THE UNIVERSITY OF MANITOBA

AXISYMMETRIC WAVE PROPAGATION IN SHELLS OF REVOLUTION

by

BRUCE W. GRAHAM

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES IN

PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN CIVIL ENGINEERING

WINNIPEG, MANITOBA

OCTOBER, 1971.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Dr. H. Cohen for his invaluable guidance and criticism throughout the preparation of this thesis.

The financial assistance of the National Research Council is gratefully acknowledged.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

As was noted in a recent paper by Cohen and Berkal [4], the propagation of waves in shells has considerable practical importance in addition to being of theoretical interest. The concept of a wave to be employed in this treatise defines a wave on a surface to be a propagating curve across which there may be discontinuities, or jumps, in certain physical quantities or their derivatives which are continuously defined elsewhere. This treatment of a wave as a singular curve with respect to some function moving on a surface relies upon techniques developed by Hadamard [8]. A thin shell may be idealized as the materialization of a surface simply by constructing surfaces parallel to some reference surface. This is analagous to the beam and flat plate being considered as a materialized line and flat surface respectively.

The basic theory used to investigate the propagation of singular surfaces in elastic media, and therefore applicable to singular curves, is discussed in the works of Hadamard, Truesdell and Toupin [19], and Thomas [18]. In particular, Thomas treats the problem of first order waves in a continuous homogeneous isotropic media employing three dimensional elasticity theory. More recently, Cohen and Suh [7] and Cohen and Berkal [4,5] have investigated strain and acceleration waves in elastic surfaces, membranes, and shells by applying the method of Hadamard to approximate linear theories of these media. Cohen and Suh treated the membrane as a two-dimensional Riemannian space while in (4,5) an extrinsic point of view treated the membrane and shell as surfaces embedded in

Euclidean space. The method employed yields the propagation velocity, the shape of the wave curve, the growth-decay equations for the magnitude of the discontinuity, any coupling effects involving higher order modes, as well as allowing a classification of the types of waves which can propagate.

Cohen [3] has examined, from an embedding viewpoint, the problem of axisymmetric wave propagation in shells of revolution within the framework of a linear shell theory based on the following postulates:

1. The shell is thin.
2. The transverse normal stress is negligible.
3. Normals to the reference surface of the shell remain normal to it and undergo no change in length during deformation. (Kirchoff hypothesis).
4. The deflections of the shell are small.

The results of this paper showed that two types of waves could propagate, each with its own characteristic velocity, similar to results from three dimensional elastic analysis. The waves were classified as longitudinal-bending (irrotational) and transverse-twisting (equiareal), the first type arising from the equations of motion describing torsionless axisymmetric displacements and the second arising from the equations governing purely torsional motions. It was found that these waves propagated as circles of latitude, hence independent of the polar angle so as to maintain axial symmetry, in the direction of the meridian curves of the shell reference surface. In addition differential equations governing the variation of the wave strengths were given and that governing bending waves was integrated giving an inverse relationship between wave strength and curve radius. Equations predicting coupling of second order longitudinal-bending waves to third order jumps in the normal displacement component were also found.

The purpose of this study is to examine this axisymmetric problem, as presented in (3), employing the method of Hadamard in conjunction with a less restrictive or higher order linear elastic shell theory. Kraus [9] suggests such a theory based on the work of Naghdi [10] and Reissner [15] which accounts for the effects of normal shear and normal stress. The inclusion of more rotatory inertia terms and normal shear strain and stress effects removes the inconsistencies inherent in more approximate theories. The theory developed may be referred to as a seven mode theory while that of (3) corresponds to three modes. The only restriction retained is that the displacements of the shell remain small so that they may be referred to the unstrained configuration whereas a finite displacement criteria would give rise to a nonlinear theory. The thinness assumption in a modified form is also delayed in the derivation.

Chapter II deals with the basic theory relevant to the problem of wave propagation in a shell of revolution. The idealization of a shell as a materialized surface and the notion of wave curves necessitates a knowledge of the differential geometry associated with these concepts. The geometry of curves is discussed first followed by its extension to surfaces, both being developed from an embedding point of view. The results for a generalized surface are then applied to a surface of revolution. The coordinate system to be employed in the shell analysis is shown to be a consequence of the materialized surface concept and the metric is derived for an orthogonal system.

The kinematics of small displacement theory, as governed by a series displacement function, are considered in relation to shells. Hooke's law is applied to shells allowing the definition of certain integral quantities, some of which may be thought of as stress resultants and couples, on the shell reference surface.

Hamilton's principle is given and then employed in conjunction with the constitutive equations to yield the equations of motion for a shell. This derivation gives seven equations of motion. The equations are shown to separate into two groups in the case of a shell of revolution under axisymmetric conditions, one group describing torsionless motions and the other describing torsional motions.

The chapter is concluded with a discussion of the kinematics of a singular curve propagating on a surface. The normal time derivative of a surface function is defined corresponding to an observer moving with the wave curve. It is shown that a wave curve in a shell of revolution, involving a discontinuity in some axially symmetric function on the shell reference surface, propagates as a circle of latitude in the direction of the shell meridians, its velocity being independent of the polar angle. In addition, the kinematic compatibility relations are given for use in the analysis of the wave propagation problem.

In Chapter III the problem of axisymmetric acceleration waves propagating into the stationary unstrained reference surface of a shell of revolution is investigated with regard to propagation conditions and velocities of propagation. It is found that the equations of motion governing torsionless motions yield two separate systems of propagation conditions, one system involving longitudinal displacement derivative jumps and the other second order normal jumps. Eigenvalue problems are presented for each of the simultaneous systems making up the propagation conditions. The solutions to these problems are shown to be the propagation velocities associated with the derivative jumps involved and are characteristic of these jumps. It is argued that these acceleration waves may be classified, according to their propagation conditions, as "longitudinal waves" and "normal waves". Longitudinal waves are further subclassified, in a physical sense, as longitudinal strain waves and longitudinal bending waves. A normal acceleration wave may be composed of a second order normal shear strain wave, normal strain wave, or normal strain rate wave. The propagation conditions give coupling effects among the modes involved in each instance.

A third wave classification, that of "transverse acceleration waves", arises from the torsional equations of motion. These equations give propagation conditions resulting in a third eigenvalue problem and the subclassifications transverse shear strain acceleration wave and transverse twisting acceleration wave.

Chapter IV is devoted to an examination of the growth-decay problem for the discontinuities across a wave curve. Once the procedure is presented the problem is analyzed first within the framework of the equations describing torsionless motions. Two simultaneous differential equations governing the variation in strength of longitudinal waves are found as well as a system of three differential equations regulating normal acceleration waves. In addition, it is shown that coupling effects exist that predict, in general, third order longitudinal strain and bending waves accompanying a normal acceleration wave. In the case of longitudinal acceleration waves, the assumed wave is generally a third order wave with respect to the displacement components normal to the reference surface.

The method is applied to the equations of motion governing torsional motions to derive the decay equations for transverse waves. A system of two differential equations is found.

The final chapter is concerned with the solution of the propagation velocity and growth-decay problems for specific examples of shells of revolution. The conical shell is examined with the hope that its simplifying of the whole propagation problem will give useful insight into the general case. Two propagation velocities are found, one corresponding to longitudinal waves and the other to both normal and transverse waves. These speeds are the same as those derived by Thomas [18] for irrotational and equiareal waves respectively and by Cohen [3] for his longitudinal-bending and transverse-twisting waves. It is suggested that the seven mode theory of this study could be reduced to a six mode theory by certain deletions. In addition it is found that longitudinal strain and bending waves can propagate independently as can the three normal wave modes and the two transverse modes. An examination of the spherical shell gives wave speeds for longitudinal and transverse waves differing from those of the cone by a common correction factor.

The growth-decay equations are solved for the conical case yielding an inverse relationship between wave strength and wave curve radius for all wave classifications identical to that obtained in (3) for bending waves. A focusing effect, which could lead to possible shell fracture, is produced. The coupling effects between longitudinal acceleration waves and third order normal waves are discussed as well as the reverse relationships. Finally, it is suggested that the acceleration wave propagation problem for a shell of revolution could be reduced to that of the conical shell by making a suitable thinness assumption.

CHAPTER II

GEOMETRY, GOVERNING EQUATIONS AND WAVE PROPAGATION

2.1 Differential Geometry

Some slight knowledge of the geometry of curves is required for the later development of the theory of surfaces.

2.1.1. Curves in Euclidean 3-Space (E^3)

2.1.1. (a) Parametric Representation

A curve in E^3 can be thought of as the path of motion of a point. At each point, u, in an open interval, I, in the real numbers, $u_1 < u < u_2$, a curve has the representation (Fig. 2.1)

$$x(u) = (x_1(u), x_2(u), x_3(u)), \tag{2.1}$$

where $x_i (i = 1, 2, 3)$ are real-valued differentiable functions. If $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ represent the natural frame field at a point then

$$\bar{x}(u) = x_1(u)\bar{e}_1 + x_2(u)\bar{e}_2 + x_3(u)\bar{e}_3, \tag{2.2}$$

which can be interpreted as the position vector of a point on the curve.

2.1.1. (b) Tangent Vector

For each number u in I, the velocity vector of x at u is the tangent vector

$$x^1(u) = \left(\frac{dx_1}{du}(u), \frac{dx_2}{du}(u), \frac{dx_3}{du}(u)\right)_{x(u)}, \tag{2.3}$$

at the point x(u).

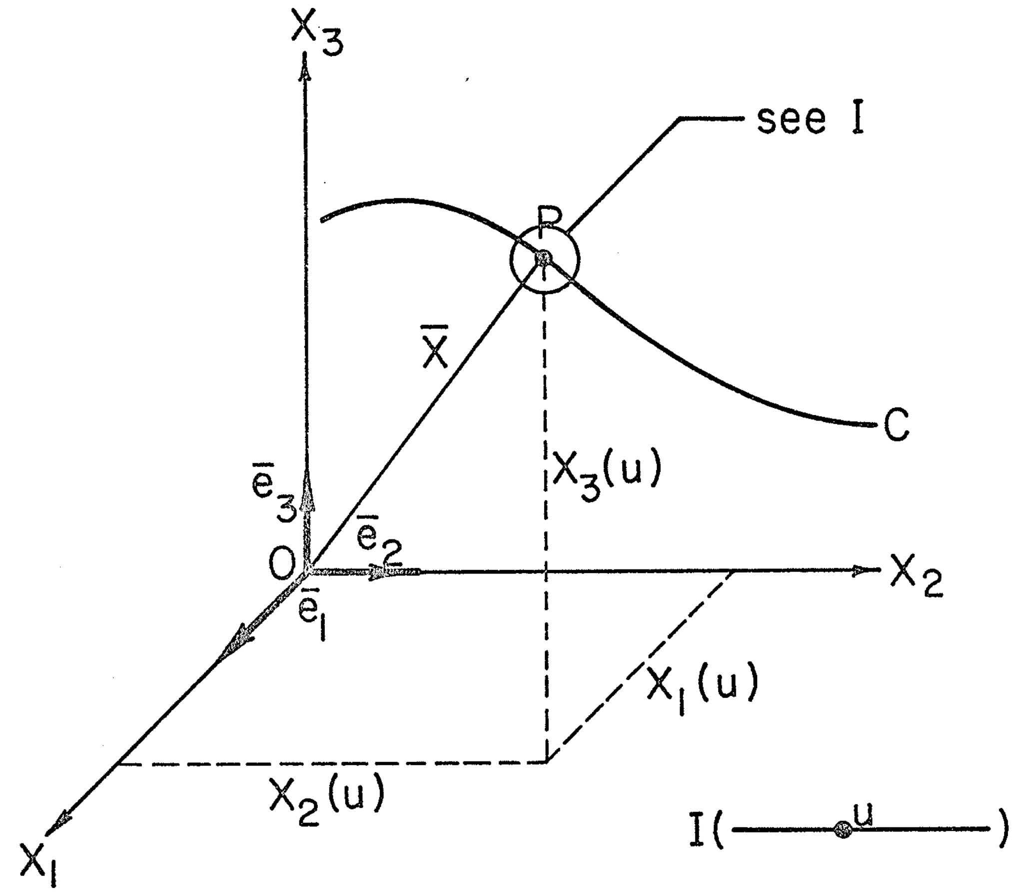

Fig. 2.1 Parametrization of a Curve

The geometrical definition (Fig. 2.2) follows from the limiting process

$$\frac{dx}{du} = \lim_{\Delta u \to o} \frac{x(u + \Delta u) - x(u)}{\Delta u} = \lim_{\Delta u \to o} \frac{\Delta x}{\Delta u}, \tag{2.4}$$

In order to reparametrize the curve x with arc length s as the new parameter, define

$$s(u) = \int_{u_1}^{u} |x^1(u)| \, du, \tag{2.5}$$

where $| x^1(u)| = (x^1(u) \cdot x^1(u))^{\frac{1}{2}} = \left[\frac{dx_1}{du}^2 + \frac{dx_2}{du}^2 + \frac{dx_3}{du}^2\right]^{\frac{1}{2}}$.

The curve x(u(s)) has unit speed as

$$| x^1(u(s)) | = \left| \frac{dx(u(s))}{du} \frac{du}{ds}(s) \right| = \frac{du}{ds}(s) \left| \frac{dx(u(s))}{du} \right|$$

$$= \frac{du}{ds}(s) \frac{ds}{du}(u(s)) = 1. \tag{2.6}$$

Therefore the velocity vector $\frac{dx}{ds}$ is a unit tangent vector.

2.1.1. (c) Frenet Formulas

The arc length parametrization of a curve will be used in this section to derive measurements of the change in shape of a curve in E^3.

As was shown in the preceding section $\frac{dx}{ds} = \bar{t}$ is a unit tangent vector on the curve x(s). The derivative $\bar{t}^1 = \frac{d\bar{t}}{ds}$ will measure the turning of the curve since $\bar{t}$ has constant unit length for each s in I. Differentiating $\bar{t} \cdot \bar{t} = 1$ results in $2\,\bar{t} \cdot \bar{t}^1 = 0$, so $\bar{t}^1$ is orthogonal to $\bar{t}$ and, therefore, normal to x(s). The curvature of x(s) is defined by the real-valued function $\kappa(s) = | \bar{t}^1(s)|$ the reciprocal of which is the

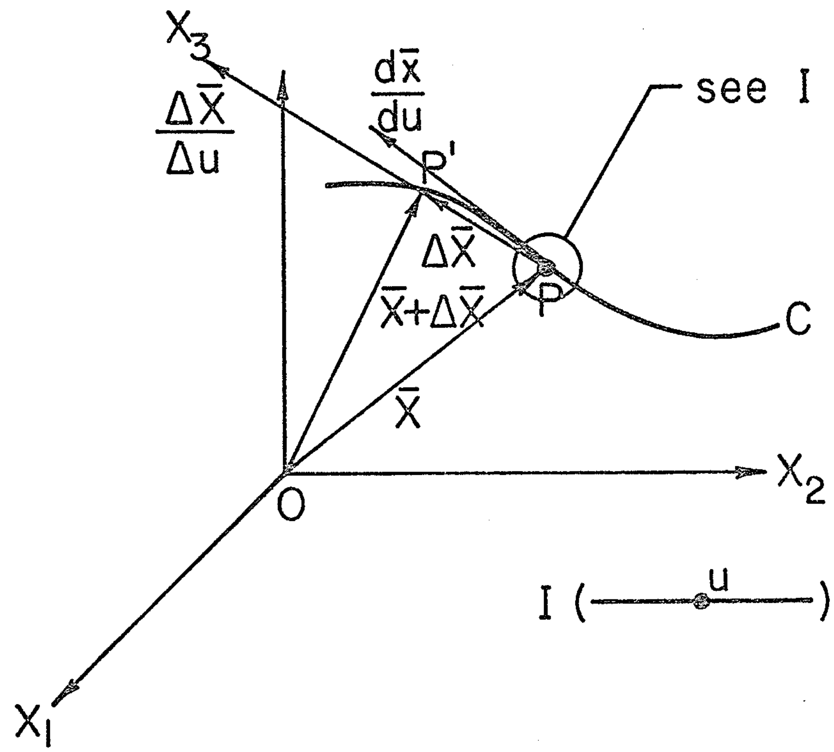

Fig. 2.2 Tangent Vector

radius of curvature. Restricting k to values greater than zero, the unit vector $\bar{N} = \frac{\bar{t}^1}{}$ gives the direction in which x(s) is turning and is defined as the principal normal to the curve. (Fig. 2.3)

The vector $\bar{b} = \bar{t} \times \bar{N}$ (a unit vector) is then defined as the binormal vector to x(s), and together with its mutually orthogonal unit vectors $\bar{t}$ and $\bar{N}$ comprises the Frenet frame field on x(s). Since the Frenet frame is intrinsic to the curve itself and not Euclidean space, it contains geometrical information unobtainable from consideration of the natural frame, i.e. Euclidean co-ordinates.

In order to make use of this information the derivatives of the three vectors forming the triad must be expressed in terms of the triad itself. As seen in the preceding discussion $\bar{t} = \frac{dx}{ds}$ giving $\bar{t}^1 = \frac{d^2x}{ds^2} = \bar{N}$. Since $\bar{b}$ is a unit vector $\bar{b} \cdot \bar{b} = 1$ and differentiation gives $2\bar{b}^1 \cdot \bar{b} = 0$. Also, differentiate $\bar{b} \cdot \bar{t} = 0$ to obtain $\bar{b}^1 \cdot \bar{t} + \bar{b} \cdot \bar{t}^1 = 0$; then $\bar{b}^1 \cdot \bar{t} = -\bar{b} \cdot \bar{t}^1 = -\bar{b} \cdot \bar{N} = 0$. It follows that $\bar{b}^1$ is orthogonal to $\bar{b}$ and $\bar{t}$ and is therefore a scalar multiple of $\bar{N}$. The torsion function of the curve x(s) is now defined such that $\bar{b}^1 = -t\bar{N}$. The torsion measures the rate of change of the plane formed by $\bar{t}$ and $\bar{n}$, the osculating plane. It can be shown by orthonormal expansion that $\bar{N}^1 = k\bar{t} + t\bar{b}$.

Summarizing the formulas as given above

$$\begin{matrix} \frac{d\bar{t}}{ds} \\ \\ \frac{d\bar{N}}{ds} \\ \\ \frac{d\bar{b}}{ds} \end{matrix} = \begin{bmatrix} 0 & k & 0 \\ & & \\ -k & 0 & t \\ & & \\ 0 & -t & 0 \end{bmatrix} \begin{Bmatrix} \bar{t} \\ \\ \bar{N} \\ \\ \bar{b} \end{Bmatrix} \qquad (2.7)$$

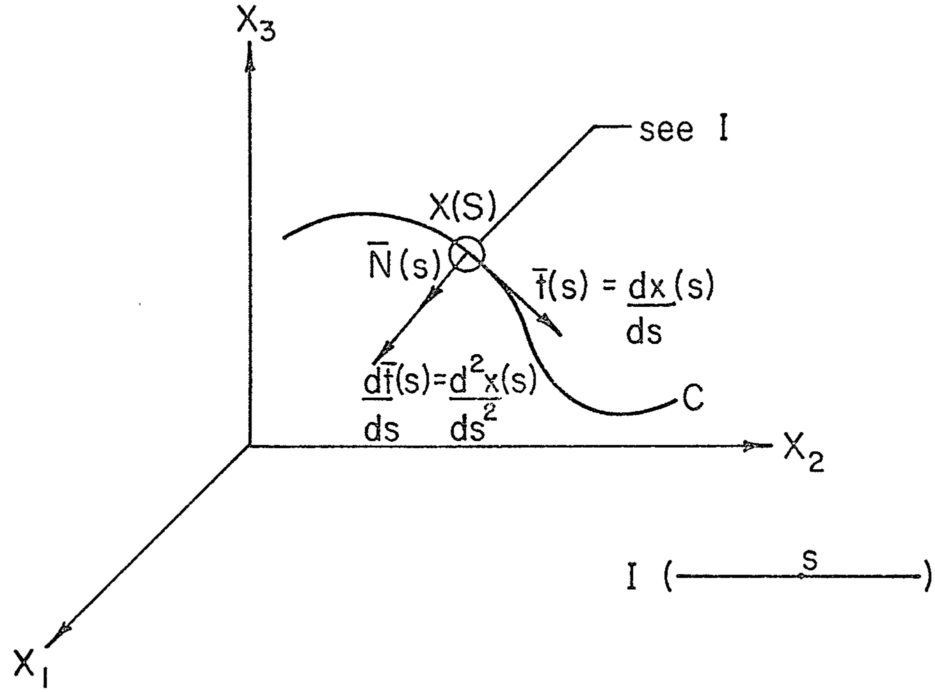

Fig. 2.3 Geometry of a Curve

2.1.2. Theory of Surfaces

The preceding results for curves can now be extended to surfaces as required for shell theory development.

2.1.2. (a) Parametric Representation of a Surface in E^3

A surface in E^3 can be written as a function of two parameters, called a coordinate patch, as follows (Fig. 2.4)

$$\gamma(\alpha_1, \alpha_2) = (x_1(\alpha_1, \alpha_2), x_2(\alpha_1, \alpha_2), x_3(\alpha_1, \alpha_2)), \tag{2.8}$$

where x_i (i = 1, 2, 3) are real-valued differentiable functions on the open set D in Euclidean two-space (E^2). Again let $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ represent the natural frame field at a point, then

$$\bar{\gamma}(\alpha_1, \alpha_2) = x_1(\alpha_1, \alpha_2)\, \bar{e}_1 + x_2(\alpha_1, \alpha_2)\, \bar{e}_2 + x_3(\alpha_1, \alpha_2)\, \bar{e}_3, \tag{2.9}$$

which, analagous to the curve, can be interpreted as the position vector of a point on the surface.

By holding α_2 constant and varying α_1, the curve $\gamma(\alpha_1, \alpha_2^o)$, called the α_1-parameter curve, is obtained. The α_2-parameter curves are obtained similarly. Together these two families of curves cover the surface $\gamma(\alpha_1, \alpha_2)$.

The velocity vector at a point on the α_i-parameter curve is given by

$$\bar{\gamma}_{,i} = \left(\frac{\partial x_1}{\partial \alpha_i}, \frac{\partial x_2}{\partial \alpha_i}, \frac{\partial x_3}{\partial \alpha_i} \right)_\gamma , \quad i = 1, 2 \tag{2.10}$$

where the comma denotes partial differentiation and the subscript γ indicates that the point of application is $\gamma(\alpha_1, \alpha_2)$. These two velocity vectors define a tangent plane on the surface.

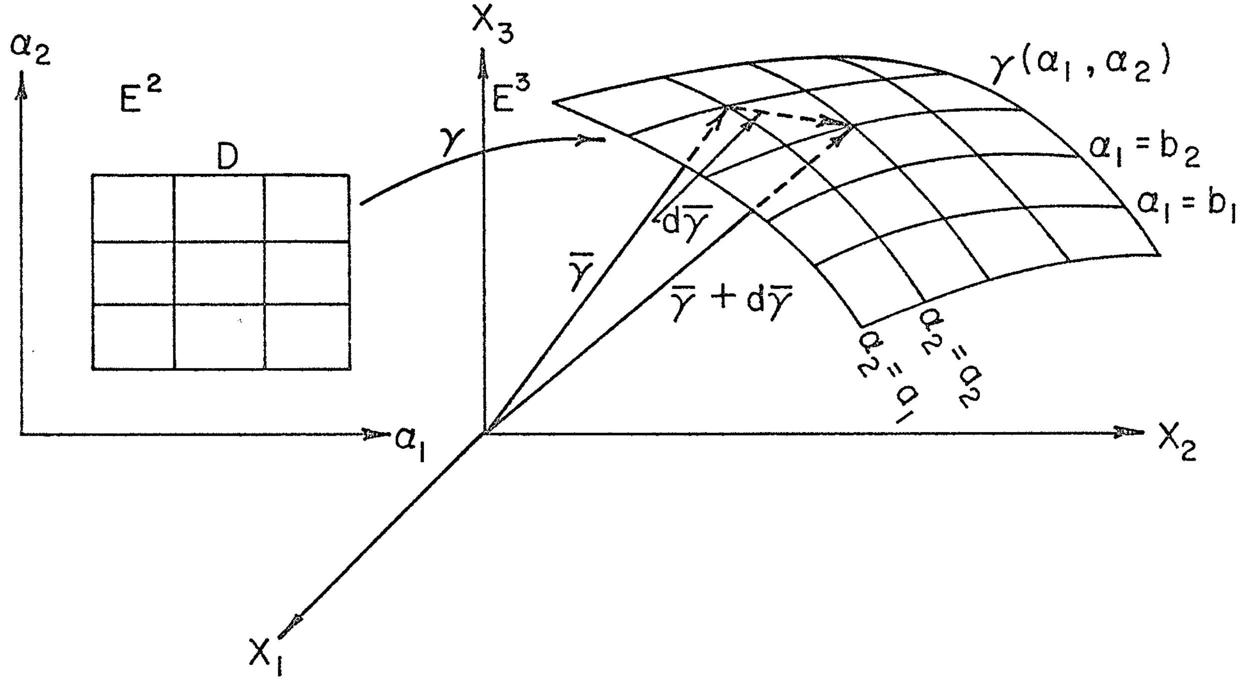

Fig. 2.4 Parametrization of a Surface

2.1.2. <u>(b) Normal Curvature of a Surface</u>

The shape operator at a point on a surface is defined as follows

$$S(\bar{v}) = -\nabla_v \bar{n}, \quad (2.11)$$

where $\nabla_v \bar{n}$ is the covariant derivative of the unit normal vector field, $\bar{n}$, of the surface with respect to the tangent vector $\bar{v}$. The shape operator measures the turning of the unit normal, and therefore that of the tangent plane, in the $\bar{v}$ direction, and can be shown to be a symmetric linear operator on the tangent plane of the surface.

The normal vector function can be constructed by the following procedure.

$$\bar{n}(\alpha_1, \alpha_2) = \frac{\bar{\gamma},_1 \times \bar{\gamma},_2}{|\bar{\gamma},_1 \times \bar{\gamma},_2|} , \quad (2.12)$$

Define:
$$\begin{aligned} \bar{\gamma},_1 \cdot \bar{\gamma},_1 &= E \\ \bar{\gamma},_1 \cdot \bar{\gamma},_2 &= F \\ \bar{\gamma},_2 \cdot \bar{\gamma},_2 &= G \end{aligned} \quad (2.13)$$

From the definition of vector cross product and dot product

$$|\bar{\gamma},_1 \times \bar{\gamma},_2| = |\bar{\gamma},_1||\bar{\gamma},_2| \sin\theta \quad (2.14)$$

$$\bar{\gamma},_1 \cdot \bar{\gamma},_2 = |\bar{\gamma},_1||\bar{\gamma},_2| \cos\theta \quad (2.15)$$

where θ is the angle between the two vectors. Substituting Eq. 2.13 into Eq. 2.15

$$\cos\theta = \frac{F}{\sqrt{EG}}$$

giving $\sin\theta = \sqrt{(EG - F^2)/EG}$

Therefore, employing Eq. (2.14)

$$\bar{n}(\alpha_1, \alpha_2) = \frac{\bar{\gamma},_1 \times \bar{\gamma},_2}{\sqrt{EG - F^2}} \quad \text{where } EG - F^2 \neq 0. \quad (2.16)$$

$EG - F^2$ is never zero since $\bar{\gamma}$, by definition, is a regular mapping, i.e. the velocity vectors are never zero.

If $\bar{u}$ is a unit vector tangent to the surface $\bar{\gamma}(\alpha_1, \alpha_2)$ at a point, then the normal curvature of $\bar{\gamma}(\alpha_1, \alpha_2)$ in the $\bar{u}$ direction is defined as

$$k(\bar{u}) = S(\bar{u}) \cdot \bar{u}. \tag{2.17}$$

If $\bar{x}(s)$ is a unit speed curve on the surface $\bar{\gamma}(\alpha_1, \alpha_2)$ with initial velocity $\bar{x}^1(0) = \bar{u}$, the Frenet apparatus of $\bar{x}$ combined with the fact that $\bar{x}^{11} \cdot \bar{n} = S(\bar{x}^1) \cdot \bar{x}^1$ yields (Figure 2.5)

$$k(\bar{u}) = S(\bar{u}) \cdot \bar{u} = \bar{x}^{11}(0) \cdot \bar{n} = \kappa(0)\bar{N}(0) \cdot \bar{n} = \kappa(0) \cos\theta. \tag{2.18}$$

The normal curvature of $\bar{\gamma}(\alpha_1, \alpha_2)$ in the $\bar{u}$ direction is then $\kappa(0)\cos\theta$, where $\kappa(0)$ is the curvature of $\bar{x}$ at $\bar{x}(0)$, the point of application of $\bar{u}$, and θ is the angle between the normal to the curve, $\bar{N}$, and the normal to the surface, $\bar{n}$. Since the normal to the curve, $\bar{N}$, gives the direction in which it is turning (Sec. 2.1.1.(a)), the sign of the normal curvature will tell which direction the surface is bending, relative to the choice of $\bar{n}$, moving along $\bar{x}(s)$.

