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WAVE PROPAGATION IN SHELLS

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ABSTRACT

The propagation of a singular curve is considered in linear elastic theories of shells. First and second order waves with respect to displacement are considered in membranes and in a bending theory. The speed of propagation, the shape of the wave, the decay equations, and certain coupling effects are established.

The method of procedure involves consideration of the discontinuities in the equations of motion and the derivatives of these equations, the dynamical compatibility equations, and the use of certain geometrical and kinematical identities.

The relevant dynamical theories of membranes and shells are derived along with the dynamical compatibility equations and any geometrical and kinematical identities required.

A discussion of time derivatives of double tensors is presented which gives a unified foundation to various definitions of time derivatives.

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CHAPTER I

INTRODUCTION

In the context of this dissertation a wave is defined to be a time persistent submanifold of the body manifold which divides the latter into two parts such that certain physical or geometrical quantities which are continuous elsewhere are discontinuous across the submanifold, although the one sided limiting values exist. If the body is considered to be three dimensional then the wave is a singular surface with respect to some quantity and it is in general propagating through the body. Certain types of bodies, called shells, can be idealized by considering a materialized surface. In this case a wave is a singular curve with respect to some quantity.

The term wave is commonly used to refer to at least two other notions, characteristic manifolds and harmonic oscillations. As pointed out by Truesdell and Toupin(1) these three definitions are in general not equivalent, although in some cases they may coincide.

The general methods used to investigate the propagation of singular surfaces in elastic media were developed in the definitive works of Hugoniot(2), Hadamard(3), and Duhem(4). The problem of wave propagation in various types of materials and under different conditions has since been dealt with by various authors. In particular, Thomas(5) has considered shockwaves or first order waves with respect to displacement in linear elastic and plastic materials. Truesdell(6,7) has investigated acceleration waves in non-linear elastic materials and he has also extended the mathematical foundations of the study of singular surfaces. Ericksen(8) has studied the propagation of acceleration waves in incompressible elastic media. More recent work involving the analysis of acceleration waves has been carried out by Chen[9,10], Bowen and Wang[11], Hill[12], Suhubi[13], and Balaban, Green, and Naghdi[14]. Green[15] has dealt with wave propagation in initially deformed bodies. B.T. Chu[16] has considered shock waves in incompressible elastic solids. Wave propagation in dissipative materials or materials with memories has been investigated by Chen and Gurtin[17], Coleman, Gurtin, Herrerar, and Truesdell[18]. McCarthey and Eringen[19] have dealt with wave propagation in micropalar visoelastic materials.

In a recent paper Cohen and Suh[20] noted the technological interest in the propagation of singular curves in membranes and shells. They then proceeded to analize shock wave propagation in a linear elastic membrane theory. Their theory, however, was developed from an intrinsic point of view.

This study attempts to extend their results by investigating the propagation of shock and acceleration waves in a linear membrane theory and in a linear bending theory of shells, both of which are developed from an embedding point of view. Mention is briefly made of a non linear membrane theory as well. The method employed for investigating singular curves yields the speed of propagation, the shape of the wave curve, and decay equations for the strength of the discontinuity. In the case of shock waves it is shown that a higher order wave with respect to the component of displacement normal to the surface accompanies the shock wave. The coupling of these waves is explicitely illustrated.

In Chapter II the relevant geometrical and kinematical aspects of moving surfaces are considered. A discussion is included of differentiation of tensors and double tensors. The very general way in which differentiation with respect to time is treated facilitates the way in which what Truesdell and Toupin call the displacement derivative is defined in Chapter III[21]. This chapter is concluded by giving the transport theorem in two dimensions. This theorem is used in Chapter IV in deriving the equations of motion.

In Chapter III the kinematics of a singular curve is discussed. The diagram on the reference surface of the wave curve is considered and the speed of propagation of the diagram is shown to be continuous and it is related to the speeds of propagation of the wave curve on either side of the wave. The consequences of "small" displacements on the discontinuities of the surface normal and the mass density are found. In addition, the relationships between the speeds of propagation of the diagram and the wave curves are found under the condition of small displacements. Then a moving singular curve is considered on a stationary surface. A special parameterization of the moving curve is shown to exist such that the velocity of a point on the curve which retains the same value of this curve parameter is normal to the curve. By referring to Chapter II derivatives of double tensors are defined corresponding to observers moving tangential to the wave curve at a fixed time and moving normal to the wave curve with the normal speed of the curve. Once these derivatives are defined the first order geometrical and kinematical compatibility relations are derived. The chapter is concluded with the derivation of all higher order compatibility relations which are required in ensuing chapters.

Chapter IV involves the derivation of the equations of motion and the dynamical compatibility equations. The Eulerian and Lagrangian forms of these equations are presented. Finally, it is shown that for "small" deformations there is no difference in these two approaches.

In Chapter V the static linear membrane theory of Green and Zerna[22] is modified to include inertia effects. After the governing equations are established the propagation of shock and acceleration waves into the stationary unstrained reference surface is considered. It is found in both cases that two types of waves can exist; each of which propagates with a characteristic constant speed. Because the speed of propagation is constant it is argued that the successive positions of the wave curve on the reference surface together with the orthogonal trajectories to these curves form a geodesic parallel net. The decay equations for the strength of the discontinuity are derived. In addition, it is found that a higher order wave with respect to the component of dispacement normal to the surface is coupled to the assumed shock or acceleration wave. Except for the latter, the results for shock waves are the same as those derived by Cohen and Suh[23] in their treatment of a linear elastic surface from an intrinsic point of view. At the conclusion of this chapter the propagation conditions are derived for a very general non-linear membrane in which acceleration waves are considered.

Chapter VI is devoted to an examination of wave propagation in bars. It is seen that is necessary to include rotary inertia effects in order to obtain a propagating wave. The suggestion is made that such effects must be included in dealing with bending moment theories of shells.

In Chapter VII the static linear theory of shells proposed by Green and Zerna[24] is modified to include inertia effects; both linear and rotary inertia terms are introduced. A motivation for the choice of the rotary inertia term employed is provided. Once the theory is presented, the problem of shock and acceleration waves propagating into the stationary, unstrained, reference surface is investigated. As in the case of membranes two types of waves are recognized to exist, each with a constant characteristic speed. Once again, the argument that the successive positions of the wave curve on the reference surface and the orthogonal trajectories to these curves form an orthogonal net is relevant. In the case of shock waves the decay equations for the strength of the discontinuity are the same as those for membranes. Higher order waves with respect to the component of displacement normal to the surface are coupled to the shock waves for transverse waves. For longitudinal waves an acceleration wave with respect to the normal component of displacement will accompany the shock wave only if it is initially excited. In the case of normal acceleration waves the assumed wave in general is an acceleration wave with respect to both components of displacement normal to the curve. The in-surface component is coupled to the surface normal component and it will therefore exist in general whether initially excited or not. This is not the case with the other component however. It will exist only if it is initially excited. On the other hand, tangential acceleration waves are in general accompanied by a higher order wave with respect to the component of displacement normal to the surface.

The final chapter presents a number of examples. The solution of the decay equations is discussed for the case of a surface of constant Gaussian curvature. A right helicoid is an example of a surface on which it is possible to find longitudinal shock waves using the membrane theory such that the acceleration wave which is in general coupled to such waves does not appear. Similarly, surfaces of revolution are such that it is possible when using the membrane theory to find transverse shock waves which do not have the usual acceleration wave coupled to them. The example of waves eminating from a point on the side of a circular cylinder is presented. The undulating nature of the acceleration wave which accompanies a tangential shock wave in the membrane theory is interesting. Finally, both shock and acceleration waves propagating on a cone are considered. The decay equations are solved and it is suggested that this example might be compared, with care, to the results of an experiment by Kenner, Goldsmith, and Sackman[25].

A comma will be used to denote partial differentiation. If any ambiguity exists as to what is being held fixed this will be removed by placing a vertical line after the derivative followed by the variables being kept fixed. Latin indices will in general assume the values 1, 2,3 and Greek indices the values 1,2.

CHAPTER II

KINEMATICS AND GEOMETRY

The notion of a materialized surface is an idealization. It approximates the physical situation which exists when two closely spaced surfaces of large extent compared to their separation define boundaries for the material. Such a physical system is called a shell. In problems involving shells it is convenient to make the above approximation. It is important then to be able to describe the geometry and motion of a moving surface.

A geometric surface, S, is defined as the point set in three dimensional Euclidean space, E_3, given by

$$x^k = x^k(u^\alpha) \tag{1}$$

where x^k are curvilinear coordinates in E_3, and where the above single valued function is defined over some domain in the space of ordered pairs of real numbers (u^1,u^2) such that the function is at least of class C^1 and the matrix $x^k_{,\alpha}$ has rank = 2 everywhere except at most on a set of measure zero. Points belonging to the set of measure zero will be called singular points and other points will be called non-singular points. Because the above matrix has rank = 2 at non-singular points the mapping $(u^1,u^2) \to (x^1,x^2,x^3)$ is 1-1 in open neighborhoods of u^α space. The parameters u^α can then be considered as local surface coordinates.

If the functions

$$v^{\alpha'} = v^{\alpha'}(u^\beta) \tag{2}$$

are introduced such that the functions are of class C^1 and the Jacobian, $J = |v^{\alpha'}{}_{,\beta}| \neq 0$ in some region of u^α space then

$$u^\alpha = u^\alpha(v^{\beta'}) \qquad (3)$$

and S can be given equivalently by

$$x^k = \bar{x}^k(v^{\beta'}) \qquad (4)$$

The parametric representation of the surface is not unique. Indeed since both u^α and $v^{\beta'}$ can be considered as surface coordinates the functions relating them can be considered as a transformation of surface coordinates.

At any time t an identification can be made between the material points constituting the shell and the geometrical points of the surface which they are thought to occupy. At some reference time, which for convenience can be taken as t = 0, the surface coordinates of the material points are U^Δ and the corresponding spatial coordinates are

$$X^K = X^K(U^\Delta) \qquad (5)$$

At time t, x^k are the spatial coordinates of the surface. The motion of the material is described by specifying the position which a material point occupies as a function of time. In other words the motion is given by

$$x^k = x^k(U^\Delta, t) \qquad (6)$$

It is assumed that the motion is continuous and continuously differentiable as many times as required everywhere over the reference surface except possibly on a curve on this surface. It is also assumed that the motion mapping is one-one. This is equivalent to assuming that mass is not created or destroyed and that mass is inpenetrable.

As has been seen the surface of material points is given at any time t by $x^k = x^k(U^\Delta,t)$. The U^Δ can be thought of as surface coordinates on the surface occupied by the material points at time t. The U^Δ were initially introduced as the surface coordinates on the surface at the reference time. The U^Δ are called convected surface coordinates. As a material point moves it always retains the same surface coordinates if convected surface coordinates are used to describe each surface. Such need not be the case. If the functions

$$u^\alpha = u^\alpha(U^\Delta,t) \tag{7}$$

are introduced such that they are of class C^1 and $J = |u^\alpha_{,\Delta}| \neq 0$, then

$$U^\Delta = U^\Delta(u^\alpha,t) \tag{8}$$

and by combining equations (6) and (8) it is seen that

$$x^k = \bar{x}^k(u^\alpha,t) \tag{9}$$

Thus new surface coordinates can be introduced by such time dependent transformations. The surface coordinates of a material point followed in its motion are given by equation (7) and in general they differ from surface to surface. The motion can then be given equivalently by

$$x^k = x^k(U^\Delta,t) \tag{10}$$

or by

$$x^k = x^k(u^\alpha,t) \text{ and } u^\alpha = u^\alpha(U^\Delta,t)$$

Obviously, transformations of surface coordinates and transformations of spatial coordinates are unrelated.

A geometric curve C on the moving surface given by $x^k = x^k(U^\Delta,t)$ is a point set on the surface given by

$$x^k = x^k(U^\Delta,t), \quad U^\Delta = U^\Delta(q,t) \tag{12}$$

where $U^\Delta = U^\Delta(q,t)$ is defined over some domain in the space of real numbers q, and where the single valued function $U^\Delta = U^\Delta(q,t)$ is at least of class C^1 and the derivatives $U^\Delta{}_{,q}$ are not both identically zero. Because of the latter requirement the mapping $q \to (u^1,u^2)$ is one-one in open neighborhoods of q space. The parameter q can then be considered as a coordinate along the curve. It is also possible to specify the surface coordinates of those points constituting the curve by eliminating q between the equations $U^\Delta = U^\Delta(q,t)$. Then these surface coordinates are given as the solution set of a condition of the form

$$g(U^\Delta,t) = 0 \tag{13}$$

where at least one component of $g_{,\Delta}$ is not equal to zero. It should also be pointed out that if different surface coordinates $u^\alpha = u^\alpha(U^\Delta,t)$ are used to describe the surface then the curve is given by

$$x^k = \bar{x}^k(u^\alpha,t) \quad \text{and} \quad u^\alpha = u^\alpha(q,t) \tag{14}$$

The u^α may alternatively be given by

$$f(u^\alpha,t) = 0 \tag{15}$$

The same conditions apply to these representations of a curve as apply to the previous forms. Important examples of curves on a surface are the coordinate curves. The u^1 coordinate curves at any time t are given by

$$u^2 = \text{constant} \equiv u_c^2, \quad x^k = (u^1, u_c^2, t) \tag{16}$$

and the u^2 coordinate curves are given by

$$u^1 = \text{constant} \equiv u_c^1, \quad x^k(u_c^1, u^2, t) \tag{17}$$

Tensor analysis is used throughout this dissertation. No attempt will be made to present a thorough development of this topic. The classical approach to tensors is nicely presented by Sokolnikoff[26]. A more modern treatment is available in an outline by Spivak[27]. Further references to tensor analysis are intended as definitions of notation or as clarification of some features, such as differentiation, which will be used later.

From the modern point of view a tensor is an entity defined without reference to coordinate systems. A tensor is defined as a real valued multilinear mapping over a product space consisting of copies of a vector space and its dual space. A consequence of the definition is that once a basis $\{\underline{e}_i | i = 1,2,\ldots n\}$ has been chosen on the n dimensional vector space and the dual basis $\{\underline{e}^j | j = 1,2,\ldots n\}$ has been associated with the dual space such that $\underline{e}^j(\underline{e}_i) = \delta_i^j$ then any tensor can be decomposed into a linear combination of certain fundamental tensors which are related to this choice of base vectors. The coefficients of such a representation are called the components of the tensor and these differ in general with the choice of bases. A tensor field is a collection of tensors, one defined at each point of a manifold with respect to the local tangent vector space and its dual space. There exists a natural choice of base vectors in both the tangent and dual spaces associated with each coordinate system on the manifold.

If V and V* are a vector space and its dual space respectively and $\underline{T}$ and $\underline{S}$ are typical tensors which for example are defined on the product spaces V x V x V* and V* x V respectively then the tensor product $\underline{T} \vee \underline{S}$ is defined as a tensor over the product space V x V x V* x V* x V. If $(\underline{v}_1,\underline{v}_2,\underline{w}_1) \varepsilon$ V x V x V*, $(\underline{w}_2,\underline{v}_3) \varepsilon$ V* x V, and $(\underline{v}_1,\underline{v}_2,\underline{w}_1,\underline{w}_2,\underline{v}_3) \varepsilon$ V x V x V* x V* x V then the tensor product is defined by

$$\underline{T} \vee \underline{S}(\underline{v}_1,\underline{v}_2,\underline{w}_1,\underline{w}_2,\underline{v}_3) \quad = \quad \underline{T}(\underline{v}_1,\underline{v}_2,\underline{w}_1)\underline{S}(\underline{w}_2,\underline{v}_3)$$

It can be shown that $\{\underline{e}^j \vee \underline{e}^k \vee \underline{e}_i \vee \underline{e}_m \vee \underline{e}^l \mid j,k,i,m,l \;=\; 1,2,\ldots n\}$ are fundamental tensors such that any tensors over V x V x V* x V* x V can be written as a linear combination of these tensors. The coefficients of such a combination form the components of the tensor with respect to the basis $\{\underline{e}_i \mid i = 1,2,\ldots n\}$. Such a tensor is said to be **covariant** of order 3 and contravariant of order 2. This example is easily generalized. A scalar or a tensor field of order 0 is defined to be a real valued function of the coordinates of the underlying manifold.

From the classical point of view a collection of coordinate systems is considered and a set of real functions of position, $t_{\cdot\, j_1 \cdots\cdots j_t}^{i_1 \ldots i_s}$, is associated with each coordinate system x^i. These quantities are the components in the x^i system of a contravariant tensor of order s and a covariant tensor of order t with respect to the transformations relating the coordinate systems considered if

$$t^{A_1 \cdots A_s}_{\cdot\, B_1 \cdots B_t} = X^{A_1}_{,\, i_1} \cdots X^{A_s}_{,\, i_s}\, x^{j_1}_{,\, B_1} \cdots x^{j_t}_{,\, B_t}\, t^{i_1 \cdots i_s}_{\cdot\, j_1 \cdots j_t}$$

where $t^{A_1 \cdots A_s}_{\cdot B_1 \cdots B_t}$ are the quantities associated with the X^A coordinate system.

The components of tensor fields as defined initially transform the same way with respect to coordinate transformations as do the tensors in the classical definition. Once base vectors have been chosen there is a one-one onto correspondence between the quantities which transform in this manner and the tensor entities themselves. The classical notion of tensors suppresses the concept of fundamental or base tensors and deals only with the coefficients. The fact that a tensor is an entity which has various representations is obscured by this point of view. It is common to think of vectors as directed line segments in Euclidean space. This is possible because any vector space of order n is isomorphic to the vector space of line segments directed from the origin of an n dimensional Euclidean space. Quite obviously, a directed line segment, which is the geometrical vector entity, is unaltered by a change of coordinates. It is evident that a given line segment can be decomposed into a linear combination of basic line segments which can be chosen in an infinite number of ways. If $\{\underline{e}_i | i = 1,2,\ldots n\}$ represents a set of basic line segments then any vector $\underline{A}$ can be written as $\underline{A} = A^i \underline{e}_i$. Although a general tensor can not be as conveniently visualized, in some circumstances there is epistomological merit in using notation which explicitly shows the base tensors. Such notation might be helpful when derivatives of tensors are discussed.

In a metric space the metric can be looked upon as a linear transformation which maps the tangent space at any point onto the corresponding dual space in a one-one fashion. Then if A is the linear transformation

$$A(\underline{e}_i) = a_{ij}\underline{e}^j \quad \text{and} \quad A^{-1}(\underline{e}^j) = a^{jk}\underline{e}_k \tag{18}$$

where a_{ij} is the positive definite non-singular matrix of the metric tensor with respect to the bases chosen and a^{kj} is the associated metric tensor defined by $a^{kj}a_{ji} = \delta^k_i$; $\delta^k_i = 1$ if $k = i$ and $\delta^k_i = 0$ if $k \neq i$. If $\underline{T}$ is a tensor over V x V* x V then

$$\underline{T} = T_{j\cdot k}^{\cdot i\cdot}\, \underline{e}^j \vee \underline{e}_i \vee \underline{e}^k$$

By virtue of the metric linear transformation V* x V* x V = A(V) x V* x V is isomorphic to V x V* x V. Then the tensor $T^{li\cdot}_{\cdot\cdot k}\, \underline{e}_l \vee \underline{e}_i \vee \underline{e}^k$ is defined such that

$$T^{li\cdot}_{\cdot\cdot k}\, \underline{e}_l \vee \underline{e}_i \vee \underline{e}^k(\underline{e}^a, \underline{e}^b, \underline{e}_c) = T_{j\cdot k}^{\cdot i\cdot}\, \underline{e}^j \vee \underline{e}_i \vee \underline{e}^k(A^{-1}(\underline{e}^a), \underline{e}^b, \underline{e}_c)$$

$$= T_{j\cdot k}^{\cdot i\cdot}\, \underline{e}^j \vee \underline{e}_i \vee \underline{e}^k(a^{ad}\, \underline{e}_d, \underline{e}^b, \underline{e}_c)$$

Therefore,

$$T^{ab}_{\cdot\cdot c} = a^{ad} T^{\cdot b\cdot}_{d\cdot c} \tag{19}$$

The metric then establishes a mapping between $T^{ab}_{\cdot\cdot c}\, \underline{e}_a \vee \underline{e}_b \vee \underline{e}^c$ and $T^{\cdot b\cdot}_{d\cdot c}\, \underline{e}^d \vee \underline{e}_b \vee \underline{e}^c$. This mapping is a isomorphism between the tensors defined over V* x V* x V and those defined over V x V* x V. This example illustrates that in a metric space the tensors defined over any n copies of the tangent space and the dual space are isomorphic to the tensors defined over any other n copies of the tangent and dual spaces.

On a surface embedded in a space of greater dimension, tensors with components $t^{\alpha_1 \ldots \alpha_p}_{\cdot\cdot\beta_1 \ldots \beta_q}$ can be defined with respect to the tangent plane or with respect to transformations of surface coordinate systems. Since the transformation of surface coordinates and the transformation of spatial coordinates are unrelated, it is possible to define on the surface, in the obvious fashion, mixed spatial and surface tensors, $t^{i_1 \ldots i_s \cdot\cdot \alpha_1 \ldots \alpha_p}_{\cdot\cdot j_1 \ldots j_t \cdot\cdot \beta_1 \ldots \beta_q}$.

When the surface under consideration is embedded in three dimensional Euclidean space the corresponding tangent plane at each point of the surface is contained in the corresponding tangent space of the embedding space. A vector in the tangent plane; that is, a surface vector, can be represented as an element of the two dimensional tangent space associated with a point on the surface or as an element of the three dimensional tangent space associated with a point in the embedding space. Given a surface vector in terms of its surface components, what are the corresponding spatial components?

To answer this question a curve on the surface is considered in the parametric form $x^k = x^k(u^\alpha(q,t),t)$. The contravariant surface vector $u^\alpha_{,q}$ is a tangent to this curve. The corresponding spatial representation of the tangent to the curve is

$$x^k_{,q} = x^k_{,\alpha} u^\alpha_{,q} \tag{20}$$

If A^α is an arbitrary contravariant surface vector then by integrating the equation $u^\alpha_{,q} = A^\alpha$ a curve is obtained such that A^α is a tangent to the curve. Then the spatial representation of this tangent vector,

$$A^i = x^i_{,q} = x^i_{,\alpha} A^\alpha \tag{21}$$

is the contravariant spatial vector associated with the surface vector A^α. As an example the u^2 coordinate curve is considered, where u^2 is the parameter along the curve. Then $\frac{\partial u^\alpha}{\partial u^2}$ is the surface representation of a tangent to the curve and $\frac{\partial x^k}{\partial u^2}$ are the spatial components of this vector. Similarly, the u^1 coordinate curve has a tangent vector given by $\frac{\partial u^\alpha}{\partial u^1}$ and the corresponding spatial representation is $\frac{\partial x^k}{\partial u^1}$.

The fact that the surface is embedded in a metric space induces a a metric upon the surface. The length of a surface vector is defined by the length of the corresponding three dimensional vector in the embedding space. An infinitesimal displacement on a surface at a fixed time, du^α, is given spatially by $dx^i = x^i_{,\alpha} du^\alpha$. Therefore the length squared of this displacement is

$$ds^2 = g_{ij} dx^i dx^j = g_{ij} x^i_{,\alpha} x^j_{,\beta} du^\alpha du^\beta = a_{\alpha\beta} du^\alpha du^\beta \tag{22}$$

where g_{ij} is the spatial metric tensor and

$$a_{\alpha\beta} = g_{ij} x^i_{,\alpha} x^j_{,\beta} \tag{23}$$

defines the induced metric tensor with respect to the surface coordinates such that $a_{\alpha\beta}$ is symmetric and positive definite. Also, the length squared of any other contravariant vector, A^α, is given by $|A^\alpha|^2 = a_{\alpha\beta} A^\alpha A^\beta$ and the cosine of the angle θ between two unit contravariant vectors A^α and B^β is given by $\cos\theta = a_{\alpha\beta} A^\alpha B^\beta$.

The existence of a surface metric tensor has the same consequences for surface tensor analysis as the existence of a metric tensor in the embedding space has on tensor analysis there. A symmetric associated metric tensor, $a^{\alpha\beta}$, can be defined such that

$$a^{\alpha\beta}a_{\beta\gamma} = \delta^{\alpha}_{\gamma} \tag{24}$$

where $\delta^{\alpha}_{\gamma} = 1$ if $\alpha = \gamma$ and $= 0$ if $\alpha \neq \gamma$ in all coordinate systems is the Kronecker delta tensor. $a^{\alpha\beta}$ and $a_{\gamma\rho}$ are involved in the processes of raising and lowering indices. For example, $t^{\alpha\cdot\gamma}_{\cdot\beta\cdot} = a^{\alpha\rho}a_{\beta\sigma}t^{\cdot\sigma\gamma}_{\rho}$. In particular, with every covariant vector A_β there is associated a unique contravariant vector $A^\alpha = a^{\alpha\beta}A_\beta$. It has been seen that $A^i = x^i_{,\alpha}A^\alpha$ is the spatial representation of A^α. Therefore, the spatial components of A_β are

$$A_i = g_{ij}A^j = g_{ij}x^j_{,\alpha}A^\alpha = g_{ij}x^j_{,\alpha}a^{\alpha\beta}A_\beta \tag{25}$$

The quantities

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\,a^{\alpha\sigma}\{a_{\beta\sigma,\gamma} + a_{\gamma\sigma,\beta} - a_{\beta\gamma,\sigma}\} \tag{26}$$

define the Christoffel symbols of the second kind on the surface. They are not tensors but transform in the following way:

$$\Gamma^{\alpha'}_{\beta'\gamma'} = \Gamma^{\alpha}_{\beta\gamma}u^{\alpha'}_{,\alpha}u^{\beta}_{,\beta'}u^{\gamma}_{,\gamma'} + u^{\alpha'}_{,\sigma}u^{\sigma}_{,\beta'\gamma'} \tag{27}$$

Multiplying this by $u^{\rho}_{,\alpha'}$ gives

$$u^{\rho}_{,\beta'\gamma'} = \Gamma^{\alpha'}_{\beta'\gamma'}u^{\rho}_{,\alpha'} - \Gamma^{\rho}_{\beta\gamma}u^{\beta}_{,\beta'}u^{\gamma}_{,\gamma'} \tag{28}$$

The analogous expression in the three dimensional space is also valid and both expressions will prove useful when discussing differentiation.

At a point P of a manifold tensors are defined with respect to the tangent and dual spaces at P. A priori, the tensors at P are unrelated to the tensors at another point Q. If mappings are established between the tangent spaces at P and Q and between the dual spaces at P and Q then these mappings can be used to induce a mapping between the tensors at P and those at Q. Since the tangent spaces at P and Q and the dual spaces at P and Q are of the same order, an infinity of isomorphisms can be established between corresponding spaces at P and Q. A rule which relates the tangent spaces and relates the dual spaces at different points of the underlying manifold is called a connexion. In general, a mapping between the tangent spaces at P and Q is associated with each curve linking the two points. Therefore, the image at Q of a tangent vector given at P will depend upon the curve traversed in going from P to Q. Similar remarks apply to the mappings between the dual spaces at P and Q.

A vector $\underline{V}_P$ at P, which may be an element of the tangent space or the dual space at P, is said to be parallel to a vector $\underline{V}_Q$ at Q with respect to a given curve passing through P and Q if $\underline{V}_P$ is mapped to $\underline{V}_Q$ as a result of the connexion along the curve. In equivalent semantics, that is to say that $\underline{V}_Q$ results from the parallel transport of $\underline{V}_P$ from P to Q along the given curve.

If $x^k = x^k(q)$ is a curve along which a connexion has been defined and $\underline{T}(q)$ is a tensor field along the curve then the absolute derivative of $\underline{T}$ along the curve is defined as

$$\frac{d\underline{T}}{dq} = \lim_{\Delta q \to 0} \frac{\underline{T}_{||}(q + \Delta q) - \underline{T}(q)}{\Delta q} \tag{29}$$

where $\underline{T}_{||}(q + \Delta q)$ is the image at $x^i(q)$ of the element of the tensor field at $x^i(q + \Delta q)$ mapped by the mapping induced by the connexion, and where $\underline{T}(q)$ is the actual element of the tensor field at $x^i(q)$. From the definition it is obvious that $\frac{d\underline{T}}{dq}$ is a tensor of the same kind as $\underline{T}$.

In order that absolute differentiation of tensors satisfy properties analogous to ordinary differentiation of functions it is necessary to impose some restrictions upon the choice of connexion. If the connexion is a linear mapping, then it is seen from the definition that the derivative of a sum is the sum of the derivatives, the derivative of a scalar is the ordinary derivative of a function, and the derivative of a tensor product obeys the rule analogous to the product rule of ordinary differentiation of functions. If $\underline{T}$ is a tensor field not only over the curve but over the n dimensional embedding manifold it would be desirable if differentiation along a given curve could be decomposed into differentiation along n independent directions in a manner analogous to the breakup of a directional derivative of a function. If a linear connexion can be found such that this last property is valid for the differentiation of both contravariant and covariant vectors; that is, for elements of the tangent spaces and dual spaces, then since the product rule is valid with respect to tensor products the differentiation of tensors would have this property as well.

In what follows it will be shown which linear connexions give absolute differentiation the properties requested. Such a connection is assumed to exist. Initially, only contravariant vectors, or elements of the tangent spaces, are considered. $\{\underline{e}_i\}$ and $\{\underline{E}_K\}$ are the coordinate base vectors of the tangent space corresponding to two arbitrary coordinate systems x^i, X^K. The absolute derivative of the base vector $\underline{e}_i$ along the j^{th} coordinate curve is a vector so that it is a linear combination of the base vectors. Thus

$$\frac{d\underline{e}_i}{dx^j} = \Gamma^k_{ij}\, \underline{e}_k \tag{30}$$

Similarly,

$$\frac{d\underline{E}_K}{dX^L} = \Gamma^M_{KL}\, \underline{E}_M \tag{31}$$

The base vectors in the different coordinate systems are related by

$$\underline{e}_i = X^K_{,i}\underline{E}_K \tag{32}$$

Therefore,

$$\Gamma^k_{ij}\underline{e}_k = \frac{d\underline{e}_i}{dx^j} = \frac{d(X^K_{,i}\underline{E}_K)}{dx^j} = X^K_{,ij}\underline{E}_K + X^K_{,i}\frac{d\underline{E}_K}{dx^j}$$

However, because of the assumed nature of the connexion the derivative $\frac{d\underline{E}_K}{dx^j}$ can be decomposed into derivatives along the X^K coordinate directions. Thus

$$\frac{d\underline{E}_K}{dx^j} = \frac{d\underline{E}_K}{dX^L}X^L_{,j} = \Gamma^M_{KL}X^L_{,j}\underline{E}_M$$

and hence

$$\Gamma^k_{ij}\underline{e}_k = X^K_{,ij}\underline{E}_K + X^K_{,i}X^L_{,j}\Gamma^M_{KL}\underline{E}_M$$

but since $\underline{E}_M = x^k_{,M}\underline{e}_k$ and the $\underline{e}_k$ are linearly independent it is seen that

$$\Gamma^k_{ij} = X^K_{,ij}x^k_{,K} + X^K_{,i}X^L_{,j}x^k_{,M}\Gamma^M_{KL} \tag{33}$$

It is easy to show that this transformation of the symbols Γ^k_{ij} called the connexion coefficients for the tangent space is transitive. That is, if these coefficients are specified in one coordinate system then the coefficients in any other system are uniquely found by the above transformation rule.

If $\underline{A} = A^i \underline{e}_i$ is an arbitrary contravariant vector and $x^i = x^i(q)$ is a curve then

$$\frac{d\underline{A}}{dq} = \frac{dA^i}{dq}\underline{e}_i + A^i \frac{d\underline{e}_i}{dq} = \left(\frac{dA^h}{dq} + \Gamma^k_{ij}\frac{dx^j}{dq}A^i\right)\underline{e}_k \tag{34}$$

must be the form of the derivative in the x^k coordinate system, and in fact since the x^i system is arbitrary this must be the form in any coordinate system. The quantity in brackets transforms as a contravariant vector. Thus if its components are zero in one coordinate system they are zero in all coordinate systems. According to the definition of the absolute derivative, the derivative of a parallel vector field must be zero.

With the foregoing in mind, arbitrary continuous functions Γ^k_{ij} are assigned in any x^i coordinate system. Using the transformation rule given in equation (33) for the coefficients of connexions for the tangent space cooresponding quantities are found in all other coordinate systems. Then a contravariant vector $\underline{A}$ is defined as undergoing parallel transport along the curve $x^i = x^i(q)$ if

$$\frac{dA^k}{dq} + \Gamma^k_{ij}\frac{dx^j}{dq}A^i = 0 \tag{35}$$

Because of the transformation law for Γ^k_{ij}, the condition is the same in every coordinate system. Using the theory of linear homogeneous first order systems of differential equations it is seen that a unique solution exists which assumes a given value at a given point on the curve. If $q = 0$ is taken as the given point and $A(0)$ as the given value at $q = 0$ it can be shown that the solution of this system of equations can be written as

$$A^k(q) \;=\; \eta^k_{\cdot j}(q)\; A^j(0)$$

where $\eta^k_{\cdot j}(q)$ is continuous and is independent of $A^i(0)$ and $\eta^k_{\cdot j}(0) = \eta^k_{\cdot j}$[28]. This provides a linear mapping between the vectors at $q = 0$ and those at any other point on the curve. Because there exists a unique solution which takes on any given value at a given point the mapping is one-one and onto. This means that the matrix $\eta^k_{\cdot j}$ is non-singular and an inverse matrix $\zeta^i_{\cdot j}$ exists. The linear transformation with the matrix representation $\eta^k_{\cdot j}$ in the x^i coordinate system is taken as the mapping between the tangent plane at point $q = 0$ and the tangent plane at any point q along the given curve. The connexion is then partially specified but nothing has yet been said about the mappings amongst the dual spaces. Because differentiation of contravariant vectors does not involve the dual spaces the specification of the total connexion can be postponed. In the present context it remains to be shown that the partial connexion defined leads to a derivative of contravariant vectors which has the properties desired.

Because the connexion is continuous the derivatives exist. The linearity of the connexion leaves only the decomposition property of the derivative to be proved. If it can be shown that the derivatives of the coordinate base vectors satisfy this property then the derivative of any contravariant vector does. The absolute derivative along the curve $x^i = x^i(q)$ of the i^{th} coordinate base vector in the x^i coordinate system is

$$\frac{d\underline{e}_i}{dq} = \lim_{q\to 0} (e^k_{i||}(q) - \delta^k_i)\, \underline{e}_k \qquad (37)$$

but $e^k_{i||}(q)\underline{e}_k$ is the image at $q = 0$ of the i^{th} base vector at q, and $e^k_{i||}(q)$ is given by

$$e^k_{i||}(q) = \zeta^k_j(q)\delta^j_i(q)$$

Therefore,

$$\frac{d\underline{e}_i}{dq} = \lim_{q\to 0} \frac{(\zeta^k_i(q) - \delta^k_i)}{q}\, \underline{e}_k = \left.\frac{d\zeta^k_i}{dq}\right|_{q=0} \underline{e}_k \qquad (38)$$

where the limit exists because $\zeta^k_i(q)$ is continuous. Differentiating the relation, $\eta^i_j\zeta^j_k = \delta^i_k$, gives

$$\frac{d\eta^i_j}{dq}\zeta^j_k + \eta^i_j\frac{d\zeta^j_k}{dq} = 0 \qquad (39)$$

If $\underline{A}$ is a vector which is everywhere along $x^i = x^i(q)$ parallel to $\underline{A}(0)$ then

$$A^k(q) = \eta^k_j(q)A^j(0)$$

and

$$-\Gamma^k_{ij}\frac{dx^j}{dq}A^i(q) = \frac{dA^k}{dq} = \frac{d\eta^k_j(q)}{dq}A^j(0) = \frac{d\eta^k_j}{dq}\zeta^j_l A^l(q) \qquad (40)$$

where the first equality comes from the condition for parallel transport of a vector. Because this is to hold for an arbitrary such vector

$$-\Gamma^k_{ij}\frac{dx^j}{dq} = \frac{d\eta^k_j}{dq}\zeta^j_i \qquad (41)$$

By combining equations (39) and (41) it is seen that

$$\eta^i_j \frac{d\zeta^j_k}{dq} = \Gamma^i_{kj} \frac{dx^j}{dq} \tag{42}$$

Multiplication by ζ^l_i gives the result

$$\frac{d\zeta^l_k}{dq} = \Gamma^i_{kj} \zeta^l_i \frac{dx^j}{dq} \tag{43}$$

Substitution into equation (38) gives

$$\frac{d\underline{e}_i}{dq} = \left(\Gamma^m_{ij} \zeta^l_m \frac{dx^j}{dq}\right)\Big|_{q=0} \underline{e}_l \tag{44}$$

but

$$\zeta^l_i(0) = \delta^l_i \text{ so that}$$

$$\frac{d\underline{e}_i}{dq} = \Gamma^l_{ij} \frac{dx^j}{dq} \underline{e}_l \tag{45}$$

Along the x^k coordinate curve,

$$\frac{d\underline{e}_i}{dx^k} = \Gamma^l_{ik} \underline{e}_l$$

therefore,

$$\frac{d\underline{e}_i}{dq} = \frac{d\underline{e}_i}{dx^k} \frac{dx^k}{dq} \tag{46}$$

The absolute derivative of an arbitrary contravariant vector $\underline{A}$ is

$$\frac{d\underline{A}}{dq} = \frac{d(A^i\underline{e}_i)}{dq} = \frac{dA^i}{dq}\underline{e}_i + A^i\frac{d\underline{e}_i}{dq} = (\frac{dA^i}{dx^j} + \Gamma^i_{kj}A^k)\frac{dx^j}{dq}\underline{e}_i$$

(47)

This is the result which was to be proved. It should be noted that the quantity in brackets transforms as a tensor of contravariant order one and covariant order one; it is called the covariant derivative of $\underline{A}$. Also it is evident from the above that if $\underline{A}$ is defined only along the curve $x^i = x^i(q)$, then

$$\frac{d\underline{A}}{dq} = (\frac{dA^i}{dq} + \Gamma^i_{jk}A^j\frac{dx^k}{dq})\underline{e}_i \tag{48}$$

The differentiation of covariant vectors involves only the mapping amongst the dual spaces of the manifold. In order to find partial connexions such that differentiation of covariant vectors satisfies the properties required, an argument is employed which duplicates the discussion given for differentiation of contravariant vectors except that the dual space and covariant vectors are substituted for the tangent space and contravariant vectors. Quantities, $\bar{\Gamma}^l_{kn}$, called the coefficients of connexion for the dual space, are arbitrarily specified in any coordinate system x^i. The corresponding quantities in any other coordinate system, X^K, are found from the transformation rule

$$\bar{\Gamma}^L_{KN} = x^k_{,KN}X^L_{,k} + x^k_{,K}x^n_{,N}X^L_{,l}\bar{\Gamma}^l_{kn} \tag{49}$$

Then a covariant vector $\underline{A}$ is defined as undergoing parallel transport along the curve $x^i = x^i(q)$ if

$$\frac{dA_k}{dq} - \bar{\Gamma}^i_{kj} A_i \frac{dx^j}{dq} = 0 \tag{50}$$

Because of the transformation law for $\bar{\Gamma}^i_{jk}$, the condition is the same in every coordinate system. The solution of this system of linear homogeneous first order differential equations establishes an isomorphism between the dual spaces at any two points and this is taken to partially specify the **connexion.** The derivatives of covariant vectors exist and satisfy the required properties when this partial connexion is used. The absolute derivative of an arbitrary covariant vector $\underline{A}$ is

$$\frac{d\underline{A}}{dq} = \frac{d(A_i\underline{e}^i)}{dq} = \frac{dA_i}{dq}\underline{e}^i + A_i \frac{d\underline{e}^i}{dq} = (\frac{dA_i}{dx^j} - \bar{\Gamma}^k_{ij}A_k) \frac{dx^j}{dq}\underline{e}^i \tag{51}$$

The quantity in brackets is a tensor of convariant order two and contravariant order zero and it is called the covariant derivative of $\underline{A}$. If $\underline{A}$ is defined only along the curve $x^i = x^i(q)$, then

$$\frac{d\underline{A}}{dq} = (\frac{dA_i}{dq} - \bar{\Gamma}^k_{ij} A_k \frac{dx^j}{dq}) \underline{e}^i \tag{52}$$

In the preceding discussion the coefficients of the connexion for the tangent space Γ^i_{jk} and for the dual space $\bar{\Gamma}^i_{jk}$ were in general chosen independent of each other. When these coefficients have been determined the connexion has been specified. The connexion has been chosen such that the derivative of a tensor product of tensors is determined by a rule analogous to the product rule of ordinary differentiation. There is another type of product called the inner product. It will be recalled that elements of the dual space simply map elements of the tangent space into real numbers and elements of the tangent space map elements of the dual space into real numbers. In particular, $\underline{e}^j(\underline{e}_i) = \delta^j_i$. Then the inner product of a covariant vector $V_i\underline{e}^i$ and a contravariant vector, $U^j\underline{e}_j$ is defined as the scalar invariant,

$$V_i\underline{e}^i(U^j\underline{e}_j) = V_iU^j\underline{e}^i(\underline{e}_j) = V_iU^j\delta^i_j = V_iU^i \tag{53}$$

The absolute derivative of an inner product is of course just the ordinary derivative. Is the inner product differentiable according to the product rule? To answer this question the following difference is considered:

$$\frac{d(V_iU^i)}{dq} - [\frac{d(V_i\underline{e}^i)}{dq}(U^j\underline{e}_j) + (V_i\underline{e}^i)\frac{d(U^j\underline{e}_j)}{dq}]$$

$$= \frac{dV_i}{dq}U^i + V_i\frac{dU^i}{dq} - [(\frac{dV_i}{dq}\underline{e}^i + V_i\frac{d\underline{e}^i}{dq})(U^j\underline{e}_j) + (V_i\underline{e}^i)(\frac{dU^j}{dq}\underline{e}_j$$

$$+ U^j\frac{d\underline{e}_j}{dq})]$$

$$= \frac{dV_i}{dq}U^i + V_i\frac{dU^i}{dq} - [\frac{dV_i}{dq}U^i - V_i\bar{\Gamma}^i_{lk}\underline{e}^l\frac{dx^k}{dq}U^j\underline{e}_j + V_i\frac{dU^i}{dq}$$

$$+ V_i\underline{e}^iU^j\bar{\Gamma}^l_{jk}\underline{e}_l\frac{dx^k}{dq}]$$

$$= V_iU^l\frac{dx^k}{dq}(\bar{\Gamma}^i_{lk} - \Gamma^i_{lk}) \tag{54}$$

If the inner product is to be differentiable according to the product rule for arbitrary vectors and for arbitrary curves, $x^i = x^i(q)$ then as seen above

$$\bar{\Gamma}^{i}_{lk} = \Gamma^{i}_{lk} \tag{55}$$

In what follows the coefficients of the connexion are chosen to satisfy this requirement.

