On the Choice of the Number of Classes in the Chi-Squared Goodness-of-Fit Test

by

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A Thesis submitted to the Faculty of Graduate Studies of

The University of Manitoba

in partial fulfilment of the requirements of the degree of

MASTER OF SCIENCE

Department of Statistics

University of Manitoba

Winnipeg, Manitoba

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES

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MASTER OF SCIENCE

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Abstract

When applying Chi-squared goodness-of-fit test on a continuous distribution, one controversial question is how to group the data and then how many classes are used. In this thesis, we are investigating the most appropriate number of classes under the grouping method of "equiprobability intervals" by computer intensive Monte Carlo simulation. For continuous distribution with location and scale parameters, we examined the patterns of the power of test against the various numbers of classes when testing null hypothesis against a general alternative hypothesis in two cases, either on a specified distribution with known parameters, or on a family of distributions with unknown parameters. Finally we concluded the most appropriate number of classes as expressed in empirical formulae for the above two cases respectively. Even if a continuous distribution is not from the distribution family with location and scale parameters, the empirical formulae are applicable.

Key words: chi-squared test, goodness-of-fit, Monte Carlo simulation.

Acknowledgements

It was 18 years ago, when I entered the university to start my bachelor's program on applied statistics in China, the first university in Canada which I can speak out the name was the University of Manitoba. It is from a poster of my school promoting international academic exchange program. At that time I never thought I would have the opportunity to study for a Master's Degree in Statistics at the University of Manitoba. But I am happy I got a chance to study here.

Compared with my fellow classmates, I usually do not have a competitive exam performance. So I am very appreciative that Dr. Jin Zhang accepted me as a student of his. Under Dr. Zhang's guidance and encouragement, I selected the topic and accomplished my simulation based study. The most important thing I learned from Dr. Zhang is the enthusiastic attitude towards research, which includes the right approach combined with persistence and patience. This will benefit me forever. I also express my sincere thanks to my advisory committee member Dr. James Fu from the Department of Statistics, and external advisory committee member Dr. Brian Fristensky from the Department of Plant Science, for their valuable advices on my thesis improvement.

My mother gives me spiritual encouragement, so that I can complete my study and thesis in two years. From my heart, I want to say to my dear late father, I have learned to recognize my weaknesses and made progressive efforts in correcting them.

I am lucky to have met Hsing-Ming Chang at the first day I came to the Department of Statistics, since then we have become best friends during the past two years. Hsing-Ming gave me help and encouragement in my thesis writing, and even helped me do lots of proof-reading.

I am appreciative of the Faculty of Science as well as our department offering me a studentship, so that I can concentrate my time and energy to complete my study and thesis in two years. Also, special thanks to Ms. Margaret Smith, she is very warm-hearted and offered a plenty of help for my registration and other personal inquiries during my studies here at the University of Manitoba.

Finally, I would like to sincerely thank all the professors who taught me in the past two years as well as my fellow classmates, with whom I spent countless hours of stressful learning but most importantly, it has been a very pleasant time.

Chapter 1

Introduction

1.1 Chi-Squared Goodness-of-Fit Test

The chi-square test, introduced by Karl Pearson in 1900, is the oldest and best known goodness-of-fit test. The idea is to reduce the goodness-of-fit problem to a multinomial setting by comparing the observed cell counts with their expected values under the null hypothesis. The most advantage of chi-squared test is that it can be applied to any type of variable: continuous, discrete, or a combination of these, and even can be extended to multivariate cases. Another advantage of chi-squared type test is that the asymptotic distribution of the test statistic is known, even for a composite hypothesis.

In the goodness-of-fit tests, obtaining the distribution of test statistics and evaluation of statistics are sometimes practically difficult. This fact may explain why people continue to be interested in chi-squared test even if they are not as efficient as other omnibus goodness-of-fit tests such as Anderson-Darling, Cramer-von Mises and Kolmogorov-Smirnov. Today chi-squared goodness-of-fit tests such as the Pearson χ^2 statistic and the likelihood ratio G^2 statistics, are still widely used by the scientific researcher and practitioner for its flexibility and easy to use in many real situations.

In order to use chi-squared goodness-of-fit test, we need to group the original data from a continuous or discrete distribution into classes. As far as we have known, grouping the data sacrifices information, especially if the underlying variable is a continuous distribution. This raises the questions on how to group data and how many classes to be used when applying chi-squared test.

In most situations, the data can be grouped by giving some cutoff points of variable of interest when applying the chi-squared goodness-of-fit test. When grouping the data by cutoff points, we need to be careful to avoid the occurrence of observed cell counts less than five, otherwise we need to merge the groups which are adjacent. In practice, the choice of classes may be arbitrary since it is often based on some previous experience.

Another approach is called "equiprobability intervals", which means the classes are divided by equalizing the cumulative probability into k groups, and consequently the expected counts are the same in all classes. Mann and Wald (1942) initiated the study of the choice of classes in the Pearson test of goodness-of-fit to a continuous distribution. They recommended that the classes be chosen to have equal probabilities under the null hypothesized distribution. The reasons for such a choice at least are: (1) The pearson test statistic χ^2 and the likelihood ratio test statistic G^2 are both unbiased (Cohen and Sackrowitz, 1975); (2) Empirical studies have shown that chi-square distribution is a more accurate approximation to the exact null distribution of χ^2 and G^2 when equiprobable cells (classes) are employed.

Mann and Wald further made recommendations on the number of equiprobable cells to be used, which we will give some more details in Chapter 2. Their work rests on large-sample approximations and on a somewhat complex minimax criterion, so that it is at best a rough guide in practice. Williams (1950) and Schorr (1974) confirms that the "best" classes are smaller than the formula given by Mann and Wald.

1.2 Motivation

It is worth doing a more detailed investigation of the most appropriate number of classes, when grouping data using the "equiprobability intervals". Although the theoretical research on this topic is a difficult field, we can put more intensively computational work into chi-squared test of continuous distributions by scrutinizing the most appropriate classes among all plausible classes.

In order to do that we need to deal with two separate cases of goodness-of-fit test, one is to test a null hypothesis with known parameters (Case 0), another is to test a null hypothesis with unknown parameters (Case 1). In the latter case, we need to estimate the parameters by maximum likelihood method before using the chi-squared goodness-of-fit test.

My study is mainly focused on the chi-squared test for the continuous distributions with location and scale parameters. In both Case 0 and Case 1, our findings of the most appropriate number of classes corresponding to various sample sizes are based on Monte Carlo simulation. We will illustrate the power gained by comparing our method with Mann and Wald's method for the choice of the number of classes when using chi-squared tests for normal distributions, exponential distributions and uniform distributions. Furthermore the second case of chi-squared test with unknown parameters of null distribution is actually to test a family of distributions.

1.3 Thesis Structure

In Chapter 1, we start with the introduction of the chi-squared goodness-of-fit test, and raise the controversial question on how to group the data when using chi-squared test for a continuous distribution. In Chapter 2, we give a brief literature review on the choice of the number of classes, which is followed by our procedures on how to find the most appropriate number of classes by using Monte Carlo simulation. In Chapter 3, the first empirical formula for the choice of the most appropriate number of classes is presented from the results of simulations. We then demonstrate the power gained by applying chi-squared test for a specified distribution with location and scale parameters such as a normal, exponential and uniform distribution. In Chapter 4, our second empirical formula for the choice of the most appropriate number of classes is presented. Then we demonstrate the power gained by applying chi-squared test for a family of distributions such as normal, exponential and uniform distributions. In Chapter 5, we conclude the two empirical formulae for the choice of the most appropriate number of classes when applying chi-squared test in either Case 0 or Case 1, and meanwhile we list some further questions which can be investigated in future work.

Chapter 2

Methodology

2.1 Pearson χ^2 Statistic and Likelihood Ratio G^2 Statistic

To test the simple (or composite) hypothesis that a random sample $X_1, \cdots, X_n$ which are independently and identically distributed from X with the distribution function $F(x)$, we wish to test the null hypothesis:

$$H_0 : F_X(x) = F_0(x) \quad \text{for all } x$$

Where $F_0(x)$ is either completely specified or less completely specified, against the general alternative hypothesis.

$$H_1 : F_X(x) \neq F_0(x) \quad \text{for some } x$$

Pearson partitioned the n observations from a population into k mutually exclusive classes. Let o_i be the observed numbers which is falling into the ith class ($i = 1,2,\cdots,k$), and p_i be the probability that an observation falls into the ith class ($i = 1,2,\cdots,k$). Sometime the p_i are completely specified by the null hypothesis as known numbers, and sometimes they are less completely specified as known functions of one or more unknown parameters. The quantities np_i are actually expected counts, where $\sum_{i=1}^{k} p_i = 1$.

The Pearson chi-squared statistic is constructed as:

$$\chi^2 = \sum_{i=1}^{k} \frac{(o_i - np_i)^2}{np_i}$$

which has approximately the $\chi^2(k-1-s)$, where s is the number of independent parameters in $F_0(x)$ which has to be estimated from the grouped data by maximum likelihood method in order to estimate all the p_i (When parameters are known, $s=0$).

Parallel to the Pearson χ^2 statistic, the log likelihood ratio statistic was firstly given by Fisher as follows,

$$G^2 = 2\sum_{i=1}^{k} o_i \ln \frac{o_i}{np_i}$$

which is asymptotically equivalent to χ^2 when the null hypothesis is true (at least if n is sufficiently large). Hence the likelihood ratio (LR) G^2 has also approximately $\chi^2(k-1-s)$, where s is the number of independent parameters in $F_0(x)$ which has to be estimated from the grouped data by maximum likelihood method in order to estimate all the p_i (When parameters are known, $s=0$).

2.2 Literature Review on the Choice of the Number of Classes

Under the context of "equiprobability intervals", the classic paper by Mann and Ward deals with the choice of the number of classes k when using the Pearson chi-square test. In their paper, the null hypothesis is assumed to specify the distribution completely, and n is large enough so that the limiting χ^2 distribution is applicable.

In order to choose the "best" value of k, some criterion is required to define what is meant by a "best" class. It seems natural to try to maximize some property of the power function of the test. The criterion set up by Mann and Wald is a little complex to describe due to the complexity of the problem itself.

They define the distance Δ between the null distribution and any alternative distribution as the maximum difference between the heights of the two cumulative distribution functions. It becomes evidence, after some examination of the problem, that there is impossible of choosing k so as to maximize the power function of χ^2 at all points along its course. So they decided to concentrate on maximizing the power at about the point where the power is 1/2. Cochran commented that it was an arbitrary but reasonable choice. The two principal properties possessed by their "best" k are as follows.

First, for a value of Δ which they determine, the power of the χ^2 test is at least 1/2 for all alternative distributions whose distance from the null distribution is at least Δ. This value Δ is a simple function of sample size and it decreases steadily with the increasing sample size.

Second, If any k other than the "best " is chosen, the power of χ^2 is less than 1/2 for at least one alternative whose distance from the null distribution exceeds Δ.

They proofed that the optimum k is given by the formula

$$k = 4\left[\frac{2(n-1)^2}{(Z(\alpha))^2}\right]^{\frac{1}{5}}$$

where $Z(\alpha)$ is the upper α-point of the standard normal distribution. The optimum values of k seems substantially higher than those customary in practice. For a test at the 5% significance level, k rises from 31 at $n = 200$ to 78 at $n = 2000$.

Table 2.1: Optimum number of classes (Mann and Wald)

Sample size: n	Number of classes: k
200	31
250	34
300	36
350	39
400	41
450	43
500	45
550	46
600	48
650	50
700	51
750	53

800	54
850	55
900	57
950	58
1000	59
1100	62
1200	64
1300	66
1400	68
1500	70
2000	78

A good exposition and critique of the Mann and Wald paper had been given by Williams (1950). Williams shows that the optimum class is a broad one, and the value of k in table 2.1 can be probably halved with little loss in sensitivity.

Another empirical rule is given by Sturges (1926) according to which

$$k = [1 + \ln n] = [1 + 2.303 \log_{10} n]$$

It seems this empirical rule is not very applicable, since the number of classes is always around 8 from $n = 200$ to $n = 2000$.

2.3 The Choice of the Number of Classes by Monte Carlo Simulation

Given nominal test size α, the actual test size α^* is checked through the Monte Carlo simulation first, and then to scrutinize the patterns of power of test with regard to power change among the various classes $k's$. In order to find the "best" classes k

in terms of the maximum power of test, the idea is to examine the power of test for all plausible classes k's under a null hypothesis against as many alternative hypotheses as possible at a fixed sample size, so that an appropriate classes k can be found from the common parts of those "best" k's corresponding to each individual alternative hypothesis. Under given sample size, what we developed is to yield the most appropriate number of classes k through summarizing various null hypotheses against a general alternative hypothesis, so that the power will be superior for most of the chi-squared goodness-of-fit tests. Then re-do the same process for another sample size, eventually we can have a whole picture of the most appropriate classes corresponding to various sample sizes.

In my study, I focused on the chi-squared test for continuous distributions with location and scale parameters, including normal distributions, exponential distributions, uniform distributions and so forth. There are two cases when using chi-squared goodness-of-fit test:

Case 0: to test a specified distribution of null hypothesis with known parameters;

Case 1: to test a family of distributions of null hypothesis with unknown parameters.

Case 0 is to test the completely specified null hypothesis, while Case 1 is to test a family of distributions. In case 1, suppose we test the null hypothesis H_0 whose distribution function composed of the independently and identically distributed $X_1, \cdots, X_n$ from a family $F(x)$ with location parameter θ_1 and scale parameter θ_2, we know that the chi-squared test statistics are invariant under the transformation, $y = \dfrac{x-\theta_1}{\theta_2}$, which is to say,

$$F(x) = F\left(\frac{x-\theta_1}{\theta_2}\right), \quad |\theta_1| < \infty, \quad \theta_2 > 0,$$

where $F(x)$ is a given function. Since the two parameters are unknown, we use the maximum likelihood estimates for θ_1 and θ_2.

For Pearson χ^2 test, the procedures of Monte Carlo simulation for selecting the most appropriate number of classes k are as follows:

First, check the actual test size (type I error) α^* from the results of simulations under a null hypothesis for all plausible classes to verify the approximately chi-square distribution. Meanwhile we can eliminate those $k's$ obtained from simulations with α^* which obviously depart from the desired test significance level α. The scopes of all the plausible $k's$ are in line with the rule of thumb recommended by Cochran. The most influential one is that np_i should be no less than five, and only few exceptions are allowed.

Second, given a sample size, compare the power of test among all different numbers of classes $k's$ under the null hypothesis against a particular alternative hypothesis. Then keeping all the conditions unchanged, choose another different alternative hypothesis. For each alternative hypothesis we can find the ideal class (it can be an adjacent interval instead of an exact point) in terms of power of test superior than that of other classes. Then the most important work is to summarize these results, and find the common parts of those ideal classes k's under various alternative hypotheses by calculating their mean value. After finding the common parts of those ideal classes, we can eventually yield the most appropriate classes k in terms of superior power under this particular null hypothesis against a general alternative hypothesis at a given sample size.

Third, for another null hypothesis, repeat the first two steps again to find the most appropriate number of classes corresponding to the null hypothesis. Finally, based on each null hypothesis against a general alternative hypothesis, we can summarize the most appropriate number of classes in terms of power superior when using chi-squared goodness-of-fit test.

Finally, for different sample sizes, repeat the first three steps again and then summarize the most appropriate number of classes under various null hypotheses against a general alternative hypothesis.

Therefore, the most appropriate number of classes k corresponding to the different sample sizes can be summarized, which can be applied in the Pearson χ^2 test or the likelihood ratio G^2 test for a null hypothesis against a completely general alternative hypothesis.

In the procedures to obtain a general rule for the number of classes when using chi-squared goodness-of-fit test, we need to have a close look at the three patterns of power of test behaviors.

(1) Increasing pattern. The power of test is increased as the number of classes increases.

(2) Decreasing pattern. The power of test is decreased as the number of classes increases.

(3) Curvature pattern. The power of test is either increased than followed by decreased, or decreased then followed by increased as the number of classes increases. In this pattern, an optimal class can be acquired.

Given significance level $\alpha = 0.05$, when testing the normal distribution under the null hypothesis for Case 0 and Case 1, the selected graphs of power of test against different numbers of classes are illustrated in figure 2.1 and figure 2.2 respectively.

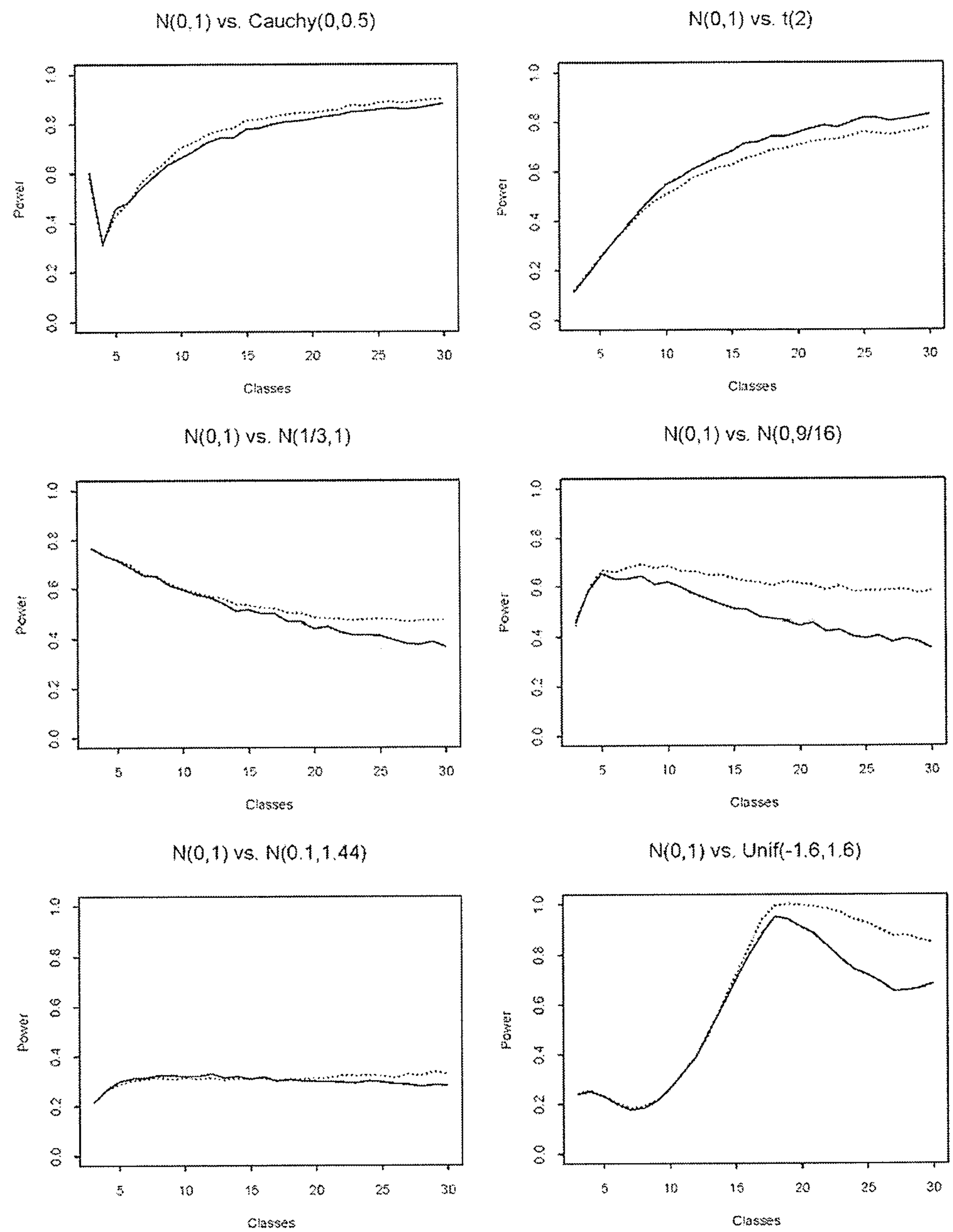

Figure2.1: the pattern of power of test against different classes at level $\alpha = 0.05$ (case 0, n=100). Solid line: Pearson x^2 statistic, dotted line: LR G^2 statistic.

