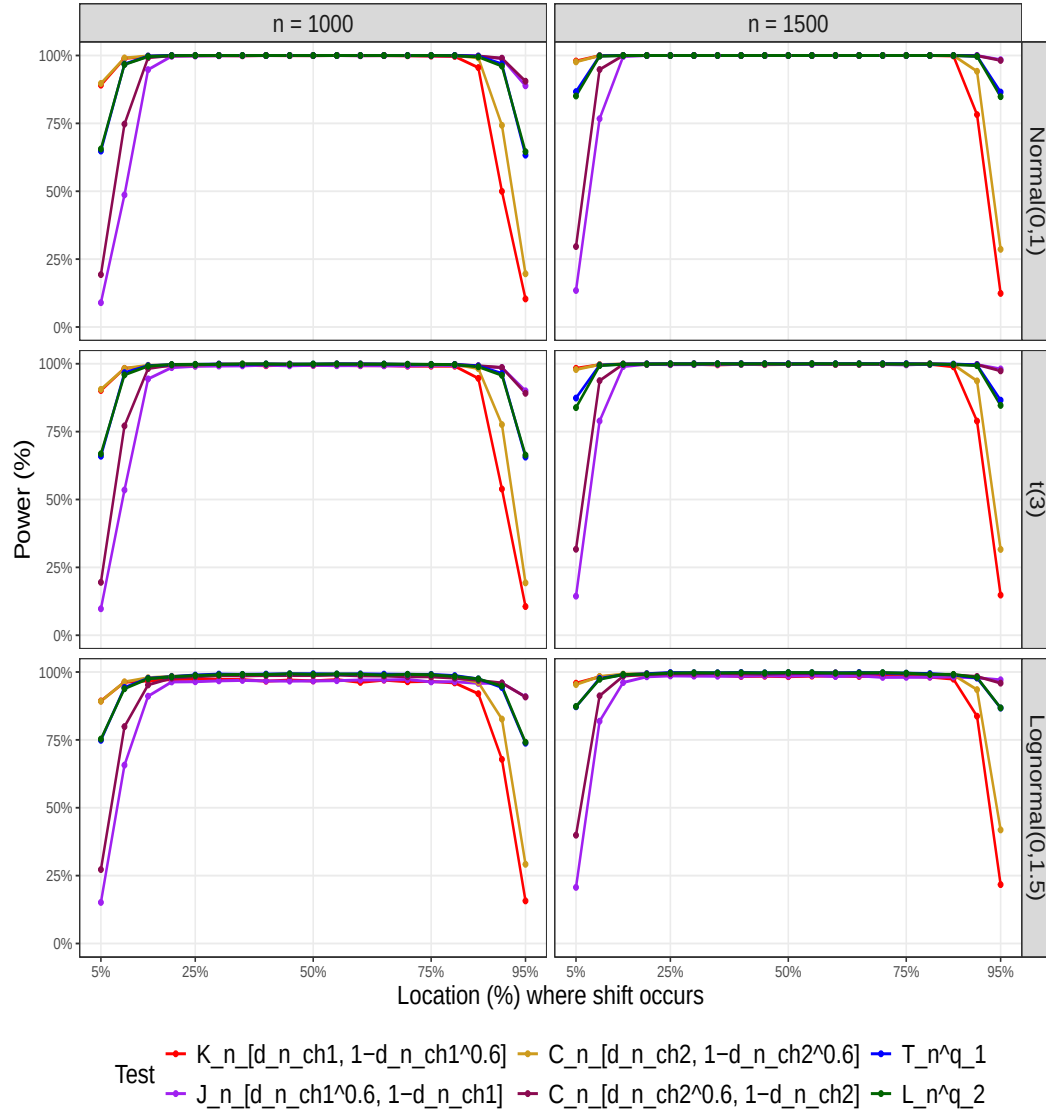


Figure 3.15: Empirical powers for the shift of 0.5σ and $n = 1000, 1500$ for the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$, $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$, $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$, $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$, $T_n^{q_1}$, and $L_n^{q_2}$ -based tests on Normal(0,1), t(3), and Lognormal(0,1.5).

In terms of the type I errors as shown in Figure 3.13, the $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ - and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -based tests with $\kappa = 0.75$ do not provide too much of an improvement on the convergence of the type I errors of the tests based on $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ and $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ of Chapter 2: they are very similar to the ones based on the latter tests. However, the major improvement is in the empirical powers for $0.25 \leq \frac{k^*}{n} \leq 0.75$: with the powers of the $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ - and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -based tests are close to those of the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests, regardless of the underlying distribution of X . Furthermore, for $0.05 \leq \frac{k^*}{n} \leq 0.25$, the (13)-based tests provide slightly higher empirical powers than the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests. These trends are notable for $n = 100, 200$, and as a result, the (13)-based tests are much more viable for these n .

From Tables 3.3–3.5, as well as the companion Tables 6.1–6.3 in Section 6.4, we observe that the empirical powers of the $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests with $\kappa = 0.75$ are mostly superior or similar to those of the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ -based tests, with the exception of the empirical powers for $n = 100$ at $\frac{k^*}{n} = 0.95$ and 0.85 . A similar observation can be made for the $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ - and $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -based tests except for $\frac{k^*}{n} = 0.05$ and 0.15 . As a result, if one has a strong suspicion that a change has occurred on an extreme tail, for a sample size close to 100, we can recommend the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ - and $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -based

tests over the $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ - and $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests. Otherwise, the $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ - and $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests are to be recommended over the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ - and $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -based tests. Notably, taking a slightly larger shift of 0.75σ (as shown in Figures 6.16–6.19 of Section 6.6) increases the rate at which the power curves converge to 1, but does not change the previously noted trends very much.

In general, the conclusions are similar from those in the previous chapter. If one has some prior general knowledge about which side of the dataset the change may occur, then we can recommend the $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ - and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -based tests with $\kappa = 0.75$ for all n regardless of the underlying distribution of the data. An exception can be made if one suspects a change around $\frac{k^}{n} = 0.05, 0.15$ or $\frac{k^*}{n} = 0.85, 0.95$: for a sample size around $n = 100$, we can recommend the $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ - and $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ -based tests respectively.*

3.4 Application to real-world datasets

In this section, the datasets are selected to best match the assumptions used to model a change in the mean in Chapters 2 and 3: the observations are independent and form two random samples with the same variances, but different means. We have compiled three datasets that have a shift, and also one example where no shift has occurred. All four of these datasets of (nearly)

independent observations with (nearly) constant variance before and after the change are introduced below.

Flow of the Nile River

The first example we have is the flow of the Nile river from the built-in dataset in R before and after a dam at Aswan was completed in 1902 (construction of the dam began in 1898). This dataset has been analyzed by many: for example, Horváth and Rice (2024) observe that a change occurs around the year 1900, which corresponds to a change at around 30%.

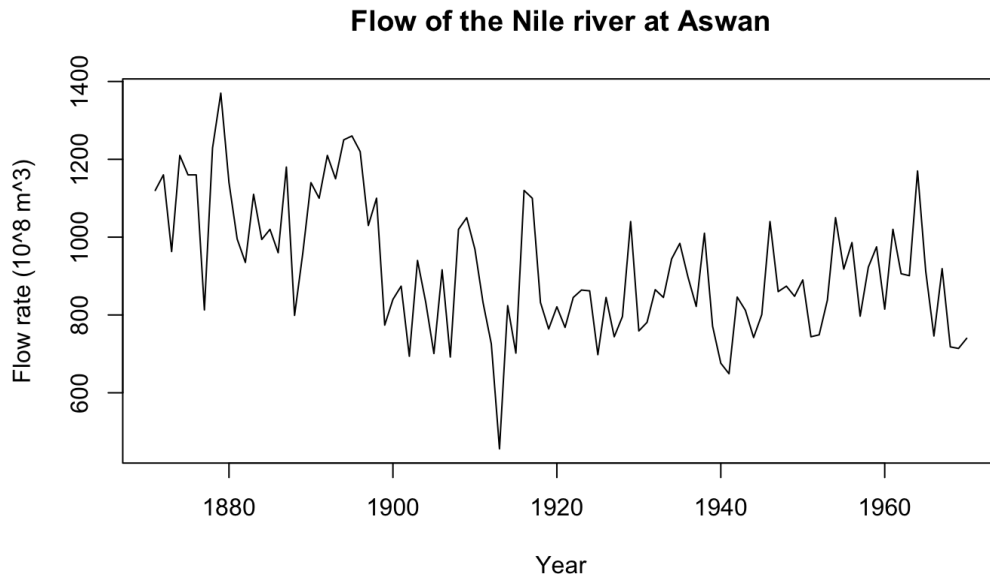


Figure 3.16: The flow of the River Nile at Aswan from 1871-1970. A dam was completed in 1902 after which the flow of the river decreased.

UK Coal Mining Disasters

This dataset from the `boot` package in R gives the number of accidents per year in British coal mines from 1851-1962 that resulted in 10 or more deaths. After 1887, legislation surrounding safety practices and training was changed in order to decrease the number of accidents per year. According to the time-series plot, this legislation indeed appears to have diminished the number of yearly accidents.

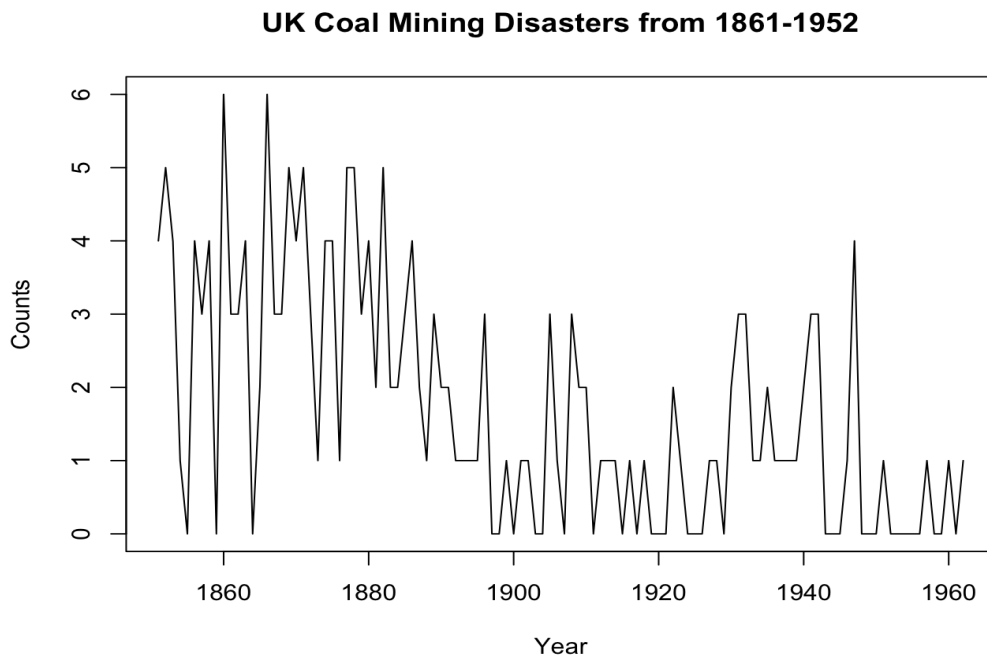
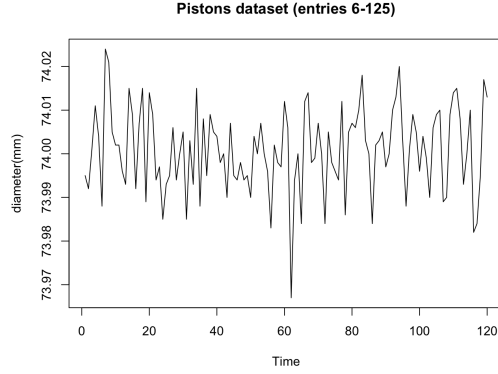


Figure 3.17: Time series plot of the number of accidents per year in British coal mines with 10 or more fatalities from 1851-1962, where starting from 1887 (index 37) legislation surrounding safety practices was changed.

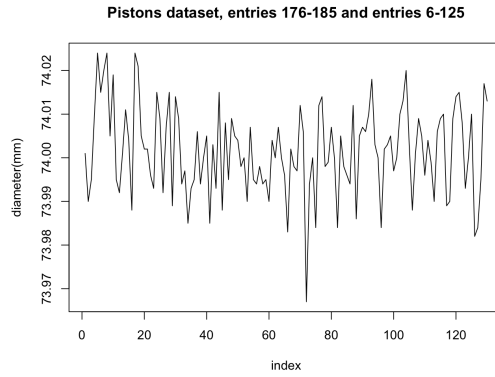
Quality Control of Piston Construction

The following dataset from the `qcc` package in R consists of 200 readings of 40 batches of piston diameters (in millimeters). The first 125 readings (the first 25 batches) correspond to a process where everything is working well and homogeneously, however the last 75 readings (the last 15 batches) correspond to the readings after a suspected change in the manufacturing process, which seems to result in a slight upward shift of the average piston diameter. The last 25 readings are particularly high, which may indicate that there were many ongoing manufacturing errors in the production of the last 5 batches of pistons. This may correspond to a second change point, which would not follow our model, and thus we cannot use the full time series data to test H_0 of (1) versus H_a of (2).

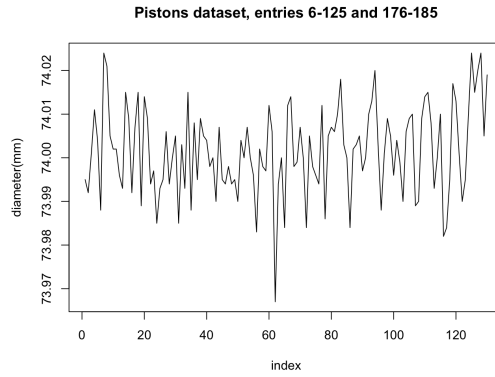
To test the sensitivity for a change on the extreme tails, we placed entries 176-185 both before and after the homogeneous process, which corresponds to roughly a change at 8% and a change at 92% respectively. We also removed the first batch of observations from the homogeneous process as the first observable appeared to be a large outlier which affected prior analyses.



(a) Pistons dataset homogeneous process, entries 6-125.



(b) Pistons dataset entries 176-185 and entries 6-125.



(c) Pistons dataset entries 6-125 and entries 176-185.

Figure 3.18: Pistons dataset: time series plots.

Scrabble Scores

This dataset is taken from the world’s number one ranked scrabble player, Nigel Richards. Nigel is famous within the community for not only dominating the English-language championships, but for also winning the Spanish and French language championships, despite not speaking the languages. He is also known for playing perfectly in every single game, with his only documented mistakes being tile transpositions of valid moves, for example, famously playing HIARNETS instead of the valid HAIRNETS against an American expert.

With this in mind, we have taken Nigel’s scores from the 2002, 2004, 2005, and 2008 North American Scrabble Championships from cross-tables.com, where volunteers and competitive scrabble players upload tournament results as well as individual games for others to observe and learn from. In total, the 2002, 2004, 2005, and 2008 championships represent a total of 113 games whereby Nigel played against the top North American players. The reason for this is that one’s scrabble score depends somewhat on the caliber of your opponent, so it is important that we only take Nigel’s scores against top-competition. Notably, we are under H_0 with this dataset. If Nigel is playing roughly the same caliber of competition, and playing the best move every time, his expected score per game should be roughly constant for every game in this dataset, and the variance of his average score should be relatively constant from game to game.

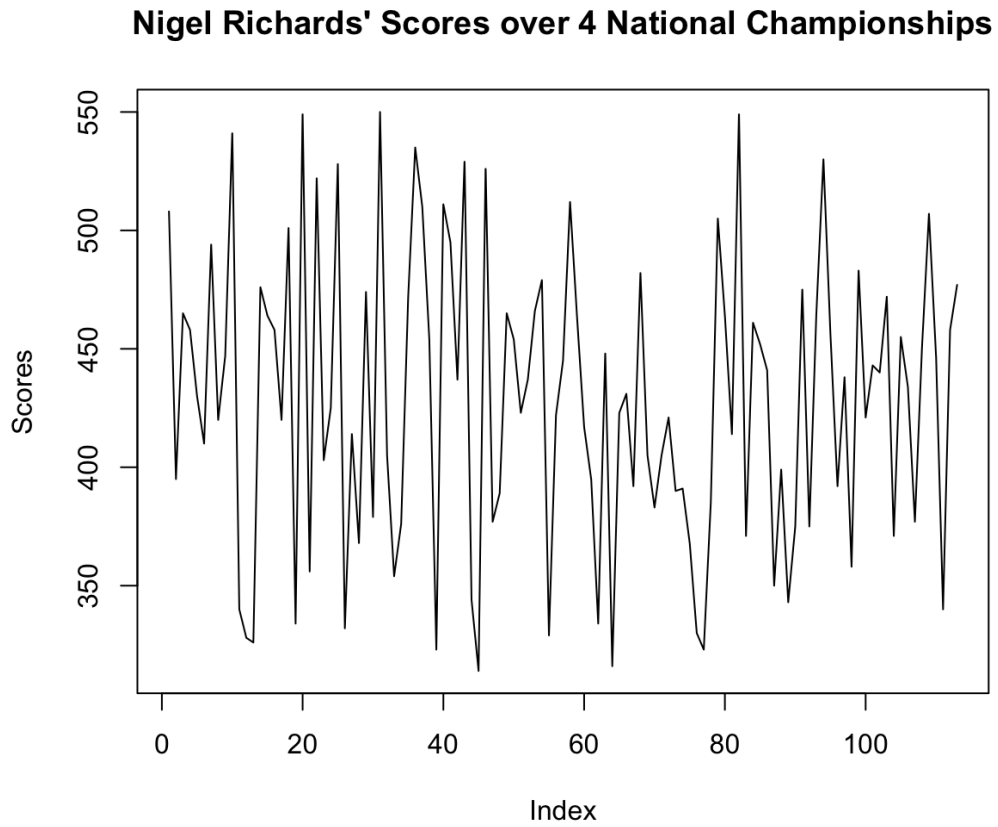


Figure 3.19: Time Series plot of Nigel Richards' scores from the 2002, 2004, 2005, and 2008 North American Scrabble Championships.

For each of the above datasets, we applied the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$, $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$, T_n^{q1-} , L_n^{q2-} , $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$, $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$, and H_n -based tests from the comparisons in our simulation studies. We report the test-statistics and the approximated p-values in Table 3.7. Notably for the T_n^{q1-} and L_n^{q2-} -based tests, Gallanosa-Martsynyuk (2026) only reported the 0.90, 0.95, and 0.98 quantiles. However, using a code provided by Gallanosa, we were able to estimate more quantiles of the limiting distributions for the T_n^{q1-} and L_n^{q2-} -based tests with $m = 100,000$ replications of an $N(0, 1)$ sample of size $n = 100,000$ (see Tables 6.4 and 6.5 in Chapter 6), which thus resulted in more accurate p-value estimations. In Table 3.6, we estimated some quantiles of the limiting distribution of $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$. For convergences (7) and (8), we can get the tabulated quantiles of the limiting distribution from Csörgő and Horváth (1993), and for the H_n -based test, we can use the Gumbel distribution in order to obtain the p-values.

Table 3.6: Estimated $1 - \gamma$ quantiles of $a(\kappa)$ of (15) with $\kappa = 0.75$, rounded to 2 decimal places

$(1 - \gamma)100\%$	Quantile	$(1 - \gamma)100\%$	Quantile	$(1 - \gamma)100\%$	Quantile
50	1.45	70	1.70	90	2.16
51	1.47	71	1.72	91	2.20
52	1.48	72	1.73	92	2.24
53	1.49	73	1.75	93	2.29
54	1.50	74	1.77	94	2.34
55	1.51	75	1.78	95	2.41
56	1.52	76	1.80	96	2.49
57	1.53	77	1.82	97	2.59
58	1.55	78	1.84	98	2.72
59	1.56	79	1.86	99	2.94
60	1.57	80	1.88	99.9	3.57
61	1.58	81	1.90	99.99	4.06
62	1.59	82	1.92		
63	1.61	83	1.95		
64	1.62	84	1.97		
65	1.63	85	2.00		
66	1.65	86	2.02		
67	1.66	87	2.05		
68	1.68	88	2.09		
69	1.69	89	2.12		

Table 3.7: Performance of the $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -, $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ -, $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -, $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ ($\kappa = 0.75$), $T_n^{q_1}$ -, $L_n^{q_2}$ -, and H_n -based tests on real-world datasets. (The p-values are written below the test statistics).

Dataset	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	$T_n^{q_1}$	$L_n^{q_2}$	H_n
Nile River Dataset	6.23	6.23	5.52	5.52	4.51	19.77	8.80
	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.001
Nile River 1871–1897 (no change)	1.18	1.90	1.21	1.78	1.24	1.23	0.87
	=0.4752	=0.1123	>0.5	≈0.25	>0.5	>0.5	=0.57
UK Coal Mining Disasters	6.05	6.05	5.58	5.58	4.67	27.49	9.32
	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.0001	<0.001
UK Coal Mining 1851–1886 (no change)	0.95	0.95	1.05	1.05	0.80	0.61	−0.18
	=0.6755	=0.6755	>0.5	>0.5	>0.5	>0.5	=0.91
Pistons Dataset (entries 176-185 and 6-125)	2.54	1.92	2.61	2.37	1.94	4.01	2.21
	=0.0222	=0.1097	≈0.03	≈0.06	≈0.1	≈0.08	=0.197
Pistons Dataset (entries 6-125 and 176-185)	1.64	3.31	2.00	3.48	2.44	6.82	4.38
	=0.202	=0.0019	≈0.15	≈0.01	≈0.02	≈0.01	=0.0247
Pistons Dataset (entries 6-125)	1.37	1.14	1.49	1.39	1.18	1.65	0.90
	=0.3413	=0.5073	≈0.46	>0.5	>0.5	≈0.49	=0.5565
Scrabble Scores	1.08	1.08	1.61	1.26	1.07	1.10	0.036
	=0.5579	=0.5579	≈0.39	>0.5	>0.5	>0.5	=0.86

In terms of detecting shifts, most of the tests are in agreement on whether or not to reject H_0 in favor of H_A . However, on the Pistons dataset where the change occurs at $\frac{k^*}{n} \approx 0.08$, only the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ - and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -based tests successfully detect the change, whilst all the other tests fail to reject H_0 in favor of H_A . This occurs despite all the tests (except the incorrectly specified versions of our tests, notably the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ - and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -based tests) agreeing to reject H_0 in favor of H_A on the same dataset except with the shift placed at roughly 92%. The comparisons on the Pistons dataset demonstrate two takeaways: the increased sensitivity on the extreme tails of the correctly specified versions of our tests over the $T_n^{q_1}$ -, $L_n^{q_2}$ -, and the H_n -based tests, as well as the importance of utilizing the correctly specified version of our tests.