Using the product rule for differentiation of tensor products the coefficients of the absolute derivative of a tensor $\underline{T} = T^{i_1 \ldots i_s}_{\cdot\cdot j_1 \ldots j_t} \underline{e}_{i_1} \vee \underline{e}^{j_1} \vee \ldots \vee \underline{e}^{j_t}$ defined along the curve $x^i = x^i(q)$ are

$$\frac{DT^{i_1 \ldots i_s}_{\cdot\cdot j_1 \ldots j_t}}{Dq} = \frac{dT^{i_1 \ldots i_s}_{\cdot\cdot j_1 \ldots j_t}}{dq} + \sum_{k=1}^{s} \Gamma^{i_k}_{ln} \frac{dx^n}{dq} T^{i_1 \ldots i_{k-1} l i_{k+1} \ldots i_s}_{\cdot\cdot j_1 \ldots\ldots\ldots j_t}$$

$$- \sum_{k=1}^{t} \Gamma^{l}_{j_k n} \frac{dx^n}{dq} T^{i_1 \ldots\ldots\ldots i_s}_{\cdot\cdot j_1 \ldots j_{k-1} l j_{k+1} \ldots j_t} \tag{56}$$

If $\underline{T}$ is a tensor over the embedding manifold then the coefficients of the covariant derivative of this tensor, a tensor of covariant order one greater than $\underline{T}$, are defined as

$$T^{i_1 \ldots i_s}_{\cdot\cdot j_1 \ldots j_t ; n} = T^{i_1 \ldots i_s}_{\cdot\cdot j_1 \ldots j_t , n} + \sum_{k=1}^{s} \Gamma^{i_k}_{ln} T^{i_1 \ldots i_{k-1} l i_{k+1} \ldots i_s}_{\cdot\cdot j_1 \ldots\ldots\ldots j_t}$$

$$- \sum_{k=1}^{t} \Gamma^{l}_{j_k n} T^{i_1 \ldots\ldots\ldots i_s}_{\cdot\cdot j_1 \ldots j_{k-1} l j_{k+1} \ldots j_t} \tag{57}$$

and the coefficients of the absolute derivative are given by

$$\frac{DT^{i_1 \ldots i_s}_{\cdot\cdot j_1 \ldots j_t}}{Dq} = T^{i_1 \ldots i_s}_{\cdot\cdot j_1 \ldots j_t ; n} \frac{dx^n}{dq} \tag{58}$$

The contraction of two indices of the tensor $\underline{T}$ associates a tensor with $\underline{T}$ which is of covariant order one less and contravariant order one less than $\underline{T}$. For example, if the first contravariant index and the last covariant index of the tensor $T^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots j_t}\, \underline{e}_{i_1} \vee \underline{e}^{j_1} \vee \cdots \vee \underline{e}^{j_t}$ are contracted then the resulting tensor is

$$(T^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots j_t}\, \underline{e}_{i_1} (\underline{e}^{j_t}))\, \underline{e}^{j_1} \vee \cdots \vee \underline{e}_{i_s} = T^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots i_1}\, \underline{e}^{j_1} \vee \cdots \vee \underline{e}_{i_s},$$

where the $T^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots i_1}$

transform as do the coefficients of a tensor of covariant order one less and contravariant order one less than $\underline{T}$. By making use of the product rules for inner products and for tensor products it is seen that the coefficients of the absolute derivative of the contracted tensor are

$$\frac{DT^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots i_1}}{Dq} = \frac{dT^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots i_1}}{dq} + \Gamma^{i_1}_{ln}\, T^{l \cdot i_2 \dots i_s}_{\cdot j_1 \dots i_1} \frac{dx^n}{dq} - \Gamma^{l}_{i_1 n}\, T^{i_1 \dots i_s}_{\cdot\cdot j_1 \cdot j_{t-1} l} \frac{dx^n}{dq}$$

$$+ \sum_{k=2}^{s} \Gamma^{i_k}_{ln}\, T^{i_1 \dots i_{k-1} l i_{k+1} \dots i_s}_{\cdot\cdot j_1 \dots\dots i_1} \frac{dx^n}{dq} - \sum_{k=1}^{s} \Gamma^{l}_{j_k n}\, T^{i_1 \dots\dots i_s}_{\cdot\cdot j_1 \cdot\cdot j_{k-1} l j_{k+1} \cdot\cdot i_1} \frac{dx^n}{dq}$$

$$= \frac{dT^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots i_1}}{dq} + \sum_{k=2}^{s} \Gamma^{i_k}_{ln}\, T^{i_1 \dots i_{k-1} l i_{k+1} \dots i_s}_{\cdot\cdot j_1 \dots\dots i_1} \frac{dx^n}{dq}$$

$$- \sum_{k=1}^{t-1} \Gamma^{l}_{j_k n}\, T^{i_1 \dots\dots i_s}_{\cdot\cdot j_1 \cdot\cdot j_{k-1} l j_{k+1} \cdot\cdot i_1} \frac{dx^n}{dq} \qquad (59)$$

A final remark about differentiation and the connexion concerns the metric tensor on a manifold where such an entity exists. It has already been mentioned that the matrix of coefficients of the metric tensor can be considered as the representation of a one-one onto linear transformation mapping the tangent onto the dual space such that

$$T(\underline{e}_i) = a_{ij}\underline{e}^j, \quad T(A^i\underline{e}_i) = A^i a_{ij}\underline{e}^j \tag{60}$$

and

$$T^{-1}(\underline{e}^j) = a^{jk}\underline{e}_k, \quad T^{-1}(A_j\underline{e}^j) = A_j a^{jk}\underline{e}_k \tag{61}$$

In general, the linear transformation T is a function of position on the manifold. If a contravariant vector field, $A^k\underline{e}_k$ is given along a curve $x^i = x^i(q)$, then the linear transformation T associates with it a covariant vector field $A_j\underline{e}^j$ such that $A_j = A^k a_{kj}$. In order to determine whether T commutes with the process of taking the absolute derivative the following difference is considered:

$$T(\frac{d}{dq}(A^k\underline{e}_k) - \frac{d}{dq}(T(A^k\underline{e}_k)) = T((\frac{dA^k}{dq} + \Gamma^k_{ij}\frac{dx^j}{dq}A^i)\underline{e}_k) - \frac{d}{dq}(A_j\underline{e}^j)$$

$$= (\frac{dA^k}{dq} + \Gamma^k_{ij}\frac{dx^j}{dq}A^i)a_{kl}\underline{e}^l - (\frac{dA_j}{dq} - \Gamma^l_{jk}\frac{dx^k}{dq}A_l)\underline{e}^j$$

$$= (\frac{dA^k}{dq} + \Gamma^k_{ij}\frac{dx^j}{dq}A^i)a_{kl}\underline{e}^l - (\frac{d}{dq}(A^n a_{nj}) - \Gamma^l_{jk}\frac{dx^k}{dq}A^n a_{nl})\underline{e}^j$$

$$= -\frac{dx^k}{dq}A^n(a_{nj,k} - \Gamma^l_{nk}a_{lj} - \Gamma^l_{jk}a_{nl})\underline{e}^j \tag{62}$$

Therefore, T and absolute differentiation commute for all curves $x^i = x^i(q)$ and all vectors A^n if and only if

$$a_{nj,k} - \Gamma^l_{nk}\, a_{lj} - \Gamma^l_{jk}\, a_{nl} = 0 \tag{63}$$

In what follows it will be convenient to consider coefficients of the connexion which are symmetric in the linear indices. Then permutation of the indices in equation (63) gives two additional equations and substraction of one of these from the sum of the other two produces a result which when multiplied by the associated metric tensor yields

$$\Gamma^i_{nk} = a^{ij}(a_{nj,k} + a_{jk,n} - a_{kn,j}) \tag{64}$$

Therefore, the connexion coefficients must be the Christoffel symbols if T and absolute differentiation are to commute. The covariant derivative and the absolute derivative of the metric tensor are then zero. Therefore when the Christoffel symbols are chosen as the connexion coefficients parallel propagation of vectors leaves the magnitude of vectors and the angles between vectors invariant.

Tensor fields or any functions defined over some region of space may also be functions of various parameters. The most frequently occuring parameter is time. It is possible to define a number of different derivatives involving time. Before investigating some of these derivatives perhaps it is best to clarify a few points.

Suppose that $A(x^i,t)$ is a function defined over some region of space, where x^i are the coordinates, and it also depends upon the time parameter t. Then for every value of t, A has a value for every point in its domain. In general, if attention is fixed on a given point in space x^i then the value of the function A at that point varies with time. Rather than record the variation of A with respect to position at a fixed time or with respect to time at a fixed position it is frequently required that the variation of A with respect to time be determined when attention is fixed on a point which traverses space as a prescribed function of time; that is, $x^i = x^i(t)$. The word "observer" is often used to refer to the point of fixation whose trajectory is a prescribed function of time.

A coordinate system is said to be fixed in space if each geometrical point of physical space is considered to have the same coordinates for all times. Coordinate systems which are got from a fixed system by means of a time dependent coordinate transformation are called moving coordinate systems. Thus if x^i is a fixed system and $x^{k'}$ is a moving system

$$x^{k'} = x^{k'}(x^k, t), \quad x^k = x^k(x^{k'}, t)$$

An observer is said to be "fixed with respect to the coordinate system $x^{k'}$" when the point of fixation is given by $x^k = x^k(x^{k'}, t)$ where $x^{k'}$ is kept constant.

The absolute and covariant derivatives of tensors have already been discussed. A tensor field $\underline{T}$ is often dependent upon the time parameter t, so that $\underline{T} = \underline{T}(x^i, t)$. In what follows various time derivatives of such a tensor field shall be investigated.

Instead of considering t as a parameter it is instructive to consider at time t the original n dimensional manifold, for which x^i are coordinates, as a surface embedded in an n+1 dimensional space, for which x^i and t are coordinates, in such a way that t = constant is the equation of the submanifold. A unique coordinate surface then passes through every point of the n+1 dimensional manifold. At each point in the n+1 dimensional space the tangent plane of the coordinate surface t = constant forms an n dimensional vector space. The dual space associated with the tangent plane is another such space. Tensors can be defined at each point of the embedding manifold over copies of these n dimensional vector spaces. The notation $\underline{T}(x^i,t)$ denotes the element at x^i,t of such a tensor field. In keeping with the point of view in which t is considered as a parameter and $\underline{T}$ is a tensor over the n dimensional manifold with coordinates x^i, it is of interest to consider the representation of the tensor in coordinate systems for the n+1 dimensional manifold which results from the system described above when transformations of the following form are undergone:

$$X^K = X^K(x^i,t), \quad K = 1,2,..n, \quad i = 1,2,..n \tag{65}$$

$$X^{n+1} = \sigma = \sigma(t) \tag{66}$$

where these relations are invertable.

The discussion of derivatives of tensors of the type described above is very similar to the earlier discussion of absolute differentiation. The first requirement is to be able to compare tensors defined at different points in the n+1 dimensional manifold. As before, the existence of a mapping amongst the various tangent planes and of a mapping amongst the corresponding dual spaces induces mappings amongst the tensors defined with respect to these vector spaces. The mappings amongst the tangent planes together with the mappings amongst the dual spaces will be referred to as connexions. The connexion between two points will in general depend upon the curve traversed in going from one point to another.

If $x^\alpha = x^\alpha(q)$, $\alpha = 1,2,..,n+1$ and $x^{n+1} = t$, is a curve in the n+1 dimensional manifold along which a connexion has been defined, and $\underline{T}(q)$ is a tensor field along the curve, the element of the field at each point being defined with respect to the local tangent plane and the corresponding dual space, then the derivative of $\underline{T}$ along the curve is defined as

$$\frac{d\underline{T}}{dq} = \lim_{\Delta q \to 0} \frac{\underline{T}_{||}(q + \Delta q) - \underline{T}(q)}{\Delta q} \tag{67}$$

where $\underline{T}_{||}(q+ \Delta q)$ is the image at $x^\alpha(q)$ of the element of the tensor field at $x^\alpha = x^\alpha(q + \Delta q)$ mapped by the mapping induced by the connexion and $\underline{T}(q)$ is the actual element of the tensor field at $x^\alpha(q)$. It is obvious from the definition that $\frac{d\underline{T}}{dq}$ is a tensor of the same kind as $\underline{T}$.

The development from this point onward is nearly the same as the previous discussion of absolute differentiation and so only an outline will be given. In order that the differentiation defined here satisfy properties analogous to those satisfied by ordinary differentiation of functions, the connexion is taken to be a linear mapping. If $\underline{T}$ is defined not only along a curve but over the n+1 dimensional embedding manifold then the connexion is restricted so that differentiation along any curve can be decomposed into differentiation along n+1 independent directions in a manner analogous to the breakup of a directional derivative of a function. Because of the linearity of the connexion, the derivative of a tensor product obeys the product rule. Therefore, if a linear connexion can be found such that differentiation of covariant and contravariant vectors has the properties required then differentiation of all tensors has these properties.

If $\{\underline{e}_i\}$ and $\{\underline{E}_K\}$ are the coordinate base vectors of the tangent plane in the x^i and X^K coordinate systems respectively, if a connexion is assumed to exist such that differentiation has the required properties, and if

$$\frac{d\underline{e}_i}{dx^\alpha} = \Gamma^j_{i\alpha}\,\underline{e}_j \qquad i = 1,2,..n;\ \alpha = 1,2,..,n+1;\ x^{n+1} = t \tag{68}$$

and

$$\frac{d\underline{E}_K}{dX^\Delta} = \Gamma^L_{K\Delta}\,\underline{E}_L \qquad K = 1,2,..n;\ \Delta = 1,2,..,n+1;\ X^{n+1} = \sigma \tag{69}$$

then

$$\Gamma^j_{i\alpha}\,\underline{e}_j = \frac{d\underline{e}_i}{dx^\alpha} = \frac{d}{dx^\alpha}(X^K_{,i}\underline{E}_K) = X^K_{,i\alpha}\,\underline{E}_K + X^K_{,i}\,\frac{d\underline{E}_K}{dx^\alpha}$$

but

$$\frac{d\underline{E}_K}{dx^\alpha} = \frac{d\underline{E}_K}{dX^\Delta}\,X^\Delta_{,\alpha} = \Gamma^L_{K\Delta}\,\underline{E}_L\,X^\Delta_{,\alpha}$$

so that

$$\Gamma^j_{i\alpha}\,\underline{e}_j = X^K_{,i\alpha}\,\underline{E}_K + X^K_{,i}X^\Delta_{,\alpha}\Gamma^L_{K\Delta}\underline{E}_L$$

$$= (X^K_{,i\alpha}x^j_{,K} + X^K_{,i}X^\Delta_{,\alpha}x^j_{,L}\Gamma^L_{K\Delta})\underline{e}_j \tag{70}$$

Therefore,

$$\Gamma^j_{i\alpha} = X^K_{,i\alpha}x^j_{,K} + X^K_{,i}X^\Delta_{,\alpha}x^j_{,L}\Gamma^L_{K\Delta} \tag{71}$$

If $\underline{A} = A^i \underline{e}_i$ is an arbitrary contravariant vector and $x^\alpha = x^\alpha(q)$ is any curve in the n+1 dimensional manifold then

$$\frac{d\underline{A}}{dq} = \frac{d}{dq}(A^i \underline{e}_i) = \frac{dA^i}{dq}\underline{e}_i + A^i \frac{d\underline{e}_i}{dq} = (\frac{dA^i}{dq} + \Gamma^i_{k\alpha} A^k \frac{dx^\alpha}{dq})\underline{e}_k$$

would be the form of the derivative in any coordinate system. The quantity in brackets, because of the transformation properties of $\Gamma^j_{i\alpha}$, transforms like a contravariant vector.

The above discussion provides the motivation for the following definition of the parallel propagation of a contravariant vector $\underline{A}$ along the curve $x^\alpha = x^\alpha(q)$. $\underline{A}$ undergoes parallel transport along the curve if

$$\frac{dA^k}{dq} + \Gamma^k_{i\alpha} A^i \frac{dx^\alpha}{dq} = 0 \tag{73}$$

The condition is the same in all the allowable coordinate systems. As before, once the $\Gamma^k_{i\alpha}$ have been chosen, the integration of these equations along the curve $x^\alpha = x^\alpha(q)$ in the n+1 dimensional manifold provides a one-one onto linear transformation amongst the tangent planes at the different points on the curve. This is taken as the partial connexion. Without duplicating the previous analysis given with respect to absolute differentiation here it suffices to say that the derivative of a contravariant vector $\underline{A}$ exists and satisfies the required properties when the differentiation is defined with respect to a partial connexion as specified. Then

$$\frac{d\underline{A}}{dq} = (\frac{dA^i}{dq} + \Gamma^i_{k\alpha} A^k \frac{dx^\alpha}{dq})\underline{e}_i = (\frac{dA^i}{dx^\alpha} + \Gamma^i_{k\alpha} A^k)\frac{dx^\alpha}{dq}\underline{e}_i \tag{74}$$

where the last equation results if $\underline{A}$ is defined everywhere and not only on the curve. The first term accounts for the change in the coefficients and the second term for the change in the base vectors.

The differentiation of covariant vectors is treated in a similar fashion. Coefficients of connexion $\bar{\Gamma}^i_{j\alpha}$, which satisfy the same transformation laws as the $\Gamma^i_{j\alpha}$, are introduced in any coordinate system and the transformation rules are used to determine the $\bar{\Gamma}^i_{j\alpha}$ in any other coordinate system. Then a covariant vector $\underline{A}$ is defined as undergoing parallel transport along the curve $x^\alpha = x^\alpha(q)$ in the n+1 dimensional manifold if

$$\frac{dA_k}{dq} - \bar{\Gamma}^i_{k\alpha} A_i \frac{dx^\alpha}{dq} = 0 \qquad (75)$$

This condition is the same in all the permissible coordinate systems because of the transformation law $\bar{\Gamma}^i_{k\alpha}$. The solution of this system of linear homogeneous first order differential equations establishes an isomorphism between the dual spaces at any two points along the curve. This mapping is taken as a partial connexion. With this partial connexion the derivative of a covariant vector $\underline{A}$ exists and satisfies the required properties. Then

$$\frac{d\underline{A}}{dq} = \left(\frac{dA_i}{dq} - \bar{\Gamma}^k_{i\alpha} A_k \frac{dx^\alpha}{dq}\right)\underline{e}^i = \left(\frac{dA_i}{dx^\alpha} - \Gamma^k_{i\alpha} A_k\right) \frac{dx^\alpha}{dq} \underline{e}^i \qquad (76)$$

where the last equation results if $\underline{A}$ is defined over the n+1 dimensional manifold and not only on the curve. The first term accounts for the change in the coefficients and the second term for the change in the base vectors.

The condition that the inner product of a covariant vector $V_i \underline{e}^i$ and and a contravariant vector $U^j \underline{e}_j$ be differentiable according to the product rule is investigated by looking at the following difference,

$$\frac{d}{dq}(V_i U^i) - [\frac{d}{dq}(V_i \underline{e}^i)(U^j \underline{e}_j) + V_i \underline{e}^i \frac{d}{dq}(U^j \underline{e}_j)] = V_i U^l \frac{dx^\alpha}{dq} (\bar{\Gamma}^i_{l\alpha} - \Gamma^i_{l\alpha}) \tag{77}$$

The detailed steps are analogous to those used in deriving equation (54). If the inner product is to be differentiable according to the product rule for arbitrary vectors and for arbitrary curves then as seen above

$$\bar{\Gamma}^i_{l\alpha} = \Gamma^i_{l\alpha} \tag{78}$$

This is taken to be the case in the ensuing discussion.

Using the product rule for differentiation of tensor products, the coefficients of the derivative of a tensor with coefficients $T^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots j_t}$ defined along the curve $x^\alpha = x^\alpha(q)$ are

$$\frac{D}{Dq} T^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots j_t} = \frac{d}{dq} T^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots j_t} + \sum_{k=1}^{s} \Gamma^{i_k}_{l\alpha} \frac{dx^\alpha}{dq} T^{i_1 \dots i_{k-1} l_{k+1} \dots i_s}_{\cdot\cdot j_1 \dots\dots\dots\dots j_t}$$

$$- \sum_{k=1}^{t} \Gamma^l_{j_k \alpha} \frac{dx^\alpha}{dq} T^{i_1 \dots\dots\dots\dots i_s}_{\cdot\cdot j_1 \cdot\cdot j_{k-1} l j_{k+1} \cdot\cdot j_t} \tag{79}$$

If the tensor is defined over the embedding manifold then the coefficients of the covariant derivative of this tensor are

$$T^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots j_t;\alpha} = \frac{d}{dx^\alpha} T^{i_1 \dots i_s}_{\cdot\cdot j_1 \dots j_t} + \sum_{k=1}^{s} \Gamma^{i_k}_{l\alpha} T^{i_1 \dots i_{k-1} l i_{k+1} \cdot\cdot i_s}_{\cdot\cdot j_1 \dots\dots\dots\dots j_t}$$

$$- \sum_{k=1}^{t} \Gamma^l_{j_k \alpha} T^{i_1 \dots\dots\dots\dots i_s}_{\cdot\cdot j_1 \cdot\cdot j_{k-1} l j_{k+1} \cdot\cdot j_t} \tag{80}$$

and the coefficients of the derivative are

$$\frac{D}{Dq} T^{i_1 \ldots\ldots i_s}_{\ldots j_1 \ldots \ldots j_t} = T^{i_1 \ldots\ldots i_s}_{\ldots j_1 \ldots\ldots j_{t;\alpha}} \frac{dx^\alpha}{dq} \tag{81}$$

The differentiation of a contracted tensor follows the same rules as established for the absolute differentiation of a contracted tensor.

If the n dimensional manifold, which has been viewed as a surface embedded in an n+1 dimensional manifold, has a metric then the matrix of coefficients of that metric can be considered to be the representation of a one-one onto linear transformation which at any point of the n+1 dimensional manifold maps the tangent plane of the surface t = constant onto the corresponding dual space. If $A^k \underline{e}_k$ is a contravariant vector field then the linear transformation T which was just described associates with it a covariant vector field $A_j \underline{e}^j$ such that $A_j = A^k a_{kj}$. In order to determine whether T commutes with the process of taking the derivative the following difference is considered

$$T(\frac{d}{dq}(A^k \underline{e}_k)) - \frac{d}{dq}(T(A^k \underline{e}_k)) = -\frac{dx^\alpha}{dq} A^i (a_{ij,\alpha} - \Gamma^l_{i\alpha} a_{lj} - \Gamma^l_{j\alpha} a_{il}) \underline{e}^j \tag{82}$$

where the details leading to this equation are analogous to those leading to equation (62). Thus T and differentiation commute for all curves and all vectors if and only if

$$a_{ij,\alpha} - \Gamma^l_{i\alpha} a_{lj} - \Gamma^l_{j\alpha} a_{il} = 0 \tag{83}$$

This can be written as

$$a_{ij,k} - \Gamma^l_{ik} a_{lj} - \Gamma^l_{jk} a_{il} = 0$$

$$a_{ij,n+1} - \Gamma^l_i a_{lj} - \Gamma^l_j a_{il}$$

where by definition $\Gamma^l_i = \Gamma^l_{i_{n+1}}$

For the sake of convenience it is assumed that

$$\Gamma^i_{jk} = \Gamma^i_{kj} \tag{84}$$

Then as already seen when considering absolute differentiation the first of these conditions is equivalent to taking the connexion coefficients Γ^i_{jk} as the Christoffel symbols. Because of their transformation rule the connexion coefficients Γ^K_{LN} are the Christoffel symbols in all permissible coordinate systems. The second of these conditions is really $\frac{n(n+1)}{2}$ independent equations for the n^2 quantities Γ^l_n. A solution to these equations, but not the only one, is

$$\Gamma^j_i = \frac{1}{2} a^{jk} a_{ik,n+1} \tag{85}$$

If a coordinate system can be found for which a_{ij} is independent of time then choosing $\Gamma^i_j = 0$ is consistant with condition (85). Then the condition for a contravariant vector to undergo parallel propagation along the coordinate curve $t = t$, x^i = constant is

$$\frac{dA^i}{dt} + \Gamma^i_j A^j \frac{dx^{n+1}}{dt} = \frac{dA^i}{dt} = 0 \tag{86}$$

or A^i = constant. Therefore, in this coordinate system the coordinate base vectors $\underline{e}_i$ at any point on the coordinate curve $t = t$, x^i = constant map into the coordinate base vectors at any other point on the curve. In some other coordinate system X^{Δ}, $\Delta = 1,2,..n+1$; $X^{n+1} = \sigma$ where $X^{K} = X^{K}(x^{i},t), \sigma = \sigma(t)$ the quantities $\Gamma^{K}_{I} \neq 0$ in general. This is seen from the transformation rules for the connexion coefficients. Under these circumstances, that is when Γ^{i}_{jk} are chosen as the Christoffel symbols and $\Gamma^{i}_{j} = 0$ and a_{ij} is independent of time, then parallel propagation along any path in the n+1 dimensional manifold is such that the magnitude of a vector and angle between two vectors are left invariant. The derivative of the metric tensor is zero.

If the Christoffel symbols are chosen for Γ^{i}_{jk} and $\Gamma^{i}_{j}=0$ but a_{ij} is not independent of time, a linear connexion has been defined which satisfies all the properties required except now differentiation along an arbitrary path and application of the linear transformation associated with the metric tensor do not commute. It is readily seen from equation (82) that these operations do commute when the curve is restricted to be of the form t = constant, $x^{i} = x^{i}(q)$. For a general path however such a connexion does not lead to parallel propagation which preserves lengths or angles. The derivative of the metric tensor is then non zero in general over an arbitrary path.

If the connexion coefficients are chosen in an arbitrary coordinate system X^{Δ}, $\Delta = 1,2,..n+1$; $X^{n+1} = \sigma(t)$, such that Γ^{K}_{LM} are the Christoffel symbols and $\Gamma^{K}_{L} = 0$ then the connexion coefficients in any other coordinate system x^{α}, $\alpha = 1,2,..n+1$ such that $x^{i} = x^{i}(X^{K},\sigma)$, $x^{n+1} = x^{n+1}(X^{n+1})$ can be found from the transformation rule. The quantities Γ^{i}_{jk} quite obviously are the Christoffel symbols. The quantities Γ^{i}_{j} are not in general zero. The transformation rule given in equation (71) yields

$$\Gamma_i^j = X^K_{,in+1}x^j_{,K} + X^K_{,i}X^{n+1}_{,n+1}x^j_{,L}\Gamma^L_K + X^K_{,i}X^M_{,n+1}x^j_{,L}\Gamma^L_{KM} \tag{87}$$

Since $\Gamma^L_K = 0$,

$$\Gamma_i^j = (X^K_{,n+1}|x^l)_{,i}\, x^j_{,K} + X^K_{,i}X^M_{,n+1}x^j_{,L}\Gamma^L_{KM} \tag{88}$$

but

$$\Gamma^L_{KM} = x^k_{,KM}X^L_{,k} + x^k_{,K}x^l_{,M}X^L_{,n}\Gamma^n_{kl} \tag{89}$$

so that

$$\Gamma_i^j = (X^K_{,n+1}|x^l)_{,i}\, x^j_{,K} + X^K_{,i}X^M_{,n+1}x^j_{,KM} + X^M_{,n+1}x^l_{,M}\Gamma^j_{il} \tag{90}$$

however,

$$(x^j_{,M}X^M_{,n+1}|x^m)_{,K} = x^j_{,KM}X^M_{,n+1} + x^j_{,M}(X^M_{,n+1}|x^i)_{,K} \tag{91}$$

and together with the chain rule these results can be combined to yield

$$\Gamma_i^j = (x^j_{,M}X^M_{,n+1}|x^m)_{,K}X^K_{,i} + (X^M_{,n+1}|x^m)x^l_{,M}\Gamma^j_{il} \tag{92}$$

By writing $x^i = x^i(X^K(x^j, x^{n+1}), x^{n+1})$ and considering the derivative with respect to x^{n+1} keeping x^j fixed it is seen that

$$0 = (x^i_{,n+1}|X^K) + x^i_{,K}(X^K_{,n+1}|x^m) \tag{93}$$

Therefore,

$$x^j_{,M}(X^M_{,n+1}|x^m) = -(x^j_{,n+1}|X^K) \tag{94}$$

When this is substituted into equation (92), the following result is proved:

$$\Gamma_i^j = -(x^j_{,n+1}|X^K)_{,i} - (x^l_{,n+1}|X^K)\Gamma^j_{il} \tag{95}$$

The $n+1^{th}$ coordinate in any of the allowable systems is always an invertible function of the time t alone. Therefore,

$$\Gamma^j_i = -\left(\underset{X^K}{v^j}\frac{dt}{dx^{n+1}}\right)_{,i} - \left(\underset{X^K}{v^l}\frac{dt}{dx^{n+1}}\right)\Gamma^j_{il} = -\underset{X^K,i}{v^j}\frac{dt}{dx^{n+1}} \tag{96}$$

where $\underset{X^K}{v^j} = x^j_{,t}|X^K$ is the velocity of the X^K coordinate system relative to the x^j system. In all coordinate systems in which $x^{n+1} = t$ this becomes

$$\Gamma^j_i = -\underset{X^K,i}{v^j} \tag{97}$$

For convenience sake, only such coordinate systems will be considered in the future.

Time derivatives are derivatives along some curve in the n+1 dimensional manifold on which the time t is taken as the parameter. Such a curve is given in general by $x^i = x^i(t)$, $t = t$. The curve specifies the observer. If the connexion is specified in some coordinate system X^Δ, $\Delta = 1,2,..n+1$; $X^{n+1} = t$, by the Christoffel symbols Γ^I_{KJ} and by $\Gamma^I_J = 0$ then the coefficients of the time derivative along the curve $X^K = X^K(t)$, $t = t$, of some contravariant vector in any coordinate system x^α, $\alpha = 1,2,..n+1$; $x^{n+1} = t$, as given by equation (79) are

$$\frac{DA^j}{Dt}\bigg|^{X^K}_{x^j = x^j(t)} = \frac{dA^j}{dt} + \Gamma^j_{ik}A^i\frac{dx^k}{dt} - \underset{X^K;i}{v^j}A^i \tag{98}$$

where the upper letter after the vertical line indicates in which coordinate system the connexion coefficients were chosen and the lower expression specifies the observer. If A^j is defined everywhere then

$$\frac{DA^j}{Dt}\bigg|^{X^K}_{xj = xj(t)} = \frac{\partial A^j}{\partial t}\bigg|_{xi} + A^j_{,k}\frac{dx^k}{dt} - \underset{X^K}{v^j}{}_{;i}A^i \tag{99}$$

It should be recalled that this means that

$$\frac{dA}{dt}\bigg|_{x^j = x^j(t)}^{X^K} = \frac{DA^j}{Dt}\bigg|_{x^j = x^j(t)}^{X^K} \underline{e}_j \qquad (100)$$

If the x^α coordinate system is taken to be the X^Δ system then

$$\frac{DA^K}{Dt}\bigg|_{X^J = X^J(t)}^{X^L} = \frac{\partial A^K}{\partial t}\bigg|_{X^L} + A^K_{;L}\frac{dX^L}{dt} \qquad (101)$$

which is consistent with the choice of connexion coefficients in the X^Δ system.

When the curve in the n+1 dimensional manifold is given in some θ^λ coordinate system, $\lambda = 1,2,..n+1$; $\theta^{n+1} = t$ by θ^β = constant, t = t then the time derivative of a contravariant vector can be expressed in the x^α coordinate system as

$$\frac{\partial A}{\partial t}\bigg|_{\theta^\beta}^{X^K} = \left(\frac{\partial A^i}{\partial t}\bigg|_{xj} + A^i_{;j}\, v^j_{\theta^\beta} - A^j\, v^i_{X^L;j}\right)\underline{e}_i \qquad (102)$$

Where the lower letter following the vertical line is to indicate that the observer is specified by keeping this quantity fixed. Furthermore, if the θ^λ coordinate system corresponds with the X^K system, the coordinate system in which the connexion coefficients were chosen such that $\Gamma^K_I = 0$, then the above equation reduces to

$$\frac{\partial A}{\partial t}\bigg|_{X^K} = \left(\frac{\partial A^i}{\partial t}\bigg|_{x^i} + A^i_{;j}\, v^j_{X^K} - A^j\, v^i_{X^K;j}\right)\underline{e}_i$$

$$= \left(\frac{\partial A^i}{\partial t}\bigg|_{x^i} + A^i_{,j}\, v^j_{X^K} - A^j\, v^i_{X^K,j}\right)\underline{e}_i \qquad (103)$$

This case is called the Lie derivative of the vector $\underline{A}$. It is evident that the preceding examples can be generalized to give the derivatives of any tensor. Once the derivatives of the base vectors in the tangent plane and the dual space are known the product rule for differentiation can be used to get the derivative of any tensor.

It should be noted that the above discussion includes absolute differentiation as the special case where t = constant, $x^i = x^i(q)$. The separate treatment of absolute differentiation is really redundant, but it was included here because of its epistomological merit.

It has been seen that time differentiation of tensors involves the specification of a path or an observer, a connexion, and the representation of the components of the derivative in different coordinate systems. An assumption is usually made that physical space can be coordinized by rectangular Cartesian coordinates which have a time independent metric associated with them. If in this coordinate system Γ^i_{jk} are chosen to be the Christoffel symbols and Γ^i_j are chosen to vanish then lengths and angles are preserved during parallel propagation. This choice of the connexion leads to the usual covariant time derivative. It is thus seen that the above assumption together with the requirement that parallel propagation preserve lengths and angles leads naturally to the usual time derivative. This assumption and the parallel propagation condition need not be included in general.

The concept of a tensor field can be broadened. If M and N are two different manifolds with coordinates x^i, $i = 1,2,..m$, and u^α, $\alpha = 1,2,..n$, respectively than double tensor fields depending on the parameter t can be defined on a manifold with coordinates x^i, u^α,t. Through each point in this m + n + 1 dimensional manifold there passes a unique pair of submanifolds from the two families of submanifolds, u^α = constant, t = constant which is m dimensional and has m dimensional tangent and dual vector spaces, V and V*, associated with it, and x^i = constant, t = constant which is n dimensional and has n dimensional tangent and dual spaces, U and U* associated with it. A double tensor field is defined such that at each point of the m + n + 1 dimensional manifold its element is a multilinear real valued mapping of a product space consisting of copies of V and V* followed by copies of U and U*. For example, if $\underline{T}$ is a real valued multilinear mapping of V x V* x V* x U* x U xU then $\underline{T}$ is said to be a double tensor of contravariant order two and covariant order one with respect to M and of contravariant order one and covariant order two with respect to N. If in some x^i, u^α, t coordinate system $\{\underline{e}_i\}$ and $\{\underline{e}^i\}$ where i = 1,2,..m are the coordinate base vectors of V and V* and $\{\underline{f}_\alpha\}$ and $\{\underline{f}^\alpha\}$ where α = 1,2,..n are the corresponding quantities of U and U* then it can be shown that the set of double tensors over V x V* x V* x U* x U x U,

$$\{\underline{e}^i \vee \underline{e}_j \vee \underline{e}_k \vee \underline{f}_\alpha \vee \underline{f}^\beta \vee f^\gamma | i,j,k \; = \; 1,2,..m \text{ and } \alpha,\beta,\gamma = 1,2,..n\}$$

where v is the analogue of the tensor product, is such that any other double tensor over this product space can be expanded as a linear combination of the double tensors in this set. Thus

$$\underline{T} \;=\; T_{i\ldots\beta\gamma}^{\;jk\alpha}\, \underline{e}^i \vee \underline{e}_j \vee \underline{e}_k \vee \underline{f}_\alpha \vee \underline{f}^\beta \vee \underline{f}^\gamma$$

When $x^{i'} = x^{i'}(x^i,t)$, $u^{\alpha'} = u^{\alpha'}(u^\alpha,t)$, and $\sigma = \sigma(t)$ then the coefficients transform according to the rule

$$T_{i'......\beta'\gamma'}^{\ \ j'k'\alpha'} = x^{i}_{,i'} x^{j'}_{,j} x^{k'}_{,k} u^{\alpha'}_{,\alpha} u^{\beta}_{,\beta'} u^{\gamma}_{,\gamma'} T_{i...\beta\gamma}^{\ \ jk\alpha}$$

As with tensors, once base vectors have been chosen there is a one-one, onto correspondence between the quantities which transform in the manner of the coefficients of double tensors and the double tensor entities themselves.

The discussion of differentiation of double tensors is completely analogous to the preceding discussion of differentiation of tensors. The first requirement in order to define derivatives is to be able to compare double tensors at different points. If mappings exist amongst the V at different points in the m + n + 1 dimensional manifold and amongst the V*, the U, and the U* then mappings are induced amongst the double tensors defined with respect to these vector spaces. The fundamental mappings will be referred to as connexions. If $\underline{T}(q)$ is a double tensor field along the curve,

$$x^i = x^i(q),\ u^\alpha = u^\alpha(q),\ t = t(q),\ i = 1,2,..m\ \alpha = 1,2,..n$$

then

$$\frac{d\underline{T}}{dq} = \lim_{\Delta q \to 0} \frac{\underline{T}_{||}(q + \Delta q) - \underline{T}(q)}{\Delta q} \tag{104}$$

where $\underline{T}_{||}(q + \Delta q)$ is the image at q of the element of the double tensor field at q + Δq mapped by the mapping induced by the connexion and $\underline{T}(q)$ is the actual element at q. $\frac{d\underline{T}}{dq}$ is a double tensor of the same kind as $\underline{T}$.

Rather than emulate the discussion of the differentiation of tensors a summary of the pertinent results will be given. The coordinate transformations in the n + m + 1 dimensional manifold shall be restricted such that $x^{i'} = x^{i'}(x^i,t), u^{\alpha'} = u^{\alpha'}(u^\alpha,t)$, $t = t$. If $T^{i_1 \ldots i_s \ldots \alpha_1 \ldots \alpha_p}_{\ldots j_1 \ldots j_t \ldots \beta_1 \ldots \beta_q}$ are the coefficients of a typical double tensor $\underline{T}$ in any acceptable coordinate system of the n + m + 1 dimensional space where $\underline{T}$ is defined on a curve $x^i = x^i(q)$, $u^\alpha = u^\alpha(q)$, $t = t(q)$ then the coefficients of the derivative along this curve are

$$\frac{DT^{i_1 \ldots i_s \ldots \alpha_1 \ldots \alpha_p}_{\ldots j_1 \ldots j_t \ldots \beta_1 \ldots \beta_q}}{Dq} = \frac{d\,T^{i_1 \ldots i_s \ldots \alpha_1 \ldots \alpha_p}_{\ldots j_1 \ldots j_t \ldots \beta_1 \ldots \beta_q}}{dq}$$

$$+ \sum_{k=1}^{s} \left(\Gamma^{i_k}_{ln} \frac{dx^n}{dq} + \Gamma^{i_k}_{l} \frac{dt}{dq}\right) T^{i_1 \ldots i_{k-1} l i_{k+1} \ldots i_s \ldots \alpha_1 \ldots \alpha_p}_{\ldots j_1 \ldots \ldots j_t \ldots \beta_1 \ldots \beta_q}$$

$$- \sum_{k=1}^{t} \left(\Gamma^{l}_{j_k n} \frac{dx^n}{dq} + \Gamma^{l}_{j_k} \frac{dt}{dq}\right) T^{i_1 \ldots \ldots i_s \ldots \alpha_1 \ldots \alpha_p}_{\ldots j_1 \ldots j_{k-1} l j_{k+1} \ldots j_t \ldots \beta_1 \ldots \beta_q}$$

$$+ \sum_{k=1}^{p} \left(\Delta^{\alpha_k}_{\sigma\gamma} \frac{du^\gamma}{dq} + \Delta^{\alpha_k}_{\sigma} \frac{dt}{dq}\right) T^{i_1 \ldots i_s \ldots \alpha_1 \ldots \alpha_{k-1} \sigma \alpha_{k+1} \ldots \alpha_p}_{\ldots j_1 \ldots j_t \ldots \beta_1 \ldots \ldots \beta_q}$$

$$- \sum_{k=1}^{q} \left(\Delta^{\sigma}_{\beta_k \gamma} \frac{du^\gamma}{dq} + \Delta^{\sigma}_{\beta_k} \frac{dt}{dq}\right) T^{i_1 \ldots i_s \ldots \alpha_1 \ldots \ldots \alpha_p}_{\ldots j_1 \ldots j_t \ldots \beta_1 \ldots \beta_{k-1} \sigma \beta_{k+1} \ldots \beta_q} \tag{105}$$

where

$$\frac{d\underline{e}_i}{dx^j} = \Gamma^k_{ij}\,\underline{e}_k \qquad \frac{d\underline{e}_i}{dt} = \Gamma^k_i\,\underline{e}_i \qquad \frac{d\underline{e}_i}{du^\alpha} = \Gamma^{*k}_{i\alpha}\,\underline{e}_k$$

$$\frac{d\underline{f}_\alpha}{du^\beta} = \Delta^\gamma_{\alpha\beta}\,\underline{f}_\gamma \qquad \frac{d\underline{f}_\alpha}{dt} = \Delta^\beta_\alpha\,\underline{f}_\beta \qquad \frac{d\underline{f}_\alpha}{dx^i} = \Delta^{*\beta}_{\alpha j}\,\underline{f}_\beta \tag{106}$$

and where these connexion coefficients are arbitrarily chosen in some coordinate system except that $\Gamma^{*k}_{i\alpha}$ and $\Delta^{*}_{\alpha j}$ have been chosen to vanish. The coefficients in any other system can be found from the transformation rules for these quantities, which in turn can be obtained in the same fashion used to obtain equation (71). In what follows M and N will be considered to be metric spaces and Γ^k_{ij}, $\Delta^\gamma_{\alpha\beta}$ will be chosen to be the Christoffel symbols of M and N respectively. $\Gamma^{*k}_{i\alpha}$, $\Delta^{*\beta}_{\alpha j}$ will be chosen to be zero in some coordinate system. Because of the transformation rules these quantities are also the Christoffel symbols and vanish respectively in any other coordinate system. Furthermore, in some coordinate system Γ^i_j, Δ^α_β will be chosen to vanish. In any other acceptable system the transformation rules require that

$$\Gamma^K_L = -\underset{x^i}{v^K}{}_{;L}, \qquad \Delta^\Sigma_\Gamma = -\underset{u^\alpha}{v^\Sigma}{}_{;\Gamma} \tag{107}$$

where $\underset{x^i}{v^K} = X^K_{,t}\big|_{x^i}$ is the velocity of the x^i system relative to the X^K system and $\underset{u^\alpha}{v^\Sigma} = U^\Sigma_{,t}\big|_{u^\alpha}$ is the velocity of the u^α system relative to the U^Σ system. As with tensors if the metric tensors are not dependent upon t then the choice of the connexion made above means that parallel propagation in the m + n + 1 dimensional manifold preserves lengths and angles. Implicit in equation (105) is the notion that differentiation of the inner product obeys the product rule.

