SOME ASPECTS OF MARKET EQUILIBRATION PROCESSES IN ECONOMIC THEORY

A Dissertation

Presented to

The Faculty of Graduate Studies

University of Manitoba

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

---

by

Constantine Alexander Nicolaou

July 1976

"SOME ASPECTS OF MARKET EQUILIBRATION PROCESSES IN ECONOMIC THEORY"

by

CONSTANTINE ALEXANDER NICOLAOU

A dissertation submitted to the Faculty of Graduate Studies of the University of Manitoba in partial fulfillment of the requirements of the degree of

DOCTOR OF PHILOSOPHY

ABSTRACT

TITLE: SOME ASPECTS OF MARKET EQUILIBRATION PROCESSES IN ECONOMIC THEORY

Equilibrium method in economic theory is based on analysis of planned behaviour, without regard to its feasibility outside equilibrium. Consequently, theorems derived by means of this method are crucially dependent on planned behaviour as it determines equilibrium, stability, market pressure in disequilibrium and description of paths outside equilibrium.

The purpose of this thesis is to examine the role of realized, as opposed to planned, behaviour in disequilibrium, and the possible consequences of taking into account the fact that planned behaviour in disequilibrium is unfeasible. More specifically, planned behaviour may not be relevant as a determinant of any of the following: the stationary state, pressure on market variables outside equilibrium, or conditions for stability of equilibrium. Nor may it be relevant as a description of disequilibrium paths. The aim is to examine realized behaviour in these contexts and to compare stationary states, stability conditions and disequilibrium paths with those derived from analysis based on planned behaviour.

The analysis employs a classification of endowment effects on demand as an analytical framework, and distinguishes feasible from unfeasible income, sales and production plans. A review, interpretation and critical evaluation of previous work on the subject is undertaken. This provides directions and points of departure for the main analysis in this thesis.

The main analysis develops explicitly dynamic disequilibrium models of production and exchange, and conducts comparisons of stationary states, stability and disequilibrium paths, with those of models in which only planned behaviour is taken into account. Models are constructed for a single market (with the commodity assumed first non-durable and later durable, and the market structure either competition or monopoly). Furthermore, aggregate disequilibrium models of production and exchange are developed, with and without inventories and exchange money. Market structure in these models is that of perfect competition.

The principal results of the analysis are that whereas the stationary state in pure exchange is sensitive to realized transactions within the market period, introduction of production considerationsin both a partial and general disequilibrium setting makes the stationary state insensitive to realized behaviour in the longer run (market period equilibrium is still sensitive). Stability is more difficult to satisfy when realized behaviour is taken into account. The model used in the equilibrium method, however, was found to be stable when the disequilibrium model was. This result, together with the invariance of the stationary state means that comparative static theorems derived by means of the equilibrium method are not changed. Finally, disequilibrium paths of explicit disequilibrium models were found to be radically different from those predicted by models based on planned behaviour.

PREFACE AND AKNOWLEDGEMENTS

My thanks are due, first, to my main supervisor, Dr. Brian Scarfe. His guidance and prompt, constructive comments and suggestions on earlier drafts of this thesis exceeded by far the call of duty and were instrumental in improving the thesis and providing direction for research.

Dr. Anthony Waterman offered a large number of useful comments which made the final draft of this thesis read much better than earlier ones.

Dr. Derek Hum read a late draft of this dissertation and offered several comments. He also provided many useful comments on an early draft of this thesis.

Finally, Barbara Spencer read a portion of this thesis and commented on the Introduction and Chapter I.

Needless to say, responsibility for errors, ommissions etc., remains with the author.

TABLE OF CONTENTS

INTRODUCTION

This investigation is concerned with the equilibration process in economic theory.More specifically, the main purpose is to examine the role of realized disequilibrium behaviour in relation to the traditional equilibrium method and the associated tatonnement process of equilibration.Consequently, the focus and concern of this thesis are with disequilibrium analysis and comparison of results to those of the equilibrium method.

In order to enlarge upon the above statement,a brief description of the equilibrium method will first be undertaken; the scope, focus, and main questions raised in this investigation will be discussed; the importance of the issues raised will be indicated; finally, an outline of the work will be given.

(a) The equilibrium method. The notion of equilibrium is fundamental, and indeed indispensable, to the method of traditional economic theory. More specifically, the great majority of "meaningful" theorems[1] in the discipline are derived from comparisons of equilibria. Indeed, some theorists would even claim that there are only theories of equilibrium behaviour in economics.[2]

---

[1] The term is due to P. Samuelson,(1947), on whose work the description of the equilibrium method in the text is largely based.

[2] Compare K. Lancaster, (1968), p.201.

The established method of equilibrium analysis involves postulation of the behavioural functions involved in the phenomenon to be studied[3] as well as the conditions that have to be satisfied for equilibrium to prevail. An examination of basic characteristics of the equilibrium position is then undertaken.[4] Once this is completed, the analysis proceeds to ask the main question. How do the values of variables involved change when something in the model under analysis changes?

The factor changed may be the initial conditions (the starting point of the analysis), an exogenous variable (a force external to the system), or an internal parameter of the model (the form of a behavioural function). Once such a parametric change is specified, the analysis proceeds to examine the change in the variables of the system. The parametric change may be permanent, intermittent or transient, and the consequent effect on the variables of the system may be investigated from several points of view. One might, for example, be interested in the short-run reaction of the system to the change; the effect on the long-term equilibrium of the system; or, the effect on some characteristic of the

---

[3] At this stage, the choice of variables to be studied and the scope and detail of the analysis are determined.

[4] It has first either been proved or assumed that equilibrium exists.

motion itself (for example, the periodicity, dampening or amplitude of the adjustment).[5]

It is clear from the above outline that the method can apply to what are conventionally called "static" as well as "dynamic" systems.[6] Indeed, it has been termed the (equilibrium) comparative dynamic method,[7] subsuming as well the well-known method of comparative statics as a special case.[8]

By way of clarification of the general method, it is useful to point out that preoccupation with the stationary (or steady) state reveals the implicit assumption that although economies are seldom ever in this state, they nevertheless tend to be "sufficiently" near it to justify description in terms of equilibrium positions.[9]

---

[5] With regard to the effect of initial conditions on the long-term equilibrium, if the system is stable the stationary (or steady) state is not changed by a change in these conditions, despite the fact that the immediate reaction to such a change may be substantial.

[6] See R. Frisch, (1936), and P. Samuelson, (1947).

[7] See F. Hahn and R. Matthews, (1967).

[8] The latter involves static systems whose behaviour over time is stationary, that is, the values of the variables are constant in equilibrium. A permanent change is made in the system and its effect is examined only from the point of view of the stationary state. The old and the new equilibria are compared and meaningful theorems on the effects of changes are derived. See P. Samuelson, (1947).

[9] Compare J. Hicks, (1965), p. 16. Disturbing as it may be, this is true for all comparative static and dynamic analyses of equilibria. It does not apply to descriptions of the process of motion towards equilibrium, of course.

A second observation involves the stability of the stationary states to be compared. Comparison of equilibria as a result of parametric change is meaningful only when the model can be assumed to be stable, so that there may be assurance that it will tend towards the new equilibrium, following a disturbance.[10] In fact, stability conditions are also the source of most meaningful theorems in a large number of models, static and dynamic. In the case of static models, stability and its conditions refer to the " corresponding" dynamic model, the stationary state of which is described by the equilibrium of the static model.[11]

Finally, for the method under discussion the stationary state values of variables are fully determined once planned behavioural functions and their parameters are specified. Disequilibrium behaviour is not of concern. Also, descriptions of the path of the system outside the stationary state are given in terms of planned behavioural functions, supplemented by initial conditions.[12]

---

[10] If one happens to believe that stability is a "natural" characteristic of the "actual" system under analysis, he may be inclined to claim that stability is a requirement that the model has to satisfy in order for it to be a good first approximation to "reality". If, however, the actual system is viewed *a priori* as internally inconsistent and therefore possibly unstable, stability remains an assumption.

[11] For a full description of the Correspondence Principle see Samuelson, (1947).

[12] The "exceptions" to this statement (such as explicit non-tatonnement analysis) are discussed in Chapters II and III. Such exceptions, however, are not part of the equilibrium method.

(b) The main issues to be investigated. It is, of course, generally accepted that systems are in disequilibrium for most (if not all) of the time. Since system behaviour in disequilibrium is the rule rather than the exception, the question arises as to the relevance and usefulness of the planned behavioural functions of the equilibrium method. In that method, these functions determine equilibrium values of the variables, degrees of market pressure in disequilibrium, stability conditions and meaningful theorems, and may also be used to describe the disequilibrium behaviour of the system. The questions that arise may then be spelt out as follows: in a situation where disequilibrium is the rule rather than the exception, are these planned functions and their parameters relevant for the determination of the stationary state? Is the behaviour implied by these functions important as a force acting on the system outside equilibrium? Is such behaviour the only determinant of stability in the model? Is it the sole determinant of meaningful theorems derivable from the analysis? And, finally, how valid can it be in the descriptions of systems in disequilibrium?

The importance of these questions is obvious. Indeed, some of them have been asked, in some form or another, by the originators of the methodological prototype discussed above.[13]

---

[13] See Chapter II for a historical review of these early contributors.

Some amplification may nevertheless assist in making the importance of these issues clearer.

(i) In the equilibrium method, only planned behaviour (as determined from solution of optimization problems of agents) and its parameters determine the stationary state. This means, in general, that realized (as opposed to planned) behaviour in disequilibrium is ignored as far as its effects on the equilibrium position are concerned. Thus, an implicit assumption of this method is that realized behaviour outside equilibrium is either irrelevant to the stationary state or (as a special case) that no agent is allowed to act outside equilibrium. The latter case of no actions in disequilibrium will be recognized as the traditional "tatonnement" process.[14]

(ii) The planned behavioural functions of the models[15] are usually specified without regard as to the feasibility of plans. However, by the definition of disequilibrium, some plans will not be feasible. In this case, it is questionable whether planned (unfeasible) behaviour is relevant at all as a force acting on the system outside equilibrium. Nevertheless, in traditional tatonnement analysis the framework is such that

---

[14] The term "tatonnement" is used in this investigation with the generally accepted meaning, to be formally defined in Chapter I. This general usage of the term has been shown to have reversed Walras' use of the term. See W. Jaffe, (1967).

[15] These planned functions could alternatively be called "*ex ante*".

it is precisely the portion of plans which is not realizable that acts as a force on the system in disequilibrium. The question is whether or not this is possible in situations where the fiction of the "referee" or "auctioneer"[16] is partially abandoned and where the system is assumed to be left to its own devices in disequilibrium.

(iii) When stability analysis is conducted in conventional theory, stability conditions are derived from planned functions.[17] Since, from (ii) above, unfeasibility of some plans is certain in disequilibrium, the question arises how realized disequilibrium behaviour may affect stability conditions of the model analyzed. It is possible that, when this behaviour is taken into account, stability conditions may be different from those derived under tatonnement assumptions.

(iv) Theorems concerning changes in equilibrium in response to parametric changes are based on stability conditions, and on the unique association of the stationary state with a set of planned behavioural functions and parameters. But, if it is possible for both the stationary state and the stability conditions to be affected by realized disequilibrium behaviour, it is a reasonable question to ask whether the theorems so derived can still be valid.

---

[16] This will be fully explained in Chapter I.

[17] As has been mentioned in footnote 12 above, there is work on stability analysis with realized disequilibrium behaviour taken into account. See K. Arrow and F. Hahn, (1971), and Chapter III infra.

(v) Finally, the behaviour of systems in disequilibrium is usually described in phase diagrams by means of planned behavioural functions. Whether such description is relevant at all depends on how closely, if at all, it approximates realized behaviour in disequilibrium. The possibility certainly exists (taking into account (ii) above) that the approximation is poor. It is thus legitimate to ask how relevant the tatonnement description is -- in the "as if" sense -- for more "realistic"processes of equilibration.[18]

In summary,the above questions show that the concern in this investigation is with realized disequilibrium behaviour and its possible effects.Such behaviour may affect the equilibrium position; may be relevant as a force on the system in disequilibrium; may affect the stability of equilibrium; may affect the theorems derivable from the equilibrium method; and may be more relevant for the description of disequilibrium states.

(c) Purpose and scope of thesis.Importance. The purpose of the present investigation is to inquire into the above issues, by means of comparative analysis of the results of the equilibrium method(and the associated tatonnement process) to those of disequilibrium analysis.More specifically, it undertakes explicit disequilibrium analysis of production and

---

[18] This point is independent of the earlier ones: realized behaviour may leave the stationary state, the stability conditions and the theorems intact, but it may still result in different paths compared to those of tatonnement.

exchange in simple models, and compares the results to those of the equilibrium method.

The elaboration of the issues given above has served to underline the importance of the questions that this investigation attempts to answer. In one statement, if it turns out that realized disequilibrium behaviour is a significant determinant of the stationary state, the disequilibrium path of the system and the stability conditions, most of conventional economic analysis will have to be modified to take this into account. If, on the other hand, it turns out that realized behaviour does not play a significant role in these matters, the usefulness and applicability of conventional theory will be reinforced.

The scope in this investigation is confined to disequilibrium models of production and exchange in a single market; and to aggregate disequilibrium models of production and exchange. Money is introduced only as a medium of exchange in the latter models. Perfect competition is assumed in most of the analysis, and monopoly in one instance. Inventory considerations are taken into account.

(d) Limitations. The considerable difficulty of the subject has basically determined[19] the limitations of the analysis presented here. Both the single-market and the aggreg-

---

[19] Together with the ability of the writer of this thesis.

ate model used constitute only specific, simple formulations. Although it does not appear the case that the conclusions of this investigation are especially sensitive to the specific assumptions and the structures of the models, the possibility nevertheless exists, and has not been investigated fully. A full investigation of the sensitivity of results to the specific assumptions of the analysis would of course be desirable, and would presumably be achieved by relaxation and change of the models' assumptions. The only justification offered here for the lack of such complete analysis is the generally used -- and accepted -- observation that research proceeds step by step. In this vein, suggestions for amplification, modifications and extensions of the analysis are given in the concluding Chapter.

(e) Contribution to knowledge. Contributions original with this thesis will be found in all of the following Chapters. The specific observations and conclusions from analysis that are deemed original are summarized in the final Chapter of this thesis. It is perhaps worth mentioning at this point that the bulk of the main conclusions are derived from the analysis in Chapters IV and V, while contributions of "secondary" importance are in Chapters I, II, and III.

(f) Outline of thesis. Chapter I provides an analytical framework for the issues addressed in this investigation.

More specifically, (a) a modern version of the general equilibrium model of production and exchange is set out, in a form amenable to manipulation. Variants of this model (most of the time more specialized) are used throughout the thesis, (b) A formal presentation of tatonnement and non-tatonnement processes is given, the former for production and exchange and the latter for pure exchange models. Moreover, a simple example of comparative statics and the Correspondence Principle serves to illustrate the importance of the equilibration process for standard comparative statics. (c) Three kinds of "income" or"endowment" effects are identified, and their role in the equilibration process is outlined. (d) An introduction to special problems of disequilibrium production is made.

Chapter II contains a review, interpretation and critical analysis of early attempts at the problem of equilibration, from L. Walras to Sir John Hicks. The framework developed in Chapter I is employed in some instances in order to fix ideas, and in an attempt to classify these contributions in terms of the kinds of endowment effects on which they concentrate.

Chapter III consists of a review, interpretation and critical evaluation of the more recent literature. For convenience, this group of works is classified into two categories. First, are the formalizations and extensions of the analysis of equilibration, which fall within the Walrasian tradition. These are considered to start with Professor Paul Samuelson's dynamic formalization of the tatonnement process in a multi-market context, and include recent analyses and proofs of stability in both the tatonnement (for production and exchange) and the non-tatonnement process (for pure exchange only). An evaluation of this group of works is undertaken, for the purpose of ascertaining whether extensions in this line of approach are desirable. A number of serious reservations are made in this respect.

The second category of recent contributions to equilibration, labelled"non-Walrasian" for convenience of exposition only, includes the following: the cobweb model; models of the trade cycle; Professor Don Patinkin's "spillover" effects; an attempt to generalize the theory of spillovers by Professor Herschel Grossman; Patinkin's interpretation of the Keynesian concept of involuntary unemployment as a disequilibrium phenomenon; Professor Robert Clower's important suggestion that quantities as well as prices should enter realized behavioural functions in proper disequilibrium analysis; and an attempt to integrate Patinkin's and Clower's contributions, by Professors Robert Barro and Herschel Grossman. The evaluation of these works brings forward points of departure for further analysis, some of which are used in this thesis.

Chapter IV is an attempt to address the issues raised in this investigation in the context of partial disequilibrium analysis of production and exchange. More specifically, a single-market model of perfect competition is developed and used to conduct comparative analysis of the tatonnement and non-tatonnement processes in this model. The comparison is conducted in terms of stationary states, stability conditions and behaviour outside equilibrium.The commodity is initially assumed non-durable.[20]

The single-market model is then extended to include inventory considerations (still under perfect competition), and comparative analysis of this and the tatonnement version is carried out for this case. Finally, a model of a monopolist producing a durable commodity and maximizing profit over time is developed, mainly to analyze and formalize the effect of inventory positions on production and sales plans.

The method of analysis in Chapter IV is partial, and therefore neglects both the phenomenon of interdependence and that of unfeasibility of production and income plans in disequilibrium. It does take into account, however, the unfeasibility of <u>sales</u> plans of producers (at expected prices) when disequilibrium in production is admitted.

---

[20] In this case, the reduced form of the mathematical structure of the model turns out identical to that of the cobweb. The similarity ends there, however, as will be seen in the analysis of that Chapter.

Chapter V develops an aggregate competitive model of disequilibrium in production and exchange. In such a setting, it is possible to take into account the phenomenon of economic interdependence. Moreover, it is also possible to examine feasibility not only of sales plans, but also of production plans as well as plans for income creation.[21]

After explicit account has been taken of disequilibrium, comparative analysis of tatonnement and non-tatonnement processes is carried out in terms of stationary states, stability conditions and out-of-equilibrium behaviour. Inventory considerations are introduced informally, and comparative analysis of this case is also conducted.

The analysis in this Chapter concentrates on a particular category of endowment effects; it incorporates quantities as well as prices in the realized excess demand functions, following Clower's suggestion; explicit dynamic analysis of disequilibrium is conducted; and, money is introduced as a medium of exchange.

The final Chapter in this thesis, Chapter VI, contains a summary of the conclusions reached in this investigation and a non-exhaustive list of suggestions for further research.

(g) <u>Definitions of terms. Notation. Tools used</u>. Terms are defined when introduced in each Chapter, in light of the

---

[21] In fact, plans for income creation are sales plans for inputs. Because of their importance for labour income, however, they will be kept as a distinct category of plans.

fact that many are technical and require formal definitions. Similarly, the notation employed will be explained as the need arises. It has been deemed useful to use a single symbol for a commodity, with superscripts denoting demand, production and sales of it, and with asterisks denoting planned magnitudes. The resulting notation is awkward to read for the first time but easy to remember thereafter, and has been judged superior to a notation that would have to have used six separate symbols to denote planned and realized demand, production and sales of a single commodity.

The mathematical tools employed in this thesis are fairly conventional, namely calculus; differential and difference equations and their stability properties; a simple application of Lyapounov's second method for analysis of stability; and, a simple application of the calculus of variations.[22]

---

[22] A remark on typing form may be in order here: the words "tatonnement" and "non-tatonnement" occur so often in this thesis, that it was thought expedient to omit the underlining in the text. Similar reasons dictated the omission of the *accent circonflexe* from the word.

CHAPTER I

A FRAMEWORK FOR THE ANALYSIS OF EQUILIBRATION

This Chapter serves as an introduction to the process of equilibration and the specific problems associated with it. More specifically, Section 1.1 contains a more or less formal description of a general model of production and exchange[1] and the assumptions usually made in connection with such a model. The model is suitable for both generalization and specialization. This is its special feature; accordingly, both the literature review of the next two Chapters and the analysis of disequilibrium production and exchange in Chapters IV and V are conducted with special variants (sometimes more generalized but most of the time more specialized) of the model set out in Section 1.1.

Section 1.2 provides the reader with a formalization of the tatonnement process of equilibration in the context of the model of production and exchange of Section 1.1. Moreover, an example of the comparative statics and the correspondence principle is given, which serves to illustrate the comparative statics method and indicate the possible importance of the equilibration process itself for the basic theorems usually derived from comparison of equilibria.

---

[1]Based in part on K. Arrow and F. Hahn, (1971).

Section 1.3 contains a formalization of the non-tatonnement process in a model of pure exchange. The latter is again a variant of the general model of Section 1.1, without production.

Section 1.4 identifies, classifies, analyzes and discusses the importance of the various "endowment effects" for the equilibration process. The analysis of the Section provides the main theme of this investigation, in the sense that the various attempts at the problem of equilibration can be, and are, classified in this thesis in terms of the kinds of effects on which they concentrate. Three basic "endowment effects" are identified. Tatonnement processes are found to deal with only the first of these while variants of non-tatonnement processes in exchange are seen to involve mostly the first and the second kind. Non-tatonnement processes in production and exchange, on the other hand, do involve all three kinds of effects.

Section 1.5 concentrates on a discussion of some special problems associated with the introduction of non-tatonnement in models of production and exchange; this discussion is necessary in an introductory framework, in view of the difficulty associated with analysis of this kind.

1.1 The General Model: Setting, Assumptions and Behavioural Functions

The economy represented by the model below has n commodities, which include producible goods and services and non-producible ones, such as labour. The agents are m individuals acting as consumers and owners of stocks, and e firms (or individuals acting as enterpreneurs) engaged in production of commodities by use of other producible and non-producible commodities, and for the purpose of selling these commodities to individuals. Technology, tastes and the initial commodity stocks are given, and all agents are price-takers.

The market stock of commodity i is the sum of the stocks owned by individuals[2] at a given time.[3] If the stock of commodity i owned by individual m is denoted by I_i^m, the total stock of this commodity in the economy is represented by

$$I_i = \sum_k I_i^k \qquad (1.1).$$

The "endowment vector" of an individual m is defined as a n-dimensional column vector with elements indicating quantities of commodity stocks owned by individual m. The matrix I (of dimensions nxm) with rows representing commodities and columns representing the endowment vectors of individuals is called the "endowment matrix" of the economy. One may write, for this matrix,

---

[2] Including those acting as enterpreneurs.

[3] Firms are supposed to hold no stocks. This does not present difficulties, since individuals acting as risk-taking enterpreneurs may be assumed to hold them instead.

$I=[I_j^k]$, $(j=1,2,\ldots n)$ and $(k=1,2,\ldots m)$ (1.1a)

The planned market demand for a commodity i (i=1,2,...n) is the sum of the demands of individuals for this commodity, q_i^{d*m}, over m, the number of individuals. It is assumed to be a (single-valued) function, continuous and twice differentiable. It is also assumed homogeneous of degree zero in all nominal prices[4], P_i (i=1,2,...n) i.e., the quantity demanded does not change when all prices change by the same proportion.[5] The market demand for commodity i may be represented by

$$Q_i^{d*} = \sum_m q_i^{d*m} = \sum_m H_i^m(P_1,\ldots P_n, I_1^m,\ldots I_n^m) = H_i(P_1,P_2,\ldots\ldots P_n,I) \quad (1.2)$$ [6]

where the asterisks indicate planned magnitudes, i.e., values that optimize behavioural functions without regard to the feasibility of the associated plans in the market. The distinction between planned and realized behavioural functions will be discussed and utilized later in this investigation. Suffice it to say, at this point, that the term "planned" is here used to identify the traditional functions indicating

---

[4] To be expressed in some unit of account.

[5] Some of these assumptions can, as is well known, be derived as consequences of more basic postulates, but this is not necessary for the purposes of the presentation in the text.

[6] Strictly speaking, it might be more proper to write out all the elements of the matrix I in the function. However, this convention of writing I only has been adopted in the literature (for example, K. Arrow and F. Hahn, (1971), p.326, T. Negishi, (1962), p.645), probably for economy of expression.

behaviour of price takers and derived from assumptions of optimization at given prices.

The specification of market demand in (1.2) applies to all commodities, consumable or not. More specifically, in the case of a primary input such as labour (which has the consumable commodity leisure as an alternative use) the demand indicates the amount of labour demanded by consumers for its alternative use, leisure. Given the total time available for labour during the period, the conventional labour supply may be derived by subtracting the demand for leisure from the total stock of labour time available. If the commodity has no alternative use in consumption the demand for it for consumption purposes is zero at all prices.

The market production (as distinct from market supply) is defined as the difference between gross outputs of the commodity by all firms and of demands for this commodity for use as intermediate input in the production of others.[7] Thus, market production is the net output volume available for final use in the economy. In the case of non-producible inputs, this net algebraic sum is negative, indicating a net demand for these inputs for production purposes. Market

---

[7] The convention of representing all inputs and outputs of a firm by a vector, with negative elements for the inputs and positive for the outputs is common in the so-called activity analysis of production. See T.C. Koopmans, (1951).

production may be represented by

$$Q_i^{o*} = \sum_e q_i^{o*e} = \sum_e F_i^e(P_1 \ldots P_n) = F_i(P_1, P_2, \ldots\ldots P_n) \qquad (1.3)$$

and is assumed to depend on all prices. As with market demand, production is assumed a function of prices, continuous, differentiable and homogeneous of degree zero at these prices. The asterisk again implies that this is the amount of planned production without concern for the feasibility of it in the market.

The market supply of commodity i is defined as the sum of market production and market stock, i.e.,

$$Q_i^{s*} = Q_i^{o*} + I_i \qquad (1.4).$$

In the case of a non-producible input such as labour the "production" is actually negative. The market supply is the net labour time that would be available to individuals for leisure if production plans were satisfied.

Finally, the market excess demand for commodity i is defined as the difference between market demand and market supply, i.e.,

$$E_i^* = Q_i^{d*} - Q_i^{s*} = E_i(P_1, P_2, \ldots\ldots P_n, I) \qquad (1.5).$$

Since both demand and supply are continuous, differentiable and zero-degree homogeneous functions of prices, so is the market excess demand function in this model.

Lastly, the notion of equilibrium in this model should be more or less formalized. If the assumption of free disposal of commodities is made, it is reasonable to suggest that equilibrium may obtain even with some excess demands negative, i.e., with excess supply in some commodities. The excess supply will, in this case, be disposed of. When there is positive excess demand, however, the system cannot be in equilibrium, since the unsatisfied portion of that demand will not be disposable in the same way as the commodities in excess supply. Thus, it seems logical to define equilibrium as a situation where

$$E_i^* = E_i(\bar{P}_1 \ldots \bar{P}_n, I) \leqq 0 \qquad (1.6)$$

i.e., a set of prices at which excess demands are either zero or negative. A negative excess demand implies that the commodity is a free good in the period under consideration, under the assumptions of the present model.[8] The above definition of equilibrium together with the assumption of free disposal are consistent with a situation in which mistaken production volumes come to existence in a period or with the case where the stock of a non-producible commodity

---

[8] K. Arrow and F. Hahn, (1971) have proved that if Walras' law holds a negative excess demand for a commodity must imply a zero price for this commodity in equilibrium. See (1971), p.23.

turns out to be a "free good".

The model to be used as framework for the analysis in this and the following Chapters is now complete, and consists of equations (1.1) to (1.6) above. Before the discussion proceeds to another Section, however, it is convenient to state here a number of additional assumptions widely used in analyses of stability.

(a) Walras' law. With respect to market excess demands for all commodities the additional assumption[9] that Walras' Law holds is made, i.e.,

$$\sum_i P_i E_i^* = 0 \qquad (1.7)$$

which means that the sum of excess demands for all commodities, evaluated at any prices, are zero.

(b) Gross substitutability. Commodities are defined as gross substitutes if

$$E_{ij}^* > 0 \quad i \neq j, \; i,j=1,2,\ldots n \qquad (1.8)$$

It should be noted that if (1.8) holds for all i and j, as is stated in the definition, all commodities are gross substitutes for one another. Moreover, since "commodities" includes non-producible ones used in the production of others, this definition extends to cases of production and exchange.

[9] Which may be derived from basic postulates, in a more detailed exposition.

It is worth noting here that gross substitutability combined with zero-degree homogeneity of the excess demand functions implies that the response of excess demand to the "own" price is always negative. Thus, from zero-degree homogeneity (using Euler's theorem) we have,

$$\sum_j E^*_{ij} P_j = 0 \quad (i=1,2\ldots n) \qquad (1.8a).$$

With positive prices, if all commodities are gross substitutes we have

$$\sum_j E^*_{ij} P_j > 0 \quad (i=1,2,\ldots n) \; i \neq j \qquad (1.8b)$$

which implies that

$$E^*_{ii} < 0 \quad (i=1,2,\ldots n) \qquad (1.8c),$$

the result mentioned above.

(c) Diagonal dominance. This concept refers to a matrix. In the present context this may be the Jacobian of excess demands, which has elements $[E^*_{ij}]$, or the Jacobian of (own and cross-) elasticities of excess demands with respect to prices. The elements of the latter may be denoted by $[e_{ij}]$, where

$$e_{ij} = E^*_{ij} P_j / E^*_i \qquad (1.9)$$

The elements of the Jacobian of excess demands are of course (being derivatives) sensitive to changes in the units of measurement of commodities, while the elements of the Jacobian of elasticities are not.

Given the above preliminaries, the Jacobian of excess demands is said to have a dominant diagonal (at the price vector where the Jacobian is evaluated) if

(i) $E^*_{ii} < 0$ for all i (1.9a)

(ii) there is a vector with elements h_i such that

$\sum_i h_i |E^*_{ii}| > \sum_{h > j \neq i} h_i |E^*_{ij}|$ all i<n.

The definition implies that there exist some units of measurement of commodities such that the diagonal term dominates the off-diagonal terms.[10]

On the other hand "elasticity diagonal dominance"[11] holds if

$e_{ii} < 0$ all $i \neq n$ (1.9b)

and $|e_{ii}| > \sum_i e_{ij}$

The definitions of diagonal dominance and of gross substitutability have been given here for convenience of the exposition of stability analysis, undertaken later in this thesis.[12]

The discussion now proceeds to an example of the tatonnement process of adjustment, as well as an application

---

[10] For more discussion on this definition see K. Arrow and F. Hahn, (1971), pp.233-235.

[11] See *ibid*, pp.292-294.

[12] Chapter III.

of comparative statics and the Correspondence Principle using a simple variant of the model of this Section.

1.2 Tatonnement Processes, Comparative Statics and the Correspondence Principle.

A tatonnement process of price adjustment provides a rule according to which price changes when the system is not in equilibrium, and remains unchanged when it is. Transactions are "permitted" only in equilibrium. An agent or institution called "the auctioneer" is burdened with the task of changing prices, and instructed to raise price when excess demand is positive, and lower price when excess demand is negative (and price is positive).

The above may be formalized as follows: Let

$$K_i = K_i(E_i^*) \qquad (i=1,2,\ldots n) \text{ with } K_i(0)=0 \text{ and } K_{iE}>0 \qquad (1.10)$$

be a sign-preserving function of E_i, where K_{iE} is the derivative of K_i with respect to the ith excess demand. A tatonnement process is, then, represented by

$$\dot{P}_i = 0, \text{ if } P_i \leq 0 \text{ and } E_i^* \leq 0,$$

and $$\dot{P}_i = K_i(E_i^*), \quad (i=1,2,\ldots n) \text{ otherwise.} \quad (1.11).$$

It is worth emphasizing that since, by assumption, no transactions are allowed in disequilibrium the endowment matrix I is a constant during this process and may be dropped from the expressions for the excess demand functions.

This disregard is possible only in a process where it is either assumed that actual disequilibrium transactions do not affect the equilibrium (or equilibria) of the model or in cases where transactions in disequilibrium are not permitted, as in the above tatonnement process. If, on the other hand, one desires to analyze a situation where transactions do take place in disequilibrium and are possibly involved in the determination of the equilibrium position, it is necessary to keep the original formulation (1.5) intact. The non-tatonnement process formalized in the next Section is a case in point.

Another way of making the above point is via the definition of an equilibrium of the model when tatonnement is assumed: given tastes, technology and the <u>initial</u> endowment matrix, an equilibrium set of prices is a non-negative price vector $(\bar{P}_1, \bar{P}_2, \ldots \bar{P}_n)$ such that excess demands for commodities are zero (or non-positive at zero price). The definition implies that the initial endowment matrix determines (along with technology and tastes), and is therefore associated with, equilibrium prices. In contrast, when a non-tatonnement process is assumed, the <u>initial</u> commodity endowment is not a determinant of equilibrium prices since this endowment is subject to constant change during equilibration.[13] This point will be amplified further in the next Section.

It may be useful to note that the tatonnement process described here is in fact one for a model of an economy involving production as well as exchange. This is clear from the specification of the excess demand function in (1.5) above: the excess demand for a commodity is the difference between demand and the sum of production and stocks of the commodity. Obviously, from the assumption that no transactions are to be carried out in disequilibrium it follows that no production or exchange takes place during equilibration, but the auctioneer is supposed to record planned magnitudes of commodities demanded and produced at each set of prices.

In the usual representation of the tatonnement process of pure exchange models the question of production does not arise. In the present representation, production cannot take place without exchange since firms do not hold sufficient stocks of commodities such as labour. The present representation of tatonnement in a full model of production and exchange thus does not seem to require additional assumptions about what happens out of equilibrium: if no exchange is to

[13] The initial endowment constrains, however, the range of possible equilibria.

take place, nothing else can.

With this final remark on the tatonnement process and its implications the discussion may now turn to a simple example of the process, including an illustration of comparison of equilibria and the Correspondence Principle. For the purposes of this illustration, the analysis of the example concentrates on a single market for one commodity and in typical partial analysis vein it regards all other commodity prices constant.

Under the above simplifications, the market demand and supply functions for commodity i may be written as

$$Q_i^{d*} = H_i(P_1^o, P_2^o, \ldots P_i, \ldots P_n^o, I, \alpha) = D_i(P_i, I, \alpha) \qquad (1.12)$$

and $$Q_i^{s*} = F_i(P_1^o, P_2^o, \ldots P_i, \ldots P_n^o) + I_i = S_i(P_i) \qquad (1.13)$$

where the superscript zero denotes constancy, the parameter α is a shift parameter representing a change in tastes and D_i, S_i are obvious functional notations for this partial model. Given the above partial specifications, the excess demand for commodity i may be written

$$E_i^* = J_i(P_i, I, \alpha) \qquad (1.14)$$

and the tatonnement process may be specified as

$$\dot{P}_i = 0 \quad \text{if } P_i \leq 0 \quad \text{and } E_i^* \leq 0 \qquad (1.15)$$

$$\dot{P}_i = K_i(E_i^*) \quad \text{otherwise} \qquad (1.16).$$

Assume that, given the endowment matrix, the shift parameter and the other commodity prices there exists an equilibrium price for commodity i, $\bar{P}_i$, such that the excess demand is either zero or negative.[14] We then have

$$E_i^* = J_i(\bar{P}_i, I, \alpha) = b \leq 0 \qquad (1.17).$$

The standard question of comparative static analysis in this simple framework is as follows: how is the equilibrium price affected by a shift in the parameter α of the system? In other words, how do two equilibria defined by different values of the parameter compare in terms of prices and quantities?

Qualitative comparative statics seeks an answer to this question in terms of the general direction of the change in equilibrium price in response to the parametric change. To answer this question, the equilibrium condition (1.17) is differentiated with respect to the variable P_i and the shift parameter α to yield

$$J_{ip} d\bar{P}_i + J_{i\alpha}\, d\alpha = 0, \text{ i.e., } \quad d\bar{P}_i/d\alpha = -J_{i\alpha}/J_{ip} \qquad (1.18),$$

where J_{ip} is the derivative of the excess demand function with respect to the price $\bar{P}_i$ in the neighbourhood of equilibrium and $J_{i\alpha}$ is similarly the derivative with respect to the shift parameter.

---

[14] Questions of existence and uniqueness of equilibrium are outside the scope of this investigation. See K. Arrow and F. Hahn, (1971) or G. Debreu, (1959), as examples of formal treatment of these questions.

The answer to the question of what happens to the equilibrium price when the "taste" parameter changes is seen to depend on the derivatives of excess demand with respect to the parameter and the equilibrium price. The derivative with respect to the parameter can be made positive, i.e., an increase in α increases demand. The derivative of the excess demand function with respect to the price is the difference of the derivatives of the demand and supply functions with respect to that price, as may be seen from the definition of the excess demand function, (1.5). Unless information is somehow obtained concerning the sign of this derivative, no theorem on the effect of tastes on equilibrium price is forthcoming. Moreover, and more importantly, when the system analyzed involves more than one equation the answer to the comparative statics questions cannot usually be given even if the signs of all derivatives are known: this is because, in the case of numerous equations, the numerators and denominators of such expressions as equation (1.18) consist of determinants of matrices, the signs of which are not determined from simple knowledge of the signs of their elements except in special cases.

The Correspondence Principle was first expounded by P. Samuelson as a partial solution to the above problem of qualitative comparisons of equilibria.[15] He pointed out that the equations describing the equilibrium stationary state of a static system are indeed the same as the equations of the stationary state of the "corresponding" dynamic system. The latter was formulated as a tatonnement process of adjustment, along the lines described above; it was then seen that information on the behaviour of the dynamic system could be used to determine the sign of the denominator of the comparative statics expressions. More specifically, if the system could be considered "stable" in some sense (to be defined below) the stability conditions would provide direct information on the sign of the denominator of expressions such as equation (1.18).

To make the above explicit in terms of the simple example of this Section, the Correspondence Principle amounts to the statement that the dynamic system describing the tatonnement process, equations (1.15) and (1.16), has as stationary state the equilibrium condition of the static model. That is to say, the price movement stops when excess demand is zero (or negative, and with the price zero in this case). This is, of course, obvious from the specification of the tatonnement process in (1.15). To complete this illustration of the use of the Correspondence Principle a definition of "stability" and some technical manipulation is needed.

---

[15] In Samuelson, (1947).

An equilibrium is asymptotically locally stable if, after a small displacement away from it the system tends to return to this equilibrium. To use this concept, the dynamic system (1.16) is linearized around the equilibrium price $\bar{P}_i$ via a Taylor series expansion[16] to obtain

$$\dot{P}_i = K_{iJ}J_{ip}(P_i - \bar{P}_i) + K_i J_i(\bar{P}_i, I, \alpha) = K_{iJ}J_{ip}P_i + C \quad (1.16L)$$

where C is the sum of the constant terms from the expansion.

Equation (1.16L) is a linear differential equation of the first order, with constant coefficients and a constant term. The complementary-function solution, which is the only one involving the time element and therefore the only one relevant for stability[17], is obtained from the homogeneous part of this differential equation, by considering the solution of

$$\dot{P}_i = K_{iJ}J_{ip}P_i \quad (1.16LH).$$

Try a solution of the form:

$$P(t) = P(0)e^{\lambda t} \quad (1.19)$$

by substituting it in (1.16LH). If it is to be a solution, the equation obtained from this substitution must hold as an identity for all t, i.e.,

---

[16] Compare Samuelson, (1947).

[17] The particular solution only involves, in this case, a constant term. See, for example, S. Ross, (1964).

$$\lambda P(0)e^{\lambda t} = K_{iJ}J_{ip}\ P(0)e^{\lambda t} \qquad (1.20)$$

from which one obtains the condition $\lambda = K_{iJ}J_{ip}$ for the equation to hold. Thus, the complementary-function solution to the dynamic equation (1.16L) may be written

$$P(t) = P(0)e^{K_{iJ}J_{ip}t} \qquad (1.21).$$

Now from the definition of a locally asymptotically stable equilibrium, the path of price over time must approach some constant equilibrium value. This implies that the time-dependent portion of the solution of the equation, i.e., the complementary function, must vanish as time approaches infinity. But this condition can be satisfied only if the quantity $K_{ij}J_{ip}$ is negative. Thus, in order for the system to be locally stable,

$$K_{iJ}J_{ip} < 0, \text{i.e.,} J_{ip} < 0 \qquad (1.22)$$

since K_{iJ} is positive by definition (equation (1.10)[18].

Once it is realized that J_{ip} is the denominator of the comparative statics expression (1.18) the usefulness of stability analysis for the derivation of comparative static theorems should be in plain view. The sign of the denominator of this expression is now definite from that analysis,

---

[18] Thus, stability is not affected by the speed of adjustment in this case. In some higher-dimensional cases this may not hold, and in the discrete time formulation of the adjustment process it does not hold. Examples will be encountered in other Chapters of this investigation.

and the answer to the question of how a change in tastes affects equilibrium price can be given unequivocally.

It should perhaps be repeated that in higher dimensional systems the numerators of comparable expressions of comparative statics are sub-determinants of the matrix of the full system. Thus, information on the sign of the denominator is necessary but not sufficient for the derivation of theorems of comparative statics in these cases.[19]

With this example of comparative statics and the Correspondence Principle complete, one may briefly inquire into the importance of the equilibration process for the derivation of comparative statics theorems. In this connection, it may be remarked that the specific process of adjustment assumed seems more or less crucial to the comparative static theorems: it may not be possible to obtain the same kind of information under other adjustment processes, such as a non-tatonnement. A full discussion of this point will have to await for Chapter III below.

The task undertaken in this Section is now complete. In the next Section, a model of non-tatonnement adjustment is formalized for a pure exchange economy.

---

[19]For further discussion on the Correspondence Principle see Arrow and Hahn ,(1971); K. Lankaster (1962), (1964); and W. Gorman (1964).

1.3 The Non-Tatonnement Process in Pure Exchange

The setting of the general model described in Section 1.1 needs modification in order to describe a model of pure exchange. More specifically, the market production level represented by equation (1.2) is set equal to zero at all prices. Thus, the excess demand functions for commodities are now dependent on demands by individuals and on stocks alone.

If exchange is to take place in goods i.e., consumable producible commodities, it must be assumed that commodity stocks have somehow come into being in the past, and that the system starts with an endowment matrix of stocks, both in terms of total quantities of each good and distribution of goods among individuals. In conventional accounts of exchange models this endowment matrix is simply postulated. This approach will be followed here. It will also be pointed out, however, that it may be useful to regard the initial endowment matrix as the result of production. It then becomes obvious that one way to conceptualize the whole process of production and exchange is in terms of the sequence: exchange in inputs, production, exchange in goods. This sequence will be discussed in Section 1.5 below.

Given the above, a non-tatonnement process of equilibration allows transactions at disequilibrium prices. The auctioneer is retained, and burdened with the tasks of price change and dissemination of information. More specifically, the auctioneer is instructed to change prices according to the same rule as in the tatonnement process, namely,

$$\dot{P}_i = 0 \quad \text{if } P_i \leq 0 \text{ and } E_i \leq 0 \qquad (1.15)$$

$$\dot{P}_i = K_i(E_i^*) \quad \text{otherwise} \qquad (1.16).$$

In addition, the auctioneer is supposed to give information as to who is willing to exchange what quantities at each step of the equilibration process. An alternative to this assumption would be a search procedure by means of which individuals willing to exchange would seek each other. In order to make this search process fruitful, however, other no less unrealistic assumptions would have to be introduced.[20]

In addition to price changes, and in contrast with the tatonnement process, transactions at disequilibrium prices change the endowment matrix of the economy. Changes in the individual elements of this matrix may be represented as follows:

$$\dot{I}_i^m = G_i^m(P_1, P_2, \ldots P_n, I), \quad (i=1,2,..n,\ m=1,2,\ldots m) \qquad (1.23)$$

where G is a function showing the changes in endowments over time. Changes in endowments in disequilibrium must obey the following conditions:[21]

---

[20] Compare Arrow and Hahn, (1971), p. 329.

[21] Compare *ibid*, p.326.

$$\sum_i P_i \dot{I}_i^m = 0 \quad \text{for all individuals } m \text{ (1.24), and}$$

$$\sum_m \dot{I}_i^m = 0 \quad \text{for all commodities } i \text{ (1.25).}$$

The first condition says that the individual must pay for any exchange he undertakes, hence the change in total value of his endowment at existing prices is zero. The second condition says that there cannot be change in the total quantity of each commodity available in the economy, since this is a model of pure exchange.

In the analysis of stability of this process, the functions representing changes in the commodity endowments are usually specified in a more detailed form than (1.23). Examples of such specifications will be given in Chapter III of this investigation. The rest of the discussion in this Section will concentrate on the basic feature of the non-tatonnement process in exchange, namely the dependence of the stationary state on disequilibrium transactions in this process.

An equilibrium for the tatonnement process of Section 1.2 was defined as a set of non-negative prices such that (given the initial endowments, technology and tastes) excess commodity demands at these prices are zero.[22] On the other hand, equilibrium when disequilibrium transactions are

---

[22] Or non-positive at zero price.

permitted is defined as a set of non-negative prices and an endowment matrix such that (given technology and tastes) excess commodity demands at these prices and endowments are zero. More formally, if there exists a non-negative vector $\bar{P}=(\bar{P}_1,\bar{P}_2,\ldots\bar{P}_n)$ and a matrix $\bar{I}$ such that

$$E_i^* = E_i(\bar{P}_1,\bar{P}_2,\ldots\bar{P}_n,\bar{I}) \leqq 0 \qquad (1.26)$$

the vector $\bar{P}$ and the matrix $\bar{I}$ define an equilibrium.

Given an initial endowment matrix I, the path of exchanges in disequilibrium will determine, at each moment, the endowment matrix at that time. If it happens that the endowment matrix and the associated price vector at any time are such that excess demands are zero[23] the system is in equilibrium. The variability of the endowment matrix during disequilibrium exchange suggests that, given an initial endowment, various price vectors are consistent with equilibrium, depending on the specific path followed by the process of adjustment. This is clearly a case where actual transactions might play a significant role in the determination of the stationary state of the system. The further questions refer to whether stability conditions, comparative static theorems and behaviour in disequilibrium are also affected. These questions will be dealt with in Chapters III, IV and V. The point to be made here is that since the endowment matrix

---

[23] Or non-positive at zero price.

is subject to change during the equilibration process various price vectors become consistent with equilibrium, and which one does become the equilibrium vector may depend crucially on the path of equilibration.

1.4 Endowment Effects: Identification, Classification and Importance

This Section identifies, classifies, analyzes and discusses the importance of various endowment effects present during the process of market equilibration. Because endowment effects are the means by which actual transactions may come to influence the stationary state, stability etc., their importance for the analysis in this investigation should be obvious: indeed, various approaches to the problem of market equilibration are classified herein in terms of the effects on which they concentrate.