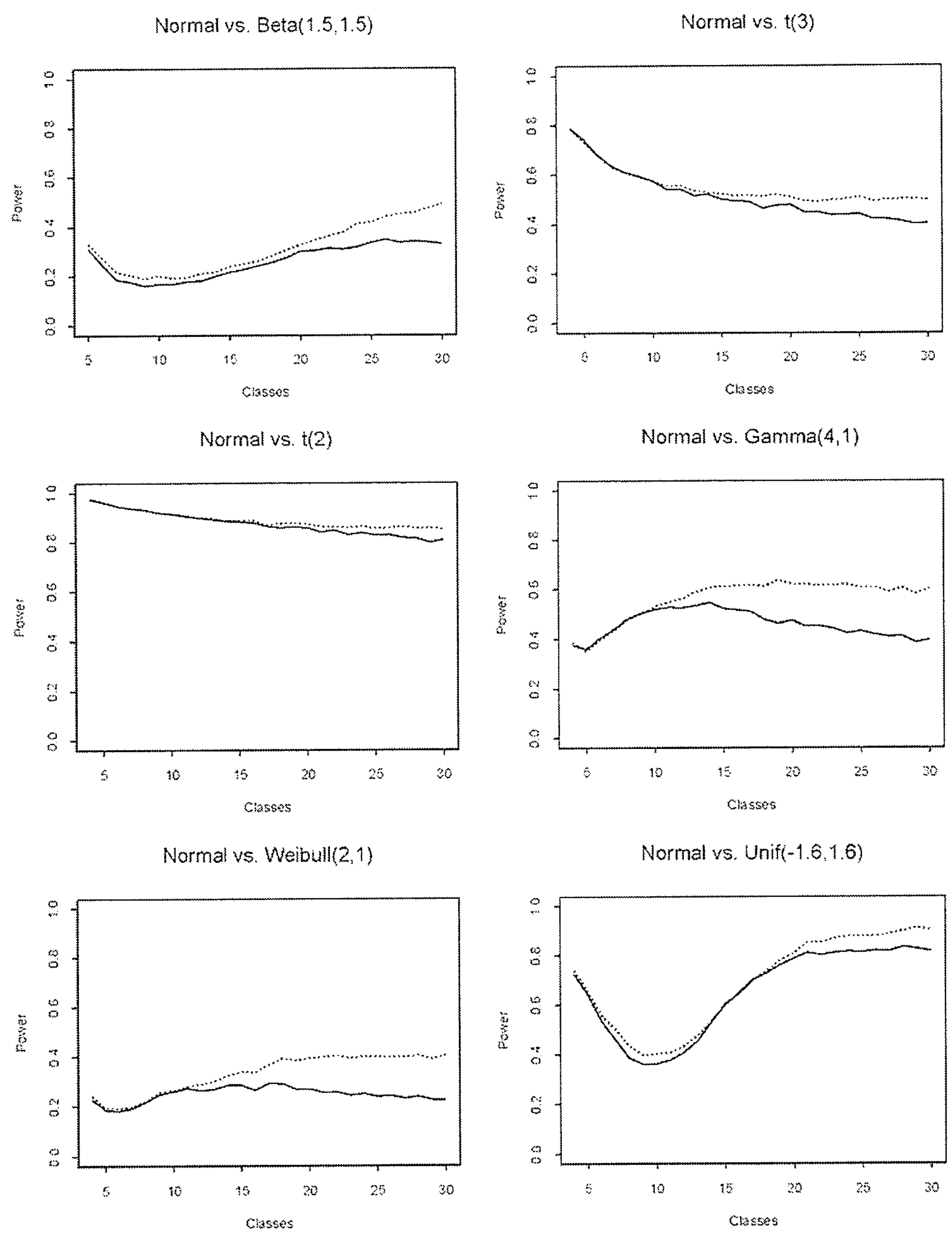

Figure2.2: the pattern of power of test against different classes at level $\alpha = 0.05$ (case 1, n=100). Solid line: Pearson x^2 statistic, dotted line: LR G^2 statistic.

Chapter 3

The Chi-Squared Test for a Specified Distribution (Case 0)

3.1 The Choice of the Number of Classes for Case 0

Through computer intensive simulation we examined various null hypotheses against a large amount of alternative hypotheses, and we summarized the most appropriate number of classes in terms of superior power of test under different sample sizes, at significance level $\alpha = 0.05$. The following table is a summary for Case 0 scenario.

Table 3.1: The most appropriate number of classes corresponding to sample size

Sample size: n	Number of classes: k
50	9
100	11
150	12
200	13
250	14
300	15
400	16
500	18
600	19
700	20
800	21
900	21
1000	22

The empirical formula for the above table is : $k = [\,2.81 + 1.23 n^{2/5}\,]$.

Employing the new formula to group data, we use the Monte Carlo simulation to examine the powers of the Pearson chi-squared χ^2 statistic and the likelihood ratio chi-squared G^2 statistic, compared with the power of the Pearson chi-squared χ^2 statistic based on Mann and Wald's formula. We illustrated the power comparison for three types of distribution: normal, exponential and uniform. The simulation size is 100000, and test significance level or the probability of type I error for the chi-squared goodness-of-fit test is $\alpha = 0.05$.

We checked the actual test size of the chi-squared tests for null hypothesis in order to filter and determine the scope of all plausible $k's$, which are very close to 0.05 for sample size from 50 to 1000. We found that the Pearson χ^2 statistic is more robust than the likelihood ratio G^2 statistic against different number of classes. For various null hypotheses H_0 against all kinds of alternatives H_1, the simulated powers of the Pearson χ^2 based on the new choice of the number of classes, of the likelihood ratio G^2 based on the new choice of the number of classes, and of the Pearson χ^2 based on the Mann and Wald's formula for the number of classes are illustrated with graphs, where the powers are plotted against the sample size n for selected values $n = 50, 100, 150, 200, 250, 300, 400$.

3.2 Chi-Squared Test for a Specified Normal Distribution

In this section, we use the chi-squared test to test specified normal distributions by choosing the number of classes we concluded.

(1) Example 1: $H_0 : N(\mu, \sigma^2)$ versus $H_1 : t(k)$

Under the null hypothesis H_0, suppose F to be a normal distribution $N(\mu, \sigma^2)$. It is reasonable to consider that

(a) F also has a symmetric distribution under the alternative hypothesis H_1, say $t(k)$, the *t*-distribution with *k* degree of freedom, and

(b) both distribution have the same mean and variance, i.e. $\mu = 0$ and $\sigma^2 = k/(k-2)$ $(\text{k} \geq 3)$.

Since $N(0,1) = t(\infty)$, testing $H_0 : \text{F} = N(\mu, \sigma^2)$ versus $H_1 : \text{F} = t(k)$ is equivalent to testing $H_0 : k = \infty$ versus $H_1 : k \neq \infty$.

The Cauchy and Logistic distributions are also typical examples of symmetric distributions which can be considered as the underlying distribution under alternative hypothesis H_1.

(2) Example 2: $H_0 : N(\mu, \sigma^2)$ versus $H_1 : \text{Logistic}(a, b)$

In this example, we assume the null hypothesis is a normal distribution $N(1,1)$. The alternative distribution is logistic(a,b) with location parameter a and scale parameter b.

Figure 3.1 shows the power comparison of the two Pearson χ^2 statistics based on the new choice of the number of classes and Mann and Wald's choice of the number of classes respectively, as well as the likelihood ratio G^2 statistic based on the new choice of the number of classes for both example 1 and 2. Obviously, with respect to the power of the Pearson χ^2 statistic, our new choice of the number of classes is better than the choice of the number of classes used by Mann and Wald's method. The power of the Pearson χ^2 statistic is quite close to the power of the likelihood ratio G^2 statistic, based on the new choice of the number of classes.

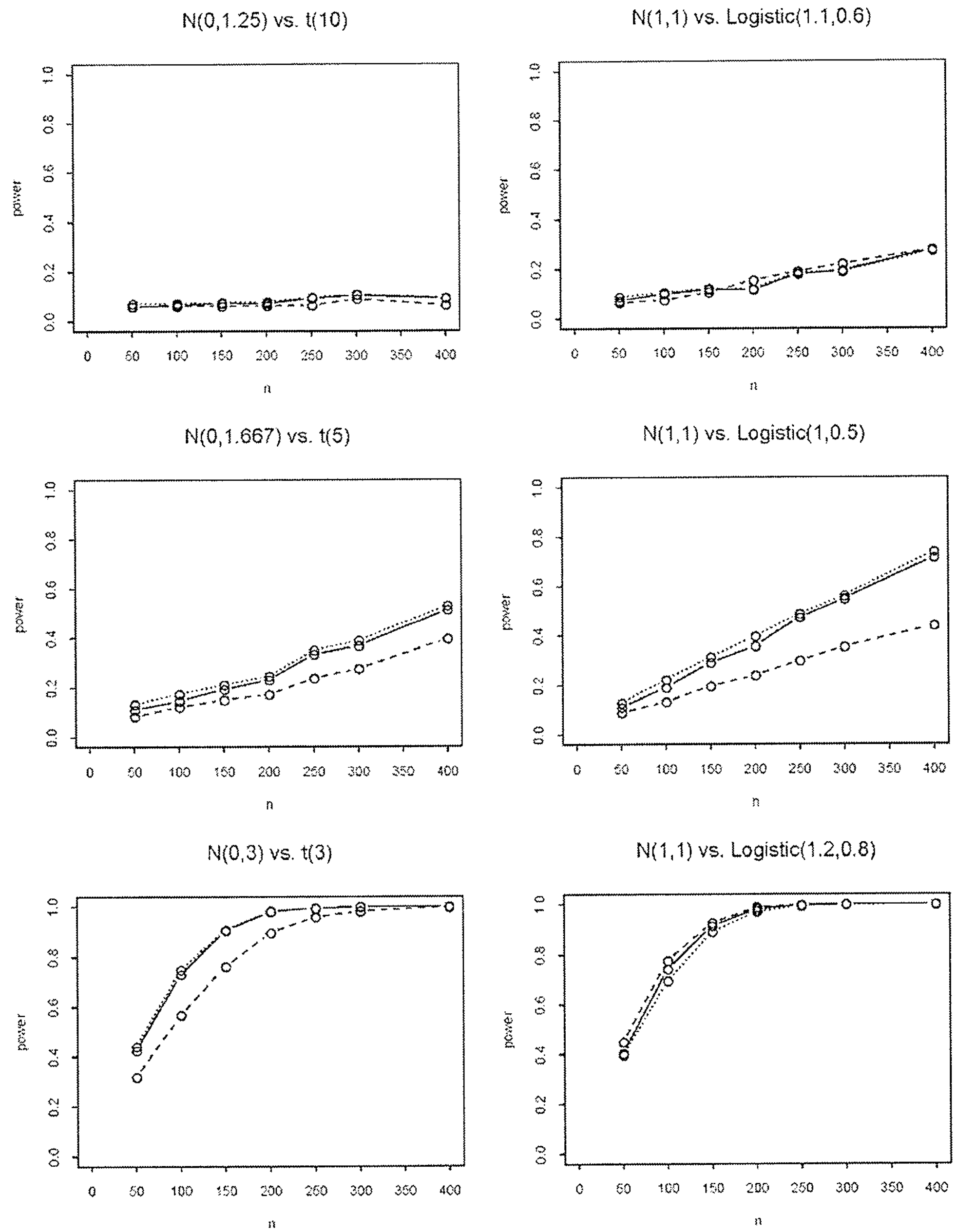

Figure 3.1: Power comparison when testing specified normal versus *t* or Logistic at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ····: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

(3) Example 3: $H_0 : N(\mu, \sigma^2)$ versus $H_1 : Gamma(a,1)$

In this example F is also assumed to be normal under null hypothesis H_0, but it has an asymmetric distribution under H_1, such as *gamma(a, 1)*, the gamma distribution with shape parameter a and scale parameter 1, which includes exponential and χ^2 distribution. We also assume that both distributions have the same mean and variance, i.e. $\mu = a$ and $\sigma^2 = a$.

Similarly, since the asymptotic distribution of gamma *(a, 1)* is normal when $a \to \infty$, testing $H_0 : \mathrm{F} = N(\mu, \sigma^2)$ versus $H_1 : \mathrm{F} = gamma(a,1)$ is equivalent to testing $H_0 : a = \infty$ versus $H_1 : a \neq \infty$.

Other asymmetric distributions, such as Lognormal, Weibull, F, and Beta distributions, were also considered as alternative distributions against the normal distribution.

(4) Example 4: $H_0 : N(\mu, \sigma^2)$ versus $H_1 : Weibull(a,b)$

In this example, the null hypothesis is a specified normal distribution, and the alternative hypothesis is $weibull(a,b)$ with shape parameter a and scale parameter b.

The simulated powers of the three chi-squared statistics for both example 3 and example 4 are plotted in Figure 3.2, which shows that our choice of the number of classes outperforms that used by Mann and Wald, with respect to the power of the

Pearson χ^2 statistic. The Power of the likelihood ratio G^2 statistic is quite close to that of the Pearson χ^2 statistic based on the new choice of the number of classes.

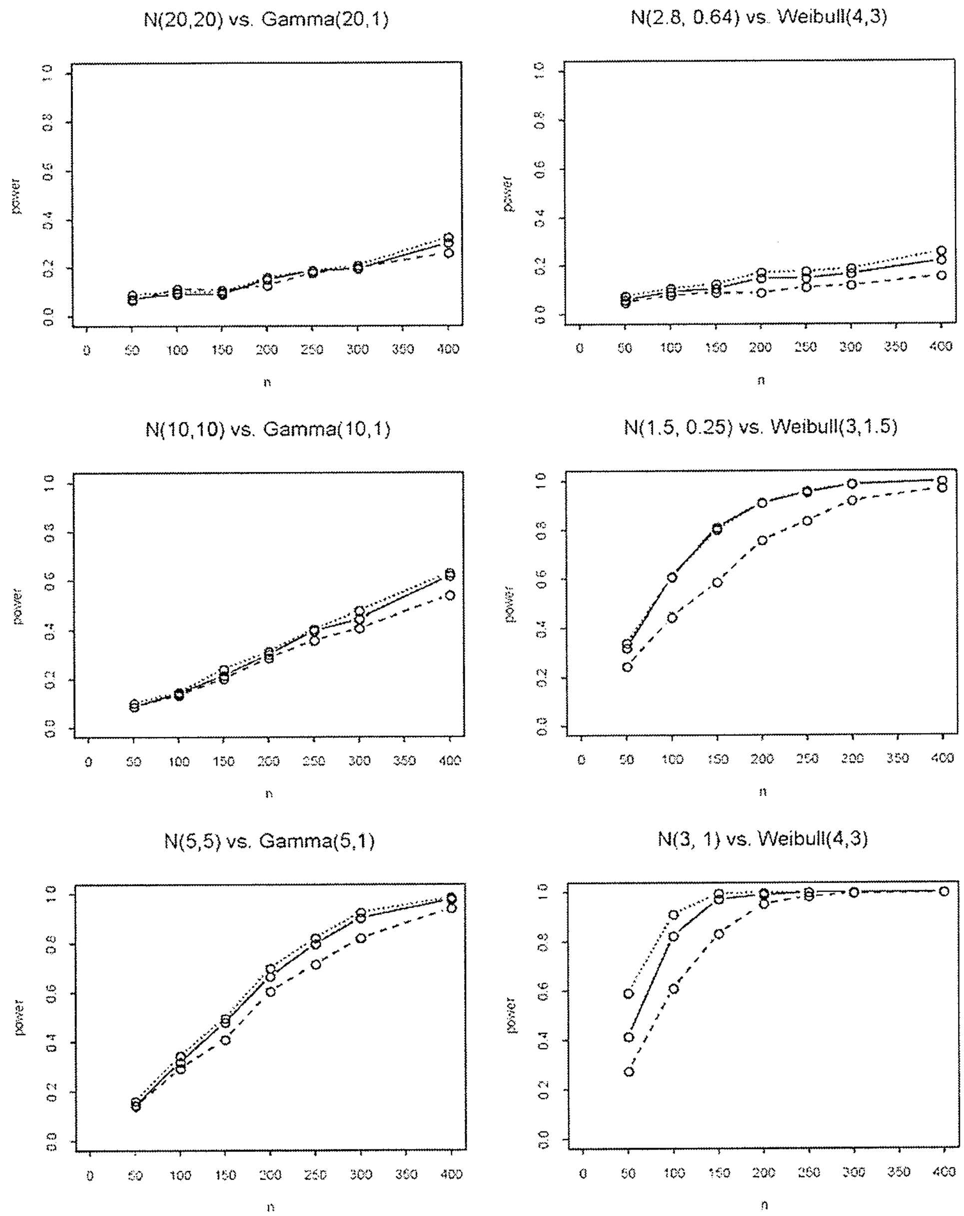

Figure 3.2: Power comparison when testing specified normal versus Gamma or Weibull at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ···: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

(5)Example 5: $H_0: N(0,1)$ versus $H_1: N(\mu, \sigma^2)$

In this example, F is assumed to be normal under null hypothesis H_0, and to be normal (or mixed normal) under alternative hypothesis H_1. Without loss of generality, we only need to consider the test for $H_0: N(0,1)$ versus $H_1: N(\mu, \sigma^2)$, or equivalently $\mathrm{H}_0: (\mu, \sigma^2) = (0,1)$ versus $\mathrm{H}_0: (\mu, \sigma^2) \neq (0,1)$.

Eight cases are considered with alternatives

(a) N(0.1, 1)

(b) N(0.4, 1)

(c) N(0, 1.5)

(d) N(0, 2)

(e) N(0.1, 2)

(f) N(0.4, 1.5)

(g) Mixed Normal: 0.5*N*(0, 1.44)+0.5*N*(0.1, 1)

(h) Mixed Normal: 0.5*N*(0.1, 1.44)+0.5*N*(0.2, 1.21)

For normal distribution models, the distribution differs from the mean and variance only. There is no shape difference in terms of skewness and kurtosis. In cases (a) and (b), the two distributions have the same variance but different means, and in cases (c) and (d) they have the same mean but different variances. In cases (e) and (f), the means and variances are both different, and in cases (g) and (h), these are two mixed normal distributions.

Figure 3.3(a) and 3.3(b) show the simulated powers of the three chi-squared statistics with respect to the above eight cases. It is evident that our choice of the number of classes is better than that used by Mann and Wald, with respect to the power of the Pearson χ^2 statistic. The Power of the likelihood ratio G^2 statistic is very close to that of the Pearson χ^2 statistic based on the new choice of the number of classes.

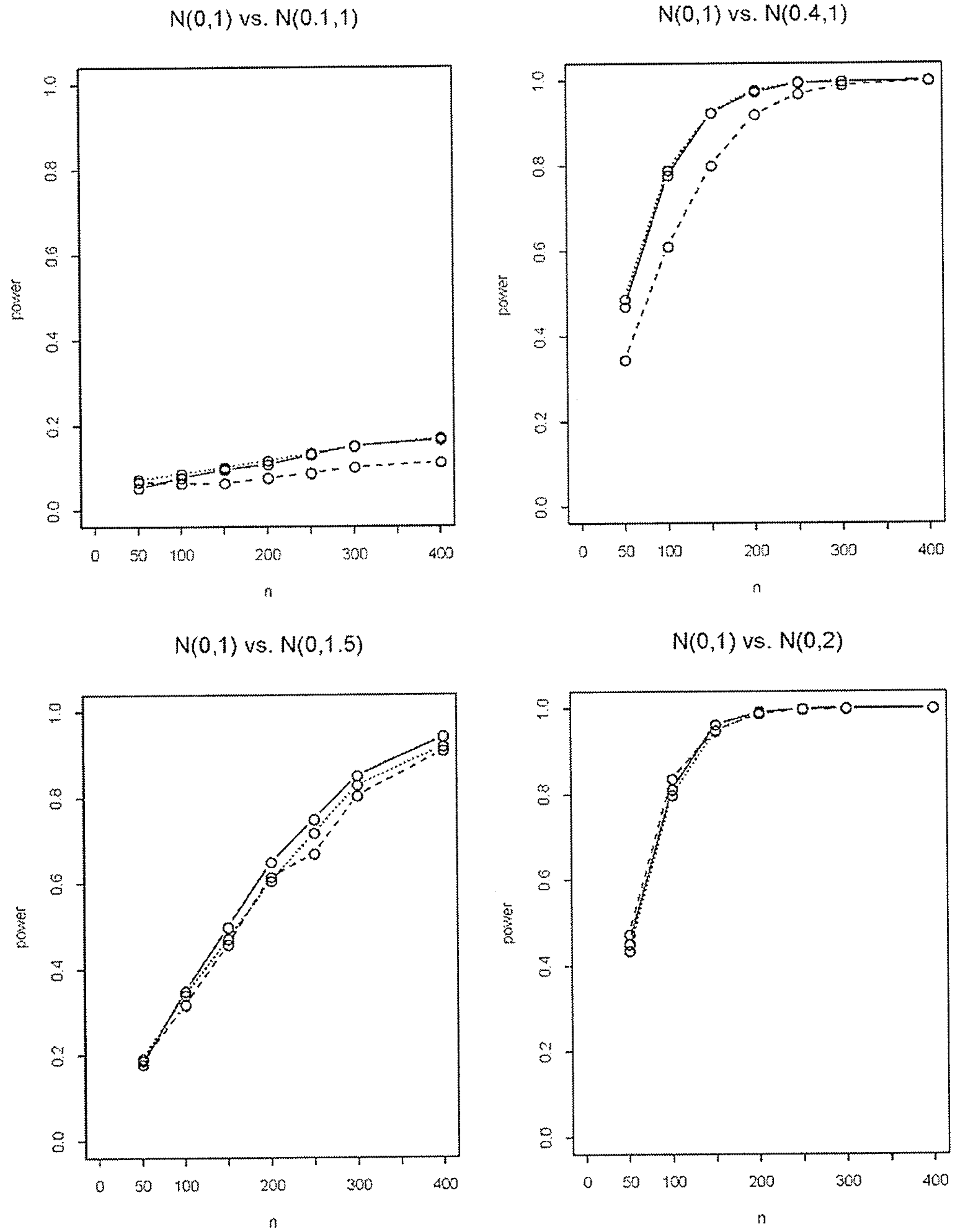

Figure 3.3(a): Power comparison when testing Standard Normal versus Normal at level $\alpha = 0.05$. —: Pearson X^2 based on the new choice of classes, ⋯: LR G^2 based on the new choice of classes, ---: Pearson X^2 based on Mann and Wald's choice of classes.

