Generalizations and Applications of Alexander Self-Duality in Combinatorial Commutative Algebra

by

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Abstract

We describe Alexander duality in three equivalent contexts: square-free monomial ideals, simple hypergraphs, and simplicial complexes. The first major result gives necessary and sufficient conditions for Alexander self-duality in each of these objects. As a consequence, we show a novel equivalence between Alexander self-duality and intersecting, 3-chromatic hypergraphs, about which Erdős and Lovász posed a number of still open questions in the 1970's. A recent topological study of Alexander self-duality gives an enumeration algorithm for all such hypergraphs, which is only realizable for small n. We describe improvements for a computational implementation of this enumeration algorithm that allow us to enumerate all intersecting hypergraphs and all non 2-colorable hypergraphs with six vertices or fewer. Further, we develop a technique called symmetric polarization to give generalized versions of these results for generalized Alexander self-duality as it is defined for monomial ideals.

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Chapter 1

Introduction

The field of Combinatorial Commutative Algebra is, in large part, a dictionary between Combinatorics and Commutative Algebra. Entries in this dictionary are typically written about a correspondence between an algebraic object that is constructed by the data encoded in a combinatorial object. One such example is the correspondence between ideals in a polynomial ring generated by square-free, degree 2 monomials and the edges of a simple graph. This correspondence–called the edge ideal of a graph–is very widely studied in Commutative Algebra, and is central to the work in this thesis. Studying correspondences of this nature allows us to ask algebraic questions and solve them combinatorially and vice versa. The purpose of this Introduction is to provide an overview of the content of this thesis, as well as the context within the larger body of mathematical knowledge that this work lies. Hence, we will delay any definitions and technical discussions until Chapter 2.

As the title might suggest, Alexander duality is central to this work. The incarnation of Alexander duality that we will use is "combinatorial" Alexander duality. Its progenitor, "topological" Alexander duality is a statement of an isomorphism between the reduced cohomology in degree i of a compact, locally contractible, nonempty, proper subspace of the n-sphere, S^n, and the degree $n-i-1$ reduced homology of its complement [12]. The combinatorial version replaces S^n with a simplex and the subspace with a simplicial complex Δ. Instead of directly taking a "complement" of Δ in the space, this version uses an "Alexander dual" simplicial complex, $\Delta^\vee$.

The purpose of Chapter 2 is to interpret the construction of this Alexander dual via two different correspondences. The first comes from noting that the independent sets of a simple hypergraph define a simplicial complex. In the other direction, given a simplicial complex we can reconstruct this exact hypergraph by noting that the edges of a hypergraph are exactly the minimal non-independent sets, i.e. the minimal non-faces of the simplicial complex. The second correspondence is the edge ideal correspondence between square-free monomial ideals and simple hypergraphs. In Section 2.1 we start with a thorough algebraic description of the nice structure of (square-free) monomial ideals and their combinatorial nature. We conclude this section by introducing a novel operation that we call "symmetric polarization". This operation gives a mapping between monomial ideals and square-free monomial ideals, which we use extensively in Chapter 4. Section 2.2 covers the requisite definitions and lemmas about hypergraphs and includes a number of examples, some of which will be pertinent in later sections. In Section 2.3 we finally introduce Alexander duality for simplicial complexes. Further, we detail the hypergraph-simplicial complex correspondence and show that under this correspondence, the "Alexander dual" of a hypergraph H is the hypergraph $H^\vee$ whose edges consist of the minimal vertex covers of H. Chapter 2 concludes with Section 2.4. In this Section we introduce the edge ideal correspondence between square-free monomial ideals and hypergraphs. Under this correspondence the "Alexander dual" of an edge ideal is called a cover ideal. In the algebraic setting, we show that this duality can be expressed entirely algebraically via the minimal primary decomposition of square-free monomial ideals. We also establish some preliminary results about the correspondence of induced subhypergraphs with their algebraic counterparts and the behavior of induced subhypergraphs under Alexander duality.

Chapter 3 is the fruitful result of asking when a hypergraph's edge ideal is equal to its cover ideal. This problem is absent from the published literature in Commutative Algebra, hence there is no prior work to report or build on. In Section 3.1 we answer this question, showing this happens exactly when a hypergraph is intersecting and not 2-colorable. Moreover, we provide a construction of every such "self-dual" hypergraph on n vertices from a hypergraph on $n-1$ vertices that is either intersecting or not 2-colorable. Next, Section 3.2 gives an account of intersecting, non 2-colorable hypergraphs as they have appeared in Combinatorics literature. Although intersecting hypergraphs are well studied in the field, very little has been written about the case when they are also not 2-colorable. Chapter 3 concludes with Section 3.3 which is a summary of the literature in (Combinatorial) Algebraic Topology involving Alexander self-duality, i.e. simplicial complexes that satisfy $\Delta = \Delta^\vee$. In this setting there is more to report on. Most relevantly, a 2019 paper by Marinko Timotijević, [20], written solely about simplicial complexes corroborates many (but not all) of the algebraic results of Section 3.1. The connection to intersecting, non 2-colorable hypergraphs that we have shown is novel. Timotijević also introduced a connected, bipartite graph where each node is a self-dual complex and each edge corresponds to an easily computable "flip" operation on the edge's endpoints. This graph is the focus of Section 4.2.

Chapter 4 is largely a generalization of Chapter 3. Monomial ideals can be viewed as a generalization of square-free monomial ideals and [15] defines a "generalized Alexander duality" for monomial ideals. The use of hypergraphs is instrumental in Chapter 3, so we introduce a corresponding generalization that we call "multihypergraphs" where sets are replaced by multisets. The purpose of Section 4.1 is to introduce these generalizations and prove many generalized versions of the main results in Chapter 3. In some cases, the symmetric polarization mapping from monomial ideals to square-free monomial ideals allows the generalized version to be reduced to a similar problem in the classical, square-free version. In Section 4.2, we use symmetric polarization in the opposite manner. In this section we continue the work of Timotijević from a computational perspective. Specifically, in [20] Timotijević reports computing the graph of self-dual simplicial complexes on up to $n = 5$ vertices. We improve on this by computing the graphs for $n = 6$ and $n = 7$, the latter of which is five orders a magnitude larger than the graph for $n = 5$. This graph however is wasteful in the sense that it considers the labels of the vertices of each simplicial complex. Specifically, simplicial complexes that are isomorphic may still be represented by different nodes of the graph. We define a similar graph where the vertices are instead the isomorphism classes of self-dual hypergraphs. This graph is much smaller, but much more computationally expensive. In this setting, symmetric polarization and a few other "computational tricks" reduce the run-time and computer memory required to compute this graph for $n \leq 7$. We note that parts of this section are slightly less formal and rigorous than the rest of this work. This is a purposeful choice as we intend to present a non-trivial computational problem and implementation in a manner that is friendly to mathematicians with little to no coding experience. For the inclined reader, we include the commented code resulting from this work in the Appendix.

Finally we conclude with Section 4.3 which details the problem that inspired the development of symmetric polarization. This problem deals with the associated primes of powers of ideals and the persistence property. We start the section with a brief literature review on the matter, which culminates–for us–in a Corollary given in [9] by Francisco, Há, and Van Tuyl. We then provide a more direct, combinatorial proof of the result that utilizes symmetric polarization and colorings of multihypergraphs.

Chapter 2

Preliminaries

In this Chapter we collect the necessary tools, objects, and results used in the main contributions of this work. These contributions do not rely on a large body of prior research, hence we directly establish most of these results. With the exceptions of the introduction of symmetric polarization in Section 2.1 and a few statements in Section 2.4, the contents of this Chapter are well-known in literature.

2.1 Algebraic Preliminaries

The theory of primary decomposition is an attempt to recover the Fundamental Theorem of Arithmetic in rings that are more general than the integers ($\mathbb{Z}$). In the more general setting, in lieu of factoring elements of $\mathbb{Z}$ as a product of atomic elements, we express ideals as the intersection of atomic ideals. The atomic elements of $\mathbb{Z}$, primes and their powers, are correspondingly replaced by prime ideals and primary ideals.

After Section 2.1, we will work only with monomial ideals, hence we do not need the full generality of primary decomposition and other tools from Commutative Algebra. Instead, we provide an account of the (largely combinatorial) structure of atomic monomial ideals. We also interpret the results of more general theories in the case of monomial ideals, which are relevant to our work.

Let k be your favorite field and set R to be the polynomial ring in $n \geq 1$ variables $R = k[x_1, \ldots, x_n]$. We fix $V = \{x_1, \ldots, x_n\}$, so equivalently $R = k[V]$. Unless stated otherwise, every ideal belongs to R.

Definition 2.1. A *monomial* in R is an element of R that is a product of variables and a non-zero scalar from the field k. If the scalar is 1 then we say the monomial is *monic*.

Definition 2.2. An ideal I is a *monomial ideal* if it is generated by monomials.

Definition 2.3. The *multidegree* (or *exponent vector*) of a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$ is the vector $(a_1, \ldots, a_n) \in \mathbb{N}^n$ denoted by $\mathrm{mdeg}(m)$.

Remark 2.4. We explicitly include $0 \in \mathbb{N}$. Consider also the partial order $\preceq$ on $\mathbb{N}^n$ where $(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n)$ exactly when $a_i \leq b_i$ for all $i \in \{1, \ldots, n\}$. Given a set of monomials $M = \{f_1, \ldots, f_s\}$ we say $f_i \in M$ is *minimal* if there is no $f_j \in M \backslash \{f_i\}$ such that $\mathrm{mdeg}(f_j) \preceq \mathrm{mdeg}(f_i)$. We say the set of monomials M is minimal if every member of M is minimal. Finally, we note that for monomials $f, g \in R$, $\mathrm{mdeg}(f) \preceq \mathrm{mdeg}(g)$ if and only if f divides g.

Proposition 2.5. *A monomial ideal I has a unique set of minimal monic generators, denoted $\mathcal{G}(I)$.*

Proof. A classic result from Hilbert's Basis Theorem [2, Theorem 7.5] is that I is finitely generated by some set of monomials. Suppose I is minimally generated by two sets of monic monomials: $M = \{f_1, \ldots, f_s\}$ and $N = \{g_1, \ldots, g_t\}$. Note that because k is a field, the monic assumption does not change the ideals that M and N generate. We will show $M = N$.

Since $f_i \in I$, there are some elements $p_1, \ldots, p_t \in R$ such that $f_i = \sum_{j=1}^{t} p_j g_j$. This sum is a monomial, so for some j we can write $f_i = p_j g_j$ (adjusting p_j by some scalar in k if necessary). We may repeat this argument for g_j and write $g_j = q_\ell f_\ell$ for some $q_\ell \in R$. Hence, $f_i = p_j q_\ell f_\ell$. By minimality of M, it must be that $i = \ell$ and $p_j q_\ell = 1$. This shows that $M \subseteq N$. An identical argument on an arbitrary element in N shows $N \subseteq M$, hence the two sets are equal. □

Definition 2.6. A monomial ideal I is *square-free* if $\mathrm{mdeg}(m) \preceq (1, 1, \ldots, 1)$ for every $m \in \mathcal{G}(I)$.

Proposition 2.7. *A polynomial belongs to a monomial ideal I if and only if each of its monomial terms belongs to I.*

Proof. If each term belongs to I we note that the finite sum of elements in I remains in I, so the polynomial also belongs to I. Now, if $\mathcal{G}(I) = \{f_1, \ldots, f_s\}$ and $\sum_{i \in T} m_i$ is a polynomial in I where each m_i is a monomial with unique multidegree, then we have $p_1, \ldots, p_s \in R$ such that

$$\sum_{i \in T} m_i = \sum_{j=1}^{s} p_j f_j.$$

Define $p_{j,i}$ to be the (possibly 0) term of p_j with multidegree $\mathrm{mdeg}(m_i) - \mathrm{mdeg}(f_j)$. For each i we consider the term on the right hand side with multidegree $\mathrm{mdeg}(m_i)$. In particular, this term is given by the sum $\sum_{j=1}^{s} f_j p_{j,i}$, and because two polynomials are equal if and only if each of their terms are equal we have

$$m_i = \sum_{j=1}^{s} f_j p_{j,i} \ .$$

This shows each term m_i belongs to I. □

Proposition 2.8. *The finite intersection of (square-free) monomial ideals is a (square-free) monomial ideal.*

Proof. It suffices to show that if I, J are two (square-free) monomial ideals, then $I \cap J$ also is. Let T be a set of polynomials that generate $I \cap J$. If $p \in T$ then $p \in I$ and $p \in J$, hence by Proposition 2.7 each term of p is in $I \cap J$. Thus p can be generated by monomials and thus is a monomial ideal.

If I, J are both square-free and $x_i^2 m$ is a monomial in $I \cap J$, then so is $x_i m$. Indeed, we have some $f \in \mathcal{G}(I)$ and $g \in \mathcal{G}(J)$ such that $f \mid x_i^2 m$ and $g \mid x_i^2 m$. Notably then $\operatorname{lcm}(f, g)$ is square-free and divides $x_i^2 m$, hence must also divide $x_i m$. This shows $I \cap J$ is square-free. □

The argument in Proposition 2.8 extends more generally to show that for any monomial ideals I, J we have

$$\mathcal{G}(I \cap J) \subseteq \{\operatorname{lcm}(f, g) \mid f \in \mathcal{G}(I),\ g \in \mathcal{G}(J)\} \ .$$

Proposition 2.9. *A monomial ideal $\mathfrak{p}$ is prime if and only if $\mathcal{G}(\mathfrak{p}) \subseteq V$.*

Proof. Suppose $x_i m \in \mathcal{G}(\mathfrak{p})$ for some monomial m with positive degree. By minimality of $\mathcal{G}(\mathfrak{p})$, neither x_i nor m belong to $\mathfrak{p}$, yet their product does. Hence $\mathfrak{p}$ is not prime. This shows the minimal generators of a prime monomial ideal must have degree 1.

Now, suppose $\mathfrak{p} = \langle x_{i_1}, \ldots, x_{i_s} \rangle$. By Proposition 2.7 it suffices to show the product of two monomials not in $\mathfrak{p}$ is also not in $\mathfrak{p}$. If $W = V \backslash \{x_{i_1}, \ldots, x_{i_s}\}$, then the monomials that are not in $\mathfrak{p}$ are precisely the monomials in $k[W] \subseteq R$, which as a subring is multiplicatively closed. □

Notation 2.10. For a subset $W = \{x_{i_1}, \ldots, x_{i_s}\} \subseteq V$ we denote:

- the prime monomial ideal $\mathfrak{p}_W = \langle x_{i_1}, \ldots, x_{i_s} \rangle$;
- the square-free monomial $m_W = x_{i_1} \cdots x_{i_s}$.

Remark 2.11. The introduction of W in Proposition 2.9 is not necessary to show the proposition, however we will later be interested in the natural ring homomorphism for which $\mathfrak{p}$ is the kernel. By Noether's First Isomorphism Theorem for rings, the image of this map will be $k[V]/\mathfrak{p} \cong k[W]$. We will also later introduce a notion of duality, by which this map will have a dual map with the same domain and image. We define both of these maps algebraically below. The exact relationship of these maps will be the subject of Theorem 2.101.

We observe that R can be viewed as an infinite dimensional k-vector space where the monic monomials define a basis. Both maps will be defined by their action on this basis, hence are both linear transformations. Thus we will only need to check that these maps respect (monomial) multiplication to show they are ring homomorphisms.

Definition 2.12. Let $W \subseteq V$ and $\mathfrak{p}_{V\backslash W}$ be the prime ideal with $\mathcal{G}(\mathfrak{p}_{V\backslash W}) = V\backslash W$. We consider the map π_W defined on (monic) monomials:

$$\pi_W(m) = \begin{cases} 0 & \text{if } \ m \in \mathfrak{p}_{V\backslash W} \\ m & \text{if } \ m \notin \mathfrak{p}_{V\backslash W}. \end{cases}$$

We linearly extend π_W to a map $\pi_W : R \twoheadrightarrow k[W]$, by defining $\pi_W\left(\sum m_i\right) = \sum \pi_W(m_i)$. Showing that π_W respects multiplication is equivalent to showing $\mathfrak{p}_{V\backslash W}$ is prime, hence the map is a homomorphism of rings. We call π_W the *projection map onto* W, and note it acts as the identity on $k[W]$.

Definition 2.13. Let L be (any) commutative ring with identity and $\mathfrak{p}$ be a prime ideal. Let $S = L\backslash\mathfrak{p}$, and construct a set of new elements

$$S^{-1} = \left\{ s^{-1} \;\middle|\; ss^{-1} = 1 \text{ and } s \in S \right\}.$$

The *localization of* L *at* $\mathfrak{p}$, denoted $L_{\mathfrak{p}}$, is the smallest ring that contains L and S^{-1}.

Example 2.14. When L is an integral domain, the set S^{-1} can be taken as a subset of the ring's field of fractions. In particular, if $R = k[x]$ and $\mathfrak{p} = \langle x \rangle$, then $R_{\mathfrak{p}}$ is the ring of rational functions whose denominator has a non-zero constant term:

$$R_{\mathfrak{p}} = \left\{ \frac{p}{q} \;\middle|\; p, q \in R[x], \text{ and } q(0) \neq 0 \right\}.$$

Moreover, R is an integral domain so $\langle 0 \rangle$ is a prime ideal. Hence $R_{\langle 0 \rangle}$ is the field of rational functions, $k(x)$.

Remark 2.15. Showing that localization is well-defined is a mostly straightforward exercise in algebra, and we refer the reader to Chapter 3 of [2] for a proper treatment. Notably, any element of R that is not in $\mathfrak{p}$ becomes a unit in $R_{\mathfrak{p}}$, hence $\mathfrak{p}$ is a unique maximal ideal in $R_{\mathfrak{p}}$. Thus, localization indeed yields a local ring.

In the case that $R = k[V]$, $W \subset V$, and $\mathfrak{p}_W = \langle W \rangle$ there is a natural inclusion $k[V] \hookrightarrow R_{\mathfrak{p}_W}$. We will modify this inclusion to define a map that is onto $k[W] \subset R_{\mathfrak{p}_W}$.

Definition 2.16. For a given subset of variables $W \subseteq V$ and monic monomial m, we factor $m = m_1 m_2$ where $m_1 \in k[W]$ and $m_2 \in k[V \backslash W]$ are both monic. Note that m_1 and m_2 are well-defined and unique. We define a linear map $\widehat{\varphi}_W : k[V] \to R_{\mathfrak{p}_W}$ on monic monomials:

$$\widehat{\varphi}_W(m) = m \cdot m_2^{-1} = m_1.$$

We now check that $\widehat{\varphi}_W$ respects (monomial) multiplication. Suppose f, g are monomials with $f = f_1 f_2$ and $g = g_1 g_2$ as above. Then $\widehat{\varphi}_W(f)\widehat{\varphi}_W(g) = f_1 g_1$. We observe that $fg = (f_1 g_1)(f_2 g_2)$ is the unique factorization for fg, hence $\widehat{\varphi}_W(f)\widehat{\varphi}_W(g) = \widehat{\varphi}_W(fg)$. Thus, $\widehat{\varphi}_W$ is a homomorphism of rings.

Further we note that the image of $\widehat{\varphi}_W$ is contained in $k[W]$, and when restricted to $k[W]$, $\widehat{\varphi}_W$ acts as the identity. Thus, we call $\varphi_W : k[V] \twoheadrightarrow k[W]$ the *localization map at W* where $\varphi_W(p) = \widehat{\varphi}_W(p)$.

Remark 2.17. We note that for a fixed $W \subseteq V$, the prime ideal denoted by $\mathfrak{p}_{V \backslash W}$ in Definition 2.12 is different from $\mathfrak{p}_W$ in Definition 2.16. In the former, $\mathcal{G}(\mathfrak{p}_{V\backslash W}) = V \backslash W$, whereas in the latter $\mathcal{G}(\mathfrak{p}_W) = W$. This follows Notation 2.10, and we choose this convention (for the subscripts on the maps) to emphasize that these maps are both onto the same image, $k[W]$.

Moreover, we note that both π_W and φ_W can be defined as evaluation homomorphisms. If $W = \{x_1, \ldots, x_r\}$ then $\pi_W(f) = f(x_1, \ldots, x_r, 0, \ldots, 0)$ and $\varphi_W(f) = f(x_1, \ldots, x_r, 1, \ldots, 1)$. However, Definition 2.12 and Definition 2.16 are more natural to generalize in Chapter 4.

Definition 2.18. An ideal $\mathfrak{b}$ is *irreducible* if it cannot be expressed as the intersection of two strictly larger ideals.

Proposition 2.19. *A monomial ideal $\mathfrak{b}$ is irreducible if and only if $\mathcal{G}(\mathfrak{b}) = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$.*

Proof. Suppose $\mathfrak{b}$ is a monomial ideal and $\mathcal{G}(\mathfrak{b})$ does not take the form in the proposition statement. Then there are coprime monomials f, g with $fg \in \mathcal{G}(\mathfrak{b})$. We note $I = \mathfrak{b} + \langle f \rangle$ and $J = \mathfrak{b} + \langle g \rangle$ are both strictly larger than $\mathfrak{b}$. A monomial in $I \cap J$ belongs to one or both of $\mathfrak{b}$ and $\langle f \rangle \cap \langle g \rangle = \langle fg \rangle \subseteq \mathfrak{b}$. Hence $I \cap J \subseteq \mathfrak{b}$. Since both I and J contain $\mathfrak{b}$, we have $\mathfrak{b} = I \cap J$, thus $\mathfrak{b}$ is not irreducible.

Now, suppose with relabeling $\mathcal{G}(\mathfrak{b}) = \{x_1^{a_1}, \ldots, x_n^{a_n}\}$ and I is any ideal that properly contains $\mathfrak{b}$. We will show that $\mathfrak{b}$ is irreducible by showing $x_1^{a_1-1} \cdots x_n^{a_n-1} \in I \backslash \mathfrak{b}$. That is, the intersection of any two ideals that properly contain $\mathfrak{b}$ cannot be $\mathfrak{b}$.

Pick any polynomial $q \in I \backslash \mathfrak{b}$. By Proposition 2.7 we may assume every term of q is not in $\mathfrak{b}$, otherwise we may subtract that term from q and the difference remains in $I \backslash \mathfrak{b}$. We pick a term of q with minimal multidegree, say $\hat{q}$ with $\operatorname{mdeg}(\hat{q}) = (b_1, \ldots, b_n)$. By minimality, there is no other term of q with multidegree $(c_1, \ldots, c_n)$ such that $c_i \leq b_i$ for each i, and because $\hat{q} \notin \mathfrak{b}$ we have $b_i < a_i$ for each i. Consider the product $q x_1^{a_1-b_1-1} \cdots x_n^{a_n-b_n-1}$ and note that by construction every term of this product belongs to $\mathfrak{b}$, except for the monomial $\hat{q} x_1^{a_1-b_1-1} \cdots x_n^{a_n-b_n-1} = x_1^{a_1-1} \cdots x_n^{a_n-1}$. So just as before, we may subtract each of those terms in $\mathfrak{b}$ from the product to see that the monomial $x_1^{a_1-1} \cdots x_n^{a_n-1}$ belongs to $I \backslash \mathfrak{b}$. □

Definition 2.20. An ideal $\mathfrak{q} \subset R$ is *primary* if $\mathfrak{q} \neq R$ and $pq \in \mathfrak{q} \implies$ either $p \in \mathfrak{q}$ or $q^r \in \mathfrak{q}$ for some integer $r > 0$.

Proposition 2.21. *A monomial ideal $\mathfrak{q}$ is primary if and only if*

$$\mathcal{G}(\mathfrak{q}) = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}, m_1, \ldots, m_t\},$$

where each m_j is a monomial in $k[x_{i_1}, \ldots, x_{i_s}]$.

Proof. Suppose $\mathcal{G}(\mathfrak{q})$ contains a monomial $x_j m$ such that there is no $r > 0$ with $x_j^r \in \mathcal{G}(\mathfrak{q})$. By minimality of $\mathcal{G}(\mathfrak{q})$, $m \notin \mathfrak{q}$. Moreover, there is no $\ell > 0$ such that $x_j^\ell \in \mathfrak{q}$, otherwise the smallest such power would be a minimal generator. Hence, $\mathfrak{q}$ is not primary.

Now, suppose $\mathcal{G}(\mathfrak{q})$ takes the form given in the proposition and that $pq \in \mathfrak{q}$. It suffices to show $p \notin \mathfrak{q} \implies$ there is some $r > 0$ with $q^r \in \mathfrak{q}$. We assume the hypothesis and pick r wisely. Let $q = g_1 + \cdots + g_\ell$ be a sum of monomials and note that by hypothesis, p has a term $\hat{f}$ not in $\mathfrak{q}$ with minimal multidegree. For any $\hat{g}$ with minimal multidegree among $\{g_1, \ldots, g_\ell\}$ the product $\hat{f}\hat{g}$ is, by minimality of the multidegree of both factors, a term in pq and thus belongs to $\mathfrak{q}$. Therefore $\hat{g}$ must be divisible by some element in $\{x_{i_1}, \ldots, x_{i_s}\}$. Moreover, every g_j is divisible by some term of q with minimal multidegree, hence every g_j is divisible by some $x_{i_j} \in \{x_{i_1}, \ldots, x_{i_s}\}$. If $r = \ell \cdot \max\{a_1, \ldots, a_s\}$, then each term of q^r is a k-scalar multiple of $\prod_{j=1}^{\ell} g_j^{b_j}$ where each $b_j \geq 0$ and $\sum_{j=1}^{\ell} b_j = r$. Note that there is some j such that $b_j \geq \max\{a_1, \ldots, a_s\}$, thus $x_{i_j}^{b_j}$ divides the term. By the inequality, the term is divisible by a minimal generator of $\mathfrak{q}$. This shows every term of q^r is in $\mathfrak{q}$. □

Definition 2.22. The *radical* of an ideal $I \subset R$ is $\sqrt{I} = \{p \in R \mid p^r \in I$ for some integer $r > 0\}$.

Remark 2.23. If $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is the smallest prime ideal that contains $\mathfrak{q}$ [2, Proposition 4.1]. Thus, if $\mathfrak{q}$ is a primary monomial ideal with $\mathcal{G}(\mathfrak{q}) = \{x_{i_1}^{a_1}, \dots, x_{i_s}^{a_s}, m_1, \dots, m_t\}$, then $\sqrt{\mathfrak{q}} = \langle x_{i_1}, \dots, x_{i_s} \rangle$.

Notation 2.24. We note that a monomial in R is entirely determined by its multidegree, hence given some $\omega \in \mathbb{N}^n$ we identify the monomial $M_\omega = x_1^{\omega_1} \cdots x_n^{\omega_n}$. Similarly, we denote the irreducible ideal $\mathfrak{b}_\omega = \langle x_i^{\omega_i} \mid 0 < \omega_i \rangle$. Note that $\sqrt{\mathfrak{b}_\omega} = \mathfrak{p}_W$ where $W = \operatorname{supp}(M_\omega) = \{x_i \mid 0 < \omega_i\}$.

Definition 2.25. For any $p \in R$ and ideal $I \subseteq R$, the *colon ideal* is $(I : p) = \{q \in R \mid pq \in I\}$.

Definition 2.26. A *minimal primary decomposition* of an ideal I is the intersection of primary ideals $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ such that no $\mathfrak{q}_i$ is contained in the intersection of the other $s - 1$ ideals, and each $\sqrt{\mathfrak{q}_j}$ is unique.

Remark 2.27. Every monomial ideal in R admits a minimal primary decomposition. More generally, [2, Theorem 7.13] shows every ideal of a Noetherian ring admits a minimal primary decomposition.

Theorem 2.28. *[2, Theorem 4.5] For any minimal primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$, define $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. The set $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ is precisely equal to $\{(I : x)$ is prime $\mid x \in R\}$. Hence, this set is independent of the particular primary decomposition.*

Definition 2.29. The set of *associated primes* of I is the set $\operatorname{Ass}(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ in Theorem 2.28.

Definition 2.30. An *irredundant irreducible decomposition* of an ideal I is the intersection of irreducible ideals $I = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_s$ such that no $\mathfrak{b}_i$ is contained in the intersection of the other $s - 1$ ideals.

Theorem 2.31. *[15, Theorem 5.27] Every monomial ideal admits a unique irredundant irreducible decomposition.*

Remark 2.32. Note that Theorem 2.31 implies that monomial ideals have unique minimal primary decompositions. In particular, an irredundant irreducible decomposition provides "finer" data than a minimal primary decomposition. To be precise, given an irredundant irreducible decomposition $I = \mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_s$, we relabel the irreducible ideals in the form $\mathfrak{b}_{i,j}$ such that any two ideals with the same first component have the same radical. That is, $\sqrt{\mathfrak{b}_{i,j}} = \sqrt{\mathfrak{b}_{i,\ell}}$. Now, we may group the irreducible ideals to give a minimal primary decomposition:

$$I = \underbrace{\mathfrak{b}_{1,1} \cap \cdots \cap \mathfrak{b}_{1,s_1}}_{\mathfrak{q}_1} \cap \underbrace{\mathfrak{b}_{2,1} \cap \cdots \cap \mathfrak{b}_{2,s_2}}_{\mathfrak{q}_2} \cap \cdots \cap \underbrace{\mathfrak{b}_{t,1} \cap \cdots \cap \mathfrak{b}_{t,s_t}}_{\mathfrak{q}_t}.$$

Hence if I is a monomial ideal, then $I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$ is a unique minimal primary decomposition. In Chapter 4 we will revisit Theorem 2.31 and work with the machinery used in [15] to prove it.

The remainder of this Section involves working with monomial ideals and square-free monomial ideals that are in different rings. The purpose of the remainder of this Section is to establish our novel tool, symmetric polarization, which maps between the two types of ideals.

Notation 2.33. For clarity we will use capital letters to denote variables and monomials in one ring, and lowercase letters for variables and monomials in the other ring. In particular we take $V = \{X_1, \ldots, X_n\}$ and for some $\omega \in \mathbb{N}^n$ we set

$$V^\omega = \{x_{1,1}, \ldots, x_{1,\omega_1}, x_{2,1}, \ldots, x_{n-1,\omega_{n-1}}, x_{n,1}, \ldots, x_{n,\omega_n}\},$$

where we view $x_{i,j}$ as the j-th "copy" of X_i. We will primarily consider monomial ideals in $k[V]$ and square-free monomial ideals in $k[V^\omega]$.

Definition 2.34. If $\omega \in \mathbb{N}^n$ and $M \in k[V]$ is a monomial with $\mathrm{mdeg}(M) \preceq \omega$, then the *polarization of* M is the square-free monomial:

$$\mathrm{Pol}(M) = \prod_{X_i^j \mid M} x_{i,j} \in k[V^\omega].$$

Definition 2.35. Let I be a monomial ideal in $k[V]$ with $\mathcal{G}(I) = \{M_1, \ldots, M_s\}$ and let $\omega = (\omega_1, \ldots, \omega_n)$ be the minimal element of $\mathbb{N}^n$ such that $\mathrm{mdeg}(M_i) \preceq \omega$ for every $M_i \in \mathcal{G}(I)$. The *polarization of* I, denoted $\mathrm{Pol}(I)$, is the ideal in $k[V^\omega]$ minimally generated by $\{\mathrm{Pol}(M_i) \mid M_i \in \mathcal{G}(I)\}$.

Example 2.36. If $I = \langle X_1^3,\ X_1^2X_2,\ X_1X_2^3\rangle$ then $V^\omega = \{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}\}$ and

$$\mathrm{Pol}(I) = \langle x_{1,1}x_{1,2}x_{1,3},\ x_{1,1}x_{1,2}x_{2,1},\ x_{1,1}x_{2,1}x_{2,2}x_{2,3}\rangle .$$

In literature, polarization is a useful tool. Of particular note, Faridi [7] proved a number of results about polarization itself and monomial ideals that polarize to ideals in correspondence with "simplicial trees". However, our interest in capturing various symmetries motivates us to treat each $x_{i,j}$ as an equal copy of x_i, so we introduce a process called symmetric polarization.

Definition 2.37. For some $\omega \in \mathbb{N}^n$ let Y be the set of square-free monomials in $k[V^\omega]$ and Z be the set of monomials in $k[V]$ that divide m_ω. We have a map $\mathcal{S}_\omega : Y \to Z$,

$$\mathcal{S}_\omega(m) = \prod_{x_{i,j} \mid m} X_i.$$

Now, we define the *symmetric polarization with respect to* ω, $\mathrm{sPol}(-;\omega) = \mathcal{S}_\omega^{-1}$, to be a (possibly) multi-valued function that maps monomials $M \in Z$ to:

$$M \mapsto \mathcal{S}_\omega^{-1}(M) = \{m \in Y \mid \mathcal{S}_\omega(m) = M\}.$$

Definition 2.38. If $\omega \in \mathbb{N}^n$ and I is a monomial ideal in $k[V]$ with minimal generators $\mathcal{G}(I) = \{M_1, \ldots, M_s\}$, then the *symmetric polarization of* I *with respect to* ω, denoted $\mathrm{sPol}(I;\omega)$, is the square-free ideal in $k[V^\omega]$ minimally generated by:

$$\{\mathrm{sPol}(M_i;\omega) \mid M_i \in \mathcal{G}(I)\}.$$

Example 2.39. If $I = \langle X_1^2 X_2, X_2^3 \rangle$ and $\omega = (3,3)$, then $V^\omega = \{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}\}$ and

$$\begin{aligned}\mathrm{sPol}(I;\omega) = \langle & x_{1,1}x_{1,2}x_{2,1},\ x_{1,1}x_{1,2}x_{2,2},\ x_{1,1}x_{1,2}x_{2,3},\\ & x_{1,1}x_{1,3}x_{2,1},\ x_{1,1}x_{1,3}x_{2,2},\ x_{1,1}x_{1,3}x_{2,3},\\ & x_{1,2}x_{1,3}x_{2,1},\ x_{1,2}x_{1,3}x_{2,2},\ x_{1,2}x_{1,3}x_{2,3},\\ & x_{2,1}x_{2,2}x_{2,3}\rangle,\end{aligned}$$

where the first nine generators are $\mathrm{sPol}(X_1^2 X_2;\omega)$ and the tenth is $\mathrm{sPol}(X_2^3;\omega)$.