It will prove useful to investigate the case where the manifold N is embedded in the manifold M. In particular, N is considered as a surface in Euclidean space. u^α are any surface coordinates and x^i are spatial coordinates which can be considered fixed in space. Then $x^i = x^i(u^\alpha,t)$. In the x^i,u^α,t manifold this situation corresponds to a submanifold Q given by $x^i = x^i(u^\alpha,t)$, $u^\alpha = u^\alpha$, $t = t$. A moving curve on the surface given by $u^\alpha = u^\alpha(q,t)$ corresponds to the two dimensional submanifold of Q, R, given by $x^i = x^i(u^\alpha,t)$, $u^\alpha = u^\alpha(q,t)$, $t = t$. If t is constant or q is constant a curve is defined on Q. As an example $T^i{}_\alpha(q,t)$ are taken as the coefficients of a double tensor defined on R. Then if the connexion is chosen such that in the x^i,u^α coordinate system Γ^i_j, Δ^α_β, $\Gamma^{*i}_{j\alpha}$ $\Delta^{*\alpha}_{\beta i}$ vanish and Γ^i_{jk}, $\Delta^\alpha_{\beta\gamma}$ are the Christoffel symbols, and if the observer is given by the curve t = constant defined above then the coefficients of the derivative of $t^i{}_\alpha(q,t)$ are given by equation (105) so that in any coordinate system X^K,U^Δ,t

$$\frac{DT^K{}_\Delta}{Dq} = T^K{}_{\Delta,q|t} + \Gamma^K_{LM}\, T^L_\Delta X^M_{,q} - \Delta^\Sigma_{\Delta\Gamma}\, T^K{}_\Sigma U^\Gamma_{,q} \tag{108}$$

If $T^K{}_\Delta$ is defined over Q then

$$\begin{aligned}\frac{DT^K{}_\Delta}{Dq}(U^\Gamma(q,t),t) &= (T^K{}_{\Delta,\Gamma} + \Gamma^K_{LM}\, T^L{}_\Delta\, X^M_{,\Gamma} - \Delta^\Sigma_{\Delta\Gamma} T^K{}_\Sigma)U^\Gamma_{,q} \\ &= T^K{}_{\Delta;\Gamma}\, U^\Gamma_{,q}\end{aligned} \tag{109}$$

where $T^K{}_{\Delta;\Gamma}$, called the total covariant derivative of $T^K{}_\Delta$, equals the quantity in brackets; and it can be shown by its transformation properties that it is a double tensor of covariant order one greater than $T^K{}_\Delta$ with respect to N. If the observer is given by q = constant and $T^K{}_\Delta$ is defined over R, then the coefficients of the derivative of $T^K{}_\Delta(q,t)$ are given by equation (105); so that

$$\frac{DT^K{}_\Delta}{Dt}\Bigg|^{x^i,u^\alpha}_{q=\text{constant}} = T^K{}_{\Delta,t}\Big|_q + \Gamma^K_{LM}T^L{}_\Delta X^M{}_{,t}\Big|_q - \Delta^\Sigma_{\Delta\Gamma}T^K{}_\Sigma U^\Gamma{}_{,t}\Big|_q - v^K_{x^i;L}\,T^L{}_\Delta + v^\Sigma_{u^\alpha;\Delta}\,T^K{}_\Sigma \tag{110}$$

where the upper letters after the vertical line indicate in which system the connexion coefficients were chosen and the lower notation refers to the observer.

In the case where N is a surface fixed in space, $x^i = x^i(u^\alpha)$. Then if $T^K{}_\Delta$ is defined over Q equation (110) becomes

$$\frac{DT^K{}_\Delta}{Dt}(U^\Gamma(q,t),t)\Bigg|^{x^i,u^\alpha}_{q\,=\,\text{constant}} = T^K{}_{\Delta,t}\Big|_{U^\Gamma} + T^K{}_{\Delta;\Gamma}\,U^\Gamma{}_{,t}\Big|_q - v^K_{x^i;L}\,T^L{}_\Delta + v^\Sigma_{u^\alpha;\Delta}\,T^K{}_\Sigma \tag{111}$$

These examples are easily generalized to take into account any double tensor.

It is usually convenient when dealing with tensors or double tensors to suppress the fundamental or base vectors and to deal only with the coefficients. Having gone through the derivation of the derivatives, the coefficients of such derivatives are now meaningful and may be interpreted term by term. In the ensuing chapters the coefficients of tensors will be used. This completes the discussion of tensors.

The discussion of differential geometry is now resumed.

In the three dimensional embedding space the permutation tensors are defined by

$$\varepsilon_{ijk} = g^{1/2} e_{ijk}, \qquad \varepsilon^{ijk} = g^{-1/2} e_{ijk} \tag{112}$$

where $e_{ijk} = 1,-1,0$ if ijk is an even or odd permutation or if any two indices are equal, and g is the determinant of the metric tensor.

On the surface, the surface permutation tensors are defined by

$$\varepsilon_{\alpha\beta} = a^{1/2} e_{\alpha\beta}, \quad \varepsilon^{\alpha\beta} = a^{-1/2} e_{\alpha\beta} \tag{113}$$

where $e_{\alpha\beta} = 1,-1,0$ depending on whether $\alpha\beta$ is an even or odd permutation or whether the indices are equal, and a is the determinant of the surface metric tensor.

Useful identities which follow from the definitions are

$$\varepsilon_{\gamma\sigma}\varepsilon^{\alpha\beta} = \delta^{\alpha}_{\gamma}\delta^{\beta}_{\sigma} - \delta^{\alpha}_{\sigma}\delta^{\beta}_{\gamma} \tag{114}$$

and

$$\varepsilon_{ijk}\varepsilon^{ipq} = \delta^{p}_{j}\delta^{q}_{k} - \delta^{q}_{j}\delta^{p}_{k} \tag{115}$$

The cross product of two spatial vectors A^i, B^j is a vector C_i given by $C_i = \varepsilon_{ijk}A^j B^k$. Two vectors are orthogonal if the angle between them is $\frac{\pi}{2}$. C_i is orthogonal to A^i and B^i since the cosine of the angle between C^i and A^i is $\cos\theta_{CA} = C_i A^i = \varepsilon_{ijk}A^i A^j B^k = 0$. Similarly, C^i is orthogonal to B^j. Thus a unit vector n_i orthogonal to the tangent plane of the surface can be found by taking the cross product of any two independent vectors in the tangent plane and dividing by their magnitudes. $x^i_{,\alpha}$ are two such vectors. Thus

$$n_i = \varepsilon_{ijk} x^j_{,1} x^k_{,2} (g^{pq} \varepsilon_{pst} x^s_{,1} x^t_{,2} \; \varepsilon_{quv} x^u_{,1} x^v_{,2})^{-1/2} \tag{116}$$

If one factor is expanded as follows,

$$\varepsilon_{ijk} \, x^j_{,1} \, x^k_{,2} = \frac{1}{2} e_{\alpha\beta} \, \varepsilon_{ijk} \, x^j_{,\alpha} \, x^k_{,\beta}$$

and the other is expanded using equation (115) then it can be shown that

$$n_i = \frac{1}{2} \varepsilon^{\alpha\beta} \, \varepsilon_{ijk} \, x^j_{,\alpha} \, x^k_{,\beta} \tag{117}$$

If this equation is multiplied by $\varepsilon_{\sigma\rho}$ and equation (114) is used then

$$\varepsilon_{\sigma\rho} \, n_i = \varepsilon_{ijk} \, x^j_{,\sigma} \, x^k_{,\rho} \tag{118}$$

If instead, equation (116) is multiplied by $g^{pi} \; \varepsilon_{stp} \; x^t_{,\sigma}$ and equation (115) is employed then

$$\varepsilon_{stp} \, x^t_{,\sigma} n^p = a^{\alpha\delta} \, \varepsilon_{\delta\sigma} \, x_{s,\alpha} \tag{119}$$

If A_α is a non zero surface vector then there exists a surface vector $B^\alpha = \varepsilon^{\alpha\beta} A_\beta$ orthogonal to A_α. This is evident from the result $\cos\theta_{AB} = a_{\alpha\gamma} B^\alpha A^\gamma = \varepsilon^{\alpha\beta} A_\beta A_\alpha = 0$. If A_β is a unit vector then so is B^α. This is easily shown by looking at the expression for the magnitudes of B^α and by using equation (114). In particular, if $u^\alpha = u^\alpha(s,t)$ is a curve on the surface then $\nu_\alpha = \varepsilon_{\alpha\beta} t^\beta$, where $t^\beta = u^\beta_{,s}$ and s is arc length, is a unit vector normal to the curve. Incidentally, t^α is a unit vector and any other tangent $u^\alpha_{,q} = \frac{ds}{dq} t^\alpha$ has a magnitude $\frac{ds}{dq} = (a_{\alpha\beta} u^\alpha_{,q} u^\beta_{,q})^{1/2}$.

Having introduced the unit normal vector to a surface it is now possible to discuss another fundamental surface tensor the elements of which constitute the coefficients of the so called second fundamental form. From the definition of the covariant derivative it is easily shown that $a_{\alpha\beta;\gamma} = 0$ and $g_{ij;\gamma} = g_{ij;k} x^k_{,\gamma} = 0$. Thus from the definition of $a_{\alpha\beta}$,

$$0 = a_{\alpha\beta;\gamma} = g_{ij}\, x^i_{;\alpha\gamma}\, x^j_{,\beta} + g_{ij}\, x^i_{;\alpha}\, x^j_{;\beta\gamma}$$

If cyclic permutations of this equation are written and one of the three equations which result is subtracted from the sum of the other two and the fact that $x^i_{;\alpha\beta} = x^i_{;\beta\alpha}$ is employed then

$$0 = g_{ij}\, x^i_{;\beta\alpha}\, x^j_{,\gamma}$$

The vectors $x^i_{;\alpha\beta}$ are orthogonal to the surface vectors $x^j_{,\gamma}$ so that

$$x^i_{;\alpha\beta} = b_{\alpha\beta} n^i \tag{120}$$

These equations known as Gauss's formulae define $b_{\alpha\beta}$ so that

$$b_{\alpha\beta} = x^i_{;\alpha\beta}\, n_i \tag{121}$$

The geometrical role of $b_{\alpha\beta}$ is exposed by the following discussion. If s is the arc length along a curve $x^i = x^i(u^\alpha(s,t)t)$ on the surface then $t^i = x^i_{,s}$ is the unit tangent to the curve and the first Frenet equation for curves in space is

$$\frac{Dt^i}{Ds} = \frac{D^2 x^i}{Ds^2} = KR^i$$

where D/Ds is defined in equation (108) and where K is the curvature and R^i is a unit vector called the principal normal vector. R^i in general does not coincide with n^i. If θ is the angle between n^i and R^i then

$$\cos\theta = n^i R_i = \frac{D^2 x^i}{Ds^2}\frac{n_i}{K} \tag{122}$$

However,

$$\frac{D^2x^i}{Ds^2} = x^i_{;\alpha\beta}\, u^\alpha_{,s}\, u^\beta_{,s} + x^i_{,\alpha}\frac{Du^\alpha_{,s}}{Ds}$$

so that

$$K\cos\theta = x^i_{;\alpha\beta}\, n_i\, u^\alpha_{,s}\, u^\beta_{,s} = b_{\alpha\beta}t^\alpha t^\beta \tag{123}$$

If the parameter along the curve is q then $u^\alpha_{,s} = u^\alpha_{,q}q_{,s}$ and

$$K_n = K\cos\theta = \frac{b_{\alpha\beta}du^\alpha du^\beta}{a_{\sigma\rho}du^\sigma du^\rho} \tag{124}$$

where $b_{\alpha\beta}$ are the coefficients of the second fundamental form and k_n is called the normal curvature corresponding to the direction t^α. It is seen from the definition that K_n depends only on the direction of the tangent. In general, there are many surface curves passing through the point under consideration which have a given tangent in common. In particular, the plane containing the normal to the surface and the given tangent direction intersects the surface in such a curve, called the normal section. K_n is obviously the curvature of a normal section.

A result which is allied to Gauss's formulae can be obtained by taking the covariant derivative of the equation $g_{ij}n^i n^j = 1$. Then

$$g_{ij}n^i n^j{}_{;\alpha} = 0$$

Therefore, $n^j_{;\alpha}$ must lie in the tangent plane and so

$$n^j_{;\alpha} = \eta^\beta_\alpha \, x^j_{,\beta} \tag{125}$$

If $g_{ij}n^i x^j_{,\beta} = 0$ is differentiated and use is made of equations (120) and (125) then

$$n^j_{;\alpha} = - b^\beta_\alpha \, x^j_{,\beta} \tag{126}$$

These are called Weingarten's equations.

Another result can be obtained in a similar fashion. The tensor derivative with respect to the arc length s of the equation $a_{\alpha\beta}u^\alpha_{,s}u^\beta_{,s} = 1$ leads to the result that $\frac{Du^\alpha_{,s}}{Ds}$ is a surface vector normal to itself, so that

$$\frac{Dt^\alpha}{Ds} = \sigma\nu^\alpha \tag{127}$$

where $t^\alpha = u^\alpha_{,s}$, σ is called the geodesic curvature of the curve, and ν^α is the unit normal to the curve. Using the facts that ν^α is a unit vector and that ν^α is normal to t^α together with equation (127) yields

$$\frac{D\nu^\alpha}{Ds} = -\sigma t^\alpha \tag{128}$$

Returning to the definition of normal curvature, it is possible to define the mean and the Gaussian curvatures at each point of a surface. These quantities reflect the geometrical characteristics of the neighborhood of the point on the surface. Given a direction t^α in the tangent plane at the point P on the surface the corresponding normal curvature is $K_n(t^\alpha) = b_{\alpha\beta}t^\alpha t^\beta$. If K_n is optimized subject to the condition that $a_{\alpha\beta}t^\alpha t^\beta = 1$, the result is the eigenvalue problem

$$(b_{\alpha\beta} - \lambda a_{\alpha\beta})t^{\beta} = 0 \tag{129}$$

Because $b_{\alpha\beta}$ and $a_{\alpha\beta}$ are real symmetric tensors the eigenvalues are real and are called the principal curvatures. The corresponding eigenvectors are called the principal directions at P. It can be shown that the eigenvectors are orthogonal if the eigenvalues are distinct and can be chosen to be orthogonal in case of duplicity. If λ_1, λ_2 are the principal curvatures then the mean curvature H is defined as

$$2H = \lambda_1 + \lambda_2 \tag{130}$$

and the Gaussian curvature is defined as

$$K = \lambda_1\lambda_2 \tag{131}$$

The characteristic equation of the eigenvalue problem given in equation (129) together with the theory of algebraic equations yields

$$K = \frac{b}{a} \tag{132}$$

and

$$2H = a^{\alpha\beta}b_{\alpha\beta} \tag{133}$$

The Riemann-Christoffel surface tensor is another tensor that frequently appears. It is defined by

$$R^{\eta}_{\cdot\alpha\beta\gamma} = \Gamma^{\eta}_{\alpha\gamma,\beta} - \Gamma^{\eta}_{\alpha\beta,\gamma} + \Gamma^{\eta}_{\beta\sigma}\,\Gamma^{\sigma}_{\alpha\gamma} - \Gamma^{\eta}_{\gamma\sigma}\,\Gamma^{\sigma}_{\alpha\beta} \tag{134}$$

This tensor appears naturally when the commutation properties of covariant differentiation are investigated. From the definition of $R^{\eta}_{\cdot\alpha\beta\gamma}$ and of covariant differentiation it is easily shown that

$$A_{\alpha;\beta\gamma} - A_{\alpha;\gamma\beta} = R^{\eta}_{\cdot\alpha\beta\gamma}A_{\eta} \tag{135}$$

This can be generalized to give

$$A_{\alpha_1..\alpha_n;\beta\gamma} - A_{\alpha_1..\alpha_n;\gamma\beta} = \sum_{i=1}^{n} R^{\eta}_{\cdot\alpha_i\beta\gamma} A_{\alpha_1..\alpha_{i-1}\eta\alpha_{i+1}..\alpha_n} \tag{136}$$

It can be shown that the covariant form of this tensor $R_{\sigma\alpha\beta\gamma} = a_{\sigma\eta}R^{\eta}_{\cdot\alpha\beta\gamma}$ has the following properties:

$$R_{\alpha\beta\gamma\sigma} = - R_{\alpha\beta\sigma\gamma} = - R_{\beta\alpha\gamma\sigma} = R_{\gamma\sigma\alpha\beta} \tag{137}$$

$$R_{\alpha\beta\gamma\sigma} + R_{\alpha\sigma\beta\gamma} + R_{\alpha\gamma\sigma\beta} = 0 \tag{138}$$

Finally, the Ricci tensor is defined by $R_{\alpha\beta} = R^{\eta}_{\cdot\alpha\beta\eta}$ (139)

Using the results already established the derivation of the Gauss-Codazzi equations, which restrict the forms of $a_{\alpha\beta}$ and $b_{\alpha\beta}$, is straight forward. In equation (135) A_{α} is taken to be $x^{i}_{,\alpha}$. However, the left hand side can be written in a different form by using equations (120) and (126). Then multiplication by n_i and $x_{i,p}$ yields

$$b_{\alpha\beta;\gamma} - b_{\alpha\gamma;\beta} = 0 \tag{140}$$

and

$$R_{\mu\alpha\beta\gamma} = b_{\alpha\beta}b_{\mu\gamma} - b_{\alpha\gamma}b_{\mu\beta} \tag{141}$$

These are the Gauss-Codazzi equations. The second of these is really one independent equation,

$$R_{1212} = b \tag{142}$$

The Gaussian curvature is then given by

$$K = \frac{R_{1212}}{a} \tag{143}$$

It is now seen that K depends only on the components of $a_{\alpha\beta}$.

Eisenhart[29] proves that if $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are given functions of u^α and $a^{\alpha\beta}$ is positive definite then sufficient conditions for the existence of a function $x^k = x^k(u^\alpha)$, that is a surface such that $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the fundamental tensors of this surface are that $a_{\alpha\beta}$ and $b_{\alpha\beta}$ satisfy the Gauss-Codazzi equations. The surface is determined to within a rigid body motion in space.

If a moving surface is considered, its motion, it will be recalled, was given by $x^k = x^k(U^\Delta, t)$, where U^Δ are convected surface coordinates. For a materialized surface keeping U^Δ fixed amounted to following a given material particle in its motion. The velocity of a material particle relative to the x^i spatial coordinate system is defined as

$$v^k = x^k_{,t}\Big|_{U^\Delta} \tag{144}$$

In general, the quantities defined as the velocities in a given spatial coordinate system form the components of a contravariant vector only with respect to the time independent coordinate transformations from the given system.

The moving surface can be described by $x^k = x^k(u^\alpha, t)$, where u^α are arbitrary surface coordinates. Then

$$r^k = x^k_{,t}\Big|_{u^\alpha} \tag{145}$$

give the rates of change of the spatial coordinates with respect to time of a geometrical point on the moving surface which always has the same label in the u^α surface coordinate system. r^k is a vector under the same conditions required for v^k to be a vector.

r^k can be considered to be a vector once a spatial coordinate system has been chosen. Therefore components can be taken tangential and normal to the surface. Thus,

$$r^k = Gn^i + T^\alpha x^i_{,\alpha} \tag{146}$$

Truesdell and Toupin (30) have shown that G is independent of the choice of surface parametrization. It is possible to chose surface coordinates θ^Δ such that $T^\Delta = 0$. Then it is easy to prove that

$$T^\alpha = -u^\alpha_{,t}\Big|_{\theta^\Delta} = u^\alpha_{,\Delta}\, \theta^\Delta_{,t}\Big|_{u^\alpha} \tag{147}$$

Thus the tangential velocity T^α is the velocity of the u^α coordinate system relative to the θ^Δ system.

In the above reference it is also stated that the necessary and sufficient conditions that $a_{\alpha\beta}$, $b_{\alpha\beta}$, G, T^α, functions of u^α with $a_{\alpha\beta}$ positive definite, be the coefficients of the first and second forms, the normal surface velocity, and the tangential velocity, respectively, of a moving surface $x^i = x^i(u^\alpha, t)$, are that $a_{\alpha\beta}$, $b_{\alpha\beta}$ satisfy the Gauss-Codazzi equations and the following two equations:

$$a_{\alpha\beta,t}\Big|_{u^\gamma} + 2T_{(\alpha;\beta)} = -2Gb_{\alpha\beta} \tag{148}$$

$$b_{\alpha\beta,t}\Big|_{u^\gamma} + T^\gamma b_{\alpha\beta;\gamma} + b^\gamma_{(\alpha}T_{\beta);\gamma} = G_{;\alpha\beta} - Gb^\gamma_\alpha b_{\gamma\beta} \tag{149}$$

When spatial vector fields are defined over a moving surface or over a curve on such a surface it is often convenient to utilize the fact that any spatial vector at a point P can be written as a linear combination of an arbitrary triad of three independent vectors at P. At a point on the surface a natural triad which is useful is the set consisting of the unit normal to the surface n^i and any two independent vectors in the tangent plane such as the vectors $x^i_{,\alpha}$. Along a curve on a surface the unit tangent $x^i_{,s}$ and the unit in-surface normal to the curve $\nu^i = x^i_{,\alpha}\nu^\alpha$ together with the surface normal n^i form a natural triad. For example, if A^i is a spatial vector field defined over a surface then

$$A^i = A^\alpha x^i_{,\alpha} + An^i \tag{150}$$

where the components are given by

$$A = A^i n_i \tag{151}$$

and

$$A^\gamma = A_i x^i_{,\beta} a^{\beta\gamma} \tag{152}$$

It should be noted that A^γ is a surface vector. If B^i is defined over the curve $x^i(u^\alpha(s,t),t)$ then

$$B^i = Qx^i_{,s} + R\nu^i + Bn^i \tag{153}$$

where

$$Q = B_i x^i_{,s}, \quad R = B^i \nu_i, \quad \text{and} \quad B = B_i n^i \tag{154}$$

In the tangent plane at a point on the surface any two independent vectors form a basis. Therefore if A^α is a surface vector defined along a curve it can be written as

$$B^{\alpha} = Eu^{\alpha}_{,s} + F\nu^{\alpha} \tag{155}$$

where $E = B_{\alpha}u^{\alpha}_{,s}$ and $F = B_{\alpha}\nu^{\alpha}$

A useful result which follows is the identity

$$g^{ij} = a^{\alpha\beta}x^{i}_{,\alpha}x^{j}_{,\beta} + n^{i}n^{j} \tag{156}$$

To prove this an arbitrary vector r_1 is considered. Then $g^{ij}r_j$ is a vector and so using equations (150),(151), and (152) it can be decomposed such that

$$g^{ij}r_j = (g^{kj}r_jx_{k,\beta}a^{\beta\alpha})x^{i}_{,\alpha} + (g^{kj}r_jn_k)n^{i}$$

$$= (a^{\beta\alpha}x^{i}_{,\alpha}x^{j}_{,\beta} + n^{i}n^{j})r_j$$

but since r_j is arbitrary the above identity is proved.

Similarly, the identity

$$a^{\alpha\beta} = u^{\alpha}_{,s}u^{\beta}_{,s} + \nu^{\alpha}\nu^{\beta} \tag{157}$$

is valid along a curve and may be proved by employing an arbitrary vector r_{α} and decomposing $a^{\alpha\beta}r_{\beta}$ using equations (153) and (154).

A double tensor $S^{k\alpha}$ can be considered as a set of spatial vectors for which α specifies the vector and k specifies the component. Then by virtue of equation (150),

$$S^{k\alpha} = S^{\beta\alpha}x^{k}_{,\beta} + S^{\alpha}n^{k} \tag{158}$$

Then taking the total covariant derivative and using equations (120) and

(126) it is seen that

$$S^{k\alpha}_{;\gamma} = (S^{\beta\alpha}_{;\gamma} - b^{\beta}_{\gamma}S^{\alpha})x^{k}_{,\beta} + (S^{\alpha}_{;\gamma} + b_{\beta\gamma}S^{\beta\alpha})n^{i} \tag{159}$$

A special surface coordinate system which is often convenient in the neighborhood of a curve on the surface is the geodesic parallel coordinate system. Before defining this coordinate system the following definition is required. A geodesic curve on the surface is a curve which satisfies the equation,

$$\frac{Du^{\alpha}_{,s}}{Ds} = 0 \tag{160}$$

where D/Ds was defined by equation (108). An equivalent definition of a geodesic, by virtue of equation (127), is a curve for which the geodesic curvature vanishes. The theory of differential equations ensures that for a fixed time there is only one geodesic passing through a given point and having a given direction. To begin the definition of the geodesic parallel coordinate system a curve C is taken on a surface S. Through every point of C a unique geodesic passes which is orthogonal to C. It can be shown that in a sufficiently small region of S containing C every point has passing through it one and only one of the normal geodesics through C. The orthogonal trajectories to the geodesics together with the family of geodesics form a coordinate net. If u^2 is chosen as the parameter which is constant along a geodesic and u^1 is chosen to be constant along the orthogonal trajectories, this u^{α} coordinate system is the geodesic parallel coordinate system.

It is now apropos to find expressions for the geodesic curvatures of the coordinate curves of an orthogonal system. Along the curves u^2 = constant $ds^2 = a_{11}du^{12}$; so that $t^\alpha = (a_{11}^{-1/2},0)$ are the components of the unit tangents. Along the orthogonal trajectories or u^1 = constant curves $ds^2 = a_{22}du_{22}$; so that the components of the unit tangents to these curves are $\nu^\alpha = (0,a_{22}^{-1/2})$. The geodesic curvature σ_1 along the u^2 = constant curve satisfies

$$\frac{Dt^\alpha}{Ds} = \sigma_1\nu^\alpha$$

and when this is expanded and use is made of equation (26) then it is found that

$$\sigma_1 = -a_{22}^{-1/2}\frac{\partial \ln(a_{11}^{1/2})}{\partial u^2} \tag{161}$$

Similarly, the geodesic curvature σ_2 of the u^1 = constant curve is

$$\sigma_2 = -a_{11}^{-1/2}\frac{\partial \ln(a_{22}^{1/2})}{\partial u^1} \tag{162}$$

Returning to the discussion of the geodesic parallel coordinate system, it is evident that since the system is orthogonal, the geodesic curvature σ_1 along the geodesics u^2 = constant is given by equation (161). However, the geodesic curvature along a geodesic curve is zero. Therefore equation (161) yields the fact that $a_{11} = a_{11}(u^1)$. It is thus possible to define a transformation of coordinates by

$$\begin{aligned} \bar{u}^1 &= \int a_{11}(u^1)^{1/2}du^1 \\ \bar{u}^2 &= u^2 \end{aligned} \tag{163}$$

so that in the new system the metric tensor is

$$\bar{a}_{11} = 1, \quad \bar{a}_{22} = a_{22}, \quad \bar{a}_{12} = \bar{a}_{21} = 0 \tag{164}$$

The geodesic curvature of the $\bar{u}^1$ = constant curves is then given by

$$\bar{\sigma}_2 = -\frac{\partial \ln(a_{22}{}^{1/2})}{\partial \bar{u}^1} \tag{165}$$

It is easily seen that u^1 = constant if and only if $\bar{u}^1$ = constant and that u^2 = constant if and only if $\bar{u}^2$= constant, so that the new coordinate curves are the same as the old coordinate curves. Furthermore, $\bar{u}^1$ is now arc length along the geodesics. This means that the orthogonal trajectories to the geodesics are curves which are the loci of points which are equidistant from the given curve C as measured along the geodesics.

In passing, it might be worthy to note that using the definition of $R_{\alpha\beta\gamma\sigma}$ and the fact that the Gaussian curvature of the surface is given by equation (143) an explicit expression can be found for K in terms of the components of $a_{\alpha\beta}$ (31). In the geodesic parallel coordinate system because $\bar{a}_{22}$ = 1 the expression for K is particularly simple. It is

$$K = -\bar{a}_{22}{}^{-1/2} \frac{\partial^2(\bar{a}_{22}{}^{1/2})}{\partial \bar{u}^{12}} \tag{166}$$

By differentiating equation (165) it is easily shown that

$$\frac{\partial \sigma_2}{\partial \bar{u}^1} = \bar{\sigma}_2{}^2 + K \tag{167}$$

The final topic treated in this chapter concerns the transport theorem on a moving surface in three space. The surface metric tensor is usually time dependent. This theorem states that if ψ is some function defined over the surface and if S is a region of this surface bounded by a curve C then

$$\frac{d}{dt}\int_S \psi d\sigma = \int_S (\psi,_t\big|_{u^\beta} + \frac{\psi}{2a} a,_t\big|_{u^\beta})d\sigma + \oint_C \dot{u}^\alpha \psi n_\alpha ds \tag{168}$$

where u^β are surface coordinates on the deformed surface, a is the determinant of the surface metric tensor, and $\dot{u}^\alpha = u^\alpha,_t\big|_{U^\Gamma}$ where U^Γ are the coordinates of the undeformed surface.

To prove this theorem, u^α are taken to be surface coordinates at any time t. The vector element of area associated with the vector product of any two surface vectors $d\bar{u}^\alpha, du^\beta$ with spatial representation $d\bar{x}^j$ and dx^k is

$$d\sigma_i = \varepsilon_{ijk} d\bar{x}^j dx^k = \varepsilon_{ijk} x^j,_\alpha x^k,_\beta d\bar{u}^\alpha du^\beta$$

Using equation (118) it is seen that

$$d\sigma_i = n_i \varepsilon_{\alpha\beta} d\bar{u}^\alpha du^\beta$$

The magnitude of the element of area is

$$d\,\sigma = \varepsilon_{\alpha\beta} d\bar{u}^\alpha du^\beta$$

This is a tensor equation and is valid in any coordinate system. In particular it is valid in a convected coordinate system U^Γ. In this system $d\bar{u}^\alpha$ and du^β have components $d\bar{u}^\Delta$, du^Σ. The element of area in the coordinate system is

$$d\,\sigma = \varepsilon_{\Delta\Sigma} d\bar{u}^\Delta du^\Sigma$$

where $\bar{A}$ is the determinate of the surface metric in the convected coordinate system.

In the reference state the vectors used to define the above area element have the representation $d\bar{U}^{\Delta}$ and dU^{Σ} in the convected coordinate system. The element of area on the reference surface corresponding to these vectors has a magnitude given by

$$d\Sigma = e_{\Delta\Sigma} A^{1/2} d\bar{U}^{\Delta} dU^{\Sigma}$$

where A is now the determinate of the surface metric tensor in the reference state in terms of the convected coordinates. Therefore,

$$\frac{d\sigma}{d\Sigma} = \left(\frac{\bar{A}}{A}\right)^{1/2} \tag{169}$$

However,

$$a_{\Delta\Sigma} = a_{\alpha\beta} u^{\alpha}_{,\Delta} u^{\beta}_{,\Sigma}$$

Then

$$\bar{A} = |a_{\Delta\Sigma}| = |a_{\alpha\beta}| J^2 = aJ^2 \tag{170}$$

where $\quad J = |u^{\alpha}_{,\Delta}|$

Substitution of (170) into (169) results in

$$\frac{d\sigma}{d\Sigma} = \left(\frac{a}{A}\right)^{1/2} J \tag{171}$$

where A is independent of time.

ψ is any function of position on the surface and of time. If S is the region of the deformed surface which is occupied by a quantity of material which is followed in its motion then equation (171) gives the result

$$\int_S \psi d\sigma = \int_{S_0} \psi \left(\frac{a}{A}\right)^{1/2} J d\Sigma$$

where S_0 is the corresponding region of the reference surface. Taking the derivative d/dt of this equation, where d/dt implies that the observer follows the material as it deforms; the region of integration on the right hand side is independent of time and hence the derivative can be taken under the integral sign. Then

$$\frac{d}{dt}\int_S \psi d\sigma = \int_{S_0} \frac{1}{A^{1/2}} \frac{d}{dt}(\psi a^{1/2} J) d\Sigma$$

$$= \int_{S_0} \frac{1}{A^{1/2}} \left(\frac{d\psi}{dt} a^{1/2} J + \frac{\psi J}{2a^{1/2}} \frac{da}{dt} + \psi a^{1/2} \frac{dJ}{dt}\right) d\Sigma \tag{172}$$

It will be shown later that

$$\frac{\partial a}{\partial u^\alpha} = 2a\Delta^\beta_{\alpha\beta} \tag{173}$$

and

$$\frac{dJ}{dt} = J \frac{\partial \dot{u}^\alpha}{\partial u^\alpha} \tag{174}$$

Then using the chain rule,

$$\frac{da}{dt} = a,_t\big|_{U^\Gamma} = a,_t\big|_{u^\beta} + \dot{u}^\beta a,_\beta \tag{175}$$

where $\dot{u}^\beta = u^\beta,_t\,|_{U^\Delta}$. Combining equations (172-175) the following result is obtained:

$$\frac{d}{dt}\int_S \psi d\sigma = \int_{S_0} \left[\frac{d\psi}{dt} + \frac{\psi}{2a} a,_t\big|_{u^\beta} + \psi(\dot{u}^\alpha,_\alpha + \Delta^\alpha_{\beta\alpha}\dot{u}^\beta)\right]\left(\frac{a}{A}\right)^{1/2} J d\Sigma$$

$$= \int_S \left(\frac{d\psi}{dt} + \frac{\psi}{2a} a,_{t}|_{u}\beta + \psi\dot{u}^{\alpha}_{;\alpha}\right)d\sigma$$

Using the chain rule again, it is evident that

$$\frac{d\psi}{dt} = \psi,_{t}|_{U}\Gamma = \psi,_{t}|_{u}\beta + \dot{u}^{\alpha}\psi;\alpha$$

Therefore,

$$\frac{d}{dt}\int_S \psi d\sigma = \int [\psi,_{t}|_{u}\beta + \frac{\psi}{2a} a,_{t}|_{u}\beta + (\dot{u}^{\alpha}\psi);_{\alpha}]d\sigma$$

If Green's theorem in two dimensions[(31)] is employed the last term can be written as

$$\int_S (\dot{u}^{\alpha}\psi);_{\alpha}d\sigma = \oint_C \dot{u}^{\alpha}\psi n_{\alpha}ds \tag{175}$$

where C is the boundary curve of S and n_{α} is the unit surface normal to C. Therefore,

$$\frac{d}{dt}\int_S \psi d\sigma = \int_S \left(\frac{\partial\psi}{\partial t}\Big|_{u}\beta + \frac{\psi}{2a}\frac{\partial a}{\partial t}\Big|_{u}\beta\right)d\sigma + \oint_C \dot{u}^{\alpha}\psi n_{\alpha}ds$$

If the region of integration S is changing in a prescribed fashion instead of deforming with the actual material an analogous result could have been arrived at by considering the region to be associated with an imaginary material which has the same velocity on C as the prescribed velocity. The result would differ in that $\dot{u}^{\alpha}n_{\alpha}$ on the right hand side would be replaced by the outward normal velocity of the boundary C.

Returning to the two results which were previously stated without proof, the proofs are now provided. First, it shall be proved that

$$\frac{dJ}{dt} = J\,\frac{\partial \dot{u}^{\alpha}}{\partial u^{\alpha}}$$

where $J = |u^{\beta}_{,\Delta}|$. From the definition, $J = J(u^{\alpha}_{,\Delta})$. Therefore,

$$\frac{dJ}{dt} = \frac{\partial J}{\partial u^{\alpha}_{,\Delta}}\;\frac{du^{\alpha}_{,\Delta}}{dt}$$

but

$$\frac{du^{\alpha}_{,\Delta}}{dt} = \frac{\partial u^{\alpha}_{,\Delta}}{\partial t}\Big|_{U^{\Gamma}} = \frac{\partial\left(\frac{\partial u^{\alpha}}{\partial t}\Big|_{U^{\Gamma}}\right)}{\partial U^{\Delta}} = \frac{\partial \dot{u}^{\alpha}}{\partial U^{\Delta}}$$

Also

$$\frac{\partial J}{\partial u^{\alpha}_{,\Delta}} = \text{cofactor } u^{\alpha}_{,\Delta} = U^{\Delta}_{,\alpha}J$$

because $U^{\Delta}_{,\alpha}$ is the inverse of $u^{\alpha}_{,\Delta}$ and $J = |u^{\alpha}_{,\Delta}|$. Therefore

$$\frac{dJ}{dt} = JU^{\Delta}_{,\alpha}\,\frac{\partial \dot{u}^{\alpha}}{\partial U^{\Delta}} = J\,\frac{\partial \dot{u}^{\alpha}}{\partial u^{\alpha}} \tag{177}$$

Second, it shall be proved that

$$a_{,\alpha} = 2a\,\Delta^{\beta}_{\alpha\beta} \tag{178}$$

Towards this end the quantities $\Delta_{\alpha\beta\gamma}$ are defined as

$$\Delta_{\alpha\beta\gamma} = \frac{1}{2}(a_{\alpha\gamma,\beta} + a_{\beta\gamma,\alpha} - a_{\alpha\beta,\gamma})$$

and the cyclic permutation is

$$\Delta_{\gamma\beta\alpha} = \frac{1}{2}(a_{\gamma\alpha,\beta} + a_{\beta\alpha,\gamma} - a_{\gamma\beta,\alpha})$$

Therefore by adding

$$\Delta_{\alpha\beta\gamma} + \Delta_{\gamma\beta\alpha} = a_{\alpha\gamma,\beta} \tag{179}$$

By definition,

$$\Delta^{\sigma}_{\alpha\beta} = a^{\sigma\gamma}\Delta_{\alpha\beta\gamma} \tag{180}$$

Using the chain rule,

$$a_{,\beta} = \frac{\partial a}{\partial a_{\alpha\gamma}} a_{\alpha\gamma,\beta} = (\text{cofactor } a_{\alpha\gamma})a_{\alpha\gamma,\beta}$$

Since $a^{\alpha\gamma}$ is the inverse of $a_{\alpha\gamma}$,

$$a_{,\beta} = a\, a^{\alpha\gamma}a_{\alpha\gamma,\beta} \tag{181}$$

Combining equations (179-181) the following result is obtained

$$a_{,\beta} = a\, a^{\alpha\gamma}(\Delta_{\alpha\beta\gamma} + \Delta_{\gamma\beta\alpha}) = a(\Delta^{\alpha}_{\alpha\beta} + \Delta^{\gamma}_{\gamma\beta}) = 2a\Delta^{\alpha}_{\alpha\beta}$$

CHAPTER III

KINEMATICS OF A SINGULAR CURVE AND GEOMETRICAL AND KINEMATICAL COMPATIBILITY RELATIONS

The phenomena which will be investigated here are concerned with the propagation on a surface of a curve across which certain physical or geometrical quantities undergo a finite discontinuity although they are continuous elsewhere on the surface. This chapter is devoted to an exposition of the relevant kinematics of such a situation.

As has been seen in the preceding chapter a deforming surface in three dimensional Euclidean space is given in parametric form by

$$x^i = x^i(u^\alpha, t) \tag{1}$$

where u^α are the surface coordinates and where the matrix $x^i_{,\alpha}$ has rank two everywhere except possibly on a curve or at isolated points. A curve on such a surface is given by

$$x^i = x^i(u^\alpha(q,t),t) \quad \text{or by } x^i = x^i(u^\alpha, t),\ f(u^\alpha, t) = 0 \tag{2}$$

where $f(u^\alpha, t) = 0$ gives the surface coordinates of the points on the curve.

The surface at some fixed time is given by

$$X^K = X^K(U^\Gamma) \tag{3}$$

where U^Γ are surface coordinates of the reference surface. This function, its derivatives, and all other geometrical quantities referring to the reference surface are continuous by hypothesis. As already noted the mapping

$$u^\alpha = u^\alpha(U^\Gamma, t) \tag{4}$$

is one-one and onto and is continuous.

In a two dimensional space a curve across which certain functions are discontinuous but continuous elsewhere, a wave curve, can be described in the following equivalent ways:

$$f(u^\alpha, t) \;=\; 0 \quad \text{or} \quad u^\alpha \;=\; u^\alpha(q,t) \tag{5}$$

where $f,_\alpha$ exists and is not zero and where q is a parameter along the curve and $u^\alpha,_q$ exists and is not zero. x^i and u^α are assumed to be continuous across this curve although derivatives of these quantities may not be. Taking the time derivative of $f(u^\alpha(q,t),t) = 0$ keeping q constant gives the result

$$f,_t\big|_{u^\alpha} + f,_\alpha\, u^\alpha,_t\big|_q \;=\; 0 \tag{6}$$

The covariant surface vector $f,_\alpha$ is a surface vector which is normal to the curve. The unit normal to the curve is

$$\nu^\pm_\alpha \;=\; f,_\alpha |f^\pm,_\beta|^{-1} \tag{7}$$

where $|f^\pm,_\beta| = (a^{\alpha\beta\pm}\, f,_\alpha\, f,_\beta)^{1/2}$. In general, allowance must be made for the possibility that the surface metric tensor might be discontinuous. The + and - denote the limiting values of quantities on opposite sides of the curve. Then combining equations (6) and (7) yields

$$u_n^\pm \;=\; \nu^\pm_\alpha\, u^\alpha,_t\big|_q \;=\; -(f,_t\big|_{u^\alpha})\,\big|f^\pm,_\beta\big|^{-1} \tag{8}$$

It should be noted that u_n^+ is the component in the ν_α direction of the curve velocity with respect to the parameter $q, u^\alpha,_t\big|_q$. However, $u_n^\pm$ is seen to be independent of the choice of q. If $a^{\alpha\beta}$ is continuous then so is u_n.

