Displacement, Velocity, Acceleration, and Energy of a Dynamical System Derived from a Damped Harmonic Oscillator Extracted from Video Frames

by

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Abstract

This dissertation reflects applying external forces to a harmonic oscillatory system based on the digital-image-based harmonic oscillator equation. This leads to solving partial differential equations (PDEs) since we consider video frame picture elements $fr(x, y)$ in each equation. By solving each equation subject to an external applied force, we can extract motion waveforms from video frames (fr's) showing a moving object with sinusoidal oscillations about an equilibrium point (e.g., up and down movements of a biker in motion). A frame fr waveform emanates from a harmonic oscillator.

After solving the PDE equation which results video frame pixels $fr(x, y)$, we can compute other quantities including velocity, acceleration, and energy of the harmonic oscillator system. In fact, role of $fr(x, y)$ in the digital-image-based harmonic oscillator equation that is the equation of motion of the oscillator is identical to the displacement of the system. As a result, the main component to computing the quantities of the oscillator is $fr(x, y)$. By taking the derivatives of $fr(x, y)$, we obtain the velocity, acceleration, and energy of the system and then investigate their features. A specific focus belongs to the energy of the oscillatory system. The main reason for this concentration is that the fluctuations in energy suggest a varying distribution of forces across the image. Higher energy regions likely represent important features of the object in the digital image, such as fluctuating edges, textures, and so on.

Keywords: Video, energy, frames, harmonic oscillator, damped systems, image, displacement, partial differential equations, video frame waveform, force, pixel

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HO	Harmonic Oscillator
$E(t)$	Energy
fr	Frame
$fr(x, y)$	Pixel
m	Mass
k	Stiffness
D	Damping Coefficient

Chapter 1

Introduction

1.1 Overview

This dissertation presents a comprehensive analysis of the relationship between the video frame waveform and harmonic oscillators, by extracting the video motion frame picture elements $fr(x, y, t)$ at frame location (x, y) at time t of a recorded harmonic oscillator. The primary aim of this research is to use harmonic oscillators in their digital-image-based format to extract their waveform under different applied forces. Harmonic oscillators (HO) have been extensively studied in measuring displacement, velocity, acceleration, and HO waveform energy. The dissertation also bridges this innovative approach with existing knowledge from the field, particularly harmonic oscillations, offering insights into the dynamic behavior of systems under varying forces and conditions.

Harmonic oscillators have been studied for their ability to model systems experiencing periodic changes. A few examples of such systems include a person biking, running, and walking. Applying external forces to the system based on the digital-image-based harmonic oscillator equation involves leveraging a mathematical model that integrates the dynamics of harmonic motion with visual data extracted from video frames. This process typically requires solving partial differential equations (PDEs) that account for the interaction between the external forces and the system's inherent oscillatory behavior. These equations govern the displacement, velocity, acceleration, and energy of the oscillator as influenced by the applied forces.

The following sections outline the various components of this study, including the mathematical modeling, video frame extraction from harmonic oscillators, the interaction of video frame motion waveform with these oscillators, and a detailed exploration of the digital-image-based harmonic oscillator quantities involving velocity, acceleration, and energy.

1.2 The Role of Harmonic Oscillators in Video Frame Motion waveform

At the core of this thesis lies the use of harmonic oscillators to extract video frame waveform. Harmonic oscillators describe systems that experience repetitive motion, typically governed by second-order differential equations. Motion waveforms are extracted from the video frames (fr's), which capture the movement of a dynamic object exhibiting sinusoidal oscillations about an equilibrium point. For instance, this could represent the up-and-down motion of a biker in motion, where the oscillatory pattern reflects a periodic displacement caused by the harmonic oscillator dynamics. In fact, the video frame waveform stems from the various forces acting on the harmonic oscillator, which, in turn, determines the homogeneity of the differential equations. An important difference is that the equations are partial differential equations because we are using $fr(x, y)$ in the digital-image-based harmonic oscillator equation 1.1.

1.3 Differential Equations and Harmonic Oscillators

As it was mentioned, harmonic oscillators are typically governed by second-order differential equations. On the other hand, when we are dealing with video frames, we are talking about pixels in a specific time with two spatial coordinates x and y and if we want to extract the motion waveform $fr(x, y)$ from the harmonic oscillator, we have to solve partial differential equations (PDEs) because we no longer have one variable. Consequently, the solutions for the harmonic oscillators used in this research are derived from partial differential equations describing harmonic motion. Each equation corresponds to a unique force applied to the oscillator. The digital-image-based harmonic oscillator equation (or equation of motion) emanates from the original differential equation of harmonic oscillators and is:

$$\boxed{m\frac{\partial^2 fr(x,y)}{\partial y^2} + D\frac{\partial fr(x,y)}{\partial y} + kfr(x,y) = 0} \tag{1.1}$$

Where m is mass, D is the damping coefficient, and k is the spring constant or stiffness. By solving the PDEs, we obtain waveform functions that represent the position of the harmonic oscillator over time. This is because instead of a simple displacement, we have motion waveform $fr(x, y)$ that are the result of solving the digital-image-based harmonic oscillator equations (PDEs).

1.4 Interaction Between the Biker Video Frame Waveform and Harmonic Oscillators

A significant part of this dissertation is dedicated to studying the interaction between the video frame waveform and harmonic oscillators. The frame (fr) waveform itself is generated as a result of the harmonic oscillator, where the underlying mathematical model predicts the sinusoidal displacement of the object over time. This harmonic oscillator-based approach allows for seamless integration of physics-based principles into the visual representation of motion. Such an approach not only enhances the realism of motion in digital imagery but also provides a robust framework for studying and simulating complex oscillatory systems in a visually intuitive manner.

We illustrate this as a moving biker recorded in a video. The interaction between the biker motion waveform and the harmonic oscillators is critical for understanding the biomechanics of biking. biking, a repetitive, cyclical motion, can be modeled as a periodic motion, much like the harmonic oscillators used in this study. By analyzing video frames with harmonic oscillators, we are able to capture and analyze the effects of periodic forces on human motion (biking in this case). This analysis has far-reaching applications in biomechanics, where understanding gait and motion is crucial for fields such as rehabilitation, prosthetics, and sports science.

1.5 Application of Harmonic Oscillator to Biker Motion Videos

The application of harmonic oscillators to biker motion videos provides a robust framework for analyzing and replicating the periodic motion of cyclists. By modeling the vertical oscillations of the biker as a harmonic oscillator, the system captures the up-and-down motion caused by the interaction of the rider's body dynamics, the bike's suspension, and the terrain. This model utilizes the principles of oscillatory mechanics, where the displacement of the biker from an equilibrium position can be expressed as a sinusoidal function over time. The extracted motion parameters allow for the creation of a realistic motion profile that reflects the actual dynamics of the cyclist.

This approach not only facilitates the visualization of motion dynamics in video frames but also provides valuable insights into biomechanical and mechanical aspects of cycling.

For instance, applying a harmonic oscillator model enables researchers to simulate and analyze how different terrains or bike designs influence the rider's oscillatory motion. Additionally, by integrating these principles with digital video processing, it becomes possible to modulate video frames to highlight or exaggerate specific motion features, aiding in performance analysis, virtual reality applications, or educational demonstrations. This fusion of physics-based modeling with video analysis offers a powerful tool for studying and enhancing the understanding of dynamic systems in motion.

The MATLAB scripts developed for this research play a pivotal role in processing the video frames and applying the harmonic oscillator. These scripts enable the extraction of the motion waveform and the application of oscillatory motion, allowing for the comprehensive analysis presented in the following chapters.

1.6 Digital-Image-Based Harmonic Oscillator Equations

The mathematical framework for analyzing harmonic oscillatory behavior in the digital-image-based systems is derived from fundamental principles of motion, specifically oscillations under external forcing and damping. In the context of this research, the displacement function $fr(x, y)$ is considered as a spatial representation of oscillatory motion applied to digital image frames. This displacement function varies across spatial coordinates x and y, influenced by time-dependent forces.

We considered the two types of external forces applied to the system. So, the governing equation for the displacement $fr(x, y)$ is given in two forms:

$$\boxed{m\frac{\partial^2 fr(x,y)}{\partial y^2} + D\frac{\partial fr(x,y)}{\partial y} + kfr(x,y) = f_0\cos(\omega t)} \tag{1.2}$$

And

$$\boxed{m\frac{\partial^2 fr(x,y)}{\partial y^2} + D\frac{\partial fr(x,y)}{\partial y} + kfr(x,y) = f_0 \exp(-j\omega t)} \tag{1.3}$$

These equations describe a damped harmonic oscillator with a mass m, damping coefficient D, and spring constant or stiffness k, subjected to an external periodic force of amplitude f_0 and angular frequency ω. The cosine term in the first form emphasizes real-valued oscillations, while the exponential representation in the second form facilitates complex analysis, commonly used for computational simplicity. The above equations are the PDEs that we should solve to obtain the displacement or video frame waveform $fr(x, y)$ and based on $fr(x, y)$ function we can obtain the other quantities including velocity, acceleration, and energy of the system.

1.7 Related Works

To provide context for the present study, this section reviews key contributions in the field that are closely related to the analysis of harmonic oscillators. The following works lay the foundation for understanding oscillatory motion, and the use of harmonic oscillators in the video frames. These works have significantly influenced the approach and framework adopted in this thesis.

Article [1] explores the damped simple harmonic motion of an oscillator and the instantaneous displacement, velocity, and acceleration are graphically represented by projecting a rotating radius vector, whose magnitude gradually decreases, onto the diameter of a circle. It also shows that the instantaneous energy of the oscillator is not a pure exponential decay function of time. The main focus of [1] is the discussion on displacement, velocity, acceleration, and energy of the oscillator and the same quantities in our method in this thesis were computed and expanded.

The core section of this research includes the two cases (1 and 2 of displacements that are the solutions of the partial differential equations and used to extract the motion waveform from video frames. The two displacements are based on a discussion in chapter 17 of [2] (P.77). We also used Hilbert transform that connects the ups and downs of a video frame waveform and reference [2] investigated Hilbert transforms and it has chapters related to this topic so, [2] can be used in the study of Hilbert transforms. Another discussion on Hilbert transform can be found in [3] which studies Hilbert transform in vibration analysis.

Another suitable reference in harmonic oscillators is [3]. It aims to explain the significance and versatility of the harmonic oscillator model in physics. It highlights how the seemingly simple system of a mass, spring, and damper is used to understand a wide range of physical phenomena. It also discusses the role of damping in energy dissipation and equilibrium with the environment, the effect of external forces on the oscillator's response, and introduces key concepts like resonance and mechanical impedance. Finally, it extends the harmonic oscillator model to more complex systems with multiple masses and springs, showing how it helps describe continuous systems like strings.