In terms of detecting the homogeneous processes, all of the tests do not reject H_0 in favor of H_A , however the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests often exhibit higher p-values than the other tests. In particular, the smaller p-values of the $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -, $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ -, $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -, and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -based tests likely have to do with some spikes in the time-series plots, which may influence the resulting test statistic. Potentially, the truncated suprema of the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ - and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -based tests include the portion of the dataset with a large spike, whereas the truncated suprema of the $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -, $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests do not.

3.5 Conclusions of Chapter 3

Using what we have learned in the previous chapter surrounding the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests to find powerful $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with controlled type I errors yielded a significant improvement over the best $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests obtained in Chapter 2. The empirical powers of our best Chapter 3 tests, namely the $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ - and $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests with $\kappa = 0.75$, are higher for a change location in the range $0.25 \leq \frac{k^*}{n} \leq 0.75$. As a result, our best $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests are much more recommendable for smaller n . Overall, the $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests found in Gallanosa and Martsynyuk (2026) represent very well-rounded tests that should be recommended especially if one has no prior knowledge about which side of the dataset a change may occur. However, if one has such knowledge, then we can recommend our $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ - and $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests with $\kappa = 0.75$ for any n , regardless of the underlying distribution of the data. A small exception occurs when $n = 100$, and we suspect a change at $\frac{k^*}{n} = 0.05, 0.15$ or $\frac{k^*}{n} = 0.85, 0.95$. In this case, the $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ - and $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ -based tests can be recommended respectively, regardless of the underlying distribution of the data.

In terms of deciding among the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ -, $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -, or the $C_n^{[t_1(n), 1-t_2(n)]}$ -

based tests, if one has no prior knowledge of where the change may have occurred, then the $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ - and $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests with $\kappa = 0.75$ are to be recommended regardless of the sample size due to having more balanced power functions. *However, if one is confident that a change is likely to have occurred on the extreme left-hand tail ($0.05 \leq \frac{k^*}{n} \leq 0.15$) of the sample, the $K_n^{[d_n, 1-d_n^{0.6}]}$ -based test is to be recommended, provided $n = 100$. If one has a similar suspicion for the extreme right-hand tail ($0.85 \leq \frac{k^*}{n} \leq 0.95$) of the dataset, then the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ -based test is to be recommended, again, provided $n = 100$.*

When compared with real-world data, the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ -, $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -, $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -, $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -, and L_n^{q2} -based tests exhibit similar trends showcased in our simulation studies. In particular, the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ - and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ -based tests perform well when one suspects a change on the right-hand side of the dataset, whereas the $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ - and $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests perform well when one suspects a change on the left-hand side of the dataset. The L_n^{q2} -based test appears to be more sensitive than the T_n^{q1} -based test with respect to their performance on these real-world datasets. This matches the conclusions in Gallanosa and Martsynyuk (2026).

Further investigations could be done with $0.5 < \kappa \leq 0.6$ as these tests have the potential to be similar to the T_n^{q1} - and L_n^{q2} -based tests, where small

improvements could possibly be had for the power for a change near the middle of the dataset, as well as for the convergence of the type I errors on the heavy-tailed, asymmetric distributions. However, since preliminary investigations appeared to decrease the rate of convergence of the power functions on the extreme tails, we did not include a thorough investigation of these tests in this paper.

Chapter 4

Conclusions

In this thesis, we studied the finite-sample performance of various non-parametric tests for at most one change in the mean based upon different renormalizations of (4) and upon a related integral statistic in various simulation settings as well as on real-world datasets. The main goal of the thesis was to obtain all-around balanced tests that are based on the convergences in distribution in (7), (8), and (15) of the trimmed suprema of an appropriately weighted tied-down partial sums process. By choosing suitable trimming sequences d_n and e_n for (7) and (8), and appropriate trimming sequences $t_1(n)$ and $t_2(n)$ as well as the parameter κ for (15), we proposed powerful tests for change in the mean with controlled type I errors for a wide variety of distributions. Thus, we obtained the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ - and $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -based tests respectively from (7) and (8), and $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ - and $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests from (15). These tests were seen to have a significantly higher power for changes that occur

near the beginning or end of the observation period compared to the related powerful $T_n^{q_1}$ - and $L_n^{q_2}$ -based tests from Gallanosa and Martsynyuk (2026). Although the latter two tests appear to be more balanced overall, if one has some prior knowledge on which side, right or left, of the dataset a change is likely to occur, then using respectively the $C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$ - or $C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$ -based tests provides an improvement on the empirical powers for a shift on that side. Whilst for the most part, these two statistics appear to outperform the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ - and $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -based tests, if one is confident that a shift will occur on the extreme left-hand tail of the sample ($0.05 \leq \frac{k^*}{n} \leq 0.15$), then using the $K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$ -based test is preferred provided $n = 100$. On the other hand, if one suspects a change on the extreme right-hand tail of the dataset ($0.85 \leq \frac{k^*}{n} \leq 0.95$), then using the $J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$ -based test will provide a slight improvement in power over the other tests, once again provided that $n = 100$. The test comparisons on the real-world dataset mostly agree with the simulation studies.

Chapter 5

Appendix: Chapter 2 Extra Figures

5.1 Companion figures to Figures 2.1–2.3

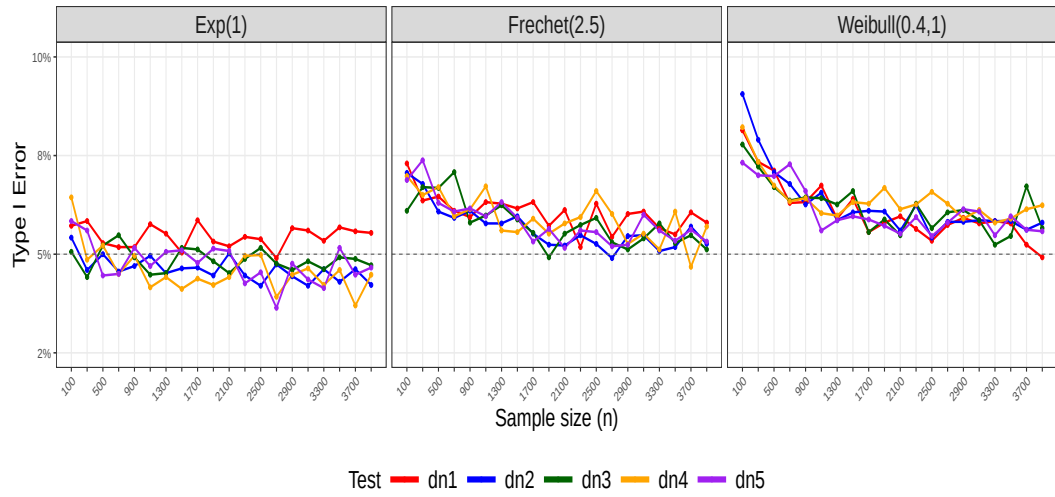


Figure 5.1: Empirical type I errors of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}$ of (20) and $e_n = d_n^{1-0.6}$, $i = \overline{1, 5}$, on Exp(1), Frechet(2.5), and Weibull (0.4,1).

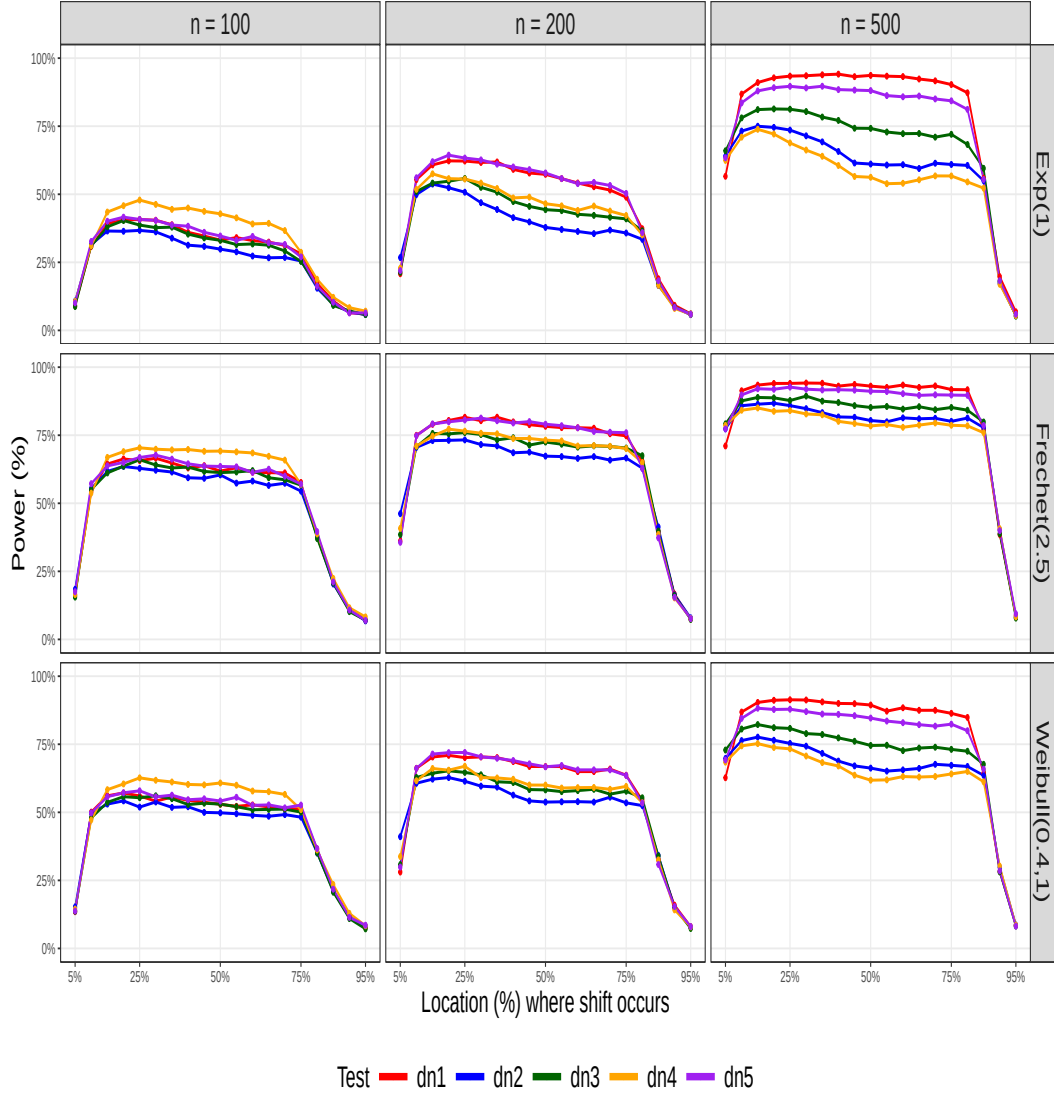


Figure 5.2: Empirical powers for $n = 100, 200, 500$ of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}$ of (20) and $e_n = d_n^{1-0.6}$, $i = \overline{1, 5}$, on $\text{Exp}(1)$, $\text{Frechet}(2.5)$, and $\text{Weibull}(0.4, 1)$.

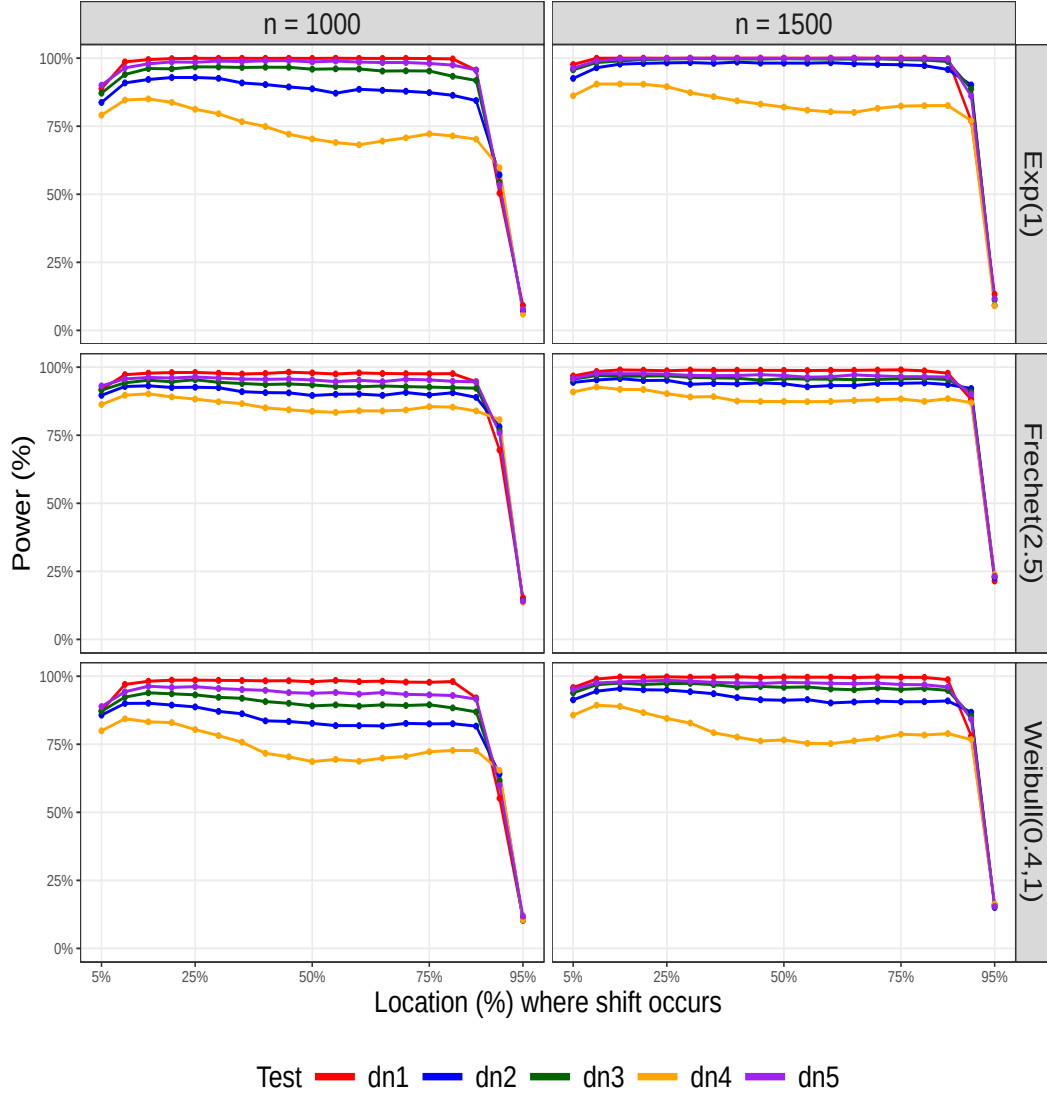


Figure 5.3: Empirical powers for $n = 1000, 1500$ of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}$ of (20) and $e_n = d_n^{1-0.6}$, $i = 1, 5$, on Exp(1), Frechet(2.5), and Weibull (0.4,1).

5.2 Companion figures to Figures 2.4–2.6

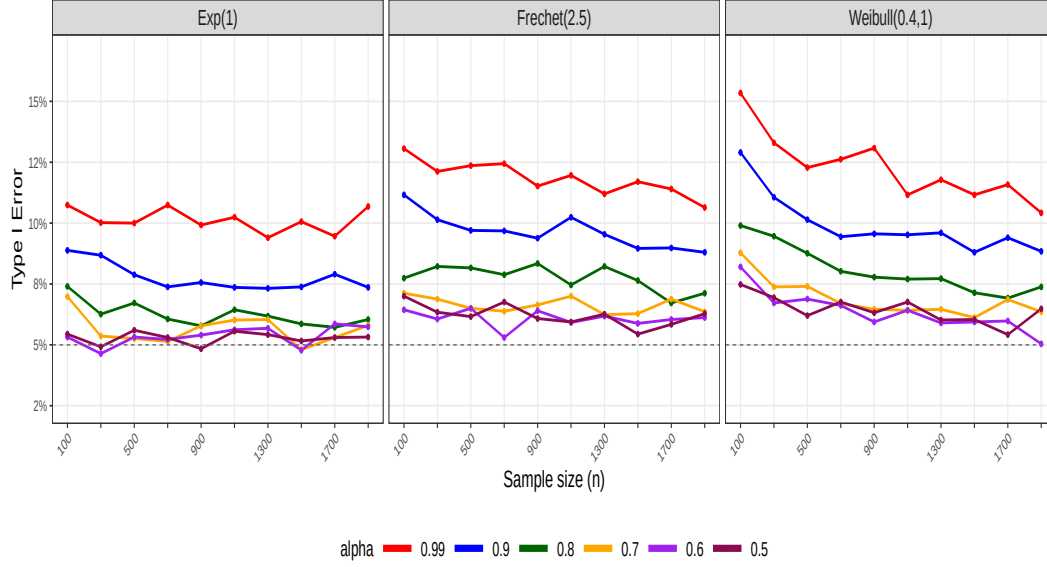


Figure 5.4: Empirical type I errors for the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-\alpha}$, for $\alpha = 0.99, 0.9, 0.8, 0.7, 0.6, 0.5$, on Exp(1), Frechet(2.5), and Weibull (0.4,1).

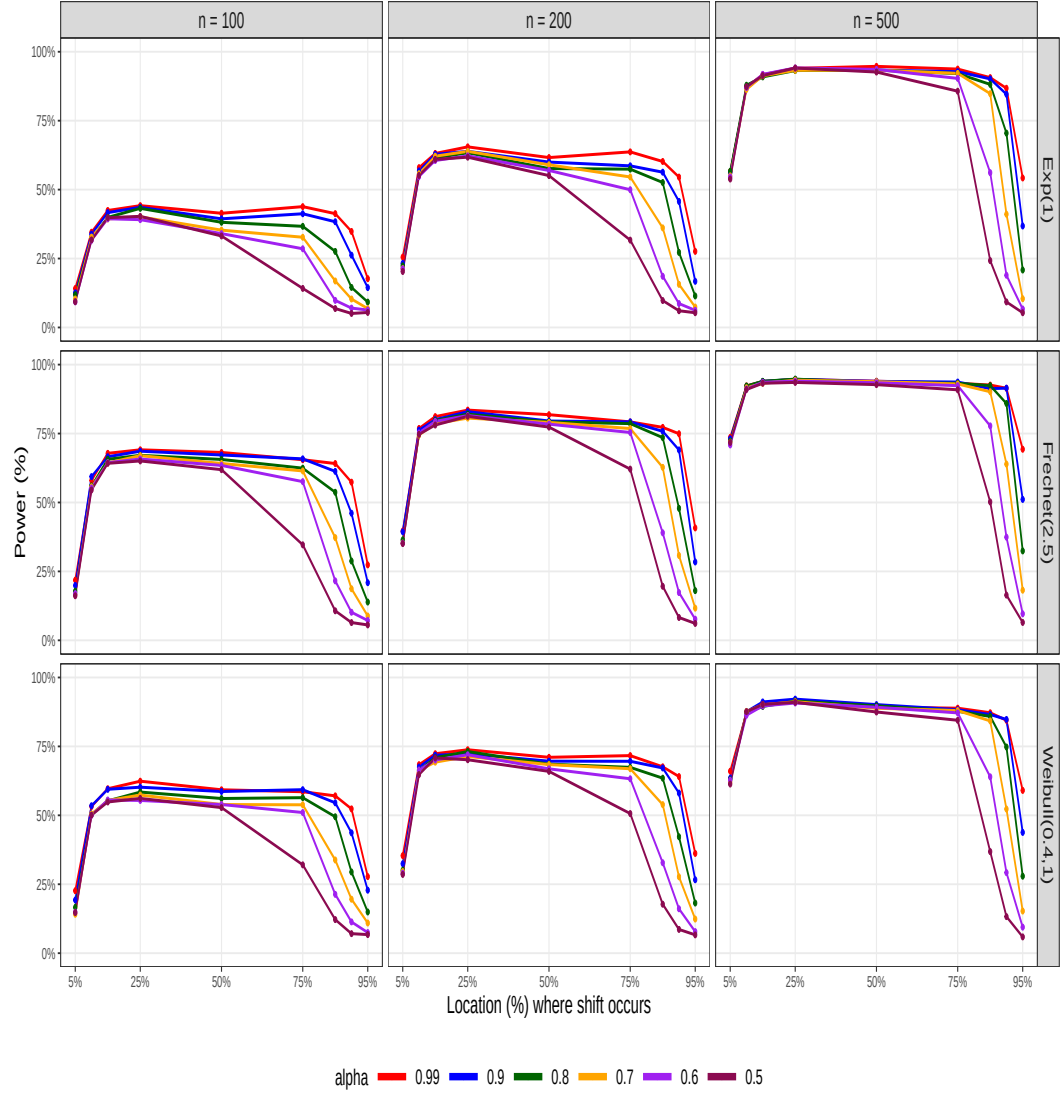


Figure 5.5: Empirical powers for $n = 100, 200, 500$ for the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-\alpha}$, for $\alpha = 0.99, 0.9, 0.8, 0.7, 0.6, 0.5$, on $\text{Exp}(1)$, $\text{Frechet}(2.5)$, and $\text{Weibull}(0.4, 1)$.