Define three more geometric quantities as follows

$$\begin{aligned} L &= S(\bar{\gamma},_1) \cdot \bar{\gamma},_1 = \bar{\gamma},_{11} \cdot \bar{n} \\ M &= S(\bar{\gamma},_1) \cdot \bar{\gamma},_2 = S(\bar{\gamma},_2) \cdot \bar{\gamma},_1 = \bar{\gamma},_{12} \cdot \bar{n} \\ N &= S(\bar{\gamma},_2) \cdot \bar{\gamma},_2 = \bar{\gamma},_{22} \cdot \bar{n} \end{aligned} \tag{2.19}$$

If arc length s is employed to parametrize (α_1, α_2), then the reparametrization $\bar{\gamma}(\alpha_1, \alpha_2) = \bar{\gamma}(\alpha_1(s), \alpha_2(s))$ is a curve on $\bar{\gamma}(\alpha_1, \alpha_2)$.

$$\frac{d\bar{\gamma}}{ds} = \frac{\partial\bar{\gamma}}{\partial\alpha_1}\frac{d\alpha_1}{ds} + \frac{\partial\bar{\gamma}}{\partial\alpha_2}\frac{d\alpha_2}{ds} = \bar{t}, \tag{2.20}$$

where $\bar{t}$ is the unit tangent vector at a point on the curve.

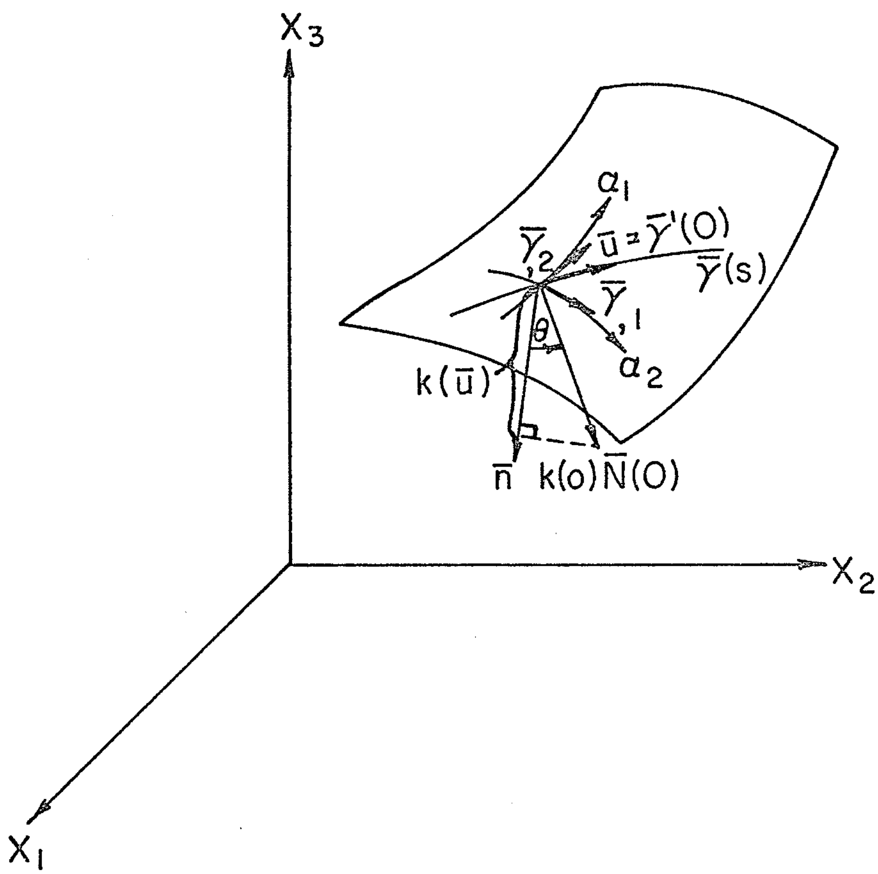

Fig. 2.5 Normal Curvature of a Surface

$$\frac{d\bar{t}}{ds} = \frac{\partial \bar{t}}{\partial \alpha_1}\frac{d\alpha_1}{ds} + \frac{\partial \bar{t}}{\partial \alpha_2}\frac{d\alpha_2}{ds}$$

$$= \bar{\gamma},_{11}\left(\frac{d\alpha_1}{ds}\right)^2 + 2\,\bar{\gamma},_{12}\frac{d\alpha_1}{ds}\frac{d\alpha_2}{ds} + \bar{\gamma},_{22}\left(\frac{d\alpha_2}{ds}\right)^2 \qquad (2.21)$$

$$= \frac{\bar{\gamma},_{11}\,d\alpha_1^{\,2} + 2\bar{\gamma},_{12}\,d\alpha_1\,d\alpha_2 + \bar{\gamma},_{22}\,d\alpha_2^{\,2}}{ds^2} = \kappa\bar{N}$$

Now consider the unit tangent vector to the curve $\bar{\gamma}(s)$.

$$\frac{d\bar{\gamma}}{ds}\cdot\frac{d\bar{\gamma}}{ds} = \frac{d\bar{\gamma}\,.\,d\bar{\gamma}}{ds^2} = 1,$$

$$d\bar{\gamma}\cdot d\bar{\gamma} = ds^2. \qquad (2.22)$$

The final expression for distance on the surface is

$$ds^2 = (\bar{\gamma},_1\,d\alpha_1 + \bar{\gamma},_2\,d\alpha_2)\cdot(\bar{\gamma},_1\,d\alpha_1 + \bar{\gamma},_2\,d\alpha_2)$$

$$= \bar{\gamma},_1\cdot\bar{\gamma},_1\,d\alpha_1^{\,2} + 2\bar{\gamma},_1\cdot\bar{\gamma},_2\,d\alpha_1\,d\alpha_2 + \bar{\gamma},_2\cdot\bar{\gamma},_2\,d\alpha_2^{\,2}. \qquad (2.23)$$

From Eq. 2.13

$$ds^2 = E\,d\alpha_1^{\,2} + 2F\,d\alpha_1\,d\alpha_2 + G\,d\alpha_2^{\,2}. \qquad (2.24)$$

Since the normal curvature of the surface is given by $\kappa\bar{N}\cdot\bar{n}$, it follows from Eqs. 2.19, 2.21, and 2.24

$$k\left(\frac{d\bar{\gamma}}{ds}\right) = \frac{L\,d\alpha_1^{\,2} + 2Md\alpha_1\,d\alpha_2 + Nd\alpha_2^{\,2}}{E\,d\alpha_1^{\,2} + 2Fd\alpha_1\,d\alpha_2 + Gd\alpha_2^{\,2}} = \frac{II}{I} \qquad (2.25)$$

where I is termed the first fundamental form, or metric, of the surface and II is called the second fundamental form. Considering Eq. 2.25 it can be seen that the normal curvature depends only on the direction of the tangent vector to the curve which is being examined on the surface.

2.1.2. (c) Principal Curvatures

The maximum and minimum values of the normal curvature, k, of a surface at a point are termed the principal curvatures of the surface at that point, and are denoted by k_1 and k_2. The directions in which these extreme values occur are called principal directions of $\bar{\gamma}(\alpha_1, \alpha_2)$ at the point and unit vectors in these directions are called principal vectors.

From Eq. 2.17, if k has its maximum value k_1 in the direction of the unit tangent vector $\bar{e}_1$

$$k_1 = k(\bar{e}_1) = S(\bar{e}_1) \cdot \bar{e}_1. \tag{2.26}$$

If $\bar{e}_2$ is a unit tangent vector orthogonal to $\bar{e}_1$, any unit tangent vector, $\bar{u}$, at the point can be written in terms of this orthonormal basis for the tangent plane to the surface (Fig. 2.6)

$$\bar{u} = \bar{u}(\theta) = \cos\theta\ \bar{e}_1 + \sin\theta\ \bar{e}_2. \tag{2.27}$$

As a result, the normal curvature at the point becomes a function on the real line such that $k(\bar{u}(\theta)) = k(\theta)$. Set $S_{ij} = S(\bar{e}_i) \cdot \bar{e}_j$ for $1 < i, j < 2$. Then

$$\begin{aligned} k(\theta) &= S(\cos\theta\ \bar{e}_1, + \sin\theta\ \bar{e}_2) \cdot (\cos\theta\ \bar{e}_1 + \sin\theta\ \bar{e}_2) \\ &= \cos^2\theta\ S_{11} + 2\cos\theta\sin\theta\ S_{12} + \sin^2\theta\ S_{22}. \end{aligned} \tag{2.28}$$

Differentiating Eq. 2.28 yields

$$\frac{dk}{d\theta}(\theta) = 2\cos\theta\sin\theta\ (S_{22} - S_{11}) + 2(\cos^2\theta - \sin^2\theta)S_{12}. \tag{2.29}$$

However, if $\theta = 0$, $\bar{u} = \bar{u}(0) = \bar{e}_1$ implying from Eq. 2.26 that $k(\theta)$ is a maximum, k_1, at $\theta = 0$, so $\frac{dk}{d\theta}(0) = 0$.

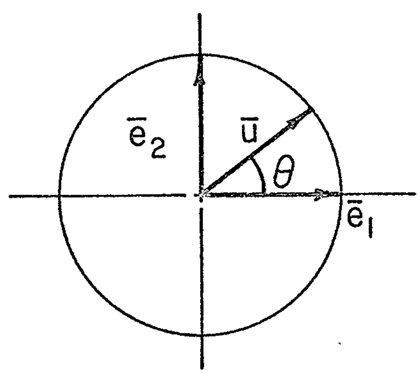

Fig. 2.6 Orthonormal Basis of a Tangent Plane

Eq. 2.29 then gives $S_{12} = 0$ and becomes from Eq. 2.26

$$\frac{dk}{d\theta}(\theta) = 2 \cos\theta \sin\theta \, (S_{22} - k_1) \qquad (2.30)$$

In order to find the other extremum set Eq. 2.30 equal to zero. Then

$$\cos\theta \sin\theta \, (S_{22} - k_1) = 0, \qquad (2.31)$$

and either (a) $S_{22} = k_1$ or (b) $\cos\theta = 0$.

In case (a) above, $S_{22} = k(\bar{e}_2) = k(\bar{e}_1)$ implies that the normal curvature at the point remains unchanged in all directions. Such a point is called an umbilic point.

Case (b) gives $\bar{u} = \bar{u}(\frac{\pi}{2}) = \bar{e}_2$ and the minimum curvature $k(\frac{\pi}{2}) = S_{22} = k(\bar{e}_2)$. It can be shown by orthonormal expansion that $S(\bar{e}_1) = S_{11}\,\bar{e}_1$ and $S(\bar{e}_2) = S_{22}\,\bar{e}_2$. If $k(\bar{e}_2) = k_2$, then it follows that

$$S(\bar{e}_1) = k_1\,\bar{e}_1, \qquad S(\bar{e}_2) = k_2\,\bar{e}_2. \qquad (2.32)$$

The prinicpal curvatures of $\gamma(\alpha_1, \alpha_2)$ at a point are the characteristic values of S, and the corresponding principal vectors are the characteristic vectors of S. The principal vectors, as such, are orthogonal.

The Gaussian curvature of a surface is defined as the real-valued function $K = \det S$ on $\gamma(\alpha_1, \alpha_2)$. The mean curvature of the surface is $H = \frac{1}{2}$ trace S. Thus, the Gaussian and mean curvatures can be expressed in terms of the principal curvatures as follows

$$K = k_1 k_2, \qquad H = \frac{k_1 + k_2}{2}. \qquad (2.33)$$

In terms of the tangent vectors, $\bar{\gamma},_1$ and $\bar{\gamma},_2$, at a point on the surface, it can be shown that

$$K = \frac{LN - M^2}{EG - F^2} \quad , \qquad H = \frac{GL + EN - 2FM}{2(EG - F^2)} \, . \tag{2.34}$$

Now, if the principal directions at a point correspond to the directions of the parametric velocity vectors, F = 0 due to the orthogonality discussed previously and from Eq. 2.19 and Eq. 2.32

$$M = S(\bar{\gamma},_1) \cdot \bar{\gamma},_2 = a\, S(\bar{e}_1) \cdot b\bar{e}_2 = a\, b\, k_1\, \bar{e}_1 \cdot \bar{e}_2 = 0 \tag{2.35}$$

where a, b are real numbers. Orthonormal expansion of the shape operator gives

$$S(\bar{\gamma},_1) \;=\; a\, \bar{\gamma},_1 + b\bar{\gamma},_2. \tag{2.36}$$

Taking the dot product of Eq. 2.36 with $\bar{\gamma},_1$ and $\bar{\gamma},_2$ and employing Eqs. 2.13, 2.19 and 2.35

$$a = \frac{L}{E}\,, \quad b = \frac{M}{G} = 0. \tag{2.37}$$

Eq. 2.36 can then be rewritten

$$cS(\bar{e}_1) \;=\; \frac{L}{E}\; c\, \bar{e}_1, \tag{2.38}$$

where c is a real number.

From Eq. 2.32 the principal curvature in the $\bar{\gamma},_1$ direction is

$$k_1 \;=\; \frac{L}{E} \;=\; \frac{1}{R_1}. \tag{2.39}$$

Similarly, application of the shape operator to the $\bar{\gamma},_2$ velocity vector gives the principal curvature in this direction as

$$k_2 \;=\; \frac{N}{G} \;=\; \frac{1}{R_2}. \tag{2.40}$$

where the inverses are called the radii of curvature.

2.1.2. (d) Gauss and Weingarten Equations

The velocity vectors at a point on the surface $\bar{\gamma}(\alpha_1, \alpha_2)$ in combination with the unit normal function at the point (defined in Sec. 2.1.2.(b)) can be thought of as a defective triad on the surface, defective in the sense that $\bar{\gamma},_1$ and $\bar{\gamma},_2$ are not unit vectors and are not necessarily orthogonal. This triad is analagous to the Frenet frame field for a curve and, considering the discussion of the merits of that system, it will be advantageous to have the derivatives of the three vectors forming the triad in terms of the triad itself.

The derivatives of the tangent velocity vectors can be written as follows

$$\begin{aligned} \bar{\gamma},_{11} &= a_1\,\bar{\gamma},_1 + a_2\,\bar{\gamma},_2 + a_3\,\bar{n} \\ \bar{\gamma},_{12} &= b_1\,\bar{\gamma},_1 + b_2\,\bar{\gamma},_2 + b_3\,\bar{n} \\ \bar{\gamma},_{22} &= c_1\,\bar{\gamma},_1 + c_2\,\bar{\gamma},_2 + c_3\,\bar{n} \end{aligned} \tag{2.41}$$

where the coefficients are real-valued functions on the surface at the point under consideration. The determination of these coefficients is carried out in Struik[17]. The final equations, called the Gauss equations, are given by

$$\begin{Bmatrix} \bar{\gamma},_{11} \\ \bar{\gamma},_{12} \\ \bar{\gamma},_{22} \end{Bmatrix} = \begin{bmatrix} \Gamma_{11}^{1} & \Gamma_{11}^{2} & L \\ \Gamma_{12}^{1} & \Gamma_{12}^{2} & M \\ \Gamma_{22}^{1} & \Gamma_{22}^{2} & N \end{bmatrix} \begin{Bmatrix} \bar{\gamma},_1 \\ \bar{\gamma},_2 \\ \bar{n} \end{Bmatrix} \tag{2.42}$$

where the Christoffel symbols are

$$\Gamma_{11}^{1} = \frac{GE,_1 - 2F\,F,_1 + FE,_2}{2D^2}$$

$$\Gamma_{11}^{2} = \frac{2E\,F,_1 - EE,_2 - FE,_1}{2D^2}$$

$$\Gamma_{12}^{1} = \frac{GE,_2 - FG,_1}{2D^2} \qquad (2.43)$$

$$\Gamma_{12}^{1} = \frac{EG,_1 - FE,_2}{2D^2}$$

$$\Gamma_{22}^{1} = \frac{2G\,F,_2 - GG,_1 - FG,_2}{2D^2}$$

$$\Gamma_{22}^{2} = \frac{EG,_2 - 2F\,F,_2 + FG,_1}{2D^2}$$

and $D^2 = EG - F^2$.

Similarly, it will be useful to complement the Gauss equations by the equations which express the derivatives of the unit normal vector in terms of the moving triad. The equations, called the Weingarten equations, are given by

$$\bar{n},_1 = \frac{FM - LG}{D^2}\,\bar{\gamma},_1 + \frac{FL - EM}{D^2}\,\bar{\gamma},_2$$

$$\bar{n},_2 = \frac{FN - GM}{D^2}\,\bar{\gamma},_1 + \frac{FM - EN}{D^2}\,\bar{\gamma},_2 \qquad (2.44)$$

where $D^2 = EG - F^2$.

2.1.2. (e) Codazzi Equations and Gauss Condition

The Gauss equations, Eq. 2.42, also termed the partial differential equations of surface theory, are not independent but must satisfy specific conditions of compatibility. These conditions are expressed by the equations

$$(\bar{\gamma},_{11}),_2 = (\bar{\gamma},_{12}),_1 \quad , \quad (\bar{\gamma},_{22}),_1 = (\bar{\gamma},_{12}),_2 . \tag{2.45}$$

Employing Eq. 2.42 gives

$$\begin{aligned} (\Gamma_{11}^{1}\,\bar{\gamma},_1 + \Gamma_{11}^{2}\,\bar{\gamma},_2 + L\bar{n}),_2 &= (\Gamma_{12}^{1}\,\bar{\gamma},_1 + \Gamma_{12}^{2}\,\bar{\gamma},_2 + M\bar{n}),_1 , \\ (\Gamma_{12}^{1}\,\bar{\gamma},_1 + \Gamma_{12}^{2}\,\bar{\gamma},_2 + M\bar{n}),_2 &= (\Gamma_{22}^{1}\,\bar{\gamma},_1 + \Gamma_{22}^{1}\,\bar{\gamma},_2 + N\bar{n}),_1 \end{aligned} \tag{2.46}$$

Carrying out the indicated differentiations and again employing the Gauss equations, Eq. 2.42, and the Weingarten equations, Eq. 2.44, results in two identities between the vectors of the triad. In order to be satisfied these identities give six scalar equations, three of which are given by

$$\begin{aligned} &\text{(a)} \quad -E\,\frac{LN - M^2}{D^2} = \Gamma_{12,1}^{2} - \Gamma_{11,2}^{2} + \Gamma_{12}^{1}\Gamma_{11}^{2} - \Gamma_{11}^{1}\Gamma_{12}^{2} + \Gamma_{12}^{2}\Gamma_{12}^{2} - \Gamma_{11}^{2}\Gamma_{22}^{2} , \\ &\text{(b)} \quad L,_2 - M,_1 = L\Gamma_{12}^{1} + M(\Gamma_{12}^{2} - \Gamma_{11}^{1}) - N\Gamma_{11}^{2} , \\ &\text{(c)} \quad M,_2 - N,_1 = L\Gamma_{22}^{1} + M(\Gamma_{22}^{2} - \Gamma_{12}^{1}) - N\Gamma_{12}^{2} . \end{aligned} \tag{2.47}$$

From Eq. 2.34 it can be seen that the left-hand side of Eq. 2.47(a) contains the Gaussian curvature of the surface. Therefore, the Gaussian curvature depends upon E, F, G and their first and second derivatives and, as such, is a bending invariant. The three scalar equations which have been omitted express this theorem of Gauss in different analytic form. Eq. 2.47(b) and (c) are known as the Codazzi or Mainardi-Codazzi equations.

A consideration of the compatibility equations associated with the Weingarten equations, Eq. 2.44, obtained by expressing $\bar{n},_{12} = \bar{n},_{21}$ and using Eq. 2.42, results in equations equivalent to the Codazzi equations. If the parametric curves of the surface are orthogonal, then

F = 0 and Eq. 2.34, Eq. 2.43, Eq. 2.47(a) give

$$-EK = \left(\frac{G,_1}{2G}\right),_1 + \left(\frac{E,_2}{2G}\right),_2 - \frac{E,_2\, E,_2}{4EG} - \frac{E,_1\, G,_1}{4EG} + \frac{G,_1\, G,_1}{4G^2} + \frac{E,_2\, G,_2}{4G^2} . \quad (2.48)$$

For the sake of convenience set

$$A_1 = \sqrt{E}, \; A_2 = \sqrt{G} . \quad (2.49)$$

Carrying out this substitution and the indicated differentiations, Eq. (2.48) becomes

$$-A_1A_2K = \frac{A_2}{A_1}\left(\frac{1}{A_2} A_{2,1}\right),_1 + \frac{A_2}{A_1}\left(\frac{A_1}{A_2^2} A_{1,2}\right) - \frac{1}{A_1A_2}(A_{1,2})^2 + \frac{1}{A_1A_2}(A_{2,1})^2 - \frac{1}{A_1^2} A_{1,1}A_{2,1} + \frac{1}{A_2^2} A_{1,2}A_{2,2} . \quad (2.50)$$

Noting that

$$\frac{A_2}{A_1}\left(\frac{1}{A_2} A_{2,1}\right),_1 = \left(\frac{1}{A_1} A_{2,1}\right),_1 - \frac{1}{A_2} A_{2,1}\left(\frac{A_2}{A_1}\right),_1 ,$$

$$\frac{A_2}{A_1}\left(\frac{A_1}{A_2^2} A_{1,2}\right),_2 = \left(\frac{1}{A_2} A_{1,2}\right),_2 - \frac{A_1}{A_2^2} A_{1,2}\left(\frac{A_2}{A_1}\right),_2 . \quad (2.51)$$

the final expression for the Gauss condition becomes

$$-A_1A_2K = \left(\frac{1}{A_1} A_{2,1}\right),_1 + \left(\frac{1}{A_2} A_{1,2}\right),_2 . \quad (2.52)$$

If the principal directions coincide with the parametric curves, then not only does F = 0 but M = 0 also. The Codazzi equation, Eq. 2.47(b) with Eq. 2.43, Eq. 2.39, and Eq. 2.40 yields

$$L,_{2} = \frac{A_1}{R_1} A_{1,2} + \frac{A_1}{R_2} A_{1,2} \quad . \tag{2.53}$$

However, employing Eq. 2.39

$$L,_{2} = (\frac{A_1^{\ 2}}{R_1}),_{2} = \frac{2A_1}{R_1} A_{1,2} - \frac{A_1^{\ 2}}{R_1^{\ 2}} R_{1,2} \quad . \tag{2.54}$$

Eqs. 2.53 and 2.54 give

$$\frac{1}{R_1} A_{1,2} - \frac{A_1}{R_1^{\ 2}} R_{1,2} = \frac{1}{R_2} A_{1,2} \quad . \tag{2.55}$$

which reduces to

$$(\frac{A_1}{R_1}),_{2} = \frac{1}{R_2} A_{1,2} \quad . \tag{2.56}$$

Similarly, Eq. 2.47(c) gives the second Codazzi condition as

$$(\frac{A_2}{R_2}),_{1} = \frac{1}{R_1} A_{2,1} \tag{2.57}$$

The three equations, Eq. 2.52, Eq. 2.56, Eq. 2.57, thus express the compatibility conditions for a surface, the principal directions of which lie along the parametric curves.

2.1.2. (f) Surface of Revolution

If a curve, C, in a plane, P, is revolved about an axis in P which does not meet C, it sweeps out a surface of revolution in E^3. The circles in the surface generated by each point of C as it is revolved are called parallels and the different positions of C as it is rotated are called meridians (Fig. 2.7).

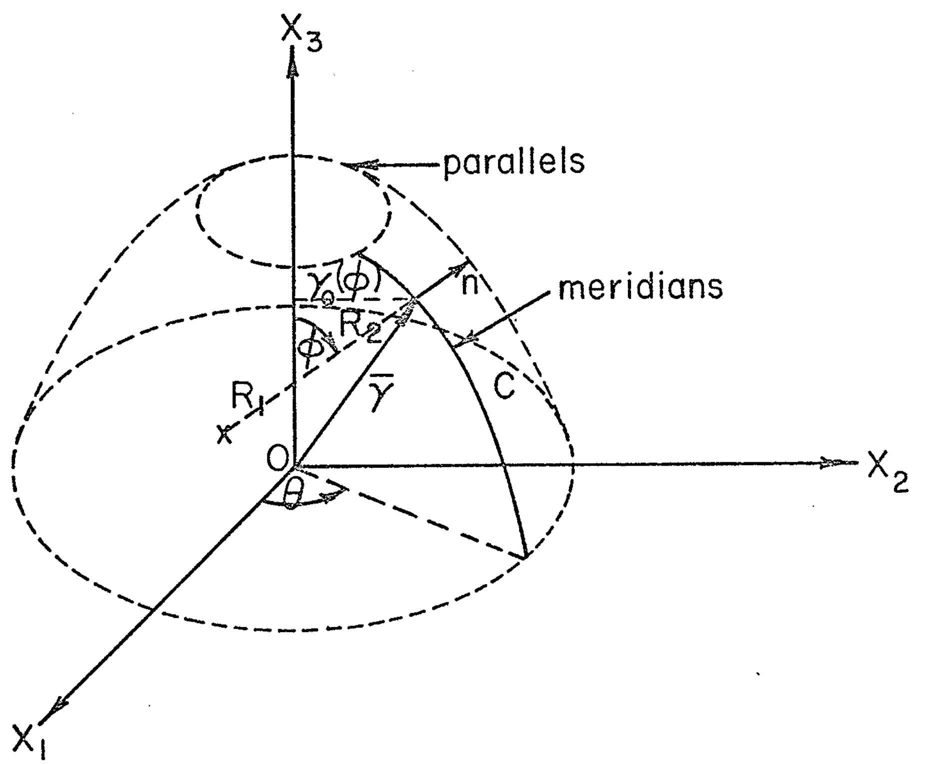

Fig. 2.7 Geometry of a Surface of Revolution

If the parameters, (α_1, α_2), of the surface are assumed to be ϕ and θ respectively, where ϕ is the angle between the axis of revolution and the normal to the surface at the point under consideration and θ is the angle of rotation from a reference axis, and $\gamma_o(\phi)$ is the perpendicular distance of the point from the axis of revolution, Eq. 2.8 gives

$$\bar{\gamma}(\phi, \theta) = (\gamma_o(\phi) \cos\theta, \gamma_o(\phi) \sin\theta, g(\phi)). \tag{2.58}$$

The parametric velocity vectors are given by Eq. 2.10 as

$$\begin{aligned} &(a) \quad \bar{\gamma},_{\phi} = (\gamma_o^{1}(\phi) \cos\theta, \gamma_o^{1}(\phi) \sin\theta, g^{1}(\phi)), \\ &(b) \quad \bar{\gamma},_{\theta} = (-\gamma_o(\phi) \sin\theta, \gamma_o(\phi) \cos\theta, 0), \end{aligned} \tag{2.59}$$

where the prime denotes differentiation with respect to the parameter involved.

Eq. 2.13 gives

$$\begin{aligned} E &= \gamma_o^{1}(\phi)^2 + g^{1}(\phi)^2, \\ F &= 0, \\ G &= \gamma_o(\phi)^2. \end{aligned} \tag{2.60}$$

The normal vector function, given by Eq. 2.16, is

$$\bar{n} = \frac{1}{\sqrt{EG}} (-g^{1}(\phi) \gamma_o(\phi) \cos\theta, -g^{1}(\phi) \gamma_o(\phi) \sin\theta, \gamma_o(\phi) \gamma_o^{1}(\phi)). \tag{2.61}$$

The acceleration vectors of the parametric curves are given by

$$\begin{aligned} \bar{\gamma},_{\phi\phi} &= (\gamma_o^{11}(\phi) \cos\theta, \gamma_o^{11}(\phi) \sin\theta, g^{11}(\phi)), \\ \bar{\gamma},_{\phi\theta} &= (-\gamma_o(\phi)^{1} \sin\theta, \gamma_o^{1}(\phi) \cos\theta, 0), \\ \bar{\gamma},_{\theta\theta} &= (-\gamma_o(\phi) \cos\theta, -\gamma_o(\phi) \sin\theta, 0). \end{aligned} \tag{2.62}$$

Employing Eq. 2.19 in combination with Eqs. 2.61 and 2.62 yields

$$L = \frac{1}{\sqrt{EG}} (g^{11}(\phi)\ \gamma_o(\phi)\ \gamma_o^{\ 1}(\phi) - g^1(\phi)\ \gamma_o(\phi)\ \gamma_o^{\ 11}(\phi)),$$

$$M = 0, \qquad (2.63)$$

$$N = \frac{g^1(\phi)\ \gamma_o(\phi)^2}{\sqrt{EG}} .$$

Since both F and M vanish the meridians and parallels, which are the parametric curves, coicide with the principle directions of the surface. The principle radii of curvature are calculated from Eqs. 2.39 and 2.40 to be

$$R_1 = \frac{[\gamma_o^{\ 1}(\phi)^2 + g^1(\phi)^2]^{3/2}}{g^{11}(\phi)\ \gamma_o^{\ 1}(\phi) - g^1(\phi)\ \gamma_o^{\ 11}(\phi)}, \qquad (2.64)$$

$$R_2 = \frac{\gamma_o(\phi)}{g^1(\phi)} [\gamma_o^{\ 1}(\phi)^2 + g^1(\phi)^2]^{\frac{1}{2}}.$$

From Eqs. 2.49, 2.60, 2.64 and the Gauss-Codazzi conditions given by Eqs. 2.52, 2.56, 2.57, it can be shown that these compatibility relations are satisfied. Therefore the surface of revolution described by the quantities A_1, A_2, R_1, and R_2 is a valid surface.