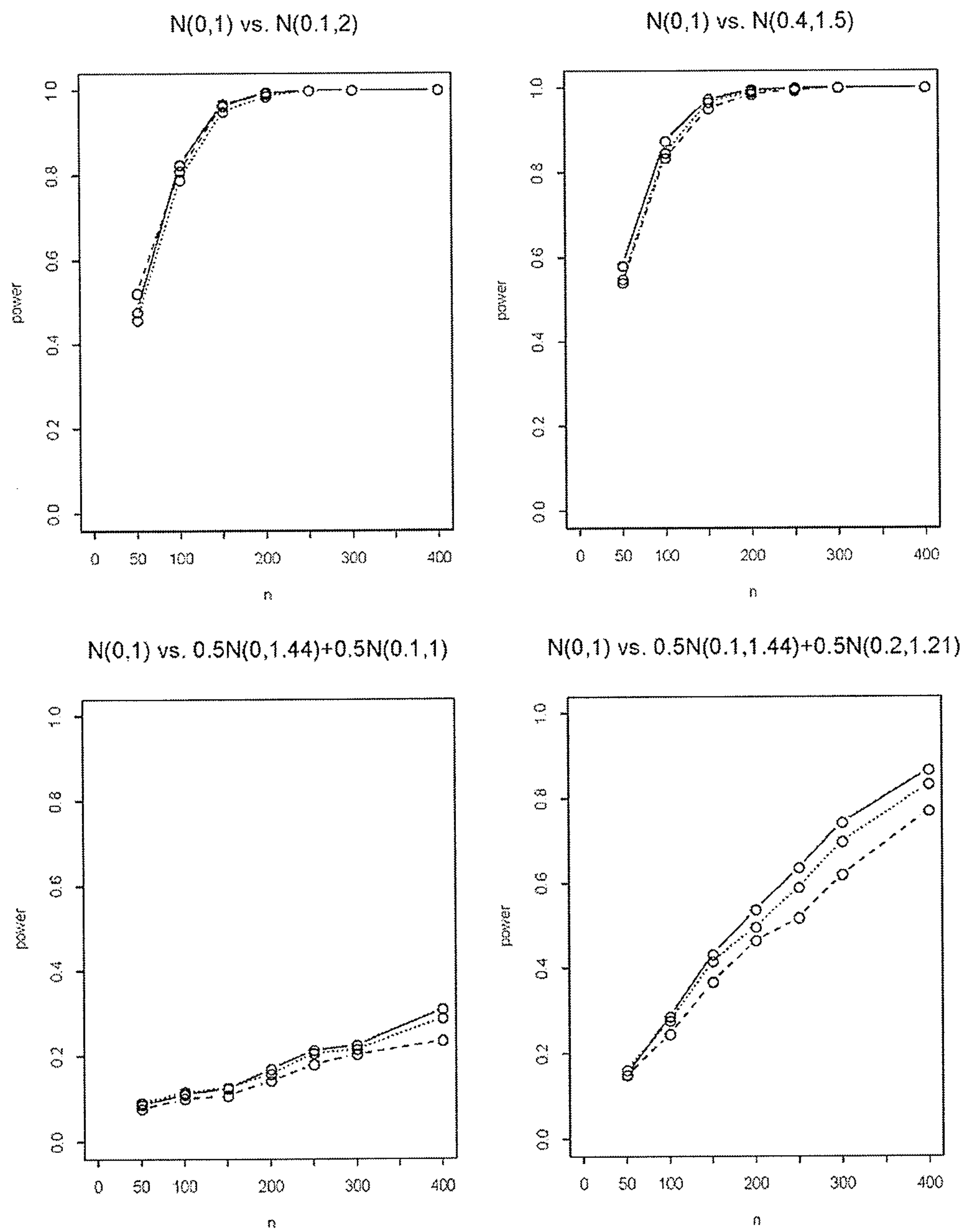

Figure 3.3(b): Power comparison when testing Standard Normal versus (mixed) Normal at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ···: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

3.3 Chi-Squared Test for a Specified Exponential Distribution

In this section, we use chi-squared test based on our findings of the choice of the number of classes to test a non-negative continuous random variable X which has a specified exponential distribution. Under the same significance level $\alpha = 0.05$, we compared the simulated powers of previously mentioned three chi-squared test statistics by using Monte Carlo simulation.

We consider the following three type alternatives against the specified exponential distribution $Exp(\theta)$ with scale parameter θ.

(a) the Weibull distribution $Weibull(a,b)$ with shape parameter a and scale parameter b.

(b) the Gamma distribution $Gamma(a,b)$ with shape parameter a and scale parameter b.

(c) the log-normal distribution $Lognormal(\mu,\sigma^2)$ where μ and σ^2 are the mean and variance of the variable's logarithm.

For each distribution we set parameters such that $E(X)=1$, i.e. $b = 1/\Gamma(1+1/a)$ in case (a), $b = 1/a$ in case (b) and $\mu = -\sigma^2/2$ in case (c).

(1) Example 1: $H_0 : Exp(1)$ versus $H_1 : \text{Weibull}(a,b)$

In this example for alternative hypothesis $Weibull(a,b)$, we choose *(a, b)=(0.6, 0.665), (0.7, 0.790), (1.2, 1.063), (1.5, 1.108).* The simulated powers for the three chi-squared test statistics are shown in figure 3.4.

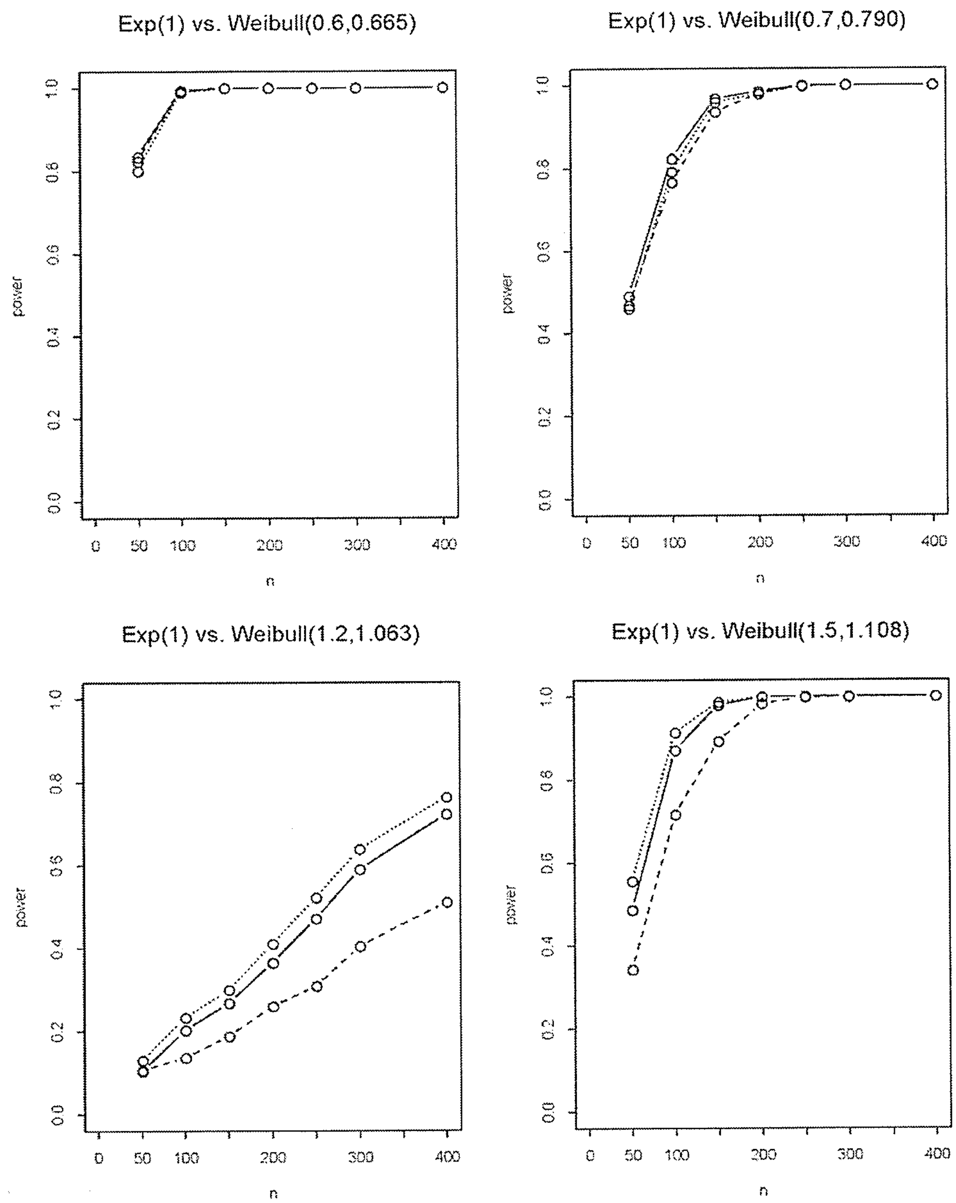

Figure 3.4: Power comparison when testing Exp(1) versus Weibull at level $\alpha = 0.05$. —: Pearson X^2 based on the new choice of classes, ···: LR G^2 based on the new choice of classes, ---: Pearson X^2 based on Mann and Wald's choice of classes.

(2) Example 2: $H_0 : Exp(1)$ versus $H_1 : \text{Gamma}(a, b)$

In this example for alternative hypothesis $\text{Gamma}(a, b)$, we choose *(a, b)=(0.5, 2), (0.7, 10/7), (1.2, 5/6), (2, 0.5).* The simulated powers for the three chi-squared test statistics are shown in figure 3.5.

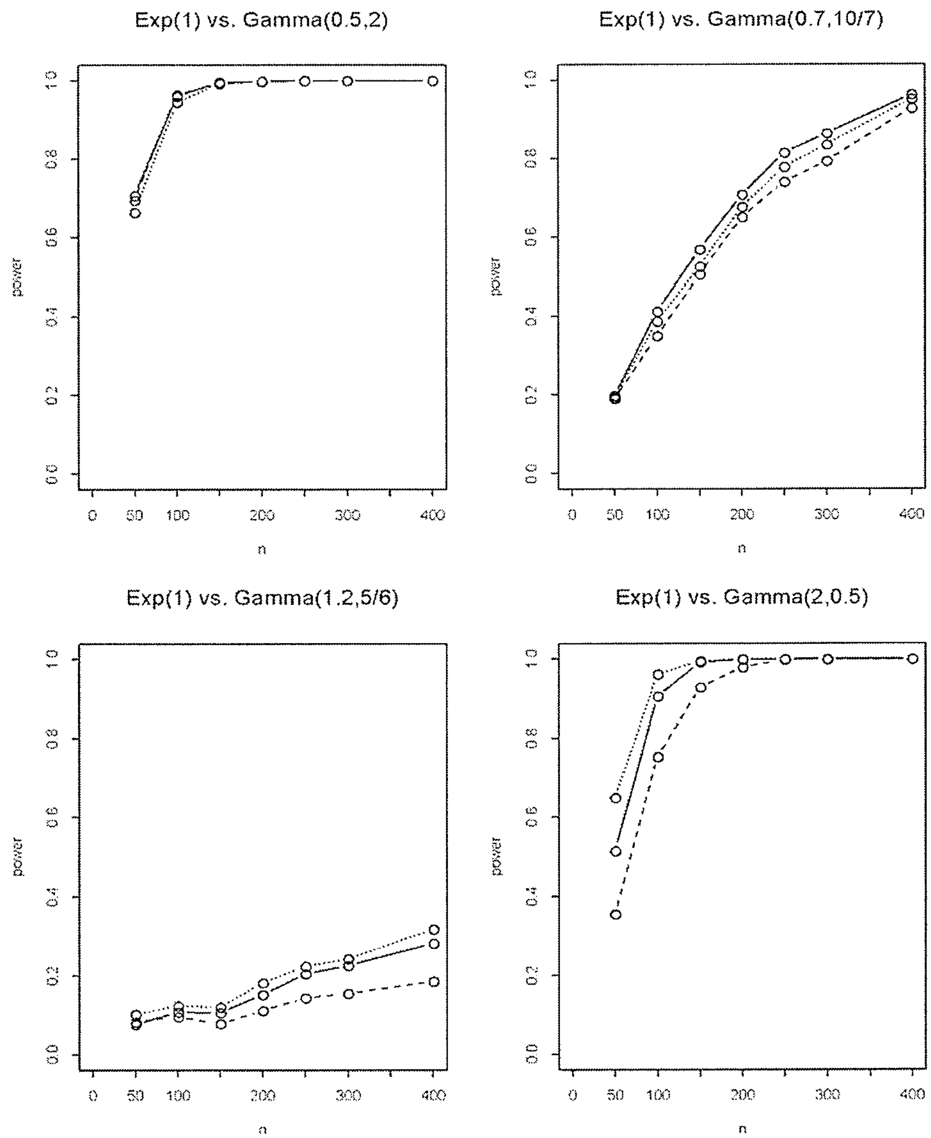

Figure 3.5: Power comparison when testing Exp(1) versus Gamma at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ⋯: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

(3) Example 3: $H_0 : Exp(1)$ versus H_1 : Lognormal(μ, σ^2)

In this example for alternative hypothesis lognormal(μ, σ^2), we choose (μ, σ^2)=*(0.5, 2), (0.7, 10/7), (1.2, 5/6), (2, 0.5).* The simulated powers for the three chi-squared test statistics are shown in figure 3.6.

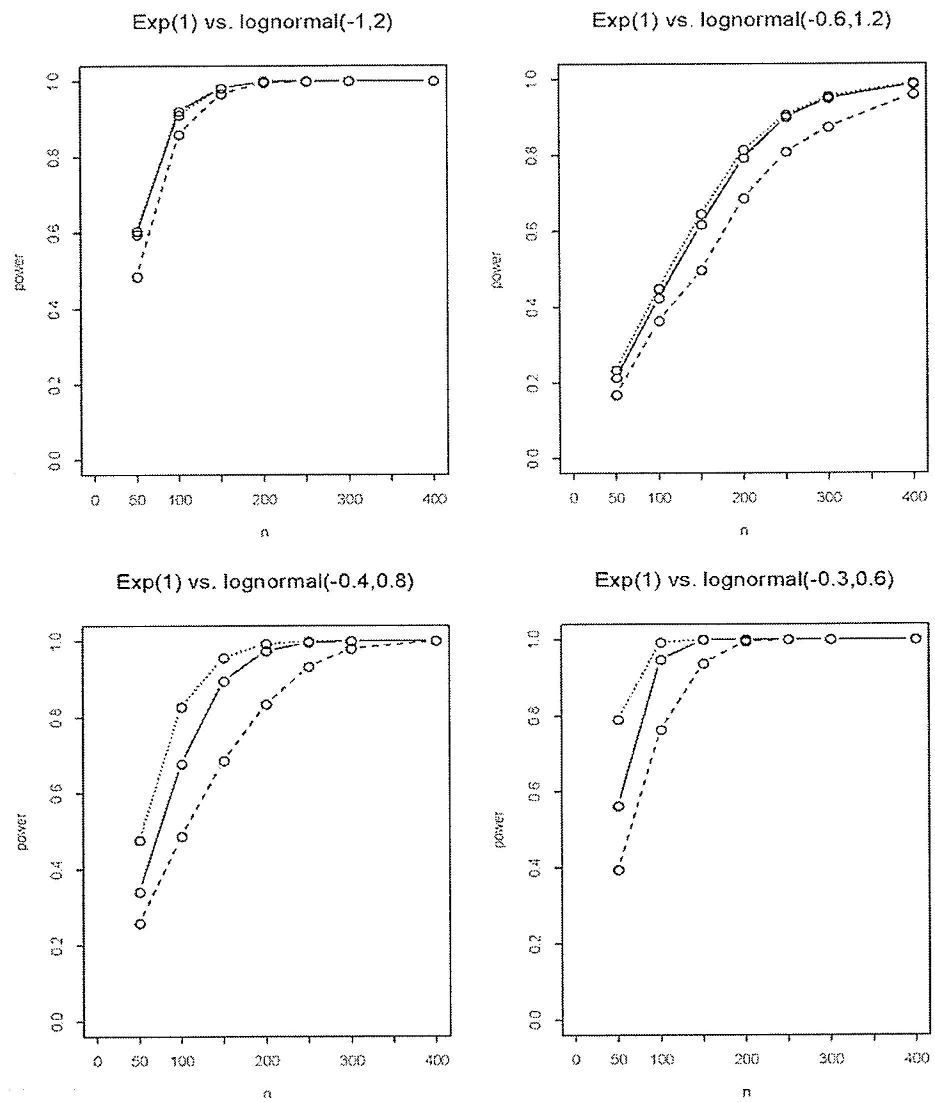

Figure 3.6: Power comparison when testing Exp(1) versus Lognormal at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ⋯: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

From the above three examples, we concluded that our choice of the number of classes is better than that used by Mann and Wald, with respect to the power of the Pearson χ^2 statistic. It seems that sometimes the likelihood ratio G^2 statistic has slightly more powerful than the Pearson χ^2 statistic based on the new choice of the number of classes.

3.4 Chi-Squared Test for a Specified Uniform Distribution

In this section, we use chi-squared test based on our findings of the choice of the number of classes to test a continuous random variable X which has a specified uniform distribution. Under the significance level $\alpha = 0.05$, we compared the simulated powers of the three chi-squared test statistics which we looked at in the previous section by using Monte Carlo simulation.

Without loss of generality, we can assume that the underlying distribution F_0 is the standard uniform U(0,1) distribution under the null hypothesis H_0. Then a natural candidate for F under the alternative hypothesis H_1 is the beta distribution beta*(p, q)* with both shape parameters p and q, which includes the uniform U(0,1) distribution, or beta(1,1) distribution. So this case is actually a parametric test for $H_0 : (p,q) = (1,1)$ versus $H_1 : (p,q) \neq (1,1)$.

(1) Example 1: $H_0 : Unif(0,1)$ versus $H_1 : \text{Beta}(p,\ q)$

In this example for alternative hypothesis, we choose (p, q)=(0.6, 0.8), (0.6, 0.6), (0.8, 0.8), (1.3, 1.3), (1.6, 1.6), (1.3, 1.6), the simulated powers for the two Pearson χ^2 test statistics and the likelihood ratio G^2 test statistic are plotted in figure 3.7.

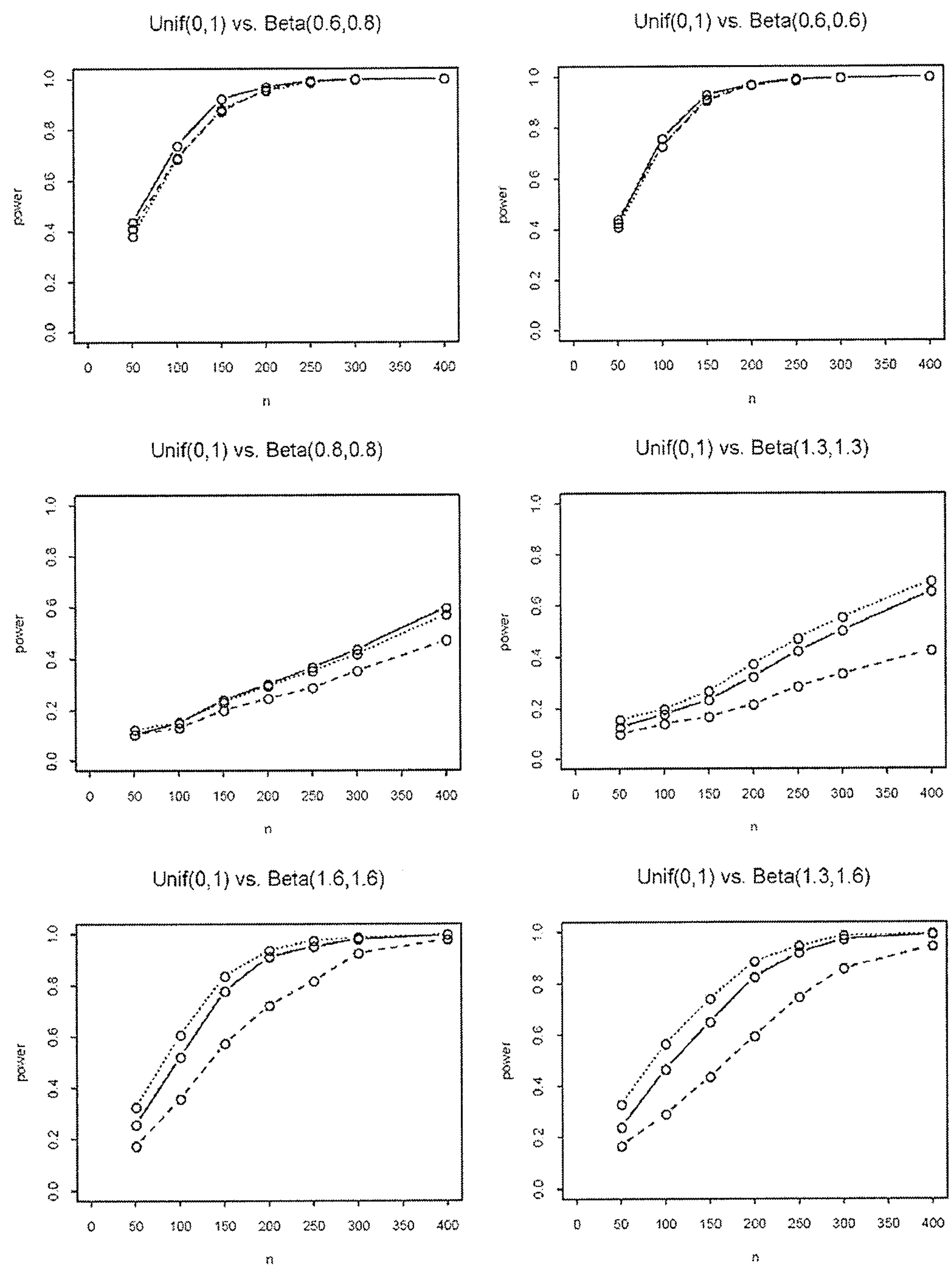

Figure 3.7: Power comparison when testing U(0,1) versus Beta at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ⋯: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

(2) Example 2: $H_0 : Unif(0,1)$ versus H_1 : Mixed Beta

In this example, the alternative hypotheses are mixed beta distributions. In figure 3.8, we illustrated the simulated powers for the three test statistics for the following four cases.