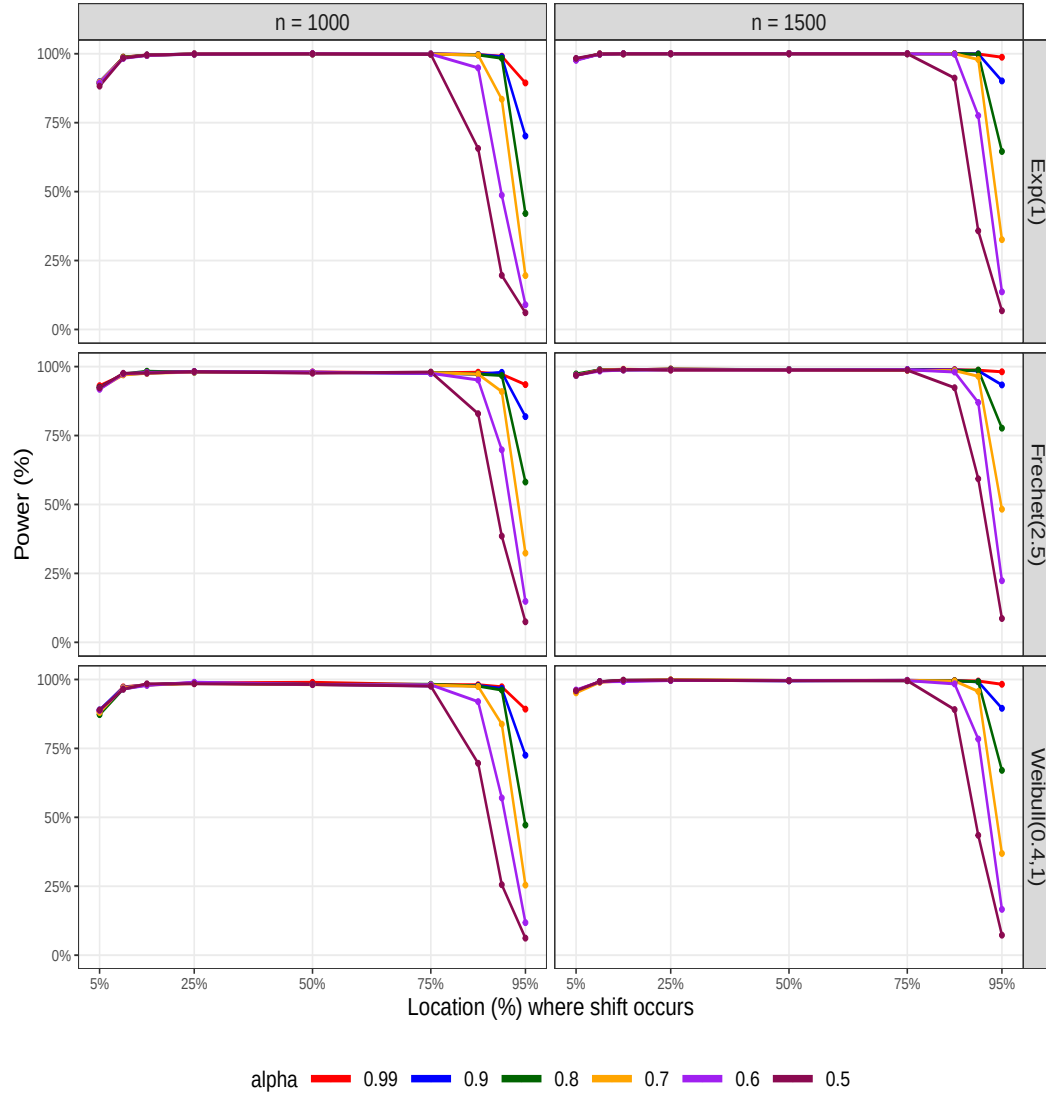


Figure 5.6: Empirical powers for $n = 1000, 1500$ for the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-\alpha}$, for $\alpha = 0.99, 0.9, 0.8, 0.7, 0.6, 0.5$, on $\text{Exp}(1)$, $\text{Frechet}(2.5)$, and $\text{Weibull}(0.4,1)$.

5.3 Companion figures to Figures 2.7–2.9

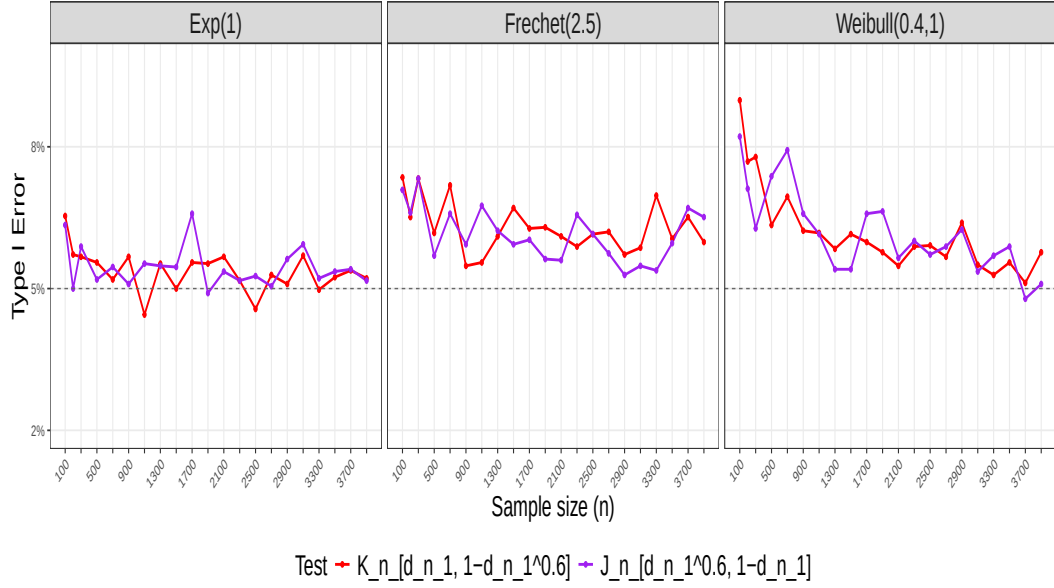


Figure 5.7: Empirical type I errors for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - vs. $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_1}$ of (20) and $e_n = d_n^{1-0.6}$ on Exp(1), Frechet(2.5), and Weibull(0.4,1).

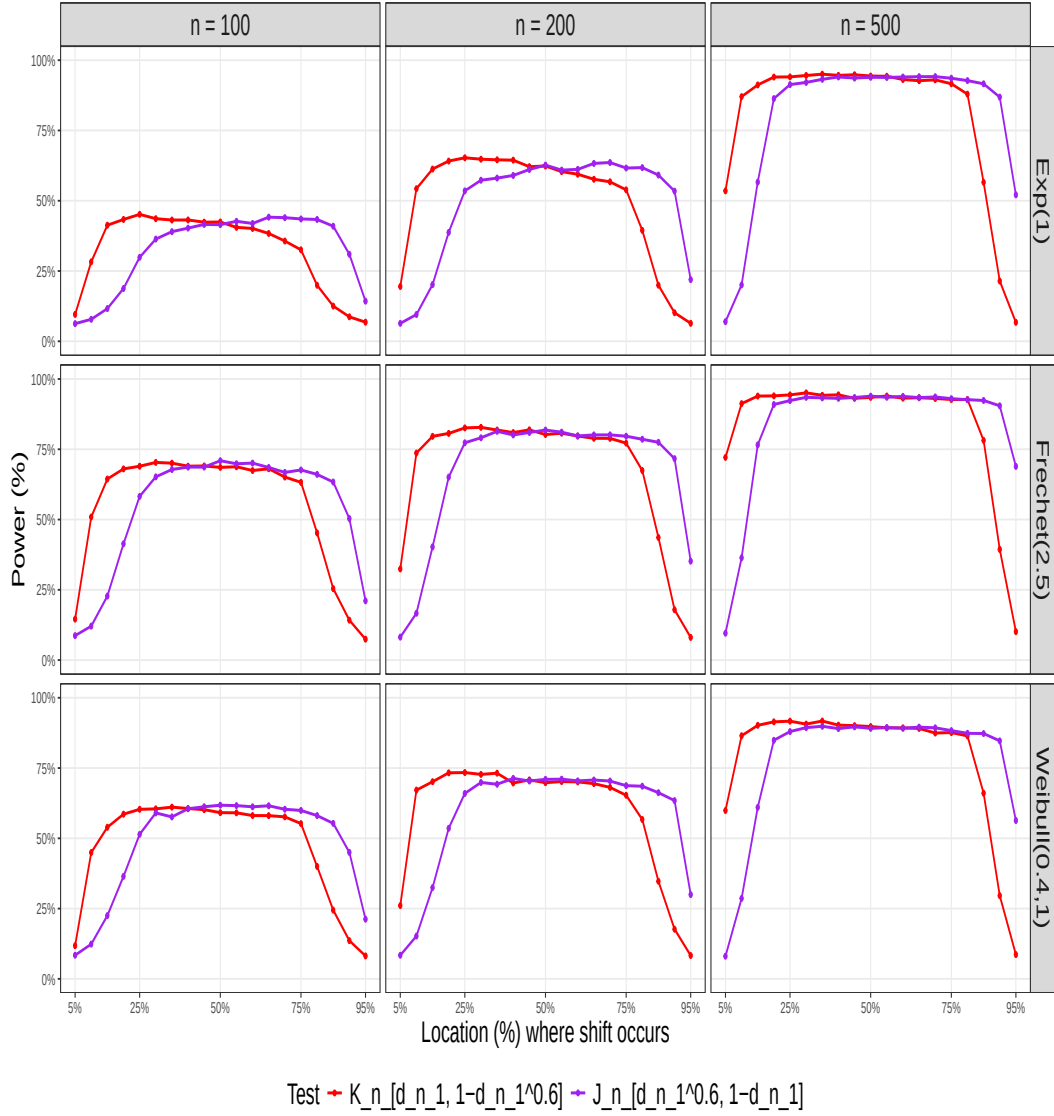


Figure 5.8: Empirical powers for $n = 100, 200, 500$ for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - vs. $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_1}$ of (20) and $e_n = d_n^{1-0.6}$ on Exp(1), Frechet(2.5), and Weibull(0.4,1).

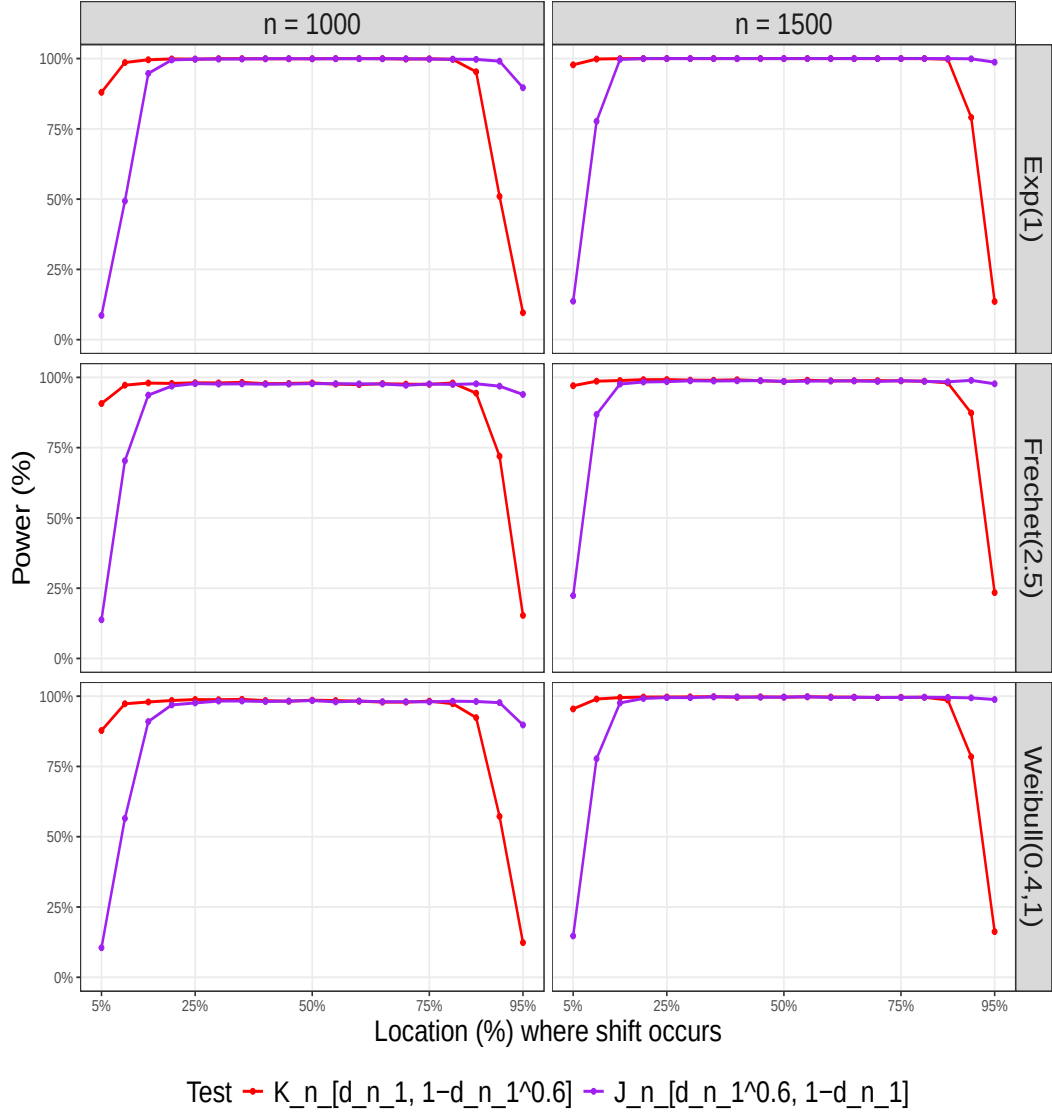


Figure 5.9: Empirical powers for $n = 1000, 1500$ for the $J_n^{[d_n, 1-d_n]}$ - vs. $K_n^{[d_n, 1-d_n]}$ - based tests with $d_n = d_{n_1}$ of (20) and $e_n = d_n^{1-0.6}$ on Exp(1), Frechet(2.5), and Weibull(0.4,1).

5.4 Companion figures to Figures 2.10–2.12

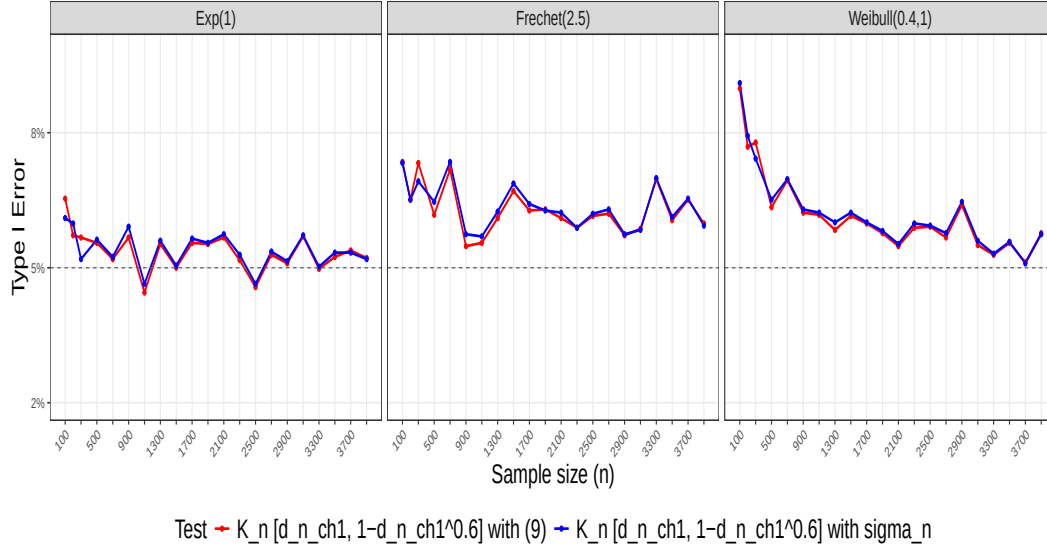


Figure 5.10: Empirical type I errors using $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9) in the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-0.6}$, on Exp(1), Frechet(2.5), and Weibull(0.4,1).

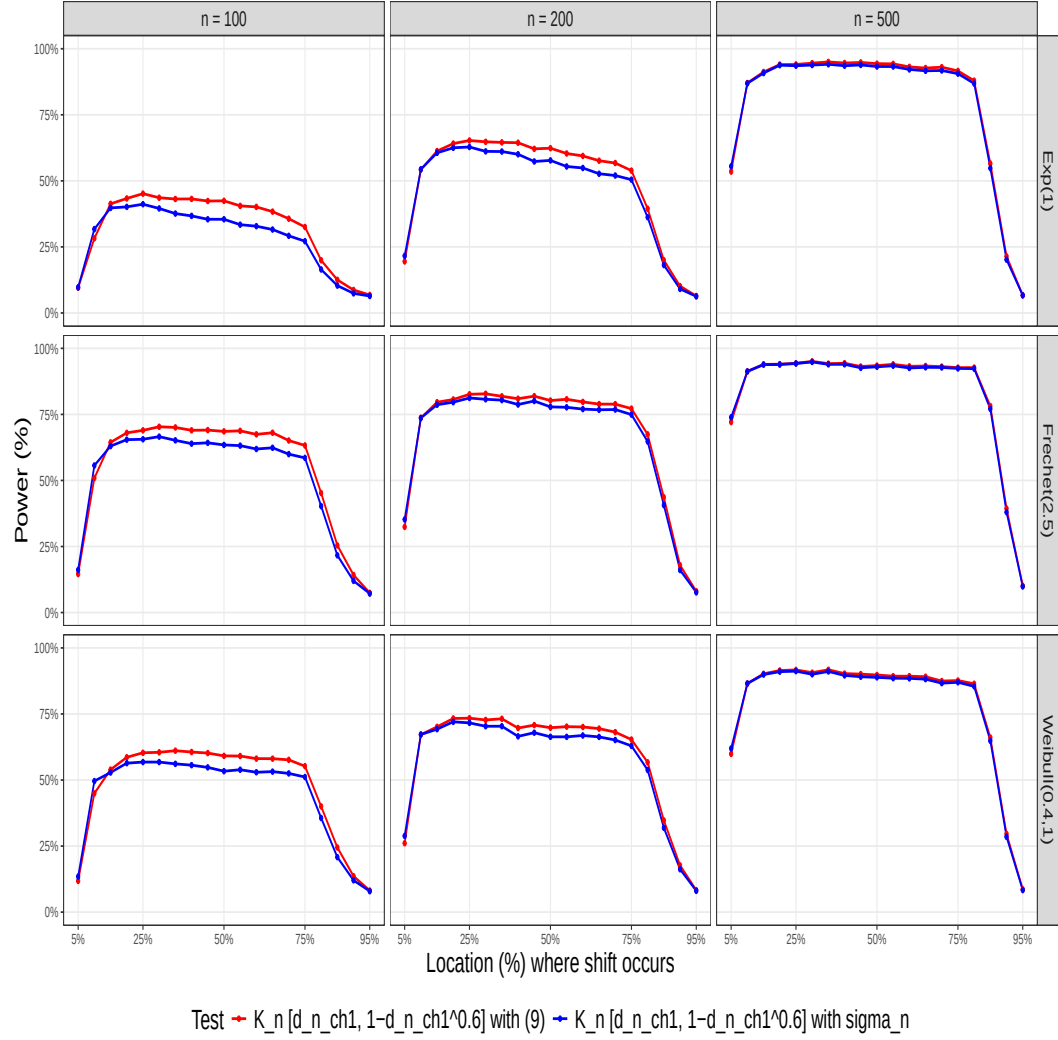
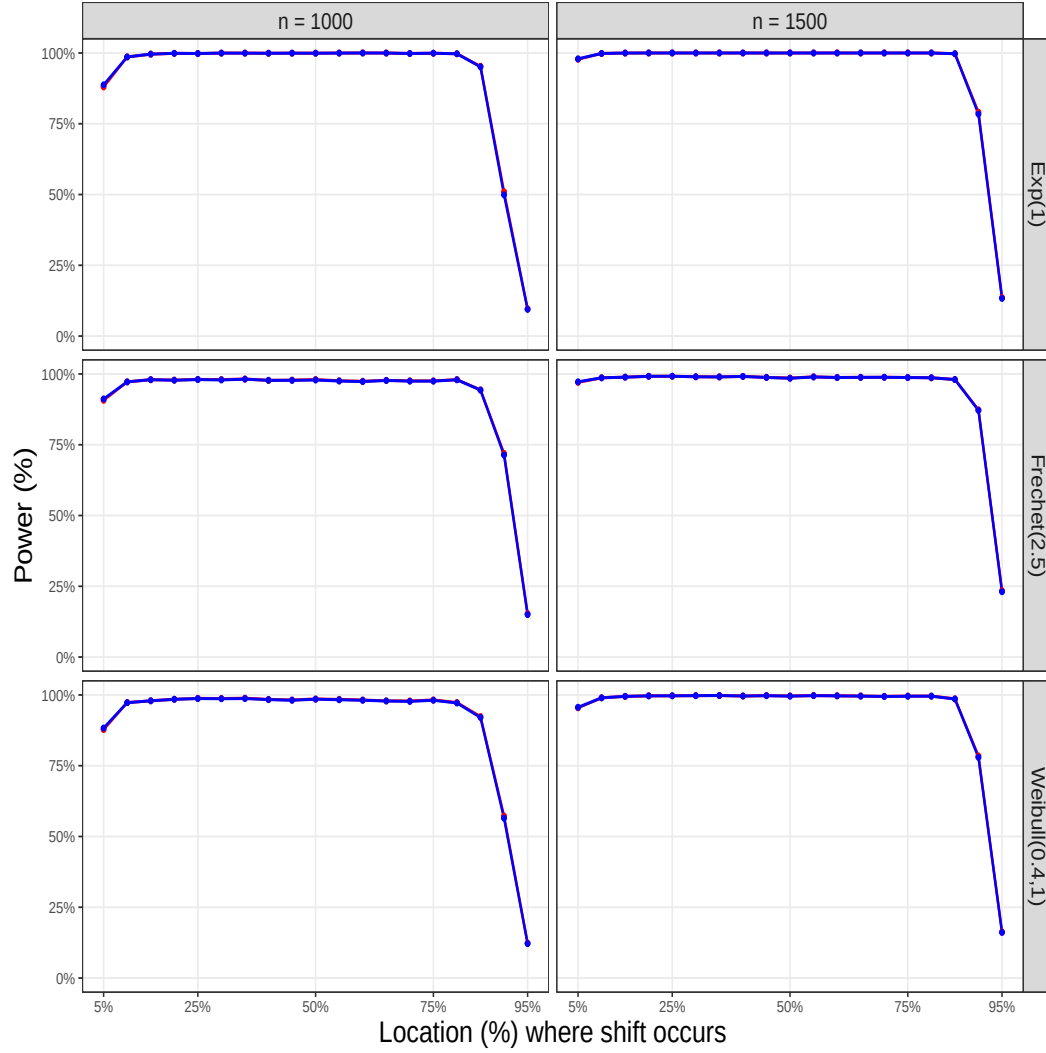


Figure 5.11: Empirical powers for $n = 100, 200, 500$ using $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9) in the $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-0.6}$, on Exp(1), Frechet(2.5), and Weibull(0.4,1).