In the three dimensional space with coordinates u^α, t the wave curve sweeps out the surface

$$f(u^\alpha, t) = 0 \tag{9}$$

Quantities which are continuous everywhere on the surface $x^i = x^i(u^\alpha, t)$ except on the wave curve $f(u^\alpha, t) = 0$ are continuous everywhere in the u^α, t space except on the surface $f(u^\alpha, t) = 0$. Hadamard's lemma[32] states that any directional derivative in this surface of a function $\psi(u^\alpha, t)$ can be written using the chain rule and one sided derivatives. Thus, if + and - are used to denote quantities on opposite sides of the surface then

$$\frac{d\psi^\pm}{dl} = \psi,_\alpha^\pm \, u^\alpha,_l + \psi,_t^\pm \, t,_l \tag{10}$$

where l is the parameter along a curve in the surface $f(u^\alpha, t) = 0$, and $\psi,_\alpha^\pm$ and $\psi,_t^\pm$ are one sided derivatives. The proof of this lemma hinges on the fact that the polygonal paths between two near points required in the standard proof of the chain rule can be found such that these paths lie exclusively on one side of the surface. As a result, the standard arguments are valid here as well.

In particular if the curve on the surface $f(u^\alpha, t)$ is given as

$$u^\alpha = u^\alpha(q,t)\big|_{q\,=\,\text{constant}}, \qquad t = t \tag{11}$$

so that in equation (10) l becomes t and if ψ is taken to be $x^i = x^i(u^\alpha, t)$ then

$$x^i,_{t|q} = (x^i,_{t|u^\alpha})^\pm + (x^i,_\alpha)^\pm \, u^\alpha,_{t|q} \tag{12}$$

The one sided velocity of a material point is defined to be $x^i_{,t|U}\Gamma^{\pm}$. Since this derivative is defined along a path entirely on one side of the surface $f(u^\alpha, t) = 0$ in u^α, t space the chain rule applies if one sided derivatives are used. Hence,

$$(x^i_{,t|U}\Gamma)^{\pm} = (x^i_{,t|u}\alpha)^{\pm} + x^i_{,\alpha}\,(u^\alpha_{,t|U}\Gamma)^{\pm} \tag{13}$$

If $\nu^{\alpha\pm}$ is the unit in-surface normal to the curve on the $\pm$ side of the curve then

$$\nu^{i^{\pm}} = x^{i^{\pm}}_{,\beta}\,\nu^{\beta^{\pm}} \tag{14}$$

The normal component of the velocity of the curve relative to the velocity of the material is given by subtracting equation (13) from equation (12) and multiplying the result by $g_{ij}\nu^{j^{\pm}}$; thus this quantity $P_n^{\pm}$ is defined by

$$P_n^{\pm} = [x^i_{,t|q} - (x^i_{,t|U}\Gamma)^{\pm}]\nu^{j^{\pm}} g_{ij} = a^{\pm}_{\alpha\beta}\nu^{\beta^{\pm}}(u^\alpha_{,t|q} - u^\alpha_{,t|U}\Gamma^{\pm}) \tag{15}$$

so that

$$P_n^{\pm} = G_n^{\pm} - V_n^{\pm} = u_n^{\pm} - v_n^{\pm} \tag{16}$$

where on the $\pm$ side of the curve $G_n^{\pm}$, $V_n^{\pm}$, $u_n^{\pm}$, and $v_n^{\pm}$ are the components in the in-surface normal direction of, the spatial velocity of the curve, the spatial velocity of a material point, the surface velocity of the curve, and the surface velocity of a material point. It should be noted that $P_n^{\pm}$ is independent of the choice of surface coordinates as $G_n^{\pm} - V_n^{\pm}$ is independent of the surface coordinates. Also $P_n^{\pm}$ is independent of the choice of q, the surface parameter, as $u_n^{\pm}$ has been shown not to depend upon q.

Since the mapping given in equation (4) is one-one, onto, and continuous there exists a curve in the reference state which is the image of the wave curve on the deformed surface. This curve will be referred to as the diagram. The surface coordinates in the reference state of the diagram are given by

$$F(U^\Gamma,t) = f(u^\alpha(U^\Gamma,t),t) = 0 \tag{17}$$

As before, the diagram sweeps out the surface $F(U^\Gamma,t) = 0$ in the three dimensional space with coordinate U^Γ and t. This surface divides U^Γ,t space into two regions. The mapping in equation (3) maps the regions "above" and "below" the dividing surface in U^Γ,t space onto the corresponding region in u^α,t space. The symbols + and - will be used to denote quantities on opposite sides of the surface. No ambiguity results because points on one side of the dividing surface in U^Γ,t space are all on one side of the dividing surface in u^α,t space.

The quantities $u^\alpha{}_{,\Gamma}$ and $u^\alpha{}_{,t|U^\Gamma}$ are in general allowed to be discontinuous across the surface $F(U^\Gamma,t) = 0$ although they are continuous elsewhere. The following one sided derivatives can then be found using the chain rule

$$F,_\Gamma{}^{\pm} = f,_\alpha u^\alpha{}_{,\Gamma}{}^{\pm} \tag{18}$$

and

$$(F,_{t|U^\Gamma})^{\pm} = f,_{t|u^\alpha} + f,_\alpha(u^\alpha{}_{,t|U^\Gamma})^{\pm} \tag{19}$$

An expression for $U_N{}^{\pm} = \nu_\Delta U^\Delta{}_{,t|q}$, where ν_Δ is the unit normal in the reference surface to the diagram and q is a parameter along the diagram, can be found in the same way as an expression for $u_n{}^{\pm}$ was found. Hence,

$$U_N^{\pm} = -F,_t^{\pm}|F,_\Delta^{\pm}|^{-1} \tag{20}$$

where $|F,_\Delta^{\pm}| = (A^{\Delta\Sigma}F,_\Delta^{\pm}F,_\Sigma^{\pm})^{1/2}$ and $A^{\Delta\Sigma}$ is the associate metric tensor for the reference surface. It can be shown that $U_N^+ = U_N^-$.

In order to prove this result the differential of $u^\alpha = u^\alpha(U^\Gamma,t)$ in any direction along the surface $F(U^\Gamma,t) = 0$ in U^Γ,t space is expressed using Hadamard's lemma. Then

$$u^{\alpha+}_{,\Gamma}dU^\Gamma + u^\alpha_{,t}\Big|^+_{U^\Delta} dt = du^\alpha$$

or

$$u^{\alpha-}_{,\Gamma}dU^\Gamma + u^\alpha_{,t}\Big|^-_{U^\Delta} dt = du^\alpha$$

Subtracting one from the other gives the result

$$[u^\alpha_{,\Gamma}]dU^\Gamma + [u^\alpha_{,t}|_{U^\Delta}]dt = 0 \tag{21}$$

where $[A] = A^+ - A^-$. Similarly, Hadamard's lemma can be used to find

$$F,^+_\Gamma dU^\Gamma + F,^+_t dt = 0 \tag{22}$$

$$F,^-_\Gamma dU^\Gamma + F,^-_t dt = 0 \tag{23}$$

Then using the method of Lagrangian multipliers equations (21) and (22) can be combined such that

$$[u^\alpha_{,\Gamma}] = \lambda^{\alpha+}F,^+_\Gamma \quad \text{and} \quad [u^\alpha_{,t}|_{U^\Gamma}] = \lambda^{\alpha+}F,^+_t \tag{24}$$

Equations (21) and (23) can be combined such that

$$[u^\alpha_{,\Gamma}] = \lambda^{\alpha-}F,^-_\Gamma \quad \text{and} \quad [u^\alpha_{,t}|_{U^\Gamma}] = \lambda^{\alpha-}F,^-_t \tag{25}$$

From equations (18) and (19) it can be shown that

$$[F,_{\Gamma}] = f,_{\alpha}[u^{\alpha},_{\Gamma}] \tag{26}$$

and

$$[F,_{t}] = f,_{\alpha}[u^{\alpha},_{t|U^{\Gamma}}] \tag{27}$$

Combining equations (25) and (26), and (25) and (27) it is evident that

$$F,_{\Gamma}^{+} - F,_{\Gamma}^{-} = [F,_{\Gamma}] = f,_{\alpha}\,\lambda^{\alpha^{-}}\,F,_{\Gamma}^{-} \tag{28}$$

or

$$F,_{\Gamma}^{+} = (1 + f,_{\alpha}\lambda^{\alpha^{-}})F,_{\Gamma}^{-} \tag{29}$$

and

$$F,_{t}^{+} - F,_{t}^{-} = [F,_{t}] = f,_{\alpha}\,\lambda^{\alpha^{-}}\,F,_{t}^{-} \tag{30}$$

or

$$F,_{t}^{+} = (1 + f,_{\alpha}\lambda^{\alpha^{-}})F,_{t}^{-} \tag{31}$$

By combining equations (20),(29), and (31) it is seen that

$$U_N^{+} = U_N^{-} \tag{32}$$

The derivation of this result requires $(1 + f,_{\alpha}\lambda^{\alpha^{-}}) > 0$. If $\nu_{\Delta}^{\pm}$ is the unit in surface normal to the diagram on the ± side of the curve then

$$\nu_{\Delta}^{+} = F,_{\Delta}^{+}|F,_{\Sigma}^{+}| = (1 + f,_{\alpha}\lambda^{\alpha^{-}})|1 + f,_{\beta}\lambda^{\beta^{-}}|^{-1}\nu_{\Delta}^{-} \tag{33}$$

If ν_{Δ}^{+} and ν_{Δ}^{-} are taken to have the same sense then $(1 + f,_{\alpha}\lambda^{\alpha^{-}}) > 0$.

It is easily seen that if $u^{\alpha},_{\Gamma}$ and $u^{\alpha},_{t|U^{\Gamma}}$ are continuous then $F,_{\Gamma}$ and $F,_{t|U^{\Gamma}}$ are continuous so that U_N is uniquely defined in any case.

A wave curve is of order n with respect to the motion if the n^{th} derivative of the motion, $x^i = x^i(U^{\Gamma},t)$, is discontinuous but all lower order derivatives are continuous. In what follows the deformed and undeformed surfaces will be assumed to be continuous. Thus zero order waves will not be considered. Shock waves and acceleration waves are first and second order waves respectively.

The analysis considered in later chapters will for the most part deal with the linear theory of elasticity in which deformations and rotations are assumed to be "small". Majuscule and minuscule letters and indices will be associated with the undeformed strain free stationary reference surface, S, and the deformed surface, s, respectively. It is the purpose of the ensuing discussion to find an expression for the ratio $P_n^{\pm}/U_N$ which is applicable to the case of "small" displacements. What is meant by "small" shall be made clear later.

By combining equations (19) and (20) it is seen that

$$|F,_{\Delta}^{\pm}|U_N \;=\; -F,_t^{\pm} \;=\; -f,_{t|u^\alpha} - f,_\alpha(u^\alpha,_{t|U^\Gamma})^{\pm} \tag{34}$$

where $|F,_{\Delta}^{\pm}| = (A^{\Delta\Sigma}F,_{\Delta}^{\pm}F,_{\Sigma}^{\pm})^{1/2}$ and $A^{\Delta\Sigma}$ is the associate metric tensor on S. u^α is any coordinate system on s. However, it can be seen that since $f(u^\alpha(q,t),t) = 0$ on the curve then

$$f,_{t|u^\alpha} + f,_\alpha\, u^\alpha,_{t|q} \;=\; f,_{t|q} \;=\; 0 \tag{35}$$

Therefore,

$$U_N \;=\; \frac{f,_\alpha\,(u^\alpha,_{t|q} - (u^\alpha,_{t|U^\Gamma})^{\pm})}{|F,_{\Delta}^{\pm}|} \tag{36}$$

The unit normal to the wave curve which lies in the surface is given by equation (7) so that

$$f,_{\alpha} = \nu^{\pm}_{\alpha} \, |f,^{\pm}_{\beta}| \tag{37}$$

where

$$|f,^{\pm}_{\beta}| = (a^{\alpha\beta^{\pm}} f,_{\alpha} f,_{\beta})^{1/2} = (a^{\alpha\beta\pm} u,^{\Delta\pm}_{\alpha} u,^{\Sigma\pm}_{\beta} F,^{\pm}_{\Delta} F,^{\pm}_{\Sigma})^{1/2} = (a^{\Delta\Sigma\pm} F,^{\pm}_{\Delta} F,^{\pm}_{\Sigma})^{1/2}$$

Therefore equations (36),(37), and (15) yield

$$U_N = \frac{|f,^{\pm}_{\beta}| \; P_n^{\pm}}{|F,^{\pm}_{\Delta}|} \tag{38}$$

As seen in Chapter II if $U^{\Sigma}_{,S}$ is the unit tangent to the diagram and S here refers to arc length then the unit normal to the diagram which lies in the reference surface is given by

$$\mathcal{V}_{\Delta} = \varepsilon_{\Delta\Sigma} \, U^{\Sigma}_{,S} \tag{39}$$

where $\varepsilon_{\Delta\Sigma}$ is the covariant permutation tensor on the reference surface which was defined in equation (II-113). Similarly, the corresponding unit normal to the wave curve is

$$\nu^{\pm}_{\alpha} = \varepsilon^{\pm}_{\alpha\beta} u^{\alpha}_{,s} \tag{40}$$

where here s is the arc length along the wave curve. Alternatively, from equations (33) and (7) it is seen that

$$\mathcal{V}_{\Delta} = F,^{\pm}_{\Delta} |F,^{\pm}_{\Sigma}| \qquad \nu^{\pm}_{\alpha} = f,_{\alpha} |f,^{\pm}_{\beta}|^{-1} \tag{41}$$

When these results are combined with equation (18) the result is

$$|F,^{\pm}_{\Sigma}| \mathcal{V}_{\Delta} = f,_{\alpha} u,^{\alpha^{\pm}}_{\Gamma} = |f,^{\pm}_{\beta}| \nu^{\pm}_{\alpha} u,^{\alpha^{\pm}}_{\Gamma} \tag{42}$$

Since $\mathcal{V}_{\Delta}$ is a unit vector, by using equations (40) and (42) it can be shown that

$$|F,_{\Phi}^{\pm}|^2 = |f,_{\beta}^{\pm}|^2 A^{\Delta\Sigma} u,_{\Sigma}^{\alpha\pm} u,_{\Sigma}^{\gamma\pm} \varepsilon_{\alpha\sigma}^{\pm} \varepsilon_{\gamma\rho}^{\pm} u,_{s}^{\sigma} u,_{s}^{\rho} \tag{43}$$

On the deformed surface convected coordinates, u^{Σ}, can be defined such that a material point on the deforming surface always bears the label it had on the reference state. Therefore, $u^{\Sigma} = U^{\Sigma}$. However, $U^{\Sigma} = U^{\Sigma}(u^{\alpha},t)$ is a one-one onto mapping and so the points on the wave curve are given by

$$u^{\Sigma} = U^{\Sigma} = U^{\Sigma}(u^{\alpha}(s,t),t) \tag{44}$$

Then using Hadamard's lemma it is easily shown that

$$u,_{s}^{\Sigma} = u,_{\alpha}^{\Sigma\pm} u,_{s}^{\alpha} \tag{45}$$

Equivalently,

$$u,_{s}^{\alpha} = u,_{s}^{\Sigma} u,_{\Sigma}^{\alpha\pm} \tag{46}$$

Then substitution from equation (46) into equation (43) demonstrates that

$$|F,_{\Phi}^{\pm}|^2 = |f,_{\beta}^{\pm}| A^{\Delta\Sigma} \bar{\varepsilon}_{\Delta\Phi}^{\pm} u,_{s}^{\Phi} \bar{\varepsilon}_{\Sigma\lambda}^{\pm} u,_{s}^{\lambda} \tag{47}$$

where $\bar{\varepsilon}_{\Delta\Phi}^{\pm}$ is the surface permutation tensor on the deformed surface in the convected coordinate system. That is,

$$\bar{\varepsilon}_{\Delta\Phi}^{\pm} = e_{\Delta\Phi}(\bar{A}^{\pm})^{1/2} = e_{\Delta\Phi}A^{1/2}\left(\frac{\bar{A}^{\pm}}{A}\right)^{1/2} = \varepsilon_{\Delta\Phi}\left(\frac{\bar{A}^{\pm}}{A}\right)^{1/2} \tag{48}$$

where $\bar{A}^{\pm}$ is the determinant of the deformed surface metric tensor, $a_{\Delta\Sigma}$, in the convected coordinate system and A of course is the determinate of the reference surface metric tensor, $A_{\Delta\Sigma}$, in the same coordinate system.

Then by combining equations (47) and (48) and noting that $u_{,s}^{\Delta\pm} = U_{,s}^{\Delta\pm}$ it is seen that

$$|F_{,\Phi}^{\pm}|^2 = |f_{,\beta}^{\pm}|^2 A^{\Delta\Sigma}\varepsilon_{\Delta\Phi}U^{\Phi}\ \varepsilon_{\Sigma\lambda}U_{,s}^{\lambda}(\frac{\bar{A}^{\pm}}{A}) \tag{49}$$

If it is noted that

$$U_{,s}^{\Phi} = U_{,S}^{\Phi}\frac{\partial S}{\partial s}\Big|_t \tag{50}$$

then by utilising equation (39) it is clear that

$$|F_{,\Phi}^{\pm}|^2 = |f_{,\beta}^{\pm}|^2\ (\frac{\partial S}{\partial s}\Big|_t)^2\ (\frac{\bar{A}^{\pm}}{A}) \tag{51}$$

Upon substitution from equation (51) into equation (39) it is evident that

$$\frac{P_n^{\pm}}{U_N} = (\frac{\partial S}{\partial s}\Big|_t)^2\ (\frac{\bar{A}^{\pm}}{A}) \tag{52}$$

The coordinate system in the embedding space is for convenience sake chosen to be rectangular Cartesian. x^i denotes the position occupied in space by a material particle on the deformed surface whose position in the reference configuration was X^i. Then the displacement, W^i, in the Cartesian system is given by

$$x^i = X^i + W^i \tag{53}$$

Signorini[33] has developed a systematic method of approximate solution in the theory of finite elastic deformation. In this approach the displacement function W^i is assumed to be an analytic function of ε which vanishes when $\varepsilon = 0$. Therefore the series expansion is

$$W^i(\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n W_n^i \tag{54}$$

where $W_n^i = \frac{1}{n!} \left. \frac{d^n W^i}{d\varepsilon^n} \right|_{\varepsilon = 0}$. If ε is chosen to be small enough then

$$W^i \approx \varepsilon W_1^i \tag{55}$$

When this is substituted into equation (53)

$$x^i = X^i + W^i(\varepsilon) \approx X^i + \varepsilon W_1^i \tag{56}$$

The series could have been extended so that higher order terms in ε were included. By taking ε small, it is sufficient to consider only the first order approximation.

It follows from the definition that

$$\bar{A} = |a^{\pm}_{\Delta\Sigma}| = \frac{1}{2} e^{\Delta\Phi} e^{\Sigma\Lambda} a^{\pm}_{\Delta\Sigma} a^{\pm}_{\Phi\Lambda}$$

$$= \frac{1}{2} e^{\Delta\Phi} e^{\Sigma\Lambda} x^{i\pm}_{,\Delta}\, x^{i\pm}_{,\Phi}\, x^{j\pm}_{,\Phi} x^{j\pm}_{,\Lambda} \tag{57}$$

where the last equation follows from

$$a^{\pm}_{\Delta\Sigma} = x^{i\pm}_{,\Delta} x^{i\pm}_{,\Sigma} \tag{58}$$

By differentiating equation (56) and substituting into equation (57) it is found that

$$\bar{A} = \frac{1}{2} e^{\Delta\Phi} e^{\Delta\Lambda} (X^i_{,\Delta} + \varepsilon W^{i\pm}_{1,\Delta})(X^i_{,\Sigma} + \varepsilon W^{i\pm}_{1,\Sigma})(X^j_{,\Phi} + \varepsilon W^{j\pm}_{1,\Phi})(X^j_{,\Lambda} + \varepsilon W^{j\pm}_{1,\Lambda}) \tag{59}$$

In order to be consistent with the approximation made above, second order and higher terms are neglected. Then

$$\bar{A}^{\pm} = \frac{1}{2} e^{\Delta\Phi} e^{\Sigma\Lambda} X^i_{,\Delta} X^i_{,\Sigma} X^j_{,\Phi} X^j_{,\Lambda} + 2 e^{\Delta\Phi} e^{\Sigma\Lambda} X^i_{,\Delta}\, X^i_{,\Sigma} X^j_{,\Phi} \varepsilon W^{j\pm}_{1,\Lambda} \tag{60}$$

Since W_1^i is a spatial vector defined over the reference surface it is possible to decompose W_1^i into components tangent to the surface and normal to the surface. Following the example given in equation (II-150) W_1^i becomes

$$W_1^i = X^i_{,\Delta}W_1^\Delta + N^i W_1 \tag{61}$$

Then equation (II-159) yields

$$W^{i\,\pm}_{1;\Sigma} = (W^{\Delta\pm}_{1;\Sigma} - B^\Delta_\Sigma W_1)X^i_{,\Delta} + (W_{1;\Sigma} + B_{\Delta\Sigma}W_1^\Delta)N^i \tag{62}$$

where $B_{\Delta\Sigma}$ are the coefficients of the second fundamental form of the reference surface. Substitution from equation (62) into equation (60) and utilisation of the orthogonality conditions for $X^i_{,\Delta}$ and N^i together with equation (II-114) yields

$$\bar{A}^{\pm} = A\{1 + 2\varepsilon(W^{\Gamma\pm}_{1;\Gamma} - B^\Gamma_\Gamma W_1)\} \tag{63}$$

The definition of the arc length of the wave curve and its diagram are

$$ds^2 = a^{\pm}_{\Delta\Sigma}dU^\Delta dU^\Sigma = a^{\pm}_{\Delta\Sigma}dU^\Delta dU^\Sigma \tag{64}$$

and

$$dS^2 = A_{\Delta\Sigma}dU^\Delta dU^\Sigma \tag{65}$$

However,

$$a^{\pm}_{\Delta\Sigma} = x^{i\pm}_{,\Delta}\, x^{i\pm}_{,\Sigma} = (X^i_{,\Delta} + W^{i\pm}_{,\Delta})(X^i_{,\Sigma} + W^{i\pm}_{,\Sigma})$$

and if $W^i \approx \varepsilon W_1^i$ then

$$a^{\pm}_{\Delta\Sigma} = (X^i_{,\Delta} + \varepsilon W^i_{1,\Delta})(X^i_{,\Sigma} + \varepsilon W^i_{1,\Sigma}) = X^i_{,\Delta}X^i_{,\Sigma} + \varepsilon(W^{i\,\pm}_{1,\Delta}X^i_{,\Sigma} + X^i_{,\Delta}W^{i\pm}_{1,\Sigma}) \tag{66}$$

where second order terms in ε have been neglected. Combining equations (65),(66) and (67) gives the result,

$$ds^2 = dS^2 + \varepsilon(W^{i}_{1,\Delta}{}^{\pm}X^{i}_{,\Sigma} + X^{i}_{,\Delta}W^{i}_{1,\Sigma}{}^{\pm})dU^{\Delta}dU^{\Sigma}$$

Therefore

$$\left(\frac{\partial s}{\partial S}\Big|_t\right)^2 = 1 + \frac{\varepsilon}{dS^2}(W_1{}^{i}_{,\Delta}X^{i}_{,\Sigma} + X^{i}_{,\Delta}W^{i}_{1,\Sigma})dU^{\Delta}dU^{\Sigma} \qquad (67)$$

By substitution from equation (62) it can be shown that

$$\left(\frac{\partial s}{\partial S}\Big|_t\right)^2 = 1 + \frac{2\varepsilon}{dS^2}(W_1{}^{\Gamma\pm}_{,\Delta} - B^{\Gamma}_{\Delta}W_1)\, A_{\Gamma\Sigma}dU^{\Delta}dU^{\Sigma} \qquad (68)$$

The results of equations (63) and (68) can be used to find from equation (52) the series expansion in ε of $P_n^{\pm}/U_N$ in which higher order terms than first order are neglected. Thus

$$\frac{P_n{}^{\pm}}{U_N} = 1 + 2\varepsilon\,\{W^{\Phi\,\pm}_{1;\Phi} - B^{\Phi}_{\Phi}W_1 - \frac{1}{dS^2}(W^{\Gamma\,\pm}_{1,\Delta} - B^{\Gamma}_{\Delta}W_1)A_{\Gamma\Sigma}dU^{\Delta}dU\} \qquad (69)$$

This is the expression which was to be found.

Another point of interest, in the same vein as the preceding discussion, is the question of the continuity of the deformed surface normal as a first order wave curve is crossed when only "small" displacements are considered. By differentiating equation (53) and then taking the jump in the resulting equation it is seen that

$$[x^{i}_{,\Delta}] = [X^{i}_{,\Delta}] + [W^{i}_{,\Delta}] = [W^{i}_{,\Delta}] \qquad (70)$$

where the spatial coordinate system is still rectangular Cartesian and where $X^i_{,\Delta}$ is, of course, continuous. When $W^i \sim \varepsilon W^i_1$ does $[x^i_{,\Delta}] = [W^i_{,\Delta}] \neq 0$ imply that the unit surface normal is discontinuous across the wave curve on the deformed surface? This is the question which the following discussion attempts to answer.

The unit normal to the deformed surface is given by equation (II-117). A convected surface coordinate system will be used here. Then,

$$n_i = \frac{1}{2}\bar{\varepsilon}^{\Delta\Gamma} \varepsilon_{ijk} x^j_{,\Delta} x^k_{,\Gamma} \tag{71}$$

where n_i is the unit normal to the deformed surface and $\bar{\varepsilon}^{\Delta\Gamma}$ is the representation in the convected coordinate system of the contravariant permutation tensor on the deformed surface.

By definition

$$\bar{e}^{\Delta\Gamma} = e^{\Delta\Gamma}\bar{A}^{-1/2} \tag{72}$$

where $e^{\Delta\Gamma}$ is the two dimensional permutation function. If $\bar{A}^{1/2}$ is replaced in equation (75) by the expression in equation (63) and the binomial expansion is employed neglecting second order and higher terms in ε than

$$\varepsilon^{\Delta\Gamma} = \varepsilon^{\Delta\Gamma}\{1 - \varepsilon(W_{1;\Phi}^{\ \ \Phi} - B^{\Phi}_{\Phi} W_1)\} \tag{73}$$

where $\varepsilon^{\Delta\Gamma} = \dfrac{e^{\Delta\Gamma}}{A^{1/2}}$ is the permutation tensor on the reference surface.

Substitution from equation (73) into equation (71) together with replacement of $x^i_{,\Delta}$ with $X^i_{,\Delta} + \varepsilon W^i_{1,\Delta}$ produces the result

$$n_i = \frac{1}{2}\varepsilon^{\Delta\Gamma}\varepsilon_{ijk}\{1 - \varepsilon(W_{1,\Phi}^{\ \ \Phi} - B^{\Phi}_{\Phi}W_1)\}\{X^j_{,\Delta} + \varepsilon W^j_{1,\Delta}\}\{X^k_{,\Gamma} + \varepsilon W^k_{1,\Gamma}\}$$

Using equations (62) and (II-117) this equation can be expanded so that to first order in ε

$$n_i = N_i - \varepsilon A^{\Lambda\Phi} X_{i,\Phi}(W_{1;\Lambda} + B_{\Sigma\Lambda} W_1^{\Sigma}) \tag{74}$$

When the jump in this equation is considered

$$[n_i] = -\varepsilon A^{\Lambda\Phi} X_{i,\Phi}[W_{1;\Lambda}] \tag{75}$$

Since $W^i \approx \varepsilon W_1^i$

$$[n_i] = -A^{\Lambda\Phi} X_{i,\Phi}[W_{;\Lambda}] \tag{76}$$

It can be concluded that a necessary and sufficient condition for the normal to the deformed surface to be continuous across a wave curve of first order or greater when $W^i \approx \varepsilon W_1^i$ is that

$$[W_{;\Lambda}] = 0 \tag{77}$$

Before leaving the notion of a "small" displacement it will prove useful to find the expression for $\frac{\rho}{\rho_o}$ which is correct to first order in ε. ρ is the mass per unit area of the deforming surface and ρ_o is the mass per unit area of the reference surface. Towards this end the conservation of mass is written as

$$\rho d\sigma = \rho_o d\Sigma \tag{78}$$

where $d\Sigma$ is the area of the reference surface which deforms into the area element $d\sigma$. Therefore,

$$\frac{\rho}{\rho_o} = \frac{d\Sigma}{d\sigma} \tag{79}$$

However, it was seen in equation (II-169) that

$$\frac{d\Sigma}{d\sigma} = (\frac{A}{\bar{A}})^{1/2} \tag{80}$$

so that

$$\frac{\rho}{\rho_o} = (\frac{A}{\bar{A}})^{1/2} \tag{81}$$

When $W^i \approx \varepsilon W_1^i$ then equation (63) gives

$$\frac{\rho}{\rho_o} = (\frac{A}{\bar{A}})^{1/2} = \{1 + 2\varepsilon(W_{1;\Gamma}^{\Gamma} - B_{\Gamma}^{\Gamma}W_1)\}^{-1/2} \tag{82}$$

so that to first order in ε

$$\frac{\rho}{\rho_o} = 1 - \varepsilon(W_{1;\Gamma}^{\Gamma} - B_{\Gamma}^{\Gamma}W_1) \tag{83}$$

The jump of this equation is

$$[\rho] = -\rho_o [W_{1;\Gamma}^{\Gamma}] \tag{84}$$

Since $W^i \approx \varepsilon W_1^i$

$$[\rho] = -\rho_o[W_{;\Gamma}^{\Gamma}] \tag{85}$$

This concludes for the moment the discussion of the consequence of "small" displacements.

In the ensuing discussion all relationships developed will refer to a moving curve on a surface which is fixed in the embedding three dimensional Euclidean space and which has continuous surface quantities. The $a_{\alpha\beta}$, $b_{\alpha\beta}$ and their derivatives are continuous everywhere. u^α is taken to be a surface coordinate system on such a surface such that a point on the surface has the same coordinates at all times. On the surface a moving curve at time t, C(t), is given by $u^\alpha = u^\alpha(q,t)$ where q is a parameter along the curve. The rate of change of surface coordinates for an observer who moves with the curve in such a way that he always is at a point of the moving curve with a fixed value of the curve parameter q is $u^\alpha_{,t}|_q$. With respect to time independent surface coordinate transformations $u^\alpha_{,t}|_q$ is a surface vector defined on the curve. As was shown in equation (II-155) such a vector can be decomposed into components tangential and normal to the curve such that

$$u^\alpha_{,t}|_q = u_n \nu^\alpha + T u^\alpha_{,q} \tag{86}$$

where ν^α is the unit normal and $u^\alpha_{,q}$ is a tangent vector to the curve, $u_n = u^\alpha_{,t}|_q \nu_\alpha$ and it was shown to be independent of the choice of q, and

$$T = u^\alpha_{,t}|_q \, u^\beta_{,q} a_{\alpha\beta}(a_{\gamma\sigma} u^\gamma_{,q} u^\sigma_{,q})^{-1/2}$$

Can one choose a curve parameter s_o such that $u^\alpha_{,t}|_{s_o} = u_n \nu^\alpha$ and such that s_o is the arc length along the curve at some reference time?

If one can find such an s_o then

$$q = q(s_o,t) \tag{87}$$

which can be inverted. Also, the chain rule requires that

$$u_n \nu^\alpha = u^\alpha_{,t}(q(s_o,t),t)|_{s_o} = u^\alpha_{,t}|_q + u^\alpha_{,q}|_t \, q_{,t}|_{s_o} \tag{88}$$

By combining equations (86) and (88) it is seen that

$$T(q,t) = - q_{,t}|_{s_o} \tag{89}$$

Differentiation of $q = q(s_o(q,t),t)$ and $q = q(s_o,t)$ with respect to t keeping q fixed and q keeping t fixed respectively and combination of the results yields

$$- q,_{t|s_o} = s_{o,t|q} (s_{o,q|t})^{-1} \tag{90}$$

Combining equations (89) and (90) yields

$$s_{o,q|t} \, T(q,t) = s_{o,t|q} \tag{91}$$

s_o must be a solution of this first order homogeneous partial differential equation. It can be shown[34] that a unique solution of equation (91) is determined by the requirement that the surface $s_o = s_o(q,t)$ representing the solution in s_o,q,t space shall contain a specified curve,

$$q = q(k), \quad t = t(k), \quad \text{and} \quad s_o = s_o(k)$$

such that

$$\begin{vmatrix} T & -1 \\ q'(k) & t'(k) \end{vmatrix} \neq 0$$

If the specified curve in s_o,q,t space is given by

$$q = q(s,t_o), \quad t = t_o, \quad s_o = s$$

where t_o is some reference time and s is the arc length along C(t) at the reference time then the determinant above becomes $q,_s$ which does not vanish identically because q and s are both acceptable parameters for the curve. Hence there is a unique solution of equation (91), $s_o = s_o(q,t)$, such that $s_o = s$ at $t = t_o$. This solution can be inverted as $s_{o,q|t} = 0$, by virtue of equation (91), implies that s_o is a constant which can not be so if s_o is a parameter along C(t).

If s_o is taken as the parameter along C(t) then

$$u^\alpha_{,t|s_o} = u_n \nu^\alpha \tag{92}$$

where C(t) is given by $u^\alpha = u^\alpha(s_o,t)$. The function $u^\alpha = u^\alpha(s_o,t)$ can be viewed geometrically as defining two one parameter families of curves on the surface. If t is taken as constant and s_o is taken as the parameter then the C(t) system of curves is obtained. Alternatively, if s_o is kept constant and t is taken as the parameter then a system of reference point paths is defined. $u^\alpha_{,s_o|t}$ is tangent to the former system and $u^\alpha_{,t|s_o}$ is tangent to the latter system of curves. Because of equation (92)

$$a_{\alpha\beta} u^\alpha_{,t|s_o} u^\beta_{,s_o|t} = 0 \tag{93}$$

Therefore, the curves given by s_o = constant are the orthogonal trajectories of the curves C(t).

If g(q,t) is some function defined on the moving curve $u^\alpha = u^\alpha(s_o,t)$ then the displacement derivative[34] of g is defined to be

$$\frac{\delta g}{\delta t}(q,t) = g_{,t|s_o} = g_{,t|q} - Tg_{,q} \tag{94}$$

where equation (89) was used. If g is defined on the surface then

$$\frac{\delta g}{\delta t} = g_{,t|u^\alpha} + g_{,\alpha} u_n \nu^\alpha \tag{95}$$

In order to define the analogue of the displacement derivative of a double tensor $\underline{\psi}$ which is defined on the two dimensional submanifold of the x^i, u^α, t space given by $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha(q,t)$, $t = t$ it is necessary to specify a connexion and an observer. The coordinate systems x^i, u^α in which the spatial and surface metrics are time indepdendent; such systems exist as the surface is assumed to be at rest in Euclidean space, is chosen to be the coordinate system in which Γ^i_j, Δ^α_β, $\Gamma^{i*}_{j\alpha}$, and $\Delta^{\alpha*}_{\beta j}$ vanish and Γ^i_{jk} and $\Delta^\alpha_{\beta\gamma}$ are the Christoffel symbols. The observer in the x^i, u^α, t manifold is specified as the curve $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha(q,t)$, $t = t$, $q = q(s_o,t)$, s_o = constant. This curve is embedded in the submanifold $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha(q,t)$, $t=t$. It is the curve in the x^i, u^α, t manifold which corresponds to the normal trajectory to the given surface curve in Euclidean space. The derivative required is now given by equation (II-105) where s_o plays the role of q. The coefficients of the derivative of a typical double tensor with coefficients $\psi^i_{\cdot\alpha}(q(s_o,t),t)$ are given by equation (II-110) so that in the x^i, u^α system and all coordinate systems obtainable from it by time independent coordinate transformations

$$\frac{D}{Dt}\psi^i_{\cdot\alpha}\Big|^{x^i,u^\alpha}_{s_o} = \psi^i_{\cdot\alpha,t}|_q + \psi^i_{\cdot\alpha,q}\, q_{,t}|_{s_o} + \Gamma^i_{jk}\psi^j_{\cdot\alpha}u_n\nu^k - \Delta^\gamma_{\alpha\beta}\psi^i_{\cdot\gamma}u_n\nu^\gamma \tag{96}$$

If $\psi^i_{\cdot\alpha}$ is defined over the the submanifold $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha$, $t = t$ then

$$\frac{D}{Dt}\psi^i_{\cdot\alpha}\Big|^{x^i,u^\alpha}_{s_o} = \psi^i_{\cdot\alpha,t}|_{u^\beta} + \psi^i_{\cdot\alpha;\beta}\nu^\beta u_n \tag{97}$$

This derivative will be referred to as the absolute derivative of the double tensor $\underline{\psi}$. The notation $\frac{D}{Dt}\psi^i_{\cdot\alpha}$ will be used.

Using equation (II-108) the derivative of $\underline{\psi}$ along the curve $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha(q,t), t = \text{constant}$ in the x^i, u^α,t manifold is given as

$$\frac{D}{Dq}\psi^{i}_{\cdot\alpha} = \psi^{i}_{\cdot\alpha,q|t} + \Gamma^{i}_{jk}\psi^{j}_{\cdot}\, x^{k}_{,q} - \Delta^{\gamma}_{\alpha\beta}\psi^{i}_{\cdot\alpha}u^{\beta}_{,q} \tag{98}$$

If $\underline{\psi}$ is defined on the submanifold $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha$, $t = t$ then

$$\frac{D}{Dq}\psi^{i}_{\cdot\alpha} = \psi^{i}_{\cdot\alpha;\beta}u^{\beta}_{,q} \tag{99}$$

The examples in the last two paragraphs are easily generalized.

A fixed surface S in three dimensional Euclidean space, $x^i = x^i(u^\alpha)$, at time t = constant corresponds to a two dimensional submanifold, Q(t), given by $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha$, t = constant, in the x^i,u^α,t manifold. As t changes this two dimensional submanifold sweeps out a three dimensional submanifold Q, $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha$, t = t. A moving curve C on S, which divides S into two parts A^+ and A^- at time t, A^+ being the region into which C is propagating, is described on S by $u^\alpha = u^\alpha(q,t)$, where q is the parameter along the curve at time t. In the x^i,u^α,t manifold this curve at t = constant corresponds to a curve R(t) embedded in Q(t) and given by $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha(q,t)$, t = constant. As t changes, this curve sweeps out a two dimensional manifold, R, embedded in Q and given by $x^i = x^i(u^\alpha)$, $u^\alpha = u^\alpha(q,t)$, t = t. Obviously, R(t) divides Q(t) into two regions $Q^+(t)$, $Q^-(t)$ and R divides Q into two parts Q^+,Q^-. At t = constant there exists a one-one onto mapping between S and Q(t) and between C and R(t). Thus Q represents the points of the x^i,u^α,t manifold swept out by S as t changes and R represents the points of this manifold swept out by C as t changes. A double tensor can be said to be defined at a point of S as a function of time in the sense that a point of S at a given time corresponds to a unique point of Q where a double tensor has been well defined.

A quantity, $\psi(u^\alpha, t)$, is defined on Q; that is ψ is defined on the surface S at all times. At any point of Q; that is, at any point swept out by S, except possibly on R, the points swept out by C, ψ is such that it is continuous and has continuous derivatives along the u^α and t coordinate directions, and also ψ and the derivatives of ψ approach unique finite limiting values denoted by superscripts + or - when a point on R, a point on C at some time, is approached along arbitrary curves entirely within Q^+ or Q^- respectively. Q^+ and Q^- are the regions covered by A^+ and A^- respectively as t changes. Then the jump or finite discontinuity in ψ is denoted by $[\psi] = \psi^+ - \psi^-$. $[\psi]$ is the difference between the two limiting values of ψ at a given time and at a given point on C. $\psi^\pm(q,t)$ and hence $[\psi(q,t)]$ are assumed to be continuous and differentiable quantities on R; that is, on the moving curve C. In passing, it can be seen that

$$[\psi\phi] = \psi^+[\phi] + [\psi]\phi^- = [\psi]\phi^+ + \psi^-[\phi] \tag{100}$$

It will be shown in the following discussion that the jumps in the derivatives of ψ along the u^α and t coordinate directions can be expressed in terms of jumps in derivatives along the in-surface normal to C and as derivatives along curves in R; that is, derivatives on the moving curve, of jumps in lower order derivatives. Thus, the jumps in the derivatives of ψ are related to the jumps in lower order derivatives. In general, a quantity can not be so specified such that the jumps in the quantity and its derivatives are arbitrarily specified. For this reason, the relations relating the jumps in the derivatives will be called compatibility relations.

Since the restrictions on the jumps are due to geometrical and/or kinematical arguments the compatibility relations bear the prefix geometrical, kinematical, or mixed.

In the x^i, u^α, t coordinate system the quantity ψ is taken to be a function $\psi(u^\alpha, t)$ defined over Q or over S for all times. The parameter along the curve C is chosen to be the arc length s. Then along C at a fixed time or along R(t), $\psi^+_{,s}$ and $\psi^-_{,s}$ have been assumed to exist. Hadamard's lemma states that any directional derivative on R of a function $\psi(u^\alpha, t)$ can be written using the chain rule and one sided derivatives. The proof of this lemma follows readily from the existance of the polygonal paths required to prove the chain rule which lie exclusively in Q^+ or in Q^-. Therefore,

$$\frac{D}{Ds} \psi^\pm = \psi^\pm_{,\alpha} u^\alpha_{,s} \tag{101}$$

If the result on one side is subtracted from the result on the other, then

$$\frac{D}{Ds} [\psi] = \frac{D}{Ds} \psi^+ - \frac{D}{Ds} \psi^- = [\psi_{,\alpha}] u_{,s} \tag{102}$$

where the first equation is a direct consequence of the definition of D/Ds. When this equation is multiplied by $a_{\beta\gamma} u^\beta_{,s}$ and identity (II-157) is used the following result is shown to be valid.

$$[\psi_{,\gamma}] = B\nu_\gamma + \frac{D}{Ds} [\psi] u_{\gamma,s} \tag{103}$$

where $B = [\psi_{,\alpha}]\nu^\alpha$. This is called the first order geometrical compatibility relation for ψ.

Similarly, the derivatives $\frac{\delta}{\delta t} \psi^+$ and $\frac{\delta}{\delta t} \psi^-$ have been assumed to exist. Hadamard's lemma is again invoked so that as suggested by equation (95)

$$\frac{\delta}{\delta t}\psi^{\pm} = \psi^{\pm}_{,t|u^{\alpha}} + \psi^{\pm}_{,\alpha}\, u_n \nu^{\alpha} \tag{104}$$

Subtraction of one equation from the other gives the result

$$[\psi_{,t|u^{\alpha}}] = \frac{\delta}{\delta t}[\psi] - Bu_n \tag{105}$$

where $\frac{\delta}{\delta t}[\psi] = \frac{\delta}{\delta t}\psi^{+} - \frac{\delta}{\delta t}\psi^{-}$ follows from definition of $\frac{\delta}{\delta t}$ and $B = [\psi_{,\alpha}]\nu^{\alpha}$.