Endowment effects have traditionally been discussed with reference to demand: the discussion in this Section begins with the demand side but in Section 1.5 it is indicated that endowment effects may originate in the supply side, at least in terms of the visible components of the system.

Starting with the demand side, the first task in this Section is the identification of endowment effects on individual -- as contrasted to market -- demand for a commodity, say commodity j. Next, endowment effects on the market demand functions are identified.

The demand functions of an individual m for commodities have been denoted by equations (1.2) in this Chapter. These functions are derived from the standard problem of utility maximization of an individual subject to his budget constraint. In what follows, a brief discussion of this problem is undertaken which serves to identify the endowment effects on demand.

Consider an individual m maximizing his utility

$$U^m = U^m(q_1^m, q_2^m, \ldots q_m^m) \qquad (1.27)$$

subject to a budget constraint

$$\sum_i P_i q_i^m = \sum_i P_i I_i^m \qquad (1.28)$$

First order conditions for a maximum are given by

$$U_i^m - \lambda P_i = 0, \ (i=1,2,\ldots n) \text{ and } \sum_i P_i (I_i^m - q_i^m) = 0 \qquad (1.29).$$

where U_i^m is the partial derivative of the utility function with respect to commodity i, λ the Lagrangean multiplier and all other symbols have already been defined in Section 1.1. These conditions form a system of (n+1) equations in (n+1) unknowns (prices and the Lagrangean multiplier) and may be solved to derive demand functions of the form of equation (1.2) above. For the purpose of identification of endowment effects it is convenient to take the total differential of these conditions with respect to all parameters, i.e., prices as well as initial commodity endowments. One then obtains[24]

$$\begin{bmatrix} U_{11}, U_{12}, \ldots\ldots\ldots U_{1n} & -P_1 \\ U_{21}, U_{22}, \ldots\ldots\ldots U_{2n} & -P_2 \\ & \\ U_{n1}, U_{n2}, \ldots\ldots\ldots U_{nn} & -P_n \\ P_1, P_2, \ldots\ldots\ldots P_n & 0 \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ \\ dq_n \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp_1 \\ \lambda dp_2 \\ \\ \lambda dp_n \\ \sum_i (I_i - q_i) dP_i + \sum_i P_i dI_i \end{bmatrix} \qquad (1.30)$$

where U_{ij} is the cross-partial derivative of the utility function U^m, and the superscript m has been dropped temporarily, for expositional convenience.

Denoting the (n+l)x(n+l) matrix by A, the vector of changes in the variables by dq and the vector of changes in the parameters by dp one may write

$$A\ dq = dp \qquad (1.30a)$$

instead of (1.30).

Using conditions (1.29) the determinant of matrix A[25] may be written

[24] The last equation in (1.30) may be multiplied by $-\lambda$ to obtain uniformity in the P_j which border the matrix. For purposes of the present exposition, however, this does not make a difference.

[25] And indeed any cofactor of this determinant which is formed by deletion of rows and columns other than the last.

$$|A| = (-1/\lambda^2)\,|U_b| \qquad (1.30b)$$

where $|U_b|$ is the bordered Hessian of the utility function U. This relation will be used presently.

The change in the quantity demanded of commodity j in response to parameter changes[26] is then given by Cramer's rule:

$$dq_j = \frac{|A,dp|}{|A|} = \frac{\sum_i |A_{ij}|\,dP_i + |A_{n+1,j}|\,[\sum_i (I_i - q_i)\,dP_i + \sum_i P_i\,dI_i]}{|A|} \qquad (1.31)$$

where $|A,dp|$ is the determinant formed by substitution of the vector dp for the ith column of A, and A_{ij} is a cofactor[27] of A. Using (1.30b) one may write, instead of (1.31),

$$dq_j = \frac{\sum_i |U_{ij}|\,\lambda dP_i}{|U_b|} + \frac{\lambda |U_{n+1,j}|}{|U_b|}\,\{\sum_i (I_i - q_i)\,dP_i + \sum_i P_i\,dI_i\} \qquad (1.32)$$

The first term in this expression is the well-known substitution term[28] of price changes. The second is an "income" or endowment term, which consists of two components: the

---

[26] One can of course consider changes of one parameter at a time. Thus, when only $dP_j \neq 0$ in (1.31), the equation shows the own price effect on demand. When $dP_k, j \neq k$ is considered, one obtains the cross-price effect.

[27] i.e., a determinant of order nxn obtained by deletion of the ith row and jth column of A and given the sign $(-1)^{i+j}$.

[28] Weighted by the changes in prices, dP_i.

endowment effect of a price change, which is the product of the outside term and the first term in the brackets[29], and the endowment effect of a change in the endowment vector, which is the product of the outside and the second bracketed term[30]. Both endowment effects have the term outside the brackets in common. Denoting the latter by C_j[31], and the substitution terms by S_{ij}[32] one may write, instead of (1.32),

$$dq_j = \sum_i S_{ij} dP_i + C_j \sum_i (I_i - q_i) dP_i + C_j \sum_i P_i dI_i \quad (j=1,2,..n)$$

(1.33).

With the mechanics of the problem out of the way, the discussion may now proceed to identify and analyze various endowment effects. The term $C_j \sum_i (I_i - q_i) dP_i$, which results from change in price, will be labelled Endowment Effect of the First Kind in this investigation. As is obvious from the expression, endowment effects of the first kind depend not only on the term C_j but also on the "degree of participation" of the individual in the market for the specific commodity whose price changes. Suppose, for example, that the price

---

[29] Weighted by price changes, again.

[30] Weighted by endowment changes.

[31] The sign of C_j is ambiguous, as is known from conventional demand theory.

[32] The sign of S_{ii} is negative, and the sign of S_{ij}, $i \neq j$ is ambiguous.

of commodity i changes; the endowment effect of this change in price is, then, equal to $C_j(I_i-q_i)dP_i$. Besides the term C_j, the strength of this effect depends on the absolute value of (I_i-q_i), which represents the "degree of participation"[33] of the individual in the market for the commodity. If, for example, this term is zero, it implies that the individual does not enter the market[34] for this commodity at all, i.e., the individual is content to consume his initial endowment of this commodity. If the term is positive it means that the individual is a seller[34] and consequently stands to be affected by a change in the price to the extent of the amount sold, which is precisely the term (I_i-q_i). If, finally, the term is negative it means that the individual is a buyer[34] and therefore stands to be affected by price change to the extent of the amount bought, again equal to (I_i-q_i).

Endowment effects of the First Kind exist irrespective of changes in the quantities of commodity stocks with which the individual is endowed: even if these quantities remain constant, a price change sets such effects in operation as long as as C_j differs from zero and the individual participates in the market. Thus, effects of the First Kind can occur quite independently of other effects, as long as there is

---

[33] That is, the difference between his actual and utility-maximizing stock of the commodity.

[34] At the initial prices where the maximum utility was calculated.

price variation. Because of this, one may wish to label these effects <u>pure price effects</u> on demand, to distinguish them from other effects, which sometimes involve price and quantity variations, and sometimes only quantity changes.

<u>Endowment Effects of the Second Kind</u> occur when a change in the composition of commodity endowments is followed by a price change. They thus operate in two steps: first, the term $\sum_i P_i dI_i$ will denote the change in the endowment <u>composition</u>, and then the re-evaluation of the new endowment will produce pure price effects on this new endowment, through the term $C_j \sum_i (I_i - q_i) dP_i$.

To make this clear, consider an individual participating in exchange of commodities, and let this individual purchase, at the initially given prices, dI_i of commodity i for dI_j of commodity j. Assuming that no credit transactions are permitted, the exchange must satisfy, from equation (1.24) above,

$$P_i dI_i = P_j dI_j \text{, i.e., } \sum_i P_i dI_i = 0 \qquad (1.24).$$

The effect of this transaction at the initial prices is thus simply to change the <u>composition</u> (but not the value) of the endowment vector of the individual. More specifically, his quantity of i is now $(I_i + dI_i)$ and of j $(I_j + dI_j)$. Thus if a price change then occurs in either commodity, the pure price effect resulting from it will not be the same as if the transaction described above had not taken place. Consider, for example, a change in the price of commodity i, occurring <u>after</u> the above transaction has been completed: since the individual was assumed a buyer of i, the term $(I_i+dI_i-q_i)$ -- which shows the changed endowment of i -- is now either negative or zero.[35] In either case, the individual is affected in a lesser degree by the subsequent price change, since the transaction by which he obtained dI_i has reduced (other things equal) his participation in the market for commodity i. Alternatively, the effect of the subsequent price change on the modified endowment may be separated into two terms, $C_i(I_i-q_i)dP_i$ and $C_i dI_i dp_i$.[36] The first is the pure price effect (of the First Kind) that the individual would experience with the price change even if he had not previously changed his endowment composition via exchange. The second term indicates the effect on demand for i due to the previous transaction, and is the Endowment Effect of the Second Kind. If the price P_i has risen, the individual buyer has gained $dI_i dP_i$ in endowment value, and the effect on his demand for i is consequently

[35] It will be zero if it is assumed that the individual purchased the optimal quantity of i, q_i, in the transaction.

[36] One of course could avoid the product of two differentials by telling the above story in terms of discrete changes.

$C_i dI_i dP_i$.[37]

The gain in the endowment value described above will affect not only the demand for commodity i but all commodity demands[38] as can be seen from (1.33). The effect of the gain on the demand for commodity j is given by $C_j dI_i dP_i$.

From the above it is obvious that endowment effects of the Second Kind operate only when first the endowment composition and subsequently the price vector are allowed to change. If there is a subsequent price change, the change in the endowment composition affects demand via endowment effects of the First Kind. Thus, endowment effects of the Second Kind could be labelled quantity and price effects, since the necessary conditions for them include both quantity and price variation.

Finally, it may be noted that the total endowment quantity of commodity j or i in the economy does not change: only redistribution is involved between individuals. Thus, the second kind of endowment effects could alternatively be called redistribution effects: this makes clear that such effects operate in exchange and they are to be distinguished

---

[37]Whether the demand for i increases or decreases from this gain in endowment depends on whether C is positive or negative, i.e., on whether the commodity is superior or inferior.

[38]Subject to non-zero values of C_j, j=1,2,...n.

from others involving net increases in endowment quantities.

The next category of endowment effects involve precisely such net changes in endowment quantities. For the moment, it is not necessary to examine how these changes come about; this will be discussed below in this Section.

Consider a change in the initial endowment of individual m, consisting of an increase of the quantity of commodity i, denoted by dI_i as before. The consequent change in the value of the endowment is $P_i dI_i$, and the effect of this endowment change on the demand for commodity j is $C_j P_i dI_i$. This kind of <u>effect</u> on demand for commodities will be labelled one <u>of the Third Kind</u>. It is obviously a <u>pure quantity effect</u>, in the sense that it is attributable to changes in the initial quantities of commodities with which the individual is endowed, and it does not require price variation to operate. Moreover, it cannot operate under redistribution of commodities among individuals at given prices, since in this case the total change in the individual endowment at these prices is zero, by (1.24). This endowment effect thus operates only when the total value of the endowment increases due to change in quantity.

It may be useful at this point to consider the meaning of the term C_j in equation (1.33): it is easily seen that it is in effect the marginal propensity to consume commodity j, i.e., it is the change in the quantity of j demanded per unit change in the value of the endowment. This latter change may be brought about by price changes (in the case of endowment effects of the first two kinds) or by physical endowment changes with prices constant (in the case of effects of the Third Kind). In either case, the effect of a unit value change in the endowment is the same and equal to C_j.

The question now is: How can a new change in the quantity of a commodity come about in an economy of pure exchange? The answer is, of course, that it cannot, since equation (1.25) prohibits this from happening, and indeed defines a pure exchange model. Such a change can then only happen when production is allowed to vary. In the sequential scheme alluded to in Section 1.3 above[39] a variation in production of commodity i will have to be reflected in a change in the initial endowment of some individual in the economy, without any compensating change elsewhere. A hidden condition for the existence of endowment effects of the third Kind is, then, <u>change in</u> volumes of <u>production</u>. It goes without saying, therefore, that such effects will not be present in pure exchange models.

This completes the identification, classification and analysis of the three basic kinds of endowment effects on

---

[39]And more fully discussed in Section 1.5 below.

demand. Some remarks as to the situations in which these effects are important follow.

Endowment effects of the First Kind are important in any situation where individuals hold stocks and prices vary. There is no need for redistribution or change in production to occur for them to operate: thus, they are present in a situation of tatonnement process of equilibration via price. Indeed, they are the only endowment effects present in that case. Clearly, such effects will turn out to be important in stability analysis of such a process. This will be seen more clearly in Chapters II and III below.

Endowment effects of the Second Kind need both commodity redistribution and price change in order to operate. They are thus non-existent in a tatonnement process, which precludes redistribution of commodities until the equilibrium price vector is established. Such effects are of course relevant in a non-tatonnement process of price change where redistribution is allowed at disequilibrium prices. Thus, such effects are important, together with those of the First Kind, in stability analysis of non-tatonnement processes, further discussed in Chapters III, IV and V.

Endowment effects of the Third Kind are relevant only when production changes are allowed in the analysis. They are thus non-existent in analyses of pure exchange models.

Before this Section is drawn to a close, it is important to indicate how the three kinds of endowment effects affect the market (as opposed to the individual) demands for goods. Summation of (1.33) over individuals yields,

$$\sum_m dq_j^m = dQ_j^{d*} = \sum_m \sum_i S_{ij}^m dP_i + \sum_m C_j^m \sum_i (I_i^m - q_i^m) dP_i + \sum_m C_j^m \sum_i P_i dI_i^m \quad (j=1,2,\ldots n)$$

(1.34)

where m is the number of individuals, as before. From this expression it is easily seen that effects of the First Kind are present in the market demand under the same conditions as with individual demand. Endowment effects of the Second Kind require that the terms C_j^m be different for each individual m, otherwise there is no effect on demand.[40] Finally, the Third Kind of effects are present under the same conditions as for individual demand.

The discussion of endowment effects is now complete. The next Section deals specifically with problems associated with introduction of production in a non-tatonnement setting.

---

[40] This point is elaborated upon in Chapter III, Section 3.1.3.

1.5 Non-Tatonnement in Models of Production and Exchange

This Section discusses problems of non-tatonnement analysis when production is taken into account. The role of endowment effects in a production and exchange model of non-tatonnement is also discussed.

The definition of a disequilibrium transaction in the exchange model is simple and straightforward, namely, an exchange at a disequilibrium price, a price which does not clear the market. A model of production and exchange, however, necessitates examination of input markets, the production process itself, and the commodity markets of the exchange model. The analogous definition of a disequilibrium "transaction" in this model might then refer to an exchange of inputs at an input price which does not clear the input market under examination; or, it could refer to an exchange of commodities at a price which does not clear the commodity market.

Disequilibrium transactions in such a model, therefore, may occur in both the input and commodity markets. But this is not all that is necessary to mention when production is introduced: if only flows created within the market period are considered, the operation of the commodity market requires that hiring of inputs and production has preceded the commodity exchange process. Otherwise, neither commodities nor income will have been created, and the commodity market is plainly not in existence as yet.

It is perhaps convenient to visualize the above sequential process as follows: individuals functioning as "enterpreneurs"[41] (whose initial endowments are more or less specialized in a non-producible commodity called "exchange money") purchase inputs from other individuals at going prices and pay in terms of "exchange money", a non-producible commodity among those in the model. Production proceeds on the basis of hired input services, and commodities come into existence. The situation now has the individual enterpreneurs holding stocks of producible commodities, and other individuals (with endowments "specialized" in these inputs) holding stocks of exchange money. The endowment of the latter group of individuals is thus now "specialized" in money, and the endowment of the enterpreneurs in producible commodities. This is in no conflict with the specifications of the model described above in this Section, and it seems plausible as a representation of the process in a model of production and exchange. As for the "specialization" with regard to enterpreneurs' and other individuals' endowments, it actually seems to correspond to "reality" much more closely than other specifications.

---

[41] If a commodity called "organizational ability" is specified, these individuals must be assumed to hold stocks of it.

The difficulty presented above with regard to the necessity for a "sequence"[42] does not appear in the tatonnement model of production and exchange, since equilibration in that model proceeds on the basis of planned, as contrasted to effective, demands and supplies. Thus, planned input supply is accompanied by a planned demand for commodities, on the basis of income planned from the supply of inputs. In the same way, planned input demand is accompanied by planned commodity supply by producers, on the basis of the planned input employment. Thus, equilibration analysis in this system does not require that any production plans[43] be carried out in disequilibrium: in fact, it requires exactly the opposite, namely that no plans be put into effect until the equilibrium price vector is found.

The situation in disequilibrium models of production and exchange, in contrast, requires output to have been

---

[42] The necessity for such a sequence may be questioned. Introduction of commodity inventory makes it possible, _prima facie_, to have disequilibrium commodity exchange before or at the same time as production, provided that the "wealth" which has been accumulated from previous periods, is allowed to vary at the time of the transaction. However, since both the commodity stock and the "wealth" stock repletion require production the introduction of stock considerations does not invalidate the statement in the text. Moreover, the presence of inventories and wealth stocks imply prior production, too.

[43] Or, for that matter, any other plans.

produced for purposes of exchange in disequilibrium. But, if output is to have been produced before the establishment of an equilibrium price vector and while input markets are in disequilibrium, the income creation plans of agents in these markets will not be realized. Moreover, chances are that production plans may not be realizable, under circumstances of excess demand for one or more inputs. Finally, sales plans may not be realizable in the sense that either the market for the commodity does not clear at the expected price or that this price has to change for the market to equilibrate.

The conclusion of the foregoing is that disequilibrium in models of production and exchange implies the possibility that income creation, production and sales plans may be unfeasible. Moreover, when interdependence of production, income, expenditure and sales is taken into account, it is easily seen that falsification of one category of plans may bring about another.

Little discussion exists on the feasibility of income and production plans in disequilibrium. An analysis of this matter will be undertaken in Chapter V. With respect to sales plans, three broad categories of disequilibrium[44] production

[44] The term is used as a synonym for "non-tatonnement".

and exchange models may be distinguished. The first category includes those in which the false production volume is put onto the market and demand is destroyed or created as the case may be by means of pure price variation so that equilibrium is temporarily established. These are pure price adjustment models. Because of the mistake involved in the estimation of production volumes, the price of the commodity must adjust to bring about equilibrium and the cost is borne by the firms involved, that is to say by the enterpreneurs. Thus, the cost of this equilibration process appears as an endowment effect of the third kind, since it implies that the income of some input-owners taking the risks of production is smaller than expected.

The second category of models corrects the discrepancy between production and demand not through price but through quantity variation. Thus, given the prices that are to stay constant the production volume is put to the market and what is not sold is accumulated in inventory (or, the excess demand is satisfied by inventory depletion).

A third category of models would have both inventory change and price change sharing the burden of equilibration in the market period. Models in this category are very scarce indeed.

In all the above models, production plans in the next period are modified to take account of past mistakes,and this variation affects demand by means of endowment effects of the Third Kind once again. Also, production plans are affected by the situation in the input markets, in the sense that, unless equilibrium prevails, some production plans may be unfeasible.

The above is only a preview of the problems associated with introduction of production in non-tatonnement processes. A fuller discussion of this matter is contained in Chapters III, IV and V. Suffice it to conclude here that the sequence of input exchange, production and commodity exchange, introduces the possibility of mistakes in production volumes which carry forward in the form of endowment Effects of the Third Kind. Moreover, feasibility of production plans in disequilibrium is not always assured.

1.6 Closing Remarks

The aim of this Chapter has been to provide an introductory framework for the analysis of equilibration problems and to present a formalization of equilibration processes, as well as to give an example of the way meaningful theorems in economics are derived. The next two Chapters use the concepts of this Chapter in a historical review of the literature. Chapters IV and V apply the general model presented here to the process of equilibration in particular settings.

CHAPTER II

REVIEW, INTERPRETATION AND CRITICAL ANALYSIS OF THE EARLY LITERATURE

This Chapter discusses the work of such early contributors to the problem of equilibration as Walras[1], Edgeworth[2], Marshall[3], Wicksteed[4], and Hicks[5]. Section 2.1 deals with Walras' treatment of the equilibration problem, both in pure exchange and in production and exchange. Section 2.2 discusses Edgeworth's contribution in this respect, while Marshall, Wicksteed and Hicks are dealt with in Sections 2.3 2.4 and 2.5 respectively. Section 2.6 summarizes the conclusions of the analysis of this Chapter. The aim of this historical review, interpretation and critical analysis of this early work is not only to provide a background for the research undertaken in this investigation but also to put the contributions in the general framework of this thesis wherever possible. Moreover, it will be seen that the analysis of Chapter V of this thesis is, in a loose sense, connected with the approach implied by Edgeworth to the problem of equilibration.

---

[1] (1954).

[2] (1881), (1925).

[3] (1920).

[4] (1933).

[5] (1946).

2.1 L. Walras

Leon Walras[6] may be considered the first theorist to face seriously the question basic to the phenomenon of market equilibration, although others before him may be said to have mentioned the concept of tatonnement in relatively the same context.[7] His preoccupation was to show that the actual market mechanism may be considered to achieve essentially the same result as that indicated by the solution to the mathematical problem of equilibrium of exchange. This was presumably to be achieved by the market through a process of tatonnement, or groping, towards the equilibrium price. In Walras' own words (or, rather, in the words of his translator):

> "What must we do in order to prove that the theoretical solution is identically the solution worked out by the market? Our task is very simple: we need only show that the upward and downward movements of prices solve the system of equations of offer and demand by a process of groping."[8]

Walras is concerned here with the empirical relevance of his system of equations for the model of commodity exchange. Unless it can be shown that the market possesses behavioural and institutional mechanisms that lead it to the equilibrium set of prices within a reasonably acceptable time span, the

---

[6] (1954).

[7] cf. W. Jaffe, (1967)

[8] Walras, (1954), p.170, (§125).

system of equations of exchange loses much of its significance as a tool descriptive of the equilibrium of exchange and its characteristics. This, of course, is not to be taken to imply that Walras was aware of all the behavioural, institutional and time-span problems associated with the question of identification of the actual market solution to that of the equation system; but he did seek to establish some correspondence between the market and the theoretical formulation.

Walras' own treatment of the process of tatonnement has been characterized as "wholly inadequate",[9] and "a swindle".[10] According to Jaffe's interpretation Walras basically asserts that the market mechanism of exchange in the case of many commodities is convergent, i.e., stable, and moreover that it tends to the set of prices established by the static set of equations of the mathematical model of exchange. It should be noted that the second statement not only implies stability, but also uniqueness of the equilibrium set of prices, and independence of that set from the path that the market follows during equilibration. Obviously, this set of propositions requires more than simple assertion, and, as will be pointed out later, proofs (which have now been furnished in the literature) require assumptions and

---

[9] Jaffe, (1967), p.5.

[10] R. Solow, (1956), pp 87-89.

behavioural and institutional specifications that never entered Walras' own discussion of the matter.

It may be, however, that the above interpretation is too harsh. One might wish to recognize that Walras was successful in showing static stability in the case of the two-commodity model of exchange. As for multi-commodity exchange, while it is true that he could prove very little about stability it is also true that the relevant paragraphs of the Elements contain interesting remarks as to conditions which were later found to be necessary for stability. At one point, for example,[11] in which he attempts to show that the multi-commodity equilibrium will be stable, we find Walras suggesting that stability would be probable if we remember that the response of the quantity demanded of a commodity to its own price is a "direct" (hence stronger) effect while the consequent changes of the other prices are only indirect (hence weaker) effects on the quantity demanded of the commodity in question. This, of course, is certainly no proof, as Jaffe notes. But it is interesting to note that later literature does contain a proof of stability of multi-commodity exchange based on diagonal dominance.[12] When expressed in terms of responsiveness of quantity demanded to prices, diagonal dominance means that the response of the

---

[11] Walras, (1954), p.172 (§130).

[12] See Chapter III, Section 3.1.1

quantity demanded to the own price is greater than the sum of the responses of the same quantity to other prices, which is what Walras alluded to. Thus, if he cannot be credited with a proof he should at least be credited with perception.

At the same time, it may be fair to state that Walras either has no conception of what trading at false prices does to equilibrium or he suspected the problems associated with it and shied away from its analysis. One does find, however, that Walras believed in a very high speed of actual market adjustment. Thus,

> "the rapidity and reliability of the practical solutions leave no room for improvement. It is a matter of daily experience that even in big markets where there are neither brokers nor auctioneers, the current equilibrium price is determined within a few minutes, and considerable quantities of merchandise are exchanged at that price within half or three-quarters of an hour. In fact, the theoretical solution would be absolutely impracticable in almost every case."[13]

It is this and related remarks in Walras' book that give rise to the question whether he allowed disequilibrium transactions to take place in his description of the equilibration of the exchange model.[14] Peter Newman,[15] for example, suggests that Walras did explicitly allow for such transactions

---

[13] Walras, (1954), p.106

[14] *Ibid*, pp. 105-106 (§§60-61), and pp. 169-172 (§§125-130).

[15] (1965) pp. 101-103.

and refers to pages 84-86 and the above passage for evidence relating to Walras' discussion on disequilibrium trading and the speed of convergence to equilibrium respectively.

While it is true that some discussion on disequilibrium transactions is there on pages 84-86 and while Walras' belief in the speed of adjustment of the actual market is evident from the above passage, one might agree with W. Jaffe that Newman's interpretation strains conjecture too much. This issue, of course, would be of interest as a subject in the history of economic thought and may be considered to reduce to the following alternative statements:

(i) Walras, being a careful theorist and having announced at the outset of his discussion of the tatonnement mechanism that his purpose was to show that the market arrives at a solution <u>identical</u> to that of the equations of exchange,[1] could not have considered transactions at disequilibrium prices, irrespective of his belief about the speed of convergence. This viewpoint is reinforced by the fact that in the model of production and exchange Walras explicitly excludes disequilibrium transactions, via use of "tickets" in disequilibrium.

---

[16] Walras, (1954), pp. 169-170 (§§124-125).

(ii) While it is true that Walras was a rigorous and careful theorist, it is also true that in the particular instance under discussion he did experience difficulty. As was pointed out, he came to assert simply that the market mechanism will converge to equilibrium. What equilibrium he did not say, as Jaffe himself notes.[17] It is then conceivable that Walras did not mean an equilibrium identical to that yielded by the equations of exchange, but one very close to it (since the speed of convergence he believed to be high, and hence the bulk of transactions would take place at equilibrium).

Related to the above issue is the question of the meaning and analytical use of the Walrasian theorem of equivalent redistributions of commodity holdings. Jaffe[18] suggests that the theorem has relevance for the Walrasian theory of tatonnement which was missed by Walras himself. He also points out that modern writers have not taken into account the result of this theorem when they state the conditions for invariance of equilibrium prices under commodity redistribution among individuals.

It may be useful, then, to examine the meaning and possible analytical use of the Walrasian theorem of "equivalent redistributions of commodity holdings", and its possible

---

[17] Jaffe,(1967), pp. 5-6

[18] Ibid, p.3

relevance to the Walrasian theory of tatonnement. A related question is that of necessary and sufficient conditions for invariance of equilibrium commodity prices in the Walrasian model of tatonnement exchange.

The theorem of "equivalent redistributions of commodity holdings" states that

> "given several commodities in a market in a state of general equilibrium, the current prices of these commodities will remain unchanged no matter in what way the ownership of the respective quantities are redistributed among the parties to the exchange, provided, however, that the value of the sum of the quantities possessed by each of these parties remains the same"[19]

It seems clear that an equivalent redistribution of commodity holdings is defined by Walras as one which has the same value, calculated at the equilibrium level of prices. It is also clear that such redistribution will not change the total quantity of each commodity available for consumption and exchange. The theorem, then, suggests that a change in the commodity composition of the initial individual endowments of the participants in the exchange does not affect the final solution of the equations of exchange (that is, it does not play a role in the determination of equilibrium prices) in the case where the value (in terms of the numeraire, and calculated at these equilibrium prices) of the endowments is the same for each individual participant, after redistribution.

---

[19] Walras, (1954), p.185.

In order to determine the possible analytical use of this theorem, consider a tatonnement process without "false trading"[20] in which the price change has the same sign as that of excess demand. Although no transactions take place before the final equilibrium price is arrived at, it should be obvious that the values of the commodity holdings of the individual participants keep changing as the system gropes toward equilibrium, for the simple reason that prices at which these values are calculated vary during the process. This is, of course, due to the operation of endowment effects of the first kind.

When a process with disequilibrium trading without transactions on credit is considered, the individuals who turn out to have gained (lost) from the disequilibrium transaction of the previous moment (or period) can be shown to have experienced an unambiguous increase (decrease) in the value of their holdings compared with that value before the disequilibrium transaction, when both are calculated at prices of the present period. This is the endowment effect of the second kind, discussed in Chapter I.

Thus, while in both processes the value of endowment of the individual changes during the process of equilibration, the difference is that without disequilibrium transactions

---

[20] i.e., transactions at disequilibrium prices.

that value changes only with the operation of endowment effects of the first kind. With disequilibrium transactions, however, when an individual is allowed to transact at a given set of prices and thus change the physical quantities of commodities in his initial endowment, the value of his new endowment is different from the value of the old endowment (with both values reckoned at a price vector different from that in which the disequilibrium transaction took place[21]). This difference in the value of the endowments is responsible for "income" or "endowment" effects of the second kind on the demand of the particular individual.

Thus, one finds that irrespective of false trading or not the value of the individual commodity endowment will change during the process of equilibration for the simple reason that prices at which the value of this endowment is calculated change during the process. This remark serves to point out that the analytical use of the "equivalent redistributions of commodity holdings" cannot be in the framework of the process of equilibration of the market: because the essential characteristic of the theorem is invariance of the endowment value calculated at a given price vector, while the essential characteristic of the equilibration process is price

---

[21] Because no transactions on credit are permitted, the endowments before and after the false trading will have the same value if reckoned at the price in which false trading took place.

change.

It may be asked why it is necessary to revalue the endowments of the individuals during the process of equilibration. The reason is, of course, that such revaluation is necessary for the determination of the quantity demanded at various prices, and is part of the conceptual experiment via which the planned demand curve is determined.

Thus, the use of the theorem is restricted to situations where the price does not change: more specifically, it may be used to show that the equilibrium price vector which results from the equations of exchange depends solely on conditions of market supply, demand and on the distribution of wealth (that is, on the total quantities of commodities, the utility functions of the individuals and on the <u>equilibrium</u> value of their endowment) and not so much on conditions of distribution of the total available supply of physical quantities of commodities among individuals, since, according to the theorem, the latter can be varied in specific ways and still leave equilibrium prices unchanged.

To make the matter clearer, it may be appropiate to consider the example of exchange of two commodities, such as bread and meat. Given the preferences of the participants in the exchange, and their endowments of physical quantities of those two commodities, we assume that an equilibrium relative price exists for a tatonnement process without disequilibrium transactions. This equilibrium price ratio may be obtained from the solution of the system of excess demand functions of the participants. The theorem then suggests that this relative price will not change even if we start off the individual participants with very different physical commodity allocations,[22] provided that the values of those new commodity allocations, calculated at the equilibrium relative prices, are the same as before. If, for example, an individual was in possession of some quantities of both commodities in the first case, given the equilibrium prices (parameters to him) he may be a supplier of bread and a demander of meat. If, however, in the redistribution of commodity holdings he is found at the beginning to hold no bread at all and meat in quantity whose equilibrium value is the same as that of initial quantity of bread and meat he had before, he will obviously end up supplying meat in exchange for bread, in equilibrium, and the equilibrium price will be the same.

The meaning of the theorem is, then, that it is the equilibrium value of the physical quantities of the initial individual endowments which, together with preferences and total supplies, determines the equilibrium prices: it does not matter what the composition of this equilibrium value is in terms of physical quantities.

---

[22]These will have to add up to the given total market quantities, however.

The above example also shows that the theorem suggests that the question of who will supply what commodity in demand for others depends on the distribution of the physical quantities.

Consider Figure 2.1, which is a standard Edgeworth box describing exchange of two commodities by two individuals behaving according to parametrically given prices. The theorem says that any initial endowment lying on the line AB will lead to the equilibrium price ratio described by the tangent of the angle Θ, i.e., all offer curves derived from initial endowments lying on this line intersect at E. During the equilibration process, the endowment values will vary for the reasons described above.

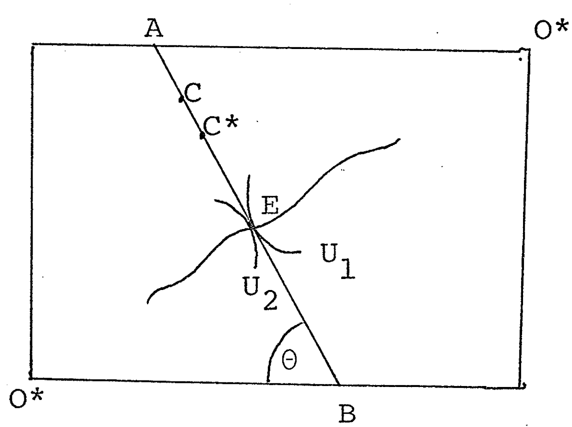


Figure 2.1

Jaffe seems to believe that the theorem may be used in connection with a tatonnement process (with or without false trading) and suggests that Walras

> "missed perceiving the relevance of his 'theorem of equivalent redistribution...' to his theory of *tatonnement*, since the proof he formulated...begs the question. He calculated the value of the equivalent commodity holdings in terms of the mathematically determined prices, which *ipso facto* guaranteed a solution at the same equilibrium prices! Had he calculated (it) at another set of prices, the solution would not have been the same."[23]

It has already been pointed out that the theorem has no use outside equilibrium, and this shows that Jaffe's criticism of Walras having missed perceiving its relevance is unjustified, although it is true that Walras could possibly have seen the importance of false trading for his scheme had he examined the theorem more closely. In order to evaluate the force of Jaffe's criticism, however, it may be instructive to examine his suggestion for calculation of the value of equivalent commodity bundles in terms of other than the equilibrium prices. Jaffe suggests that in this case the solution would not have been the same. Actually, under an equivalent redistribution of commodity holdings there cannot be a solution satisfying the equilibrium conditions of the markets unless the prices come to be the mathematically determined prices. This may be seen in Figure 2.1 where the

[23] Jaffe, (1967), p.3.

evaluation of the bundles C,C* in terms of another price ratio does not lead both parties onto the contract curve (a unique equilibrium is assumed in the Figure).

Jaffe's statement that calculation of the values of the endowment at another set of prices would yield another equilibrium price vector seems to imply that the value of the endowment is calculated at one set of prices, and afterwards the equilibrium vector is found on the basis of the given money value of the endowment. This is clearly inappropriate, since the value of the endowment should be calculated at the set of ruling prices, and thus its value in equilibrium depends on the equilibrium price vector. Thus, Walras' calculation of the value of equivalent redistributions at the initial equilibrium prices was meant to check whether the market excess demands would be zero at these prices, after redistribution had taken place. He would then be able to claim, as he did, that equilibrium prices do not change.[24]

There is, however, an implication that may be drawn from the theorem as regards alternative sufficient conditions for invariance of the initial solution prices of the model of <u>tatonnement</u> exchange. One such sufficient condition is shown by the theorem to be the invariance of the equilibrium values of individual endowments.[25] Another one is invariance of the

[24] Equilibrium is assumed to be unique.

[25] Together with unchanged total quantities of commodities.

physical quantities in the endowment of each individual.

The second of these sufficient conditions is quite strong: not only does it require that the total market quantity of each commodity be constant, but it also demans a fixed distribution of quantities of all commodities among individual participants.

The first condition has the same requirement with regard to the total market quantities, but it is a little weaker on their distribution, since it suggests that the latter may be varied as long as the equilibrium value of each individual endowment remains constant.

On the other hand, neither of the above conditions is necessary for invariance of the equilibrium prices. This is so because it is conceivable that in a non-tatonnement process with false trading the equilibrium values of the endowments may change but the consequences of these changes for excess demand may cancel one another. We then have the same equilibrium prices but different endowment values (at these prices) for some or all individuals. It may be worthwhile to stress that the cancelling out refers to the consequences of <u>different</u> endowments on excess demands; the endowments have different equilibrium values from the initial ones. The total market excess demands, however, may be the same if the consequences of these different endowments cancel out.[26]

---

[26] Contrast with Jaffe's elimination of the endowment changes themselves: Jaffe,(1967), p.3.

In terms of Figure 2.1, it is possible that from an endowment off the line AB there may exist a price line of slope θ taking the market participants to a point on the contract curve other than E but with the same marginal rate of substitution as that of E.

Jaffe, however, claims that the condition provided by the theorem of equivalent redistributions is not only sufficient but also necessary, when he states that:

> "If follows from this theorem that if the values of... assets are affected...and do not remain unchanged, the prices...must also be affected".[27]

This implication is incorrect.[28]

If neither of the above conditions is necessary but only sufficient, the question arises whether it is preferable in any sense to use the one rather than the other. In this respect, it seems proper to use the weakest sufficient condition available, although this does not imply that the stronger one is improper or incorrect. Thus, Jaffe's criticism of D. Patinkin is couched in an unjustifiable manner when the former says that

---

[27] Ibid, pp. 2-3.

[28] In a later passage Jaffe has the weaker, and correct statement that when the endowment values change "equilibrium... is unlikely to remain unchanged" (ibid, p.4, italics mine). If by "equilibrium" he means not only the equilibrium commodity distribution but also the final price by which equilibrium was arrived at, this statement is in contradiction to that quoted in the text.

> "Patinkin...leaves one with the impression that what is needed is constancy in the quantities of the physical components of the individual endowment collections, when in fact what is required for a unique equilibrium is constancy in the values of these separate endowments..."[29]

As mentioned above, neither condition is necessary. Jaffe implies that his condition should be used by Patinkin because it is necessary, while he should have suggested that his condition is a weaker sufficient condition, hence preferable.

One may conclude from the above discussion that the theorem of equivalent redistributions is most relevant with regards to equilibrium positions, namely it describes sufficient conditions for invariance of equilibrium prices to some distributional changes in the endowments. While it may be true that it gives the weakest conditions available, it is not correct that it gives necessary conditions. Therefore, it could have helped Walras only marginally as regards the question of disequilibrium transactions; moreover, the fact that its conditions for price invariance are weaker than others does not render other conditions improper or useless.

The theorem of equivalent commodity holdings has been examined at some length because it provides conditions under which endowment effects of the second kind cannot operate on demand and consequently cannot change the equilibrium price

---

[29] Ibid, pp.16-17.

vector despite disequilibrium transactions. This, then, is one case where actual transactions outside equilibrium do not affect the stationary state of the model: the case is clearly interesting because of precisely this feature.

There is, as a matter of fact, a case where the theorem's conditions on price invariance under commodity redistribution are indeed necessary, as well as sufficient,(i.e., unless redistributions satisfy the conditions of the theorem equilibrium prices do change, and if they do satisfy these conditions prices remain the same). This is the case where the utility functions of all participants in exchange are homogeneous and strictly convex. If this is so, it is well known that the marginal rate of substitution depends only on the ratio of commodities. Moreover, strict convexity of indifference surfaces guarantees that to each marginal rate of substitution there corresponds only one commodity ratio. The contract curve of the exchange model in this case exhibits the characteristic that to each commodity ratio there corresponds a given marginal rate of substitution and vice versa. Under these conditions, it is necessary and sufficient for invariance of equilibrium prices that any commodity redistribution among individuals be equivalent as defined by Walras.[30]

Consider, as an illustration of the above, Figure 2.2 be-

---

[30]Thus, in this particular case Jaffe would be right to regard the conditions of the theorem both necessary and sufficient.

low. To each point on the contract curve there corresponds a unique equilibrium price ratio, since each point is associated with a different commodity ratio. Let the initial commodity distribution be on the line AB, at point C. Then, any equivalent redistribution, such as C*, will not affect equilibrium prices, while any non-equivalent one will have to affect them, since the offer curves from a point off AB, such as point D, will intersect on another point of the contract curve, such as E' rather than at point E.

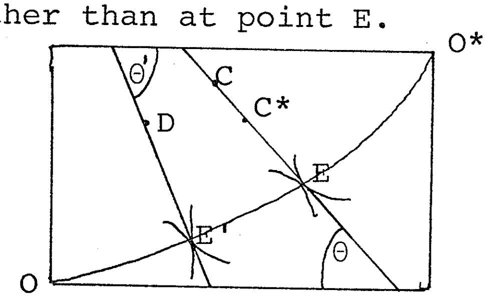


Figure 2.2

2.2 F.Y. Edgeworth

In his early writings[31] Edgeworth may at best be interpreted as unclear in his own thoughts on the matter of transactions outside equilibrium. His main concern seems to be the question of "determinate contract", i.e., an indefinite number of "final settlements". Some definitions of terms are required for proper discussion of Edgeworth's concerns and analysis.

---

[31] F. Edgeworth, (1881).

A "contract" is defined as an action by an agent taken with the consent of others affected by this action[32]; the "field of competition" with reference to contracts consists of all the individuals willing and able to recontract about the articles under consideration[33]; a "settlement" is a contract which cannot be varied with the consent of all the parties to it; a "final settlement" is one which cannot be varied with recontract within the field of competition; and, contract is indeterminate when there are an indefinite number of final settlements.[34]

The essential ambiguity in Edgeworth's early work refers to the question whether he allowed the contracts at each stage of the equilibration process to be carried out: "recontract" i.e., may be interpreted to mean either that contracts are annulled and others take their place in the process of equilibration; or that they are carried out and another set of contracts takes place. In the first case it seems clear that the process is similar to the tatonnement of Walras. In the second, the further question arises whether new sets of contracts take place within the specific market period, or the next one.

---

[32] ibid, pp. 16-17.

[33] ibid, p.18.

[34] ibid, p.19.

To make things clear, assume that the system starts the market period with given tastes, technology and initial endowments. Let contracts be formed (as contrasted to "carried out") at disequilibrium prices. The first possibility in interpreting the term "recontract" is to take it to mean that in disequilibrium contracts may be annulled and others formed within the period; the equilibration process will then stop when a "final settlement" is reached.

The above is obviously similar to the Walrasian tatonnement process, with the possible exception that the price setting during equilibration seems to be done by those who find it profitable to enter into new contracts, rather than the Walrasian auctioneer.

The second possibility is to interpret the term "recontract" to mean that contracts formed at disequilibrium prices are carried out after they are formed. Two cases now arise: either no more contracts are entered into in the given market period or not. In the first case, the system must be considered to proceed to economic activity within the period on the basis of the initial disequilibrium contracts. Obviously, equilibrium is not attained in the market period under this interpretation. However, it is possible to postulate that the system starts the next market period with exactly the same initial conditions as before, with the exception that traders now know that the previous period's contracts can be improved upon. Sufficient repetition of this process may be considered legitimate for analysis of determination of what could be called the "normal", as opposed to "market period" price.

In the second case, after the first series of contracts has been formed and carried out it is possible to visualize a new set of contracts during the same market period. This is, of course, the genuine process of non-tatonnement within the market period, as defined in Chapter I above.

The discussion may now return to Edgeworth's work. In the Mathematical Psychics his main concern seems to be the conditions of determinate contract. He starts his analysis with "the simplest case of contract"[35], involving two persons and two commodities, and constructs the now widely used contract curve in the model of exchange. He then shows that many equilibria are possible in this situation, i.e., contract is indeterminate, since all points on that portion of the contract curve which is enclosed by the indifference curves passing through the initial endowment are eligible for equilibria. The final outcome depends on the bargaining ability of the participants, since no price taking can be assumed in this context. In this early discussion of indeterminate equilibrium Edgeworth does not specify whether the participants in exchange enter

[35] Edgeworth, (1881) pp.20-30.

into a series of contracts: however, he may have adopted the possibility of disequilibrium transactions in that model quite early in his thought. This possibility must be recognised, since one finds Edgeworth remarking (in a comment on Jevons' equations of exchange):

> "Why, indeed, should an isolated couple exchange every portion of their respective commodities at the same rate of exchange? or what meaning can be attached to such a law (the Jevonian Law of Indifference) in their case? The dealing of an isolated couple would be regulated not by the theory of exchange...(under perfect competition) but by the theory of the simple contract.[36] (The theory of bilateral exchange) (parenthetical statements inserted).

Here Edgeworth may be taken to mean that, since the outcome of the process of exchange between an isolated couple is indeterminate in any case, there is no reason to impose the Jevonian rule which, in the theory of multilateral exchange, is necessary for uniqueness of the equilibrium price.

With his analysis of the case of "simple contract" complete Edgeworth turns to the case of increasing the "field of competition", the number of traders in the market. He states that his purpose is to prove that indeterminateness of contract gradually vanishes as the number of traders in the market increases, since "recontracting" is now possible. His discussion has provided the beginning of analysis of the core of an economy, designed to prove that the set of possible

[36] Edgeworth (1881), p.109.

equilibrium points shrinks to a single point as the number of traders increases.[37] It is not clear, however, what Edgeworth meant by "recontracting" in this case: it is possible to interpret the term either as a variant of the tatonnement process[38] or as a recontract from period to period, all periods starting with the same initial endowments.[39] In the first case, one may say that as far as disequilibrium transactions are concerned, Edgeworth followed in the path of Walras. In the second case, it may be said that he adopted a discussion of determination not of the market period equilibrium price but of a more "normal" long-run price, by adopting a repetitive process of actual disequilibria.

Textual evidence that the first interpretation may be true is furnished by such statements of Edgeworth's as the following:

> "You might suppose each dealer to write down his demand, how much of an article he would take at each price, without attempting to conceal his requirements; and these data having been furnished to a sort of market-machine, the price to be passionlessly evaluated".[40]

---

[37] See, for example, K. Vind, (1965).

[38] As did N. Kaldor, (1933-34).

[39] It is of course also possible to interpret the term as actual "formation and completion" of contracts within the market period. In this case, even with competition the equilibrium is indeterminate, as it depends on the path of actual transactions, and Edgeworth is obviously wrong in his claim of determinateness, if he is so interpreted. However, there is no textual evidence for this interpretation, while there is some such for recontract from period to period as will be seen below.

[40] Edgeworth, (1881), p.30.

This is in reference to Walras' tatonnement process, in the theory of multilateral exchange. Edgeworth does not raise issues with regard to the absence of disequilibrium transactions in this Walrasian framework: his object is, rather, to prove that equilibrium is determinate not by the Walrasian method but by an analysis of perfect competition with "recontract".[41]

On the other hand, textual evidence exists that supports the second interpretation: for example,

> "So a landlord on expiry of lease recontracts, it may be, with a new tenant",[42] or,

> "let us imagine a simple case -- Robinson Crusoe contracting with Friday. The articles of contract: wages... labour".[43]

In both the above cases the nature of the commodity exchanged is such that it is possible to postulate that individuals return at the beginning of each market period with the same initial commodity endowment.