Test —■— $K_n[d_n, 1-d_n \cdot e_n^{-0.6}]$ with (9) —●— $K_n[d_n, 1-d_n \cdot e_n^{-0.6}]$ with σ_n

Figure 5.12: Empirical powers for $n = 1000, 1500$ using $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9) in the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_1}$ of (20) and $e_n = d_{n_1}^{1-0.6}$, on Exp(1), Frechet(2.5), and Weibull(0.4,1).

5.5 Companion figures to Figures 2.13–2.15

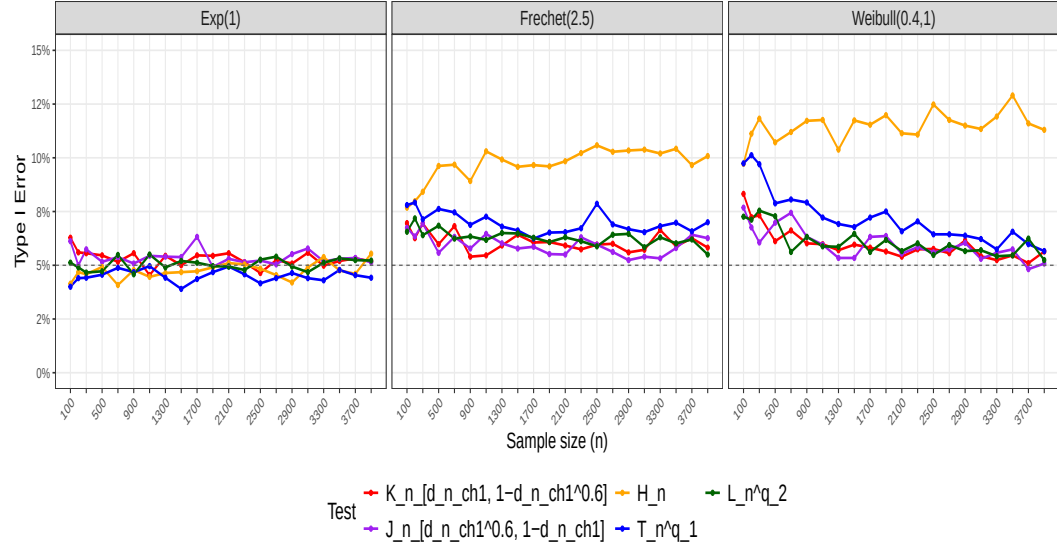


Figure 5.13: Empirical type I errors for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_n^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$ -, and H_n -based tests on Exp(1), Frechet(2.5), and Weibull(0.4,1).

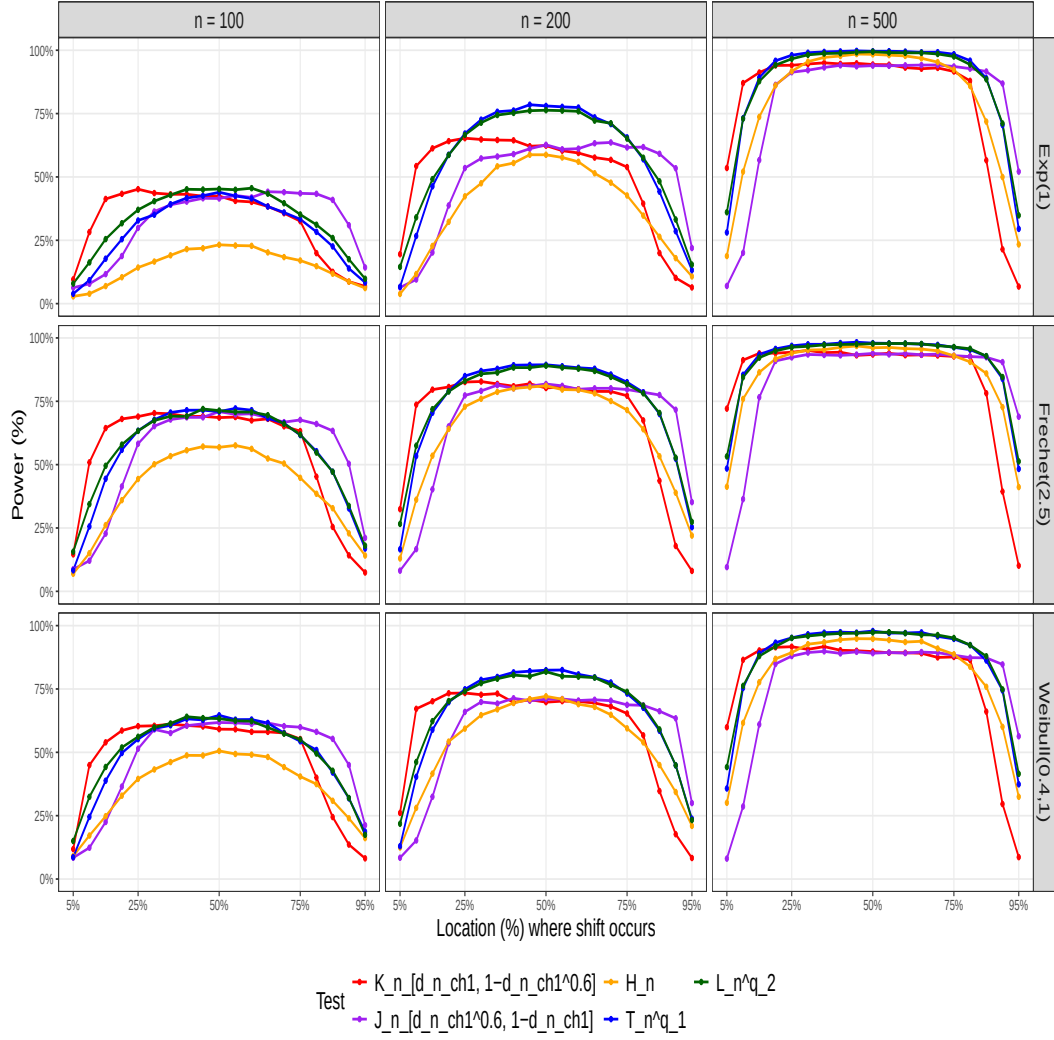


Figure 5.14: Empirical powers for $n = 100, 200, 500$ for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_n^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$ -, and H_n -based tests on Exp(1), Frechet(2.5), and Weibull(0.4,1), for a shift of 0.5σ

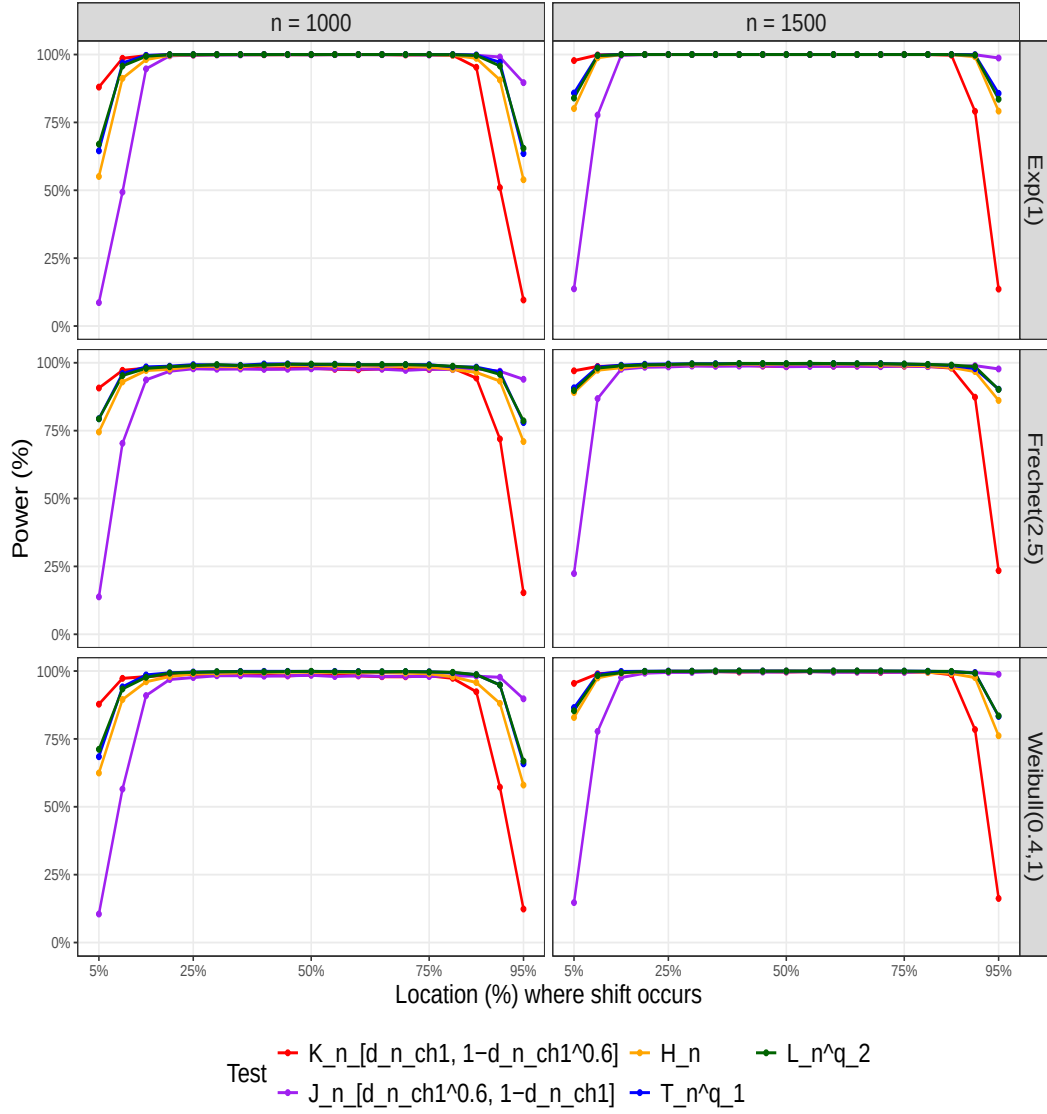


Figure 5.15: Empirical powers for $n = 1000, 1500$ for the $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_n^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$ -, and H_n -based tests on Exp(1), Frechet(2.5), and Weibull(0.4,1), for a shift of 0.5σ

Chapter 6

Appendix: Chapter 3 Extra Figures and Tables

6.1 Companion figures to Figures 3.1–3.3

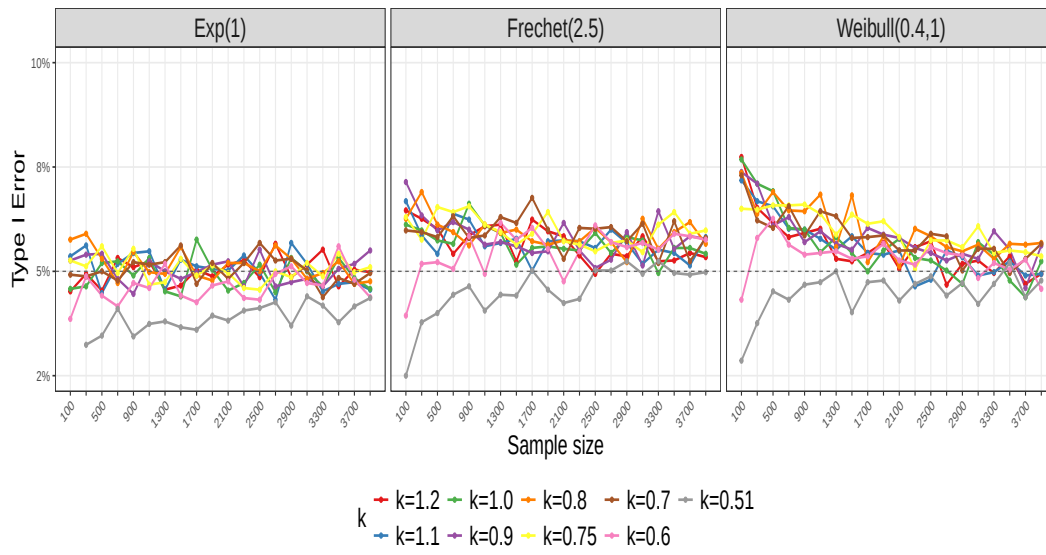


Figure 6.1: Empirical type I errors of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_{ch1}}$ of (20), $t_2(n) = 1 - d_{n_{ch1}}/e_n$, $e_n = d_{n_{ch1}}^{1-0.6}$, and $\kappa = 1.2, 1.1, 1.0, 0.9, 0.8, 0.75, 0.7, 0.6, 0.51$ across Exp(1), Frechet(2.5), and Weibull(0.4,1).

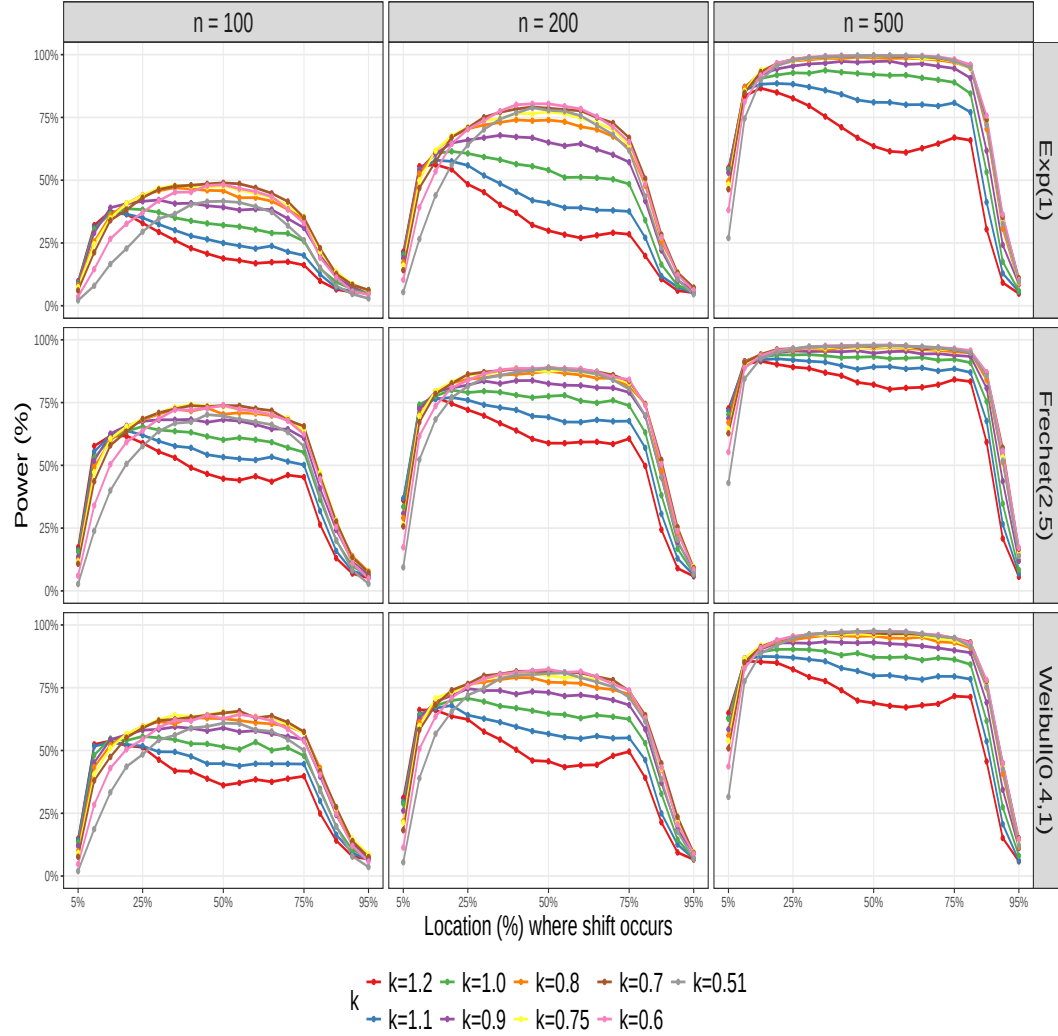


Figure 6.2: Empirical powers for $n = 100, 200, 500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_1}$ of (20), $t_2(n) = 1 - d_{n_1}/e_n$, $e_n = d_{n_1}^{1-0.6}$, and $\kappa = 1.2, 1.1, 1.0, 0.9, 0.8, 0.75, 0.7, 0.6, 0.51$ across $\text{Exp}(1)$, $\text{Frechet}(2.5)$, and $\text{Weibull}(0.4, 1)$.

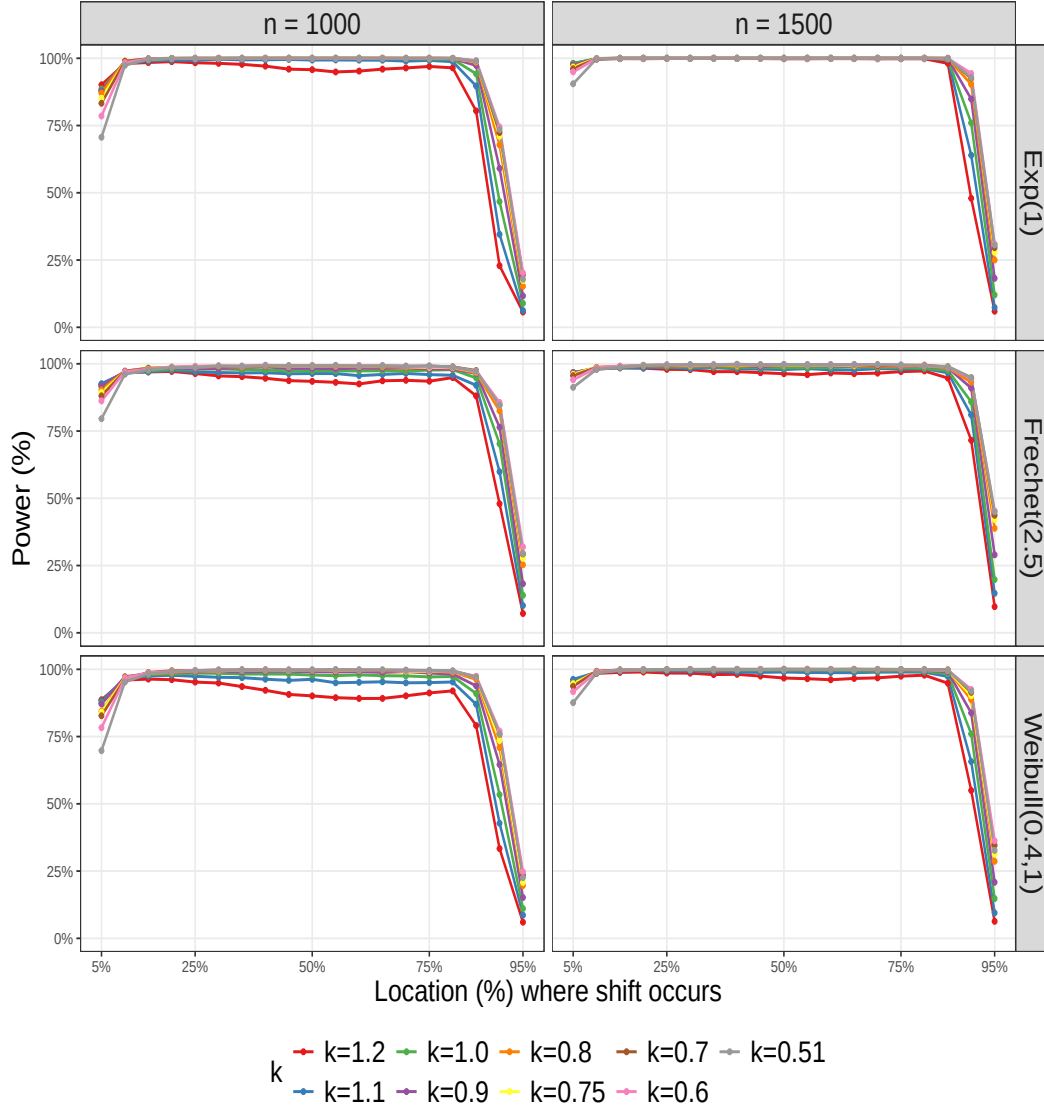


Figure 6.3: Empirical powers for $n = 1000, 1500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based test with $t_1(n) = d_{n_1}$ of (20), $t_2(n) = 1 - d_{n_1}/e_n$, $e_n = d_{n_1}^{1-0.6}$, and $\kappa = 1.2, 1.1, 1.0, 0.9, 0.8, 0.75, 0.7, 0.6, 0.51$ across Exp(1), Frechet(2.5), and Weibull(0.4,1).