Remark 2.40. Given a fixed monomial ideal I, one question we ask is when does I admit a monomial ideal J and $\omega \in \mathbb{N}^t$ such that $I = \mathrm{sPol}(J;\omega)$? In general, $\omega = (1, \ldots, 1)$ and $J = I$ works. We now characterize every such possibility.

Definition 2.41. Given a finite set X, the group of all bijections from X to X is the *symmetric group* on X, denoted S_X. When $X = \{1, \ldots, n\}$, we will sometimes use S_n to denote S_X.

Definition 2.42. For a monomial $M = X_{i_1}^{a_{i_1}} \cdots X_{i_s}^{a_{i_s}}$ and permutation $\sigma \in S_n$ we define $\sigma(M) = X_{\sigma(i_1)}^{a_{\sigma(i_1)}} \cdots X_{\sigma(i_s)}^{a_{\sigma(i_s)}}$. If I is a monomial ideal, then we define $\sigma(I)$ to be the ideal with minimal generators $\mathcal{G}(\sigma(I)) = \{\sigma(M) \mid M \in \mathcal{G}(I)\}$. We say I is σ-*stable* if $\sigma(I) = I$.

Note, this is the extension of the natural group action of S_n on V to a group action on monomial ideals. As such, we have that if $\sigma, \sigma' \in S_n$ then $\sigma(\sigma'(I)) = (\sigma\sigma')(I)$.

Lemma 2.43. *Let $I \subseteq k[V]$ be a monomial ideal, σ_{ij} be the transposition of X_i and X_j, and $\sim_I$ be the relation on V defined by $X_i \sim_I X_j$ if and only if I is σ_{ij}-stable. Then $\sim_I$ is an equivalence relation.*

Proof. If (1) is the identity permutation then I is (1)-stable so $\sim_I$ is reflexive. The transposition σ_{ij} is exactly σ_{ji} so $\sim_I$ is symmetric. Finally, suppose $X_i \sim_I X_j$ and $X_j \sim_I X_\ell$. Then

$$\begin{aligned} \sigma_{i\ell}(I) &= (\sigma_{ij}\sigma_{j\ell}\sigma_{ij})(I) \\ &= (\sigma_{ij}\sigma_{j\ell})(I) \\ &= \sigma_{ij}(I) \\ &= I. \end{aligned}$$

Hence I is $\sigma_{i\ell}$-stable, and $X_i \sim_I X_\ell$. $\square$

Definition 2.44. If $W = \{x_1, \ldots, x_n\}$ and $I \subseteq k[W]$ is a monomial ideal, then a partition $W = W_1 \cup \cdots \cup W_t$ is *I-admissible* if every W_i is a subset of an equivalence class of $W/\sim_I$.

Notation 2.45. Given a monomial ideal I and an I-admissible partition $W = W_1 \cup \cdots \cup W_t$ we set $\omega = (|W_1|, \ldots, |W_t|) \in \mathbb{N}^t$ and relabel the elements of $W \rightsquigarrow V^\omega$ such that $W_i \rightsquigarrow \{x_{i,1}, \ldots, x_{i,\omega_i}\}$.

Example 2.46. Let $W = \{x_1, \ldots, x_6\}$ and

$$\begin{aligned} I = \langle & x_1x_2x_4,\ x_1x_2x_5,\ x_1x_2x_6, \\ & x_1x_3x_4,\ x_1x_3x_5,\ x_1x_3x_6, \\ & x_2x_3x_4,\ x_2x_3x_5,\ x_2x_3x_6, \\ & x_4x_5x_6\rangle. \end{aligned}$$

We have $W/\sim_I = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}\}$, and hence $W = \{x_1, x_2\} \cup \{x_3\} \cup \{x_4, x_5\} \cup \{x_6\}$ is an I-admissible partition (among several others). For our own sanity later in Example

2.48, we will substitute letters in lieu of the first index when relabeling W as:

$$\begin{aligned}\{x_1, x_2\} &\rightsquigarrow \{x_1, x_2\};\\ \{x_3\} &\rightsquigarrow \{y_1\};\\ \{x_4, x_5\} &\rightsquigarrow \{w_1, w_2\};\\ \{x_6\} &\rightsquigarrow \{z_1\}.\end{aligned}$$

Theorem 2.47. *Let $I \subseteq R = k[W]$ be a square-free monomial ideal and V^ω be a relabeling of W as given by Notation 2.45 for some I-admissible partition of W. If*

$$J = \left\langle \prod_{x_{i,j} \mid m_\ell} X_i \;\middle|\; m_\ell \in \mathcal{G}(I) \right\rangle$$

then $I = \mathrm{sPol}(J; \omega)$. Moreover, if there are some J', ω', and relabeling $V^{\omega'} \rightsquigarrow W$ such that $I \rightsquigarrow \mathrm{sPol}(J'; \omega')$, then there must be some I-admissible partition of W such that J' is constructed as above.

Proof. After relabeling, we note there is a natural action by the group $G_\omega = S_{\omega_1} \times \cdots \times S_{\omega_t}$ on R where each $\sigma = (\sigma_1, \ldots, \sigma_t) \in G_\omega$ acts on each variable by $\sigma(x_{i,j}) = x_{i,\sigma_i(j)}$. Showing that I is σ-stable for every $\sigma \in G_\omega$ follows from the popular exercise in Group Theory showing that S_n is generated by its transpositions. Now, consider a monomial $M \in k[X_1, \ldots, X_t]$ such that $\mathrm{mdeg}(M) \preceq \omega$. If m is some square-free monomial in I with $\mathcal{S}_\omega(m) = M$, then $\mathrm{sPol}(M; \omega)$ is the orbit of m under G_ω, hence by G_ω-stability, $\mathrm{sPol}(M; \omega) \subseteq I$. In particular, if $m_i \in \mathcal{G}(I)$ and $\sigma \in G_\omega$ then $\sigma(m_i) \in \mathcal{G}(I)$. It then follows that:

$$\begin{aligned}\mathrm{sPol}(J; \omega) &= \langle \mathrm{sPol}(\mathcal{S}_\omega(m_i); \omega) \mid m_i \in \mathcal{G}(I)\rangle\\ &= \langle m_i \mid m_i \in \mathcal{G}(I)\rangle\\ &= I.\end{aligned}$$

Given any J', ω', and $\rightsquigarrow$ as described in the theorem statement, we partition

$$V^{\omega'} = \mathrm{sPol}(X_1; \omega') \cup \cdots \cup \mathrm{sPol}(X_s; \omega').$$

By assigning each $\mathrm{sPol}(X_i) \rightsquigarrow W_i$, we recover a partition $W = W_1 \cup \cdots \cup W_s$. It now suffices to show that any pair $x_{\ell_1}, x_{\ell_2} \in W_i$ has $x_{\ell_1} \sim_I x_{\ell_2}$. By assignment, we have some $x_{i,j_1} \rightsquigarrow x_{\ell_1}$ and $x_{i,j_2} \rightsquigarrow x_{\ell_2}$. We set σ to be the element of $G_{\omega'}$ with the transposition $(j_1\ j_2)$ in the i-th entry and the identity in all other entries, so that $\sigma \rightsquigarrow (\ell_1\ \ell_2)$. We note that any $\mathrm{sPol}(-; \omega')$ is necessarily $G_{\omega'}$-stable, so in particular $\mathrm{sPol}(J', \omega')$ is σ-stable. Applying $\rightsquigarrow$ to this statement gives I is $(\ell_1\ \ell_2)$-stable. $\square$

Example 2.48. Continuing with Example 2.46, our relabeling gives

$$\begin{aligned} I = \langle & x_1x_2w_1,\ x_1x_2w_2,\ x_1x_2z_1, \\ & x_1y_1w_1,\ x_1y_1w_2,\ x_1y_1z_1, \\ & x_2y_1w_1,\ x_2y_1w_2,\ x_2y_1z_1, \\ & w_1w_2z_1\rangle. \end{aligned}$$

Hence J, as given by Theorem 2.47 is:

$$\begin{aligned} J = \langle & X^2W,\ X^2W,\ X^2Z, \\ & XYW,\ XYW,\ XYZ, \\ & XYW,\ XYW,\ XYZ, \\ & W^2Z\rangle. \end{aligned}$$

That is, $J = \langle X^2W,\ X^2Z,\ XYW,\ XYZ,\ W^2Z\rangle$.

Remark 2.49. Recall that in Example 2.39 we calculated $\mathrm{sPol}(\langle X_1^2X_2, X_2^3\rangle\,; (3,3))$, which is a square-free monomial ideal that is isomorphic to the ideal I in Example 2.48. Hence taking J from Example 2.48, we have $\mathrm{sPol}(\langle X_1^2X_2, X_2^3\rangle\,; (3,3)) \cong \mathrm{sPol}(J; (2,1,2,1))$. Further every other pair of a monomial ideal and element of $\mathbb{N}^t$ that would symmetrically polarize to a square-free monomial ideal isomorphic to these two comes from the other I-admissible partitions mentioned in Example 2.46.

Lemma 2.50. *If $M \in k[V]$ is a monomial with $\mathrm{mdeg}(M) \preceq \omega$ and M' is the monomial with $\mathrm{mdeg}(M') = \omega - \mathrm{mdeg}(M)$, then*

$$\mathrm{sPol}(M'; \omega) = \left\{ m_{V^\omega \setminus W} \;\middle|\; m_W \in \mathrm{sPol}(M; \omega) \right\}.$$

Proof. We first observe that $\mathcal{S}_\omega$ is multiplicative for square-free monomials m_1, m_2 with $\gcd(m_1, m_2) = 1$. That is, $\mathcal{S}_\omega(m_1m_2) = \mathcal{S}_\omega(m_1)\mathcal{S}_\omega(m_2)$. Thus, for every $m_W \in \mathrm{sPol}(M; \omega)$ we have

$$\mathcal{S}_\omega(m_{V^\omega}) = \mathcal{S}_\omega(m_W)\mathcal{S}_\omega(m_{V^\omega \setminus W}).$$

Hence, $MM' = M\mathcal{S}_\omega(m_{V^\omega \setminus W})$ and $m_{V^\omega \setminus W} \in \mathrm{sPol}(M'; \omega)$. The argument for the other necessary containment is essentially identical. $\square$

2.2 Combinatorial Preliminaries

Combinatorial Commutative Algebra is the product of combining two different fields, neither of which has entirely consistent conventions for neither notation, names, nor definitions. In this section, we introduce the combinatorial definitions as well as choices in terminology and notation that are intended to be concise yet provide insight and clarity. Throughout, we will repeatedly use the same hypergraphs, H_1 and $Z(n)$ respectively introduced in Example 2.57 and Example 2.60 to illustrate examples (or non-examples) of most of the following definitions.

Definition 2.51. A *hypergraph* H is a pair of non-empty, finite sets (V, E) called vertices and edges such that:

- E is a subset of the power set of V;
- no pair of edges $e, e' \in E$ satisfy $e \subset e'$.

Definition 2.52. A *subhypergraph of* H *on* W is a pair $H_W = (W, F)$ such that $\emptyset \neq W \subseteq V$ and $F = \{e \in E \mid e \subseteq W\}$.

Remark 2.53. For topological reasons that will be apparent in Section 3.3, our definition of a hypergraph does allow singleton edges and isolated vertices, i.e. vertices that do not belong to an edge. Note that a subhypergraph is entirely determined by two data: the parent hypergraph and a subset of the parent's vertices. Hence, the notation H_W in Definition 2.52 is unambiguous. Many authors refer to hypergraphs as "simple hypergraphs" and subhypergraphs as "induced subhypergraphs". In this work, we will only be concerned with these objects, hence we drop the extra adjectives for semantic succinctness. Unless otherwise specified, we always assume $H = (V, E)$ and $V = \{x_1, \ldots, x_n\}$.

Definition 2.54. A hypergraph is *proper* if every vertex $x_i \in V$ is contained in some $e \in E$.

Definition 2.55. A hypergraph is r*-uniform* if $|e| = r$ for all $e \in E$.

Notation 2.56. On occasion, it will be convenient to verbally specify the size of an edge. If an edge has r vertices, we say it is an r*-edge*. For example, an r-uniform hypergraph is a hypergraph with only r-edges.

Example 2.57. The set of 2-uniform hypergraphs are commonly known as undirected graphs (without loops or repeated edges.) These hypergraphs are the easiest to represent pictorially. We do so by drawing a dot or variable to indicate a vertex. We use lines connecting two vertices to indicate an edge containing those two vertices.

Below, we depict

$$H_1 = (\{x_1, x_2, x_3, x_4\},\ \{\ \{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\ \}).$$

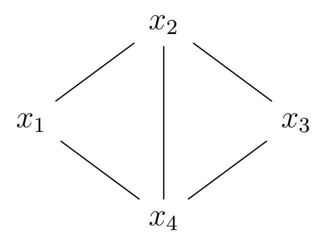


Figure 2.1: H_1

Definition 2.58. The 2-uniform hypergraph $C_n = (\{x_1, \ldots, x_n\}, \{\{x_i, x_j\} \mid j - i \equiv 1 \mod n\})$ is called the *cycle graph with n vertices.*

Definition 2.59. A hypergraph H is *intersecting* if for every pair of edges $e, e' \in E$, $e \cap e' \neq \emptyset$.

Example 2.60. Depicting hypergraphs with larger edges is less straightforward. We borrow notation from [3] where each edge is again depicted by a line connecting two of its member vertices. If that edge contains more than the two vertices, its other members are listed along the representative line. We depict this with the following family of intersecting hypergraphs for $n \geq 2$:

$$Z(n) = (\{z, x_1, \ldots, x_n\},\ \{\ \{x_1, \ldots, x_n\}, \{z, x_1\}, \ldots, \{z, x_n\}\ \}).$$

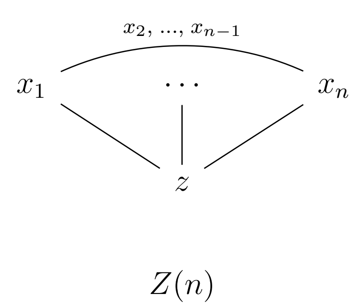


Figure 2.2: $Z(n)$

Example 2.61. The complete r-uniform hypergraph on n vertices, K_n^r, with edge set $E = \{e \subseteq V \mid |e| = r\}$ is r-uniform. If $r > \frac{n}{2}$ then K_n^r is also intersecting.

Definition 2.62. Given a hypergraph H, a set $W \subseteq V$ is *independent* if there is no $e \in E$ such that $e \subseteq W$. Further, W is *maximally independent* if there is no independent set W' such that $W \subsetneq W'$.

Definition 2.63. An *s-coloring* of a hypergraph is a partition of its vertices $V = V_1 \cup \cdots \cup V_s$ such that each V_i is independent.

Definition 2.64. The *chromatic number* of a hypergraph, $\chi(H)$, is the smallest integer s such that an s-coloring of H exists.

Remark 2.65. If an s-coloring of a hypergraph exists, we say it is s-colorable. H has a 1-edge if and only if it is not s-colorable for any $s \in \mathbb{N}$. In this case, we choose the convention $\chi(H) = \infty$.

Example 2.66. The maximal independent sets of H_1 from Example 2.57 are $\{x_1, x_3\}, \{x_2\}$, and $\{x_4\}$. Hence, there is no 2-coloring of H_1 and there is a unique 3-coloring of H_1. Indeed, $\chi(H_1) = 3$.

Example 2.67. The maximal independent sets of $Z(n)$ from Example 2.60 are $\{z\}$ and $\{x_1, \ldots, x_n\} \backslash \{x_i\}$ for $1 \leq i \leq n$. Hence, there is no 2-coloring of $Z(n)$. Any coloring of this hypergraph must have $\{z\}$ as one component. The remaining n vertices can be colored by any partition consisting of two non-empty sets. If these sets have cardinalities n_1 and n_2, then $n = n_1 + n_2$ and there are $\binom{n}{n_1} = \binom{n}{n_2}$ possibilities for a partition consisting of two sets with these sizes. By symmetry, the following sum counts twice the number of these partitions:

$$\binom{n}{1} + \cdots + \binom{n}{n-1} = 2^n - 2.$$

The equality follows by recalling that 2^n is the sum of the n-th row of Pascal's triangle. Thus, the number of 3-colorings of $Z(n)$ is exactly $2^{n-1} - 1$. Because a 3-coloring of $Z(n)$ exists and there are no such 2-colorings, we have $\chi(Z(n)) = 3$.

Definition 2.68. A hypergraph with more than one vertex is *critically s-chromatic* if $\chi(H) = s$ and $\chi(H_{V \backslash \{x_i\}}) < s$ for every $x_i \in V$.

Proposition 2.69. *If a hypergraph is critically chromatic then it is proper.*

Proof. We show the contrapositive; if H is not proper then there is a vertex x_i that does not belong to any edge of H. Thus H and $H_{V \backslash \{x_i\}}$ share the same edge set, and if W is independent in $H_{V \backslash \{x_i\}}$ then $W \cup \{x_i\}$ is independent in H. Thus any s-coloring of $H_{V \backslash \{x_i\}}$ yields a s-coloring of H, so H is not critically chromatic. $\square$

Example 2.70. The hypergraph H_1 from Example 2.57 is not critically chromatic. Indeed, the subhypergraph given by removing the vertex x_3 is C_3, the smallest graph that is not 2-colorable.

On the other hand, $Z(n)$ from Example 2.60 is critically 3-chromatic. The subhypergraph given by removing z from $Z(n)$ has a single edge which is 2-colorable. If x_i is removed, the remaining edges are the 2-edges in $Z(n)$ less the edge $\{z, x_i\}$. This subhypergraph, a so-called star, is 2-colored by the partition $V \backslash \{x_i\} = \{z\} \cup \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$.

Definition 2.71. A *vertex cover* of a hypergraph H is a subset $W \subseteq V$ such that $W \cap e \neq \emptyset$ for every $e \in E$. Further, W is a *minimal vertex cover* if no proper subset of W is a vertex cover.

Example 2.72. The minimal vertex covers of H_1 from Example 2.57 are $\{x_2, x_4\}$, $\{x_1, x_3, x_4\}$, and $\{x_1, x_2, x_3\}$.

Example 2.73. The minimal vertex covers of $Z(n)$ from Example 2.60 are exactly the edges of the hypergraph. This property will be the focus of Chapter 3.

Lemma 2.74. *W is a (minimal) vertex cover if and only if its complement, $\overline{W} = V \backslash W$, is a (maximal) independent set.*

Proof. If W is a vertex cover, then each $e_i \in E$ has some $x_i \in W \cap e_i$. Thus, $x_i \notin \overline{W}$, so it cannot be that $e_i \subseteq \overline{W}$. Hence $\overline{W}$ is independent. On the other hand, if $\overline{W}$ is independent then each $e_i \in E$ has some $x_i \in e_i$ with $x_i \notin \overline{W}$. Hence, $x_i \in W$, which shows W meets every e_i and is thus a vertex cover.

Further, if W is a minimal vertex cover, then for every $x_i \in W$ the set $W \backslash \{x_i\}$ is not a vertex cover. Hence the complement $\overline{W} \cup \{x_i\}$ is not independent. This shows no set that contains $\overline{W}$ is independent, so $\overline{W}$ is maximally independent. Similarly, if $x_i \in W$ and $\overline{W}$ is maximally independent, then no $\overline{W} \cup \{x_i\}$ is independent. Thus, no $W \backslash \{x_i\}$ is a vertex cover and thus W is a minimal vertex cover. □

We conclude this section by introducing "hypergraph expansions". In Chapter 4 we will use the s-th expansion of H to understand certain properties of (the s-th power of) a square-free monomial ideal related to H.

Definition 2.75. The *s-th expansion* of a hypergraph $H = (V, E)$, denoted H^s, is a hypergraph with vertex set $W = \{x_{1,1}, \ldots, x_{1,s}, x_{2,1}, \ldots, x_{n-1,s}, x_{n,1}, \ldots, x_{n,s}\}$ and edge set $F_1 \cup F_2$, where

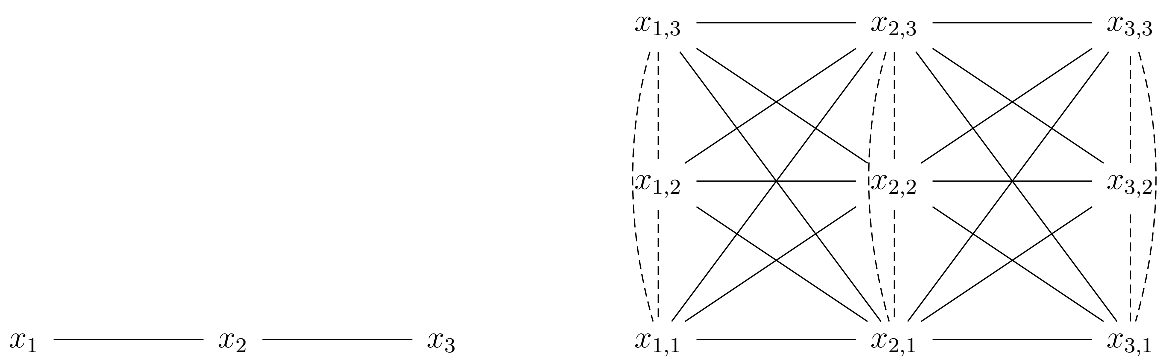
$$F_1 = \{\{x_{i_1,j_1}, \ldots, x_{i_t,j_t}\} \mid \{x_{i_1} \ldots, x_{i_t}\} \in E\}$$

consists of every possible "copy" of an edge in E and

$$F_2 = \{\{x_{i,a}, x_{i,b}\} \mid 1 \leq a \neq b \leq s\}$$

consists of every possible size 2 edge with members that are "copies" of the same vertex.

Example 2.76. We consider the third expansion of P_2, the path with 2 edges. The solid lines represent edges in F_1 whereas the dashed lines represent edges in F_2.



P_2 $(P_2)^3$

Figure 2.3: P_2 and $(P_2)^3$

Remark 2.77. Hypergraph expansions grow very quickly; an edge with r vertices has s^r copies in an s-th expansion. Below, we depict the second expansion of the hypergraph that contains a single 3-edge: $H_3 = (\{x_1, x_2, x_3\},\ \{\ \{x_1, x_2, x_3\}\ \})$. For clarity, we suppress the label on each of the eight 3-edges by instead using colors. A blue edge contains the vertex $x_{2,2}$, whereas a purple edge contains the vertex $x_{2,1}$. Each of the dashed lines are the 2-edges from F_2.

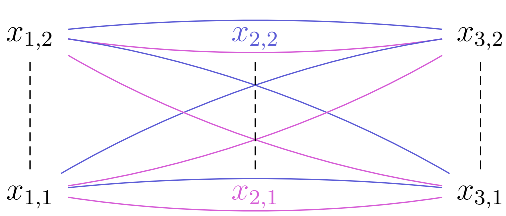


Figure 2.4: $(H_3)^3$

2.3 Simplicial Complexes and Alexander Duality

We start this Section by briefly defining simplicial complexes and use them to define Alexander duality in its first incarnation. Then we detail the correspondence between hypergraphs and simplicial complexes, which consequently provides a second version of Alexander duality for hypergraphs.

Definition 2.78. A *simplicial complex* Δ on a vertex set V is a collection of subsets of V closed under taking subsets, that is if $f \in \Delta$ and $g \subseteq f$, then $g \in \Delta$.

Definition 2.79. An element of a simplicial complex is a *face* and a maximal face is a *facet.*

Remark 2.80. A simplicial complex is entirely determined by its facets. Indeed, the faces of the simplicial complex are precisely the collection of subsets of the facets.

Definition 2.81. The *Alexander dual of a simplicial complex* Δ, denoted $\Delta^\vee$, is the simplicial complex on the same vertex set consisting of the complements of the non-faces of Δ, that is:

$$\Delta^\vee = \left\{ \overline{f} \;\middle|\; f \in V \text{ and } f \notin \Delta \right\}.$$

Proposition 2.82. *Taking the Alexander dual of a simplicial complex is an involution, that is:*

$$(\Delta^\vee)^\vee = \Delta.$$

Proof. We first observe that $(\Delta^\vee)^\vee = \left\{ f \;\middle|\; \overline{f} \notin \Delta^\vee \right\}$. Note that $\overline{f} \notin \Delta^\vee$ implies that $f \in \Delta$, and hence $(\Delta^\vee)^\vee = \{f \mid f \in \Delta\} = \Delta$. □

Definition 2.83. A simplicial complex Δ is called *sub-dual* if $\Delta \subseteq \Delta^\vee$, and *super-dual* if $\Delta^\vee \subseteq \Delta$. In turn, Δ is called *self-dual* if it is both sub-dual and super-dual, and *transient* if it is neither sub-dual nor super-dual.

Simplicial complexes are a useful tool to track the data of independent sets of a hypergraph, which gives rise to a correspondence between the two families of objects.

Definition 2.84. Given a hypergraph H, the set of independent sets form a simplicial complex, called the *independence complex* of H:

$$\Delta(H) = \{W \subseteq V \mid W \text{ is independent in } H\}.$$

Remark 2.85. The independence complex of a hypergraph is indeed a simplicial complex as the subset of an independent set is necessarily independent. Moreover, the minimal non-faces of $\Delta(H)$ (i.e. the smallest sets that are not independent in H) are precisely the edges of H. Thus, the facets of the Alexander dual, $\Delta(H)^\vee$, are the complements of the edges of H.

Proposition 2.86. *The independence complex gives a bijective correspondence between simplicial complexes and hypergraphs.*

Proof. This is the logical conclusion of combining Remarks 2.80 and 2.85 as well as Proposition 2.82. In particular, Proposition 2.82 shows that taking the Alexander dual is a bijective correspondence of simplicial complexes. In turn, Remark 2.80 means a simplicial complex is determined both by its minimal non-faces as well as its facets. In the case of independence complexes, Remark 2.85 notes the edges are precisely the minimal non-faces, hence both H and $\Delta(H)$ are entirely and uniquely determined by the same data. $\square$

Example 2.87. For small n it is convenient to use a Hasse diagram to visualize a simplicial complex, and in our case also the non-faces. We do so by representing subsets of V as a string of its elements, and lines between these sets represent the relation "contains one fewer/more element." The depictions below show the entire partially ordered set 2^V, so we use black to represent faces of a simplicial complex and gray for the non-faces.

Recall that in Example 2.57 the maximal independent sets of H_1 are $\{x_1, x_3\}$, $\{x_2\}$, and $\{x_4\}$. These are the facets of $\Delta(H_1) = \{ \{x_1, x_3\}, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \emptyset \}$.

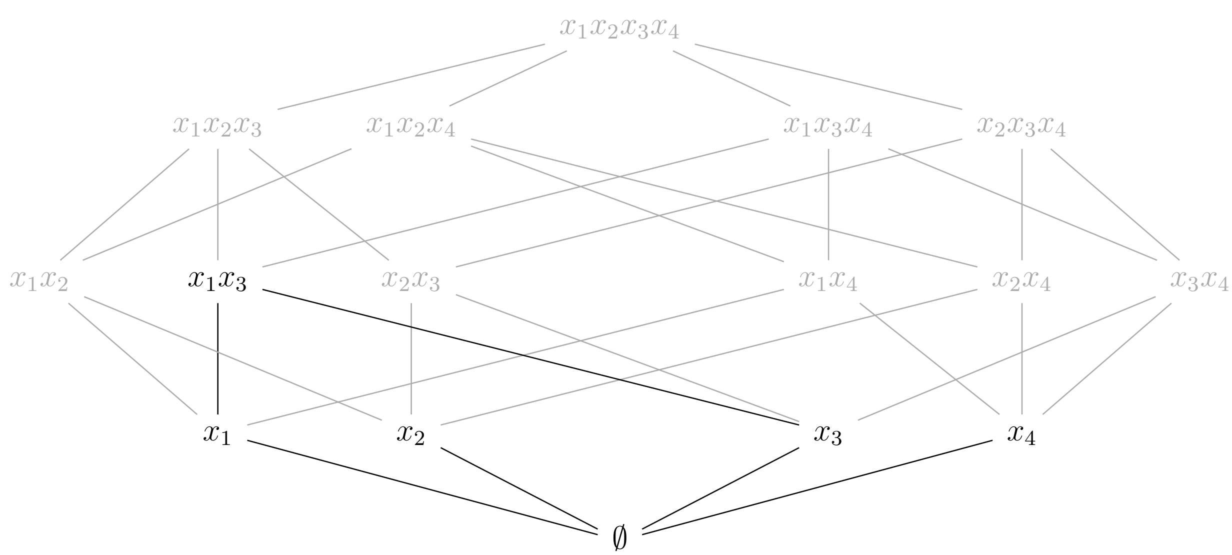


Figure 2.5: $\Delta(H_1)$

Finding the Alexander dual from this diagram is a straightforward task; we list the complements of the non-faces from top to bottom and left to right:

$$\Delta(H_1)^\vee = \{\ \emptyset, \{x_4\}, \{x_3\}, \{x_2\}, \{x_1\}, \{x_3, x_4\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_1, x_3\}, \{x_1, x_2\}\ \}.$$

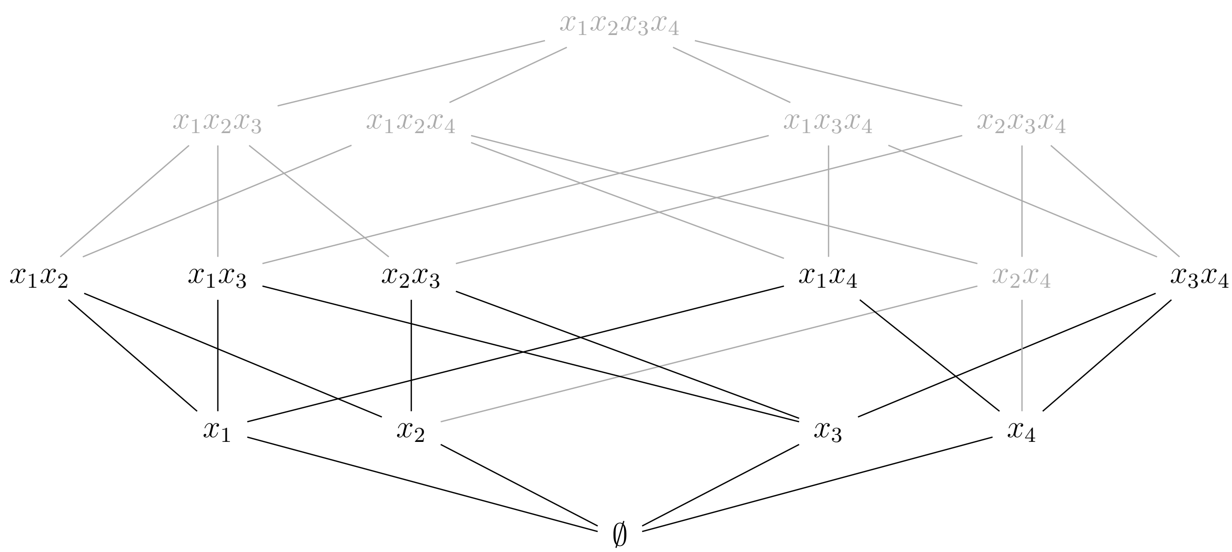


Figure 2.6: $\Delta(H_1)^\vee$

Indeed, we can observe that the facets of $\Delta(H_1)^\vee$ are exactly the complements of the edges of H_1. It suffices to recall that H_1 has every possible 2-edge except for $\{x_1, x_3\}$, the complement of $\{x_2, x_4\}$. Finally, we note that $\Delta(H_1)$ is sub-dual, and equivalently $\Delta(H_1)^\vee$ is super-dual.

Proposition 2.88. $\Delta(H)^\vee$ *is the independence complex of the hypergraph whose edges consist of the minimal vertex covers of* H*.*

Proof. Let $H^\vee$ denote the hypergraph for which $\Delta(H)^\vee$ is the independence complex, that is $\Delta(H^\vee) = \Delta(H)^\vee$. By Remark 2.85 the edges of $H^\vee$ are complements of the facets of $\Delta(H^\vee)^\vee = \Delta(H)$, or rather the complements of the maximal independent sets of H. Thus by Proposition 2.74 the edges of $H^\vee$ are the minimal vertex covers of H. □

Definition 2.89. The *Alexander dual of a hypergraph* H is $H^\vee = (V, F)$ where F is the set of minimal vertex covers of E.

Example 2.90. Taking H_1 from Example 2.57, the edges of $H_1^\vee$ are the minimal non-faces of $\Delta(H_1)^\vee$, or rather the minimal vertex covers of H_1 :

$$H_1^\vee = (\{x_1, x_2, x_3, x_4\},\ \{\ \{x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_3\}\ \}).$$

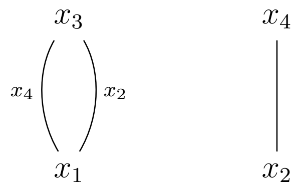


Figure 2.7: $H_1^{\vee}$

2.4 Edge Ideals and Cover Ideals

This Section is essentially an algebraic continuation of Section 2.1 that makes use of the tools introduced in Section 2.2 and Section 2.3. We start with a correspondence between square-free monomial ideals and hypergraphs. Under this correspondence, Alexander duality takes yet a third form. Remark 2.97 provides a thorough account of Alexander duality's behavior under the three-way correspondence between simplicial complexes, hypergraphs, and square-free monomial ideals.