A reverse procedure may be utilized to obtain a more convenient form for the surface of revolution. If the above four quantities are specified as functions of the parameter ϕ and $A_1 = R_1$, $A_2 = \gamma_o(\phi)$, the first fundamental form is given by Eq. 2.25 as

$$ds^2 = R_1^{\ 2}\ d\phi^2 + \gamma_o(\phi)^2\ d\theta^2. \qquad (2.65)$$

The quantity $\gamma_o(\phi)$ can be shown to be equal to $-R_2 \sin\phi$ for a surface of revolution. The Codazzi conditions are then identically satisfied and the Gauss condition yields the result

$$\gamma_o(\phi)^1 = -R_1 \cos \phi. \tag{2.66}$$

Eq. 2.66 can be proved geometrically. Thus, the quantities specify a valid surface except for its position in space. Since all quantities are functions of the parameter ϕ, Eqs. 2.49, 2.60, and 2.64 together with the previous assumptions specify the function $g(\phi)$. Eq. 2.58 is then satisfied which shows that the surface described is a surface of revolution.

If arc length along a meridian is employed as a reparametrization of the ϕ-parameter curves (meridians), Eqs. 2.5, 2.59(a), and 2.60 yield

$$ds(\phi) = \sqrt{E}\ d\phi. \tag{2.67}$$

Since $\sqrt{E} = A_1 = R_1$, the final result is

$$\frac{d\phi}{ds} = \frac{1}{R_1}, \tag{2.68}$$

where s is the arc length along a meridian and is not to be confused with the s employed in the surface metric given by Eq. 2.65.

2.1.3. Shell Coordinates

A shell may be idealized as the materialization of a surface. This approximates the real situation which exists when two surfaces, closely spaced relative to their extent, form the boundaries of the material.

As the solving of problems in three-dimensional elasticity is extremely difficult in most cases, simplifying assumptions have been made which have led to theories describing the behaviour of thin elastic shells. One of the many theories which have been developed will be utilized in this treatise.

The prime assumptions upon which linear thin elastic shell theory is based are that the shell is thin and that its deflections are small. Novozhilov[12] gives, as his criteria for thinness, the limitation

$$\max \left(\frac{h}{R}\right) < \frac{1}{20} , \tag{2.69}$$

where h is the shell thickness and R will be the minimum radius of curvature of some reference surface. However, Kraus[9] gives his maximum ratio as one tenth, so no generally accepted definition is available.

The second assumption, that the displacements of a point are small, permits all quantities describing the shell space to be written in terms of some reference surface and allows a linear theory to be developed.

The reference surface mentioned above is generally taken to be the middle surface of the shell or the surface that has been materialized by constructing parallel surfaces equidistant above and below it. This choice is suited to an elastically homogeneous material which will be assumed here.

In this sense, a point in the shell space may be written in terms of the curvilinear coordinates, or parameters, of the reference surface and the surface normal function. This representation from Eqs. 2.8 and 2.12 is

$$\bar{R}(\alpha_1, \alpha_2, \zeta) = \bar{\gamma}(\alpha_1, \alpha_2) + \zeta\bar{n}(\alpha_1, \alpha_2) , \frac{-h}{2} < \zeta < \frac{h}{2} . \tag{2.70}$$

From Eq. 2.22 it can be seen that a curve in $(\alpha_1, \alpha_2, \zeta)$ space can also be parametrized with respect to arc length giving the shell metric as

$$ds^2 = d\bar{R} \cdot d\bar{R}. \tag{2.71}$$

However, $$d\bar{R} = \frac{\partial\bar{\gamma}}{\partial\alpha_1} d\alpha_1 + \frac{\partial\bar{\gamma}}{\partial\alpha_2} d\alpha_2 + \frac{\zeta\partial\bar{n}}{\partial\alpha_1} d\alpha_1 + \frac{\zeta\partial\bar{n}}{\partial\alpha_2} d\alpha_2 + d\zeta\bar{n}. \tag{2.72}$$

Substituting this expression into Eq. 2.71 and then employing Eqs. 2.13, 2.19, and the Weingarten equations, Eq. 2.44, noting that on a curve, $\bar{\gamma}(s)$, $S(\frac{d\bar{\gamma}}{ds}) = -\frac{d\bar{n}}{ds}(s)$, yields

$$\begin{aligned} ds^2 = {} & \{E - 2L\zeta + \frac{1}{D^2} (GL^2 + EM^2 - 2FLM) \zeta^2 \} d\alpha_1^2 \\ & + \{G - 2N\zeta + \frac{1}{D^2} (EN^2 + GM^2 - 2FMN) \zeta^2 \} d\alpha_2^2 + d\zeta^2 \\ & +2 \{F - 2M\zeta + \frac{1}{D^2} [M(GL + EN) - F(LN + M^2)]\zeta^2\} d\alpha_1 d\alpha_2. \end{aligned} \tag{2.73}$$

If the parametric curves of the reference surface are chosen to coincide with its principle directions, $F = 0$ and $M = 0$ from Sec. 2.2.2.(c) and the shell metric, Eq. 2.73, becomes, using Eqs. 2.39, 2.40 and 2.49

$$ds^2 = A_1^2 (1 - \frac{\zeta}{R_1})^2 d\alpha_1^2 + A_2^2 (1 - \frac{\zeta}{R_2})^2 d\alpha_2^2 + d\zeta^2 . \tag{2.74}$$

Choosing one of the parallel surfaces a distance ζ from the reference surface, the arc length along an α_i-parameter curve on this surface will be, from Eq. 2.74

$$ds_i(\zeta) = A_i(1 - \frac{\zeta}{R_i}) d\alpha_i, \qquad i = 1, 2 \tag{2.75}$$

The principle radii of curvature of this parallel surface will be simply $R_1 - \zeta$ and $R_2 - \zeta$. From Eq. 2.74 $A_1^1 = |\bar{R},_1| = A_1(1 - \frac{\zeta}{R_1})$ and $A_2^1 = A_2 (1 - \frac{\zeta}{R_2})$ on this surface.

The Codazzi equations, Eqs. 2.56 and 2.57 give

$$[A_1(1 - \frac{\zeta}{R_1})]_{,2} = (1 - \frac{\zeta}{R_2}) A_{1,2} \quad ,$$

$$[A_2(1 - \frac{\zeta}{R_2})]_{,1} = (1 - \frac{\zeta}{R_1}) A_{2,1} \quad . \qquad (2.76)$$

2.2 Constitutive Equations

The constitutive equations of a material involve the relationships which describe its behaviour at a point as it deforms. These relationships comprise the strain-displacement relations, or kinematics of the material, and the stress-strain relations.

2.2.1. Kinematics

The assumption of small displacements, as mentioned in Sec. 2.1.3., allows the deletion of terms involving the products of displacements or of their derivatives in the derivation of the strain-displacement relations from three-dimensional elasticity theory.

It can be shown that for a curvilinear orthogonal coordinate system the above small displacement assumption yields the following linear strain-displacement relations(16).

$$\varepsilon_i = \frac{\partial}{\partial \alpha_i} \left(\frac{u_i}{\sqrt{g_i}}\right) + \frac{1}{2g_i} \sum_{k=1}^{3} \frac{\partial g_i}{\partial \alpha_k} \frac{u_k}{\sqrt{g_k}} \quad , \quad i = 1, 2, 3 \qquad (2.77)$$

$$\gamma_{ij} = \frac{1}{\sqrt{g_i g_j}} \left[g_i \frac{\partial}{\partial \alpha_j} \left(\frac{u_i}{\sqrt{g_i}}\right) + g_j \frac{\partial}{\partial \alpha_i} \left(\frac{u_j}{\sqrt{g_j}}\right)\right] . \quad i = 1, 2, 3, \; i \neq j$$

In shell coordinates, or $(\alpha_1, \alpha_2, \zeta)$ space, the displacement vector at a point of the shell space will be given by

$$\bar{U}(\alpha_1, \alpha_2, \zeta) = U_1(\alpha_1, \alpha_2, \zeta)\ \bar{t}_1 + U_2(\alpha_1, \alpha_2, \zeta)\ \bar{t}_2 + W(\alpha_1, \alpha_2, \zeta)\bar{n}, \qquad (2.78)$$

where the vectors $\bar{t}_1$ and $\bar{t}_2$ are unit vectors along the parametric curves of the parallel surface in which the point lies.

The quantities in Eq. 2.77 which are expressed in a spatial coordinate system, become, in shell coordinates, using Eq. 2.74

$$\begin{aligned} &\alpha_1 = \alpha_1, &&\alpha_2 = \alpha_2, &&\alpha_3 = \zeta, \\ &u_1 = U_1, &&u_2 = U_2, &&u_3 = W, \\ &g_1 = A_1(1 - \frac{\zeta}{R_1}), &&g_2 = A_2(1 - \frac{\zeta}{R_2}), &&g_3 = 1. \end{aligned} \qquad (2.79)$$

As was seen in Sec. 2.1.3., a point in the shell space is simply on a parallel surface and, as such, may be written in terms of the reference surface. It follows, then, that a real-valued function on the shell space may be written in terms of a function on the reference surface. This may be accomplished by expanding the function in a Taylor-Maclaurin series about the reference surface, $\zeta = 0$. The region and nature of convergence of these series have not been established as yet, however.

The displacement components of Eq. 2.78 under such an expansion become, in a truncated series,

$$\begin{aligned} U_1(\alpha_1, \alpha_2, \zeta) &= u_1(\alpha_1, \alpha_2) + \zeta\beta_1(\alpha_1, \alpha_2), \\ U_2(\alpha_1, \alpha_2, \zeta) &= u_2(\alpha_1, \alpha_2) + \zeta\beta_2(\alpha_1, \alpha_2), \\ W(\alpha_1, \alpha_2, \zeta) &= w(\alpha_1, \alpha_2) + \zeta w^1(\alpha_1, \alpha_2) + \frac{\zeta^2}{2} W^{11}(\alpha_1, \alpha_2). \end{aligned} \qquad (2.80)$$

where $\beta_1 = \dfrac{\partial U_1}{\partial \zeta}$, $\beta_2 = \dfrac{\partial U_2}{\partial \zeta}$, $w^1 = \dfrac{\partial W}{\partial \zeta}$, $w^{11} = \dfrac{\partial^2 W}{\partial \zeta^2}$. (2.81)

In a strictly physical sense, the quantities u_1, u_2, and w represent the components of the displacement of a point on the reference surface, while the quantities β_1 and β_2 represent the rotations of the normal to the reference surface in the directions of the α_1 and α_2 parameter curves, respectively. The quantity w^1 represents the normal strain of the reference surface and w^{11} represents the rate of change of this strain along the normal to the reference surface.

Now, employing Eq. 2.77 and substituting into it Eqs. 2.76, 2.79 and 2.80, the strain-displacement relations become

$$\varepsilon_1 = \frac{1}{1 - \zeta/R_1} \left(\varepsilon_1{}^o + \zeta\varepsilon_1{}^1 + \frac{\zeta^2}{2} \varepsilon_1{}^{11}\right),$$

$$\varepsilon_2 = \frac{1}{1 - \zeta/R_2} \left(\varepsilon_2{}^o + \zeta\varepsilon_2{}^1 + \frac{\zeta^2}{2} \varepsilon_2{}^{11}\right),$$

$$\varepsilon_n = W^1 + \zeta W^{11},$$

$$\gamma_{12} = \frac{1}{1 - \zeta/R_1} \left(\beta_1{}^o + \zeta\beta_1{}^1\right) + \frac{1}{1 - \zeta/R_2} \left(\beta_2{}^o + \zeta\beta_2{}^1\right), \qquad (2.82)$$

$$\gamma_{1n} = \frac{1}{1 - \zeta/R_1} \left(\mu_1{}^o + \zeta\mu_1{}^1 + \frac{\zeta^2}{2} \mu_1{}^{11}\right),$$

$$\gamma_{2n} = \frac{1}{1 - \zeta/R_2} \left(\mu_2{}^o + \zeta\mu_2{}^1 + \frac{\zeta^2}{2} \mu_2{}^{11}\right).$$

where

$$\varepsilon_1{}^o = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} - \frac{w}{R_1}, \quad \varepsilon_2{}^o = \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} - \frac{w}{R_2},$$

$$\varepsilon_1{}^1 = \kappa_1 - \frac{w^1}{R_1}, \qquad \varepsilon_2{}^1 = \kappa_2 - \frac{w^1}{R_2},$$

$$\varepsilon_1{}^{11} = \frac{-w^{11}}{R_1}, \qquad \varepsilon_2{}^{11} = \frac{-w^{11}}{R_2},$$

$$\beta_1^{\,o} = \frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{u_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\,, \qquad \beta_2^{\,o} = \frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2} - \frac{u_2}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\,,$$

$$\beta_1^{\,1} = \frac{1}{A_1}\frac{\partial \beta_2}{\partial \alpha_1} - \frac{\beta_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\,, \qquad \beta_2^{\,1} = \frac{1}{A_2}\frac{\partial \beta_1}{\partial \alpha_2} - \frac{\beta_2}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\,,$$

$$\mu_1^{\,o} = \frac{1}{A_1}\frac{\partial w}{\partial \alpha_1} + \frac{u_1}{R_1} + \beta_1\,, \qquad \mu_2^{\,o} = \frac{1}{A_2}\frac{\partial w}{\partial \alpha_2} + \frac{u_2}{R_2} + \beta_2\,,$$

$$\mu_1^{\,1} = \frac{1}{A_1}\frac{\partial w^1}{\partial \alpha_1}\,, \qquad \mu_2^{\,1} = \frac{1}{A_2}\frac{\partial w^{11}}{\partial \alpha_2}\,,$$

$$\mu_1^{\,11} = \frac{1}{A_1}\frac{\partial w^{11}}{\partial \alpha_1}\,, \qquad \mu_2^{\,11} = \frac{1}{A_2}\frac{\partial w^{11}}{\partial \alpha_2}\,, \qquad (2.83)$$

$$\kappa_1 = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\,, \qquad \kappa_2 = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\,.$$

Also set

$$\tau = \beta_1^{\,1} + \beta_2^{\,1}\,, \qquad \gamma_{12}^{\,o} = \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right)\,.$$

2.2.2. Stress-Strain Relations

From elasticity theory, the stress-strain relations for a homogeneous isotropic material are given by Hooke's law. In a material of this nature the total state of stress at a point is defined by the stress components in three mutually perpendicular directions. Therefore, at each point in the shell space, these directions will be chosen to coincide with the shell coordinates at that point. Hooke's law is given by

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu\,(\sigma_2 + \sigma_n)\right],$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu\,(\sigma_1 + \sigma_n)\right],$$

$$\varepsilon_n = \frac{1}{E}[\sigma_n - \nu(\sigma_1 + \sigma_2)] , \qquad (2.84)$$

$$\gamma_{12} = \frac{t_{12}}{G} , \quad \gamma_{1n} = \frac{t_{1n}}{G} , \quad \gamma_{2n} = \frac{t_{2n}}{G} ,$$

where σ_1, σ_2, and σ_n are the normal stresses along the three orthogonal directions of the triad at a point; ε_1, ε_2, and ε_n are the normal strains in these directions; t_{12}, t_{1n}, and t_{2n} are the shearing stresses; γ_{12}, γ_{1n}, and γ_{2n} are the corresponding shearing strains; E is Young's modulus, ν is Poisson's ratio, and G is the shear modulus.

The first three equations of Eq. 2.84 may be solved for the normal stresses to yield

$$\sigma_1 = \frac{\bar{E}}{1 - \nu^2} [(1 - \nu^2)\varepsilon_1 + \nu(1 + \nu)(\varepsilon_2 + \varepsilon_n)] ,$$

$$\sigma_2 = \frac{\bar{E}}{1 - \nu^2} [(1 - \nu^2)\varepsilon_2 + \nu(1 + \nu)(\varepsilon_1 + \varepsilon_n)] , \qquad (2.85)$$

$$\sigma_n = \frac{\bar{E}}{1 - \nu^2} [(1 - \nu^2)\varepsilon_n + \nu(1 + \nu)(\varepsilon_1 + \varepsilon_2)] ,$$

where $\bar{E} = \frac{1 - \nu}{1 - \nu - 2\nu^2} E$.

In order to simplify succeeding derivations a set of relations, the first ten of which may be thought of as stress resultants and couples on the reference surface of a differential element of the shell space, will be defined.

$$\begin{Bmatrix} N_i \\ N_{ij} \\ Q_i \end{Bmatrix} = \int_{h/2}^{h/2} \begin{Bmatrix} \sigma_i \\ t_{ij} \\ t_{in} \end{Bmatrix} (1 - \frac{\zeta}{R_j}) \, d\zeta ,$$

$$\begin{Bmatrix} M_i \\ M_{ij} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_i \\ t_{ij} \end{Bmatrix} (1 - \frac{\zeta}{R_j})\zeta \, d\zeta, \qquad \begin{matrix} i,j = 1,\ 2 \\ i \neq j \end{matrix} \qquad (2.86)$$

$$\begin{Bmatrix} S_i \\ P_i \\ T_i \\ A \\ B \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} t_{in} & (1 - \zeta/R_j)\zeta \\ \sigma_i & (1 - \zeta/R_j)\zeta^2/2 \\ t_{in} & (1 - \zeta/R_j)\zeta^2/2 \\ \sigma_n & (1 - \zeta/R_1)\ (1 - \zeta/R_2) \\ \sigma_n & (1 - \zeta/R_1)\ (1 - \zeta/R_2)\zeta \end{Bmatrix} d\zeta,$$

In order to carry out the integration indicated in Eqs. 2.86, certain manipulations must be employed which make use of expansions of the type

$$\log(\frac{1 - x}{1 + x}) = -2\ (x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots + \frac{x^n}{n} + \ldots), \quad x^2 < 1 \qquad (2.87)$$

If $x = \zeta/R_i$, $i = 1, 2$, the only additional restriction encountered is that now $(\zeta/R_i)^2 < 1$. These manipulations are explained in Appendix A, the final results being

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} d\zeta = h\ [1 + \frac{h^2}{12R_j}\ (\frac{1}{R_j} - \frac{1}{R_i})\ ],$$

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} \zeta d\zeta = \frac{h^3}{12} (\frac{1}{R_j} - \frac{1}{R_i}) \quad , \quad \begin{matrix} i,j = 1,2 \\ i \neq j \end{matrix}$$

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} \zeta^2 d\zeta = \frac{h^3}{12} [1 + \frac{3h^2}{20R_j} (\frac{1}{R_j} - \frac{1}{R_i})] , \quad (2.88)$$

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} \zeta^3 d\zeta = \frac{h^5}{80} (\frac{1}{R_i} - \frac{1}{R_j}) ,$$

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} \zeta^4 d\zeta = \frac{h^5}{80} [1 + \frac{5h^2}{28R_j} (\frac{1}{R_j} - \frac{1}{R_i})] .$$

Substituting the last three of Eqs. 2.84, Eqs. 2.85 and Eqs. 2.82 into Eqs. 2.86, then making use of the results given by Eqs. 2.88, yields the relationships between the stress resultants and couples and the quantities given by Eqs. 2.83 as follows

$$N_i = \frac{Eh}{1-\nu^2} \{(1 - \nu^2)\varepsilon_i{}^o + \nu(1 + \nu)(\varepsilon_j{}^o + w^1)$$

$$+ \frac{h^2}{24} [(1 - \nu^2)\varepsilon_i{}^{11} + \nu(1 + \nu)(\varepsilon_j{}^{11} - \frac{2w^{11}}{R_j})]$$

$$+ \frac{h^2}{12} (\frac{1}{R_i} - \frac{1}{R_j}) (1 - \nu^2)(\frac{\varepsilon_i{}^o}{R_i} + \varepsilon_i{}^1 + \frac{3h^2}{40R_i} \varepsilon_i{}^{11})\} ,$$

$$N_{ij} = Gh \{ \gamma_{ij}{}^o + \frac{h^2}{12} (\frac{1}{R_i} - \frac{1}{R_j}) (\frac{\beta_i{}^o}{R_i} + \beta_i{}^1)\} , \quad i,j = 1,2, \; i \neq j$$

$$M_i = \frac{Eh^3}{12(1-\nu^2)} \{ (1 - \nu^2)\varepsilon_i{}^1 + \nu(1 + \nu)(\varepsilon_j{}^1 + w^{11} - \frac{w^1}{R_j})$$

$$+ (\frac{1}{R_i} - \frac{1}{R_j}) (1 - \nu^2) [\varepsilon_i{}^o + \frac{3h^2}{20} (\frac{\varepsilon_i{}^1}{R_i} + \frac{\varepsilon_i{}^{11}}{2})] \},$$

$$M_{ij} = \frac{Gh^3}{12} \{ \tau + (\frac{1}{R_i} - \frac{1}{R_j}) (\beta_i{}^o + \frac{3h^2}{20} \frac{\beta_i{}^1}{R_i}) ,$$

$$Q_i = Gh \{\mu_i{}^o + \frac{h^2}{24} \mu_i{}^{11} + \frac{h^2}{12} (\frac{1}{R_i} - \frac{1}{R_j})(\frac{\mu_i{}^o}{R_i} + \mu_i{}^1 + \frac{3h^2}{40R_i} \mu_i{}^{11})\} ,$$

$$S_i = \frac{Gh^3}{12} \{ \mu_i{}^1 + (\frac{1}{R_i} - \frac{1}{R_j}) [\mu_i{}^o + \frac{3h^2}{20} (\frac{\mu_i{}^1}{R_i} + \frac{\mu_i{}^{11}}{2})] \} ,$$

$$T_i = \frac{Gh^3}{24} \{\mu_i{}^o + \frac{3h^2}{40} \mu_i{}^{11} + \frac{3h^2}{20} (\frac{1}{R_i} - \frac{1}{R_j})(\frac{\mu_i{}^o}{R_i} + \mu_i{}^1 + \frac{5h^2}{56R_i} \mu_i{}^{11})\} ,$$

$$P_i = \frac{\bar{E}h^3}{12(1-\nu^2)} \{ (1 - \nu^2)\varepsilon_i{}^o + \nu(1 + \nu)(\varepsilon_j{}^o + w^1) \tag{2.89}$$

$$+ \frac{3h^2}{40} [(1 - \nu^2)\varepsilon_i{}^{11} + \nu(1 + \nu)(\varepsilon_j{}^{11} - 2 \frac{w^{11}}{R_j})]$$

$$+ \frac{3h^2}{20} (\frac{1}{R_i} - \frac{1}{R_j})(1 - \nu^2)(\frac{\varepsilon_i{}^o}{R_i} + \varepsilon_i{}^1 + \frac{5h^2}{56R_i} \varepsilon_i{}^{11})\} ,$$

$$A = \nu(N_1 + N_2 - \frac{M_1}{R_1} - \frac{M_2}{R_2})$$

$$+ Eh [(1 + \frac{h^2}{12R_1R_2}) w^1 - \frac{h^2}{12} (\frac{1}{R_1} + \frac{1}{R_2})w^{11}],$$

$$B = \nu(M_1 + M_2 - \frac{2P_1}{R_1} - \frac{2P_2}{R_2})$$

$$+ \frac{Eh^3}{12} [- (\frac{1}{R_1} + \frac{1}{R_2})w^1 + (1 + \frac{3h^2}{20R_1R_2}) w^{11}] .$$

2.3. Equations of Motion

It will now be assumed that the displacements of a point in the shell space are time dependent. This is accomplished by merely adding an additional parameter, t, to Eqs. 2.80.

The theorem of minimum potential energy states that of all displacements satisfying the given boundary conditions for a material, those which satisfy the equilibrium equations make the potential energy an absolute minimum. In other words the potential energy functional will be stationary when the displacements are those of the equilibrium state. For the dynamic problem the kinetic energy must also be included in the functional, adding a time dependency. In this form the theorem is known as Hamilton's principle, stated as follows

$$\delta \int_{t_o}^{t_1} (\pi - T)\, dt = 0. \tag{2.90}$$

where π is the potential energy of the deformed material and T is its kinetic energy.

If it is assumed that there are no external loading functions or body forces acting upon the material, the potential energy is given by the strain energy density function

$$P = \frac{1}{2} \int_V \sigma_{ij}\, \varepsilon_{ij}\, dV, \qquad i,j = 1,2,3 \tag{2.91}$$

where the summation convention is in effect.

The kinetic energy is given by the expression

$$T = \frac{1}{2} \int_V \rho(\dot{U}_1^{\,2} + \dot{U}_2^{\,2} + \dot{W}^2)\, dV, \tag{2.92}$$

where U_1, U_2, and W are the displacement components from Eqs. 2.78 and 2.80 and where the dot denotes partial differentiation with respect to time.