6.2 Companion figure to Figures 3.4–3.9

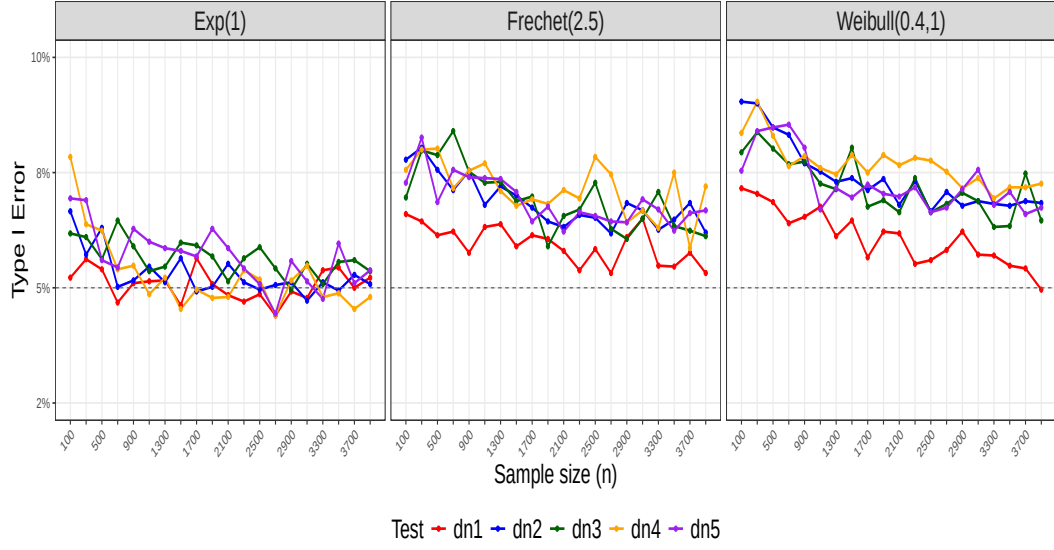


Figure 6.4: Empirical type I errors of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_n$, and $e_n = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 0.75$, across Exp(1), Frechet(2.5), and Weibull(0.4,1).

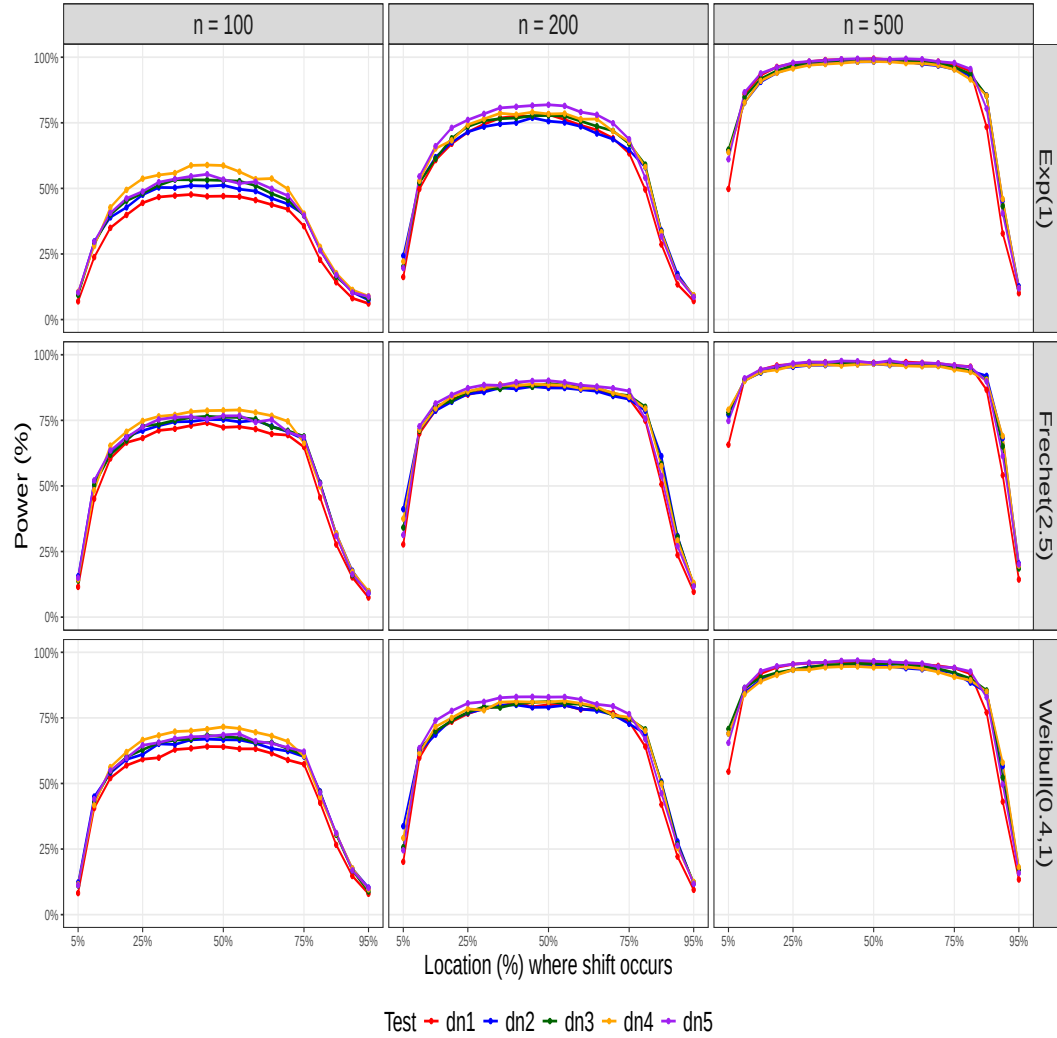


Figure 6.5: Empirical powers for $n = 100, 200, 500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_n$, and $e_n = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 0.75$, across $\text{Exp}(1)$, $\text{Frechet}(2.5)$, and $\text{Weibull}(0.4, 1)$.

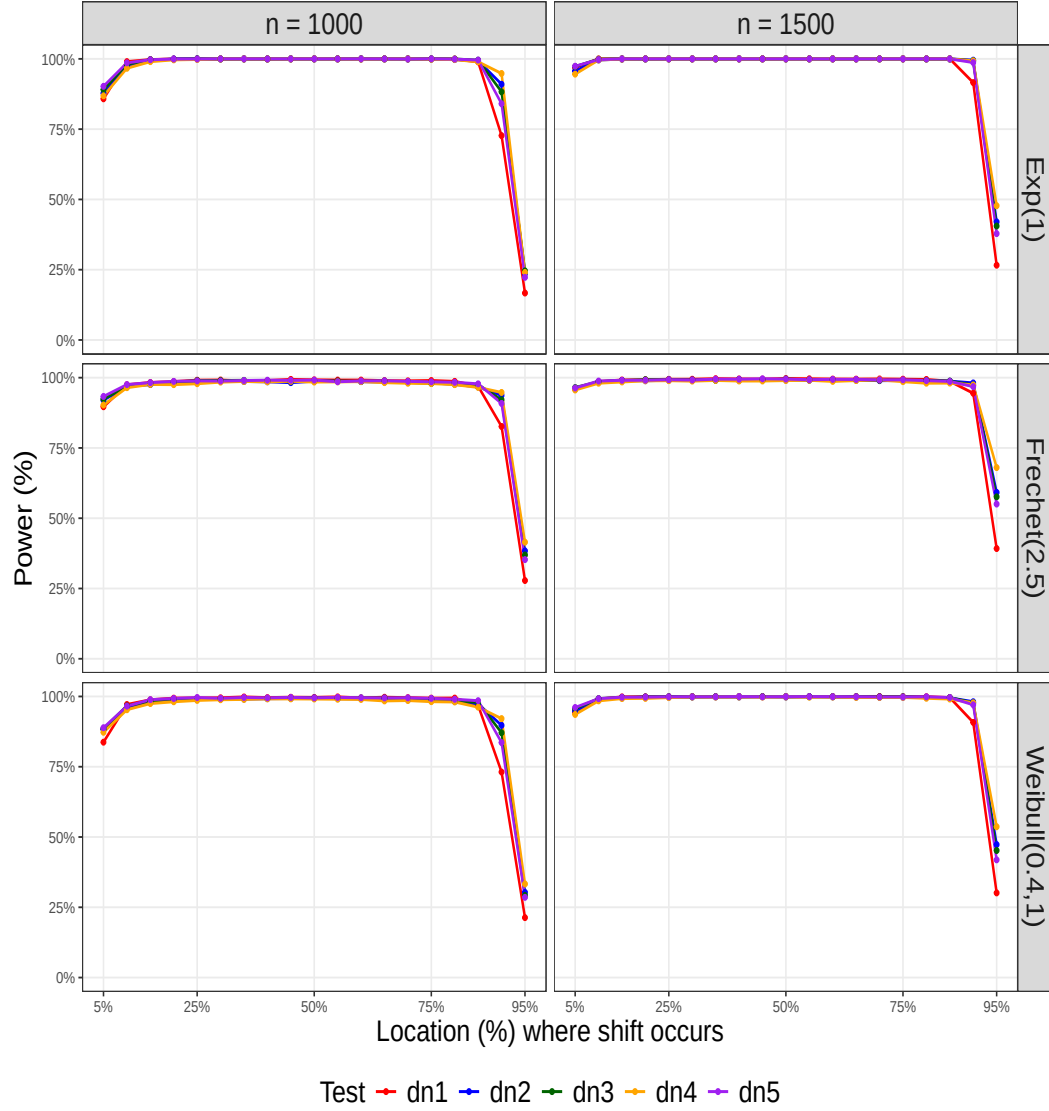


Figure 6.6: Empirical powers for $n = 1000, 1500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_n$, and $e_n = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 0.75$, across Exp(1), Frechet(2.5), and Weibull(0.4,1).

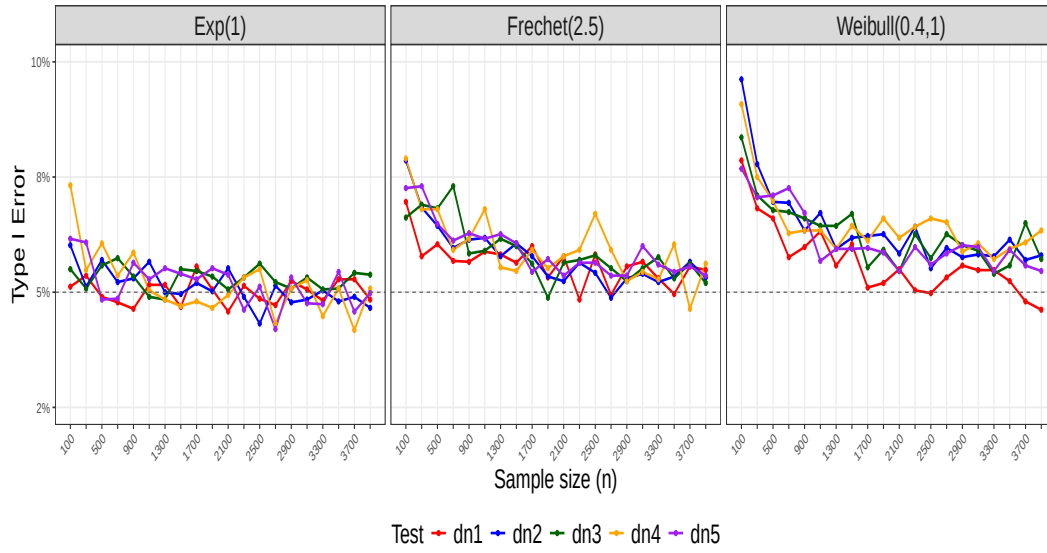


Figure 6.7: Empirical type I errors of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_n$, and $e_n = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 1.2$, across Exp(1), Frechet(2.5), and Weibull(0.4,1).

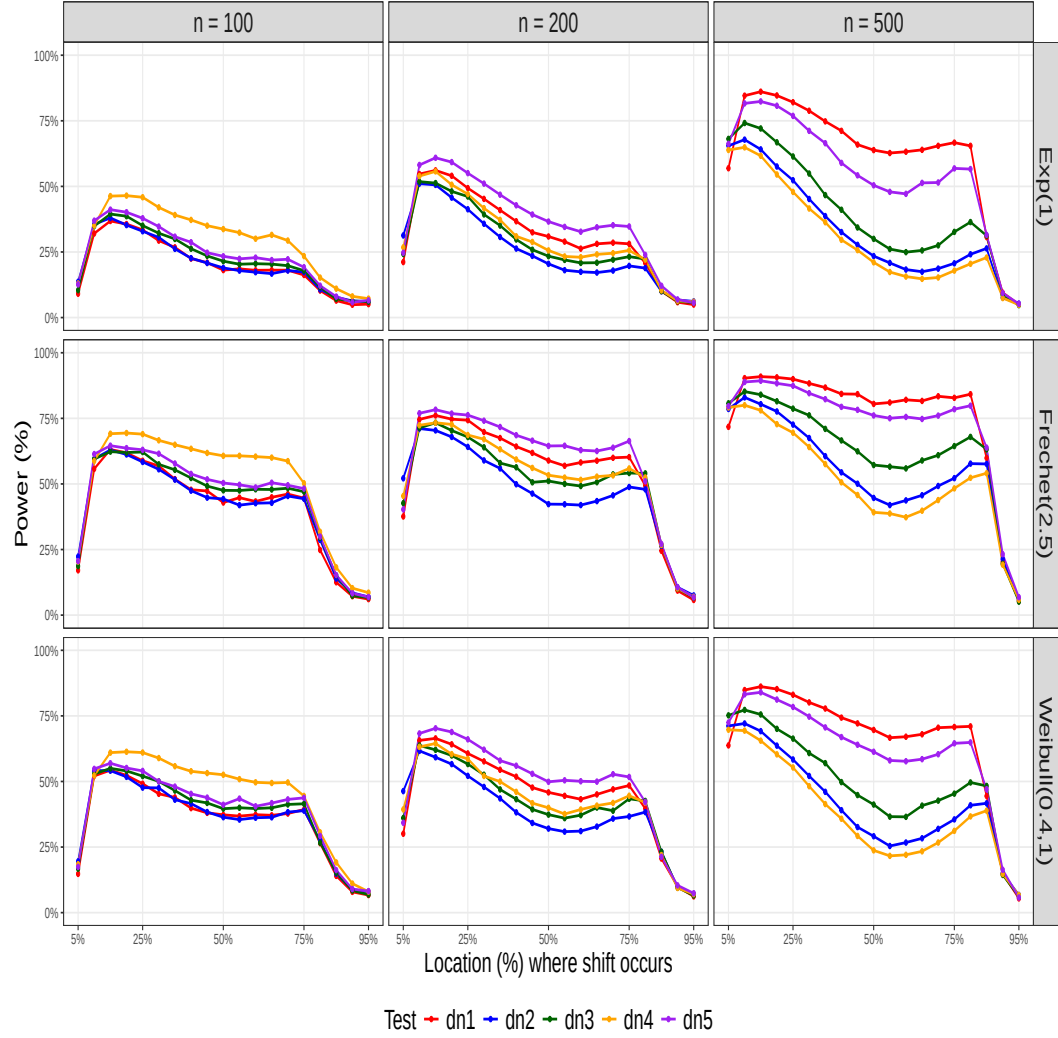


Figure 6.8: Empirical powers for $n = 100, 200, 500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_n$, and $e_n = d_{n_i}^{1-0.6}$, $i = \overline{1, 5}$, and $\kappa = 1.2$, across Exp(1), Frechet(2.5), and Weibull(0.4,1).

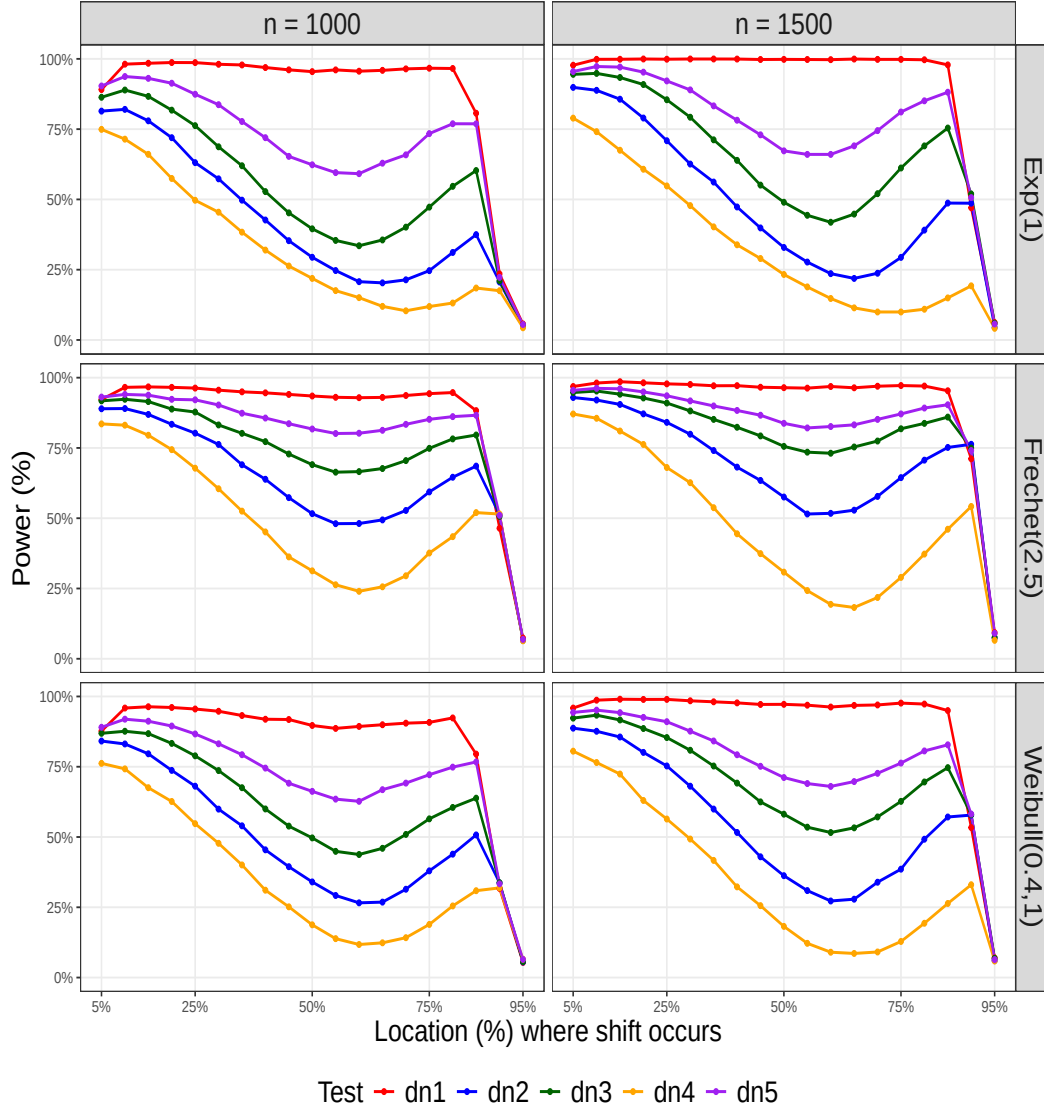


Figure 6.9: Empirical powers for $n = 1000, 1500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests tests with $t_1(n) = d_{n_i}$ of (20), $t_2(n) = 1 - d_{n_i}/e_n$, and $e_n = d_{n_i}^{1-0.6}$, and $\kappa = 1.2$, across Exp(1), Frechet(2.5), and Weibull(0.4,1).

6.3 Companion figures to Figures 3.10–3.12

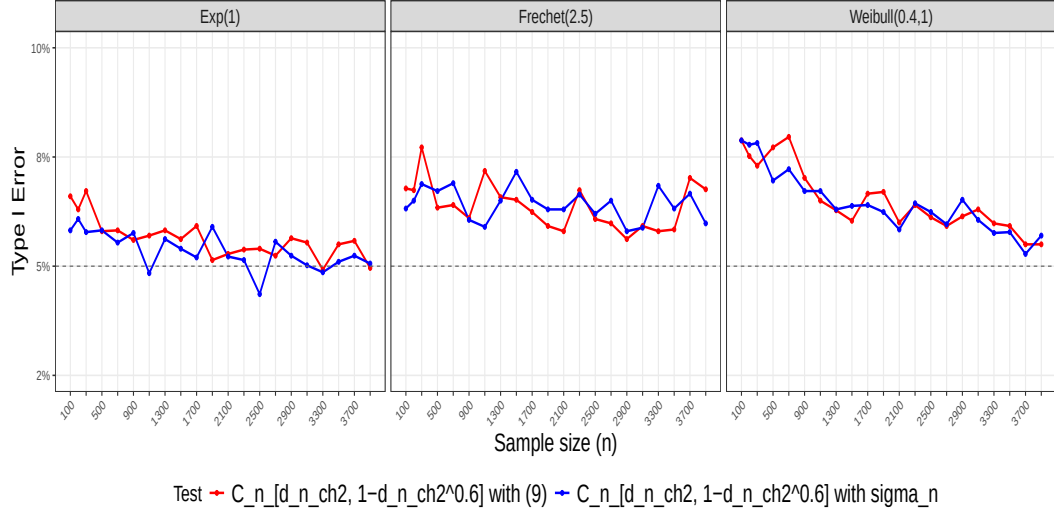


Figure 6.10: Empirical type I errors of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, and $e_n = d_{n_6}^{1-0.6}$ comparing $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9) on Exp(1), Frechet(2.5), and Weibull(0.4,1).