This is called the first kinematical compatibility relation for ψ.

If the $\underline{\psi}(u^{\alpha},t)$ is taken to be a double tensor defined over Q; that is, over S for all times, then the derivatives $\frac{D}{Ds}\psi^{+}$ and $\frac{D}{Ds}\psi^{-}$ as defined by equation (98) have been assumed to exist. The first term on the right hand side is just an ordinary derivative so that Hadamard's lemma yields

$$\psi^{\pm}_{,s} = \psi^{\pm}_{,\gamma}\, u^{\gamma}_{,s} \tag{106}$$

When this is substituted into equation (98) evaluated on both sides of R and the two resulting equations are differenced then it is found that

$$\frac{D}{Ds}[\psi] = \frac{D}{Ds}\psi^{+} - \frac{D}{Ds}\psi^{-} = [\psi_{;\gamma}]u^{\gamma}_{,s} \tag{107}$$

where the first equation is a consequence of the definition of D/Ds and where; denotes the total covariant derivative. The indices on ψ have been suppressed. As before, multiplication by $a_{\alpha\beta}u^{\beta}_{,s}$ and the use of identity (II-157) gives the first geometrical compatibility relation for the double tensor $\underline{\psi}$. Thus

$$[\psi;_{\alpha}] = \lambda\nu_{\alpha} + \frac{D}{Ds}[\psi]u_{\alpha,s} \tag{108}$$

Similarly, the derivatives $\frac{D}{Dt}\psi^{+}$, $\frac{D}{Dt}\psi^{-}$, as defined by equation (96), have been assumed to exist. The first two terms on the right hand side of equation (96) can be written using Hadamard's lemma as

$$\frac{\delta}{\delta t}\psi^{\pm} = \frac{\partial}{\partial t}\psi^{\pm} + \psi^{\pm}_{;\gamma}\, u_n \nu^{\gamma} \tag{109}$$

When this is substituted into equation (96) evaluated on both sides of R and the two resulting equations are differenced then it is found that

$$\frac{D}{Dt}[\psi] = \frac{D}{Dt}\psi^{+} - \frac{D}{Dt}\psi^{-} = [\psi,_{t|u}{}^{\alpha}] + [\psi;_{\gamma}]\nu^{\gamma}u_n \tag{110}$$

Therefore,

$$[\psi,_{t|u}{}^{\alpha}] = \frac{D}{Dt}[\psi] - Bu_n \tag{111}$$

where B is defined following equation (103). This is called the first kinematical compatibility relation for the double tensor $\underline{\psi}$. The indices on ψ and B have been suppressed.

The identities developed for the double tensor $\underline{\psi}$ of course apply to the special case where ψ is a scalar or function.

The higher order compatibility relations are obtained by an iterative procedure from the lower order relations. Before proceding further it will prove useful to obtain expressions for $\frac{D}{Dt}u^{\alpha}_{,s}$ and $\frac{D}{Dt}\nu^{\alpha}$, which will be involved in deriving the higher order relations. From the definition of D/Dt in equation (96), the interchange of the order of differentiation in the first term, and the chain rule applied to the last term, it is possible to write

$$\frac{D}{Dt} u^{\alpha}_{,s} = \frac{D}{Ds} (u^{\alpha}_{,t|s}) + \frac{D}{Ds} (u^{\alpha}_{,s}) s_{,t|s_o} \tag{112}$$

Combining equations (112),(86),(89), (II-127), and (II-128) yields

$$\frac{D}{Dt} u^{\alpha}_{,s} = (T_{,s} - \sigma u_n) u^{\alpha}_{,s} + u_{n,s} \nu^{\alpha} \tag{113}$$

where σ is the geodesic curvature of the wave curve, and s is the arc length along the surve.

It can be shown that $T_{,s} = \sigma u_n$. To do this the expression for T given in equation (86) is considered. Then

$$T_{,s} = (u_{\alpha,s} u^{\alpha}_{,t|s})_{,s} = u_{\alpha,ss} u^{\alpha}_{,t|s} + u_{\alpha,s} (u^{\alpha}_{,t|s})_{,s} \tag{114}$$

From equation (II-127) it is seen that

$$u_{\alpha,ss} = (\sigma \nu_{\alpha} + \Delta^{\gamma}_{\alpha\beta} u_{\gamma,s} u^{\beta}_{,s})$$

Substitution of this into equation (114) yields

$$T_{,s} = \sigma u_n + u_{\gamma,s} \{ (u^{\gamma}_{,s})_{,t|s} + \Delta^{\gamma}_{\beta\alpha} u^{\beta}_{,s} u^{\alpha}_{,t|s} \} \tag{115}$$

where equation (II-110) indicates that the expression in brackets is the tensor $\frac{D}{Dt} u^{\gamma}_{,s}\Big|_s$. Therefore

$$T_{,s} = \sigma u_n + u_{\gamma,s} \frac{D}{Dt} u^{\gamma}_{,s}\Big|_s \tag{116}$$

However, the derivative of the equation which indicates that $u^{\alpha}_{,s}$ is a unit vector yields

$$a_{\alpha\beta}\frac{D}{Dt}u^{\alpha}_{,s}\Big|_{s}u^{\beta}_{,s} = 0$$

and when this is substituted into equation (116) the result is

$$T_{,s} = \sigma u_{n} \tag{117}$$

It is now seen that

$$\frac{D}{Dt}u^{\alpha}_{,s} = u_{n,s}\nu^{\alpha} \tag{118}$$

By differentiating the conditions that ν^{β} is a unit vector and that it is normal to the unit tangent and by using equation (II-155) to decompose $\frac{D}{Dt}\nu^{\alpha}$ it is seen that

$$\frac{D}{Dt}\nu^{\alpha} = -u_{n,s}u^{\alpha}_{,s} \tag{119}$$

In deriving the second order compatibility relations the indices on double tensors will be suppressed for convenience sake unless they are directly involved in the derivation. To begin with, if $\phi^{i_1\ldots i_s\ldots\alpha_1\ldots\alpha_p}_{\ldots j_1\ldots j_t\ldots\beta_1\ldots\beta_q}$ is an arbitrary double tensor then $\phi;_{\beta}$ is substituted for ψ in the first order geometrical compatibility relation (108). Then

$$[\phi;_{\beta\alpha}] = [\phi;_{\beta\gamma}]\nu^{\gamma}\nu_{\alpha} + \frac{D}{Ds}[\phi;_{\beta}]u_{\alpha.s} \tag{120}$$

Equation (II-136) provides an expression for $\phi;_{\alpha\beta} - \phi;_{\beta\alpha}$. By taking the jump in this equation, multiplying the result and equation (120) by ν^{β}, and adding these two equations it is evident that

$$[\phi;_{\alpha\beta}]\nu^{\beta} = \tilde{\lambda}\nu_{\alpha} + \frac{D}{Ds}[\phi;_{\beta}]\nu^{\beta}t_{\alpha} + \sum_{k=1}^{q}[\phi^{\alpha_1\ldots\ldots\ldots\alpha_p}_{\ldots\beta_1.\beta_{k-1}\eta\beta_{k+1}\ldots\beta_q}]R^{\eta}_{.\beta_k\alpha\beta}\nu^{\beta}$$

$$- \sum_{k=1}^{p} [\phi^{\cdot\cdot\alpha_1\ldots\alpha_{k-1}\eta\alpha_{k+1}\ldots\alpha_p\cdot\cdot}_{\cdot\cdot\cdot\cdot\beta_1\ldots\ldots\ldots\ldots\beta_q}] R^{\cdot\alpha_k\cdot\cdot}_{\cdot\eta\alpha\beta} \nu^\beta \tag{121}$$

where $\tilde{\lambda} = [\phi;_{\alpha\beta}]\nu^\alpha\nu^\beta$ and $t_\alpha = u^\alpha_{,s}$. The combination of equations (108), (II-127), and (II-128) yields

$$\frac{D}{Ds}[\phi;_\beta] = (-\sigma\lambda + \frac{D^2}{Ds^2}[\phi])t_\beta + (\frac{D\lambda}{Ds} + \sigma\frac{D}{Ds}[\phi])\nu_\beta \tag{122}$$

where $\lambda = [\phi;_\alpha]\nu^\alpha$ and σ is the geodesic curvature of the curve. The second order geometrical compatibility relation is obtained by substituting from equation (121) into equation (120) and then substituting from equation (122) into the result. Then

$$[\phi;_{\beta\alpha}] = \tilde{\lambda}\nu_\beta\nu_\alpha + (\frac{D\lambda}{Ds} + \sigma\frac{D}{Ds}[\phi])(t_\beta\nu_\alpha + \nu_\beta t_\alpha) + (-\sigma\lambda + \frac{D^2}{Ds^2}[\phi])t_\beta t_\alpha$$

$$+ \sum_{k=1}^{q} [\phi^{\cdot\cdot\alpha_1\ldots\ldots\ldots\ldots\alpha_p}_{\cdot\cdot\cdot\cdot\beta_1\cdot\beta_{k-1}\eta\beta_{k+1}\cdot\cdot\beta_q}] R^{\eta}_{\cdot\beta_k\beta\gamma}\nu^\gamma\nu_\alpha - \sum_{k=1}^{p} [\phi^{\cdot\cdot\alpha_1\cdot\cdot\alpha_{k-1}\eta\alpha_{k+1}\cdot\alpha_p}_{\cdot\cdot\cdot\beta_1\ldots\ldots\ldots\beta_q}] \cdot$$

$$R^{\cdot\alpha_k\cdot}_{\cdot\eta\beta\gamma}\nu^\gamma\nu_\alpha \tag{123}$$

This is called the second order geometrical compatibility relation for the double tensor ϕ. It should be remembered that the indices on ϕ, λ, and $\tilde{\lambda}$ have been suppressed.

In order to obtain the second order mixed compatibility relation for the double tensor ϕ, $\phi;_\beta$ is substituted for ψ in the first order kinematical relative, equation (111). Then

$$[(\phi,_t);_\beta] = [(\phi;_\beta),_t] = \frac{D}{Dt}[\phi;_\beta] - [\phi;_{\beta\gamma}]\nu^\gamma u_n \tag{124}$$

where the first equality follows from the definitions of the differentiations. The derivative of equation (108) together with equations (118) and (119) yields

$$\frac{D}{Dt}[\phi;_\beta] = \left(\frac{D^2}{DtDs}[\phi] - \lambda u_{n,s}\right)t_\beta + \left(\frac{D}{Dt}\lambda + \frac{D}{Ds}[\phi]\, u_{n,s}\right)\nu_\beta \tag{125}$$

Now by substituting from equations (123) and (125) into equation (124) it is seen that

$$\begin{aligned}
[(\phi;_\beta),_t] = {} & \left(\frac{D}{Dt}\lambda - \tilde{\lambda} u_n + \frac{D}{Ds}[\phi]\, u_{n,s}\right)\nu_\beta + \left(- u_n\sigma \frac{D}{Ds}[\phi]\right. \\
& \left. - \frac{D}{Ds}(u_n\lambda) + \frac{D^2}{DtDs}[\phi]\right)t_\beta \\
& - u_n \sum_{k=1}^{q} [\phi^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_{k-1}\eta\beta_{k+1}\dots\beta_q}] R^{\eta}{}_{\beta_k\beta\gamma}\nu^\gamma \\
& + u_n \sum_{k=1}^{p} [\phi^{\alpha_1 \dots \alpha_{k-1}\eta\alpha_{k+1}\dots\alpha_p}{}_{\beta_1 \dots \beta_q}] R^{\alpha_k}{}_{\eta\beta\gamma}\nu^\gamma
\end{aligned} \tag{126}$$

This result is called the second order mixed compatibility relation for the double tensor ϕ. The indices on ϕ, λ, and $\tilde{\lambda}$ have been suppressed.

The second order kinematical compatibility relation is obtained by substituting $\phi,_t$ for ψ in equation (111), the first order kinematical compatibility relation. Then

$$[\phi,_{tt}] = \frac{D}{Dt}[\phi,_t] - [(\phi,_t);_\beta]\nu^\beta u_n \tag{127}$$

Differentiation of equation (111) gives the result

$$\frac{D}{Dt}[\phi,_t] = \frac{D^2}{Dt^2}[\phi] - \frac{D}{Dt}\lambda u_n - \lambda \frac{D}{Dt} u_n \tag{128}$$

Substitution of this result and the result of equation (126) leads to the expression,

$$[\phi,_{tt}] = \tilde{\lambda}u_n^2 - 2u_n \frac{D}{Dt}\lambda - \lambda\frac{D}{Dt}u_n - u_n u_{n,s}\frac{D}{Ds}[\phi] + \frac{D^2}{Dt^2}[\phi] \tag{129}$$

This identity is known as the second order kinematical compatibility relation for the double tensor ϕ.

The higher order compatibility relations which will be derived below will not be completely general. The restriction placed upon them are motivated by the purposes for which they are required.

First, the third order geometrical compatibility relation for a double tensor ϕ will be derived for the case in which $[\phi] = 0$. In the second order geometrical relation, equation (123), the index α is changed to γ and then $\phi;_\alpha$ is substituted for ϕ. Then

$$\begin{aligned}[\phi;_{\alpha\beta\gamma}] &= [\phi;_{\alpha\sigma\eta}]\nu^\sigma\nu^\eta\nu_\beta\nu_\gamma + \{\frac{D^2}{Ds^2}[\phi;_\alpha] - \sigma[\phi;_{\alpha\eta}]\nu^\eta\}t_\beta t_\gamma \\ &+ \{\frac{D}{Ds}([\phi;_{\alpha\eta}]\nu^\eta) + \sigma\frac{D}{Ds}[\phi;_\alpha]\}(t_\gamma\nu_\beta + t_\beta\nu_\gamma) + [\phi;_\eta]R^{\cdot\eta}_{\cdot\alpha\beta\delta}\nu^\delta\nu_\gamma \\ &+ \sum_{k=1}^{q}[\phi^{\cdot\cdot\alpha_1\cdots\cdots\cdots\cdots\alpha_p}_{\cdots\beta_1\cdot\beta_{k-1}\eta\beta_{k+1}\cdot\cdot\beta_q;\alpha}]R^{\cdot\eta}_{\cdot\beta_k\beta\sigma}\nu^\sigma\nu_\gamma \\ &+ \sum_{k=1}^{p}[\phi^{\cdot\cdot\alpha_1\cdot\cdot\,\alpha_{k-1}\eta\alpha_{k+1}\cdot\cdot\alpha_p}_{\cdots\beta_1\cdots\cdots\cdots\cdots\cdots\beta_q;\alpha}]R^{\alpha_k}_{\cdot\eta\beta\sigma}\nu^\sigma\nu_\gamma\end{aligned} \tag{130}$$

Equation (II-136) provides an expression for $\phi;_{\alpha\beta\gamma} - \phi;_{\beta\alpha\gamma}$. The jump is taken of this equation with $[\phi] = 0$. Then the resulting equation and equation (130) are multiplied by $\nu^\alpha\nu^\gamma$ and the former is subtracted from the latter. This provides an expression for $[\phi;_{\alpha\sigma\eta}]\nu^\sigma\nu^\eta$ which can be substituted into equation (130). Thus

$$[\phi;_{\alpha\beta\gamma}] = \{\tilde{\tilde{\lambda}}\nu_\alpha + \frac{D}{Ds}([\phi;_{\delta\eta}]\nu^\eta)t_\alpha\nu^\delta + \sigma\frac{D}{Ds}[\phi;_\delta]t_\alpha\nu^\delta\}\nu_\beta\nu_\gamma$$

$$+ [\phi;_\eta]R^{\eta}_{\cdot\sigma\alpha\delta}\nu^\delta\nu^\sigma\nu_\beta\nu_\gamma + 2\sum_{k=1}^{q}[\phi^{\cdot\cdot\alpha_1\cdots\cdots\cdots\cdots\alpha_p}_{\cdots\beta_1\cdot\beta_{k-1}\eta\beta_{k+1}\cdot\cdot\beta_{q;\delta}}]R^{\eta}_{\cdot\beta_k\alpha\sigma}\nu^\sigma\nu^\delta\nu_\beta\nu_\gamma$$

$$- 2\sum_{k=1}^{p}[\phi^{\cdot\cdot\alpha_1\cdots\alpha_{k-1}\eta\alpha_{k+1}\cdot\cdot\alpha_p}_{\cdots\beta_1\cdots\cdots\cdots\cdots\beta_{q;\delta}}]R^{\alpha_k}_{\cdot\eta\alpha\sigma}\nu^\sigma\nu^\delta\nu_\beta\nu_\gamma + \{\frac{D^2}{Ds^2}[\phi;_\alpha]$$

$$- \sigma[\phi;_{\alpha\eta}]\nu^\eta\}t_\beta t_\gamma + \{\frac{D}{Ds}([\phi;_{\alpha\eta}]\nu^\eta) + \frac{D}{Ds}[\phi;_\alpha]\}(t_\gamma\nu_\beta + t_\beta\nu_\gamma)$$

$$+ [\phi;_\eta]R^{\eta}_{\cdot\alpha\beta\delta}\nu^\delta\nu_\gamma + \sum_{k=1}^{q}[\phi^{\cdot\cdot\alpha_1\cdots\cdots\cdots\cdots\alpha_p}_{\cdots\beta_1\cdot\beta_{k-1}\eta\beta_{k+1}\cdot\cdot\beta_{q;\alpha}}]R^{\eta}_{\cdot\beta_k\beta\sigma}\nu^\sigma\nu_\gamma$$

$$- \sum_{k=1}^{p}[\phi^{\cdot\cdot\alpha_1\cdots\alpha_{k-1}\eta\alpha_{k+1}\cdot\cdot\alpha_p\cdots}_{\cdots\beta_1\cdots\cdots\cdots\cdots\cdots\beta_{q;\alpha}}]R^{\alpha_k}_{\cdot\eta\beta\sigma}\nu^\sigma\nu_\gamma \tag{131}$$

where $\tilde{\tilde{\lambda}} = [\phi;_{\alpha\beta\gamma}]\nu^\alpha\nu^\beta\nu^\gamma$.

From the first and second order geometrical compatibility equations, equations (108) and (123), with $[\phi] = 0$ together with (II-127) and (II-128) it is evident that

$$[\phi;_\alpha] = \lambda\nu_\alpha \tag{132}$$

$$\frac{D}{Ds}[\phi;_\alpha] = \frac{D}{Ds}\lambda\,\nu_\alpha - \lambda\sigma t_\alpha \tag{133}$$

$$\frac{D^2}{Ds^2}[\phi;_\alpha] = (\frac{D^2}{Ds^2}\lambda - \lambda\sigma^2)\nu_\alpha + (-2\sigma\frac{D}{Ds}\lambda - \lambda\frac{D}{Ds}\sigma)t_\alpha \tag{134}$$

$$[\phi;_{\beta\gamma}] = \tilde{\lambda}\nu_\beta\nu_\gamma - \sigma\lambda t_\beta t_\gamma + \frac{D}{Ds}\lambda\,(t_\beta\nu_\alpha + \nu_\beta t_\alpha) \tag{135}$$

$$\frac{D}{Ds}[\phi;_{\beta\gamma}] = (\frac{D}{Ds}\tilde{\lambda} + 2\sigma\frac{D}{Ds}\lambda)\nu_\beta\nu_\gamma + (-\lambda\frac{D}{Ds}\sigma - 3\sigma\frac{D}{Ds}\lambda)t_\gamma t_\beta + (-\tilde{\lambda}\sigma - \lambda\sigma^2 + \frac{D^2}{Ds^2}\lambda)$$

$$(t_\beta\nu_\beta + t_\gamma\nu_\beta) \tag{136}$$

Combining equations (130) to (136) yields the third order geometrical compatibility relation for the case where the double tensor ψ satisfies the condition $[\phi] = 0$,

$$[\psi;_{\alpha\beta\gamma}] = \tilde{\tilde{\lambda}}\nu_\alpha\nu_\beta\nu_\gamma + (-\lambda\frac{D}{Ds}\sigma - 3\sigma\frac{D}{Ds}\lambda)\,t_\alpha t_\beta t_\gamma$$

$$+ (\frac{D}{Ds}\tilde{\lambda} + 2\sigma\frac{D}{Ds}\lambda)(t_\alpha\nu_\beta\nu_\gamma + t_\beta\nu_\alpha\nu_\gamma + t_\gamma\nu_\alpha\nu_\beta)$$

$$+ \lambda\nu_\eta R^{\eta}_{\cdot\alpha\beta\delta}\nu^\delta\nu_\gamma + \sum_{k=1}^{q}\lambda^{\cdot\cdot\alpha_1\cdots\cdot\alpha_p}_{\cdots\beta_1\cdot\eta\cdot\cdot\beta_q}(2R^{\eta}_{\cdot\beta_k\alpha\sigma}\nu_\beta + R^{\eta}_{\cdot\beta_k\beta\sigma}\nu_\alpha)\nu^\sigma\nu_\gamma$$

$$- \sum_{k=1}^{p}\lambda^{\cdot\alpha_1\cdots\eta\cdot\cdot\alpha_p}_{\cdots\beta_1\cdots\cdots\beta_q}(2R^{\alpha_k}_{\cdot\eta\alpha\sigma}\nu_\beta + R^{\alpha_k}_{\cdot\eta\beta\sigma}\nu_\alpha)\nu^\sigma\nu_\gamma \tag{137}$$

If ϕ is an arbitrary double tensor and $\phi;_{\beta\gamma}$ is substituted for ϕ in equation (111) then

$$[(\phi;_{\beta\gamma}),_t] = \frac{D}{Dt}[\phi;_{\beta\gamma}] - [\phi;_{\beta\gamma\sigma}]\nu^\sigma u_n \tag{138}$$

If instead, $(\phi;_\beta),_t$ is substituted for ϕ in equation (111) then

$$[(\phi;_\beta),_{tt}] = \frac{D}{Dt}[(\phi;_\beta),_t] - [(\phi;_{\beta\gamma}),_t]\nu^\gamma u_n \tag{139}$$

Substitution from (138) into (139) leads to the conclusion that

$$[(\phi;_{\beta}),_{tt}] = \frac{D}{Dt}[(\phi;_{\beta}),_{t}] - \frac{D}{Dt}[\phi;_{\beta\gamma}]\nu^{\gamma}u_n + [\phi;_{\beta\gamma\sigma}]\nu^{\sigma}\nu^{\gamma}u_n^2 \tag{140}$$

To this point no restrictions have been imposed upon the foregoing relations. If ϕ is assumed to be a spatial tensor or a scalar and $[\phi] = 0$ then equation (140) takes a special form. Substitution from equations (126), (135), and (137) into equation (140) together with the use of equations (118) and (119) yields the third order mixed compatibility relation for a spatial tensor or scalar ϕ for which $[\phi] = 0$,

$$\begin{aligned}[(\phi;_{\beta}),_{tt}] = {} & \{\tilde{\tilde{\lambda}}\, u_n^2 - \frac{D}{Dt}\tilde{\lambda}\, u_n - 2u_{n,s}\frac{D}{Ds}\lambda\, u_n + \frac{D^2}{Dt^2}\lambda - \frac{D}{Dt}(\tilde{\lambda}u_n) \\ & - \frac{D}{Ds}(u_n\lambda)u_{n,s}\}\nu_{\beta} \\ & + \{u_n^2\frac{D}{Ds}\tilde{\lambda} + 2\sigma\frac{D}{Ds}\lambda u_n^2 + \tilde{\lambda}u_{n,s}u_n + \lambda\sigma u_{n,s}u_n - \frac{D^2}{DtDs}u_n - u_{n,s}\frac{D}{Dt}\lambda \\ & + \tilde{\lambda}u_n u_{n,s} - \frac{D^2}{DtDs}(u_n\lambda)\}t_{\beta}\end{aligned} \tag{141}$$

If ϕ is a double tensor such that $[\phi] = 0$ and $[\phi;_{\alpha}] = 0$ and if u_n is constant then equations (138) and (139) take special forms. With these restrictions in mind, it is seen that substitution from equations (123) and (137) into equation (138) together with the use of equations (118) and (119) gives the following third order mixed compatibility relation for the double tensor ϕ,

$$[(\phi;_{\beta\gamma}),_{t}] = (\frac{D}{Dt}\tilde{\lambda} - u_n\tilde{\tilde{\lambda}})\nu_{\beta}\nu_{\gamma} - u_n\frac{D}{Ds}\tilde{\lambda}\,(t_{\beta}\nu_{\gamma} + t_{\gamma}\nu_{\beta}) + u_n\sigma\tilde{\lambda}t_{\beta}t_{\gamma} \tag{142}$$

With the same restrictions, equation (139) upon substitution from equation (126) and (142) becomes, when equations (118) and (119) are employed,

$$[(\phi;_{\beta}),_{tt}] \;=\; (-\,2u_n \frac{D}{Dt}\tilde{\lambda} + u_n^2\tilde{\tilde{\lambda}})\nu_\beta + u_n^2 \frac{D}{Ds}\tilde{\lambda}t_\beta \tag{143}$$

This is another of the third order compatibility relations required for the double tensor ϕ, when $[\phi] = 0$, $[\phi;_\alpha] = 0$, and u_n is constant.

The third order kinematical compatibility relation for the double tensor ϕ under the conditions of the preceding paragraph is obtained by first substituting $\phi,_{tt}$ for ϕ in equation (111). Then

$$[\phi,_{ttt}] \;=\; \frac{D}{Dt}[\phi,_{tt}] - [(\phi;_\gamma),_{tt}]\nu^\gamma u_n \tag{144}$$

Substitution from equations (129) and (143) and use of equations (118) and (119) yields

$$[\phi,_{ttt}] \;=\; -\,u_n^3\tilde{\tilde{\lambda}} + 3u_n^2\frac{D}{Dt}\tilde{\lambda} \tag{145}$$

This is the third order kinematical compatibility relation for the double tensor ϕ when $[\phi] = 0$, $[\phi;_\beta] = 0$, and u_n is constant.

The remaining relations which will be employed are the fourth order geometrical and mixed compatibility relations for a scalar or spatial tensor ϕ under the conditions that $[\phi] = 0$, $[\phi;_\alpha] = 0$ and u_n is constant. The derivation is begun by changing the index α to ρ and then replacing ϕ with $\phi;_{\alpha\beta\gamma}$ in equation (108). Then,

$$[\phi;_{\alpha\beta\gamma\rho}] \;=\; [\phi;_{\alpha\beta\gamma\sigma}]\nu^\sigma\nu_\rho + \frac{D}{Ds}[\phi;_{\alpha\beta\gamma}]t_\rho \tag{146}$$

Equation (II-136) provides an expression for $[\phi;_{\alpha\beta\rho\gamma} - \phi;_{\alpha\beta\rho\gamma}]$. If this equation and equation (146) are multiplied by ν^β and the former is subtracted from the latter then an expression for $[\phi;_{\alpha\beta\rho\gamma}]\nu^\gamma$ in terms of $[\phi;_{\alpha\beta\gamma\sigma}]\nu^\gamma\nu^\sigma$ results. This result is substituted into equation (146). Once again equation (II-136) is appealed to in order to get an expression for $[\phi;_{\alpha\beta\rho\gamma} - \phi;_{\alpha\rho\beta\gamma}]$. This result and the modified form of equation (146) then yield an expression for $[\phi;_{\alpha\beta\gamma\sigma}]\nu^\gamma\nu^\sigma$ in terms of $[\phi;_{\alpha\beta\gamma\sigma}]\nu^\beta\nu^\gamma\nu^\sigma$ which in turn is substituted back into the modified form of equation (146). A final repetition of the process gives $[\phi;_{\alpha\beta\rho\gamma}]$ in terms of $\tilde{\tilde{\tilde{\lambda}}} = [\phi;_{\alpha\beta\rho\gamma}]\nu^\alpha\nu^\beta\nu^\rho\nu^\gamma$, $[\phi;_{\alpha\beta}]$, and $[\phi;_{\alpha\beta\gamma}]$. Therefore this equation can be put into the required form with the help of equations (123), (137), (II-127), and (II-128). Thus

$$\begin{aligned}
[\phi;_{\alpha\beta\rho\gamma}] = {} & \tilde{\tilde{\tilde{\lambda}}}\, \nu_\alpha\nu_\beta\nu_\rho\nu_\gamma + \left(\frac{D}{Ds}\tilde{\tilde{\lambda}} + 3\,\frac{D}{Ds}\,\tilde{\lambda}\sigma\right)(t_\alpha\nu_\beta\nu_\rho\nu_\gamma + t_\beta\nu_\alpha\nu_\rho\nu_\gamma + t_\rho\nu_\alpha\nu_\beta\nu_\gamma \\
& + t_\gamma\nu_\alpha\nu_\beta\nu_\rho) + \left(-\tilde{\tilde{\lambda}}\sigma + \frac{D^2}{Ds^2}\,\tilde{\lambda} - 2\tilde{\lambda}\sigma\right)(t_\alpha t_\beta\nu_\rho\nu_\gamma + t_\alpha t_\beta\nu_\beta\nu_\gamma + t_\alpha t_\gamma\nu_\beta\nu_\rho \\
& + t_\beta t_\rho\nu_\alpha\nu_\gamma + t_\beta t_\gamma\nu_\rho\nu_\alpha + t_\rho t_\gamma\nu_\alpha\nu_\beta) + \left(-3\sigma\,\frac{D}{Ds}\,\tilde{\lambda} - \tilde{\lambda}\,\frac{D}{Ds}\,\sigma\right)(\nu_\gamma t_\alpha t_\beta t_\rho \\
& + \nu_\rho t_\alpha t_\beta t_\gamma + \nu_\beta t_\alpha t_\gamma t_\rho + \nu_\alpha t_\beta t_\gamma t_\rho) + 3\tilde{\lambda}\sigma^2 t_\alpha t_\beta t_\gamma t_\rho \\
& + 2\tilde{\lambda}\nu_\eta R^{\eta}_{\cdot\alpha\beta\sigma}\nu^\sigma\nu_\rho\nu_\gamma + \tilde{\lambda}\nu_\eta R^{\eta}_{\cdot\alpha\rho\sigma}\nu^\sigma\nu_\beta\nu_\gamma + \tilde{\lambda}\nu_\eta R^{\eta}_{\cdot\beta\rho\sigma}\nu^\sigma\nu_\alpha\nu_\gamma
\end{aligned} \tag{147}$$

This is the required fourth order geometrical compatitility relation for the scalar or spatial tensor ϕ where $[\phi] = 0$ and $[\phi;_\alpha] = 0$.

The next compatibility relation which is found for a scalar or spatial tensor is obtained by replacing ϕ by $\phi;_{\alpha\beta}$ in equation (129) where u_n is taken to be constant and $[\phi] = 0$, $[\phi;_\alpha] = 0$. Then,

$$[(\phi;_{\alpha\beta}),_{tt}] = [\phi;_{\alpha\beta\gamma\delta}]\nu^\gamma\nu^\delta u_n^2 - 2u_n \frac{D}{Dt}([\phi;_{\alpha\beta\gamma}]\nu^\gamma) + \frac{D^2}{Dt^2}[\phi;_{\alpha\beta}] \tag{148}$$

Using equations (123), (137), (147), (118), and (119) it is evident that

$$\begin{aligned}[(\phi;_{\alpha\beta}),_{tt}] = {} & (\tilde{\tilde{\tilde{\lambda}}}u_n^2 + \frac{D^2}{Dt^2}\tilde{\lambda} - 2u_n \frac{D}{Dt}\tilde{\tilde{\lambda}})\nu_\alpha\nu_\beta + (u_n^2 \frac{D}{Ds}\tilde{\tilde{\lambda}} + 3\sigma u_n^2 \frac{D}{Ds}\tilde{\lambda} \\ & - 2u_n \frac{D^2}{DtDs}\tilde{\lambda})(t_\alpha\nu_\beta + t_\nu\nu_\alpha) + (-\tilde{\tilde{\lambda}}\sigma u_n^2 + u_n^2 \frac{D^2}{Ds^2}\tilde{\lambda} \\ & - 2u_n^2\tilde{\lambda}\sigma^2 + 2u_n \frac{D}{Dt}(\tilde{\lambda}\sigma))t_\alpha t_\beta + 2\tilde{\lambda}u_n^2\nu_\eta R^{\eta}_{\cdot\alpha\beta\sigma}\nu^\sigma\end{aligned} \tag{149}$$

Again this result is valid for a scalar or spatial tensor such that $[\phi] = 0$, $[\phi;_\alpha] = 0$, and u_n is constant.

Another relation is obtained by substituting $\phi;_{\alpha\beta\gamma}$ for ϕ in equation (111). The same conditions used in the preceding paragraph are assumed to apply here. Then,

$$[(\phi;_{\alpha\beta\gamma}),_t] = \frac{D}{Dt}[\phi;_{\alpha\beta\gamma}] - [\phi;_{\alpha\beta\gamma\delta}]\nu^\delta u_n \tag{150}$$

Substition from (137) and (147) together with use of equations (118) and (119) yields

$$\begin{aligned}[(\phi;_{\alpha\beta\gamma}),_t] = {} & (\frac{D}{Dt}\tilde{\tilde{\lambda}} - u_n\tilde{\tilde{\tilde{\lambda}}})\nu_\alpha\nu_\beta\nu_\gamma + (\frac{D^2}{DtDs}\tilde{\lambda} - u_n \frac{D}{Ds}\tilde{\tilde{\lambda}} - 3u_n\sigma \frac{D}{Ds}\tilde{\lambda}) \\ & (t_\alpha\nu_\beta\nu_\gamma + t_\gamma\nu_\alpha\nu_\beta + t_\gamma\nu_\alpha\nu_\beta) + (\frac{D}{Dt}(-\tilde{\lambda}\sigma) + u_n\sigma\tilde{\tilde{\lambda}} \\ & - u_n \frac{D^2}{Ds^2}\tilde{\lambda} + 2u_n\tilde{\lambda}\sigma^2)(t_\alpha t_\beta\nu_\gamma + t_\alpha t_\gamma\nu_\beta + t_\beta t_\gamma\nu_\alpha) \\ & + (3u_n\sigma \frac{D}{Ds}\tilde{\lambda} + u_n\tilde{\lambda} \frac{D}{Ds}\sigma)(t_\alpha t_\beta t_\gamma) - 2u_n\tilde{\lambda}\nu_\eta R^{\eta}_{\cdot\alpha\beta\sigma}\nu^\sigma\nu_\gamma \\ & - u_n\tilde{\lambda}\,\nu_\eta R^{\eta}_{\cdot\alpha\gamma\sigma}\nu^\sigma\nu_\beta - u_n\tilde{\lambda}\nu_\eta R^{\eta}_{\cdot\beta\gamma\sigma}\nu^\sigma\nu_\alpha\end{aligned} \tag{151}$$

This compatibility relation applies to a scalar or spatial tensor such that $[\phi] = 0$, $[\phi;_\alpha] = 0$, and u_n is constant.

The final compatibility relation which is to be found is the relation concerning $(\phi;_\alpha),_{ttt}$. To obtain it, under the same conditions as applied in the previous paragraph, $(\phi;_\alpha),_t$ is substituted for ϕ in equation (129). Then

$$[(\phi;_\alpha),_{ttt}] = [(\phi;_{\alpha\beta\gamma}),_t]\nu^\beta\nu^\gamma u_n^2 - 2u_n \frac{D}{Dt}[(\phi;_{\alpha\beta}),_t]\nu^\beta + \frac{D^2}{Dt^2}[(\phi;_\alpha),_t] \tag{152}$$

Then using equations (126), (142), (151), (118), and (119) the result is the required compatibility relation for a scalar or spatial tensor such that $[\phi] = 0$, $[\phi;_\alpha] = 0$, and u_n is a constant. That is,

$$\begin{aligned}[(\phi;_\alpha),_{ttt}] = {} & \left(u_n^2 \frac{D}{Dt}\tilde{\tilde{\lambda}} - u_n^3\tilde{\tilde{\tilde{\lambda}}} - 2u_n \frac{D^2}{Dt^2}\tilde{\lambda} + 2u_n^2 \frac{D}{Dt}\tilde{\tilde{\lambda}} - u_n \frac{D^2}{Dt^2}\tilde{\lambda}\right)\nu_\alpha \\ & + \left(u_n^2 \frac{D^2}{DtDs}\tilde{\lambda} - u_n^3 \frac{D}{Ds}\tilde{\tilde{\lambda}} - 3u_n^3\sigma \frac{D}{Ds}\tilde{\lambda} + 2u_n^2 \frac{D^2}{DtDs}\tilde{\lambda}\right)t_\alpha\end{aligned} \tag{153}$$

If ϕ is a spatial tensor or a scalar such that all derivatives up to but not including fifth order are continuous then it can easily be shown that

$$[\phi;_{\alpha\beta\gamma\rho t}] = -\tilde{\tilde{\tilde{\tilde{\lambda}}}}u_n\nu_\alpha\nu_\beta\nu_\gamma\nu_\rho \tag{154}$$

and

$$[\phi;_{\alpha\beta ttt}] = -\tilde{\tilde{\tilde{\tilde{\lambda}}}}u_n^3\nu_\alpha\nu_\beta \tag{155}$$

where $\tilde{\tilde{\tilde{\tilde{\lambda}}}}$ is defined in the obvious way.

CHAPTER IV

EQUATIONS OF MOTION AND DYNAMICAL COMPATIBILITY RELATIONS

Physical systems occupy non zero volumes of Euclidean space. When the volume occupied by the continuous system is bounded by two closely spaced surfaces of large extent compared to their separation then such a system is called a shell. The approach adopted here is to idealize a shell by treating it as a materialized surface. The equations of motion of such a surface are obtained by the direct approach[36].

It is postulated that in an inertial frame of reference, in a rectangular Cartesian coordinate system, the equations of motion of a region r of the deforming materialized surface bounded by a closed curve c are

$$\int_c t^i ds + \int_r \rho f^i d\sigma = \frac{d}{dt}\int_r \rho v^i d\sigma \qquad (1)$$

and

$$\int_c (m^i + e^{ijk}z^j t^k)ds + \int_r (l^i + e^{ijk}z^j f^k)\rho d\sigma = \frac{d}{dt}\int_r (\rho e^{ijk}z^j v^k + \rho j^i)d\sigma \qquad (2)$$

where the region r may or may not contain a singular curve but where quantities are continuous everywhere else and have finite limits as such a curve is approached. The quantities which occur in equations (1) and (2) are:

t^i - the force per unit length of c exerted on r by the material outside c

f^i - the externally applied body force per unit mass

ρ - the mass per unit area of the deforming surface

v^i - the velocity of a material point

m^i - the couple per unit length of c exerted on r by the material outside c

z^i - the position of a point on the surface relative to the origin of the coordinate system

l^i - the externally applied body couple per unit mass

j^i - the intrinsic angular momentum per unit area

Equations (1) and (2) express the conservation of linear and angular momentum.

If equations (1) and (2) are applied to an infinitesimal triangle in a region of the surface not containing a singular curve then the stress vector associated with a surface curve that has a unit normal vector ν^α is given by[37]

$$t^k(\nu^\alpha) = t^{k\delta}\nu_\delta \tag{3}$$

where $t^{k\delta}$ is the stress vector acting on the u^δ = constant coordinate curve. It can be seen that

$$t^k(-\nu^\alpha) = -t^k(\nu^\alpha) \tag{4}$$

Similarly,

$$m^k(\nu^\alpha) = m^{k\delta}\nu_\delta, \quad m^k(-\nu^\alpha) = -m^k(\nu^\alpha) \tag{5}$$

These equations are tensor equations and are valid in any spatial and in any surface coordinate system.

Substitution from these equations into equations (1) and (2) and the application of Green's theorem[38], yields when the region of integration is considered to be arbitrary,

$$t^{k\delta}{}_{;\delta} + \rho f^k = \rho \frac{d}{dt} v^k \tag{6}$$

$$m^{k\delta}{}_{;\delta} + \rho l^k + e^{kij} z^i{}_{,\delta} t^{j\delta} = \rho \frac{d}{dt} j^k \tag{7}$$

If equation (II-105) is employed to define a tensor time derivative $\frac{D}{Dt}$ in which the observer in x^i, u^α, t space is defined by U^Δ = constant, $x^i = x^i(U^\Delta, t)$, $u^\alpha = u^\alpha(U^\Delta, t)$, $t = t$ where U^Δ are convected coordinates, and the connexion is given in a fixed spatial system and the U^Δ system by having $\Gamma^i_j, \Delta^\Sigma_\Gamma, \Gamma^{i*}_{j\Delta}, \Delta^{\Sigma*}_{\Gamma i}$ vanish and $\Gamma^i_{jk}, \Delta^\Sigma_{\Gamma\Lambda}$ be the Christoffel symbols, then

$$\frac{D}{Dt} A^k = A^k{}_{,t}\big|_{U^\Gamma} + \Gamma^k_{ij} A^i (x^j{}_{,t}\big|_{U^\Gamma}) \tag{8}$$

Equations (6) and (7) can be written so that they are valid in any spatial coordinate system which is related to the given Cartesian system by a time independent transformation, and in any surface system. Then,

$$t^{k\delta}{}_{;\delta} + \rho f^k = \rho \frac{D}{Dt} v^k \tag{9}$$

$$m^{k\delta}{}_{;\delta} + \rho l^k + g^{kl} \varepsilon_{lij} x^i{}_{,\delta} t^{j\delta} = \rho \frac{D}{Dt} j^k \tag{10}$$

The restriction on the choice of spatial coordinates is due to the fact that v^k transforms as a vector under only such transformations.

If equations (1) and (2) are applied to a region of the deforming surface, r, with a singular curve, δ, given by $u^{\alpha} = u^{\alpha}(q,t)$ passing through it, then it can be shown that restrictions are imposed upon the jumps in several physical quantities. The curve δ divides the region r into two regions r^{+} and r^{-} in each of which all quantities have been assumed to be continuous. c^{+} and c^{-} designate the curves which together with δ bound r^{+} and r^{-} respectively. $\nu^{\pm}_{\alpha}$ is the unit normal to δ and it is chosen to be in the direction of propagation. δ is assumed to be propagating into r^{+}.

The surface integrals over r are in general improper integrals because of the possible discontinuity of the integral at δ. The values of such integrals are given by taking the sum of the integrals over r^{+} and r^{-}. By employing the two dimensional transport theorem, equation (II-168), an expression can be found for the time derivative of an integral over r^{+} or r^{-}. This is so because all quantities are continuous over these regions. Thus for an arbitrary function ψ it is clear that

$$\frac{d}{dt} \int_{r^{\pm}} \psi d\sigma = \int_{r^{\pm}} (\psi,_{t}\big|_{U^{\beta}} + \frac{\psi}{2a} a,_{t}\big|_{U^{\beta}}) d\sigma + \int_{c^{\pm}} u^{\alpha},_{t}\big|_{U^{\Delta}} \nu_{\alpha} \psi ds + \int_{\delta} - u^{\alpha},_{t}\big|_{q} \nu^{\pm}_{\alpha} \psi^{\pm} ds \tag{11}$$

where a is the determinant of the metric tensor and where the normal to the bounding curves is always taken in the sense of the propagation of δ.