The volume, dV, of a differential element of the shell space is found, employing Eq. 2.75, to be

$$dV = A_1(1 - \zeta/R_1)A_2(1 - \zeta/R_2)\, d\alpha_1\, d\alpha_2\, d\zeta. \tag{2.93}$$

Taking the variation[2] of the integrand of Eq. 2.91 and noting that the stress-strain relations, Eqs. 2.85, may be written in the form $\sigma_{ij} = C_{ij}\,\varepsilon_{ij}$, the following result is obtained,

$$\delta(\tfrac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij}) = \tfrac{1}{2}\,\sigma_{ij}\,\delta\varepsilon_{ij} + \tfrac{1}{2}\,\varepsilon_{ij}\,\delta\sigma_{ij}. \tag{2.94}$$

However,

$$\delta\sigma_{ij} = C_{ij}\,\delta\varepsilon_{ij}. \tag{2.95}$$

Substituting Eq. 2.95 into Eq. 2.94 and using the expression for σ_{ij} gives

$$\delta(\tfrac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij}) = \sigma_{ij}\,\delta\varepsilon_{ij}. \tag{2.96}$$

The variation of Eq. 2.91, employing Eqs. 2.93 and 2.96 and integrating over the time interval involved, becomes

$$\delta \int_{t_o}^{t_1} P\, dt = \int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \int_{\zeta} (\sigma_1\,\delta\varepsilon_1 + \sigma_2\,\delta\varepsilon_2 + \sigma_n\,\delta\varepsilon_n + t_{12}\,\delta\gamma_{12}$$

$$+ t_{1n}\,\delta\gamma_{1n} + t_{2n}\,\delta\gamma_{2n})A_1A_2(1 - \zeta/R_1)(1 - \zeta/R_2)$$

$$d\alpha_1\, d\alpha_2\, d\zeta dt. \tag{2.97}$$

Now, substitution of the strain-displacement relations, given by Eqs. 2.82 and 2.83, and the stress resultant and couple definitions, given by Eqs. 2.86, into Eq. 2.97 yields

$$\delta \int_{t_o}^{t_1} P\,dt = - \int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \{ (\frac{\partial A_2 N_1}{\partial \alpha_1} + \frac{\partial A_1 N_{21}}{\partial \alpha_2} + N_{12} \frac{\partial A_1}{\partial \alpha_2} - N_2 \frac{\partial A_2}{\partial \alpha_1} - \frac{Q_1 A_1 A_2}{R_1}) \delta u_1$$

$$+ (\frac{\partial A_2 N_{12}}{\partial \alpha_1} + \frac{\partial A_1 N_2}{\partial \alpha_2} + N_{21} \frac{\partial A_2}{\partial \alpha_1} - N_1 \frac{\partial A_1}{\partial \alpha_2} - \frac{Q_2 A_1 A_2}{R_2}) \delta u_2$$

$$+ [\frac{\partial A_2 Q_1}{\partial \alpha_1} + \frac{\partial A_1 Q_2}{\partial \alpha_2} + A_1 A_2 (\frac{N_1}{R_1} + \frac{N_2}{R_2})] \delta w \qquad (2.98)$$

$$+ (\frac{\partial A_2 M_1}{\partial \alpha_1} + \frac{\partial A_1 M_{21}}{\partial \alpha_2} + M_{12} \frac{\partial A_1}{\partial \alpha_2} - M_2 \frac{\partial A_2}{\partial \alpha_1} - Q_1 A_1 A_2) \delta \beta_1$$

$$+ (\frac{\partial A_2 M_{12}}{\partial \alpha_1} + \frac{\partial A_1 M_2}{\partial \alpha_2} + M_{21} \frac{\partial A_2}{\partial \alpha_1} - M_1 \frac{\partial A_1}{\partial \alpha_2} - Q_2 A_1 A_2) \delta \beta_2$$

$$+ (\frac{\partial A_2 S_1}{\partial \alpha_1} + \frac{\partial A_1 S_2}{\partial \alpha_2} + A_1 A_2 (\frac{M_1}{R_1} + \frac{M_2}{R_2}) - A A_1 A_2) \delta w^1$$

$$+ (\frac{\partial A_2 T_1}{\partial \alpha_1} + \frac{\partial A_1 T_2}{\partial \alpha_2} + A_1 A_2 (\frac{P_1}{R_1} + \frac{P_2}{R_2}) - B A_1 A_2) \delta w^{11} \} d\alpha_1\, d\alpha_2\, dt$$

$$+ \int_{t_o}^{t_1} \int_{\alpha_1} (N_{21}\, \delta u_1 + N_2\, \delta u_2 + Q_2\, \delta w + M_{21}\, \delta \beta_1 + M_2\, \delta \beta_2$$

$$+ S_2\, \delta w^1 + T_2\, \delta w^{11})\, A_1 d\alpha_1\, dt$$

$$+ \int_{t_o}^{t_1} \int_{\alpha_2} (N_1\, \delta u_1 + N_{12}\, \delta u_2 + Q_1\, \delta w + M_1\, \delta \beta_1 + M_{12}\, \delta \beta_2$$

$$+ S_1\, \delta w^1 + T_1\, \delta w^{11})\, A_2\, d\alpha_2\, dt.$$

Examining the definition of kinetic energy, Eq. 2.92, the expression becomes, upon utilization of Eqs. 2.80 and 2.93 followed by integration over the thickness,

$$\int_{t_o}^{t_1} T\,dt = \frac{\rho}{2}\int_{t_o}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\Big\{ h\,\dot{u}_1^{\,2} + \frac{h^3}{12}\dot{\beta}_1^{\,2} + h\,\dot{u}_2^{\,2} + \frac{h^3}{12}\dot{\beta}_2^{\,2}$$

$$+ h\dot{w}^2 + \frac{h^3}{12}\dot{w}w^{11} + \frac{h^3}{12}\dot{w}^{1\,2} + \frac{h^5}{320}\dot{w}^{11\,2}$$

$$- \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(\frac{2h^3}{12}\dot{u}_1\dot{\beta}_1 + \frac{2h^3}{12}\dot{u}_2\dot{\beta}_2 + \frac{2h^3}{12}\dot{w}\dot{w}^1 + \frac{h^5}{80}\dot{w}\dot{w}^{11}\right)$$

$$+ \frac{1}{R_1R_2}\left(\frac{h^3}{12}\dot{u}_1^{\,2} + \frac{h^5}{80}\dot{\beta}_1^{\,2} + \frac{h^3}{12}\dot{u}_2^{\,2} + \frac{h^5}{80}\dot{\beta}_2^{\,2}\right. \tag{2.99}$$

$$\left. + \frac{h^3}{12}\dot{w}^2 + \frac{h^5}{80}\dot{w}\dot{w}^{11} + \frac{h^5}{80}\dot{w}^{1\,2} + \frac{h^7}{28(64)}\dot{w}^{11\,2}\right)\Big\}A_1A_2\,d\alpha_1\,d\alpha_2\,dt,$$

where ρ is assumed constant.

Consider the leading term of the integrand of Eq. 2.99 as an example of the manipulations involved in taking the variation of this equation. The procedure is as follows:

$$\delta\dot{u}_1^{\,2} = 2\,\dot{u}_1\,\delta\dot{u}_1.$$

But $\frac{\partial}{\partial t}(\dot{u}_1\,\delta u_1) = \ddot{u}_1\,\delta u_1 + \dot{u}_1\,\delta\dot{u}_1,$

which gives $\delta\dot{u}_1^{\,2} = 2\frac{\partial}{\partial t}(\dot{u}_1\,\delta u_1) - 2\ddot{u}_1\,\delta u_1.$

This result, when substituted into the integrand, gives

$$\delta\int_{t_o}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\dot{u}_1^{\,2} = 2\int_{t_o}^{t_1}\int_{\alpha_1}\int_{\alpha_2}\left[\frac{\partial}{\partial t}(\dot{u}_1\,\delta u_1) - \ddot{u}_1\,\delta u_1\right]A_1A_2\,d\alpha_1\,d\alpha_2\,dt. \tag{2.100}$$

Integrating the first term on the right hand side of Eq. 2.100 with respect to t will give the initial conditions for the deformation of the shell. If the variations in the displacements are assumed to vanish at the arbitrary times t_o and t_1, the initial conditions are set and this term is zero.

Applying this technique to the remainder of Eq. 2.99 yields the variation of this equation to be

$$
\begin{aligned}
\delta \int_{t_o}^{t_1} T\,dt = - \int_{t_o}^{t_1} \int_{\alpha_1} \int_{\alpha_2} \{ & \rho h \left[\left(1 + \frac{h^2}{12R_1R_2}\right) \ddot{u}_1 - \frac{h^2}{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{\beta}_1 \right] \delta u_1 \\
& + \rho h \left[\left(1 + \frac{h^2}{12R_1R_2}\right) \ddot{u}_2 - \frac{h^2}{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{\beta}_2 \right] \delta u_2 \\
& + \rho h \left[\left(1 + \frac{h^2}{12R_1R_2}\right) \ddot{w} - \frac{h^2}{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{w}^1 + \frac{h^2}{24}\left(1 + \frac{3h^2}{20R_1R_2}\right) \ddot{w}^{11} \right] \delta w \\
& + \frac{\rho h^3}{12} \left[-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{u}_1 + \left(1 + \frac{3h^2}{20R_1R_2}\right) \ddot{\beta}_1 \right] \delta \beta_1 \qquad (2.101) \\
& + \frac{\rho h^3}{12} \left[-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{u}_2 + \left(1 + \frac{3h^2}{20R_1R_2}\right) \ddot{\beta}_2 \right] \delta \beta_2 \\
& + \frac{\rho h^3}{12} \left[-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{w} + \left(1 + \frac{3h^2}{20R_1R_2}\right) \ddot{w}^1 - \frac{3h^2}{40}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{w}^{11} \right] \delta w^1 \\
& + \frac{\rho h^3}{24} \left[\left(1 + \frac{3h^2}{20R_1R_2}\right) \ddot{w} - \frac{3h^2}{20}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{w}^1 + \frac{3h^2}{40}\left(1 + \frac{5h^2}{28R_1R_2}\right) \ddot{w}^{11} \right] \delta w^{11} \\
& \} A_1 A_2 \, d\alpha_1 \, d\alpha_2 \, dt.
\end{aligned}
$$

Substitution of Eqs. 2.98 and 2.101 into the equation of Hamilton's principle, Eq. 2.90, yields an identity when the coefficients of the arbitrary variational displacements under the area integral are each zero.

These seven coefficients give the equations of motion for the shell. The two line integrals in Eq. 2.98 give the boundary conditions for the shell as their integrands must also satisfy the identity. However, neither these conditions nor the initial conditions are required for the analysis procedure to be employed.

The equations of motion are then given by

$$\frac{\partial A_2 N_1}{\partial \alpha_1} + \frac{\partial A_1 N_{21}}{\partial \alpha_2} + N_{12}\frac{\partial A_1}{\partial \alpha_2} - N_2 \frac{\partial A_2}{\partial \alpha_1} - Q_1 \frac{A_1 A_2}{R_1}$$

$$- A_1 A_2\, \rho h \left[\left(1 + \frac{h^2}{12 R_1 R_2}\right) \ddot{u}_1 - \frac{h^2}{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{\beta}_1 \right] = 0 ,$$

$$\frac{\partial A_2 N_{12}}{\partial \alpha_1} + \frac{\partial A_1 N_2}{\partial \alpha_2} + N_{21}\frac{\partial A_2}{\partial \alpha_1} - N_1 \frac{\partial A_1}{\partial \alpha_2} - Q_2 \frac{A_1 A_2}{R_2}$$

$$- A_1 A_2\, \rho h \left[\left(1 + \frac{h^2}{12 R_1 R_2}\right) \ddot{u}_2 - \frac{h^2}{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{\beta}_2 \right] = 0 ,$$

$$\frac{\partial A_2 Q_1}{\partial \alpha_1} + \frac{\partial A_1 Q_2}{\partial \alpha_2} + A_1 A_2 \left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right)$$

$$- A_1 A_2\, \rho h \left[\left(1 + \frac{h^2}{12 R_1 R_2}\right) \ddot{w} - \frac{h^2}{12}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{w}^{1} + \frac{h^2}{24}\left(1 + \frac{3h^2}{20 R_1 R_2}\right) \ddot{w}^{11} \right] = 0,$$

$$\frac{\partial A_2 M_1}{\partial \alpha_1} + \frac{\partial A_1 M_{21}}{\partial \alpha_2} + M_{12}\frac{\partial A_1}{\partial \alpha_2} - M_2 \frac{\partial A_2}{\partial \alpha_1} - Q_1 A_1 A_2$$

$$- A_1 A_2 \frac{\rho h^3}{12} \left[-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{u}_1 + \left(1 + \frac{3h^2}{20 R_1 R_2}\right) \ddot{\beta}_1 \right] = 0 ,$$

$$\frac{\partial A_2 M_{12}}{\partial \alpha_1} + \frac{\partial A_1 M_2}{\partial \alpha_2} + M_{21}\frac{\partial A_2}{\partial \alpha_1} - M_1 \frac{\partial A_1}{\partial \alpha_2} - Q_2 A_1 A_2 \qquad (2.102)$$

$$- A_1 A_2 \frac{\rho h^3}{12} \left[-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \ddot{u}_2 + \left(1 + \frac{3h^2}{20 R_1 R_2}\right) \ddot{\beta}_2 \right] = 0 ,$$

$$\frac{\partial A_2 S_1}{\partial \alpha_1} + \frac{\partial A_1 S_2}{\partial \alpha_2} + A_1 A_2 \left(\frac{M_1}{R_1} + \frac{M_2}{R_2}\right) - AA_1 A_2$$

$$- A_1 A_2 \frac{\rho h^3}{12} \left[-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{w} + \left(1 + \frac{3h^2}{20R_1R_2}\right)\ddot{w}^1 - \frac{3h^2}{40}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{w}^{11}\right] = 0 ,$$

$$\frac{\partial A_2 T_1}{\partial \alpha_1} + \frac{\partial A_1 T_2}{\partial \alpha_2} + A_1 A_2 \left(\frac{P_1}{R_1} + \frac{P_2}{R_2}\right) - BA_1 A_2$$

$$- A_1 A_2 \frac{\rho h^3}{24} \left[\left(1 + \frac{3h^2}{20R_1R_2}\right)\ddot{w} - \frac{3h^2}{20}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\ddot{w}^1 + \frac{3h^2}{40}\left(1 + \frac{5h^2}{28R_1R_2}\right)\ddot{w}^{11}\right] = 0.$$

The preceding equations of motion, Eqs. 2.102, in combination with the strain-displacement relations, Eqs. 2.82, and the stress resultant-strain relations, Eqs. 2.89, form the basis for a high order linear shell theory which accounts for the effects of transverse normal stress, for the effects of transverse shear strain, and for the effects of rotatory inertia. The theory outlined will be a closer approximation to the exact three dimensional elasticity theory than will the classical theories which ignore these effects with the exception of a few rotatory inertia terms.

2.4. Governing Equations for Shell of Revolution

In order to facilitate usage in subsequent chapters, it is advantageous to write the kinematical relations and the dynamical relations, or the equations of motion, in terms of the parameters applicable to a shell of revolution. The reference surface parameters employed will be those of Sec. 2.1.2.(f) where the parameters α_1 and α_2 are the angles ϕ and θ respectively.

The representation of the surface of revolution to be used will be that given by Eq. 2.65, the metric. In addition, set

$$R_1 = \gamma_\phi ,$$

$$R_2 = \gamma_\theta . \qquad (2.103)$$

Then, Eq. 2.66 becomes

$$\frac{d\gamma_o(\phi)}{d\phi} = -\gamma_\phi \cos\phi ,$$

and, $$\gamma_o(\phi) = -\gamma_\theta \sin\phi . \qquad (2.104)$$

The Gaussian and mean curvatures, from Eqs. 2.33, are

$$K = \frac{1}{\gamma_\theta \gamma_\phi} = \frac{\sin\phi}{\gamma_o(\phi)\gamma_\phi} , \; 2H = \gamma_\theta + \gamma_\phi ,$$

and set $$k = \frac{\cos\phi}{\gamma_o(\phi)} = -\frac{\gamma_o(\phi)^1}{\gamma_o(\phi)\gamma_\phi} , \qquad (2.105)$$

where k is the geodesic curvature of a circle of latitude and ^{1} denotes differentiation with respect to the parameter involved.

If the ϕ parameter curve is reparametrized employing arc length along the curve as the new parametrization, then Eq. 2.68 applies. It will also be assumed that the displacement components, given by Eq. 2.78, of a point in the shell space are independent of the polar angle, θ, giving rise to an axisymmetric situation since θ is the angle of rotation of the generating curve. Eqs. 2.80, with time dependency added, become

$$U_\phi\ (\phi(s),\ \zeta,\ t) = u_\phi\ (s,\ t) + \zeta\ \beta_\phi\ (s,\ t),$$

$$U_\theta\ (\phi(s),\ \zeta,\ t) = u_\theta\ (s,\ t) + \zeta\ \beta_\theta\ (s,\ t), \qquad (2.106)$$

$$W\ (\phi(s),\ \zeta,\ t) = w(s,\ t) + \zeta w^{1}(s,\ t) + \frac{\zeta^2}{2}\ w^{11}(s,\ t).$$

Using Eqs. 2.68, 2.103, 2.104 and 2.105, the strain-displacement equations, Eqs. 2.82 and 2.83, are obtained as follows:

$$\varepsilon_\phi = \frac{1}{1 - \zeta/\gamma_\phi}\ (\varepsilon_\phi{}^{o} + \zeta\varepsilon_\phi{}^{1} + \frac{\zeta^2}{2}\ \varepsilon_\phi{}^{11})\ ,$$

$$\varepsilon_\theta = \frac{1}{1 - \zeta/\gamma_\theta}\ (\varepsilon_\theta{}^{o} + \zeta\varepsilon_\theta{}^{1} + \frac{\zeta^2}{2}\ \varepsilon_\theta{}^{11})\ ,$$

$$\varepsilon_n = w^{1} + \zeta w^{11}\ , \qquad (2.107)$$

$$\gamma_{\phi\theta} = \frac{1}{1 - \zeta/\gamma_\phi}\ (\beta_\phi{}^{o} + \zeta\beta_\phi{}^{1}) + \frac{1}{1 - \zeta/\gamma_\theta}\ (\beta_\theta{}^{o} + \zeta\beta_\theta{}^{1})\ ,$$

$$\gamma_{\phi n} = \frac{1}{1 - \zeta/\gamma_\phi}\ (\mu_\phi{}^{o} + \zeta\mu_\phi{}^{1} + \frac{\zeta^2}{2}\ \mu_\phi{}^{11})\ ,$$

$$\gamma_{\theta n} = \frac{1}{1 - \zeta/\gamma_\theta}\ (\mu_\theta{}^{o} + \zeta\mu_\theta{}^{1} + \frac{\zeta^2}{2}\ \mu_\theta{}^{11})\ .$$

where

$$\varepsilon_\phi{}^{o} = \frac{\partial u_\phi}{\partial s} - \frac{w}{\gamma_\phi}\ , \qquad \varepsilon_\theta{}^{o} = -\ ku_\phi - \frac{w}{\gamma_\theta}\ ,$$

$$\varepsilon_\phi{}^{1} = \kappa_\phi - \frac{w^{1}}{\gamma_\phi}\ , \qquad \varepsilon_\theta{}^{1} = \kappa_\theta - \frac{w^{1}}{\gamma_\theta}\ ,$$

$$\varepsilon_\phi{}^{11} = -\ \frac{w^{11}}{\gamma_\phi}\ , \qquad \varepsilon_\theta{}^{11} = -\ \frac{w^{11}}{\gamma_\theta}\ ,$$

$$\beta_\phi{}^{o} = \frac{\partial u_\theta}{\partial s}\ , \qquad \beta_\theta{}^{o} = ku_\theta\ ,$$

$$\beta_\phi{}^{1} = \frac{\partial \beta_\theta}{\partial s}\ , \qquad \beta_\theta{}^{1} = k\beta_\theta\ , \qquad (2.108)$$

$$\mu_\phi{}^o = \frac{\partial w}{\partial s} + \frac{u_\phi}{\gamma_\phi} + \beta_\phi \ , \qquad \mu_\theta{}^o = \frac{u_\theta}{\gamma_\theta} + \beta_\theta \ ,$$

$$\mu_\phi{}^1 = \frac{\partial w^1}{\partial s} \ , \qquad \mu_\theta{}^1 = 0 \ ,$$

$$\mu_\phi{}^{11} = \frac{\partial w^{11}}{\partial s} \ , \qquad \mu_\theta{}^{11} = 0 \ ,$$

$$\kappa_\phi = \frac{\partial \beta_\phi}{\partial s} \ , \qquad \kappa_\theta = k\beta_\phi \ ,$$

$$\tau = \beta_\phi{}^1 + \beta_\theta{}^1 \ , \qquad \gamma_{\phi\theta}{}^o = \frac{\partial u_\theta}{\partial s} + ku_\theta \ .$$

Employing Eqs. 2.105 the stress resultant-displacement relations, Eqs. 2.89, become

(a)
$$N_\phi = C \{ (1 - \nu^2)\varepsilon_\phi{}^o + \nu(1 + \nu)(\varepsilon_\theta{}^o + w^1) + \frac{\alpha}{2}[(1 - \nu^2)\varepsilon_\phi{}^{11} + \nu(1 + \nu)(\varepsilon_\theta{}^{11} - 2\gamma_\phi Kw^{11})] + \alpha K(\gamma_\theta - \gamma_\phi)(1 - \nu^2)(\gamma_\theta K\varepsilon_\phi{}^o + \varepsilon_\phi{}^1 + \frac{9}{10}\alpha\gamma_\theta K\varepsilon_\phi{}^{11})\} \ ,$$

(b)
$$M_\phi = B \{ (1 - \nu^2)\varepsilon_\phi{}^1 + \nu(1 + \nu)(\varepsilon_\theta{}^1 + w^{11} - \gamma_\phi Kw^1) + K(\gamma_\theta - \gamma_\phi)(1 - \nu^2) [\varepsilon_\phi{}^o + \frac{9}{5}\alpha(\gamma_\theta K\varepsilon_\phi{}^1 + \frac{\varepsilon_\phi{}^{11}}{2})]\} \ ,$$

(c)
$$Q_\phi = \mu \{ \mu_\phi{}^o + \frac{\alpha}{2}\mu_\phi{}^{11} + \alpha K(\gamma_\theta - \gamma_\phi)(\gamma_\theta K\mu_\phi{}^o + \mu_\phi{}^1 + \frac{9}{10}\alpha\gamma_\theta K\mu_\phi{}^{11})\} \ ,$$

(d)
$$S_\phi = \mu\alpha\{ \mu_\phi{}^1 + K(\gamma_\theta - \gamma_\phi)[\mu_\phi{}^o + \frac{9}{5}\alpha(\gamma_\theta K\mu_\phi{}^1 + \frac{\mu_\phi{}^{11}}{2})]\} \ ,$$

(e)
$$T_\phi = \mu \frac{\alpha}{2} \{ \mu_\phi{}^o + \frac{9}{10}\alpha\mu_\phi{}^{11} + \frac{9}{5}\alpha K(\gamma_\theta - \gamma_\phi)(\gamma_\theta K\mu_\phi{}^o + \mu_\phi{}^1 + \frac{15}{14}\alpha\gamma_\theta K\mu_\phi{}^{11})\} \ ,$$

(f) $$P_\phi = B \{ (1 - \nu^2)\varepsilon_\phi{}^o + \nu(1 + \nu)(\varepsilon_\theta{}^o + w^1)$$

$$+ \frac{9}{10} \alpha [(1 - \nu^2)\varepsilon_\phi{}^{11} + \nu(1 + \nu)(\varepsilon_\theta{}^{11} - 2\gamma_\phi Kw^{11})]$$

$$+ \frac{9}{5} \alpha K (\gamma_\theta - \gamma_\phi)(1 - \nu^2)(\gamma_\theta K\varepsilon_\phi{}^o + \varepsilon_\phi{}^1 + \frac{15}{14} \alpha\gamma_\theta K\varepsilon_\phi{}^{11}) \} ,$$

(g) $$A = \nu [N_\phi + N_\theta - K(\gamma_\theta M_\phi + \gamma_\phi M_\theta)] \quad (2.109)$$

$$+ Eh [(1 + \alpha K)w^1 - 2\alpha KH w^{11}],$$

(h) $$B = \nu [M_\phi + M_\theta - 2K(\gamma_\theta P_\phi + \gamma_\phi P_\theta)]$$

$$+ E\alpha h [- 2 K H w^1 + (1 + \frac{9}{5} \alpha K)w^{11}],$$

where

$$C = \bar{E}h/(1 - \nu^2), \quad B = \bar{E}h^3/12(1 - \nu^2), \quad \mu = Gh, \quad \alpha = h^2/12. \quad (2.110)$$

Interchanging the subscripts ϕ and θ in the first six equations yields an additional six expressions. An interesting property of the above equations, which will become significant upon examination of the equations of motion, is that, with the exception of Q_θ, S_θ, and T_θ, all can be shown to be dependent upon the displacement components U_ϕ and W by substituting Eqs. 2.108. Q_θ, S_θ, and T_θ are dependent upon the U_θ component.

The remaining four resultant-displacement relations are obtained by again interchanging subscripts in the following two equations.

$$N_{\phi\theta} = \mu \{\gamma_{\phi\theta}{}^o + \alpha K(\gamma_\theta - \gamma_\phi)(\gamma_\theta K\beta_\phi{}^o + \beta_\phi{}^1) \} ,$$

$$M_{\phi\theta} = \mu\alpha \{ \tau + K(\gamma_\theta - \gamma_\phi)(\beta_\phi{}^o + \frac{9}{5} \alpha\gamma_\theta K\beta_\phi{}^1)\} , \quad (2.111)$$

Similarly, it can be shown that these expressions are dependent only upon the U_θ component of displacement.

A consideration of the equations of motion, Eqs. 2.102, noting the preceding discussion on dependencies, suggests a division of the equations into two distinct groups. Reducing the equations to those of a shell of revolution by means of Eqs. 2.68, 2.103, 2.104, and 2.105 and then substituting Eqs. 2.109 and 2.111 results in a grouping of five equations dependent upon U_ϕ and W and a grouping of two equations dependent upon U_θ. The equations of motion, grouped accordingly, then become

$$\text{(a)} \quad \frac{\partial N_\phi}{\partial s} + k(N_\theta - N_\phi) - \gamma_\theta KQ_\phi = \rho h[(1 + \alpha K)\ddot{u}_\phi - 2\alpha K H \ddot{\beta}_\phi] ,$$

$$\text{(b)} \quad \frac{\partial M_\phi}{\partial s} + k(M_\theta - M_\phi) - Q_\phi = \rho\alpha h [- 2K H \ddot{u}_\phi + (1 + \frac{9}{5} \alpha K)\ddot{\beta}_\phi], \qquad (2.112)$$

$$\text{(c)} \quad \frac{\partial Q_\phi}{\partial s} - kQ_\phi + K(\gamma_\theta N_\phi + \gamma_\phi N_\theta) = \rho h[(1 + \alpha K)\ddot{w} - 2\alpha K H \ddot{w}^1 + \frac{\alpha}{2} (1 + \frac{9}{5} \alpha K)\ddot{w}^{11}] ,$$

$$\text{(d)} \quad \frac{\partial S_\phi}{\partial s} - kS_\phi + K(\gamma_\theta M_\phi + \gamma_\phi M_\theta) - A = \rho\alpha h [- 2K H \ddot{w} + (1 + \frac{9}{5} \alpha K)\ddot{w}^1 - \frac{9}{5} \alpha KH \ddot{w}^{11}],$$

$$\text{(e)} \quad \frac{\partial T_\phi}{\partial s} - kT_\phi + K(\gamma_\theta P_\phi + \gamma_\phi P_\theta) - B = \frac{\rho\alpha h}{2}[(1 + \frac{9}{5} \alpha K)\ddot{w} - \frac{18}{5} \alpha K H\ddot{w}^1 + \frac{9}{10} \alpha(1 + \frac{15}{7} \alpha K)\ddot{w}^{11}]$$

and

$$\text{(a)} \quad \frac{\partial N_{\phi\theta}}{\partial s} - k(N_{\phi\theta} + N_{\theta\phi}) - \gamma_\phi KQ_\theta = \rho h[(1 + \alpha K)\ddot{u}_\theta - 2\alpha K H \ddot{\beta}_\theta], \qquad (2.113)$$

$$\text{(b)} \qquad \frac{\partial M_{\phi\theta}}{\partial s} - k(M_{\phi\theta} + M_{\theta\phi}) - Q_{\theta} = \rho\alpha h \quad - 2KH \quad \ddot{u}_{\theta} + (1 + \frac{9}{5}\alpha K)\ddot{\beta}_{\theta} \quad .$$

Eqs. 2.112 describe torsionless axisymmetric motions since none of the resultants or couples involved is related to the in-plane shear stress which causes twisting. It follows that Eqs. 2.113 govern purely torsional motions since the in-plane resultants and couples contained in these two equations are dependent only upon the in-plane shear stress.

2.5. Elastic Waves

In the context of this treatise, a wave will be defined as a moving curve, propagating on the shell reference surface at rest in its undeformed position, across which there may be a discontinuity in some field variable or in its derivatives. In this sense, a wave is a singular curve with respect to the discontinuous quantity. A discussion of the theory employed to investigate wave propagation under this definition can be found in the works of Hadamard [8], Truesdell and Toupin[19], and Thomas[18].

In a two dimensional space a propagating wave curve can be described by

$$\alpha_i = \alpha_i(q, t), \qquad i = 1, 2 \tag{2.114}$$

where q is a parameter along the curve and α_{i}, q exists and is not zero. The curve velocity in each coordinate direction with respect to the parameter q being held constant is given by $\alpha_{i,t}$. However, in the three dimensional space with coordinates α_{i}, q the wave curve sweeps out a surface, $\bar{\gamma}(\alpha_1, \alpha_2, t)$. Quantities which are continuous everywhere on the shell reference surface, except on the wave curve, are continuous everywhere in α_i, t space except on the surface $\bar{\gamma}(\alpha_1, \alpha_2, t)$. Hadamard's lemma states that any directional derivative in this surface of a function $\psi(\alpha_1, \alpha_2, t)$ can be written using the chain rule and one sided derivatives. If + and - are used to denote quantities on opposite sides of the surface then, with the aid of Eq. 2.114, the normal time derivative is defined to be

$$\frac{\delta\psi^{\pm}}{\delta t} = \frac{\partial\psi^{\pm}}{\partial t} + \frac{\partial\psi^{\pm}}{\partial\alpha_1}\frac{\partial\alpha_1}{\partial t} + \frac{\partial\psi^{\pm}}{\partial\alpha_2}\frac{\partial\alpha_2}{\partial t} , \qquad (2.115)$$

where the ± quantities are one sided derivatives. In actuality this derivative gives the rate of change with respect to time of the function involved as seen by an observer moving with the frame of reference of the wave curve and can be applied to quantities defined everywhere on the reference surface or to those defined only on the wave curve. Thus, if the function ψ is any field quantity or one of its derivatives, Eq. 2.115 applies across the wave curve.

If the field quantity $\psi(\alpha_1, \alpha_2, t)$ on the shell reference surface is independent of the α_2 parameter and $\psi^{(m)} = \partial^m \psi/\partial\alpha_1{}^m$, then employing Eq. 2.114 gives Eq. 2.115 as

$$\frac{\delta\psi^{(m)\pm}}{\delta t} = \frac{\partial\psi^{(m)\pm}}{\partial t} + \frac{\delta\alpha_1}{\delta t}\psi^{(m+1)\pm} , \qquad (2.116)$$

where, as was noted previously, $\delta\alpha_1/\delta t$ is the velocity of the wave curve in the direction of the α_1 coordinate curve.