(a) 0.5Beta(0.7, 0.8)+0.5Beta(1.1, 1.1);

(b) 0.5Beta(0.6, 0.8)+0.5Beta(1.3, 1.3)

(c) 0.5Beta(0.8, 0.8)+0.5Unif(-0.1, 1.1)

(d) 0.5Beta(0.9, 0.9)+0.5N(0.5, 0.078)

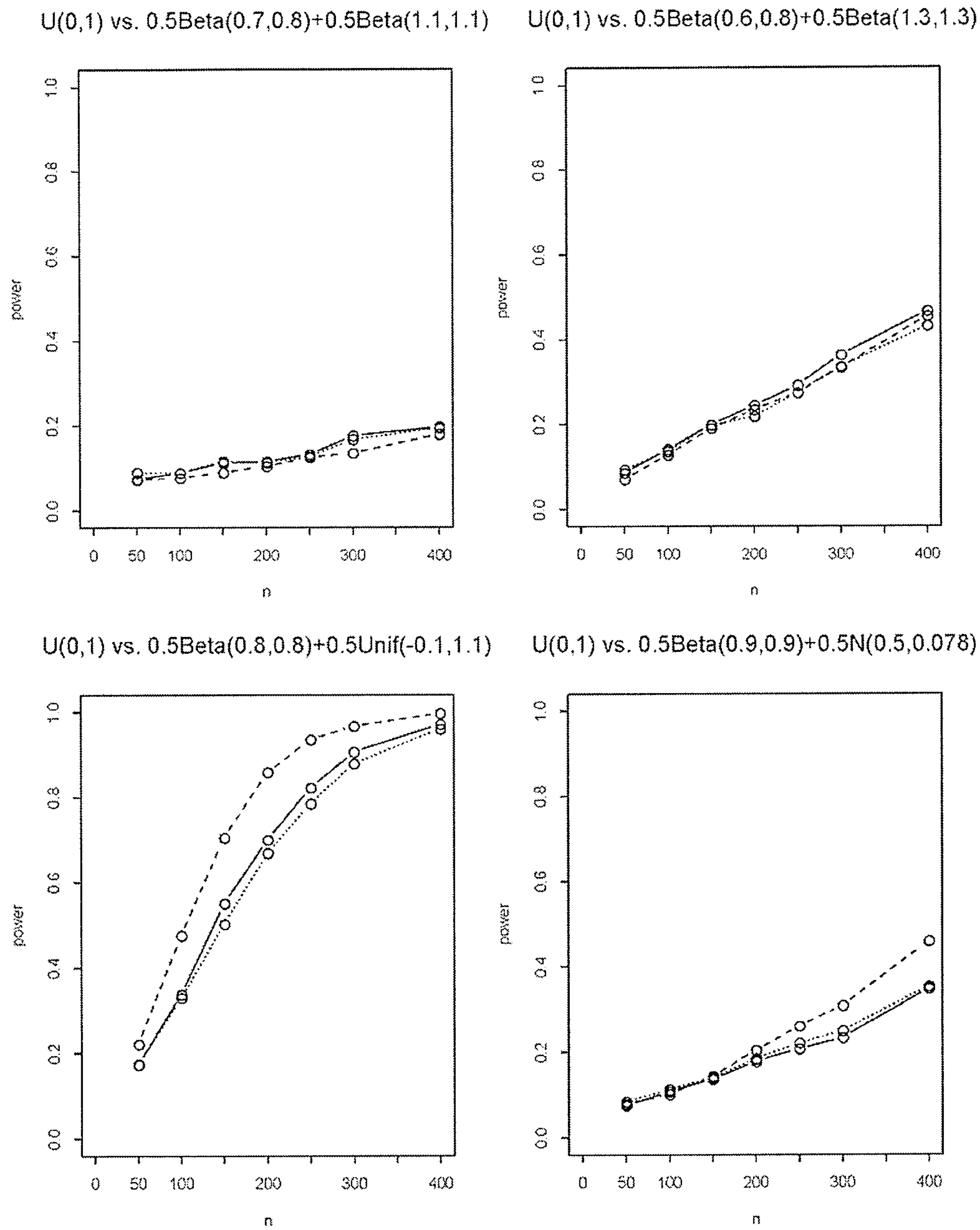

Figure 3.8: Power comparison when testing U(0,1) versus mixed Beta at level $\alpha = 0.05$. —: Pearson X^2 based on the new choice of classes, ····: LR G^2 based on the new choice of classes, ---: Pearson X^2 based on Mann and Wald's choice of classes.

Figure 3.7 illustrates that our choice of the number of classes is better than that used by Mann and Wald, with respect to the power of the Pearson χ^2 statistic. While the last two cases displayed in figure 3.8 can be seen as counterexamples, the power of the Pearson χ^2 statistic based on Mann and Wald's approach is somewhat better than that of the Pearson χ^2 statistic based on our new choice of the number of classes.

In summary, with respect to the power of the Pearson χ^2 statistic, our choice of the number of classes outperforms that used by Mann and Wald in most cases for testing a specified uniform distribution.

Chapter 4

The Chi-Squared Test for a Family of Distributions (Case 1)

4.1 The Choice of the Number of Classes for Case 1

Through computer intensive simulation, we examined various null hypotheses against a large amount of alternative hypotheses, and we summarized the most appropriate number of classes in terms of superior power of test under different sample sizes, at significance level $\alpha = 0.05$. The following table is a summary for Case 1 scenario.

Table 4.1: The most appropriate number of classes corresponding to sample size

Sample size: n	Number of classes: k
50	10
100	12
150	14
200	16
250	18
300	20
400	22
500	24
600	26
700	27
800	29
900	30
1000	32

The empirical formula for the above table is : $k = [\, 2n^{2/5} \,]$.

Employing the new formula to group data, we used the Monte Carlo simulation to examine the powers of the Pearson chi-squared χ^2 statistic and the likelihood ratio chi-squared G^2 statistic, compared with the power of the Pearson chi-squared χ^2 statistic based on Mann and Wald's formula. We illustrated the power of test for the three distribution families: normal, exponential and uniform. The simulation size is 100000, and the significance level or the probability of type I error for the chi-squared goodness-of-fit test is $\alpha = 0.05$.

We checked the actual test size of the chi-squared tests for null hypothesis in order to filter and determine the scope of plausible $k's$, which are very close to 0.05 for sample size from 50 to 1000. We also found that the Pearson χ^2 statistic is more robust than the likelihood ratio G^2 statistic against different number of classes. For various null hypotheses H_0 against all kinds of alternatives H_1, the simulated powers for χ^2 statistic and G^2 statistic based on the new choice of the number of classes, and χ^2 statistic based on Mann and Wald's formula for the choice of the number of classes are illustrated with graphs, where the powers are plotted against the sample size n for selected values $n = 50, 100, 150, 200, 250, 300, 400$.

4.2 Chi-Squared Test for Normality

In this section, the chi-squared test is used to test a family of normal distributions by choosing the number of classes we recommended.

For normal distribution family $N(\mu, \sigma^2)$ with mean $\mu \in (-\infty, \infty)$ and standard deviation $\sigma > 0$, which are both unknown parameters. In order to apply goodness-of-fit test, first we need to estimate μ and σ by maximum likelihood method from sample. Usually we simply use sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and sample standard deviation $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$ to be the estimate of μ and σ respectively. It is easy to see that the chi-squared test statistics for null hypothesis of normal distributions are invariant under any affine transformation $y = \frac{x - \hat{\mu}}{\hat{\sigma}}$, so the problem becomes testing $F_0(x) = \Phi((x - \hat{\mu}) / \hat{\sigma})$ with $\Phi(x)$ which denotes the cumulative distribution function of the standard normal distribution *N(0, 1)*.

(1) Example 1: Normal versus *Beta(p, q)*

In the first example of testing normality, the alternative distribution is beta distribution *Beta(p, q)* with parameters *p* and *q*, which is symmetric when *p=q* and includes the standard uniform distribution $U(0,1) = Beta(1,1)$. We also know that the limiting normal distribution is $Beta(\infty, \infty)$.

The skewness and kurtosis for any normal distribution are $r_1 = 0$ and $r_2 = 3$, but for the beta distribution,

$$r_1 = \frac{2(q-p)}{p+q+2}\sqrt{\frac{p+q+1}{pq}} \quad \text{and} \quad r_2 = \frac{3(p+q+1)}{p+q+3}\left[\frac{2(q-p)^2}{pq(p+q+2)}+1\right],$$

Where $r_1 = 0$ and $r_2 < 3$ if $p=q$ (Johnson et al., 1994; Zhang and Wu, 2002).

For the cases of (p, q)=(2, 1), (2, 2), (2, 5.51), (5, 3), (5, 5), (5, 13.4), which correspond to various departures from normality with different combinations of skewness and kurtosis.

r_1	-0.5657	0	0.6487	-0.3098	0	0.4431
r_2	2.4	2.143	3	2.585	2.538	3

where r_1=0 for cases 2 and 5 but r_2=3 for cases 3 and 6.

The powers of the Pearson χ^2 statistic and the likelihood ratio G^2 statistic based on our choice of the number of classes, compared with that of the Pearson χ^2 statistic based on Mann and Wald's approach for the choice of the number of classes are plotted in figure 4.1. With respect to the power of the Pearson χ^2 statistic, our choice of the number of classes is better than that used by Mann and Wald. In the second case, when testing Normal versus Beta(2, 2), the power of test has an increasing pattern which means the power of test is increased as the number of classes increases. Also, the pattern of the power of test of the Pearson χ^2 statistic and the likelihood ratio G^2 statistic are very consistent.

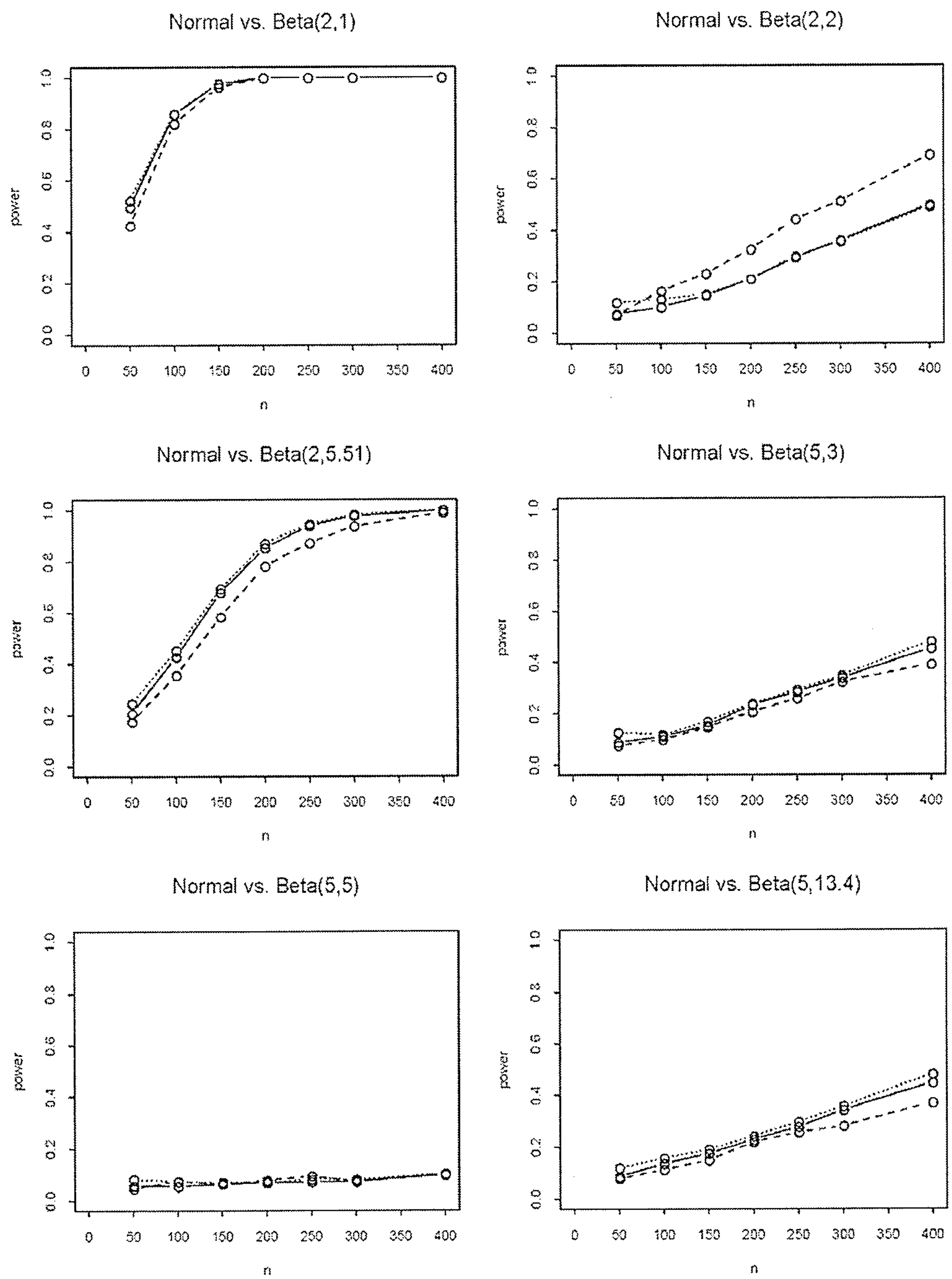

Figure 4.1: Power comparison when testing Normal versus Beta at level $\alpha = 0.05$. —: Pearson X^2 based on the new choice of classes, ···: LR G^2 based on the new choice of classes, ---: Pearson X^2 based on Mann and Wald's choice of classes.

(2) Example 2: Normal versus *t(k)*

In the second example, the alternative distribution is *t(k)*, the *t* distribution with *k* degrees of freedom, which is symmetric and includes Cauchy distribution, i.e. $t(1)$ and the standard normal distribution $N(0,1) = t(\infty)$.

The skewness and kurtosis for *t(k)* are $r_1 = 0$ and $r_2 = 6/(k-4)+3$ (Johnson et al., 1994), so the departure from normality comes only from kurtosis. For different values of k=3, 5, 9 with corresponding $r_2 = \infty, 9, 4.2$, the powers of the three statistics are plotted in figure 4.2.

(3) Example 3: Normal versus *Gamma(a,b)*

In the third example, the alternative distribution is *Gamma(a,b)*, the gamma distribution with shape *a* and scale parameter *b*, which is an asymmetric distribution. In fact, the family of gamma distributions includes exponential and chi-squared distributions, as well as normal distribution when *a* goes into infinity.

The skewness and kurtosis for *Gamma(a,b)* are $r_1 = 2/\sqrt{a}$ and $r_2 = 6/a+3$ (Johnson et al., 1994), which depend on the shape parameter only. Without loss of generality, we need to consider *Gamma(a,1)* only. We choose *a=16, 9, 4* with corresponding $r_1 = \frac{1}{2}, \frac{2}{3}, 1$ and $r_2 = \frac{27}{8}, \frac{11}{3}, \frac{9}{2}$ for illustration. The power comparison of example 3 is also exhibited in figure 4.2.

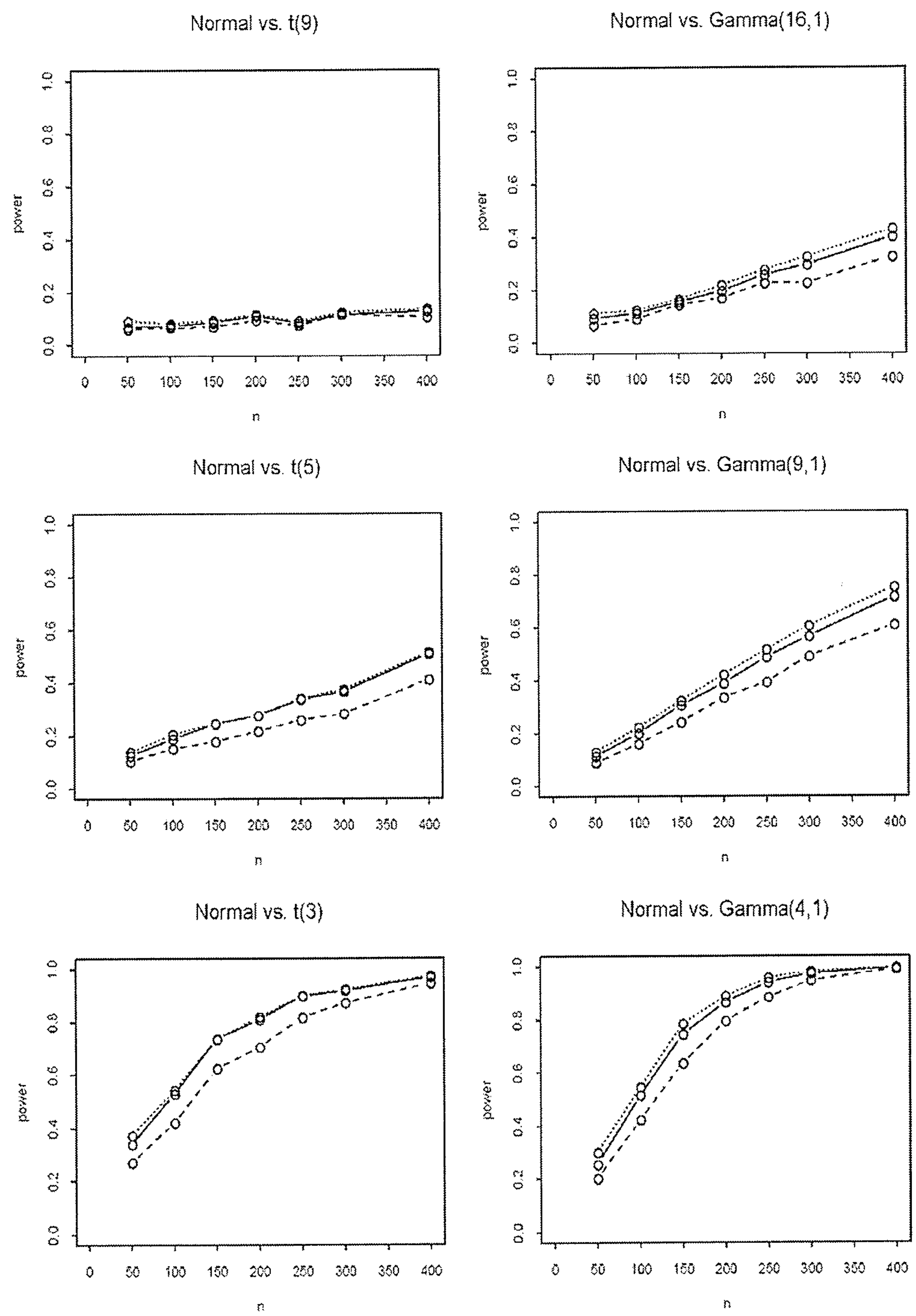

Figure 4.2: Power comparison when testing Normal versus t or Gamma at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ···: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

(4) Example 4: Normal versus *Weibull (a,b)*

In the fourth example, the alternative distribution is *Weibull(a,b)*, the Weibull distribution with shape parameter a and scale parameter b.

The skewness and kurtosis of the Weibull distribution are complicated, so we skip to calculate their values for a specified Weibull distribution. Without loss of generality, we only need to consider *Weibull(a,1)*. In figure 4.3 the simulated powers for the three statistics are from the value of a=*2, 3, 5* respectively.

(5) Example 5: Normal versus *Lognormal*(μ, σ^2)

In the fifth example, the alternative distribution is *Lognormal*(μ, σ^2), the lognormal distribution with parameter μ and σ^2 of the variable's logarithm.

The skewness and kurtosis for *Lognormal*(μ, σ^2) are $r_1 = (s+2)/\sqrt{(s-1)}$ and $r_2 = s^4 + 2s^3 + 3s^2 - 3$ with $s = e^{\sigma^2}$ (Johnson et al., 1994), which depend on σ only. We selected (μ, σ^2)=(0, 0.1), (0, 0.2), (0, 0.3) with r_1=0.302, 0.614, 0.950 and r_2=3.16, 3.68, 4.64, respectively.

We may also consider other alternative distributions, such as Logistic and F distributions, however the situations are similar to above examples, from which we concluded that our choice of the number of classes outperforms that used by Mann and Wald, with respect to the power of the Pearson χ^2 statistic. The simulated powers of the three statistics are also plotted in figure 4.3.

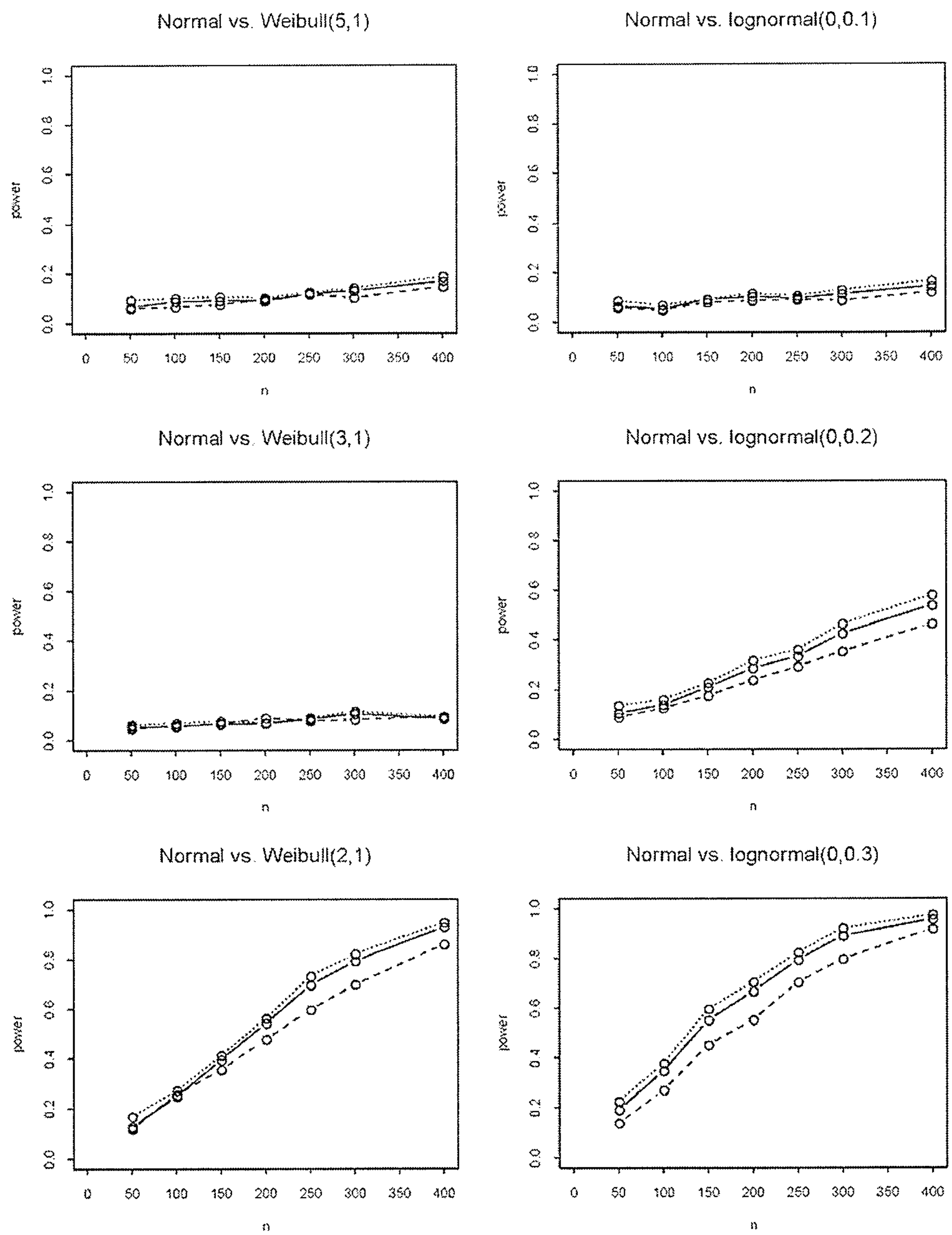

Figure 4.3: Power comparison when testing Normal versus Weibull or Lognormal at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ⋯: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

(6) Example 6: Normal versus Mixed normal

In the last example, the alternative distribution is mixed normal, which can generate more complicated and diverse distributions. We illustrated four mixed normal alternative distributions:

(a)$0.5N(1, 4)+0.5N(2, 16)$, (b) $0.3N(1, 4)+0.7N(3, 25)$,

(c) $0.5N(1, 4)+0.5N(3, 25)$, (d) $0.5N(1, 9)+0.5N(6, 100)$.