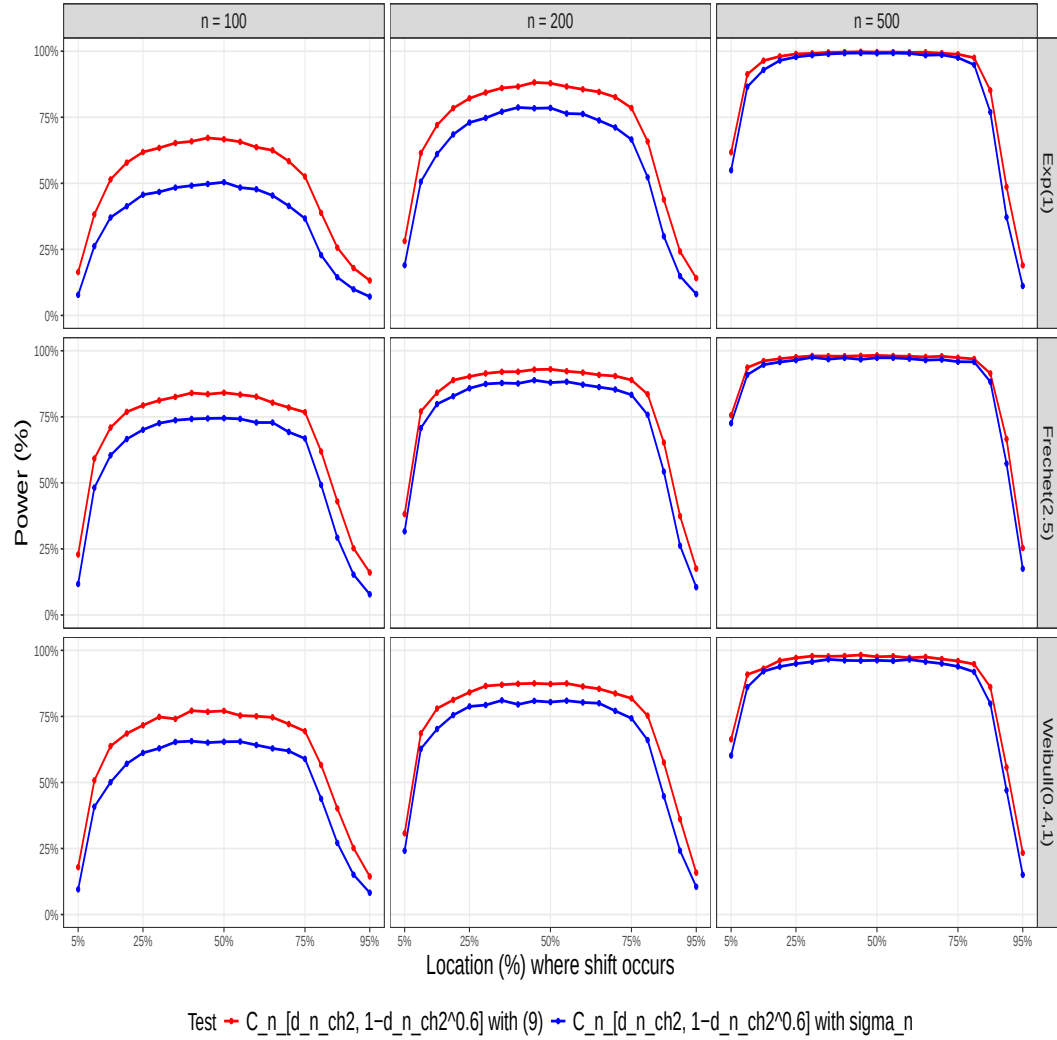
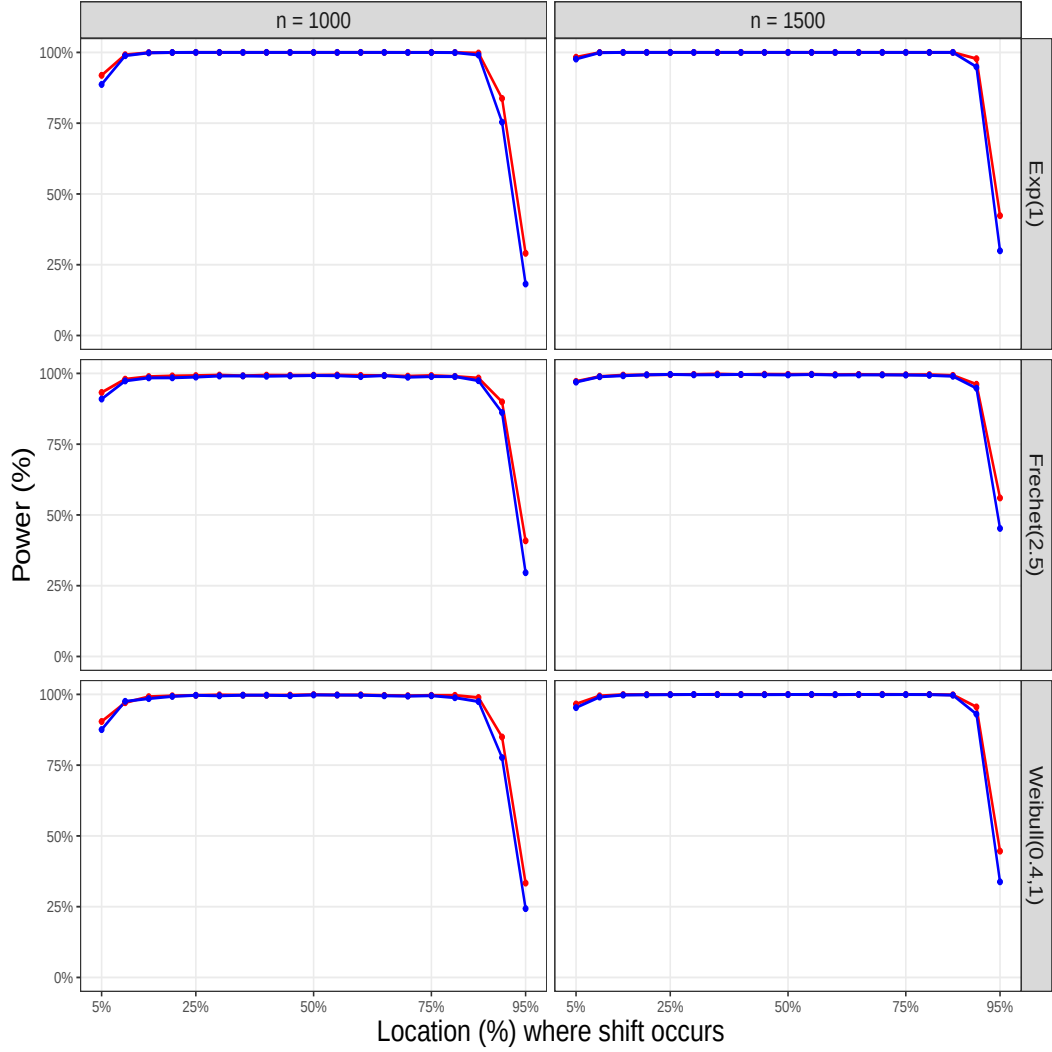


Figure 6.11: Empirical powers for $n = 100, 200, 500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, $e_n = d_{n_6}^{1-0.6}$ comparing $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9) on Exp(1), Frechet(2.5), and Weibull(0.4,1).



Test $\rightarrow C_n[d_n, 1-d_n^{0.6}]$ with (9) $\rightarrow C_n[d_n, 1-d_n^{0.6}]$ with σ_n

Figure 6.12: Empirical powers for $n = 1000, 1500$ of the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, $e_n = d_{n_6}^{1-0.6}$ comparing $\hat{\sigma}_n$ vs. $\hat{\sigma}_{[nt+1]}$ of (9) on Exp(1), Frechet(2.5), and Weibull(0.4,1).

6.4 Companions to Tables 3.3–3.5 and Figures 3.13–3.17

Table 6.1: Empirical powers for a change at k^*/n on Exponential(1), with $\Delta = 0.5\sigma$

n	Test statistic	$\frac{k^*}{n} = 0.05$	0.15	0.25	0.50	0.75	0.85	0.95
100	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.0956	0.4130	0.4516	0.4244	0.3254	0.1254	0.0680
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0628	0.1164	0.2988	0.4150	0.4352	0.4094	0.1428
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.0778	0.3706	0.4568	0.5040	0.3668	0.1444	0.0710
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0642	0.1396	0.3578	0.5064	0.4450	0.3654	0.1252
	$T_n^{q_1}$	0.0388	0.1774	0.3280	0.4394	0.3330	0.2258	0.0834
	$L_n^{q_2}$	0.0792	0.2546	0.3702	0.4524	0.3518	0.2590	0.0982
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.1950	0.6128	0.6528	0.6236	0.5388	0.1996	0.0640
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0638	0.2018	0.5350	0.6268	0.6164	0.5914	0.2194
200	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.1902	0.6104	0.7298	0.7852	0.6658	0.2994	0.0806
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0726	0.2864	0.6684	0.7786	0.7062	0.6030	0.2276
	$T_n^{q_1}$	0.0664	0.4628	0.6712	0.7802	0.6560	0.4418	0.1320
	$L_n^{q_2}$	0.1450	0.4912	0.6658	0.7632	0.6514	0.4826	0.1540
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.5352	0.9120	0.9406	0.9436	0.9162	0.5656	0.0674
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0698	0.5664	0.9132	0.9392	0.9358	0.9160	0.5210
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.5492	0.9290	0.9782	0.9920	0.9756	0.7694	0.1114
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1018	0.7502	0.9704	0.9920	0.9762	0.9356	0.5452
500	$T_n^{q_1}$	0.2808	0.8928	0.9802	0.9958	0.9842	0.8896	0.2950
	$L_n^{q_2}$	0.3606	0.8776	0.9660	0.9942	0.9752	0.8852	0.3482
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.8800	0.9958	0.9980	0.9994	0.9992	0.9530	0.0960
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0862	0.9472	0.9980	0.9992	0.9982	0.9972	0.8964
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.8868	0.9986	1.0000	1.0000	0.9998	0.9908	0.1820
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1794	0.9878	0.9998	1.0000	0.9998	0.9980	0.8984
	$T_n^{q_1}$	0.6452	0.9970	0.9998	0.9998	1.0000	0.9984	0.6352
	$L_n^{q_2}$	0.6698	0.9934	0.9996	1.0000	0.9998	0.9968	0.6546
1000	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9778	0.9998	1.0000	1.0000	1.0000	0.9976	0.1360
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1370	0.9970	1.0000	1.0000	1.0000	1.0000	0.9872
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9766	1.0000	1.0000	1.0000	1.0000	1.0000	0.2992
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.2960	0.9998	1.0000	1.0000	1.0000	1.0000	0.9822
	$T_n^{q_1}$	0.8582	1.0000	1.0000	1.0000	1.0000	1.0000	0.8570
	$L_n^{q_2}$	0.8396	0.9998	1.0000	1.0000	1.0000	1.0000	0.8354
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9778	0.9998	1.0000	1.0000	1.0000	0.9976	0.1360
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1370	0.9970	1.0000	1.0000	1.0000	1.0000	0.9872
1500	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9766	1.0000	1.0000	1.0000	1.0000	1.0000	0.2992
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.2960	0.9998	1.0000	1.0000	1.0000	1.0000	0.9822
	$T_n^{q_1}$	0.8582	1.0000	1.0000	1.0000	1.0000	1.0000	0.8570
	$L_n^{q_2}$	0.8396	0.9998	1.0000	1.0000	1.0000	1.0000	0.8354
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9778	0.9998	1.0000	1.0000	1.0000	0.9976	0.1360
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1370	0.9970	1.0000	1.0000	1.0000	1.0000	0.9872
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9766	1.0000	1.0000	1.0000	1.0000	1.0000	0.2992
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.2960	0.9998	1.0000	1.0000	1.0000	1.0000	0.9822

Table 6.2: Empirical powers for a change at k^*/n on Frechet(2.5), with $\Delta = 0.5\sigma$

n	Test statistics	$\frac{k^*}{n} = 0.05$	0.15	0.25	0.50	0.75	0.85	0.95
100	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.1458	0.6442	0.6896	0.6856	0.6324	0.2542	0.0744
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0870	0.2278	0.5824	0.7092	0.6764	0.6334	0.2108
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.1174	0.6044	0.7010	0.7448	0.6682	0.2924	0.0782
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0778	0.2774	0.6428	0.7502	0.6932	0.5996	0.1974
	$T_n^{q_1}$	0.0828	0.4442	0.6336	0.7092	0.6166	0.4718	0.1688
	$L_n^{q_2}$	0.1564	0.4954	0.6332	0.7132	0.6188	0.4718	0.1802
200	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.3242	0.7960	0.8260	0.8024	0.7718	0.4362	0.0806
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0816	0.4026	0.7730	0.8182	0.7964	0.7748	0.3520
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.3166	0.7982	0.8580	0.8796	0.8332	0.5428	0.1052
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1088	0.5140	0.8310	0.8782	0.8334	0.7798	0.3468
	$T_n^{q_1}$	0.1658	0.7048	0.8496	0.8940	0.8254	0.6998	0.2530
	$L_n^{q_2}$	0.2660	0.7188	0.8304	0.8904	0.8174	0.7038	0.2740
500	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.7210	0.9392	0.9434	0.9346	0.9270	0.7816	0.1012
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0956	0.7658	0.9232	0.9388	0.9300	0.9236	0.6890
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.7258	0.9472	0.9646	0.9734	0.9586	0.8826	0.1752
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1412	0.8558	0.9604	0.9690	0.9576	0.9390	0.7102
	$T_n^{q_1}$	0.4846	0.9346	0.9688	0.9794	0.9628	0.9278	0.4830
	$L_n^{q_2}$	0.5324	0.9222	0.9634	0.9774	0.9642	0.9282	0.5126
1000	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9070	0.9798	0.9808	0.9802	0.9754	0.9438	0.1532
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1378	0.9370	0.9778	0.9778	0.9764	0.9772	0.9392
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9094	0.9842	0.9870	0.9924	0.9888	0.9742	0.2960
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.2744	0.9696	0.9842	0.9908	0.9876	0.9784	0.9388
	$T_n^{q_1}$	0.7940	0.9848	0.9930	0.9930	0.9928	0.9840	0.7798
	$L_n^{q_2}$	0.7936	0.9794	0.9896	0.9952	0.9912	0.9818	0.7862
1500	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9704	0.9888	0.9920	0.9856	0.9876	0.9804	0.2344
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.2238	0.9762	0.9848	0.9856	0.9882	0.9842	0.9772
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9692	0.9916	0.9964	0.9946	0.9942	0.9898	0.4524
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.4276	0.9872	0.9954	0.9942	0.9932	0.9894	0.9718
	$T_n^{q_1}$	0.9078	0.9910	0.9954	0.9962	0.9950	0.9902	0.9026
	$L_n^{q_2}$	0.8978	0.9888	0.9938	0.9970	0.9956	0.9916	0.9018

Table 6.3: Empirical powers for a change at k^*/n on Weibull(0.4,1), with $\Delta = 0.5\sigma$

n	Test statistic	$\frac{k^*}{n} = 0.05$	0.15	0.25	0.50	0.75	0.85	0.95
100	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.1182	0.5394	0.6032	0.5914	0.5524	0.2446	0.0816
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0842	0.2252	0.5142	0.6178	0.5992	0.5532	0.2124
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.0954	0.5010	0.6120	0.6540	0.5892	0.2710	0.0826
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0830	0.2462	0.5652	0.6666	0.5980	0.5154	0.1990
	$T_n^{q_1}$	0.0864	0.3880	0.5520	0.6460	0.5436	0.4204	0.1874
	$L_n^{q_2}$	0.1500	0.4422	0.5614	0.6330	0.5508	0.4274	0.1744
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.2608	0.7014	0.7340	0.6980	0.6532	0.3472	0.0830
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0840	0.3246	0.6596	0.7098	0.6874	0.6624	0.2998
200	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.2416	0.7020	0.7876	0.8042	0.7428	0.4478	0.1052
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.0966	0.4180	0.7512	0.8078	0.7412	0.6672	0.2888
	$T_n^{q_1}$	0.1296	0.5890	0.7486	0.8242	0.7320	0.5850	0.2382
	$L_n^{q_2}$	0.2182	0.6230	0.7410	0.8182	0.7388	0.5904	0.2328
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.5990	0.9020	0.9168	0.8976	0.8764	0.6606	0.0864
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.0806	0.6102	0.8798	0.8912	0.8832	0.8728	0.5634
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.6020	0.9208	0.9494	0.9622	0.9390	0.7982	0.1506
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.1166	0.7780	0.9358	0.9584	0.9346	0.8980	0.5886
500	$T_n^{q_1}$	0.3574	0.8904	0.9518	0.9786	0.9476	0.8624	0.3738
	$L_n^{q_2}$	0.4418	0.8798	0.9512	0.9732	0.9512	0.8790	0.4152
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.8778	0.9794	0.9880	0.9858	0.9824	0.9236	0.1234
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1050	0.9096	0.9760	0.9844	0.9798	0.9810	0.8976
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.8756	0.9850	0.9966	0.9980	0.9952	0.9746	0.2432
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.2056	0.9584	0.9936	0.9974	0.9946	0.9816	0.9008
	$T_n^{q_1}$	0.6844	0.9856	0.9964	0.9988	0.9970	0.9862	0.6582
	$L_n^{q_2}$	0.7116	0.9770	0.9950	0.9990	0.9968	0.9868	0.6682
1000	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9546	0.9950	0.9970	0.9964	0.9960	0.9864	0.1624
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1472	0.9760	0.9950	0.9968	0.9952	0.9956	0.9880
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9532	0.9976	0.9990	0.9994	0.9994	0.9972	0.3378
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.3206	0.9930	0.9988	0.9994	0.9990	0.9972	0.9722
	$T_n^{q_1}$	0.8654	0.9984	0.9998	1.0000	0.9996	0.9974	0.8328
	$L_n^{q_2}$	0.8536	0.9938	0.9988	0.9996	0.9996	0.9986	0.8348
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9546	0.9950	0.9970	0.9964	0.9960	0.9864	0.1624
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1472	0.9760	0.9950	0.9968	0.9952	0.9956	0.9880
1500	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9532	0.9976	0.9990	0.9994	0.9994	0.9972	0.3378
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.3206	0.9930	0.9988	0.9994	0.9990	0.9972	0.9722
	$T_n^{q_1}$	0.8654	0.9984	0.9998	1.0000	0.9996	0.9974	0.8328
	$L_n^{q_2}$	0.8536	0.9938	0.9988	0.9996	0.9996	0.9986	0.8348
	$K_n^{[d_{n_{ch1}}, 1-d_{n_{ch1}}^{0.6}]}$	0.9546	0.9950	0.9970	0.9964	0.9960	0.9864	0.1624
	$J_n^{[d_{n_{ch1}}^{0.6}, 1-d_{n_{ch1}}]}$	0.1472	0.9760	0.9950	0.9968	0.9952	0.9956	0.9880
	$C_n^{[d_{n_{ch2}}, 1-d_{n_{ch2}}^{0.6}]}$	0.9532	0.9976	0.9990	0.9994	0.9994	0.9972	0.3378
	$C_n^{[d_{n_{ch2}}^{0.6}, 1-d_{n_{ch2}}]}$	0.3206	0.9930	0.9988	0.9994	0.9990	0.9972	0.9722

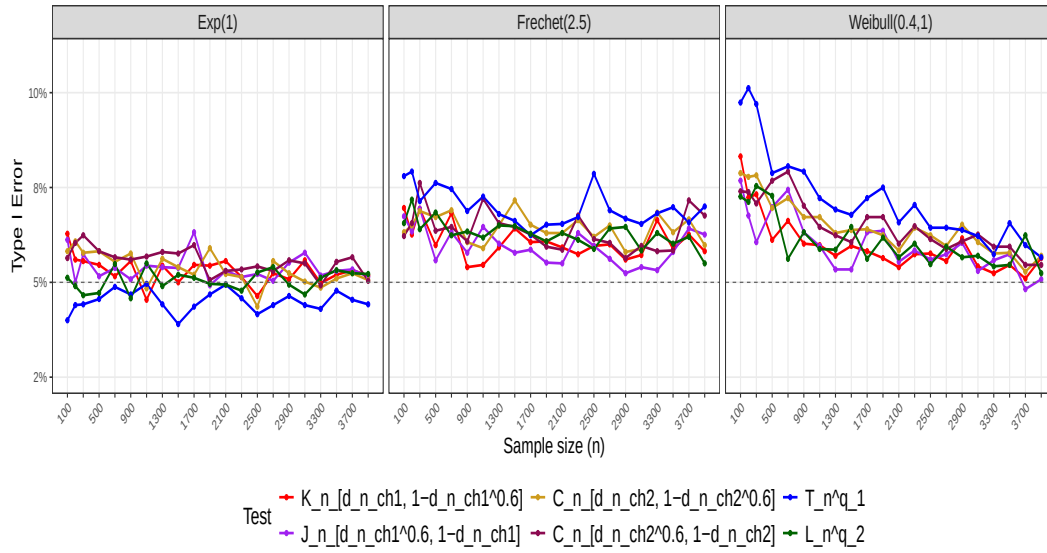


Figure 6.13: Empirical type I errors for the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_{n_{ch1}}^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$ -based tests on Exp(1), Frechet(2.5), and Weibull(0.4,1).

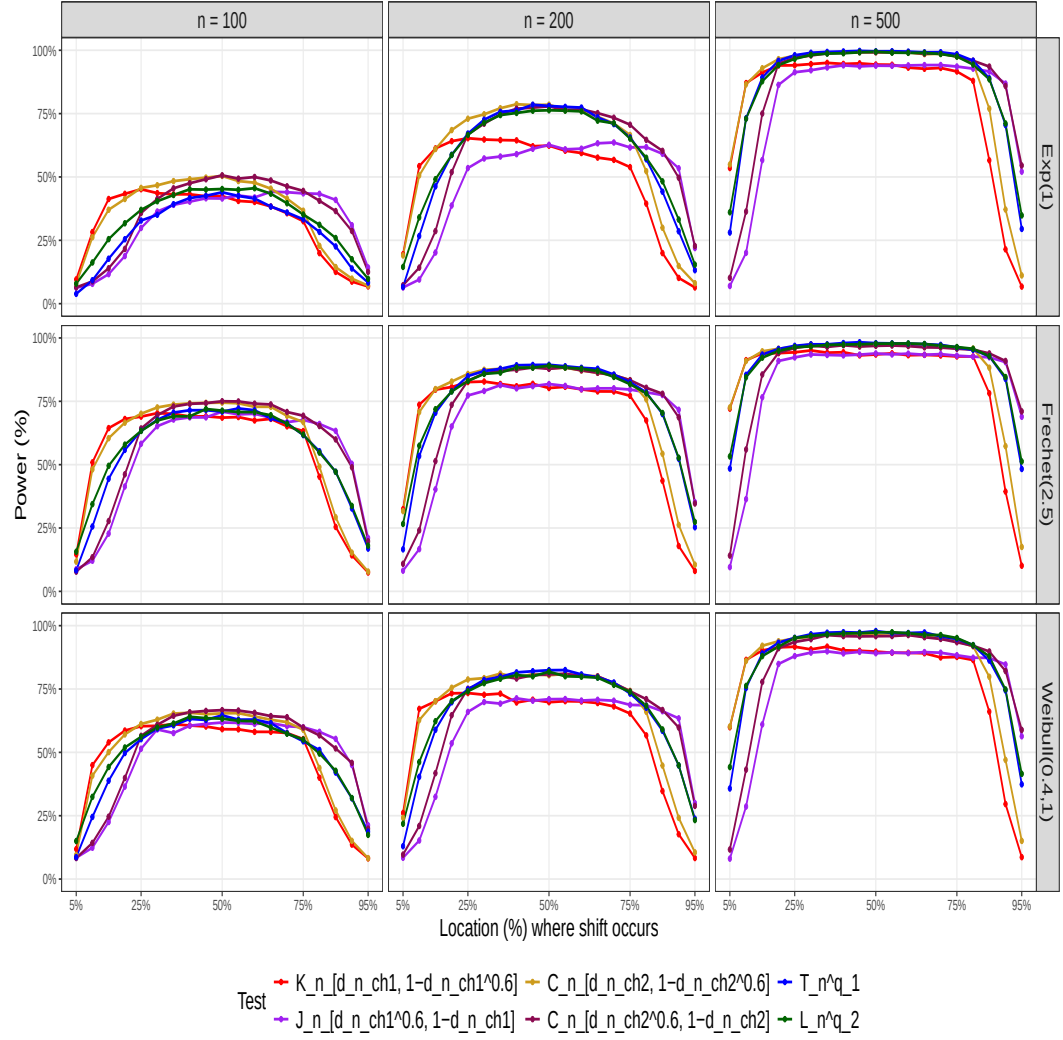


Figure 6.14: Empirical powers for a shift of 0.5σ and $n = 100, 200, 500$ for the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_{n_{ch1}}^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$ -based tests on Exp(1), Frechet(2.5), and Weibull(0.4,1).