This function can be written in terms of the parameters of the reference surface of a shell of revolution, becoming $\psi(s, t)$ when arc length is used as a reparametrization of the ϕ parameter curve. Eq. 2.116 then shows that, for functions associated with a shell of revolution and independent of the polar angle or θ parameter curve, the direction of wave propagation will be along the ϕ parameter curve reparametrized with respect to s. In other words, the wave curve propagates along the meridians of the shell with velocity independent of the polar angle since the third term of Eq. 2.115 vanishes. Restricting the field quantity, $\psi(s, t)$, to dependency only upon the ϕ parameter curve implies that the wave curve will be a θ parameter curve or a parallel curve.

The jump, or discontinuity, in $\psi^{(m)}$ across the wave curve is quantitatively defined to be

$$[\psi^{(m)}] = \psi^{(m)+} - \psi^{(m)-} , \tag{2.117}$$

where $\psi^{(m)+}$ and $\psi^{(m)-}$ are the limiting values of $\psi^{(m)}$ as the wave curve is approached from ahead and from behind the direction of wave propagation respectively. The wave curve is said to be of order m with respect to the quantity ψ if $[\psi^{(m)}] \neq 0$, $[\psi^{(k)}] = 0$, $k = 0, 1, \ldots.\ m - 1$.

From the definition of the quantities involved it can be shown that

$$\frac{\delta}{\delta t} [\psi^{(m)}] = [\frac{\delta\psi^{(m)}}{\delta t}] . \tag{2.118}$$

The kinematical conditions of compatibility, obtained by manipulating Eqs. 2.116 and 2.118, are given by

$$[\frac{\partial^n \psi^{(m)}}{\partial t^n}] = \sum_{k=o}^{n} \frac{n!\ (-G)^{n-k}}{(n-k)!\ k!} \frac{\delta^k}{\delta t^k} [\psi^{(n-k+m)}], \tag{2.119}$$

where $G = \delta\alpha_1/\delta t$, the velocity of propagation.

Eq. 2.119 holds for all n and any order wave provided G is constant, and in particular for n = 1 for first order waves and n = 1, 2 for second order waves if G is not constant.

Other restrictions on the possible discontinuities in the derivatives will result from a consideration of the equations of motion. These conditions may be termed dynamical conditions of compatibility since they depend on the dynamical equations which govern the behaviour of the shell space. When these dynamical compatibility equations are combined with the kinematical conditions of compatibility, as given by Eq. 2.119, important information concerning wave classification, speeds of wave propagation, wave growth and decay, and coupling between various types of waves, or discontinuities, is obtained. This theory will be applied to shells of revolution in the succeeding chapters.

CHAPTER III

WAVE PROPAGATION CONDITIONS

3.1. Analysis Procedure

The shells considered here will be shells of revolution which will be subject to the linear shell theory developed in Chapter II. It will also be assumed that all disturbances are axisymmetric and are small so that the approximations of linear elasticity are valid. Under these conditions the equations of motion given by Eqs. 2.112 and the kinematical conditions of compatibility given by Eq. 2.119 are consistent with the approximations being made.

The problem to be dealt with is the following. A curve, singular with respect to a quantity of given order as defined in Sec. 2.5, is assumed to be propagating across the initially stationary and undeformed reference surface of a shell of revolution. This singular curve divides the reference surface into two regions in each of which all quantities and their derivatives to whatever order required are continuous and all these quantities have unique finite limits as a point on the curve is approached from either side. The reference surface, and therefore the shell space, remains continuous and there is no discontinuity in the surface normal at the singular curve or in any other aspect of the surface geometry. The problem then is one of determining the propagation conditions, or speed of propagation, of this wave curve.

The equations of motion and the constitutive relations are valid at each point not on the wave curve. In addition, each term in these equations approaches a finite limit as a point on the wave curve is approached from one side since each can be written in terms of the displacement functions, given by Eqs. 2.106, on the reference surface. Therefore, by taking the jumps in the equations of motion, the jumps or discontinuities in the quantities occurring in these equations are related to one another. If the constitutive relations are used to relate all terms in the equations of motion to the displacements and the derivatives of the displacements, then taking the jumps in the equations of motion produces relationships involving the jumps in the derivatives of the displacements. The kinematical compatibility equations can then be applied to these jumps resulting in equations which will hopefully be soluble for the speed of propagation of the advancing wave curve relative to the material at rest in front of it.

As was noted in Sec. 2.4, in which the generalized constitutive equations and equations of motion for a shell were reduced to those applicable to a shell of revolution under axisymmetric conditions, these equations are separable into two uncoupled groupings, one dependent upon the U_ϕ and W components of the displacement and their derivatives and the other dependent upon the U_θ component and its derivatives. These two groups will be kept distinct in the analysis of the problem outlined above as it is to be expected that each will give rise to separate discontinuity conditions across a wave curve. As was also pointed out in Sec. 2.4, the first grouping above applies to torsionless axisymmetric motions, the second to purely torsional motions.

The problem to be investigated will be restricted to second order axisymmetric displacement waves (m = 2), henceforth to be termed acceleration waves, for which the dynamical jump conditions are not required. These conditions would have to be determined by the procedure employed in Thomas[18] involving the conservation of linear momentum across a wave curve. Thus, the restriction to acceleration waves will yield useful results without the complication of determining these dynamical jump conditions.

3.2. Longitudinal and Normal Waves

Waves which are associated with discontinuities in the derivatives of the U_ϕ component of the displacement will be called longitudinal waves since the direction of propagation is along a tangent to the ϕ parameter curve as is the direction of U_ϕ. Similarly, the term normal waves will be associated with waves dependent upon jumps in the derivatives of the normal displacement component, W. The discussion in this section will centre around the dynamical equations describing torsionless axi-symmetric motions, Eqs. 2.112.

3.2.1. Longitudinal Waves

The problem of second order axisymmetric displacement waves as governed by Eqs. 2.112 (a) and (b) will be considered in this section.

Employing the definition of a discontinuity, Eq. 2.117, and the kinematic compatibility relations, Eqs. 2.119, it can be shown that acceleration waves satisfy the following conditions

$$[u_\phi] = [\tilde{u}_\phi] = [\dot{u}_\phi] = [\beta_\phi] = [\tilde{\beta}_\phi] = [\dot{\beta}_\phi] = 0 ,$$

$$[w] = [\tilde{w}] = [\dot{w}] = [w^1] = [\tilde{w}^1] = [\dot{w}^1] = [w^{11}] = [\tilde{w}^{11}] = [\dot{w}^{11}] = 0,$$

$$[\tilde{u}_\phi] \neq 0, \quad [\tilde{\tilde{\beta}}_\phi] \neq 0, \quad [\tilde{\tilde{w}}] \neq 0, \quad [\tilde{\tilde{w}}^1] \neq 0, \quad [\tilde{\tilde{w}}^{11}] \neq 0, \tag{3.1}$$

$$\text{where} \quad [\tilde{\psi}] = [\frac{\partial \psi}{\partial s}], \quad [\tilde{\tilde{\psi}}] = [\frac{\partial^2 \psi}{\partial s^2}] \; . \tag{3.2}$$

All geometric functions on the deforming reference surface and their first derivatives are continuous across the wave curve. This follows from the fact that the displacements are continuous across the curve (there is no break in the reference surface) and their first derivatives are continuous also.

Taking the jump in Eq. 2.112 (a) across a wave curve, assuming that the material density remains constant as in the derivation of the equations of motion and noting the above discussion, gives

$$[\frac{\partial N_\phi}{\partial s}] + k([N_\theta] - [N_\phi]) - \gamma_\theta K[Q_\phi] = \rho h[(1 + \alpha K)[\ddot{u}_\phi] - 2\alpha KH [\ddot{\beta}_\phi]] \tag{3.3}$$

Substitution of the three stress resultant-displacement relations, Eqs. 2.109 (a), (b), and (c), and the appropriate strain-displacement definitions from Eqs. 2.108 into Eq. 3.3 will result in an equation involving the jumps in the derivatives of the displacement components when Eqs. 3.1 are utilized.

As an example consider the leading term in the jump of the equation of motion being examined. To carry out the necessary manipulations, first recall from Eq. 2.109 (a) that

$$N_\phi = C\{ (1-\nu^2)\varepsilon_\phi{}^o + \nu(1+\nu)(\varepsilon_\theta{}^o + w^1)$$
$$+ \frac{\alpha}{2}[(1-\nu^2)\varepsilon_\phi{}^{11} + \nu(1+\nu)(\varepsilon_\theta{}^{11} - 2\gamma_\phi Kw^{11})] \tag{3.4}$$
$$+ \alpha K(\gamma_\theta - \gamma_\phi)(1-\nu^2)(\gamma_\theta K\varepsilon_\phi{}^o + \varepsilon_\phi{}^1 + \frac{9}{10}\alpha\gamma_\theta K\varepsilon_\phi{}^{11})\} .$$

Differentiation of the above equation with respect to s and then taking its jump, recalling that all geometric quantities and their first partial derivatives with respect to s will be continuous for acceleration waves and that the shell material is elastically homogeneous yields

$$[\frac{\partial N_\phi}{\partial s}] = C\{ (1-\nu^2)[\tilde{\varepsilon}_\phi{}^o] + \nu(1+\nu)([\tilde{\varepsilon}_\theta{}^o] + [\tilde{w}^1])$$
$$+ \frac{\alpha}{2}[(1-\nu^2)[\tilde{\varepsilon}_\phi{}^{11}] + \nu(1+\nu)([\tilde{\varepsilon}_\theta{}^{11}] - 2\gamma_\phi K[\tilde{w}^{11}])] \tag{3.5}$$
$$+ \alpha K(\gamma_\theta - \gamma_\phi)(1-\nu^2)(\gamma_\theta K[\tilde{\varepsilon}_\phi{}^o] + [\tilde{\varepsilon}_\phi{}^1] + \frac{9}{10}\alpha\gamma_\theta K[\tilde{\varepsilon}_\phi{}^{11}])\},$$

where the convention set up in Eq. 3.2 applies.

From the strain-displacement definitions given by Eqs. 2.108

$$\varepsilon_\phi{}^o = \frac{\partial u_\phi}{\partial s} - \frac{w}{\gamma_\phi}, \qquad \varepsilon_\theta{}^o = -ku_\phi - \frac{w}{\gamma_\theta},$$
$$\varepsilon_\phi{}^1 = \frac{\partial \beta_\phi}{\partial s} - \frac{w^1}{\gamma_\phi}, \tag{3.6}$$
$$\varepsilon_\phi{}^{11} = -\frac{w^{11}}{\gamma_\phi}, \qquad \varepsilon_\theta{}^{11} = -\frac{w^{11}}{\gamma_\theta}.$$

Differentiating these relations with respect to s and then taking their jumps gives

$$[\tilde{\varepsilon}_\phi{}^{o}] = [\tilde{\tilde{u}}_\phi] - \frac{1}{\gamma_\phi}[\tilde{w}] \ , \qquad [\tilde{\varepsilon}_\theta{}^{o}] = -k[\tilde{u}_\phi] - \frac{1}{\gamma_\theta}[\tilde{w}] \ ,$$

$$[\tilde{\varepsilon}_\phi{}^{1}] = [\tilde{\tilde{\beta}}_\phi] - \frac{1}{\gamma_\phi}[\tilde{w}^{1}] \ , \tag{3.7}$$

$$[\tilde{\varepsilon}_\phi{}^{11}] = -\frac{1}{\gamma_\phi}[\tilde{w}^{11}] \ , \qquad [\tilde{\varepsilon}_\theta{}^{11}] = -\frac{1}{\gamma_\theta}[\tilde{w}^{11}]$$

However, in the case of acceleration waves Eqs. 3.1 apply and the jumps become

$$[\tilde{\varepsilon}_\phi{}^{o}] = [\tilde{\tilde{u}}_\phi] \ , \qquad [\tilde{\varepsilon}_\theta{}^{o}] = 0 \ ,$$

$$[\tilde{\varepsilon}_\phi{}^{1}] = [\tilde{\tilde{\beta}}_\phi] \ , \quad [\tilde{\varepsilon}_\phi{}^{11}] = 0, \qquad [\tilde{\varepsilon}_\theta{}^{11}] = 0 \ . \tag{3.8}$$

Substituting Eqs. 3.8 into Eq. 3.5 and again employing Eqs. 3.1 results in

$$[\frac{\partial N_\phi}{\partial s}] = C\,[\,(1-\nu^2)\,[\tilde{\tilde{u}}_\phi] + \alpha K(\gamma_\theta - \gamma_\phi)(1-\nu^2)(\gamma_\theta\, K[\tilde{\tilde{u}}_\phi] + [\tilde{\tilde{\beta}}_\phi])\,] \tag{3.9}$$

By proceeding in the same fashion with the remaining terms in Eq. 3.3 and then substituting the expressions obtained back into this equation, it can be shown that

$$C\,[(1-\nu^2)[\tilde{\tilde{u}}_\phi] + \alpha K(\gamma_\theta - \gamma_\phi)(1-\nu^2)(\gamma_\theta\, K[\tilde{\tilde{u}}_\phi] + [\tilde{\tilde{\beta}}_\phi])\,] =$$

$$\rho h\,[(1+\alpha K)[\ddot{u}_\phi] - 2\alpha KH\,[\ddot{\beta}_\phi]]. \tag{3.10}$$

The kinematic compatibility relations, Eqs. 2.119, yield for $m = 0$, $n = 2$ when applied to acceleration waves

$$[\frac{\partial^2 \psi}{\partial t^2}] = G^2\,[\tilde{\tilde{\psi}}]. \tag{3.11}$$

Substituting into Eq. 3.10 for the two expressions of this type and then collecting coefficients gives the final relationship between the jumps in the derivatives of the longitudinal displacement components as follows

$$[C(1 - \nu^2) + C\,\alpha\,\gamma_\theta\,K^2(1 - \nu^2)(\gamma_\theta - \gamma_\phi) - \rho h(1 + \alpha K)G^2]\,[\tilde{\tilde{u}}_\phi]$$

$$+ [\,C\,\alpha\,K(1 - \nu^2)(\gamma_\theta - \gamma_\phi) \quad + \quad 2\rho\alpha h\,K\,H\,G^2]\,[\tilde{\tilde{\beta}}_\phi] \;=\; 0. \qquad (3.12)$$

Taking the jump in Eq. 2.112 (b) across the wave curve results in

$$[\frac{\partial M_\phi}{\partial s}] + k\,([M_\theta] - [M_\phi]) - [Q_\phi] = \rho\alpha h[\,-\,2\,K\,H[\ddot{u}_\phi] + (1 + \frac{9}{5}\,\alpha K)\,[\ddot{\beta}_\phi]] \quad (3.13)$$

Employing the stress resultant-displacement relations, Eqs. 2.109 (b) and (c) alternating subscripts in (b) to obtain M_θ, the strain-displacement definitions, Eqs. 2.108, and the conditions set by Eqs. 3.1 gives the following results when the procedure outlined in the preceding example is used

$$[\frac{\partial M_\phi}{\partial s}] = \;B\,[(1 - \nu^2)[\tilde{\tilde{\beta}}_\phi] + K(\gamma_\theta - \gamma_\phi)(1 - \nu^2)([\tilde{\tilde{u}}_\phi] + \frac{9}{5}\,\alpha\gamma_\theta\,K\,[\tilde{\tilde{\beta}}_\phi])],$$

$$[M_\theta] \;=\; [M_\phi] \;=\; [Q_\phi] \;=\; 0. \qquad (3.14)$$

Substituting Eqs. 3.14 and the kinematic compatibility conditions imposed by Eq. 3.11 into Eq. 3.13 yields the propagation condition

$$[B\,K\,(1 - \nu^2)(\gamma_\theta - \gamma_\phi) \quad + \quad 2\rho\alpha h\,K\,H\,G^2]\,[\tilde{\tilde{u}}_\phi]$$

$$+ [B(1 - \nu^2) + \frac{9}{5}\,B\,\alpha\gamma_\theta K^2(1 - \nu^2)(\gamma_\theta - \gamma_\phi) - \rho\alpha h(1 + \frac{9}{5}\,\alpha K)G^2][\tilde{\tilde{\beta}}_\phi] = 0. \qquad (3.15)$$

In order for a longitudinal wave curve to propagate Eqs. 3.12 and 3.15, the propagation conditions relating the derivative jumps, must both be satisfied. This criteria results in a set of two simultaneous homogenous equations in two unknowns. Since the equations are linear in form, a nontrivial solution exists if, and only if, their coefficient determinant vanishes. In other words, this new condition must be satisfied if the wave curve is to propagate in the shell.

The determinant is given by

$$\begin{vmatrix} a_{11} - b_{11}G^2 & a_{12} + b_{12}G^2 \\ a_{21} + b_{21}G^2 & a_{22} - b_{22}G^2 \end{vmatrix} = 0 \ , \qquad (3.16)$$

where

$$\begin{aligned} a_{11} &= C(1 - \nu^2) + C\alpha\gamma_\theta\, K^2(1 - \nu^2)(\gamma_\theta - \gamma_\phi) \ , \\ b_{11} &= \rho h(1 + \alpha K) \ , \\ a_{12} &= C\alpha K(1 - \nu^2)(\gamma_\theta - \gamma_\phi) \ , \\ b_{12} &= b_{21} = 2\rho\alpha h\ K\ H \ , \qquad (3.17) \\ a_{21} &= BK(1 - \nu^2)(\gamma_\theta - \gamma_\phi) \ , \\ a_{22} &= B(1 - \nu^2) + \frac{9}{5} B\alpha\gamma_\theta\, K^2(1 - \nu^2)(\gamma_\theta - \gamma_\phi) \ , \\ b_{22} &= \rho\alpha h(1 + \frac{9}{5}\alpha K) \ . \end{aligned}$$

Since all the quantities in Eqs. 3.17 are specified by the geometry of the shell, the solution of the above determinant becomes an eigenvalue problem in G^2, the speed of propagation being G. Expanding the determinant results in a quadratic equation in G^2 given by

$$(b_{11}b_{22} - b_{12}b_{21})G^4 - (a_{11}b_{22} + a_{22}b_{11} + a_{12}b_{21} + a_{21}b_{12})G^2 + (a_{11}a_{22} - a_{12}a_{21}) = 0 \quad . \tag{3.18}$$

Solving this equation for G^2, that is the real roots, then gives the speed or speeds with which a longitudinal wave curve will propagate. The solution for the general shell of revolution involved here would necessitate a prohibitive amount of algebra. For this reason the equation will be left as such and solutions will be given in a later chapter for specific shells of revolution.

The classification "longitudinal wave" arises from the fact that the propagation conditions, Eqs. 3.12 and 3.15, involve only the relationships between the derivative jumps of the components of the longitudinal displacement. The jumps in the derivatives of these two components each represent a wave curve.

Physically, the wave curve associated with the jump in $\tilde{\tilde{u}}_\phi$ can be interpreted as a longitudinal strain acceleration wave as $\tilde{u}_\phi$ is the longitudinal strain of the shell reference surface. Similarly, since β_ϕ represents a rotation of the surface normal in the longitudinal direction, $\tilde{\beta}_\phi$ is the bending of the shell in this direction and the discontinuity in $\tilde{\tilde{\beta}}_\phi$ can be thought of as a longitudinal bending acceleration wave. These two wave subclassifications propagate with the same velocity, as shown, and are coupled by the propagation conditions. It should be noted that setting one of the jumps equal to zero does not imply that the other also vanishes since the coupling depends upon the geometry of the shell under consideration.

As the coefficients of Eq. 3.18 depend upon the geometry of the shell involved, this implies that the speed of propagation of a longitudinal wave in this generalized shell of revolution depends upon the geometry of the reference surface at each point and will vary as these geometric quantities vary over the surface.

3.2.2. Normal Waves

The propagation of axisymmetric acceleration waves will be considered within the framework of Eqs. 2.112 (c), (d), and (e) in this section. The discontinuity conditions for acceleration waves as given by Eqs. 3.1 are still applicable here as is the discussion in Sec. 3.2.1. of geometric functions, and their derivatives, on the reference surface.

The jump in Eq. 2.112 (c) across the wave curve is given by

$$[\frac{\partial Q_\phi}{\partial s}] - k[Q_\phi] + K(\gamma_\theta\, [N_\phi] + \gamma_\phi\, [N_\theta]\,) = \rho h[(1 + \alpha K)\, [\ddot{w}] - 2\alpha K\, H\, [\ddot{w}^1] + \frac{\alpha}{2}\,(1 + \frac{9}{5}\,\alpha K)\, [\ddot{w}^{11}]] \qquad (3.19)$$

Substituting the strain-displacement definitions, Eqs. 2.108, into the stress resultant-displacement relations, Eqs. 2.109 (a) and (c) with the subscripts interchanged in (a) to give N_θ and (c) differentiated with respect to s to give $\partial Q_\phi/\partial s$, and then taking the jumps in these quantities across an acceleration wave, for which Eqs. 3.1 come into effect, gives

$$[\frac{\partial Q_\phi}{\partial s}] = \mu \; [[\tilde{w}] + \frac{\alpha}{2} [\tilde{\tilde{w}}^{11}] + \alpha K(\gamma_\theta - \gamma_\phi)(\gamma_\theta K \; [\tilde{w}] + [\tilde{\tilde{w}}^1]$$

$$+ \frac{9}{10} \alpha\gamma_\theta \; K \; [\tilde{\tilde{w}}^{11}])] \; , \qquad (3.20)$$

$$[Q_\phi] = [N_\phi] = [N_\theta] = 0.$$

The kinematic compatibility conditions, Eqs. 3.11, give

$$[\ddot{w}] = G^2[\tilde{\tilde{w}}] \; , \quad [\ddot{w}^1] = G^2[\tilde{\tilde{w}}^1] \; , \; [\ddot{w}^{11}] = G^2[\tilde{\tilde{w}}^{11}] \; . \qquad (3.21)$$

Substituting Eqs. 3.20 and 3.21 into Eq. 3.19 and then collecting terms yields the propagation condition

$$[\mu + \mu\alpha\gamma_\theta \; K^2(\gamma_\theta - \gamma_\phi) - \rho h(1 + \alpha K)G^2] \; [\tilde{\tilde{w}}]$$

$$+[\mu\alpha \; K(\gamma_\theta - \gamma_\phi) + 2\rho\alpha h \; K \; H \; G^2] \; [\tilde{\tilde{w}}^1] \qquad (3.22)$$

$$+[\mu\frac{\alpha}{2} + \frac{9}{10}\mu\alpha^2\gamma_\theta \; K^2(\gamma_\theta - \gamma_\phi) - \rho\frac{\alpha}{2} h \; (1 + \frac{9}{5} \alpha K)G^2] \; [\tilde{\tilde{w}}^{11}] = 0 \; .$$

Employing the same procedure with Eq. 2.112 (d) and the applicable stress resultant-displacement relations gives the propagation condition

$$[\mu\alpha K(\gamma_\theta - \gamma_\phi) + 2\rho\alpha h \; K \; H \; G^2] \; [\tilde{\tilde{w}}] \qquad (3.23)$$

$$+[\mu\alpha + \frac{9}{5} \mu\alpha^2\gamma_\theta \; K^2(\gamma_\theta - \gamma_\phi) - \rho\alpha h \; (1 + \frac{9}{5} \alpha K)G^2] \; [\tilde{\tilde{w}}^1]$$

$$+ [\frac{9}{10}\mu\alpha^2 \; K \; (\gamma_\theta - \gamma_\phi) + \frac{9}{5} \rho\alpha^2 h \; K \; H \; G^2] \; [\tilde{\tilde{w}}^{11}] = 0 \; .$$

Again applying the same procedure to Eq. 2.112 (e) results in the third propagation condition

$$[\mu\frac{\alpha}{2} + \frac{9}{10}\mu\alpha^2\gamma_\theta \; K^2(\gamma_\theta - \gamma_\phi) - \rho\frac{\alpha}{2} h \; (1 + \frac{9}{5} \alpha K)G^2] \; [\tilde{\tilde{w}}]$$

$$+ \left[\frac{9}{10} \mu \alpha^2 K(\gamma_\theta - \gamma_\phi) + \frac{9}{5} \rho \alpha^2 h K H G^2 \right] [\tilde{\tilde{w}}^1] \quad (3.24)$$

$$+ \left[\frac{9}{20} \mu \alpha^2 + \frac{27}{28} \mu \alpha^3 \gamma_\theta K^2(\gamma_\theta - \gamma_\phi) - \frac{9}{20} \rho \alpha^2 h (1 + \frac{15}{7} \alpha K) G^2 \right] [\tilde{\tilde{w}}^{11}] = 0 .$$

The three propagation conditions relating the jumps in the derivatives of the normal displacement components, Eqs. 3.22, 3.23, and 3.24, must then be met in order to have a normal wave propagating in the shell. This gives rise to a system of three linear homogenous equations in three unknowns. As in the previous section the result is an eigenvalue problem in G^2 since the coefficient determinant of the system must vanish in order to have nontrivial solutions.

The determinant can be set up as follows

$$\begin{vmatrix} a_{11} - b_{11}G^2 & a_{12} + b_{12}G^2 & a_{13} - b_{13}G^2 \\ a_{21} + b_{21}G^2 & a_{22} - b_{22}G^2 & a_{23} + b_{23}G^2 \\ a_{31} - b_{31}G^2 & a_{32} + b_{32}G^2 & a_{33} - b_{33}G^2 \end{vmatrix} = 0 \quad (3.25)$$

where

$$a_{11} = \mu + \mu \alpha \gamma_\theta K^2(\gamma_\theta - \gamma_\phi) ,$$

$$b_{11} = \rho h (1 + \alpha K) ,$$

$$a_{12} = \mu \alpha K(\gamma_\theta - \gamma_\phi) ,$$

$$b_{12} = b_{21} = 2\rho\alpha h K H ,$$

$$a_{13} = \mu \frac{\alpha}{2} + \frac{9}{10} \mu \alpha^2 \gamma_\theta K^2(\gamma_\theta - \gamma_\phi) ,$$

$$b_{13} = b_{31} = \rho \frac{\alpha}{2} h (1 + \frac{9}{5} \alpha K) ,$$

$$
\begin{aligned}
a_{21} &= \mu\alpha K(\gamma_\theta - \gamma_\phi) ,\\
a_{22} &= \mu\alpha + \frac{9}{5}\mu\alpha^2\gamma_\theta K^2(\gamma_\theta - \gamma_\phi),\\
b_{22} &= \rho\alpha h(1 + \frac{9}{5}\alpha K) ,\\
a_{23} &= \frac{9}{10}\mu\alpha^2 K(\gamma_\theta - \gamma_\phi) ,\\
b_{23} &= \frac{9}{5}\rho\alpha^2 h\, K\, H ,\\
a_{31} &= \mu\alpha/2 + \frac{9}{10}\mu\alpha^2\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) ,\\
a_{32} &= \frac{9}{10}\mu\alpha^2 K(\gamma_\theta - \gamma_\phi) ,\\
b_{32} &= \frac{9}{5}\rho\alpha^2 h\, K\, H ,\\
a_{33} &= \frac{9}{20}\mu\alpha^2 + \frac{27}{28}\mu\alpha^3\gamma_\theta K^2(\gamma_\theta - \gamma_\phi),\\
b_{33} &= \frac{9}{20}\rho\alpha^2 h\,(1 + \frac{15}{7}\alpha K) .
\end{aligned}
\qquad (3.26)
$$

This determinant will yield a cubic equation in G^2 upon expansion, the real roots of which will be the propagation velocities of a normal wave curve in a shell. This cubic can be solved by computer for specific shell geometries(19), the algebra involved in the general case again being prohibitive.

The general classification "normal wave" is applied to all three derivative jumps in the three propagation conditions since they are all components of the normal displacement at a point in the shell space. Each of these derivative jumps is itself a wave curve. All three will propagate with the same velocity. This velocity, G^2, is a function of the shell's reference surface geometry as the coefficients of the cubic equation are real-valued functions of this geometry.

These three normal displacement waves can be further subclassified if they are examined in a purely physical sense. $\tilde{w}$ represents a rotation or shearing strain in the normal direction allowing an interpretation of the jump in its second derivative, $[\tilde{\tilde{w}}]$, as a normal shear strain acceleration wave. The quantity w^1 is the strain at a point on the reference surface in the normal direction. The jump $[\tilde{\tilde{w}}^1]$ can then be thought of as a normal strain acceleration wave. Difficulties are encountered with the third component of the normal displacement, w^{11}, since its physical interpretation is not clear. w^{11} is the change in the strain in the normal direction and, as such, its derivative jump, $[\tilde{\tilde{w}}^{11}]$, will be taken as an acceleration wave representing this quantity.

As can be seen from the propagation conditions, one of these normal waves may propagate alone or it may be coupled to waves of the other two types depending upon the geometry of the shell being considered.