Figure 4.4 shows the simulated powers of the three statistics.

In summary, with respect to the power of the Pearson χ^2 statistic, our choice of the number of classes outperforms that used by Mann and Wald. The power of the likelihood ratio G^2 statistic is very close to that of the Pearson χ^2 statistic based on our new choice of the number of classes.

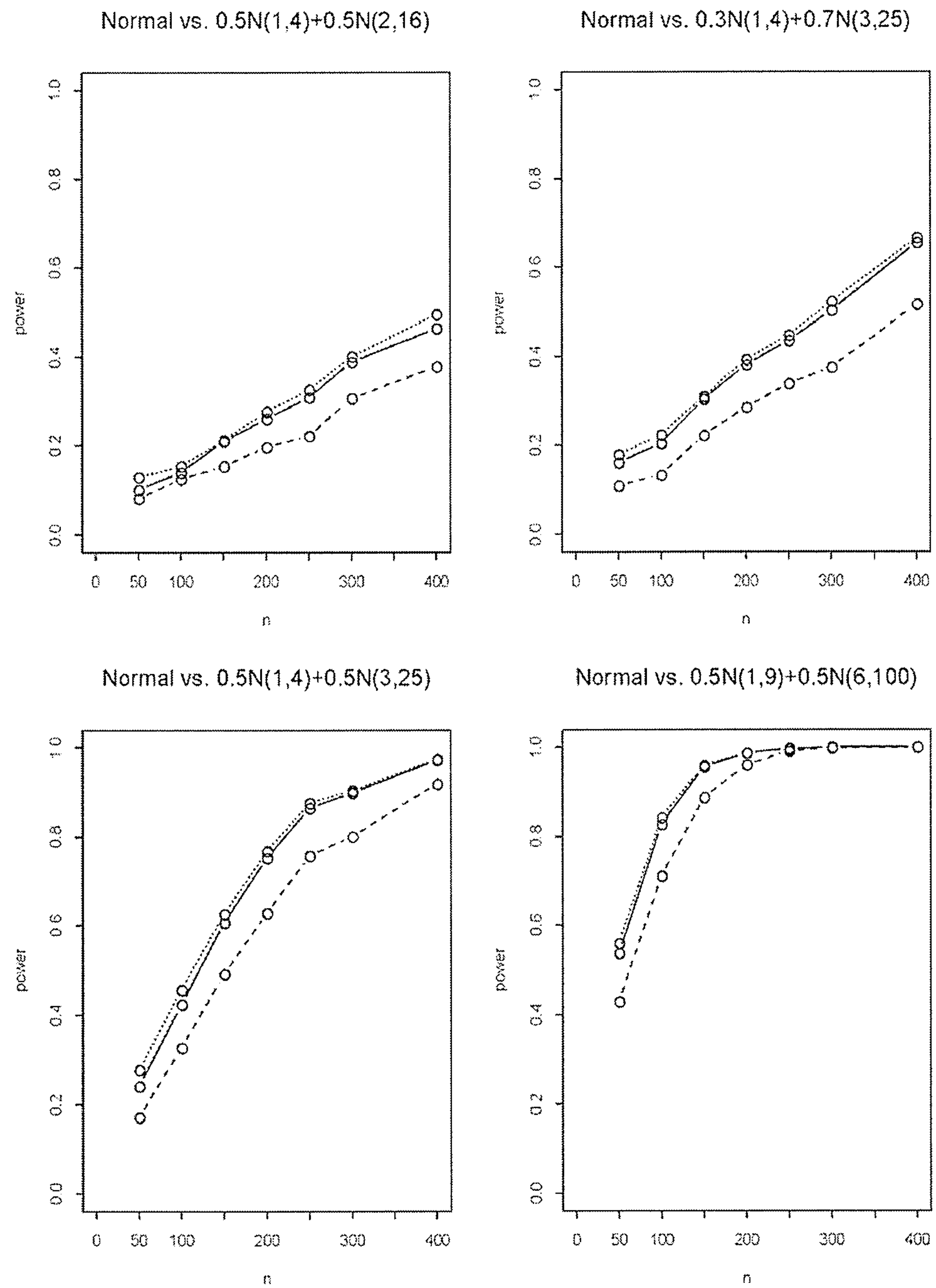

Figure 4.4: Power comparison when testing Normal versus mixed Normal at level $\alpha = 0.05$. —: Pearson X^2 based on the new choice of classes, ···: LR G^2 based on the new choice of classes, ---: Pearson X^2 based on Mann and Wald's choice of classes.

4.3 Chi-Squared Test for Exponentiality

In this section, we use chi-squared test based on our choice of the number of classes to test a non-negative continuous random variable X which comes from a family of exponential distributions. Under the test significance level $\alpha = 0.05$, we compared the powers of two Pearson χ^2 statistics and the likelihood ratio G^2 statistic by using Monte Carlo simulation.

For exponential distribution family $Exp(\theta)$ with unknown scale parameter, first we need to estimate θ by maximum likelihood method from sample. It is easy to see that $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the maximum likelihood estimate of θ. The chi-squared test statistics for null hypothesis of exponential distributions are invariant under any affine transformation $y = x/\hat{\theta}$, so the problem becomes testing $F_0(x) = \Psi(x/\hat{\theta})$ with $\Psi(x)$ denoting the cumulative distribution function of the standard exponential distribution $Exp(1)$.

We still consider the following three type alternatives against the exponential distribution family $Exp(\theta)$ with scale parameter θ.

(a) the Weibull distribution $Weibull(a,b)$ with shape parameter a and scale parameter b.

(b) the Gamma distribution $Gamma(a,b)$ with shape parameter a and scale parameter b.

(c) the log-normal distribution $Lognormal(\mu, \sigma^2)$ where μ and σ^2 are the mean and variance of the variable's logarithm.

(1) Example 1: $H_0 : Exp(\theta)$ versus $H_1 : \text{Weibull}(a, b)$

In the first example the alternative hypothesis is *Weibull(a,b)*, we choose (*a, b*)=(1.5, 1), (1.1, 1), (0.8, 1), (0.8, 3). The simulated powers for the three statistics are plotted in figure 4.5.

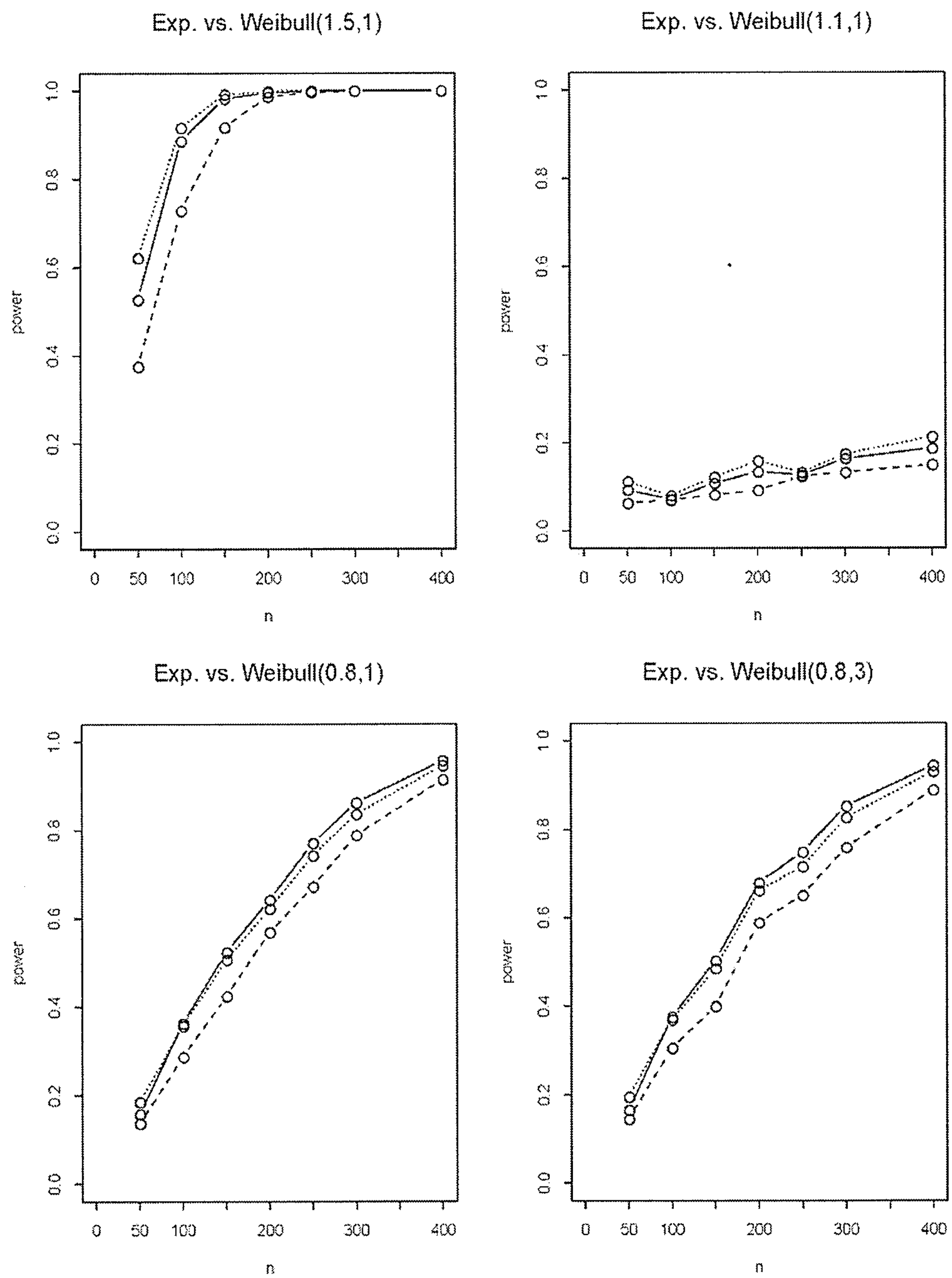

Figure 4.5: Power comparison when testing Exponential versus Weibull at level $\alpha = 0.05$. —: Pearson X^2 based on the new choice of classes, ···: LR G^2 based on the new choice of classes, ---: Pearson X^2 based on Mann and Wald's choice of classes.

(2) Example 2: $H_0 : Exp(\theta)$ versus $H_1 : \text{Gamma}(a, b)$

In the second example the alternative hypothesis is *Gamma(a, b)*, we choose (*a, b*)=(2, 1), (1.5, 1), (0.9, 1), (0.8, 1/3). The simulated powers for the three statistics are plotted in figure 4.6.

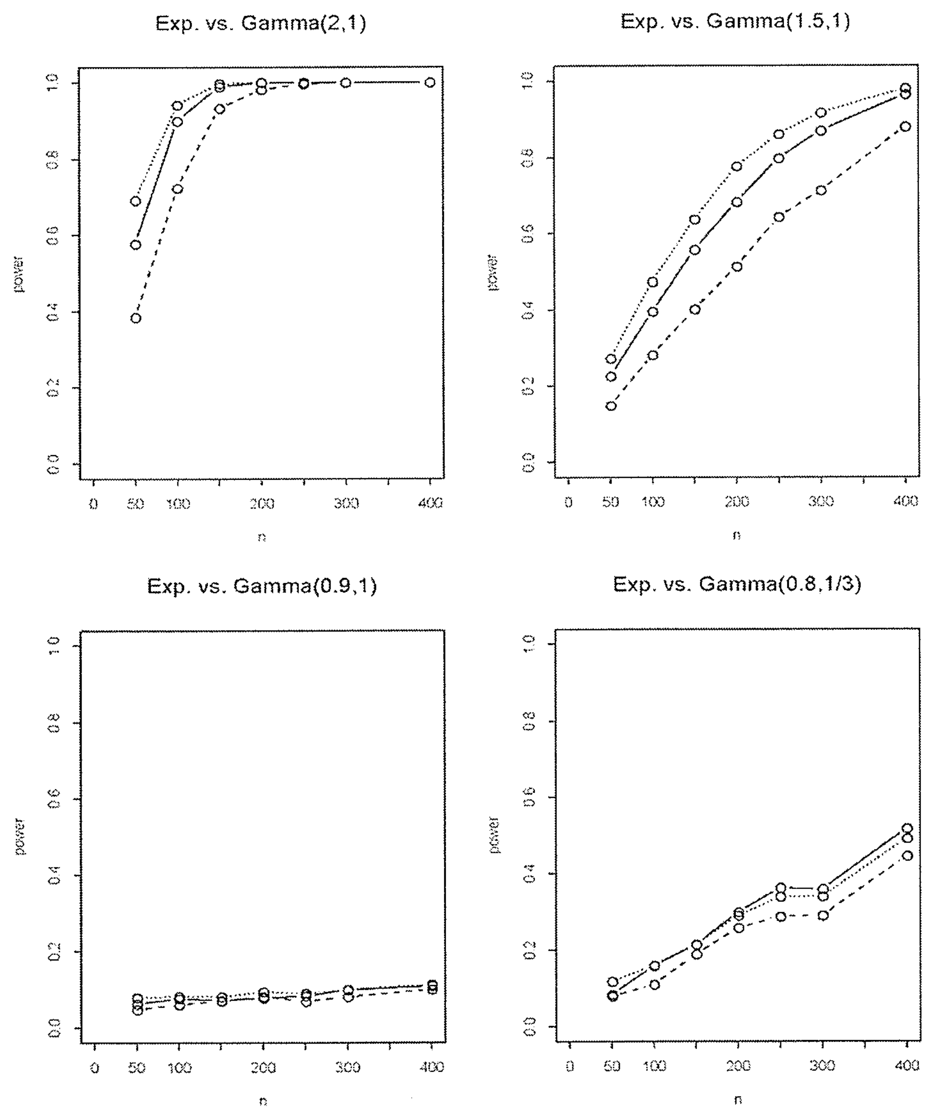

Figure 4.6: Power comparison when testing Exponential versus Gamma at level $\alpha = 0.05$. —: Pearson X^2 based on the new choice of classes, ⋯: LR G^2 based on the new choice of classes, ---: Pearson X^2 based on Mann and Wald's choice of classes.

(3) Example 3: $H_0 : Exp(\theta)$ versus H_1 : Lognormal(μ, σ^2)

In the third example the alternative hypothesis is $Lognormal(\mu, \sigma^2)$, we choose $(\mu, \sigma^2) = (-1.2, 2), (-0.6, 1.3), (-0.3, 0.8), (0.8, 0.8)$. The simulated powers for the three statistics are graphed in figure 4.7.

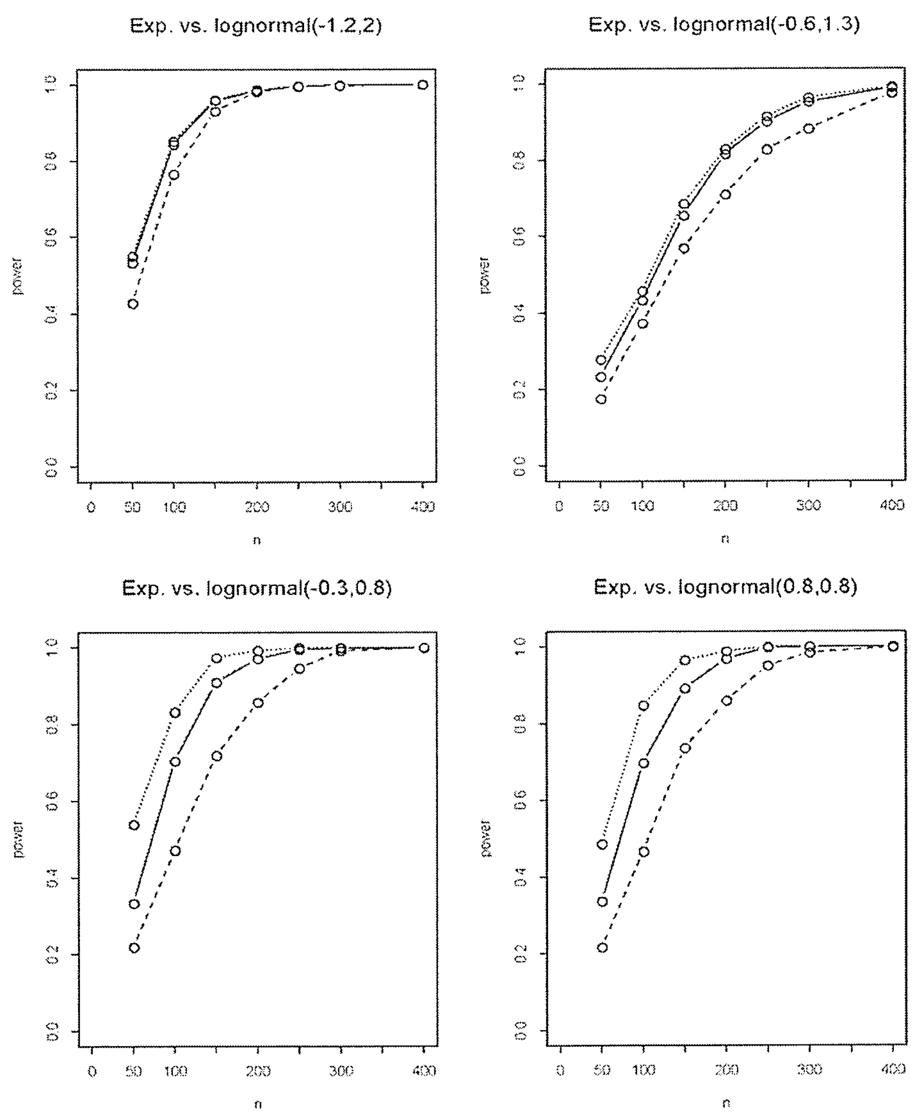

Figure 4.7: Power comparison when testing Exponential versus Lognormal at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ⋯: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

From the above three examples, we concluded that our choice of the number of classes is better than that used by Mann and Wald, with respect to the power of the Pearson χ^2 statistic. The power of the likelihood ratio G^2 statistic is close to that of the Pearson χ^2 statistic based on our choice of the number of classes, and sometimes the likelihood ratio G^2 statistic is slightly more powerful than the Pearson χ^2 statistic.

4.4 Chi-Squared Test for a Family of Uniform

In this section, chi-squared test are used to test the null hypothesis of a family of continuous uniform distributions with unknown parameters based on our choice of the number of classes. Under the test significance level $\alpha = 0.05$, we compared the powers of two Pearson χ^2 statistics and the likelihood ratio G^2 statistic by using Monte Carlo simulation.

In order to use chi-squared goodness-of-fit test, first we need to estimate the unknown parameters of start point a and end point b from a uniform distribution by maximum likelihood method. Suppose X from the continuous uniform distribution having X_1, $\mathrm{X}_2, \cdots, \mathrm{X}_n$ samples. Then adjusted MLE (unbiased) of a and b is computed from the following formulae:

$\hat{a} = \max(x) \cdot (n+1) - n \cdot b$,

$$\hat{b} = \frac{\max(x) \cdot n \cdot (n+1) - \min(x) \cdot (n+1)}{n^2 - 1}$$

The chi-squared test statistics for null hypothesis of uniform distributions are invariant under any affine transformation $y = (x - \hat{a})/(\hat{b} - \hat{a})$, so the problem becomes testing $F_0(x) = \Theta((x - \hat{a})/(\hat{b} - \hat{a}))$ with $\Theta(x)$ denoting the cumulative distribution function of the standard uniform distribution *Unif(0, 1)*.

(1) Example 1: $H_0 : Unif(a,b)$ versus $H_1 : \text{Beta}(p,q)$

Beta is a frequently used alternative hypothesis when testing uniform distribution. For alternative hypothesis *Beta(p, q)*, we choose (*p, q*)=(0.8, 0.6), (0.8, 0.8), (1.2, 1.3), (1.7,1.6) to illustrate the simulated powers for two Pearson χ^2 and the likelihood ratio G^2 test statistics, which are plotted in figure 4.8.

(2) Example 2: $H_0 : Unif(a,b)$ versus $H_1 : \text{Mixed Beta}$

In this example, we choose four cases of mixed Beta to show the simulated powers for the three test statistics as shown in figure 4.9.