By combining equations (1),(2),(3),(5), and (11) and by taking the limit as c^{+} and c^{-} approach the segment of δ which was contained within the original region r it is seen that the areas of r^{+} and r^{-} approach zero so that only the line integrals are non zero, and since the result applies to an arbitrary segment of δ it can be concluded that

$$[t^{k\alpha}\nu_\alpha] = [-\rho v^k P_n] \tag{12}$$

$$[m^{k\alpha}\nu_\alpha] = [-\rho j^k P_n] \tag{13}$$

It should be noted that equations (12) and (13) are tensor equations and are valid in any spatial coordinate system obtainable from the initial fixed rectangular Cartesian coordinate system by a time independent transformation. The surface coordinate system is completely arbitrary. These two equations are called the dynamic compatibility equations. The first of these expresses the conservation of linear momentum at a curve of discontinuity. The discontinuity in the stress forces across the singular curve must equal the net momentum flux from the curve. The second equation expresses the conservation of angular momentum at a curve of discontinuity. The discontinuity in the moments attributed to the couple stresses must equal the net flux of angular momentum carried by the intrinsic angular momentum of the mass elements away from the curve.

Since equations (9) and (10) are vector equations it is possible to decompose each of these equations into components tangential and normal to the surface as was shown in equation (II-158). Therefore with the aid of equations (II-158), (II-159), (II-118) and (II-119) it can be shown that equations (9) and (10) become

$$t^{\beta\alpha}{}_{;\alpha} - b^\beta_\alpha t^\alpha + \rho f^\beta = \rho \frac{D}{Dt} v^i x^j_{,\gamma} g_{ij} a^{\beta\gamma} \tag{14}$$

$$t^\alpha_{;\alpha} + b_{\alpha\beta} t^{\alpha\beta} + \rho f = \rho \frac{D}{Dt} v^i n_i \tag{15}$$

$$m^{\beta\alpha}{}_{;\alpha} - b^{\beta}_{\alpha}t^{\alpha} + \rho l^{\beta} + a^{\beta\gamma}\varepsilon_{\gamma\alpha}t^{\alpha} = \rho \frac{D}{Dt} j^{i} x^{j}_{,\gamma} g_{ij} a^{\beta\gamma} \tag{16}$$

$$m^{\alpha}_{;\alpha} + b_{\alpha\beta}m^{\alpha\beta} + \rho l + \varepsilon_{\alpha\beta}t^{\beta\alpha} = \rho \frac{D}{Dt} j^{i} n_{i} \tag{17}$$

Equations (14) and (16) express the conservation of the tangential components of linear and angular momentum respectively. Equations (15) and (17) likewise express the conservation of the normal components of these quantities.

In deriving equations (9) and (10) the stresses, couple stresses, and mass density were defined with respect to the deforming surface. Such need not be the case. It is possible to define these quantities with respect to the reference surface. This will be done in the ensuing paragraphs and the form of the equations of motion in terms of these quantities will be found as well.

As before, minuscule letters and indices are to refer to the defomred surface and majuscule letters and indices are to refer to the reference surface. u^{α} and U^{Δ} are arbitrary coordinates on the respective surfaces, the only restriction is that U^{Δ} be fixed in the reference surface for all times. Equation (II-7) is valid and can be inverted.

If $u^{\alpha} = u^{\alpha}(s,t)$ picks out the surface coordinates of the points of a curve c on the deformed surface, where the parameter s is arc length along the curve, then the corresponding points on the reference surface, given by $U^{\Delta} = U^{\Delta}(u^{\alpha}(s,t),t)$, define a curve C. The arc length along this curve is denoted by S. As shown in Chapter II the unit normals to c and C are given respectively by

$$\nu_\alpha = \varepsilon_{\alpha\beta} u^\beta_{,s}, \qquad \mathcal{V}_\Delta = \varepsilon_{\Delta\Sigma} U^\Sigma_{,S} \tag{18}$$

In a convected coordinate system,

$$\nu_\Delta = \bar{\varepsilon}_{\Delta\Sigma} U^\Sigma_{,s} \tag{19}$$

where $\varepsilon_{\Delta\Sigma}$ refers to the reference surface and $\bar{\varepsilon}_{\Delta\Sigma}$ refers to the deformed surface. Equations (18) and (19) together with equation (II-112) yield

$$\nu_\Delta = \mathcal{V}_\Delta \left(\frac{\bar{A}}{A}\right)^{1/2} S_{,s} \tag{20}$$

where A and $\bar{A}$ are the determinants of the metric tensors on the reference and deformed surfaces respectively in the convected coordinate system. If a is the determinant of the metric tensor of the deformed surface in an arbitrary coordinate system then by combining equations (II-170), (18), and (20) it is seen that

$$\nu_\alpha ds = U^\Gamma_{,\alpha} \left(\frac{a}{A}\right)^{1/2} J \mathcal{V}_\Gamma dS \tag{21}$$

where $J = |u^\alpha_{,\Delta}|$.

The stress tensor $T^{k\Sigma}$ is defined through the relation

$$t^{k\alpha} \nu_\alpha ds = T^{k\Sigma} \mathcal{V}_\Sigma dS \tag{22}$$

Combining equations (20) and (22) yields

$$t^{k\alpha} = u^\alpha_{,} T^{k\Sigma} \left(\frac{A}{a}\right)^{1/2} J^{-1} \tag{23}$$

where $J^{-1} = |U^\Delta_{,\alpha}|$. Similarly, the couple stress tensor $M^{k\Sigma}$ is defined by the requirement that

$$m^{k\alpha} \nu_\alpha ds = M^{k\Sigma} \mathcal{V}_\Sigma dS \tag{24}$$

so that

$$m^{k\alpha} = u^{\alpha}_{,\Sigma} M^{k\Sigma} (\frac{A}{a})^{1/2} J^{-1} \tag{25}$$

Before substituting from equations (23) and (25) into the equations of motion preliminary results are required. If $A^{k\alpha}$ is an arbitrary double tensor of the type indicated and the spatial coordinate system is rectangular Cartesian then combining the definition of the total covariant derivative of $A^{k\alpha}$ with the result of equation (II-178) yields

$$A^{k\alpha}_{;\alpha} = a^{-1/2} (a^{1/2} A^{k\beta})_{,\beta} \tag{26}$$

Another preliminary result to be established is the fact that

$$(u^{\alpha}_{,\Gamma} |U^{\Delta}_{,\beta}|)_{,\alpha} = 0 \tag{27}$$

To prove this result the differentiation is carried out yielding the expression,

$$u^{\alpha}_{,\Gamma\Sigma} U^{\Sigma}_{,\alpha} |U^{\Delta}_{,\beta}| + |U^{\Delta}_{,\beta}|_{,\alpha} u^{\alpha}_{,\Gamma} \tag{28}$$

but the rules for differentiating a determinant can be invoked to show that

$$|U^{\Delta}_{,\beta}|_{,\alpha} = V_{\Delta}{}^{\beta} U^{\Delta}_{,\beta\alpha} \tag{29}$$

where $V_{\Delta}{}^{\beta}$ is the cofactor of the element U, so that expression (28) becomes

$$u^{\alpha}_{,\Gamma\Sigma} U^{\Sigma}_{,\alpha} |U^{\Delta}_{,\beta}| + V_{\Delta}{}^{\beta} U^{\Delta}_{,\beta\alpha} u^{\alpha}_{,\Gamma} \tag{30}$$

By differentiating the relation $U^{\Delta}_{,\beta} u^{\beta}_{,\Gamma} = \delta^{\Delta}_{\Sigma}$ it is seen that

$$U^{\Delta}_{,\gamma\alpha} = - U^{\Sigma}_{,\gamma} U^{\Delta}_{,\beta} u^{\beta}_{,\Sigma\Gamma} U^{\Gamma}_{,\alpha} \tag{31}$$

and together with the rule for expanding a determinant,

$$U^{\Delta}_{,\gamma} V^{\beta}_{\Delta} \;=\; |U^{\Sigma}_{,\sigma}|\,\delta^{\beta}_{\gamma} \tag{32}$$

this shows that expression (30) which is the left hand side of equation (27) equals zero which of course is equation (27).

Now by employing equation (26), equation (9) can be written in rectangular Cartesian coordinates as

$$a^{-1/2}(a^{1/2}t^{k\beta})_{,\beta} + \rho f^{k} \;=\; \frac{d}{dt}\,v^{k} \tag{33}$$

Then substitution from equation (23) and use of equations (27), (III-81) and (II-170) produces the result,

$$A^{-1/2}\;(A^{1/2}T^{k\Sigma})_{,\Sigma} + \;\rho_{o}(f^{k} - \frac{d}{dt}\,v^{k}) \;=\; 0 \tag{34}$$

By virtue of equation (26) this result becomes

$$T^{k\Delta}_{\;\;;\Delta} + \rho_{o}(f^{k} - \frac{D}{Dt}\,v^{k}) \;=\; 0 \tag{35}$$

where the mass density, ρ_o, and the covariant differentiation are now with respect to the reference surface. This equation although derived for the case where the spatial coordinates are rectangular Cartesian is a tensor equation at a point on the reference surface and hence remains valid in any allowable spatial coordinate system.

If v^k is the velocity in the rectangular Cartesian system Z^a and θ^k is any other allowable spatial system then the velocity in the θ^k system referred to a point on the reference surface is given by

$$v^{k} \;=\; \frac{\partial\theta^{k}}{\partial Z^{a}}\,v^{a} \tag{36}$$

where $v^a = z^a_{,t}\big|_U \Gamma$, z^a being the position of the material point on the derforming surface. $\frac{D}{Dt}$ was defined and its form in the preferred frame was given in equation (8). Its form in any other allowable system is given by (II-105) and from the fact that the allowable systems are fixed with respect to the preferred system it is evident that for a vector such as v^k defined on the reference surface

$$\frac{D}{Dt} v^k = v^k_{,t}\big|_U \gamma = \frac{\partial\theta^k}{\partial Z^a} z^a_{,tt}\big|_U K \tag{37}$$

The equation governing angular momentum in which quantities are defined with respect to the reference state is obtained in an analogous fashion. Thus the tensor form of this equation is

$$M^{k\Gamma}{}_{;\Gamma} + G^{kl}\varepsilon_{lij} \frac{\partial X^i}{\partial Z^a} z^a_{,\Sigma} T^{j\Sigma} + \rho_0(l^k - \frac{D}{Dt} j^k) = 0 \tag{38}$$

where Z^a are the rectangular Cartesian coordinates of the point on the reference surface whose position on the deforming surface is given in rectangular Cartesian coordinates as z^i. As before, in an allowable coordinate system θ^k the derivative of the vector j^k defined on the reference surface by

$$j^k = \frac{\partial\theta^k}{\partial Z^a} j^a \tag{39}$$

where j^a is the representation in a rectangular Cartesian system, is

$$\frac{D}{Dt} j^k = j^k_{,t}\big|_U \Gamma \tag{40}$$

The aim now is to decompose equations (35) and (38) into components tangential and normal to the reference surface. By following the example of equations (II-158), and (II-159) equation (35) becomes

$$T^{\Delta\Gamma}{}_{;\Gamma} - B^{\Delta}_{\Gamma}T^{\Gamma} + \rho_o F^{\Delta} = \rho_0 v^{\Delta}{}_{,t}\big|_{U^{\Sigma}} \tag{41}$$

and

$$T^{\Delta}_{;\Delta} + B_{\Delta\Gamma}T^{\Delta\Gamma} + \rho_0 F = \rho_0 v_{,t}\big|_{U^{\Sigma}} \tag{42}$$

where

$$v^{\Delta} = \frac{D}{Dt} v^i X^j_{,\Sigma} G_{ij} A^{\Sigma\Delta} \quad \text{and} \quad v = \frac{D}{Dt} v^i N_i$$

and G_{ij} is the spatial metric tensor at a point on the reference surface. $F^k = f^k$ so that $F^{\Delta}_{,}F$ denote the components with respect to the reference surface.

Equation (38) is somewhat more involved due to the second term. When equation (II-158) is used

$$\frac{\partial X^k}{\partial Z^a} = (A^{\Sigma\Delta}G_{ij}\frac{\partial X^i}{\partial Z^a}X^j_{,\Sigma})X^k_{,\Delta} + (\frac{\partial X^i}{\partial Z^a}N_i)N^k \tag{43}$$

and when this is substituted into equation (38) and equation (II-158) (II-159), (II-118), and (II-119) the resulting equations are

$$M^{\Delta\Sigma}{}_{;\Sigma} - B^{\Delta}_{\Sigma}M^{\Sigma} + \varepsilon^{\Delta\Gamma}G_{pq}X^p_{,a}z^a_{,\Sigma}X^q_{,\Gamma}T^{\Sigma} + \rho_0 L^{\Delta} = \rho_0 J^{\Delta}_{,t}\big|_{U^{\Sigma}} \tag{44}$$

and

$$M^{\Delta}_{;\Delta} + B_{\Delta\Gamma}M^{\Delta\Gamma} + \varepsilon_{\Delta\Lambda}A^{\Gamma\Delta}G_{pq}X^p_{,a}z^a_{,\Sigma}X^q_{,\Gamma}T^{\Lambda\Sigma} + \rho_0 L = \rho_0 J_{,t}\big|_{U^{\Sigma}} \tag{45}$$

where $j^k = J^k$ and $l^k = L^k$ so that $J^{\Delta}_{,}J, L^{\Delta}_{,}L$ refer to components with respect to the reference surface. Equations (41),(42),(44), and (45) are the required equations. They do not lend themselves to an intuitively attractive physical interpretation.

In order to derive the dynamical compatibility equations in a form in which quantities refer to the reference surface, equations (1) and (2) with the aid of equations (III-81),(3),(5),(22), and (24) are rewritten so that the integrations are over a region of the reference surface R or over its boundary C. Then

$$\int_C T^{k\Delta}\nu_\Delta dS + \int_R \rho_0 F^k d\Sigma = \frac{d}{dt}\int_R \rho_0 v^k d\Sigma \tag{46}$$

and

$$\int_C (M^{k\Delta}\nu_\Delta + e^{kij}z^i T^{j\Delta}\nu_\Delta)dS + \int_R \rho_0(L^k + e^{kij}z^i F^j)d\Sigma$$

$$= \frac{d}{dt}\int (e^{kij}z^i v^j + J^k)\rho_0 d\Sigma \tag{47}$$

where dS is an element of arc length of C and $d\Sigma$ is an element of area of R, and where the spatial coordinate system is rectangular Cartesian. These equations are to be valid when applied to the mass occupying a region of the reference surface R which is traversed by a segment of the diagram Λ of a wave curve. Because of the possible discontinuities in the integrands the derivatives on the right hand sides of equations (46) and (47) can not be taken under the integral signs. As was done when dealing with the deforming surface, the surface integrals are divided into the sum of two integrals. The domains of these integrals are the regions R^+ and R^-, the portions of R not yet reached by Λ and already traversed by Λ respectively. C^+ and C^- denote the curves which together with Λ bound R^+ and R^- respectively.

The unit normal to any of the boundary curves will be taken in the sense of the propagation of Λ. On R^+ and R^- all quantities have been assumed to be continuous. Therefore the transport theorem, equation (II-168), can be employed on integrals over R^+ and R^-. On the stationary reference surface and in any surface coordinate system on it which is fixed, the metric tensor is not explicity dependent upon time. Also, the boundary of R, $C = C^+ + C^-$, always consists of the same material points and so C^+ and C^- are fixed on the reference surface. If Λ is given by $U^\Delta = U^\Delta(q,t)$ then the process by which equations (12) and (13) were derived can be emulated to give

$$[T^{k\Delta}]v_\Delta = -\rho_0 U_N [v^k] \tag{48}$$

and

$$[M^{k\Delta}]v_\Delta = -\rho_0 U_N [J^k] \tag{49}$$

These equations which are vector equations can be decomposed into components tangential and normal to the reference surface. Then they become

$$[T^{\Delta\Gamma}]v_\Gamma = -\rho_0 U_N [v^\Delta] \tag{50}$$

$$[T^\Gamma]v_\Gamma = -\rho_0 U_N [v] \tag{51}$$

and similarly,

$$[M^{\Delta\Gamma}]v_\Gamma = -\rho_0 U_N [J^\Delta] \tag{52}$$

$$[M^\Gamma]v_\Gamma = -\rho_0 U_N [J] \tag{53}$$

where v^k and j^k are given by equations (36) and (39).

It was briefly mentioned in Chapter III that Signorini[39] has developed a systematic method of approximate solution in the theory of finite elastic deformation. In this approach it is assumed that body forces and moments are analytic functions of a parameter ε which vanish when $\varepsilon = 0$. Thus

$$f^k = \varepsilon f_1^k + O(\varepsilon^2)^k, \quad l^k = \varepsilon l_1^k + O(\varepsilon^2)^k \qquad (54)$$

where f_1^k and l_1^k are the coefficients of the first order terms in ε. The displacement in a rectangular Cartesian coordinate system is defined by

$$x^i = X^i + W^i \qquad (55)$$

W^i is also assumed to be an analytic function of ε which vanishes when $\varepsilon = 0$. Therefore

$$W^i(\varepsilon) = \sum_{n=1}^{\infty} \varepsilon^n W_n^i \qquad (56)$$

where $W_n^i = \frac{1}{n!}\frac{d^n}{d\varepsilon^n} W^i\Big|_{\varepsilon=0}$. The theory of linear elasticity results if ε is chosen to be small enough and if terms in the various expansions of order two or greater in ε are neglected. Then for example

$$W^i \approx \varepsilon W_1^i$$

The stress and couple stress tensors $t^{k\alpha}, m^{k\alpha}, T^{k\Delta}, M^{k\Delta}$ are assumed analytic functions of $x^i_{,\Delta}$, $x^i_{,\Delta\Sigma}$, and possibly of higher order derivatives with respect to fixed surface coordinates on the stationary reference surface such that the functions vanish when the arguments take on their reference surface values.

It is the intent of the ensuing discussion to show that in the case of a linear elastic medium the equations of motion and the dynamical compatibility equations with respect to the deforming surface and those with respect to the reference surface are equivalent. That is, the two sets of equations are the same up to and including first order in ε.

The spatial coordinate system will be taken as rectangular Cartesian. Then equation (55) is valid. The notation $O(\varepsilon^n)$ will mean terms of order n and greater in ε. Sometimes indices will be assigned to this notation indicating that each of the terms implicitly represented carries the assigned indices.

The first step involves finding the expansions for those quantities that are encountered in the equations of motion and the dynamical compatibility equations. To first order in ε it is evident that

$$v^i = \varepsilon W^i_{1},_t\Big|_{U^\Delta} \tag{57}$$

and

$$x^i,_\Delta = X^i,_\Delta + \varepsilon W^i_1,_\Delta \quad \text{and} \quad x^i,_{\Delta\Sigma} = X^i,_{\Delta\Sigma} + \varepsilon W^i_1,_{\Delta\Sigma} \tag{58}$$

In equations (III-63),(III-68), (III-69), (III-74), and (III-83) it was shown that

$$\left(\frac{\bar{A}^\pm}{A}\right)^{1/2} = 1 + O(\varepsilon)^\pm \tag{59}$$

$$S,_s = 1 + O(\varepsilon) \tag{60}$$

$$\frac{P_n^{\ \pm}}{U_N} = 1 + O(\varepsilon)^\pm \tag{61}$$

$$n_i = N_i + O(\varepsilon)_i \tag{62}$$

and

$$\rho = \rho_0 + O(\varepsilon) \tag{63}$$

The rate at which the singular curve progresses through the material, U_N, is independent of the deformation process and hence of ε.

From their definitions expansions can be found for a number of geometrical surface quantities. Thus,

$$a_{\Delta\Sigma} = A_{\Delta\Sigma} + O(\varepsilon)_{\Delta\Sigma} \tag{64}$$

$$a^{\Delta\Sigma} = A^{\Delta\Sigma} + O(\varepsilon)^{\Delta\Sigma} \tag{65}$$

$$\bar{\Delta}^{\Gamma}_{\Delta\Sigma} = \Delta^{\Gamma}_{\Delta\Sigma} + O(\varepsilon)^{\Gamma}_{\Delta\Sigma} \tag{66}$$

where the Christoffel symbols on the left hand side are with respect to the deformed surface and in a convected coordinate system and the Christoffel symbols on the right hand side are with respect to the reference surface. Also,

$$b_{\Delta\Sigma} = B_{\Delta\Sigma} + O(\varepsilon)_{\Delta\Sigma} \tag{67}$$

and

$$b^{\Sigma}_{\Delta} = B^{\Sigma}_{\Delta} + O(\varepsilon)^{\Sigma}_{\Delta} \tag{68}$$

Equation (21) can be expanded to give

$$\nu_\Delta = V_\Delta(1 + O(\varepsilon)) \tag{69}$$

By substituting from equation (58) into the arguments of the stresses and couple stresses and by writing the Taylor expansion of these functions taking the term of order zero to vanish by hypothesis then it is easily seen that to first order in ε

$$t^{k\Delta} = \varepsilon t_1^{k\Delta} \tag{70}$$

$$m^{k\Delta} = \varepsilon m_1^{k\Delta} \tag{71}$$

and

$$T^{k\Delta} = \varepsilon T_1^{k\Delta} \tag{72}$$

$$M^{k\Delta} = \varepsilon M_1^{k\Delta} \tag{73}$$

If substitution is made from equations (70-73) into equations (23) and (25) the results are that to first order in ε

$$t^{k\Delta} = T^{k\Delta}, \quad m^{k\Delta} = M^{k\Delta} \tag{74}$$

The last preliminary result required comes from the fact that the intrinsic angular momentum at a point is found from a constitutive relation of the form

$$j^k = j^k(x^i_{,t\Delta}) \tag{75}$$

such that $j^k(0) = 0$. Since $x^i_{,t\Delta} = \varepsilon W^i_{1,t\Delta}$ to first order in ε, the Taylor series about 0 yields a series in ε which to first order is

$$j^k = \varepsilon j_1^k \tag{76}$$

By employing the preliminary results it is now possible to prove the main results. All quantities in equation (9), (10), (35) and (38) are expanded in terms of ε and second order and higher terms are neglected. All equations are written in rectangular Cartesian spacial coordinates and convected surface coordinates. Equations (57-73) gives the pertinent expansions. When equations (74) are employed it is easily seen that the equations of motion to first order in ε are the same whether taken with respect to the deformed or with respect to the reference surface. To first order the equations in any allowable coordinate system are

$$T^{k\Delta}_{;\Delta} + \rho_0 F^k = \rho_0 W^k{}_{,tt}|U^\Sigma \tag{77}$$

$$M^{k\Delta}_{;\Delta} + G^{kl}\varepsilon_{lij}X^i_{,\Sigma}T^{j\Sigma} + \rho_0 L^k = \rho_0 J^k_{,t}|U^\Delta \tag{78}$$

where

$$T^{k\Delta} = \varepsilon T^{k\Delta}_1, \quad F^k = \varepsilon F^k_1, \quad W^k = \varepsilon W^k_1$$

and similarly for the other quantities. In addition, the covariant differentiation is with respect to the reference surface.

To first order in ε the dynamical compatibility relations given by equations (12) and (13) can be shown to be equivalent to those given by equations (48) and (49). This is accomplished by employing equations (69-74) and (61). The results to first order in ε are

$$[T^{k\Delta}]v_\Delta = -\rho_0 U_N [W^k_{,t}|U^\Delta] \tag{79}$$

$$[M^{k\Delta}]v_\Delta = -\rho_0 U_N [J^k] \tag{80}$$

where $T^{k\Delta} = \varepsilon T^{k\Delta}_1$ and similarly for $M^{k\Delta}$, W^k, and J^k.

The equation of motion and the dynamical compatibility equations are spatial vector equations and hence can be decomposed into components tangential and normal to the deformed surface or tangential and normal to the reference surface. It is easily shown that if a vector $A^k = \varepsilon A_1^k$ then to first order in ε it makes no difference which surface is used. This is done using equations (II-150), (II-151), (II-152),(58),(62), and (65).

Each term in equations (77-80) satisfies the condition described above. Therefore, the decomposition of these equations to first order in ε does not depend upon the surface used. In the ensuing chapters the linear theory of elasticity will be used unless otherwise specified so that equations (77-80) are the required equations. To first order in ε equations (77-80) when decomposed give

$$T^{\Delta\Gamma}{}_{;\Gamma} - B^{\Delta}_{\Gamma}T^{\Gamma} + \rho_0 F^{\Delta} = \rho_0 W^{\Delta}{}_{,tt|U^{\Sigma}} \tag{81}$$

$$T^{\Delta}{}_{;\Delta} + B_{\Delta\Gamma}T^{\Delta\Gamma} + \rho_0 F = \rho_0 W_{,tt|U^{\Sigma}} \tag{82}$$

$$M^{\Delta\Gamma}{}_{;\Gamma} - B^{\Delta}_{\Gamma}M^{\Gamma} + A^{\Delta\Gamma}\varepsilon_{\Gamma\Sigma}T^{\Sigma} + \rho_0 L^{\Delta} = \rho_0 J^{\Delta}{}_{,t|U^{\Sigma}} \tag{83}$$

$$M^{\Gamma}_{;\Gamma} + B_{\Delta\Gamma}M^{\Delta\Gamma} + \varepsilon_{\Delta\Lambda}T^{\Lambda\Delta} + \rho_0 L = \rho_0 J_{,t|U^{\Sigma}} \tag{84}$$

$$[T^{\Delta\Gamma}]\nu_{\Gamma} = -\rho_0 U_N [W^{\Delta}{}_{,t|U^{\Sigma}}] \tag{85}$$

$$[T^{\Delta}]\nu_{\Delta} = -\rho_0 U_N [W_{,t|U^{\Sigma}}] \tag{86}$$

$$[M^{\Delta\Gamma}]\nu_{\Gamma} = -\rho_0 U_N [J^{\Delta}] \tag{87}$$

$$[M^{\Delta}]v_{\Gamma} = -\rho_0 U_N [J] \tag{88}$$

where $T^{\Delta\Gamma} = \epsilon T_1^{\Delta\Gamma}$ and similarly for the other stresses and for the components of W^k and J^k.

For the sole purpose of comparing with other theories a new surface tensor is defined[40] such that

$$N^{\Delta\Sigma} = A^{\Delta\Lambda}\epsilon_{\Lambda\Gamma}M^{\Gamma\Sigma}, \quad M^{\Delta\Sigma} = -A^{\Delta\Lambda}\epsilon_{\Lambda\Gamma}N^{\Gamma\Sigma}$$

Substitution into equations (83),(84), and (87) yields

$$N^{\Lambda\Gamma}{}_{;\Gamma} + A^{\Lambda\Sigma}\epsilon_{\Delta\Sigma}B^{\Delta}_{\Gamma}M^{\Gamma} - T^{\Lambda} + C^{\Lambda} = -\rho_0 A^{\Lambda\Sigma}\epsilon_{\Delta\Sigma}J^{\Delta}{}_{,t|U^{\Gamma}} \tag{89}$$

where

$$C^{\Lambda} = -A^{\Lambda\Sigma}\epsilon_{\Delta\Sigma}L^{\Delta}$$

$$M^{\Gamma}{}_{;\Gamma} - \epsilon_{\Lambda\Sigma}B^{\Lambda}_{\Gamma}N^{\Sigma\Gamma} + \epsilon_{\Delta\Lambda}T^{\Lambda\Delta} + \rho_0 L = \rho_0 J_{,t|U^{\Sigma}} \tag{90}$$

and

$$[N^{\Delta\Sigma}]v_{\Sigma} = \rho_0 U_N A^{\Delta\Sigma}\epsilon_{\Gamma\Sigma}[J^{\Gamma}] \tag{91}$$

This change of variable is introduced in order to be able to more readily compare these equations with those of other theories. These equations are merely new representations of the same physical restrictions.

In summary then, the equations of motion and dynamical compatitility equations used in the linear theory of shells are equations (81),(82), (85),(86), and (88-91).

CHAPTER V

WAVE PROPAGATION IN A LINEAR THEORY OF MEMBRANES

A membrane is defined as a shell in which no couple stresses or shearing stresses can exist. In this sense a membrane is the two dimensional generalization of an extensible string. The membranes considered here will be subject to the linear theory developed in Chapter IV. This means that equations (IV-81), (IV-82), (IV-89), (IV-90), (IV-85), (IV-86), (IV-91), and (IV-88) are equations of motion and dynamical compatibility equations which are consistent with the approximation being made and hence can be used. It should be remembered that by hypothesis the reference surface with respect to which these equations are written is stationary and strain free, and all geometrical quantities which refer to this surface are continuous everywhere except possibly at isolated points. These singular points will not be dealt with in the ensuing discussion. The results derived therefore will apply to regions of the reference surface not containing such points.

The problem posed is the following one: A singular curve of given order, as defined in Chapter III, is assumed to be moving across a previously stationary, strain free, membrane surface. The disturbance is assumed to be small so that the approximations of linear elasticity are valid. The curve divides the surface into two regions in each of which all quantities and their derivatives to whatever order required are continuous and all these quantities have unique finite limits as a point on the curve is approached from one side. The surface itself remains continuous although the possibility that the deforming surface has a "kink" or discontinuity in the surface normal at the singular curve is retained when dealing with shock waves. Discontinuities in other aspects of the surface geometry at the singular curve are possibilities for waves of any order. Of course, all geometric quantities on the reference surface are continuous everywhere except possibly at isolated points. The problem then is to determine the speed of propagation of the singular curve, the shape of the curve, and the equations governing the growth or decay of some quantity which measures the size of the disturbance across the wave curve of given order, from the equations of motion, the dynamical compatibility relations, and the constitutive relations by employing whatever geometrical and kinematical compatibility identities are required.

The constitution relations used in the linear membrane theory given here are those of Green and Zerna[41]. Here they shall be taken as being postulated. Then

$$T^{\Delta\Gamma} = T^{\Gamma\Delta} = DH^{\Delta\Gamma\Sigma\Lambda}\alpha_{\Sigma\Lambda} \tag{1}$$

where

$$\alpha_{\Lambda\Theta} = \frac{1}{2}(W_{\Lambda;\Theta} + W_{\Theta;\Lambda} - 2B_{\Lambda\Theta}W), \tag{2}$$

W^{Λ}, W are the components of displacement tangential and normal to the reference surface,

$$H^{\Delta\Gamma\Sigma\Lambda} = \frac{1}{2}\{(1-\eta)(A^{\Delta\Lambda}A^{\Gamma\Sigma} + A^{\Delta\Sigma}A^{\Gamma\Lambda}) + 2\eta A^{\Delta\Gamma}A^{\Sigma\Lambda}\} \tag{3}$$

and

$$D = \frac{Eh}{(1-\eta^2)} \tag{4}$$

h being the constant thickness of membrane, E the three dimensional

Young's modulus and η Poisson's ratio.

The fact that a membrane is being considered means that

$$T^{\Delta} = M^{\Delta} = J^{\Delta} = C^{\Delta} = 0 = N^{\Delta\Sigma} \quad \text{and} \quad L = J = 0$$

Therefore the equations of motion (IV-81), (IV-82), and (IV-90) become

$$T^{\Delta\Gamma}{}_{;\Gamma} + \rho_0 F^{\Delta} = \rho_0 W^{\Delta}{}_{,tt}\big|_{U^{\Sigma}} \tag{5}$$

$$B_{\Delta\Gamma}T^{\Delta\Gamma} + \rho_0 F = \rho_0 W_{,tt}\big|_{U^{\Sigma}} \tag{6}$$

$$T^{\Delta\Gamma} = T^{\Gamma\Delta} \tag{7}$$

and equation (IV-89) is identically satisfied. The only dynamical compatibility relations which are not identically zero are equations (IV-85) and (IV-86) and these become

$$[T^{\Delta\Gamma}]\nu_{\Gamma} = -\rho_0 U_N [W^{\Delta}{}_{,t}\big|_{U^{\Sigma}}] \tag{8}$$

and

$$0 = -\rho_0 U_N [W_{,t}\big|_{U^{\Sigma}}] \tag{9}$$

The dynamical compatibility equations superficially refer to the diagram, the image of the wave curve on the reference surface. U_N for instance is the speed of propagation of the diagram in the reference surface. However, the diagram and the wave curve are coincident in the situation envisaged here. This is because the wave curve is propagating into the reference surface at rest and unstrained. Since $W^i = 0$ on the virgin side of the wave curve it is seen from equation (III-69) that

$$P_n^{+} = U_N \tag{10}$$

The geometrical and kinematical compatibility equations which will be employed are given by equations (III-108), (III-111), (III-123), and (III-129). In Chapter III minuscule indices were used as these identities were derived for any surface on which all geometric quantities are continuous. In their applications here these relations are to refer to the diagram and the reference surface so that majuscule indices will be used.

Before preceding with the analysis a word should be said about the method to be pursued. The equations of motion and the constitutive relations are valid at each point of the surface not on the wave curve. Furthermore, each term in these equations approaches a finite limit as a point on the wave curve is approached from one side. Therefore, by taking the jumps in the equations of motion, the jumps or discontinuities in the quantities occuring in these equations are related to one another. If the constitutive relations are used to relate all terms in the dynamical compatibility relations and in the equations of motion to the displacements and the derivatives of the displacements, with the exception of the externally applied forces and moments which incidentally are assumed to be continuous, then taking the jumps in the equations of motion produces relationships involving the jumps in the derivatives of the displacements. As is seen from the geometrical, kinematical, and mixed compatibility identities, if the order of the derivatives of a double tensor ϕ is raised by one and the jump is taken then the number of quantities introduced which are geometrically and kinematically speaking unrelated to the jumps in lower order derivatives equals the number of components of the double tensor. These quantities have been denoted above by $[\phi],\lambda,\tilde{\lambda}$ et cetera. Thus the jumps in the equations of motion and the dynamical compatibility equations provide equations involving the above quantities, where the displacement is the double tensor ϕ and the speed of propagation is U_N. It is hoped that these equations can be solved to yield the value of U_N and an expression for the growth or decay of the amplitude of the disturbance.

It should be pointed out that if a spatial vector equation denoted by

$$A^i = 0 \tag{11}$$

is decomposed with respect to the reference surface such that

$$A^i = A\, X^i_{,\Delta} + AN^i = 0 \tag{12}$$

so that

$$A^\Delta = 0, \quad A = 0 \tag{13}$$

then the jumps of equations (13) are equivalent to the jump of equation (12), because $X^i_{,\Delta}$ and N^i are continuous. If the total covariant derivative is taken of equation (12) then

$$A^i{}_{;\Sigma} = A^\Delta_{;\Sigma} X^i_{,\Delta} + A_{;\Sigma} N^i = 0 \tag{14}$$

so that

$$A^\Delta_{;\Sigma} = 0 \quad \text{and} \quad A_{;\Sigma} = 0 \tag{15}$$

and the jumps of equations (15) are equivalent to the jump of equation (14). Similar remarks are valid for the higher order derivatives of equation (11).

The following investigation will be concerned with the propagation of schock waves with respect to the displacement in the linear theory of membranes. A shock wave with respect to the displacement is a singular curve in the membrane surface such that $[W^i] = 0$ but at least one component of $W^i{}_{,t|U}\Sigma$ or $W^i{}_{;\Delta}$ is discontinuous across this curve. It is assumed that the curve is persistent with respect to time and that it is propagating into the reference surface which is unstrained and at rest.

The analysis is begun by noticing that the dynamical compatibility relation given by equation (9) together with identity (III-111) indicates that

$$\omega^3 = 0 \tag{16}$$

where $\omega^3 = [W_{;\Delta}]\nu^\Delta$. As a result of the fact that W^3 is continuous, equation (III-108) indicates that $[W^3_{;\Delta}] = 0$. This is the condition, given by equation (III-77) that the normal to the deformed surface be continuous across the wave curve.

Combining equations (8),(1),(2), and (3) with identities (III-108) and (III-111) and employing the decomposition of the quantity

$$\omega_\Delta = \omega\nu_\Delta + \hat{\omega}t_\Delta \tag{17}$$

where $t_\Delta = U_{\Delta,S}$ and $\omega^\Delta = [W^\Delta{}_{;\Sigma}]\nu^\Sigma$ yields

$$(D - \rho_0 U_N^2)\omega = 0 \tag{18}$$

$$\left(\frac{D}{2}(1-\eta) - \rho_0 U_N^2\right)\hat{\omega} = 0 \tag{19}$$

These are called the propagation conditions for a shock wave.

It is seen from identities (III-108) and (III-111) that $[W^i,_{t|U}{}^{\Gamma}] = 0$ if $[W^i;_\Delta] = 0$. Therefore, for a shock wave $[W^i;_\Delta] \neq 0$ for at least one component. Equation (14) indicates that this is equivalent to saying that at least one component of $[W^\Sigma;_\Delta] \neq 0$ or $[W;_\Delta] \neq 0$. From identity (III-108) it can be concluded that at least one of $\omega, \hat{\omega}$, or ω^3 must be non zero for a shock wave. It has already been shown in equation (16) however, that $\omega^3 = 0$; therefore, at least one of ω and $\hat{\omega}$ does not vanish.

If $\omega \neq 0$, then from equations (18) and (4) it is seen that

$$U_N^2 = \frac{D}{\rho_0} = \frac{Eh}{\rho_0(1-\eta^2)} \tag{20}$$

When this value is substituted into equation (19) it is found that

$$\hat{\omega} = 0 \tag{21}$$

If $\hat{\omega} \neq 0$, then from equations (19) and (4) it is seen that

$$U_N^2 = \frac{D(1-\eta)}{2\rho_0} = \frac{Eh}{2\rho_0(1+\eta)} \tag{22}$$

Upon substitution into equation (18) it is clear that

$$\omega = 0 \tag{23}$$

It should be pointed out that in both cases U_N^2 is a positive real valued constant. U_N is the speed of the diagram. In the problem being considered it is also, as shown in equation (10), the speed of the wave curve relative to the material on the side of the wave curve not yet effected by the propagating disturbance and at rest.

In the first case, when $\omega_\Delta = \omega\nu_\Delta$, the application of identity (III-108) indicates that

$$[W_{\Delta;\Gamma} - W_{\Gamma;\Delta}] = 0 \tag{24}$$

but since $W_{\Delta;\Sigma}{}^{+} = 0$ this means that immediately behind the wave front

$$W_{\Delta;\Gamma}^{-} - W_{\Delta;\Gamma}^{-} = 0 \tag{25}$$

that is to say, since it can be shown that $W_{[\Delta;\Sigma]}$ is the in-surface rotation of the material, the material behind the wave is not rotating. The wave is said to be irrotational. In addition, it is clear from the definition of the velocity that

$$[v^i] = [W^{\Delta}_{,t}\big|_U \Gamma]X^i_{,\Delta} + [W_{,t}\big|_U \Gamma]N^i \tag{26}$$

and then using identity (III-111) and equations (16) and (21) it can be shown that

$$v^{i^-} = U_N \omega \mathcal{V}^i \tag{27}$$

where $\mathcal{V}^i = X^i_{,}\mathcal{V}^{\Delta}$. Of course, $v^{i^+} = 0$. Because the velocity behind the wave is in the direction of propagation, an irrotational wave is also called a longitudinal wave. Finally, from equation (III-85) and identity (III-108) it is clear that

$$\rho^- = \rho_0(1 + \omega) \quad \text{and} \quad \rho^+ = \rho_0 \tag{28}$$

This relates the density behind the wave, ρ^-, to the density in front of the wave front ρ_0.

In the second case where $\omega_\Delta = \hat{\omega} t_\Delta$, identity (III-108) applied to the jump in the two dimensional analogue of the cubical dilatation

$$[W^{\Delta}{}_{;\Delta}] = 0 \tag{29}$$

but since $W^{\Delta}_{;\Delta}{}^{+} = 0$ this means that

$$W^{\Delta -}_{;\Delta} = 0 \tag{30}$$

Because of equations (III-79), (III-88), and (29) the area of an element immediately behind the curve is unchanged from its area on the reference surface. Such a wave is called an equiareal wave. Equation (III-111) when applied to equation (26) and subject to the conditions of equations (16) and (23) yields

$$v^{i^-} = U_N \hat{\omega} X^i_{,S} \tag{31}$$

because $v^{i^+} = 0$. Because the velocity behind the wave is in a direction tangential to the wave curve, an equiareal wave is also called a transverse wave. Finally, the discontinuity in the mass density is given by equations (III-85) and (III-108), so that

$$\rho^- = \rho^+ = \rho_0 \tag{32}$$

Thus, for an equiareal wave the mass density is continuous.

As has already been mentioned, for both types of possible waves the speed of propagation of the diagram, U_N, is constant. Also in the problem being considered the actual wave curve and the diagram are coincident. It is possible to predict the shape of the wave curves for all times once the curve is known at some reference time. This information, as will be shown, is a direct consequence of the fact that U_N is constant.

It was demonstrated in Chapter II by the discussion surrounding equations (III-86 to III-92) that given a moving curve on a surface, $U^{\Delta} = U^{\Delta}(q,t)$ where q is the parameter along the curve at time t, then it is possible to find a time dependent transformation of the curve parameter

$$q = q(s_0,t) \tag{33}$$

such that

$$U^{\Delta}_{,t}\big|_{s_0} = U_N \nu^{\Delta} \tag{34}$$

where U_N is the speed of the curve along the normal direction to the curve ν^{Δ}, and such that s_0 is the arc length along the curve at some reference time. Here, the curve is taken to be the diagram on the reference surface and hence U_N is the speed of propagation of this curve. The function $U^{\Delta} = U^{\Delta}(s_0,t)$ can be viewed geometrically as defining two one parameter families of curves on the reference surface. Members of the family of diagrams are given by taking t to be a constant. A tangent to these curves is the vector, $U^{\Delta}_{,s_0}\big|_t$. Members of the other family of curves are specified by choosing s_0 to be a constant. Then $U^{\Delta}_{,t}\big|_{s_0}$ is a tangent vector to these curves. It was shown in Chapter III that $U^{\Delta}_{,t}\big|_{s_0}$ and $U^{\Delta}_{,s_0}\big|_t$ are orthogonal. Thus the family of curves given by s_0 = constant is the family of orthogonal trajectories to the family of diagrams or wave curves.