3.3. Transverse Waves

The problem of second order axisymmetric displacement waves will be considered in relation to the equations describing purely torsional motions, Eqs. 2.113, in this section. The term transverse will be applied to motions perpendicular to the direction of propagation of the wave curve and tangential to the reference surface. Since the direction of wave propagation in a shell of revolution is in the direction of the ϕ-parameter curves, a transverse motion will be in the direction of the θ-parameter curves. Acceleration waves governed by these transverse motion equations satisfy the following conditions as can be seen from the definition of a second order discontinuity.

$$[u_\theta] = [\tilde{u}_\theta] = [\beta_\theta] = [\tilde{\beta}_\theta] = 0 \,,$$
$$[\tilde{\tilde{u}}_\theta] \neq 0 \,, \qquad [\tilde{\tilde{\beta}}_\theta] \neq 0 \,. \tag{3.27}$$

where the definition, Eq. 3.2, is employed.

The kinematic compatibility relations, Eqs. 2.119, give for $m = 0$, $n = 2$,

$$[\ddot{u}_\theta] = G^2 [\tilde{\tilde{u}}_\theta] \,, \quad [\ddot{\beta}_\theta] = G^2 [\ddot{\beta}_\theta] \,. \tag{3.28}$$

The discussion in Sec. 3.2.1. on the continuity of the geometric quantities on the reference surface and that of their first derivatives across an acceleration wave curve is also valid here. The jump in Eq. 2.113 (a) across a wave curve is then given by

$$\left[\frac{\partial N_{\phi\theta}}{\partial s}\right] - k\,([N_{\phi\theta}] + [N_{\theta\phi}]) - \gamma_\phi K[Q_\theta] = \rho h[(1 + \alpha K)[\ddot{u}_\theta] - 2\alpha K\, H\, [\ddot{\beta}_\theta]] . \tag{3.29}$$

The stress resultant-displacement equations, Eq. 2.109 (c) with θ substituted for the ϕ subscript and Eq. 2.111 (a) differentiated to give $\partial N_{\phi\theta}/\partial s$ and subscripts interchanged to give $N_{\theta\phi}$, are written in terms of the reference surface displacements using the strain-displacement definitions, Eqs. 2.108. Taking the jumps in these quantities and then applying the conditions for an acceleration wave, given by Eqs. 3.27, yields

$$\left[\frac{\partial N_{\phi\theta}}{\partial s}\right] = \mu[[\tilde{\tilde{u}}_\theta] + \alpha K(\gamma_\theta - \gamma_\phi)(\gamma_\theta K[\tilde{\tilde{u}}_\theta] + [\tilde{\tilde{\beta}}_\theta])] \,,$$
$$[N_{\phi\theta}] = [N_{\theta\phi}] = [Q_\theta] = 0. \tag{3.30}$$

Substituting these relations into Eq. 3.29 and using the kinematic compatibility relations, Eq. 3.28, gives the propagation condition

$$[\mu + \mu\alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) - \rho h(1 + \alpha K)G^2] [\tilde{\tilde{u}}_\theta]$$

$$+ [\mu\alpha K(\gamma_\theta - \gamma_\phi) + 2\rho\alpha h K H G^2][\tilde{\tilde{\beta}}_\theta] = 0 . \tag{3.31}$$

Taking the jump in Eq. 2.113 (b) and proceeding in the same fashion with Eqs. 2.109 (c) and 2.111 (b) yields the second propagation condition as

$$[\mu\alpha K(\gamma_\theta - \gamma_\phi) + 2\rho\alpha h K H G^2] [\tilde{\tilde{u}}_\theta]$$

$$+ [\mu\alpha + \frac{9}{5}\mu\alpha^2\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) - \rho\alpha h(1 + \frac{9}{5} \alpha K)G^2] [\tilde{\tilde{\beta}}_\theta] = 0 . \tag{3.32}$$

A transverse wave will propagate if, and only if, these two propagation conditions are satisfied. As in the previous section, Sec. 3.2, these two linear homogenous equations in two unknowns lead to an eigenvalue problem in G^2.

The coefficient determinant is given by

$$\begin{bmatrix} a_{11} - b_{11}G^2 & a_{12} + b_{12}G^2 \\ a_{21} + b_{21}G^2 & a_{22} - b_{22}G^2 \end{bmatrix} = 0 . \tag{3.33}$$

where

$$\begin{aligned} a_{11} &= \mu + \mu\alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) , \\ b_{11} &= \rho h(1 + \alpha K) , \\ a_{12} &= a_{21} = \mu\alpha K(\gamma_\theta - \gamma_\phi) , \\ b_{12} &= b_{21} = 2\rho\alpha h K H , \\ a_{22} &= \mu\alpha + \frac{9}{5}\mu\alpha^2\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) , \\ b_{22} &= \rho\alpha h(1 + \frac{9}{5} \alpha K). \end{aligned} \tag{3.34}$$

Expanding the determinant gives Eq. 3.18 where the coefficients are given by Eqs. 3.34 above. The real roots of this quadratic will be the velocities with which a transverse wave can propagate in a shell of revolution. As in Sec. 3.2 this velocity will be a function of the shell geometry for the general case being considered.

Since the derivative jumps in the propagation conditions involve only displacement components in the θ direction, the classification "transverse wave" is employed. Both waves represented by the two derivative jumps will propagate with the same velocity, that given by the eigenvalue problem. These two derivative jumps can be further subclassified. The quantity $\tilde{u}_\theta$ represents the in-plane shear strain on the shell reference surface. The jump in its derivative, $[\tilde{\tilde{u}}_\theta]$, corresponds to a transverse shear strain acceleration wave in this sense. β_θ represents the rotation of a normal to the reference surface in the direction of the θ-parameter curves or parallel curves. Its first derivative, $\tilde{\beta}_\theta$, can then be thought of as a twisting in the longitudinal direction and the discontinuity in its second derivative, $[\tilde{\tilde{\beta}}_\theta]$, corresponds to a transverse twisting acceleration wave. As in the cases of longitudinal and normal waves the propagation conditions govern whether or not one wave is capable of propagating independent of the other. As was noted previously, this is dependent upon the geometry of the shell.

CHAPTER IV

GROWTH - DECAY EQUATIONS

4.1. Analysis Procedure

As in Chapter III the shells investigated in this chapter will be shells of revolution subject to the restrictions of the small displacement shell theory developed in Chapter II. Again axisymmetric displacements only will be allowed. Therefore, the equations of motion, the constitutive equations, and the kinematic compatibility equations of Secs. 2.4 and 2.5 are also applicable here as they were derived in accordance with these conditions.

In Chapter III the problem of a second order singular curve propagating across a shell reference surface was analyzed so as to determine the types of displacement waves which may propagate, information concerning coupling between the various wave modes possible, and the velocities with which these modes propagate. As was seen, the propagation conditions separated into three distinct systems involving seven possible wave modes, each system yielding velocities of propagation from the solution of an eigenvalue problem. Additional information dealing with the coupling between wave modes and the growth or decay in the discontinuities in the displacement derivatives across the wave curve can be obtained by finding the equations which govern the change in magnitude of the discontinuities with respect to time, that is, the total time derivatives of the jumps.

As was noted, the equations of motion and the constitutive relations are valid at each point in the shell space not on the wave curve and each quantity in these equations approaches a finite limit on each side of the curve. Differentiating the equations of motion with respect to time and then taking the jumps in the resulting equations across a wave curve yields relations involving the discontinuities in the partial time derivatives of the stress resultant quantities in the equations of motion. Substituting the constitutive relations in order to relate all the stress resultants to the displacements and taking the indicated jumps gives relationships governing the jumps in the partial time derivatives of the displacements and their arc length derivatives. Restricting the possible discontinuities to those of acceleration waves and employing the kinematic compatibility equations results in the required growth-decay equations. Making use of the wave velocity information and the propagation conditions contained in Chapter III gives the final growth-decay equations. The equations, in this form, give the desired information concerning the coupling of acceleration waves to higher order waves and the governing differential equations of wave strength variation.

Again, the analysis of the problem will be split into two distinct categories as dictated by the separation of the equations of motion. It is to be expected that the equations describing torsionless axisymmetric motions and those describing purely torsional motions will result in two systems of growth-decay equations which are independent of each other. Only acceleration waves will be considered in order to remain consistent with and to make use of the results of Chapter III.

The kinematic compatibility relations, given by Eq. 2.119, are valid for second order waves with $n = 3$ only if the propagation velocity, G, is a constant. This fact must be accounted for in any applications of the growth-decay equations as the eigenvalue problems of the preceding chapter yielded wave speeds, in the generalized case, which were dependent upon real-valued geometric quantities on the shell reference surface. This would imply that these speeds are nonconstant for the generalized shell. As a result the growth-decay equations of this chapter will be applicable to shells of revolution whose propagation velocities are constant for the wave modes involved.

4.2. Longitudinal and Normal Waves

The growth-decay characteristics of second order axisymmetric displacement waves will be discussed in this section within the framework of the equations of motion describing torsionless axisymmetric motions, Eqs. 2.112. These equations gave rise to propagation conditions involving only the longitudinal and normal components of the displacement, hence the classifications. The longitudinal and normal acceleration waves which will be encountered in the ensuing analysis satisfy the conditions set down in Eqs. 3.1 which were derived from the kinematic compatibility relations and the definition of a second order jump. The physical interpretations attributed to each longitudinal and normal wave mode in Secs. 3.2.1 and 3.2.2 are also carried over to this chapter.

Before proceeding any further it is appropriate to write down the kinematic compatibility relations which will be employed. Second order and third order relations will be required applicable to acceleration waves. Eq. 2.119 gives for $m = 1$, $n = 1$

$$[\dot{\tilde{u}}_\phi] = -G[\tilde{\tilde{u}}_\phi] \;, \quad [\dot{\tilde{\beta}}_\phi] = -G[\tilde{\tilde{\beta}}_\phi] \;, \quad [\dot{\tilde{w}}] = -G[\tilde{\tilde{w}}] \;,$$

$$[\dot{\tilde{w}}^1] = -G[\tilde{\tilde{w}}^1] \;, \quad [\dot{\tilde{w}}^{11}] = -G[\tilde{\tilde{w}}^{11}] \;. \tag{4.1}$$

For m = 2, n = 1

$$[\dot{\tilde{\tilde{u}}}_\phi] = -G[\tilde{\tilde{\tilde{u}}}_\phi] + \frac{\delta}{\delta t}[\tilde{\tilde{u}}_\phi] \;, \quad [\dot{\tilde{\tilde{\beta}}}_\phi] = -G[\tilde{\tilde{\tilde{\beta}}}_\phi] + \frac{\delta}{\delta t}[\tilde{\tilde{\beta}}_\phi],$$

$$[\dot{\tilde{\tilde{w}}}] = -G[\tilde{\tilde{\tilde{w}}}] + \frac{\delta}{\delta t}[\tilde{\tilde{w}}] \;, \quad [\dot{\tilde{\tilde{w}}}^1] = -G[\tilde{\tilde{\tilde{w}}}^1] + \frac{\delta}{\delta t}[\tilde{\tilde{w}}^1] \;, \tag{4.2}$$

$$[\dot{\tilde{\tilde{w}}}^{11}] = -G[\tilde{\tilde{\tilde{w}}}^{11}] + \frac{\delta}{\delta t}[\tilde{\tilde{w}}^{11}] \;.$$

For m = 0, n = 3

$$[\dddot{u}_\phi] = -G^3[\tilde{\tilde{\tilde{u}}}_\phi] + 3G^2 \frac{\delta}{\delta t}[\tilde{\tilde{u}}_\phi] \;, \quad [\dddot{\beta}_\phi] = -G^3[\tilde{\tilde{\tilde{\beta}}}_\phi] + 3G^2 \frac{\delta}{\delta t}[\tilde{\tilde{\beta}}_\phi] \;,$$

$$[\dddot{w}] = -G^3[\tilde{\tilde{\tilde{w}}}] + 3G^2 \frac{\delta}{\delta t}[\tilde{\tilde{w}}] \;, \quad [\dddot{w}^1] = -G^3[\tilde{\tilde{\tilde{w}}}^1] + 3G^2 \frac{\delta}{\delta t}[\tilde{\tilde{w}}^1] \;,$$

$$[\dddot{w}^{11}] = -G^3[\tilde{\tilde{\tilde{w}}}^{11}] + 3G^2 \frac{\delta}{\delta t}[\tilde{\tilde{w}}^{11}] \;, \tag{4.3}$$

where G is restricted to a constant value.

The first growth-decay equation is derived by applying the procedure outlined in Sec. 4.1 to the first equation of motion, Eq. 2.112 (a). A consequence of the small displacement theory that is being utilized is that all geometric quantities and their derivatives defined on the shell reference surface are those of the undeformed surface. Therefore, these quantities are functions of position only and are independent of time. Differentiation of Eq. 2.112 (a) with respect to time, making use of the preceding information, gives

$$\frac{\partial^2 N_\phi}{\partial t \partial s} + k\left(\frac{\partial N_\theta}{\partial t} - \frac{\partial N_\phi}{\partial t}\right) - \gamma_\theta K \frac{\partial Q_\phi}{\partial t} = \rho h\left[(1 + \alpha K)\frac{\partial^3 u_\phi}{\partial t^3} - 2\alpha K\, H \frac{\partial^3 \beta_\phi}{\partial t^3}\right] . \quad (4.4)$$

Taking the jump in this equation across a wave curve, recalling that for acceleration waves all geometric quantities on the reference surface are continuous across the wave curve since there can be no break in the shell, gives

$$\left[\frac{\partial \dot{N}_\phi}{\partial s}\right] + k\left([\dot{N}_\theta] - [\dot{N}_\phi]\right) - \gamma_\theta K[\dot{Q}_\phi] = \rho h\left[(1 + \alpha K)[\dddot{u}_\phi] - 2\alpha K\, H\,[\dddot{\beta}_\phi]\right], \quad (4.5)$$

where the dot denotes partial time differentiation of the quantity.

Employing the appropriate constitutive relations as given by the stress resultant-displacement equations and the displacement definitions, Eqs. 2.109 and 2.108 respectively, then makes it possible to express Eq. 4.5 in terms of the jumps in the displacement derivatives alone. The easiest method of doing this is to proceed term by term in Eq. 4.5 making the required substitutions, taking the indicated jumps, applying the acceleration wave conditions and kinematic compatibility conditions, and then substituting back into Eq. 4.5.

Differentiating the expression for N_ϕ given by Eq. 2.109 (a) first with respect to arc length and then with respect to time and taking its jump yields

$$[\frac{\partial \dot{N}_\phi}{\partial s}] = C \{ (1 - \nu^2) [\frac{\partial \dot{\varepsilon}_\phi^o}{\partial s}] + \nu(1 + \nu) ([\frac{\partial \dot{\varepsilon}_\theta^o}{\partial s}] + [\dot{\tilde{w}}^1])$$

$$+ \frac{\alpha}{2} [(1 - \nu^2) [\frac{\partial \dot{\varepsilon}_\phi^{11}}{\partial s}] + \nu(1 + \nu)([\frac{\partial \dot{\varepsilon}_\theta^{11}}{\partial s}] - 2\gamma_\phi K[\dot{\tilde{w}}^{11}]) \quad (4.6)$$

$$+ \alpha K(\gamma_\theta - \gamma_\phi)(1 - \nu^2)(\gamma_\theta K[\frac{\partial \dot{\varepsilon}_\phi^o}{\partial s}] + [\frac{\partial \dot{\varepsilon}_\phi^1}{\partial s}] + \frac{9}{10} \alpha\gamma_\theta K[\frac{\partial \dot{\varepsilon}_\phi^{11}}{\partial s}] \}.$$

Performing the operations indicated on the displacement definition terms in the right hand side of the above equation with the aid of Eqs. 2.108, the acceleration wave conditions given by Eqs. 3.1, and the kinematic compatibility conditions given by Eqs. 4.1 and 4.2 results in

$$[\frac{\partial \dot{\varepsilon}_\phi^o}{\partial s}] = [\dot{\tilde{u}}_\phi] - \gamma_\theta K[\dot{w}] + \frac{w}{\gamma_\phi^2} [\dot{\tilde{\gamma}}_\phi] = -G[\tilde{\tilde{u}}_\phi] + \frac{\delta}{\delta t} [\tilde{u}_\phi] + \gamma_\theta K\, G[\tilde{\tilde{w}}] ,$$

$$[\frac{\partial \dot{\varepsilon}_\theta^o}{\partial s}] = k\, G[\tilde{u}_\phi] + \gamma_\phi K\, G[\tilde{w}] , \qquad [\dot{\tilde{w}}^1] = - G[\tilde{\tilde{w}}^1] ,$$

$$[\frac{\partial \dot{\varepsilon}_\phi^{11}}{\partial s}] = \gamma_\theta K\, G[\tilde{\tilde{w}}^{11}] , \qquad [\frac{\partial \dot{\varepsilon}_\theta^{11}}{\partial s}] = \gamma_\phi K\, G[\tilde{w}^{11}] , \qquad (4.7)$$

$$[\dot{\tilde{w}}^{11}] = - G[\tilde{\tilde{w}}^{11}] ,$$

$$[\frac{\partial \dot{\varepsilon}_\phi^1}{\partial s}] = - G [\tilde{\tilde{\beta}}_\phi] + \frac{\delta}{\delta t} [\tilde{\beta}_\phi] + \gamma_\theta K\, G[\tilde{\tilde{w}}^1] .$$

Substituting these relations into Eq. 4.6 gives

$$[\frac{\partial \dot{N}_\phi}{\partial s}] = C \{ (1 - \nu^2)(- G[\tilde{\tilde{u}}_\phi] + \frac{\delta}{\delta t} [\tilde{u}_\phi] + \gamma_\theta K\, G[\tilde{w}])$$

$$+ \nu(1+\nu)(k\, G\, [\tilde{u}_\phi] + \gamma_\phi K\, G\, [\tilde{w}] - G\, [\tilde{w}^1]) \qquad (4.8)$$

$$+ \frac{\alpha}{2}\, (\gamma_\theta K(1-\nu^2)G + 3\gamma_\phi K\nu(1+\nu)G)\, [\tilde{w}^{11}]$$

$$+ \alpha K(\gamma_\theta - \gamma_\phi)(1-\nu^2)\, \gamma_\theta\, K(-\, G\, [\tilde{u}_\phi] + \frac{\delta}{\delta t}\, [\tilde{u}_\phi] + \gamma_\theta K\, G\, [\tilde{w}]\,)$$

$$-\, G\, [\tilde{\tilde{\beta}}_\phi] + \frac{\delta}{\delta t}\, [\tilde{\tilde{\beta}}_\phi] + \gamma_\theta K\, G\, [\tilde{w}^1] + \frac{9}{10}\, \alpha\gamma_\theta{}^2\, K^2\, G\, [\tilde{w}^{11}]\} \; .$$

Differentiating with respect to time the expressions for N_ϕ, N_θ, and Q_ϕ, given by Eqs. 2.109 (a) and (c) with subscripts interchanged in (a) to give N_θ, and following the procedure just outlined will yield the following results

$$[\dot{N}_\phi] = C\, [\, -\, (1-\nu^2)G\, [\tilde{u}_\phi] + \alpha K(\gamma_\theta - \gamma_\phi)(1-\nu^2)(-\, \gamma_\theta K\, G[\tilde{u}_\phi] - G\, [\tilde{\tilde{\beta}}_\phi])]\; ,$$

$$[\dot{N}_\theta] = -\, C\, \nu\, (1+\nu)G\, [\tilde{u}_\phi] \qquad (4.9)$$

$$[\dot{Q}_\phi] = \mu\, [\, -\, G\, [\tilde{w}] - \frac{\alpha}{2}\, G\, [\tilde{w}^{11}] + \alpha K(\gamma_\theta - \gamma_\phi)(-\, \gamma_\theta K\, G[\tilde{w}] - G[\tilde{w}^1]$$

$$-\, \frac{9}{10}\, \alpha\gamma_\theta K\, G\, [\tilde{w}^{11}])] \quad ,$$

Substituting these terms, as given by Eqs. 4.8 and 4.9, back into Eq. 4.5 and making use of the third order kinematic compatibility relations given by Eqs. 4.3 results in a growth-decay equation involving the displacement derivative jumps only. Upon collecting the coefficients of the jumps the equation can be written as follows

$$[\, -\, C\, (1-\nu^2)(1 + \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) + \rho h(1+\alpha K)G^2]G\, [\tilde{\tilde{u}}_\phi]$$

$$+ C\,k\,(1 - \nu^2)(1 + \alpha\gamma_\theta\, K^2(\gamma_\theta - \gamma_\phi))G\,[\tilde{u}_\phi]$$

$$+ [\, C(1 - \nu^2)(1 + \alpha\gamma_\theta\, K^2(\gamma_\theta - \gamma_\phi) - 3\rho h(1 + \alpha K)G^2]\, \frac{\delta}{\delta t}\, [\tilde{u}_\phi]$$

$$- [\, C\alpha K\, (\gamma_\theta - \gamma_\phi)(1 - \nu^2) + 2\rho\alpha h\, K\, H\, G^2]\, G\, [\tilde{\tilde{\beta}}_\phi] \qquad (4.10)$$

$$+ Ck\alpha K(\gamma_\theta - \gamma_\phi)(1 - \nu^2)G\,[\tilde{\tilde{\beta}}_\phi] + \; C\alpha K(\gamma_\theta - \gamma_\phi)(1 - \nu^2)$$

$$+ 6\rho\alpha h\, K\, H\, G^2]\, \frac{\delta}{\delta t}\, [\tilde{\tilde{\beta}}_\phi]$$

$$+ [(C(1 - \nu^2) + \mu)\, \gamma_\theta\, K\, (1 + \alpha\gamma_\theta\, K^2(\gamma_\theta - \gamma_\phi)) + C\gamma_\phi K\nu(1 + \nu)]G[\tilde{w}]$$

$$+ [\, - C\nu(1 + \nu) + \alpha\gamma_\theta\, K^2(\gamma_\theta - \gamma_\phi)(C(1 - \nu^2) + \mu)]G\, [\tilde{w}^1]$$

$$+ [(C(1 - \nu^2) + \mu)\, \frac{\alpha}{2}\, \gamma_\theta K(1 + \frac{9}{5}\, \alpha\gamma_\theta\, K^2(\gamma_\theta - \gamma_\phi))$$

$$+ 3\, C\, \frac{\alpha}{2}\, \gamma_\phi\, K\, \nu(1 + \nu)]G\, [\tilde{w}^{11}] = 0.$$

It is now possible to derive four additional growth-decay equations involving arc length derivative jumps in the components of the longitudinal and normal displacements by utilizing the four remaining equations of motion in conjunction with the method which was employed for the first equation. All four of the equations obtained contain terms in the five displacement components as does the equation above. These additional equations are found to be.

$$- [\, B\, K\, (\gamma_\theta - \gamma_\phi)(1 - \nu^2) + 2\rho\alpha h\, K\, H\, G^2]G\, [\tilde{\tilde{u}}_\phi] + BkK(\gamma_\theta - \gamma_\phi)(1 - \nu^2)G[\tilde{u}_\phi]$$

$$+ [\, B\, K\, (\gamma_\theta - \gamma_\phi)(1 - \nu^2) + 6\rho\alpha h\, K\, H\, G^2]\, \frac{\delta}{\delta t}\, [\tilde{u}_\phi]$$

$$+ \left[- B(1 - \nu^2)(1 + \frac{9}{5} \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) + \rho\alpha h(1 + \frac{9}{5} \alpha K)G^2\right]G \, [\tilde{\tilde{\beta}}_\phi] \qquad (4.11)$$

$$+ Bk(1 - \nu^2)(1 + \frac{9}{5} \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi))G \, [\tilde{\beta}_\phi]$$

$$+ \left[B(1 - \nu^2)(1 + \frac{9}{5} \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) - 3\rho\alpha h(1 + \frac{9}{5} \alpha K)G^2\right] \frac{\delta}{\delta t} \, [\tilde{\beta}_\phi]$$

$$+ [(B(1 - \nu^2) + \mu\alpha)\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) + \mu]G \, [\tilde{w}] + [B\gamma_\theta K(1 - \nu^2)(1 + \frac{9}{5} \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi))$$

$$+ 2 B\gamma_\phi K\nu(1 + \nu) + \mu\alpha K(\gamma_\theta - \gamma_\phi)]G \, [\tilde{w}^1] + [- B\nu(1 + \nu) + \frac{9}{10} \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)(B(1 - \nu^2) + \mu\alpha) + \mu\frac{\alpha}{2}] \, G[\tilde{w}^{11}] = 0.$$

$$- [(C(1 - \nu^2) + \mu)\gamma_\theta K (1 + \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) + C\gamma_\phi K\nu(1 + \nu)]G \, [\tilde{u}_\phi]$$

$$- [(C(1 - \nu^2) + \mu)\alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) + \mu]G \, [\tilde{\beta}_\phi]$$

$$+ [- \mu - \mu\alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) + \rho h(1 + \alpha K)G^2]G \, [\tilde{\tilde{w}}]$$

$$+ k \mu (1 + \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi))G \, [\tilde{w}] + [\mu + \mu\alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) - 3\rho h(1 + \alpha K)G^2] \frac{\delta}{\delta t} \, [\tilde{w}]$$

$$- [\mu\alpha K(\gamma_\theta - \gamma_\phi) + 2\rho\alpha h K H G^2]G \, [\tilde{\tilde{w}}^1] + k \mu\alpha K(\gamma_\theta - \gamma_\phi)G \, [\tilde{w}^1]$$

$$+ [\mu\alpha K(\gamma_\theta - \gamma_\phi) + 6\rho\alpha h K H G^2] \frac{\delta}{\delta t} \, [\tilde{w}^1] \qquad (4.12)$$

$$+ [- \mu\frac{\alpha}{2} - \frac{9}{10} \mu\alpha^2\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) + \rho\frac{\alpha}{2} h(1 + \frac{9}{5} \alpha K)G^2]G \, [\tilde{\tilde{w}}^{11}]$$

$$+ [\ k\mu\frac{\alpha}{2} + \frac{9}{10} k\,\mu\alpha^2\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi)]G\ [\tilde{\tilde{w}}^{11}]$$

$$+ [\ \mu\frac{\alpha}{2} + \frac{9}{10}\mu\alpha^2\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi) - 3\rho\frac{\alpha}{2}h\ (1 + \frac{9}{5}\alpha K)G^2]\ \frac{\delta}{\delta t}\ [\tilde{w}^{11}] = 0\ ,$$

$$[\ -\alpha\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi)(\mu + C(1 - \nu^2) + C\nu(1 + \nu^2)]\ G\ [\tilde{u}_\phi]$$

$$- [\alpha\ K(\gamma_\theta - \gamma_\phi)(\mu + \frac{9}{5}B\gamma_\theta{}^2\ K^2(1 - \nu^2)(1 + \nu) - C\nu(1 - \nu^2))$$

$$+ B\gamma_\phi\ K\ \nu\ (1 + \nu)^2 + B\gamma_\theta\ K(1 - \nu^2)(1 + \nu)]G\ [\tilde{\tilde{\beta}}_\phi]$$

$$- [\mu\alpha\ K(\gamma_\theta - \gamma_\phi) + 2\rho\alpha h\ K\ H\ G^2]G\ [\tilde{\tilde{w}}] + k\mu\alpha K(\gamma_\theta - \gamma_\phi)G\ [\tilde{\tilde{w}}] \qquad (4.13)$$

$$+ [\mu\alpha\ K(\gamma_\theta - \gamma_\phi) + 6\rho\alpha h\ K\ H\ G^2]\ \frac{\delta}{\delta t}\ [\tilde{\tilde{w}}] + [\ -\mu\alpha - \frac{9}{5}\mu\alpha^2\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi)$$

$$+ \rho\alpha h(1 + \frac{9}{5}\alpha K)G^2]G\ [\tilde{\tilde{w}}^1] + [k\mu\alpha + \frac{9}{5}k\mu\alpha^2\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi)]G\ [\tilde{\tilde{w}}^1]$$

$$+ [\mu\alpha + \frac{9}{5}\mu\alpha^2\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi) - 3\rho\alpha h(1 + \frac{9}{5}\alpha K)G^2]\ \frac{\delta}{\delta t}\ [\tilde{\tilde{w}}^1]$$

$$- [\ \frac{9}{10}\mu\alpha^2 K(\gamma_\theta - \gamma_\phi) + \frac{9}{5}\rho\alpha^2 h\ K\ H\ G^2]G\ [\tilde{\tilde{w}}^{11}] + \frac{9}{10}k\mu\alpha^2\ K(\gamma_\theta - \gamma_\phi)G\ [\tilde{\tilde{w}}^{11}]$$

$$+ [\ \frac{9}{10}\mu\alpha^2 K(\gamma_\theta - \gamma_\phi) + \frac{27}{5}\rho\alpha^2 h\ K\ H\ G^2]\ \frac{\delta}{\delta t}\ [\tilde{\tilde{w}}^{11}] = 0.$$

$$- [(\mu\frac{\alpha}{2}\gamma_\theta\ K + B\gamma_\theta K(1 - \nu^2)(1 + 2\nu))(1 + \frac{9}{5}\alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) + B\gamma_\phi K\nu(1 + \nu)(1 + 2\nu)$$

$$- B\ K\ \nu(\gamma_\theta - \gamma_\phi)(1 - \nu^2)]G\ [\tilde{u}_\phi]$$

$$+ [\ -\mu\frac{\alpha}{2} - \frac{9}{5}\alpha\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi)(\mu\ \frac{\alpha}{2} + B(1 - \nu^2)(1 + 2\nu) + B\nu(1 - \nu^2))$$

$$+ B\nu(1 + \nu^2)]G\ [\tilde{\tilde{\beta}}_\phi]$$

$$+ \left[- \mu\frac{\alpha}{2} - \frac{9}{10}\mu\alpha^2\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) + \rho\frac{\alpha}{2} h \left(1 + \frac{9}{5}\alpha K\right)G^2\right]G\,[\tilde{\tilde{w}}]$$

$$+ \left[k\mu\frac{\alpha}{2} + \frac{9}{10} k\,\mu\alpha^2\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)\right]G\,[\tilde{w}]$$

$$+ \left[\mu\frac{\alpha}{2} + \frac{9}{10}\mu\alpha^2\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) - 3\rho\frac{\alpha}{2} h \left(1 + \frac{9}{5}\alpha K\right)G^2\right] \frac{\delta}{\delta t}[\tilde{w}] \qquad (4.14)$$

$$- \left[\frac{9}{10}\mu\alpha^2 K(\gamma_\theta - \gamma_\phi) + \frac{9}{5}\rho\alpha^2 h\,K\,H\,G^2\right]G\,[\tilde{\tilde{w}}^1] + \frac{9}{10} k\mu\alpha^2 K(\gamma_\theta - \gamma_\phi)G\,[\tilde{w}^1]$$

$$+ \left[\frac{9}{10}\mu\alpha^2 K(\gamma_\theta - \gamma_\phi) + \frac{27}{5}\rho\alpha^2 h\,K\,H\,G^2\right] \frac{\delta}{\delta t}[\tilde{w}^1]$$

$$+ \left[- \frac{9}{20}\mu\alpha^2 - \frac{27}{28}\mu\alpha^3\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) + \frac{9}{20}\rho\alpha^2 h\left(1 + \frac{15}{7}\alpha K\right)G^2\right]G\,[\tilde{\tilde{w}}^{11}]$$

$$+ \left[\frac{9}{20} k\,\mu\alpha^2 + \frac{27}{28} k\,\mu\alpha^3\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)\right]G\,[\tilde{w}^{11}]$$

$$+ \left[\frac{9}{20}\mu\alpha^2 + \frac{27}{28}\mu\alpha^3\gamma_\theta K^2(\gamma_\theta - \gamma_\phi) - \frac{27}{20}\rho\alpha^2 h\left(1 + \frac{15}{7}\alpha K\right)G^2\right] \frac{\delta}{\delta t}[\tilde{w}^{11}] = 0 .$$

These five equations then are the growth-decay equations for torsionless axisymmetric acceleration waves. It is advantageous at this point to examine the Eqs. 4.10 to 4.14 inclusive with reference to the propagation conditions and possible wave velocities of Secs. 3.2.1 and 3.2.2. As was shown in these two sections longitudinal and normal acceleration waves had separate propagation conditions which were uncoupled as opposed to the coupling of modes which has occurred in the growth-decay equations. Since the eigenvalue problems for the determination of the propagation velocities of the two types of waves were separate because of this, it is a valid assumption,as will be seen later, that these propagation velocities will also be distinct as an inspection of the coefficients involved in the two problems will verify. This implies that a longitudinal acceleration wave cannot be accompanied by a normal acceleration wave.