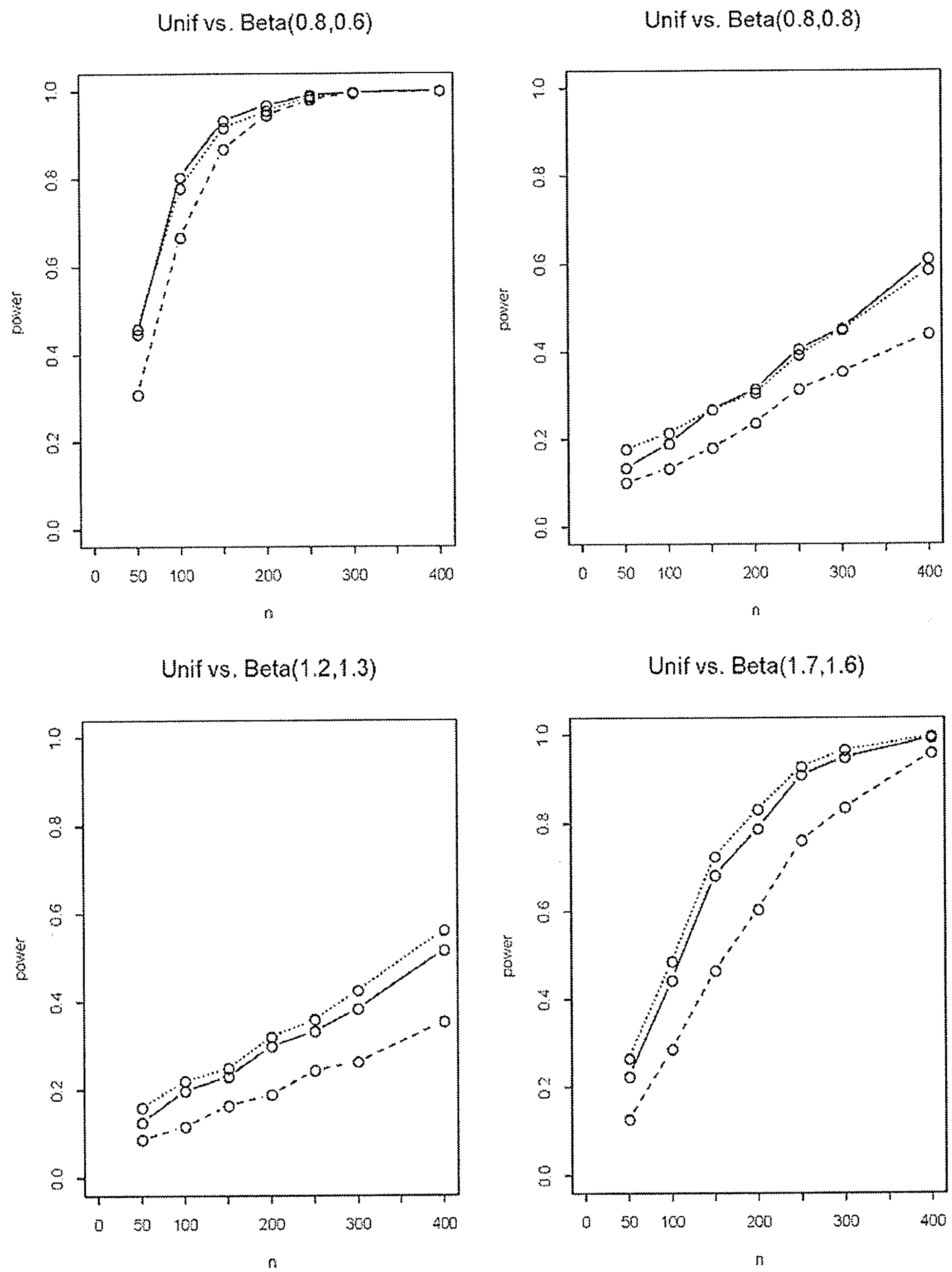

Figure 4.8: Power comparison when testing Uniform versus Beta at level $\alpha = 0.05$. —: Pearson X^2 based on the new choice of classes, ⋯: LR G^2 based on the new choice of classes, ---: Pearson X^2 based on Mann and Wald's choice of classes.

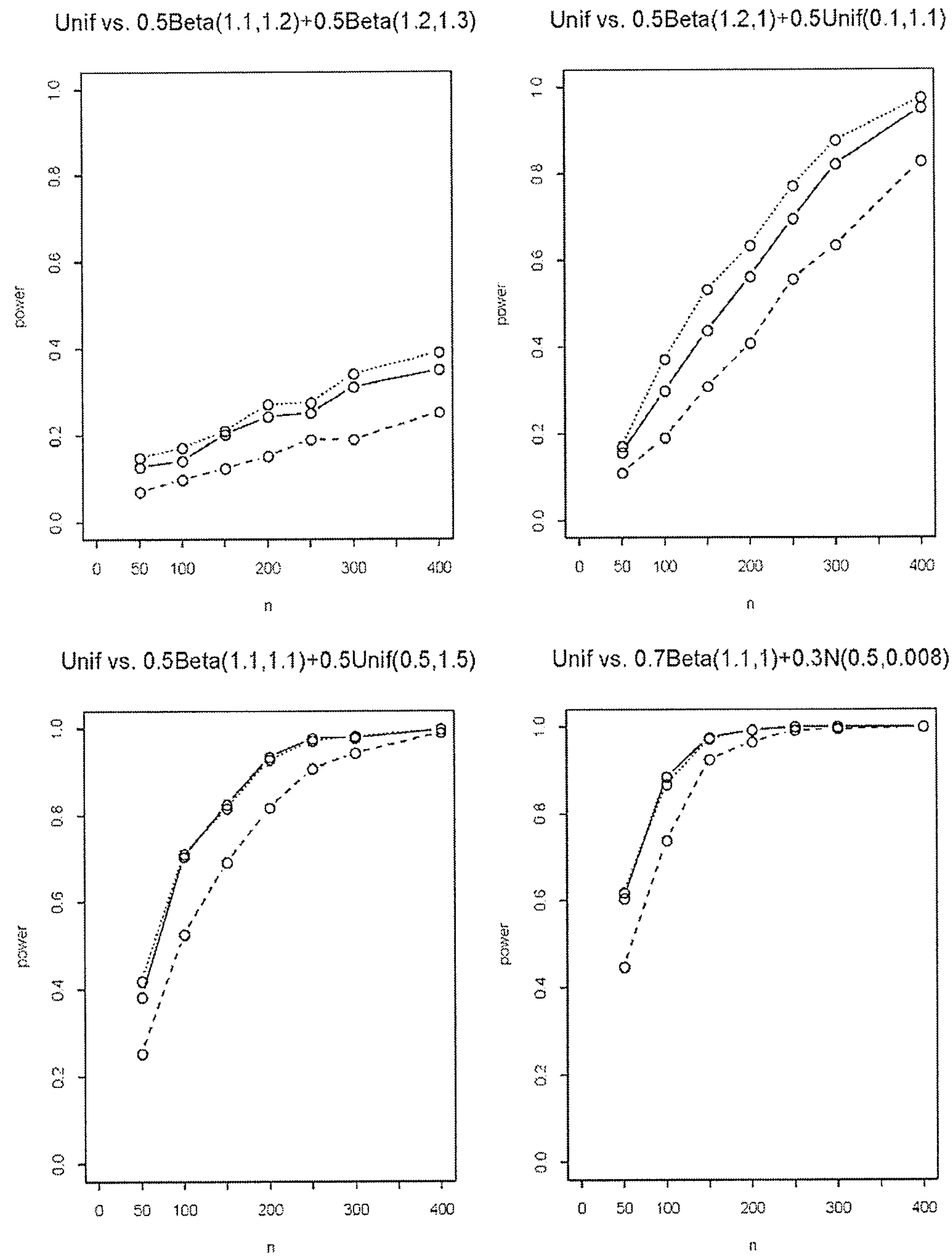

Figure 4.9: Power comparison when testing Uniform versus mixed Beta at level $\alpha = 0.05$. —: Pearson χ^2 based on the new choice of classes, ⋯: LR G^2 based on the new choice of classes, ---: Pearson χ^2 based on Mann and Wald's choice of classes.

From the above two examples, we concluded that our choice of the number of classes outperforms that used by Mann and Wald, with respect to the power of the Pearson χ^2 statistic. Based on our new choice of the number of classes, the power of the likelihood ratio G^2 statistic is very close to or sometimes slightly more powerful than that of the Pearson χ^2 statistic.

Chapter 5

Conclusions and Further Questions

5.1 Conclusions

Based on the findings from our computer intensive simulation, we summarized two empirical formulae for the number of classes when applying the chi-squared Pearson χ^2 or likelihood ratio G^2 to test null hypotheses under either fully specified distributions (Case 0) or not fully specified distributions (Case 1). Case 0 is to test null hypothesis of a specified distribution with known parameters, and Case 1 is to test null hypothesis of a family of distributions with unknown parameters. Given significance level or type I error $\alpha = 0.05$, the empirical formulae for the most appropriate number of classes k from our simulation results are as follows:

Case 0: $k = [\, 2.81 + 1.23 n^{2/5} \,]$

Case 1: $k = [\, 2n^{2/5} \,]$

The empirical formulae are expressed in a simple way which is only related with the sample size n at significance level $\alpha = 0.05$. For continuous distribution with location and scale parameters such as normal, exponential, and uniform distributions, we have illustrated the power gained by comparing our selected number of classes with the theoretical "best" number of classes from the Mann and Wald's approach. Even if test significance level α is not 0.05, we have verified by using simulation that the above formulae are still effective because the "best" k is less sensitive to the change of α, especially when α is between 0.01 and 0.1 .

Although we illustrated the new empirical formulae by demonstrating the examples from continuous distributions with location and scale parameters, the formulae themselves can be extended to grouped data when testing a more general simple or composite hypothesis. Our findings basically conform to the judgment of Williams that the value of k from the Mann and Wald's formula can probably be halved in practice. Mann and Wald's formula as well as Williams's judgment is the most accepted guide on applying the Pearson chi-squared test for any continuous distribution.

Our results are implied to use a standardized procedure when applying chi-squared goodness-of-fit test to test a general continuous distribution. First, the data will be grouped by "equiprobability intervals", that means the classes will be divided by equalizing the cumulative probability into k groups. Second, given sample size n, the most appropriate classes k can be calculated from the new empirical formulae according to Case 0 or Case 1. Third, the cutoff points of classification are determined by the percentage points of a specified null distribution in Case 0, or by the percentage points of the underlying null distribution by plugging in the MLE of parameters in

Case 1. This procedure will make the chi-squared goodness-of-fit test more easy to use and more effective.

5.2 Further Questions

Due to time limitations, we have to leave the following discussions in future study.

(1) Based on the new findings of the choice of the number of classes, we can further do some computer simulations which are extended to test a more general continuous distribution from the distribution with location and scale parameters, to examine the extent of power improvement.

(2) In our Monte Carlo simulation study, the most appropriate number of classes in the chi-squared goodness-of-fit test is based on the grouping method of "equiprobability intervals", which is applying balanced partitions of the cumulative probability, and it is more applicable than using unbalanced partitions to group data in chi-squared goodness-of-fit test. A further comparison of the power of test with equal intervals of measurement or equal observed counts in each class can be demonstrated, which are generally the grouping method of "unequiprobability intervals".

(3) Based on the new empirical formulae, we can evaluate the overall power efficiency of the chi-squared Pearson χ^2 or likelihood ratio G^2 statistics comparing with other goodness-of-fit test statistics such as Anderson-Darling, Cramer-von Mises and Kolmogorov-Smirnov etc. by using Monte Carlo simulation.

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Appendix 1: Checking the Test Size α for Various Classes k

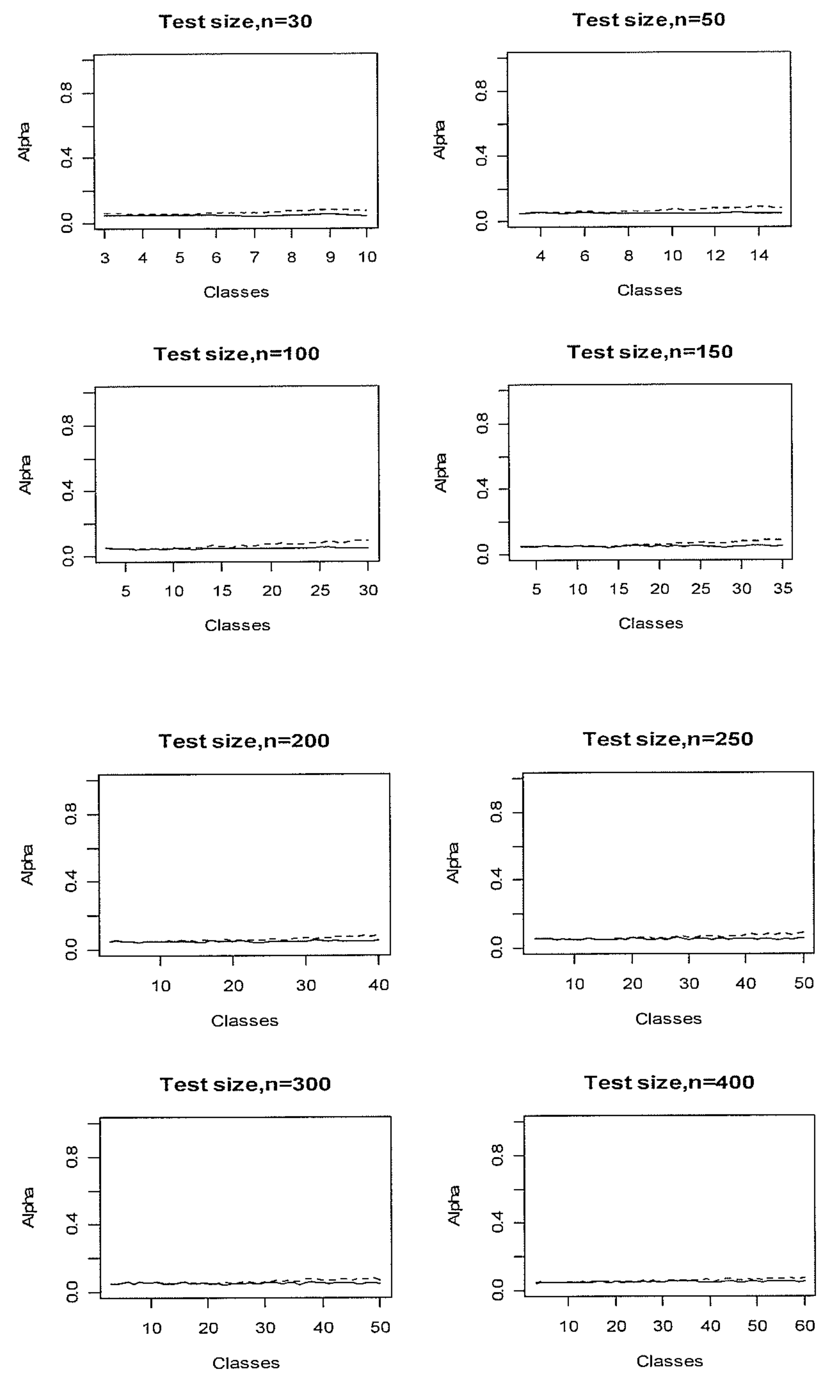

Figure 1: Test size for Normal(0, 1). —: Pearson chi-squared statistic, ---: LR chi-squared statistic.

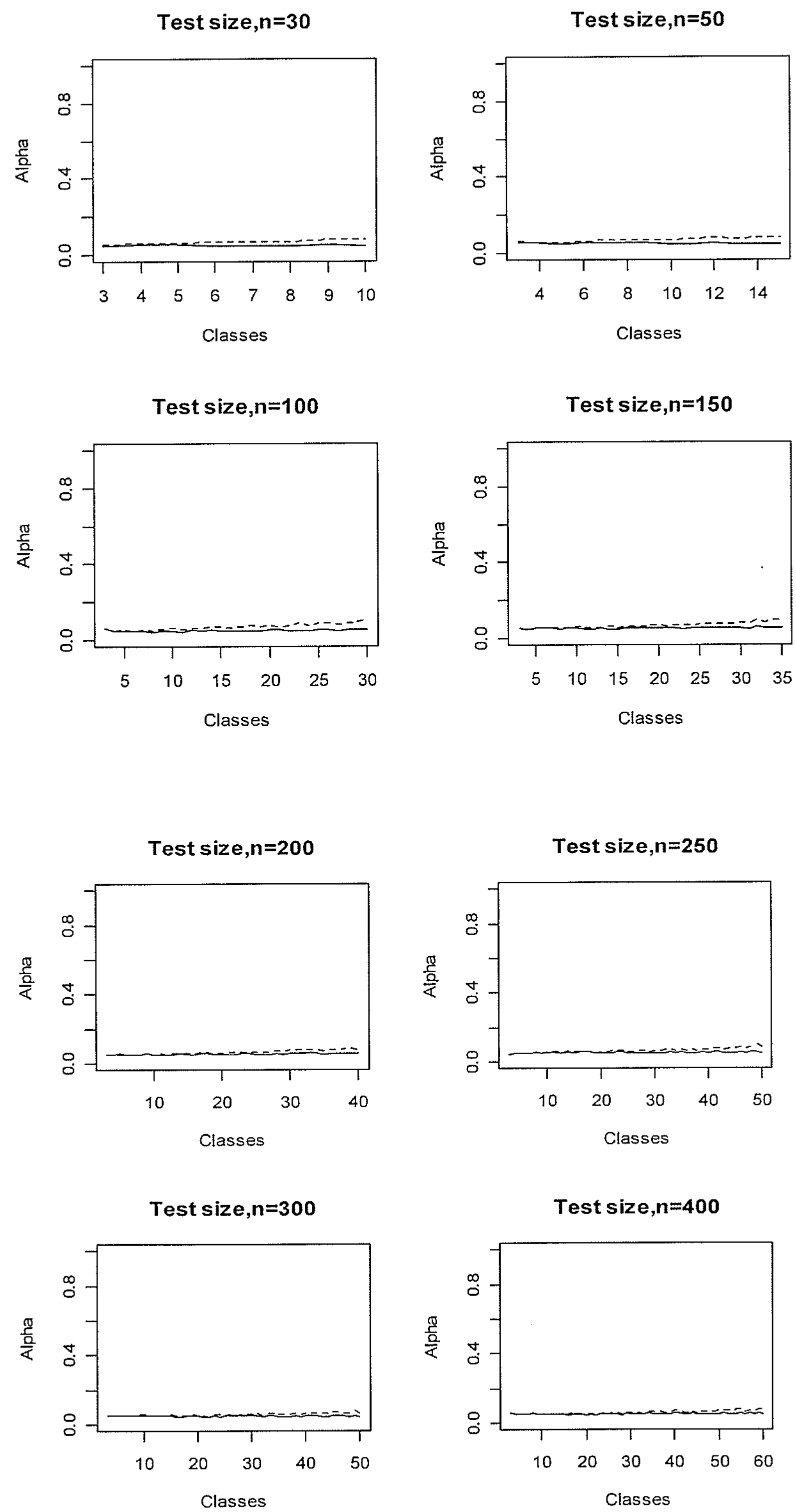

Figure 2: Test size for Exp(1). —: Pearson chi-squared statistic, ---: LR chi-squared statistic.

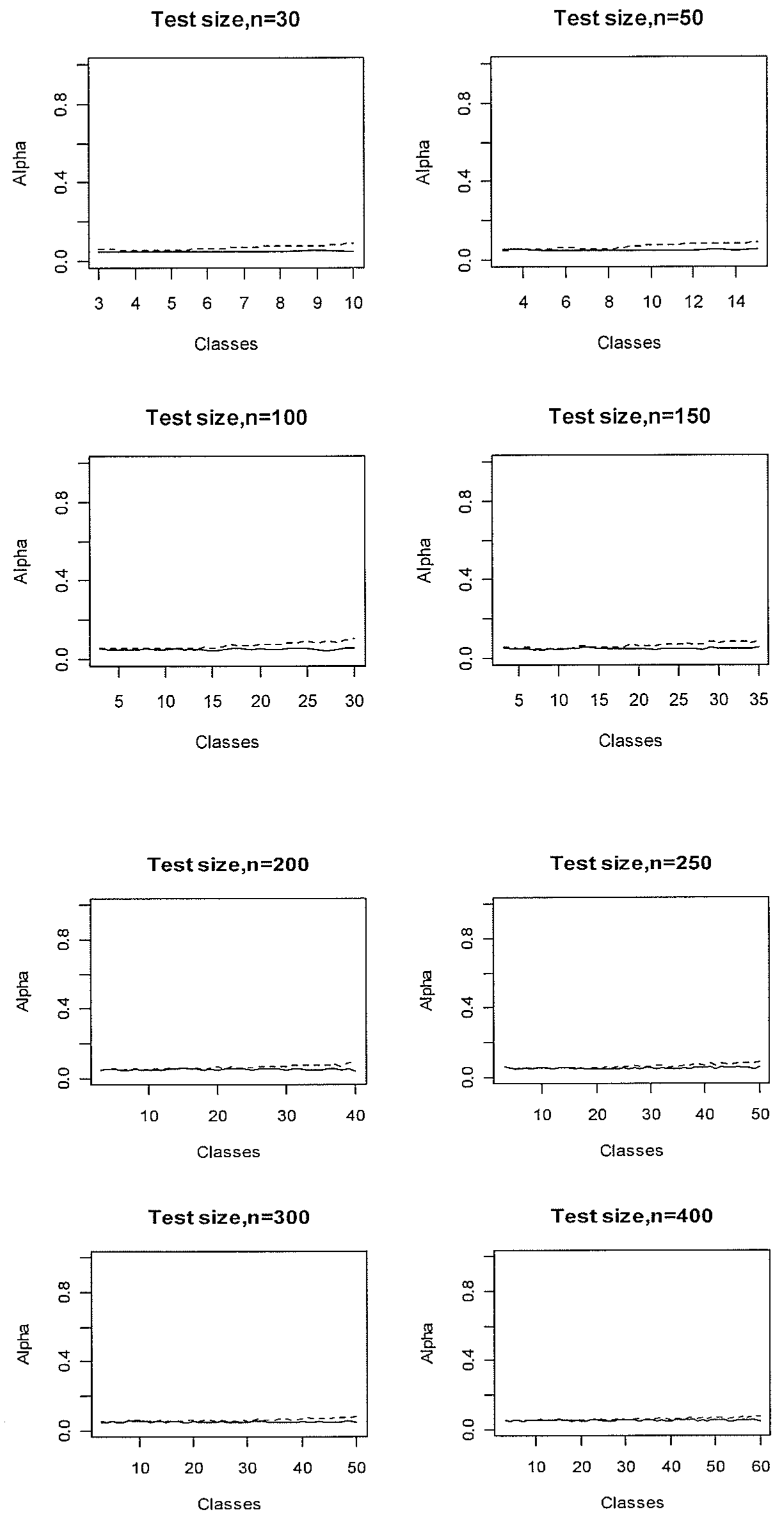

Figure 3: Test size for Unif(0, 1). —: Pearson chi-squared statistic, ---: LR chi-squared statistic.