Definition 2.91. Given a hypergraph H, the *edge ideal* of H, denoted $I(H)$, is the square-free monomial ideal with minimal generators:

$$\mathcal{G}(I(H)) = \{m_e \mid e \in E\}.$$

Remark 2.92. The edge ideal gives a bijective correspondence between hypergraphs on $V = \{x_1, \ldots, x_n\}$ and square-free monomial ideals in $R = k[V]$; both the edges and minimal generators uniquely determine a given hypergraph or monomial ideal.

Definition 2.93. Given a hypergraph H, the *cover ideal* of H is the square-free monomial ideal given by the minimal primary decomposition:

$$J(H) = \bigcap_{e \in E} \mathfrak{p}_e.$$

Proposition 2.94. *If m_W is a square-free monomial in $J(H)$ then W is a vertex cover of H, hence the minimal generators of $J(H)$ correspond to the minimal vertex covers of H. That is,*

$$J(H) = \langle m_f \mid f \text{ is a minimal vertex cover of } H\rangle.$$

Proof. For each edge $e \in E$, $m_W \in \mathfrak{p}_e$. Thus there is some $x_{i_e} \in e$ such that $x_{i_e} \mid m_W$, or rather $\{x_{i_e}\} \subseteq e \cap W$. Thus W is a vertex cover of H. □

Proposition 2.95. *A square-free monomial m_W is in $I(H)$ if and only if W is a non-face of $\Delta(H)$, and by duality a square-free monomial m_W is in $J(H)$ if and only if W is a non-face of $\Delta(H^\vee)$.*

Proof. Suppose $m_W \in I(H)$. For some edge $e \in E$, $m_e \mid m_W$ thus W contains the edge e and is not independent.

If W is not independent, then there is some $e \in E$ with $e \subseteq W$. Thus, $m_W = m_e m_{W \setminus e} \in I(H)$. For the dual version, recall that the edges of $H^\vee$ are the minimal vertex covers of H. □

Example 2.96. We give the minimal generators and minimal primary decompositions for the edge and cover ideals of H_1 in Example 2.57:

$$\begin{aligned} I(H_1) &= \langle x_1x_2, x_1x_4, x_2x_3, x_2x_4, x_3x_4 \rangle \\ &= \langle x_2, x_4 \rangle \cap \langle x_1, x_3, x_4 \rangle \cap \langle x_1, x_2, x_3 \rangle \end{aligned}$$

$$\begin{aligned} J(H_1) &= \langle x_1, x_2 \rangle \cap \langle x_1, x_4 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_3, x_4 \rangle \\ &= \langle x_2x_4, x_1x_3x_4, x_1x_2x_3 \rangle . \end{aligned}$$

Remark 2.97. Definition 2.89 and Proposition 2.94 show that edge ideals and cover ideals are Alexander duals in the sense that $J(H) = I(H^\vee)$, and in turn $I(H) = J(H^\vee)$. Moreover, we note that this notion of Alexander duality can be described algebraically for square-free monomial ideals. In particular, m_W is a minimal generator of the edge ideal if and only if $\mathfrak{p}_W$ is a component of the minimal primary decomposition of the corresponding cover ideal. In Chapter 4 we will make use of an algebraic generalization of Alexander duality for monomial ideals where minimal generators are instead in correspondence with components of the irredundant irreducible decomposition of the "generalized dual".

We have equivalently described Alexander duality for three different objects, as well as bijective correspondences between these objects: simplicial complexes, hypergraphs, and square-free monomial ideals. The figure below summarizes each of these relationships. The dotted line is included to emphasize the easier computation between simplicial complexes and hypergraphs. In particular, finding the facets of the Alexander dual of the independence complex of H can be done by listing the complements of the edges of H.

$$\begin{tikzcd}
\text{Simplicial Complexes:} & \Delta(H) \arrow[r, leftrightarrow, "\vee"] \arrow[d, leftrightarrow] & \Delta(H)^\vee \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow, dotted, "\overline{e}"'] & {} \arrow[d, leftrightarrow, "\text{Independence Complexes}"] \\
\text{Hypergraphs:} & H \arrow[r, leftrightarrow, "\vee"] \arrow[d, leftrightarrow] & H^\vee \arrow[d, leftrightarrow] & {} \arrow[d, leftrightarrow, "\text{Edge Ideals}"] \\
\text{Square-free Monomial Ideals:} & I(H) \arrow[r, leftrightarrow, "\vee"] & J(H) & {}
\end{tikzcd}$$

Figure 2.8: Alexander Duality and Correspondences

We have shown that the following problems–that describe moving between the left and the right of the diagram–above are equivalent:

- finding “the Alexander dual”;
- finding the minimal non-faces of a simplicial complex;
- finding the minimal vertex covers of a hypergraph;
- finding the maximal independent sets of a hypergraph;
- finding the minimal primary decomposition of a square-free monomial ideal.

The utility of Alexander duality extends far beyond the problems above; for example the following theorem shows finding the chromatic number of H and a certain ideal containment problem are equivalent.

Theorem 2.98. *[9, Theorem 1.1] If H is a hypergraph and $J = J(H)$ its its cover ideal then*

$$\chi(H) = \min\left(\left\{s \;\middle|\; m_V^{s-1} \in J^s\right\}\right).$$

Proof. We will show that an s-coloring of H exists if and only if the membership $m_V^{s-1} \in J^s$ holds. Note that if W is a vertex cover of H, then the monomial $\frac{m_V}{m_W} = m_{\overline{W}}$ corresponds to an independent set.

($\implies$) Suppose $V = \overline{W_1} \cup \cdots \cup \overline{W_s}$ is an s-coloring of H where each W_i is necessarily a vertex cover. Then $m_V = m_{\overline{W_1}} \cdots m_{\overline{W_s}} = \frac{m_V}{m_{W_1}} \cdots \frac{m_V}{m_{W_s}}$, and so $\frac{m_V^s}{m_V} = m_{W_1} \cdots m_{W_s}$. Hence m_V^{s-1} is the product of s monomials, each corresponding to a vertex cover, and thus belongs to J^s.

($\impliedby$) If $m_V^{s-1} \in J^s$, then there are s vertex covers $W_1, \ldots, W_s$ such that $m_V^{s-1} = m_{W_1} \cdots m_{W_s}$. Reversing the prior argument, we get that $m_V = m_{\overline{W_1}} \cdots m_{\overline{W_s}}$, hence $V = \overline{W_1} \cup \cdots \cup \overline{W_s}$ is an s-coloring of H. □

Algebraically, Alexander duality gives a map between square-free monomial ideals. In Theorem 2.101 we show how this mapping relates to the ring homomorphisms π_W and φ_W introduced in Definition 2.12 and Definition 2.16, respectively. However we first must show that these homomorphisms also map one square-free monomial ideal to another. In general, the image of an ideal need not even be an ideal, for example consider the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

Lemma 2.99. *If I is a (square-free) monomial ideal and $W \subseteq V$, then $\pi_W(I)$ is either a (square-free) monomial ideal or 0, and $\varphi_W(I)$ is either a (square-free) monomial ideal or $k[W]$.*

Proof. We start with the projection map. We partition the minimal generators of I, $\mathcal{G}(I) = A \cup B$ such that $A \subseteq k[W]$ and $B \cap k[W] = \emptyset$. That is, π_W acts as the identity on A and as the zero map on B. By definition,

$$\begin{aligned}
\pi_W(I) &= \left\{ \pi_W \left(\sum_{m_i \in \mathcal{G}(I)} p_i m_i \right) \middle| \, p_i \in R \right\} \\
&= \left\{ \sum_{m_i \in A} \pi_W(p_i)\pi_W(m_i) + \sum_{m_j \in B} \pi_W(p_j)\pi_W(m_j) \middle| \, p_i, p_j \in R \right\} \\
&= \left\{ \sum_{m_i \in A} \pi_W(p_i) m_i \middle| \, p_i \in R \right\} \\
&= \left\{ \sum_{m_i \in A} p_i m_i \middle| \, p_i \in k[W] \right\}.
\end{aligned}$$

The last step follows because π_W is surjective and the final set is the definition of the ideal generated by A in $k[W]$. Hence, if A is empty, then $\pi_W(I) = 0$, otherwise $\pi_W(I)$ is a monomial ideal. If I is square-free then every monomial in A is square-free so $\pi_W(I)$ is also square-free.

Now we consider the localization map. A similar argument gives:

$$\begin{aligned}
\varphi_W(I) &= \left\{ \varphi_W \left(\sum_{m_i \in \mathcal{G}(I)} p_i m_i \right) \middle| \, p_i \in R \right\} \\
&= \left\{ \sum_{m_i \in \mathcal{G}(I)} p_i \varphi_W(m_i) \middle| \, p_i \in k[W] \right\}.
\end{aligned}$$

Thus, $\varphi_W(I)$ is the ideal generated by $\varphi_W(\mathcal{G}(I))$ in $k[W]$. We note that φ_W maps (square-free) monomials not in $k[V \backslash W]$ to (square-free) monomials in $k[W]$. Hence, if $\mathcal{G}(I) \cap k[V \backslash W]$ is empty then $\varphi_W(I)$ is a (square-free) monomial ideal. Otherwise, $\varphi_W(\mathcal{G}(I))$ contains 1 and $\varphi_W(I) = k[W]$. □

Remark 2.100. In Lemma 2.99 the set of monomials, A, is minimal hence $\mathcal{G}(\pi_W(I)) = A$. The minimal generators of $\varphi_W(I)$ are not clear from the proof of the lemma, though we will now show they take a nice combinatorial form.

Theorem 2.101. *If H is a hypergraph and $W \subseteq V$ then $I(H_W) = \pi_W(I(H))$ and $J(H_W) = \varphi_W(J(H))$. In other words, the following diagram (of maps between square-free monomial ideals) commutes.*

$$
\begin{tikzcd}
I(H) \arrow[r, "\pi_W"] \arrow[d, leftrightarrow, "\vee"'] & I(H_W) \arrow[d, "\vee"] \\
J(H) \arrow[r, "\varphi_W"] & J(H_W)
\end{tikzcd}
$$

Proof. Let $\{e_1, \ldots, e_s\}$ be the edges of H. We rearrange labels so that $\{e_1, \ldots, e_a\}$ are the edges contained by W, i.e. the edges of H_W. Note that by Remark 2.100 $\mathcal{G}(\pi_W(I(H))) = \{m_{e_1}, \ldots, m_{e_a}\}$ hence $I(H_W) = \pi_W(I(H))$.

We now show that $J(H_W) = \varphi_W(J(H))$ by showing its dual statement:

$$\pi_W(I(H)) = (\varphi_W(J(H)))^\vee.$$

If $\{f_1, \ldots, f_\ell\}$ are the minimal vertex covers of H then

$$I(H) = \langle m_{e_1}, \ldots, m_{e_s} \rangle = \mathfrak{p}_{f_1} \cap \cdots \cap \mathfrak{p}_{f_\ell};$$

and

$$J(H) = \langle m_{f_1}, \ldots, m_{f_\ell} \rangle = \mathfrak{p}_{e_1} \cap \cdots \cap \mathfrak{p}_{e_s}.$$

By Lemma 2.99 we have:

$$\begin{aligned} \varphi_W(J(H)) &= \langle \varphi_W(\{m_{f_1}, \ldots, m_{f_\ell}\}) \rangle \\ &= \langle m_{f_1 \cap W}, \ldots, m_{f_\ell \cap W} \rangle . \end{aligned}$$

If $\{g_1, \ldots, g_b\}$ are the minimal elements of $\{f_1 \cap W, \ldots, f_\ell \cap W\}$, then $\varphi_W(J(H))$ is minimally generated as $\langle m_{g_1}, \ldots, m_{g_b} \rangle$.

Let $1 \leq i \leq a$. We observe that the monomial m_{e_i} belongs to the intersection $\mathfrak{p}_{f_1} \cap \cdots \cap \mathfrak{p}_{f_\ell}$, and in particular e_i intersects each $f_j \cap W$. That is to say

$$m_{e_i} \in \mathfrak{p}_{f_1 \cap W} \cap \cdots \cap \mathfrak{p}_{f_\ell \cap W} \subseteq \mathfrak{p}_{g_1} \cap \cdots \cap \mathfrak{p}_{g_b} = (\varphi_W(J(H)))^\vee.$$

This shows $\pi_W(I(H)) \subseteq (\varphi_W(J(H)))^\vee$.

If $m_Z \in (\varphi_W(J(H)))^\vee$ is a square-free monomial then Z meets each g_j, hence meets every $f_j \cap W$. In other words $m_Z \in \mathfrak{p}_{f_1} \cap \cdots \cap \mathfrak{p}_{f_\ell} = I(H)$, or equivalently, Z contains an edge of H. We lastly observe that because Z is a subset of W, the edges of H that Z contains are also edges of the subhypergraph H_W. This means $m_Z \in \pi_W(I(H))$, hence $\pi_W(I(H)) = (\varphi_W(J(H)))^\vee$ and the proof is complete. ∎

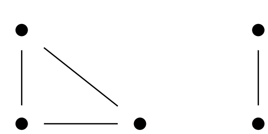
Remark 2.102. In the proof of Theorem 2.101, we explicitly address the cases when the image of π_W and φ_W are 0 and $k[W]$ respectively. These occur simultaneously and precisely when the edge set of H_W is empty. Technically this does not satisfy our definition of a hypergraph, however the argument holds throughout by following the convention that the edge ideal of a hypergraph on W with no edges is 0, and its cover ideal is $k[W]$. Note also the empty set generates the 0 ideal and the empty intersection of ideals is the entire ring $k[W]$.

We conclude this Section and Chapter with a brief consideration of the symmetries of hypergraphs and square-free monomial ideals.

Remark 2.103. The natural action of the symmetric group S_n on V extends to subsets of V. If $\sigma \in S_n$ and $W = \{x_{i_1}, \ldots, x_{i_s}\} \subseteq V$ then $\sigma(W) = \{x_{\sigma(i_1)}, \ldots, x_{\sigma(i_s)}\}$. This action extends further to collections of subsets of V, which are precisely the data that define hypergraphs, square-free monomial ideals, and simplicial complexes. Thus, this action induces a large number of isomorphisms of these objects. We will later be interested in counting the number of these objects "up to isomorphism". The action S_n on V provides the $n!$ isomorphisms that there are to consider, but we must account for when a collection of subsets is fixed by a permutation.

Definition 2.104. An element $\sigma \in S_n$ is an *automorphism of a hypergraph* H if for every edge $e \in E$ the image $\sigma(e)$ is also an edge of H. The *automorphism group* of a hypergraph, $\mathrm{Aut}(H)$, is the collection of every automorphism of H and forms a subgroup of S_n.

Example 2.105. The automorphism group of the graph below is $S_3 \times S_2$.



Lemma 2.106. *For a hypergraph H and permutation of its vertices σ, the following are equivalent:*

(i) $\sigma \in \mathrm{Aut}(H)$*;*

(ii) $I(H)$ *is* σ*-stable;*

(iii) $\sigma \in \mathrm{Aut}(H^\vee)$*;*

(iv) $J(H)$ *is* σ*-stable.*

Proof. ($i \implies ii$) If $\sigma \in \mathrm{Aut}(H)$, then:

$$\begin{aligned}\sigma(I(H)) &= \left\langle m_{\sigma(e)} \mid e \in H \right\rangle \\ &= \langle m_e \mid e \in H \rangle \\ &= I(H).\end{aligned}$$

($ii \implies i$) If $\sigma(I(H)) = I(H)$ then by Proposition 2.5, $\mathcal{G}(\sigma(I(H))) = \mathcal{G}(I(H))$. So, for any edge $e \in H$, $m_{\sigma(e)}$ is a generator of $I(H)$ and thus $\sigma(e)$ is an edge of H.

By duality we have also shown ($iii \iff iv$). It suffices to show that ($i \implies iv$), as its dual statement is ($iii \implies ii$). By definition,

$$\sigma(J(H)) = \left\langle m_{\sigma(f)} \mid f \text{ a minimal vertex cover of } H \right\rangle .$$

We show that if f is a minimal vertex cover of H and $\sigma \in \mathrm{Aut}(H)$, then $\sigma(f)$ is a minimal vertex cover of H. We note that because $\sigma \in \mathrm{Aut}(H)$, so is σ^{-1}. If e is any edge, $\sigma^{-1}(e)$ is also an edge. Thus, f meets $\sigma^{-1}(e)$ and consequently $\sigma(f)$ meets e. Because f is minimal, every $f \backslash \{x_i\}$ is disjoint with some edge $g \in E$. Consequently $\sigma(f)\backslash\{x_{\sigma(i)}\}$ is disjoint with the edge $\sigma(g)$. Hence $\sigma(f)$ is a minimal vertex cover of H. $\square$

Chapter 3

On Self-Duality

The purpose of this Chapter is to describe when objects are equal to their own Alexander dual, i.e. self-dual. In Section 3.1 we characterize self-duality primarily from an algebraic perspective. In Sections 3.2 and 3.3 we provide an account of what is known about self-dual objects in Combinatorics and Algebraic Topology, respectively.

3.1 An Algebraic Characterization of Self-Duality

We recall that a simplicial complex Δ is self-dual if and only if it is both sub-dual and super-dual, i.e.

$$\Delta = \Delta^\vee \iff \Delta \subseteq \Delta^\vee \text{ and } \Delta^\vee \subseteq \Delta.$$

We start by describing the hypergraphs and square-free monomial ideals corresponding to when Δ is a sub-dual or super-dual independence complex.

Lemma 3.1. *If H is a hypergraph then the following are equivalent:*

(i) $\Delta(H)$ *is super-dual;*

(ii) H *is intersecting;*

(iii) $I(H) \subseteq J(H)$.

Proof. ($i \implies ii$) If $\Delta(H)^\vee \subseteq \Delta(H)$ then every facet of $\Delta(H)^\vee$ is independent in H. Thus, the complements of the facets of $\Delta(H)^\vee$ are vertex covers of H. We recall from Remark 2.85 that the complements of the facets of $\Delta(H)^\vee$ are the edges of H, so every edge of H is a vertex cover of H. Hence every pair of edges of H has non-empty intersection and H is intersecting.

($ii \implies iii$) If H is intersecting then every edge of H is a vertex cover of H and thus contains a minimal vertex cover of H. Hence every minimal generator of $I(H)$ is divisible by some minimal generator of $J(H)$. The inclusion $I(H) \subseteq J(H)$ immediately follows.

($iii \implies i$) Suppose $I(H) \subseteq J(H)$ and W is a non-face of $\Delta(H)$. By Proposition 2.95 $m_W \in I(H)$. By assumption we have $m_W \in J(H)$ and thus W is a non-face of $\Delta(H^\vee) = \Delta(H)^\vee$. Since every non-face of $\Delta(H)$ is also a non-face of $\Delta(H)^\vee$, it follows that every face of $\Delta(H)^\vee$ is also a face of $\Delta(H)$. Thus $\Delta(H)^\vee \subseteq \Delta(H)$ and $\Delta(H)$ is super-dual. □

Lemma 3.2. *If H is a hypergraph then the following are equivalent:*

(i) *$\Delta(H)$ is sub-dual;*

(ii) *$\chi(H) \geq 3$;*

(iii) *$J(H) \subseteq I(H)$.*

Proof. ($i \implies ii$) If $\Delta(H) \subseteq \Delta(H)^\vee$ then every independent set of H is independent in $H^\vee$. In other words, every independent set in H does not contain a vertex cover of H, hence no independent set of H is a vertex cover. Thus, if W is independent in H then there is some $e_W \in E$ such that $e_W \cap W = \emptyset$. This shows that H is not 2-colorable; if $V = W \cup W'$ and W is independent, then W' must contain e_W and cannot be independent.

($ii \implies iii$) If H is not 2-colorable then every pair of independent sets W_1, W_2 has $W_1 \cup W_2 \subsetneq V$. Hence every pair of vertex covers has non-empty intersection; $\overline{W_1} \cap \overline{W_2} \supsetneq \emptyset$. Thus $H^\vee$ is intersecting, so by Lemma 3.1 $I(H^\vee) \subseteq J(H^\vee)$. By duality, this means $J(H) \subseteq I(H)$.

($iii \implies i$) If $J(H) \subseteq I(H)$ then $I(H^\vee) \subseteq J(H^\vee)$. Applying Lemma 3.1 gives $\Delta(H^\vee)$ is super-dual, hence $\Delta(H)$ is sub-dual. □

Corollary 3.3. *Intersecting hypergraphs and non 2-colorable hypergraphs are in bijective correspondence via Alexander duality.*

Proof. This is an immediate consequence of applying Theorem 2.101 to Lemmas 3.1 and 3.2. □

Theorem 3.4. *If H is a hypergraph then the following are equivalent:*

(i) *$\Delta(H)$ is self-dual;*

(ii) *H is intersecting and $\chi(H) \geq 3$;*

(iii) *$I(H) = J(H)$;*

(iv) Every edge of H is a vertex cover of H, and every vertex cover of H contains an edge of H;

(v) For every $W \subseteq V$, exactly one of W or $\overline{W}$ belongs to $\Delta(H)$.

Proof. The equivalence of (i), (ii), and (iii) is straightforward by combining Lemma 3.1 and Lemma 3.2.

We note that every edge of H being a vertex cover of H is equivalent to H being intersecting, hence by Lemma 3.1 we have $I(H) \subseteq J(H)$. Because the square-free monomials in $J(H)$ correspond to vertex covers of H and the square-free monomials in $I(H)$ correspond to sets that contain edges of H, we observe that $J(H) \subseteq I(H) \iff$ every vertex cover of H contains an edge of H. Thus (iv) is equivalent to (iii). The equivalence of (i) and (v) is given by [20, Theorem 2.4 (iii)]. □

Corollary 3.5. *If H is a self-dual hypergraph and not of the form $(V, \{x_i\})$ then $\chi(H) = 3$. Further, if H has more than one vertex and is proper, then H is critically 3-chromatic.*

Proof. If H is self-dual and has a 1-edge, then it must be of the from $(V, \{x_i\})$, otherwise if H has any other edge then it is not intersecting and consequently not self-dual.

We now give a proper 3-coloring of any self-dual hypergraph with edges of size 2 or larger. Pick any edge e and a vertex $z \in e$ and color z blue. Next, color every other vertex in e red. Finally, color every remaining vertex green. There is no edge that can be colored entirely by blue or red, because such an edge would have to be a subset of e. Also, no edge may be colored by green, because every edge must meet e, which has no green vertices. Finally, because only z is blue, this coloring induces a valid 2-coloring of $H_{V \setminus \{z\}}$. If H is proper, then every vertex can be chosen for z which implies H is critically chromatic. □

Corollary 3.6. *If H is a self-dual hypergraph then the facets of $\Delta(H)$ are exactly the complements of the edges of H.*

Proof. Edges are, by definition, the minimal non-faces of $\Delta(H)$. Hence by Theorem 3.4 (v) their complements are the maximal faces of $\Delta(H)$. □

Example 3.7. As mentioned in Example 2.73, the edges and minimal vertex covers coincide for every hypergraph in the family $Z(n)$, and hence the family consists of self-dual hypergraphs. Alternatively, we can arrive at the same conclusion by combining the observations that this family is intersecting in Example 2.60 and (critically) 3-chromatic in Example 2.67.

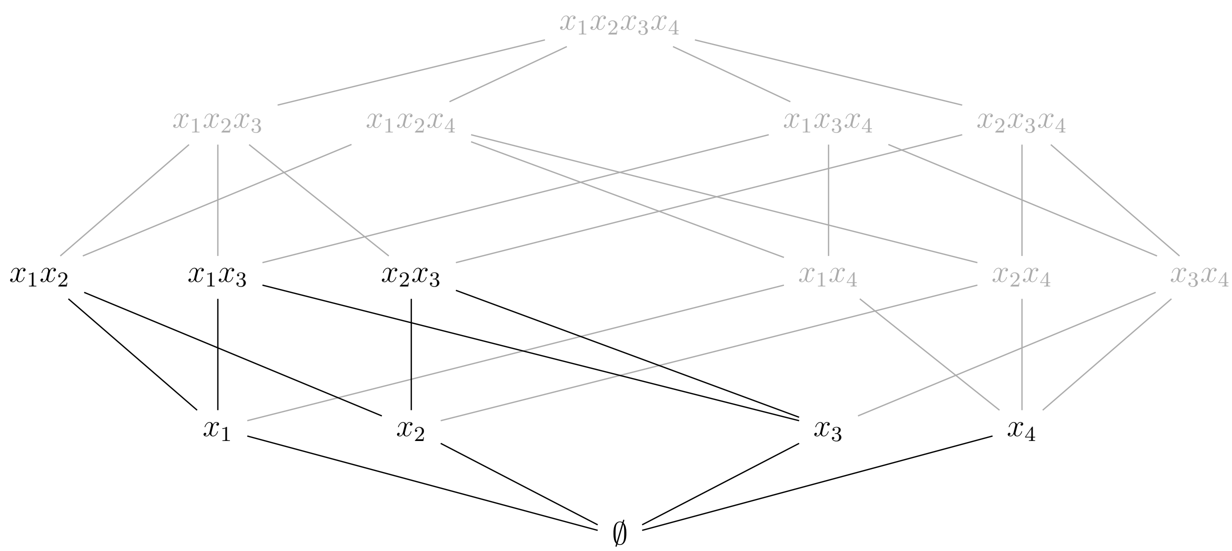


Figure 3.1: $\Delta(Z(3))^\vee$

Example 3.8. If $r \geq 1$ and $n = 2r - 1$ the complete hypergraph K_n^r is self-dual. To see this, note that K_n^r is intersecting because $r > \frac{2r-1}{2} = \frac{n}{2}$. It suffices to show that a vertex cover must have more than $r - 1$ vertices. Indeed, if $W \subseteq V$ and $|W| < r$ then $\left|\overline{W}\right| \geq r$, so there is an edge that W does not meet and W is not a vertex cover.

Example 3.9. The hypergraph on 7 vertices with 7 edges given by the lines of the Fano plane is self-dual. Note that in this finite geometry each line consists of three points. The lines are depicted as 6 lines and one circle in the figure below.

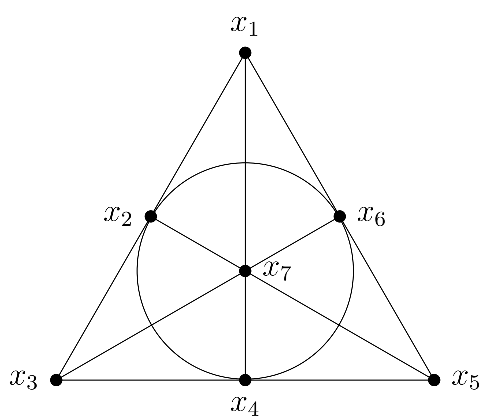


Figure 3.2: Fano Plane

Explicitly, we have:

$$H_F = (\{x_1, \ldots, x_7\},\ \{\ \{x_1, x_2, x_3\}, \{x_3, x_4, x_5\}, \{x_1, x_5, x_6\}, \{x_1, x_4, x_7\},\\ \{x_2, x_5, x_7\}, \{x_3, x_6, x_7\}, \{x_2, x_4, x_6\}\ \}).$$

The Fano plane serves as an example (or counter-example) in many areas of mathematics. For example, the Fano plane is the smallest finite projective plane. We refer the reader to [18] for a brief account of what this means. Some of the details in the paragraph titled "Hall's Magical Labelling" in [18] will be relevant in Section 4.2. We note that the title of this article gives a mnemonic for the lines of a Fano plane on the vertex set $\{A, E, H, R, T, W, Y\}$.

Remark 3.10. We recall the three axioms of projective planes:

(i) *two distinct points are contained in a unique line;*

(ii) *two distinct lines intersect in a unique point;*

(iii) *every point is contained in at least three lines and every line contains at least three points.*

If we take the edges of $Z(n)$ from Example 2.60 to be the lines of a finite geometry, then this geometry satisfies (i) and (ii) but fails (iii). Such geometries are known as "degenerate" projective planes. Axioms (ii) and (iii) guarantee that the lines of a finite projective plane define the edge set of a proper, intersecting hypergraph. Moreover, axiom (ii) means every edge of this hypergraph is a minimal vertex cover, a necessary but insufficient condition for self-duality.

It is well known (*cf.* [1]) that for finite projective planes there is an integer $r \geq$ called the *order* of the projective plane such that:

- there are $r^2 + r + 1$ points;
- there are $r^2 + r + 1$ lines;
- each line contains $r + 1$ points;
- each point belongs to $r + 1$ lines.

For the Fano plane, $r = 2$. Moreover, the Fano plane is the only finite projective plane with order 2. We next show that the Fano plane is the only finite projective plane whose lines give the edge set of a self-dual hypergraph.

Proposition 3.11. *If H is a hypergraph whose edges are the lines of a finite projective plane with order $r > 2$, then H is not self-dual.*

Proof. By Theorem 3.4 (iv) it suffices to find an independent vertex cover, because no vertex cover of a self-dual hypergraph is independent. Pick an edge $e \in E$ and let $W = e \backslash \{x_i\}$ for some $x_i \in e$. Of the $r^2 + r + 1$ edges in E, each of which has size $r + 1$:

- r of them intersect e at x_i, these are the lines that are not e and contain x_i, hence they do not meet W and only meet each other at x_i;
- r^2 of them intersect W at one vertex, these are the lines that do not contain x_i;
- one of them (namely e) intersects W at r vertices.

To find a vertex cover it suffices to add a collection of r vertices to W, one from each of the r edges described by the first bullet. Call this collection W'. However, to ensure $W \cup W'$ is independent, all r points in W' cannot come from a single edge described by the second bullet. There are r^r distinct choices for picking W', and only r^2 of those choices are contained by an edge described by the second bullet. Hence, if $r > 2$ we have given an independent vertex cover. □

We now show that self-dual hypergraphs can be constructed from smaller, intersecting hypergraphs, but first recall that a square-free monomial m_W denotes the product of the elements in W, as introduced in Notation 2.10.

Theorem 3.12. *If $H = (V, E)$ is an intersecting hypergraph then $\psi(H) := I(H) + zJ(H)$ is self-dual in $R[z]$.*

Proof. We first note that $\psi(H)$ is a square-free monomial ideal, hence is the edge ideal of a unique hypergraph on $V \cup \{z\}$. For succinctness we will also refer to this hypergraph as $\psi(H) = (V \cup \{z\}, T)$.

Let

$$F' = \{f \subset V \mid f \text{ a minimal vertex cover of } H\},$$

and define $F = F' \backslash E$ to be the set of minimal vertex covers of H that are not edges of H. Thus, $\psi(H) = I(H) + zJ(H)$ is minimally generated by $\{m_e \mid e \in E\} \cup \{zm_f \mid f \in F\}$. We use the shorthand $zF = \{zf = f \cup \{z\} \mid f \in F\}$, and define the edge set of $\psi(H)$ to be $T = E \cup zF$.

We use Theorem 3.4 (iv) to show $\psi(H)$ is self-dual. Consider an edge $e \in E \subseteq T$. By our intersecting hypothesis, e meets every other edge $e' \in E$. Every other edge in T belongs to zF and thus contains a vertex cover of H, which must intersect e. Thus, each edge of the form $e \in E$ is a vertex cover of $\psi(H)$. Now, consider an edge $zf \in zF \subseteq T$. Clearly this edge meets every other edge in zF at z. Further, f is a minimal vertex cover of H so zf meets every edge of the form $e \in E$. We have shown every edge in T is a vertex cover of $\psi(H)$.

It remains to show that every vertex cover of $\psi(H)$ contains an edge of $\psi(H)$. Let W be a vertex cover of $\psi(H)$ and first assume $z \in W$. Since no $e \in E$ contains z, $W \backslash \{z\}$ must intersect each $e \in E$. That is, $W \backslash \{z\}$ is a vertex cover of H, and must contain some minimal vertex cover of H, say W'. If W' is also an edge of H then $W' \in E$, otherwise $W' \cup \{z\} \in zF$ is a subset of W. Hence, W contains some edge in T. Now, suppose $z \notin W$. We observe W must intersect each element of zF somewhere other than z. Further, W must intersect the edges in E that are minimal vertex covers of H. Thus, W meets every vertex cover of H and thus is a vertex cover of the Alexander dual of H. Hence, W contains some minimal vertex cover of the Alexander dual of H, which by definition is an edge of H. Thus every vertex cover of $\psi(H)$ contains an edge in T, and thus $\psi(H)$ is self-dual. $\square$

Corollary 3.13. *Every self-dual hypergraph can be constructed by ψ in Theorem 3.12.*

Proof. Suppose H is self-dual on $V \cup \{z\}$. Since H is intersecting, so is H_V. It suffices to show that $H = \psi(H_V)$. Because $I(H) = J(H)$, the diagram in Theorem 2.101 collapses to:

$$\begin{tikzcd} & I(H_V) \arrow[dd, "\vee"] \\ I(H) \arrow[ur, "\pi_V"] \arrow[dr, "\varphi_V"'] & \\ & J(H_V)\ . \end{tikzcd}$$

We have $\psi(H_V) = I(H_V) + zJ(H_V) = I(H_V) + z\varphi_V(I(H))$ and consider the minimal generators of $\psi(H_V)$. If $m_e \in \mathcal{G}(I(H_V))$, then e is an edge in H that does not contain z. Thus, $m_e = \varphi_V(m_e)$ and m_e is a minimal generator of $\psi(H_V)$. Every other minimal generator of $\psi(H_V)$ is of the form $z\varphi_V(m_e)$ for some edge of H, say e, that does contain z. In this case, the monomial $z\varphi_V(m_e)$ is precisely m_e, hence $I(H)$ and $\psi(H_V)$ have the same minimal generators and $H = \psi(H_V)$. $\square$

Corollary 3.14. *Every intersecting hypergraph on n vertices not of the form $(V, \{x_i\})$ is a subhypergraph of a unique 3-chromatic, intersecting hypergraph on $n+1$ vertices.*

Proof. Theorem 3.12 shows that ψ is a map from intersecting hypergraphs on n vertices to self-dual hypergraphs on $n+1$ vertices. Corollary 3.13 shows that ψ has inverse π_V.