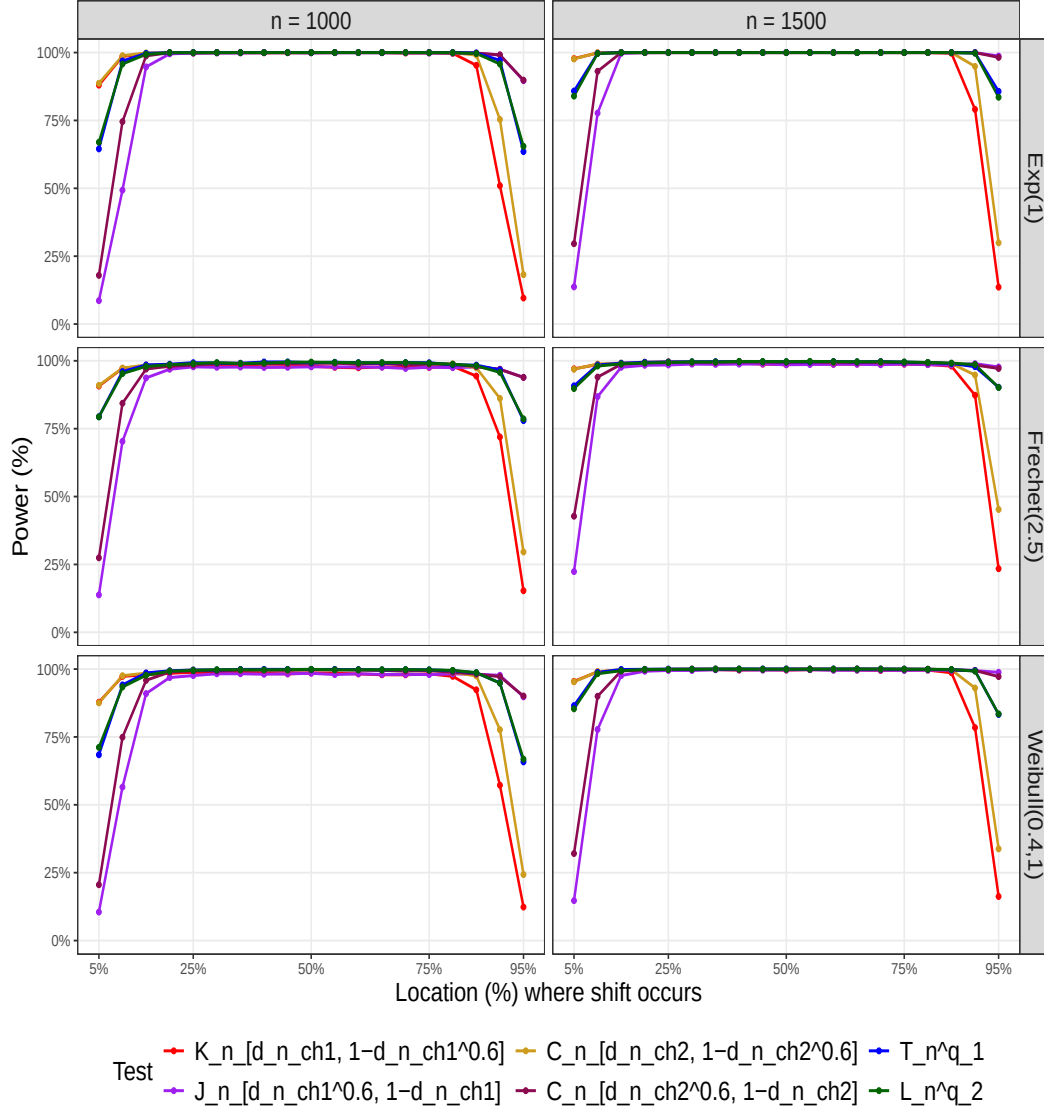


Figure 6.15: Empirical powers for a shift of 0.5σ and $n = 1000, 1500$ for the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, $J_n^{[d_n/e_n, 1-d_n]}$ - and $K_n^{[d_n, 1-d_n/e_n]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_{n_{ch1}}^{1-0.6}$, along with the T_n^{q1} -, L_n^{q2} -based tests on $\text{Exp}(1)$, $\text{Frechet}(2.5)$, and $\text{Weibull}(0.4, 1)$.

6.5 Companions to Table 3.6

Table 6.4: Estimated $1 - \gamma$ quantiles of $\sup_{0 < t < 1} \frac{|B(t)|}{q_1(t)}$

$(1 - \gamma)100\%$	Quantile	$(1 - \gamma)100\%$	Quantile	$(1 - \gamma)100\%$	Quantile
50	1.40	70	1.60	90	1.93
51	1.41	71	1.61	91	1.96
52	1.41	72	1.62	92	1.99
53	1.42	73	1.63	93	2.02
54	1.43	74	1.65	94	2.06
55	1.44	75	1.66	95	2.10
56	1.45	76	1.67	96	2.16
57	1.46	77	1.69	97	2.22
58	1.47	78	1.70	98	2.31
59	1.48	79	1.72	99	2.46
60	1.49	80	1.73	99.9	2.86
61	1.50	81	1.75	99.99	3.23
62	1.51	82	1.76		
63	1.52	83	1.78		
64	1.53	84	1.80		
65	1.54	85	1.82		
66	1.55	86	1.84		
67	1.56	87	1.86		
68	1.57	88	1.88		
69	1.58	89	1.90		

Table 6.5: Estimated $1 - \gamma$ quantiles of $\int_0^1 \frac{B^2(t)}{q_2(t)} dt$

$(1 - \gamma)100\%$	Quantile	$(1 - \gamma)100\%$	Quantile	$(1 - \gamma)100\%$	Quantile
50	1.61	70	2.23	90	3.58
51	1.64	71	2.27	91	3.71
52	1.66	72	2.32	92	3.85
53	1.69	73	2.36	93	4.02
54	1.71	74	2.40	94	4.21
55	1.74	75	2.45	95	4.45
56	1.77	76	2.50	96	4.73
57	1.80	77	2.55	97	5.13
58	1.83	78	2.60	98	5.63
59	1.86	79	2.66	99	6.56
60	1.88	80	2.72	99.9	9.77
61	1.91	81	2.78	99.99	13.16
62	1.94	82	2.85		
63	1.98	83	2.92		
64	2.01	84	2.99		
65	2.05	85	3.08		
66	2.08	86	3.16		
67	2.11	87	3.25		
68	2.15	88	3.35		
69	2.19	89	3.46		

6.6 Matches to Figures 3.14–3.15: powers for 0.75σ shift

6.6. MATCHES TO FIGURES 3.14–3.15: POWERS FOR 0.75σ SHIFT 123

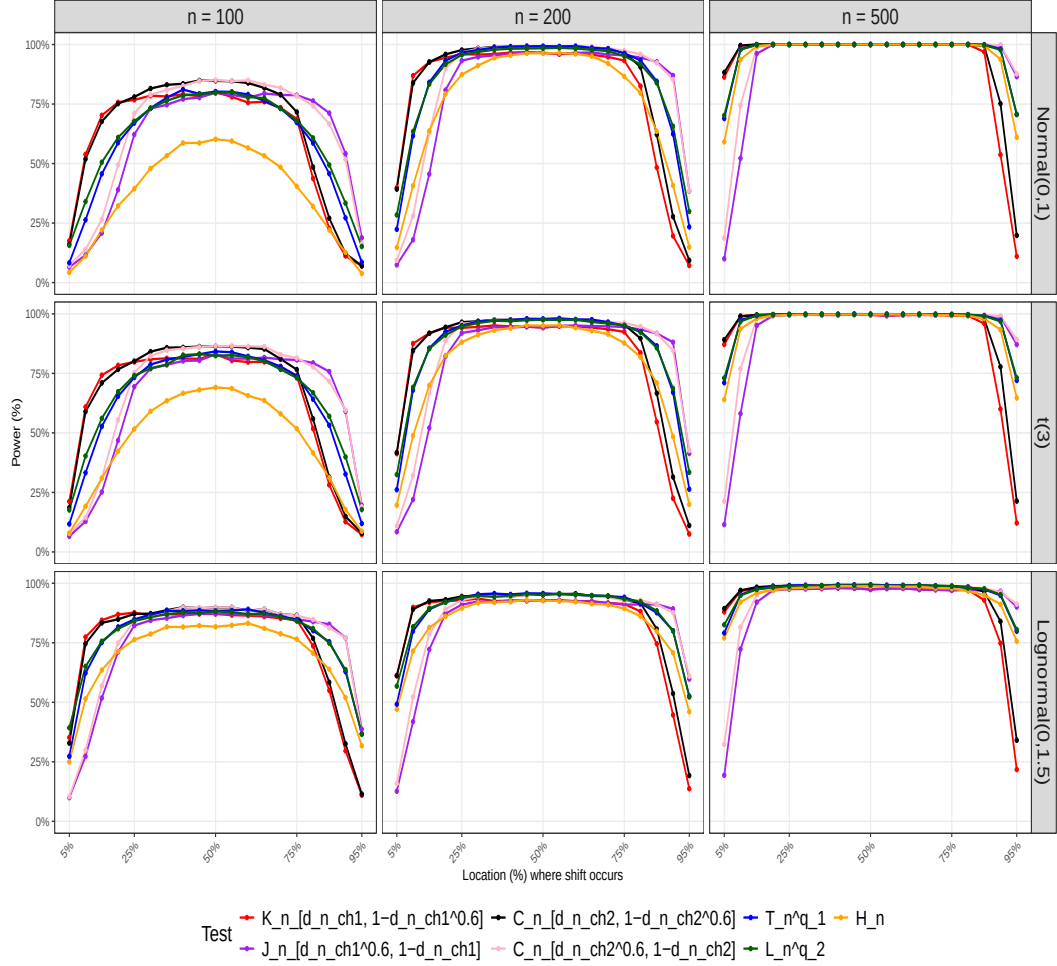


Figure 6.16: Empirical powers for a shift of 0.75σ and $n = 100, 200, 500$ for the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, $J_n^{[d_n, 1-d_n]}$ - and $K_n^{[d_n, 1-d_n]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_{n_{ch1}}^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$, and H_n -based tests on Normal(0,1), $t(3)$, and Lognormal(0,1.5).

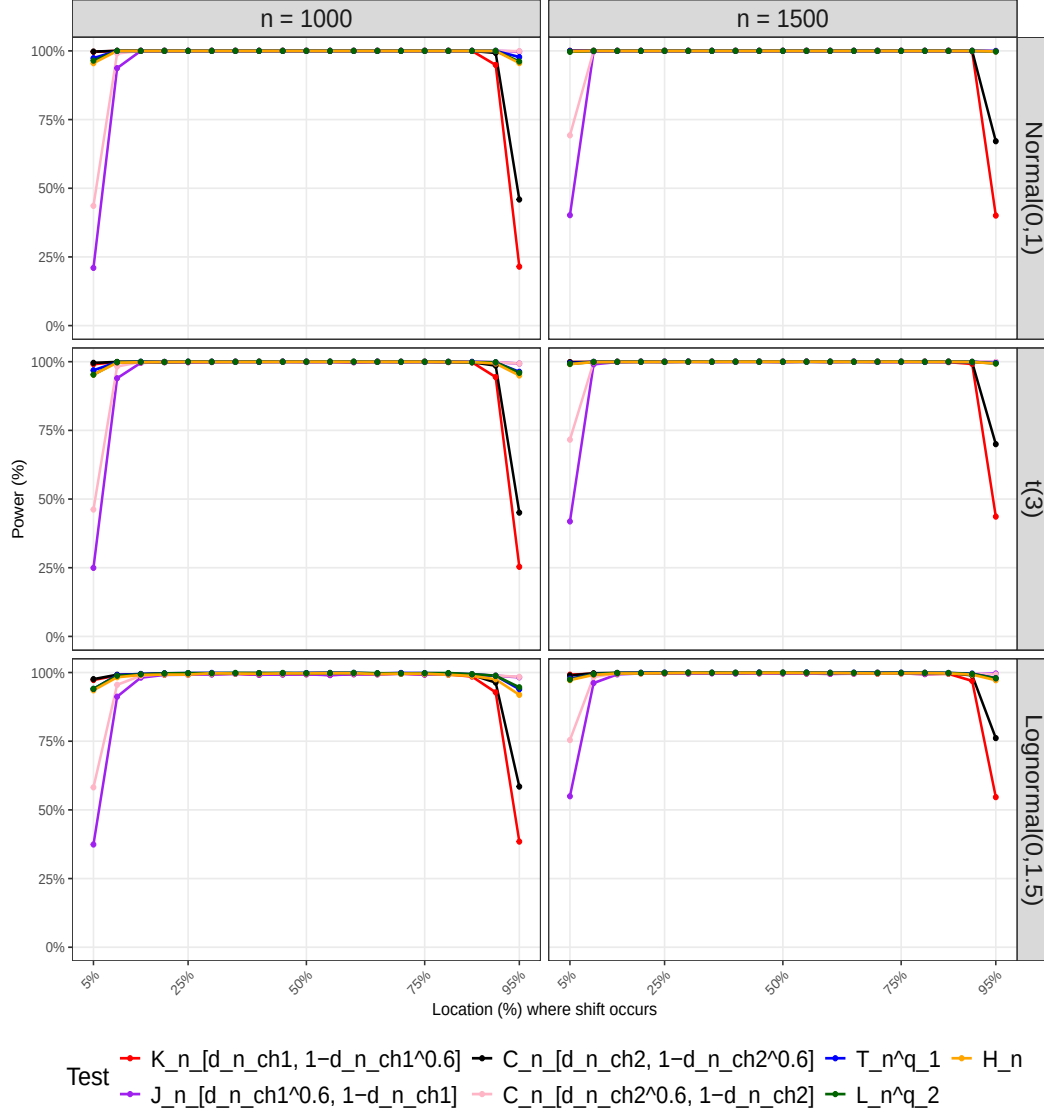


Figure 6.17: Empirical powers for a shift of 0.75σ and $n = 1000, 1500$ for the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, $J_n^{[\frac{d_n}{e_n}, 1-d_n]}$ - and $K_n^{[d_n, 1-\frac{d_n}{e_n}]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_{n_{ch1}}^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$, and H_n -based tests on Normal(0,1), $t(3)$, and Lognormal(0,1.5).

6.6. MATCHES TO FIGURES 3.14–3.15: POWERS FOR 0.75σ SHIFT 125

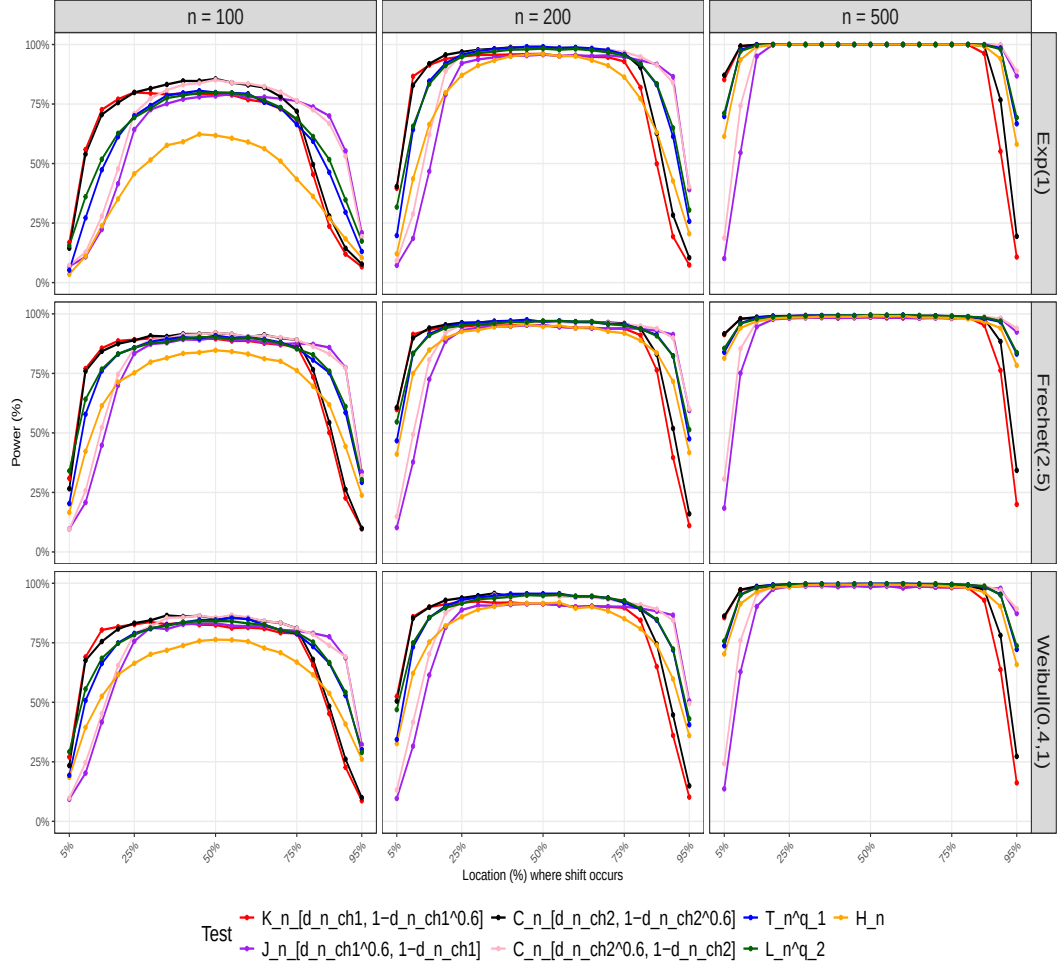


Figure 6.18: Empirical powers for a shift of 0.75σ and $n = 100, 200, 500$ for the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, $J_n^{[d_n, 1-d_n]}$ - and $K_n^{[d_n, 1-d_n/e_n]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_{n_{ch1}}^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$, and H_n -based tests on $\text{Exp}(1)$, $\text{Frechet}(2.5)$, and $\text{Weibull}(0.4,1)$.

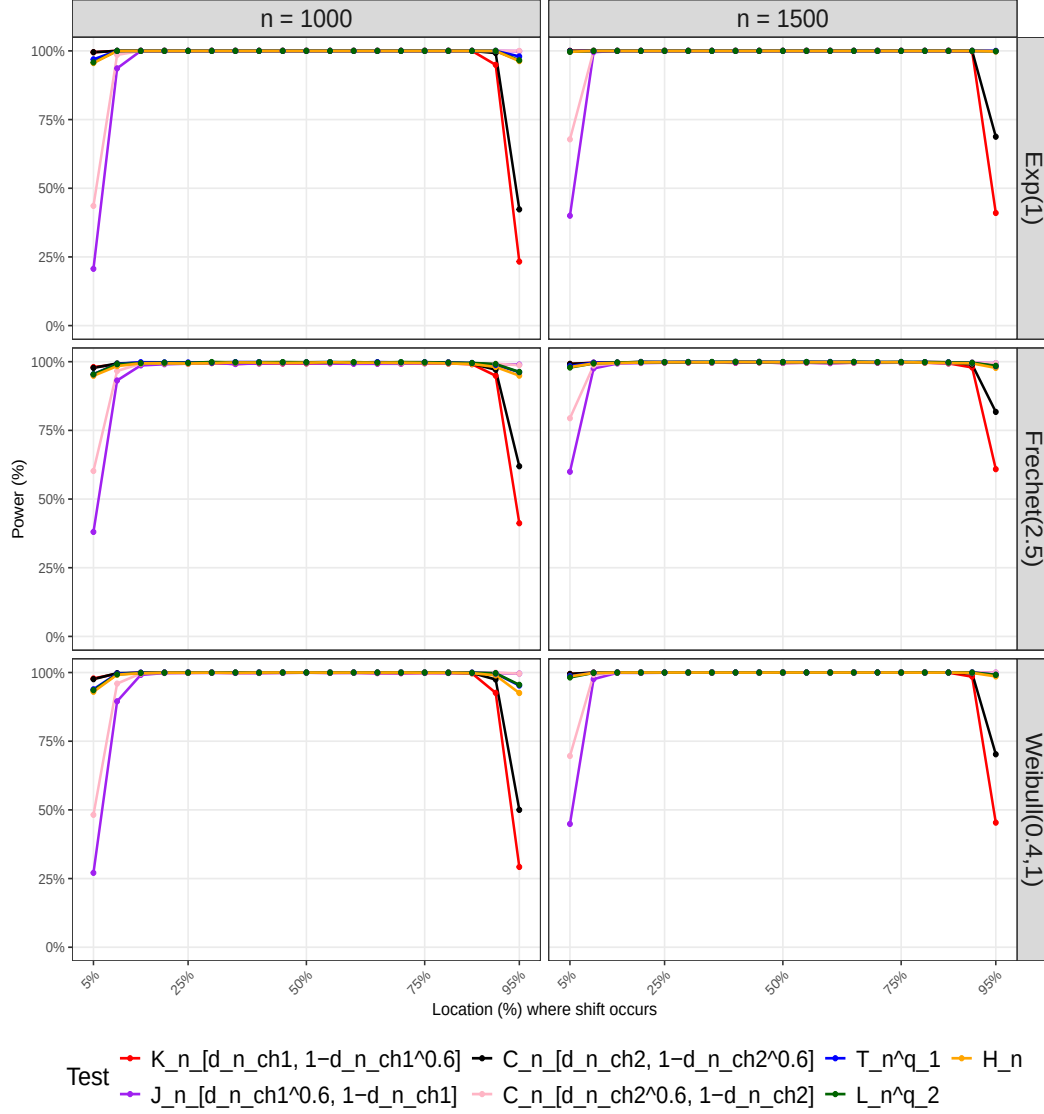


Figure 6.19: Empirical powers for a shift of 0.75σ and $n = 1000, 1500$ for the $C_n^{[t_1(n), 1-t_2(n)]}$ -based tests with $t_1(n) = d_{n_6}$, $t_2(n) = 1 - d_{n_6}/e_n$, $J_n^{[d_n/e_n, 1-d_n]}$ - and $K_n^{[d_n, 1-d_n/e_n]}$ -based tests with $d_n = d_{n_{ch1}}$ of (20) and $e_n = d_{n_{ch1}}^{1-0.6}$, along with the $T_n^{q_1}$ -, $L_n^{q_2}$, and H_n -based tests on $\text{Exp}(1)$, $\text{Frechet}(2.5)$, and $\text{Weibull}(0.4, 1)$.