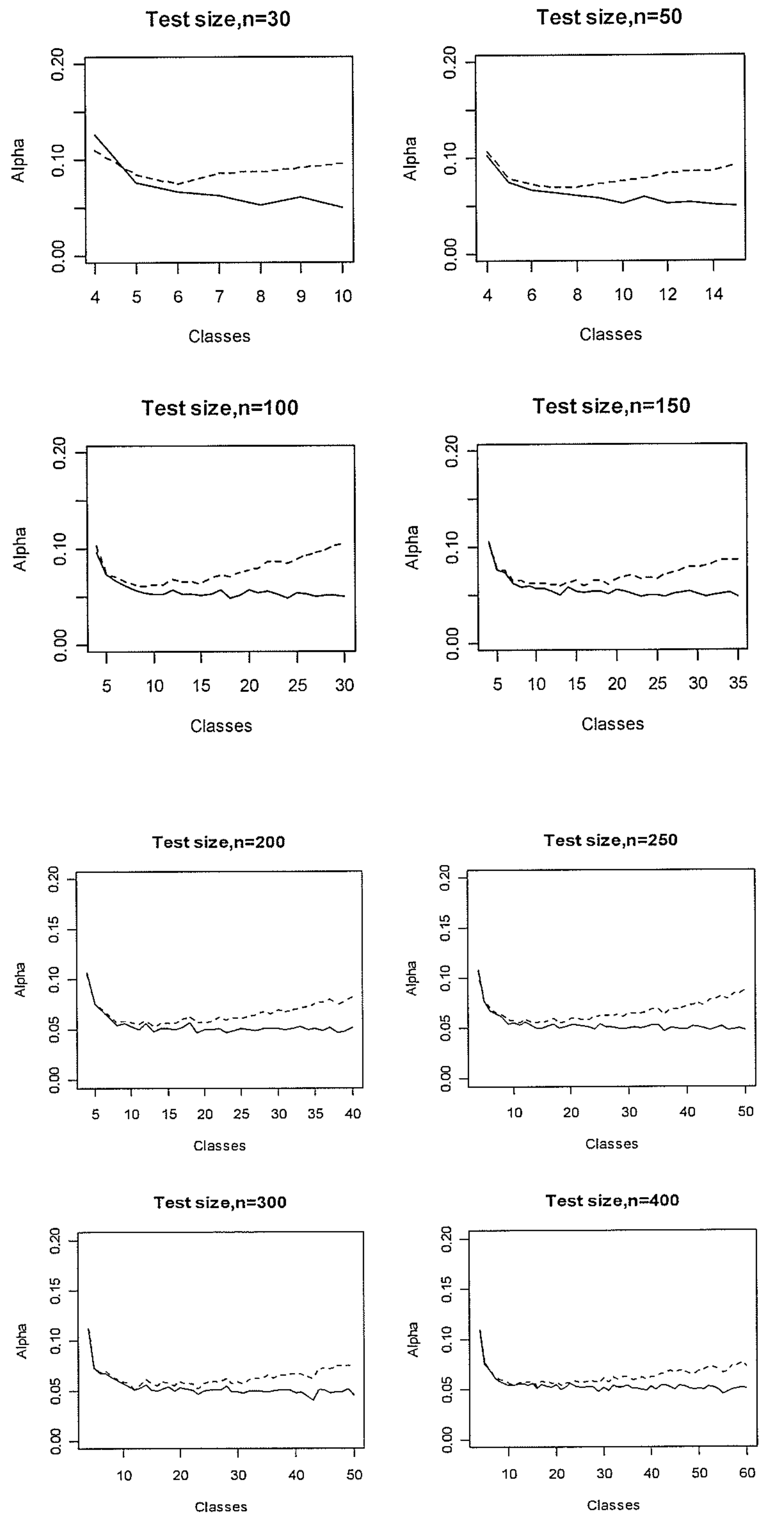

Figure 4: Test size for a family of Normal distribution. —: Pearson chi-squared statistic, ---: LR chi-squared statistic.

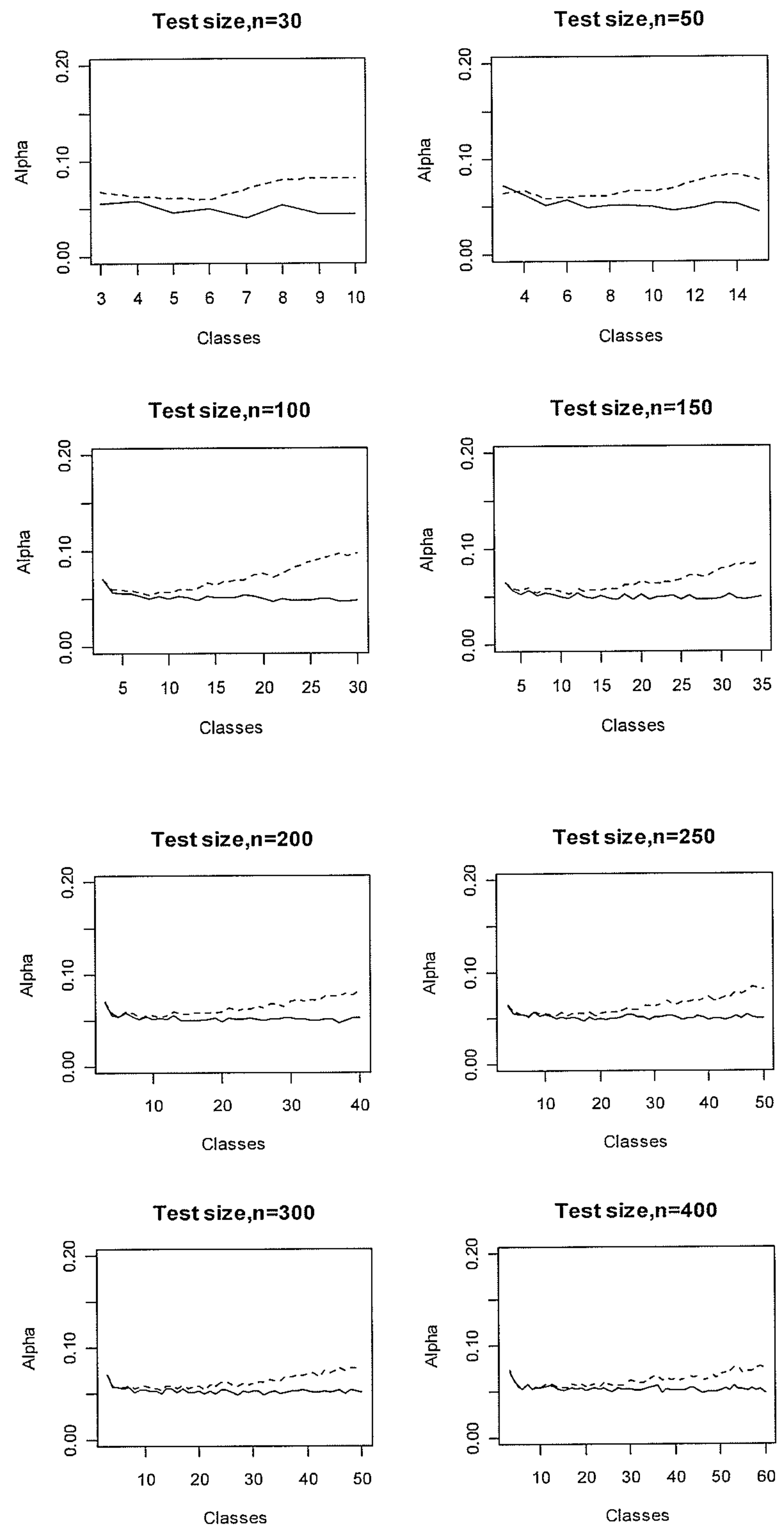

Figure 5: Test size for a family of Exponential distribution. —: Pearson chi-squared statistic, ---: LR chi-squared statistic.

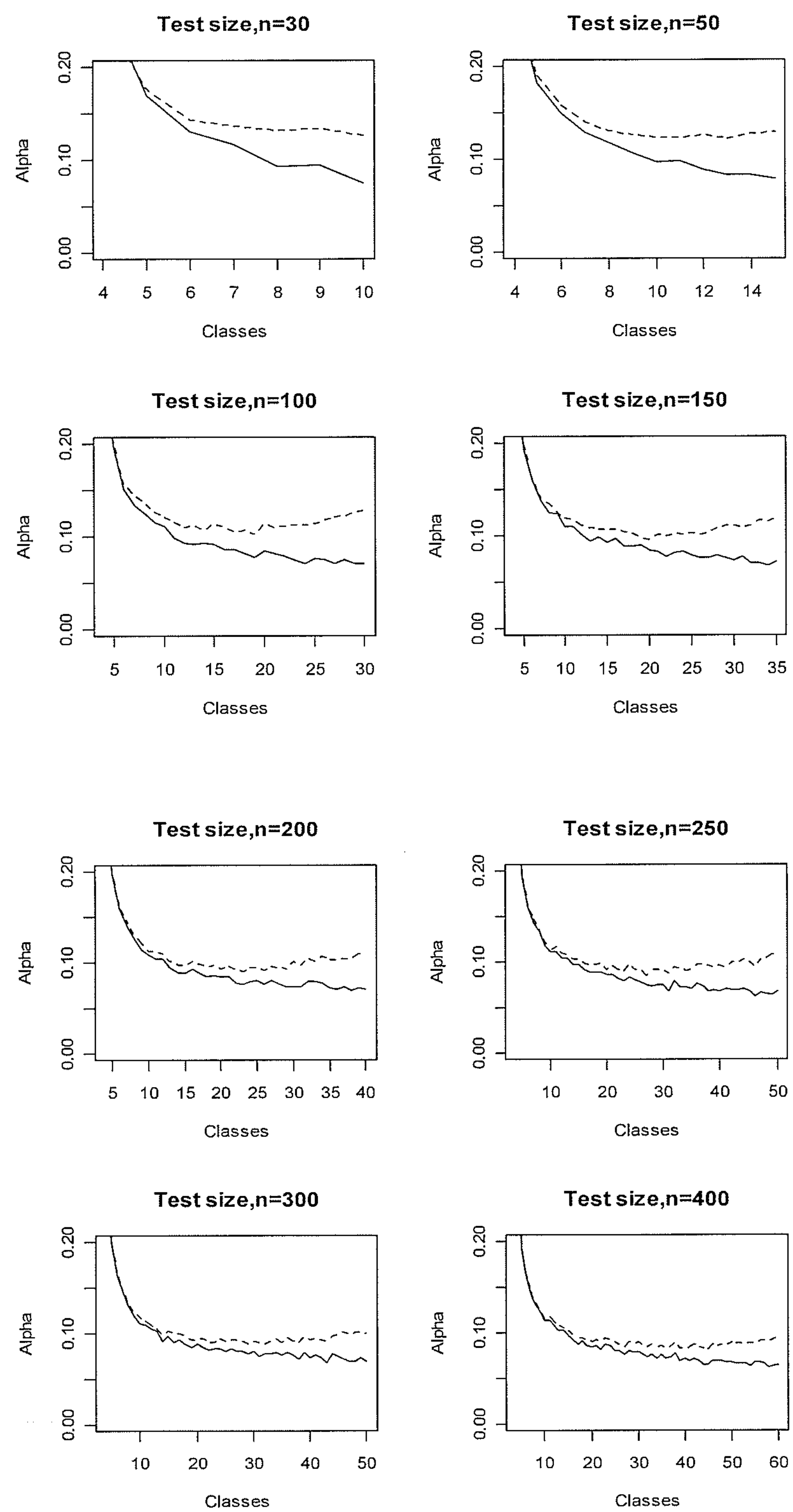

Figure 6: Test size for a family of Uniform distribution. —: Pearson chi-squared statistic, ---: LR chi-squared statistic.

Appendix 2: Selected S+/R Codes

```
#--------------------------------------------------------------
#   Checking the test size alpha for various classes k's
#
#                          Case 0
#---------------------------------------------------------------

#1. Norm(0,1)

f =function(n,grp)
{
    x=rnorm(n,0,1)
    X =matrix(0,2,grp)
    for (k in 3:grp)
    {
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
    }
    X

}


ff =function(n,grp,alpha, N, label=' ')
 {
 u=v=rep(0,grp)
 for (i in 1:N)
 {
    X=f(n,grp)
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
 }
   PX=u/N ; PG =v/N
   plot(3:grp,
PX[3:grp],type="l",ylim=c(0,1),main=label,ylab='Alpha',xlab='Classes' )
   lines(3:grp,PG[3:grp],type="l",lty=2)
   return(list(Pearson=PX,LR=PG))
```

```
 }

par(mfrow=c(2,2))
ff(30,10,0.05,100000, 'Test size,n=30')
ff(50,15,0.05,100000, 'Test size,n=50')
ff(100,30,0.05,100000,'Test size,n=100')
ff(150,35,0.05,100000, 'Test size,n=150')

windows()
par(mfrow=c(2,2))
ff(200,40,0.05,100000, 'Test size,n=200')
ff(250,50,0.05,100000, 'Test size,n=250')
ff(300,50,0.05,100000 , 'Test size,n=300')
ff(400,60,0.05,100000 , 'Test size,n=400')


#2. Exp(1)

f =function(n,grp)
{
    x=rexp(n,1)
    X =matrix(0,2,grp)
    for (k in 3:grp)
    {
    o=1:(k-1)
    xqtl=qexp((1:(k-1))/k,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
    }
    X

}


ff =function(n,grp,alpha, N, label=' ')
  {
  u=v=rep(0,grp)
  for (i in 1:N)
  {
    X=f(n,grp)
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
```

```
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
  }
   PX=u/N ; PG =v/N
   plot(3:grp,
PX[3:grp],type="l",ylim=c(0,1),main=label,ylab='Alpha',xlab='Classes' )
   lines(3:grp,PG[3:grp],type="l",lty=2)
   return(list(Pearson=PX,LR=PG))
  }

par(mfrow=c(2,2))
ff(30,10,0.05,100000 , 'Test size,n=30')
ff(50,15,0.05,100000 , 'Test size,n=50')
ff(100,30,0.05,100000, 'Test size,n=100')
ff(150,35,0.05,100000, 'Test size,n=150')

windows()
par(mfrow=c(2,2))
ff(200,40,0.05,100000, 'Test size,n=200')
ff(250,50,0.05,100000, 'Test size,n=250')
ff(300,50,0.05,100000, 'Test size,n=300')
ff(400,60,0.05,100000, 'Test size,n=400')


#3. Unif(0,1)

f =function(n,grp)
{
    x=runif(n,0,1)
    X =matrix(0,2,grp)
    for (k in 3:grp)
    {
    o=1:(k-1)
    xqtl=qunif((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
    }
    X
}
```

```
ff =function(n,grp,alpha, N, label=' ')
 {
 u=v=rep(0,grp)
 for (i in 1:N)
 {
    X=f(n,grp)
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
 }
   PX=u/N ; PG =v/N
   plot(3:grp,
PX[3:grp],type="l",ylim=c(0,1),main=label,ylab='Alpha',xlab='Classes' )
   lines(3:grp,PG[3:grp],type="l",lty=2)
   return(list(Pearson=PX,LR=PG))
 }

par(mfrow=c(2,2))
ff(30,10,0.05,100000, 'Test size,n=30')
ff(50,15,0.05,100000, 'Test size,n=50')
ff(100,30,0.05,100000,'Test size,n=100')
ff(150,35,0.05,100000, 'Test size,n=150')

windows()
par(mfrow=c(2,2))
ff(200,40,0.05,100000, 'Test size,n=200')
ff(250,50,0.05,100000, 'Test size,n=250')
ff(300,50,0.05,100000, 'Test size,n=300')
ff(400,60,0.05,100000, 'Test size,n=400')
```

```
#----------------------------------------------------------------
#   Checking the test size alpha for various classes k's
#
#                        Case 1
#----------------------------------------------------------------

#1. A family of Normal distributions

f =function(n,grp)
{

    x=rnorm(n,0,1)
    x=(x-mean(x))/sqrt(var(x))
    X =matrix(0,2,grp)
    for (k in 4:grp)
    {
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,0,1)
    for (i in 1:(k-1))   o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
    }
    X
}


ff =function(n,grp,alpha, N, label=' ')
 {
 u=v=rep(0,grp)
 for (i in 1:N)
 {
    X=f(n,grp)
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,1,4:grp-3)))
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,1,4:grp-3)))
 }
   PX=u/N ; PG =v/N
   plot(4:grp, PX[4:grp],type="l",ylim=c(0,0.2),
main=label,ylab='Alpha',xlab='Classes' )
   lines(4:grp,PG[4:grp],type="l",lty=2)
   return(list(Pearson=PX,LR=PG))
 }
```

```
par(mfrow=c(2,2))
ff(30,10,0.05,100000, 'Test size,n=30')
ff(50,15,0.05,100000, 'Test size,n=50')
ff(100,30,0.05,100000,'Test size,n=100')
ff(150,35,0.05,100000, 'Test size,n=150')

windows()
par(mfrow=c(2,2))
ff(200,40,0.05,100000, 'Test size,n=200')
ff(250,50,0.05,100000, 'Test size,n=250')
ff(300,50,0.05,100000, 'Test size,n=300')
ff(400,60,0.05,100000, 'Test size,n=400')



#2. A family of Exponential distributions

f =function(n,grp)
{

    x=rexp(n,1)
    x=x/mean(x)
    X =matrix(0,2,grp)
    for (k in 3:grp)
    {
    o=1:(k-1)
    xqtl=qexp((1:(k-1))/k,1)
    for (i in 1:(k-1))   o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
    }
    X
}


ff =function(n,grp,alpha, N, label=' ')
 {
 u=v=rep(0,grp)
 for (i in 1:N)
 {
    X=f(n,grp)
```

```
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,3:grp-2)))
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,3:grp-2)))
  }
  PX=u/N ; PG =v/N
  plot(3:grp, PX[3:grp],type="l",ylim=c(0,0.2),
main=label,ylab='Alpha',xlab='Classes' )
  lines(3:grp,PG[3:grp],type="l",lty=2)
  return(list(Pearson=PX,LR=PG))
 }

par(mfrow=c(2,2))
ff(30,10,0.05,100000, 'Test size,n=30')
ff(50,15,0.05,100000, 'Test size,n=50')
ff(100,30,0.05,100000,'Test size,n=100')
ff(150,35,0.05,100000, 'Test size,n=150')

windows()
par(mfrow=c(2,2))
ff(200,40,0.05,100000, 'Test size,n=200')
ff(250,50,0.05,100000, 'Test size,n=250')
ff(300,50,0.05,100000, 'Test size,n=300')
ff(400,60,0.05,100000, 'Test size,n=400')


#3. A family of Uniform distributions

f =function(n,grp)
{

    y=runif(n,0,1)
    b=(max(y)*n*(n+1)-min(y)*(n+1))/(n^2-1)
    a=max(y)*(n+1)-n*b
    x=(y-a)/(b-a)

    X =matrix(0,2,grp)
    for (k in 4:grp)
    {
    o=1:(k-1)
    xqtl=qunif((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
```

```
    }
    X
}


ff =function(n,grp,alpha, N, label=' ')
 {
 u=v=rep(0,grp)
 for (i in 1:N)
 {
    X=f(n,grp)
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,1,4:grp-3)))
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,1,4:grp-3)))
 }
   PX=u/N ; PG =v/N
   plot(4:grp, PX[4:grp],type="l",ylim=c(0,0.2),
main=label,ylab='Alpha',xlab='Classes' )
   lines(4:grp,PG[4:grp],type="l",lty=2)
   return(list(Pearson=PX,LR=PG))
 }

par(mfrow=c(2,2))
ff(30,10,0.05,100000, 'Test size,n=30')
ff(50,15,0.05,100000, 'Test size,n=50')
ff(100,30,0.05,100000,'Test size,n=100')
ff(150,35,0.05,100000, 'Test size,n=150')

windows()
par(mfrow=c(2,2))
ff(200,40,0.05,100000, 'Test size,n=200')
ff(250,50,0.05,100000, 'Test size,n=250')
ff(300,50,0.05,100000, 'Test size,n=300')
ff(400,60,0.05,100000, 'Test size,n=400')
```

```
#------------------------------------------------------------
#              The pattern of power of test
#
#                        Case 0
#-------------------------------------------------------------


#1. N(0,1) vs. cauchy(0,0.5)

f =function(n,grp)
{

    x=rcauchy(n,0,0.5)
    X =matrix(0,2,grp)
    for (k in 3:grp)
    {
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
    }
    X
}


ff =function(n,grp,alpha, N)
 {
 u=v=rep(0,grp)
 for (i in 1:N)
 {
    X=f(n,grp)
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
 }
   PX=u/N ; PG =v/N
   plot(3:grp, PX[3:grp],type="l",ylim=c(0,1),main='N(0,1) vs.
cauchy(0,0.5)',ylab='Power',xlab='Classes' )
   lines(3:grp,PG[3:grp],type="l",lty=2)
   return(list(Pearson=PX,LR=PG))
 }
```

```
par(mfrow=c(3,2))
ff(100,30,0.05,100000)


#2. N(0,1) vs. Unif(-1.6,1.6)

f =function(n,grp)
{
    x=runif(n,-1.6,1.6)
    X =matrix(0,2,grp)
    for (k in 3:grp)
    {
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
    }
    X
}


ff =function(n,grp,alpha, N)
 {
 u=v=rep(0,grp)
 for (i in 1:N)
 {
    X=f(n,grp)
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,3:grp-1)))
 }
   PX=u/N ; PG =v/N
   plot(3:grp, PX[3:grp],type="l",ylim=c(0,1),main='N(0,1) vs.
Unif(-1.6,1.6)',ylab='Power',xlab='Classes' )
   lines(3:grp,PG[3:grp],type="l",lty=2)
   return(list(Pearson=PX,LR=PG))
 }

ff(100,30,0.05,100000)
```

```
#------------------------------------------------------------
#               The pattern of power of test
#
#                         Case 1
#-------------------------------------------------------------

#1. Normal vs. Beta(1.5,1.5)

f =function(n,grp)
{
    x=rbeta(n,1.5,1.5)
    x=(x-mean(x))/sqrt(var(x))
    X =matrix(0,2,grp)
    for (k in 5:grp)
    {
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
    }
    X
}


ff =function(n,grp,alpha, N)
 {
 u=v=rep(0,grp)
 for (i in 1:N)
 {
    X=f(n,grp)
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,1,1,5:grp-3)))
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,1,1,5:grp-3)))
 }
   PX=u/N ; PG =v/N
   plot(5:grp, PX[5:grp],type="l",ylim=c(0,1), main='Normal vs.
Beta(1.5,1.5)',ylab='Power',xlab='Classes' )
   lines(5:grp,PG[5:grp],type="l",lty=2)
   return(list(Pearson=PX,LR=PG))
 }
```

```
par(mfrow=c(3,2))
ff(100,30,0.05,100000)


#2. Normal vs. Unif(-1.6,1.6)

f =function(n,grp)
{
    x=runif(n,-1.6,1.6)
    x=(x-mean(x))/sqrt(var(x))
    X =matrix(0,2,grp)
    for (k in 4:grp)
    {
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1,k]=sum((obs-e)^2/e)
    X[2,k]=2*sum(obs*log((obs+0.00000001)/e))
    }
    X
}


ff =function(n,grp,alpha, N)
 {
 u=v=rep(0,grp)
 for (i in 1:N)
 {
    X=f(n,grp)
    u=u+(X[1,]>=qchisq(1-alpha,c(1,1,1,4:grp-3)))
    v=v+(X[2,]>=qchisq(1-alpha,c(1,1,1,4:grp-3)))
 }
   PX=u/N ; PG =v/N
   plot(4:grp, PX[4:grp],type="l",ylim=c(0,1), main='Normal vs.
Unif(-1.6,1.6)',ylab='Power',xlab='Classes' )
   lines(4:grp,PG[4:grp],type="l",lty=2)
   return(list(Pearson=PX,LR=PG))
 }

ff(100,30,0.05,100000)
```

```
#--------------------------------------------------------------------
#      Power comparison for specified normal distributions
#
#                        Case 0
#--------------------------------------------------------------------


#1.N(0,1.25) vs. t(10)

f =function(n,k)
{
   x=rt(n,10)
   X =rep(0,2)
   o=1:(k-1)
   xqtl=qnorm((1:(k-1))/k,0,sqrt(5/4))
   for (i in 1:(k-1))   o[i]=sum(x<=xqtl[i])
   obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
   e=n/k
   X[1]=sum((obs-e)^2/e)
   X[2]=2*sum(obs*log((obs+0.00000001)/e))
   return( X,x)
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
   g=f(n,k)
   u=u+(g$X[1]>=qchisq(1-alpha,k-1))
   v=v+(g$X[2]>=qchisq(1-alpha,k-1))
   z=chisq.gof(g$x,n.classes=ceiling(3.765 *
   (length(g$x)^(2/5))),dist="normal",mean=0,sd=sqrt(5/4), n.param.est=0)
   zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N ; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,9,0.05,100000),ff(100,11,0.05,100000),ff(150,12,0.05,100000),ff(200,13,
   0.05,100000),ff(250,14,0.05,100000),ff(300,15,0.05,100000),
```

```
    ff(400,16,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="N(0,1.25) vs. t(10)",    xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)



#2. N(1,1) vs. Logistic(1.1,0.6)

f =function(n,k)
{
    x=rlogis(n,1.1,0.6)
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,1,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return( X,x)
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-1))
    v=v+(g$X[2]>=qchisq(1-alpha,k-1))
    z=chisq.gof(g$x,n.classes=ceiling(3.765 *
    (length(g$x)^(2/5))),dist="normal",mean=1,sd=1, n.param.est=0)
    zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N ; PC=zz/N
```

```
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,9,0.05,100000),ff(100,11,0.05,100000),ff(150,12,0.05,100000),ff(200,13,
    0.05,100000),ff(250,14,0.05,100000),ff(300,15,0.05,100000),
    ff(400,16,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="N(1,1) vs. Logistic(1.1,0.6)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)


#3. N(0,1) vs. 0.5N(0,1.44)+0.5N(0.1,1)

f =function(n,k)
{
    x1=rnorm(n,0,1.2)
    x2=rnorm(n, 0.1,1)
    y=runif(n,0,1)
    x=x1*(y<0.5)+x2*(y>=0.5)

    X =rep(0,2)
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,0,1)

    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return( X,x)
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
```

```
  for (i in 1:N)
 {
   g=f(n,k)
   u=u+(g$X[1]>=qchisq(1-alpha,k-1))
   v=v+(g$X[2]>=qchisq(1-alpha,k-1))
   z=chisq.gof(g$x,n.classes=ceiling(3.765 *
   (length(g$x)^(2/5))),dist="normal",mean=0,sd=1, n.param.est=0)
   zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N ; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,9,0.05,100000),ff(100,11,0.05,100000),ff(150,12,0.05,100000),ff(200,13,
   0.05,100000),ff(250,14,0.05,100000),ff(300,15,0.05,100000),
   ff(400,16,0.05,100000))

w=matrix(V,7,4,T)

a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="N(0,1) vs. 0.5N(0,1.44)+0.5N(0.1,1)",xlab="n", ylab="power",
   type="b", lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)



#---------------------------------------------------------------------------
#      Power comparison for specified exponential distributions
#
#                        Case 0
#---------------------------------------------------------------------------

#1. Exp(1) vs. Weibull(1.5,1.108)

f =function(n,k)
{
   x=rweibull(n,shape=1.5,scale=1.108)
   X =rep(0,2)
```

```
    o=1:(k-1)
    xqtl=qexp((1:(k-1))/k,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return( X,x)
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-1))
    v=v+(g$X[2]>=qchisq(1-alpha,k-1))
    z=chisq.gof(g$x, n.classes=ceiling(3.765 * (length(g$x)^(2/5))),
    dist="exponential",rate=1.0,n.param.est=0)
    zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N ; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,9,0.05,100000),ff(100,11,0.05,100000),ff(150,12,0.05,100000),ff(200,13,
    0.05,100000),ff(250,14,0.05,100000),ff(300,15,0.05,100000),
    ff(400,16,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Exp(1) vs. Weibull(1.5,1.108)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)
```