The evidence in the early writings on whether Edgeworth did examine disequilibrium transactions is thus not definite.

Ten years later[44] Edgeworth returned to the model of isolated exchange (barter) and adopted A. Marshall's account of equilibration in that model[45] which involves ex-

[41] Edgeworth, (1881), pp.30-31.

[42] Ibid, p.17.

[43] Ibid, p.28.

[44] Edgeworth, (1925), pp.313-319.

[45] Marshall, (1920).

change in disequilibrium prices during the market period. At the same time, however, he repeated his statement that

> "the essential condition of indeterminateness is the absence of competition".[46]

Clearly implied in this is the statement that if the number of traders increases in the exchange model equilibrium will be determinate.

However, if disequilibrium transactions are allowed, (i.e., if contracts are formed, carried out, and others formed and carried out in the same market period) equilibrium will not be determinate in Edgeworth's sense. Thus, Edgeworth may be interpreted as having failed to see the fact that disequilibrium transactions may cause indeterminateness in multi-lateral exchange.

Alternatively, it may be suggested that Edgeworth had his mind fixed either on the case of contracts which could be annulled within the period (tatonnement), or he was, from that time in his research, moving towards a theory of determination of normal price, as the alternative interpretation discussed above suggests.

It is seen that, in his effort to describe the equilibrating mechanism of the market Edgeworth may either be considered to have adopted the tatonnement of Walras or, more appropriately, may be interpreted as having by-passed the

---

[46]Edgeworth, op.cit., p.317.

problem of market period price determination in favour of what may be called Edgeworthian normal price formation.[47] Textual evidence of his later work[48] seems to support the second interpretation.[49] For example,

> "...two kinds of higgling may be distinguished as appropriate respectively to short and long periods. First, we may suppose the intending buyers and sellers to remain in communication without actually making exchanges, each trying to get at the dispositions of the others...By this preliminary tentative process a system of bargains complying with the conditions of equilibrium is, as it were, rehearsed before it is actually performed. Or, second, one may suppose a performance to take place before such rehearsal is completed. On the first day in our example (of equilibration in the labour market) a set of hirings are made which prove not to be in accordance with the disposition of the parties. These contracts terminating within the day, the parties encounter each other the following day[3] (fn.3: they recontract, in the phraseology of Mathematical Psychics) with dispositions the same as the first day -- like combatants armis animisque refecti,[4] -- in all respects as they were at the beginning of the first encounter, except that they have obtained by experience the knowledge that the system of bargains entered into on the first occasion does not fit the real dispositions of the parties. The second plan of higgling was supposed in the example (of the labour market) -- the plan which is more appropriate to "normal" price"[50] (Parenthetical statements inserted)

Once this second interpretation of Edgeworth is accepted as more appropriate, it may be useful to comment on the implications, usefulness and possibility of extension of this approach.

---

[47]Compare D. Walker, (1973).

[48]Edgeworth, (1925), pp.333,452,453.

[49]First suggested by D. Walker, in an attempt to dispel the common interpretation that Edgeworth had not considered disequilibrium in any different manner from Walras. See D. Walker, (1973).

[50]Edgeworth, (1925,Vol.I) p.40.

The method involves a sequence of market periods in which disequilibrium prevails. Repetition of market periods having the same initial conditions accumulates enough experience for the participants to be able finally to settle on "normal" prices, which would in theory be determined by the equations of the model under a tatonnement process.

The above needs certain conditions to be satisfied in order to become relevant to the process of equilibration: specifically, it must be assumed that the disequilibrium of each period does not affect the initial conditions of the future periods. This is obviously a strong assumption, violated in cases of "carryover" of any sort from the disequilibrium of the one period to the situation of the other periods. Moreover, the speed of adjustment must be relatively high, if the assumption of identical initial endowments in each period is to be defensible.

Edgeworth, of course, made no such assumptions for the process of normal price formation attributed to him in this interpretation. The approach suggested by him is nevertheless obviously useful, especially in a model of production and exchange where the market period may reasonably be assumed to involve disequilibrium: it is not reasonable to suggest that because the input markets are in disequilibrium no production takes place. Thus the approach in which the labour and other input markets open at disequilibrium prices, hirings are made and production is undertaken on the basis of such hirings seems to have merit.

In Chapter V of this investigation a disequilibrium model of production and exchange is developed along the lines of the Edgeworthian repetitive process, with initial conditions of each period sometimes independent and sometimes dependent on the situation of disequilibrium of the previous periods.

2.3 A. Marshall

Marshall saw the problem of indeterminacy of equilibrium and in his typical way tried to find simplified sets of assumptions that would make the supply and demand apparatus insensitive to disequilibrium transactions. His first approach to the problem was to assume that "every dealer... has a perfect knowledge of the circumstances of the market", and to conclude from this that no buying or selling would happen at prices other than the equilibrium one. This, of course, assumes away the problem and is not very different from Walras' approach. Marshall, however, proceeded to say that

> "It is not indeed necessary...that any dealers should have a thorough knowledge of the circumstances of the market."[51]

---

[51] A. Marshall, (1920), p.334.

and accepted the possibility that a certain volume of transactions would take place at disequilibrium prices. He asserted, however, that the final price under these circumstances "would be probably close" to the equilibrium price as derived from the solution of the initial supply and demand equations, and justified his position by suggesting that, for any particular market, the marginal utility of the commodity falls with the increase in the quantity held by the individual while the marginal utility of money is relatively insensitive to the amounts held of money and this commodity. In effect, Marshall postulated constancy (or approximate constancy) of the marginal utility of money.

This Marshallian assumption has generated considerable debate in the literature.[52] The reasons for the debate may be found in the lack of clarity of the assumption and in the multiple use to which it was put. More specifically, Marshall did not make clear precisely what he assumed to be constant, and as a consequence some writers interpreted him saying that the marginal utility of the numeraire is constant, while others interpreted such constancy to refer to the marginal utility of general purchasing power. Moreover, it was not

[52] See V. Pareto, (1927), Appendix, art.56 ff, p.585 ff; E. Wilson, (1935), (1939); P. Samuelson, in O. Lange, (1942), pp. 75-91; J. Hicks, (1946), pp.39-40; H. Liebhafsky, (1961); R. Bilas, (1965); N. G. Roegen, (1968); C. Higgins and H. Liebhafsky, (1968); D. Walker (1969); E. Wilson, (1935) and (1939).

made clear with respect to what the marginal utility of income was assumed constant: when constancy refers to the marginal utility of the numeraire it is reasonable to imply that the utility is constant with respect to increases of the quantity of the commodity. When, however, it is the marginal utility of general purchasing power that is considered constant the question arises as to whether it is constant with respect to increases in that purchasing power or with respect to changes in prices.

The multiple uses to which Marshall put the assumption aggravated the problem of ambiguity. The assumption is used in at least three distinct cases: to "prove" the law of demand, to discuss the concept of consumer surplus and finally to have the commodity market equilibrate with false trading without ambiguity as to the final equilibrium price. Thus, one passage reads,

> "The richer a man becomes the less is the marginal utility of money to him".[53]

This is in conjunction to the derivation of the law of demand where Marshall says that in stating the law he has not considered changes in the marginal utility of money. Also, in his discussion of the consumer surplus[54] he points out that the substance of his argument would not be affected if he

---

[53] Marshall, (1920), pp.96 and 690.

[54] *Ibid*, p.132.

took account of changes in the marginal utility of money which presumably happen as a consequence of changes in the prices of commodities. In the associated footnote he argues that the neglected elements would generally belong to the second order of small quantities, and in the Mathematical Appendix[55] he makes the point that the marginal utility of money can be kept the same throughout (presumably while prices are changing) if one assumes that the consumer's expenditure on any one commodity is only a small part of his total expenditure. According to Marshall, this condition takes care of Giffen's case, mentioned in his text.[56] In stating the condition that the consumer spends a small percentage of his income on each commodity Marshall refers the reader to his discussion of market equilibration with false trading.[57] In that discussion the marginal utility of money is not allowed to change appreciably and the assumption is justified on the basis of the condition that the consumer spends a small part of his total resources on the commodity in question. In

---

[55] Ibid, Note VI, p.842.

[56] Ibid, pp.132-133.

[57] Ibid, p.334. The actual reference in Marshall's note is to Book V, Chapter III, Paragraph 3, but this must be a typographical mistake. I have taken the reference to read: Book V, Chapter II, Paragraph 3, which is the discussion on market equilibration, and where the condition of a small percentage of expenditure on each commodity is mentioned.

the associated Appendix on barter[58], Marshall discusses the exchange of two commodities for one another for the case of two individuals and more than two individuals and comes to the conclusion that the final equilibrium rate of exchange is indeterminate, because it depends on the path of false trading. He then proceeds to suggest that the uncertainty of the final equilibrium price depends indirectly on the fact that one commodity is traded for another instead of being sold for money. This is because the marginal utilities of the commodities exchanged are functions of the amounts of commodities held, while money can be thought of as having approximately constant marginal utility. To show that this is the case Marshall suggests that if one of the commodities is available in large quantities in the endowments of both sides of the market, its marginal utility to the participants in the exchange can be considered constant and the consequence of this is that the final equilibrium price with false trading is not dependent on the path of that trading. In the mathematical note to this Appendix[59] he becomes more specific and writes out the contract curve equation in the case of constant marginal utility of the one commodity. This equation shows that the contract curve is a straight line parallel to the one commodity axis and the final equilibrium price along that curve is

---

[58] Ibid, Appendix F.

[59] Ibid, Note XII, pp.844-845.

a constant, dependent on the net amount exchanged of the commodity whose utility varies with its quantity.

Since the interest here is in the assumption of constant marginal utility as it can be used in the theory of market equilibration with false trading, most of the discussion which follows concentrates on this use of the assumption. In this connection, it will first be shown that if constancy of the marginal utility of income is taken to mean constancy of the equilibrium marginal utility per dollar with respect to a change in income,[60] this result can be obtained if the marginal utility of one commodity consumed in equilibrium is constant. It then follows that pure price and pure quantity endowment effects for all other commodities except the one whose marginal utility is constant are zero, and consequently the law of demand holds for these commodities without exceptions. Moreover, close examination of the Slutsky equation of value theory shows that Marshall's condition of a small percentage of expenditure on each commodity has use, since under the above assumption equilibration with false trading may follow an erratic path, but the final equilibrium price is not affected by that path.

In order to show the above, it is useful to recall the formulation of the individual's problem of utility maximization, discussed in Section 1.4. From the formulation of this problem (equations 1.30) it is clear that the change in the

---

[60] Given a utility index.

value of the Lagrangean multiplier in response to a change in the endowment quantity of commodity i is given by

$$d\lambda/dI_i = \frac{P_i|U|}{-1/\lambda^2|U_b|} \qquad (2.1)$$

where $|U|$ is the determinant of the Hessian of the utility function. Also, it will be recalled that the equation expressing the change in the quantity demanded in response to price and endowment changes is

$$dq_j = \frac{\sum_i |U_{ij}|\lambda dp_i}{|U_b|} + \frac{\lambda|U_{n+1,j}|}{|U_b|}\{\sum_i(I_i-q_i)dP_i+\sum_i P_i dI_i\} \quad (1.32).$$

Clearly, if one column of the unbordered Hessian in (2.1) is zero, the change in the equilibrium marginal utility per dollar which results from a change in the endowment of the consumer is zero. But a zero column in the Hessian implies constancy of the marginal utility of the respective commodity, both with respect to its own quantity and with respect to changes in the quantities of the other commodities. The commodity in question is then independent in the Paretian sense from all the other commodities.[61] Thus, it has been shown that constancy in the marginal utility per dollar with respect to a change in the endowment can be obtained if a good is independent of all others in the Paretian sense and it has a constant marginal utility with respect to its own quantity.

[61] The discussion requires that the utility index be known. Otherwise, a non linear monotonic transformation of it will change both the value of the derivative of λ and the values of the partials of the utility function.

If this is the case one can also show that pure price effects of the first kind and pure quantity effects of the third kind are zero for all commodities other than the one which has constant marginal utility. From equation (1.32), the numerator of the expression for the marginal propensity to consume commodity i in response to a change in the value of the endowment involves the determinant $U_{n+1,j}$, which contains a column of zeros (except for the marginal propensity to consume the commodity with the constant marginal utility). Given that, it follows that any increase in the endowment value will be entirely devoted to an increase in the consumption on the commodity with the constant marginal utility. This is to be expected, since one commodity in the optimizing bundle of the consumer has a constant marginal utility per dollar: an increase in expenditure is properly channeled into this commodity, since increase in expenditure of the other commodities would lower the equilibrium marginal utility per dollar that can be obtained and would thus result in a non-optimal bundle. Thus, the additional remark has to be made that the above results only hold when the commodity with a constant marginal utility is contained in the optimizing bundle of the consumer. Moreover, it may be useful to add that when the increase in income is in terms of a commodity other than the one with constant marginal utility the individual consumer will exchange that additional quantity of the commodity with the one that has constant marginal utility for him.

Another remark that can be made with regard to Equation (1.32) is that the Law of Demand is seen to hold unambiguously for all commodities other than the one with the constant marginal utility, since pure price endowment effects are zero. Moreover, Equation (1.32) explains Marshall's repeated reference to a small percentage of expenditure on one commodity as one condition under which constancy of the marginal utility of money is a valid approximation. In this case, it is not the term C_j that is equal to zero but the term $(I_i - q_i)$ which is made arbitrarily small. The unacceptable result that an increase in the endowment leads to an increase in the consumption of one commodity only is avoided. But we can still say, from Equation (1.32) that a change in the own price results under most circumstances in a change of opposite direction in the quantity demanded, which is a statement on the approximate validity of the law of demand. The concept of the consumer surplus seems also justifiable on the basis of this assumption. What cannot be justified is the statement relevant to uniqueness of equilibrium in exchange when false trading takes place: the effects of such trading can only be zero if the term C_j is zero and not if the expenditure on the commodity is small. This explains why Marshall had to use the constancy assumption when it came to the discussion of uniqueness of equilibrium prices, while in other places he contented himself with the condition that the expenditure on the commodity in question was only a small percentage of total expenditure of the consumer.

The consequences of the assumption of constant marginal utility of one commodity for the shape of the contract curve and for the equilibrium price vector in the case of exchange of two commodities can be seen from the equation of the contract curve in that case. Let I_1 and I_2 be the quantities available of commodities 1 and 2, and let q_j^k be the equilibrium quantity of commodity j (j=1,2) for individual k (k=1,2). We must have, then,

$$q_1^1+q_1^2=I_1$$

$$q_2^1+q_2^2=I_2 \qquad (2.2)$$

and the individual utility functions can be written

$U = U\ (q_1^1,q_2^1)$ for the first individual, and

$V = G\ (q_1^2,q_2^2) = V\ (q_1^1,q_2^1)$ for the second. (2.3)

In equilibrium we must have

$$-\frac{dq_2}{dq_1} = \frac{U_1}{U_2} = \frac{V_1}{V_2} \qquad (2.4)$$

Where U_i,V_i are the first partials of the utility functions, and the superscripts of q have been omitted for simplicity.

Equation (2.4) is Marshall's equation for the contract curve. Given the Marshallian assumption of constancy

$$U_2 = a \qquad V_2 = b$$

$$U_{21} = U_{12} = 0 \quad \text{and} \quad V_{21} = V_{12} = 0 \qquad (2.5)$$

the equation of the contract curve becomes a function of q_1 only, yielding $q_1 = J$ (or constant), which shows that the contract curve is a straight line parallel to the axis of the commodity with the constant marginal utility. More importantly, the marginal rate of substitution along the contract curve is a constant given by $U_1(J)$. Since the equilibrium price ratio must equal the marginal rate of substitution it follows that the equilibrium price ratio is constant, and thus it does not matter which point on the contract curve is the final equilibrium point. The power of the Marshallian assumption in ensuring a unique equilibrium price ratio with or without false transactions is now in plain view. It is unfortunate that the assumption has such objectionable implications for the theory of demand, mentioned above.

It may be worthwhile at this point to discuss an alternative interpretation of the Marshallian constancy assumption.[62] To avoid the implication that all increases in income are spent on the commodity with constant marginal utility it may be stipulated that the assumption refers to individuals whose income is such that their budget includes a number of

---

[62] Georgescu - Roegen, (1968), p.180.

commodities which have the same marginal utility per dollar and are needed in small quantities only. Similarity in marginal utility per dollar implies that the consumer is indifferent between various commodities at the margin of his budget. The stipulation that the commodities are needed only in small quantities implies that their marginal utility declines sharply after these quantities are obtained. Therefore, each increase in income is now spent on some additional new commodity. Each new commodity has roughly the same marginal utility per dollar as the last, hence the marginal utility of money remains approximately constant to the purchaser.

This assumption would not violate the basic result of equilibrium price invariance in the case of exchange with false trading if it were true that the number of items which enter the budget of each consumer is invariant to false trading. If this were so the income effects on the commodities already in the budget of the consumer would be zero and since the effect of false trading is always an income effect there would be no change in the equilibrium price of those commodities. However, although the income effect with respect to the commodities included in the budget is zero, it is not zero with regard to the commodities not included in the budget at this level of income. Hence, a gain in income will cause additional demand of the marginal commodities and a loss in income will reduce the demand for those or other commodities. Therefore, this interpretation of Marshall's constancy of marginal utility of money does not yield the desired invariance of the final equilibrium price.

Finally, the following remark may be made with regard to the comparative statics of Marshall's scheme: since in equilibrium the marginal utility per dollar for each consumer is equal to the constant marginal utility per dollar for the one commodity, an increase in income that takes the form of any commodity whose marginal utility is variable will, in the general equilibrium system of exchange, result in a lower relative price for that commodity. If, however, the increase takes the form of a change in the quantity of the commodity with the constant marginal utility, equilibrium prices do not get affected.

This concludes the discussion of Marshall's assumption on the constancy of the marginal utility of money. The case is an example of his typical mode of theorizing: the object was to guarantee a certain result and this is precisely what the assumption did.

2.4 P. Wicksteed

Of all the early contributors to the problem of market equilibration, Wicksteed is the most neglected by modern writers. Yet not only did he correctly perceive the problems of a typical market for a commodity during equilibration.

He also stayed clear of the barter formulation which occupied most of the other economists of his time. Moreover, he used one market with money as medium of exchange in his discussion of the problem of equilibration, and with this partial equilibrium analysis avoided the various problems of the Walrasian model of exchange in which everyone is in the tatonnement of every market during the period of equilibration. This partial equilibrium approach permits Wicksteed to consider a fairly realistic market which does not need an <u>intellectus</u> <u>angelicus</u> for the function of price adjustment. Rather, the process of price formation proceeds via "intelligent estimates" by the sellers, of the price which would produce equilibrium. This attempt on the part of the sellers fixes the actual price at any moment. Thus, as far as the problem of price formation is concerned, the process is postulated to be the sellers' business, by means of a series of estimates of the equilibrium price, the current market price being the latest estimate. Buyers are price takers in this framework.

At the beginning of his analysis Wicksteed asserts, without elaboration, that the estimated (actual) market price will tend to approach the equilibrium price. He does, however, add that this statement neglects "certain secondary reactions" whose discussion he puts off for a later part of his analysis.

Before proceeding with a short discussion of Wicksteed's approach, it may be useful to note that he also examines the possibility of different terms of exchange to different buyers and disposes of it with the assumption of competition among sellers and dealings among the buyers. It is in the seller's interest to endeavour to form

> "the most accurate possible estimate of the equilibrating price, and to ask nothing above it, unless some mistake on the part of his rivals enables him to do so safely."[63]

With those two details out of the way we can now examine Wicksteed's discussion of disequilibrium price formation. He suggests, without much analysis, that if

> "the sellers collectively made an error in their judgement and named something below or above the true equilibrium price..."[64]

the stream of customers coming into the market would have to transact at these prices, that is accept them as parametrically given and form their purchasing plans accordingly. This has the effect of "destroying" or "creating" demands for the commodity in question, since with a lower price there may be more customers who may buy more of the same commodity and with a higher price fewer people may buy less of the commodity than they would at the equilibrium price. If it is assumed,

---

[63] P. Wicksteed, (1933), p.226.

[64] <u>Ibid</u>, p.222.

as is by Wicksteed, that customers appear in the market only once, conclude their transaction and do not return, this destruction or creation of demand is final. Thus the sellers are obliged, during the final stages of the market period, to set a price which is not necessarily near the initial equilibrium solution price of this market, in order to clear the remaining stock or to shut off excess demands. Wicksteed's main point of difference with the previous contributors, of course, is that the customers do not stay in the market waiting for the equilibration process to come to an end (as they do in Walras' and Marshall's cases). This characteristic could be called "imperfect market attendance" and, in my opinion, is one of the most important steps towards a realistic examination of the problem of equilibration in exchange in that it attaches some significance to the time element in the process. Wicksteed understands this perfectly well, as is shown by the following quotation:

> "This process will always and necessarily occupy time. The persons potentially consituting the market will not all be present at the same time, and therefore the composition of the collective scale...must be a matter of estimating conjecture. The transactions actually conducted at any moment will be determined in relation to the anticipated possibilities of transactions at other moments."[65]

Wicksteed is also aware that, in the case of disequilibrium transactions, although the market may clear the trans-

---

[65] *Ibid*, p.236.

actors are not in full equilibrium since they have acquired commodities at different prices. Hence the marginal utility per dollar for each transactor is not the same for the commodity in question.

The foregoing remarks, it is hoped, have placed Wicksteed in his proper position as far as the efforts towards analysis of the equilibration process are concerned. His contribution is in sharp contrast with the previous contributors who consistently assume that the participants in the exchange remain in the market for the whole duration of the equilibration period. In this connection, it would be interesting to ascertain whether it is the assumption of the constant marginal utility of money or the assumption of all the transactors being present during the period of equilibration that is more necessary for the truth of Marshall's statement that the market would probably close near the equilibrium price, with the equilibrium amount of quantity transacted.[66] That is to say, if traders in the Marshallian scheme are allowed to enter the market, purchase whatever quantity they wish at the going price and depart, will the final equilibrium price, the price at which the market clears, be unaffected by such disequilibrium transactions?

This and the related general question of equilibration in a market with imperfect attendance seem interesting direc-

---

[66] Marshall, (1920), p.278.

tions for research. They will not be pursued in this investigation, however.

2.5 J.R. Hicks

As far as a single market for a commodity is concerned, Hicks follows in Marshall's footsteps with a few clarifications. He then concerns himself briefly with the question of generalization of Marshall's assumption.[67]

With regard to the single market Hicks shows by example the simple fact that the effect of trading at disequilibrium is an income effect.[68] He then repeats Marshall's assertion that if the expenditure on the commodity in question[69] is a small portion of total expenditure on goods, disequilibrium transactions will not affect the position of the demand curve appreciably since the accompanying income effects will not be great.

In an attempt to generalize this result (of insensitivity of equilibrium price to transactions at false prices) for many markets Hicks suggests two distinct possibilities that can serve his purpose. First, if the transactions at false prices are limited the accompanying income effects will be small. Hicks argues that such transactions must be limited if any intelligence in price fixing is shown.

---

[67] (1946), pp.127-129.

[68] Of the second kind, in our terms.

[69] Which depends, in our terms, on the degree of market participation $(I_i - q_i)$.

The statement about limited volume of transactions at false prices can safely be considered an assertion, especially in the case where different buyers reveal their intentions to buy at different intervals through the market day and not at the same time. That is to say, when a price is fixed and the purchasers of the commodity appear in the market at successive intervals, (as with Wicksteed) even the most intelligent price fixing may not prevent a sizeable number of large disequilibrium transactions from taking place. Thus, Hicks obviously has in mind the case in which all the buyers and sellers of the commodity are in the market at the same time. In this case, if a false price is set the buyers will indicate their intention to buy at this price simultaneously. The total excess demand (or supply) will show itself almost immediately. It may then be reasonable to assume, with Hicks, that transactions at false prices will be limited in volume since the sellers will immediately attempt to correct the "false" price.

A second reason why the final equilibrium price may not be affected by disequilibrium transactions according to Hicks is that some offsetting must be expected to happen. Every transaction at a disequilibrium price yields gain to one person and loss to another in the form of an endowment effect of the second kind and consequently shifts the market demand and supply curves. If the effect of the gain is exactly the same as that of the loss then the total effect on excess demand in the market is zero and the equilibrium price is not affected. This result can be expected to hold in approximation when the two sides of the market (buyers and sellers) are similar in their distribution of expenditure among goods (i.e., in the propensities to spend, C_j^k).

This statement, like the first one of Hicks, again depends on the crucial assumption of the presence of all traders in the market during the whole of the market day. If only the sellers remain, as is usual in the operation of a more realistic market model, it is only the effect on their supply curves that will remain in the excess demand, hence there is no possibility of offsetting these effects on the equilibrium price.

With endowment effects of the second kind effectively assumed away, Hicks proceeded to analyze the stability of a multi-commodity model of exchange; in this, we find him taking off from Walras. Using what has been labelled as a comparative statics method he discussed conditions of perfect and imperfect stability. This analysis was conducted in terms of an implicit tatonnement process, but it is interesting to note that endowment effects (this time of the first kind) were again found to be the only possible source of instability even in the tatonnement model.[70] Such instability

[70]See J. Hicks, (1946), pp.316-317, and below in the text.

could be caused if strong asymmetry exists between buyers and sellers in the endowment effect coefficients, C_j^k, that is, if those who have a large excess supply of a commodity also have a large propensity to consume that commodity[71] and vice versa.

The attempt by Hicks to generalize the Walrasian analysis did not result in a dynamic formulation of the equations of exchange under tatonnement assumptions: rather, Hicks defined stability in a very special non-dynamic manner, and his work on this matter was therefore subject to criticism by later contributors to the problem of equilibration.[72] However, a number of conditions have been found under which the Hicksian stability conditions are equivalent to more "appropriate" dynamic conditions; Moreover, Hicks' proof that endowment effects of the first kind are the only possible source of instability in tatonnement seems to survive in the full analysis of local dynamic stability of equilibrium.[73] It seems useful, therefore, to give a brief discussion of Hicks' analysis of stability of multiple markets.

---

[71] See J. Hicks, (1946), p.316.

[72] See P. Samuelson, (1947), and Chapter III in this investigation.

[73] See Arrow and Hahn, (1971), pp.296-298 for a formal exposition of the importance of such endowment effects for local dynamic stability.

Hicks[74] defines stability in multiple exchange to mean that a rise in the price of a commodity (starting from an equilibrium) must create excess supply for it. The question arises, however, whether this happens in the face of all other commodity prices given (in which case other markets might be affected by the rise in the price of the commodity under examination) or whether it is supposed to hold even after some or all of the other prices have been adjusted to neutralize the effects of the initial price rise on the excess demands in these markets. If the first condition only is satisfied, that is, if a rise in the commodity price creates excess supply with all other prices constant the system is called *imperfectly stable*. Similarly, if a rise in the price creates excess supply in the respective market even after some other prices have been adjusted to keep the situation unchanged in other markets, equilibrium in this market is *imperfectly stable*.

A system of commodity exchange is *not unstable* if a rise in the price of a commodity creates excess supply in the respective market even after all other prices have been adjusted.[75] Finally, a system is *perfectly stable* if a rise in price creates excess supply in the respective market under all possible adjustments, i.e., with one or two or more prices adjusted to neutralize the effects of the initial price rise.

---

[74] (1946), pp.66-71, and 315-317.

[75] This is a statement on the value of partial derivative of excess demand with respect to prices.

The above definitions will now be formalized in terms of the framework given in Chapter I (Section 1.1) and the derivation of the relevant stability conditions will be carried out. Consider the excess demand functions of a pure exchange system, given in Chapter I,

$$E_i^* = E_i(P_1, P_2, \ldots . P_n, I) \quad (i=1,2,.n)\,(1.5)$$

where production levels at all prices are set to zero, in accordance with the definition of pure exchange. Take the total differential of this equation system[76] to obtain

$$\begin{bmatrix} dE_1^* \\ dE_2^* \\ \\ dE_n^* \end{bmatrix} = \begin{bmatrix} E_{11}, E_{12}, \ldots\ldots & E_{1n} \\ E_{21}, E_{22}, \ldots\ldots & E_{2n} \\ \\ E_{n1}, E_{n2}, \ldots\ldots & E_{nn} \end{bmatrix} \begin{bmatrix} dP_1 \\ dP_2 \\ \\ dP_n \end{bmatrix} \qquad (2.6)$$

or,

$$dE = JdP \qquad (2.6)$$

where dE and dP are column vectors and J is the Jacobian of the excess demand equations (1.5). Using the definition of market excess demand (in pure exchange) as the difference between demand and the commodity stock, and noting that this stock is constant in the exchange model, the change in the market <u>excess</u> demands E_i^* is equal to the change in the market

---

[76] Keeping all endowment quantities, i.e., the endowment matrix I, unchanged, in accordance with tatonnement assumptions.

<u>demands</u>, Q_i^{d*}. But the latter can be written, from equation (1.34) (taking note of the assumption that I does not change):

$$dQ_i^{d*} = \sum_{mk}\sum S_{ki}^m \; dP_k + \sum_m C_i^m \sum_k (I_k^m - q_k^m)\, dP_k \qquad (i=1,2,..n)\; (2.7)$$

where all terms have been defined in Chapter I. In matrix form, this system of equations may be written,

$$\begin{bmatrix} dE_1^* \\ dE_2^* \\ \cdot \\ \cdot \\ \cdot \\ dE_n^* \end{bmatrix} = \begin{bmatrix} \sum_m S_{11}^m, \sum_m S_{12}^m, \cdots \sum_m S_{1n}^m \\ \sum_m S_{21}^m, \sum_m S_{22}^m, \cdots \sum_m S_{2n}^m \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \sum_m S_{n1}^m, \sum_m S_{n2}^m, \cdots \sum_m S_{nn}^m \end{bmatrix} \begin{bmatrix} dP_1 \\ dP_2 \\ \cdot \\ \cdot \\ \cdot \\ dP_n \end{bmatrix} + \begin{bmatrix} \sum_m C_1^m (I_1^m - q_1^m), \ldots\ldots \sum_m C_1^m (I_n^m - q_n^m) \\ \sum_m C_2^m (I_1^m - q_1^m), \ldots\ldots \sum_m C_2^m (I_n^m - q_n^m) \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \sum_m C_n^m (I_1^m - q_1^m), \ldots\ldots \sum_m C_n^m (I_n^m - q_n^m) \end{bmatrix} \begin{bmatrix} dP_1 \\ dP_2 \\ \cdot \\ \cdot \\ \cdot \\ dP_n \end{bmatrix}$$

(2.7)

or, in more compact notation,

$$dE=SdP+CdP \qquad (2.7)$$

where S is the matrix of pure substitution terms and C that of pure price endowment effects.[77]

With the mechanics of the problem complete, the formalization of the Hicksian definitions and the stability conditions for these may be given. In the equation system (2.6) above consider a change in the price of commodity 1, all other prices constant. The change in the excess demand for the commodity is, then, obtained from this system by setting all dP_i (except dP_1) equal to zero and solving for dE_1^* to obtain

[77]As defined in Chapter I.

$$dE_1^* = E_{11} dP_1 \qquad (2.8)$$

from which it is obvious that a fall (rise) in prices will create excess demand (supply) only if

$$E_{11} < 0 \qquad (2.9).$$

For all markets to satisfy this condition, we must have

$$E_{ii} < 0 \quad \text{for all } i=1,2,\ldots n \qquad (2.10),$$

i.e., all the first order principal minors of J in (2.6) must be negative.

Now consider the same experiment of a change in price of commodity 1, with the additional consideration that the price of commodity 2 is adjusted to maintain equilibrium in the respective market.[78] The system of equations (2.6) may again be used, with dP_1, dP_2 different from zero and all other prices unchanged. Moreover, the excess demand in the second market will remain unchanged, by the definition of the experiment. Thus, the first two equations of (2.6) are reduced to

$$\begin{aligned} dE_1^* &= E_{11} dP_1 + E_{12} dP_2 \\ 0 &= E_{21} dP_1 + E_{22} dP_2 \end{aligned} \qquad (2.11)$$

[78] The system is assumed to be in equilibrium before the experiment starts.

In order for a fall in P_1 to cause excess demand in the respective market under these circumstances, it is easily seen[79] that we must have

$$(E_{11}E_{22} - E_{21}E_{12})/E_{22} < 0 \qquad (2.12)$$

Since, from condition (2.10) E_{22} is negative, the system is stable in this case if the determinant in the numerator is positive. This determinant is one of the second order principal minors of the matrix J in equation (2.6). Repetition of this experiment for commodities other than the first yields the result that for stability all the second order principal minors of J must be positive.

The same type of experiment may now be conducted, this time with two other prices adjusted to take care of the discrepencies caused in other markets from a change in the price of the commodity under consideration. One then finds that all third order principal minors of J must be negative, for stability.

The condition for perfect stability emerges naturally out of the above series of experiments: this is that all principal minors of J must alternate in sign, with positive sign for those of even order.

Having established these stability conditions, Hicks was able to deduce the comparative statics laws of the model of pure exchange using the information on the signs of the

---

[79] By solving (2.11) for dE_1^*/dP_1.

principal minors.[80] Moreover, using the result that the substitution matrix S in equation (2.7) is symmetric and negative definite,[81] he was able to claim that the only possible source of instability (in his scheme of definitions) resides in the pure price endowment effect matrix C of equations (2.7) above. As has already been mentioned, this result resurfaces in analysis of local dynamic stability.

This concludes the discussion of Hicks' efforts towards analysis of stability in multiple markets of tatonnement exchange. Before the discussion proceeds to modern dynamic analysis of this problem in the following Chapter, a summary of the early contributions will be undertaken, in the next Section.

1.6 Summary of Early Contributions to Equilibration.

The attempts at analysis of stability discussed in this Chapter have some common characteristics: for example, they are chronologically early attempts; they all lack explicit dynamic formulation of the problem of market equilibration; the most important concern in most of them is the question of how the actual market may approximate the solution values of prices obtained from the system of equations of exchange; most of them deal, or attempt to deal, with the

---

[80](1945), pp.73-77, and 317-319.

[81]From second order-conditions for utility maximization.

question of disequilibrium transactions and how these might affect the equilibrium position and, finally, most of them contain some discussion of stability.

In an attempt to summarize the discussion of this Chapter, it may be said that Walras was concerned, first, with the question whether the actual market solution of the equations of exchange approximated the mathematical solution of the equation system; second, he attempted to prove (and succeeded, in the two-commodity case) that the system of equations is stable under some conditions; third, he may be interpreted to have been unaware of the possible effects of transactions in disequilibrium; and, finally, he developed some conditions under which equilibrium prices would remain invariant to commodity redistributions among participants in the exchange.

Edgeworth spent some time with the model of barter, but was mainly concerned to prove that there exists a relationship between the "field of competition" and "determinateness" of equilibrium. With regards to disequilibrium transactions, different interpretations are possible, though it seems reasonable to take the position that Edgeworth adopted a view of the equilibration process in which the system adjusts not in the market period but in a sequence of such periods. In this process, disequilibrium transactions do take place within each period, but something must be specified about effects of these transactions carrying over from period to period.

Marshall saw the effect of disequilibrium transactions on equilibrium price and established conditions under which any commodity redistribution via disequilibrium exchange would have no effect on equilibrium. It is unfortunate that in a general equilibrium framework these conditions are unacceptable since they imply objectionable properties of demand functions. For partial analysis purposes, however, Marshall's solution seems acceptable.

Wicksteed confined himself to partial analysis, but he did bring in the aspect of time in the equilibration process, an aspect totally neglected by previous (and later) writers. His insistence on what I have called "imperfect market attendance" is certainly justified, and provides a pointer for research in the problem of equilibration in exchange.

Finally, Hicks is classified with the early contributors for the sole reason that his approach was not explicitly dynamic. However, his contribution to the analysis of tatonnement exchange may be judged superior to all those before him. He defines stability and develops conditions for it; he shows that the only source of instability resides in the terms connected with pure price endowment effects; and his analysis yields a full range of comparative static results.

The Hicksian contribution to non-tatonnement exchange is much less: it consists of the conjectures that, first, the volume of disequilibrium transactions would be small and therefore may be neglected without considerable danger; and, second, that some offsetting may be expected to happen between the gains and losses experienced via endowment effects of the second kind on excess demands. Such offsetting would be due to similar marginal propensities to spend on commodities (similar C_j^m) between those experiencing gains and those having suffered losses from disequilibrium exchanges. However, the Hicksian contribution to non-tatonnement exchange is not used any further in the stability and other analysis developed by Hicks and for this reason it is of no consequence for the rest of his work that his contribution to non-tatonnement consists of a set of assumptions.

The next Chapter discusses more recent analysis of equilibration for models of production and exchange.

CHAPTER III

REVIEW, INTERPRETATION, AND CRITICAL EVALUATION OF MODERN WORK ON MARKET EQUILIBRATION.

This Chapter is a continuation of the review undertaken in Chapter II; it covers work on stability and equilibration processes after the contribution of J. Hicks[1]. More specifically, Section 3.1 deals with formalizations and extensions in the "Walrasian tradition" of analysis, i.e., in the framework of the traditional general equilibrium model of production and exchange. The discussion in this Section includes dynamic formalizations of the Walrasian tatonnement process; various conditions for "local stability"[2]; an enumeration of situations in which the Hicksian conditions for perfect stability are equivalent to dynamic stability conditions proper; and "global stability" analysis. The Section also contains a review of extensions in the direction of non-tatonnement analysis of the pure exchange model. A summary and evaluation of this strand of "Walrasian" analyses are undertaken in the last part of this Section.

---

[1] (1946).

[2] Stability concepts are defined in the appendix to this Chapter.

Section 3.2 discusses a group of works which may be classified as (not necessarily conscious) attempts at disequilibrium analysis in the general context of models of production and exchange. Some of these models are partial such as the well-known "cobweb" model; others are aggregate, such as cycle models; and others finally belong to the family of general equilibrium models, such as those used by D. Patinkin,[3] R. Clower,[4] and H. Grossman.[5] Thus, it is not the type of model used that constitutes the common characteristic of the works discussed in this Chapter. Rather it is the attempt, conscious or otherwise, at some form of analysis involving disequilibrium. Neither is the equilibration process in every case specified in terms of price change. In some instances, quantities are burdened with adjustment (as in the case of cycle models) and in other cases (as in the work of Barro and Grossman)[6] no adjustment takes place at all. Perhaps this justifies classification of these models as "non-Walrasian." A summary and evaluation is undertaken in the last part of this Section. Finally, Section 3.3 contains the conclusions of the analysis in this Chapter.

---

[3] (1965).

[4] (1965).

[5] (1971).

[6] (1971).

The discussion of formalizations and extensions of the model in the Walrasian tradition is included in this investigation for obvious reasons: formal stability analyses of both tatonnement and non-tatonnement processes are clearly attempts to establish conditions under which equilibrium is "attainable," and thus comparative statics has meaning. In the process, it is discovered that, at least in pure exchange models, the equilibrium position is affected by disequilibrium transactions. On the other hand, the works included in Section 3.2 also concern themselves in some sense with the same questions, though in different frameworks of analysis. Moreover, in some of these works there are useful pointers at directions of research, some of which are taken up in the present investigation. Thus, it is both useful and proper to review these contributions.

This Chapter is, like the previous one, in the nature of a review. Nevertheless, it is deemed that some of the expositions, and in particular the Sections on evaluation do constitute contributions to knowledge in the subject. The discussion in this Chapter uses, to the extent possible, the general framework set out in Chapter I of this thesis. An appendix with definitions, local stability analysis and use of Lyapounov's second method for global stability is at the end of the Chapter: these are the main tools used in this Chapter together, of course, with differential equation systems.

3.1 Formalizations and Extensions in the "Walrasian Tradition"

Section 3.1.1 below deals with the formalization and stability analysis of the tatonnement process; both local and global stability are discussed. Section 3.1.2 contains contributions in the nature of extensions to the Walrasian tatonnement, since these involve analysis of non-tatonnement processes. Finally, Section 3.1.3 provides a summary and an evaluation of both tatonnement and non-tatonnement works in this Walrasian framework.

3.1.1 Formalizations in the Walrasian Tradition

(a) Samuelson's formulation and local stability analysis. The first formalization of the standard Walrasian model of production and exchange in a fully dynamic framework was carried out by Samuelson,[7] in a framework of tatonnement. He used the dynamic specification similar to that of equation (1.11)[8]

---

[7](1941), (1944), and (1947), p.270ff.

[8]The difference is in the lack of concern for negative prices. If the non-negativity condition on prices is included -- as it should be -- the additional specification of the tatonnement given in equation (1.11) is required. See Arrow, Block and Hurwicz (1959), and Mackenzie, (1960), for a treatment of the problem.

above. More specifically, he postulated that prices outside equilibrium would move according to the rule:

$$\dot{P}_i = K_i(E_i^*) \qquad (i=1,2,\ldots n) \qquad (3.1)$$

where all symbols have been defined in Chapter I, and E_i^* may be interpreted as excess demand in the model of production and exchange. Samuelson then linearized the system around equilibrium (described by the prices $\bar{P}_i$, i=1,2,...n) to obtain

$$\dot{P}_i = E_i^*(\bar{P}) + K_{iE} \sum_j E_{ij}(P_j - \bar{P}_j) = K_{iE} \sum_j E_{ij} P_j + C \quad (i=1,2,\ldots n) \quad (3.2)$$

where $\bar{P}$ denotes the equilibrium price vector and C the sum of constant terms of the expansion, while K_{iE} and E_{ij} are, respectively, the partial derivative of the K_i function with respect to the ith excess demand and the partial derivative of the ith excess demand with respect to the jth price. The solution of the homogeneous part of this equation system is of interest for stability. Samuelson further simplified his analysis by assuming that all K_{iE} equal unity[9], thus reducing the homogeneous part of the system to

$$\dot{P}_i = \sum_j E_{ij} P_j \qquad (i=1,2,\ldots n) \qquad (3.3).$$

A trial solution to this system is of the form,

$$P_i(t) = A_i e^{\lambda t} \qquad (i=1,2,\ldots n) \qquad (3.4).$$

Substitution of the trial solution (3.4) in (3.3) yields the

---

[9] It was later pointed out by Arrow and Hurwicz, (1958), that this in fact is not a restrictive assumption, since it involves change in units of measurement of commodities only.

characteristic polynomial of the Jacobian matrix J of the system of excess demand equations[10]. The characteristic polynomial is of degree n in λ. Thus, the solution to the dynamic system (3.3) may be written,

$$P_i(t) = \sum_j B_{ij} e^{\lambda_j t} \qquad (i=1,2,\ldots n) \qquad (3.5)^{11}$$

where B_{ij} are constants dependent on the Jacobian and the initial conditions, and λ_j is the jth characteristic root of J.

For local asymptotic stability it is necessary and sufficient that the real parts of the characteristic roots of the Jacobian matrix J be negative. Samuelson was able to show by example that the Hicks conditions for perfect stability are in general neither necessary nor sufficient for dynamic stability[12].

The contribution of Samuelson is obviously important. For the first time, the problem of multimarket tatonnement adjustment is cast in a "proper" dynamic form and subject to formal analysis of local asymptotic stability. Necessary and sufficient conditions are derived for such stability, too. This concept of stability, however, refers to small displacements of the system from the equilibrium position: more

---

[10] This matrix has been defined in Chapter II, equation (2.6).

[11] If not all the characteristic roots of the polynomial are distinct the solution is written in a somewhat different form. The stability of the system is not modified, however. See, for example, P. Samuelson (1947), p. 271n.

[12] (1947), pp. 271-274, and also (1944).

general global stability analysis had to await the work of Arrow and Hurwicz[13].

(b) Hicksian and true dynamic stability. The question of the relationship of Hicks' conditions to Samuelson's may be expressed as follows: since the Hicks conditions for perfect stability require that the matrix J be negative definite, and the true dynamic conditions require its characteristic roots to have negative real parts, when does a negative definite matrix also have characteristic roots with negative real parts? And furthermore, when do characteristic roots with negative real parts imply negative definiteness of the matrix?

Samuelson contributed to this matter by remarking that

(i) If the Jacobian of excess demands is symmetrical, i.e., if $E_{ij}=E_{ji}$ (the response of the ith excess demand to the jth price equals the response of the jth excess demand to the ith price) the Hicksian and the dynamic stability conditions are equivalent, i.e., the one set implies the other[14].

Further, L. Metzler[15] proved that the Hicksian conditions

(ii) are necessary (but not sufficient) if the equilibrium is to be dynamically stable for all possible

---

13 (1958).

14 (1947), p. 271.

15 (1945).

speeds of adjustment, K_{iE}.

(iii) are equivalent to the dynamic conditions when all commodities are gross substitutes[16].

T. Negishi[17] attributes to Samuelson the following:

(iv) If J is quasi-negative definite (i.e., if J+J'/2 is negative definite, where J' is the transpose of J) both Hicksian and dynamic conditions hold.

And finally, Negishi[18], Arrow-Hurwicz[19] and Hahn[20] proved that

(v) If all commodities are gross substitutes and either Walras' Law or homogeneity of demand functions with respect to prices hold, not only are the Hicksian and dynamic conditions equivalent, but they necessarily hold i.e., the equilibrium is locally stable under gross substitutability and Walras' Law (or homogeneity).

As is obvious from the above array of results, the Hicksian stability conditions generated considerable

---

[16]As defined in Chapter I above.

[17](1962), p. 643.

[18](1958).

[19](1958).

[20](1958).

interest. It was not initially noticed, however[21], that Hicks' statement on the role of pure price effects for stability[22] carries untouched into the dynamic analysis: for example, it has been shown in Chapter I that gross substitutability and homogeneity imply that the response of excess demand to the own commodity price is negative, i.e., the commodity is "normal". Income effects (pure price effects) alone can change this situation, precisely as Hicks remarked. Moreover, Hicks anticipated the analysis based on the assumption of gross substitutability when he remarked that a moderate degree of substitutability is all that is required for stability of multi-commodity exchange[23].

It is thus interesting that although Hicks' method lacked rigour and proper formulation of the dynamic mechanism it nevertheless established, in a "practical" manner, the basic conditions that have to hold if stability is to be assured.

(c) Other dynamic conditions for local stability. Relations among various conditions. Contributions which followed Samuelson's initial formalization of Walras were not only

---

[21] Until 1971, in Arrow and Hahn.