The unit vector to the orthogonal trajectories of the wave curves is found by dividing $U^{\Delta}_{,t}\big|_{s_0}$ by the magnitude of this vector, but from equation (34) its magnitude is U_N, so that

$$\nu^{\Delta} = \frac{1}{U_N} U^{\Delta}_{,t}\big|_{s_0} = U^{\Delta}_{,Q}\big|_{s_0} \tag{35}$$

is the unit tangent vector to the orthogonal trajectories, where $Q = U_N t$ defines a new parameter along the orthogonal trajectories which must be the arc length and where ν^{Δ} is of course the unit normal vector to the diagram or wave curve. It was seen in equation (III-119) that

$$\frac{D}{Dt}\nu^{\Delta} = -U_{N,S}\,t^{\Delta} \tag{36}$$

where $t^{\Delta} = U^{\Delta}_{,S}$, but U_N is a constant here so that

$$\frac{D}{Dt}\nu^{\Delta} = 0 \tag{37}$$

Therefore,

$$\frac{D}{DQ}\left(U^{\Delta}_{,Q}\Big|_{s_0}\right) = \frac{D}{DQ}\nu^{\Delta} = \frac{1}{U_N}\frac{D}{Dt}\nu^{\Delta} = 0 \tag{38}$$

where Q is the arc length along the orthogonal trajectories to the diagrams and $\frac{D}{DQ}$ is defined by equation (II-105) with the same connexion as used in the definition of the derivative in equation (III-96). Equation (38) is the condition, as given by equation (II-160), that the orthogonal trajectories to the diagrams be geodesic curves on the reference surface. The discussion surrounding equations (II-160) to (II-163) indicates that the diagrams and their orthogonal trajectory form a geodesic parallel coordinate system on the reference surface. Thus if the diagram or wave curve is given at some reference time and the orthogonal geodesics are drawn on the reference surface then the wave curve after an interval of time Δt is given by the locus of points a distance along the geodesics $U_N \Delta t$ from the reference curve. This curve is orthogonal to the geodesics as well.

Another aspect of wave propagation to be investigated in the linear theory of membranes involves finding the equations which govern the growth or decay of some quantity which measures the magnitude of the disturbance which is propagating. The wave strength M is defined by

$$M = (\omega_\Delta \omega^\Delta)^{1/2} \tag{39}$$

therefore if $\omega_\Delta = \omega v_\Delta$

$$M = |\omega| \tag{40}$$

and if

$$\omega_\Delta = \hat{\omega} t_\Delta$$

$$M = |\hat{\omega}| \tag{41}$$

This definition is quite natural and M = 0 if and only if the wave ceases to be of first order.

The decay equations come from considering the conservation of the surface components of linear momentum. Thus by combining equations (1), (2),(3), and (5) with identities (III-108), (III-123), and (III-129) and by employing the decomposition of the quantity $\tilde{\omega}^\Delta = [W^\Delta_{;\Sigma\Gamma}]v^\Sigma v^\Gamma$

$$\tilde{\omega}^\Delta = \tilde{\omega} v^\Delta + \hat{\tilde{\omega}} t^\Delta \tag{42}$$

it can be shown that

$$\frac{D}{4}\{4\tilde{\omega} - 2\Sigma(1-\eta)\omega + 2(1+\eta)(\frac{D}{DS}\hat{\omega} - \omega\Sigma)\} = \rho_0\{U_N^2\tilde{\omega} - 2U_N\frac{D}{Dt}\omega\} \tag{43}$$

and

$$\frac{D}{4}\{2(1-\eta)\hat{\tilde{\omega}} - 4\Sigma\hat{\omega} + 2(1+\eta)(\frac{D}{DS}\omega + \hat{\omega}\Sigma)\} = \rho_0\{U_N^2\hat{\tilde{\omega}} - 2U_N\frac{D}{Dt}\hat{\omega}\} \tag{44}$$

where Σ is the geodesic curvature of the wave curve on the reference surface.

From this point, the two types of waves are treated separately. First an irrotational or longitudinal wave is considered; thus $\hat{\omega} = 0$. Substitution from equation (20) into equation (43) and use of the fact that $\hat{\omega} = 0$ yields

$$\frac{D}{DQ}\omega = \frac{1}{U_N}\frac{D}{Dt}\omega = \frac{1}{2}\Sigma\omega \tag{45}$$

Substitution from equation (40) gives

$$\frac{D}{DQ}M = \frac{1}{2}\Sigma M \tag{46}$$

For the same type of wave, $\hat{\omega} = 0$, substitution from equation (20) into equation (44) produces the result,

$$\frac{D}{DS}\omega = \hat{\tilde{\omega}} \tag{47}$$

The case of equiareal or transverse waves is now considered; therefore, $\omega = 0$. Substitution from equation (22) and use of the fact that $\omega = 0$ allows equation (44) to be written as

$$\frac{D}{DQ}\hat{\omega} = \frac{1}{U_N}\frac{D}{Dt}\hat{\omega} = \frac{1}{2}\Sigma\hat{\omega} \tag{48}$$

If equation (41) is employed then equation (48) becomes equation (46).

If substitution from equation (22) is carried out in equation (43) for the case where $\omega = 0$ it is seen that

$$\frac{D}{Ds}\hat{\omega} = -\tilde{\omega} \tag{49}$$

It has been seen that the equations governing the growth or decay of the wave strength, M, are identical for longitudinal and transverse waves.

The solution of this equation will be considered in an ensuing chapter which will deal with examples.

The final aspect to be investigated concerns the derivatives of the component of the displacement normal to the curve, that is W. It has already been seen that W, $W;_\Delta$, and $W,_t$ are continuous. Are all derivatives of W continuous? To answer this question, equations (1),(2),(3), and (6) are combined with identities (III-108) and (III-129) to yield

$$D\{(1-\eta)B_{\Delta\Gamma}\omega^\Delta\nu^\Gamma + \eta B^\Delta_\Delta\omega\} = \rho_0 U_N^2\tilde{\omega}^3 \qquad (50)$$

In the case of longitudinal waves where $\omega^\Delta = \omega\nu^\Delta$ equation (50) becomes

$$\tilde{\omega}^3 = (B_{\Delta\Gamma}\nu^\Delta\nu^\Gamma + \eta B_{\Delta\Gamma}t^\Delta t^\Gamma)\omega = \{(1-\eta)K(\nu^\Gamma) + 2\eta H\}\omega \qquad (51)$$

where $K(\nu^\Gamma)$, given in equation (II-124), is the normal curvature in the direction normal to the wave curve and H, given in equation (II-133), is the mean curvature of the reference surface, at the point being considered.

In the case of transverse waves where $\omega^\Delta = \hat{\omega}t^\Delta$ equation (50) becomes

$$\tilde{\omega}^3 = 2B_{\Delta\Gamma}t^\Delta\nu^\Gamma\hat{\tilde{\omega}} \qquad (52)$$

In general then, a singular curve which is a first order wave with respect to the in-surface components of displacement is a second order wave with respect to the component of displacement normal to the reference surface.

If the wave curve had been assumed to be an acceleration wave, that is, a second order wave with respect to displacement, then the dynamical compatibility equations (8) and (9) become identities and yield no information. Equation (50), which comes from consideration of equation (6) and which is still valid, now gives the result,

$$\tilde{\omega}^3 = 0 \tag{53}$$

The propagation conditions for acceleration waves come from considering equation (5). By considering equations (1),(2),(3), and (5) with identities (II-123) and (II-129) and by employing equation (42) it can be shown that

$$(D - \rho_0 U_N^2)\tilde{\omega} = 0 \tag{54}$$

and

$$\left(\frac{D}{2}(1-\eta) - \rho_0 U_N^2\right)\hat{\tilde{\omega}} = 0 \tag{55}$$

Thus it is seen that two kinds of acceleration waves are possible in this linear theory of membranes, longitudinal waves when $\hat{\tilde{\omega}} = 0$ and transverse waves when $\tilde{\omega} = 0$. For longitudinal waves identity (III-129) applied to the acceleration yields

$$W^{\bar{i}}_{,tt}\Big|_U \Sigma = -U_N^2 \omega \nu^i, \quad W^{\overset{+}{i}}_{,tt}\Big|_U \Sigma = 0$$

and for transverse waves

$$W^{\bar{i}}_{,tt}\Big|_U \Sigma = - U_N^2 \hat{\tilde{\omega}} X^i_{,S}, \quad W^{\overset{+}{i}}_{,tt}\Big|_U \Sigma = 0$$

These waves propagate with the same characteristic speeds as the corresponding shock waves. The speeds of propagation for longitudinal and transverse waves are given by equations (20) and (22) respectively. Thus, the argument about the shape and successive positions of the wave carries over from the discussion of shock waves.

In order to find the decay equations and the relation of $\tilde{\tilde{\omega}}^3$ to $\tilde{\omega}$ and $\hat{\tilde{\omega}}$, equations (5) and (6) are differentiated with respect to time, that is, the partial derivative with respect to time is taken. The jump in the resulting equations then allows the required results to be found. The jump in the partial derivative with respect to time of equation (5) combined with equations (1),(2),(3), and (42) and used together with identities (III-142) and (III-145) gives

$$\frac{D}{DQ}\tilde{\omega} = \frac{1}{2}\Sigma\tilde{\omega} \tag{56}$$

for longitudinal waves and

$$\frac{D}{DQ}\hat{\tilde{\omega}} = \frac{1}{2}\Sigma\hat{\tilde{\omega}} \tag{57}$$

for transverse waves. The derivative of equation (6) together equations (1),(2),(3), and (42) together with identities (III-126) and (III-145) yields

$$\tilde{\tilde{\omega}}^3 = (B_{\Delta\Gamma}\nu^\Delta\nu^\Gamma + \eta B_{\Delta\Gamma}t^\Delta t^\Gamma)\tilde{\omega} = \{(1-\eta)K(\nu^\Gamma) + 2\eta H\}\tilde{\omega} \tag{58}$$

where $\tilde{\tilde{\omega}}^3 = [W;_{\alpha\beta}]\nu^\alpha\nu^\beta$ for longitudinal waves, and

$$\tilde{\tilde{\omega}}^3 = 2B_{\Delta\Gamma}t^\Delta\nu^\Gamma\hat{\omega} \tag{59}$$

for transverse waves.

When dealing with acceleration waves in membranes which in general are non linear, an analysis is followed which emulates the analysis of acceleration waves in three dimensional elastic bodies[42].

The equations of motion of a membrane with respect to the reference surface are by virtue of equations (IV-35) and (IV-38)

$$T^{k\Sigma}{}_{;\Sigma} + \rho_0(F^k - \frac{\partial X^k}{\partial Z^a} z^a_{,tt}\Big|U^\Gamma) = 0 \qquad (60)$$

$$\varepsilon_{lmr} X^l_{,a} z^a_{,\Gamma} T^{m\Gamma} = 0 \qquad (61)$$

where the spatial system is fixed in an inertial frame and the surface system is fixed in the reference surface which in turn is at rest in the inertial frame. The dynamical compatibility relation is given by equation (IV-48), so that

$$[T^{k\Gamma}]\nu_\Gamma = - U_N \rho_0 X^k_{,a} [z^a_{,t}] \qquad (62)$$

If the constitutive relation for the membrane is postulated to be of the following form

$$T^{k\Gamma} = h^{k\Gamma}(z^a_{,\Delta}) \qquad (63)$$

then

$$T^{k\Gamma}{}_{;\Gamma} = A^{k.\Gamma\Delta}_{.a} z^a_{,\Delta\Gamma} + \Gamma^k_{ij} T^{i\Gamma} X^j_{,\Gamma} + \Gamma^\Gamma_{\Delta\Gamma} T^{k\Delta} \qquad (64)$$

where

$$A^{k.\Gamma\Delta}_{.a} = \frac{\partial h^{k\Gamma}}{\partial z^a_{,\Delta}}$$

If a second order wave with respect to z^a is assumed to be propagating into the stationary, strain-free reference surface then the wave curve and the diagram coincide, and

$$P_n = U_N \qquad (65)$$

Substitution is made from equations (63) and (64) into equations (60), (61) and (62).

Then if the jump is taken of these equations the result is

$$A^{k\Gamma\Delta}_{\cdot a}[z^{a}_{;\Delta\Gamma}] = A^{k\Gamma\Delta}_{\cdot a}[z^{a}_{,\Delta\Gamma}] = \rho_0 X^{k}_{,a}[z^{a}_{,tt}] \tag{66}$$

and equation (62) leads to an identity as does equation (61). By employing identities (III-123) and (III-129) it is seen that

$$A^{k\ \Gamma\Delta}_{\cdot i}s^{i}\nu_{\Delta}\nu_{\Gamma} = \rho_0 U_N^2 s^k \tag{67}$$

where

$$s^a = [z^{a}_{;\Delta\Sigma}]\nu^{\Delta}\nu^{\Sigma},\ s^k = X^{k}_{,a}s^a,\ A^{k\ \Gamma\Delta}_{\cdot i} = Z^{a}_{,i}A^{k\ \Gamma\Delta}_{\cdot a}$$

so that

$$(Q^{k}_{\cdot i} - \delta^{k}_{i}\rho_0 U_N^2)s^i = 0 \tag{68}$$

where

$$Q^{k}_{\cdot i} = A^{k\ \Gamma\Delta}_{\cdot i}\nu_{\Gamma}\nu_{\Delta}$$

This is the propagation condition for an acceleration wave in a membrane which in general is non linear. Because s^i can not vanish for an acceleration wave the speed of propagation can be found from this condition.

In order to proceed further a specific function must be chosen in equation (63). This is not done here.

CHAPTER VI

WAVE PROPAGATION IN BARS

Before going on to discuss the propagation of singular curves in the linear bending theory of shells perhaps there is epistomological merit in examining the problem of a propagating singularity in a bending theory of bars. This problem does not contain the geometrical complexities of the former one and hence might illuminate some features obscurred in the shell problem. For completeness sake an outline is provided of an elementary derivation of the equations governing the mechanical behavior of the bar.

For convenience sake, the bar is taken to be inextensible. Then a bar initially along the x axis in the x,y plane undergoes a deflection $y(x,t)$, a rotation caused by bending of $\psi(x,t)$ radians, and a distortion due to shearing of $\beta(x,t)$ radians. The linear and angular deflections are assumed to be small.

It can be shown that the constitutive equations are as follows

$$M(x,t) = EI\psi,_{x} \tag{1}$$

$$Q(x,t) = k'\mu A(x)\beta(x,t) \tag{2}$$

where

$M(x,t)$ is the bending moment about the z axis at the point x,

E and μ are Young's and the shear modulus respectively,

$Q(x,t)$ is the shearing force,

k' is a numerical factor which depends upon the shape of the cross section,

$I(x)$ is the area moment of inertia about the z axis, and

$A(x)$ is the cross sectional area.

The equations of motion are provided by a variational principal which is

$$\delta \int_{t_2}^{t_1} (T + W)dt = 0 \tag{3}$$

T is the kinetic energy given by

$$T = \frac{1}{2} \int_o^L m(x)(y,_t)^2 dx + \frac{1}{2} \int_o^L (\psi,_t)^2 J(x)dx \tag{4}$$

where $m(x)$ is the mass per unit length and $J(x)$ is the mass moment of inertia per unit length about the z axis. Then

$$J(x) = \rho(x)I(x) = \frac{m(x)}{A(x)} I(x) = k^2 m(x) \tag{5}$$

where $k(x)$ is the radius of gyration about the z axis, $\rho(x)$ is the mass density and W is given by

$$W = - V + \int p(x,t)y(x,t)dx \tag{6}$$

where $p(x,t)$ is the body force and

$$V = \frac{1}{2} \int EI \ \psi,_x^2 dx + \frac{1}{2} \int k'\mu A\beta^2 dx \tag{7}$$

Combining equations (3),(4),(5),(6), and (7) it is apparent that

$$\delta \int_{t_1}^{t_2} \int (\frac{1}{2} y,_t^2 + \frac{1}{2} k^2 \psi,_t^2) m dxdt + \delta \int_{t_1}^{t_2} \int (py - \frac{1}{2} EI\psi,_x^2 - \frac{1}{2} k'\mu A\beta^2) dxdt \tag{8}$$

However, since the linear and angular deflections are assumed to be small the total angle of deflection is approximately equal to $y,_x$. Therefore

$$y,_x = \beta + \psi \tag{9}$$

Substituting from equation (9) for β into equation (8) gives the result,

$$\int_{t_1}^{t_2}\int \{y,_t\delta(y,_t) + \psi,_t\delta(\psi,_t)k^2\}m\,dxdt - \int_{t_1}^{t_2}\int EI\psi,_x\delta(\psi,_t) + k'\mu A(y,_t - \psi)$$

$$\delta(y,_t - \psi)dxdt + \iint p\delta y\,dxdt = 0$$

Once integration by parts has been performed and the boundary conditions imposed then it is evident that

$$\{k'\mu A(y,_x - \psi)\},_x - my,_{tt} + p = 0 \tag{10}$$

$$(EI\psi,_x),_x + k'\mu A(y,_x - \psi) - k^2m\psi,_{tt} = 0 \tag{11}$$

Equation (10) represents conservation of linear momentum and equation (11) is the equation governing angular momentum.

Substitution from equation (11) into equation (10) requires that

$$-(EI\psi,_x),_{xx} + (k^2m\psi,_{tt}),_x - my,_{tt} + p = 0 \tag{12}$$

Now if the shearing deflection is small and is neglected then

$$y,_x = \psi$$

and furthermore if A,I,k and m are taken to be constants then

$$-EIy,_{xxxx} + k^2my,_{xxtt} - my,_{tt} + p = 0 \tag{13}$$

The second term which is due to the rotary inertia is often neglected relative to the term due to the linear inertia. Then equation (13) becomes

$$- EIy,_{xxxx} - my,_{tt} + p = 0 \tag{14}$$

Jaunzemis[44] has associated propagating singular surfaces in three dimensional bodies with oscillatory waves in the limit as the wave length approaches zero. Can singularities propagating in bars whose motion is governed by either equation (13) or equation (14) be associated with plane oscillatory waves in the limit as the wavelength approaches zero?

To begin to answer this question trial oscillatory solutions of the following form:

$$y = B \cos \frac{2\pi}{\lambda} (x - vt) \tag{15}$$

are substituted into equation (14) with p = 0. Then solutions are of this form if

$$v^2 = \frac{EI}{m} (\frac{2\pi}{\lambda})^2 \tag{16}$$

Thus v, the phase velocity, approaches infinity as the wavelength λ approaches zero. Any disturbance which resembles a discontinuity would be a linear combination of such solutions forming a localized wave packet centered about a short wavelength. The propagation of this wave pocket is associated with the group velocity of the packet. The group velocity of a wave pocket is

$$U = \frac{d}{dl} \omega. \tag{17}$$

where $\omega = lv$ and $l = \frac{1}{\lambda}$. Therefore from equation (16) and (17),

$$U = 2v = \frac{4\pi}{\lambda}\left(\frac{EI}{m}\right)^{1/2} \tag{18}$$

As the wavelength λ approaches zero U is unbounded and approaches infinity. This result is physically unreasonable as the energy is propagated with the wave pocket.

The dilemma is resolved by noting that it can be shown that the approximations made in ignoring the rotary inertia terms in the derivation of equation (14) are invalid at small wavelengths[45]. Equation (13) results from ignoring the shear deformation but taking rotary inertia into effect. When a trial oscillatory solution of the form given in equation (15) is substituted into equation (13) a solution is obtained only if

$$v^2 = \frac{E}{\rho}\left(1 + \frac{\lambda^2}{4\pi^2 k^2}\right)^{-1} \tag{19}$$

Employing equation (17) it is seen that the group velocity is

$$U = \left(\frac{E}{\rho}\right)^{1/2}\left(1 + \frac{\lambda^2}{4\pi^2 h^2}\right)^{-1/2} + \left(\frac{E}{\rho}\right)^{1/2} \frac{\lambda^2}{4\pi^2 k^2}\left(1 + \frac{\lambda^2}{4\pi^2 k^2}\right)^{-3/2} \tag{20}$$

In the limit as the wavelength, λ, approaches zero the group velocity, U, approaches the value $(E/\rho)^{1/2}$. For that matter the phase velocity v also approaches $(E/\rho)^{1/2}$ in the limit. It has been seen therefore that the inclusion of rotary inertia in the equation of motion allows for a more realistic result physically and indeed makes possible the existence of oscillatory waves and wave pockets in the limit as the wavelength goes to zero.

In order to ascertain whether a connection can be established between oscillatory waves in the short wavelength limit and propagating singularities in bars, the latter must now be investigated. In the case where rotary inertia was neglected the equation of motion in the integrated form is taken to apply to all regions of the bar including regions containing singular cross sectional planes. Thus

$$\int_{x_1}^{x_2} - EIy,_{xxxx} dx = \frac{\partial}{\partial t}\int_{x_1}^{x_2} my,_t dx \tag{21}$$

If the singularity is at $x = \xi$ then

$$\int_{x_1}^{x_2} - EIy,_{xxxx} = \frac{\partial}{\partial t}\int_{x_1}^{\xi} my,_t dx + \frac{\partial}{\partial t}\int_{\xi}^{x_2} my,_t dx$$

or

$$- EIy,_{xxx}\Big|_{x_1}^{x_2} = my,_t(\xi^-)\xi,_t - my,_t(\xi^+)\xi,_t + \int_{x_1}^{x_2} my,_{tt} dx$$

In the limit as x_1 and x_2 approach ξ the dynamical compatibility equation,

$$EI[y,_{xxx}] = mG[y,_t] \tag{22}$$

where $G = \xi,_t$ is the speed of propagation of the singularity, results.

The first order kinematical compatitility relation follows from the definition of the displacement derivative of a function A given by

$$\frac{\delta A}{\delta t} = A,_t\big|_x + GA,_x \tag{23}$$

Therefore,

$$[A,_t] = \frac{\delta}{\delta t}[A] - G[A,_x] \tag{24}$$

If A = y and [y] = 0 then

$$[y,_t] = - G[y,_x] \tag{25}$$

Letting A = $y,_t$ in equation (24) gives

$$[y,_{tt}] = \frac{\delta}{\delta t}[y,_t] - G[y,_{xt}]$$

but if A = $y,_x$ then

$$[y,_{tx}] = \frac{\delta}{\delta t}[y,_x] - G(y,_{xx}] \tag{26}$$

so that

$$[y,_{tt}] = - \frac{\delta G}{\delta t}[y,_x] - 2G\frac{\delta}{\delta t}[y,_x] + G^2[y,_{xx}] \tag{27}$$

This is the second order kinematical compatibility equation.

Taking the jump in equation (14) and using identity (27) it is evident that

$$- EI[y,_{xxxx}] = m(- \frac{\delta G}{\delta t}[y,_x] - 2G\frac{\delta}{\delta t}[y,_x] + G^2[y,_{xx}]) \tag{28}$$

Combining equations (22) and (25) yields

$$EI[y,_{xxx}] = - mG^2[y,_x] \tag{29}$$

Equation (28) and (29) are the only equations available to describe the phenomenum using this model. These two equations involve the five a priori independent unknown variables $[y,_x]$, $[y,_{xx}]$, $[y,_{xxx}]$, $[y,_{xxxx}]$, and G. Thus it is apparent that the system is underdetermined. If second order waves are studied instead, then $[y,_x] = 0$, and equation (29), means that $[y,_{xxx}] = 0$. The remaining equation involves three unknowns and so is underdetermined. If third order waves are investigated, then equation (29) indicates that such waves can not propagate. Equation (28) likewise shows that fourth order waves can not propagate. By differentiating equation (14) as many times as necessary and taking the jump in the result it is evident that all waves of order three or greater with respect to y can not propagate. It has been pointed out that in the case of first and second order waves that the constraints imposed by the physics of the situation, that is, equations (28) and (29), are not sufficient to determine a unique speed of propagation. It appears therefore that the model chosen does not adequately define the actual physical situation. The fact that all waves of third order or greater can not propagate is expected if the relationship between the plane waves of infinitesimal wavelength and propagating discontinuities is valid.

What happens when rotary inertia is not neglected? If the relationship between plane waves of infinitesimal wavelength and propagating singularities is valid it would be expected that waves of some order at least could propagate and that the speed of propagation would be the same for both viewpoints. In order to examine the situation when rotary inertia is taken into account and singularities are assumed to be propagating it is necessary first to develop the dynamical compatibility equation corresponding to equation (13). It is postulated that the equation of motion in integrated form which is applicable to any region of the bar, including one containing a singular cross sectional surface at $x = \xi$ is

$$\int_{x_1}^{x_2} - EIy,_{xxxx} dx = \frac{\partial}{\partial t}\int_{x_1}^{\xi} my,_t dx + \frac{\partial}{\partial t}\int_{\xi}^{x_2} my,_t dx$$

$$- k^2 m \left(\frac{\partial}{\partial t}\int_{x_1}^{\xi} y,_{xxt}dx + \frac{\partial}{\partial t}\int_{\xi}^{x_2} y,_{xxt}dx\right)$$

or

$$- EIy,_{xxx}\Big|_{x_1}^{x_2} = my,_t(\xi^-)\xi,_t - my,_t(\xi^+)\xi,_t + \int_{x_1}^{x_2} my,_{tt}dx$$

$$- k^2my,_{xxt}(\xi^-)\xi,_t + k^2my,_{xxt}(\xi^+)\xi,_t - k^2m\int_{x_1}^{x_2} y,_{xxtt}dx$$

In the limit as x^1 and x^2 approach ξ this becomes

$$- EI[y,_{xxx}] = - mG[y,_t] + k^2mG[y,_{xxt}] \tag{30}$$

To find the required compatibility equations $A = y,_{xx}$ is substituted into equation (24), so that

$$[y,_{xxt}] = \frac{\delta}{\delta t}[y,_{xx}] - G[y,_{xxx}] \tag{31}$$

If $A = y,_{xxt}$ in equation (24) then

$$[y,_{xxtt}] = \frac{\delta}{\delta t}[y,_{xxt}] - G[y,_{xxxt}] \tag{32}$$

Also if $A = y,_{xxx}$ in equation (24) then

$$[y,_{xxxt}] = \frac{\delta}{\delta t}[y,_{xxx}] - G[y,_{xxxx}] \tag{33}$$

Substitution from equations (33) and (31) into equation (32) yields

$$[y,_{xxtt}] = \frac{\delta^2}{\delta t^2}[y,_{xx}] - \frac{\delta}{\delta t}(G[y,_{xxx}]) - G\frac{\delta}{\delta t}[y,_{xxx}] + G^2[y,_{xxxx}].$$

If identities (34),(27),(31), and (25) are employed in equation (30) and in the jump in equation (13) the results are

$$(-\ EI + k^2 mG^2)[y,_{xxx}] = 0 \tag{35}$$

so that

$$G^2 = \frac{E}{\rho} \tag{36}$$

and

$$[y,_{xxx}] = \text{constant} \tag{37}$$

along the x axis, respectively. The jump in equation (13) yields the same speed of propagation for fourth order waves. By differentiating equation (13), the speed of propagation for any order wave is given by equation (36). This is the same result obtained for oscillatory waves of infinitesimal wavelength.

The important point, however, is that the inclusion of the rotary inertia term in the equation of motion is what made possible wave propagation of any kind. When dealing with the propagation of singular curves in the linear bending theory of shells, the above discussion indicates that a term analogous to the rotary inertia term used here should be included. This will be done.

CHAPTER VII

WAVE PROPAGATION IN A LINEAR BENDING THEORY OF SHELLS

In Chapter V the propagation of singularities was investigated by employing a linear theory to describe a membrane, that is a shell in which couple and shearing stresses can not exist. In this chapter these limitations of the shell will be eliminated. The theory of shells which will be used in the ensuing discussion will be the linear bending theory of shells proposed by Green and Zerna[46] modified, however, by the inclusion of dynamic contributions to the equations of motion including rotary inertia effects. The approach used by Green and Zerna is the one in which the three dimensional equations of classical elasticity are integrated over the thickness of the material bounded by two surfaces separated by a small distance compared to their dimensions. The equations of motion thus obtained are identical to those derived in Chapter IV in equations (IV-14) to (IV-17) if the twisting couple stresses, externally applied twisting couples, and the normal component of the intrinsic angular momentum are set equal to zero. In the linear approximation discussed in Chapter IV these equations can be written equivalently with respect to the reference surface as is done in equations (IV-81), (IV-82), (IV-89), and (IV-90) with $L = J = 0$ and $M^{\Delta} = 0$.

Green and Zerna make approximations in their bending theory. The approximations which effect the equations of motion are the approximation that the thickness of the shell is small compared to the minimum radius of curvature of the so-called middle surface of the shell, and the approximation that the shearing stress in the shell space and the derivative of the shearing stress with respect to a dimensionless parameter along the normal to the surface are of the same order of magnitude, or that the latter is of greater order of magnitude than the former. When the integrations of the three dimensional equations of motion are carried out after having taken account of the foregoing approximations the results are that the term involving the resultant shearing stress T^{Δ} is neglected in equation (IV-81) and in equation (IV-89) the term involving $N^{\Delta\Sigma}$ is neglected; equations (IV-82) and (IV-88) remain unaltered. There appears to be little motivation for making these approximations when dealing solely with the materializaed surface point of view. By appealing to the Green and Zerna approach the circumstances under which it is legitimate to neglect those terms neglected in the equations of motion are better defined. The dynamical compatibility relations derived in Chapter IV and given by equations (IV-85), (IV-86), (IV-88), and (IV-91) are retained unaltered except that equation (IV-88) is an identity because M^{Δ} and J have been assumed to vanish.

The equations of motion and the dynamical compatibility relations in the linear bending theory of shells employed in the ensuing discussion are the following equations:

$$T^{\Delta\Gamma}{}_{;\Gamma} + \rho_0 F^{\Delta} \;=\; \rho_0 W^{\Delta}{}_{,tt}\big|_{U^{\Sigma}} \tag{1}$$

$$T^{\Gamma}{}_{;\Gamma} + B_{\Delta\Gamma} T^{\Delta\Gamma} + \rho_0 F \;=\; \rho_0 W_{,tt}\big|_{U^{\Sigma}} \tag{2}$$

$$N^{\lambda\Gamma}{}_{;\Gamma} - T^{\lambda} + C^{\lambda} \;=\; -\rho_0 A^{\lambda\Lambda} \varepsilon_{\Delta\Lambda} J^{\Delta}{}_{,t}\big|_{U^{\Sigma}} \tag{3}$$

$$T^{\Delta\Gamma} = T^{\Gamma\Delta} \tag{4}$$

and

$$[T^{\Delta\Gamma}]v_\Gamma = -U_N\rho_0[W^\Delta_{,t}|_U^\Sigma] \tag{5}$$

$$[T^\Gamma]v_\Gamma = -U_N\rho_0[W^3_{,t}|_U^\Sigma] \tag{6}$$

$$[N^{\lambda\Sigma}]v_\Sigma = U_N\rho_0 A^{\lambda\Lambda}\varepsilon_{\Delta\Lambda}[J^\Delta] \tag{7}$$

The constitutive equations in the present theory can be regarded as being postulated. In fact, the constitutive relations for $T^{\Delta\Gamma}$ and $N^{\lambda\Sigma}$ are those given by Green and Zerna, and the expression for J^Δ is the expression for the angular momentum found by integration over the thickness of the shell less the angular momentum due to the moment of the resultant linear momentum after integration over the thickness. The constitutive relations are postulated to be equations (V-1) to (V-4) together with

$$N^{\Delta\Gamma} = - BH^{\Delta\Gamma\Sigma\Lambda}W_{;\Sigma\Lambda} \tag{8}$$

where

$$B = \frac{Eh^3}{12(1-\eta^2)} \tag{9}$$

and

$$J^\Delta = A\varepsilon^{\Delta\Gamma}(W_{,t}|_U^\Sigma)_{;\Gamma} \tag{10}$$

where

$$A = \frac{h^2}{12} \tag{11}$$

It is noticed that

$$B = AD \tag{12}$$

There might be epistomological merit in reviewing briefly the motivation for equation (10). In deriving the constitutive relations given in equations (V-1) and (8) Green and Zerna made use of approximations equivalent to assuming a special form for the displacement of any point in the three dimensional shell. If u^{α} and u are the components of the displacement tangential and normal to the middle surface of the deformed shell and θ^3 is the height above the middle surface of a point in the shell, then the displacement of this point is assumed to be given by

$$u_{\alpha}(\theta^{\beta},\theta^3) = W_{\alpha}(\theta^{\beta}) - \theta^3 W,_{\alpha}(\theta^{\beta}) \tag{13}$$

$$u(\theta^{\beta},\theta^3) = W(\theta^{\beta}) \tag{14}$$

where θ^{β},θ^3, $\beta = 1,2$ form a shell coordinate system[47] and $\theta^3 = 0$ gives the middle surface.

In Chapter IV all variables were expanded in series form in a parameter ε, and a linear theory involved neglecting terms beyond first order in ε. Such a theory was considered a good approximation if the displacements were small. In a rectangular Cartesian coordinate system the displacement is taken such that

$$u^i = \varepsilon u_1^i \tag{15}$$

where u_1^i does not depend upon ε. As was mentioned in Chapter IV, to first order in ε the decomposition of a spatial vector which is homogeneous of the first order in ε into components tangential and normal to a surface does not depend upon whether the deformed or reference surface is employed; the componenets are the same in either case. If the reference surface is used, then the velocity of a point in the three dimensinnal shell is

$$v^i = u^{\Delta}{}_{,t}\Big|_{X^K} X^i{}_{,\Delta} + u_{,t}\Big|_{X^K} N^i \tag{16}$$

where X^K are material coordinates.

The total angular momentum about a point fixed in space of an element of the deformed shell is

$$\int^{h/2} (\underline{\xi}_0 + \underline{\eta} + \theta^3\underline{n}) X \underline{v} \sigma dS d\theta^3 \tag{17}$$

where - h/2

dS is the area of the middle surface of the element of shell being considered

σ is the volume mass density

h is the thickness of the shell

$\underline{\xi}_0$ is a point on the element dS

$\underline{\eta}$ is the component of displacement relative to $\underline{\xi}_0$ tangential to the surface of a point in the volume element being considered

$\underline{n}$ is the unit normal to the middle surface

and

θ^3 is the height about the reference surface of a point in the element.

Since dS is infinitesimal in extent the magnitude of $\underline{\eta}$ is small so that the total angular momentum is

$$\underline{\xi}_0 X \int_{-h/2}^{h/2} \underline{v} \sigma dS d\theta^3 + \int_{-h/2}^{h/2} \theta^3 \underline{n} X \underline{v} \sigma dS d\theta^3 \tag{18}$$

The first term is the moment of the resultant linear momentum of the element and the remaining term from the materialized surface point of view is the intrinsic angular momentum associated with the point ξ_0 of the middle surface.

Now to first order in ε the first integral in expression (18), the linear momentum, can be written in components tangential and normal to the middle surface in the reference or deformed state; it matters not which, by substituting from equation (16). Then if p^i is the linear momentum per unit mass, the components tangential and normal to the surface are

$$p_\Delta = \frac{1}{h}\int_{-h/2}^{h/2} (W_{\Delta},_t\big|_{X^K} - \theta^3(W,_\Delta),_t\big|_{X^K})d\theta^3 = W_{\Delta},_t\big|_{X^K} \qquad (19)$$

and

$$p^3 = \frac{1}{h}\int_{-h/2}^{h/2} W,_t\big|_{X^K}\, d\theta^3 = W,_t\big|_{X^K} \qquad (20)$$

In a similar fashion, to first order in ε the intrinsic angular momentum, the second integral in expression (18), can be decomposed into components tangential and normal to the middle surface by substituting from equation (16). Then if J^i is the intrinsic angular momentum per unit mass, the components tangential and normal to the surface are

$$J^\Delta = \frac{1}{h}\int_{-h/2}^{h/2} -\varepsilon^{\Delta\Sigma}\theta^3(W_\Sigma,_t\big|_{X^K} - \theta^3(W,_\Sigma),_t\big|_{X^K})d\theta^3 = \frac{h^2}{12}\varepsilon^{\Delta\Sigma}(W,_\Sigma),_t\big|_{X^K} \qquad (21)$$

and

$$J = 0 \qquad (22)$$

The discussion in this chapter to this point has established the linear bending theory of shells which will be employed in dealing with the problem of the propagation of singular curves in shells. A more precise definition of the problem to be solved follows. A singular curve of a given order, as defined in Chapter III, is assumed to be propagating on a materialized surface. The curve divides the surface into two regions in each of which all quantities and their derivatives to whatever order required are assumed to be continuous. All these quantities have unique finite limits as a point on the curve is approached from one side. The region into which the curve is propagating is taken to be stationary and strain free. The disturbance which is propagating is assumed to be small so that the linear bending theory of shells presented in this chapter is applicable. All geometric quantities occuring in the governing equations will be taken to refer to the reference surface. All geometric quantities on the reference surface are continuous everywhere except possibly at isolated points. The deforming surface is assumed to be continuous everywhere. However, in general discontinuities in various geometrical quantities exist across the wave curve. The problem then is to determine the speed of propagation of the singular curve, the shape of the curve, and the equations governing the growth or decay of some quantity which measures the size of the disturbance across the wave curve of given order. This information is to be deduced if possible from the equations of motion, the dynamical compatibility relations, the constituive relations, and whatever geometrical and kinematical compatibility identities are required.

In the situation imagined the wave curve on the deforming surface and the diagram on the reference surface coincide so that it is evident from equation (III-69) that

$$P_n^+ = U_N \tag{23}$$

It is this speed of propagation relative to the material at rest in front of the advancing wave curve that will be found.

The method to be pursued will be the same as that used in Chapter V for the case of membranes except that the governing equations are different in this case. The discussion given there which pertains to the procedure followed is applicable here as well. The remarks surrounding equations (V-11) to (V-15) are of equal importance to this chapter.

Two types of waves will be investigated. Shock waves or waves of order one with respect to the displacement will be considered first. Then acceleration waves or second order waves with respect to the displacement will be analyzed. In the case of shock waves, at least one component of $W^i_{;\Delta}$ is discontinuous across the wave front. It will be assumed however that $[W_{;\Delta}] = 0$ where W is the component normal to the surface. From the point of view where a shell is considered as a materialized surface it appears that the above assumption is restrictive and that only shock waves where $[W_{;\Delta}] = 0$ will be taken into account. Shock waves where $[W_{;\Delta}] \neq 0$ appear to be neglected. Some insight into this question is obtained by looking into what happens when a shell is regarded as a three dimensional body. Then the wave curve on the materialized surface $f(\theta^\Delta, t) = 0$, where θ^Δ are surface coordinates, is associated with a wave surface in the three dimensional body which in a shell coordinate system θ^K is given by $f(\theta^\Delta, t) = 0$ where $\theta^3 = 0$ gives the middle surface. The condition that the deformed shell be continuous is that the displacement u^i, the tangential and normal components of which are given by equations (13) and (14), be continuous. A necessary and sufficient condition that u_α and u be continuous for all points in the three dimensional shell is that $[W_\Delta]$, [W], and $[W,_\Delta]$ all vanish. This is apparent from equations (13) and (14). Thus what appears to be a restriction from the materialized surface point of view on the type of shock waves being considered is a necessary condition for a three dimensional shell to be continuous and hence a necessary condition for a shock wave.

The geometrical and kinematical compatibility relations employed in the ensuing discussion are equations (III-108), (III-111), (III-123), (III-126), (III-129), (III-137), (III-141), (III-142), (III-145), (III-147), (III-149), (III-154), and (III-155). These compatibility relations will be regarded as referring to the diagram on the reference surface and in their application will be written using majuscule symbols.

The analysis, employing the linear theory of shells adopted in the preceding discussion, of first order singular curves or shock waves with respect to the displacement and such that $[W;_\Delta] = 0$ is begun by taking the dynamical compatibility equation, equation (5), and by combining it with equations (V-1) to (V-3). Then if identities (III-108) and (III-111) are employed, the propagation conditions are achieved. They turn out to be identical to the propagation conditions for shock waves in membranes, that is, equations (V-18) and (V-19). Therefore, two types of waves exist each with its characteristic speed of propagation. The discussion surrounding equations (V-18) to (V-38) is carried over intact to apply to shock waves in the linear bending theory.

Similarly, if equations (V-1) to (V-3) are combined with equation (1) and identities (III-123) and (III-126) are used then the decay equations result. Since the same equations have been used for shock waves in membranes and in the bending theory, the decay equations for longitudinal and transverse shock waves are the same. The discussion surrounding equations (V-39) to (V-49) is again applicable for shock waves in this linear bending theory.

At this point the analysis of shock waves in shells departs from the analysis of shock waves in membranes. It has already been seen that W, $W;_{\Delta}$, and $W,_{t|}U^{\Sigma}$ are continuous; the last of these following from (III-111). Are all derivatives of W continuous? The answer lies in investigating the consequences of the remaining dynamical compatibility equations and equations of motion.

Equation (7) can be combined with equations (8), (V-3), and (10) together with identities (II-114), (III-123), and (III-126) to yield

$$(B - \rho_0 U_N^2 A)\tilde{\omega}^3 = 0 \tag{24}$$

but because of equation (12) this is equivalent to

$$(D - \rho_0 U_N^2)\tilde{\omega}^3 = 0 \tag{25}$$

In the case of longitudinal waves the coefficient vanishes and nothing can be said about $\tilde{\omega}^3$. In the case of transverse waves the coefficient does not vanish and $\tilde{\omega}^3 = 0$. $\tilde{\omega}^3 = [W;_{\Delta\Sigma}]\nu^{\Delta}\nu^{\Sigma}$ of course, and is associated with an acceleration wave with respect to W.