Two important references for addressing topics in digital images are [4] and [5]. These references covers a wide range of topics from video frame picture elements or pixels ($fr(x, y)$), digital image fundamentals to object recognition.

Chapter 2

Displacement

2.1 Video Frames and Spatial-Temporal Representation

In video processing, a video frame refers to a single, static image within a sequence, captured at a specific moment in time. When these frames are displayed consecutively at a rapid rate-typically between 24 and 60 frames per second (fps)-the human eye perceives them as continuous motion, creating what we recognize as video. Each video frame is composed of a grid of tiny picture elements, or *pixels*, which are the fundamental units of a digital image. These pixels encode visual data in terms of color and intensity at specific spatial coordinates within the frame. Table 2.1 contains some symbols used in the following sections.

Each pixel within a frame can be denoted by the function $fr(x, y, t)$ [4], where:

• x and y represent the spatial coordinates of the pixel within the frame

• t signifies the time at which the frame was captured, effectively serving as a temporal marker

The function $fr(x, y, t)$ provides the color and intensity values of a pixel at position (x, y) for a fixed time t, capturing the spatial details of the scene in that frame. Thus, the

Symbol	**Definition**
fr	frame
$fr(x, y, t)$	pixel at location (x, y) and time t
f_0	Amplitude of the applied force
t	time
$x(t)$	displacement
A	amplitude
ω	angular frequency
ϕ	phase constant
m	mass
D	damping coefficient
k	stiffness

Table 2.1: Symbols

progression of these frames across time, as t varies, conveys dynamic information, allowing for the visual representation of motion.

In extending from two-dimensional spatial data (pixels within a frame) to a three-dimensional spatial-temporal domain, we introduce the concept of *voxels*, or volume elements. While pixels represent the smallest unit of spatial information in a two-dimensional plane, voxels serve as the equivalent in three-dimensional space, incorporating both spatial and temporal dimensions. Each voxel can be conceptualized as a pixel with depth, adding a third coordinate, z, which represents time or depth within the scene.

For a video sequence, voxels allow us to model a three-dimensional representation, where each voxel, denoted by $v(x, y, z, t)$, represents a small unit of data across both space and time. Here:

- x, y, and z correspond to spatial coordinates within a scene
- t represents the frame or time at which that specific voxel's data was captured

Thus, by treating each frame as a two-dimensional slice of this larger spatial-temporal volume, we can analyze changes not only across the width and height of each frame but also across frames, enabling a deeper analysis of motion and object persistence over time.

The concept of pixels provides a foundation for understanding the basic structure of individual video frames, while the concept of voxels extends this into a three-dimensional space. This voxel-based approach enhances our capacity to interpret motion dynamics, depth information, and other complex features within video data, laying the groundwork for advanced video analysis techniques such as object tracking, segmentation, and depth estimation. This approach bridges individual frames and the broader spatial-temporal analysis, critical for applications in computer vision, 3D reconstruction, and video analytics. With these explanations for pixels and voxels it can be obvious that a ***selected row*** in a video frame shows a ***moving object*** as it contains all the spatial and temporal information about the scene. The main reason that we select a ***particular row*** of a video frame for our analysis is capturing a moving object of the scene as a source of a motion waveform so that the selected row represents the moving parts as depicted in figure 2.1 . As a result, the relationship between a moving object in a video frame and a motion waveform can be described in the way that a motion waveform shows different pixels in the selected row which are changing as the object moves.

Figure 2.1: Sample biker video frame with a selected row (red line)

2.2 Harmonic Displacement

In physics and engineering, harmonic displacement describes the periodic motion of an object oscillating about an equilibrium position. This oscillation, governed by harmonic or sinusoidal motion, is essential for understanding systems where forces work to restore an object back to its original position, such as in mechanical vibrations, wave propagation, and even atomic motion within solids. Harmonic displacement forms the basis of simple harmonic motion (SHM), where the object's position changes over time in a predictable, repeating pattern. The displacement $x(t)$ of an object in simple harmonic motion is mathematically described by the function:

$$x(t) = A\cos(\omega t + \phi) \tag{2.1}$$

where:

- A is the amplitude, representing the maximum displacement from the equilibrium position
- ω is the angular frequency, defining the rate of oscillation in radians per second
- t is time, which dictates the progression of motion

- ϕ is the phase constant, indicating the initial displacement at $t = 0$

$x(t)$ satisfies the following differential equation that is the equation of motion for simple harmonic oscillators which describes the object in an oscillatory motion:

$$m\frac{d^2x(t)}{dt^2} + D\frac{dx(t)}{dt} + kx(t) = 0 \tag{2.2}$$

Wher m is mass, D is the damping coefficient, and k is the stiffness. This equation is a general form of the harmonic oscillator equation that we solve it to compute harmonic displacement and it can be subject to varying applied forces.

In harmonic oscillators, energy alternates between kinetic and potential forms as the object moves. At maximum displacement $|x(t)| = A$, potential energy is at its peak, while kinetic energy is zero. As the object passes through the equilibrium position $x = 0$, kinetic energy reaches its maximum, while potential energy is momentarily zero. This continuous exchange between kinetic and potential energy is governed by the total energy of the system, which remains constant for undamped oscillations.

All the other quantities of the harmonic oscillator can be computed from displacement $x(t)$ so that by getting the first derivative of the displacement we can reach the velocity $v(t)$ and by getting the second derivative of the displacement we can have the acceleration $a(t)$. Finally, the energy can be computed as the sum of kinetic and potential energies which in turn are proportional to $v(t)$ and $a(t)$.

2.2.1 Digital-Image-Based Version of The Harmonic Oscillator Equation

In this section we explore the general equation of harmonic oscillator, presented in the previous part, based on the pixels ($fr(x, y)$) in a digital image or a video frame. In this new approach we will consider $fr(x, y)$ instead of $x(t)$ in the motion equation of a harmonic oscillator so that we can extract the oscillatory motion of pixels by solving the pixel-based version of the harmonic oscillator equation. One of the main differences in this new approach in comparison to the simple harmonic oscillator equation is that we need to solve partial differential equations (PDEs)-partial derivatives on digital images. The new form of the pixel-based harmonic oscillator equation would be:

$$\boxed{m\frac{\partial^2 fr(x, y)}{\partial y^2} + D\frac{\partial fr(x, y)}{\partial y} + kfr(x, y) = 0} \tag{2.3}$$

We considered the oscillation in the y direction since the objects oscillates in this direction in the video frames. As it has been mentioned, the two cases can be considered in this equation. The first case is when the system is subject to the applied force $f_0 \cos(\omega t)$. The second case is when we apply the complex exponential force $f_0 \exp(-j\omega t)$. So, the right-hand side of the equation 2.3 equals these two forces. The next section will explain the two cases.

2.3 Two Forms of Displacement

As it has been mentioned, two forms of partial differential equations for a harmonic oscillator have considered. The two differential equations come from the two different types of the force applied to the harmonic oscillator which in turn determines homogeneity of the equations. As a result, the two different forces lead to two different displacements as the solutions of equations. In fact, the solutions are the video frame waveform $fr(x, y, t)$.

1. **Case 1:**

   The first case is when the right-hand side of the equation 2.3 equals $f_0 \cos(\omega t)$ so that we should solve the following PDE:

$$\boxed{m\frac{\partial^2 fr(x,y)}{\partial y^2} + D\frac{\partial fr(x,y)}{\partial y} + kfr(x,y) = f_0 \cos(\omega t)} \tag{2.4}$$

   In this case the equation is inhomogeneous since the applied force is $f_0 cos(\omega t)$ and the right side of the equation 2.4 is not zero. ω is a circular frequency. f_0 was considered 255 that is the maximum value of pixels (picture elements) in any video frame. This value of f_0 corresponds to a maximum height in motion waveform recorded in a video frame. t is the time. D is the average value of pixels in the specific row of the video frame. In the video, m is the estimated mass of the person. Since a typical adult human has a mass of about 70 kg, I considered 70,000 gr for m. We also have the following definitions:

$$\frac{\partial fr(x,y)}{\partial y} = fr(x,y+1) - fr(x,y) \tag{2.5}$$

$$\frac{\partial^2 fr(x,y)}{\partial y^2} = fr(x+1,y) - 2fr(x,y) + fr(x-1,y) \tag{2.6}$$

   Substituting these into the equation 2.4 gives:

$$m(fr(x+1,y)-2fr(x,y)+fr(x-1,y))+D(fr(x,y+1)-fr(x,y))+kfr(x,y) = f_0 \cos(\omega t) \tag{2.7}$$

   Solution Strategy:

   We will look for a solution of the form:

$$fr(x,y,t) = fr^h(x,y) + fr^p(x,y,t) \tag{2.8}$$

Where $fr^h(x,y)$ is the solution to the homogeneous equation (when the right side is zero) and $fr^p(x,y,t)$ is a particular solution to the non-homogeneous equation.
The homogeneous equation is:

$$m(fr(x+1,y)-2fr(x,y)+fr(x-1,y))+D(fr(x,y+1)-fr(x,y))+kfr(x,y) = 0 \tag{2.9}$$

By separating variables for the homogeneous equation we assume a separable solution of the form:

$$fr^h(x,y) = X(x)Y(y) \tag{2.10}$$

This leads to two separate equations for $X(x)$ and $Y(y)$. For $X(x)$: $X(x+1) - 2X(x) + X(x-1) = -\frac{\lambda}{m}X(x)$, where λ is a separation constant. This equation has the characteristic form:

$$r^2 - 2r + (1 + \frac{\lambda}{m}) = 0 \tag{2.11}$$

With solutions:
$r_1 = 1 + \sqrt{1 - \frac{\lambda}{m}}$
$r_2 = 1 - \sqrt{1 - \frac{\lambda}{m}}$
So, the general solution for $X(x)$ is:

$$X(x) = A_1 r_1^x + A_2 r_2^x \tag{2.12}$$

Where A_1 and A_2 are constants determined by initial conditions.