[22] Discussed in Chapter II above.

[23] (1946), pp.72-73.

concerned with the question of the Hicksian versus the dynamic conditions; there was also the aim to develop other, readily comprehensible conditions that would guarantee that the characteristic roots of the Jacobian had negative real parts. The reason is that the conditions on the characteristic roots are not too illuminating in terms of economic implications; nor are they easy to compute.

Newman[24] has provided not only an excellent summary of the stability conditions that were found but also an extension of this strand of works. A modified tabulation of the results presented by Newman is given here. Three kinds of Jacobian excess demand matrices are considered: a matrix without additional conditions on its elements, a matrix satisfying gross substitutability and a symmetric matrix. Results are tabulated in each case. The symbol S_{ij} means that condition i implies condition j, but not vice versa. If for a specific i and j we have S_{ij} <u>and</u> S_{ji} the two conditions are equivalent and the symbol E_{ij} is used to denote this fact. Finally, the symbol $\bar{S}_{ij}$ means that the condition i does not imply condition j.

Table I below relates four conditions, namely: (i) the condition that the characteristic roots of the matrix have negative real parts (Routh-Hurwicz condition, as it is called),

---

[24] (1959-1961), and (1961).

(ii) the condition that the matrix be negative quasi-definite, (iii) the condition that the matrix possess a dominant diagonal, and (iv) the condition that principal minors alternate in sign (that is, the condition that the matrix be Hicksian, as it has come to be called).

Entry in this Table should be only from each row. The symbol in the row and the relevant column establishes the relationship among the four conditions. In the case where both S_{ij} and S_{ji} are found, equivalence is recorded at the right-hand side of the table.

Table I

Relationship Between Stability Conditions and Other Characteristics of a Matrix

	Neg. Real Parts	Neg. Quasi Definite	Dominant Diagonal	Altern. Princ. Minors	Equivalence
(1) Negative Real Parts		$\bar{S}_{12}$	$\bar{S}_{13}$	$\bar{S}_{14}$	No
(2) Negative Quasi-Definite	S_{21}		$\bar{S}_{23}$	S_{24}	Equivalent
(3) Dominant Diagonal	S_{31}	$\bar{S}_{32}$		S_{34}	Conditions
(4) Alternating Principal Minors	$\bar{S}_{41}$	$\bar{S}_{42}$	$\bar{S}_{43}$		Emerge

From the table, results S_{21} and S_{24} are due to Samuelson[25] while result S_{31}, that if the system possesses a

[25] (1947), p.438 and p.141, respectively.

dominant diagonal it is stable, is due to Newman.[26] As may be seen from the Table, the results relating Hicksian stability to true dynamic stability are also tabulated, in the fourth row.

Tables II and III below give relationships of stability conditions for Metzlerian and symmetric matrices respectively. With respect to results, the equivalence E_{41} in Table II is due to Metzler.[27] In Table III, the equivalence E_{41} is due to Samuelson.[28]

Table II

Relationship Between Stability Conditions and Other Characteristics of a Metzlerian Matrix

	Neg. Real Parts	Neg. Quasi Definite	Dominant Diagonal	Altern. Princ. Minors	Equivalence
(1) Negative Real Parts		$\bar{S}_{12}$	S_{13}	S_{14}	E_{31}
(2) Negative Quasi-Definite	S_{21}		S_{23}	S_{24}	E_{41}
(3) Dominant Diagonal	S_{31}	$\bar{S}_{32}$		S_{34}	E_{43}
(4) Alternating Principal Minors	S_{41}	$\bar{S}_{42}$	S_{43}		

[26] (1959-1961), Theorem 10, p.6.

[27] (1945), pp.285-290.

[28] (1941), pp.110-111.

Table III

Relationship Between Stability Conditions and Other Characteristics of a Symmetric Matrix

	Neg. Real Parts	Neg. Quasi-Definite	Dominant Diagonal	Altern Princ. Minors	Equivalence
(1) Negative Real Parts		S_{12}	$\bar{S}_{13}$	S_{14}	E_{21}
(2) Negative Quasi-Definite	S_{21}		$\bar{S}_{23}$	S_{24}	E_{41}
(3) Dominant Diagonal	S_{31}	S_{32}		S_{34}	E_{42}
(4) Alternating Principal Minors	S_{41}	S_{42}	$\bar{S}_{43}$		

(c) Contributions on global stability. The contributions discussed above in this Section have the common characteristic that they analyze conditions for local stability of equilibrium.[29] The method consists of examination of a linearized version of the model under analysis. Thus, results are valid only for the small neighbourhood for which the linearization is a tolerable approximation of the non-linear system. Moreover, most of the contributions mentioned above did not utilize, in their derivations of stability conditions, basic assumptions involved in the model of production and

[29] With the exception of Negishi (1958), Arrow and Hurwicz, (1958) and Hahn, (1958) for the proof stated as (v) above.

exchange such as Walras' Law, homogeneity of demand functions, etc.

Analysis (and proofs of global stability) under tatonnement assumptions was first carried out by Arrow and Hurwicz, and Arrow, Block and Hurwicz.[30] In the following, the proof for the case of three commodities, and that for the case of any number of commodities, will be outlined. Not only are these results important in themselves, they also provide illustrations of how basic assumptions such as Walras' Law, homogeneity, and the rather strong postulate of gross substitutability (all defined in Chapter I) may be used to secure stability results.

The system of tatonnement equations used to illustrate the three-commodity case is a modified version of equation (3.1) above, with all speeds of adjustment equal to unity:[31]

$$\dot{P}_i = E_i^* \qquad (i=1,2,3) \qquad (3.6).$$

The assumptions are:

(i) Gross substitutability of all commodities at all prices,

(ii) Zero-degree homogeneity of all excess demands,

---

[30] For a three-commodity model, the proof of global stability was given in Arrow and Hurwicz, (1958). For a multi-commodity one, the proof is in Arrow, Block and Hurwicz, (1959).

[31] Any constant speeds of adjustment will also do, but the analysis will not differ: all speeds may be reduced to unity by suitable choice of units of measurement of commodities. See Arrow and Hurwicz, (1958), p.525.

(iii) Walras' Law,

(iv) Positive prices at all times,

(v) Existence of Equilibrium.

Under these conditions, it can be proved that equilibrium is unique.[32] Moreover, from homogeneity it follows that the prices may be normalized to make $P_3=1$ always. Finally, from Walras' Law if the two markets (for commodities 1 and 2) are in equilibrium, so is the third. Thus, it is sufficient to consider the first two markets, the adjustment equations of which may be written (given the price normalization)

$$\dot{p}_1=E_1^*(p_1,p_2) \;, \quad \text{and} \quad \dot{p}_2=E_2^*(p_1,p_2) \qquad (3.7)$$

where p_i are the relative prices of commodities 1 and 2.

The adjustment process may now be illustrated in a phase diagram in relative price space, in Figure 3.1 below.

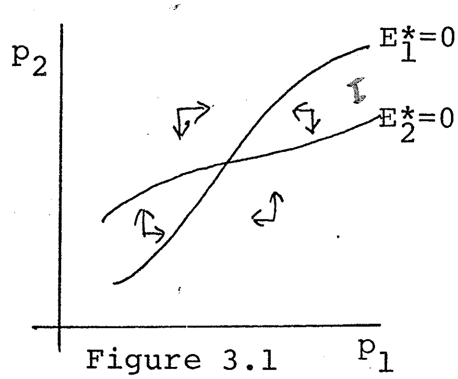


Figure 3.1

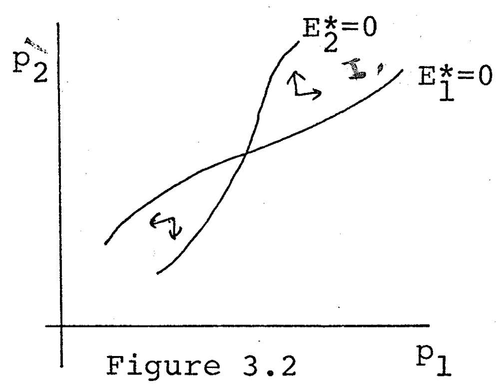


Figure 3.2

The equations $E_1^*=0$ and $E_2^*=0$ are plotted in this diagram, and

---

[32]See, for example, Takayama, (1974), pp. 321-325. The stability discussion in the text is also based on this as well as Arrow and Hurwicz (1958).

are the loci of equilibria in the first and second market respectively. From the assumptions of homogeneity and gross substitutability it follows that to the right of the equation $E_1^*=0$ (in a direction of increasing p_1) the situation is one of excess supply, and vice versa for the left side. On the other hand, the same two assumptions imply that to the right of the equation $E_2^*=0$ there is excess demand in the market for commodity 2, and vice versa for the other side. Given the dynamic rule for price adjustment, the arrows in the Figure indicate the direction of price movements in each of the four sections of phase space. Equilibrium is obviously stable, given the way in which the curves are drawn to cut each other. It remains to show that it is impossible for them to cut in another way.

To prove that this is so, suppose the $E_1^*=0$ curve cuts the $E_2^*=0$ curve from above (see Figure 3.2). Then, in section I of the phase space the situation in both markets is one of positive excess demand, and, moreover, prices p_1 and p_2 are both above their equilibrium values. But if this is the case, the situation in the third market must also be positive excess demand, by the assumption of gross substitutability. Positivity of all excess demands violates Walras' Law. Hence, it is impossible for the curves to cut as in Figure 3.2. This proves global stability in the three-commodity case.

For the case of any number of commodities, proof of global stability utilizes equation (3.1) above (i.e., variable speeds of adjustment)

$$\dot{P}_i = K_i(E_i^*) \qquad (i=1,2,\ldots n) \qquad (3.1)$$

and the same assumptions as the proof for the three commodity case. A sketch of the proof will be provided here.[33]

It is first proved, by use of Walras' Law, that the norm (length in this case) of the price vector is constant, that is, it moves, during the equilibration process, along the surface of a sphere of given radius (the radius depends on the initial price vector). It is then proved that under homogeneity and gross substitutability the sphere can only contain one equilibrium price vector. Finally, it is shown that Walras' Law, homogeneity and gross substitutability imply that the weak axiom of revealed preference holds for the economy as a whole, that is, the values of excess demand at any prices, weighed by the equilibrium price vector and summed are positive.

The distance between the price vector and the equilibrium price vector at any point in time is then used[34] to prove global stability of the system. This distance is found to diminish with time, and thus (with some discussion that shows

---

[33] Based on Takayama, (1974), pp. 325-329, and Negishi, (1962), pp. 629-636.

[34] This is, of course, a Lyapounov function for the system. See the Appendix to this Chapter.

that the price vector cannot be bounded away from equilibrium) the proof is complete.

It is perhaps worth remarking that Walras' Law is an essential assumption in both the above proofs: for example, in the three-commodity case it is this law that requires that the two curves cut "the right way" in the phase plane; and in the multi-commodity case it is Walras' Law that requires that the price vector moves only on a quite restricted area, namely the surface of a sphere. This, of course, is not to be taken to mean that the other assumptions are less important for the stability proofs: for example, homogeneity combined with gross substitutability establishes the fact that when all relative prices are above equilibrium the numeraire commodity must be in (positive) excess demand, which is a vital part of the proof that the curves have to cut the "right way" in the three-commodity case.

With the above discussion of the proofs of global stability complete, one might wish to discuss the various developments following the above contributions. This would take the present review off its main course, however[35]; thus, in

---

[35]A brief indication of what followed may be given in this footnote: Scarf, (1960), furnished examples in which absence of gross substitutability was assumed, and instability was shown; explicit use of the second method of Lyapounov was made, first by Mackenzie, (1960), who acknowledged the idea to Arrow; non-negativity of prices was examined by various contributors; expectations were introduced into the analysis by Enthoven and Arrow, (1956), Arrow and Nerlove, (1958), Arrow

bringing the discussion in this subsection to a close a number of selected results given by Arrow and Hahn[36] will be mentioned, without proofs.

First, it is proved[37] that a two-commodity economy with isolated equilibria[38] is globally stable under the tatonnement rule of (nominal) price change. The proof uses only Walras' Law, the assumption that equilibrium exists, a distance (Lyapounov) function between actual and equilibrium price vectors, and technical assumptions on the differential equation system of tatonnement.

It appears that in the two-commodity model the requirement regarding slopes of excess demand functions is not really necessary; at the same time, it must be remarked[39] that the fact that such proof of stability cannot be given for higher dimensions means that it may not be too sensible, in stability analysis, to devote the most attention to the two-commodity model of the economy.

---

and Hurwicz (1962) and Negishi, (1964); and Uzawa, (1959-1960) formulated the discrete time process and proved the main results.

[36] (1971), Chapter XII.

[37] Ibid., p. 283.

[38] For the definition, see ibid.

[39] Compare ibid., pp. 282-283.

Another proof furnished in a succint way by Arrow and Hahn establishes global stability of an equilibrium for a tatonnement with constant speeds under the assumption that the weak axiom of revealed preference holds for the economy.[40]

Finally, a number of results on the assumption of diagonal dominance are furnished, some for pure exchange and some for both production and exchange. The strength of the relevant theorems is found to depend on the specific definition of diagonal dominance adopted. In summary, Arrow and Hahn[41] prove, among other things, that an economy with "elasticity" diagonal dominance[42] is stable, and an economy with "absolute" diagonal dominance in some units of measurement at all prices is also stable.

3.1.2 Extensions to Non-Tatonnement in the Walrasian Model of Pure Exchange

After considerable work had been done on the stability conditions of the tatonnement, it became increasingly clear that the stability results could not be obtained without such strong assumptions as gross substitutability among all

---

[40] ibid., pp. 285-286.

[41] ibid., pp. 292-296.

[42] For definitions see Chapter I .

commodities at all prices. Research then turned towards the examination of the non-tatonnement process (as defined in Chapter I) with the purpose of establishing whether allowance of disequilibrium transactions made a difference to the stability requirements. The contributions confined themselves to the pure exchange model, for reasons partially explained in Chapter I. Moreover, while trade was allowed out of equilibrium in these models, the institution of the auctioneer was nevertheless retained for the reason that unless this is done some explicit account of the manner of price setting in the system must be given. It is not logically consistent to suggest that price takers undertake to change prices under "market pressure"[43]; therefore the auctioneer is still necessary.[44]

The first result in this line of analysis was published[45] by Negishi[46]. It was shown that the non-tatonnement process described in Chapter I is stable if universal gross substitutability at all prices <u>and</u> endowment distributions along with

---

[43] This point was first made by Koopmans (1957), and further developed by Arrow, (1959). See also Arrow and Hahn, (1971) p. 325.

[44] There is a strand of analyses of market operation without the auctioneer. See Fisher, (1970), (1972); Diamond, (1971); and Cook and Veendrop, (1975). This group of works will not be discussed in this thesis.

[45] The contributions to non-tatonnement analysis are so close to one another that it seems useful to distinguish "publish" from "discover" in this review.

[46] (1961).

Walras' Law are assumed. Thus, it turned out that the consideration of actual out of equilibrium transactions in the model of exchange does not seriously modify stability conditions established by means of tatonnement analyses. This is clearly an interesting result. It was of course also pointed out that equilibrium depends on the specific equilibration path and the associated transactions. Discussion of this and other points will be undertaken in the next part of this Section.

A year later, Hahn and Negishi[47] proved that, if the tatonnement process satisfies the so-called "Hahn process" of disequilibrium exchange stability can be obtained under weaker assumptions.

The Hahn process of exchange specifies that those who are in the "short end" of the market at any time not only are able to carry out their optimizing plans at these prices but they do so at all times. That is to say, if there is excess demand in the market no seller is found with more than he is willing to hold at the given price vector, while if there is excess supply no buyer is found with less than he would be willing to hold. Thus, this is a specific form for the functions describing changes in stocks in non-tatonnement, (equations (1.23) in Chapter I).

With the above specifications of the non-tatonnement

---

[47] (1962).

and the additional assumptions of: (a) Walras' Law; (b) homogeneity; (c) the stipulation that each individual has some of each commodity at the beginning of the process; (d) constant speeds of adjustment; and (e) some technical assumptions as to the nature of the demand functions, Hahn and Negishi proved, via the second method of Lyapounov, that the system approached an equilibrium (i.e., they proved "quasi-stability").

Finally, Uzawa[48] formulated the "Edgeworth barter process" of exchange as another specific formalization of the equations describing stock changes in non-tatonnement. According to this process, exchange is to take place if and only if at least one individual gains by exchange and no individual loses. Thus the distribution of stocks following a disequilibrium exchange is more satisfactory to at least one individual and less satisfactory to no one (at prices where exchange took place).

On the basis of this specification, and by means of other assumptions[49] Uzawa proved that the exchange non-tatonnement system moves towards an equilibrium.

---

[48](1962).

[49] *Ibid.*, pp. 226-227. As Negishi, (1962), p. 661, remarks, this proof is not directly dependent on such assumptions as Walras' Law and homogeneity. However, these assumptions may be regarded as the rationale underlying the formulation of the Edgeworth exchange process.

3.1.3 Summary and Evaluation of Contributions in the Walrasian Tradition

The content of this part is sufficiently accurately described in the title. First, a summary of the results in 3.1.1 and 3.1.2 is given and subsequently an evaluation of the contributions is undertaken, in the light of the main questions raised in this thesis.

(a) Summary. Contributions started with explicitly dynamic formulations of the Walrasian tatonnement in multi-commodity models, and derivation of conditions for true local dynamic stability. Other work followed which gave more readily comprehensible conditions for stability than the mathematical requirement on the characteristic roots. In the process, the relation of the Hicksian to true dynamic conditions was clarified. Finally, a number of researchers proved local stability by means of such postulates as Walras' Law, homogeneity, gross substitutability and existence of equilibrium. The group of the above works was solely concerned with Walrasian tatonnement, as defined in Chapter I.

Global stability analysis by means of distance functions followed these works. The main assumptions were gross substitutability at all prices, Walras' Law, homogeneity and existence of equilibrium. Stability was also proved for the case of diagonal dominance. Finally, use of Lyapounov functions in global stability analysis was made explicit. The framework of these works was still the Walrasian tatonnement model, mostly of production and exchange, but sometimes of pure exchange.

Finally, in the framework of the pure exchange model this time, contributions came which showed that actual disequilibrium behaviour in exchange does not modify stability of the system: more specifically, it was proved that the system would be stable with disequilibrium transactions if it was stable without them (that is, in tatonnement). Moreover, it was proved that disequilibrium transactions made it actually easier to achieve stability, if one was willing to postulate specific rules for disequilibrium exchange.

(b) Evaluation. The formidable set of analytical tools used in both the formalizations and the stability analysis discussed above can scarcely fail to impress the reader of the contributions reviewed in Sections 3.1.1 and 3.1.2. Nevertheless, the literature on stability of competitive equilibrium is vulnerable to criticism, as will be argued in the following paragraphs. Some of these objections have already been made, while others are original with this investigation. Indication will be provided for those points that have already been made in the literature.

It may be more convenient to begin the evaluation with the positive aspects and classify them into those applying to analyses of tatonnement and analyses of non-tatonnement processes. Thus, with reference to tatonnement, one may make at least three positive remarks:

(i) we now have a full analysis of pure price effects in equilibration, for the model of production and exchange. This is certainly useful (but see critical point (ii) with reference to these effects below).

(ii) we have learned things which were unknown before: the proper formalization and stability analysis of Walrasian tatonnement is now complete; Walras' conjecture that diagonal dominance implies stability has been proved correct; Hicks' proof that income (pure price) effects are the only source of instability has been found to hold in proper dynamic settings; in short, the contributions have certainly advanced knowledge.

With reference to non-tatonnement, positive points include:

(iii) it is now known that the suspicion that disequilibrium transactions might possibly cause instability in the model of exchange is totally unfounded. Indeed, the reverse is true, since equilibration is hampered by restriction of the adjustment process to one of no transactions in disequilibrium. (But see critical point as to the assumptions involved in non-tatonnement, (i) below.)

(iv) The contention (known since Edgeworth and Marshall) that disequilibrium transactions affect the final equilibrium position has now been formally stated and taken into account in analyses and discussions of non-tatonnement equilibration. Moreover, some attempt has been made to find conditions under which invariance of equilibrium prices to disequilibrium transactions obtains. The theorem of equivalent redistributions of commodity holdings[50] may be considered to provide sufficient (and sometimes necessary[51]) conditions for such invariance. The Marshallian assumption of constant marginal utility of money[52] provides another condition. Finally, another rather obvious condition may be mentioned here: it is that all individuals have constant and identical marginal propensities to consume C_j^m. In this case, a transfer of commodities in disequilibrium exchange from one individual to another and a consequent price change does not affect the excess market demand function because the gain experienced by one individual increases the excess demand by exactly the amount it is decreased due to the loss experienced by the other individual. More specifically, the term C_j^m in equation (1.34)[53]

---

[50] See Chapter II.

[51] As has been pointed out in the discussion of the theorem in Chapter II.

[52] See Chapter II.

[53] Chapter I.

for the excess market demand need not be under the summation sign with these assumptions: it becomes constant and may be taken outside the summation sign, which implies that the pure price effects on demand are identical for all individuals per dollar change in the evaluation of their endowments. But endowment effects of the second kind can only operate on excess demand if pure price effects are different. This proves that constant identical C_j^m is a sufficient (though very strong) condition for invariance of equilibrium prices in non-tatonnement.

It may be useful to provide an example of equal and constant C_j^m. The discussion will be confined to exchange of two commodities by two groups and homogeneous utility functions. If the utility functions of the two participant groups in exchange are identical and homogeneous of degree r in the variables x and y (the two commodities) it is well known that their first partial derivatives are homogeneous of degree r-1 in the same variables. Hence, the rate of substitution is homogeneous of degree zero in the same variables. This is the well known dependence of the marginal rate of substitution on the ratio of the commodities alone.

Under these circumstances it can be easily shown that the contract curve must be the diagonal in the Edgeworth box. Let $\bar{x}$ and $\bar{y}$ represent the amounts of commodities X and Y of the participant group 1, and $\bar{x}$ and $\bar{y}$ the total quantities available. Then, group 2 combinations can be expressed as

$\bar{y} - y$ and $\bar{x} - x$. Thus, the rate of substitution for group 1 is a function of the ratio y/x while the rate of substitution for group 2 is dependent on the ratio of $(\bar{y}-y)/(\bar{x}-x)$. Along the contract curve the marginal rate of substitution must be equal for both groups: hence, the ratios y/x and $(\bar{y}-y)/(\bar{x}-x)$ must be equal[54]. But this will hold if y/x is equal to $\bar{y}/\bar{x}$ that is, when the contract curve is the diagonal of the box[55]. And, finally, when the contract curve is a straight line the ratio of the two commodities is constant, hence the marginal rate of substitution along the curve is constant. Obviously, under such circumstances it does not matter where on the contract curve the parties equilibrate since the price ratio is the same along it. This ratio is dependent solely on the quantities of the commodities available, $\bar{x}$ and $\bar{y}$, and it thus follows that any change in the quantities available changes the equilibrium price ratio in an unambiguous way.

A number of <u>criticisms</u> may now be cited, and made, with reference to the contributions dealt with in 3.1.1 and 3.1.2. Thus, <u>with reference to stability analysis of tatonnement</u>:

(i) the development of the <u>tatonnement</u> exchange model

---

[54] The requirement of a diminishing marginal rate of substitution ensures that there exists a one to one relationship between the marginal rates and the commodity bundles.

[55] $$\frac{\bar{y}-y}{\bar{x}-x} = \frac{y}{x} \quad \text{implies} \quad \frac{\bar{y}-y}{y} = \frac{\bar{x}-x}{x}, \quad \text{i.e.,} \quad \frac{\bar{y}}{y} - 1 = \frac{\bar{x}}{x} - 1$$

i.e., $\frac{\bar{y}}{y} = \frac{\bar{x}}{x}$, i.e., $\frac{\bar{y}}{\bar{x}} = \frac{y}{x}$, Q.E.D.

has gone through a series of formalizations that were gradually found necessary for its logical consistency. These formalizations have served to make transparent the fact that the institutional specifications required deny the model relevance for the impersonal perfectly competitive market, as will be shown presently.

The necessity of institutional specifications becomes obvious when the essential characteristics of tatonnement exchange are recognized, namely,[56]

(a) one price ruling at each point in time (so that buying and selling intentions to be aggregated all refer to the same price)

(b) a procedure whereby the price is made known to the participants

(c) collection and aggregation of buying and selling intentions at the 'cried' price and

(d) application of some rule of price change in order to reach equilibrium .

These obviously cannot be generated by the impersonal forces of competition: what is needed is some institution to facilitate the process and apply the rules. The literature has used a number of concepts, including "intellectus angelicus", "secretary of the market", "umpire", "auctioneer", etc.

---

[56] On this and all the preceeding points, compare D.A. Walker, (1973a).

It has been pointed out that this institution is not merely a concept to be used for elucidation of the operation of the impersonal competitive mechanism, but a necessary addition to the model if it is to exhibit the _tatonnement_ properties[57]. It has also been remarked that if this institution is to be a real one and not a market machine, it will have some capacity for rational behaviour and some degree of freedom, which it can use instead of the mechanical rule of price change in the differential equations of the _tatonnement_ model[58].

The first of the above remarks casts doubt on the capacity of the impersonal competitive model to achieve its own equilibrium position, since the necessary institutional addition of the price setter makes one wonder how much of the competitive mechanism remains, while the second remark reinforces any doubts that one might have regarding this issue since the central institution of the price setter may have an interest beyond the mechanical application of the Walrasian pricing rule. But if this is so, the basic question arises of the incentive under which this authority would operate, and of interests which it would serve. In short, perfect competition with central authority does not seem to be a happy combination.

---

[57] Koopmans (1957).

[58] Cf. Walker, _loc_. _cit_., p. 359.

(ii) The impresive proofs of local and global stability of tatonnement are beset by the common characteristic of having to assume away the existence of Giffen goods. In other words, basic assumptions of this strand of works have effectively assumed the result, by specifying well-known conditions that make pure price endowment effects not inoperable but essentially harmless to stability. But if the above claims are correct the value of these contributions reduces to finding conditions equivalent to the requirement that endowment effects of the first kind be harmless for equilibration.

The claims advanced in the above paragraph may now be substantiated. Most of the stability proofs are based on the assumptions of zero-degree homogeneity and gross substitutability. It has already been shown in Chapter I that gross substitutability coupled with homogeneity imply that the own-price effect on demand is negative. Hence, these two assumptions together in fact rule out Giffen goods. The pure price effects on excess demands, that is, may be "perverse" but they are never of a magnitude sufficient to cause problems.

It has thus been established that the formidable array of tools used in the stability analysis of tatonnement exchange (and production and exchange) has resulted only in enumeration of conditions under which Giffen goods do not exist. But this has indeed been suspected all along as a stability requirement, for example by Hicks[59].

(iii) The analysis of tatonnement effectively abstracts from endowment effects other than pure price. While this is not a serious point, since research must proceed in steps, it nevertheless has to be borne in mind.

(iv) The "low brow" question may be raised as to which market takes precedence in terms of equilibration in a multi-market system. Are all markets to be equilibrated at the same time? Then, this implies that all individuals are somehow able to participate in a <u>global</u> tatonnement process, which takes place at a certain locale and encompasses all markets at the same time. The process is surely fanciful[60].

The above <u>objections</u> refer to the tatonnement process. <u>With respect to non-tatonnement</u>:

(i) All analyses of stability with disequilibrium transactions assume (implicitly) that all individuals remain in the market for the duration of the equilibration process. This may be called here the assumption of "perfect market attendance". The question immediately arises, with regard to this assumption, why individuals remain in the market during the whole equilibration process if, at the same time, they are

---

[59] (1946). See also Chapter II, *infra*.

[60] One may nevertheless accept this difficulty, along with the difficulty associated with "auctioneer".

allowed to trade at disequilibrium prices. The assumption relevant to this question is that price expectations of individuals in non-tatonnement are static, and are held with certainty[61]. But, if this is the case, there is contradiction between this and the assumption of perfect market attendance. Why would anyone stay in the market after having traded, even though he expects prices to remain constant? Consider, for example, a Hahn process of exchange. Let an individual be at the "short end" of the markets given the price vector at a point in time; then, it is questionable that this individual has any incentive to remain in the market under these conditions.

But if the possibility of exit of individuals from markets in disequilibrium is recognised, it is obvious that the non-tatonnement analysis has to be modified along lines similar to those hinted at by Wicksteed[62]. This has not been discussed in the literature.

It is of course possible to suggest that individuals remain in the market because of the incentive of speculation. But then, some assumption other than static expectations should be introduced to make speculative incentives operable, and

---

[61] See Arrow and Hahn, (1971)

[62] (1933). See also Chapter II, <u>infra</u>.

again the analysis of equilibration in that case will not be the same as that which has been offered in the works discussed. Neither will it apply to all markets, since not all markets abound with speculators.

(ii) The assumption of perfect attendance is responsible for endowment effects of the second kind becoming operative during equilibration: if an individual changes his endowment at disequilibrium prices and leaves the market, the effect on aggregate demand from the re-evaluation of his endowment due to a subsequent price change will not materialize. Thus, it is seen that the quantity-price effects have been given unjustified importance in the analysis, due to the implicit assumption here criticised.

(iii) Ironically, while most neoclassical analysis has not considered effects of income redistribution, the non-tatonnement analyses have focussed on it. As it turns out, redistribution effects in non-tatonnement exchange may not even materialize in any operational sense, as has been argued in (iii) above. Moreover, because the non-tatonnement analyses have been confined to pure exchange models they have completely ignored pure quantity effects, which, as the analysis in this investigation shows[63], are more important than either those of the second or the first kind, in an analysis of disequilibrium

---

[63]In Chapter V.

in production and exchange.

(v) Finally, among the positive points raised for the analyses of non-tatonnement was the claim that it turns out to be "more stable" than the tatonnement process, at least under certain specifications. The question is, however, whether this stability is at all useful in terms of information relevant for comparative statics theorems. With respect to this question, it may be remarked that if a stable non-tatonnement system is perturbed when in equilibrium it is only by chance that it will return to the same equilibrium. This is of course due to the fact that the initial endowment matrix is not uniquely associated with any equilibrium price vector in non-tatonnement processes. But if more than one price ratio is associated with a specific cluster of parameters nothing can be said on the effect that changes in those parameters will have on equilibrium prices. Thus, despite the strong results on stability of non-tatonnement, "meaningful" theorems will still have to be deduced from traditional tatonnement processes, where actual transactions are assumed to play no role in the determination of equilibrium.

The evaluation of contributions in the Walrasian tradition is now complete. The next Section in this Chapter is devoted to examination of attempts at analysis of disequilibrium in both production and exchange.

3.2 "Non Walrasian" Attempts at Disequilibrium Analysis

This Section critically reviews a number of different contributions relevant to disequilibrium in production and exchange. The term "Non-Walrasian" is here used for classificatory convenience only, that is, in order to distinguish the material of this from the works discussed in Section 3.1.

3.2.1 The Cobweb Model for a Single Market

The literature on the cobweb model[64] seems to have been concerned with questions rather different from those addressed in the present investigation. This will become obvious from the remarks to be made in this Section. Nevertheless, analysis of a non-durable commodity model of production and exchange, where production is carried out "in disequilibrium, has the same mathematical structure, in reduced form. This analysis is carried out in Chapter IV; the aim of this Section is to briefly discuss the traditional "cobweb" model as a point of departure for that analysis.

As it is well known, the cobweb model is usually specified with quantity supplied dependent on the price of the previous market period. The equilibrium condition of this model then imposes equality of the quantity supplied and the quantity demanded in each market period. Since the quantity supplied is determined by the price in the previous period, the market period supply curve

---

[64] See Ezekiel, (1937); Buchanan, (1939); Samuelson, (1947); Nerlove, (1958); Ackerman, (1957).

is in effect a vertical line in price-quantity space; there is no room for variation in the quantity supplied during the market period. Hence, in order for equilibrium to be established, the quantity demanded has to vary with price change in order to clear the market. On the basis of this condition a difference equation in price is derived from which the stability condition is deduced that the slope of the long-period supply curve with reference to the price axis must be less in absolute value than the slope of the demand curve with respect to that axis.

The analysis briefly discussed above is conducted without any mention as to what happens during equilibration in each market period. Given the fact that the equilibrium price for each such period is determined by solution of the market demand and given market supply equations, it is reasonable to assume that the process of equilibration in the market period follows the tatonnement process. But then it follows that stability of this process requires a downward-sloping demand curve for the commodity.[65] This requirement is to be found nowhere in the cobweb literature.

The following points may be added with respect to the cobweb model:

(i) The cobweb model can legitimately be considered

---

[65] Given that the market supply curve is vertical.

as one of production and exchange, with "disequilibrium" in the production side. The model thus belongs to the first category of disequilibrium production and exchange models, as discussed in Chapter I.

(ii) The conventionally derived stability conditions for the cobweb model are not sufficient for its stability: it is also required that the demand curve slope downward at all times.

(iii) It is possible to discuss a model of similar structure, with a non-tatonnement process in each market period. An attempt will be made in Chapter IV.

(iv) It is possible to introduce inventory considerations in the structure of this model. This is done in Chapter IV.

In summary, it is here claimed that the analysis of the cobweb model in the literature has not only missed an essential stability condition involved in the cobweb process but also the opportunity to examine this model in terms of its implications for disequilibrium in production, and in production and exchange. This gap will be filled, it is hoped, by the analysis of Chapter IV of this thesis. In the process, some answers will be obtained to the basic questions asked in this investigation, with reference to a partial analysis model of production and exchange.

3.2.2 Analyses of the Cycle

The discussion in this Section refers to such models as have been developed by Hicks[66], Samuelson[67], and the subsequent literature on mathematical models of the business cycle.

The models of this family have properly been labelled[68] fix-price models, since prices are assumed constant throughout with quantities taking the brunt of equilibration process. In terms of the present investigation, the relevant question to ask is whether the contributions under discussion are indeed full analyses of disequilibrium, in the sense that they contain a description of how the system tends towards its state of "rest", both in the market period and in the "longer-run".

In order to answer this question, consider, for example, a fix-price model of the aggregate economy from the family of multiplier-accelerator models.[69] Aggregate expenditure is the sum of consumption and investment at fixed prices; its components are found to depend on incomes of both the present and the previous period. Since production and income are identical,

---

[66](1950).

[67](1939).

[68]By Hicks, (1965).

[69]Samuelson, (1939). Strictly examined, such models are one-commodity, as has often been pointed out.

expenditure in fact depends on production volumes of both the present and the previous periods. Moreover, without inventories, commodity supply in each period equals commodity production.

Consider, now, the condition that is invariably imposed on such models in order to derive their dynamic equations. This condition is that expenditure equal production in each and every market period. In a fix-price model it is not possible to have shrinkage or expansion of the expenditure components via price change; thus, the only other degree of freedom is the income (production) level. Moreover, the consumption component of aggregate demand in each market period is indeed fixed and unchangeable, since it depends on income created in the previous period.

How is the equilibration process to be examined under such circumstances? Since production is required to equal demand, and since the latter partially depends on the former, it is possible to interpret the model by saying that it achieves equilibrium in the market period by means of a sort of "tatonnement in quantity" rather than in price. That is to say, the production volume may be experimentally varied, and the consequent situation examined from the point of view of clearance of the market. The level of output that produces a demand equal to it is selected, and market period equilibrium is achieved. In the next market period, it will be necessary to repeat this process and find the new[70] production level at which demand equals supply.

That the variation of quantity of production within the market period may indeed be interpreted to be in the nature of a tatonnement process in quantity I do not think it can be doubted. The family of models discussed here then resemble, in terms of equilibration processes, the cobweb model: in the latter, price is "wrong" outside the stationary state, and volumes of production dependent on this price are also "wrong". During the market period, demand is brought to equal supply by sufficient price change, in a tatonnement process. In cycle models of the category discussed the market period tatonnement is in terms of quantity produced.

Other interpretations of cycle models[71], in fact formally equivalent to the one advanced above, are possible. Following Hicks[72], for example, one may effectively assume perfect foresight on the part of producers. As a consequence, no tatonnement process is now required during the market period, since by means of the assumption of perfect foresight the

---

[70] The output volume will change unless the system is in its "stationary state".

[71] I owe this point, and the associated reference to Hicks, to my advisor B.L. Scarfe.

[72] (1959).

system achieves "temporary equilibrium" during this period, without auctioneering. Another interpretation would be to assume that producers have full knowledge of the circumstances of the market, a la Marshall.[73] It is obvious that both alternative interpretations achieve, in fact, the result of a "quantity tatonnement" without disequilibrium actions permitted, as has been argued above.[74]

There is one additional remark to be made with regard to the models of the cycle discussed in this Section. Such models totally ignore considerations of input markets, and the consequent possibility that some production levels may be unfeasible at current (and fixed) prices of inputs and outputs. These models thus not only by-pass the problem of market-period equilibration and the consequent possibility of disequilibrium in that period, but they also do not take into account the possible unfeasibility of output levels. The model developed in Chapter V of this investigation takes

---

[73] See the discussion on Marshall, in Chapter II.

[74] In a forthcoming book (to be published in 1976) B.L. Scarfe has a fix-price model with inventories which takes explicitly into account the real possibility that (without tatonnement or perfect foresight or perfect knowledge) supply and demand in the market period may not be equal. Inventory accumulation or decumulation then takes place, which in turn affects income and expenditure levels. While the model is one of genuine disequilibrium, it is interesting to observe that in fact its formal structure can be collapsed to that of the Metzlerian model of the inventory cycle. Thus, it may be that despite ignoring market period problems, some formulations of cycle models may not be all that far from proper disequilibrium analysis. This point is certainly worth investigating.

explicit account of both problems.

It may perhaps be thought that the present interpretation of cycle models is too harsh; but examination of the original works cited above will show that, in actual fact, the contributors to the trade cycle literature cited do not seem to be aware of the implications of assuming that, in each period, supply is set to equal demand.

3.2.3 "Spillovers" in the Theory of Multi-Market Equilibration

The concept of spillovers from the one market to the other in a general equilibrium system during its process of equilibration is due to Patinkin[75] who credits Friedman for pointing out to him the possibility. According to this idea, the pressure on the price of a particular good to change should come not only from the excess demand within that market, but also, from excess demands in all other markets of the system. In the original work in which Patinkin suggests the reformulation of the dynamic adjustment mechanism to take into account the spillover he uses the argument to suggest a generalization of the Samuelsonian formulation of the Walrasian adjustment mechanism which, as we saw above, is a *tatonnement* process. In the same paragraph, however, we find Patinkin suggesting that the spillover effects are due to frustrated de-

---

[75] (1952).

mand from other markets where it has been impossible to satisfy it. Thus,

"The pressure on the price of say, 'shoes', comes from two sources: first, the buyers who have not succeeded obtaining all the shoes they wish to purchase at the given prices; second, those who have not succeeded in spending all they intended to (at the given prices) on clothing, and redirect part of this unspent income to the new market." [76]

And, a little later,

"This type of market pressure has become a familiar phenomenon of the post-war period...clearly this characteristic pressure of semi-permanent disequilibrium should also be taken into account in the disequilibria studied in dynamic analysis. And one possible way of doing so is to generalize in the manner indicated above." [77]

Patinkin thus seems to be unclear about the fact that his proposed generalization of the Walrasian adjustment mechanism (which would make the rate of change of each price dependent on the excess demand in all markets) does not really describe the phenomenon of unsatisfied demand in some markets

---

[76] D. Patinkin, <u>Ibid.</u>, page 41.

[77] D. Patinkin, <u>Ibid.</u>, pages 41-42.

spilling over others. Without transactions outside equilibrium, and with perfect market attendance, there seems to be no logical reason for spillovers in the excess demands of the various markets while the process of equilibration is taking its time. This is so because individual participants in exchange are not even given a chance to find out whether their optimum plans are feasible given the conditions in the market. On the other hand, in a process where disequilibrium transactions are allowed the notion of spillovers is legitimate but the mechanism of the price adjustment in this situation is different from that of Walrasian tatonnement, as generalized by Patinkin. This is so because in a non-tatonnement process we must have, as mentioned above, some rules governing transfer of commodities among individuals at disequilibrium prices, as well as perfect attendance.

In any case, Patinkin did not carry the idea of spillover further into his analysis nor did he make any use of results from it. Thus, this lack of clear delineation of the conditions under which the spillovers are meaningful did not have any consequence for his work.

An attempt at synthesis of the spillover concept with the "dual decision hypothesis" of R. Clower[78] has been made by

---

[78] (1965). Clower's contribution is discussed in 3.2.5.

H. Grossman[79]. Grossman remarked that while Patinkin provided the concept of spillovers, he gave no analysis of the determination of its magnitude or its effects. On the other hand, Clower's "dual decision hypothesis" provides an analytical framework for analysis in one market when another is in disequilibrium and therefore actual transacted quantities differ from planned ones[80]. Grossman then proceeded to generalize the Clower framework in a multi-commodity model, and to incorporate the spillover coefficients suggested by Patinkin.

In the process of determining the effective constraints imposed on individuals by disequilibria in some markets, Grossman found himself on difficult ground: which actual transactions in which markets are to be considered first, and which later? Is the disequilibrium in market A to affect transactions in market B or vice versa? Grossman assumes, in effect, that actual transactions in all markets are simultaneously determined, and so are effective demands in these markets (the latter depending on actual transactions)[81]. He recognises that this assumption in fact implies "dissemination of information" about actual transactions, and confesses that such information and the consequent determination of

---

[79](1971).

[80]Cf. *ibid.*, pp. 948-949.

[81]*Ibid.*, pp. 952-953.

relationships between effective demands of individuals and in the aggregate)

> "could only emerge from a recursive process involving a sequence of individual and market experiments. The present paper considers a special case in which this recursive process is collapsed into an instantaneous interaction." [82]

But this set of assumptions seems to deny the reasons for which spillovers happen, and indeed almost implies a tatatonnement process of a sort. Consider a multimarket system in disequilibrium: the assumptions suggest that individuals are able to estimate the feasible transactions that they can undertake in each market, simultaneously for all markets. But, the amount of transactions that would be feasible in each market depends on the amount of spillovers present in the system. Thus, in order to determine spillovers, one needs their values. This is obviously circular, and for this reason the collapsing of the process into an instantaneous interaction is a necessary and vital assumption of Grossman's analysis.

But if the above remarks are correct, the process analyzed by Grossman could be viewed as a "tatonnement" in terms of what is feasible in each market during disequilibrium. It is in this sense that the Grossman contribution may be viewed as an extension of the Walrasian tatonnement. The question of

---

[82] Ibid.

the actual sequence of transactions, it seems, is important if spillovers are to make sense in disequilibrium analysis.

3.2.4 D. Patinkin on Involuntary Unemployment

In his discussion of the concept of involuntary unemployment[83] Patinkin attempted a sort of disequilibrium analysis of the aggregate model in the framework of his model. His starting point was that when a firm cannot sell the profit-maximising quantity of sales in the market for some time, it will eventually have to "take some step to bring current output -- and consequently current input -- into line with current sales. And this is the beginning of involuntary unemployment."[84]

This remark clearly implies that, at least in disequilibrium, labour demand is sensitive to quantities (of inventories) apart from prices. The same point was made later by Clower, in a more general context[85]. Patinkin, however, did not formally introduce a modification into the functions of his model to take this into account. Thus, in the account of the story by Patinkin[86] firms cut down on production volume

---

83 (1965), Chapter XIII.

84 *Ibid.*, p. 318.

85 See Section 3.2.5 below.

86 (1965), p. 318 ff.

when they find out that they cannot sell expected quantities. No discussion follows of how the excess inventory that has accumulated during the previous periods of excess supply is to be corrected. Neither is mention made of the fact that the adjustment in production to bring it into line with current sales is not enough. Moreover, it is difficult to rationalize why, in the price-taking environment of Patinkin's model, firms would find themselves with accumulated inventories rather than sell at lower prices.

In Chapter IV of this investigation an analysis of a price-setter holding inventories over time suggests that it is optimal for him to adjust both price and production volume in response to a maladjustment in inventory. Thus, a formal justification is provided for modification not only of the excess demand functions (to incorporate quantities as well as prices) but also for the view that both quantities and prices would be adjusted in these circumstances[87].

It is thus seen that the point raised by Patinkin is taken up to some extent in this investigation, with some modifications[88].

---

[87] In a situation of oligopolistic rivalry and the consequent uncertainty the price change may indeed be quite low. The model developed in Chapter IV concentrates on a price setter not experiencing interdependence in the market for his products.

[88] See Chapter V.

3.2.5 Clower's Concepts of "Notional" and "Effective" Demands

It may be fair to state at the outset of the discussion that Clower's contribution to the fundamental question of disequilibrium behaviour is one of the most important: this is because it was he who first provided a specific suggestion for modifying planned functions so as to make them relevant for behaviour outside equilibrium.

In an early contribution to the matter[89], Clower concentrated on the fact that when a market is in disequilibrium only the short end of it can in fact be satisifed. Thus, while both the quantity supplied and quantity demanded at any price represent planned magnitudes, chosen as optimal given the price, only the short-end quantity (i.e., the quantity demanded in a situation of excess supply, and vice versa) can be realized in disequilibrium. This, of course, is the basis for the Hahn process of exchange in non-tatonnement, described earlier in this Chapter. Clower used this point to advance an interpretation of Keynesian economics different from the commonly accepted one.

Later[90], Clower returned much better prepared with his interpretation of the Keynesian method of analysis. He finds that traditional analysis gives no information about realized

---

[89] Clower, (1960).

[90] Clower, 1965.

("effective") as distinct from planned ("notional") transactions in disequilibrium, and in general it assumes that actual transactions do not influence the equilibration process. In other words, the pressure on the price to equilibrate does not come from realized disequilibrium transactions in conventional analyses of disequilibrium. A special case where this holds is that of the tatonnement process, where no disequilibrium transactions occur at all.

In Clower's view, Keynes disagreed with precisely this aspect of traditional theory, namely that it is not a general theory of disequilibrium states. Thus, he isolated a number of points on the basis of which he conducted his criticism. One of these points was the relevance of Walras' Law in the process of price formation (i.e., as a force acting outside equilibrium). Keynes, according to Clower, may be interpreted as either having accepted the law, in which case the modern "neo-keynesian" interpretation of his contribution is correct; or as having used a more general theory of disequilibrium price formation, in which case his contribution should indeed be examined in a different light from traditional interpretations.

This investigation is not concerned with matters of exegesis in the history of economic analysis, except insofar as they are immediately relevant to the aims spelled out in the introduction. For this reason, Clower's discussion from now on will be conducted in abstraction from the points of interpretation concerning Keynes' work.