Taking Eq. 4.10 into consideration, if the wave propagating in the shell is a longitudinal acceleration wave for which the propagation conditions, Eqs. 3.12 and 3.15, will permit both longitudinal strain and bending waves to propagate together, then the derivative jumps in the normal displacement components will vanish. The result is a differential equation governing the variation in wave strength of both longitudinal strain and bending acceleration waves and involving third order jumps in each of the modes. The equation would appear as follows

$$[- C(1 - \nu^2)(1 + \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) + \rho h(1 + \alpha K)G^2]G\ [\tilde{\tilde{u}}_\phi]$$

$$+ Ck(1 - \nu^2)(1 + \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi))G\ [\tilde{u}_\phi] + [C(1 - \nu^2)(1 + \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi))$$

$$- 3\rho h(1 + \alpha K)G^2] \frac{\delta}{\delta t} [\tilde{u}_\phi]$$

$$- [C\alpha K(\gamma_\theta - \gamma_\phi)(1 - \nu^2) + 2\rho\alpha h\ K\ H\ G^2]G\ [\tilde{\tilde{\beta}}_\phi] + Ck\alpha K(\gamma_\theta - \gamma_\phi)(1 - \nu^2)G\ [\tilde{\beta}_\phi]$$

$$+ [C\alpha K(\gamma_\theta - \gamma_\phi)(1 - \nu^2) + 6\rho\alpha h\ K\ H\ G^2] \frac{\delta}{\delta t} [\tilde{\beta}_\phi] = 0 . \qquad (4.15)$$

Under the same conditions the growth-decay equation, Eq. 4.11, yields a second differential equation relating the same wave modes as the above equation. An inspection of the coefficients of the third order discontinuities in Eqs. 4.10 and 4.11 reveals that they are identical to the coefficients of the corresponding second order modes in the longitudinal propagation conditions, Eqs. 3.12 and 3.15. The assumption that a third order jump in a displacement component propagating with a second order jump in the same component is subject to the same propagation conditions as the second order discontinuity leads to the elimination of the third order modes in Eqs. 4.10 and 4.11 when longitudinal acceleration wave conditions are applied. This results in a system of two simultaneous differential equations which will hopefully be soluble when the propagation velocity, as determined in Sec. 3.2.1, is substituted. This solution will be a measure of the variation in magnitude of a second order longitudinal strain or bending wave as it propagates. Again the solution will have to be limited to specific shells of revolution as the generalized case becomes algebraically prohibitive for the advantage to be gained.

The remaining three growth-decay equations form a system of three simultaneous differential equations involving the second order jumps in the three normal displacement components when the conditions for the propagation of normal acceleration waves are applied and the assumption concerning the propagation of third order discontinuities with their corresponding second order discontinuities is invoked. Longitudinal acceleration waves would not be associated with the wave curve in this instance. The solution to these equations would govern the strengths of normal strain and normal shear strain acceleration waves as well as that of the wave associated with the normal strain rate in the normal direction.

Application of the longitudinal and normal acceleration wave conditions to the growth-decay equations as dictated by the respective propagation conditions thus yields two separate systems of partial differential equations. Reversing the procedure employed to obtain these equations, setting longitudinal acceleration wave jumps equal to zero in the first two growth-decay equations and the normal acceleration jumps to zero in the last three, gives some interesting results. Again using Eq. 4.10 as an example, it becomes

$$[- C(1 - \nu^2)(1 + \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) + \rho h(1 + \alpha K)G^2] [\tilde{\tilde{u}}_\phi]$$

$$- [C\alpha K(\gamma_\theta - \gamma_\phi)(1 - \nu^2) + 2\rho\alpha h K H G^2] [\tilde{\tilde{\beta}}_\phi] \tag{4.16}$$

$$+ [(C(1 - \nu^2) + \mu)\gamma_\theta K (1 + \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) + C\gamma_\phi K\nu(1 + \nu)] [\tilde{\tilde{w}}]$$

$$+ [- C\nu(1 + \nu) + \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)(C(1 - \nu^2) + \mu)] [\tilde{\tilde{w}}^1] + [(C(1 - \nu^2) + \mu)$$

$$\frac{\alpha}{2} \gamma_\theta K(1 + \frac{9}{5} \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) + 3C \frac{\alpha}{2} \gamma_\phi K (1 + \nu)] [\tilde{\tilde{w}}^{11}] = 0 .$$

It can be seen from this equation that normal acceleration waves are accompanied by third order longitudinal strain and bending waves. Substituting the propagation velocity for these acceleration waves and the growth-decay solution equations governing the strength of the normal displacement jumps into Eq. 4.16 will give an expression regulating the strength of these two third order longitudinal modes. Eq. 4.11, when subjected to the same analysis, yields a second equation coupling the third order longitudinal jumps and the normal acceleration waves resulting in a simultaneous system of two equations in the two third order quantities. Thus, the magnitudes of these jumps may be found as functions of the normal wave strengths.

Applying longitudinal acceleration wave conditions to Eqs. 4.12, 4.13, and 4.14 shows that these acceleration waves are coupled to third order normal waves. Again a system of equations is obtained, this time three equations governing the three third order normal displacement waves. The solution of this system will then yield these third order normal modes as functions of the longitudinal strain and bending acceleration waves.

4.3. Transverse Waves

The growth-decay problem will be dealt with in this section under the auspices of the equations governing purely torsional motions, Eqs. 2.113. As was seen previously, the terms in these two equations are dependent only on the transverse components of the displacement and, as such, resulted in propagation conditions for a second order singular curve involving only acceleration waves with respect to these two quantities. These transverse acceleration waves are subject to the conditions of Eqs. 3.27 as well as to the physical interpretations of each of the two discontinuities given in Sec. 3.3. The analysis procedure employed here parallels that of Sec. 4.2 for longitudinal and normal waves, restricting the problem to second order axisymmetric wave curves only.

For second order singular curves, the kinematic compatibility relations, Eq. 2.119, yield the following second and third order conditions. For m = 1, n = 1

$$[\dot{\tilde{u}}_\theta] = -G\,[\tilde{\tilde{u}}_\theta]\,, \quad [\dot{\tilde{\beta}}_\theta] = -G\,[\tilde{\tilde{\beta}}_\theta]\,. \tag{4.17}$$

For m = 2, n = 1

$$[\dot{\tilde{\tilde{u}}}_\theta] = -G\,[\tilde{\tilde{\tilde{u}}}_\theta] + \frac{\delta}{\delta t}\,[\tilde{\tilde{u}}_\theta]\,, \quad [\dot{\tilde{\tilde{\beta}}}_\theta] = -G\,[\tilde{\tilde{\tilde{\beta}}}_\theta] + \frac{\delta}{\delta t}\,[\tilde{\tilde{\beta}}_\theta]\,. \tag{4.18}$$

For m = 0, n = 3

$$[\dddot{u}_\theta] = -G^3\,[\tilde{\tilde{\tilde{u}}}_\theta] + 3G^2\,\frac{\delta}{\delta t}\,[\tilde{\tilde{u}}_\theta]\,, \quad [\dddot{\beta}_\theta] = -G^3\,[\tilde{\tilde{\tilde{\beta}}}_\theta] + 3G^2\,\frac{\delta}{\delta t}[\tilde{\tilde{\beta}}_\theta]\,. \tag{4.19}$$

Applying the analysis procedure, as explained in Sec. 4.1 and utilized in Sec. 4.2, to the first equation of motion, Eq. 2.113 (a), results in a growth-decay equation governing transverse shear strain and twisting acceleration waves. The previous discussion involving the time dependency of geometric quantities and of their derivatives on the shell reference surface and the continuity of these functions across wave curves is also applicable in this section. Differentiating Eq. 2.113 (a) with respect to time and then taking its jump across a singular curve gives

$$[\frac{\partial \dot{N}_{\phi\theta}}{\partial s}] - k\,([\dot{N}_{\phi\theta}] + [\dot{N}_{\theta\phi}]) - \gamma_\phi\, K[\dot{Q}_\theta] = \rho h[(1 + \alpha K)\,[\ddot{u}_\theta] - 2\alpha K\, H\,[\ddot{\beta}_\theta]] \tag{4.20}$$

By utilizing the constitutive relations as given by Eqs. 2.108 and 2.109, the acceleration wave conditions as given by Eqs. 3.27, and the kinematic compatibility relations as given by Eqs. 4.17, 4.18 and 4.19 the above equation can be shown to be a growth-decay equation relating the transverse displacement derivative jumps only. Eq. 4.20 then becomes

$$\begin{aligned} &[\, -\mu - \mu\alpha\gamma_\theta\, K^2(\gamma_\theta - \gamma_\phi) + \rho h(1 + \alpha K)G^2]G\,[\tilde{\tilde{u}}_\theta] + [k\,\mu + k\mu\alpha\gamma_\theta\, K^2(\gamma_\theta - \gamma_\phi)]G[\tilde{u}_\theta] \\ &+ [\,\mu + \mu\alpha\gamma_\theta\, K^2(\gamma_\theta - \gamma_\phi) - 3\rho h(1 + \alpha K)G^2]\,\frac{\delta}{\delta t}\,[\tilde{u}_\theta] \\ &- [\,\mu\alpha\, K(\gamma_\theta - \gamma_\phi) + 2\rho\alpha h\, K\, H\, G^2]G\,[\tilde{\tilde{\beta}}_\theta] + k\,\mu\alpha K(\gamma_\theta - \gamma_\phi)G\,[\tilde{\beta}_\theta] \\ &+ [\,\mu\alpha\, K(\gamma_\theta - \gamma_\phi) + 6\rho\alpha h\, K\, H\, G^2]\,\frac{\delta}{\delta t}\,[\tilde{\beta}_\theta] = 0\,. \end{aligned} \tag{4.21}$$

The remaining equation of motion results in a second growth-decay equation involving the second and third order modes of the preceding expression when the same procedure is employed. This equation is given as follows

$$- [\ \mu\alpha\ K(\gamma_\theta - \gamma_\phi) + 2\rho\alpha h\ K\ H\ G^2]G\ [\tilde{\tilde{u}}_\theta] + k\ \mu\alpha K(\gamma_\theta - \gamma_\phi)G\ [\tilde{u}_\theta]$$
$$+ [\ \mu\alpha\ K(\gamma_\theta - \gamma_\phi) + 6\rho\alpha h\ K\ H\ G^2]\ \frac{\delta}{\delta t}\ [\tilde{u}_\theta]$$
$$+ [\ - \mu\alpha - \frac{9}{5}\mu\alpha^2\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi) + \rho\alpha h(1 + \frac{9}{5}\alpha K)G^2]G\ [\tilde{\tilde{\beta}}_\theta] \quad (4.22)$$
$$+ [\ k\ \mu\alpha + \frac{9}{5}k\ \mu\alpha^2\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi)]G\ [\tilde{\beta}_\theta]$$
$$+ [\ \mu\alpha + \frac{9}{5}\mu\alpha^2\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi) - 3\rho\alpha h(1 + \frac{9}{5}\alpha K)G^2]\ \frac{\delta}{\delta t}\ [\tilde{\beta}_\theta] = 0\ .$$

Eqs. 4.21 and 4.22 are then the growth-decay equations for axisymmetric transverse acceleration waves. The assumption that the propagation of third order jumps propagating with their corresponding second order jumps is governed by relations identical to those governing the propagation of the second order discontinuities in combination with the propagation conditions for transverse waves, Eqs. 3.31 and 3.32, allows Eqs. 4.21 and 4.22 to be rewritten as follows

$$[\ k\ \mu + k\ \mu\alpha\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi)]G\ [\tilde{u}_\theta] + [\ \mu + \mu\alpha\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi)$$
$$- 3\rho h(1 + \alpha K)G^2]\ \frac{\delta}{\delta t}\ [\tilde{u}_\theta]$$
$$+ k\ \mu\alpha\ K(\gamma_\theta - \gamma_\phi)G\ [\tilde{\beta}_\theta] + [\mu\alpha\ K(\gamma_\theta - \gamma_\phi) + 6\rho\alpha h\ K\ H\ G^2]\ \frac{\delta}{\delta t}\ [\tilde{\beta}_\theta] = 0\ ,$$
$$k\ \mu\alpha\ K(\gamma_\theta - \gamma_\phi)G\ [\tilde{u}_\theta] + [\ \mu\alpha\ K(\gamma_\theta - \gamma_\phi) + 6\rho\alpha h\ K\ H\ G^2]\ \frac{\delta}{\delta t}\ [\tilde{u}_\theta] \quad (4.23)$$
$$+ [\ k\ \mu\alpha + \frac{9}{5}k\mu\alpha^2\gamma_\theta\ K^2(\gamma_\theta - \gamma_\phi)]G\ [\tilde{\beta}_\theta]$$

$$+ \left[\mu\alpha + \frac{9}{5}\mu\alpha^2\gamma_0\, K^2(\gamma_\theta - \gamma_\phi) - 3\rho\alpha h\left(1 + \frac{9}{5}\alpha K\right)G^2 \right] \frac{\delta}{\delta t}\,[\tilde{\tilde{\beta}}_\theta] = 0 \ .$$

The solution to this system of two simultaneous differential equations, obtained after the speed of propagation as found in Sec. 3.3 is substituted, will govern the variation in strength of the transverse shear strain and twisting acceleration waves. This solution will have to be limited to specific cases as the generalized case dealt with here becomes overly complex algebraically. As can be seen, transverse waves are completely uncoupled from all longitudinal and normal modes.

CHAPTER V

EXAMPLES

5.1. Introduction

The propagation and growth-decay problems which have been discussed, employing a high order linear theory of shells, have dealt with a second order singular curve propagating into the stationary unstrained reference surface of a shell of revolution. Due to the orthogonality of the coordinate system being employed and the axial symmetry of the problem, this singular curve, as was seen, propagates as a circle of latitude in the direction of the shell meridians. The propagation conditions for this curve resulted in eigenvalue problems involving the velocities with which the curve may propagate, depending upon which displacement quantity the curve is singular with respect to. The propagation velocities were required in the solution of the partial differential equations arising from the growth-decay problem. The growth-decay equations for the curve gave relations governing the variation in magnitude of the displacement singularities as they propagate and coupling these second order singularities to third order singularities. As was pointed out, the solutions to these two problems would be extremely cumbersome to obtain for the generalized shell of revolution. It proves to be advantageous to choose one or two specific cases in order to facilitate further discussion in these areas. The example which yields the simplest and perhaps most significant results is that of the conical shell of revolution. The spherical case will also be employed whenever it is thought to be of interest and does not become overly complex in comparison to the conical case. The propagation problem and the growth-decay problem, each involving all three wave classification types, will be kept separate in the ensuing analysis.

5.2. Geometry

The reference surface of a conical shell (Fig. 5.1) is geometrically flat since its principal curvature in the direction of a meridian, or s-parameter curve, is zero. Employing Eq. 2.105, the Gaussian curvature then becomes

$$K = \frac{1}{\gamma_\theta \gamma_\phi} = 0 . \tag{5.1}$$

For a spherical shell,(Fig. 5.2), with a reference surface of radius γ, the principal radii of curvature, recalling the results of Sec. 2.1.2 (f), are both equal to this radius. Therefore

$$K = \frac{1}{\gamma^2} . \tag{5.2}$$

5.3. Wave Propagation Conditions

This section will deal with the solution of the eigenvalue problems of Chapter III for longitudinal, normal, and transverse waves for conical shells and with the solution for longitudinal and transverse waves in the spherical case.

The conditions which longitudinal waves must satisfy in order to propagate are given by Eqs. 3.12 and 3.15. The eigenvalue problem associated with these conditions is given by the quadratic equation,

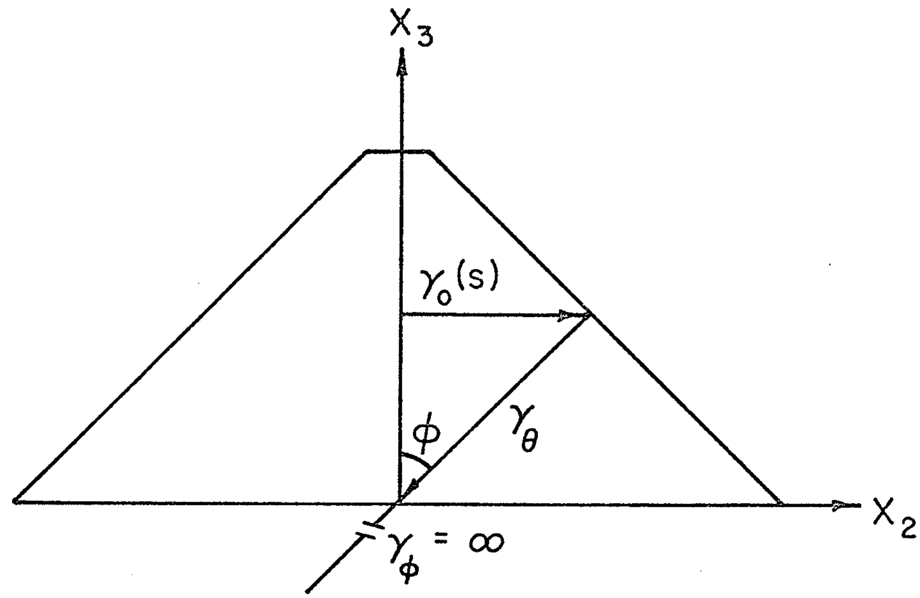

Fig. 5.1 Geometry of a Conical Shell

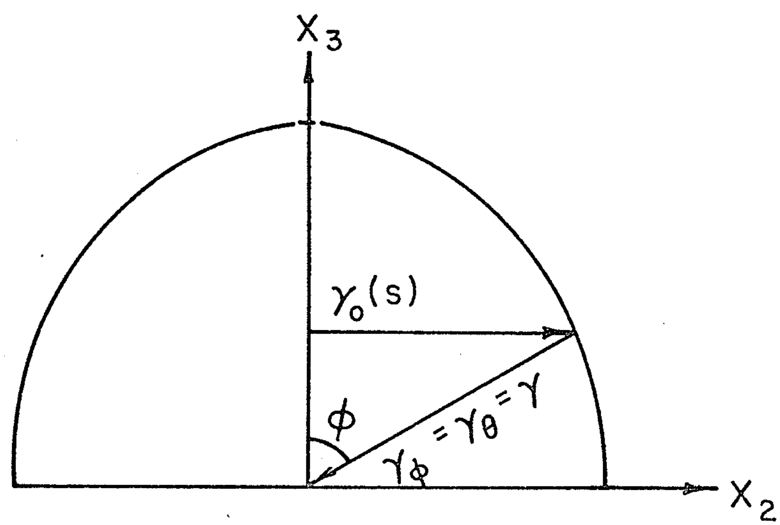

Fig. 5.2 Geometry of a Spherical Shell

Eq. 3.18. Substituting Eq. 5.1 into the coefficient relations for the determinant, Eqs. 3.17, and then employing the results in the quadratic gives, for a conical shell,

$$\rho^2 \alpha h^2 G^4 - 2C\rho\alpha h(1 - \nu^2)G^2 + C^2\alpha(1 - \nu^2)^2 = 0 . \tag{5.3}$$

The solution to this equation may be found using the quadratic formula and can be shown to consist of one root

$$G^2 = \frac{C(1 - \nu^2)}{\rho h} . \tag{5.4}$$

Thus, both longitudinal strain and longitudinal bending acceleration waves propagate with this characteristic velocity in a conical shell. Since the material density and the shell thickness were assumed constant, as well as the material being elastically homogeneous, the wave speed is a constant. A consideration of the propagation conditions shows that they are both identically satisfied and that the two wave modes are uncoupled allowing one to propagate independent of the other.

The normal wave problem is more complex due to the presence of three simultaneous equations which give rise to a cubic equation in the propagation velocity. However, for a cone the solution simplifies considerably as the associated determinant, Eq. 3.25, appears as follows

$$\begin{vmatrix} \mu - \rho h G^2 & 0 & \frac{\alpha}{2}(\mu - \rho h G^2) \\ 0 & \alpha(\mu - \rho h G^2) & 0 \\ \frac{\alpha}{2}(\mu - \rho h G^2) & 0 & \frac{9}{20}\alpha^2(\mu - \rho h G^2) \end{vmatrix} = 0. \tag{5.5}$$

Since the value of a determinant is zero if all the elements of a row or column are zero

$$G^2 = \frac{\mu}{\rho h} \quad . \tag{5.6}$$

The propagation conditions, Eqs. 3.22, 3.23, and 3.24, are identically satisfied by this constant velocity and, as a result, it can be shown that there are no coupling effects among the three normal wave subclassifications.

The quadratic equation associated with the transverse wave case, with the aid of Eq. 5.1 and the coefficient relations, Eqs. 3.34, becomes

$$\rho^2\alpha\, h^2\, G^4 - 2\mu\rho\alpha h\, G^2 + \mu^2\alpha = 0 \ , \tag{5.7}$$

which has one root given by

$$G^2 = \frac{\mu}{\rho h} \quad . \tag{5.8}$$

This wave velocity again yields an identity when substituted into the propagation conditions, Eqs. 3.31 and 3.32, which, as for the previous two wave types, predict an absence of coupling between the wave modes involved for a conical shell.

It should be pointed out here that, for this example, normal and transverse waves propagate with the same constant speed, distinct from that of longitudinal waves. The two propagation velocities, as given by Eqs. 5.4 and 5.6, agree with those of Thomas[18] for irrotational and equivoluminal waves respectively as derived from three dimensional elasticity theory. In light of the normal and transverse wave velocities coinciding, it is interesting to note that if the discontinuity in the normal strain rate, $[\tilde{w}^{11}]$, and its coefficients are set to zero in

Eqs. 3.22, 3.23, and 3.24 the resulting propagation conditions governing normal waves have coefficients identical to those of the equations governing transverse waves. This outcome could be expected if the $[w^{11}]$ term were ignored in the derivation of the constitutive relations and the equations of motion for a generalized shell given in Chapter II. A six mode linear shell theory would be the result as opposed to the seven mode theory which has been employed.

It was shown in Sec. 2.5 that an axisymmetric singularity propagates as a parallel curve in the direction of the meridians in a shell of revolution. Thus, if a wave curve is given on the reference surface of the cone at a reference time, say $t = 0$, the position of this curve after a time t is given by the parallel on the cone a distance Gt from the given curve as measured along a meridian in the units of the propagation velocity. This is a consequence of the constant velocities of the three types of waves and will be valid for any shell of revolution with this property.

Eqs. 5.4, 5.6, and 5.8 show that longitudinal acceleration waves propagate with a velocity different from that of normal and transverse waves implying from the propagation conditions, as noted in Sec. 4.2, that the second order discontinuities in the normal and transverse displacement components vanish across a longitudinal wave curve and vice-versa for second order normal and transverse wave curves. The propagation conditions for each of the three wave types predict no coupling effects among any of the seven modes involved. Thus, in order for a longitudinal wave to propagate, either the strain jump or the bending jump or both must be non-zero but one may vanish. The same reasoning applies to normal and transverse waves and the modes which comprise each of them.

A consideration of the propagation problem applied to a spherical shell, as an example of a more generalized shell of revolution, yields additional useful information. Eq. 5.2 applies in this case as $\gamma_\phi = \gamma_\theta = \gamma$. Employing these relations in conjunction with the coefficient terms of the associated determinant for longitudinal wave propagation, Eqs. 3.17, and substituting the resulting expressions into Eq. 3.18 gives the following quadratic in G^2

$$[\rho^2\alpha h^2((1 + \alpha K)(1 + \tfrac{9}{5}\alpha K) - 4\alpha K)]G^4 \tag{5.9}$$
$$- [C\rho\alpha h(1 - \nu^2)((1 + \alpha K) + (1 + \tfrac{9}{5}\alpha K))]G^2 + C^2\alpha(1 - \nu^2) = 0 .$$

The solution to this equation involves two real roots which can be shown to be

$$G^2_{1,2} = \frac{C(1 - \nu^2)\,[(1 + \frac{7}{5}\alpha K) \pm (\frac{4}{25}\alpha^2K^2 + 4\alpha K)^{\frac{1}{2}}]}{\rho h\,[1 - \frac{6}{5}\alpha K + \frac{9}{5}\alpha^2K^2]} \tag{5.10}$$

Thus, two propagation velocities are predicted for longitudinal waves. Substitution of these velocities into the propagation conditions, Eqs. 3.12 and 3.15, gives two simultaneous equations coupling longitudinal strain acceleration waves to longitudinal bending acceleration waves. This implies that in a spherical shell one is unable to propagate independent of the other and that specifying the magnitude of one automatically dictates the strength of the other.

Normal waves will be deleted here due to the complexity which would be introduced and the difficulty encountered in solving the cubic equation introduced by the eigenvalue problem. As discussed in Sec. 3.2.2, this problem could be handled by computer[20] for a given spherical shell. Results paralleling those for second order longitudinal waves would be expected.