For $Y(y)$:

$$D\frac{Y(y+1)-Y(y)}{Y(y)} = \lambda - k \tag{2.13}$$

This can be solved by setting $q = 1 + \frac{\lambda-k}{D}$ which gives:

$$Y(y) = C.q^y = C.(1 + \frac{\lambda - k}{D})^y \tag{2.14}$$

Where C is a constant. Finally, the general homogeneous solution is:

$$\boxed{fr^h(x,y) = (A_1 r_1^x + A_2 r_2^x).C(1 + \frac{\lambda - k}{D})^y} \tag{2.15}$$

Particular Solution:

Since the non-homogeneous term is $f_0 \cos(\omega t)$, we assume a particular solution of the form:

$$fr^p(x,y,t) = A(x,y)\cos(\omega t) + B(x,y)\sin(\omega t) \tag{2.16}$$

Substituting $fr^p(x,y,t)$ into the original equation and equating coefficients of $\cos(\omega t)$ and $\sin(\omega t)$, we get two separate equations.

For the $\cos(\omega t)$ terms:

$$m(A(x+1,y)-2A(x,y)+A(x-1,y))+D(A(x,y+1)-A(x,y))+kA(x,y) = f_0 \tag{2.17}$$

For the $\sin(\omega t)$ terms:

$$m(B(x+1,y)-2B(x,y)+B(x-1,y))+D(B(x,y+1)-B(x,y))+kB(x,y) = 0 \tag{2.18}$$

The equation for $B(x,y)$ is identical to the homogeneous equation, so we can write:

$$B(x,y) = B_1 r_1^x + B_2 r_2^x \tag{2.19}$$

For $A(x,y)$, we seek a particular form that matches the forcing term f_0. If $A(x,y)$ is constant, say $A(x,y) = A$, this simplifies to finding a solution for A such that:

$$kA = f_0$$

Hence:

$$A = \frac{f_0}{k}$$

(if k is not 0).

Eventually, the complete solution in the first case is:

$$\boxed{fr(x,y,t) = (A_1 r_1^x + A_2 r_2^x).C(1 + \frac{\lambda - k}{D})^y + \frac{f_0}{k}\cos(\omega t) + (B_1 r_1^x + B_2 r_2^x)\sin(\omega t)} \tag{2.20}$$

In which A_1, A_2, B_1, B_2, and C are constants determined by boundary or initial conditions.

2. **Case 2:**

In this case, the applied force to the system would be in the form of a complex exponential. This forcing term is $f_0 \exp(-j\omega t)$ so, the second case arises when the right-hand side of the equation 2.3 equals $f_0 \exp(-j\omega t)$ so that we should solve the following PDE:

$$\boxed{\frac{m\partial^2 fr(x,y)}{\partial y^2} + D\frac{\partial fr(x,y)}{\partial y} + kfr(x,y) = f_0 \exp(-j\omega t)} \tag{2.21}$$

Where j is the imaginary unit. Substituting the partial derivatives into the original equation gives:

$$m(fr(x{+}1,y){-}2fr(x,y){+}fr(x{-}1,y)){+}D(fr(x,y{+}1){-}fr(x,y)){+}kfr(x,y) = f_0\exp(-j\omega t) \tag{2.22}$$

Solution Structure:

Since the right-hand side is a complex exponential, we can assume a particular solution of the form:

$$fr^p(x,y,t) = A(x,y)\exp(-j\omega t) \tag{2.23}$$

Where $A(x,y)$ is a spatial function that we need to determine. By substituting into the digital-image based harmonic oscillator equation 2.21 we will have the following terms:

$$m\frac{\partial^2 fr^p(x,y,t)}{\partial y^2} = m(A(x+1,y) - 2A(x,y) + A(x-1,y))\exp(-j\omega t) \tag{2.24}$$

The first derivative with respect to y is:

$$D\frac{\partial fr^p(x,y,t)}{\partial y} = D(A(x,y+1) - A(x,y))\exp -j\omega t \tag{2.25}$$

And for the term $kfr^p(x,y,t)$:

$$kfr^p(x,y,t) = kA(x,y)\exp(-j\omega t) \tag{2.26}$$

Substituting these into the original equation 2.21:

$$\begin{aligned} &m(A(x+1,y) - 2A(x,y) + A(x-1,y))\exp(-j\omega t) + D(A(x,y+1) - A(x,y))\exp(-j\omega t) \\ &+ kA(x,y)\exp(-j\omega t) = f_0\exp(-j\omega t) \end{aligned} \tag{2.27}$$

Since $\exp(-j\omega t)$ is non-zero, we can divide through by $\exp(-j\omega t)$ to get:

$$m(A(x+1,y)-2A(x,y)+A(x-1,y))+D(A(x,y+1)-A(x,y))+kA(x,y)=f_0 \tag{2.28}$$

This is now a recurrence relation for $A(x, y)$. The approach to solving this recurrence relation will be similar to that of a homogeneous solution, but now we have a constant forcing term f_0 on the right-hand side. This is a two-dimensional difference equation. To solve it, let's assume constant boundary conditions or that the solution has a steady-state form where $A(x, y)$ does not vary significantly over large value of x and y so $A(x, y) \approx A$ so, we will simplify the equation as follows:

$$m(A-2A+A)+D(A-A)+kA=f_0 \tag{2.29}$$

This reduces to:

$$kA = f_0$$

So, $A = \frac{f_0}{k}$ (provided $k \neq 0$). Consequently, the final expression for the particular solution is:

$$fr^p(x,y,t) = A\exp(-i\omega t) = \frac{f_0}{k}\exp(-j\omega t) \tag{2.30}$$

And the final expression for the full solution $fr(x, y, t) = fr^h(x, y) + fr^p(x, y, t)$ will be:

$$\boxed{fr(x,y,t) = (A_1 r_1^x + A_2 r_2^x)Y(y) + \frac{f_0}{k}\exp(-j\omega t)} \tag{2.31}$$

Term $Y(y)$:

This term corresponds to the solution of the spatial part of the equation, which is governed by the differential operator with respect to y:

$$m\frac{\partial^2 Y(y)}{\partial y^2} + D\frac{\partial Y(y)}{\partial y} + kY(y) = 0 \tag{2.32}$$

We assume a solution of the form $Y(y) = \exp(\gamma y)$ where γ is a constant to be determined. By substituting $Y(y)$ into the differential equation 2.32 we can get:

$$m\gamma^2 \exp(\gamma y) + D\gamma \exp(\gamma y) + k \exp(\gamma y) = 0 \tag{2.33}$$

Which leads to:

$$m\gamma^2 + D\gamma + k = 0 \tag{2.34}$$

And has the following solutions:

$$\gamma_1 = \frac{-D + \sqrt{D^2 - 4mk}}{2m}$$

$$\gamma_2 = \frac{-D - \sqrt{D^2 - 4mk}}{2m}$$

The general solution for $Y(y)$ is a linear combination of the two exponential solutions:

$$Y(y) = C_1 \exp(\gamma_1 y) + C_2 \exp(\gamma_2 y) \tag{2.35}$$

Where C_1 and C_2 are constants determined by boundary or initial conditions.

Finally, the full solution for $fr(x, y, t)$ in this case becomes:

$$\boxed{fr(x, y, t) = (A_1 r_1^x + A_2 r_2^x)(C_1 \exp(\gamma_1 y) + C_2 \exp(\gamma_2 y)) + \frac{f_0}{k} \exp(-j\omega t)} \tag{2.36}$$

In conclusion, we extracted the video frame motion waveform $fr(x, y, t)$ for the two cases which included long and detailed solutions since we had to solve the partial differential equations (PDEs). The resulting $fr(x, y, t)$ will be used in the following chapters to extract the other quantities of the oscillator from and also to measure motion energy or in other words measurement of motion waveform energy $E(t)$ in video frames.

Chapter 3

Video Frame Waveform Case Studies

3.1 Overview

In chapter 2, we introduced the two forms of motion waveform displacements $fr(x, y, t)$ in a video fram fr as a result of the two external forces applied to a harmonic oscillator and we describe two cases separately in chapter 3. In each case, the process of investigating includes extracting a video frame raw motion waveform, injecting Hilbert envelope on the motion waveform, deriving harmonic oscillator waveform $fr(x, y, t)$, and the motion energy for each case.

3.2 First Displacement Model

Displacement:

We use equation 2.4 as our first case using:

$$\boxed{fr(x,y,t) = (A_1 r_1^x + A_2 r_2^x).C(1 + \frac{\lambda - k}{D})^y + \frac{f_0}{k}\cos(\omega t) + (B_1 r_1^x + B_2 r_2^x)\sin(\omega t)} \tag{3.1}$$

General Structure:

Formula 3.1 has the following components:

Spatial Dependence:

• Term $A_1 r_1^x + A_2 r_2^x$ suggests a spatial variation along each video frame x-axis, with r_1 and r_2 representing decay factors or scaling parameters and A_1 and A_2 being constants that adjust amplitude.

• $C(1 + \frac{\lambda - k}{D})^y$ introduces a spatial variation along each video frame y-axis.

Temporal Dependence:

• $\frac{f_0}{k}\cos(\omega t)$ is a harmonic oscillation term, with ω being the angular frequency and f_0 representing a forcing amplitude. This contributes to time-dependent oscillations.

• $(B_1 r_1^x + B_2 r_2^x)\sin(\omega t)$ is another oscillatory component, modified by spatial terms r_1^x and r_2^x, coupled with sine oscillations in time.

Physical Interpretation:

The function 3.1 represents a wave propagating in a $2D$ space with both spatial (x, y) and temporal t variations. The spatial terms modulate the amplitude or decay along the x-axis and y-axis, while the oscillatory components reflect periodic behavior over time. This structure is typical in systems involving vibrations, wave equations, or modulated fields, where the interplay between exponential terms (suggesting growth or decay) and trigonometric terms (suggesting oscillations) is used to model phenomena involving electromagnetic waves in a medium, acoustic or mechanical oscillations, and modulated signals in communication systems.

Waveform Interpretation:

The waveform 3.1 shows spatially modulated oscillations:

• In the x-direction, the oscillations are weighted by $A_1 r_1^x + A_2 r_2^x$.

• In the y-direction, the growth or decay factor $C(1 + \frac{\lambda - k}{D})^y$ influences the amplitude.

• The temporal behavior combines cosine and sine oscillations at frequency ω, modulated spatially by $(B_1 r_1^x + B_2 r_2^x)$.

3.2.1 Waveform and Energy Plots

In this section, we use a biker video to illustrate some plots for our solution for the first case 3.1 including raw motion waveform of the biker, Hilbert envelope, video frame waveform $fr(x, y, t)$, and also a sample energy computation of a biker video frame for specific columns.