#2. Exp(1) vs. Gamma(2,1/2)

```
f =function(n,k)
{
   x=rgamma(n,shape=2,rate=2)
   X =rep(0,2)
   o=1:(k-1)
   xqtl=qexp((1:(k-1))/k,1)
   for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
   obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
   e=n/k
   X[1]=sum((obs-e)^2/e)
   X[2]=2*sum(obs*log((obs+0.00000001)/e))
   return( X,x)
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
   g=f(n,k)
   u=u+(g$X[1]>=qchisq(1-alpha,k-1))
   v=v+(g$X[2]>=qchisq(1-alpha,k-1))
   z=chisq.gof(g$x,    n.classes=ceiling(3.765 *
   (length(g$x)^(2/5))),dist="exponential",rate=1.0,n.param.est=0)
   zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N ; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,9,0.05,100000),ff(100,11,0.05,100000),ff(150,12,0.05,100000),ff(200,13,
   0.05,100000),ff(250,14,0.05,100000),ff(300,15,0.05,100000),
   ff(400,16,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Exp(1) vs. Gamma(2,0.5)",xlab="n", ylab="power", type="b",
```

```
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)


#3. Exp(1) vs. lognormal(-0.3,0.6)

f =function(n,k)
{
    x=rlnorm(n,-0.3,sqrt(0.6))
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qexp((1:(k-1))/k,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return( X,x)
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-1))
    v=v+(g$X[2]>=qchisq(1-alpha,k-1))
    z=chisq.gof(g$x, n.classes=ceiling(3.765 *
    (length(g$x)^(2/5))),dist="exponential",rate=1.0, n.param.est=0)
    zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N ; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,9,0.05,100000),ff(100,11,0.05,100000),ff(150,12,0.05,100000),ff(200,13,
    0.05,100000),ff(250,14,0.05,100000),ff(300,15,0.05,100000),
    ff(400,16,0.05,100000))
```

```
w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Exp(1) vs. lognormal(-0.3,0.6)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)




#---------------------------------------------------------------------------
#      Power comparison for specified uniform distributions
#
#                              Case 0
#---------------------------------------------------------------------------

#1. Unif(0,1) vs. Beta(0.6,0.8)

f =function(n,k)
{

    x=rbeta(n,0.6,0.8)
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qunif((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return( X,x)
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-1))
```

```
    v=v+(g$X[2]>=qchisq(1-alpha,k-1))
    z=chisq.gof(g$x,n.classes=ceiling(3.765 *
    (length(g$x)^(2/5))),dist="unif",min=0,max=1, n.param.est=0)
    zz=zz+(z$p.value<=0.05)
  }
  PX=u/N ; PG =v/N ; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,9,0.05,100000),ff(100,11,0.05,100000),ff(150,12,0.05,100000),ff(200,13,
    0.05,100000),ff(250,14,0.05,100000),ff(300,15,0.05,100000),
    ff(400,16,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Unif(0,1) vs. Beta(0.6,0.8)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)


#2. Unif(0,1) vs. Beta(0.6,0.6)

f =function(n,k)
{

    x=rbeta(n,0.6,0.6)
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qunif((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return( X,x)
}


ff =function(n,k,alpha, N)
```

```
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-1))
    v=v+(g$X[2]>=qchisq(1-alpha,k-1))
    z=chisq.gof(g$x,n.classes=ceiling(3.765 *
    (length(g$x)^(2/5))),dist="unif",min=0,max=1, n.param.est=0)
    zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N ; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,9,0.05,100000),ff(100,11,0.05,100000),ff(150,12,0.05,100000),ff(200,13,
    0.05,100000),ff(250,14,0.05,100000),ff(300,15,0.05,100000),
    ff(400,16,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Unif(0,1) vs. Beta(0.6,0.6)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)
```

#3. Unif(0,1) vs. 0.5Beta(0.9,0.9)+0.5N(0.5,0.078)

```
f =function(n,k)
{
    x1=rbeta(n,0.9,0.9)
    x2=rnorm(n,0.5,0.28)
    y=runif(n,0,1)
    x=x1*(y<0.5)+x2*(y>=0.5)
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qunif((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
```

```
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return( X,x)
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-1))
    v=v+(g$X[2]>=qchisq(1-alpha,k-1))
    z=chisq.gof(g$x,n.classes=ceiling(3.765 *
    (length(g$x)^(2/5))),dist="unif",min=0,max=1, n.param.est=0)
    zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N ; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,9,0.05,100000),ff(100,11,0.05,100000),ff(150,12,0.05,100000),ff(200,13,
    0.05,100000),ff(250,14,0.05,100000),ff(300,15,0.05,100000),
    ff(400,16,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Unif(0,1) vs. 0.5Beta(0.9,0.9)+0.5N(0.5,0.078)",xlab="n",
    ylab="power", type="b", lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)
```

```
#-----------------------------------------------------------------------
#       Power comparison for a family of normal distributions
#
#                         Case 1
#------------------------------------------------------------------------

#1. Normal vs. Beta(5,3)

f =function(n,k)
{
   x=rbeta(n,5,3)
   x=(x-mean(x))/sqrt(var(x))
   X =rep(0,2)
   o=1:(k-1)
   xqtl=qnorm((1:(k-1))/k,0,1)
   for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
   obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
   e=n/k
   X[1]=sum((obs-e)^2/e)
   X[2]=2*sum(obs*log((obs+0.00000001)/e))
   return(list(X=X, x=x))
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
   g=f(n,k)
   u=u+(g$X[1]>=qchisq(1-alpha,k-3))
   v=v+(g$X[2]>=qchisq(1-alpha,k-3))
   z=chisq.gof(g$x,n.classes=ceiling(3.765 * (length(g$x)^(2/5))), n.param.est=2)
   zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,10,0.05,100000),ff(100,12,0.05,100000),ff(150,14,0.05,100000),ff(200,16,
   0.05,100000),ff(250,18,0.05,100000),ff(300,20,0.05,100000),
   ff(400,22,0.05,100000))
```

```
w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Normal vs. Beta(5,3)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)


#2. Normal vs. t(9)

f =function(n,k)
{
    x=rt(n,9)
    x=(x-mean(x))/sqrt(var(x))
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return(list(X=X, x=x))
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-3))
    v=v+(g$X[2]>=qchisq(1-alpha,k-3))
    z=chisq.gof(g$x,n.classes=ceiling(3.765 * (length(g$x)^(2/5))), n.param.est=2)
    zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N; PC=zz/N
  V=c(PX,PG,PC,n)
  V
```

```
 }

V=c(ff(50,10,0.05,100000),ff(100,12,0.05,100000),ff(150,14,0.05,100000),ff(200,16,
    0.05,100000),ff(250,18,0.05,100000),ff(300,20,0.05,100000),
    ff(400,22,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Normal vs. t(9)",xlab="n", ylab="power", type="b", lty=1,ylim=c(0,1),
    xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)


#3. Normal vs. 0.5N(1,4)+0.5N(2,16)

f =function(n,k)
{
    x1=rnorm(n,1,2)
    x2=rnorm(n,2,4)
    y=runif(n,0,1)
    x=x1*(y<0.5)+x2*(y>=0.5)
    x=(x-mean(x))/sqrt(var(x))
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qnorm((1:(k-1))/k,0,1)
    for (i in 1:(k-1))   o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return(list(X=X, x=x))

}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
```

```
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-3))
    v=v+(g$X[2]>=qchisq(1-alpha,k-3))
    z=chisq.gof(g$x, n.classes=ceiling(3.765 * (length(g$x)^(2/5))),n.param.est=2)
    zz=zz+(z$p.value<=0.05)
  }
   PX=u/N ; PG =v/N; PC=zz/N
   V=c(PX,PG,PC,n)
   V
  }

V=c(ff(50,10,0.05,100000),ff(100,12,0.05,100000),ff(150,14,0.05,100000),ff(200,16,
    0.05,100000),ff(250,18,0.05,100000),ff(300,20,0.05,100000),
    ff(400,22,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Normal vs. 0.5N(1,4)+0.5N(2,16)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)



#-----------------------------------------------------------------------------
#      Power comparison for a family of exponential distributions
#
#                        Case 1
#-----------------------------------------------------------------------------


#1. Exp vs. Weibull(1.5,1)

f =function(n,k)
{
    x=rweibull(n,shape=1.5,scale=1)
    x=x/mean(x)
    X =rep(0,2)
    o=1:(k-1)
```

```
    xqtl=qexp((1:(k-1))/k,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return(list(X=X, x=x))

}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
   g=f(n,k)
   u=u+(g$X[1]>=qchisq(1-alpha,k-2))
   v=v+(g$X[2]>=qchisq(1-alpha,k-2))
   z=chisq.gof(g$x,n.classes=ceiling(3.765 * (length(g$x)^(2/5))),
   dist="exponential",rate=1.0, n.param.est=1)
   zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,10,0.05,100000),ff(100,12,0.05,100000),ff(150,14,0.05,100000),ff(200,16,
   0.05,100000),ff(250,18,0.05,100000),ff(300,20,0.05,100000),
   ff(400,22,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Exp. vs. Weibull(1.5,1)",xlab="n", ylab="power", type="b",
   lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)
```

#2. Exp vs. Gamma(2,1)

```
f =function(n,k)
{

    x=rgamma(n,shape=2,rate=1)
    x=x/mean(x)
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qexp((1:(k-1))/k,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return(list(X=X, x=x))
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-2))
    v=v+(g$X[2]>=qchisq(1-alpha,k-2))
    z=chisq.gof(g$x, n.classes=ceiling(3.765 *
    (length(g$x)^(2/5))),dist="exponential",rate=1.0, n.param.est=1)
    zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }


V=c(ff(50,10,0.05,100000),ff(100,12,0.05,100000),ff(150,14,0.05,100000),ff(200,16,
    0.05,100000),ff(250,18,0.05,100000),ff(300,20,0.05,100000),
    ff(400,22,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
```

```
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Exp. vs. Gamma(2,1)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)


#3. Exp vs. Lognormal(0.8,0.8)

f =function(n,k)
{
    x=rlnorm(n,0.8,sqrt(0.8))
    x=x/mean(x)
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qexp((1:(k-1))/k,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return(list(X=X, x=x))
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-2))
    v=v+(g$X[2]>=qchisq(1-alpha,k-2))
    z=chisq.gof(g$x,n.classes=ceiling(3.765 * (length(g$x)^(2/5))),
    dist="exponential",rate=1.0, n.param.est=1)
    zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }
```

```
V=c(ff(50,10,0.05,100000),ff(100,12,0.05,100000),ff(150,14,0.05,100000),ff(200,16,
    0.05,100000),ff(250,18,0.05,100000),ff(300,20,0.05,100000),
    ff(400,22,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Exp. vs. lognormal(0.8,0.8)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)



#---------------------------------------------------------------------------
#       Power comparison for a family of uniform distributions
#
#                       Case 1
#---------------------------------------------------------------------------


#1. Unif vs. Beta(0.8,0.6)

f =function(n,k)
{
    x=rbeta(n,0.8,0.6)
    b=(max(x)*n*(n+1)-min(x)*(n+1))/(n^2-1)
    a=max(x)*(n+1)-n*b
    x=(x-a)/(b-a)
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qunif((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return(list(X=X, x=x))
}


ff =function(n,k,alpha, N)
```

```
 {
  u=v=zz=0
  for (i in 1:N)
 {
   g=f(n,k)
   u=u+(g$X[1]>=qchisq(1-alpha,k-3))
   v=v+(g$X[2]>=qchisq(1-alpha,k-3))
   z=chisq.gof(g$x,n.classes=ceiling(3.765 *
   (length(g$x)^(2/5))),dist="unif",min=0,max=1, n.param.est=2)
   zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,10,0.05,100000),ff(100,12,0.05,100000),ff(150,14,0.05,100000),ff(200,16,
   0.05,100000),ff(250,18,0.05,100000),ff(300,20,0.05,100000),
   ff(400,22,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Unif vs. Beta(0.8,0.6)",xlab="n", ylab="power", type="b",
   lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)


#2. Unif vs. Beta(0.8,0.8)

f =function(n,k)
{
   x=rbeta(n,0.8,0.8)
   b=(max(x)*n*(n+1)-min(x)*(n+1))/(n^2-1)
   a=max(x)*(n+1)-n*b
   x=(x-a)/(b-a)
   X =rep(0,2)
   o=1:(k-1)
   xqtl=qunif((1:(k-1))/k,0,1)
   for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
   obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
```

```
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return(list(X=X, x=x))
}


ff =function(n,k,alpha, N)
 {
  u=v=zz=0
  for (i in 1:N)
 {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-3))
    v=v+(g$X[2]>=qchisq(1-alpha,k-3))
    z=chisq.gof(g$x,n.classes=ceiling(3.765 *
    (length(g$x)^(2/5))),dist="unif",min=0,max=1, n.param.est=2)
    zz=zz+(z$p.value<=0.05)
 }
  PX=u/N ; PG =v/N; PC=zz/N
  V=c(PX,PG,PC,n)
  V
 }

V=c(ff(50,10,0.05,100000),ff(100,12,0.05,100000),ff(150,14,0.05,100000),ff(200,16,
    0.05,100000),ff(250,18,0.05,100000),ff(300,20,0.05,100000),
    ff(400,22,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
plot(a,c,main="Unif vs. Beta(0.8,0.8)",xlab="n", ylab="power", type="b",
    lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)


#3. Unif vs. 0.7Beta(1.1,1)+0.3N(0.5,0.0081)

f =function(n,k)
{
    x1=rbeta(n,1.1,1)
```

```
    x2=rnorm(n,0.5,0.09)
    y=runif(n,0,1)
    x=x1*(y<0.7)+x2*(y>=0.7)
    b=(max(x)*n*(n+1)-min(x)*(n+1))/(n^2-1)
    a=max(x)*(n+1)-n*b
    x=(x-a)/(b-a)
    X =rep(0,2)
    o=1:(k-1)
    xqtl=qunif((1:(k-1))/k,0,1)
    for (i in 1:(k-1))    o[i]=sum(x<=xqtl[i])
    obs=c(o[1], o[2:(k-1)]-o[1:(k-2)],n-o[k-1])
    e=n/k
    X[1]=sum((obs-e)^2/e)
    X[2]=2*sum(obs*log((obs+0.00000001)/e))
    return(list(X=X, x=x))
  }


ff =function(n,k,alpha, N)
  {
   u=v=zz=0
   for (i in 1:N)
  {
    g=f(n,k)
    u=u+(g$X[1]>=qchisq(1-alpha,k-3))
    v=v+(g$X[2]>=qchisq(1-alpha,k-3))
    z=chisq.gof(g$x,n.classes=ceiling(3.765 *
    (length(g$x)^(2/5))),dist="unif",min=0,max=1, n.param.est=2)
    zz=zz+(z$p.value<=0.05)
  }
   PX=u/N ; PG =v/N; PC=zz/N
   V=c(PX,PG,PC,n)
   V
  }

V=c(ff(50,10,0.05,100000),ff(100,12,0.05,100000),ff(150,14,0.05,100000),ff(200,16,
   0.05,100000),ff(250,18,0.05,100000),ff(300,20,0.05,100000),
   ff(400,22,0.05,100000))

w=matrix(V,7,4,T)
a=w[,4]
b=w[,2]
c=w[,1]
d=w[,3]
```

```
plot(a,c,main="Unif vs. 0.7Beta(1.1,1)+0.3N(0.5,0.008)",xlab="n", ylab="power",
    type="b", lty=1,ylim=c(0,1), xlim=c(0,400),lab=c(8,5,8))
  lines(a,b,type="b",lty=2)
  lines(a,d,type="b",lty=3)
```