Chapter 7

Appendix: the Initial Struggle in Chapter 2

For the plots below, define

$$d_{n_1*} = \frac{n^{0.985}(\log n)^{-1.45}}{n+1}, \quad d_{n_2*} = \frac{n^{0.3}(\log n)^{0.3}}{n+1}, \quad d_{n_3*} = \frac{n^{0.35}}{n+1},$$
$$d_{n_4*} = \frac{(\log n)^{1.2}}{n+1}, \quad \text{and} \quad d_{n_5*} = \frac{(\log(\log(n)))^4}{n+1}.$$

and let $e_n = d_n^{1-0.99}$. This should give the reader a good summary of what we were initially dealing with in Chapter 2 with respect to the (8)-based tests, for a wide variety of starting values and convergence speeds for d_n .

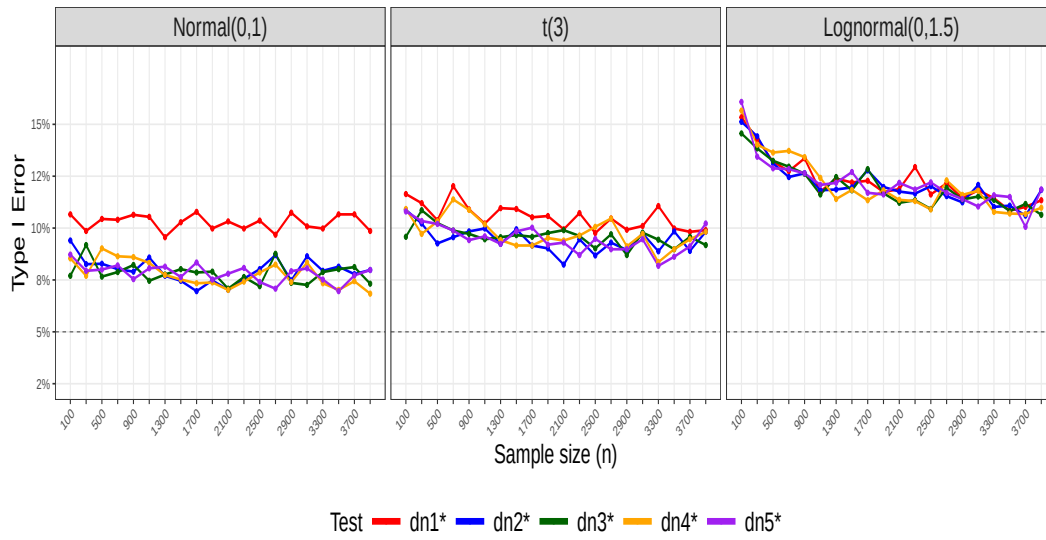


Figure 7.1: Empirical type I errors of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}^*$ and $e_n = d_n^{1-0.99}$, $i = \overline{1, 5}$, on Normal(0,1), t(3), and Lognormal(0,1.5).

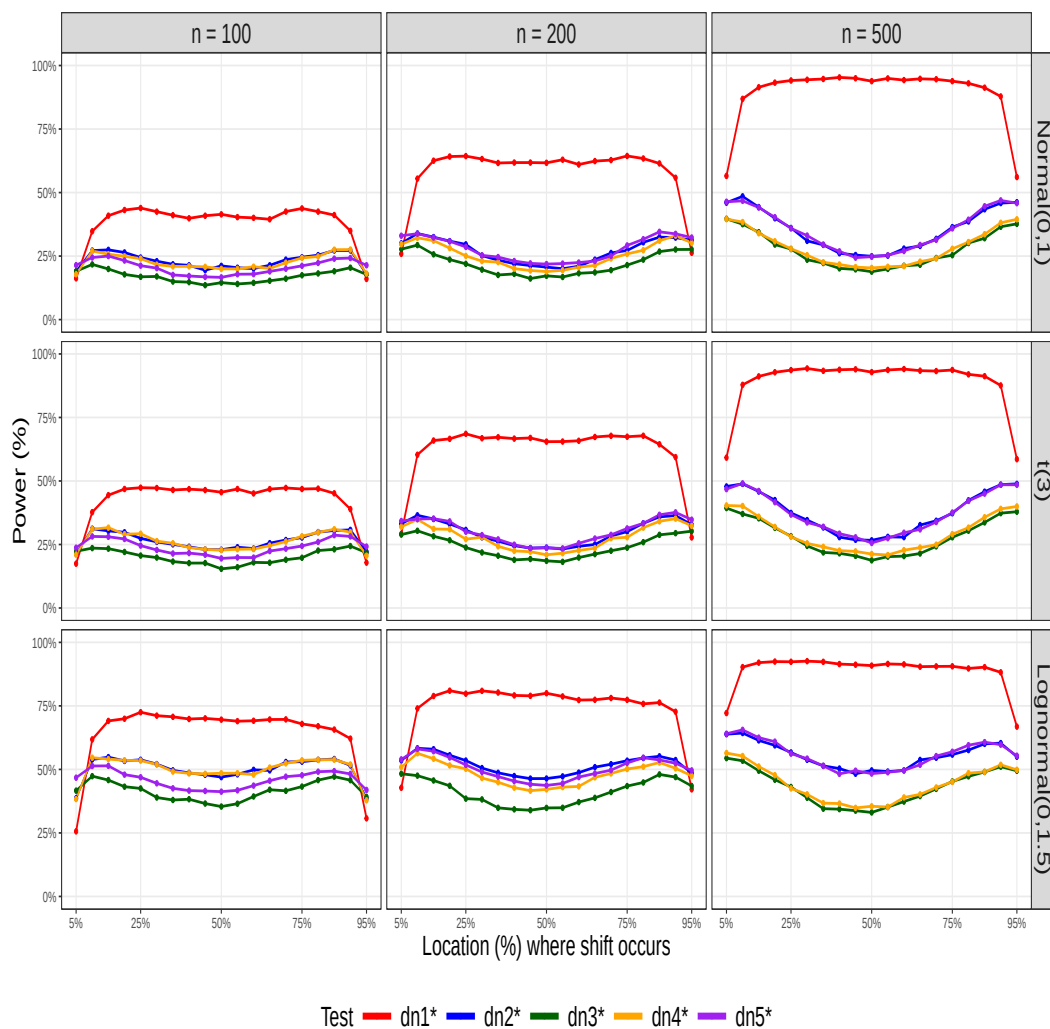


Figure 7.2: Empirical powers for $n = 100, 200, 500$ of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}^*$ and $e_n = d_n^{1-0.99}$, $i = \overline{1, 5}$, on Normal(0,1), $t(3)$, and Lognormal(0,1.5).

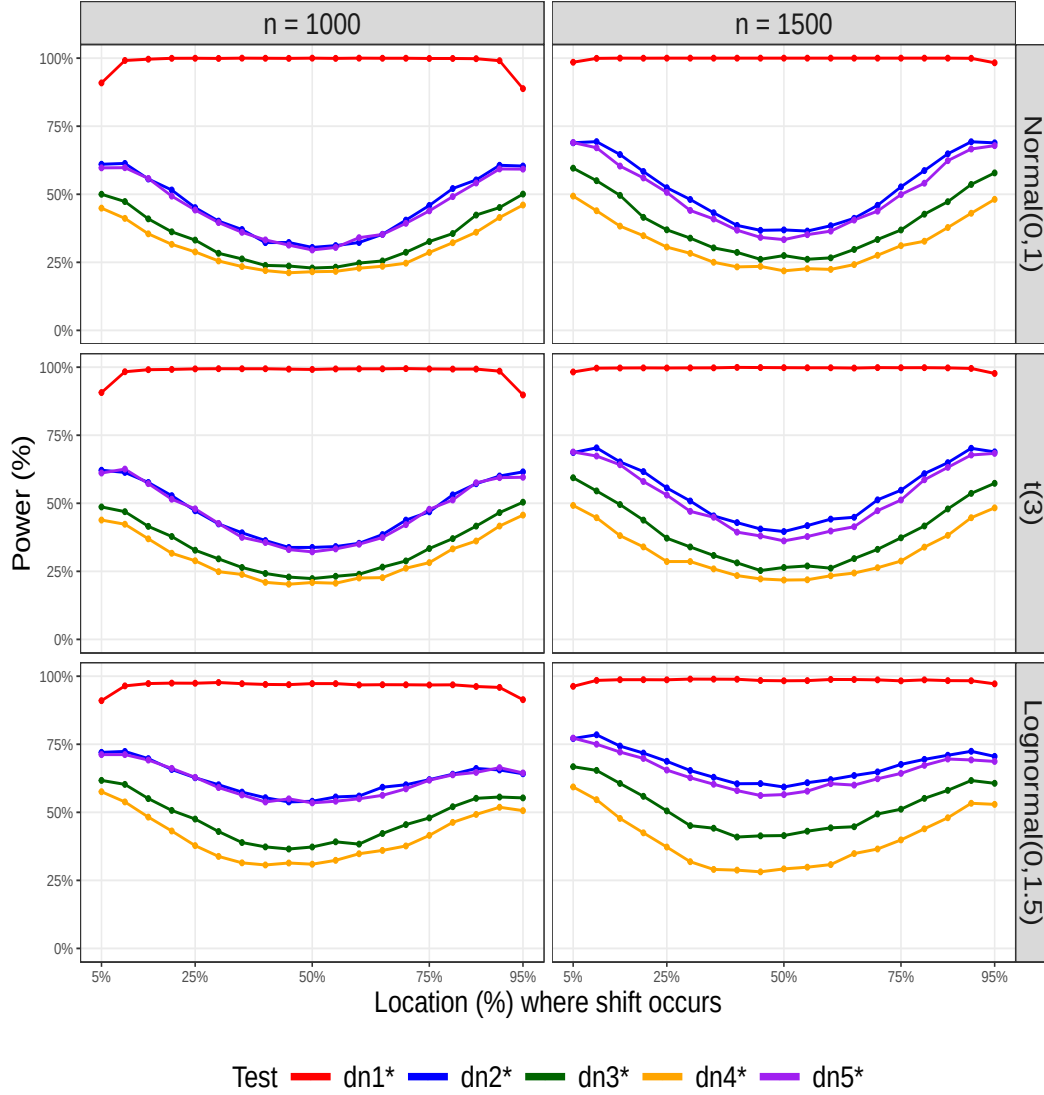


Figure 7.3: Empirical powers for $n = 1000, 1500$ of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}^*$ and $e_n = d_n^{1-0.99}$, $i = \overline{1, 5}$, on Normal(0,1), $t(3)$, and Lognormal(0,1.5).

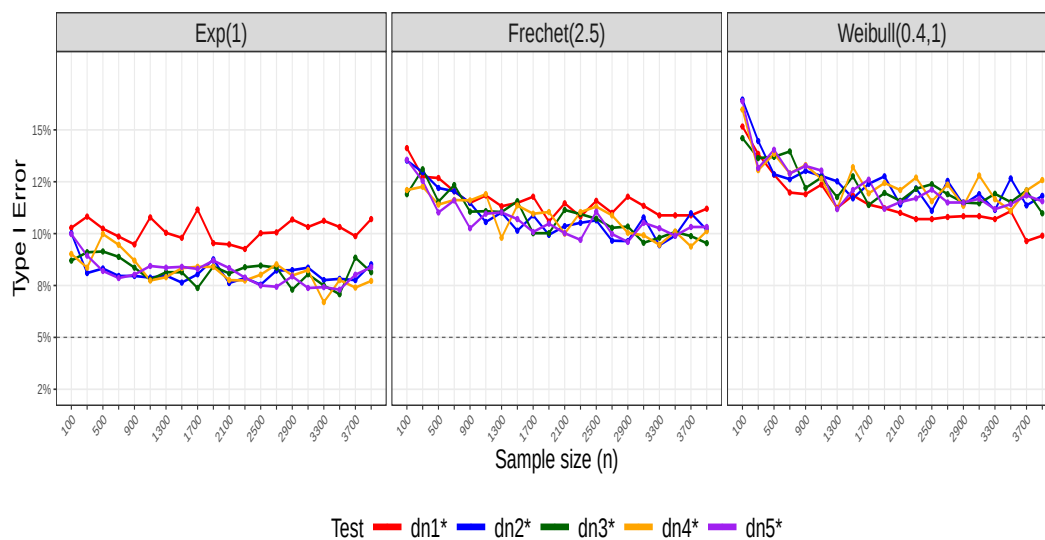


Figure 7.4: Empirical powers of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}^*$ and $e_n = d_n^{1-0.99}$, $i = \overline{1, 5}$, on Exp(1), Frechet(2.5), and Weibull(0.4,1).

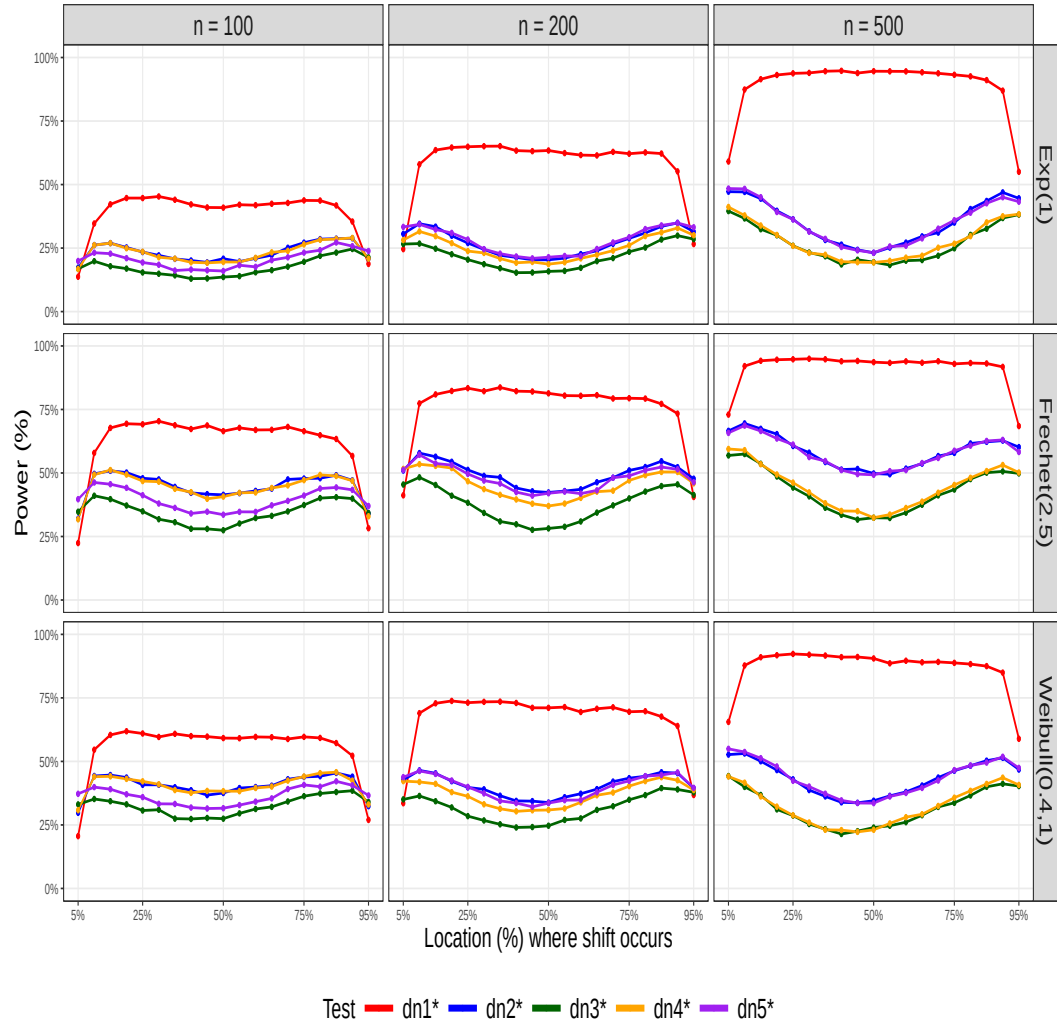


Figure 7.5: Empirical powers for $n = 100, 200, 500$ of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}^*$ and $e_n = d_n^{1-0.99}$, $i = \overline{1, 5}$, on $\text{Exp}(1)$, $\text{Frechet}(2.5)$, and $\text{Weibull}(0.4,1)$.

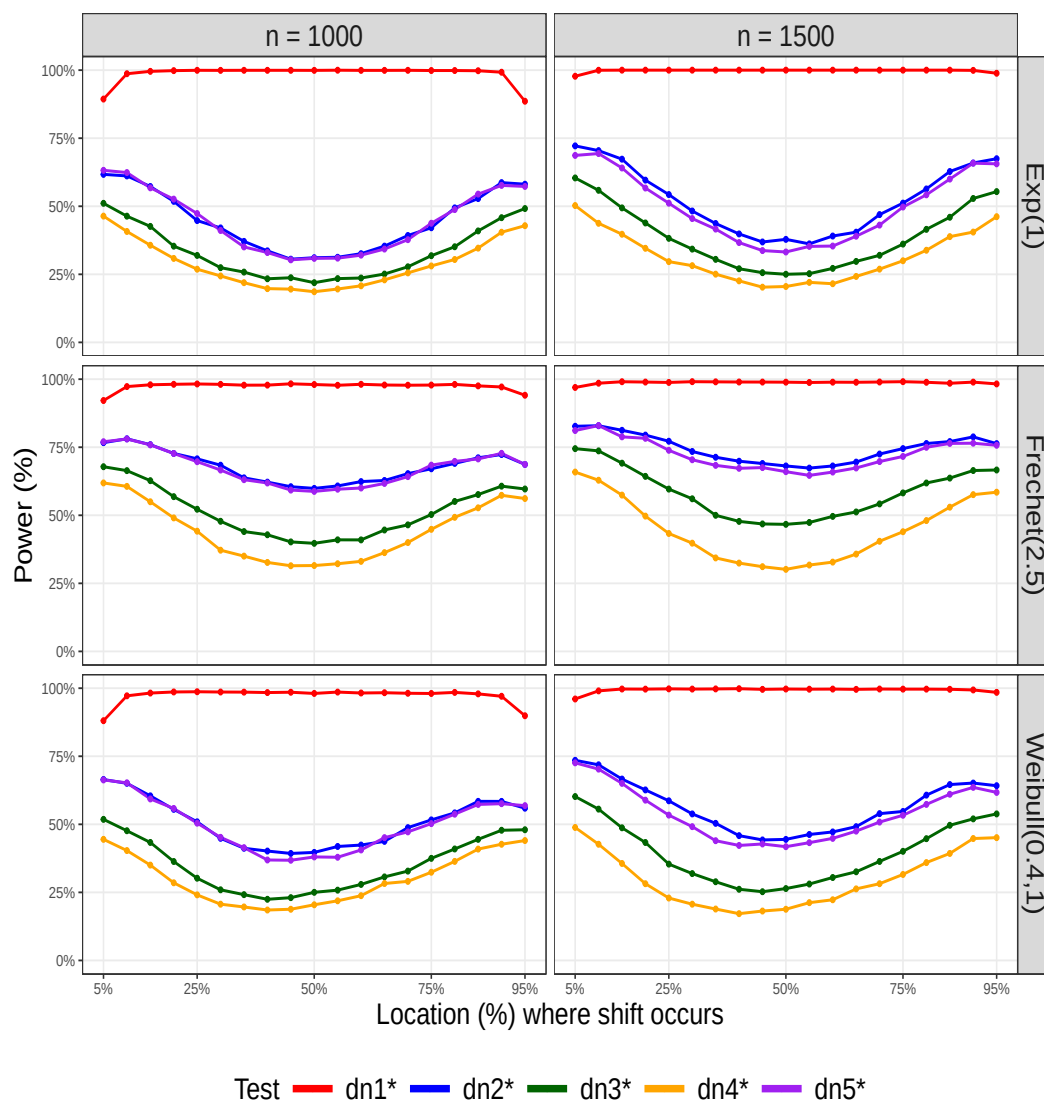


Figure 7.6: Empirical powers for $n = 1000, 1500$ of the $K_n^{[d_n, 1 - \frac{d_n}{e_n}]}$ -based test with $d_n = d_{n_i}^*$ and $e_n = d_n^{1-0.99}$, $i = \overline{1, 5}$, on Exp(1), Frechet(2.5), and Weibull(0.4,1).

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