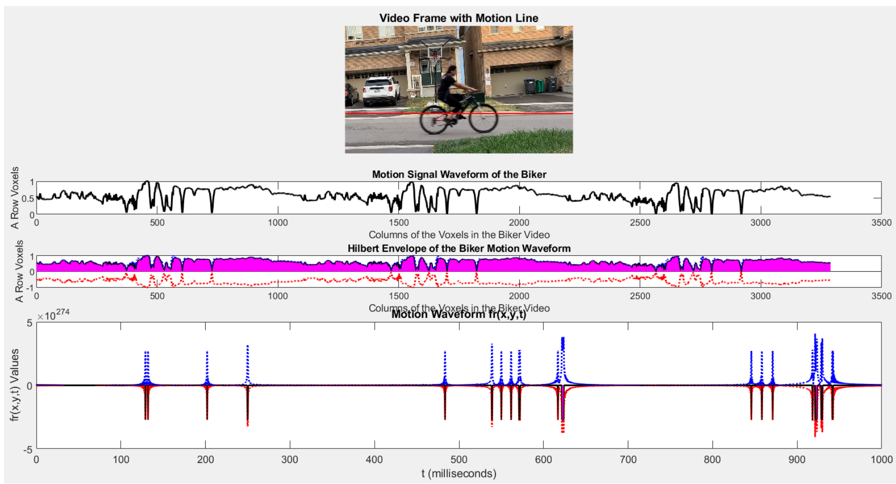

Figure 3.1: The first case plots

As can be seen, figure 3.1 depicts three subplots for a biker. From the top, the biker video frame displays a red line going through a moving part of the object. The first subplot shows the raw motion waveform of the biker that is the trend of the voxels in the sequence of the video frames.

The second subplot is illustrating Hilbert envelope of the motion waveform with area under the waveform in magenta. The third subplot is the video frame motion waveform solution of the first case or in other words it is the plot of 3.1. This function is a combination of spatially modulated terms, exponential decay or growth, and time-dependent oscillations. At any fixed time, the plot will appear as a $2D$ spatial surface over $x$and y, with amplitudes influenced by the spatial modulation terms. As time elapses, the biker surface will oscillate up and down producing a pulsating wave, with the oscillation amplitude depending on spatial position.

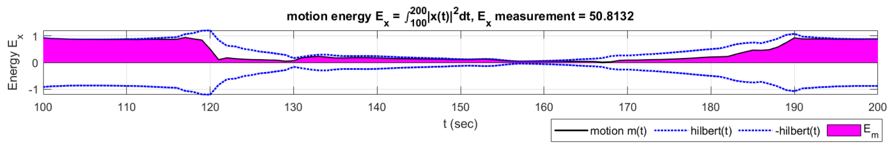

Figure 3.2: Biker energy between columns 100 and 200

Figure 3.2 provides a visualization of the energy E_x of the biker video frame (showed in figure 3.1) between columns 100 and 200. The black solid line (motion $m(t)$) represents the original motion signal $m(t)$ (or $x(t)$, which is modulated and shown as the central waveform. The motion signal fluctuates around the zero line, exhibiting periodic behavior over time. The blue dotted line (Hilbert transform of $m(t)$) represents the Hilbert transform of the signal $m(t)$, which is used to compute the envelope of the signal. The Hilbert transform introduces a phase shift to the signal, helping to capture the amplitude variations.
The other blue dotted line (negative Hilbert transform) shows the negative Hilbert transform, which provides the lower envelope of the signal. It complements the Hilbert transform, creating a pair of curves that define the upper and lower bounds of the modulated signal's amplitude. The magenta shaded region (energy E_x) represents the motion energy E_x, which is computed as the integral of the squared envelope of the signal over the time interval from 100 to 200 seconds:

$$\boxed{E_x = \int_{100}^{200} |x(t)|^2 \, dt}$$

This region illustrates how the energy fluctuates with respect to time as the signal varies. The plot effectively shows how the energy varies as the motion signal oscillates over time. The Hilbert transform helps to visualize how the signal's envelope evolves, and the energy measurement reflects the overall magnitude of the signal's variation within the selected time frame. The same approach depicted in figure 3.3 for the same frame between columns 300 to 400.

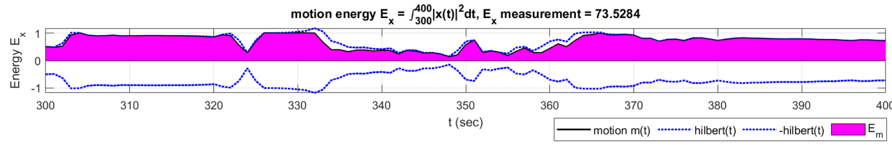

Figure 3.3: Biker energy between columns 300 and 400

Energy Calculation and Interpretation:

• Energy Measurement: The energy is calculated using the square of the signal's envelope area over the specific time interval. The energy value E_x is measured as approximately 50.8132 J for columns 100 to 200 and 73.5284 J for columns 300 to 400.

• The energy peaks and dips correspond to the amplitude maximal variations of the signal, and the magenta area shows where the signal has higher energy (larger fluctuations) and where it is lower (smaller fluctuations).

3.3 Second Displacement Model

Displacement:

We obtained the solution of equation 2.21 as our second case in chapter 2 as the following:

$$\boxed{fr(x,y,t) = (A_1 r_1^x + A_2 r_2^x)(C_1 \exp(\gamma_1 y) + C_2 \exp(\gamma_2 y)) + \frac{f_0}{k}\exp(-j\omega t)} \tag{3.2}$$

General Structure:

Formula 3.2 describes a spatially and temporally varying video frame function $fr(x,y,t)$ with the following components:

• A product of spatially varying terms $(A_1 r_1^x + A_2 r_2^x)$ and $(C_1 \exp(\gamma_1 y) + C_2 \exp(\gamma_2 y))$

• A temporally varying term $\frac{f_0}{k}\exp(-j\omega t)$

This function describes a spatially modulated wave field influenced by both radial and exponential spatial factors, combined with oscillatory temporal behavior.

Spatial Interpretation:

• Radial term $(A_1 r_1^x + A_2 r_2^x)$ defines how the amplitude varies with radial distance, which could correspond to wave decay, amplification, or superposition of multiple radial components. r represents radial distance from some origin in a $2D$ or $3D$ coordinate system. x_1 and x_1 exponents determine how the radial amplitude decays or grows. A_1 and A_2 are amplitude coefficients.

• Exponential term $(C_1 \exp(\gamma_1 y) + C_2 \exp(\gamma_2 y))$ indicates how the wave's amplitude changes exponentially along the y-axis, representing spatial damping or amplification. y represents the position along the vertical axis. γ_1 and γ_2 show growth/decay rates in the y-direction. C_1 and C_2 are coefficients for exponential terms.

Temporal Interpretation:

• Term $\frac{f_0}{k} \exp(-j\omega t)$ describes the oscillatory behavior of the wave as a function of time, characteristic of harmonic motion. The use of $\exp(-j\omega t)$ implies that the field supports a sinusoidal time dependence (when interpreted in the real or imaginary parts).

Physical Context:

The formula represents a spatially and temporally varying system, describing a wave or oscillation. The radial term $(A_1 r_1^x + A_2 r_2^x)$ shows how the amplitude changes with distance r, modeling growth, decay, or superposition of two radial components. The exponential terms $(C_1 \exp(\gamma_1 y) + C_2 \exp(\gamma_2 y))$ describe amplitude variations along the y-axis, suggesting damping or amplification. The temporal term $\frac{f_0}{k} \exp(-j\omega t)$ represents harmonic motion with frequency ω, driven by an external force. Together, the formula captures spatial and temporal dynamics of the system.

Waveform Interpretation:

The waveform in figure 3.4 is spatially modulated, with $(A_1 r_1^x + A_2 r_2^x)$ shaping its amplitude radially, possibly due to interference between two components. The exponential terms $(C_1 \exp(\gamma_1 y) + C_2 \exp(\gamma_2 y))$ introduce damping or amplification in the y-direction, modifying the spatial profile. The temporal term $\exp(-j\omega t)$ adds a sinusoidal oscillation in time, creating a dynamically evolving wave with a modulated amplitude across space.

3.3.1 Waveform Plots

In this part, we will show some plots of our solution for the second case 3.2 including raw motion waveform of the biker, Hilbert envelope, and video frame waveform $fr(x, y, t)$.

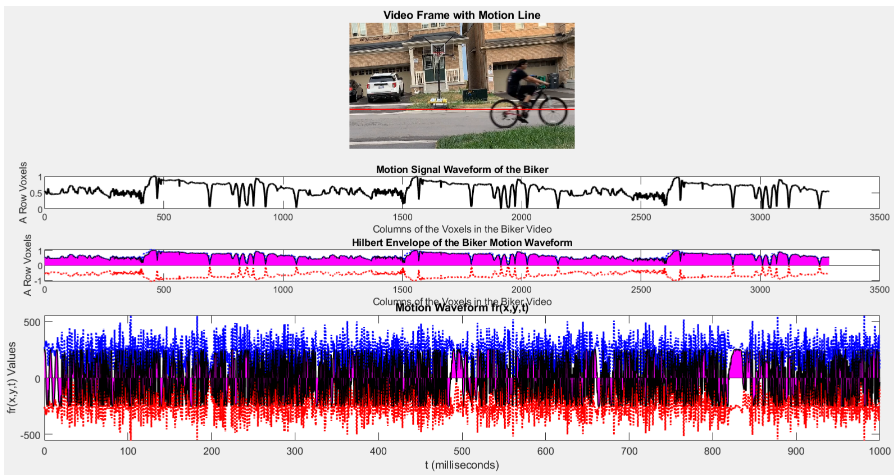

Figure 3.4: The second case plots

In figure 3.4, the top subplot shows a single frame from the biker video, with a red line drawn horizontally across a specific row (row number: 421). This line indicates the row of pixel data being analyzed for video frame waveform extraction. The red line is drawn accurately across the biker, whose motion is being studied. The second subplot on the x-axis shows columns of the video frame (pixel positions along the row in the horizontal direction) and along th y-axis illustrates normalized pixel intensity values from the selected row (values range from 0 to 1). The black waveform in the second plot shows the intensity values of the selected row and peaks and troughs indicate variations in brightness along the row. The motion signal in the second subplot shows distinct variations, caused by the biker's movement and surrounding objects.

In the third subplot x-axis is the same as the second subplot (columns of the video frame) and y-axis shows normalized pixel intensity values. The black line in the third subplot depicts the original motion waveform (identical to the second subplot) and the blue dotted line is the positive envelope of the motion waveform (computed using the Hilbert transform). The red dotted line in the third subplot is the negative envelope of the motion waveform (symmetrical to the positive envelope). The bottom subplot along the x-axis is the time and in the y-axis direction shows the computed values of $fr(x, y, t)$, representing the solution motion waveform 3.2. The black line in the fourth subplot shows the motion waveform (oscillatory behavior), the blue dotted line is the positive envelope of the waveform, the red dotted line is the negative envelope of the waveform, and the magenta shading shows filled regions under the waveform and highlight variations in amplitude.