In an attempt to construct a model in which quantities as well as prices enter the behavioural functions Clower formulated the "dual decision hypothesis" according to which actual behaviour is the same as planned if other variables (such as income, for example) are not falsified. If such variables are falsified, however, the individual must be assumed to proceed to optimization *given* the realized values of these variables. Clower carried out this discussion in terms of the labour market (assuming it to be in a state of excess supply) and suggested that unemployed individuals would have to re-evaluate their positions in the commodity market in accordance with the dual decision hypothesis. As a consequence, the *effective* demand in that market may reasonably be expected to be less than the demand calculated on the basis of planned functions. Now, Walras' Law suggests that the excess supply in the labour market will, in a two-commodity model, be reflected with excess demand of equal magnitude (at current prices) in the commodity market. While Clower does not in any way suggest that Walras' Law does not hold for planned demands the fundamental question raised is whether it is reasonable to take this planned excess demand as an indicator of equilibrating pressure in the commodity market under the circumstances.

The claim in this respect is, than, that planned ("notional", as called by Clower) excess demand may not be operative at all in terms of pressure on prices outside equilibrium. The total of realized excess demands, on the other hand, may not be zero. That is, it is possible to have excess realized supply in one market which is not reflected in excess realized demand in other markets. This, of course, does not mean that Walras' Law does not hold under the circumstances, since the Law only refers to planned, not realized excess demands in a system.

Clower's main point then is that once income appears as independent variable in the market excess demand functions -- or, more generally, once transactions quantities at false prices appear as arguments -- traditional price theory ceases to shed any light on the dynamic stability of a market economy[91].

The above points do not in any way imply instability of the system; the mere fact that realized excess demands do not obey Walras' Law is certainly not sufficient for that. However, the analysis of equilibration in terms of planned rather than realized functions is dealt a rather serious blow with this contention.

Clower concentrates, in his example, on the case of

---

[91] Compare Clower, (1965), p. 123.

excess supply of labour and its consequence, namely falsification of planned labour income. Without analysis for the case of excess demand for labour, however, he slips into saying that orthodox tatonnement analysis is valid when labour is on its supply curve[92]. The implication of this statement is that as long as labour is on its supply curve all planned incomes are realized, hence planned excess demands are indeed relevant as indicators of disequilibrium pressure. This, however, is not correct. For when labour is in excess demand, planned commodity production exceeds the realized magnitude, since planned labour demand exceeds realized amounts of labour hired. But this implies that planned commodity excess supply is in excess of realized excess supply, and Walras' Law is again not relevant for description of the actual disequilibrium situation.

Finally, Clower does not build an explicit model with which to consider pure disequilibrium analysis of an economy under his own formulation of excess demands dependent on quantities: he therefore does not derive stability theorems (or any other theorems) in his otherwise important work.

In Chapter V of this investigation, the point made by Clower regarding quantities as arguments in excess demands is

---

[92]That is, no incomes are falsified in this case. As will be showed below -- and in more detail in Chapter V, some incomes will always be falsified outside equilibrium, hence the statement is false.

combined with a Hahn(constrained)process of exchange and with Edgeworth's scheme of sequential equilibration. The result is an explicitly dynamic disequilibrium model of production and exchange. In formulating the model and examining its properties, a number of slips in Clower's discussion (such as the one mentioned above) are corrected. When the model is extended to include exchange money and commodity inventories, Patinkin's contribution, as formalized in the work of Chapter IV, is brought in, too.

3.2.6 Barro and Grossman: Integration of Patinkin's and Clower's Contributions.

Barro and Grossman[93] note that Patinkin is interested in the effects that deficient commodity demand has on production and unemployment, while Clower is concerned with the effects of unemployment on effective commodity demand. The possibility for integration is obvious, after this rather incisive remark. Barro and Grossman proceed to an informal integration of Patinkin and Clower, and they produce a scheme in which the relation between planned and realized excess demands is brought out. They consider, for example, the case of excess realized supplies in both the labour and commodity markets, as well as the case of realized excess demand in these markets.

The main purpose of Barro and Grossman is to give a

---

[93] (1971).

method of analysis which shows the situation in all markets, given a particular (constant) price vector. Given the situation, implications exist for disequilibrium pressure on prices. However, Barro-Grossman do not explicitly investigate these implications in a dynamic model[94]. Thus, the claim in their introduction that their purpose is to "develop a generalized analysis of both booms and depressions as disequilibrium phenomena"[95] can at best be interpreted to mean that they describe instantaneous situations of excess demand, and others of excess supply, given a constant price vector. It thus seems that the claim in their introduction should be taken with a grain of salt.

The main reason why Barro and Grossman have not analyzed dynamic adjustment in disequilibrium seems to reside in the fact that their work lacks explicit modelling of the disequilibrium process in dynamic terms.

3.2.7 Summary and Evaluation of the "Non-Walrasian" Disequilibrium Strands of Analysis

A summary and evaluation of the works discussed in this Section (3.2) may be useful, in view of its length. The summary of works is given first and an evaluation is attempted below, in the light of the aims of this thesis.

---

[94] Compare *ibid.*, p. 84.

[95] *Ibid.*, p. 83.

The literature on the cobweb model seems to have implicitly assumed tatonnement in exchange during the market period. It has not been noted, in this respect, that an additional stability condition is required. The cobweb model can be made one of genuine disequilibrium in both production and exchange, and inventories can and will be introduced explicitly in Chapter IV of this thesis.

Conventional cycle models involve, like the cobweb, some kind of tatonnement process or equivalent assumptions in the equilibration process for the market period. They also ignore supply side considerations. They will not be followed up in this thesis.

Patinkin's concept of spillovers has no use in the tatonnement process; it is obviously applicable in non-tatonnement, with plans not fully realized in some markets. Patinkin does not seem to have been clear about this, while the use of the concept by Grossman suffers from his collapsing into an instantaneous process what is in effect a sequential phenomenon.

Patinkin's attempt at analysis of involuntary unemployment due to deficient commodity demand was hampered by his model, which is one of the best representatives of modern neoclassical systems of Walrasian parentage. However, the idea of involuntary unemployment as a disequilibrium phenomenon is worthy of attention. This idea is in part formalized in Chapter IV in the framework of a price setter with inventory, maximizing profits over time. Further, the formalized dependence of labour demand and price on the inventory position of the firm (in Chapter IV) is used in the analysis of Chapter V for description of situations of general disequilibrium.

Clower's "notional" and "effective" demands are by far the most important of the ideas discussed in this Section. His point that in disequilibrium it is possible that planned excess demands may not be relevant as forces acting on the system is also of great value as a suggestion of the reasons why a theory of behaviour outside equilibrium must be sought. The analysis of disequilibrium in Chapter V incorporates Clower's basic points (with modifications to take care of a slip or two) and builds on them.

Finally, Barro and Grossman have succeeded in showing, in a scheme interrelating Patinkin and Clower, the relation between planned and effective magnitudes in various markets given a disequilibrium price vector which remains constant.

3.3 Closing Remarks and Main Conclusions on the Modern Work In Market Equilibration.

This chapter has reviewed and critically interpreted modern contributions in the general subject of market equilibration. Two main strands have been detected and dealt with separately, the "Walrasian" and the "non-Walrasian" one.

The main conclusion that seems to emerge from the review and analysis of the "Walrasian" strand seems to be that although it has advanced knowledge in some sense, it nevertheless suffers from basic criticisms and weaknesses that seem difficult to take care of: the assumption of perfect market attendance; the effective exclusion of Giffen goods from the models analyzed; the troublesome concept of the auctioneer in view of the widely held belief that the perfectly competitive price system solves the problem of allocation costlessly; the lack of comparative static theorems once disequilibrium transactions are allowed.

With regard to the "Non-Walrasian" group of works discussed, the general impression created seems to be that some of them provide a much less formal but also richer framework for analysis of disequilibrium: thus, Patinkin's involuntary unemployment; Clower's notional and effective demands; the cobweb model as one of disequilibrium analysis of production in a partial setting; seem to be much more directly relevant to analysis of actual equilibration processes. As has already been mentioned, the next two Chapters of this thesis will be building upon these ideas in an attempt to handle the problem of disequilibrium analysis.

CHAPTER IV

PARTIAL DISEQUILIBRIUM ANALYSIS OF PRODUCTION AND EXCHANGE

The critical summary and interpretation of the literature in the subject of equilibration of economic models, which was the subject matter of the two previous Chapters, has made it clear that there is little by way of analysis of production and exchange in which trading is permitted out of equilibrium, i.e., when the non-tatonnement process is considered.[1] One of the purposes of the present Chapter is to provide an introduction to the special problems of non-tatonnement analysis when production is admitted into the model.

The scope of this Chapter is partial analysis involving production and exchange. "Partial" should be taken to mean that the analysis concentrates on one market, with consequent neglect of the general interdependence present in the system as a whole. Section 4.1 provides a description of the analytical setting used in this Chapter, and also points out factors typically neglected in partial analysis. Since the next Chapter in this investigation, Chapter V, deals with general analysis of equilibration the remarks as to the abstractions used in the partial analysis here also

---

[1]The reader may recall that theorems on the non-tatonnement process involved a pure exchange economy.

serve to outline, in part, the scope of the general analysis in the next Chapter.

The main body of this Chapter consists of Sections 4.2, 4.3 and 4.4, in which analysis and discussion of simple production cum exchange models is conducted. The analysis in these Sections attempts to provide answers to three basic questions, some of which have already been asked in the analysis of pure exchange. The questions refer to the stationary states of the models as well as to their equilibration processes and could be briefly formulated as follows:

a. Does the stationary state of the tatonnement model change when non-tatonnement is considered, and if so how?

b. Does it become more difficult to satisfy stability conditions under non-tatonnement equilibration?

c. How does the out-of-equilibrium behaviour of the model change when non-tatonnement equilibration is taken into account?

An attempt to answer these questions is made in Sections 4.2 and 4.3, for simple models of price-taker producers and for the cases of durable and non-durable commodity.

The introduction of production and consequently of inventory considerations necessitates some analysis of the way in which inventory positions may modify production plans and input demand functions of producers. Analysis of this problem, and derivation of a theorem on price, output and inventory policy by a price setter with sufficient knowledge of his demand conditions is carried out in Section 4.4. This theorem is found helpful not only in partial but also in general analysis of equilibration, in Chapter V[2]. This is another sense in which the present Chapter serves as a bridge to the following one.

The general tenor of the conclusions to the analysis of this Chapter is that introduction of non-tatonnement considerations in a partial analysis setting does not seem to modify the stationary state of the model of tatonnement. Stability conditions are more difficult to satisfy than in the tatonnement process, at least in the simple formulations used in this Chapter. Since it is found that the tatonnement process is always stable when the non-tatonnement is, comparative statics as known will still hold. Finally, the out-of-equilibrium behaviour of the models is changed in important ways when non-tatonnement is introduced.

4.1 Partial Disequilibrium Analysis: the Setting and the Assumptions; Endowment Effects; Abstractions

This Section contains a brief description of the analytical setting of partial disequilibrium analysis to be used in this Chapter, as well as the main general assumptions employed in the models used. The relationship of

---

[2]Section 5.3.d.

these simple partial models to the general model described in the introduction is given. Moreover, a short discussion of endowment effects as they operate in this framework is included. Finally, attention is drawn to basic abstractions of this analysis from questions of interdependence between production and demand; and from questions of unfeasibility of production plans of producers and employment plans of individuals.

(a) Setting and assumptions. Briefly, the analysis in this Chapter confines itself to one commodity, all other commodity prices and quantities held constant in the process. Input markets are abstracted from, and input prices are held constant also; it is assumed that inputs are always available in sufficient quantity to permit production plans to be realized.

In line with conventional partial analysis, production volumes depend upon unchanging cost conditions, demand conditions and stock positions of producers.[3] These determinants of production will be briefly considered in the following paragraphs.

---

[3] In the model presented in the introduction, inventories of all commodities are assumed to be held by individuals. Here, in contrast, they are held by producers. A reconciliation of this apparent discrepancy may be achieved by assuming that although ownership of stocks of the commodity discussed is with the individual agents, management of production proceeds by taking inventory volumes into account nevertheless.

Cost conditions must remain unaltered if input prices and technology are assumed constant, and this is in effect the assumption of the analysis below.

"Demand conditions" refers to information on revenue possibilities. In the model employed in this Chapter, information on demand conditions is based on prices, which are expected by producers with certainty (point expectations). Moreover, expectations of prices are static, that is, are given by

$$P_{et} = P_{t-1}$$

where P_{et} is the expected price at time t and P_{t-1} is the actual price at time t-1. Demand conditions (as expressed in prices) are therefore expected to repeat themselves without change.[4]

---

[4]A simple defence of this assumption is that it is a natural first approximation to reality. However, one could also point out that there exist results which suggest that static expectations hypotheses are not more favourable to stability than more complicated expectations functions: for example, Enthoven and Arrow (1956) have shown that if a multimarket exchange system is stable with static expectations it will also be stable with a simple formulation of extrapolative expectations as long as adjustment speeds of prices are sufficiently low. Moreover, Arrow and Nerlove (1958) have proved that a multimarket exchange system with adaptive expectations, irrespective of elasticity of expectations, is stable if and only if the system with static expectations is stable. Negishi (1964) has in turn proved stability of the multimarket exchange system with adaptive expectations in a non-tatonnement process, irrespective of elasticity values. And, finally, Nerlove (1958) has shown that a linear variant of the model of Section 4.3 of this investigation with adaptive expectations is stable under less stringent circumstances than those required when static expectations are assumed.

The assumption that expectations are certain does not guarantee that they are correct; thus, in the model of this Chapter, expectations are incorrect in each market period except when "full equilibrium" prevails. As a matter of fact, "full equilibrium" or the stationary state in the model of this Chapter cannot obtain unless the market clears within the period at the expected price. Another way to express this is to point out that, in disequilibrium, planned sales by producers (at the expected price) are not equal to realized ones. This discrepancy between planned and realized sales is the main disequilibrium feature of the models of this Chapter.

Inventory positions in the commodity produced are relevant as determinants of production levels in cases of both production to stock and production to order, and irrespective of whether or not the commodity is durable. In the case of production to stock, if the commodity is durable it is here assumed that a positive inventory level is desired by the individual firm for purposes of satisfying the speculative and/or the transaction motives. This inventory level is assumed to be given or may vary with price. When the commodity is non-durable and production is to stock, the desired inventory level at the end of each market period is set equal to zero (since positive inventory holding beyond this period is excluded by definition in this case).

When production is to order, the inventory position may be indicated by the volume of orders outstanding at each time (order backlog), in both the case of durable and non-durable commodity.

Given that production to stock is generally larger than production to order, the analysis below will concentrate on this case.[5]

With the discussion of the determinants of production plans now complete, a number of other features of the models discussed in this Chapter may now be mentioned.

The market structure in most of the following analysis is that of perfect competition. However, when behaviour of price takers out of equilibrium is discussed an element of price-setting power is found not only admissible in the analysis of market disequilibrium but also desirable, for reasons to be discussed below. Thus, a small portion of the analysis is conducted within the market structure of monopoly (Section 4.4 below).

Finally, the time horizon of disequilibrium models of production and exchange is naturally longer than the single market period of the pure exchange model, for reasons to be

---

[5]Since the backlog of orders is basically a stock variable (the same as inventory level is for production to stock) generalization of the discussion in the text to cover the case of production to order seems possible. Such generalization is not undertaken in the present investigation, however.

discussed below in this Section.

The description of the setting of the analysis above has, it is hoped, made clear the relationship of the models to be used in this Chapter to the general model presented in Chapter I: this Chapter sets all commodity prices but one constant, and the analysis concentrates on one market, as in the example given in Chapter I.

(b) Endowment effects in partial disequilibrium analysis. The analysis in this Chapter is one of production and exchange in disequilibrium. As mentioned in Chapter I, it is reasonable to proceed on the assumption that production preceeds exchange, based on some expectations as to future demand conditions. If this is the case, however, the possibility of "mistaken" production volumes arises. Thus, introduction of production considerations in disequilibrium implies not only "false" transactions in input and commodity markets but also false production volumes, in the sense that they cannot be sold at the expected ("planned") price.

The discussion in Chapter I distinguished three categories of disequilibrium models of production and exchange. In terms of that classification, the models of perfect competition in this Chapter are pure price adjustment models. The "false" production volume is put on the market and the price must vary sufficiently to clear it.

The model of the monopolist discussed in Section

4.4, however, presents the case of a mixture of pure price and pure quantity adjustment. A fuller description of it must await for Section 4.4 below.

Given the above description, and the discussion of the general setting of the analysis of this Chapter, endowment effects as they operate in this setting may be briefly mentioned.

Pure price endowment effects will operate whenever there is a change in price; therefore, they are relevant in the analysis that follows.

Pure quantity effects on demand can only operate when the link between production and demand is explicitly considered; it will be seen below that this link is ignored in partial analysis, hence production volumes which turn out false will be allowed to affect only production plans, but not commodity demand.

Finally, quantity-price effects are present whenever non-tatonnement in exchange is allowed. These effects will have to be taken into account in such cases, below.

(c) Abstractions from interdependence and feasibility. The above described setting of partial disequilibrium analysis obviously abstracts from a large number of phenomena. The discussion which follows singles out two fundamentally important abstractions, namely, the question of feasibility of production plans of producers and of "employment" plans of individuals; and the dependence of demands on production levels.

First, the abstraction of partial disequilibrium analysis from input markets ignores the possibility that either production or income creation plans may be unfeasible in disequilibrium.[6] More specifically, since input markets are not to be assumed in equilibrium all the time, production plans may not be feasible if there exists excess demand for inputs and no inventories of such inputs are available to clear the markets. On the other hand, in a situation of excess supply of inputs production plans are feasible but input employment (and the consequent income creation) plans cannot all be feasible. Such infeasibility of plans must reflect itself on demand.[7]

Second the effects of variation of production volume on incomes and demand (pure quantity effects) are ignored in the analysis of this Chapter. Thus, demand conditions are unaffected by such variations during disequilibrium. Demand is assumed to repeat itself, unchanged, from period to period, irrespective of production volume.

---

[6]Since planned and realized production magnitudes are equal, the convention of representing planned magnitudes with an asterisk is not followed here. Sales, however, are to be taken as realized sales, when the price at which they occur differs from the expected one

[7]To the extent that wealth stocks are used to sustain demand, this may not hold fully.

4.2 Price-Quantity Formation: Price-taking in the Case of Non-durables

This Section contains an analysis of market equilibration in the simple case of production of a non-durable commodity.[8] The model is set out first, and the simplest possible case of tatonnement in both production and exchange is discussed for reference purposes. Then, the assumptions of tatonnement equilibration in production and exchange are dropped one after the other and the consequences studied. The main questions asked in this Section have already been summarized in the introduction to this Chapter and they relate to the modifications that have (or do not have) to be made to the stationary state, the stability conditions the out-of-equilibrium behaviour and the comparative statics of the simple model analyzed here when disequilibrium is allowed in production and in both production and exchange.

(a) The structure of the model. Consider a perfectly competitive industry whose demand conditions may be

---

[8] In mathematical structure, the reduced form of the model of this Section when non-tatonnement is assumed in production is identical to the well-known cobweb model. See Ezekiel, (1957); Buchanan, (1939); Samuelson, (1947); Nerlove, (1958); Ackerman, (1957). The analysis of this Section, however, has a totally different scope from that of the above literature, namely, a comparative analysis of stability, stationary states and out of equilibrium behaviour of models with tatonnement and non-tatonnement processes. Moreover, the stability analysis offered in this Section for the non-linear variant of the cobweb model is not available in the literature.

represented by demand function which remains unchanged from period to period and is of the form:

$$Q_t^d = H\ (P_t) \qquad (4.1)$$

where Q_t^d is planned market demand for the industry's output at time t and P_t is the money price for the industry's product, while H_p will denote the derivative of the quantity demanded with respect to price.

It is legitimate to write the demand function in this simple form with prices of other commodities and inputs held constant, tastes also held constant and given the partial analysis assumption that variations in the industry's level of production do not influence the industry's demand for its product. The more general analysis of Chapter V will take some of the above effects into account. It is of course possible even in partial analysis to have endowment effects of the second kind present and the demand for the industry's product in any period will then also depend on the specific distribution of commodity endowments among individuals and the change in these endowments in the process of non-tatonnement in exchange.

On the production side, the representative firm in this industry[9] is assumed to face given production conditions

[9]The results of this Section are not changed in any important respect if the fiction of the representative firm is dropped, but the assumption is retained for the ease of exposition that it entails. Dropping the assumption implies

and to behave as a price taker in an input market where money wages are constant and all plans of the firm are realizable. The input markets, that is, are assumed to be in equilibrium during the partial analysis of this Section, in contrast to the analysis of the next Chapter where both money wage changes and labour shortages are taken into account in the analysis. Production in period t-1 takes the whole period to complete and is based on the price expected to rule in period t, P_{et}. In accordance with static and certain expectations, $P_{et}=P_{t-1}$. Production is to stock, which becomes available for sale in the next period, and may be written (taking into account the expectations hypothesis)

$$Q^{o}_{t-1} = F(P_{t-1}) \qquad (4.2)$$

where Q^{o}_{t-1} denotes aggregate production during period t-1. Since this becomes available for sale in period t, and since the commodity is here assumed to be non-durable the supply in that period is set equal to the production at t-1, i.e.,

$$Q^{s}_{t} = Q^{o}_{t-1} \qquad (4.3)$$

which implies an optimum inventory level constant at zero at the end of the market period for the non-durable case. This will be modified when the analysis is extended to a durable commodity in the next Section.

---

that the distribution of output of the industry among firms may be allowed to vary. Since the only relevant magnitude for purposes of the present analysis is the industry supply, the distribution does not make a difference to the results of this Section.

Finally, market equilibrium implies the condition

$$Q_t^d = Q_t^s \qquad (4.4).$$

The simple model of this Section, consisting of equations (4.1) to (4.4) above, is complete and the next steps involve analysis of it under various adjustment assumptions.

(b) Tatonnement in production and exchange. The simplest case of auctioneering in both production and exchange (t-t for economy of expression) will be briefly examined. The auctioneer must be assumed to conduct the iteration for the equilibrium price before any production takes place. Since the same price will be quoted to both sides of the market each time, the lag between production and supply is not relevant in the tatonnement process. The auctioneer may then be assumed to proceed on the basis of the rule

$$P_{t+1} - P_t = k\ (Q_t^d - Q_t^s) = k[H(P_t) - F(P_t)] \qquad (4.5)$$

where k is the speed of price adjustment, and the lag between production and supply has been dropped. Linearization of this equation around the equilibrium point yields[10]

$$P_{t+1} - (1 + kH_p - kF_p)P_t = 0 \qquad (4.6)$$

and the solution of this version is of the form

$$P_t = A\lambda^t \text{ where } \lambda = 1 + k(H_p - F_p) \qquad (4.7)$$

Stability requires that the absolute value of λ be less than

[10]Assuming that the equilibrium is at P=0. This involves a simple transformation of axes, which is useful in the stability analysis of this model via Lyapounov's second method, conducted in the Appendix to this Chapter.

unity, which requires that the following condition must hold:

$$-2/k < (H_p - F_p) < 0 \qquad (4.8).$$

Leaving the left-hand side of the inequality aside for a moment, the condition that the difference of the slopes of demand and supply be negative is the well-known one that the slope of the excess demand function be negative for stability. The left-hand side requirement is one on the speed. The system may be made stable by choice of small speed of adjustment if this is necessary, and this is of no consequence since in the tatonnement process here described the speed of adjustment is not a behavioural parameter but part of the rule on the basis of which the auctioneer operates.[11]

The stationary state price reached by the auctioneer is the solution to equation (4.5) after substitution of a uniform price for all P. Thus, the stationary state price is that which equates supply and demand in each period without necessity of price change between periods, since the above substitution yields

$$\bar{P}-\bar{P} = k[H(\bar{P}) - F(\bar{P})] = H(\bar{P}) - F(\bar{P}) = 0^{12} \qquad (4.9)$$

---

[11] It may be useful to mention that if a continuous adjustment rule were specified, the speed of adjustment would not be involved at all in the stability conditions, which would then simply be the right-hand side of (4.8). This was the case, for example, of the analysis of a single market in Chapter I (see equation 1.22).

[12] k is positive, and the product of k and $H(\bar{P})-F(\bar{P})$ is zero. Hence, $H(\bar{P})-F(\bar{P})$ must be zero.

where $\bar{P}$ is the equilibrium price. Comparative statics theorems follow for both price and quantity if the system is stable.

The above completes the tatonnement analysis of the simple model to be used in this Section. The analysis now proceeds with relaxation of the tatonnement equilibration assumptions and study of the consequences for the results of the model.

(c) Non-tatonnement in production. First, let the equilibration in production follow a process of non-tatonnement, while exchange still follows the tatonnement rules. This will be called the nt-t model, for economy of expression.

How is non-tatonnement in production to be combined with tatonnement in exchange? The production at time t-1 must be assumed to proceed on the basis of the price expected for the period t. Under static expectations, this price is the same as that of period t-1. In period t, production from t-1 becomes available for sale, and the commodity market model that results is exactly the same as one meets in the stability analysis of exchange, only that one commodity is involved here instead of many. The tatonnement process in exchange then solves, in each period, the equation

$$\bar{Q}^{o}_{t-1} = Q^{d}_{t} = H(P_t) \qquad (4.10)$$

where the bar over the supply variable denotes constancy of supply for period t in which exchange takes place.

The condition for stability of the tatonnement process in this simple case now has to be derived. The condition is of course obvious, and does not require formulation of the tatonnement process in mathematical terms:[13] it is simply that

$$H_p < 0 \qquad (4.10a).$$

Unless this condition is satisfied, the tatonnement process in exchange during the market period cannot lead to an equilibrium price, and questions referring to how the system equilibrates in the longer-run are obviously irrelevant.

Let it be assumed that condition (4.10a) is indeed satisfied. The system then equilibrates in period t by price variation sufficient to make the quantity demanded equal to the quantity supplied. If it happens that the price required to clear the market in period t is not equal to the price at t-1, expectations of producers as to their revenues are falsified, and this difference between planned and realized values of sales affects subsequent production plans. The price in period t is used for formulation of production plans for t+1, and a different level of commodity supply is made available in t+1 with the process described above repeating itself.

---

[13]The example worked out in Chapter I for the illustration of the correspondence principle is sufficient for this formulation. See equations (1.16L) to (1.22).

The paths of production and prices over time in this model may be obtained if equations (4.1), (4.2) and (4.3) are substituted into (4.4) thus yielding the equation of motion for this system:

$$H(P_t) = F(P_{t-1}) \qquad (4.11).$$

(i) Local Stability

Linearization of this equation around equilibrium yields[14]

$$H_p P_t = F_p P_{t-1} \qquad (4.12)$$

the solution of which is

$$P_t = A\lambda^t \qquad (4.13)$$

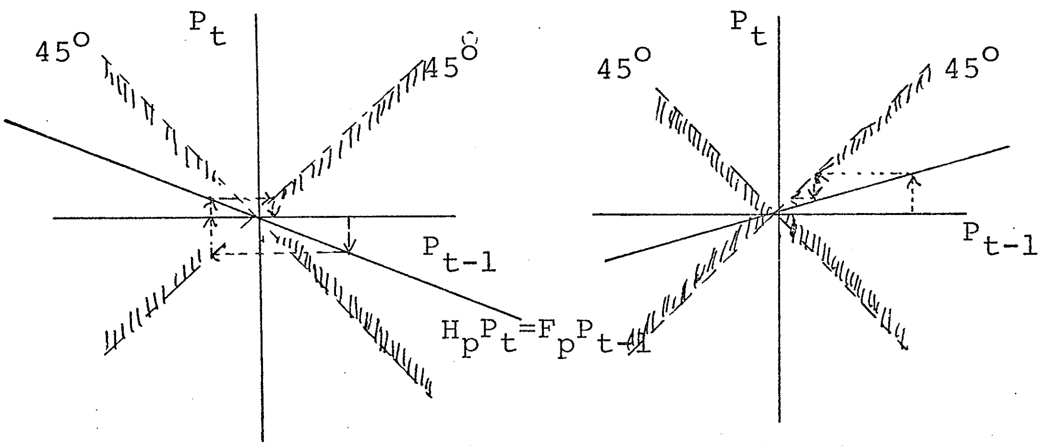
where A is the initial value of price and λ equals F_p/H_p. Stability is obtained as long as the absolute value of λ is less than unity.[15] Figure 4.1 illustrates the relationship between the prices in two consecutive periods indicated by equation (4.12). The area of stability of the system is contained in the shaded region between the two 45° lines in the Figure, since the slope of equation (4.12) should not exceed unity in absolute value. In Figure 4.1(a) it has been assumed that H_p and F_p are of opposite sign and as a

---

[14] On the assumption that the stationary state price is zero.

[15] Equation (4.12) is identical to the equation of the cobweb model. Results are not the same, however, due to the additional requirement imposed on the sign of H_p for stability in exchange.

consequence the slope of (4.12) is negative. The motion to equilibrium is then cyclical, as illustrated. If the responsiveness of both demand and supply to price are of the same sign the slope of (4.12) is positive and the approach to equilibrium is from one side, as in Figure 4.1(b).



(a) Figure 4.1 (b)

(ii) Global Stability

The technique of linearization around the equilibrium point allows derivation of conditions for stability in the small, since the linearized form of the equation of the system is an approximation of the non-linear function only in the neighbourhood of the equilibrium point. Global stability analysis of the system under discussion may be conducted in terms of a Lyapounov function, with more general conclusions as to the requirements for stability.

Such analysis is conducted in the Appendix to this Chapter. It is there shown that the local stability conditions may be violated and the system may yet be stable.

(iii) The Stationary State

The final question in the formal analysis of the dynamic equation (4.11) refers to the stationary state. The requirement for the stationary state is that price does not change from period to period, i.e.,

$$H(\bar{P})-F(\bar{P}) = 0 \qquad (4.9)$$

which is the same result as that obtained from the stationary state of the system with tatonnement, described by equations (4.5) to (4.9) above. Thus, the assumption of non-tatonnement in production does not change the stationary state of the full tatonnement model.

(d) Comparison of results. With the formal analysis of the nt-t model complete, the discussion may now proceed to compare the results of this model to those of the original model of tatonnement in both production and exchange. As it stands, the former model explains the formation of the equilibrium quantity of output in an environment of non-tatonnement in production with exchange in each period taking place according to tatonnement rules. The price variation exhibited by the fundamental dynamic equation of the system, (4.11), is not in response to the tatonnement process in exchange but in response to variation in the production volume from period to period. Thus, the price variation is in the nature of a comparative static change in the endowment quantity of the exchange model in each period. The quantity variation, however, is genuinely caused by the non-tatonnement process in production since production volume variations are real and the "wrong" volumes do get produced in each period until equilibrium is reached, with the already mentioned consequences on the revenue plans of producers.

With regard to the stationary state, it has already been mentioned that it is the same for both the t-t and nt-t models.

The local stability conditions of the model with nt-t may now be compared to those of the t-t model. These conditions are reproduced here for convenience:

$-2/k<(H_p-F_p)<0$ i.e., $F_p>H_p$ [16] (4.8) for the t-t model

$-H_p>F_p>H_p$ i.e., $|F_p|<|H_p|$ (4.13a) for the nt-t model.

All that the first stability condition requires is that the slope of the demand function be smaller than the slope of the supply function. The second stability condition, however, requires that the value of F_p lie between $-H_p$ and H_p. Obviously, the condition (4.8) for stability in the tatonnement model is always satisfied when the other stability condition (4.13a) holds, but not vice versa. Thus, it may be concluded that the introduction of non-tatonnement in production in this simple model renders stability more difficult to satisfy than in tatonnement in both production and exchange.

---

[16] Assuming that the L.H.S. of the inequality is always satisfied via small speed values: see above.

Given that the stationary state of the two models is the same and that the t-t model is always stable when the nt-t is stable, it follows that traditional comparative static theorems do not have to be changed when non-tatonnement in production is introduced in the model of this Section. This is obviously a useful result: the question is whether it is robust to complications. An attempt to answer this question will be made in the next Section where inventory considerations will be introduced into the model.

Finally, a comparison may be made of the out-of-equilibrium price and quantity variation in the t-t model versus that observed in the nt-t model. The purpose for such comparison is to see how close the tatonnement path of price and quantity adjustment comes to the path of the nt-t model. This is because the path of the t-t model, which is traced by the auctioneer, has no claim to "realism" except in so far as it approximates paths of more "acceptable" processes.

Consider, first, the possibility of cyclical behaviour in the t-t model. For this to occur, the value of λ in equation (4.7) above must be negative. i.e., the following inequality must be satisfied:

$$H_p - F_p < -1/k \qquad (4.18)$$

However, the value of k in this model can be made small enough to violate this inequality, thus precluding cyclical behaviour in the tatonnement process. Since the value of k has to be made small to satisfy the stability conditions also, and since the speed in that model is not a behavioural variable it is legitimate to set it sufficiently small. As a matter of fact, it may be noted that the problems arising with the value of this speed disappear in a tatonnement model with continuous adjustment and such a model cannot exhibit cyclical behaviour. Thus, it is legitimate to say that the typical tatonnement path would be one of price and quantity approaching their equilibrium values from one side of the equilibrium, that from which the system starts out.

In contrast with the above conclusion, the possibility of cyclical behaviour in the nt-t model is real: in fact, in the "normal" case of a downward-sloping demand and a positively sloped supply curve, cyclical approach to equilibrium is the only course of events to be expected in the present formalization of equilibration in a single market, as has already been pointed out in the formal analysis of the dynamic equation of the system, (4.11). Thus, it turns out that the tatonnement process path of quantity and price towards equilibrium is not a faithful representation of the process formalized in the nt-t model of this Section.

Before a summary of the above comparisons is given it may be useful to drop the assumption of tatonnement in exchange and briefly and informally analyze a model where non-tatonnement is also allowed in exchange, the nt-nt model for economy of expression.

The introduction of the nt process in exchange makes the demand of each period sensitive to the path of transactions during equilibration in that period. As a consequence, the equilibrium price that clears the market in each period is not uniquely determined from the available supply and the preferences of the buyers, and this indeterminancy makes formal use of the model of this Section impossible for this analysis. In a full analysis of this problem, the conditions for stability of the modified nt-nt model may be different from those discussed above. Moreover, the path of quantity and price through time may be different. However, the stationary state is not different from that of the t-t model.

To indicate the reason for this claim, it may be useful to remind the reader that the demand function is assumed not to change from period to period. This means that, at least at the opening of each period, the demand is represented by

$$Q_t^d = H(P_t) \tag{4.1}$$

This constancy implies that at the beginning of each period the initial endowments of individuals are the same as they were at the beginning of every other period.[17]

---

[17]The fact that production changes from period to period does not matter: it has been assumed that variations in the production volume do not affect demand.

The proof that the stationary states of the t-t and nt-nt models are the same may now proceed. It will first be shown that, if the variables of the nt-nt system attain the values corresponding to the stationary state of the t-t model, they will remain there. Secondly, it will be shown that if the variables do not have these values, the system cannot have an unchanging volume of production, except by chance.

If the opening price of the system is the price at which quantity supplied equals quantity demanded in the t-t model, no false transactions can take place in the nt-nt version. The equilibrium price is the price at which all transactions will be carried out. In the next period, quantity supplied will be the same, demand will be the same, and the price will be the same, too. Thus, the system will remain in this position.

Now suppose that given a price other than that of the stationary state, production proceeds according to the long-term supply function. In the exchange process, the only thing that can make the system stationary at this price is a series of "false" transactions which shift <u>the demand for the period</u> in such a way that the equilibrating price is the price of the previous period. Thus, production is sold at the price expected, and consequently the same volume reappears in the next period. In order for the system to remain at this price-quantity combination for more than one period, however, the same fortuitous sequence of "false" transactions must happen so as to shift again the period's demand by the above mentioned amount. Clearly, this is too stringent a condition to be satisfied for any length of time: if it is not, price will be different and production will move.

Thus, it is now plain that the stationary state of the t-t model is invariant even when non-tatonnement is introduced not only in production, but also in exchange.

(e) Summary of results. A summary of the results of this Section is now in order. Regarding the comparisons between the tatonnement and the nt-t models what we have in effect is that in the present formalization the removal of tatonnement from the production side leaves the stationary state and the associated comparative statics theorems intact; it makes the stability conditions more difficult to satisfy; and it produces a path of quantity and price which the path of the tatonnement process does not approximate. Thus, while the comparative statics of the traditional analysis can still be used, any analysis of the equilibration process itself should concentrate on the paths of models such as that of nt-t in this Section rather than that of tatonnement. Furthermore, removal of the tatonnement from the exchange process as well as from production may not have consequences for the validity of the traditional comparative static theorems since the stationary state is not modified.

The above results are obviously important and the question arises whether they can withstand complications of the simple formalization of this Section. The analysis of the next Section, in this vein, introduces inventory considerations in the model.

4.3 Price-Quantity Formation for a Durable in Perfect Competition

The analysis in this Section introduces inventory considerations[18] in the price-taker model of Section 4.2. All other assumptions of that model are retained. The discussion here begins with an examination of the reasons for which the price-taker producers of this model would desire to hold inventories. Following this, the necessary modifications on the supply of the commodity in each market period are effected, and a formal specification of the general nt-t model with inventories is given. Comparative analysis of stability, stationary state and out of equilibrium behaviour is then conducted for four variants of the general model, these

[18] Durable commodity considerations have been touched upon in a similar model by M. Ezekiel (1937); G. Ackerman (1957); and M. Nerlove (1958). None of the above incorporates the inventory analysis of this section explicitly, however: thus, no specification of the desired inventory function is made, and the actual inventory position of firms is not formally incorporated. Consequently, little of the analysis that follows has been touched upon, if at all.

latter obtained either by variation of the specific value of the speed of inventory adjustment or by abstraction from certain variables endogenous to the model.

The method of analysis in this Section is the same as that used for the analysis of the non-durable commodity model of Section 4.3, namely dynamic analysis using difference equations, but the order of the system to be discussed here is higher, since the introduction of inventories adds one dynamic equation to the formal model.

(a) Model formulation and structure. The main reasons for which a producer may be induced to hold finished goods inventory are usually classified as the speculative motive, the transactions motive and the desire to smooth production over time, also called the buffer stock motive.

The speculative motive refers to the desire to hold inventory because specific price expectations make such inventory holding profitable. Thus, an expectation of price increase sufficient to compensate for storage costs may induce inventory accumulation. The transactions motive refers to the need to hold inventory to meet unexpected increases in demand. Thus, if demand increases unexpectedly, the seller stands to lose both potential revenue and customer goodwill if he is sold out. Finally, the buffer stock motive suggests that if the firm expects production variations and if the unit cost of production increases with increased volume, it may be optimal to accumulate inventory in periods of low demand and liquidate it when demand is high.

It will be recalled that the model to be discussed here is based, inter alia, on the assumptions of static price expectations and perfect competition. As it turns out, these two assumptions are incompatible with inventory holding, at least from a strictly formal point of view. The assumption of static price expectations suggests that the speculative and the buffer stock motive cannot operate in this model, while the assumption of perfect competition with the consequent impersonal character of the market suggests that there is no customer goodwill to be lost when the producer is sold out.

Despite the formal incompatability of inventory holding with the above assumptions, however, it is a matter of fact that inventories are held by producers in durable commodity industries approximating the perfectly competitive model. The reasons, of course, must lie in the fact that the assumptions of perfect competition and static price expectations do not fully and strictly hold under the circumstances. The argument advanced here on the applicability of these assumptions is not empirical but refers to the logical consistency of the assumptions themselves, as will be seen in the discussion that follows.

With regard to the assumption of perfect competition, it has been remarked[19] that a logical gap exists in the formulations of perfectly competitive models, in that there is no one to make decisions with respect to prices once the fiction of the auctioneer is removed. Since decisions with respect to prices are required outside equilibrium, the implication is that perfect competition is logically consistent with equilibrium alone. Outside equilibrium, the individual perfect competitor faces a downward-sloping demand for his product and he is aware of this fact, though considerable uncertainty may exist with regard to the shape and position of it, since actions of other sellers affect it. For example, in a situation of excess supply every firm knows that it faces a downward-sloping demand curve unless all competitors follow its price change. Since, in disequilibrium, there is no reason why a uniform price should rule at all times, the possibility is real and as a consequence the firm finds itself in the position of a price-setter. Moreover, since full equilibrium can safely be regarded as the exception rather than the rule in actual markets, it is reasonable to assume that perfectly competitive producers adopt attitudes inconsistent with the formal characteristics of the perfectly competitive structure.

---

[19] K. J. Arrow, (1959).

For example, they will necessarily hold inventories during a period of falling demands, and they may also wish to hold inventories beyond it, due to the transactions motive.

The assumption of static price expectations may be considered to include the additional assumption that sellers have some notion of a "normal" range within which prices should fluctuate. If this is the case, firms may elect to accumulate or deplete inventories in response to changing market conditions as they are reflected in the selling price of each period. Behaviour of this sort amounts to a situation where the burden of equilibration is not borne totally by price changes but also by variations in quantity which are deemed desirable by firms.[20] Strictly speaking, this admits the speculative motive via "the back door", despite the assumption of static price expectations, but there seems to be no inconsistency in this since the fact that a "normal price range" is perceived makes the actual price vary within this range, as will become obvious from the formal analysis of the model, below.

---

[20]The behaviour of firms here is strictly analogous to that encountered in "fix-price" models of inventory adjustment. In these models, the implicit assumption is that price change is not desirable, and the burden falls onto quantity. Similarly here, price adjustment towards the limits of the "normal price range" is not desirable, and is therefore substituted by quantity adjustment. Given that perfect competitors would behave as price-setters outside equilibrium, the analogy seems complete.

The introduction of inventory considerations in this Section may thus proceed on the basis of either the transactions or the speculative motive or both. For reasons of simplicity, and as a first approximation, only the speculative motive will be employed in the formal specification of the desired inventory function.[21] More specifically, desired inventory will be assumed to be a function of the prevailing price, rising with a lower price and vice versa. The closer the price approaches the limits of the perceived "normal" price range, the greater the inventory accumulation or decumulation is.

If inventories are held, the commodity supply in the market in each period will consist of the production of the previous period and a (positive or negative) fraction of the difference between actual and desired inventories. Denoting actual inventory by I_t and the desired inventory by I_t^*, the modified commodity supply function for the market becomes:

$$Q_t^s = Q_{t-1}^o + k(I_t - I_t^*) \tag{4.19}$$

[21] A simple formalization of the transactions motive would make desired inventory dependent on the actual sales of the previous period. Since the latter have to equal demand, and since demand in this model depends on price, the desired inventory would end up a function of price, with a one-period lag. The response of desired inventory to the volume of sales would be positive, and the response to the price could be negative or positive, depending on the response of the quantity demanded to price. If demand is normally sloped, desired inventory varies negatively with the previous period price.

where k is the speed of inventory adjustment.[22] Desired inventory (demand) is specified as

$$I_t^* = I^*(P_t), \qquad I_p^* < 0 \tag{4.20}$$

The main difference that introduction of these inventory considerations seems to make to the structure of the non-durable model of Section 4.2 is on the supply side. Whereas previously supply was always identical with last period's production, the introduction of inventory now makes possible the divergence between production and sales. This divergence may be shown graphically, for one period, in Figure 4.2. In that Figure, Q^o is the relation on the basis of which production levels are decided, given the expected commodity price. In the non-durable model, this also denotes quantities that would be inelastically supplied given the expected prices. On the other hand, the durable commodity model has a distinct period supply curve, based on considerations of actual as well as desired inventory and labelled Q^s in the Figure. In the figure, $\overline{P}$ is the price at which the variable desired inventory is made equal to the actual inventory on hand. Obviously, the greater the amount of inventory on hand the lower the price necessary to induce the firm to hold this inventory. Consequently, the curve Q^s shifts

---

[22] Writing the supply function in this form implies that inventory adjustments are effected through sales alone, rather than production. Moreover, the whole discussion in the text abstracts from inventory carrying costs, to be discussed in the following section.

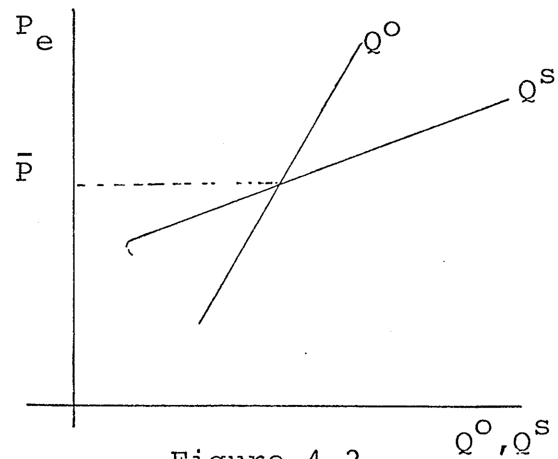


Figure 4.2

downwards with actual inventory accumulation and vice versa.

The above discussion makes it clear that the effective supply of the commodity in each period will be a function of past events that have contributed to inventory accumulation or depletion. Suppose, for example, that (Figure 4.3) P_{t-1} is the price on the basis of which production plans proceeded in period t-1. The commodity supply at t in the non-durable model would be a vertical line S_t in the Figure. In the durable model developed here, however, the commodity supply may be either greater or less than the amount of production, depending on the history of past periods and the consequent actual inventory positions of the firms. Two possibilities are depicted in the Figure: the curve Q^{s1} and the curve Q^{s2}, the latter implying more actual inventory than Q^{s1}. With the first curve, commodity supply at P_{t-1} equals cd, while with the second it equals cg.

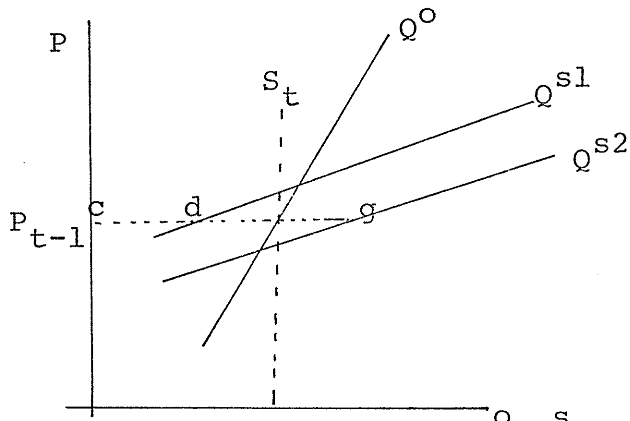


Figure 4.3

The model may now be completed with the definition of inventory change, as follows:

$$I_t - I_{t-1} = Q^o_{t-2} - Q^s_{t-1} \qquad (4.21).$$

Inventory at the beginning of period t is equal to inventory at the beginning of period t-1 plus production that became available at the beginning of t-1 minus the sales during the period t-1.