If equation (3) is used to obtain an expression for the shearing stresses which is then substituted into equation (6) and the resulting equation is then combined with equations (8), (10), and (V-3) and identities (III-137), (III-141), and (III-111) are employed then the result is

$$(B - \rho_0 A U_N^2)\tilde{\tilde{\omega}}^3 + 2\rho_0 A U_N \frac{D}{Dt}\tilde{\omega}^3 - B\Sigma\tilde{\omega}^3 = 0 \qquad (26)$$

where $\tilde{\tilde{\omega}}^3 = [W;_{\Delta\Sigma\Gamma}]\nu^\Delta\nu^\Sigma\nu^\Gamma$. In the case of longitudinal waves the coefficient of the first term vanishes and the equation reduces to

$$\frac{D}{DQ}\tilde{\omega}^3 = \frac{1}{2}\Sigma\tilde{\omega}^3 \qquad (27)$$

where Q and the derivative were defined in Chapter V. In the case of transverse waves equation (26) yields

$$\tilde{\tilde{\omega}}^3 = 0 \qquad (28)$$

The final equation to be dealt with is equation (2). Again the shearing stresses are replaced by an expression obtained from equation (3). Then the resulting equation is combined with equations (8), (10), (V-3), and (V-1) together with identities (III-147), (III-149), (III-108), and (III-111) to get for transverse waves

$$\tilde{\tilde{\tilde{\omega}}}^3 = \frac{D}{(B-\rho_0 A U_N{}^2)}\{(1-\eta)B_{\Delta\Sigma}\nu^\Delta t^\Sigma\}\hat{\omega} \qquad (29)$$

where from equation (V-4), (9), (11), and (V-22) it is seen that

$$\frac{D}{(B-\rho_0 A U_N{}^2)} = \frac{24}{h^2(1+\eta)} \qquad (30)$$

For longitudinal waves the process described yields

$$A(\Sigma\tilde{\tilde{\omega}}^3 - 2\frac{D}{DQ}\tilde{\tilde{\omega}}^3 - \frac{D^2}{DS^2}\tilde{\omega}^3 + \frac{D^2}{DQ^2}\tilde{\omega}^3 + 2\Sigma\frac{D}{DQ}\tilde{\omega}^3 - \Sigma^2\tilde{\omega}^3 + 2\frac{D}{DQ}\Sigma\tilde{\omega}^3$$

$$+ (1-\eta)\tilde{\omega}^{3} R_{\Delta\Sigma\Lambda\Gamma}\nu^{\Delta}\nu^{\Gamma}A^{\Sigma\Lambda}) + B_{\Delta\Sigma}\nu^{\Delta}\nu^{\Sigma}\omega + \eta B_{\Delta\Sigma}t^{\Delta}t^{\Sigma}\omega = \tilde{\omega}^{3} \tag{31}$$

This equation allows the determination of $\tilde{\tilde{\omega}}^{3}$ in terms of ω and $\tilde{\omega}^{3}$. If an acceleration wave with respect to W is not initially excited then $\tilde{\tilde{\omega}}^{3}$ can be found in terms of ω.

In the remaining part of this chapter the propagation of a second order singular curve with respect to the displacement, or an acceleration wave, subject to the same linear bending theory will be investigated. Again the curve will be taken to be propagating into the stationary, strain free reference surface. In this case

$$P_{\mathbf{n}} = U_{N} \tag{32}$$

as the wave curve and the diagram are coincident. For a second order wave $[W^{i}] = 0$, $[W^{i}_{;\Delta}] = 0$ and at least one component of $[W^{i}_{;\Delta\Sigma}] \neq 0$. By virtue of the discussion surrounding equations (V-11) to (V-15) this is equivalent to saying that at least one component of $[W^{\Delta}_{;\Sigma\Gamma}]$ or $[W_{;\Sigma\Gamma}]$ does not vanish but $[W^{\Delta}_{;\Sigma}]$ and $[W_{;\Sigma}]$ do vanish.

Because acceleration waves are being considered, equation (5), the dynamical compatibility relation which led to the propagation conditions in the case of shock waves, is an identity. The propagation conditions for second order waves are given by equation (7) and the jump in equation (1). If equation (7) is combined with equations (8) and (9) together with identities (III-123) and (III-126) then the result is

$$(D - \rho_{0}U_{N}^{2})\tilde{\omega}^{3} = 0 \tag{33}$$

where equation (12) has been used as well. Equation (1) together with equations (V-1) to (V-3), and (V-17) and identities (III-123) and (III-129) yields

$$(D - \rho_0 U_N^2)\tilde{\omega} = 0 \tag{34}$$

$$(\frac{D}{2}(1-\eta) - \rho_0 U_N^2)\hat{\tilde{\omega}} = 0 \tag{35}$$

Equations (33), (34), and (35) are the propagation conditions for acceleration waves as it is easily seen from identity (III-123) that $[W;_{\Delta\Sigma}] = 0$ if and only if $\omega^3 = 0$ and $[W;^{\Delta}_{\Gamma\Sigma}] = 0$ if and only if $\tilde{\omega} = \hat{\tilde{\omega}} = 0$. As in the previous problems considered two cases arise if a non trivial solution to these equations is to exist.

If the speed of propagation is given by equation (V-20) then equations (33) and (34) are satisfied identically. Equation (35) then requires that $\hat{\tilde{\omega}} = 0$. In order to have an acceleration wave at least one of $\tilde{\omega}^3$ and $\tilde{\omega}$ must be non zero. When identity (III-129) is applied to the acceleration and the fact that $W^{i+}_{,tt}\big|_U \Sigma = 0$ then it is found that

$$W^{i-}_{,tt}\big|_U \Sigma = - U_N^2(\tilde{\omega}v^i + \tilde{\omega}^3 N^i) \tag{36}$$

The acceleration is normal to the wave curve, so such a wave will be referred to as a normal wave.

The second case arises when the speed of propagation is given by equation (V-22). Then $\tilde{\omega} = \tilde{\omega}^3 = 0$, as seen from equations (33) and (34). Equation (35) becomes an identity, and in order to have an acceleration wave $\hat{\tilde{\omega}}$ must not vanish. Because $W^{i+}_{,tt}\big|_U \Sigma = 0$ identity (III-129) applied to the acceleration yields

$$W^{i-}_{,tt}\big|_U \Sigma = - U_N^2 \hat{\tilde{\omega}} X^i_{,S} \tag{37}$$

The acceleration behind the wave front is tangential to the wave curve. Such waves will be referred to as tangential waves.

It is evident that in both cases the speed of propagation of the wave curve or diagram is constant. Thus the argument advanced in Chapter V for determining the shape of the wave curves is valid here as well. The net formed by the successive positions of the diagram or wave curve on the reference surface and by the orthogonal trajectories to these curves determines a geodesic parallel coordinate system on the reference surface. The orthogonal trajectories to the diagrams were shown to be geodesic curves. Therefore, given a wave curve at some time, the wave curve after a time interval Δt is the locus of points a distance $U_N \Delta t$ from the initial curve as measured along the orthongonal geodesics to the initial curve.

Equation (3) yields an expression for the shearing stresses which is substituted into equation (6). The jump in the resulting equation combined with equations (8), (10), and (V-3) and identities (III-137), (III-141), and (III-11) results in equation (26).

In the case where normal acceleration waves are considered this equation reduces to

$$\frac{D}{DQ} \tilde{\omega}^3 = \frac{1}{2} \Sigma \tilde{\omega}^3 \tag{38}$$

On the other hand if tangential acceleration waves are considered $\tilde{\omega}^3 = 0$ and $D \neq \rho U_N^2$ so that

$$\tilde{\tilde{\omega}}^3 = 0 \tag{39}$$

where of course $\tilde{\tilde{\omega}}^3 = [W;_{\Delta\Sigma}]\nu^\Delta \nu^\Sigma$.

Again equation (3) gives an expression for the shearing stresses which is used in equation (2). The jump in the resulting equation is combined with equations (8), (10), (V-3), and (V-1) and identities (III-147), (III-149), (III-108), and (II-111) to obtain for tangential waves

$$\tilde{\tilde{\tilde{\omega}}}^3 = 0 \tag{40}$$

The result of the same process for normal waves is given by equation (31) in which ω is set equal to zero. The resulting equation relates $\tilde{\tilde{\omega}}^3$ to $\tilde{\omega}^3$.

In the case of tangential waves, where it has been proven that $\overset{3}{\omega}, \tilde{\omega}^3$, $\tilde{\tilde{\omega}}^3$, and $\tilde{\tilde{\tilde{\omega}}}^3$ all vanish, if equation (2) is differentiated with respect to time and the jump in the resulting equation is taken then $\tilde{\tilde{\tilde{\tilde{\omega}}}}^3$ can be related to in-surface quantities. Equation (3) again provides the expression for the shearing stresses in equation (2). Equation (2) differentiated, combined with equations (8), (10), and (V-1) to (V-3) together with identities (III-154), (III-155), (III-126), (III-111), and (III-145) yields for tangential acceleration waves,

$$\tilde{\tilde{\tilde{\tilde{\omega}}}}^3 = \frac{D(1-\eta)}{(B-\rho_0 AU_N^2)} \; B_{\Delta\Sigma} t^{\Delta}{}_{v} \hat{\tilde{\omega}}^{\Sigma} \tag{41}$$

where equations (V-4), (9), and (11) give

$$\frac{D}{(B-\rho_0 AU_N{}^2)} = \frac{24}{h^2(1+\eta)} \tag{42}$$

Finally, the decay equation for the in-surface quantities are obtained by differentiating equation (1) with respect to time and taking the jump in the resulting equation. Then together with equation (V-1) to (V-3) and identities (III-142), (III-111), and (III-145) the resulting equation yields in the case of normal acceleration waves

$$2\,\frac{D}{DQ}\,\tilde{\omega} - \Sigma\tilde{\omega} \;=\; (B_{\Delta\Sigma}v^{\Delta}v^{\Sigma} + \eta B_{\Delta\Sigma}t^{\Delta}t^{\Sigma})\tilde{\omega}^{3} \tag{43}$$

and

$$\frac{D}{DS}\,\tilde{\omega} - \hat{\tilde{\omega}} \;=\; 2\frac{(1-\eta)}{(1+\eta)}\;B_{\Delta\Sigma}v^{\Delta}t^{\Sigma}\tilde{\omega}^{3} \tag{44}$$

In the case of tangential waves the results are

$$\frac{D}{DQ}\,\hat{\tilde{\omega}} \;=\; \frac{1}{2}\,\Sigma\hat{\tilde{\omega}} \tag{45}$$

and

$$\frac{D}{DS}\,\hat{\tilde{\omega}} \;=\; -\,\tilde{\omega} \tag{46}$$

Because

$$[W^{i}_{;\Delta\Sigma}] \;=\; (\tilde{\omega}v^{i} + \hat{\tilde{\omega}}X^{i}_{,S} + \tilde{\omega}^{3}N^{i})v_{\Delta}v_{\Sigma}$$

a measure of the strength of the discontinuity is given by the quantity

$$M \;=\; (\tilde{\omega}^{2} + \hat{\tilde{\omega}}^{2} + \tilde{\omega}^{3\,2})^{1/2} \tag{47}$$

In the case of tangential waves, it is seen that

$$M \;=\; |\hat{\tilde{\omega}}|$$

so that from equation (45)

$$\frac{D}{DQ}\,M \;=\; \frac{1}{2}\,\Sigma M \tag{49}$$

In the case of normal waves,

$$M \;=\; (\tilde{\omega}^{2} + \tilde{\omega}^{3\,2})^{1/2} \tag{50}$$

where $\tilde{\omega}^{3}$ is found from equation (38) and $\tilde{\omega}$ is found from equation (43).

It should be noticed that a normal acceleration wave has a component normal to the surface only if such a component is initially excited. An in-surface component is present, as seen from equation (43), even if it is not initially excited.

CHAPTER VIII

EXAMPLES

The problems which have been discussed, employing either a linear membrane theory or a linear bending theory of shells, have dealt with a singular curve propagating into the stationary unstrained reference surface; hence, the wave curve and the diagram have coincided. It was seen that the speed of propagation relative to the undisturbed material is constant. As pointed out in Chapter V, this means that on the reference surface the successive positions of the wave curve and the orthogonal trajectories to these curves form a geodesic parallel net in which the orthogonal trajectories are the geodesics. Thus if a wave curve is given on the reference surface at a reference time, the position of the wave curve after a time interval Δt is given by the totality of points a distance $U_N \Delta t$ from the given curve as measured along the geodesics passing through the initial curve and in the distance of propagation.

It proves to be advantagous to choose a geodesic parallel coordinate system on the reference surface in which the successive positions of the wave curve are given by taking U^1 constant and the geodesic which are the orthogonal trajectories are given by taking U^2 constant. U^1 is chosen to be the arc length along the geodesics.

An equation which appears in all the problems discussed when dealing with decay is the equation of the form

$$\frac{D}{DQ} g = \frac{1}{2} \Sigma g \tag{1}$$

where $Q = Nt$ is the arc length along the geodesics measured from a reference position, Σ is the geodesics curvature of the wave curve, and g represents some quantity which measures an aspect of the strength of the wave. In the geodesic parallel coordinate system this becomes

$$\frac{\partial}{\partial U^1} g = -\frac{1}{2} g \frac{\partial}{\partial U^1} \ln(A_{22}{}^{1/2}) \tag{2}$$

where Σ was replaced by the expression from equation (II-165) and where U^1 is arc length along the geodesics. Integrated this becomes

$$\frac{g}{g_0} = \left(\frac{A_{22}^0}{A_{22}}\right)^{1/4} \tag{3}$$

where

A_{22} is a component of the metric tensor in the geodesic parallel coordinate system, $g_0 = g(U_0^1, U^2)$, $A_{22}^0 = A_{22}(U_0^1, U^2)$, and $U^1 = U_0^1$ is the reference curve. This result was obtained by Cohen and Suh[48] where g represented wave strength.

As shown in equation (II-167)

$$\frac{\partial}{\partial U^1}\Sigma = \Sigma^2 + K \tag{4}$$

where K is the Gaussian curvature of the reference surface. This equation can be integrated fairly easily in the case where K is a constant. If the reference curve is taken as $U^1 = 0$ then Cohen and Suh showed that for $K = 0$

$$\Sigma = \frac{\Sigma_0}{1-\Sigma_0 U^1} \tag{5}$$

and hence equation (1) yields

$$\frac{g}{g_0} = \frac{1}{(1-\Sigma_0 U^1)^{1/2}} \tag{6}$$

Rather than integrate equation (4) and then substitute into equation (1), in the case where $K>0$, equation (II-166) is integrated and the result is substituted into equation (3). Then

$$A_{22}^{1/2} = C\cos(K^{1/2}U^1) + D\sin(K^{1/2}U^1)$$

When $U^1 = 0$

$$A_{22}^{0\ 1/2} = C$$

and from equation (II-165)

$$\Sigma_0 = -\frac{1}{A_{22}^{0\ 1/2}}\frac{\partial A_{22}^{0\ 1/2}}{\partial U^1} = -\frac{DK^{1/2}}{C}$$

so that

$$D = -C\Sigma_0 K^{1/2}$$

therefore

$$\frac{g}{g_0} = \left(\frac{1}{\cos(K^{1/2}U^1) - K^{1/2}\Sigma_0\sin(K^{1/2}U^1)}\right)^{1/2} \tag{7}$$

Similarly, if $K<0$ then

$$\frac{g}{g_0} = \left(\frac{1}{\cosh(|K|^{1/2}U^1) - |K|^{1/2}\Sigma_0\sinh(|K|^{1/2}U^1)}\right)^{1/2} \tag{8}$$

where in both equations (7) and (8) $U^1 = 0$ is the reference wave curve. Therefore, on a surface of constant Gaussian curvature once the geodesic curvature of the reference curve is given, a solution of equation (1) is obtained from equation (6),(7), or (8). U^1 is length along the geodesics.

When shock or acceleration waves with respect to the displacement are considered in the linear membrane theory the first non continuous derivative of the normal component of displacement is found to be at least one order greater than the order of the wave. The coupling that is introduced by the governing equations of the membrane manifests itself in part in equations (V-51) and (V-52) in the case of shock waves and in equations (V-58) and (V-59) in the case of acceleration waves. Thus shock waves are in general accompanied by an acceleration wave with respect to the component of displacement normal to the surface, and acceleration waves are accompanied by third order waves with respect to the component of displacement normal to the surface.

The condition for the vanishing of the accompaning wave for a longitudinal wave is

$$B_{\Delta\Sigma}\nu^{\Delta}\nu^{\Sigma} + \eta B_{\Delta\Sigma}t^{\Delta}t^{\Sigma} = 0 \tag{9}$$

as is seen from equations (IV-51) and (V-58), and for a transverse wave it is

$$B_{\Delta\Sigma}U^{\Delta}_{,S}\nu^{\Sigma} = 0 \tag{10}$$

which follows from equations (V-52) and (V-59). t^{Δ} and ν^{Δ} are the unit tangent and unit in-surface normal to the wave curve. It has been seen that the wave curves and the geodesic normals to the wave curves form a geodesic parallel net on the reference surface. t^{Δ} and ν^{Δ} are then the tangents to the parametric curves.

Condition (9) to be satisfied requires a geodesic parallel coordinate system such that the parametric curves form an orthogonal net of asymptotic directions. The direction t^{Δ} is defined to be asymptotic if $A_{\Delta\Sigma}t^{\Delta}t^{\Sigma} = 0$. Such a net of asymptotic directions is possible if and only if the surface is a minimal surface; that is, the mean curvature of the surface is identically 0[49]. It can also be proven that the asymptotic lines are real and distinct only if the Gaussian curvature is negative[50]. In addition, of course, one family of curves must be geodesics. An example of uncoupling of this nature occurs in the case of right helicoids. These are the only conoids that are minimal surfaces[51].

The parametric equations of a right helicoid are

$$X^1 = U^1 \cos U^2, \; X^2 = U^1 \sin U^2, \; X^3 = aU^2 + b \tag{11}$$

where X^i are rectangular Cartesian coordinates and a and b are constants. The curves U^2 = constant are straight lines perpendicular to the x^3 axis, and the curves U^1 = constant are helices about the x^3 axis with radius U^1. Then the metric tensor is given by $A_{\Delta\Sigma} = X^i_{,\Delta} X^i_{,\Sigma}$, so that

$$\begin{aligned} A_{11} &= 1 \\ A_{12} &= 0 \\ A_{22} &= a^2 + (U^1)^2 \\ |A_{\Delta\Sigma}| &= a^2 + (U^1)^2 \end{aligned} \tag{12}$$

It can be proven that if the parametric curves form an orthogonal net, so that $A_{12} = 0$, then a necessary and sufficient condition for U^2 = constant to be geodesic curves is that A_{11} be a function of U^1 only[52].

In the example above U^2 = constant therefore are geodesics and hence the U^1, U^2 parametric curves form a geodesic parallel coordinate system. The wave curves correspond to the helices U^1 = constant and they propagate along the straight lines. U^1 is the arc length along the straight lines.

The coefficients of second fundamental form are given by[53]

$$B = |A_{\Gamma\Lambda}|^{-1/2} \; \underline{X}_{,1} \times \underline{X}_{,2} \cdot \underline{X}_{,\Delta\Sigma}$$

so that

$$B_{22} = 0$$
$$B_{11} = 0 \tag{13}$$
$$B_{12} = a(U^{12} + a^2)^{-1/2}$$

Equation (1) is the equation for the decay of the wave strength of a shock wave or for that matter an acceleration wave in a membrane. Substitution from equations (12) into equation (3) means that

$$\frac{M}{M_0} = \left(\frac{a^2 + U_0^{12}}{a^2 + U^{12}}\right)^{1/4} \tag{14}$$

Since $B_{11} = B_{22} = 0$, for longitudinal shock waves equation (V-51) gives

$$\tilde{\omega}^3 = 0$$

and for transverse shock waves equation (V-52) gives

$$\hat{\omega}^3 = 2a(U^{1\,2} + a^2)^{-1}\hat{\omega} = 2a\frac{(a^2 + U_0^{1\,2})^{1/4}}{(a^2 + U^{1\,2})^{5/4}}\hat{\omega}_0 \tag{15}$$

where $M = |\hat{\omega}|$ for this kind of wave.

The wave strength of an acceleration wave in a membrane is also given by equation (14). For a transverse acceleration wave $\tilde{\tilde{\omega}}^3$ is related to $\hat{\tilde{\omega}}_0$ in the same way that $\tilde{\omega}^3$ is related to $\hat{\omega}_0$ in equation (15) and for longitudinal acceleration waves $\tilde{\tilde{\omega}}^3 = 0$ because $B_{11} = B_{22} = 0$.

In order that condition (10) be satisfied the coordinate curves of some geodesic parallel coordinate system must form a conjugate net. Two directions p^Δ and t^Σ are conjugate by definition if they satisfy $B_{\Delta\Sigma}p^\Delta t^\Sigma = 0$.

In addition, the coordinate curves must form an orthogonal net. However, the only possible orthogonal conjugate net on a surface are the lines of curvature of the surface. One family of parametric curves must be geodesics as well. An example of uncoupling of this kind occurs on any surface of revolution.

The parametric equations of a surface of revolution are

$$X^1 = U^1 \cos U^2, \quad X^2 = U^1 \sin U^2, \quad X^3 = \phi(U^1) \tag{16}$$

where $\phi(U^1)$ is any single valued smooth function of U^1. The curves U^1 = constant curves are the meridians or generating curves of the surface. Then

$$\begin{aligned} A_{11} &= 1 + \phi'^2 \\ A_{12} &= 0 \\ A_{22} &= U^{12} \end{aligned} \tag{17}$$

By virtue of the theorem quoted previously, the curves U^2 = constant are geodesics because A_{11} is a function of U^1 alone. By defining new coordinates by the transformations

$$\begin{aligned} \bar{U}^1 &= \int_{U_0^1}^{U^1} (1 + \phi'^2)^{1/2} dU^1 \\ \bar{U}^2 &= U^2 \end{aligned} \tag{18}$$

a geodesic parallel coordinate system is obtained such that $\bar{U}^1$ is the arc length along the geodesics given by $\bar{U}^2$ = constant. The metric tensor is

$$\begin{aligned} \bar{A}_{11} &= 1 \\ \bar{A}_{12} &= 0 \\ \bar{A}_{22} &= A_{22} = U^{12} = f(\bar{U}^1) \end{aligned} \tag{19}$$

where $f(\bar{U}^1)$ is obtained by inverting equation (18).

Therefore, on a surface of revolution the circles given by taking $\bar{U}^1$ = constant are chosen to be the wave curves. These waves propagate along the geodesics given by $\bar{U}^2$ = constant.

The coefficients of the second fundamental form are given in the original coordinate system by

$$\begin{aligned} B_{11} &= \phi^{11}(1 + \phi^{12})^{-1/2} \\ B_{12} &= 0 \\ B_{22} &= U^1\phi^1(1 + \phi^{12})^{-1/2} \end{aligned} \tag{20}$$

so that

$$\begin{aligned} \bar{B}_{11} &= \phi^{11}(1 + \phi^{12})^{-3/2} \\ \bar{B}_{12} &= 0 \\ \bar{B}_{22} &= U^1\phi^1(1 + \phi^{12})^{-1/2} \end{aligned} \tag{21}$$

Equation (1), the equation for the wave strength of a shock or acceleration, is solved by substituting from equation (19) into equation (3) so that

$$\frac{M}{M_0} = \left(\frac{U^1_0}{U^1}\right)^{1/2} = \left(\frac{f(\bar{U}^1_0)}{f(\bar{U}^1)}\right)^{1/4} \tag{22}$$

Because $\bar{B}_{12} = 0$, for transverse shock waves equation (V-52) indicates that

$$\tilde{\omega}^3 = 0 \tag{23}$$

and for longitudinal shock waves equation (V-51) yields

$$\tilde{\omega} = \{\phi^{11}(1 + \phi^{12})^{-3/2} + \eta \frac{\phi^1}{U^1}(1 + \phi^{12})^{-1/2}\}\omega \tag{24}$$

where $|\omega| = M$ for this type of wave.

In the case of acceleration waves in membranes equation (22) is still valid as the equation for the wave strength. For transverse acceleration waves

$$\tilde{\tilde{\omega}}^3 = 0 \tag{25}$$

and for longitudinal acceleration waves $\tilde{\tilde{\omega}}^3$ is related to $\tilde{\omega}$ in the same way that $\tilde{\omega}^3$ is related to ω in equation (24).

A special type of geodesic parallel coordinate system is the geodesic polar coordinate system. In a small enough neighborhood of a point P there passes through each point one and only one geodesic which also passes through P. The orthogonal trajectories to the geodesics through P together with these geodesics then forms a net which can be used as a coordinate system in the neighborhood of P. This coordinate system is called a geodesic polar coordinate system with its pole at P. U^1 can be chosen to be the arc length along the geodesics measured from P and U^2 can be chosen to be the angle at P between the geodesics. A necessary and sufficient condition(54) that a geodesic parallel coordinate system such that

$$dS^2 = dU^{12} + A_{22}(U^1, U^2)dU^{22} \tag{26}$$

be a geodesic polar coordinate system with its pole at $U^1 = 0$ and with U^2 the angle between the geodesics is that

$$A_{22}^{1/2}\Big|_{U^1=0} = 0, \quad \frac{\partial A_{22}^{1/2}}{\partial U^1}\Big|_{U^1=0} = 1 \tag{27}$$

Geodesic polar coordinates can be associated with singular curves which owe their existence to intial disturbances in a very small neighborhood of the pole.

With the foregoing in mind, the example is presented of waves eminating from a point P on the side of a circular cylinder. A circular cylinder of radius a is considered with the axis of symmetry taken as the X^3 axis. The remaining rectangular Cartesian coordinates are chosen such that the point P lies on the X^1 axis and has coordinates (a,0,0). The wave curves which are being sought are associated with a geodesic polar coordinate system with its pole at P. The problem then is to find such a coordinate system.

Parametric equations of the cylinder are

$$X^1 = a \cos U^1, \quad X^2 = a \sin U^1, \quad X^3 = U^2 \tag{28}$$

Then the curves U^1 = constant are the straight line generators of the cylinder and the U^2 = constant curves are the circular intersections with the X^3 = constant planes. It is easily seen that the metric tensor is

$$A_{11} = a^2, \quad A_{12} = 0, \quad \text{and} \quad A_{22} = 1 \tag{29}$$

and from equation (II-26) it is clear that the Christoffel symbols vanish on the surface. The coefficients of the second fundamental form are

$$B_{11} = -a, \quad B_{12} = 0, \quad B_{22} = 0 \tag{30}$$

The geodesics on the surface which pass through P are given by equation (II-160) which in this example degenerates to

$$\frac{\partial^2 U^\Delta}{\partial s^2} = 0$$

The solutions are

$$U^1 = cS + d$$
$$U^2 = eS + f$$

where c,d,e,f are constants and S is the arc length measured from P. The condition that the geodesics pass through P then means that d = f = 0 and the geodesics are given by

$$g(U^1,U^2) = \frac{U^2}{U^1} = k \qquad (31)$$

where k is on arbitrary constant and $k = \infty$ is taken to mean $U^1 = 0$.

To find the orthongonal trajectories to these geodesics it is noted that

$$g_{,\Delta}dU^{\Delta} = 0$$

where dU^{Δ} is tangent to the geodesics so that $f_{,\Delta}$ is normal to this family. Thus

$$dU^1 = -\frac{g_{,2}}{g_{,1}}dU^2$$

If δU^{Δ} is a differential along the orthogonal trajectories to the geodesics then

$$A_{\Delta\Sigma}dU^{\Delta}\delta U^{\Sigma} = 0$$

or

$$A_{11}\left(-\frac{g_{,2}}{g_{,1}}\right)dU^2\delta U^1 + A_{22}dU^2\delta U^2 = 0$$

so that

$$a^2U^1\delta U^1 + U^2\delta U^2 = 0$$

Integration gives the equation of the orthogonal trajectories to the geodesics. Thus

$$a^2U^{12} + U^{22} = \text{constant} \tag{32}$$

are the required curves. Therefore, the new coordinates defined by

$$\begin{aligned} \bar{U}^1 &= a^2U^{12} + U^{22} \\ \bar{U}^2 &= \frac{U^2}{U^1} \end{aligned} \tag{33}$$

are such that $\bar{U}^2$ = constant are the geodesics through P and $\bar{U}^1$ = constant are the orthogonal trajectories.

The metric tensor in the new coordinate system is easily shown to be

$$\bar{A}_{11} = \frac{1}{4\bar{U}^1}, \quad \bar{A}_{12} = 0, \quad \bar{A}_{22} = \frac{a^2\bar{U}^1}{(a^2 + \bar{U}^{22})^2} \tag{34}$$

If another coordinate transformation is effected such that

$$\begin{aligned} \underline{U}^1 &= \int \left(\frac{\bar{U}^1}{2}\right)^{-1/2} d\bar{U}^1 = \bar{U}^{1/2} \\ \underline{U}^2 &= \bar{U}^2 \end{aligned} \tag{35}$$

then the new metric tensor is

$$\underline{A}_{11} = 1, \quad \underline{A}_{12} = 0, \quad A_{22} = \frac{a^2\underline{U}^{12}}{(a^2 + \underline{U}^{22})^2} \tag{36}$$

so that $\underline{U}^1$ is the arc length measured from P along the geodesics, $\underline{U}^2$ = constant.

A final coordinate transformation is required to have one of the coordinates be a measure of the angle, at the pole, between geodesics. If $\hat{U}^\Delta$ are new coordinates defined such that this is so then the metric tensor in this system must satisfy equations (27). If the coordinate transformation is of the form

$$\hat{U}^1 = \underline{U}^1, \quad \hat{U}^2 = \hat{U}^2(\underline{U}^2) \tag{37}$$

then the new metric tensor is

$$\hat{A}_{11} = 1, \quad \hat{A}_{12} = 0, \quad \hat{A}_{22} = \left(\frac{\partial \underline{U}^2}{\partial \hat{U}^2}\right)^2 \underline{A}_{22} \tag{38}$$

so that

$$\hat{A}_{22}^{\;1/2} = \frac{\partial \underline{U}^2}{\partial \hat{U}^2} \frac{a\hat{U}^1}{(a^2+\underline{U}^{22})}$$

and

$$\frac{\partial}{\partial \hat{U}^1} \hat{A}_{22}^{\;1/2} = \frac{\partial}{\partial} \frac{\underline{U}^2}{\hat{U}^2} \frac{a}{(a^2+\underline{U}^{22})}$$

In order to satisfy conditions (27)

$$\frac{d}{d} \frac{\underline{U}^2}{\hat{U}^2} \frac{a}{(a^2+\underline{U}^{22})} = 1$$

which can be solved to get

$$\underline{U}^2 = a \tan \hat{U}^2 \tag{39}$$

so that the new metric ternsor is

$$\hat{A}_{11} = 1, \quad \hat{A}_{12} = 0, \quad \hat{A}_{22} = \hat{U}^{12} \tag{40}$$

The coordinate transformation between the original coordinate and the final coordinates is

$$U^1 = \frac{\hat{U}^1 \cos \hat{U}^2}{a}, \quad U^2 = \hat{U}^1 \sin \hat{U}^2 \tag{41}$$

so that the parametric equations of the circular cylinder, in the case where the parameter curves form a geodesic coordinate system about P such that $\hat{U}^1$ is the arc length along the geodesics measured from P and $\hat{U}^2$ the angle at P between the geodesics, are

$$X^1 = a \cos \left(\frac{\hat{U}^1 \cos \hat{U}^2}{a}\right), \quad X^2 = a \sin \left(\frac{\hat{U}^1 \cos \hat{U}^2}{a}\right), \quad X^3 = \hat{U}^1 \sin \hat{U}^2 \tag{42}$$

Finally, the coefficients of the second fundamental form in the $\hat{U}^\Delta$ system are

$$\hat{B}_{11} = -\frac{\cos^2 \hat{U}^2}{a}, \quad \hat{B}_{12} = \frac{\hat{U}^1 \sin \hat{U}^2 \cos \hat{U}^2}{a},$$

$$\hat{B}_{22} = -\frac{\hat{U}^{12} \sin^2 \hat{U}^2}{a} \tag{43}$$

The wave curves are given by setting $\hat{U}^1$ = constant in equation (42). The geodesics are given by setting $\hat{U}^2$ = constant in these equations, and they are all the helices through P together with the straight line generator and circle through P.

Using the linear membrane theory for shock or acceleration waves equation (1) becomes the equation for the wave strength M for both longitudinal and transverse waves. Then the solution is given by equation (3) so that

$$\frac{M}{M_0} = \left(\frac{\hat{u}^1_0}{\hat{U}^1}\right)^{1/2} \tag{44}$$

where $\hat{U}^1_0$ is the reference wave curve and M_0 is the wave strength on this curve.

As has been seen, in general the first discontinuous derivative of the component of displacement normal to the surface is at least one order greater than the order of the wave. In the case of a transverse shock wave this coupling is evidenced by the equation

$$\tilde{\omega}^3 = 2B_{\Delta\Sigma}t^{\Delta}v^{\Sigma}\hat{\omega} \tag{45}$$

In the example being considered this means that

$$\tilde{\omega}^3 = 2\hat{\omega}_0 \left(\frac{\hat{U}^1_0}{\hat{U}^1}\right)^{1/2} \frac{\sin \hat{U}^2 \cos \hat{U}^2}{a} \tag{46}$$

where $\hat{\omega} = M$ for this type of wave and hence $\hat{\omega}_0 = M_0$. For a transverse acceleration wave $\tilde{\omega}^3$ is related to $\hat{\tilde{\omega}}$ in the same way that $\tilde{\omega}^3$ is related to $\hat{\omega}$ in equation (46).

If, on the other hand, a longitudinal shock wave is considered the coupling mentioned above is given by

$$\tilde{\omega}^3 = (B_{\Delta\Sigma}v^{\Delta}v^{\Sigma} + \eta B_{\Delta\Sigma}U^{\Delta}_{,S}U^{\Sigma}_{,S})\omega \tag{47}$$

In this example this means that

$$\tilde{\omega}^3 = -\frac{1}{a}[\cos^2 \hat{U}^2 + \eta \sin^2 \hat{U}^2]\omega_0 \left(\frac{\hat{U}^1_0}{\hat{U}^1}\right)^{1/2} \tag{48}$$

where $\omega = M$ for this type of wave so that $\omega_0 = M_0$. For a longitudinal acceleration wave $\tilde{\omega}^3$ is related to $\tilde{\omega}$ in the same way that $\tilde{\omega}^3$ is related to ω.

In the case of transverse shock waves propagating into an undisturbed reference state equation (46) indicates that the normal component of acceleration just behind the wave is dependent upon the angle $\hat{U}^2$. On the straight line and on the circle through P the normal component of acceleration vanishes and it changes sign from one quadrant to another. This component of acceleration is of the same magnitude but of opposite sign at corresponding points on helices with the same rate of twist but with different orientations.

For a longitudinal shock wave equation (48) indicates that the normal component of the acceleration just behind the wave is dependent upon the angle $\hat{U}^2$, but it always has the same sign. On corresponding helices with the same rate of twist but with different orientations this component is the same.

The final example which will be considered is a cone of revolution which is given by the following parametric equations:

$$\begin{aligned} X^1 &= U^1 \tan\alpha \cos U^2 \\ X^2 &= U^1 \tan\alpha \sin U^2 \\ X^3 &= U^1 + b \end{aligned} \tag{49}$$

where X^i are rectangular Cartesian spatial coordinates, X^3 is axis of the cone, α is the half angle of the cone and $X^3 = b$ is the location of the apex. The metric tensor is then

$$A_{11} = 1 + \tan^2\alpha\ , \quad A_{12} = 0, \quad A_{22} = U^{12}\tan^2\alpha \tag{50}$$

so that

$$|A_{\Delta\Sigma}| = U^{12}\tan^2\alpha(1+\tan^2\alpha)$$

Because of the theorem stated previously, the fact that A_{11} does not depend upon U^2 means that U^2 = constant are geodesics. $A_{12} = 0$ means the coordinate curves form an orthongonal net. U^2 = constant are the straight line generators and U^1 = constant are the circular intersections with the planes X^3 = constant.

A geodesic parallel coordinate system such that one coordinate is arc length along the geodesics is given by the coordinate transformation defined by the equations,

$$\bar{U}^1 = (1 + \tan^2\alpha)^{1/2} U^1$$
$$\bar{U}^2 = U^2 \tag{51}$$

Then the new metric tensor is

$$\bar{A}_{11} = 1, \quad \bar{A}_{12} = 0, \quad \bar{A}_{22} = \frac{\bar{U}^{12}\tan^2\alpha}{(1+\tan^2\alpha)} \tag{52}$$

In the example being considered the wave curves correspond to the circles $\bar{U}^1$ = constant and these propagate along the straight lines.

The coefficients of the second fundamental form are

$$\bar{B}_{11} = 0, \quad \bar{B}_{12} = 0, \quad \bar{B}_{22} = \frac{\bar{U}^1\tan\alpha}{(1+\tan^2\alpha)} \tag{53}$$

The solution to equation (1) given by equation (3), in this example becomes

$$\frac{g}{g_0} = \left(\frac{\bar{U}^1_0}{\bar{U}^1}\right)^{1/2} = \left(\frac{U^1_0}{U^1}\right)^{1/2} = \left(\frac{X^3_0 - b}{X^3 - b}\right)^{1/2} \tag{54}$$

where $\bar{U}^1_0$ is the distance of the reference circle from the apex as measured along a generator. If r_0 is the radius of the reference circle it is easily seen that $\bar{U}^1_0 = r_0 \csc\alpha$ so that a convenient form of the above is

$$\frac{g}{g_0} = \left(\frac{r_0 \csc\alpha}{r_0 \csc\alpha + \bar{U}^1 - \bar{U}^1_0}\right)^{1/2} \tag{55}$$

To this point the type of wave being considered or the nature of the material has not been specified other than the fact that a shock or acceleration wave is being considered in a linear membrane or linear bending theory. The first case to be considered will be the membrane theory. It suffices to treat shock waves only, as the result for acceleration waves can be obtained by replacing "ω" quantities by "$\tilde{\omega}$" quantities.

If longitudinal shock waves in membranes are being dealt with then equation (54) with g replaced by ω is the determining equation for ω and since the wave strength $M = |\omega|$ it also determines M if g is replaced by M. From equation (V-51) it is seen that

$$\tilde{\omega}^3 = \frac{\eta}{\tan\alpha}\,\omega_0\,\frac{\bar{U}_0^{1\;1/2}}{\bar{U}^{1\;3/2}} \tag{56}$$

If transverse shock waves in membrane are considered equation (54) with g replaced by $\hat{\omega}$ is the determining equation for $\hat{\omega}$ and because $M = |\hat{\omega}|$ in this case it also determines M if g is replaced by M. Because $\bar{B}_{12} = 0$

$$\tilde{\omega}^3 = 0 \tag{57}$$

In the case where longitudinal shock waves are considered in the linear bending theory of shells equations (VII-27) and (V-45) have the form of equation (1) and so that their solutions are given by replacing g by $\tilde{\omega}^3$ and ω respectively in equation (54), so that

$$\frac{\tilde{\omega}^3}{\tilde{\omega}_0^3} = \left(\frac{\bar{U}_0^1}{\bar{U}^1}\right)^{1/2} \tag{58}$$

and

$$\frac{\omega}{\omega_0} = \left(\frac{\bar{U}_0^1}{\bar{U}^1}\right)^{1/2} \tag{59}$$

It should be noted that $\tilde{\omega}^3$ is independent of ω. Therefore, an acceleration wave with respect to the normal component of displacement accompanies the shock wave only if such a wave is independently established. The coupling between the normal component of displacement and the in-surface components of displacement manifests itself by relating higher order derivatives. The wave strength $M = |\omega|$ in this case and so it is given by equation (3) as well.

For a transverse shock wave in the linear bending theory of shells equation (V-48) has the same form as equation (1) so that from equation (54)

$$\frac{\hat{\omega}}{\omega} = \left(\frac{\bar{U}^1_0}{\bar{U}^1}\right)^{1/2} \tag{60}$$

For this type of wave the wave strength is given by $M = |\tilde{\omega}|$, therefore M is also given by taking $g = M$. It is also evident from equation (VII-29) and the fact that $\bar{B}_{12} = 0$ that

$$\tilde{\tilde{\tilde{\omega}}}^3 = 0 \tag{61}$$

Thus the coupling between the normal and the in-surface components of displacement is such that the lowest order wave with respect to the normal component of displacement that accompanies the shock wave is of at least fifth order.

The situation in the case of transverse acceleration waves is analogous to the situation existing for the same type of shock waves. Thus

$$\frac{\hat{\tilde{\omega}}}{\tilde{\omega}_0} = \left(\frac{\bar{U}^1_0}{\bar{U}^1}\right)^{1/2} \tag{62}$$

and in this case $|\hat{\tilde{\omega}}| = M$, also

$$\tilde{\tilde{\tilde{\tilde{\omega}}}}^3 = 0 \tag{63}$$

The final situation to be examined is the case of longitudinal acceleration waves in the linear bending theory of shells. Equation (VII-38) is of the form of equation (1) so that

$$\frac{\tilde{\omega}^3}{\tilde{\omega}_0^3} = \left(\frac{U_0^1}{\bar{U}^1}\right)^{1/2} \tag{64}$$

Equation (VII-43) in the example being considered because

$$\frac{\partial \tilde{\omega}}{\partial \bar{U}^1} = -\frac{\partial}{\partial \bar{U}^1} \ln \bar{A}_{22}{}^{1/4} \tilde{\omega} + \frac{\eta}{2\bar{U}^1 \tan\alpha} \tilde{\omega}^3$$

Using the integrating factor $\bar{A}_{22}{}^{1/4}$ this equation can be integrated to give

$$\tilde{\omega} = \{\tilde{\omega}_0 + \tilde{\omega}_0^3 \frac{\eta}{2} \qquad \cot\alpha \qquad \ln \left(\frac{\bar{U}^1}{\bar{U}_0^1}\right)\} \left(\frac{\bar{U}_0^1}{\bar{U}^1}\right)^{1/2} \tag{65}$$

Thus in the case of a longitudinal acceleration waves the wave is in general an acceleration wave with respect to the component of displacement normal to the surface and also with respect to the component of displacement normal to the wave curve but in the surface. At least one of these conditions must be present for a longitudinal acceleration wave to exist at all. The acceleration wave with respect to the normal component exists only if it is created by the initial disturbance. $\tilde{\omega}^3$ is uneffected by what is happening to the surface component. On the other hand, because of the coupling the acceleration wave with respect to the in-surface component of displacement exists regardless of whether or not it is created by the initial disturbance. There are two effects involved in the change of $\tilde{\omega}$. The factor $(\bar{U}_0/\bar{U}^1)^{1/2}$ accounts for the geometrical effects. The factor $\ln(\bar{U}^1/\bar{U}^1_0)$ accounts for the contribution to $\tilde{\omega}$ due to the coupling. Thus if $\tilde{\omega}_0$ is initially zero the coupling builds up the wave until a point is reached where the geometrical effects start to dominate and the wave then decays to zero as $\bar{U}^1$ goes to infinity.

In order to correlate the results of this type of analysis with emperical results a singular curve is considered as an idealization of a thin layer of material across which rapid changes take place. With this in mind, the results of the current example of singular curves propagating in cones of revolution might be compared with the experimental data of Keiner, Goldsmith, and Sackman[55]. They produced an impulse at the truncated end of a cone with a half angle of 20°. The radius of the middle surface of the cone at the truncated end was .125 inches. The speed of propagation as calculated for a longitudinal wave using the above analysis is in close agreement with the experimental value. The predicated relative wave strength, which is the same for shock waves in both the membrane and linear bending theories, drops off more slowly than the experimental data.

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