Corollary 3.5 shows that the image of ψ is 3-chromatic and intersecting if it is not of the form $(V \cup \{z\}, \{x_i\})$. Finally we note that $\pi_V((V \cup \{z\}, \{x_i\})) = (V, \{x_i\})$, the excluded form. $\square$

Corollary 3.15. *If H is a hypergraph with $\chi(H) \geq 3$ then $\psi^\vee(H) := J(H) + zI(H)$ is self-dual in $R[z]$.*

Proof. Note that $\psi^\vee(H) = \psi(H^\vee)$. Since $\chi(H) \geq 3$, Lemma 3.2 gives that $J(H) \subseteq I(H)$. In turn then, $I(H^\vee) \subseteq J(H^\vee)$, so $H^\vee$ is intersecting. So by Theorem 3.12 the hypergraph $\psi^\vee(H) = \psi(H^\vee)$ is self-dual. $\square$

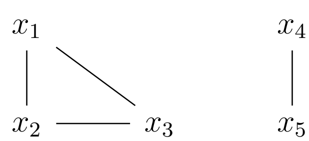
Proposition 3.16. *$\psi(H)$ is a proper hypergraph if and only if H is proper and not self-dual.*

Proof. If H is self-dual then $\psi(H) = I(H) + zI(H) = I(H)$, so no edge of $\psi(H)$ contains z. Similarly, if H is not proper and no edge contains x_i, then no minimal vertex cover contains x_i, hence no edge of $\psi(H)$ contains x_i. In either case, $\psi(H)$ is not proper.

On the other hand suppose $\psi(H)$ is not proper. We first consider the situation where no edge of $\psi(H)$ contains z. The domain of ψ is intersecting hypergraphs so by Lemma 3.1, $I(H) \subseteq J(H)$. If $m \in \mathcal{G}(J(H))$ then $zm \in \mathcal{G}(\psi(H))$ if and only if $m \notin \mathcal{G}(I(H))$. This shows $J(H) \subseteq I(H)$ thus H is self-dual. Finally if no edge of $\psi(H)$ contains x_i, then none of its subhypergraphs (including H) has an edge that contains x_i, and so H is not proper. $\square$

Example 3.17. $H = (V, \{V\})$ is trivially intersecting with $I(H) = \langle m_V \rangle$ and $J(H) = \mathfrak{p}_V$. If $n = 1$ then $I(H) = J(H)$, otherwise $\psi(H) = I(H) + z\mathfrak{p}_V = \langle m_V, zx_1, \ldots, zx_n \rangle$. For $n \geq 2$ this is exactly $Z(n)$ from Example 2.60.

Example 3.18. Let H be the 3-chromatic graph below. We construct a self-dual hypergraph using $\psi^\vee$.



$$I(H) = \langle x_1x_2,\ x_1x_3,\ x_2x_3,\ x_4x_5 \rangle$$

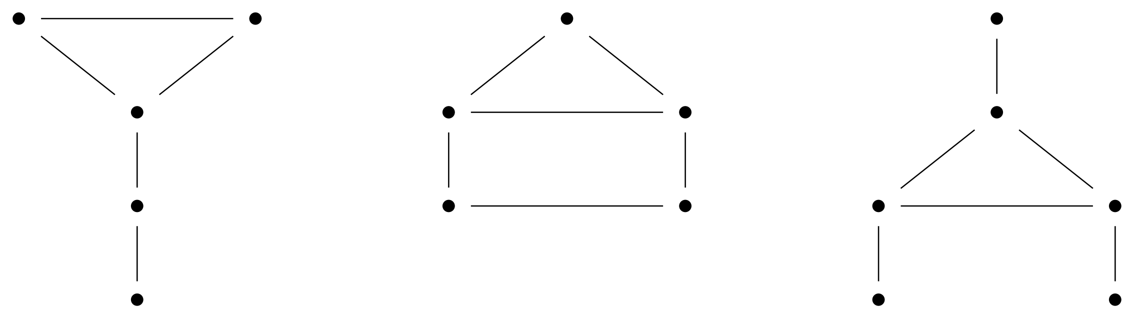
We note that the minimum vertex covers of a disconnected graph are given by every possible union taking one minimal vertex cover from each component. In this case, we have:

$$J(H) = \langle x_1x_2x_4,\ x_1x_2x_5,\ x_1x_3x_4,\ x_1x_3x_5,\ x_2x_3x_4,\ x_2x_3x_5 \rangle\,.$$

Taking x_6 in the place of z we have a 3-uniform self-dual hypergraph with edge ideal:

$$\begin{aligned}\psi^\vee(H) = \langle & x_1x_2x_4,\ x_1x_2x_5,\ x_1x_3x_4,\ x_1x_3x_5,\ x_2x_3x_4, \\ & x_2x_3x_5,\ x_1x_2x_6,\ x_1x_3x_6,\ x_2x_3x_6,\ x_4x_5x_6\rangle.\end{aligned}$$

Remark 3.19. We observe that in Example 3.18, H is a non 2-colorable r-uniform hypergraph where every minimal vertex cover has size $r+1$. Applying $\psi^\vee$ to any hypergraph that satisfies this description gives a self-dual $r+1$-uniform hypergraph. In Section 4.2 we will be able to show that for $r = 2$ there are exactly six such non-isomorphic graphs, namely H from Example 3.18, K_4^2, C_5, and the three depicted below.



In general not every self-dual $r+1$-uniform hypergraph admits such a construction. For example, a seventh (and in fact, the only other) self-dual 3-uniform hypergraph is the Fano plane H_F. For any choice of W with $|W| = 6$, the image $\varphi_W(H_F)$, i.e. every pre-image of $\psi^\vee$, is isomorphic to

$$\langle x_1x_4,\ x_2x_5,\ x_3x_6,\ x_1x_2x_3,\ x_1x_5x_6,\ x_2x_4x_6,\ x_3x_4x_5\rangle\,.$$

Remark 3.20. If $W = \{x_2, \ldots, x_6\}$ and we take H from Example 3.18 then we have

$$\begin{aligned}\varphi_W(\psi^\vee(H)) = \langle & x_2x_4,\ x_2x_5,\ x_3x_4,\ x_3x_5,\ x_2x_3x_4, \\ & x_2x_3x_5,\ x_2x_6,\ x_3x_6,\ x_2x_3x_6,\ x_4x_5x_6\rangle,\end{aligned}$$

where the minimal generators are:

$$\varphi_W(\psi^\vee(H)) = \langle x_2x_4,\ x_2x_5,\ x_3x_4,\ x_3x_5,\ x_2x_6,\ x_3x_6,\ x_4x_5x_6\rangle.$$

This gives the edge ideal of a hypergraph that is (necessarily) not 2-colorable, but different than H. We emphasize that ψ and $\psi^\vee$ are both invertible *for a fixed choice of z.*

We next study, up to isomorphism, the number of pre-images that ψ and $\psi^\vee$ have. For example, if we choose W' by removing one of x_2 or x_3 from $\{x_1, \ldots, x_6\}$, then $\varphi_{W'}(\psi^\vee(H))$ is isomorphic to $\varphi_W(\psi^\vee(H))$ above. Similarly, if we choose W'' by removing one of x_4 or x_5 from $\{x_1, \ldots, x_6\}$, then $\varphi_{W''}(\psi^\vee(H))$ is isomorphic to H. Hence, up to isomorphism, $\psi^\vee(H)$ has only 2 pre-images.

Notation 3.21. If $W = V \backslash \{x_i\}$, we take $\pi_i = \pi_W$.

Lemma 3.22. *If $\sigma \in S_n$ and m is a monomial then $\sigma(\pi_i(m)) = \pi_{\sigma(i)}(\sigma(m))$.*

Proof. If $x_i \mid m$ then $\pi_i(m) = 0$ and $x_{\sigma(i)} \mid \sigma(m)$, so then both $\sigma(\pi_i(m))$ and $\pi_{\sigma(i)}(\sigma(m))$ are 0. On the other hand, if $x_i \nmid m$ then $\pi_i(m) = m$ and $x_{\sigma(i)} \nmid \sigma(m)$, so both expressions are equal to $\sigma(m)$. $\square$

Theorem 3.23. *If H is a self-dual hypergraph, then $\pi_i(I(H)) \cong \pi_j(I(H))$ if and only if x_i and x_j are in the same orbit of* $\mathrm{Aut}(H)$*.*

Proof. We take $I = I(H)$ throughout.

($\Longleftarrow$) Suppose that $\sigma \in \mathrm{Aut}(H)$ (or equivalently by Theorem 2.106 that I is σ-stable) and that $\sigma(x_i) = x_j$. By Lemma 3.22 we have

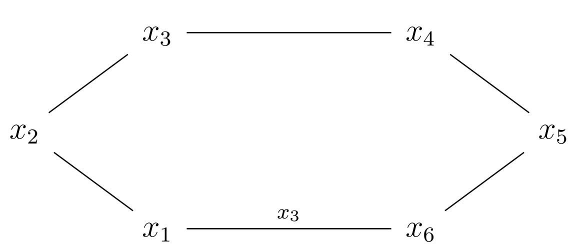
$$\sigma(\pi_i(I)) = \pi_{\sigma(i)}(\sigma(I)) = \pi_j(I).$$

Hence σ defines an isomorphism between $\pi_i(I)$ and $\pi_j(I)$.

($\Longrightarrow$) Suppose $\pi_i(I) \cong \pi_j(I)$. Then there is some $\sigma \in S_n$ such that $\sigma(\pi_i(I)) = \pi_j(I)$. It must be that $\sigma(x_i) = x_j$ (otherwise these ideals are not in the same subring of R) so we apply Lemma 3.22 to get $\pi_j(I) = \pi_j(\sigma(I))$. Taking $z = x_j$ when applying ψ to both sides gives $I = \sigma(I)$, hence $\sigma \in \mathrm{Aut}(H)$. $\square$

Remark 3.24. We note that the result in Theorem 3.23 does not generally hold. In particular the ($\Longrightarrow$) direction does require self-duality, or rather π_j to have inverse ψ. Consider the following counter-example:

$$I(H) = \langle x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_5x_6,\ x_1x_3x_6 \rangle\,.$$



H

We observe that $\pi_1(I(H))$ and $\pi_6(I(H))$ both result in a path with 5 vertices, but $\mathrm{Aut}(H)$ is trivial, so x_1 and x_6 are not in the same orbit.

Corollary 3.25. *Up to isomorphism, the number of pre-images that ψ and $\psi^\vee$ have is the number of orbits when the automorphsim group of (the hypergraph corresponding to) the maps' images acts on V.*

Lemma 3.26. *If H is self-dual and $W = V\backslash\{x_i\}$, then $\mathrm{Aut}(H_W)$ is exactly the isotropy subgroup of $\mathrm{Aut}(H)$ that fixes x_i.*

Proof. Suppose $\sigma \in \mathrm{Aut}(H_W)$, clearly $\sigma(i) = i$. By Lemma 2.106 both $I(H_W)$ and $J(H_W)$ are σ-stable. Moreover as a ring homomorphsim σ respects addition. We take $z = x_i$ when applying ψ, so

$$\begin{aligned}\sigma(\psi(I(H_W))) &= \sigma(I(H_W) + x_i J(H_W)) \\ &= \sigma(I(H_W)) + x_{\sigma(i)}\sigma(J(H_W)) \\ &= I(H_W) + x_i J(H_W) \\ &= \psi(I(H_W)).\end{aligned}$$

Hence σ belongs to the isotropy subgroup of $\mathrm{Aut}(H)$ that fixes x_i.

Now, suppose σ belongs to the isotropy subgroup of $\mathrm{Aut}(H)$ that fixes x_i. We note $I(H_W) = \pi_i(I(H))$, so applying Lemma 3.22 gives

$$\begin{aligned}\sigma(\pi_i(I(H))) &= \pi_{\sigma(i)}(\sigma(I(H))) \\ &= \pi_i(I(H)).\end{aligned}$$

Thus $I(H_W)$ is σ-stable, so $\sigma \in \mathrm{Aut}(H_W)$. □

Remark 3.27. We note that Lemma 3.26 also does not hold when the self-dual assumnption is dropped. The counter-example from Remark 3.24 shows this.

3.2 Self-Duality in Combinatorics

In Section 3.1 we primarily discussed self-duality of square-free monomial ideals from an algebraic point of view. The result of Theorem 3.4 provides a novel connection to 3-chromatic, intersecting hypergraphs (i.e. self-dual hypergraphs) which have been studied in Combinatorics. In this section, we provide a summary of what is known about self-dual hypergraphs as they have been studied in Combinatorics in the form of 3-chromatic, intersecting hypergraphs.

Self-dual hypergraphs first appeared in Combinatorics literature in [6], a paper published by Erdős and Lovász in 1975 that is famous for introducing what is now known as the Lovász Local Lemma. The primary focus of this paper however was finding certain bounds for r-uniform, intersecting hypergraphs. The first half of the paper is about "simple" hypergraphs, where the intersection of every pair of edges has size 1. The second half of the paper is about self-dual hypergraphs which includes four constructions of some r-uniform, self-dual hypergraphs and four relevant theorems each of which we include below.

Remark 3.28. Each of the following is a family of uniform, self-dual hypergraphs given by [6].

(i) K^r_{2r-1}, the complete r-uniform hypergraph on $2r-1$ vertices, e.g. Example 3.8.

(ii) Let $V = \{x_1, \ldots, x_{2r-2}\}$ and label each of the $\frac{1}{2}\binom{2r-2}{r-1}$ partitions $V = W_{i,1} \cup W_{i,2}$ where $|W_{i,1}| = |W_{i,2}| = r-1$, and let $Y = \{y_1, \ldots, y_{\frac{1}{2}\binom{2r-2}{r-1}}\}$. Then the hypergraph $(V \cup Y, E)$ is r-uniform and self-dual where

$$E = \{e \subseteq V \mid r = |e|\} \cup \{y_i \cup W_{i,j} \mid j \in \{1,2\}\}.$$

(iii) Let H be an $r-1$-uniform self-dual hypergraph and take r new vertices $W = \{z_1, \ldots, z_r\}$. Then $(V \cup W, E')$ is r-uniform, 3-chromatic, and intersecting where

$$E' = W \cup \{e \cup \{z_i\} \mid e \in E \text{ and } z_i \in W\}.$$

(iv) Let $S = (W, F)$ be an s-uniform, self-dual hypergraph with $W = \{1, \ldots, n\}$. Also let $H_1, \ldots, H_n$ each be t-uniform, self-dual hypergraphs with disjoint vertex sets $V_1, \ldots, V_n$ and edge sets $E_1, \ldots, E_n$. Then $H = (V, E)$ with $V = V_1 \cup \cdots \cup V_n$ and

$$E = \left\{e_{i_1} \cup \cdots \cup e_{i_s} \;\middle|\; e_{i_j} \in E_{i_j} \text{ and } i_1, \ldots, i_s \in F\right\}$$

is st-uniform and self-dual.

We would refer the reader to [6] for the proof that (*ii*), (*iii*), and (*iv*) indeed produce self-dual hypergraphs, however they say only that the matter is straightforward.

Theorem 3.29. *[6, Theorem 6] If $m(r)$ denotes the minimum number of edges of an r-uniform, self-dual hypergraph, then $m(r) \leq 7^{\frac{r-1}{2}}$ for infinitely many r.*

The proof of Theorem 3.29 follows by iterating the construction given by Remark 3.28 (*iv*) with the Fano plane.

Theorem 3.30. *[6, Theorem 7] If $M(r)$ denotes the maximum number of edges of an r-uniform, self-dual hypergraph, then $r!(e-1) \leq M(r) \leq r^r$.*

Theorem 3.31. *[6, Theorem 8] If $N(r)$ denotes the maximum number of vertices of an r-uniform, self-dual hypergraph, then*

$$\frac{1}{2}\binom{2r-2}{r-1} + 2r - 2 \leq N(r) \leq \frac{r}{2}\binom{2r-1}{r-1}.$$

Erdős and Lovász also considered the possible sizes of pair-wise edge intersections of uniform, self-dual hypergraphs.

Definition 3.32. The *intersection spectrum* of a hypergraph is the set

$$\mathcal{I}(H) = \{|e \cap f| \mid e, f \in E\}.$$

Theorem 3.33. *[6, Theorem 9] If r is large enough and H is r-uniform and self-dual, then $|\mathcal{I}(H)| \geq 3$.*

In [6], Erdős and Lovász mention that 3 is the best lower bound that they could prove for the size of the intersection spectrum of self-dual, r-uniform hypergraph as r goes to infinity, as given by Theorem 3.33. Contrarily, to this day there is no known example with $|\mathcal{I}(H)| < \frac{r-1}{2}$. Moreover, they also comment on the largest intersection of any two edges, where there is no known example with $\max(\mathcal{I}(H)) < r-2$. In 2022, Bucić, Glock, and Sudakov showed that the cardinality of the intersection spectrum of a self-dual, r-uniform hypergraph goes to infinity as r does with the following theorem.

Theorem 3.34. *[5, Theorem 2] The intersection spectrum of a self-dual, r-uniform hypergraph has size at least $\Omega\left(\frac{\sqrt{r}}{\log(r)}\right)$.*

3.3 Self-Duality in Algebraic Topology

In [20], Timotijtević gives an account of self-dual simplicial complexes, much of which we encapsulated algebraically in Section 3.1. In one result, Timotijtević shows a bijection between sub-dual complexes with vertex set V and self-dual complexes with vertex set $V \cup \{z\}$. He shows this bijection by constructing a so-called "root operator" and its inverse the "upgrade operator," which are equivalent to the maps φ_V and $\psi^\vee$, respectively. Further, he provides topological descriptions of these maps which we recount after providing the relevant definitions.

Definition 3.35. If Δ is a simplicial complex and f is a face of Δ, then the *link* of f is

$$link_\Delta(f) = \{W \in \Delta \mid W \cap f = \emptyset \text{ and } W \cup f \in \Delta\}.$$

Definition 3.36. If Δ is a simplicial complex with vertex set V, the *cone* of Δ, $C\Delta$, is the simplicial complex with vertex set $V \cup \{z\}$ and facets $\{f \cup \{z\} \mid f \text{ a facet of } \Delta\}$.

Timotijtević shows that if Δ is sub-dual on V then $\psi(\Delta) = \Delta^\vee \cup C\Delta$, and if Δ' is self-dual on $V \cup \{z\}$ then $\varphi_V(\Delta') = link_{\Delta'}(z)$. He also considers the (computational) problem of knowing if two self-dual simplicial complexes are isomorphic, where he mentions that one need only compare the preimages of $\psi^\vee$. This is advantageous because their vertex sets have size $n-1$, so there are only $(n-1)!$ isomorphisms to consider rather than the initial $n!$. In Section 4.2 we will consider a number of computational problems, including further improvements on this one.

A major result for self-dual simplicial complexes that Timotijtević provides that cannot be viewed as a consequence of Section 3.1 is the construction of a connected graph $\mathbb{G}(n)$ where the nodes are the self-dual simplicial complexes on n vertices. By graph, we mean $\mathbb{G}(n)$ is a 2-uniform hypergraph. We will refer to the vertices of $\mathbb{G}(n)$ as nodes in order to avoid confusion with the vertices of its constituent (corresponding) hypergraphs. These vertices are then connected by an operation that takes one self-dual object and returns another. This operation is most easily described in the language of independence complexes.

Definition 3.37. Let H be a self-dual hypergraph and e be an edge. The *flip of H with respect to e*, denoted $F_e(H)$, is the hypergraph with independence complex:

$$\Delta(F_e(H)) = e \cup (\Delta(H) \backslash \overline{e}).$$

That is, one maximally independent set is replaced with its complement.

Essentially, a facet of the complex is swapped for its complement. It is straightforward to see that this operation has an inverse $F_{\overline{e}}$ and that the set $e \cup (\Delta(H) \backslash \overline{e})$ still satisfies the property equivalent to self-duality given in Theorem 3.4 (v). However, the fact that this is actually a simplicial complex should also be checked. We note that by Corollary 3.6, every proper subset of the new facet, e, is already contained by $\Delta(H) \backslash \overline{e}$. Moreover, removing a facet from any simplicial complex yields a simplicial complex.

Definition 3.38. The *graph of self-dual hypergraphs* of dimension n, $\mathbb{G}(n)$, is the graph where each vertex is a self-dual hypergraph with n vertices and the neighborhood of each self-dual hypergraph H is $\{F_e(H) \mid e \in E\}$. That is, two hypergraphs are adjacent exactly when they differ by a single flip.

Example 3.39. We give a visual depiction of $\mathbb{G}(3)$. There are exactly 4 self-dual complexes on 3 vertices. The Hasse diagram in the middle, labeled Δ_T, corresponds to the (hyper)graph K_3^2. The remaining three diagrams are labeled Δ_i each corresponding to the self-dual hypergraph with the singleton edge $\{x_i\}$. The dashed lines with arrows depict the flip F_e, and therefore the (directed) edges of $\mathbb{G}(3)$. In particular, these start at e as an edge (i.e. minimal non-face) and end at e as a facet. The dashed lines without arrows display relevant complements within the Hasse diagram. Finally, each color represents each edge of $\mathbb{G}(3)$ in its (undirected) entirety.

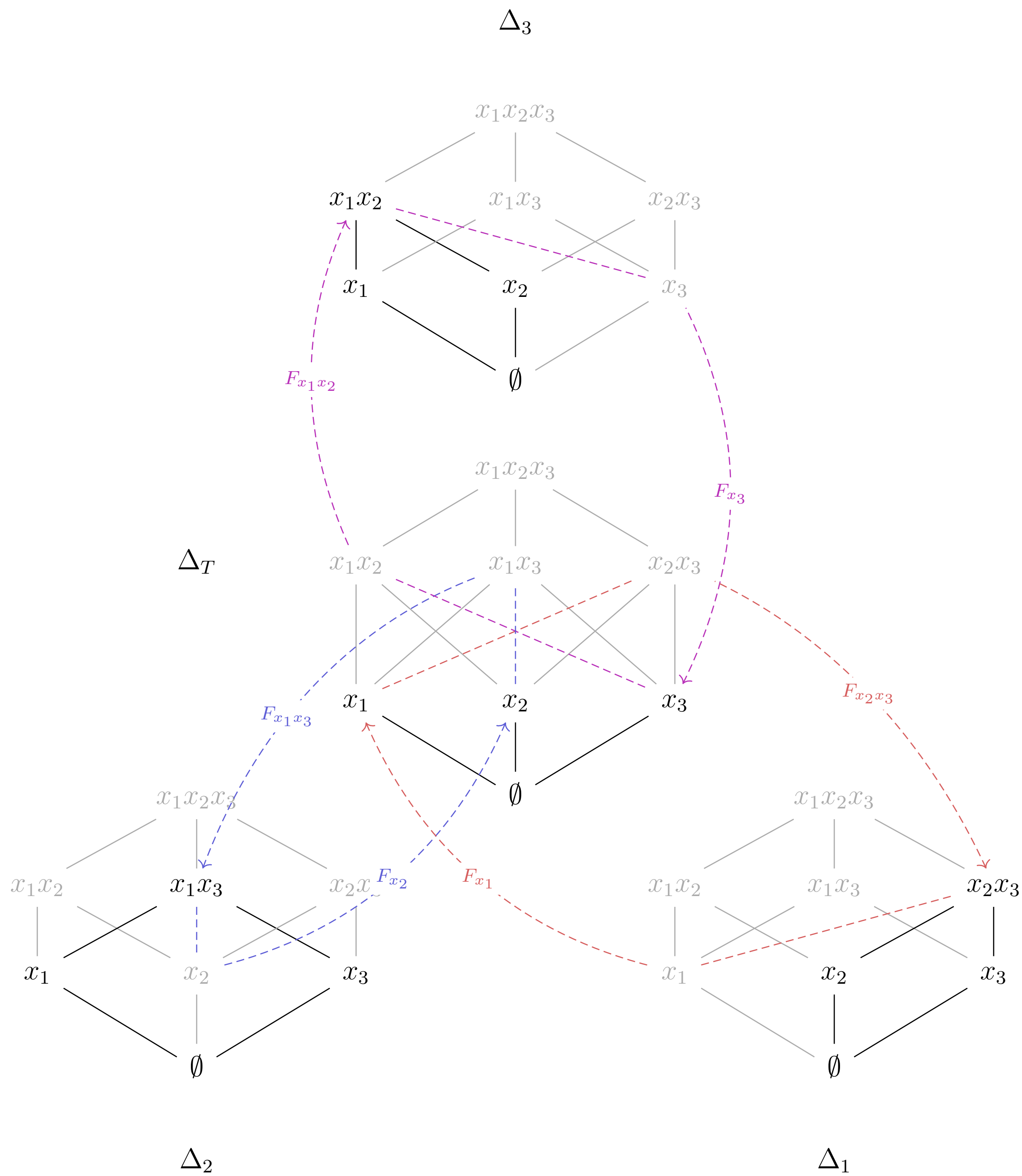


Figure 3.3: $\mathbb{G}(3)$

Below we depict $\mathbb{G}(4)$ in much less detail. Each blue vertex is a self-dual hypergraph with a singleton edge, each orange vertex is isomorphic to $Z(3)$, and each purple vertex is isomorphic to a K_3^2 adjoined with an isolated vertex. Hence, up to isomorphism, $Z(3)$ is the only proper self-dual hypergraph on 4 vertices. We explore more about $\mathbb{G}(n)$ for larger n in Section 4.2.

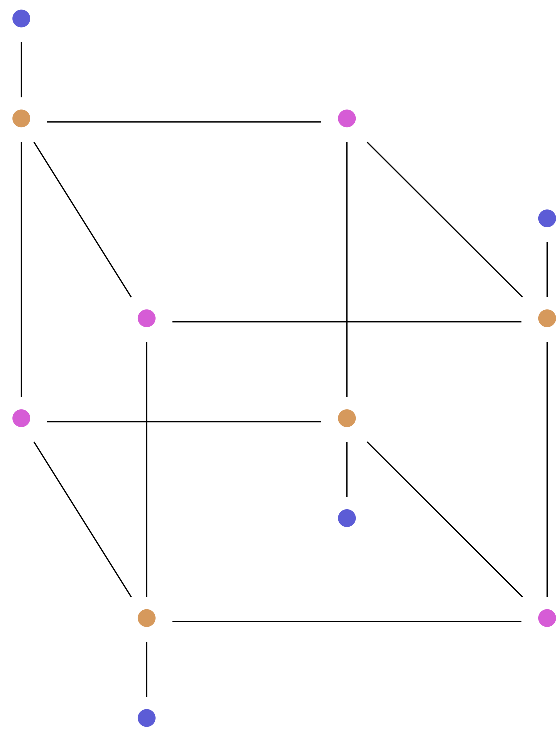


Figure 3.4: $\mathbb{G}(4)$

Below, we give a depiction of $\mathbb{G}(5)$. As with the depiction above, each color represents an isomorphism class. We note that between the 81 nodes, there are only seven colors. One focus of Section 4.2 is to reduce this "waste".

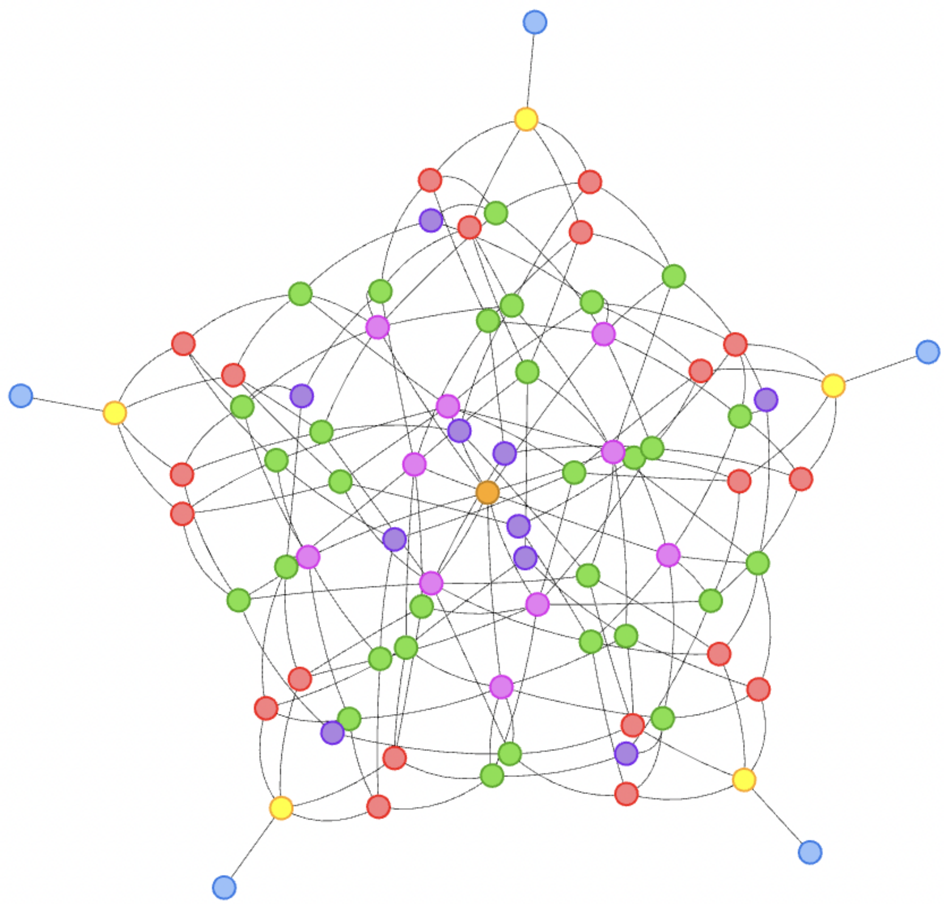


Figure 3.5: $\mathbb{G}(5)$

Lemma 3.40. *[20, Corollary 3.4] The graph $\mathbb{G}(n)$ is bipartite and connected for any $n \geq 1$.*

Lemma 3.41. *[20, Proposition 3.3] If H, H' are self-dual hypergraphs on n vertices, then every shortest path from H to H' in $\mathbb{G}(n)$ is given by a sequence of flips with respect to complements of the elements of $\Delta(H) \backslash \Delta(H')$. That is, if $\Delta(H) \backslash \Delta(H') = \{W_1, \ldots, W_s\}$ is ordered such that $W_i \subseteq W_j$ only if $i \geq j$, then the sequence*

$$(H, H_1 = F_{\overline{W_1}}(H), \ldots, F_{\overline{W_i}}(H_{i-1}), \ldots, F_{\overline{W_s}}(H_{s-1}) = H')$$

is a shortest path.

Remark 3.42. Self-dual hypergraphs are interesting objects in their own right, however they also seem to be related to certain moduli spaces. In particular, Gianae Panina's work outlines these connections. In [17], Panina describes the construction of an $n-3$ dimensional combinatorial manifold that is homeomorphic to the moduli space of a planar polygonal linkage. In [10], Panina and her former Masters student Galashin show that the rules governing this construction are truly governed by a self-dual simplicial complex, and that one can drop the planar polygonal linkage and generalize the construction for any "quasilinkage" i.e. self-dual complex. In particular, they build a regular CW-complex where each r-cell corresponds to a cyclically oriented $n-r$-coloring of a given self-dual hypergraph. In [16] Nekarov and Panina use subsets of self-dual complexes to give various compactifications of the moduli space of n-punctured rational curves that generalize the Deligne-Mumford-Knudsen compactification.

Lastly, we note that while Section 3.2 and Section 3.3 summarize the work of other authors, the contents of Section 3.1 are essentially all original. In particular, we are the first to show there is a connection between the work discussed in Section 3.2 and Section 3.3.

Chapter 4

On Monomial Ideals

4.1 On Generalized Self-Duality

We generalize some of the results in Section 3.1 by replacing the partially ordered set 2^V and partial order $\subseteq$ with a subset of $\mathbb{N}^n$ and the partial order $\preceq$ introduced in Remark 2.4. In this setting, the notion of being a minimal vertex cover is no longer useful for our purposes. Instead, the important property that we are concerned with is being the complement of a maximal independent set.

Definition 4.1. A *multiset* is a triple $\mathcal{W} = (W, \omega, n)$ where $W = \{X_{i_1}, \ldots, X_{i_s}\}$ is a set with $1 \leq i_j \leq n$ and $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{N}^n$ such that $X_i \in W$ if and only if $0 < \omega_i$. We call

- W the *support* of $\mathcal{W}$;
- ω the *multidegree* of $\mathcal{W}$;
- n the *dimension* of $\mathcal{W}$.

Definition 4.2. We say the multiset $\mathcal{W} = (W, \omega, n_1)$ *precedes* the multiset $\mathcal{V} = (V, \upsilon, n_2)$ if $n_1 = n_2$ and $\omega \preceq \upsilon$. We denote this by $\mathcal{W} \preceq \mathcal{V}$.

Example 4.3.

$$(\{X_1, X_2\},\ (2,3,0),\ 3) \preceq (\{X_1, X_2, X_3\},\ (3,3,1),\ 3)$$

Notation 4.4. Note that any multiset can be recovered entirely from its multidegree. Indeed, for some vector of natural numbers, ω, take $W = \{X_i \mid 0 < \omega_i\}$, and n to be the number of entries in ω. Hence throughout this chapter, we will interchangeably use:

- $\mathcal{V} = (V, \upsilon, n)$ as υ;

- $\mathcal{W} = (W, \omega, n)$ as ω;
- $\mathcal{N} = (f, \eta, n)$ as η;
- (e, ϵ, n) as ϵ.

We will similarly apply this convention to multisets with explicit numerical values. For example, the statement in Example 4.3 would be $(2, 3, 0) \preceq (3, 3, 1)$.

Notation 4.5. We will frequently wish to consider a multiset with "a single element", so δ^i will always be the element of $\mathbb{N}^n$ with 1 as the i-th entry, and 0 in every other entry. For example if $n = 4$, then $\delta^2 = (0, 1, 0, 0)$.