To recapitulate, the complete model consists of the following equations:

$$Q^d_t = H(P_t) \qquad (4.1)$$

$$Q^o_{t-1} = F(P_{t-1}) \qquad (4.2)$$

$$Q^d_t = Q^s_t \qquad (4.4)$$

$$Q^s_t = Q^o_{t-1} + k(I_t - I^*_t) \qquad (4.19)$$

$$I^*_t = I^*(P_t) \qquad (4.20)$$

$$I_t - I_{t-1} = Q^o_{t-2} - Q^s_{t-1} \qquad (4.21).$$

This is a set of six equations in six unknowns, price and demand (P and Q^d) production and supply (Q^o and Q^s) and desired and actual inventory (I and I*). The model may be collapsed to two fundamental dynamic equations in price and actual inventory by substitution,[23] with the following result:

$$H(P_t) = F(P_{t-1}) + k[I_t - I^*(P_t)] \qquad (4.22)$$

$$I_t - I_{t-1} = -k[I_{t-1} - I^*(P_{t-1})] \qquad (4.23).$$

(b) Tatonnement. It is fairly obvious that in a process of tatonnement in both production and exchange inventories have no role to play, since production is not undertaken before equilibrium is reached for each period, and thus there is no possibility of falsification of expected prices and quantities. The behaviour of a model with inventories in this case will be identical in all respects[24] with that of the non-durable commodity model in tatonnement. Therefore, analysis of the t-t process will not be conducted in this Section, but the model of the previous Section will

---

[23]Equations (4.2) and (4.20) into (4.19) and the result into (4.4) together with (4.1) into (4.4) yield the difference equation in price, while (4.2) and (4.20) into (4.19) and the result into (4.21) together with (4.2) into (4.21) yield the inventory equation.

[24]The implicit assumption is, of course, that the equilibrium price established by the tatonnement process will be considered to persist in the future. In this case, there is no reason for inventory creation for production smoothing purposes, and production equals equilibrium sales. Neither is there a reason, under these circumstances, for changing inventories due to the transaction or the speculative motive.

be used as basis for comparisons.

(c) Non-tatonnement in production. The discussion now proceeds to formal manipulation of the nt-t model, as summarized in equations (4.22) and (4.23). Linearization of this system in the neighbourhood of equilibrium and summation of the constant terms arising from that linearization results in the following system of difference equations of the first order, with constant coefficients and constant terms:

$$\begin{bmatrix} H_p+kI^*_p & -k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_t \\ I_t \end{bmatrix} - \begin{bmatrix} F_p & 0 \\ kI^*_p & 1-k \end{bmatrix} \begin{bmatrix} P_{t-1} \\ I_{t-1} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad (4.24)$$

where the t subscripts denote time, while the letter subscripts denote partial derivatives and the constants c_1 and c_2 summarize the remaining terms of the linear approximation around the equilibrium levels of price and inventory, $\bar{P}$ and $\bar{I}$.[25] An alternative summary notation of the above is

$$Ax_t - Bx_{t-1} = c \quad (4.25)$$

where x and c are column vectors and A,B are matrices, all with obvious interpretation from (4.24). A trial complementary function solution to the homogeneous part of this system is of the form:

$$P_t = a\lambda^t \quad \text{and} \quad I_t = b\lambda^t \quad (4.26)$$

---

[25] Thus, c_1 equals $(H_p+kI^*_p)\bar{P} - k\bar{I} - F_p\bar{P} - H(\bar{P}) + F(\bar{P}) + k[\bar{I} - I^*(\bar{P})]$ while c_2 is equal to $k\bar{I} - kI^*_p\bar{P} - k[\bar{I} - I^*(\bar{P})]$.

where a and b are arbitrary constants dependent on initial conditions and λ is to be found from the solution to the equation

$$|\lambda A - B| = 0 \qquad (4.27).$$

On the other hand, a particular solution to the system is obtained by setting all prices and inventories equal to their equilibrium values in (4.22) and (4.23) which reduces the system to

$$H(\bar{P}) = F(\bar{P}) + k[\bar{I} - I^*(\bar{P})] \qquad (4.28)$$

and $$-k[\bar{I} - I^*(\bar{P})] = 0 \qquad (4.29)$$

As is well known, the stability of the system depends on the behaviour of the complementary function. From the form of that solution in (4.26) stability is assured if and only if λ lies in the unit circle of the complex plane, i.e., its modulus is less than unity.[26] To find the values of λ, expansion of (4.27) is required which yields a polynomial of the second degree in λ of the form

$$a_o\lambda^2 + a_1\lambda + a_2 = 0$$

where $a_o = 1$, $a_1 = (kH_p - F_p)/(H_p - kI^*_p) - 1$,

and $a_2 = [F_p(1-k)]/(H_p + kI^*_p)$ (4.30)

According to Schur's conditions,[27] the roots of the above

---

[26]The modulus of λ is defined as its length.

[27]See Chipman, (1951), pp.119-120. According to Chipman, these necessary and sufficient conditions that the roots of a polynomial lie in the unit circle of the complex plane were first derived by Schur, (1917), and Cohn, (1922). It was subsequently shown by Herglotz, (1924), that the Cohn

polynomial in (4.30) will lie within the (complex) unit circle if and only if the following inequalities are satisfied:

$$\begin{aligned} &\text{(i)} \quad a_o^2 > a_2^2 \\ &\text{(ii)} \quad (a_o^2 - a_2^2)^2 > (a_1 a_2 - a_o a_1)^2 \end{aligned} \qquad (4.31).$$

Samuelson has further simplified these conditions (in the particular case of the second degree polynomial under study) to the following (with $a_o = 1$):

$$\begin{aligned} &\text{(i)} \quad 1+a_1+a_2 > 0, && \text{i.e., } a_1 > -(1+a_2) \\ &\text{(ii)} \quad 1-a_2 > 0 && \text{i.e., } a_2 < 1 \\ &\text{(iii)} \quad 1-a_1+a_2 > 0 && \text{i.e., } a_1 < (1+a_2). \end{aligned} \qquad (4.31a)$$

A little more manipulation of these conditions is necessary for the purposes of the analysis that follows. Conditions (i) and (iii) above may be reduced to $|a_1| < 1+a_2$, and for this to hold the right-hand side must be positive, which implies $|a_2| < 1$. Conditions (4.31a) above are, then, equivalent to

$$\begin{aligned} &\text{(iv)} \quad |a_1| < 1+a_2 \\ &\text{(v)} \quad |a_2| < 1 \end{aligned} \qquad (4.31b).$$

Finally, the conditions as written by Samuelson, (4.31a), define three linear inequalities in the a_1, a_2

---

conditions are equivalent to the Hurwicz conditions that the roots lie in the left-hand side of the complex plane. Schur had given the relevant transformation that proved equivalence. Finally, Samuelson, (1941), used the same transformation to derive, from the Routh-Hurwicz conditions, the conditions that the roots lie within the unit circle of the complex plane.

plane, shown in the Figure 4.4 below:

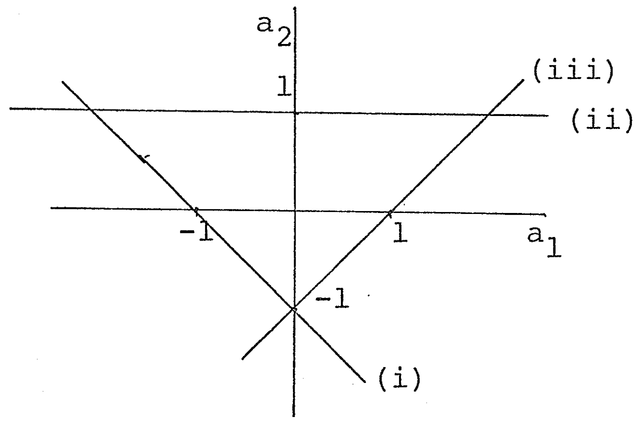


Figure 4.4

From the Figure, the values of a_1 and a_2 must lie in the triangle formed by the three inequalities.

The stability conditions of the model are now complete, and the discussion will proceed to examination of special cases of the general model developed above.

(i) Sub-model with the actual inventory position ignored

A very much simplified situation arises when it is assumed, contrary to the full specification of (4.22) and (4.23), that the effects of actual inventory variation on the output supply are ignored. This implies dropping the variable I_t in equation (4.22) and substituting a constant (which may as well be zero). This heroic assumption allows the analysis to proceed to a partial examination of the system, where only the effects of the desired inventory holding on supply are considered. The sub-model so derived serves only as a step towards more general sub-models that follow. With this assumption, then, equation (4.22) is modified as follows:

$$H(P_t) + kI^*(P_t) = F(P_{t-1}) \qquad (4.22a).$$

The dynamic equation of this sub-model then differs from that of the non-durable model of the previous Section by the term $kI^*(P_t)$ on the left-hand side. The economic interpretation of this equation is fairly clear, namely total demand now consists of two components, that of demand for consumption $H(P_t)$ and that of a portion of demand for inventory $kI^*(P_t)$. The supply side is not modified because of the simplifying assumption that it is precisely the effects of actual inventory variation on supply which are ignored.

Solution of the linearized form of (4.22a) by the usual method yields,

$$P_t = A\lambda^t \quad \text{where } \lambda = F_p/H_p{+}kI^*_p \qquad (4.32)$$

The stability condition for this sub-model is that the absolute value of λ be less than unity.[28] The stability condition of the non-durable nt-t model, on the other hand, was that the absolute value of F_p/H_p be less than unity.

---

[28]Lyapounov function analysis of this system is of course also possible. Equation (4.22a) writes $G(P_t) = F(P_{t-1})$ where G is the sum of functions H and KI*. Explicit solution for P_t yields $P_t{=}n(P_{t-1})$ and Lyapounov function analysis similar to that of Section 4.3 yields $|n(P_t)|<|P_t|$ as a necessary and sufficient condition for global stability.

Comparison of the two conditions indicates that the model with inventories can satisfy the stability condition more easily than the non-durable commodity model as long as the demand function is downward-sloping, i.e., as long as H_p is negative. But this latter condition must be satisfied for stability of exchange in tatonnement.

Thus, in the case of a "normal" demand function, the desired inventory demand which also slopes downward with respect to price makes the elasticity of total demand for the commodity greater than previously and thus helps satisfy stability.

The discussion of this model is offered only as a preliminary illustrative step: no reference to it will be made in later sections involving comparisons.

(ii) Sub-model with zero speed of adjustment

For completeness of analysis, it may be worth noting that the sub-model derivable by setting k=0 in equations (4.22) and (4.23) above is, in effect, the non-durable commodity model of the previous Section. Moreover, the stability conditions of that model may be derived by setting k=0 in the general expressions for stability, equations (4.31b) above.

(iii) Full adjustment of inventory within each period (k=1)

Here, it is assumed that the inventory discrepancy is fully corrected within each period, i.e., desired and actual inventory coincide at the end of each market period. Thus, the speed of adjustment k is set equal to unity.

In the price-taker model of this Section, the case of full inventory adjustment is actually the most plausible logically and the most probable in practice: there seems to be no reason why price-takers accumulating inventories because of a version of the speculative motive would choose to adjust inventories only partially towards the desired amount.

Setting the speed of adjustment equal to unity in the system (4.22) and (4.23) yields the simpler system

$$H(P_t) = F(P_{t-1}) + I_t - I^*(P_t) \qquad (4.33)$$

and

$$I_t = I^*(P_{t-1}) \qquad (4.34)$$

On the other hand, the stability conditions for this system are,[29] from (4.30) and (4.31b) with k=1:

---

[29]As in previous cases, stability analysis may be done via a Lyapounov function instead of the more usual and more restrictive method used in the text. As is mentioned later in the text, the dynamic system under discussion reduces to a single equation, (4.36), which may be written

$$G(P_t) = J(P_{t-1})$$

where G and J are functional notations for the sums of functions H and I* and F and I* respectively. The above equation can be explicitly solved for P_t to yield

$$P_t = x(P_{t-1})$$

and Lyapounov function analysis similar to that conducted in the Appendix yields that a necessary and sufficient condition for global stability of the system is

$$|x(P_t)| < |P_t|$$

as before.

$$\left| \frac{F_p + I^*_p}{H_p + I^*_p} \right| < 1, \text{ or,}^{30} \; -(H_p + 2I^*_p) > F_p > H_p \qquad (4.35).$$

Moreover, the system (4.33) and (4.34) reduces to a single equation in price as follows:

$$H(P_t) + I^*(P_t) = F(P_{t-1}) + I^*(P_{t-1}) \qquad (4.36).$$

Finally, setting k=1 in the linearized version (4.24) yields the linearized form of (4.36) which becomes (neglecting constants)

$$(H_p + I^*_p)P_t = (F_p + I^*_p)\, P_{t-1} \qquad (4.36a).$$

It may be useful to discuss the economic meaning of the fundamental dynamic equation of this sub-model. The left-hand side of (4.36) denotes the total demand for the commodity at time t, both for consumption and for desired inventory purposes. The right-hand side denotes the total supply in period t, both from production and from actual inventories, I_t. The amount of production is dependent on the price of the previous period, and so is the amount of actual inventories since the latter are always equal to desired inventory amounts at the end of each period. Thus, actual inventory at the beginning of period t is equal to desired inventory of the period t-1, which depends on the price at t-1.[31]

---

[30]Taking into account the condition $H_p + I^*_p < 0$ for stability in exchange.

[31]It is thus not legitimate to "deduct the schedule of excess supply from storage from the demand schedule from

(iv) Partial adjustment of inventory ($0<k<1$)

For completeness of analysis only,[32] a few remarks may be made with regard to this case. The system of equations describing movements of inventories and prices in this case is that of the original general dynamic equations (4.22) and (4.23), and the stability conditions are given by equations (4.31).

In the "normal" case of positively sloped production and negatively sloped demand with respect to price it may be shown that the value of the coefficient a_2 increases as the speed of adjustment increases.[33] When k attains the value of unity, the value of a_2 is zero. Thus, the effect of this speed on stability is that the lower it is the more restrained the value of a_1 must be. This may be seen more clearly in Figure 4.4 above: At values of a_2 less than zero (k between zero and one) the value of a_1 must lie within a range

---

consumption in order to arrive at the demand" for the product, as M. Nerlove (1958) does, since the schedule of excess supply for storage depends on both the previous and the current price in the present formalization. The lack of formal specification of the desired inventory function and the speed of adjustment of inventories in the literature of the cobweb model is responsible for a number of ambiguities and misunderstandings regarding the role that inventories play in the cobweb process.

[32]Since the plausibility of the case $k=1$ has already been argued.

[33]This can be shown by taking the derivative of a_2 with respect to k, from (4.30).

narrower than the interval [1,-1].

(d) Comparisons. Part (c) of this Section has discussed four sub-models of the general model described. As has already been mentioned, the sub-model in which inventory positions are ignored, case (i), will not be used in comparisons. Of the other three, the model with partial inventory adjustment, case (iv), will be dropped in favour of the model assuming full adjustment within the period, (iii). Finally, the model with zero speed is in fact the non-durable non-tatonnement-in-production, tatonnement-in-exchange model of the previous Section. It will be remembered that this model has been compared with the full tatonnement case in that Section, and it was found that the stationary state was the same, stability conditions were more difficult to satisfy with non-tatonnement, comparative statics were the same, and the out-of-equilibrium behaviour of the non-tattonement was not approximated by that of the tatonnement model.

The comparisons in this Section include, in fact, the above comparisons, since the model presented here has the non-durable case as a sub-model. To recapitulate, the stability conditions were,

$F_p > H_p$ (4.8) for the tatonnement,

$-H_p > F_p > H_p$ (4.13a) for k=0 in non-tatonnement production (non-durable)

and $-(H_p + 2I^*_p) > F_p > H_p$ (4.35) for the durable case with full adjustment (k=1).

Once these conditions are presented in this manner, it is obvious that, as before, introduction of non-tatonnement in production makes stability more difficult to satisfy (4.8 is satisfied when 4.13a is but not vice versa). However, when inventories are considered as in the present analysis, it is less "difficult" to satisfy stability than in the case of no inventories (4.35 is always satisfied when 4.13a is, but not vice versa).

The conclusion thus emerges that stability conditions for non-tatonnement production with inventories considered are more difficult to satisfy than conditions for tatonnement in production and exchange, though it is true that introduction of inventory considerations makes stability easier to satisfy than in the non-durable commodity case.

With the discussion of comparative stability conditions reasonably complete, the analysis now turns to the stationary state of the general model, and to comparison of that with the stationary state reached by the tatonnement model. For this purpose, the particular solution of the general model of this Section is relevant, since if the model is stable the complementary function component will tend to zero as the system approaches the stationary state.

Equations (4.28) and (4.29) above describe a particular solution to the model. From (4.29) however, it is obvious that since k is greater than zero the actual inventory equals the desired one in equilibrium, i.e., $\bar{I} = I^*(\bar{P})$. Substitution of this equation in (4.28) yields

$$H(\bar{P})=F(\bar{P}) \tag{4.37}$$

for the stationary state price of the system. But this equation is identical to that of the non-durable tatonnement model, i.e., equation (4.9) in Section 4.3 above.

Thus, it turns out that the introduction of inventories in the nt-t model of production and exchange does not in any way change the stationary state of the system. This result was also derived for the nt-t model in the non-durable commodity case and thus is robust to this complication. The result is of course of fundamental importance to the usefulness of comparative statics analysis derivable from the t-t model since unless the stationary state of that model survives non-tatonnement complications the comparative statics of traditional analysis would have to be thoroughly re-examined. It is obviously satisfactory to find that no such re-examination is necessary, at least as far as complications examined so far are concerned.

The final question in terms of comparative analysis of tatonnement and non-tatonnement processes refers to the behaviour of the models out of equilibrium. It will be recalled that the behaviour of the t-t model of Section 4.3 was judged to be as an unsatisfactory "approximation" since the nt-t model without inventories would, under normal assumptions, display cyclical behaviour in its approach to equilibrium while the t-t model does not. With respect to the model of this Section, cyclical behaviour is again to be expected normally[34] and thus the same remarks with respect to the behaviour of the t-t model are in order, namely that the out-of-equilibrium behaviour of the model with tatonnement is not a satisfactory indication of how markets will equilibrate over time in the absence of this device and therefore that this part of the conventional tatonnement analysis cannot be relied upon.

The discussion and comparative analysis of the model of this Section is now complete. Before a summary of the results is attempted it may be desirable to discuss briefly how the results of the above analysis might be modified if the assumption of tatonnement in exchange were dropped from the model.

As in the case of the non-durable commodity model of Section 4.3, the removal of tatonnement from exchange would affect the out-of-equilibrium behaviour of the model, in the sense that the path of prices and quantities towards equilibrium will now be dependent on the specific exchanges undertaken at disequilibrium prices, because endowment effects of

---

[34] At least as long as demand and supply are "normally" sloped and as long as the desired inventory response to price is less than the response of production in absolute value.

the second kind may shift demand for consumption.[35] Moreover, the stability conditions of the model may be different. The stationary state, however, will remain the same as that of the t-t model for precisely the same reasons as argued in Section 4.2 above with respect to the non-durable commodity model. Thus, the further conclusion is obtained that the stationary state and the consequent comparative statics theorems of traditional analysis are robust with respect to removal of tatonnement from both production and exchange, at least in the simple formalization of the single-market model used in this Section.

(e) Summary of results. A summary of the main characteristics of the model and the results of this Section is now in order. The model examined here is one in which inventory adjustments are all intended and take place through sales variations in response to price fluctuations. Production does not fluctuate to take care of inventory adjustments from period to period, mainly due to the fact that producers are price takers. No inventory-carrying costs are considered. Comparison of stability conditions, stationary state and out of equilibrium behaviour with those of the non-durable commodity model as well as to those of the t-t model

---

[35] It is assumed that the demand for inventories, while essentially speculative in character, does not exhibit endowment effects. Moreover, perfect attendance of buyers throughout the non-tatonnement process is implied, as usual.

of Section 4.2 suggests that stability is easier to satisfy in this model, as compared with the model without inventory. Stability is still more difficult to satisfy than in tatonnement, however. The stationary state is the same as that of the t-t model, and it does not seem to change even when tatonnement is removed from the exchange side also. Finally, the out-of-equilibrium behaviour of the model of this Section cannot be expected to be reasonably approximated by the t-t model.

Thus, the same general results seem to persist, namely that while traditional equilibrium comparisons may still be valid despite the absence of tatonnement from markets, stability and out-of-equilibrium behaviour may be quite different in the absence of Walras' ingenious device.

4.4 Price-Quantity Formation for a Durable Produced Under Monopolistic Conditions

The analysis in the present Section concentrates on discussion of the equilibration process for a durable when the market structure is the polar case of monopoly. First, a number of points are made with regard to the meaning of the tatonnement and non-tatonnement process in this setting. A formal specification and solution of a simple intertemporal maximization problem follows, together with interpretation of the profit-maximization rules so derived. Finally, an attempt is made to compare the stationary state of this profit-maximization problem with that of the conventional "tatonnement" model of monopoly.

The method of analysis in this Section employs simple calculus of variations analysis in order to obtain necessary conditions for maximization of the monopolist's profit over the "planning" period.

As with the previous Section, the main aim of this one is to examine in what sense the introduction of inventories modifies the equilibration process and the consequent variation of production, input demand, and price during this process.

(a) Preliminaries. When the market structure is one of monopoly, it is no longer possible to entertain the analytical fiction of the auctioneer setting prices during the equilibration process: the producer himself who takes the demand schedule as given rather than the price, sets the "proper" price in these conditions, "proper" taken to mean that it is expected to maximize his profit under the circumstances. Even if one assumes, as in this Section, that the monopolist knows his cost conditions, it is difficult to visualize him with complete knowledge of the demand schedule for his product. Nevertheless, this is the assumption in the conventional static theory of monopoly.

It seems reasonable to label the situation in which the monopolist has full knowledge of his demand schedule one of "tatonnement", since in this case the producer as price setter has all the information that an auctioneer would finally collect if he was to proceed with price iteration. A non-tatonnement process, it follows, is one in which the monopolist does not have full demand information. In this latter case, the possibility of discrepancy between demand and production must be admitted[36] and inventories come into play.

Thus, the tatonnement model in this Section is that of conventional monopoly analysis of static theory, and its stationary state involves equality of the marginal costs of production to marginal revenue from sales. In this process, inventories can play no role, and may safely be assumed equal to their desired value (which for simplicity may be set equal to zero).

The non-tatonnement model, in contrast, must assume that after demand has unexpectedly changed it takes considerable time for the monopolist to proceed, by trial and error, to his maximum profit position, i.e., it takes time and mistakes for the monopolist to estimate his demand curve. Inventories may fluctuate during this process, and even when the estimate of the demand has been completed there is no reason to suppose that inventories will be at the desired level.

---

[36] Compare K. J. Arrow, (1959), p.44.

The analysis of this Section will concentrate on the period after the demand has been fully estimated, but inventories are in disequilibrium because of the preceding trial and error process. This procedure is justified as a first approximation since to discuss the trial and error period one must plunge into a situation in which mistakes are observed and corrected sequentially.

(b) Model structure. With the foregoing preliminaries out of the way, the formulation of the simple non-tatonnement model of the monopolist is now in order.

Let $Q^{o}(t)$, $Q^{s}(t)$, $P(t)$, $I(t)$ and $I^{*}(t)$ represent production, sales, price, actual and desired inventories respectively, in a framework of continuous time. Cost and revenue conditions are then represented by the following relations:

$$C(t) = C\,[Q^{o}(t),\ I(t)] \qquad (4.38)$$

and $$Q^{s}(t) = S[P(t)] \qquad (4.39).$$

The first relation represents total cost of production and inventory, while the second is the sales, or (what is the same) the demand function for the monopolist's product. Inventories at any time are, by definition, equal to initial inventory plus the time integral of the difference between production and sales up to that time.

Assume that desired inventories are given and constant. In the stationary state actual inventories will equal the desired amount, leaving a conventional monopoly problem of equating production Q^o to sales Q^s given (4.38) and (4.39).

If, however, demand conditions change unexpectedly, inventories will have to change during the trial and error period in order to accommodate the difference between demand and production. The monopolist is now assumed to have estimated the new demand schedule, but is still faced with a gap between desired and actual inventory and the inventory replenishment or liquidation process is thus set up. The problem for the producer is to maximize profits during the period of inventory adjustment[37] by manipulating the variables at his disposal, namely production and sales. The total profit of the firm during the adjustment period is denoted by

$$Z=\int_0^A z(t)e^{-rt}dt=\int_0^A [R(t)-C(t)]e^{-rt}dt=\int_0^A [Q^s(t)P(t)-C(t)]e^{-rt}dt \qquad (4.40)$$

where Z is total discounted profit for the adjustment period, z is profit at time t, r is a discount factor, R(t) is revenue at time t, and the period of adjustment has been assumed to extend from t=0 to t=A.

---

[37]The period of inventory adjustment may in another problem be considered a variable, its optimal length to be determined by profit considerations. This is not the concern in this thesis, however, since the analysis here concentrates on the qualitative question of how production and sales plans are affected by the inventory position. A variation in the period of inventory adjustment, on the other hand, will affect only the strength of this relationship, not its direction.

Z is a function of production and sales or, what is the same, of production and price. The firm may then be assumed to use production and prices as control variables over time in order to maximize its total profit, given a constant adjustment period and static demand and cost expectations.[38] The problem is thus maximization of Z by choice of paths of the variables P(t) and $Q^o(t)$. Calculus of Variations methods may be used to determine the necessary conditions for maximum Z. These conditions will yield two differential equations [in $Q^o(t)$ and P(t)] describing the profit-maximizing paths of production and price through the period.

(c) Solution, interpretation and comparisons. To simplify the situation, one may set the discount rate constant at zero and specify demand and cost functions of the simplest possible form:

$$C(t) = aQ^o(t)^2 + b[I(t)-I^*]^2 \qquad (4.41)$$

$$Q^s(t) = c-dP(t) \qquad (4.42)$$

where a,b,c, and d are cost and demand parameters and I* is optimal inventory, at which the total variable inventory

---

[38]Alternatively, it might be assumed that the firm expects demand to be rising over time at the rate of growth g and costs to be increasing at the rate h. The expression for z(t) would then have to be modified as follows:

$$z(t)=[Q^s(t)P(t)\ e^{gt} - C(t)e^{ht}]e^{-rt}=Q^s(t)P(t)e^{it}-c(t)e^{it}.$$

There seems to be no reason to expect fundamentally different results in this case provided the integral still converges.

costs are at a minimum.[39] The above specification then has conventional U-shaped total inventory costs, and rising total and marginal production costs. The revenue function is of course standard, too.

Application of the method of Calculus of Variations to the problem as stated in (4.40) -- with the simple specificiations given in (4.41) and (4.42) -- yields the following differential equations for production and price if profit is to be a maximum for the adjustment period:[40]

$$Q^{o}(t) = \frac{b}{a}\,[I(t) - I^{*}] \tag{4.43}$$

$$P(t) = b\,[I(t) - I^{*}] \tag{4.44}$$

An interpretation of these optimal decision rules is in order. Assume, for the sake of exposition, that actual inventory falls short of the optimal amount at the beginning of the adjustment period. The rules then provide that both price and production should be falling during the period of adjustment, towards their stationary state values. This implies that both production and price were initially stepped up to levels above these stationary values, in

---

[39] Solution of the problem is also possible in terms of the general specifications (4.38), (4.39) and (4.40). The interpretation of the optimal rules so derived is a little more difficult, however, and for this reason the analysis in the text is carried out in terms of specific functions.

[40] For the detail of the solution see the Appendix to this Chapter, part (b).

order to take care of the inventory deficiency. Thus, the price will be higher than its stationary state value, and consequently sales will be lower. This will obviously contribute towards inventory repletion, other things equal. Moreover, production will be higher than its stationary state value. Since, in that state, production is to equal sales, it follows that, other things equal, production will exceed sales during the adjustment period. The excess of production over sales will again contribute towards inventory repletion.

Some of the foregoing may be illustrated with reference to the diagram of the elementary static problem. In Figure 4.5, the stationary state solution is at E. With the initial inventory level below optimal, the firm's price and volume of production lie, at any point during the adjustment period, above their stationary state values, since their rates of change per unit of time are negative in this case. In the Figure, P(t) and $Q^o(t)$ are two such values. It is easily seen that production exceeds sales by the quantity $Q^o(t) - Q^s(t)$, the excess being allocated to inventory. Also, the marginal cost of sales is GA and the marginal revenue GB, which is greater than the cost by AB. Finally, the marginal cost of production is HC, greater than the marginal revenue of that volume (if sold) by CD.

The change in marginal revenue of sales and marginal cost of production, however, will be equal at any point in time. This requires proof and explanation.

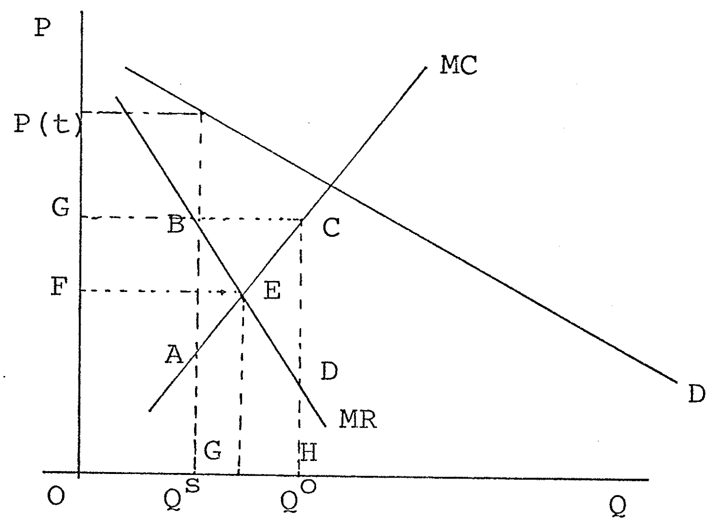


Figure 4.5

From the demand function of the problem one finds that the rate of change of marginal revenue at any point in time is equal to[41]

$$\frac{dMR}{dt} = 2\dot{P} \qquad (4.45)$$

while the rate of change of marginal cost <u>of production</u> per unit of time is

$$\frac{dMC}{dt} = 2a\dot{Q}^o \qquad (4.46).$$

Finally, the change in total cost due to changes in the amount of inventory at time t is equal to

$$\frac{dTC}{dI(t)} = 2b[I(t)-I^*] \qquad (4.47).$$

---

[41]By time differentiation of the marginal revenue and substitution of sales by price from the demand function.

But then, the rules concerning the optimal paths of production and price through time may be written, by simple manipulation of (4.43) and (4.44) and taking into account (4.45) to (4.47),

$$2a\dot{Q}^{o} = \frac{dMC}{dt} = 2\dot{P} = \frac{dMR}{dt} = 2b[I(t)-I^{*}] = \frac{dTC}{dI(t)} \qquad (4.48)$$

which is equivalent to the statement that both the change in marginal production cost and in marginal revenue of sales at any point in time should equal the change in total costs related to inventory variation.[42] The inventory variation will of course equal the difference between sales and production at each moment, the difference being nonzero as long as the price and production volume diverge from their stationary state values.

The economic interpretation of this equality of rates of change of marginal production costs and sales revenue with the cost due to inventory change is as follows: the firm which finds itself with inventories different from the optimal is faced with additional inventory costs which will persist as long as the inventory discrepancy is present. It thus pays to eliminate the discrepancy over time by using both avenues possible, namely both a change in production volume and a change in sales. The change in production

---

[42] Since the rate of change of marginal cost and revenue are equal at all points in time, marginal cost and marginal revenue themselves will be equal at all times if their initial values coincide. This is the case pictured in Figure 4.5.

proceeds to the point where the change in marginal cost is equal to the economy achieved from the change in inventory at the margin. Moreover sales are changed to the point at which the change in marginal revenue is equal to the same cost saving. In this way, inventory change does not reduce the profit that would be achieved at each "point", while at the same time it helps remove permanently the discrepancy in inventory which is associated with cost burden over time.

With the foregoing explanation of the decision rules sufficiently complete for the purposes of this Section, the analysis may now turn to an investigation of the stationary state of this model and comparison with the stationary state of the tatonnement model of conventional analysis. The decision rules above suggest that the stationary state is reached when inventories are at their optimal value I*. But in this case the problem of this Section reduces to that of tatonnement monopoly, since under static expectations there is no reason to make production diverge from sales.[43]

Thus, the stationary state of the non-tatonnement version of this Section is again identical with that of the

---

[43] Indeed, if it is made to diverge, profit will be less than maximum, since the problem has been reduced to single period profit maximization. If, however, demand and/or costs are expected to change over time it may pay the monopolist to allocate production over time differently from sales. See A. Smithies, (1939).

tatonnement model.

The out-of-equilibrium behaviour of the non-tatonnement model (apart from the trial and error period) does differ from that of the tatonnement, however, since a demand change necessitates inventory adjustment and this implies that both production and price are different from their stationary state values for the duration of the adjustment process. Thus, when inventory is in excess, price rises and production increases as adjustment proceeds to completion.

The implication of the above for the monopolist's demand for inputs is obvious, namely the latter depends on the inventory position of the firm, among other factors. If inventories are above normal, production and input demand are not at the level that the static one-period maximization problem of monopoly suggests, but lower. Moreover, sales are higher than the static model level, since price has been set lower.

The reader will recall that the perfect competitor in the case of durables (Section 4.3) behaved similarly with respect to commodity supply: when inventories were above the desired level the supply was increased, as is clear from equation (4.19) of the previous Section. As already noted in that Section, production was not used to accommodate inventory discrepancies. The analysis here suggests that this is also the case, at least for monopolistic producers facing inventory carrying costs.

4.5 Partial Disequilibrium Analysis of Production and Exchange: Summary of Conclusions

Given the simple formulations used in this Chapter, the conclusions that emerge may be summarized in the statement that removal of tatonnement from production, with the consequent possibility of "unfeasible" sales plans, does not change the stationary state. The out-of-equilibrium behaviour is different with non-tatonnement, and therefore cannot be approximated by that of the tatonnement model. In terms of stability conditions, non-tatonnement processes require more stringent ones in the partial model of production and exchange under perfect competition.

The implication of the foregoing for comparative statics is that there is no change to be expected from introduction of non-tatonnement in production.

There are some obvious directions for further research, and development of the analysis of this Chapter. In the case of the perfectly competitive market, an extension to a multi-market system might be the first step. Further, input markets could be introduced; the dependence of demand on the volumes of production could be examined in this system.

In the case of monopoly, it is clear that the analysis of this Chapter constitutes only a small step towards a full discussion of price and quantity adjustment in this market structure. It is set out here mainly because it provides a simple justification of the intuitively obvious statement that when inventories are out of equilibrium both sales and production are affected.

The line of development chosen in this investigation is to extend the single perfectly competitive market models of this Chapter to incorporate input markets and the dependence of demand for commodities on income and the volume of production. The multi-market framework is abstracted from by aggregation. This extension constitutes the subject matter of the next Chapter.

CHAPTER V

GENERAL DISEQUILIBRIUM ANALYSIS OF PRODUCTION AND EXCHANGE

The focus in this Chapter is disequilibrium in an aggregate model of general interdependence, with endowment effects of the first kind assumed "well-behaved",[1] and those of the third kind focussed upon. The relationship of this aggregate model to the general framework of Chapter I is explained below.

The main aim here is to examine the basic questions raised in this thesis in a context where plan interdependence, and feasibility, as well as pure-quantity effects are important. The analysis is conducted in terms of dynamic systems expressed in differential equations.

The following few paragraphs refresh the reader's memory with respect to the strands in the literature which the analysis of this Chapter takes as points of departure. Following this, an outline of the contents of this Chapter is given.

The state of knowledge as regards the equilibration process under non-tatonnement in production and exchange is not satisfactory, as has already been discussed in Chapter III.

---

[1]That is, not causing problems of instability.

To refresh the reader's memory as to the state of the arts in the subject, R. Clower's attempts at an alternative interpretation of Keynes' contribution[2], as well as D. Patinkin's attempt at analysis of involuntary unemployment in Chapter XIII of his Money, Interest and Prices[3] may be viewed as the initial efforts to handle the problem at hand. As has been mentioned in Chapter III, Clower's main contribution was to point out that excess demand functions in a non-tatonnement economy should contain realized incomes as independent variables. Since planned excess demand functions in a Walrasian economy do not take into account quantities, Clower was able to show that Walras' Law is not relevant for purposes of identification of market pressure in a non-tatonnement economy when the situation is one of unemployment, i.e., when labour is off its supply curve. He did not proceed, however, to an analysis of the equilibration process in these circumstances.

D. Patinkin also attempted, as mentioned in Chapter III, a disequilibrium analysis of an aggregate economy but did not advance it in any appreciable degree mainly because he did not incorporate realized production volumes in the excess demand functions in a formal way. He did point out,

---

[2] R. Clower, (1960) and (1965).

[3] (1965).

however, that involuntary unemployment could arise if in the process of equilibration producers reacted to the accumulation of inventories by lowering production and demand for labour.[4] Since this remark implies that the excess demand for labour is sensitive to quantities rather than to prices alone, the modification to the labour demand function is similar to that suggested by Clower,[5] but it was not formally carried out by Patinkin.

Finally, Barro and Grossman[6] attempted an integration of Clower's and Patinkin's points but did not produce an internally consistent model with proper interrelationships. As a consequence their analysis of disequilibrium positions is fragmentary, while that of the equilibration process itself is non-existent.

The investigation in this Chapter starts with Section 5.1, where the general setting and the assumptions of the model to be used are outlined. Moreover, matters such as plan feasibility and of pure-quantity effects are discussed.

---

[4](1965), p. 318 ff.

[5]Clower concentrated on the case of excess supply of labour where labour's planned income does not materialize and consequently effective demand for commodities has to take into account realized rather than planned income. Patinkin examined the case of excess commodity supply where selling plans, and consequently income of producers are not realized, with consequent modifications in the demand for labour by these producers.

[6](1971).

Section 5.2 contains the analysis of the simplest possible model in a tatonnement as well as a non-tatonnement framework. The purpose of this Section is mainly to serve as introduction to the later parts of this Chapter, and also to illustrate in a concrete situation both the problems of plan feasibility and endowment effects in disequilibrium.

Section 5.3 contains the main analysis of this Chapter. An aggregate model with exchange money is developed, and tatonnement and non-tatonnement analysis is conducted in the case of the commodity being non-durable. Inventory considerations are introduced informally, and a graphical analysis of the durable-commodity case is attempted.

Finally, Section 5.4 contains comparisons and the conclusions of the analysis of this Chapter.

5.1 General Disequilibrium Analysis: the Setting and the Assumptions; Endowment Effects

The present Section contains a brief description of the analytical setting and of the main assumptions employed in the general disequilibrium analysis conducted in this Chapter. A number of remarks are made with regard to the abstractions used, and finally endowment effects as they operate in this framework are discussed.

(a) Setting and assumptions. The model to be employed in this Chapter is one of disequilibrium in production and exchange. The focus of the analysis is on interdependence of production, income and exchange in disequilibrium, and it is for this reason convenient to ignore distributional questions within this model.

To effectively rule out questions of distribution in general it is assumed, first, that production conditions for all producers in the economy are identical; it thus becomes possible to discuss the production side of the economy as if only one producer exists.[7] Second, all producible commodities in this economy are postulated to be produced under identical technical conditions; it thus does not matter what the allocation of resources among various commodities is, since the per-unit of commodity requirements in resources are identical for all commodities. Neither does it matter which of these commodities consumers prefer more or less, since cost conditions are identical also, and all commodities must have the same equilibrium price.

In effect, the above two assumptions are equivalent to the assumption that the production side of the economy consists of one producer producing one commodity.

On the consumption side, it is assumed that all individuals are alike, in the sense that the distribution of wealth among them does not affect the aggregate excess demand

[7]With his possibilities suitably enlarged.

for commodities. This condition is satisfied if the marginal propensities to consume[8] are independent of the amount of endowment held by each individual and, moreover, are equal for each individual. (The marginal propensity to consume, that is, is the same irrespective of the level of wealth that an individual holds, and the same for all individuals).

Under the above assumptions on the consumption side, the analysis may be conducted as if the economy has only one consuming unit. Such an economy has been labelled "Hicksian", and it has been proved that it possesses a unique equilibrium.[9]

With respect to non-producible commodities, it is here assumed that labour is the only one with an alternative use in consumption. All others have no alternative use, (i.e., they have a transfer price equal to zero) and are constant in quantity, hence their supply is insensitive to price variations.

Money as a non-producible and infinitely durable commodity with utility is also introduced in the model of this Chapter. It is then treated as "exchange money" only, as will be seen below (part c of this Section).

---

[8] Denoted by C_j^k in Chapter I.

[9] See Arrow and Hahn, (1971), pp. 217-221.

Finally, the producible commodity in the model to be used is sometimes considered non-durable (Section 5.3.c) and sometimes durable (Section 5.3.d).

The setting described above will be recognized as that of a short-run aggregate model of the economy. The relevant problem of choice in this economy (with money absent) is between the commodity and leisure. The single relative price to be determined is that of the real wage, that is, the price of leisure in terms of the other consumable commodity. All agents in this simple economy are assumed to be price takers.

As is proper in an analysis of general interdependence, demand is here properly connected with production levels in the economy, via income creation. The effect of variations of production volume on income and demand has been labelled a pure quantity endowment effect in this investigation; it is further discussed below (Section 5.1.c).

The market for labour is considered explicitly in the present analysis. Thus questions of feasibility of plans for production and of plans for labour employment and income creation are given full attention here, in addition to feasibility of sales plans discussed in Chapter IV. When production or income plans are unfeasible under the circumstances of the market, planned incomes differ from realized incomes and in this sense there are again pure-quantity endowment effects in operation, further discussed below (Section 5.1.c).

Finally, disequilibrium transactions are allowed in both the labour and the commodity market of the model. Expectations are assumed static and certain. Problems of demand deficiency in the Keynesian tradition are assumed away, in favour of a simpler "classical" framework, to be fully described in the next Section.

This, then, is briefly summarized as an aggregate short-run model of the economy, with leisure and one other commodity as the subjects of choice; perfect competition; static certain expectations; no demand deficiencies; and explicit disequilibrium.

(b) Some remarks on the abstractions employed. The discussion of the setting and of the assumptions just concluded above has outlined the major abstractions used in the analysis of this Chapter. Among the simplifications employed is the set of assumptions that effectively renders all distributional questions irrelevant in this framework, namely, the assumptions of one producible commodity, identical tastes and identical production conditions. Another major simplification is the neglect of "Keynesian" considerations of insufficiency of demand; finally, other abstractions include perfect competition, static price expectations, and "exchange" money. Some remarks on each of these abstractions will be offered in what follows.

With regard to the neglect of distributional questions, the abstraction allows the analysis of interdependence of input and commodity markets to be much sharper than otherwise. A multi-commodity general disequilibrium analysis of production and exchange would have to consider a number of input as well as commodity markets. Different producers and individuals might also be assumed to make decisions under differing production and choice sets. This addition would surely make the analysis quite general, and for that reason perhaps less productive in the sense that it would be relatively much more difficult to focus attention on one specific aspect of the economy in this framework.

Moreover, analytical convenience suggests that research proceeds better step by step: if, for example, it is found that in the present framework no major problems arise in the equilibration process of the model, such problems might then be expected to be found in settings where distributional effects are important (or, in settings where other simplifications of the model are removed).

Finally, distributional effects of disequilibrium transactions in pure exchange have already been discussed in the literature:[10] it is thus natural to concentrate on

---

[10]See Chapter III above.

questions relatively untouched, such as the effects of interdependence of input and commodity markets, and the pure-quantity effects of unfeasible plans in disequilibrium. This, as has been mentioned above, is the main focus of this Chapter.

With regard to neglect of "Keynesian" considerations of demand insufficiency, the only justification provided is that of the analytical simplicity achieved once such complications are neglected. Asset markets can then be ignored, and money as a medium of exchange (with utility of its own) may be introduced in the model and considered as a substitute for the producible commodity only. Clearly, a future direction of research lies in incorporation of other assets as money substitutes.

Finally, static expectations and perfect competition may be briefly discussed. With regard to the first abstraction from (presumably) more plausible mechanisms of expectation formation, it may be remarked that the initial development of tatonnement analysis reviewed in Chapter III above had to proceed along these lines originally; only later were discussions on alternative expectations hypotheses taken up. Thus, the abstraction of static expectations is only a first step in disequilibrium analysis. As for the assumption of price-taking, it is well known that despite its faults, it still pervades most analyses of equilibration. Thus, no apologies will be offered in this regard; analysis of price setter behaviour is plainly much more difficult to do, given the state of disequilibrium theory.

(c) Endowment effects. In order to discuss endowment effects in the setting described above, it may be useful to provide the reader with a preliminary glimpse at the process of equilibration under non-tatonnement in the model of this Section. Briefly, the market period opens with given commodity and input prices; entrepreneurs hold mainly stocks of money and non-producible resources other than labour; other individuals hold mainly labour stocks; transactions in labour at the initial prices take place; payments are made in terms of money at the given prices; production of commodities proceeds on the basis of the labour hired,[11] and exchange of commodities for money follows. Pressure on prices from surpluses or deficits in the markets during each period is reflected in a change of these prices at the end of the market period. Thus, the new market period opens with different initial prices, and the process repeats itself. It will be remembered that this has some affinity to the process loosely

---

[11]And with the use of other non-producible inputs.

described by Edgeworth in his discussion of the labour market.[12]

Consider now the possibility of pure-price endowment effects in the labour market: these cannot operate in this framework simply because the assumption is that the market opens with a given wage and all feasible transactions are completed at this wage. On the other hand, pure-price effects in the commodity market will operate as long as there is something to substitute the producible commodity with, such as exchange money, for example. Such endowment effects are assumed "well-behaved" in this Chapter.

With respect to endowment effects of the second kind, it is obvious that in the labour market they cannot operate because no price change and consequent re-evaluation of stocks occurs within each period. In order for them to appear, one would have to assume perfect attendance in non-tatonnement exchange in the labour market until equilibrium is reached, an assumption rather implausible, and therefore easy to reject.