For transverse acceleration waves, substituting Eq. 5.2 into the coefficient relations for the eigenvalue problem, Eqs. 3.34, and again utilizing Eq. 3.18 gives

$$[\rho^2 \alpha h^2 \ ((1 + \alpha K)(1 + \frac{9}{5} \alpha K) - 4\alpha K)]G^4 \qquad (5.11)$$

$$- [\ \mu\rho\alpha h \ ((1 + \alpha K) + (1 + \frac{9}{5} \alpha K))]G^2 + \mu^2 \alpha \ = \ 0 \ .$$

As was the case for longitudinal waves this quadratic has two real roots given by

$$G^2_{1,2} = \frac{\mu[(1 + \frac{7}{5} \alpha K) \pm (\frac{4}{25} \alpha^2 K^2 + 4\alpha K)^{\frac{1}{2}}]}{\rho h\ [\ 1 - \frac{6}{5} \alpha K + \frac{9}{5} \alpha^2 K^2]} \qquad (5.12)$$

The correction factor, or the factor by which the propagation velocity for a cone must be multiplied to yield its spherical counterpart, is the same as that found for the longitudinal case in Eq. 5.10. The propagation conditions given by Eqs. 3.31 and 3.32 then predict coupling between second order transverse shear and transverse twisting waves when the above wave velocities are substituted.

Since K is constant for a spherical shell, the propagation velocities given by Eqs. 5.10 and 5.12 will also be constant. This will also hold for normal waves since the coefficients of the determinant are constant. Thus the discussion concerning the position of a wave curve on a cone relative to an initial position is also applicable in this example.

Longitudinal and transverse waves again propagate with different velocities, given respectively by Eqs. 5.10 and 5.12. Thus, the transverse wave modes will vanish across a longitudinal curve and vice-versa. However, because of the coupling effects arising from the propagation conditions, a longitudinal acceleration wave is made up of a strain wave and a bending wave and a transverse wave would be comprised of both a shear and a twisting wave. These results should also be applicable to normal acceleration waves since, for the conical shell, they propagated with the same velocity as transverse waves and also because of the fact a six mode shell theory would give propagation conditions for normal waves identical in form to those for transverse waves. A proof, however, would require the solution of the eigenvalue problem associated with transverse acceleration waves.

Setting all terms involving $(\alpha K)^{\frac{1}{2}}$ to zero in Eqs. 5.10 and 5.12 yields results identical to those obtained for the cone. It should also be noted that a cone satisfied this condition and therefore the propagation conditions for the three types of waves will be identical to those associated with a conical shell. Making the assumption that $(\alpha K)^{\frac{1}{2}}$, or $h/12\sqrt{\gamma_\theta \gamma_\phi}$, is negligible would then reduce the problem for a generalized shell of revolution to that presented for a conical shell. A decision on the validity of such an assumption, based on the accuracy of the resulting approximation compared to an exact solution from three dimensional elasticity theory, would then have to be made.

5.4. Growth-Decay Equations

The significance of the growth-decay equations as the governing differential equations of the strength of each wave mode and as a set of relations predicting the coupling of second and third order wave curves was discussed in Chapter IV. In this section the solutions of these equations will be found for the special case of a conical shell. It should be remembered that these equations were restricted to shells of revolution having constant propagation velocities.

As was seen, the growth-decay equations separated into two categories as a result of their derivation from the equations of motion. The five equations associated with torsionless axisymmetric motions, Eqs. 4.10 to 4.14 inclusive, will be analyzed first in accordance with the conditions imposed by the propagation problem.

Recall from Eq. 4.10 that for a generalized shell of revolution

$$\begin{aligned}
&[-C(1-\nu^2)(1+\alpha\gamma_\theta K^2(\gamma_\theta-\gamma_\phi)) + \rho h(1+\alpha K)G^2]G\,[\tilde{\tilde{u}}_\phi] \\
&+ Ck(1-\nu^2)(1+\alpha\gamma_\theta K^2(\gamma_\theta-\gamma_\phi))G\,[\tilde{u}_\phi] + [C(1-\nu^2)(1+\alpha\gamma_\theta K^2(\gamma_\theta-\gamma_\phi)) \\
&\qquad - 3\rho h(1+\alpha K)G^2]\,\frac{\delta}{\delta t}\,[\tilde{u}_\phi] \\
&-[C\alpha K(\gamma_\theta-\gamma_\phi)(1-\nu^2) + 2\rho\alpha h\,K\,H\,G^2]G\,[\tilde{\tilde{\beta}}_\phi] + Ck\alpha K(\gamma_\theta-\gamma_\phi)(1-\nu^2)G[\tilde{\beta}_\phi] \\
&+[C\alpha K(\gamma_\theta-\gamma_\phi)(1-\nu^2) + 6\rho\alpha h\,K\,H\,G^2]\,\frac{\delta}{\delta t}\,[\tilde{\beta}_\phi] \\
&+[(C(1-\nu^2)+\mu)\gamma_\theta K(1+\alpha\gamma_\theta K^2(\gamma_\theta-\gamma_\phi)) + C\gamma_\phi K(1+\nu)]G\,[\tilde{w}] \qquad (4.10) \\
&+[-C\nu(1+\nu) + (C(1-\nu^2)+\mu)\alpha\gamma_\theta K^2(\gamma_\theta-\gamma_\phi)]G\,[\tilde{w}^1]
\end{aligned}$$

$$+ [(C(1 - \nu^2) + \mu) \frac{\alpha}{2} \gamma_\theta K(1 + \frac{9}{5} \alpha\gamma_\theta K^2(\gamma_\theta - \gamma_\phi)) + 3C\frac{\alpha}{2} \gamma_\phi K\nu(1 + \nu)]G[\tilde{\tilde{w}}^{11}] = 0 .$$

Since the second order normal wave modes vanish across a longitudinal acceleration wave and vice-versa, the following conditions apply for longitudinal and normal acceleration waves respectively.

$$[\tilde{\tilde{w}}] = [\tilde{\tilde{w}}^1] = [\tilde{\tilde{w}}^{11}] = 0, \quad [\tilde{\tilde{u}}_\phi] \neq 0, \quad [\tilde{\tilde{\beta}}_\phi] \neq 0. \tag{5.13}$$

$$[\tilde{\tilde{u}}_\phi] = [\tilde{\tilde{\beta}}_\phi] = 0, \quad [\tilde{\tilde{w}}] \neq 0, \quad [\tilde{\tilde{w}}^1] \neq 0, \quad [\tilde{\tilde{w}}^{11}] \neq 0 . \tag{5.14}$$

As was pointed out in the preceding section, for a cone one of the associated modes must be non-zero in order for a wave to propagate but not all need be.

By employing Eqs. 5.1 and 5.13, Eq. 4.10 can be rewritten as follows for longitudinal waves

$$[- C(1 - \nu^2) + \rho h G^2]G [\tilde{\tilde{u}}_\phi] + Ck(1 - \nu^2)G [\tilde{u}_\phi] \tag{5.15}$$

$$+ [C(1 - \nu^2) - 3\rho h G^2] \frac{\delta}{\delta t} [\tilde{u}_\phi] = 0 .$$

Substituting the propagation velocity, given by Eq. 5.4, into the above equation yields

$$\frac{2}{G} \frac{\delta}{\delta t} [\tilde{u}_\phi] = k[\tilde{u}_\phi] , \tag{5.16}$$

and $[\tilde{\tilde{u}}_\phi]$ is arbitrary.

However, from Eq. 2.116 it can be seen that

$$G = \frac{\delta s}{\delta t} , \tag{5.17}$$

giving

$$G \frac{\delta}{\delta s} = \frac{\delta}{\delta t} \quad . \tag{5.18}$$

This result, in combination with the third of Eqs. 2.105 and Eq. 2.68, gives Eq. 5.16 as

$$\frac{1}{[\tilde{u}_\phi]} \frac{\delta}{\delta s} [\tilde{u}_\phi] = - \frac{1}{2\gamma_o(s)} \frac{\delta \gamma_o}{\delta s} (s) \quad . \tag{5.19}$$

Integrated, this equation becomes

$$\frac{[\tilde{u}_\phi]}{[\tilde{u}_\phi]_1} = \left(\frac{\gamma_{o1}}{\gamma_o}\right)^{\frac{1}{2}} \quad , \tag{5.20}$$

where the subscript 1 indicates evaluation on a wave curve of radius γ_{o1}.

Proceeding in the same fashion with the second growth-decay equation, Eq. 4.11, and employing Eq. 2.110, results in

$$[- C(1 - \nu^2) + \rho h \; G^2] G [\tilde{\tilde{\beta}}_\phi] + Ck(1 - \nu^2) G \; [\tilde{\tilde{\beta}}_\phi] \tag{5.21}$$

$$+ [\; C(1 - \nu^2) - 3\rho h \; G^2] \frac{\delta}{\delta t} [\tilde{\tilde{\beta}}_\phi] = 0 \; ,$$

which is identical to Eq. 5.15 for a strain wave. Therefore, the equation governing the strength of the bending wave will be given by

$$\frac{[\tilde{\tilde{\beta}}_\phi]}{[\tilde{\tilde{\beta}}_\phi]_1} = \left(\frac{\gamma_{o1}}{\gamma_o}\right)^{\frac{1}{2}} \quad , \tag{5.22}$$

where the convention used in Eq. 5.20 applies.

When Eqs. 5.1 and 5.13 are substituted into the three remaining growth-decay equations, Eqs. 4.12, 4.13, and 4.14, the following set of relations are obtained,

$$\text{(a)} \qquad -\mu\,[\tilde{\tilde{\beta}}_\phi] + (-\mu + \rho h\, G^2)[\tilde{\tilde{w}}] + \frac{\alpha}{2}\,(-\mu + \rho h\, G^2)[\tilde{\tilde{w}}^{11}] = 0\,,$$

$$\text{(b)} \qquad C\nu(1 + \nu^2)[\tilde{\tilde{u}}_\phi] = \alpha(\mu - \rho h\, G^2)\,[\tilde{\tilde{w}}^{1}]\,, \qquad (5.23)$$

$$\text{(c)} \qquad [-\mu + 2C\nu(1 + \nu^2)][\tilde{\tilde{\beta}}_\phi] + (-\mu + \rho h\, G^2)[\tilde{\tilde{w}}] + \frac{9}{10}\,\alpha\,(-\mu + \rho h\, G^2)\,[\tilde{\tilde{w}}^{11}] = 0\,.$$

Solving Eqs. 5.23 (a) and (c) simultaneously gives

$$-(\mu - \rho h\, G^2)\,[\tilde{\tilde{w}}] = \mu + \frac{5}{2}\,C\nu(1 + \nu^2)[\tilde{\tilde{\beta}}_\phi] \qquad (5.24)$$

$$\frac{\alpha}{5}\,(\mu - \rho h\, G^2)\,[\tilde{\tilde{w}}^{11}] = C\nu(1 + \nu^2)[\tilde{\tilde{\beta}}_\phi]\,.$$

If the wave propagating on the reference surface of the cone is a normal acceleration wave, then Eqs. 5.14 apply and employing these relations in combination with Eq. 5.1 allows the growth-decay equations, Eqs. 4.10 and 4.11, to be written respectively as follows

$$-\,[C(1 - \nu^2) - \rho h\, G^2]\,[\tilde{\tilde{u}}_\phi] = C\nu(1 + \nu)\,[\tilde{\tilde{w}}^{1}]\,, \qquad (5.25)$$

$$\alpha\,[\,C(1 - \nu^2) - \rho h\, G^2]\,[\tilde{\tilde{\beta}}_\phi] = \mu\,[\tilde{\tilde{w}}] + \alpha\,[-C\nu(1 + \nu) + \frac{\mu}{2}]\,[\tilde{\tilde{w}}^{11}]\,.$$

The same method yields, for Eq. 4.12,

$$(-\mu + \rho h\, G^2)G\,[\tilde{\tilde{w}}] + k\mu G\,[\tilde{w}] + (\mu - 3\rho h\, G^2)\,\frac{\delta}{\delta t}\,[\tilde{w}] \qquad (5.26)$$

$$+\,\frac{\alpha}{2}\,(-\mu + \rho h\, G^2)G\,[\tilde{\tilde{w}}^{11}] + k\,\mu\frac{\alpha}{2}\,G\,[\tilde{w}^{11}] + \frac{\alpha}{2}\,(\mu - 3\rho h\, G^2)\,\frac{\delta}{\delta t}\,[\tilde{w}^{11}] = 0\,.$$

Substituting the propagation velocity, Eq. 5.6, into this equation gives

$$kG[\tilde{w}] - 2\frac{\delta}{\delta t}[\tilde{w}] + \frac{\alpha}{2}kG[\tilde{w}^{11}] - \alpha\frac{\delta}{\delta t}[\tilde{w}^{11}] = 0 , \qquad (5.27)$$

and $[\tilde{\tilde{w}}]$ and $[\tilde{\tilde{w}}^{11}]$ are arbitrary.

Applying this procedure to Eq. 4.14 results in a differential equation involving the same jumps as above, given by

$$kG[\tilde{w}] - 2\frac{\delta}{\delta t}[\tilde{w}] + \frac{9}{10}\alpha kG[\tilde{w}^{11}] - \frac{9}{5}\alpha\frac{\delta}{\delta t}[\tilde{w}^{11}] = 0 , \qquad (5.28)$$

with the third order discontinuities being arbitrary.

Eqs. 5.27 and 5.28, when solved simultaneously, result in

$$\frac{2}{G}\frac{\delta}{\delta t}[\tilde{w}] = k[\tilde{w}] , \qquad (5.29)$$

and

$$\frac{2}{G}\frac{\delta}{\delta t}[\tilde{w}^{11}] = k[\tilde{w}^{11}] . \qquad (5.30)$$

Similarly, it can be shown that Eq. 4.13 becomes

$$\frac{2}{G}\frac{\delta}{\delta t}[\tilde{w}^{1}] = k[\tilde{w}^{1}] . \qquad (5.31)$$

These three differential equations are identical in form to Eq. 5.16 implying that the solutions will be identical, thus having the form

$$\frac{[\tilde{\psi}]}{[\tilde{\psi}]_1} = \left(\frac{\gamma_{o1}}{\gamma_o}\right)^{\frac{1}{2}} . \qquad (5.32)$$

It is interesting to note that Eqs. 5.23 (b) and 5.24 predict that a longitudinal strain acceleration wave will be accompanied by a third order normal strain wave and that a second order longitudinal bending wave will give rise to a third order normal shear strain wave as well as to a third order wave representative of the normal strain rate. Eqs. 5.25 show a normal strain acceleration wave having an associated third order longitudinal strain wave and a coupling among the second order normal shear strain and strain rate discontinuities and a third order longitudinal bending wave. Substituting the second order wave strengths, as specified by the solutions of the form given by Eq. 5.32, into these equations gives relations expressing the third order jumps as functions of the initial values of their associated second order waves.

As the growth-decay problem for transverse waves is distinct from that for torsionless motions, it suffices to say that one or both of the following conditions must be met in order to have a transverse acceleration wave propagating,

$$[\tilde{u}_\theta] \neq 0, \quad [\tilde{\tilde{\beta}}_\theta] \neq 0 \;. \tag{5.33}$$

Using this criteria, Eq. 5.1, and the propagation velocity for transverse waves, Eq. 5.8, the first growth-decay equation, Eq. 4.21, can be shown to have a solution of the form given by Eq. 5.32 for transverse shear strain waves. Similarly, Eq. 4.22 yields a solution of the same form governing the strength of a transverse twisting wave. The third order jumps, in this case, are arbitrary.

Note that the magnitudes of the second order modes are all dictated by the same relationship as functions of the initial strengths. This relationship, Eq. 5.32, indicates that when the radius of the wave curve vanishes there will be a focusing effect which may produce shell fracture. It also indicates diminishing intensity as the radius of the curve increases.

The growth-decay equations have not been examined with regard to the spherical case because of the complexity of the resulting differential and coupling equations.

Another interesting result can be obtained by setting all terms in αK to zero in the growth-decay equations. The differential equations governing the variation in wave strengths then become identical to those of the conical shell example. The equations coupling the acceleration waves to third order modes become only slightly more complicated than those of the cone with the addition of a few terms to some of the coefficients. For example, under normal acceleration wave conditions Eq. 4.10 becomes

$$[- C(1 - \nu^2) + \rho h\ G^2]\ [\tilde{\tilde{u}}_\phi] + [\ C\gamma_\theta\ K(1 - \nu^2) + C\gamma_\phi\ K\ (1 + \nu)]\ [\tilde{\tilde{w}}] \quad (5.34)$$

$$- C\nu(1 + \nu)\ [\tilde{\tilde{w}}^1] + 3\ C\ \frac{\alpha}{2}\gamma_\phi\ K\ (1 + \nu)\ [\tilde{\tilde{w}}^{11}] \ = \ 0.$$

Making the assumption that $(\alpha K)^{\frac{1}{2}}$ is negligible would then reduce the entire wave propagation problem for a generalized shell of revolution, that is both the propagation velocity and growth-decay problems, to that of a conical shell, differing from it only in the third order coupling equations.

BIBLIOGRAPHY

1. Berkal, A. B., Wave Propagation in Shells, Ph.D. Thesis, University of Manitoba, 1971.

2. Bliss, G. A., Lectures on the Calculus of Variations, The University of Chicago Press, Chicago, 1946.

3. Cohen, H., "Axisymmetric Waves in Shells of Revolution", Proceedings of the IASS, Calgary, 1972.

4. Cohen, H. and A. B. Berkal, "Wave propagation in elastic membranes", Jour. of Elasticity, Vol. 2, No. 1, March 1972.

5. Cohen, H. and A. B. Berkal, "Wave propagation in elastic shells", Jour. of Elasticity, Vol. 2, No. 1, March 1972.

6. Cohen, H. and A. B. Berkal, "Wave propagation in conical shells", Jour. Appl. Mech., Vol. 39, No. 4, Dec. 1972.

7. Cohen, H. and S. L. Suh, "Wave propagation in elastic surfaces", Jour. Math. and Mech., Vol. 19, No. 12, 1970.

8. Hadamard, J., Lecons sur la Propagation des Ondes et les Equations de l'Hydrodynamique, Herman, Paris, 1903, or Chelsea, New York, 1949.

9. Kraus, H., Thin Elastic Shells, John Wiley & Sons, New York, 1967.

10. Naghdi, P. M., "On the theory of thin elastic shells", Q. Appl. Math., Vol. 14, 1957.

11. Naghdi, P. M., "A new derivation of the general equations of elastic shells", Int. Jour. Engg. Science, Vol. 1, 1963.

12. Novozhilov, V. V., The Theory of Thin Elastic Shells, P. Noordhoff, Groningen, 1959.

13. O'Neill, Barrett, Elementary Differential Geometry, Academic Press, New York, 1966.

14. Reissner, E., "On a variational theorem in elasticity", Jour. Math. Phys., Vol. 29, 1950.

15. Reissner, E., "Stress strain relations in the theory of thin elastic shells", Jour. Math. Phys., Vol. 31, 1952.

16. Sokolnikoff, I.S., Mathematical Theory of Elasticity, 2nd ed., McGraw-Hill, New York, 1956.

17. Struik, D. J., Differential Geometry, Addison-Wesley Press, Cambridge, Mass., 1950.

18. Thomas, T. Y., Plastic Flow and Fracture in Solids, Academic Press, New York, 1961.

19. Truesdell, C. and R. A. Toupin, "The classical field theories" in Encyclopedia of Physics, S. Flugge, Ed., Springer-Verlag, Berlin, 1960, Vol. 3, 1.

20. IBM, IBM Application Program, 3rd ed., H20-0205-2, IBM Technical Publications Department, White Plains, N. Y., 1967.

APPENDIX A

This Appendix contains the procedure utilized to carry out the integrations indicated in the definitions of the stress resultant and couple quantities. Approximate series expansions of the following type will be used

$$\log \left(\frac{1-x}{1+x}\right) = -2 \sum_{n=1,3,5}^{\infty} \frac{x^n}{n}, \quad x^2 < 1 \tag{A.1}$$

The first of the required integrations appears as follows

$$\int_{-h/2}^{h/2} \frac{(1-\zeta/R_i)}{(1-\zeta/R_j)} d\zeta = \frac{R_j}{R_i} \int_{-h/2}^{h/2} \frac{(R_i-\zeta)}{(R_j-\zeta)} d\zeta, \quad \begin{matrix} i,j = 1,2 \\ i \neq j \end{matrix} \tag{A.2}$$

However,

$$\frac{(R_i-\zeta)}{(R_j-\zeta)} = \frac{(R_i-\zeta)+(R_j-\zeta)-(R_j-\zeta)}{(R_j-\zeta)} = 1 + \frac{(R_i-R_j)}{(R_j-\zeta)}, \tag{A.3}$$

and

$$d(R_j-\zeta) = -d\zeta . \tag{A.4}$$

Substituting Eqs. A.3 and A.4 into Eq. A.2 and integrating yields

$$\frac{R_j}{R_i}\left[\zeta - (R_i - R_j)\log(R_j-\zeta)\right]_{-h/2}^{h/2} = \frac{R_j h}{R_i} - \left(R_j - \frac{R_j^2}{R_i}\right)\log\left(\frac{1-h/2R_j}{1+h/2R_j}\right) \tag{A.5}$$

The limiting values of the above expression must be found as one or the other of the radii of curvature approaches infinity. If $R_i \to \infty$ and R_j is finite the left hand side of Eq. A.2 becomes

$$R_j \int_{-h/2}^{h/2} \frac{1}{(R_j - \zeta)} d\zeta = - R_j \log \left(\frac{1-h/2R_j}{1+h/2R_j}\right) , \tag{A.6}$$

which agrees with Eq. A.5.

Setting R_i finite and letting $R_j \to \infty$ in Eq. A.2 gives

$$\int_{-h/2}^{h/2} (1 - \zeta/R_i) d\zeta = h . \tag{A.7}$$

However, Eq. A.5 runs into difficulties at this point becoming an indeterminate form. If Eq. A.1 is applied to the right hand side of Eq. A.5 the resulting expression is

$$\frac{R_j h}{R_i} - \left(R_j - \frac{R_j^2}{R_i}\right) \log \left(\frac{1-h/2R_j}{1+h/2R_j}\right) = h \left[1 + \frac{h^2}{12R_j} \left(\frac{1}{R_j} - \frac{1}{R_i}\right)\right] , \tag{A.8}$$

where the series expansion has been truncated at the second term.

This expansion, when employed in the right hand side of Eq. A.6, gives

$$- R_j \log \left(\frac{1-h/2R_j}{1+h/2R_j}\right) = h \left(1 + \frac{h^2}{12R_j^2}\right) . \tag{A.9}$$

Putting R_j finite and allowing $R_i \to \infty$ in Eq. A.8 and vice-versa shows that this approximate expression yields the required results, that is Eqs. A.9 and A.7 respectively.

The solution to the left hand side of Eq. A.2 then becomes

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} d\zeta = h \left[1 + \frac{h^2}{12R_j} \left(\frac{1}{R_j} - \frac{1}{R_i}\right)\right] . \tag{A.10}$$

The second integration has the form

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} \zeta d\zeta = \frac{R_j}{R_i} \int_{-h/2}^{h/2} \zeta(1 + \frac{R_i - R_j}{R_j - \zeta}) \, d\zeta , \tag{A.11}$$

which can be shown to give

$$(R_j - \frac{R_j^2}{R_i}) \int_{-h/2}^{h/2} \frac{\zeta}{(R_j - \zeta)} = (R_j - \frac{R_j^2}{R_i}) \int_{-h/2}^{h/2} (\frac{R_j}{R_j - \zeta} - 1) d\zeta. \tag{A.12}$$

Employing Eq. A.4, integrating, and then expanding by means of Eq. A.1 allows the right hand side of the above expression to be rewritten

$$(R_j - \frac{R_j^2}{R_i}) [- h - R_j \log (\frac{1-h/2R_j}{1+h/2R_j})] = \frac{h^3}{12} (\frac{1}{R_j} - \frac{1}{R_i}) , \tag{A.13}$$

where the series is truncated at the second term.

Checking the limits, using a procedure similar to that utilized for the preceding case, yields for $R_i \to \infty$

$$\int_{-h/2}^{h/2} \frac{\zeta}{1 - \zeta/R_j} d\zeta = R_j [- h - R_j \log (\frac{1-h/2R_j}{1+h/2R_j})] = \frac{h^3}{12R_j} , \tag{A.14}$$

for $R_j \to \infty$

$$\int_{-h/2}^{h/2} \zeta(1 - \zeta/R_i) d\zeta = \frac{h^3}{12R_i} . \tag{A.15}$$

These results agree with Eq. A.13 confirming that it is a valid approximation to the integral solution sought.

The next integral expression of interest is

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} \zeta^2 d\zeta = \frac{R_j}{R_i} \int_{-h/2}^{h/2} \zeta^2 (1 + \frac{R_i - R_j}{R_j - \zeta}) \, d\zeta \;, \qquad \text{(A.16)}$$

which becomes

$$\frac{R_j}{R_i} \left[\frac{h^3}{12} + (R_i - R_j) \int_{-h/2}^{h/2} \frac{\zeta^2}{R_j - \zeta} d\zeta \right] = \frac{R_j}{R_i} \left[\frac{h^3}{12} + (R_i - R_j) \int_{-h/2}^{h/2} \right.$$

$$\left. (\frac{R_j \zeta}{R_j - \zeta} - \zeta) d\zeta \right] = \frac{R_j}{R_i} \left[\frac{h^3}{12} + (R_i - R_j) \int_{-h/2}^{h/2} \frac{R_j \zeta}{R_j - \zeta} d\zeta \right] . \qquad \text{(A.17)}$$

Noting that this equation is similar in form to Eq. A.12, it can be shown to give

$$\frac{R_j}{R_i} \frac{h^3}{12} + \frac{R_j^{\,2}}{R_i} (R_i - R_j) \left[- h - R_j \log \left(\frac{1-h/2R_j}{1+h/2R_j}\right) \right]$$

$$= \frac{h^3}{12} \left[1 + \frac{3h^2}{20R_j} \left(\frac{1}{R_j} - \frac{1}{R_i}\right)\right] \;, \qquad \text{(A.18)}$$

where the series expansion has been truncated at the third term in this instance. A check of this result at the limits used previously provides confirmation.

The fourth integral is written as follows

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} \zeta^3 d\zeta = \frac{R_j}{R_i} \int_{-h/2}^{h/2} \zeta^3 (1 + \frac{R_i - R_j}{R_j - \zeta}) \, d\zeta \;. \qquad \text{(A.19)}$$

This expression gives

$$(R_j - \frac{R_j^{\,2}}{R_i}) \int_{-h/2}^{h/2} \frac{\zeta^3}{R_j - \zeta} d\zeta = (R_j - \frac{R_j^{\,2}}{R_i}) \int_{-h/2}^{h/2} (\frac{R_j \zeta^2}{R_j - \zeta} - \zeta^2) d\zeta \;, \qquad \text{(A.20)}$$

which can be compared to Eq. A.17 to yield

$$(R_j - \frac{R_j^2}{R_i}) \ [- \frac{h^3}{12} + R_j^2 (- h - R_j \log (\frac{1-h/2R_j}{1+h/2R_j}))] = \frac{h^5}{80} (\frac{1}{R_i} - \frac{1}{R_j}), \quad (A.21)$$

where the series, Eq. A.1, is truncated at the third term. A check of the limits also confirms this result.

The final integral expression to be considered appears as

$$\int_{-h/2}^{h/2} \frac{(1 - \zeta/R_i)}{(1 - \zeta/R_j)} \zeta^4 d\zeta = \frac{R_j}{R_i} \int_{-h/2}^{h/2} \zeta^4 (1 + \frac{R_i - R_j}{R_j - \zeta}) d\zeta , \quad (A.22)$$

which becomes, upon integrating the first term and making the usual manipulation in the second,

$$\frac{R_j}{R_i} [\frac{h^5}{80} + (R_i - R_j) \int_{-h/2}^{h/2} (\frac{R_j \zeta^3}{R_j - \zeta} - \zeta^3) d\zeta]$$

$$= \frac{R_j}{R_i} [\frac{h^5}{80} + (R_i - R_j) \int_{-h/2}^{h/2} \frac{R_j \zeta^3}{R_j - \zeta} d\zeta] . \quad (A.23)$$

The integral in this equation is of the form of the integral on the left hand side of Eq. A.20. Eq. A.23 can then be shown to yield

$$\frac{R_j}{R_i} \{ \frac{h^5}{80} + (R_i - R_j) R_j [- \frac{h^3}{12} + R_j^2 (- h - R_j \log (\frac{1-h/2R_j}{1+h/2R_j}))]\}$$

$$= \frac{h^5}{80} [1 + \frac{5h^2}{28R_j} (\frac{1}{R_j} - \frac{1}{R_i})] , \quad (A.24)$$

where the log series has been truncated at the fourth term. A check of the left side of Eq. A.22 as the radii of curvature approach infinity proves the validity of this result.