We now introduce generalized versions of a number of set operations.

Definition 4.6. If $\mathcal{W} \preceq \mathcal{V}$ then the *complement of $\mathcal{W}$ with respect to $\mathcal{V}$*, denoted $\mathcal{V} \backslash \mathcal{W}$, is the multiset determined by $\upsilon - \omega$.

Note that the support of $\mathcal{V} \backslash \mathcal{W}$ need not be $V \backslash W$. When the multiset $\mathcal{V}$ is clear from context we may denote $\mathcal{V} \backslash \mathcal{W} = \overline{\mathcal{W}}$. For example, if we take the pair of multisets from Example 4.3, the complement is $(1, 0, 1)$.

Definition 4.7. The *intersection* of two multisets $\mathcal{W} \cap \mathcal{N}$ is the multiset determined by $(\min(\omega_1, \eta_1), \ldots, \min(\omega_n, \eta_n))$. Note, $M_{\omega \cap \eta} = \gcd(M_\omega, M_\eta)$, where each M is the monomial with the multidegree given by its subscript, as introduced in Notation 2.24.

Lemma 4.8. *If $\omega \preceq \upsilon$ and $\eta \preceq \upsilon$ and all complements are taken with respect to υ, then the following are equivalent:*

- $\omega \preceq \eta$;
- $M_\omega \mid M_\eta$;
- $\omega \cap \eta = \omega$;
- $\overline{\eta} \preceq \overline{\omega}$;
- $M_{\overline{\eta}} \mid M_{\overline{\omega}}$;
- $\overline{\omega} \cap \overline{\eta} = \overline{\eta}$.

Definition 4.9. The *union* of two multisets, $\mathcal{W} \uplus \mathcal{N}$, is the multiset determined by $\omega + \eta$.

Continuing on, we generalize the notion of a hypergraph in a manner that that is compatible with viewing monomial ideals as a generalization of square-free monomial ideals. To this end, we introduce an "edge ideal" correspondence between monomial ideals and this generalization.

Definition 4.10. A *multihypergraph* is a pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = (V, \upsilon, n)$ is a multiset called the *base multiset* and $\mathcal{E} = \{\epsilon_1, \ldots, \epsilon_s\}$ is a set of multisets called *multiedges*, each of which precede $\mathcal{V}$ and no pair of distinct multiedges $\epsilon, \epsilon' \in \mathcal{E}$ satisfies $\epsilon \preceq \epsilon'$.

Remark 4.11. Note that because each multiedge precedes $\mathcal{V}$, every multiset in the definition of a multihypergraph has the same dimension. As with hypergraphs, we will always assume $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = (V, \upsilon, n)$, unless otherwise specified.

Definition 4.12. A multiset ω is *independent* in $\mathcal{H}$ if $\omega \preceq \upsilon$ and there is no $\epsilon \in \mathcal{E}$ such that $\epsilon \preceq \omega$. We say ω is *maximally independent* in $\mathcal{H}$ if there is no $\omega' \neq \omega$ with $\omega \preceq \omega'$ such that ω' is independent in $\mathcal{H}$.

Definition 4.13. The *edge ideal* of a multihypergraph, denoted $I(\mathcal{H})$, is the monomial ideal in $k[V]$ with minimal generators

$$\mathcal{G}(I(\mathcal{H})) = \{M_\epsilon \mid \epsilon \in \mathcal{E}\}.$$

Example 4.14. Let $\mathcal{H} = ((2,3),\ \{(2,0),(1,1),(0,2)\})$. Then $I(\mathcal{H}) = \langle X_1^2, X_1X_2, X_2^2 \rangle$. Note also that the maximally independent multisets of $\mathcal{H}$ are $(1,0)$ and $(0,1)$.

Definition 4.15. The *cover ideal* of a multihypergraph, denoted $J(\mathcal{H})$, is the monomial ideal in $k[V]$ with minimal generators

$$\mathcal{G}(J(\mathcal{H})) = \{M_\omega \mid \mathcal{V} \backslash \mathcal{W} \text{ is maximally independent in } \mathcal{H}\}.$$

Example 4.16. We consider the same multihypergraph from Example 4.14, where $J(\mathcal{H}) = \langle X_1X_2^3, X_1^2X_2^2 \rangle$. Note that while the base multiset of a multihypergraph does not affect its edge ideal, it does affect the cover ideal. If instead, $\mathcal{H}'$ is the multihypergraph with base multiset $(2,2)$ and the same multiedge set as $\mathcal{H}$, then $J(\mathcal{H}') = \langle X_1X_2^2, X_1^2X_2 \rangle$

Proposition 4.17. *If ω is the complement of an independent multiset in $\mathcal{H}$, then for every $\varepsilon \in \mathcal{E}$, we have $\omega + \varepsilon \not\preceq \upsilon$.*

Proof. By independence we have that for all $\varepsilon \in \mathcal{E}$, $\varepsilon \not\preceq \overline{\omega}$. By substituting $\overline{\omega} = \upsilon - \omega$ and then rearranging, this is exactly $\omega + \varepsilon \not\preceq \upsilon$. □

Definition 4.18. [15, Proposition 5.23] If I is a monomial ideal and $\upsilon \in \mathbb{N}^n$ such that each $M_\omega \in \mathcal{G}(I)$ divides M_υ, then the *generalized Alexander dual of I with respect to υ*, denoted $I^{[\upsilon]}$, is the monomial ideal that satisfies $M_\omega \in I^{[\upsilon]}$ if and only if $M_{\upsilon-\omega} \notin I$ for all $\omega \preceq \upsilon$.

In Definition 4.18 we cite a proposition–not a definition–in [15] because this equivalent formulation is more suitable for our purposes. We note however, that Theorem 2.31 is proven by using generalized Alexander duality.

Lemma 4.19. *If $\mathcal{H}$ is a multihypergraph with base multiset υ, then $I(\mathcal{H})^{[\upsilon]} = J(\mathcal{H})$.*

Proof. For every $\omega \preceq \upsilon$ the following are equivalent:

- $M_{\upsilon-\omega} \in J(\mathcal{H})$;
- ω is independent in $\mathcal{H}$;
- there is no $\epsilon \in \mathcal{E}$ with $\epsilon \preceq \omega$;
- $M_\omega \notin I(\mathcal{H})$;
- $M_{\upsilon-\omega} \in I(\mathcal{H})^{[\upsilon]}$.

$\square$

Lemma 4.20. *[15, Theorem 5.24] If I is a monomial ideal such that for each $M_\omega \in \mathcal{G}(I)$, $\omega \preceq \upsilon$, then $(I^{[\upsilon]})^{[\upsilon]} = I$.*

Notation 4.21. Although symmetric polarization (defined in Definition 2.37) is defined on monomials and monomial ideals, it may act equivalently on multisets. For some multiset $\omega \preceq \upsilon$ we denote

$$\mathrm{sPol}(\omega; \upsilon) = \{W \mid m_W \in \mathrm{sPol}(M_\omega; \upsilon)\}.$$

Similarly, we may apply symmetric polarization to a multihypergraph to yield a hypergraph:

$$\mathrm{sPol}(\mathcal{H}; \upsilon) = \left(V^\upsilon, \bigcup_{\varepsilon \in \mathcal{E}} \mathrm{sPol}(\varepsilon; \upsilon) \right).$$

We now make extensive use of symmetric polarization to build up to a generalized version of Theorem 2.101, namely Theorem 4.30.

Lemma 4.22. *For a multihypergraph $\mathcal{H}$, if $H = \mathrm{sPol}(\mathcal{H};\upsilon)$ then the following diagram of maps between monomial ideals commutes.*

$$\begin{tikzcd}
I(\mathcal{H}) \arrow[r, bend left, "\mathrm{sPol}(-;\upsilon)"] \arrow[d, leftrightarrow, "{[\upsilon]}"'] & I(H) \arrow[l, bend left, "\mathcal{S}_\upsilon"] \arrow[d, leftrightarrow, "\vee"] \\
J(\mathcal{H}) \arrow[r, bend left, "\mathrm{sPol}(-;\upsilon)"] & J(H) \arrow[l, bend left, "\mathcal{S}_\upsilon"]
\end{tikzcd}$$

Proof. It follows directly from definitions that $\mathcal{S}_\upsilon$ and $\mathrm{sPol}(-;\upsilon)$ are inverses. Similarly, the edges of H are defined to be precisely the image of $\mathcal{E}$ under $\mathrm{sPol}(-;\upsilon)$. Since both edge ideals are generated by the monomials corresponding to these sets of edges and multiedges, we have $\mathrm{sPol}(I(\mathcal{H});\upsilon) = I(H)$. The Alexander dual maps being involutions follow from Lemma 4.20 and Remark 2.97. It now suffices to show that ω is maximally independent in $\mathcal{H}$ if and only if every member of $\mathrm{sPol}(\upsilon - \omega;\upsilon)$ is a minimal vertex cover of H.

Suppose that every $f \in \mathrm{sPol}(\upsilon - \omega;\upsilon)$ is a minimal vertex cover of H. Thus $V^\upsilon \backslash f$ is maximally independent in H. By Lemma 2.50, $\mathcal{S}_\upsilon(m_{V^\upsilon \backslash f}) = M_\omega$, and by independence there is no $e \in E = \bigcup_{\varepsilon \in \mathcal{E}} \mathrm{sPol}(\varepsilon;\upsilon)$ such that $e \subseteq V^\upsilon \backslash f$. So, there is no $\mathcal{S}_\upsilon(m_e) = M_\varepsilon$ such that $M_\varepsilon \mid M_\omega$. That is, ω is independent in $\mathcal{H}$. For maximality we consider any $\delta^i \preceq \upsilon - \omega$ and show there is some $\varepsilon \in \mathcal{E}$ such that $\varepsilon \preceq \omega + \delta^i$. For any $f \in \mathrm{sPol}(\upsilon - \omega;\upsilon)$ there is some $x_{i,j} \in f$ such that $x_{i,j} \in \mathrm{sPol}(\delta^i;\upsilon)$ because $\delta^i \preceq \upsilon - \omega$. By maximal independence of $V^\upsilon \backslash f$, there is some edge $e \in E$ such that $e \subseteq (V^\upsilon \backslash f) \cup \{x_{i,j}\}$. Thus there is some multiedge of $\mathcal{H}$, ε with $M_\varepsilon = \mathcal{S}_\upsilon(m_e)$, with $M_\varepsilon \mid \mathcal{S}_\upsilon(m_{(V^\upsilon \backslash f) \cup \{x_{i,j}\}}) = M_{\omega + \delta^i}$. This shows $\varepsilon \preceq \omega + \delta^i$, hence ω is maximally independent in $\mathcal{H}$.

Now, suppose that ω is maximally independent in $\mathcal{H}$. It suffices to show that every member of $\mathrm{sPol}(\omega;\upsilon)$ is maximally independent in H i.e. (by Lemma 2.50) every member of $\mathrm{sPol}(\upsilon - \omega;\upsilon)$ is a minimal vertex cover of H. Every edge of H belongs to $\mathrm{sPol}(\varepsilon;\upsilon)$ for some $\varepsilon \in \mathcal{E}$. So if $W \in \mathrm{sPol}(\omega;\upsilon)$ and there is some $e \in E$ with $e \subseteq W$ then $\mathcal{S}_\upsilon(m_e) \mid \mathcal{S}_\upsilon(m_W)$. Or rather there is some $\varepsilon \in \mathcal{E}$ with $M_\varepsilon \mid M_\omega$ hence ω is not independent, a contradiction. For maximality, consider some $x_{i,j} \in V^\upsilon \backslash W$. We will show some edge $e \in E$ is contained by $W \cup \{x_{i,j}\}$. Since $x_{i,j} \notin W$ and $W \cup \{x_{i,j}\} \subseteq V^\upsilon$, we have that $\omega + \delta^i \preceq \upsilon$. Hence there is some multiedge $\varepsilon \in \mathcal{E}$ with $\varepsilon \preceq \omega + \delta^i$. Hence, for every $W' \in \mathrm{sPol}(\omega + \delta^i;\upsilon)$ there is some $e \in \mathrm{sPol}(\varepsilon;\upsilon)$ with $e \subseteq W'$. So in particular $W \cup \{x_{i,j}\}$ contains some edge e of H. This means every element of $\mathrm{sPol}(\omega;\upsilon)$ is maximally independent in H. $\square$

Definition 4.23. If $\mathcal{H}$ is a multihypergraph and $\omega \preceq \upsilon$, then the *submultihypergraph* on ω

is $\mathcal{H}_\omega = (\omega, \mathcal{F})$ where

$$\mathcal{F} = \{\varepsilon \in \mathcal{E} \mid \varepsilon \preceq \omega\}\,.$$

We briefly recall the projection and localization maps on square-free monomials (and square-free monomial ideals) introduced in Definition 2.12 and Definition 2.16, as we now lead towards defining generalized versions of these maps.

Corollary 4.24. *Composing the diagram in Lemma 4.22 twice with the diagram in Theorem 2.101 gives the following diagram, where $W = V^\omega$.*

$$\begin{tikzcd}
I(\mathcal{H}) \arrow[r, "\mathrm{sPol}(-;\upsilon)"] \arrow[d, leftrightarrow, "{[\upsilon]}"'] & I(H) \arrow[r, "\pi_W"] \arrow[d, leftrightarrow, "\vee"] & I(H_W) \arrow[r, "\mathcal{S}_\omega"] \arrow[d, leftrightarrow, "\vee"'] & I(\mathcal{H}_\omega) \arrow[d, leftrightarrow, "{[\omega]}"] \\
J(\mathcal{H}) \arrow[r, "\mathrm{sPol}(-;\upsilon)"'] & J(H) \arrow[r, "\varphi_W"'] & J(H_W) \arrow[r, "\mathcal{S}_\omega"'] & J(\mathcal{H}_\omega)
\end{tikzcd}$$

Example 4.25. Let $\mathcal{H} = ((3,3), \{(3,0),(2,1)\})$ and $\omega = (2,2)$. For simplicity we will use $V = \{X, Y\}$ and $V^\upsilon = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Hence, $I(\mathcal{H}) = \langle X^3, X^2Y\rangle$ and $J(\mathcal{H}) = \langle X^2, XY^3\rangle$. Further,

$$\begin{aligned}
\mathrm{sPol}(I(\mathcal{H});\upsilon) = \langle\, & x_1x_2x_3,\ x_1x_2y_1,\ x_1x_3y_1,\ x_2x_3y_1 \\
& x_1x_2y_2,\ x_1x_3y_2,\ x_2x_3y_2 \\
& x_1x_2y_3,\ x_1x_3y_3,\ x_2x_3y_3 \,\rangle,
\end{aligned}$$

and $\mathrm{sPol}(J(\mathcal{H});\upsilon) = \langle\, x_1x_2,\ x_1x_3,\ x_2x_3,\ x_1y_1y_2y_3,\ x_2y_1y_2y_3,\ x_3y_1y_2y_3 \,\rangle$.

Continuing, we have $W = \{x_1, x_2, y_1, y_2\}$, so $\pi_W(\mathrm{sPol}(I(\mathcal{H});\upsilon)) = \langle x_1x_2y_1,\ x_1x_2y_2\rangle$ and

$$\begin{aligned}
\varphi_W(\mathrm{sPol}(J(\mathcal{H});\upsilon)) &= \langle x_1x_2,\ x_1,\ x_2,\ x_1y_1y_2,\ x_2y_1y_2,\ y_1y_2\rangle \\
&= \langle x_1,\ x_2,\ y_1y_2\rangle\,.
\end{aligned}$$

Lastly, $\mathcal{S}_\omega(\pi_W(\mathrm{sPol}(I(\mathcal{H});\upsilon))) = \langle X^2Y\rangle$ and $\mathcal{S}_\omega(\varphi_W(\mathrm{sPol}(J(\mathcal{H});\upsilon))) = \langle X, Y^2\rangle\,.$

Remark 4.26. We consider the compositions

$$\Pi = \mathcal{S}_\omega \circ \pi_W \circ \mathrm{sPol}(-;\upsilon)$$

and

$$\Phi = \mathcal{S}_\omega \circ \varphi_W \circ \mathrm{sPol}(-;\upsilon).$$

Note that $\mathrm{sPol}(-;\upsilon)$ is set-valued and its image need not be contained in a fiber of $\mathcal{S}_\omega \circ \pi_W$ or a fiber of $\mathcal{S}_\omega \circ \varphi_W$, as illustrated in Example 4.25. In particular, if $\eta \preceq \upsilon$ then the set $\Pi(M_\eta)$ has size either 1 or 2. If there is some $1 \leq i \leq n$ such that both $\upsilon_i - \omega_i$ and η_i are non-zero then there is some $m_f \in \mathrm{sPol}(M_\eta;\upsilon)$ where $f \not\subseteq W$, so $0 \in \Pi(M_\eta)$. Further, if $\eta \preceq \omega$ then $M_\eta \in \Pi(M_\eta)$.

For the other composition, we have $\Phi(M_\eta) = \{M_\varepsilon \mid \eta - ((\upsilon - \omega) \cap \eta) \preceq \varepsilon \preceq \eta \cap \omega\}$. The bounds come from considering the extreme cases when applying φ_W to elements in $\mathrm{sPol}(M_\eta;\upsilon)$. For the lower bound, we consider a monomial $m_f \in \mathrm{sPol}(M_\eta;\upsilon)$ such that that $\varphi_W(m_f)$ has minimal degree. This is given by maximizing the size of $f \cap (V^\upsilon \backslash W)$, which can be done by setting $f_i = \{x_{i,\upsilon_i}, x_{i,\upsilon_i-1}, \ldots, x_{i,\upsilon_i-\eta_i+1}\}$ and $f = \cup_{i=1}^n f_i$. We note that $f \cap W = \cup_{i=1}^n f_i \cap W$ and $f_i \cap W = \{x_{i,\omega_i}, \ldots, x_{i,\upsilon_i-\eta_i+1}\}$ taking the convention that if $\omega_i < \upsilon_i - \eta_i + 1$ then $\{x_{i,\omega_i}, \ldots, x_{i,\upsilon_i-\eta_i+1}\} = \emptyset$. Since $\varphi_W(m_f) = m_{f\cap W}$, the monomial $\mathcal{S}_\omega(m_f)$ has multidegree $(|f_1 \cap W|, \ldots, |f_n \cap W|)$. We note that

$$\begin{aligned}|f_i \cap W| &= \max(\omega_i - (\upsilon_i - \eta_i + 1) + 1, 0)\\ &= \max(\eta_i - (\upsilon_i - \omega_i), 0)\\ &= \eta_i - \min(\upsilon_i - \omega_i, \eta_i).\end{aligned}$$

Thus $\mathrm{mdeg}(\mathcal{S}_\omega(m_f)) = \eta - ((\upsilon - \omega) \cap \eta)$. We note that although the f we constructed need not be a unique subset of V^υ that minimizes the degree of $\varphi_W(m_f)$, any other such minimal f' must have that $f \cap (V^\upsilon \backslash W) = f' \cap (V^\upsilon \backslash W)$. It follows that $\mathrm{mdeg}(\mathcal{S}_\omega(m_{f'})) = \eta - ((\upsilon - \omega) \cap \eta)$.

For the upper bound, we consider $g \subseteq V^\upsilon$ that maximizes the degree of $\varphi_W(m_g)$. Taking $g = \{x_{i,j} \mid j \leq \eta_i\}$ achieves this, and $g \cap W = \{x_{i,j} \mid j \leq \min(\eta_i, \omega_i)\}$. Hence $\mathrm{mdeg}(\mathcal{S}_\omega(m_g)) = \eta \cap \omega$.

For all of the ε with $\eta - ((\upsilon - \omega) \cap \eta) \preceq \varepsilon \preceq \eta \cap \omega$, we take $\theta = \eta \cap \omega - \varepsilon$ and $h = \{x_{i,j} \mid 1 + \theta_i \leq j \leq \eta_i + \theta_i\}$. We then have $\mathcal{S}_\omega(\varphi_W(m_h)) = M_\varepsilon$. Informally, h is constructed by "shifting" each g_i towards f_i by the amount dictated by ε_i.

Although these compositions are multi-valued, we will choose the minimal elements of their image when defining generalized versions of π_W and φ_W. This choice is so that the generalized versions of these maps give the generators of the ideals their images "should" be.

Definition 4.27. We define π_ω as a map from monomials that divide M_υ to monomials that divide M_ω (or zero) as:

$$\pi_\omega(M_\eta) = \begin{cases} 0 & \text{if } \eta \not\preceq \omega \\ M_\eta & \text{if } \eta \preceq \omega \ . \end{cases}$$

Definition 4.28. We define φ_ω as a map from monomials that divide M_υ to monomials that divide M_ω (or one) as $\varphi_\omega(M_\eta) = M_{\eta-((\upsilon-\omega)\cap\eta)}$.

Remark 4.29. The maps π_ω and φ_ω are not (generally) ring homomorphisms like their square-free counterparts. We emphasize that they are only maps between monomials, they do not respect multiplication as their domains and images are not closed under multiplication.

If I is a monomial ideal, we define $\pi_\omega(I) = \langle \pi_\omega(M) \mid M \in \mathcal{G}(I)\rangle$ and similarly $\varphi_\omega(I) = \langle \varphi_\omega(M) \mid M \in \mathcal{G}(I)\rangle$.

Theorem 4.30. *For any multihypergraph $\mathcal{H}$ and $\omega \preceq \upsilon$, the following diagram commutes.*

$$\begin{tikzcd}
I(\mathcal{H}) \arrow[r, "\pi_\omega"] \arrow[d, leftrightarrow, "{[\upsilon]}"'] & I(\mathcal{H}_\omega) \arrow[d, leftrightarrow, "{[\omega]}"] \\
J(\mathcal{H}) \arrow[r, "\varphi_\omega"'] & J(\mathcal{H}_\omega)
\end{tikzcd}$$

Proof. This follows as π_ω and φ_ω are defined to be the minimal elements in the images of the corresponding compositions in the diagram given by Corollary 4.24. $\square$

We now move towards generalizing Theorem 3.4, by giving essentially identical combinatorial criterion for self-dual multihypergraphs. First however, we must consider what a multihypergraph coloring is.

Definition 4.31. The *chromatic number* of a multihypergraph, $\chi(\mathcal{H})$, is the smallest s such that $\mathcal{V} = \mathcal{W}_1 \uplus \cdots \uplus \mathcal{W}_s$ can be written as the union of s multisets that are independent in $\mathcal{H}$. We call such a union an s-coloring of $\mathcal{H}$.

Theorem 4.32. *For a multihypergraph $\mathcal{H}$*

$$\chi(\mathcal{H}) = \min\left\{ s \;\middle|\; M_\upsilon^{s-1} \in J(\mathcal{H})^s \right\}.$$

Proof. Suppose $\mathcal{V} = \mathcal{W}_1 \uplus \cdots \uplus \mathcal{W}_s$ is an s-coloring of $\mathcal{H}$, and let ω_i be the multidegree of $\mathcal{W}_i$. Thus we have

$$\begin{aligned}
\upsilon &= \omega_1 + \cdots + \omega_s \\
(s-1)\upsilon &= s\upsilon - \omega_1 - \cdots - \omega_s \\
(s-1)\upsilon &= (\upsilon - \omega_1) + \cdots + (\upsilon - \omega_s).
\end{aligned}$$

Hence,

$$M_\upsilon^{s-1} = M_{\upsilon-\omega_1} \cdots M_{\upsilon-\omega_s} \in J(\mathcal{H})^s.$$

Just as in Theorem 2.98, the argument can be reversed to show that $M_\upsilon^{s-1} \in J(\mathcal{H})^s$ if and only if there is an s-coloring of $\mathcal{H}$. $\square$

Lemma 4.33. *If $\mathcal{H}$ is a multihypergraph then $\chi(\mathcal{H}) > 2$ if and only if $J(\mathcal{H}) \subseteq I(\mathcal{H})$.*

Proof. The following are equivalent:

- $J(\mathcal{H}) \subseteq I(\mathcal{H})$;
- the complement of every independent multiset in $\mathcal{H}$ precedes a multiedge;
- for every independent ω, $\upsilon - \omega = \overline{\omega}$ precedes a multiedge;
- there is no 2-coloring of $\mathcal{H}$;
- $\chi(\mathcal{H}) > 2$.

$\square$

In addition to a generalized version of hypergraph coloring, we must also introduce a generalized notion of intersecting.

Definition 4.34. A multihypergraph $\mathcal{H}$ is *intersecting* if for every pair of edges $\epsilon, \epsilon' \in \mathcal{E}$ $\epsilon + \epsilon' \not\preceq \upsilon$.

Lemma 4.35. *A multihypergraph $\mathcal{H}$ is intersecting if and only if $I(\mathcal{H}) \subseteq J(\mathcal{H})$.*

Proof. The ideal containment $I(\mathcal{H}) \subseteq J(\mathcal{H})$ is equivalent to saying that every multiedge of $\mathcal{H}$ is the complement of an independent multiset. Equivalently, every pair of multiedges $\epsilon, \eta \in \mathcal{E}$ satisfies $\eta \not\preceq \overline{\epsilon}$. By substituting $\overline{\epsilon} = \upsilon - \epsilon$ and rearranging, this is exactly $\epsilon + \eta \not\preceq \upsilon$. $\square$

Theorem 4.36. *If $\mathcal{H}$ is a multihypergraph then the following are equivalent:*

(i) *$\mathcal{H}$ is self-dual;*

(ii) *$\mathcal{H}$ is intersecting and $\chi(\mathcal{H}) \geq 3$;*

(iii) *$I(\mathcal{H}) = J(\mathcal{H})$;*

(iv) *Every multiedge of $\mathcal{H}$ is the complement of an independent multiset of $\mathcal{H}$, and the complement of every independent multiset of $\mathcal{H}$ contains a multiedge of $\mathcal{H}$;*

(v) *For every $\omega \preceq \upsilon$ exactly one of $M_\omega, M_{\upsilon-\omega}$ is in $I(\mathcal{H})$.*

Proof. The equivalence of (i), (ii), and (iii) follows from Lemma 4.33 and Lemma 4.35. In the proofs of these lemmata we noted that the two conditions of (iv) are equivalent to $I(\mathcal{H}) \subseteq J(\mathcal{H})$ and $J(\mathcal{H}) \subseteq I(\mathcal{H})$ respectively.

Finally, we observe $(iii) \iff (v)$ by applying Lemma 4.19 to see that (iii) is equivalently $I(\mathcal{H}) = (I(\mathcal{H}))^{[v]}$. The definition of $I^{[v]}$ yields the equivalence. $\square$

Example 4.37. We explicitly consider generalized self-duality in the simplest setting possible, namely in $k[X]$. Let $I = \langle X^r \rangle$ for some $r \geq 0$. We recall that a square-free ideal of the form $\langle X_i \rangle$ is self-dual for any choice of V. As a monomial ideal, I is the edge ideal of many multihypergraphs with multiedge $(r) \in \mathbb{N}^1$. These different multihypergraphs depend solely on a choice of base multiset. However, this base multiset highly impacts the behavior of the cover ideal.

Since we are working with just one variable in this case, every multihypergraph has a base multiset υ such that $(r) \preceq \upsilon$. We note that $\eta = (r-1)$ is always maximally independent in $\mathcal{H}$, so $J(\mathcal{H}) = \langle x^{\upsilon-\eta} \rangle$ is generated by its complement. Hence we have $J(\mathcal{H}) = \langle X^{\upsilon-r+1} \rangle$ and we observe that this means $I(\mathcal{H}) = J(\mathcal{H})$ if and only if $r = \upsilon - r + 1$, i.e. $\upsilon = 2r - 1$.

Remark 4.38. The equivalent condition in Example 4.37 of $\langle X^r \rangle$ being self-dual in $k[X]$ is reminiscent of Example 3.8. This is no coincidence, and we now consider the consequences of the choice of $\upsilon = (n)$ on symmetric polarization:

$$\begin{aligned} \operatorname{sPol}(I(\mathcal{H}); \upsilon) &= \langle \operatorname{sPol}(X^r; \upsilon) \rangle \\ &= \langle \{ m_W \mid \mathcal{S}_\upsilon(m_W) = X^r \text{ and } W \subseteq \{x_1, \dots, x_n\} \} \rangle \\ &= I(K_n^r), \end{aligned}$$

where K_n^r is the complete r-uniform hypergraph on $V = \{x_1, \dots, x_n\}$.

Example 4.39. We consider the highly symmetric hypergraph $Z(n)$, where in particular, $\operatorname{Aut}(Z(n)) \cong S_n$. Indeed, $I(Z(n))$ is the symmetric depolarization of $\langle X^n, ZX \rangle$, which is self-dual with respect to $\omega = (n, 1)$.

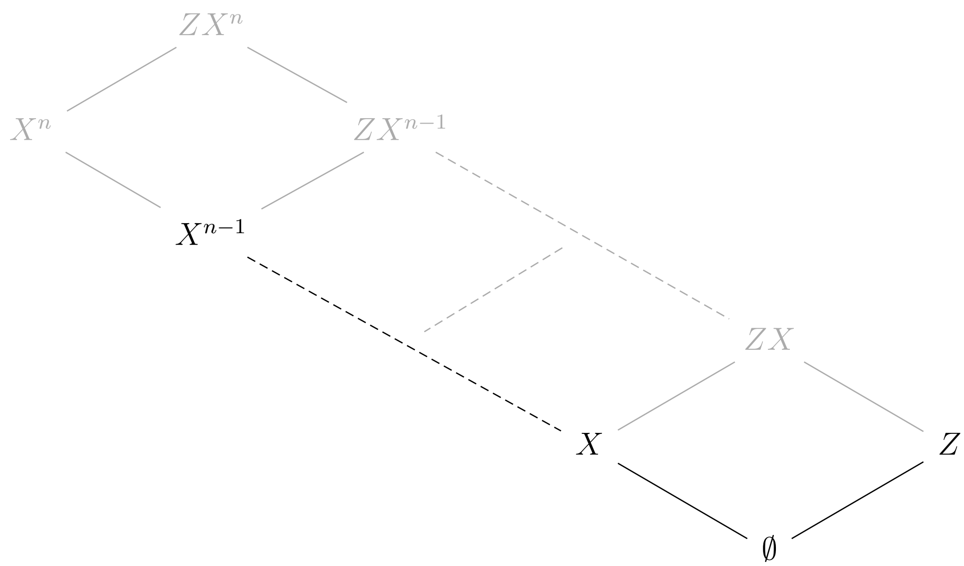


Figure 4.1: Hasse Diagram of Example 4.39

We make a visual comparison between this diagram and the Hasse diagram of $\Delta(Z(3))$ in Example 3.7. We may view the monomial ideal in fewer variables as capturing a large portion of the "horizontal" symmetry of the square-free monomial ideal, without removing the "vertical" symmetry responsible for Alexander duality.

Lemma 4.40. *If $\mathcal{H}$ is an intersecting multihypergraph, then $\Psi(\mathcal{H}) := I(\mathcal{H}) + ZJ(\mathcal{H})$ is self-dual with respect to $(\upsilon, 1)$ in $R[Z]$.*

Proof. We closely follow the argument given in the proof of Theorem 3.12. We first note that $\Psi(\mathcal{H})$ is a monomial ideal, hence is the edge ideal of a unique multihypergraph with base multiset determined by $(\upsilon, 1)$ where each multiset has dimension $n+1$. For succinctness we will also refer to this multihypergraph as $\Psi(\mathcal{H})$. All complements denoted with a $\overline{\square}$ are with respect to $(\upsilon, 1)$, otherwise we will use $\upsilon - \square$. We also recall that $\mathrm{mdeg}(Z) = \delta^{n+1}$.

Let $\mathcal{F}'$ be the set of complements of maximal independent multisets of $\mathcal{H}$, and set $\mathcal{F} = \mathcal{F}' \backslash \mathcal{E}$ to be the elements of $\mathcal{F}'$ that are not multiedges of $\mathcal{H}$. The multiedges of $\Psi(\mathcal{H})$ are:

- $\mathcal{E}$, the multiedges of $\mathcal{H}$ (in dimension $n+1$ with $\epsilon_{n+1} = 0$), and
- $Z\mathcal{F} = \{Z\eta = \eta \uplus \delta^{n+1} \mid \eta \in \mathcal{F}\}$.

We take $\mathcal{T} = \mathcal{E} \cup Z\mathcal{F}$.

First, we show that for each $\tau \in \mathcal{T}$, the complement $\overline{\tau}$ is independent in $\Psi(\mathcal{H})$. If $\epsilon \in \mathcal{E} \subseteq \mathcal{T}$ then by the intersecting hypothesis, $M_\epsilon \in J(\mathcal{H})$ so $\overline{\epsilon}$ is independent in $\mathcal{H}$. Thus no $\epsilon' \in \mathcal{E}$ precedes $\overline{\epsilon}$. We also show that $\overline{\epsilon}$ is not preceded by any $Z\eta \in Z\mathcal{F}$. Since any such

$\overline{\eta}$ is maximally independent in $\mathcal{H}$, $(\epsilon_1, \ldots, \epsilon_n) \not\preceq \upsilon - \eta$, or equivalently $\eta \not\preceq \upsilon - (\epsilon_1, \ldots, \epsilon_n)$. Thus $Z\eta \not\preceq \overline{\epsilon}$, so $\overline{\epsilon}$ is independent in $\Psi(\mathcal{H})$.

We now pick some $Z\eta \in Z\mathcal{F}$ and show $\overline{z\eta}$ is also independent in $\Psi(\mathcal{H})$. If $Z\eta' \in Z\mathcal{F}$ then $(Z\eta')_{n+1} = 1$ and $(\overline{Z\eta})_{n+1} = 0$, hence $Z\eta' \not\preceq \overline{z\eta}$. If $\epsilon' \in \mathcal{E}$ then by definition, $(\epsilon'_1, \ldots, \epsilon'_n) \not\preceq \upsilon - \eta$, hence $\epsilon' \not\preceq \overline{Z\eta}$. We have thus shown that for every $\tau \in \mathcal{T}$ that the complement $\overline{\tau}$ is independent in $\Psi(\mathcal{H})$.