Chapter 4

Velocity, Acceleration, and Energy of a Biker Video Frame

4.1 Overview

Up to this point, we have focused on extracting the video frame motion waveform $fr(x, y, t)$ under two scenarios involving applied forces. Specifically, the system was subjected to two distinct external forces, and the video frame motion waveform $fr(x, y, t)$ was computed by solving the digital-image-based harmonic oscillator equation. This provided a foundational understanding of how the applied forces influenced the motion patterns captured in the video frames. We can expand this discussion by investigating additional physical quantities associated with the system. In this chapter, we will delve deeper into the analysis by calculating key parameters such as velocity, acceleration, and energy, all derived from the extracted $fr(x, y, t)$ in both cases. These quantities provide deeper insight into the dynamics of the system, offering a more comprehensive understanding of the motion characteristics.

Moreover, energy distribution measurements across different spatial locations of the biker will be examined for each scenario. This analysis will highlight how energy varies not only with respect to time but also spatially, offering a clearer picture of the impact of external forces on the system's behavior. By incorporating these additional analyses, we aim to paint a holistic picture of the dynamics at play, enabling a more robust interpretation.

4.2 Velocity

The first quantity for discussion is velocity. We start with the first case of $fr(x, y, t)$ that is the following equation and we will extract the velocity from it:

$$\boxed{fr(x, y, t) = (A_1 r_1^x + A_2 r_2^x).C(1 + \frac{\lambda - k}{D})^y + \frac{f_0}{k}\cos(\omega t) + (B_1 r_1^x + B_2 r_2^x)\sin(\omega t)} \tag{4.1}$$

To compute the velocity $v(x, y, t)$ from the displacement function 4.1, we need to take the partial derivative of $fr(x, y, t)$ with respect to time t. So, we need differentiate the time-dependent terms with respect to t that are $\frac{f_0}{k}\cos(\omega t)$ and $(B_1 r_1^x + B_2 r_2^x)\sin(\omega t)$. Taking the partial derivative with respect to t leads to:

$$v(x, y, t) = \frac{\partial fr(x, y, t)}{\partial t} = -\frac{f_0}{k}\omega\sin(\omega t) + \omega(B_1 r_1^x + B_2 r_2^x)\cos(\omega t) \tag{4.2}$$

The final expression for the velocity will be:

$$\boxed{v(x, y, t) = -\frac{f_0}{k}\omega\sin(\omega t) + \omega(B_1 r_1^x + B_2 r_2^x)\cos(\omega t)} \tag{4.3}$$

The velocity $v(x, y, t)$ of the system exhibits oscillatory behavior driven by sinusoidal and cosinusoidal components. Its amplitude is influenced by both spatial parameters (r_1^x, r_2^x) and time-independent coefficients (B_1, B_2, $\frac{f_0}{k}$). The term proportional to $-\sin(\omega t)$ represents a phase-shifted contribution from the displacement cosine term, while the $\cos(\omega t)$ component reflects the direct contribution from the sine term in displacement. The angular frequency ω dictates the oscillation speed, and the velocity alternates between positive and negative values, indicating the system's periodic change in motion direction. The interaction of spatial and temporal variables reveals a coupling between the system's geometry and its dynamic oscillatory behavior. Figure 4.1 shows the velocity plot and reveals a smooth, oscillatory curve whose specific amplitude and shape depend on both spatial and temporal factors, illustrating the dynamic interplay of the system's oscillatory properties.

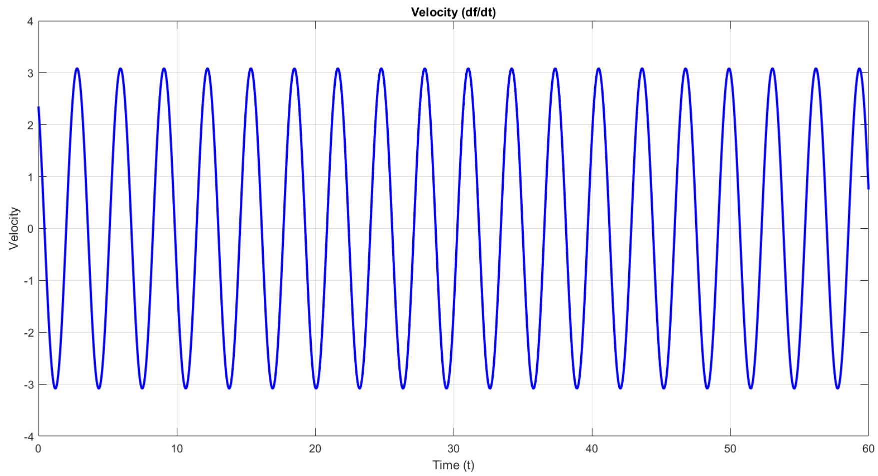

Figure 4.1: Velocity of the first case

The second case of $fr(x, y, t)$ is the following equation and we will extract the velocity from it:

$$\boxed{fr(x, y, t) = (A_1 r_1^x + A_2 r_2^x)(C_1 \exp(\gamma_1 y) + C_2 \exp(\gamma_2 y)) + \frac{f_0}{k} \exp(-j\omega t)} \tag{4.4}$$

The velocity $v(x, y, t)$ is the partial derivative of $fr(x, y, t)$ with respect to time:

$$v(x, y, t) = \frac{\partial fr(x, y, t)}{\partial t} \tag{4.5}$$

The only time-dependent term in 4.4 is $\frac{f_0}{k} \exp(-j\omega t)$ and the other terms are time-independent and treated as constants during differentiation so, the partial derivative would be:

$$v(x, y, t) = \frac{\partial fr(x, y, t)}{\partial t} = -j\omega \frac{f_0}{k} \exp(-j\omega t) \tag{4.6}$$

The final expression for the velocity is:

$$\boxed{v(x,y,t) = -j\omega\frac{f_0}{k}\exp(-j\omega t)} \tag{4.7}$$

The velocity $v(x, y, t)$ exhibits oscillatory behavior with a complex exponential form, indicating harmonic motion with angular frequency ω. Its amplitude, $\omega\frac{f_0}{k}$, is constant and determined by the system's parameters. The velocity is out of phase with the displacement by $\frac{\pi}{2}$, reflecting the relationship between position and velocity in oscillatory systems. In real terms, the velocity alternates sinusoidally between positive and negative values, crossing zero when the displacement reaches its extrema. This periodic motion highlights the continuous energy exchange between kinetic and potential forms.

Figure 4.2 depicts the velocity 4.7 that is a sinusoidal waveform oscillating over time. The figure is a smooth, periodic sine wave that oscillates symmetrically about zero, corresponding to the harmonic nature of the velocity. The maximum and minimum values of the velocity are determined by $\omega\frac{f_0}{k}$, representing the strength of oscillations. The amplitude remains constant over time. The oscillation frequency is determined by the angular frequency ω, dictating how rapidly the velocity cycles through positive and negative values. The velocity is shifted by $\frac{\pi}{2}$ relative to the displacement, meaning the velocity crosses zero when the displacement is at its maximum or minimum. It crosses zero twice during each oscillation cycle, indicating the moments when the system momentarily changes direction. Overall, the plot reflects the oscillatory dynamics of the system, with regular and predictable variations in the velocity over time.

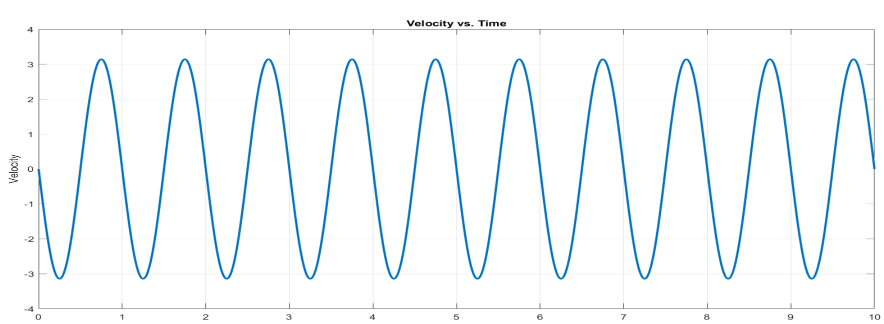

Figure 4.2: Velocity of the second case

4.3 Acceleration

The second quantity to compute is acceleration. We start with the first case of $fr(x, y, t)$ that is the equation 4.1 and we will extract the acceleration. To compute the acceleration, we take the partial derivative of the velocity 4.3 with respect to time, t.

Differentiating term by term:

$$\frac{\partial}{\partial t}(-\frac{f_0}{k}\omega \sin(\omega t)) = -\frac{f_0}{k}\omega^2 \cos(\omega t) \tag{4.8}$$

$$\frac{\partial}{\partial t}(\omega(B_1 r_1^x + B_2 r_2^x)\cos(\omega t)) = -\omega^2(B_1 r_1^x + B_2 r_2^x)\sin(\omega t) \tag{4.9}$$

By combining the terms:

$$a(x, y, t) = \frac{\partial v(x, y, t)}{\partial t} = -\frac{f_0}{k}\omega^2 \cos(\omega t) - \omega^2(B_1 r_1^x + B_2 r_2^x)\sin(\omega t) \tag{4.10}$$

Consequently, the final simplified acceleration would be:

$$\boxed{a(x, y, t) = -\omega^2(\frac{f_0}{k}\cos(\omega t) + (B_1 r_1^x + B_2 r_2^x)\sin(\omega t))} \tag{4.11}$$

The acceleration 4.11 is a combination of sinusoidal and cosinusoidal components, governed by the angular frequency ω and scaled by ω^2. It reflects the influence of two oscillatory behaviors, one proportional to $\cos(\omega t)$, scaled by $\frac{f_0}{k}$, and another proportional to $\sin(\omega t)$, modulated by the spatial factors $B_1 r_1^x + B_2 r_2^x$. The negative sign indicates that the acceleration opposes the direction of motion, characteristic of harmonic oscillatory systems, with its magnitude and phase depending on both temporal and spatial factors.

Figure 4.3 shows the acceleration 4.11. It displays a periodic waveform resulting from the combination of sine and cosine terms. Its shape depends on the relative amplitudes of the components $\frac{f_0}{k}$ and $B_1 r_1^x + B_2 r_2^x$. The cosine term introduces peaks aligned with $\cos(\omega t)$, while the sine term modulates the waveform's troughs and peaks based on spatial contributions. Overall, the acceleration plot would oscillate with the angular frequency ω, and its amplitude would vary in space due to the contributions from r_1^x and r_2^x. The waveform might be asymmetric if the sine and cosine terms have significantly different weights, but it remains smooth and continuous, characteristic of harmonic oscillations.