---

[12]See Chapter II above. One might wish to specify that, in this story, the production process itself does not take too long a time. Alternatively, a lag structure could be adopted, according to which production becomes available for sale one period later: if the same kind of lag is assumed between income and consumption, the situation is formally very similar to that of the text. The framework put forward in the text seems the most convenient for the treatment of the problem in continuous time, which will be employed in this Chapter.

In the commodity market, the assumption of identical individuals, in terms of propensities to consume, precludes second-kind effects whatever the assumptions on price change. (The effect of identical and homogeneous utility functions on these effects has already been discussed in Chapter III.[13])

Endowment effects of the pure-quantity[14] variety are the main focus of the analysis of this Chapter: they operate every time income is created by the process of production of the commodity. As mentioned above, falsification of plans for income creation brings forward endowment effects on demand which make it differ from the planned amount. A more detailed discussion of this point will be undertaken in Section 5.2.c.

In summary, pure-price effects have been retained but assumed to cause no instability, and pure-quantity effects are focussed upon; second-kind effects are abstracted from in the following analysis.

---

[13] Section 3.1.3. The assumption of equal and constant C_j is effectively the same as that of identical and homogeneous functions.

[14] It may be objected that unless payment is in commodities, no quantity of commodity is involved. However, the term "pure quantity effect" is easily extended to cover payment in terms of exchange money. Money would be considered another commodity in this case. See Chapter I.

5.2 An Aggregate Disequilibrium Model without Money

This Section provides a simple, in fact the simplest possible, aggregate model of an economy in disequilibrium. In such a simple setting, it is relatively easy to provide illustrations of feasible and unfeasible plans; the operation of endowment effects of the third kind; and the relevance of realized rather than planned behavioural functions when the system is in disequilibrium. Illustration of these matters is the main purpose of the analysis of this Section: the discussion here provides a step towards the analysis of disequilibrium in a slightly more "relevant" framework, dealt with in Section 5.3.

Part (a) below deals with the formulation of the model, for a non-durable commodity. Part (b) provides an analysis of the tatonnement process in production and exchange. Part (c) gives a graphical illustration of endowment effects and feasibility of plans, while Part (d) contains the disequilibrium analysis in this simple framework. Finally, Part (e) contains closing remarks.

(a) Formulation of the model. On the production side, producers are assumed to attempt to maximize the aggregate planned quasi-rent on the (fixed) inputs other than labour, given the real wage at which the market opens. The problem is, in the aggregate,

$$\text{Maximize } Y_t^{r*} = Q_t^{o*} - (W_t/P_t)L_t^{d*} \tag{5.1}$$

$$\text{subject to } Q_t^{o*} = f(L_t^{d*}),\ f_L > 0,\ f_{LL} < 0 \tag{5.2}$$

where Y^{r*} is total planned quasi-rent, Q^{o*} planned real production, W and P the nominal wage and price,[15] L^{d*} the total planned demand for labour and f denotes the production function, with f_L , f_{LL} its first and second derivatives. The production set is thus assumed strictly convex.[16]

Asterisks have been used to denote planned magnitudes, as mentioned above. Because the distinction between planned and realized magnitudes is important in the discussion of this Chapter, asterisks will be used throughout for planned magnitudes. Corresponding realized (as opposed to planned) magnitudes will be denoted by the same symbols, but without asterisks.

First order conditions for local maximum of the above problem yield equations from which planned labour demand and production are derived, as functions of the real wage.[17] The convexity of the production function (5.2) which makes the

---

[15]Expressed in terms of a unit of account. In what follows, the terms "wage" and "price" will be used in this way. "Real wage", on the other hand, will denote the relative price of labour.

[16]That is, whenever two points a and b belong to it, the points on the line ab are interior points of it.

[17]Assuming the maximum exists.

problem soluble, also guarantees that the functions below have the "right" slopes. One may write,

$$Q_t^{o*} = F^*(w_t), \quad F_w^* < 0 \tag{5.3}$$

and $$L_t^{d*} = D^*(w_t), \quad D_w^* < 0 \tag{5.4}$$

where w denotes the real wage.

This completes the production side of the model. One additional point needs to be made, however, which is useful in other parts of the analysis: this refers to the planned quasi-rent shown (in real terms) in equation (5.1). This magnitude is properly regarded as part of the planned income in the economy, and is seen to depend on the real wage alone (since planned production depends on the real wage, too). More specifically, total planned quasi-rent may be written, using (5.2) and (5.4),

$$Y_t^{r*} = f\ [D^*(w_t)]\ - w_t D^*(w_t) \tag{5.5}$$

and its derivative with respect to the real wage is negative, since it may be written

$$dY^{r*}/dw = (f_L - w) D_w^* - D^*(w)\ ^{18} \tag{5.6}.$$

In fact, equation (5.5) is the wage-price frontier for this simple economy that is relevant to the producers as owners of

---

[18] From the first-order conditions for a maximum, $f_L = w$.

fixed inputs. Assuming only one fixed input,[19] its rental is readily derived from (5.5), and varies inversely with the real wage. It is thus not legitimate to consider the quasi-rent as a constant in an analysis such as the present one, and the consequences of treating it as a parameter are not unimportant, as will be seen below.[20]

Individuals as consumers are assumed to maximize the aggregate utility function[21]

$$U_t = U(Q_t^{d*}, L_t^{s*}) \tag{5.7}$$

where Q_t^{d*} and L_t^{s*} are planned commondity demand and labour supply, respectively.[22] The budget constraint for this maximization problem includes labour as well as quasi-rent income. Total planned income is equal to

$$Y_t^* = Y_t^{r*} + Y_t^{w*} = Y_t^{r*} + w_t L_t^{s*} \tag{5.8}$$

---

[19] A harmless assumption in the present framework.

[20] Contrast with R. Clower, (1965), pp. 105 ff., where this assumption is made. This leads to neglect of pure-quantity endowment effects in at least one case, as will be shown in (b) below.

[21] The utility function is assumed to represent a strictly convex preference ordering, that is: (i) when bundle a is preferred to bundle b, any bundle on the line ab other than b is preferred to b, and (ii) when the individual is indifferent between a and b bundles on the line ab other than a or b are preferred to a or b.

[22] The quantity of fixed inputs other than labour does not enter the utility function since these inputs are assumed to have no consumption use.

where Y^* is total planned income, and Y^{w*} total planned labour income both in real terms. The planned quasi-rent income is a function of the real wage and may be written as

$$Y_t^{r*} = Y_t^{r*}(w) \qquad Y_w^{r*} < 0 \qquad (5.9).$$

Thus the budget constraint which implies consumption of all planned income may be written

$$Q_t^{d*} = Y_t^* = Y_t^{r*}(w_t) + w_t L_t^{s*} \qquad (5.10).$$

The individual problems of maximization of utility subject to a budget constraint are in the aggregate equivalent to maximization of (5.7) subject to (5.10).[23] The first-order conditions for a local maximum yield equations from which the planned labour supply and the planned commodity demand at all real wages are derived. The budget constraint requires that planned income equal planned demand for the commodity, and in this sense planned labour supply and planned demand for the commodity are related. One may write,

$$L_t^{s*} = S^*(w_t) \qquad S_w^* \gtreqless 0 \qquad (5.11)$$

$$Q_t^{d*} = H^*(w_t) \qquad H_w^* \gtreqless 0 \qquad (5.12)$$

for planned labour supply and commodity demand, respectively. The ambiguity in the signs of the derivative of these functions is typical in choice problems with constraints, as is

[23]The proof for the "Hicksian" economy under study has been furnished by Arrow and Hahn, (1971), pp. 217-221.

well-known. In this simple model, it does not seem to matter, as it will be explained presently. In the model that includes exchange money, the slopes have been assumed "normal", thus effectively ruling out "misbehaviour" of pure price endowment effects.

The basic structure of the model has now been described. It may be convenient, before proceeding to analysis with this model, to summarize here the main behavioural functions involved. These are,

$$Q_t^{o*} = F^*(w_t) \qquad F_w^* < 0 \qquad (5.3)$$

$$Q_t^{d*} = H^*(w_t) \qquad H_w^* \gtreqless 0 \qquad (5.12)$$

$$L^{s*} = S^*(w_t) \qquad S_w^* \gtreqless 0 \qquad (5.11)$$

$$L_t^{d*} = D^*(w_t) \qquad D_w^* < 0 \qquad (5.4)$$

and they represent commodity production (which is identical with commodity supply in this model, since no inventory considerations are yet introduced); commodity demand; labour supply; and labour demand. All functions denote planned magnitudes. Consideration of plan feasibility and the consequent distinction between planned and realized behaviour will be undertaken below in this Section (part c).

The above planned functions can be combined to give excess demands and supplies for the two markets in this model. More specifically, one may write,

$$E_c^* = Q_t^{d*} - Q_t^{o*} = H^*(w_t) - F^*(w_t) = E_c^*(w_t), E^*_{cw} > 0^{24} \quad (5.15)$$

and $$E_L^* = L_t^{d*} - L_t^{s*} = D^*(w_t) - S^*(w_t) = E_L^*(w_t), E_{Lw}^* < 0^{24} \quad (5.16)$$

where E_c^* and E_L^* represent excess demand for the commodity and labour respectively, and E_{cw}^* and E_{Lw}^* their derivatives with respect to the real wage.

Finally, if equation (5.1) is inserted in (5.10) above one obtains

$$Q_t^{d*} - Q_t^{o*} = w_t \; (L_t^{s*} - L_t^{d*}) \quad (5.17)$$

which is Walras' Law for this model economy, since it says that the excess demand for commodities is equal to the excess supply of labour evaluated in terms of the commodity.

It now remains to explain why the slopes of the labour supply and commodity demand do not matter in this simple model. From the original problems of producers and consumers above we have, when both markets are in equilibrium, that the marginal physical product of labour (in production) and the marginal rate of substitution of the commodity for leisure (in consumption) must both equal the real wage, that is, equal each other. From the alternative problem of maximization of aggregate utility $U = U(Q,L)$ subject to $Q = F(L)$ (where Q and L denote commodity and labour, as before) first

---

[24] The lack of ambiguity in the signs of E_{cw}^* and E_{Lw}^* will be explained presently.

order conditions require the same equality. Thus, when the original system is in equilibrium, the alternative problem has a maximum and vice versa. But the maximum of the alternative problem has been proved unique (under the assumption of strict convexity of both the production and choice sets).[25] Thus, the equilibrium of the original problem is also unique.

Moreover, the convexity of sets in the alternative maximization problem guarantees that when the wage is below its equilibrium value in the original problems, there is excess demand for labour and excess supply for the commodity.[26] It thus guarantees that, irrespective of the slopes of labour supply and commodity demand, the slopes of excess functions are "normal", as specified in equations (5.15) and (5.16) above.

(b) Tatonnement analysis. Given Walras' Law, the excess demand in only one market need be examined, and the adjustment mechanism of the tatonnement process may be written,

$$\dot{w}_t = k[E_L^*(w_t)] \quad \text{or} \quad \dot{w}_t = -k[E_C^*(w_t)] \qquad (5.18)$$

---

[25] See T. C. Koopmans, (1957), p. 32, proposition 3.2.

[26] This is because the alternative problem of maximization of utility subject to the production function is actually the two original problems superimposed on each other. This is seen more easily in a diagram. See, for example, T. C. Koopmans (1957), pp. 18 ff.

where k is the speed of adjustment. Solution of the linearized form of this simple differential equation[27] yields the condition for local stability as follows:

$$E^*_{Lw} < 0 \qquad \text{or} \qquad E^*_{Cw} > 0 \qquad (5.19).$$

Stability is obviously satisfied, given the specifications of the behavioural functions.

The discussion may now proceed to an introduction to non-tatonnement analysis, by providing a graphical illustration of the problem of feasibility of plans in disequilibrium, and the consequent endowment effects of the Third Kind.

(c) Plan feasibility and endowment effects. The discussion of the model up to this point has been confined to planned magnitudes. This is justified as long as the process of equilibration is that of tatonnement, as was the case. Once agents are allowed to "act" in disequilibrium, however, the situation changes, as has already been mentioned above. Here, an illustration of the unfeasibility of production and/or income plans will be provided. (Sales plans are always realized in the model of this Section. See (d) below.) Moreover, the operation of pure-quantity effects due to falsification of income plans will be discussed.

Consider disequilibrium in the model of production

---

[27] An example of the procedure has been given in Chapter I, in relation to the illustration of the Correspondence Principle.

and exchange of this Section. The market period opens with a real wage, transactions in labour take place at this wage and production proceeds. The irrelevance of planned excess demands in the commodity market resides in the fact that <u>actual</u> supply and demand for commodities will, in fact, depend on the transactions <u>completed</u> in the labour market, while the corresponding <u>planned</u> magnitudes depend on <u>planned</u> transactions in the labour market. Thus, when the labour market transactions differ from planned (which they must if this market is in disequilibrium), it is plainly obvious that realized supply and demand in the commodity market will be different from planned. Under such circumstances, it may be desirable to consider realized rather than planned excess demand as a factor affecting the situation in the commodity market.

To illustrate the above, Figure 5.1 shows the production function $Q=F(L)$ and the indifference curves U^i derived from the utility function $U=U(Q,L)$, as well as a price line AHJ whose angle with the horizontal axis is w and represents the real wage assumed to rule when the labour market opens. From the Figure, producers plan to hire L^{d*} quantity of labour and produce $0B=Q^{o*}$ quantity of output, in order to maximize their quasi-rents. Individuals base their plans on

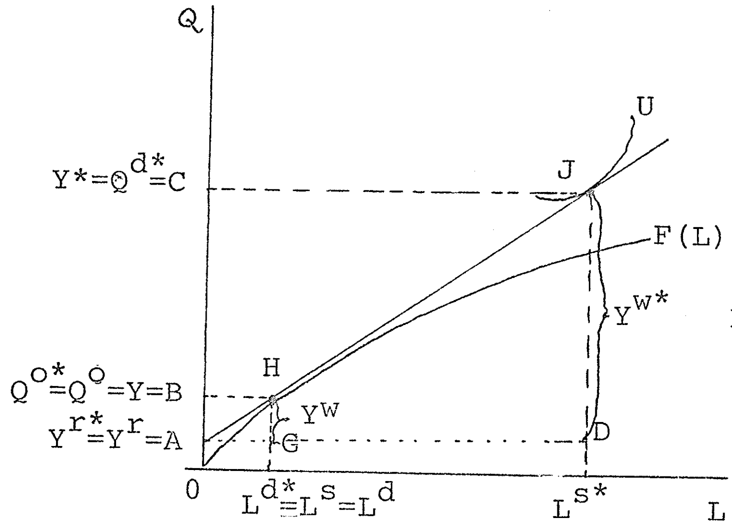


Figure 5.1

real planned income[28] from quasi-rents which equals OA, and the wage rate indicated by the tangent of the angle w. Their planned labour supply is equal to L^{s*}, and their planned income and demand for the commodity $OC=Q^{d*}$.

The situation in the labour market is clearly one of excess planned labour supply at the given real wage. If one considers planned magnitudes, this excess supply is matched by excess commodity demand of the same value at the given wage,[29] an illustration of Walras' Law. The concern here, however, is to examine plan feasibility. In this respect, it is clear that planned labour demand is certainly feasible at the given wage; hence, so is planned commodity production and planned maximum quasi-rent. Thus, planned labour demand,

---

[28] Evaluated in units of the commodity.

[29] The excess planned commodity demand is BC, equal to the planned excess labour supply GD multiplied by the real wage, that is, w.

production and quasi-rent are identical to their realized magnitudes, and this is indicated by the equalities $L^{d*} = L^d$, $Q^{o*} = Q^o$ and $Y^{r*} = Y^r$ in the Figure.

Planned income from labour, however, is unfeasible at the given wage: realized employment falls short of planned by the quantity $L^{s*} - L^{d*}$, and actual employment is in fact $L^{d*} = L^s$. Realized income from labour thus is equal to GH and total realized income is, of course, equal to total output OB, which falls short of planned total income by the amount BC (the amount of planned labour income not realized). Some of the foregoing may be summarized in the relation

$$Q^{o*} = Q^o = Y = Y^{r*} + wL^s < Y^* \quad \text{when } L^{d*} < L^{s*} \qquad (5.20)$$

which says that when transactions in labour and production are carried out in a situation of excess labour supply the consequence is that realized income is less than planned, due to falsification of labour income plans.

Consider now Figure 5.2 with a real wage at which there is excess planned demand for labour. Planned demand for labour, and hence planned production and planned quasi-rent income are all unfeasible under the circumstances. The available labour quantity being L^{s*}, realized labour demand must be equal to it; realized production will be Q^o=0B, which is also realized total income. The latter falls short of planned income by the amount of planned quasi-rent income which is unfeasible. That is to say, producers plan total income equal to planned production Q^{o*}=OF. Planned quasi-rent is 0C and planned labour costs are CF. In fact, they can only hire L^{s*} and produce Q^{o}=0B with a realized quasi-rent eqaul to 0A. On the other hand, the realized income of individuals from labour is equal to planned, that is, GH.

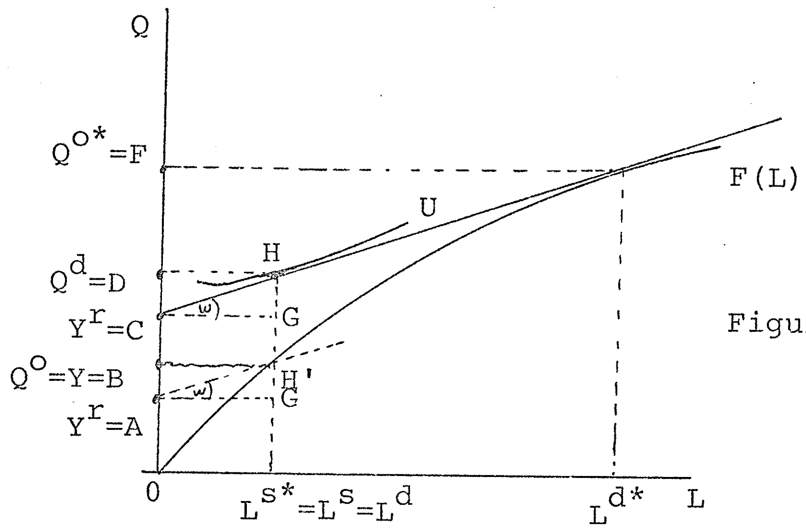


Figure 5.2

Under excess demand for labour, then, production plans are partially falsified, and quasi-rent income plans are falsified, too.[30]

In summary, the above illustrations have shown that production proceeds while the labour market is in

---

[30] Contrast this and the following discussion with that of R. Clower (1965). Because Clower treats quasi-rent as constant and always equal to its equilibrium value, he can identify a difference between planned and realized income only in the case of excess labour supply. The discussion in the text, with proper treatment of quasi-rent as a variable, has shown that planned and realized income will differ also when there is excess demand for labour. Moreover, Clower does not discuss unfeasibility of production plans, since he concentrates on the case of excess labour supply.

disequilibrium, total realized income will always fall short of its planned magnitude whether there is excess supply of or excess demand for labour. Furthermore, actual production will fall short of planned production when there is excess demand for labour.

Thus, the effects of disequilibrium transactions in the labour market, followed by production, have been shown to operate on income and production. The effects on income are recognized as those of the third kind. Consider, for example, the case of excess labour supply: individuals plan to sell labour services and obtain income in some means of payment[31] with which to obtain the producible commodity. Not all plans are realized, however, and thus the endowment terms of some individuals in the quantity of means of payment are less than planned. True, labour in these same endowments is still available, since it did not get employed, but labour is not accepted in exchange for commodities in the commodity market.

The effects of disequilibrium on market excess demand for commodities are thus pure quantity effects. Moreover,

[31] At this point, one may assume that payment is in the form of IOU's in commodities (alternatively, one could assume that producers have "circulating capital" with which they pay labour; this capital may be in the form of "wage goods", that is, it may consist of producible commodity stocks). Later in this Chapter, exchange money will be introduced.

the effects of unfeasibility of production on excess commodity demand are straightforward, namely realized supply falls short of planned.

(d) Non-tatonnement analysis. With the above illustration of the question of plan feasibility and the consequent endowment effects in disequilibrium, the analysis may now proceed to non-tatonnement equilibration in the simplest possible case. It will be assumed that technology and tastes do not change during the period required for equilibration of the system. Thus, the conditions of demand for and supply of labour will be the same in every market period[32] during equilibration. The excess demand for labour in each period will be assumed to exert pressure on the nominal wage (that is, the value of the wage in terms of a unit of account) in the standard way, that is,

$$\dot{W} = k_1 \; E_L^* \; (w_t) \qquad (5.21)$$

where $\dot{w}$ is the time-derivative of the nominal wage and w_t is the real wage as before.

In the commodity market, it is realized excess demand for commodities that will be considered as a factor affecting

[32] Excess demand for labour cannot be satisfied through inventory; neither does excess supply add to inventory. However, excess demand for labour in one market period will be expected to result in a change in the wage. See below in the text.

the commodity price change. That this is a logical choice follows from the discussion in part (c) above, where it was shown that realized magnitudes may differ from planned (since not all planned magnitudes are feasible in disequilibrium), hence pure-quantity endowment effects on realized production and demand must now be considered.

Realized production in each market period will depend on the situation in the labour market, and the consequent transactions in labour between producers and individuals in disequilibrium. Thus, in the case of excess labour supply, realized production will equal its planned magnitude. In the case of excess demand for labour realized production will proceed only to the extent that labour is available. In short, employment and the consequent production level will always be constrained by the short end of the labour market.[33] Thus, realized production may be written

$$Q_t^o \begin{cases} = f(L_t^{d*}) = f[D^*(w_t)] = F^*(w_t) \text{ if } \bar{w}_t > w, \text{ with } F_w^* < 0 \\ = f(L_t^{s*}) = f[S^*(w_t)] = G(w_t) \text{ if } w_t < \bar{w}, \text{ with } G_w > 0 \end{cases} \quad (5.22)$$

where G is a functional symbol. The first part of this equation shows that realized production is equal to planned if

[33] This reminds one of the Hahn process in non-tatonnement exchange, discussed in Chapter III, and of the same point made by Clower, also discussed in Chapter III.

the real wage is above equilibrium,[34] that is, when labour is in excess supply. The second part says that realized production equals the maximum amount that can be produced given the quantity of labour available when the labour is in excess demand. The quantity of labour available in this case is that of planned labour supply, as denoted by L_t^{s*} above.

The response of realized output to a change in the real wage then depends on whether the situation is one of excess demand or excess supply of labour. When there is excess supply, a fall in the real wage towards its equilibrium value increases the amount of labour optimally demanded by producers, and therefore increases realized production. When there is excess demand, in contrast, a fall in the real wage reduces the amount of labour supplied[35] and consequently the level of realized production. This explains why the response of production to the real wage is positive in this case, while it is negative at real wages above equilibrium.

One component of the realized excess demand for commodities has thus been specified in the above discussion. It remains to discuss the realized demand for the commodity.

---

[34] The value at which the demand for labour equals the supply. Due to the assumption of unchanged technology and tastes, this equilibrium remains unchanged throughout the equilibration period.

[35] The assumption here is, of course, that the response of labour supply to the real wage is positive.

In the simple abstract model of this Section there is not much choice as to the specification of the demand for commodities: the choice problem of individuals in this economy was confined to leisure versus consumption of the single commodity. Thus, whatever amount of income forthcoming from sales of labour services, it will be "spent"[36] on the commodity. In other words, realized commodity demand must be assumed equal to income: when income is less than planned, so is demand.

But the value of realized production is identically equal to income created in the process. Thus it turns out that in this simplest case realized excess commodity demand is identically zero, since realized production equals realized demand at all times.

It follows from the foregoing that in the present model the sales plans of producers are never falsified (the model in Section 5.3 allows for such falsification).

Before proceeding to a discussion of the implications of equality of demand to income for the process of adjustment, it may be useful to bring together all the components

---

[36] Again, the form of payment of labour does not make a difference to the argument: if payment was in IOU's, they will all be spent on commodity consumption. If it was in terms of commodities from "circulating capital", repletion of that capital and consumption by producers will exhaust production. The case of payment in exchange money will be examined in the next Section.

of the model for easy reference. Thus,

Planned Labour Demand: $L_t^{d*} = D^*(w_t)$, $D_w^* < 0$, (5.4)

Planned Labour Supply: $L_t^{s*} = S^*(w_t)$, $S_w^* \gtreqless 0$, (5.11)

Planned Excess Labour Demand: $E_L^* = E_L^*(w_t)$, $E_{Lw}^* < 0$ (5.16)

Realized Commodity Production: $Q_t^o = F^*(w_t)$ or $G(w_t)$, $F_w^* < 0$, $G_w > 0$ (5.22)

Realized Commodity Demand: $Q_t^d = Y \equiv Q_t^o$ (5.23)

Realized Excess Commodity Demand: $E_c = Q^d - Q^o$ (5.24)

Wage Adjustment Rule: $\dot{W} = k_1 \, [E_L^*(w_t)]$ (5.21)

Price Adjustment Rule: $\dot{P} = k_2 \, [E_c]$ (5.25).

Of the two equations describing adjustment in this framework, only one is in fact operative: the value of realized excess commodity demand (as distinct from planned excess demand) in disequilibrium is identically zero. Thus there is no pressure for price change in the commodity market. Walras' Law still holds as far as planned excess demands are concerned, and in this sense, excess planned supply in the labour market is still matched by excess planned demand in the commodity market according to equation (5.17) above. However, the planned excess demand in the commodity market cannot legitimately be considered as a determinant of pressure on the commodity price, since it is purely a theoretical demand: realized demand is necessarily constrained by realized income, that is by the value of realized production in disequilibrium.

It may be thought that realized excess demand in the model under discussion is equal to zero because of the fact that total demand equals income. However, this is not the assumption responsible for the lack of pressure on the commodity price under the circumstances: demand was supposed to equal income also in the tatonnement case, examined in (b) above, but excess planned demand for commodities was not equal to zero outside equilibrium. It is thus the consideration of realized rather than planned income and demand that is responsible for the zero value of realized excess demand for commodities at all prices. But consideration of realized magnitudes is required in disequilibrium; it thus follows that it is disequilibrium analysis which brings about this result.

(e) Concluding remarks: To recapitulate, the main result of the above analysis is that (under disequilibrium production) disequilibrium in the labour market will fail to produce a corresponding disequilibrium in the market for the commodity.[37] The implication of this result is that the labour market is the only one in the system where

[37] Contrast this general statement with Clower's remark that "Contrary to the findings of traditional theory, excess demand may fail to appear anywhere in the economy under conditions of less than full employment"(Clower, 1965,p.122). Clower confines his statement to the case of excess labour supply, because his treatment of quasi-rent as constant deprives his analysis of the general result. When the illegitimate assumption of constant quasi-rent is dropped, the general statement of the text above holds.

equilibration can take place. Thus changes in the nominal wage from market period to market period are necessary in order to bring the system to equilibrium. "Flexibility" of the nominal wage (as expressed in the speed of response of this wage to the excess demand in the labour market) is a key factor in the ability of the system to "attain" equilibrium. As for stability, the assumption with regard to the responsiveness of excess demand for labour to the wage ($E^*_{Lw} < 0$, in equation 5.16) assures that the system is locally stable.

The stationary state of the non-tatonnement model is the same as that of the tatonnement model of (b) above. The out-of-equilibrium behaviour in non-tatonnement is confined to the nominal wage, however.

The model presented here serves the function of making clear some basic problems associated with equilibration under non-tatonnement in production and exchange. Thus no further discussion of it is necessary, and the analysis proceeds, in the next Section, with formulation of a slightly more general model with exchange money, and later with inventories. Comparisons relevant to the main questions raised in this investigation[38] will be made with reference to that extended model.

---

[38]See Chapter I.

5.3 A Disequilibrium Model with Exchange Money

The analysis here extends the model discussed in 5.2 to include one more commodity: non-producible "exchange money", the stock of which is exogenously given and infinitely durable. The producible commodity is still assumed non-durable until Part (d) of this Section, where inventory considerations are introduced explicitly, but informally, in the model.

Part (a) below deals with the formulation of the model necessary in order to include exchange money as an additional commodity. Part (b) gives an analysis of the tatonnement process of equilibration in this model. Part (c) proceeds to disequilibrium analysis on the assumption that the commodity is non-durable, while Part (d) introduces the possibility that the commodity may be durable, and discusses the consequences of inventory considerations in the model. Finally, Part (e) contains closing remarks.

The main difference from the model of the previous Section lies in the range of choice as to how income may be "spent" in this model: it is now possible to demand not only the producible commodity but also "exchange money". In fact, the latter is a substitute commodity in the preferences of the typical individual, as will be seen presently.

The consequence of this increase in the range of choice is that it is now possible for commodity demand to differ from income. Total demand for the commodity and money must, in this model, equal the sum of total income and money balances, as will be seen in the specification of the budget constraint below. It is not necessary, however, that commodity demand equal income and demand for money equal money balances, unless the system is in equilibrium. Thus the consequence is that when commodity demand differs from income (because demand for money differs from money balances) it also differs from production, since the latter is identically equal to income.

As a result, sales plans of producers at expected prices may not be feasible, except in equilibrium. The commodity is assumed initially to be non-durable, that is, supply equals production. In this case, the excess supply or demand in the commodity market is taken care of by means of price variation, in the manner analyzed in Chapter IV. When inventory considerations are introduced, the excess commodity supply or demand is corrected by both price and inventory changes.

Unfeasibility (and falsification) of sales plans has consequences for production plans. In the case of non-durable commodity, the effects on production operate through price. When the commodity is durable, the effects on production come from price changes as well as from the fact that inventories are different from their optimal values outside equilibrium.

(a) Formulation of the model. The introduction of exchange money as an additional commodity may be made in the standard way,[39] through the utility function. The problem of utility maximization is then solved to obtain demand functions for money and the commodity, as well as the supply function for labour. The problem may be written, in the aggregate,

$$\text{Maximize} \quad U = U(Q^{d*}, L^{s*}, m^{d*}) \qquad (5.26)$$

$$\text{subject to} \quad Q^{d*} + m^{d*} = Y^{r*} + wL^{s*} + m^{s} \qquad (5.27)$$

where m^{d*} is the real amount of money demanded (measured in units of the commodity) and m^s is the total real quantity of money in the economy. The partial derivative of the utility function with respect to real money balances is assumed positive. The budget constraint has, on the left-hand side, the planned expenditure on the commodity and money, and on the right-hand side, the incomes planned from quasi-rent and labour as well as the money stock.

First-order conditions for this maximum provide equations from which the demand for commodities, the demand for money and the supply of labour (all planned functions) may be derived. These will, in general, depend on real money

[39] Compare D. Patinkin (1965).

balances and the real wage.

In order to simplify the exposition below, it will be assumed here that real balances do not affect the plans to supply labour, that is to say, the derivative of labour supply with respect to m is zero in this model.[40] Slopes of functions are assumed "normal", that is, pure-price effects are taken to be well-behaved.

Given the above remarks, the functions derived from the first-order conditions of the maximization problem above may be written,

$$Q_t^{d*} = H^*(w_t, m_t^s) \quad , \; H_w^* > 0, \; H_m^* > 0 \qquad (5.28)$$

$$m_t^{d*} = N^*(w_t, m_t^s) \quad , \; N_w^* > 0, \; N_m^* > 0 \qquad (5.29)$$

$$L_t^{s*} = S^*(w_t) \quad , \; S_w^* > 0 \qquad (5.11)$$

where N is a functional symbol and all other symbols have been explained before.

The production side of the model here is the same as in the model of the previous Section, summarized in equations (5.1) to (5.4). Of these, only the planned production

---

[40] If money balances are allowed to affect the labour supply, the excess demand for labour ceases to be homogeneous of degree zero in the nominal wage and the commodity price. Equilibrium in the labour market then depends not on the value of the real wage but on the specific values of the nominal wage and the commodity price. Since this dependence will occur with tatonnement as well as with non-tatonnement, abstraction from it is not expected to make a difference to the results of the comparisons sought here.

and planned demand for labour are useful here, and will be reproduced together with the other equations of the model to be analyzed. The complete model consists of

Labour Demand: $L^{d*} = D^*(w_t)$, $D^*_w < 0$, (5.4)

Labour Supply: $L^{s*}_t = S^*(w_t)$, $S^*_w > 0$, (5.11)

Commodity Production: $Q^{o*}_t = F^*(w_t)$, $F^*_w < 0$, (5.3)

Commodity Demand: $Q^{d*}_t = H^*(w_t, m^s_t)$, $H^*_w > 0$, $H^*_m > 0$, (5.28)

Demand for Money: $m^{d*}_t = N^*(w_t, m^s_t)$, $N^*_w > 0$, $N^*_m > 0$, (5.29)

and Money Supply (in real terms): $m^s = m^s$ (5.30).

Before proceeding, it is useful to show Walras' Law in the context of planned excess demands in this model, and to write out these planned excess demands, as they will be useful later. First, Walras' Law may be shown to hold for the sum of planned excess demands in this model if the value of quasi-rent income from (5.1) is substituted into equation (5.27) to yield,

$$(Q^{d*} - Q^{o*}) + (m^{d*} - m^s) + w(L^{d*} - L^{s*}) = 0 \qquad (5.31).$$

Thus, in tatonnement analysis at least, only two of the markets need be used. These will be the commodity and the labour market.

Second, the excess demands for commodities and labour may be written

$$E^*_L = E^*_L(w_t) \ , \ E^*_{Lw} < 0 \qquad (5.16)$$

and $$E^*_c = Q^{d*} - Q^{o*} = E^*_c(w_t, m^s_t) \ , \ E^*_{cw} > 0, \ E^*_{cm} > 0, \qquad (5.32)$$

where the signs of the partial derivatives with respect to the real wage and the real money supply are directly derived from the assumptions governing demand and supply.

It should be noted that the derivatives of planned excess demands are with respect to real magnitudes. Since these are the nominal magnitudes divided by the commodity price, it is easy to derive partial derivatives of the excess demands with respect to the nominal wage and the commodity price. These derivatives will be needed in the dynamic analysis that follows and are,

$$E^*_{LW} = E^*_{Lw} / P \ , \ E^*_{LP} = -E^*_{Lw} \ (W/P^2) \ ,$$
$$E^*_{cW} = E^*_{cw} / P \ , \ E^*_{cP} = -(E^*_{cw} + E^*_{cm} M^S) \ (W/P^2) \qquad (5.32a)$$

where capital letters signify nominal magnitudes. Thus, M^S denotes nominal money balances and E^*_{cW} the partial derivative of excess commodity demand with respect to the money wage.

The signs of these derivatives (given the assumptions of the model under analysis) are,

$$E^*_{LW} < 0, \ E^*_{LP} > 0 \ , \ E^*_{cW} > 0 \ \text{ and } E^*_{cP} < 0 \qquad (5.32b).$$

(b) Tatonnement analysis. The nominal wage is as usual assumed to respond to the excess demand in the labour market, and the nominal price to that of the commodity market. The adjustment equations are, then,

$$\dot{W} = k_1 \ [E^*_L \ (w_t)] \qquad (5.21)$$

and $$\dot{P} = k_2 \ [E^*_c \ (w_t, m^s_t)] \qquad (5.33).$$

Linearization of this differential equation system around the equilibrium wage and price and neglect of the constant terms of this linearization[41] results in the system (where the signs of the derivatives are indicated)

$$\begin{bmatrix} \dot{W} \\ \\ \dot{P} \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} \overset{(-)}{E^*_{LW}} & \overset{(+)}{E^*_{LP}} \\ \\ \overset{(+)}{E^*_{CW}} & \overset{(-)}{E^*_{CP}} \end{bmatrix} \begin{bmatrix} W \\ \\ P \end{bmatrix} \qquad (5.34)$$

or, in short-hand notation,

$$\dot{V} = KAV \qquad (5.34)$$

where $\dot{V}$ is the vector of time-derivatives of W and P; K is the diagonal matrix of speeds; A is the Jacobian of excess demands; and V the vector with elements W and P.

Stability conditions require that the trace of KA be negative and the determinant of KA positive.[42] Given that E^*_{LW} and E^*_{CP} are both negative (as has been established in (5.32b) above), the trace condition is satisfied for any speeds of adjustment k_1, k_2. As for the determinant of A,

---

[41]As it has already been pointed out, this neglect has no consequences for the stability analysis of the model, since it merely implies a transformation of axes.

[42]According to the Routh-Hurwicz conditions. See Chapter III above, and, for example, Samuelson (1947).

its sign is also consistent with stability, that is, positive.[43] Thus, the system is stable for any speeds of adjustment. Moreover, its approach to equilibrium cannot be cyclical,[44] that is, both price and wage must approach it from one side.

The dynamic movement of wages and prices during the tatonnement process may be illustrated in a simple phase diagram in wage-price space. In Figure 5.3, the line CEC is the commodity-market-equilibrium locus, that is, the set of combinations of wages and prices at which planned excess demand for the commodity is zero.[45] Points above this line represent combinations of wages and prices at which there exists excess demand for commodities while point below represent excess supply of commodities.[46]

---

[43]Using 5.32a, one finds that the determinant of KA is equal to the expression $k_1 k_2 E^*_{Lw} E^*_{cm}(-M^S\bar{W}/\bar{P}^3)$, where bars denote equilibrium values. The expression for the determinant is unambiguously positive, given the assumptions of the analysis.

[44]The off-diagonal elements of KA are of the same sign, and this precludes complex characteristic roots, as examination of the well-known formula for these roots shows.

[45]Strictly speaking, the straight line in the Figure is the linear approximation of the true line.

[46]This follows from the sign of either E^*_{cP} or E^*_{cW}.

The line OEL is the labour-market-equilibrium locus. Points above this line indicate excess supply of labour, and vice versa for points below.

Examination of the algebraic expressions for the slopes of these loci in (W,P) space will show that they must cut in the way illustrated, given the assumptions of the model.[47]

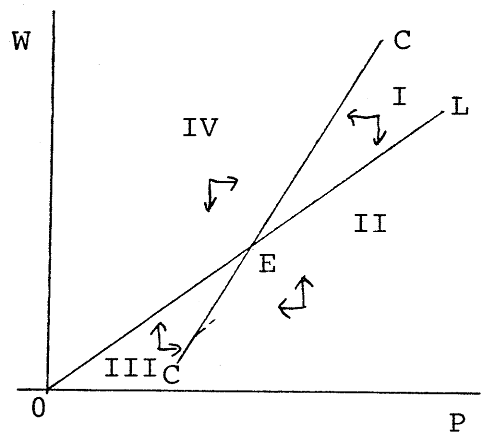


Figure 5.3

The diagram may now be used to illustrate the out-of-equilibrium movements of prices and wages traced by the auctioneer, while every participant in the market sits and waits for equilibrium to be established. The reason for such an examination is to provide the basis for comparison of this path with the path of non-tatonnement. If it turns out that the tatonnement path "approximates" the more "acceptable"

---

[47]The slope of the line OEL is $-E^*_{LP}/E^*_{LW}$ evaluated at equilibrium, which equals $\bar{W}/\bar{P}$. On the other hand, the slope of CEC is $-E^*_{CP}/E^*_{CW}$ which equals $\bar{W}/\bar{P} + E^*_{cm}M^S/E^*_{CW}\ P$. Thus, the slope of the labour-market locus is less than the slope of the commodity locus.

non-tatonnement movement, so much the better for tatonnement analysis of the process of adjustment itself. If not, such analysis will have to be abandoned in favour of non-tatonnement analysis of movement towards equilibrium.

Section I in Figure 5.3 is an area where both the wage and the price are too high for equilibrium. There is excess supply in both markets, and both wage and price tend to fall, as the arrows in this Section indicate. Section III in the Figure presents exactly the opposite problem, with too low prices and excess demands. Finally, Section II is one of excess commodity supply and excess labour demand, while Section IV has the opposite situation.[48]

In conclusion, the tatonnement process in this model is stable under the given assumptions, and does not cycle in the approach to equilibrium. Moreover, it can be shown that the dynamics of tatonnement will bring and keep the system to a situation of either excess demand or excess supply in both markets considered.[49]

---

[48]The money market is unambiguously in excess demand when the system is in I, and in excess supply when in III. It is not clear that the situation is in that market when the system is either in II or IV.

[49]The proof of this statement is based on the fact that most paths of the differential equation system tend to approach the "dominant characteristic vector"(the vector associated with the largest root). In the case examined, this vector lies in between the CEC and OEL lines. Thus, most paths will be approaching equilibrium via either Section I or III in the diagram.

(c) Non-tatonnement equilibration: non-durable commodity. As before, the introduction of disequilibrium labour purchases and production in each market period requires no modification of the labour market components of the model. It is the commodity market that has to be modified and this modification will be carried out first, followed by analysis of the non-tatonnement process in production and exchange. The commodity is explicitly assumed non-durable in this discussion. Thus, sufficient price variation is assumed to clear the commodity market in each period.

It has already been shown that when false transactions in the labour market are carried out, the total realized income from production following these transactions falls short of its planned magnitude because of falsification of either wage or quasi-rent income. In any event, the determination of realized demand for commodities is in fact based on realized income, which is a constant for purposes of utility maximization in this case, and a source of pure-quantity effects on demand.

The problem of utility maximization becomes,

$$\text{Maximize } U_t = U(L_t^s, Q_t^d, m_t^d) \tag{5.35}$$

$$\text{subject to } Q_t^d + m_t^d = Y_t + m_t^s \tag{5.36}$$

where the amount of labour supplied is a constant, and so is total income from labour and quasi-rent. The choice is now between the commodity and money, given a constant income and an initial money stock. From the first-order conditions for a maximum of this problem one obtains the realized demand functions for money and the commodity as follows:

$$m_t^d = N(Y, m^s) \ , \ N_Y > 0 \ , \ N_m > 0 \qquad (5.37)$$

$$Q_t^d = H(Y, m^s) \ , \ H_Y > 0 \ , \ H_m > 0 \qquad (5.38)$$

where the slopes of the realized functions have been assumed "normal".

The demand functions thus derived are seen to depend not only on prices (via M^s/P) as is the case of demand functions of traditional theory, but also on quantities, namely income realized from sales of input services.[50] This integration of quantities and prices in the demand functions has now been achieved. In fact, the realized demand for commodities (5.37) is easily seen to correspond to the traditional "Keynesian" expenditure function where the "real balance effect" introduced by Patinkin has been incorporated. Thus, expenditure depends on "real income" and the value of real money balances. The main point, however, is that "real income" as a determinant of expenditure is <u>realized</u> rather than planned income.

---

[50] This, it will be remembered, was the complaint voiced by Clower (1965) where he suggested a solution to it by means of the "dual decision hypothesis". As has been pointed out in Chapter III and shown in this Chapter, planned income is never realized in disequilibrium. Hence, the concept of "duality" in the decision-making process loses its meaning. The truth remains, however, that realized incomes must enter demand functions, to take account of <u>pure-quantity</u> effects in disequilibrium.

In the model under discussion, realized income -- which is identically equal to realized production -- is a function of the real wage ruling at the time when transactions in inputs took place. This has been formalized in equation (5.22) above, which may be used to substitute the real wage instead of income in the realized demand functions. With this substitution one obtains, for the demand for commodities,

$$Q_t^d = \begin{cases} H\ [F^*\ (w_t)\ ,\ m_t^s\ ] & \text{for } w > \bar{w} \\ H\ [G\ (w_t)\ ,\ m_t^s\ ] & \text{for } w < \bar{w} \end{cases} \qquad (5.39)$$

The derivatives of this function with respect to the real wage and with respect to the real money supply are as follows:

$$H_w = H_Y F_w^* < 0 \quad \text{for } w > \bar{w}$$
$$H_w = H_Y G_w > 0 \quad \text{for } w < \bar{w} \qquad (5.40)$$

and $H_m > 0$,

that is, a rise in the real wage when it is above its equilibrium value reduces commodity demand, because employment and income of labour is reduced. On the other hand, a rise in the wage when below its equilibrium value makes more labour available for employment and the increased income results to a higher demand. Finally, a rise in the real balances increases demand for commodities.

The demand side of the commodity market has now been specified. The supply side is as in the equation (5.22), depicting realized production.

Walras' Law cannot be shown to hold with respect to the realized excess demands in this model: as is obvious from equation (5.36) above, the sum of realized excess demands for money and commodities is identically zero at all prices.[51] Thus, either positive or negative excess demand in the labour market fails to be reflected in the other markets of the system.[52] The consequence of this is that the labour market must equilibrate by means of its own devices.

It will be recalled that this same conclusion was reached in the case of the previous Section, where demand was always equal to income. Thus it has again been shown that it is not the assumption of income-demand equality that throws the burden of equilibration on the labour market; it is rather the fact that realized and not planned excess demands are considered in the disequilibrium analysis of this and the previous Section.

Though Walras' Law does not hold with respect to all excess demands, it certainly holds for the sum of the realized excess demand for money and the commodity, as has already

---

[51]Substitution of Q^o for Y in that equation makes this obvious.

[52]This is again a generalization of Clower's claim (1965,p.122) that excess labour supply fails to be reflected in the other markets of the system. See also Section 5.2(c) above.

been pointed out by means of equation (5.36). It is thus possible to ignore what happens in the money market of the system, since if it is in equilibrium, the commodity market will also be in equilibrium.

The model to be analyzed here, thus, consists of the equations for the labour market, and those for the commodity market, just developed. These equations are reproduced here for easy reference:

Labour Demand: $L^{d*} = D^*(w_t)$, $D^*_W < 0$, (5.4)

Labour Supply: $L^{s*} = S^*(w_t)$, $S^*_W > 0$, (5.11)

Commodity Demand: $Q^d = H[F^*(w_t) \text{ or } G(w_t), m^s_t]$ (5.39)

and Commodity Supply: $Q^o = [F^*(w_t) \text{ or } G(w_t)]$, (5.22)

where labour demand and supply are planned and commodity demand and supply are realized. As mentioned above, the money market may be omitted.

The excess demands for labour and the commodity may be written,

$$E^*_L = E^*_L(w_t), \qquad E^*_{LW} < 0 , \qquad E^*_{LP} > 0 \qquad (5.16)$$

$$E_c = Q^d - Q^o = H[F^*(w_t) \text{ or } G(w_t), m^s] - [F^*(w_t) \text{ or } G(w_t)] \qquad (5.40)$$

and the responses of excess demand for commodities to the nominal wage and the price of the commodity are[53]

$$E_{cW} > 0 \text{ and } E_{cP} < 0 \quad \text{for } w > \bar{w}$$
$$E'_{cW} < 0 \text{ and } E'_{cP} \gtreqless 0 \quad \text{for } w < \bar{w} \qquad (5.41).$$

where the prime on the derivatives distinguishes them from the same derivatives when the wage is above its equilibrium value.