It suffices to show that if $\mathcal{W}$ is independent in $\Psi(\mathcal{H})$ then there is some $\tau \in \mathcal{T}$ such that $\tau \preceq \overline{\omega}$. Suppose $\mathcal{W}$ is independent in $\Psi(\mathcal{H})$, then every $\tau \in \mathcal{T}$ has $\tau \not\preceq \omega$.

We first consider if $\omega_{n+1} = 0$. Note that because each $\epsilon \in \mathcal{E}$ has $\epsilon_{n+1} = 0$ we have $(\epsilon_1, \ldots, \epsilon_n) \not\preceq (\omega_1, \ldots, \omega_n)$. Thus $(\omega_1, \ldots, \omega_n)$ is independent in $\mathcal{H}$. Hence there is some $\eta \in \mathcal{F}'$ with $(\omega_1, \ldots, \omega_n) \preceq \upsilon - \eta$, so then $\eta \preceq \upsilon - (\omega_1, \ldots, \omega_n)$. By hypothesis $(\overline{\omega})_{n+1} = 1$, so then $Z\eta \preceq \overline{\omega}$.

Now, suppose $\omega_{n+1} = 1$. If η is the complement of a maximally independent multiset in $\mathcal{H}$ then either $(\eta, 0) \in \mathcal{E}$ or $Z\eta \in \mathcal{F}$. In both cases, by independence of ω and by hypothesis, $\eta \not\preceq (\omega_1, \ldots, \omega_n)$. So $\upsilon - (\omega_1, \ldots, \omega_n) \not\preceq \upsilon - \eta$. This shows that $\upsilon - (\omega_1, \ldots, \omega_n)$ precedes no maximally independent multiset in $\mathcal{H}$ and therefore must be preceded by some multiedge of $\mathcal{H}$, say $(\epsilon_1, \ldots, \epsilon_n)$. So, $\epsilon = (\epsilon_1, \ldots, \epsilon_n, 0) \in \mathcal{E}$, and we have $\epsilon \preceq \overline{\omega}$. Thus the complement of every independent multiset in $\Psi(\mathcal{H})$ is preceded by some multiedge in $\Psi(\mathcal{H})$. $\square$

Remark 4.41. We recall that Corollary 3.13 to Theorem 3.4 provides a classification of self-dual square-free monomial ideals as the image of ψ. One may hope a similar Corollary to Lemma 4.40 holds, but we observe otherwise as Example 4.42 gives a self-dual multihypergraph where every entry of the base multiset is larger than 1.

Example 4.42. We recall Example 2.39: if $I = \langle X_1^2 X_2, X_2^3 \rangle$ and $\omega = (3, 3)$, then $V^\omega = \{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}\}$ and

$$\begin{aligned}\mathrm{sPol}(I;\omega) = \langle &x_{1,1}x_{1,2}x_{2,1},\ x_{1,1}x_{1,2}x_{2,2},\ x_{1,1}x_{1,2}x_{2,3},\\ &x_{1,1}x_{1,3}x_{2,1},\ x_{1,1}x_{1,3}x_{2,2},\ x_{1,1}x_{1,3}x_{2,3},\\ &x_{1,2}x_{1,3}x_{2,1},\ x_{1,2}x_{1,3}x_{2,2},\ x_{1,2}x_{1,3}x_{2,3},\\ &x_{2,1}x_{2,2}x_{2,3}\rangle.\end{aligned}$$

The reader may also notice this is isomorphic to the ideal given by $\psi^\vee(H)$ in Example 3.18. Indeed I symmetrically polarizes to a self-dual, square-free monomial ideal, however we also note it is itself self-dual with respect to $\omega = (3, 3)$. We provide the relevant Hasse diagram below.

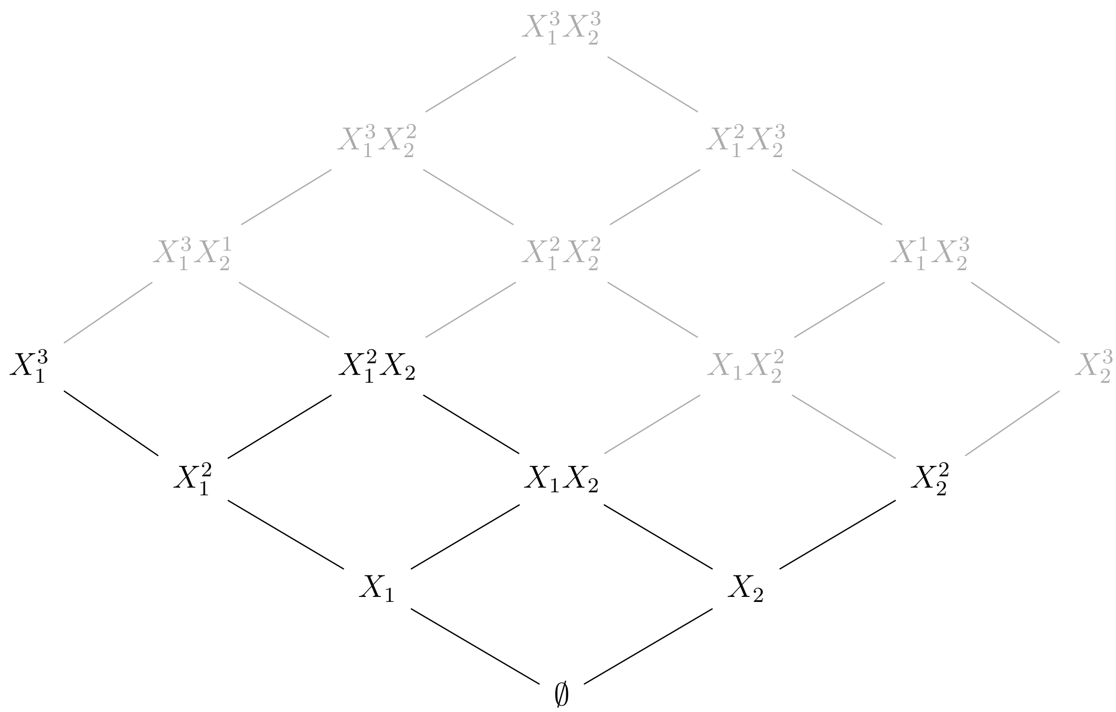


Figure 4.2: Hasse Diagram of Example 4.42

Remark 4.43. We continue the train of thought from Remark 4.41. Example 4.42 shows that not every self-dual multihypergraph is the image of Ψ, so perhaps this is not a general enough approach. One obvious potential generalization of Lemma 4.40 would be that if $\mathcal{H}$ is intersecting, then $\Psi'(\mathcal{H}) = I(\mathcal{H}) + X_i J(\mathcal{H})$ is self-dual with base multiset $\upsilon + \delta^i$. However, Lemma 4.44 shows this cannot be.

Lemma 4.44. *If $\mathcal{H}$ is a self-dual multihypergraph, then υ must have an odd entry.*

Proof. If every entry of υ is even, then $\omega = \frac{\upsilon}{2} \in \mathbb{N}^n$. However, $\omega = \upsilon - \omega$, so M_ω cannot satisfy exactly one of $M_\omega, M_{\upsilon-\omega} \in I(\mathcal{H})$ thus $\mathcal{H}$ cannot be self-dual. $\square$

Lemma 4.45. *A multihypergraph $\mathcal{H}$ is intersecting if and only if* $\mathrm{sPol}(\mathcal{H};\upsilon)$ *is an intersecting hypergraph.*

Proof. For each $\varepsilon \in \mathcal{E}$, set $e_{\min}(\varepsilon) = \{x_{i,j} \mid 1 \leq j \leq \varepsilon_i\}$ and $e_{\max}(\varepsilon) = \{x_{i,j} \mid \upsilon_i - \varepsilon_i < j \leq n\}$. Note that for any two edges in $\mathrm{sPol}(\mathcal{H};\upsilon)$, say $e \in \mathrm{sPol}(\varepsilon;\upsilon)$ and $f \in \mathrm{sPol}(\eta;\upsilon)$ the smallest possible $|e \cap f|$ is given by $|e_{\min}(\varepsilon) \cap e_{\max}(\eta)|$.

If $\mathrm{sPol}(\mathcal{H};\upsilon)$ is intersecting, then in particular for every $\varepsilon, \eta \in \mathcal{E}$ the intersection $e_{\min}(\varepsilon) \cap e_{\max}(\eta)$ is non-empty. Hence, there is some i such that $\{x_{i,1}, \ldots, x_{i,\varepsilon_i}\} \cap \{x_{i,\upsilon_i-\eta_i+1}, \ldots, x_{i,\upsilon_i}\}$ is non-empty. That is, $\upsilon_i - \eta_i + 1 \leq \varepsilon_i$, or rather $\upsilon_i < \varepsilon_i + \eta_i$. Thus, $\mathcal{H}$ is intersecting.

If $\mathcal{H}$ is intersecting, then for each pair $\varepsilon, \eta \in \mathcal{E}$, the intersection $e_{\min}(\varepsilon) \cap e_{\max}(\eta)$ is non-empty. Thus the pairs of edges of the hypergraph $\mathrm{sPol}(\mathcal{H}; \upsilon)$ with the smallest intersection size have non-empty intersections, i.e. $\mathrm{sPol}(\mathcal{H}; \upsilon)$ is intersecting. $\square$

Corollary 4.46. *A multihypergraph $\mathcal{H}$ is not 2-colorable if and only if $\chi(\mathrm{sPol}(\mathcal{H}; \upsilon)) > 2$.*

Corollary 4.47. *A multihypergraph $\mathcal{H}$ is a self-dual multihypergraph if and only if $\mathrm{sPol}(\mathcal{H}; \upsilon) \subseteq k[V^\upsilon]$ is a self-dual multihypergraph.*

We could prove the following Theorem with a similar analysis as the proofs of Theorem 3.12 and Lemma 4.40, however we have developed enough knowledge of self-dual multihypergraphs to avoid some of this tedium.

Theorem 4.48. *If $\mathcal{H}$ is an intersecting multihypergraph and $r \geq 1$ then the multihypergraph with edge ideal*

$$\Psi_r(\mathcal{H}) := I(\mathcal{H}) + Z^r J(\mathcal{H})$$

and base multiset $(\upsilon, 2r-1)$ is self-dual.

Proof. We will apply Corollary 4.47 by showing that $\Psi_r(\mathcal{H})$ symmetrically polarizes to a self-dual hypergraph, L. We note that $\Psi_r(\mathcal{H})$ is a monomial ideal such that each minimal generator M_ω divides $M_{(\upsilon,2r-1)}$. Hence we will refer to $\Psi_r(\mathcal{H})$ as a multihypergraph on $(\upsilon, 2r-1)$.

This multihypergraph $\Psi_r(\mathcal{H})$ is intersecting. For any pair of its multiedges, say ε and η, we consider three possible cases. If both $\pi_{(\upsilon,0)}(\varepsilon)$ and $\pi_{(\upsilon,0)}(\eta)$ belong to $\mathcal{E}$, then $\varepsilon + \eta \not\leq (\upsilon, 2r-1)$ by the definition of $\mathcal{H}$ being intersecting. If $\pi_{(\upsilon,0)}(\varepsilon) \in \mathcal{E}$ and $\eta = (\eta', r)$ where η' is the complement of a maximal independent set of $\mathcal{H}$, then by Proposition 4.17 $\varepsilon + \eta \not\leq (\upsilon, 2r-1)$. Finally, if both $\varepsilon = (\varepsilon', r)$ and $\eta = (\eta', r)$ then $\varepsilon + \eta \not\leq (\upsilon, 2r-1)$ follows by inspecting the last entry.

We now consider the hypergraphs $H = \mathrm{sPol}(\mathcal{H}; (\upsilon, 2r-1))$ and $L = \mathrm{sPol}(\Psi_r(\mathcal{H}); (\upsilon, 2r-1))$. Both are intersecting by Lemma 4.45. If we take $\omega = (\upsilon, 2r-2)$ then we have the following relationships among edge and cover ideals, $I(H) = \mathrm{sPol}(I(\mathcal{H}); (\upsilon, 2r-1))$ and

$$\begin{aligned}
I(L) &= \mathrm{sPol}[I(\mathcal{H}) + Z^r J(\mathcal{H}); (\upsilon, 2r-1)] \\
&= \mathrm{sPol}[I(\mathcal{H}); (\upsilon, 2r-1)] + \mathrm{sPol}[Z^r J(\mathcal{H}); (\upsilon, 2r-1)] \\
&= I(H) + \mathrm{sPol}[Z^r J(\mathcal{H}); \omega] + z_{2r-1}\,\mathrm{sPol}[Z^{r-1} J(\mathcal{H}); \omega] \\
&= I(H) + J(H)[\mathrm{sPol}(Z^r; \omega) + z_{2r-1}\,\mathrm{sPol}(Z^{r-1}; \omega)].
\end{aligned}$$

In the last two lines, note that the symmetric polarizations are with respect to ω. Further, we separate z_{2r-1} from the other z_i to help us consider the subhypergraph $L' =$

$\pi_{V^v \cup \{z_1,\ldots,z_{2r-2}\}}(L)$. As the subhypergraph of an intersecting hypergraph, L' is intersecting. It suffices show that $I(L) = \psi(L') = I(L') + z_{2r-1}J(L')$. This is sufficient to conclude the proof as this shows L is self-dual and since L is also the symmetric polarization of $\Psi_r(\mathcal{H})$, Lemma 4.40 implies $\Psi_r(\mathcal{H})$ is self-dual.

We now calculate $I(L')$. From Theorem 2.101 $I(L') = \pi_{V^v \cup \{z_1,\ldots,z_{2r-2}\}}(I(L))$, and by the calculation above this is $I(L') = I(H) + J(H)\,\mathrm{sPol}(Z^r;\omega)$. So, every edge of L' is either an edge of H or a minimal vertex cover of H that does not contain an edge of H adjoined with a size r subset of $\{z_1,\ldots,z_{2r-2}\}$.

To calculate $J(L')$, suppose W is a minimal vertex cover of L'. Since $\pi_{V^v}(I(L')) = I(H)$, there is some minimal vertex cover of H, say f, such that $m_f \mid \varphi_{V^v}(m_W)$ i.e. $f \subseteq W$. We consider two cases for W.

If W intersects every minimal vertex cover of H, then W is a vertex cover of $H^\vee$ In other words, W contains an edge of H. Hence, by minimality, W contains no z_i as W intersects every edge of L' somewhere in V^v.

Otherwise, there is some minimal vertex cover f' of H such that $W \cap f' = \emptyset$. Then it must be that W meets every size r subset of $\{z_1,\ldots,z_{2r-2}\}$. By minimality of W (as a vertex cover of L') this means $W = f \cup \{z_{i_1},\ldots,z_{i_{r-1}}\}$.

Thus for every $m_W \in \mathcal{G}(J(L'))$, either $m_W \in I(H)$ or $m_W \in J(H)\,\mathrm{sPol}(Z^{r-1};\omega)$. Lastly, we observe that $J(H)\,\mathrm{sPol}(Z^{r-1};\omega) \subseteq J(L')$ because if f is a vertex cover of H then $f \cup \{z_{i_1},\ldots,z_{i_{r-1}}\}$ meets every edge of L'. Thus $I(L) = I(L') + z_{2r-1}J(L')$. $\square$

Example 4.49. The multihypergraph $\mathcal{H} = ((4),\{(3)\})$ is intersecting with maximally independent multiset (2). Hence $J(\mathcal{H}) = \langle X^2\rangle$ and $\Psi_2(\mathcal{H}) = \langle X^3, X^2Z^2\rangle$ is self dual on $(4,3)$.

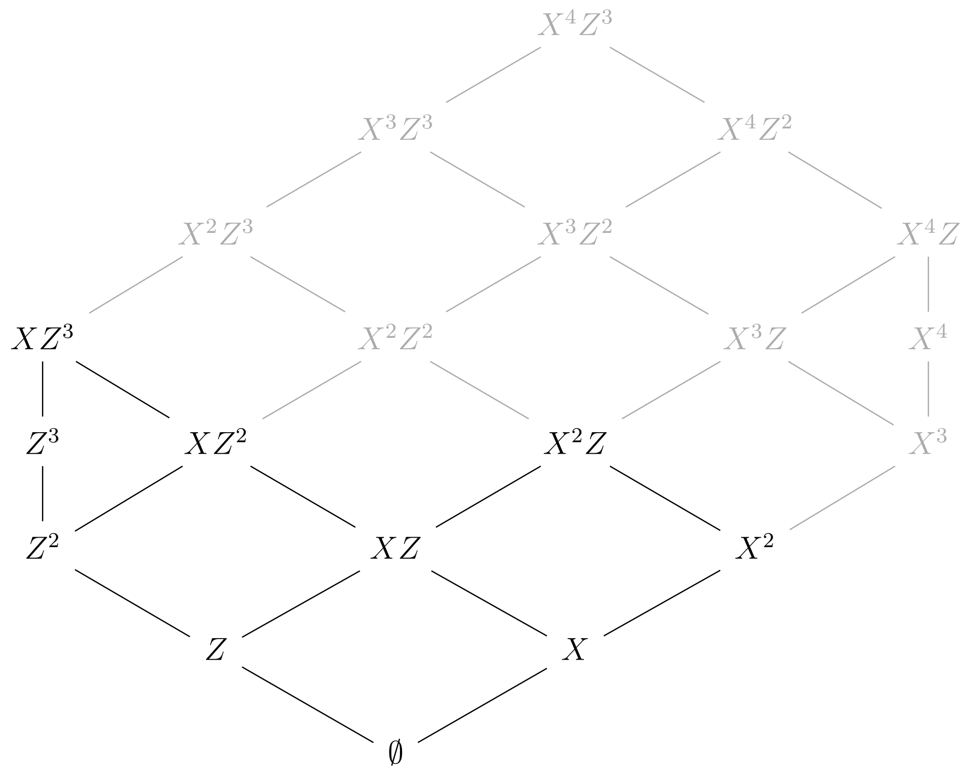


Figure 4.3: $\Psi_2(\mathcal{H})$

Example 4.50. The multihypergraph $\mathcal{H} = ((2,2), \{(2,1),(1,2)\})$ is intersecting, with maximally independent multisets $(0,2),(1,1)$, and $(2,0)$. Hence $J(\mathcal{H}) = \langle X_1^2, X_1X_2, X_2^2\rangle$ and $\Psi_2(\mathcal{H}) = \langle X_1^2X_2, X_1X_2^2, X_1^2Z^2, X_1X_2Z^2, X_2^2Z^2\rangle$ is self dual on $(2,2,3)$.

We re-examine Remark 4.41, with the hope that Theorem 4.48 provides a classification of self-dual monomial ideals as the image of Ψ_r. Again, this hope fails as we will show in Example 4.51.

Example 4.51. The multihypergraph

$$\mathcal{K} = ((2,2,3), \{\ (2,1,0),(1,2,0),(1,1,2),(0,2,1),(2,0,3)\ \})$$

is self-dual. As a counter-example, it suffices to observe that only the third entry of the base multiset of $\mathcal{K}$ is odd, but there are multiedges where the third entry is not 2. Thus, $\mathcal{K}$ cannot be the image of any Ψ_r. Moreover, we observe that $\pi_{(2,2,0)}(\mathcal{K}) \cong \mathcal{H}$ from Example 4.50.

Definition 4.52. If $\mathcal{H}$ is a multihypergraph then $\mathcal{D}(\mathcal{H})$ is the partially ordered set of independent multisets in $\mathcal{H}$.

Remark 4.53. We now consider a generalization of the flip operation introduced in Definition 3.37 for self-dual monomial ideals. Just like in the square-free case, it is convenient to work with the (downwards-closed) poset of independent multisets which determines a multihypergraph. If $\mathcal{H}$ is a self-dual multihypergraph, $\mathcal{D} = \mathcal{D}(\mathcal{H})$, and ω is maximal in $\mathcal{D}$ then for all $\eta \preceq \upsilon$, exactly one of η or $\upsilon - \eta$ is in $(\mathcal{D}\backslash\{\omega\}) \cup \{\upsilon - \omega\}$. However, it need not be the case that $(\mathcal{D}\backslash\{\omega\}) \cup \{\upsilon - \omega\}$ is the set of independent sets of a multihypergraph. In particular, the poset $(\mathcal{D}\backslash\{\omega\}) \cup \{\upsilon - \omega\}$ is not generally downwards closed with respect to $\preceq$.

Definition 4.54. Let $\mathcal{H}$ be a self-dual multihypergraph, ω be a maximally independent multiset of $\mathcal{H}$ not of the form $\frac{\upsilon-\delta^i}{2}$ for some i, and $\mathcal{D}$ be the set of independent multisets in $\mathcal{H}$. *The flip of $\mathcal{H}$ with respect to ω*, denoted $F_\omega(\mathcal{H})$, is the self-dual multihypergraph with independent multisets $F_\omega(\mathcal{D}) = (\mathcal{D}\backslash\{\omega\}) \cup \{\upsilon - \omega\}$.

We now give two examples to show why maximally independent multisets of the form $\frac{\upsilon-\delta^i}{2}$ are prohibited in Definition 4.54. In particular, we will show that the poset resulting from $F_{\frac{\upsilon-\delta^i}{2}}$ is not downwards closed, and hence does correspond to a multihypergraph.

Example 4.55. Let $\mathcal{H} = ((2r-1), \{(r)\})$ be the self-dual multihypergraph from Example 4.37. In this case $\mathcal{D}(\mathcal{H}) = \{(0), (1), \dots, (r-1)\}$ which has unique maximal element $(r-1)$. So, $F_{(r-1)}(\mathcal{D}(\mathcal{H})) = \{(0), (1), \dots, (r-2), (r)\}$, but it cannot be that (r) is independent in a multihypergraph and $(r-1)$ isn't.

Example 4.56. We give another counterexample showing that the failure in Example 4.55 is not an artifact of working in only one variable. Let $\mathcal{H} = ((2,3), \{(2,1), (1,2), (0,3)\})$. The independent multisets are represented in the Hasse diagram below.

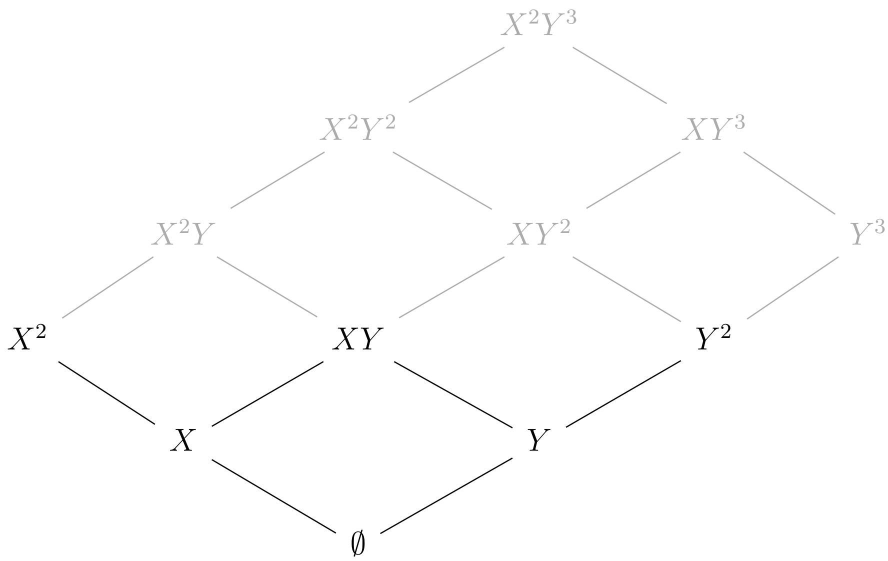


Figure 4.4: Hasse Diagram of $\mathcal{D}(\mathcal{H})$

Calculating $F_{(2,0)}(\mathcal{D}(\mathcal{H}))$ yields the self-dual multihypergraph with multiedge set $\{(2,0),(1,2)\}$. Similarly, calculating $F_{(0,2)}(\mathcal{D}(\mathcal{H}))$ yields the self-dual multihypergraph with multiedge set $\{(0,2)\}$. However, $F_{(1,1)}(\mathcal{D}(\mathcal{H}))$ yields a partially ordered set displayed in the diagram below, where the red line shows where downwards containment fails.

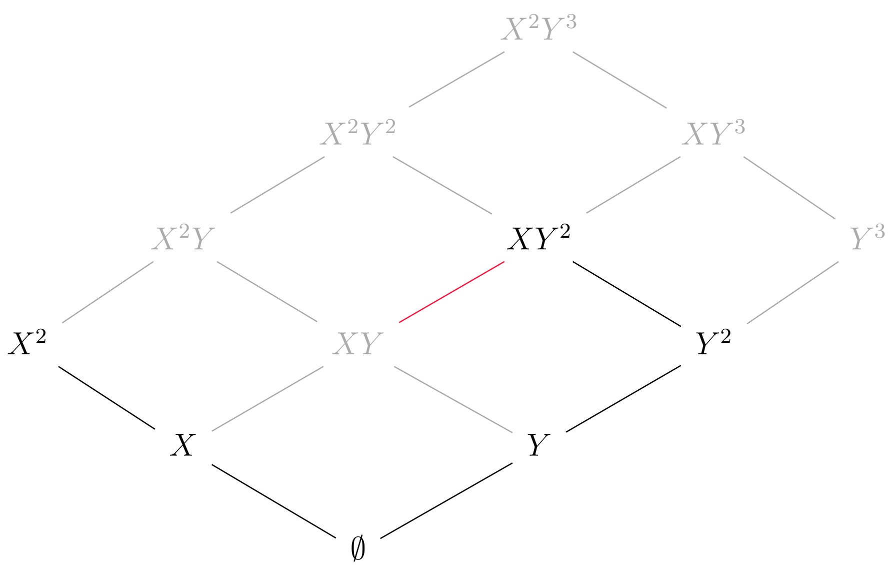


Figure 4.5: Hasse Diagram of $F_{(1,1)}(\mathcal{D}(\mathcal{H}))$

Lemma 4.57. *Let $\mathcal{H}$ be a self-dual multihypergraph and $\omega \in \mathcal{D}(\mathcal{H})$ be maximal. If υ has more than one odd entry, or υ has exactly one odd entry (υ_i) and $\omega \neq \frac{\upsilon-\delta^i}{2}$, then $F_\omega(\mathcal{D})$ is the set of independent multisets of a self-dual multihypergraph.*

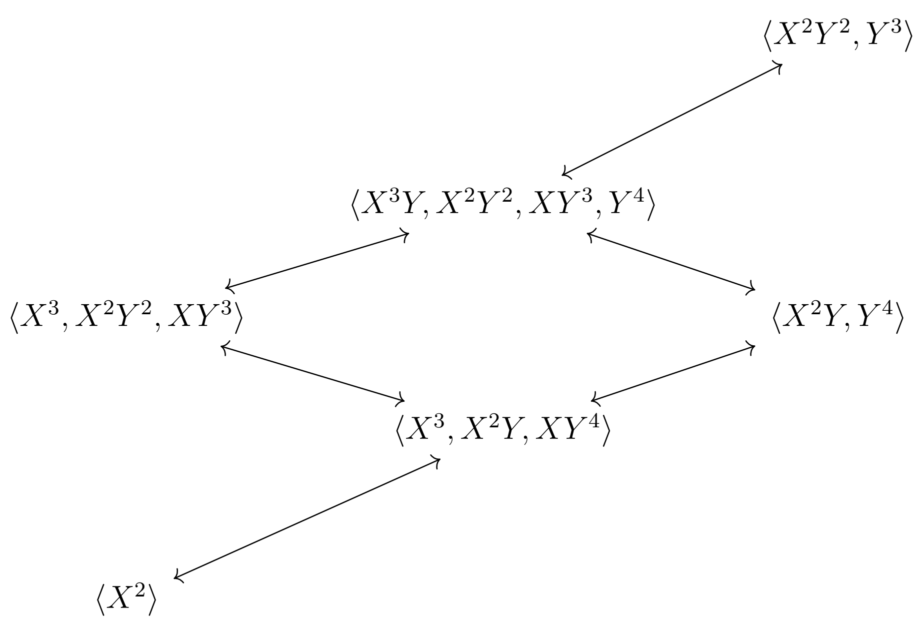
Proof. By Remark 4.53 it suffices to show that $F_\omega(\mathcal{D})$ is the set of independent sets of a multihypergraph i.e., $F_\omega(\mathcal{D})$ is downwards-closed with respect to $\preceq$. Since $\mathcal{D}$ is downwards-closed and ω is maximal in $\mathcal{D}$ then $\mathcal{D}\backslash\{\omega\}$ is downwards closed. We now consider the multiset that we will add, $\upsilon - \omega$. Note $\upsilon - \omega$ is the complement of a maximal independent multiset in $\mathcal{H}$ so by self-duality, $\upsilon - \omega$ is a multiedge of $\mathcal{H}$. Thus for every unit vector ε that satisfies $\varepsilon \preceq \upsilon - \omega$, the multiset $\upsilon - \omega - \varepsilon \in \mathcal{D}$. It follows that $F_\omega(\mathcal{D})$ fails to be downwards closed if and only if ω, the multiset we have removed from $\mathcal{D}$, is also $\upsilon-\omega-\varepsilon$. In this case, $\upsilon = 2\omega+\varepsilon$ which is exactly the case that $\omega = \frac{\upsilon-\delta^i}{2}$. □

Proposition 4.58. *If $\mathcal{H}$ is a self-dual hypergraph and υ has exactly one odd entry, say υ_i, then $\frac{\upsilon-\delta^i}{2} \in \mathcal{D}(\mathcal{H})$.*

Proof. If $\omega = \frac{\upsilon-\delta^i}{2}$, then the complement is $\overline{\omega} = \frac{\upsilon+\delta^i}{2}$. Thus $\omega \preceq \overline{\omega}$, and since only one can belong to the downwards closed $\mathcal{D}(\mathcal{H})$, it must be ω. □

Definition 4.59. The *graph of self-dual multihypergraphs* with base multiset υ, $\mathbb{G}(\upsilon)$, is the graph where each vertex is a self-dual multihypergraph and the neighborhood of each self-dual multihypergraph $\mathcal{H}$ is $\left\{ F_{\overline{\varepsilon}}(\mathcal{H}) \;\middle|\; \varepsilon \in \mathcal{E}\backslash\{\frac{\upsilon+\delta^i}{2}\} \right\}$. That is, two multihypergraphs are adjacent exactly when then they differ by a single flip.

Example 4.60. There are six self-dual multihypergraphs on the base multiset $(3, 4)$. Below we display the graph structure of $\mathbb{G}((3, 4))$ by representing each node with its edge ideal.



$\mathbb{G}((3, 4))$

Figure 4.6: $\mathbb{G}((3, 4))$ with Ideals

Below, we depict the same graph in more detail by representing each node with its Hasse diagram of independent multisets. Each thick line represents an edge in $\mathbb{G}((3,4))$. The endpoints of the thick lines show the maximally independent multiset that is removed/added by the depicted flip. The dots in red show the maximally independent multisets of the form $\frac{(3,4)-(1,0)}{2}$.

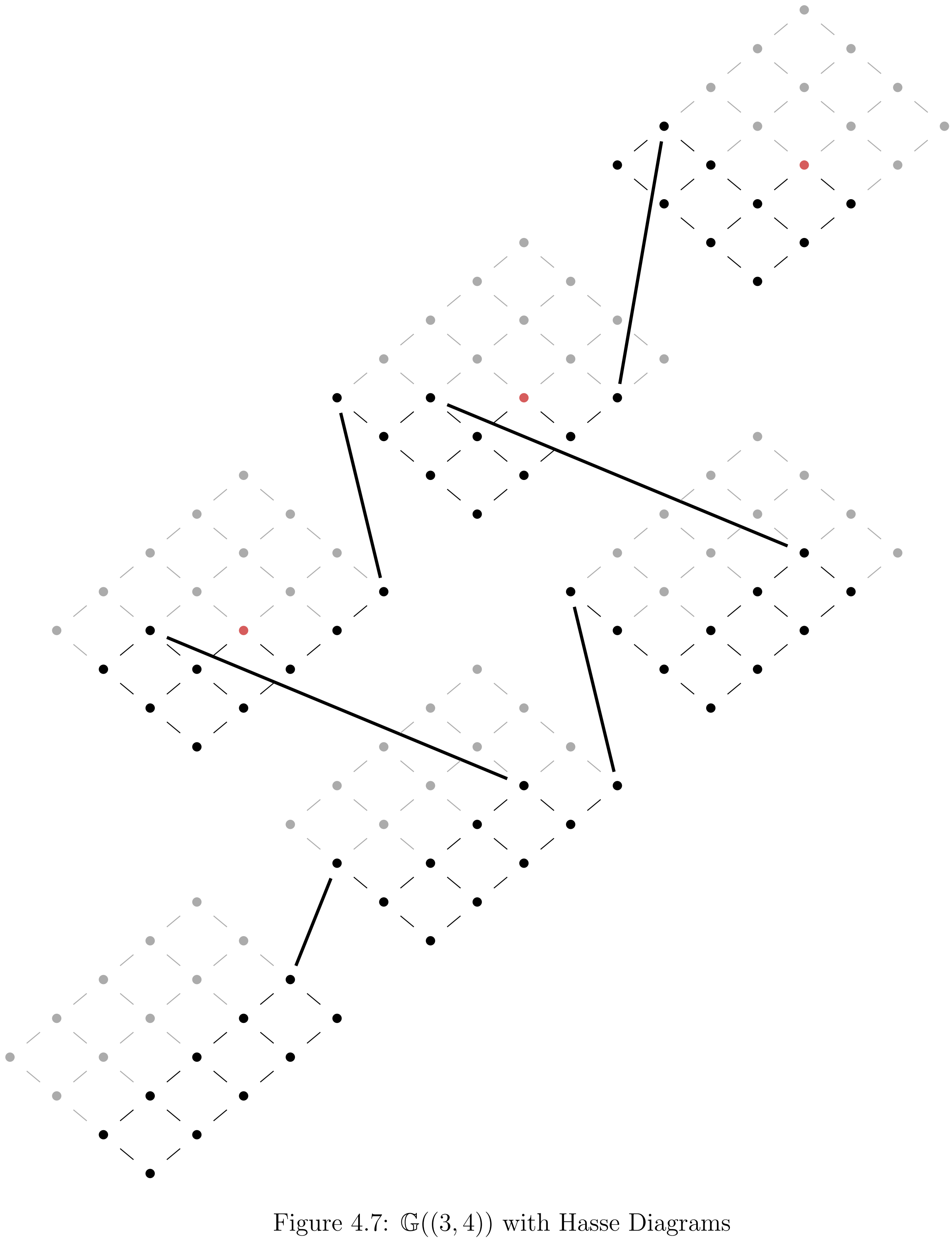


Figure 4.7: $\mathbb{G}((3,4))$ with Hasse Diagrams

We now generalize Lemma 3.40 and Lemma 3.41.