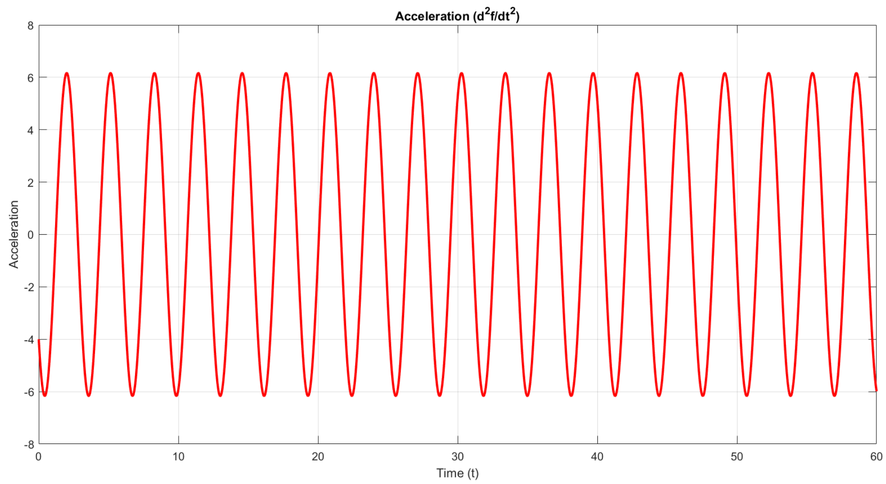

Figure 4.3: Acceleration of the first case

Now, we will compute the acceleration in the second case of $fr(x, y, t)$ (4.4) that is the partial derivative of 4.7 and we will extract the acceleration from it. To compute the acceleration $a(x, y, t)$ from the velocity, we differentiate $v(x, y, t)$ with respect to time t:

$$\frac{\partial}{\partial t}(\exp(-j\omega t)) = -j\omega \exp(-j\omega t) \tag{4.12}$$

Thus, the final acceleration is:

$$\boxed{a(x, y, t) = -\omega^2 \frac{f_0}{k} \exp(-j\omega t)} \tag{4.13}$$

The acceleration is a complex-valued oscillatory function characterized by an exponential dependence on $-j\omega t$, which combines oscillation in both magnitude and phase. The factor $-\omega^2 \frac{f_0}{k}$ scales the amplitude of the oscillations, with ω^2 indicating a quadratic dependence on angular frequency. The negative sign reflects a phase inversion relative to the driving force. The real and imaginary parts of the acceleration describe sinusoidal variations, highlighting harmonic motion with periodicity determined by ω. This formulation is particularly useful in analyzing systems with sinusoidal driving forces in the context of complex phasors. Figure 4.4 shows the acceleration 4.13. It illustrates a sinusoidal oscillation if its real or imaginary part is taken separately. The real part corresponds to $-\omega^2 \frac{f_0}{k} \cos(\omega t)$, while the imaginary part corresponds to $-\omega^2 \frac{f_0}{k} \sin(\omega t)$. These components oscillate with a frequency ω and an amplitude scaled by $\omega^2 \frac{f_0}{k}$. If both real and imaginary parts are plotted simultaneously, the result would be a circular or elliptical trajectory in the complex plane, showcasing the harmonic oscillatory nature of the system. The periodic nature of the acceleration is evident, with peaks and troughs occurring at intervals of $\frac{2\pi}{\omega}$.

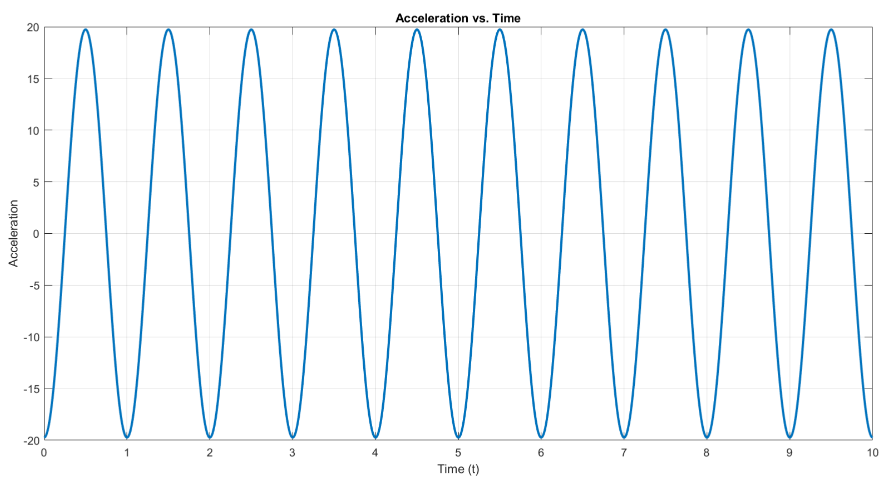

Figure 4.4: Acceleration of the second case

4.4 Energy

In this section, we discuss the energy in the first case 4.1 and the second case 4.4 of the video frame waveform. The expressions for the two cases will be extracted and we also provide the measurements of the energy for different locations of the biker video frame 2.1.

To compute the energy, we need to calculate the kinetic energy (K.E.) and the potential energy (P.E.) in both cases. We start with the first case. In the first case, K.E. would be:

$$K.E. = \frac{1}{2}m\left(\frac{\partial fr(x,y,t)}{\partial t}\right)^2 \tag{4.14}$$

We already have the expression for $\frac{\partial fr(x,y,t)}{\partial t}$ that is the velocity in the first case:

$$\frac{\partial fr(x,y,t)}{\partial t} = -\frac{f_0}{k}\omega\sin(\omega t) + \omega(B_1r_1^x + B_2r_2^x)\cos(\omega t) \tag{4.15}$$

So, K.E. in the first case will be:

$$K.E. = \frac{1}{2}m\left(-\frac{f_0}{k}\omega\sin(\omega t) + \omega(B_1r_1^x + B_2r_2^x)\cos(\omega t)\right)^2 \tag{4.16}$$

The potential energy in an oscillatory system is typically given by:

$$P.E. = \frac{1}{2}k(fr(x,y,t))^2 \tag{4.17}$$

Which in the first case P.E. is:

$$P.E. = \frac{1}{2}k\left((A_1r_1^x + A_2r_2^x).C(1+\frac{\lambda-k}{D})^y + \frac{f_0}{k}\cos(\omega t) + (B_1r_1^x + B_2r_2^x)\sin(\omega t)\right)^2 \tag{4.18}$$

The total energy of the system is the sum of the K.E. and P.E. that is as the following in the first case:

$$\boxed{\begin{aligned} E(t) &= \frac{1}{2}m\left(-\frac{f_0}{k}\omega\sin(\omega t) + \omega(B_1 r_1^x + B_2 r_2^x)\cos(\omega t)\right)^2 \\ &+ \frac{1}{2}k\Bigg[(A_1 r_1^x + A_2 r_2^x)\cdot C\left(1 + \frac{\lambda - k}{D}\right)^y \\ &\quad + \frac{f_0}{k}\cos(\omega t) + (B_1 r_1^x + B_2 r_2^x)\sin(\omega t)\Bigg]^2 \end{aligned}} \tag{4.19}$$

This formula gives the total energy of the system as a function of time. It represents a combination of kinetic and potential energy terms in a dynamic system influenced by oscillatory components and spatial interactions. The first term corresponds to the kinetic energy, characterized by a squared velocity term that involves sinusoidal and cosine oscillations dependent on time t and angular frequency ω. It incorporates contributions from both a driving force f_0 and spatial coefficients B_1, B_2, r_1^x, and r_2^x. The second term represents the potential energy, encompassing complex spatial and temporal dependencies, including a correction factor $C\left(1 + \frac{\lambda-k}{D}\right)^y$ and harmonic oscillations involving $\cos(\omega t)$ and $\sin(\omega t)$. This energy formulation highlights intricate interactions between temporal oscillations, spatial contributions, and nonlinear corrections, reflecting a system with coupled mechanical and spatial dynamics.

Figure 4.5 illustrates energy 4.19.

The energy plot 4.5 of $E(t)$ exhibits a periodic pattern influenced by the oscillatory terms $\sin(\omega t)$ and $\cos(\omega t)$. The kinetic energy term contributes oscillations dominated by the angular frequency ω, with amplitude modulated by the driving force f_0 and spatial coefficients B_1, B_2, r_1^x, and r_2^x. Similarly, the potential energy term introduces periodic variations, incorporating a combination of spatial corrections $C\left(1 + \frac{\lambda-k}{D}\right)^y$ and harmonic time-dependent contributions. The interaction of these terms could produce a composite waveform, with regions of constructive and destructive interference, leading to varying amplitude envelopes. The plot would likely show periodic energy peaks and troughs, with

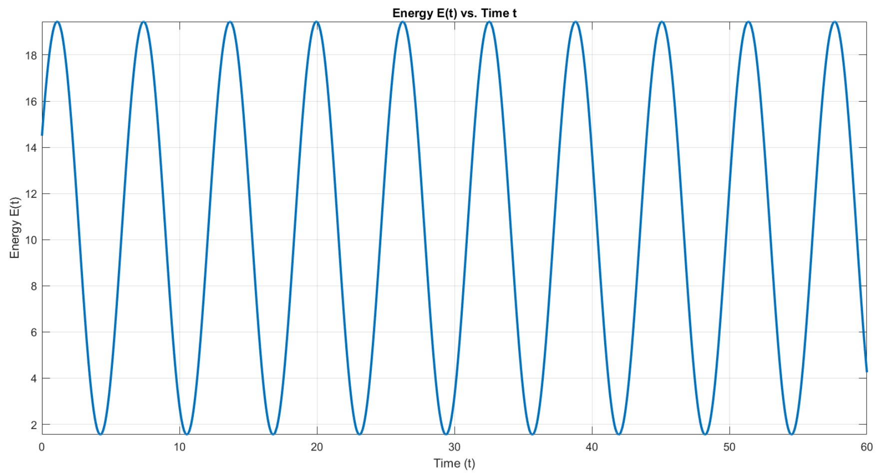

Figure 4.5: Energy of the first case

overall characteristics reflecting the interplay between spatial factors and temporal oscillations. Table 4.1 shows the energy measurement in the first case for different locations of the biker.By different locations, we mean different columns of the biker video frame that are the moving parts of the frame.