Proposition 4.61. *If $\mathcal{H}_s$ and $\mathcal{H}_t$ are nodes in $\mathbb{G}(\upsilon)$ then a shortest path between these two* *nodes has length* $|\mathcal{D}(\mathcal{H}_s)\backslash\mathcal{D}(\mathcal{H}_t)|$ *.*

Proof. Let $\mathcal{D}_i = \mathcal{D}(\mathcal{H}_i)$. We first observe that in general if $n = |\mathcal{D}_s\backslash\mathcal{D}_t|$ and $\omega \in \mathcal{D}_s$ is maximal, then $|F_\omega(\mathcal{D}_s)\backslash\mathcal{D}_t| \in \{n-1, n+1\}$. If $\omega \in \mathcal{D}_s\backslash\mathcal{D}_t$ then by self-duality $\upsilon - \omega \in \mathcal{D}_t$, hence $F_\omega(\mathcal{D}_s)\backslash\mathcal{D}_t = (\mathcal{D}_s\backslash\mathcal{D}_t)\backslash\{\omega\}$. Otherwise, if $\omega \notin \mathcal{D}_s\backslash\mathcal{D}_t$ then by self-duality $\upsilon - \omega \notin \mathcal{D}_t$, hence $F_\omega(\mathcal{D}_s)\backslash\mathcal{D}_t = (\mathcal{D}_s\backslash\mathcal{D}_t) \cup \{\upsilon - \omega\}$. It follows that any path between the two nodes has length at least $|\mathcal{D}_s\backslash\mathcal{D}_t|$.

We also note that if $\mathcal{D}_s\backslash\mathcal{D}_t$ is non-empty, it contains a maximal element of $\mathcal{D}_s$. Suppose otherwise and take some maximal $\omega \in \mathcal{D}_s\backslash\mathcal{D}_t$. Since ω is not maximal in $\mathcal{D}_s$ there is some $\eta \in \mathcal{D}_s \cap \mathcal{D}_t$ with $\omega \preceq \eta$ but $\omega \notin \mathcal{D}_t$, a contradiction because $\mathcal{D}_t$ is downwards closed.

Thus, to construct a shortest path from $\mathcal{H}_s$ to $\mathcal{H}_t$, at each step $\mathcal{H}_i$ it suffices to pick $\mathcal{H}_{i+1} = F_\omega(\mathcal{H}_i)$ where ω is maximal in $\mathcal{D}_i\backslash\mathcal{D}_t$. □

Corollary 4.62. *The graph* $\mathbb{G}(\upsilon)$ *is bipartite.*

Proof. From the proof of Proposition 4.61 we have that for any sequence of adjacent nodes $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_\ell$ the size of the set $\mathcal{D}_1\backslash\mathcal{D}_i$ has the same parity as i. Hence, if $\mathcal{H}_1 = \mathcal{H}_\ell$, then ℓ is odd and this sequence traverses $\ell - 1$ edges. Thus every cycle is even. □

4.2 On the Enumeration of Intersecting Hypergraphs and Non-2-Colorable Hypergraphs

The graph of self-dual graphs of dimension n, introduced in Definition 3.38 and denoted $\mathbb{G}(n)$, enumerates every self-dual hypergraph of a given size. In this section we explore this enumeration from a computational perspective. We begin by showing that the maps π_W and φ_W can be used to further enumerate every intersecting hypergraph and every non-2-colorable hypergraph, respectively. Though we first briefly recall Notation 3.21 where $\pi_i = \pi_{V\setminus\{x_i\}}$.

Proposition 4.63. *Every intersecting hypergraph with n vertices has an edge ideal given by $\pi_{n+1}(I(H))$ for some self-dual hypergraph H with $n+1$ vertices. Similarly, every non-2-colorable hypergraph with n vertices has an edge ideal given by $\varphi_{n+1}(I(H))$ for some self-dual hypergraph H with $n+1$ vertices.*

Proof. The first part of the proposition follows from Corollary 3.14. The second part follows by applying Corollary 3.3 and Theorem 2.101 to the first part of the proposition. □

Remark 4.64. In [20], Timotijtević both introduces the graph $\mathbb{G}(n)$ and uses Wolfram Alpha to compute $\mathbb{G}(n)$ for $n \leq 5$. The author only describes this as an "elementary computer assisted calculation" so, beyond this, their computational methodology is entirely opaque. We develop and describe in detail a Python implementation that is capable of enumerating $\mathbb{G}(n)$ for $n \leq 7$. This implementation, however, is not an elementary pursuit as the number of nodes in $\mathbb{G}(n)$ grows quite quickly: for $n = 5$ there are 81 nodes but for $n = 7$ there are 1,422,564 nodes.

Algorithm 4.65. The following algorithm, *Breadth First Search* (BFS), is a standard graph-traversing algorithm in Computer Science.

```
#Breadth First Search
def Breadth_First_Search(root):
  visited_nodes = list(root)
  nodes_to_process = list()
  current_node = root

  while len(nodes_to_process) > 0:
    for neighbor in neighbors(current_node):
      if neighbor not in visited_nodes:
        visited_nodes.append(neighbor)
```

```
11         nodes_to_process.append(neighbor)
12     current_node = nodes_to_process.pop(0)
13   return visited_nodes
```

The basic idea of BFS is that, by the end, each node is stored in the list `visited_nodes`. The algorithm starts with the input of a given node called `root`, the first `current_node`. At each step, the current node's neighbors are either appended to both lists if they haven't yet been visited, or nothing is done if they have already been visited. The step concludes at line 12 by updating the current node to be node at the front of `nodes_to_process`. This continues until every node has been processed.

A nearly identical algorithm called *Depth First Search* (DFS) similarly traverses graphs. The only change required is that line 12 of DFS is instead:

`current_node = nodes_to_process.pop()`,

which removes the node at the end of the list i.e. the most recently added node. Both algorithms have the same time and space complexity, but traverse the nodes of the graph in a different order. BFS will first traverse the nodes closest to the root and then expand outwards, while DFS behaves in the opposite manner. Given that the graph is connected, both algorithms traverse the entirety of a graph, otherwise they only traverse the connected component that `root` belongs to.

The names for both algorithms come historically from the case when the (directed) graph is a tree that is storing data in the nodes. In this case, lines 9 and 10 need not be included. The algorithms traditionally stop when a specific datum is found or the entire tree has been traversed, hence they are "searches".

Remark 4.66. Algorithm 4.65 provides the high-level structure of our enumeration algorithm. However, this algorithm requires an implementation of the `neighbors` function, and does not detail how nodes are stored or compared to each other. The specifics of our implementation can be found in the Appendix, but we provide a general overview.

We first consider a "naive approach" of storing each node as the list of facets of its simplicial complex. In this setting, the implementation of `neighbors` used by Algorithm 4.65 is straightforward. With access to `current_node` we have a list of its facets, hence every neighbor of the current node can be computed by performing the flip operation on each facet. In particular, this computes list of facets of each neighbor. For small n this approach is sufficient, however when n becomes large, line 9 becomes exorbitantly expensive for two reasons. First, it is run many more times as there are many more complexes and moreover the complexes have more facets. Secondly, the list `visited_nodes` also becomes large, so checking if a complex is in the list takes longer.

To address this inefficiency, we make use of so-called "dictionaries" that are widely used because they typically provide much faster access to data when compared to other data structures like lists.

Definition 4.67. In Computer Science, a *dictionary* is a data structure that represents a function with a finite domain and range. In particular, elements of the domain are called *keys* and elements of the range are called *values*, and additionally both the keys and the values must be able to be stored in a finite way.

Notation 4.68. We will borrow Python's notation for dictionaries. If `d` is a dictionary that represents the function `f`, then `d = {key_1: f(key_1), ... , key_n: f(key_n)}`. We may also refer to (and assign) key-value pairs with the notation `d[key_i] = f(key_i)`.

Example 4.69. We store $\mathbb{G}(n)$ as a dictionary. Each node has an associated integer called `id`. The keys of the dictionary are all of the `id`'s, and the value for each node is another dictionary of the form detailed in Example 4.70

Example 4.70. We store each node of $\mathbb{G}(n)$ as a dictionary. Recall that every node itself is a simplicial complex. It is also useful to store various other data at each node, such as a list of neighbors' `id`'s. Each simplicial complex is stored as a list of its facets. For example, if `d3` is a dictionary that stores $\mathbb{G}(3)$ and the node that stores Δ_T from Example 3.39 has the `id` 0, then `d3[0] = {'complex': [{1}, {2}, {3}], 'neighbors': [1, 2, 3]}`.

Example 4.71. Putting Examples 4.69 and 4.70 together, $\mathbb{G}(3)$ is stored as the following dictionary.

```
d3 = {
    0: {'complex': [{1}, {2}, {3}], 'neighbors': [1, 2, 3]},
    1: {'complex': [{2, 3}], 'neighbors': [0]},
    2: {'complex': [{1, 3}], 'neighbors': [0]},
    3: {'complex': [{1, 2}], 'neighbors': [0]}
     }
```

Remark 4.72. In practice, the dictionaries we use for each node are more complicated, but Example 4.71 suffices to show the utility of dictionaries in storing data.

We now return to our purpose for introducing dictionaries in the first place. By using a dictionary for `visited_nodes` in lieu of a list, line 9 of Algorithm 4.65 need not check equality of `neighbor` with every single other simplicial complex. In particular, the keys of this dictionary are certain numerical invariants of simplicial complexes. Each value of the dictionary is a list of nodes whose simplicial complexes all have the numerical invariant of the corresponding key.

For example, if the numerical invariant is the number of facets, then the dictionary `visited_nodes` in Example 4.71 would be `{1: [1, 2, 3], 3: [0]}`. In short, the use of this dictionary removes the waste of testing the equality of two simplicial complexes if they do not share the same invariants given by the keys. In practice, the key we use is a vector that encodes each of the following invariants:

- A vector where the i-th entry is the number of facets of size i;
- A vector where the i-th entry is the number of facets that x_i belongs to;
- A vector where the i-th entry is the number of pairwise facet intersections of size i.

Remark 4.73. Using dictionaries and invariants as described in Remark 4.72, we are able to enumerate $\mathbb{G}(7)$. Although these improvements address one of the issues raised in Remark 4.66, nothing can be done to address the issue of the sheer size of $\mathbb{G}(n)$ for larger n. However, the graph $\mathbb{G}(n)$ is highly redundant when considering self-dual simplicial complexes up to isomorphism. For example, of the 81 nodes in $\mathbb{G}(5)$, there are only 7 simplicial complexes that are unique up to isomorphism.

In an effort to reduce these redundancies, we now consider certain symmetries of $\mathbb{G}(n)$.

Proposition 4.74. *There is an injective group homomorphism* $S_n \hookrightarrow \mathrm{Aut}(\mathbb{G}(n))$.

Proof. We recall from Remark 2.103 that the action of S_n naturally extends to an action on hypergraphs. That is, $\sigma \mapsto (H \mapsto \sigma(H))$. In particular this action preserves self-duality and hence maps one node of $\mathbb{G}(n)$ to another. To check that the image is an automorphsim of $\mathbb{G}(n)$, consider an edge $\{H, H'\}$. There must be some edge of H, e, such that $F_e(H) = H'$. It then follows that $F_{\sigma(e)}(\sigma(H)) = \sigma(H')$, so then $\{\sigma(H), \sigma(H')\}$ is an edge of $\mathbb{G}(n)$.

To see that this mapping is injective, we need only consider n nodes of $\mathbb{G}(n)$, in particular let $H_i = (V, \{x_i\})$. Suppose σ_1, σ_2 map to the same automorphism of $\mathbb{G}(n)$. Then for each i, $\sigma_1(H_i) = \sigma_2(H_i)$. Evaluating these actions gives $H_{\sigma_1(i)} = H_{\sigma_2(i)}$, hence the two permutations agree on every i and are equal. Finally, checking that this is a group homomorphism is shown by noting that both groups have the same operation, composition:

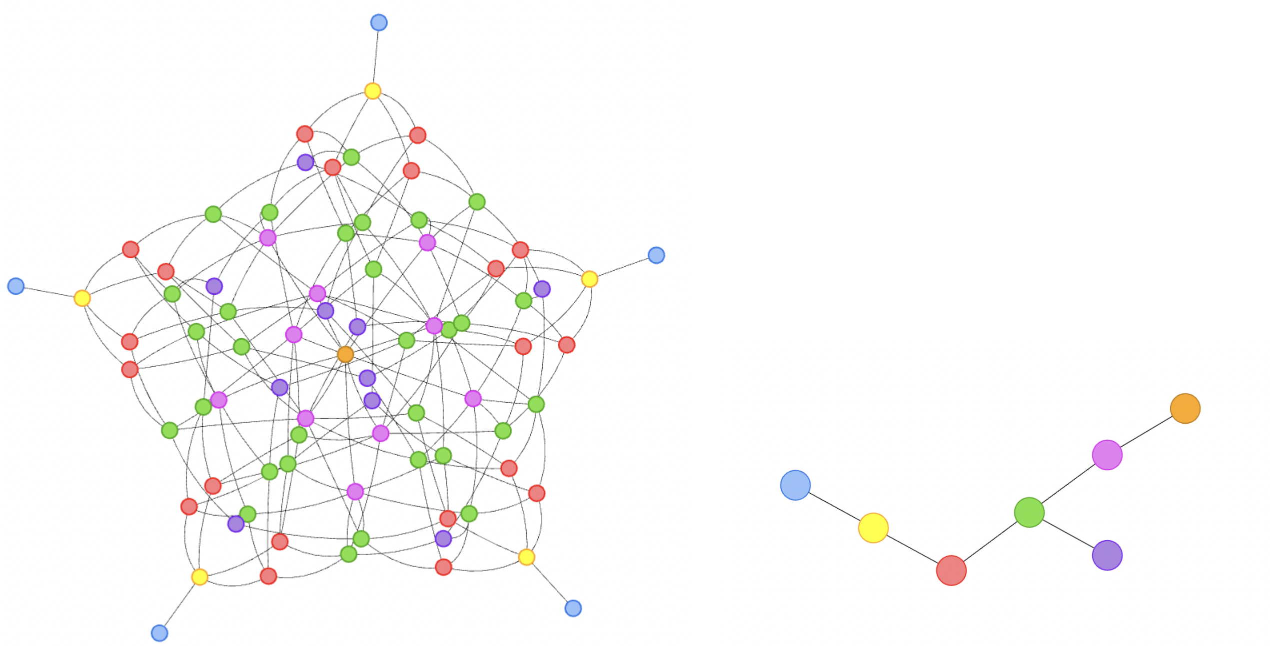
$$\sigma\tau \mapsto (H \mapsto \sigma \circ \tau(H)) = (H' \mapsto \sigma(H')) \circ (H \mapsto \tau(H)).$$

$\square$

Remark 4.75. The injective homomorphism in Proposition 4.74 yields a natural S_n-action on $\mathbb{G}(n)$. Under this action, the orbit of a node H is the set of hypergraphs that are isomorphic to H and by the Orbit-Stabilizer Theorem has size $\frac{n!}{|\mathrm{Aut}\, H|}$.

Definition 4.76. The *graph of self-dual hypergraphs of dimension n up to isomorphism*, denoted $\mathbb{G}_{\cong}(n)$, is the graph where each node is an orbit described by Remark 4.75, and edge set $\{\{[H_1], [H_2]\} \mid \{H_1, H_2\}$ is an edge of $\mathbb{G}(n)\}$.

Example 4.77. We show both $\mathbb{G}(5)$ and $\mathbb{G}_{\cong}(5)$ below, with the same coloring as in Example 3.39. Each node of $\mathbb{G}_{\cong}(5)$ has a unique color, and represents every node of that exact color in $\mathbb{G}(5)$.



$\mathbb{G}(5)$ $\qquad\qquad$ $\mathbb{G}_{\cong}(5)$

Figure 4.8: $\mathbb{G}(5)$ and $\mathbb{G}_{\cong}(5)$

For $n \leq 5$, each $\mathbb{G}_{\cong}(n)$ is a tree and hence bipartite, however this does not hold for larger n. Next, we show $\mathbb{G}_{\cong}(6)$, where the yellow nodes represent the 3-uniform hypergraphs with 6 vertices. Notably, there is a 3-cycle of blue nodes just to the right of the yellow nodes.

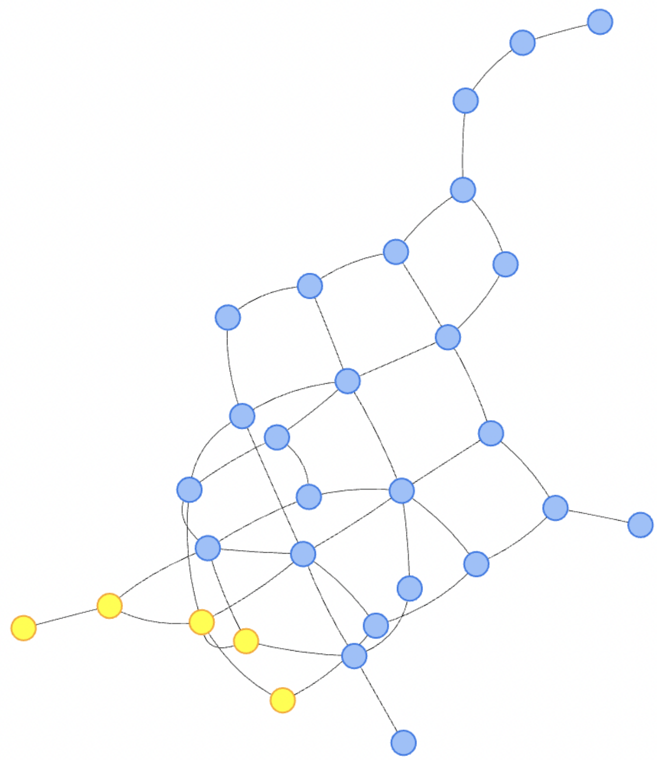


Figure 4.9: $\mathbb{G}_{\cong}(6)$

We now consider the problem of computing $\mathbb{G}_{\cong}(n)$, and show that it can be computed without first computing all of $\mathbb{G}(n)$.

Algorithm 4.78. We consider a slight modification of BFS, where we may start at any node. In particular we restrict the condition of line 9 in Algorithm 4.65 even further so that `neighbor` is appended to the lists if it is not isomorphic to any node in `visited_nodes`.

```
#Breadth First Search up to Isomorphism
def Breadth_First_Search_Iso(root):
  visited_nodes = list(root)
  nodes_to_process = list()
  current_node = root

  while len(nodes_to_process) > 0:
    for neighbor in neighbors(current_node):
```

```
    new_representative = True
    for representative in visited_nodes
      if isomorphic(representative, neighbor):
          new_representative = False
    if new_representative:
      visited_nodes.append(neighbor)
      nodes_to_process.append(neighbor)
  current_node = nodes_to_process.pop(0)
return visited_nodes
```

Lemma 4.79. *Algorithm 4.78 produces the nodes of a connected subgraph of $\mathbb{G}(n)$ that contains exactly one representative of each node of $\mathbb{G}_{\cong}(n)$.*

Proof. By construction, it is straightforward to see that Algorithm 4.78 gives a list of nodes that induces a connected subgraph of $\mathbb{G}(n)$ that contains at most one representative of each isomorphism class. It suffices to show that this list contains one representative of each isomorphism class. Suppose by contradiction that there is some isomorphism class $[H]$ with no representative in this list. Then there exists some $H' \in [H]$ as well as N' in the list with minimal distance between them in $\mathbb{G}(n)$, say d. Because $\mathbb{G}(n)$ is bipartite, every neighbor of N' has either distance $d-1$ or $d+1$ to H'. Further, every neighbor of N' is either in the list or isomorphic to some node in the list. In the former case, by minimality, the distance between the neighbor and H' is $d+1$. Hence, there must be some neighbor of N', say M', with distance $d-1$ to H' and $\sigma \in S_n$ such that $M = \sigma(M')$ is in the list. It follows that the distance between M and $\sigma(H')$ is also $d-1$, but this contradicts the minimality of the distance between N' and H'. □

Remark 4.80. Lemma 4.79 shows that Algorithm 4.78 enumerates exactly one of every self-dual hypergraph on n vertices up to isomorphism. This however, requires checking if many pairs of hypergraphs are isomorphic. In general, for each check there are $n!$ isomorphisms to consider. Timotijević mentions that in the case of self-dual hypergraphs, this can be reduced to $(n-1)!$. We now describe an approach to the isomorphism problem that significantly reduces the number of permutations to consider, requires fewer comparisons, and is also more memory efficient. The mathematical description of this approach is significantly more informative than its implementation in code is, however we refer the interested reader to the function `isomorphic_reps` in the Appendix for our implementation.

For any hypergraph H, we take the coarsest possible $I(H)$-admissible partition of its vertices, i.e. the equivalence classes of $\sim_{I(H)}$. Applying Theorem 2.47 gives a monomial ideal and hence a multihypergraph $\mathcal{H}$ that symmetrically polarizes to H. If each equivalence class is a singleton, then there are still $n!$ permutations to consider, however if just one pair of vertices can be composed we have recovered Timotijtević's $(n-1)!$. We note also that a single multiedge ε represents $\prod_i \binom{\upsilon_i}{\varepsilon_i}$ edges of the hypergraph it symmetrically polarizes to, hence storing multihypergraphs is often more memory efficient.

Our approach extends further beyond reducing the number of variables via multihypergraphs as we may efficiently eliminate more permutations from consideration. If we are checking if two multihypergraphs $\mathcal{H}, \mathcal{H}'$ satisfy $\sigma(\mathcal{H}) = \mathcal{H}'$, then every entry in the base multisets must have $\upsilon_{\sigma(i)} = \upsilon'_i$. Hence we may partition the variables $V = V_1, \ldots, V_{\max(\upsilon)}$ where each $V_i = \{X_i \mid \upsilon_i = i\}$ as well as $V = V'_1, \ldots, V'_{\max(\upsilon')}$ where each $V_i = \{X_i \mid \upsilon'_i = i\}$. If there is some i such that $|V_i| \neq |V'_i|$ then there is no permutation such $\sigma(\upsilon) = \upsilon'$ and the two multihypergraphs are not isomorphic. Thus we need only compute the actions of permutations that satisfy $\sigma(V_i) = V'_i$. In our implementation we eliminate more permutations by constructing finer partitions based on several invariants of each variable. Specifically, we have implemented the partitions $V = \bigcup V_{i,j,\ell}$ where

$$V_{i,j,\ell} = \left\{ X_i \;\middle|\; \upsilon_i = i,\ |\{\varepsilon \in \mathcal{E} \mid \varepsilon_i > 0\}| = j,\ \sum_{\varepsilon \in \mathcal{E}} \varepsilon_i = \ell \right\}.$$

Thus, the number of permutations to consider is:

$$\prod_{i,j,\ell} |V_{i,j,\ell}|!\,.$$

Remark 4.81. In the table below we give the number of vertices and edges in $\mathbb{G}(n)$ and $\mathbb{G}_{\cong}(n)$ for $3 \leq n \leq 7$. With our current implementation, enumerating either $\mathbb{G}(8)$ or $\mathbb{G}_{\cong}(8)$ seems entirely hopeless due to the sheer size of these graphs.

n	$\lvert V(\mathbb{G}(n))\rvert$	$\lvert E(\mathbb{G}(n))\rvert$	$\lvert V(\mathbb{G}_{\cong}(n))\rvert$	$\lvert E(\mathbb{G}_{\cong}(n))\rvert$
3	4	3	2	1
4	12	16	3	2
5	81	185	7	6
6	2,646	11,276	30	46
7	1,422,564	10,552,451	716	3,127

Table 4.1: Sizes of $\mathbb{G}(n)$ and $\mathbb{G}_{\cong}(n)$

4.3 A Note on the Associated Primes of Powers of Cover Ideals

The problem that we consider in this section directly inspired our definition and development of symmetric polarization. The problem deals with the associated primes from Definition 2.29. The story starts in 1979 with [4] when Broadmann showed that for a Noetherian ring R and ideal $I \subset R$, there is some n_0 such that for all $n \geq n_0$, $\mathrm{Ass}(R/I^n) = \mathrm{Ass}(R/I^{n_0})$. This stability gives rise to the following property.

Definition 4.82. An ideal $I \subseteq R$ has the *persistence property* if for every $\mathfrak{p} \in \mathrm{Ass}(R/I^n)$ we have $\mathfrak{p} \in \mathrm{Ass}(R/I^{n+i})$ for all $i \in \mathbb{N}$ and $n \geq 1$.

For the remainder of the section, we will take $R = k[V]$ which is indeed Noetherian.

Remark 4.83. Showing that an ideal satisfies the persistence property is generally a difficult task. In the following proposition, we give a minimal primary decomposition of the powers of the edge ideal of $Z(n)$. Since this graph is self-dual, $\mathrm{Ass}(R/I(Z(n))) = \{\mathfrak{p}_e \mid e \in E\}$. The minimal primary decomposition shows that for $r \geq 2$, $\mathrm{Ass}(R/I(Z(n))^r) = \{\mathfrak{p}_e \mid e \in E\} \cup \{\mathfrak{p}_{\{z,x_1,\ldots,x_n\}}\}$. Thus this ideal satisfies the persistence property.

We first introduce some notation used in the following proposition. For any $\omega \in \mathbb{N}^n$, we set $|\omega| = \sum_{i=1}^n \omega_i$, and $\beta = |\mathrm{supp}(\omega)|$. For an integer $r > 1$, we define

$$\mathcal{X}_r = \bigcup_{\ell=1}^{r-2} \bigcup_{\substack{2\leq\beta\leq r-\ell,\\ |\omega|=r+\ell(\beta-1)}} z^{r-\ell} M_\omega.$$

We take the convention that $\mathcal{X}_2 = \emptyset$.

As an example, if $n = 3$ we have

$$\mathcal{X}_4 = \left\{\sigma(z^3x_1x_2^4,\ z^3x_1^2x_2^3,\ z^3x_1x_2x_3^4,\ z^3x_1x_2^2x_3^3,\ z^3x_1^2x_2^2x_3^2,\ z^2x_1x_2^5,\ z^2x_1^2x_2^4,\ z^2x_1^3x_2^3) \;\middle|\; \sigma \in S_3\right\},$$

where each σ acts by permuting $\{x_1, x_2, x_3\}$. The first two monomials are given by $\ell = 1$ and $\beta = 2$, the next three are given by $\ell = 1$ and $\beta = 3$, and the remaining three are given by $\ell = 2$ and $\beta = 2$.

Proposition 4.84. *Let $I = I(Z(n)) = \langle zx_1, zx_2, \ldots, zx_n, x_1\cdots x_n\rangle$, $\omega \in \mathbb{N}^n$, $n > 2$, and $r > 1$. I^r has a minimal primary decomposition given by*

$$I^r = \left(\bigcap_{e\in E} \mathfrak{p}_e^r\right) \cap \langle z^r, x_1^r, \ldots, x_n^r, \mathcal{X}_r\rangle.$$

Proof. By Proposition 2.21 and Remark 2.23 it follows that each component is primary and that the radical of each component of the intersection is unique. Thus it suffices to show that I^r is equal to the given intersection and no component of the intersection contains the intersection of the remaining components.

We start with minimality. Note that z^r belongs to every component except for the prime power corresponding to $\{x_1, \dots, x_n\}$. We also see that $zx_1^{r-1}\cdots x_n^{r-1}$ belongs to every prime power, but not $\langle z^r, x_1^r, \dots, x_n^r, \mathcal{X}_r\rangle$, as every monomial in $\mathcal{X}_r$ is divisible by z^2. Lastly, for each i, we wish to find a monomial m that does not belong to $\mathfrak{p}^r_{\{z,x_i\}}$ but does belong to every other component. If $r = 2$ we choose $m = x_1^2\cdots x_{i-1}^2x_{i+1}^2\cdots x_n^2$. If $r > 2$ we choose $m = z^2x_1^{r-1}\cdots x_{i-1}^{r-1}x_{i+1}^{r-1}\cdots x_n^{r-1}$. We note that for each $j \neq i$, zx_j^{r-1} divides m and belongs to $\mathfrak{p}^r_{\{z,x_j\}}$. Now fix $s, t \neq i$ with $s \neq t$[1], and consider $m' = z^2x_s^{r-1}x_t^{r-1}$. Observe that m' divides m and belongs to $\mathfrak{p}^r_{\{x_1,\dots,x_n\}}$. Further yet, m' belongs to $\mathcal{X}_r$ with $\ell = r-2$, which forces $\beta = 2$ hence $|\omega| = 2r-2$.

We now show I^r is a subset of the intersection. If $m \in I^r$, then we have $m = m_{e_1}\cdots m_{e_r}M$. For each $e \in E$ we have some $y_i \in e \cap e_i$, hence $m = y_1\cdots y_rM'$ belongs to $\mathfrak{p}^r_e$. It remains to show that m belongs to $\langle z^r, x_1^r, \dots, x_n^r, \mathcal{X}_r\rangle$. Let ℓ be the number of e_i that are the edge $\{x_1, \dots, x_n\}$. If $\ell = 0$, then z^r divides m, if $\ell = r$ every x_j^r divides m, and if $\ell = r-1$, then the remaining edge $e_i = \{z, x_j\}$ gives x_j^r divides m. Otherwise, with some relabeling we have $m = m_{e_1}\cdots m_{e_{r-\ell}}(x_1\cdots x_n)^\ell M$. Now, each e_i contains z so for some ω', we have $m = M_{\omega'}z^{r-\ell}(x_1\cdots x_n)^\ell M$. Note that $|\omega'| = r-\ell$. We set $\omega_i = \omega_i' + \ell$ if $\omega_i' \neq 0$ and $\omega_i = 0$ otherwise. Thus, $|\omega| = r - \ell + \beta\ell$ and $m = M_\omega z^{r-\ell}M''$. This shows that $m \in \langle \mathcal{X}_r\rangle$.

Finally, we assume m belongs to the intersection and show $m \in I^r$. Let s be the largest power of z that divides m. We consider several cases:

- If $s \geq r$, then z^r divides m and further yet, $m \in \mathfrak{p}^r_{\{x_1 \dots, x_n\}}$ gives some ω with $|\omega| = r$ such that M_ω divides m. Together, this means $z^rM_\omega \in I^r$ is a divisor of m.

- If $s = 0$, then each x_i^r divides m, hence $m = (x_1\cdots x_n)^rM \in I^r$.

- If $s = 1$, then $m \in \mathfrak{p}^r_{\{z,x_i\}}$ gives x_i^{r-1} divides m, moreover $m \in \langle z^r, x_1^r, \dots, x_n^r, \mathcal{X}_r\rangle$ gives some x_i^r that divides m. Hence $m = zx_i(x_1\cdots x_n)^{r-1}M \in I^r$.

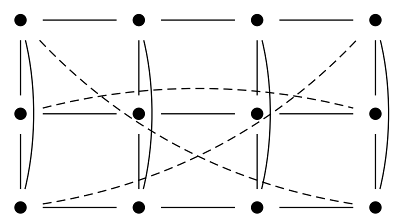
- If $2 \leq s < r$, then $m \in \mathfrak{p}^r_{\{z,x_i\}}$ for each i means $(x_1\cdots x_n)^{r-s}$ divides m. Moreover, some monomial $z^{r-\ell}M_\omega \in \mathcal{X}_r$ divides m. It suffices to show that there are at least s x_i's (with multiplicity) in $\frac{m}{(x_1\cdots x_n)^{r-s}}$, that is, $|\omega| - \beta(r-s) \geq s$. Since $|\omega| = r + \ell(\beta - 1)$, this is equivalent to $(r-s-\ell)(1-\beta) \geq 0$. The difference $1-\beta$ is necessarily negative and

[1] This choice requires the condition $n > 2$; the claim still holds for $n = 2$ though the proof is a bit different here.

by maximality of s we have $s \geq r - \ell$, thus $0 \geq r - s - \ell$, and the requisite inequality holds.

$\square$

The associated primes of edge ideals of graphs are well studied, and in particular in [14] Martínez-Bernal, Morey, and Villearal showed that the edge ideals of graphs satisfy the persistence property. In [8] Fransisco, Há, and Van Tuyl posited a conjecture that would show the same for cover ideals of graphs. However four years later Kaiser, Stehlík and Škrekovski showed in [13] that the cover ideal of the graph below fails the persistence property.



In [11], Há and Sun showed that this counter-example can be generalized by adding arbitrarily more "columns" to the graph above. They also explored other obvious generalizations and showed their cover ideals do have the persistence property. Finally, the only other known square-free monomial ideals that fail the persistence property are given by [3]. In the work, Bela, Favacchio, and Tran construct two hypergraphs by making certain additions to the graph above, and show that the cover ideals of both also fail the persistence property.

In [9] Fransisco, Há, and Van Tuyl study the associated primes of powers of cover ideals of hypergraphs, which results in the following Theorem. The reader may first wish to recall the definition of hypergraph expansions (i.e. Definition 2.75).

Theorem 4.85. *[9, Corollary 4.5] For a hypergraph H and $s \geq 2$, the following are equivalent:*

(i) $\mathfrak{p}_W \in \operatorname{Ass}(R/J(H)^s)$;

(ii) there exists some $\omega \preceq (s, \ldots, s)$ *with* $\operatorname{supp}(\omega) = W$ *such that* $(H^s)_{V^\omega}$ *is critically* $s+1$*-chromatic.*

Corollary 4.86. *If H is a proper self-dual hypergraph, then* $\mathfrak{p}_V \in \operatorname{Ass}(I(H)^2)$.

Proof. We have $I(H) = J(H)$ and H is critically 3-chromatic by Corollary 3.5. $\square$

Remark 4.87. Theorem 4.85 was proven by Fransisco, Há, and Van Tuyl by showing that for a hypergraph H, $(J(H)^s)^{[s]}$ depolarizes to the s-th secant power of $I(H^s)$. The latter ideal has a combinatorial (chromatic) description given by Sturmfels and Sullivant in [19]. The purpose of the work that follows is to provide a more direct, combinatorial proof of Theorem 4.85.

Definition 4.88. A multihypergraph $\mathcal{H}$ with base multiset υ such that $|\upsilon| > 1$ is called *critically s-chromatic* if $\chi(\mathcal{H}) = s$ and $\chi(\mathcal{H}_{\upsilon-\delta^i}) < s$ for every unit vector $\delta^i \preceq \upsilon$.

Definition 4.89. Fix a hypergraph H and $\upsilon \in \mathbb{N}^n$. The *pre-symmetric polarization of H with respect to υ* is the multihypergraph $\operatorname{psPol}(H;\upsilon) = (\upsilon, \mathcal{F}_1 \cup \mathcal{F}_2)$ where $\mathcal{F}_1 = \{e \mid e \in E\}$ and $\mathcal{F}_2 = \{2\delta^i \mid \upsilon_i \geq 2\}$. Note in the definition of $\mathcal{F}_1$ we interpret sets $e \subseteq V$ as multisets that are preceded by $(1, \ldots, 1)$.

Remark 4.90. By construction we have:

$$I(H^s) = \operatorname{sPol}(I[\operatorname{psPol}(H; (s, \ldots, s))]; (s, \ldots, s));$$

and for any $0 \neq \upsilon \preceq (s, \ldots, s)$, we have:

$$I((H^s)_{V^\upsilon}) = \operatorname{sPol}(I[\operatorname{psPol}(H;\upsilon)];\upsilon).$$

The purpose of this construction is to give a coloring argument about (sub)hypergraphs of H^s in terms of its pre-symmetric polarization which has vastly fewer edges to work with.

Lemma 4.91. *Fix a hypergraph H on n vertices and $0 \neq \upsilon \in \mathbb{N}^n$. Any $\eta \preceq \upsilon$ is (maximally) independent in $\mathcal{H} = \operatorname{psPol}(H;\upsilon)$ if and only if every $f \in \operatorname{sPol}(\eta;\upsilon)$ is (maximally) independent in H.*

Proof. If η is independent in $\operatorname{psPol}(H;\upsilon)$ then the multiedge set $\mathcal{F}_2$ imposes that $\eta \preceq (1, \ldots, 1)$. The remaining multiedges of $\mathcal{H}$ belong to $\mathcal{F}_1$, all also precede $(1, \ldots, 1)$, and are in direct correspondence with the edges of H. Thus, the problem of deciding if η is independent in $\mathcal{H}$ is equivalent to the problem of deciding if $f \in \operatorname{sPol}(\eta;\upsilon)$ is independent in H. □

Corollary 4.92.

$$\operatorname{sPol}(J[\operatorname{psPol}(H;\upsilon)];\upsilon) = J((H^s)_{V^\upsilon})$$

and

$$\chi(\operatorname{psPol}(H;\upsilon)) = \chi((H^s)_{V^\upsilon}).$$

The first statement follows by noting it is dual to the conclusion of Remark 4.90. The statement about chromatic numbers follows because the both the vertices and (maximally) independent (multi)sets of $(H^s)_{V^\upsilon}$ and $\operatorname{psPol}(H;\upsilon)$ are in correspondence.

Example 4.93. Let $\upsilon = (1, 1, 1, 2)$.

$$H = (\{x_1, x_2, x_3, x_4\},\ \{\ \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4\}\ \}).$$

We have $V^\upsilon = \{x_{1,1}, x_{2,1}, x_{3,1}, x_{4,1}, x_{4,2}\}$,

$$\begin{aligned}(H^2)_V^\upsilon = (V^\upsilon, \{\ \{x_{4,1}, x_{4,2}\}, &\{x_{1,1}, x_{2,1}, x_{4,1}\}, \{x_{1,1}, x_{2,1}, x_{4,2}\}, \{x_{1,1}, x_{3,1}, x_{4,1}\}, \\ &\{x_{1,1}, x_{3,1}, x_{4,2}\}, \{x_{2,1}, x_{3,1}, x_{4,1}\}, \{x_{2,1}, x_{3,1}, x_{4,2}\}\ \}),\end{aligned}$$

and

$$\mathcal{H} = \mathrm{psPol}(H; \upsilon) = (\upsilon, \{(0, 0, 0, 2), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 1)\}).$$

The maximal independent multisets of $\mathcal{H}$ are

$$\{(1, 1, 1, 0), (1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1)\}.$$

No pair of these sum to an element of $\mathbb{N}^n$ that is preceded by υ, so $\mathcal{H}$ is not 2-colorable. Therefore $\chi((H^2)_{V^\upsilon}) > 2$ and the maximal independent sets of $(H^2)_{V^\upsilon}$ are:

$$\{x_{1,1}, x_{2,1}, x_{3,1}\}, \{x_{1,1}, x_{4,1}\}, \{x_{1,1}, x_{4,2}\}, \{x_{2,1}, x_{4,1}\}, \{x_{2,1}, x_{4,2}\}, \{x_{3,1}, x_{4,1}\}, \{x_{3,1}, x_{4,2}\}.$$

Here are four (of many other possible) 3-colorings of $\mathcal{H}$:

$$\begin{aligned}(1, 1, 1, 2) &= (1, 0, 0, 0) \uplus (0, 1, 0, 1) \uplus (0, 0, 1, 1)\\ (1, 1, 1, 2) &= (0, 1, 0, 0) \uplus (1, 0, 0, 1) \uplus (0, 0, 1, 1)\\ (1, 1, 1, 2) &= (0, 0, 1, 0) \uplus (0, 1, 0, 1) \uplus (1, 0, 0, 1)\\ (1, 1, 1, 2) &= (0, 0, 0, 1) \uplus (0, 1, 0, 1) \uplus (1, 0, 1, 0).\end{aligned}$$

Note that, for every possible unit vector $\delta^i \preceq \upsilon$ there is a 3-coloring with δ^i as a component. Thus for every such δ^i, $\chi(\mathcal{H}_{\upsilon-\delta^i}) = 2$ because we have two independent sets that sum to $\upsilon - \delta^i$. Therefore both $\mathcal{H}$ and $(H^2)_\upsilon$ are critically 3-chromatic. This brings us back to Theorem 4.85.

Lemma 4.94. *For a hypergraph H and $s \geq 2$, the following are equivalent:*

(i) $\mathfrak{p}_W \in \mathrm{Ass}(R/J(H)^s)$ *;*

(ii) there exists some $\upsilon \preceq (s, \ldots, s)$ *with* $\mathrm{supp}(\upsilon) = W$ *such that* $(H^s)_{V^\upsilon}$ *is critically*

$s+1$-*chromatic;*

(iii) there exists some $\upsilon \preceq (s,\dots,s)$ *with* $\operatorname{supp}(\upsilon) = W$ *such that* $\mathcal{H} = \operatorname{psPol}(H;\upsilon)$ *is critically* $s+1$-*chromatic.*

Proof. $(i \iff ii)$ is Theorem 4.85. $(ii \iff iii)$ is clear from Corollary 4.92. However, we provide an alternative proof by directly showing $(i \iff iii)$.

We note that (i) is equivalent to the existence of some monomial $M_{\hat{\upsilon}} \in R$ with $M_{\hat{\upsilon}} \notin J(H)^s$ such that $\mathfrak{p}_W = (J(H)^s : M_{\hat{\upsilon}})$. In other words (i) holds if and only if there is some $M_{\hat{\upsilon}} \in R$ such that:

(i_a) $M_{\hat{\upsilon}} \notin J(H)^s$;

(i_b) $x_j M_{\hat{\upsilon}} \in J(H)^s$ for every $x_j \in W$.

By Theorem 4.32 we have that (iii) holds if and only if there exists some $\upsilon \preceq (s,\dots,s)$ such that:

(iii_a) $M_\upsilon^{s-1} \notin J(\mathcal{H})^s$;

(iii_b) $M_\upsilon^s \in J(\mathcal{H})^{s+1}$;

(iii_c) $\chi(\mathcal{H}_{\upsilon-\delta^i}) = s$ for every $x_i \in W$.

We take $\upsilon = (s,\dots,s) - \hat{\upsilon}$ and show $(i_a \iff iii_a)$ as well as $(i_b \iff iii_b$ and $iii_c)$. Note that because of the multiedges of $\mathcal{H} = \operatorname{psPol}(H,\upsilon)$ given by $\mathcal{F}_2$, every independent set η in $\mathcal{H}$ has $\eta \preceq (1,\dots,1)$. Applying Lemma 4.91 immediately gives that H and $\mathcal{H}$ have the same independent (multi)sets.

$(i_a \iff iii_a)$ The condition (i_a) holds if and only if there are no s independent sets in H, say $f_1,\dots,f_s$, such that $M_{\hat{\upsilon}} = \frac{m_V}{m_{f_1}} \cdots \frac{m_V}{m_{f_s}}$. Treating each f_i as an element in $\mathbb{N}^n$, this condition is:

$$\begin{aligned}
\hat{\upsilon} &= (V - f_1) + \cdots + (V - f_s)\\
(s,\dots,s) - \upsilon &= sV - f_1 - \cdots - f_s\\
-\upsilon &= -f_1 - \cdots - f_s\\
(s-1)\upsilon &= (\upsilon - f_1) + \cdots + (\upsilon - f_s).
\end{aligned}$$

Every summand of the last line is the complement of an independent multiset in $\mathcal{H}$, and so the last line is equivalent to $M_\upsilon^{s-1} \in J(\mathcal{H})^s$. This shows $(i_a \iff iii_a)$.

($i_b \iff iii_b$ and iii_c) The condition (i_b) holds if and only if there are s independent sets in H, say $f_1, \ldots, f_s$, such that $x_j M_{\hat{\upsilon}} = \frac{m_V}{m_{f_1}} \cdots \frac{m_V}{m_{f_s}}$. This is equivalently:

$$\begin{aligned}
\delta^j + \hat{\upsilon} &= (V - f_1) + \cdots + (V - f_s) \\
\delta^j + (s-1)\upsilon &= (\upsilon - f_1) + \cdots + (\upsilon - f_s) \\
s\upsilon &= (\upsilon - \delta^j) + (\upsilon - f_1) + \cdots + (\upsilon - f_s).
\end{aligned}$$

This last line implies (iii_b). However, we note that it also gives $\upsilon - \delta^j = f_1 + \cdots + f_s$, which gives an s-coloring of $\mathcal{H}_{\upsilon - \delta^j}$. Because the condition on x_j in (i_b) is equivalent to the condition on δ^j in (iii_c) we are done. $\square$

We conclude the Chapter, and this work by possible extensions of this work in the future, starting with a conjecture that builds on Lemma 4.94.

Conjecture 4.95. *The edge ideal of every non 2-colorable hypergraph satisfies the persistence property.*

Sketch of Proof. Every edge ideal of every non 2-colorable hypergraph is the cover ideal of an intersecting hypergraph. It suffices to show that H is intersecting, $\mathrm{psPol}(H;\upsilon)$ is critically s-chromatic, and that f is maximally independent in H implies $\mathrm{psPol}(H;\upsilon + f)$ is critically $s+1$-chromatic.

If $\mathrm{psPol}(H;\upsilon)$ is critically s-chromatic then for each δ^i we have independent $f_1, \ldots, f_{s-1}$ with

$$(s-1)\upsilon = (\upsilon - \delta^i) + (\upsilon - f_1) + \cdots + (\upsilon - f_{s-1}).$$

So then,

$$\begin{aligned}
(s-1)(\upsilon + f) &= (\upsilon - \delta^i) + (\upsilon + f - f_1) + \cdots + (\upsilon + f - f_{s-1}) \\
s(\upsilon + f) &= \upsilon + f + (\upsilon - \delta^i) + (\upsilon + f - f_1) + \cdots + (\upsilon + f - f_{s-1}) \\
s(\upsilon + f) &= (\upsilon + f - \delta^i) + (\upsilon + f - f) + (\upsilon + f - f_1) + \cdots + (\upsilon + f - f_{s-1}).
\end{aligned}$$

Or rather, every $\mathrm{psPol}(H;\upsilon + f - \delta^i)$ is s-colorable. It follows that $\mathrm{psPol}(H;\upsilon + f)$ is $s+1$-colorable as every δ^i is independent.

It remains to show (by using the intersecting property!) that $\mathrm{psPol}(H;\upsilon + f)$ not s-colorable, i.e. there is no independent $f_1, \ldots, f_s$ with

$$(s-1)(\upsilon + f) = (\upsilon + f - f_1) + \cdots + (\upsilon + f - f_s).$$

Aside from Conjecture 4.95, there are a number of possible future directions for this work.

Primarily, Theorem 3.4 allows us to restate the open problems that Erdős and Lovász asked about uniform, intersecting, and 3-chromatic hypergraphs in terms of Commutative Algebra and Algebraic Topology. Given our limited time and lack of true expertise, we were unable to make any substantive progress on the matter. However, the connection itself invites a large swath of algebraic and topological approaches available in reasoning about (uniform) self-dual hypergraphs.

Understanding more about the structure of the graphs $\mathbb{G}(n)$ and $\mathbb{G}_{\cong}(n)$ seems to be a promising approach considering that the graphs give and explicit construction of every hypergraph there is to consider. Anecdotally, in Figure 4.9 we see that all of the uniform hypergraphs form a connected component, but this does not hold for $\mathcal{G}_{\cong}(7)$. Moreover, these nodes are all far away from the "root" (hypergraph with a singleton edge) used in our algorithm. With this root, we are able to find uniform self-dual hypergraphs in say, $\mathbb{G}_{\cong}(8)$, much faster with DFS than BFS. A more sophisticated graph search algorithm may be able to prioritize finding the uniform self-dual hypergraphs, rather than enumerating all of $\mathbb{G}_{\cong}(n)$ for $n \geq 8$.

In the case of multihypergraphs, the implementations of Algorithm 4.65 and Algorithm 4.78 can easily be modified to enumerate the graphs $\mathbb{G}(\upsilon)$ and an analogous (though yet undefined) $\mathbb{G}_{\cong}(\upsilon)$. Based on the discussion of Remark 2.49, if υ' corresponds to a finer (I-admissible) partition than υ, then each node of $\mathbb{G}(\upsilon)$ corresponds to a node in $\mathbb{G}(\upsilon')$. Thus, in a certain sense, each $\mathbb{G}(\upsilon)$ provides a "coarse" view of the structure of $\mathbb{G}(\sum \upsilon_i)$ and is easier to compute. Perhaps these coarse views can be combined in a way that provides a more efficient enumeration of $\mathbb{G}_{\cong}(n)$ entirely.

Finally, Alexander duality is itself, a statement about homology. Given the amount of structure imposed by self-duality, we expect there is correspondingly nice structure in the homology of these objects. Because homology is well studied in Commutative Algebra, there are many techniques available to use in studying invariants like Betti numbers and Castelnuovo–Mumford regularity.

Appendix

The following code is written in Python and implements Algorithm 4.65 on line 358 as `search_by_equals` and Algorithm 4.78 on line 435 as `search_by_iso`. The first 357 lines are responsible for the majority of the computation via many “helper methods” that the search functions call. The package `itertools` is imported and used to efficiently work with permutations. Each method has at least a comment (in green, starting with ‘#’) that describes its purpose.

```
1 import itertools
2
3 # ============================================================
4 # The following code works mostly with simplicial complexes.
5 # All complexes are assumed to be self-dual,
6 # hence facets are complements of edges in the corresponding hypergraph.
7 # Only the facets are stored.
8 # The facets are stored as a list of sets using Python's implementations.
9 # The variable n always refers to the number of vertices in the complex.
10 # The vertex set is {1, 2, ..., n}.
11 # ============================================================
12
13
14 # If every vertex is not contained by some facet.
15 def is_proper(simplicial_complex,n):
16     for i in range(1,n+1):
17         missing = False
18         for facet in simplicial_complex:
19             if i not in facet:
20                 missing = True
21         if not missing:
22             return False
23     return True
24
25
```

```python
# Adds 'face' if it is not contained by any facet.
def add_if_maximal(simplicial_complex, face):
    if len(simplicial_complex) > 0:
        for facet in simplicial_complex:
            if face.issubset(facet):
                return
    simplicial_complex.append(face)


# Adds 'facet' and removes any former facets contained by 'facet'.
def add_facet(simplicial_complex, facet):
    copy_complex = list(simplicial_complex)
    for copy_facet in copy_complex:
        if copy_facet.issubset(facet):
            simplicial_complex.remove(copy_facet)
    simplicial_complex.append(facet)


# Returns complement of 'facet'.
def get_complement(facet, n):
    complement = set()
    for i in range(1,n+1):
        if i not in facet:
            complement.add(i)
    return complement


# Performs the flip F_{facet}.
def flip(simplicial_complex, facet, n):
    simplicial_complex.remove(facet)
    for i in facet:
        face = set(facet)
        face.remove(i)
        add_if_maximal(simplicial_complex, face)
    complement = get_complement(facet, n)
    add_facet(simplicial_complex, complement)


# Returns the h vector.
def h_vector(simplicial_complex,n):
    h = [0 for i in range(0,n)]
    for facet in simplicial_complex:
        i = len(facet)
```

```
            h[i] = h[i] + 1
    return tuple(h)


# Tests equality of 2 complexes.
def equal_complexes(complex_1, complex_2):
    if len(complex_1) != len(complex_2):
        return False

    for facet in complex_1:
        if facet not in complex_2:
            return False
    return True


# Returns a vector where the i-th entry is the number of facets the i-th vertex belongs to.
def vertex_distribution(simplicial_complex, n):
    distribution = [0 for i in range(0,n)]
    for facet in simplicial_complex:
        for vertex in facet:
            distribution[vertex-1] = distribution[vertex-1] + 1
    return tuple(distribution)


# Returns a sorted list of the sum of each index over a multihypergraph's facets.
def sorted_edge_sum(iso_rep):
    sums = [0 for i in range(0,len(iso_rep[0]))]
    for facet in iso_rep:
        for i in range(0, len(facet)):
            sums[i] = sums[i] + facet[i]
    return tuple(sorted(sums))


# Tests if complex is fixed under the transposition of vertices i and j.
def is_stable_transposition(simplicial_complex, i, j):
    for facet in simplicial_complex:
        if i in facet and j not in facet:
            copy = set(facet)
            copy.remove(i)
            copy.add(j)
            if copy not in simplicial_complex:
```

```python
                return False
        if j in facet and i not in facet:
            copy = set(facet)
            copy.remove(j)
            copy.add(i)
            if copy not in simplicial_complex:
                return False
    return True


# Gives the equivalence classes described by ~_I in Lemma 2.43.
def find_symmetric_vertices(simplicial_complex, n):
    vertices = [i for i in range(2,n+1)]
    equivalence_classes = [[1]]
    distribution = vertex_distribution(simplicial_complex, n)

    for vertex in vertices:
        matched = False
        for equiv_class in equivalence_classes:
            representative = equiv_class[0]
            if distribution[vertex-1] == distribution[representative-1]:
                if is_stable_transposition(simplicial_complex, vertex, representative):
                    equiv_class.append(vertex)
                    matched = True
                    break
        if not matched:
            equivalence_classes.append([vertex])
    return equivalence_classes


# Calculates omega and the map between variables W -> V in Notation 2.45.
def get_eqivalence_class_map(simplicial_complex, n):
    equivalence_classes = find_symmetric_vertices(simplicial_complex, n)
    map_vertex_to_eqiv_class = {}
    t = len(equivalence_classes)
    omega = []

    for i in range(0,t):
        equivalence_class = equivalence_classes[i]
        omega.append(len(equivalence_class))

```

```
        for vertex in equivalence_class:
            map_vertex_to_eqiv_class[vertex] = i
    return (omega, map_vertex_to_eqiv_class)

# Returns a list of vectors in N^t that "polarize" to the given simplicial complex via the given map.
def get_ppm_complex(simplicial_complex, map_vertex_to_eqiv_class, t):
    ppm_complex = []
    for facet in simplicial_complex:
        ppm_facet = [0 for i in range(0,t)]
        for vertex in facet:
            equivalence_class = map_vertex_to_eqiv_class[vertex]
            ppm_facet[equivalence_class] = ppm_facet[equivalence_class]+1
        if ppm_facet not in ppm_complex:
            ppm_complex.append(ppm_facet)

    return ppm_complex


# Returns a list of vectors in N^t that "polarize" to the given simplicial complex.
def get_ppm_complex_no_map(simplicial_complex,n):

    base_multiset, map_vertex_to_eqiv_class = get_eqivalence_class_map(simplicial_complex, n)
    t = len(base_multiset)

    ppm_complex = []
    for facet in simplicial_complex:
        ppm_facet = [0 for i in range(0,t)]
        for vertex in facet:
            equivalence_class = map_vertex_to_eqiv_class[vertex]
            ppm_facet[equivalence_class] = ppm_facet[equivalence_class]+1
        if ppm_facet not in ppm_complex:
            ppm_complex.append(ppm_facet)

    return (base_multiset,ppm_complex)

# Returns \sigma(facet), where oermutation is stored as a dictionary.
def permute_facet(facet, permutation):
    new_facet = list(facet)
```

```python
    for i in range(0, len(facet)):
        new_facet[permutation[i]] = facet[i]
    return new_facet

# Tests if a simplicial complex is isomorphic to the (polarization of the) given reference.
def isomoprhic_complexes(simplicial_complex, reference_ppm_complex, n):
    base_multiset_1, eqivalence_class_map = get_eqivalence_class_map(simplicial_complex, n)
    base_multiset_2 = reference_ppm_complex[0]
    reference_ppm_complex = reference_ppm_complex[1]

    t = len(base_multiset_1)

#   Isomorphic ppmc's must have \omega's that are permutations of each other.
    if sorted(base_multiset_1) != sorted(base_multiset_2):
        return False

    ppm_complex = get_ppm_complex(simplicial_complex, eqivalence_class_map, t)

#   Isomorphic ppmc's must have the same number of m-facets.
    if len(ppm_complex) != len(reference_ppm_complex):
        return False
    if equal_complexes(ppm_complex, reference_ppm_complex):
        return True

#   The next portion finds a permutation between the two complexes, if one exists.
#   Rather than consider every permutation, we consider only permutations \sigma, where \omega_i = \omega_{\sigma(i)}.
#   verticies_by_base is a partition of the verticies stored as a list of lists, where the j-th list contains the verticies such that \omega_i = j.
    verticies_by_base = [[] for i in range(0,max(base_multiset_1))]
    reference_verticies_by_base = [[] for i in range(0,max(base_multiset_1))]

    for vertex in range(0,t):
        verticies_by_base[base_multiset_1[vertex]-1].append(vertex)
        reference_verticies_by_base[base_multiset_2[vertex]-1].append(vertex)
```

```

#   Remove empty lists to avoid the considering the empty permutation.
    verticies_by_base = [l for l in verticies_by_base if l!=[] ]
    reference_verticies_by_base = [l for l in reference_verticies_by_base if l!=[] ]

#   permutations stores the elements of the direct product of symmetric groups, each permuting a list in verticies_by_base
    block_permutations = []
    for l in verticies_by_base:
        block_permutations.append(itertools.permutations(l))
    permutations = itertools.product(*block_permutations)

#   For every permutation of verticies_by_base, construct perm_dic, a permuttation of the vertex set sending elements of verticies_by_base to reference_verticies_by_base.
    for permutation in permutations:
        perm_dic = {}
        for block in zip(permutation,reference_verticies_by_base):
           for i in range(0, len(block[0])):
                perm_dic[block[0][i]] = block[1][i]
        matches = True
        i = 0
        #Check if this premutation is an isomorphism.
        while matches and i < len(ppm_complex):
            facet = permute_facet(ppm_complex[i], perm_dic)
            if facet not in reference_ppm_complex:
                matches = False
            i = i+1

        if matches:
            return True

    return False


#   Computes the intersection spectrum of the facets of a simplicial complex.
def intersection_spectrum(simplicial_complex, n):
    spectrum = [0 for i in range(0,n)]
    for i in range(0,len(simplicial_complex)-1):
        for j in range(i+1,len(simplicial_complex)):
```

```
            intersection_size = len(simplicial_complex[i].intersection(
    simplicial_complex[j]))
            spectrum[intersection_size] = spectrum[intersection_size] +
    1
    return tuple(spectrum)

#   Computes the intersection spectrum analog for multihypergraphs.
def multi_intersection_spec(multi_edges,n):
    spectrum2 = [0 for i in range(0,n)]
    t = len(multi_edges[0])
    for i in range(0,len(multi_edges)):
        for j in range(i+1,len(multi_edges)):
            intersection_size2 = 0
            for k in range(0,t):
                intersection_size2 = intersection_size2 + min(
    multi_edges[i][k],multi_edges[j][k])
            spectrum2[intersection_size2] = spectrum2[intersection_size2
    ] + 1

    return tuple(spectrum2)


#   Determines if two multihypergraphs are isomorphic.
def isomorphic_reps(iso_rep, reference_ppm_complex):
    base_multiset_1 = iso_rep[0]
    ppm_complex = iso_rep[1]


    base_multiset_2 = reference_ppm_complex[0]
    reference_ppm_complex = reference_ppm_complex[1]

    t = len(base_multiset_1)

    if len(ppm_complex) != len(reference_ppm_complex):
        return False
    if equal_complexes(ppm_complex, reference_ppm_complex):
        return True

    vertex_blocks = {}
    reference_vertex_blocks = {}

    #Compute vertex invariants that must be shared.
    for vertex in range(0,t):
```

```
     supp_i = base_multiset_1[vertex]
     supp_j = base_multiset_2[vertex]
     sum_i = 0
     sum_j = 0
     num_dividing_i = 0
     num_dividing_j = 0
     for edges in zip(ppm_complex,reference_ppm_complex):
         i = edges[0][vertex]
         j = edges[1][vertex]
         if i>0:
             sum_i = sum_i + i
             num_dividing_i = num_dividing_i + 1
         if j>0:
             sum_j = sum_j + j
             num_dividing_j = num_dividing_j + 1

     if (supp_i,sum_i,num_dividing_i) not in vertex_blocks:
         vertex_blocks[(supp_i,sum_i,num_dividing_i)] = []
     vertex_blocks[(supp_i,sum_i,num_dividing_i)].append(vertex)

     if (supp_j,sum_j,num_dividing_j) not in reference_vertex_blocks:
         reference_vertex_blocks[(supp_j,sum_j,num_dividing_j)] = []
     reference_vertex_blocks[(supp_j,sum_j,num_dividing_j)].append(
vertex)

 #Stores pairs of sets of vertices that share invariants.
 matched_blocks = []
 for id_triple in vertex_blocks:
     if id_triple not in reference_vertex_blocks:
         return False
     matched_blocks.append((vertex_blocks[id_triple],
reference_vertex_blocks[id_triple]))

 block_permutations = []
 for block in matched_blocks:
      block_permutations.append(itertools.permutations(block[0]))
 permutations = itertools.product(*block_permutations)

 #Compute permutations and test equality under their actions.
 for permutation in permutations:
     perm_dic = {}

     for block in zip(permutation,matched_blocks):
```

```
            for i in range(0, len(block[0])):
                perm_dic[block[0][i]] = block[1][1][i]


        matches = True
        i = 0
        while matches and i<len(ppm_complex):
            facet = permute_facet(ppm_complex[i], perm_dic)
            if facet not in reference_ppm_complex:
                matches = False
            i = i+1

        if matches:
            return True

    return False

#   Implementation of Algorithm 4.65.
#   root is the complex [n-1] has id 0.
def search_by_equals(n):
    next_id = 1
    root_h_vec = [0 for i in range(0,n)]
    root_h_vec[n-1] = 1
    root_h_vec = tuple(root_h_vec)

    root_distribution = [1 for i in range(0,n)]
    root_distribution[n-1] = 0
    root_distribution = tuple(root_distribution)

    root_int_spec = tuple([0 for i in range(0,n)])

    root = {"complex": [set([i for i in range(1,n)])],
            "neighbors":{},
            "h_vec count": root_h_vec,
            "distribution": root_distribution,
            }
    visited = []
    to_visit = [0]
    graph = {0: root}
    ids_by_h_vecs_and_distributions = {(root_h_vec, root_distribution,
   root_int_spec):set([0])}

    while len(to_visit) > 0:
```

```

        complex_id = to_visit.pop(0)
        visited.append(complex_id)
        simplicial_complex = graph[complex_id]["complex"]
        known_neighbors = graph[complex_id]["neighbors"]
        known_neighbor_facets = known_neighbors.values()

        copy_complex = list(simplicial_complex)

        for facet in known_neighbor_facets:
            copy_complex.remove(facet)

        for facet in copy_complex:

            new_complex = list(simplicial_complex)
            flip(new_complex, facet, n)

            new_h_vecs = h_vector(new_complex, n)
            new_distribution = vertex_distribution(new_complex, n)
            new_int_spec = intersection_spectrum(new_complex,n)
            new_complex_id = -1

            if (new_h_vecs, new_distribution, new_int_spec) not in ids_by_h_vecs_and_distributions:
                ids_by_h_vecs_and_distributions[(new_h_vecs, new_distribution, new_int_spec)] = set()

            for old_id in ids_by_h_vecs_and_distributions[(new_h_vecs, new_distribution, new_int_spec)]:
                if equal_complexes(new_complex, graph[old_id]["complex"]):
                    new_complex_id = old_id
                    break

            complement = get_complement(facet, n)

            if new_complex_id == -1:
                new_complex_id = next_id
                to_visit.append(new_complex_id)
                next_id = next_id + 1

                graph[new_complex_id] = {
                "complex": new_complex,
```

```
                "neighbors": {complex_id:complement},
                "h_vec count": new_h_vecs,
                "distribution": new_distribution
                }

            else:
                graph[new_complex_id]["neighbors"][complex_id] = complement

            ids_by_h_vecs_and_distributions[(new_h_vecs, new_distribution, new_int_spec)].add(new_complex_id)
            known_neighbors[new_complex_id] = facet

    return graph


#   Implementation of Algorithm 4.78.
def search_by_iso(n):
    next_id = 1

    root = {"complex": [set([i for i in range(1,n)])], "neighbors": set(), "sorted_base": tuple([1,n-1])}
    visited = []
    to_visit = [0]
    graph = {0: root}
    ppm_root = ([n-1,1], [[n-1,0]])
    root_h_vec = h_vector(root["complex"],n)
    root_sums = sorted_edge_sum(ppm_root[1])
    root_spec = multi_intersection_spec(ppm_root[1],n)

    iso_classes_by_sorted_support_and_h_vecs = {(tuple(sorted(ppm_root[0])),
                                                  root_h_vec,root_sums, root_spec):[0]}
    iso_reps = {0:ppm_root}
    iso_reps_by_base = {tuple([n-1,1]):[0]}


    count = 0
    while len(to_visit) > 0:
        count = count + 1
        complex_id = to_visit.pop()
        visited.append(complex_id)
```

```python
        simplicial_complex = graph[complex_id]["complex"]

        for facet in simplicial_complex:
            found = False
            new_complex = list(simplicial_complex)
            flip(new_complex, facet, n)

            iso_rep = get_ppm_complex_no_map(new_complex,n)
            support = tuple(sorted(iso_rep[0]))
            h_vec = h_vector(new_complex,n)
            edge_sums = sorted_edge_sum(iso_rep[1])
            spec = multi_intersection_spec(iso_rep[1],n)

            if (support,h_vec,edge_sums,spec) not in iso_classes_by_sorted_support_and_h_vecs:
                iso_classes_by_sorted_support_and_h_vecs[(support,h_vec,edge_sums,spec)] = []

            for old_id in iso_classes_by_sorted_support_and_h_vecs[(support,h_vec,edge_sums,spec)]:
                old_iso_rep = iso_reps[old_id]
                if isomorphic_reps(iso_rep, old_iso_rep):
                    graph[old_id]["neighbors"].add(complex_id)
                    graph[complex_id]["neighbors"].add(old_id)
                    found = True
                    break
            if not found:
                new_complex_id = next_id
                to_visit.append(new_complex_id)
                iso_reps[new_complex_id] = get_ppm_complex_no_map(new_complex,n)
                iso_classes_by_sorted_support_and_h_vecs[(support,h_vec,edge_sums,spec)].append(new_complex_id)
                graph[new_complex_id] = {"complex": new_complex,"neighbors": set([complex_id]), "sorted_base":support}
                graph[complex_id]["neighbors"].add(new_complex_id)

                if tuple(iso_rep[0]) not in iso_reps_by_base:
                    iso_reps_by_base[tuple(iso_rep[0])] = []
                iso_reps_by_base[tuple(iso_rep[0])].append(new_complex_id)


```

```
                next_id = next_id + 1
    return graph
```

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