Columns	**Energy Measurements** $E(t)$
$400 - 450$	911.0423
$450 - 500$	866.3905
$500 - 550$	747.2625
$550 - 600$	829.2567
$600 - 650$	856.852
$650 - 700$	923.433
$700 - 750$	910.6817

Table 4.1: Energy measurements of the biker in the first case

The fluctuations in energy suggest a varying distribution of forces across the image. Areas with higher energy likely represent important features of the biker or bike, such as edges, textures, or regions with high contrast. The drop in energy between **500 and 550 columns** suggests that certain parts of the image may be less detailed or more uniform. The overall trend of higher energy towards the edges could indicate the presence of significant structural or texture features, such as the outline of the biker, bike components, or reflections that are more pronounced in certain parts of the image.

Now, we will compute the energy expression for the second case 4.4. To compute the energy, we use the following formula that is the sum of k.E. and P.E.:

$$E(t) = \frac{1}{2}m\left(\frac{\partial fr(x,y,t)}{\partial t}\right)^2 + \frac{1}{2}k(fr(x,y,t))^2 \tag{4.20}$$

So that:

$$\frac{\partial fr(x,y,t)}{\partial t} = -j\omega\frac{f_0}{k}\exp(-j\omega t) \tag{4.21}$$

As a result, the kinetic energy is:

$$K.E. = \frac{1}{2}m\left(-j\omega\frac{f_0}{k}\exp(-j\omega t)\right)^2 \tag{4.22}$$

By simplifying:

$$K.E. = \frac{1}{2}m\left(\omega\frac{f_0}{k}\right)^2 \tag{4.23}$$

And the potential energy is:

$$P.E. = \frac{1}{2}k\left((A_1r_1^x + A_2r_2^x)(C_1\exp(\gamma_1 y) + C_2\exp(\gamma_2 y)) + \frac{f_0}{k}\exp(-j\omega t)\right)^2 \tag{4.24}$$

Finally, the expression for the total energy $E(t)$ in the second case will be:

$$E(t) = \frac{1}{2}m\left(\omega\frac{f_0}{k}\right)^2 + \frac{1}{2}k\left((A_1r_1^x + A_2r_2^x)(C_1\exp(\gamma_1 y) + C_2\exp(\gamma_2 y)) + \frac{f_0}{k}\exp(-j\omega t)\right)^2 \tag{4.25}$$

The energy 4.25 consists of two main components. A time-independent kinetic energy term and a time- and space-dependent potential energy term. The kinetic energy is determined by the oscillation frequency ω, the driving force amplitude f_0, and the stiffness constant k. It remains constant over time. The potential energy, on the other hand, is more complex, involving spatial coefficients A_1, A_2, r_1^x, and r_2^x, and exponential terms $C_1 \exp(\gamma_1 y) + C_2 \exp(\gamma_2 y)$ that account for spatial variation along the y-axis. Additionally, it includes a harmonic time-dependent term $\frac{f_0}{k} \exp(-j\omega t)$, introducing oscillatory behavior. Together, these components reflect the interplay of spatial structure, temporal dynamics, and system parameters.

Figure 4.6 depicts the energy 4.25. It exhibits a combination of constant and oscillatory behaviors. The first term, representing the kinetic energy, remains constant and contributes a fixed baseline to the plot. The second term, the potential energy, introduces time-varying oscillations due to the harmonic dependence on $\exp(-j\omega t)$, These oscillations are modulated by spatial coefficients A_1, A_2, r_1^x, r_2^x and exponential factors $C_1 \exp(\gamma_1 y)$ and $C_2 \exp(\gamma_2 y)$, which add layers of complexity. The resulting plot likely displays periodic peaks and troughs superimposed on the constant baseline, with the oscillation amplitude influenced by the spatial and exponential terms. Overall, the energy plot reflects the dynamic interaction of harmonic motion and spatial modulation.

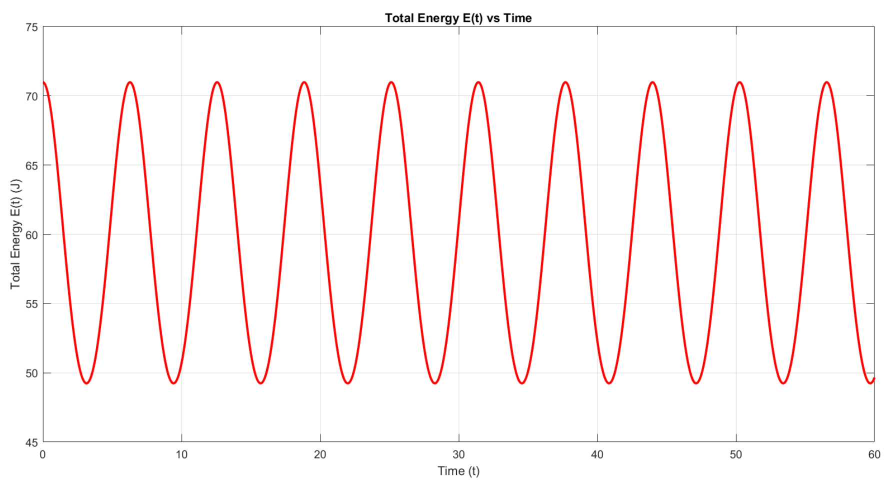

Figure 4.6: Energy of the second case

Table 4.1 shows the energy measurement in the first case for different locations of the biker.By different locations, we mean different columns of the biker video frame that are the moving parts of the frame.

Columns	**Energy Measurements $E(t)$ (absolute value)**
$400 - 450$	684.0142
$450 - 500$	695.949
$500 - 550$	729.5593
$550 - 600$	719.5914
$600 - 650$	709.688
$650 - 700$	676.7656
$700 - 750$	693.6365

Table 4.2: Energy measurements of the biker in the second case

The variations in energy measurements highlight regions in the biker image with different levels of detail or contrast. Higher energy values indicate areas with significant visual features, such as edges, textures, or prominent contrasts, while lower energy values likely correspond to areas with less detail, such as smoother or less textured regions. Higher energy regions (e.g., 500 to 550 columns) are likely related to areas of the biker or bike that are more visually significant, with sharper edges or higher contrast. Lower energy regions (e.g., 650 to 700 columns) might correspond to parts of the background or areas with less pronounced details or less variation in pixel intensity.

Overall, these energy values give insight into the distribution of visual information across the image, helping to highlight important features, textures, or shapes that contribute to the overall image of the biker.

Chapter 5

Conclusion, Recommendations, and Future Work

5.1 Conclusions

In conclusion, this dissertation demonstrates the application of harmonic oscillator theory to the video frames, particularly focusing on human motion. By exploring extraction of displacement from the digital-image-based harmonic oscillator equation 2.3, the research offers innovative insights into dynamic system behaviors, with potential applications in fields such as biomechanics and robotics. Additionally, the detailed examination of the other quantities including velocity, acceleration, and energy of the oscillators contributes to a broader understanding of oscillatory motion and its diverse practical uses. This work not only bridges theoretical concepts with real-world applications but also paves the way for future studies in video frame waveform and harmonic analysis.

Chapter 2 delves into the exploration of two distinct forms of displacement that arise from solving the partial differential equations governing the behavior of a harmonic oscillator. Each case represents a unique response of the system under different applied forces, showcasing the versatility and complexity of oscillatory motion. These displacements play a pivotal role in later analyses, as they serve as foundational components for extracting other critical quantities, such as velocity, acceleration, and energy, which describe the motion waveform of video frames. Specifically, Chapter 2 introduces a system subject to a periodic force as the first case and another system responding to a complex exponential force as the second case. The focus of this chapter lies in methodically solving the PDEs and deriving the video frame waveform function $fr(x, y, t)$, which serves as the starting point for further analysis in subsequent chapters. Establishing this function is essential, as it provides a mathematical description of the motion within the video frames, a crucial first step in understanding and interpreting the dynamic behaviors captured in the video.

Chapter 3 shifts the focus to the analysis and interpretation of the video frame waveform $fr(x, y, t)$ through detailed case studies. These studies examine the structural and physical properties of the waveform derived in Chapter 2, with particular emphasis on the first and second cases of displacement. A key component of this chapter involves presenting and analyzing plots that visualize the video frame motion waveform and the corresponding processing steps required to refine the data. The processing pipeline begins with the extraction of the raw motion signal from the video frames, which is then subjected to a Hilbert envelope to enhance its interpretability. Following this, the processed waveform for the biker video frame motion $fr(x, y, t)$ is plotted and carefully illustrated. This chapter provides comprehensive insights into the features of these plots, emphasizing the underlying physics and waveform behaviors that characterize the video frame motion.

Additionally, Chapter 3 includes a detailed measurement of the energy associated with the biker video frame. Energy computations are carried out for specific columns in the frame, focusing on the ranges of columns 100-200 and 300-400 as representative samples. These measurements serve to bridge the theoretical framework developed in earlier sections with practical, quantifiable observations, highlighting the connection between the displacements and their energy representations.

Chapter 4 continues the discussion by expanding on the computation of critical quantities associated with the video frame motion, including velocity, acceleration, and energy. The video frame waveform $fr(x, y, t)$, derived in Chapter 2, serves as the foundation for these computations. By applying differentiation to $fr(x, y, t)$, we systematically calculate the velocity and acceleration for both the periodic and complex exponential force cases. This approach not only reinforces the theoretical framework but also demonstrates its application to real-world motion analysis.

Energy, being a key quantity that combines both kinetic and potential components, receives particular attention in Chapter 4. For the biker video frame, energy measurements are carried out exclusively for the moving parts of the frame, spanning columns 400 to 750. This range corresponds to the dynamically active portions of the video frame, where motion is most pronounced. The analysis includes a thorough examination of energy fluctuations within this range, providing a detailed interpretation of how these variations correspond to the physical behavior of the system. By tying these fluctuations back to the underlying displacements and applied forces, this chapter offers a comprehensive understanding of the energy dynamics within the video frame motion.

5.2 Recommendations and Future Work

This thesis provides a comprehensive framework for analyzing the motion of video frames using harmonic oscillator principles, with a particular focus on displacement, velocity, acceleration, and energy. Based on the findings and limitations encountered in this work, the following recommendations and avenues for future research are proposed.

5.2.1 Recommendations

• Improved Waveform Analysis Techniques:

While the motion waveform was successfully processed using the Hilbert envelope, exploring advanced signal processing methods, such as wavelet transforms or adaptive filtering, could provide further insights into subtle variations in motion.

• Enhanced Visualization:

The plots presented in this work effectively conveyed key motion characteristics. However, employing advanced visualization tools, such as interactive 3D plots or motion heatmaps, could improve the interpretability and accessibility of the results.

• Precision in Energy Calculations:

While energy fluctuations were analyzed within a specified range of columns, increasing the spatial resolution or incorporating more precise force-displacement models could yield more accurate energy profiles, particularly for highly dynamic regions.

5.2.2 Future Work

• Application to Multidimensional Motion:

Expanding the framework to account for three-dimensional motion would make it applicable to a broader range of real-world scenarios.

• Integration with Machine Learning:

Leveraging machine learning models to analyze the extracted video frame motion waveform could enable automated identification of patterns and anomalies, providing deeper insights into complex motion behaviors.

• Real-Time Analysis:

Developing methods to apply the derived framework in real-time video analysis could make it highly relevant for applications such as motion detection, video surveillance, and biomechanics.

• Inclusion of Environmental Factors:

Incorporating environmental variables, such as friction or air resistance, into the model could further enhance its accuracy and applicability to real-world systems.

• Exploration of Energy Transfer Mechanisms:

The energy calculations focused on kinetic and potential energy components. Investigating energy dissipation or transfer mechanisms, such as damping effects, could provide a more holistic understanding of the system's dynamics.

These recommendations and future research directions aim to build upon the contributions of this thesis, addressing its limitations and broadening its scope to enhance its scientific and practical relevance.

Appendix A

Appendix-A Main Script (MATLAB)

A.1 MATLAB Main Script for The First Case

Following section provides a sample script of producing figure 3.1 in MATLAB for the first case..

```
clc, clear, close all;

% Create a VideoReader object
v = VideoReader('biker.mp4');

% Define the number of frames to process
NumFrames = 1:53;

% Define the row number to be selected
row_number = 421;

% Create a figure for all subplots, including the video
    display
figure;
```

```
% Loop through the specified frames
for i = NumFrames
    % Clear the figure at the start of each iteration
    clf;

    % Video Frame Display - allocate the top 2 out of 6 rows for video frame display
    subplot(6,1,1:2); % First 2 rows for the video frame
    frame = read(v, i); % Read the current frame
    imshow(frame); % Display the frame
    title('Video Frame with Motion Line');
    hold on;
    % Draw a red line across the specified row
    line([1, size(frame, 2)], [row_number, row_number], 'Color', 'red', 'LineWidth', 1);
    hold off;

    % Convert frame to double precision
    img = im2double(frame);

    % Select the specified row
    x0 = img(row_number, :);

    % Subplot 1 - Non-modulated Motion Signal
    subplot(6,1,3); % Third row for non-modulated motion signal
    plot(1:length(x0), x0, 'k', 'linewidth', 1.5);
```

```
xlabel('Columns of the Voxels in the Biker Video');
ylabel('A Row Voxels');
title('Motion Signal Waveform of the Biker');

% Subplot 2 - Hilbert Envelope and Energy
subplot(6,1,4); % Fourth row for Hilbert envelope and
    energy
env0 = abs(hilbert(x0));
plot(1:length(x0), x0, 'k', 'linewidth', 1.5); hold on;
plot(1:length(x0), env0, 'b:', 'linewidth', 1.5);
plot(1:length(x0), -env0, 'r:', 'linewidth', 1.5);
area(1:length(x0), x0, 'FaceColor', 'm'); hold off;
xlabel('Columns of the Voxels in the Biker Video');
ylabel('A Row Voxels');
title('Hilbert Envelope of the Biker Motion Waveform');

% Subplot 3 - Modulated Motion Waveform with Harmonic
    Displacement
[fr_vals, t] = pde_biker_compute_fr_17_21(img,
    row_number); % Call the function to compute fr and
    time vector
env = abs(hilbert(fr_vals));

subplot(6,1,5:6); % Last two rows for the final plot
plot(t, fr_vals, 'k', 'linewidth', 1.5); hold on;
plot(t, env, 'b:', 'linewidth', 1.5);
plot(t, -env, 'r:', 'linewidth', 1.5);
area(t, fr_vals, 'FaceColor', 'm'); hold off;
```

```
    xlabel('t (milliseconds)');
    ylabel('fr(x,y,t) Values');
    title('Motion Waveform fr(x,y,t)');
    % Pause to allow viewing each frame and plot update
    pause(0.1);

end
```

Appendix B

Appendix-B Function Script (MATLAB)

B.1 MATLAB Function Script for The First Case

Following section provides a sample of function script for the first case $fr(x, y, t)$ made in MATLAB.

```
function [fr_val, t] = pde_biker_compute_fr_17_21(img, row_number)
    % Select the specified row
    selected_row = img(row_number,:);

    % Normalize selected_row
    selected_row = double(selected_row);  % Convert to double for calculations
    selected_row = (selected_row - min(selected_row)) / (max(selected_row) - min(selected_row)); % Normalize to [0, 1]
    selected_row = 2 * selected_row - 1;  % Normalize to [-1, 1]

```

```
% Constants
m = 70000; % mass
f0 = 255;
k = 1;
A1 = 1; A2 = 1;
B1 = 1; B2 = 1;
C = 1;

% Frequency vector
lambda = abs(atan(2 * selected_row - 1)); % Ensure lambda is non-negative and scaled appropriately

% Calculate r1 and r2
r1 = 1 + sqrt(max(0, 1 - lambda / m)); % Avoid complex values
r2 = 1 - sqrt(max(0, 1 - lambda / m));

% Set up the time vector t
t = linspace(0, 1000, length(selected_row)); % Spread 1000 ms over the length of selected_row

% Calculate fr(x, y, t)
fr_val = (A1 * r1.^selected_row + A2 * r2.^selected_row) * C .* (1 + (lambda - k) ./ mean(selected_row)).^421 ...
        + (f0 / k) * cos(lambda .* t) ...
        + (B1 * r1.^selected_row + B2 * r2.^selected_row) .* sin(lambda .* t);
```

```
end
```

Appendix C

Appendix-C Main Script (MATLAB)

C.1 MATLAB Main Script for The Second Case

Following section provides a sample script of producing figure 3.4 in MATLAB for the second case.

```
clc, clear, close all;

% Create a VideoReader object
v = VideoReader('biker.mp4');

% Define the number of frames to process
NumFrames = 1:53;

% Define the row number to be selected
row_number = 421;

% Create a figure for all subplots, including the video
    display
figure;
```

```
% Loop through the specified frames
for i = NumFrames
    % Clear the figure at the start of each iteration
    clf;

    % Video Frame Display - allocate the top 2 out of 6 rows
        for video frame display
    subplot(6,1,1:2); % First 2 rows for the video frame
    frame = read(v, i); % Read the current frame
    imshow(frame); % Display the frame
    title('Video Frame with Motion Line');
    hold on;
    % Draw a red line across the specified row
    line([1, size(frame, 2)], [row_number, row_number], '
       Color', 'red', 'LineWidth', 1);
    hold off;

    % Convert frame to double precision
    img = im2double(frame);

    % Select the specified row
    x0 = img(row_number, :);

    % Subplot 1 - Non-modulated Motion Signal
    subplot(6,1,3); % Third row for non-modulated motion
       signal
    plot(1:length(x0), x0, 'k', 'linewidth', 1.5);
```

```
    xlabel('Columns of the Voxels in the Biker Video');
    ylabel('A Row Voxels');
    title('Motion Signal Waveform of the Biker');

    % Subplot 2 - Hilbert Envelope and Energy
    subplot(6,1,4); % Fourth row for Hilbert envelope and energy
    env0 = abs(hilbert(x0));
    plot(1:length(x0), x0, 'k', 'linewidth', 1.5); hold on;
    plot(1:length(x0), env0, 'b:', 'linewidth', 1.5);
    plot(1:length(x0), -env0, 'r:', 'linewidth', 1.5);
    area(1:length(x0), x0, 'FaceColor', 'm'); hold off;
    xlabel('Columns of the Voxels in the Biker Video');
    ylabel('A Row Voxels');
    title('Hilbert Envelope of the Biker Motion Waveform');

    % Subplot 3 - Modulated Motion Waveform with Harmonic Displacement
    fr_vals = pde_biker_compute_fr_17_22(img, row_number); % Call the function to compute fr
    env = abs(hilbert(fr_vals));
    t = linspace(0, 1000, length(fr_vals)); % Spread 1000 ms over the length of fr_vals

    subplot(6,1,5:6); % Last two rows for the final plot
    plot(t, fr_vals, 'k', 'linewidth', 1.5); hold on;
    plot(t, env, 'b:', 'linewidth', 1.5);
    plot(t, -env, 'r:', 'linewidth', 1.5);
```

```
    area(t, fr_vals, 'FaceColor', 'm'); hold off;
    xlabel('t (milliseconds)');
    ylabel('fr(x,y,t) Values');
    title('Motion Waveform fr(x,y,t)');

    % Pause to allow viewing each frame and plot update
    pause(0.1);
end
```

Appendix D

Appendix-D Function Script (MATLAB)

D.1 MATLAB Function Script for The Second Case

Following section provides a sample of function script for the second case $fr(x, y, t)$ made in MATLAB.

```
function fr_val = pde_biker_compute_fr_17_22(img, row_number)
    % Select the specified row
    selected_row = img(row_number,:);

    % Normalize selected_row
    selected_row = double(selected_row);  % Convert to double for calculations
    selected_row = (selected_row - min(selected_row)) / (max(selected_row) - min(selected_row)); % Normalize to [0, 1]
    selected_row = 2 * selected_row - 1;  % Normalize to [-1, 1]

```

```
    % Constants
    m = 70000; % mass
    f0 = 255;
    k = 1;
    D=mean(selected_row);
    A1 = 1; A2 = 1;
    C1 = 1; C2 = 1;
    gamma1=(-D+sqrt(D^2-4*m*k))/2*m;
    gamma2=(-D-sqrt(D^2-4*m*k))/2*m;

    % Frequency vector
    lambda = atan(2 * selected_row - 1); % angular frequency
        , lambda = w

    % Calculate r1 and r2
    r1 = 1 + sqrt(1 - lambda / m);
    r2 = 1 - sqrt(1 - lambda / m);

    % Set up the time vector t
    t = linspace(0, 1000, length(selected_row)); % Spread
        1000 ms over the length of selected_row

    % Calculate fr(x, y, t) as per the given formula
    fr_val = (A1 * r1.^selected_row + A2 * r2.^selected_row)
         * C1*(exp(gamma1*421))+C2*(exp(gamma2*421)) ...
            + (f0 / k) * exp(-1i*lambda.*t);

end
```

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