The discussion may now proceed to the equilibration process, by specifying the wage and price adjustments in the standard ways:

$$\dot{W} = k_1 \, [E^*_L] \qquad (5.42)$$
$$\dot{P} = k_2 \, [E_c] \qquad (5.43).$$

The linearization of this system around the equilibrium must take care of the fact that the derivatives of the excess demand for commodities with respect to the wage and price are not the same on both sides of the equilibrium point, as is clear from (5.41) above. Thus, a "left-hand"

---

[53]From (5.40), $E_{cW} = (H_Y F^*_W - F^*_W)/P > 0$, and $E_{cP} = W[F^*_W(1-H_Y) - E_{cm}M^S/P^2$ which is less than zero at least as long as $H_Y \leqq 1$. Since H_Y is in fact the marginal propensity to spend out of real income, the assumption concerning its value is eminently reasonable.

The expressions for E'_{cW} and E'_{cP} are identical to the above but with the term G_W _in lieu_ of F^*_W. Since G_W is of opposite sign to that of F^*_W, the derivative E'_{cW} is negative, while while E'_{cP} is of ambiguous sign. This sign ambiguity causes problems for the stability of the system, as will be seen below.

and a "right-hand" derivative must be used, and each must be supposed to hold when the proper values of the real wage are considered. The linearized system may be written[54]

$$\begin{bmatrix} \dot{W} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} \overset{(-)}{E^*_{LW}} & \overset{(+)}{E^*_{LP}} \\ \overset{(+)}{E_{cW}} & \overset{(-)}{E_{cP}} \end{bmatrix} \begin{bmatrix} W \\ P \end{bmatrix} \quad \text{for } w > \bar{w} \qquad (5.44)$$

or, in short-hand notation,

$$\dot{V} = KBV \qquad \text{for } w > \bar{w} \qquad (5.44)$$

and, for the case of the real wage being below its equilibrium,

$$\begin{bmatrix} \dot{W} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} \overset{(-)}{E^*_{LW}} & \overset{(+)}{E^*_{LP}} \\ \overset{(-)}{E'_{cW}} & \overset{(?)}{E'_{cP}} \end{bmatrix} \begin{bmatrix} W \\ P \end{bmatrix} \quad \text{for } w < \bar{w} \qquad (5.45),$$

or, in short-hand notation,

$$\dot{V} = KDV \qquad \text{for } w < \bar{w} \qquad (5.45).$$

The signs of the elements of the Jacobians B and D are given in this notation.

The mechanics of the formulation of the process of non-tatonnement in this simple model are now complete, and stability may be examined. The values of the determinants

---

[54]Neglecting constant terms.

of KB and KD must be positive, and their traces negative. As it turns out, the values of the determinants of KB and KD are in fact equal to the common value:

$$-k_1 k_2 W E^*_{LW} E_{cm} M^S / P^3 \qquad (5.46)$$

which is unambiguously positive, given the assumptions made in the formulation of the model.[55] Thus, the first of the stability conditions is satisfied by both determinants.

On the other hand, the trace condition is unambiguously satisfied only by KB, whose main diagonal elements are both negative, as may be seen from (5.44). The trace of KD, however, is not unambiguously negative, since the sign of E'_{cP} is ambiguous.

The possibility of instability thus arises when the real wage is below its equilibrium value, and depends on the sign of the trace of KD, that is,

$$k_1 E^*_{LW} + k_2 E'_{cP} \qquad (5.47) .$$

The economics of this case may now be briefly considered. Given the specification of the model under analysis,

[55] In fact, the value of the determinants of KB and KD is almost identical with the value of the determinant of KA in the tatonnement analysis in part (b) above. The only difference is that it is the partial derivative of the planned excess demand for commodities with respect to real balances that is involved in the value of the determinant of KA, while here it is the derivative of the realized function, E_{cm} .

it is clear what effect a change in the commodity price has on the realized excess demand for commodities: on the one hand, a rise in price lowers the real wage and makes for conditions of excess demand for labour. Production falls as a consequence of less labour available to employ, and the supply side of the commodity market tends to create conditions for excess commodity demand. Realized income and real money balances, however, both fall as a consequence of the rise in price, and therefore demand for commodities also falls.

If it happens that the fall in realized commodity demand is greater than the fall in the commodity supply, the response of excess demand to price, E'_{cP} , is negative, and there is no possibility of instability. If, however, the fall in realized demand is less (in absolute value) than the fall in the commodity supply, the response of excess demand to the price is positive, and the possibility of an unstable system is present.

Even when the response of excess demand for commodities to price is positive, it is still possible that the trace of the matrix D as shown in equation (5.47) be negative, thus assuring stability. The determining factor in this case is obviously the relative size of the speeds of adjustment of price and wage. Thus, as is clear from (5.47), if the speed of adjustment in the nominal wage is high relative to the speed of price adjustment, the possibility of instability may never materialize.

Casual empricism does not seem to support the suggestion that the speed of adjustment of wages is greater than that of prices; the possibility of instability raised above may thus become a reality, once the basic condition of the negativity of E_{cP} fails to be satisfied. Some more discussion of this point is found towards the end of this Section.

The stability analysis of the system is now complete, and the discussion may turn to the question of the path of adjustment towards equilibrium, assuming that the system is stable. If the initial real wage is above equilibrium, the approach cannot be cyclical,[56] while if the system finds itself with real wage below equilibrium, it may approach equilibrium in a cyclical fashion.[57]

Finally, the question of the stationary state of this disequilibrium model may be raised: under the assumption that the commodity is a non-durable, the stationary state of the model does not change when disequilibrium behaviour is taken explicitly into account: the labour market will be

---

[56] Both off-diagonal elements of B are of the same sign, and this precludes complex characteristic roots.

[57] The off-diagonal elements of D are not the same sign, and the possibility of complex roots exists.

in equilibrium only when the real wage attains the same value as that required for equilibrium in the tatonnement version of the model. At this real wage, planned and realized income are equal, and probably the only adjustment necessary would be a change in nominal wages and prices such that -- while the real wage does not change -- the price of the commodity adjusts so that demand for the commodity equals supply. By Walras' Law as applied to commodity and money demands, the money market will also be in equilibrium under these conditions.

Before proceeding to a graphical illustration of most of the above points and to an example of a parameter change within this disequilibrium model, it may be desirable to remind the reader of the assumption that the producible commodity in this model is a non-durable. The implication is that the market for it is "cleared"[58] during each period, and thus there are no carry-overs of inventories from one market period to another. Such carry-overs will be introduced informally in the next part of this Section.

Most of the foregoing may be summarized and

---

[58]The speed of adjustment of price, that is, may assume to be high enough to achieve this effect. Since the commodity is non-durable, it is plausible to think in these terms. One could calculate the speed of adjustment necessary to equalize the given commodity supply to the variable commodity demand, via variation of real balances. The required value is $k_2 = p^2 / E_{cm} M^s$.

illustrated in a diagram similar to that used for the tatonnement process. In Figure 5.4 below, the line OEL is identical to the labour market equilibrium locus of the tatonnement model.[59] The commodity market equilibrium locus now has a positive slope for real wages above equilibrium[60] and possibly a positive slope for values below equilibrium.[61] In the Figure, this latter possibility is illustrated by three possible portions of CEC below equilibrium real wage, namely, EC, EC' and EC".

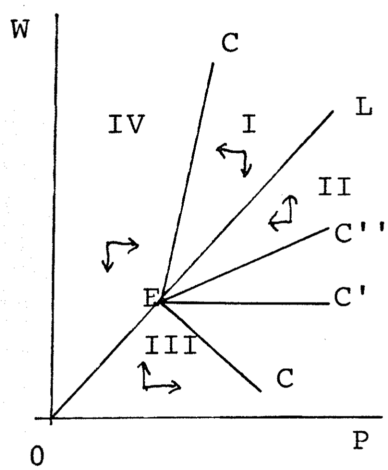


Figure 5.4

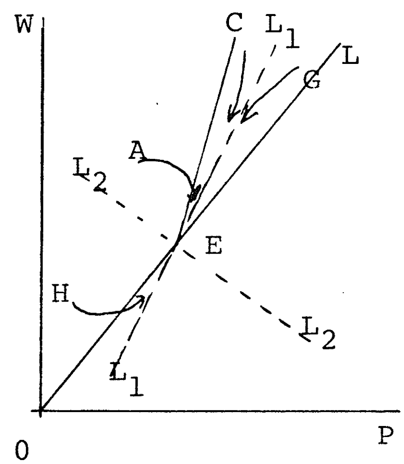


Figure5.5

The ambiguity of the slope of the lower portion of CEC relates to the possibility of instability in the system.

---

[59] As before, the slope is equal to $-E^*_{LP}/E^*_{LW} = \bar{w}$.

[60] Equal to $-E_{cP}/E_{cW}$, that is, equal to $\bar{w} - \dfrac{E_{cm}\, M^s}{F^*_W\,(1-H_Y)}$, which is greater than $\bar{w}$.

[61] Equal to $-E'_{cP}/E'_{cW}$. This slope may be positive, but it is always less than the equilibrium real wage, since, with the above partial derivatives evaluated in the neighbourhood of equilibrium the slope equals $\bar{w}\,[1-E_{cm}M^S/G_W(1-H_Q)]$.

In order for this possibility to exist, the slope must be positive, as is the portion EC".[62]

The W,P space in the Figure is again divided into four sections with the familiar circumstances of excess demands for labour and the commodity and the consequent tendencies in wage and price movements, summarized by the arrows. The situation is similar but not identical to that of the tatonnement process analyzed in the previous section, since the approach to equilibrium may be different and the possibility of instability is present.

Figures 5.5 and 5.6 may be used to illustrate the approach to equilibrium in non-tatonnement. In Figure 5.5, the characteristic vectors of matrix B above are shown as lines L_1L_1 for the dominant and L_2L_2 for the other vector. Since matrix B applies in the case of excess supply of labour, the characteristic vectors are relevant only for this case. Thus, if the system is initially in section I, it will remain there, while if in section IV to the right of L_2L_2 , say point A, it will enter section I eventually. If, however, the initial conditions are as at point H, the system will cross to section III where matrix D becomes relevant.

Once in Section III, the system may cycle or approach

---

[62] Still, instability need not necessarily obtain, as has been discussed above.

equilibrium asymptotically or exhibit instability. In Figure 5.6(a), the cyclical movement is illustrated which eventually brings the economy to Section I and to equilibrium. Figure 5.6(b) shows the non-cyclical approach to equilibrium,[63] and 5.6(c) the unstable case.

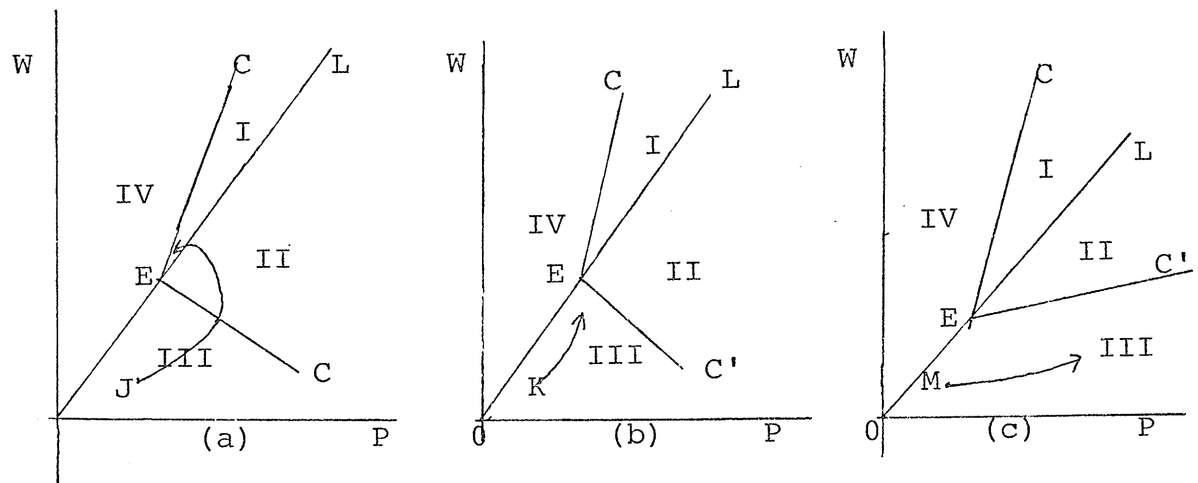


Figure 5.6

The movements of the relevant variables in the above illustrated cases may now be discussed. When the system starts with unemployment and excess supply (Section I), both prices and wages fall but the real wage and production movements may be in either direction, i.e., the real wage may fall and production may rise (as in the path from A to E in Figure 5.5) or the real wage rise and output fall (as in the path from G to E). When the initial position is excess demand for commodities and excess supply of labour (Section IV)

---

[63] The dominant characteristic vector in this case may be positively or negatively sloped. The path in the Figure assumes a positive slope.

prices rise and wages fall, output rises and the real wage falls. The system may then either eliminate excess commodity demand first, in which case it enters Section I, or the excess labour supply first by entering Section III. With the initial position in Section III, i.e., with excess demand in both commodities and labour, the possibilities depicted in Figures 5.6 arise. Thus,(Figure 5.6(a)) with prices and wages rising the real wage may rise or fall for some time and production will move in the same direction. When excess commodity demand is eliminated and excess supply appears (Section II), the real wage will unambiguously rise since prices will start falling, and production will unambiguously rise, too. Elimination of the excess demand for labour and appearance of excess supply will be the last stage of equilibration in this case of cyclical approach. Once in Section III, however, the system may approach equilibrium by means of wage-price inflation, with rising real wage and output (Figure 5.6(b)). Finally, it may exhibit tendencies to instability, with prolonged wage and price inflation and recession in output, until either parameters or speeds of adjustment change so that the real wage rises, thus bringing the economy to Section II.

The reader is reminded that the possibility of instability (once it exists) becomes greater the less the speed of wage adjustment is in relation to the speed of price adjustment. Thus, instability is probable to happen in a situation in which labour is the loser in the wage-price race.

Moreover, it may be of interest to note that in most of the cases discussed above, the system must finally approach equilibrium from Section I of the diagrams, which means wage and price deflation. As will be seen in the following paragraph, price and wage deflation may be avoided by means of an increase in the money supply. In the absence of policy, however, and in the presence of some downward inflexibility, the process of equilibration may be painful.

It is obviously not legitimate to identify the dynamic movement of this model economy to the "real world" without some elaboration of the simple model used in this Section. Nevertheless, it is interesting to see how parametric shocks would work themselves out in this simple framework. In the next paragraph, the effect of a change in the money supply is briefly examined.

A change in the nominal stock of money will displace the commodity equilibrium locus along OEL, to the right for an increase and to the left for a decrease in M^s. Thus, for an increase in money supply, the path of equilibration will be as in Figure 5.6, with the results of the "crude" quantity theory holding between equilibria. If the system happened to be in Section I when the money supply changed, some of the deflation necessary for equilibration will be avoided.

With the analysis of the non-durable commodity model complete, the discussion in part (d) below introduces non-durability and inventories in an informal way.

(d) Disequilibrium analysis. Non-durable commodity. As has already been mentioned, introduction of inventories in the model implies that the commodity market no longer has to clear by means of price variation sufficient to make demand equal to production in each period. Inventory movement may take up some of the discrepancy, and some price change may also happen.

To the extent that inventory positions are changed to accommodate part of the excess supply or demand in the commodity market, one must expect that inventory positions will affect plans for production and demand for labour in the next period. It thus becomes the case that production and sales plans, as well as demand for labour, are now dependent on the inventory position which reflects the past history in the commodity market.

The formal modification necessary to the model in order to take account of inventory effects on the dynamic movement of the system is thus not the simplest one, since it involves making labour demand and supply of the commodity dependent on the integral of past excess flow demands. Such formal modification will not be attempted but instead the obvious changes in results will be discussed, with the help of diagrammatic analysis.

Consider Figure 5.7(a), an exact duplicate of those illustrating the non-durable commodity case. Any point on the line OL in this Figure indicates equilibrium in the labour market. If firms are in stock equilibrium, the same line describes labour market equilibrium in the model of the present section.

Now suppose that firms have inventories in excess of the optimum amount: then, at what was before the equilibrium real wage these firms will demand less labour,[64] and as a consequence, the line OL in the Figure indicates excess supply of labour rather than equilibrium in the labour market. As a matter of fact, the demand for labour will be lower than before at any real wage level if firms are in an excess inventory position, and the opposite is true if there is inventory deficiency.

This implies that the labour market equilibrium locus depends directly on the inventory position, and rotates

---

[64] Because production will be less than sales in this case. Compare the analysis of Section 4.4 <u>infra</u>.

clockwise if inventories are in excess and counter-clockwise if inventories are deficient. Moreover, this dependence on inventories means that the slope of the locus at any time depends on the past history of equilibration of the economy. Finally, it has already been mentioned that in stock-flow equilibrium where flow demand equals flow supply and inventories are optimal, the locus is that of the non-durable model.

The dependence of the labour market equilibrium locus on the inventory position and consequently on the path of equilibration is the first consequence of introducing inventory considerations in the model under analysis here. The second consequence concerns effective demand. The latter has been assumed to depend on realized income which is identical in value to realized production. But realized production equals planned production when the real wage is above equilibrium and firms are in the optimum inventory position: hence $w > \bar{w}$ and inventories are in excess planned and realized production will be less than they would be with stock equilibrium. Since realized output is one of the determinants of demand for commodities, that demand will be less than what it would be if firms were in stock equilibrium.

The above remarks with regard to the consequences of inventory considerations on realized demand have been confined to the case where the real wage is above its long-run, stock-flow equilibrium value. For levels of the real

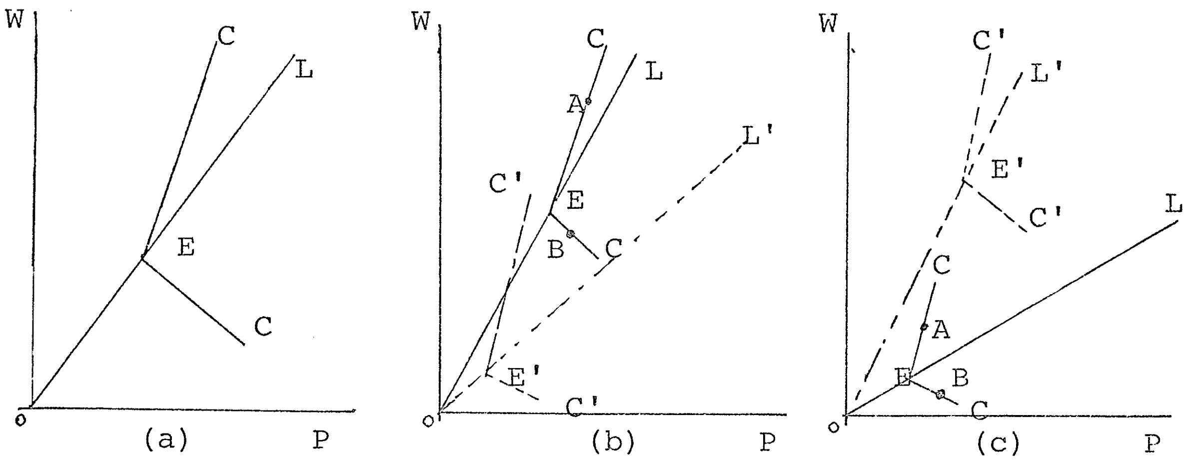


Figure 5.7

wage below this value, the planned production will again be affected by inventory positions but since realized production is less than planned, there will probably not be any consequences on realized demand income and commodity stock disequilibrium.

The third consequence of introducing inventory considerations involves commodity supply as distinct from production. If there is excess inventory, planned commodity supply will be assumed to exceed production by the amount of planned inventory liquidation, while with inventory deficiency planned supply will fall short of planned production. The fact that planned production will not materialize at wage levels below equilibrium complicates matters a little, and the situation is better explained with the help of

Figure 5.7.

In Figure 5.7(b), solid lines represent the original loci and dotted lines the new loci under the assumption of a sudden inventory shock which has brought firms into a commodity stock surplus position.[65] The labour market equilibrium locus rotates to OL' from OL and the commodity market equilibrium locus shifts to C'E'C' from CEC.

The rotation in the labour market locus has been sufficiently explained both in this Section and in Chapter IV. The shift in the commodity market locus is due both to endowment effects of the third kind operating on demand through realized income, and to the effects of inventory positions on supply, both mentioned above. At point A in the Figure, production after the inventory shock has taken place is less than before, and so is income and demand. Supply is the same as before. The result is excess commodity supply at A, and a lower price level is needed to equilibrate the market, that is, CE shifts to C'E'. At point B, on the other hand, production and income are the same as before,[66] and so is demand, but sales are higher due to inventory liquidation. Hence there is again excess supply at B, and

---

[65] One may choose to think of this situation as caused by a fall in the optimum amount of inventories of producers.

[66] Since realized production is less than planned, the fact that planned production is less than before probably does not matter.

the commodity locus has shifted to E'C' from EC.

In summary, inventory surpluses rotate the labour market locus clockwise and shift the commodity market locus to the left.

Now consider Figure 5.7(c) for the case of inventory deficiency. The labour market locus rotates counter-clockwise as expected. For the commodity market locus, at point A, production and income are higher than before and so is demand for commodities, while supply is either the same as before or less. Hence there is excess commodity demand after the shock which produced inventory deficiency, and the locus shifts to C'E' from CE. On the other hand, at point B, production is the same as before and so is income and demand.[67] Supply is either the same or less than before. Thus, point B is either an equilibrium again, or a point of excess demand, in which case EC shifts to E'C'.

In summary, inventory deficiencies rotate the labour market locus counter-clockwise and shift all or portion of the commodity market locus to the right.

With the consequences of introduction of inventories clear at least in diagrammatic terms, the investigation now proceeds to analysis of equilibration in this case. In

[67] The fact that planned production is higher than before is immaterial, since it is not realizable.

Figure 5.8 the initial position of full stock-flow equilibrium in the system was originally at E, but a decrease in the demand for commodities[68] has autonomously shifted the commodity locus to C'E'C'. At point E there now exists excess supply of commodities though the labour market is still in equilibrium.

For the sake of simplicity in exposition, let the system stay at point E for one period. This implies price rigidity and adjustment in the commodity market solely via inventory variation. At the end of the period, inventories are in excess and firms reduce their planned production at the going real wage OW; full employment of labour now requires a fall in the real wage equal to the difference between the slopes of OL and OL'. At the initial price level, the fall in the money wage necessary to achieve this result is equal to Ea. The unemployment associated with this situation has been called "involuntary",[69] since it occurs in a situation in which the equilibrium real wage indicated by OL rules but the labour supplied at this real wage cannot all find employment. Since OL is clearly not "too high" for stock-flow equilibrium, it is appropriate to attribute the unemployment at E not to a real wage that is too high but to

---

[68]Implying an algebraically equal shift in the demand money.

[69]Compare Patinkin, (1965), Chapter XIII.

conditions of demand deficiency and price-wage sluggishness. If price-wage movements were instantaneous, the system would have moved to point E' by wage-price deflation and no change in employment would occur.

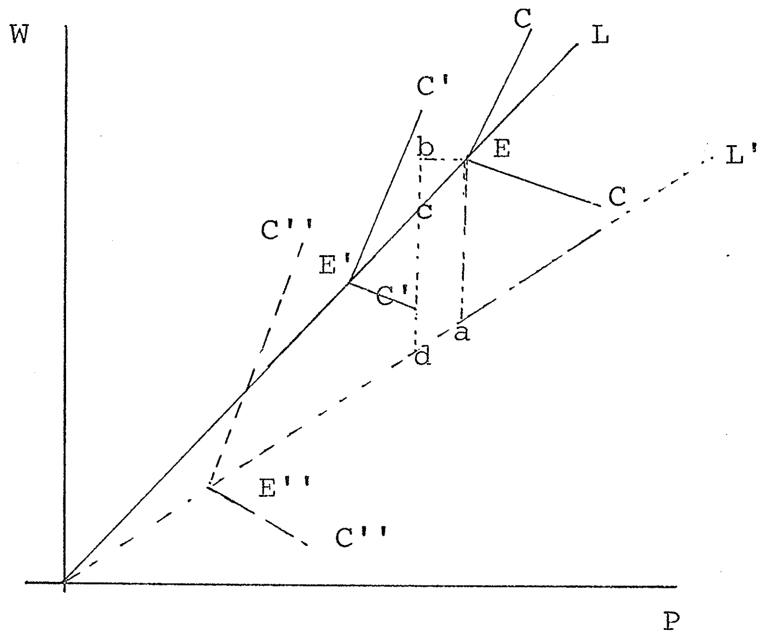


Figure 5.8

The foregoing explanation of involuntary unemployment is thus properly attributed to sluggishness in the adjustment required from the price system in the face of a fall in aggregate demand, and seems close to Keynes' view of the matter in Chapter 19 of the "General Theory".[70] Notice that even a fall in the money wage by Ea may not increase employment for any appreciable length of time, since the output implied by OL' cannot be sold, inventories accumulate and

[70] (1936).

the line OL' rotates clockwise again causing unemployment in the system. The fall of the money wage by Ea did not affect realized excess demand in any positive way, and thus the wage change alone did not bring the system to equilibrium neither did it correct the unemployment situation. It is only through both wage and price falls that the situation may be corrected, the price fall stimulating money demand for commodities and reducing aggregate excess commodity supply. The system may then move to both commodity and labour market equilibrium, and finally to stock equilibrium, too.[71] The path towards equilibrium is not as simple as before, however, because of inventory repercussions on the economy, and to this analysis the investigation now turns.

Consider again Figure 5.8, and assume that the economy had been operating for some time in stock-flow

---

[71] cf ibid, pp.260 ff. The analysis in Keynes is not explicitly in terms of inventories, but the general argument seems very close to the above analysis, especially on page 261 where he allows enterpreneurs to expect that a reduction in money wages will allow them to sell more output than before at an increased profit. This assumption is equivalent to the static expectations assumption in the text, and Keynes' enterpreneurs are disappointed at the proceeds from realized output except in the case where the marginal propensity to consume is unity. Hence, unemployment is again created. Finally, the effect of falling wages and prices is the same here as in Keynes (pp. 267 ff.), though for reasons different than Keynes' since the demand side of the model here is "classical" in flavour.

equilibrium at point E, but the demand for commodities shifts, in period O,[72] to C'E'C'. The resulting commodity excess supply is only partially corrected via price change, the price falling by Eb, and partially by inventory accumulation. Period One thus opens with a higher real wage,[73] and employment and production fall due to this higher wage. But this is not the only effect in this economy: inventory accumulation has affected planned production negatively, and the latter is less at all real wages than it was with stock equilibrium. The situation is reflected in the excess supply of labour: if the money wage falls to c, it will only reduce the portion of unemployment which is due to the higher real wage. A further fall to d is required to reduce the unemployment caused by inventory accumulation.

The fall in production resulting from inventory accumulation shifts the commodity locus further to C''E''C'' intensifying the excess commodity supply situation in period One. Both price and wage now fall in that period, in accordance with the adjustment assumptions set out above, and the economy moves southwest in terms of price and wage configuration. The inventory accumulation of period one, however,

---

[72] It will be found more convenient to conduct the following analysis in discrete time, for purposes of simpler illustration.

[73] The slope of the line from the origin to b.

again rotates OL' clockwise and shifts C''E''C'' to the left, cancelling out some or all of the effects of price adjustment. If it cancels all these effects, the system is unstable. If it does not, the wages and prices finally "catch up" with an equilibrium point such as E'' where there exists flow but not stock equilibrium, since inventories are still in excess. Gradual inventory liquidation at E'' shifts the commodity locus to the right and the labour locus is rotated counter-clockwise, bringing the economy to a situation of excess commodity and labour demand. Excess inventories are now liquidated faster and soon become deficient, and the situation completely reverses itself to one of boom and inflationary tendencies, with description that should be obvious given the analysis up to this point.

The final stock-flow equilibrium position of the economy is at E'. The stationary state solution is thus again insensitive to the complications introduced in this Section, but the path to equilibrium is again affected. The situation now seems one of cyclical approach, if the system is stable. The stability of this modified system is seen to depend on all the factors affecting the stability of the simpler model and in addition on speeds having to do with inventory adjustment. If the inventory adjustment proceeds with high speed, the effects of that adjustment on production and demand may be de-stabilizing. In terms of the above diagrammatic analysis, the higher such speeds are, the more the loci of labour and commodity markets rotate and shift in response to stock disequilibrium.

The behaviour of real wages in this cyclical movement is of importance. Returning to Figure 5.8, the real wage may initially rise as the price falls to b, but thereafter may be rising or falling, depending on relative speeds of response of the wage and price. It is true, however, that it will eventually be falling as the system approaches E" and in this sense it may be said that the real wage in this model behaves pro-cyclically, at least for some time in the recession. The same is true of the boom situation.

It is obvious that the analysis in this part can be carried further, and indeed it could have been carried out in more detail. However, enough has been done to provide a basis for the comparisons sought in this thesis.

(e) Closing remarks. The disequilibrium model with exchange money developed in this Section is the one on the basis of which comparisons between tatonnement and non-tatonnement equilibration will be carried out. The relevant comparisons and the conclusions are contained in the next section.

5.4 Comparisons and Conclusions

The analysis of this Chapter has concentrated on pure quantity endowment effects as they operate because of plan falsification in an aggregate model where interdependence is taken into account. All plans may be falsified in this framework namely production, income and commodity sales plans. Pure price effects have been assumed well-behaved, and those of the second kind have been assumed away.

The comparisons between tatonnement and non-tatonnement will be carried out in terms of the model of (b), (c), and (d) of Section 5.3, that is, the disequilibrium model with exchange money, including the durable and the non-durable case.

Given the assumption of well-behaved pure price effects, exemplified in "normal" slopes of functions, the tatonnement model was found stable under any speeds of adjustment; its approach towards equilibrium could not be cyclical, that is, the wage and price had to approach equilibrium values either from above or from below, depending on initial conditions. Finally, its stationary state was determined in the labour market, given the assumptions employed.

The non-tatonnement model with the non-durable commodity presented the possibility of instability despite well-behaved pure price effects. This possibility was greater the faster the speed of adjustment of price in relation to the speed of adjustment in the wage. Cyclical approach to equilibrium was also a possibility; finally, the stationary state of this model was the same as that of the tatonnement version.

The non-tatonnement model with a durable commodity was not formally analyzed, but some tentative results emerge from its graphical analysis. Stability in this model seems a complicated question, dependent on speeds of adjustment of prices, wages and commodity inventories; the approach to equilibrium seems almost certainly cyclical; and the stationary state seems to be the same as in all other cases.

In conclusion, despite the modifications and complications introduced into the analysis of equilibration by considerations of disequilibrium, the stationary state seems insensitive to these. The requirements for stability, however, if worked out on the basis of tatonnement, certainly misrepresent what is required for stability with disequilibrium transactions. The tatonnement process only needs well-behaved pure price effects, while introduction of disequilibrium and inventory considerations makes stability more difficult to satisfy, and certainly makes speeds of adjustment relevant to it. Finally, the approach towards equilibrium in the disequilibrium case is almost certainly cyclical, whereas the possibility of cyclical behaviour does not exist in the tatonnement process.

In summary, apart from the fact that the stationary state of the tatonnement process (and the consequent comparative statics) remains the same, the rest of the analysis of equilibration under tatonnement does not approximate, in any satisfactory measure, the out-of-equilibrium behaviour and the stability requirements when disequilibrium is explicitly considered.

CHAPTER VI

SUMMARY OF CONCLUSIONS. SUGGESTIONS FOR FURTHER RESEARCH

This Chapter provides a summary of conclusions reached in this investigation, and an outline of suggestions for further research on the subject.

The conclusions summarized here are -- unless otherwise stated -- original with this thesis. These conclusions fall mainly into two general categories. First, there are those which are not based on formal analysis, but are in the nature of perceptions; observations regarding the relative importance of factors involved in the problem under analysis; and critical points and evaluative remarks on the existing contributions. Second, there are those conclusions which are derived from the main analysis in this thesis, and therefore constitute answers to the questions posed in this investigation.

Both categories of conclusions are deemed important, for different reasons: the first category provides insights to the problem at hand; assists in the choice of direction that the analysis should take; and suggests important points of departure for further analysis. The second category of conclusions have obvious usefulness, since they provide direct answers to the questions raised in this investigation.

(a) Perceptions, observations, and critical points. To begin with,[1] it is of considerable importance to realize that the comparative static (or dynamic) method is based on the determination of the stationary state by planned behavioural functions and on the stability conditions based on these functions.[2] If it turns out that realized rather than planned behaviour determines the stationary state or the stability conditions or both, it may be the case that traditional comparative statics will have to take this into account.

The question as to whether planned behaviour is relevant outside equilibrium arises because, by definition, some of the (income, sales, and production) plans implied by this behaviour are plainly not realizable in disequilibrium.

Moreover, if realized behaviour determines the out-of-equilibrium path of the economy, the additional question arises whether the path traced by planned functions is at all an approximation of the disequilibrium path.

A useful framework in which effects of actual behaviour may be discussed is provided by the classification of endowment effects into "pure-quantity", "pure-price" effects, and a combination of the two, the "quantity-price" effects. Most

---

[1]Most of the observations that follow are contained in Chapter I of the thesis.

[2]Clearly, this realization is not original with this investigation, except perhaps for the way in which the problem is spelled out.

of the work on equilibration processes can be categorized in terms of the kinds of effects it concentrates upon.

When disequilibrium production is considered explicitly, the possibility arises that not only prices but also quantities may turn out to be "false". This important observation leads to a classification of disequilibrium models of production and exchange in terms of the way in which "false" production volumes are disposed of. Three categories are distinguished: first are those analyses where the false quantity is absorbed by demand by means of sufficient price change; second, there is a strand of works where the discrepancy between supply and demand is taken care of by means of inventory variations; and, finally, there exists the possibility for an approach where the discrepancy is corrected partially by price change and partially by inventory change. The analysis in this investigation has proceeded with models that correct the discrepancy either by price change alone, or by a combination of inventory and price change. It is obvious that this approach, which has both prices and inventories change in the face of a "false" production volume, is superior.

With regard to early contributions on the subject of market equilibration,[3] L. Walras may be interpreted to have

---

[3] Most of the points made below are contained in Chapter II of this thesis.

been concerned mainly with the role of pure-price effects.[4] His conjecture that diagonal dominance is a sufficient condition for tatonnement stability has now been proved correct. On the other hand, his search for conditions that make equilibrium prices invariant to quantity-price effects points towards a direction of research that has not been followed by modern works on equilibration.

Edgeworth's concept of "sequential equilibration"[5] in the determination of "normal" price has been judged useful, and has in fact served as a point of departure for the analysis in Chapter V of this investigation.

Marshall concerned himself briefly with realized transactions in a partial equilibrium setting, and established a condition under which the equilibrium price is insensitive to such transactions.

Wicksteed took the route of "imperfect market attendance" in the equilibration process. From a casual-empiricism point of view, this is an eminently reasonable approach to take, though it has not been followed up. It seems reasonable to expect that quantity-price endowment effects may not be important at all in this process, since traders do not stay in the market to experience a price change subsequent

---

[4] This point is due to W. Jaffe, (1967).

[5] This interpretation of Edgeworth is due to Walker, (1973).

to their transactions.

Finally,[6] Hicks' establishment of the fact that pure-price effects are the main problem in the stability analysis of tatonnement exchange is an important contribution. Despite the fact that he established it in a "non-dynamic" setting, it re-surfaces in full dynamic analysis. In fact, most of the proofs of dynamic stability in multi-market systems are effectively based on assumptions that make pure-price effects unable to cause instability. Such exclusion of Giffen goods is not the most attractive feature of this strand of analyses.

Work on non-tatonnement processes of equilibration has been confined to pure exchange models. Extension to include production is obviously desirable. In any event, it has been found that quantity-price effects, which are given the upper hand in the non-tatonnement process of exchange, do not in fact make stability harder to achieve.[7] The stationary state is, however, sensitive to the path that actual behaviour takes.

The contributions on stability of non-tatonnement exchange, however, suffer from a serious defect: they assume that all agents remain in the market for the duration of the

---

[6]Most of the following conclusions are contained in Chapter III of this thesis.

[7]This stability result is due to K. Arrow and F. Hahn, (1971).

equilibration process, despite the fact that transactions are allowed at all prices. While it is reasonable to have market participants stay in the market for the whole duration of the equilibration process when no transactions are allowed outside equilibrium, the same assumption in non-tatonnement analysis is questionable. Why, indeed, would an individual who has succeeded in obtaining his optimal commodity bundle (at a disequilibrium price vector) remain in the market any further?

There are two possible solutions to this contradictory situation. One is, of course, to adopt Wicksteed's "imperfect attendance" concept, and proceed to analysis of the non-tatonnement process on the basis of a variable number of traders. A consequence of this is that quantity-price endowment effects might lose much of their importance. If this is shown to be so, the concentration of present contributions on these effects will in fact turn out to have been partially a waste of effort.

The other possible solution is to postulate that traders remain in the market for speculative reasons. But, in this case, not only will the analysis have to include specific assumptions on expectations, but also its scope and the questions asked will be different.

Neither of the above seems a pleasant prospect for the contributions on the subject of non-tatonnement in pure exchange. This is the main reason why analysis of this thesis does not use them as points of departure.

From the strand of works in other than the pure Walrasian tradition, there are promising points of departure. Thus, the partial disequilibrium analysis of production and exchange in this thesis constructs a model whose reduced form is similar to that of the cobweb model. For general disequilibrium analysis, the points of departure are Clower's distinction of notional versus effective demands; the feasibility of plans in disequilibrium; and the role of inventories as determinants of disequilibrium behaviour, hinted at by D. Patinkin. These considerations are examined in the context of an "Edgeworthian" process of "sequential equilibration", with attention paid to the operation of pure-quantity and other endowment effects.

(b) Main conclusions of the analysis undertaken. The main results of the analysis undertaken in this thesis may now be stated. In the context of partial disequilibrium analysis[8] with one (durable or non-durable) commodity under perfect competition, where only sales plans may be unfeasible, the result is that introduction of disequilibrium in production (with the consequent "false" volumes of output) does not change the stationary state of the model as compared to that of the model with tatonnement. The importance of this result

---

[8]These results are derived in Chapter IV.

is obvious: comparative static analysis can continue to use the traditional method of deriving theorems, since introduction of disequilibrium in production does not make for change in the equilibria that this analysis concentrates upon. (There remains the question of information from stability conditions, on which see below.)

When disequilibrium in exchange during each market period is considered in addition, the claim in this thesis is that the stationary state still remains the same (though nothing has been said about its stability). Thus, the importance of the quantity-price endowment effects is reduced to almost nil: it will be recalled from Chapter III and (a) above that they require rather unreasonable assumptions in order to operate, but since it also turns out that their operation itself is insignificant from the point of view of derivation of theorems, there is little left in them worth examining.

With regard to the stability conditions under disequilibrium behaviour, stability is invariably more difficult to satisfy than in the case where tatonnement is assumed. Since non-tatonnement in pure exchange is stable under the same conditions as the tatonnement process, the conclusion follows that it is the introduction of production considerations in disequilibrium which makes stability more difficult.

With regard to the behaviour of the system outside equilibrium, the conclusion is that the out-of-equilibrium path traced by the tatonnement process model is a gross misrepresentation of the path actually traced when disequilibrium is taken into account in production. The latter path tends to be cyclical most of the time, while the tatonnement path is unlikely to exhibit cycles. Care should be taken, therefore, not to mistake the insensitivity of the stationary state to disequilibrium as licence to use other features of the tatonnement model, such as its path of approach to equilibrium.

Finally, the fact that stability conditions are different when explicit disequilibrium is allowed does not seem to change comparative static theorems derivable from tatonnement analysis, for the reason that when the more stringent stability conditions for the disequilibrium model are satisfied, so are the conditions for stability of tatonnement, on the basis of which comparative analysis derives signs of variable changes.

In the partial analysis of monopoly in disequilibrium, the conclusion is again that the stationary state remains the same. Moreover, production and price are both sensitive to the stock position of the producer, as would be expected by intuition. The out-of-equilibrium behaviour of the disequilibrium model again differs from that of the tatonnement version.

When general analysis of disequilibrium is undertaken,[9] the equilibration process is found to be crucially dependent on feasibility of plans for income creation and plans for production in addition to the sales plans. Endowment effects of the pure-quantity variety are important in this respect. Realized behaviour turns out different from planned, and this has consequences for the equilibration of the system, too. Thus, for example, the labour market has to equilibrate largely by means of its own devices when realized behaviour is taken into account.

Despite the above, the stationary state of the disequilibrium model seems to be the same as that of the model with tatonnement. Stability is again more difficult to satisfy than in tatonnement, as before. Comparative statics theorems do not seem to be affected by this, however; and finally, the out-of-equilibrium behaviour of the tatonnement model is far from a faithful representation of the path followed under disequilibrium.

This string of conclusions is thus seen to carry intact from the partial disequilibrium analysis. Such insensitivity may be considered an indication of the strength of results obtained.

---

[9] Most of the conclusions which follow are derived in Chapter V of this thesis.

(c) Suggestions for further research. This could be made a long section. The reason lies not only in the simplicity and abstractness of the analysis employed in this thesis, but also in the fact that search of the literature has brought up a number of points that are judged to be worth following up.

Thus, the search for conditions for invariance of equilibrium prices in exchange, first undertaken by Walras and continued by Marshall seems a worthwhile direction of research. Results in this area would be expected to produce conditions under which the equilibration process with false transactions does not bear on the equilibrium itself.[10]

Wicksteed's concept of "imperfect market attendance" is certainly worth pursuing for mechanisms of exchange, since casual empirical observation suggests its plausibility.

With regard to models of the cycle, it may be worthwhile to examine in detail the question whether they allow an implicit tatonnement assumption in the process of derivation of their basic dynamic equations.

With regard to the analysis developed in Chapter IV, a possible extension lies in the consideration of a multi-

---

[10] One may remark that since quantity-price effects require "unreasonable" assumptions it may not be worthwhile to search for conditions under which they are inoperative. However, it has been mentioned that perfect market attendance is not unreasonable for speculators in markets, and for these markets, therefore, quantity-price effects will still be relevant.

market system, keeping the assumption of no dependence from input markets. This would provide some insight of the effects of interdependence (via prices) of demands and production levels in a disequilibrium model of production, with tatonnement exchange.

The analysis of non-tatonnement in exchange coupled with disequilibrium in production should be formalized. Results from this line of research would establish, as mentioned above, the relative importance of quantity-price effects.

Finally, the analysis of monopoly with inventories offered in this thesis is only a simple beginning to a consideration of quantity-price dynamics.

With regard to the general disequilibrium analysis of Chapter V, the first item in the agenda seems to be formal introduction of inventories in the simple model used. Further, more "realistic" expectation formation hypotheses could be attempted; intertemporal choice could be introduced as a possibility; an attempt to extend the analysis to a multi-commodity framework could be made; other assets alternative to money could be introduced, "Keynesian" problems of demand deficiencies could be examined; and price setting, and some price rigidity or sluggishness in price adjustment could be assumed.

It is not claimed here that the simple model developed in Chapter V is amenable to all the above extensions and modifications without major change: neither is the list of extensions and modifications exhaustive. The purpose has only been to indicate that the analysis of disequilibrium is in fact in an infant stage, and therefore a multitude of improvements are possible in a number of directions.

BIBLIOGRAPHY

Arrow, K. J., (1959), "Towards a Theory of Price Adjustment," in The Allocation of Economic Resources. Essays in Honour of B. F. Haley, by M. Abramovitz et. al., Stanford: Stanford University Press, 1959.

Arrow, K. J., and Hahn, F. H., (1971), General Competitive Analysis. San Francisco: Holden Day, Inc., 1971.

Arrow, K. J., and Hurwicz, L., (1958), "On the Stability of Competitive Equilibrium, I." Econometrica, 27 (1958), 522-552.

Arrow, K. J., Block, H. D., and Hurwicz, L., (1959), "On the Stability of Competitive Equilibrium, II," Econometrica, 27 (1959), 82-109.

Arrow, K. J., and Hurwicz, L., (1962), "Competitive Stability Under Weak Gross Substitutability: Non-Linear Price Adjustment and Adaptive Expectations," International Economic Review, III (1962), 233-235.

Arrow, K. J., and Nerlove, M., (1958), "A Note on Expectation and Stability," Econometrica, 26 (1958), 297-305.

Ackerman, G., (1957), "The Cobweb Theorem: A Reconsideration," Quarterly Journal of Economics, 71 (1957), 151-160.

Barro, R. J., and Grossman, H. I., (1971), "A General Disequilibrium Model of Income and Employment," American Economic Review, LXI (1971), 82-93.

Bilas, R., (1965), "Liebhafsky and the Constant Marginal Utility of the Numeraire: A Comment," Canadian Journal of Economics and Political Science, XXXI (1965), 117-118.

Buchanan, N. S., (1939), "A Reconsideration of the Cobweb Theorem," Journal of Political Economy, 47 (1939), 67-81.

Chipman, J. S., (1951), The Theory of Intersectoral Money Flows and Income Formation. Baltimore: John Hopkins Press. 1951.

Clower, R., (1960), "Keynes and the Classics: A Dynamical Perspective," Quarterly Journal of Economics, 74 (1960), 318-323.

Clower, R., (1965), "The Keynesian Counterrevolution: A Theoretical Appraisal," in Hahn, F. H., and Brechling, F. P. R., (editors), The Theory of Interest Rates. Proceedings of a Conference Held by the International Economics Association, London: MacMillan, 1965.

Cohn, A., (1922), "Uber die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise," Mathematische Zeitschrift, 14 (1922), 110-148.

Cook, W. D., and Veendrop, E. C. H., (1975), "Six Markets in Search of an Auctioneer," Canadian Journal of Economics, VIII (1975), 238-257.

Danese, A. E., (1965), Advanced Calculus. Boston: Allyn and Bacon, 1965.

Debreu, G., (1959), The Theory of Value: An Axiomatic Approach of Economic Equilibrium. Cowles Foundation monograph No. 17, New York: J. Wiley and Sons Inc., 1959.

Diamond, P. A., (1971), "A Model of Price Adjustment," Journal of Economic Theory, 3 (1971), 156-168.

Edgeworth, F. Y., (1881), Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences. London: C. Kegan, Paul & Co., 1881.

Edgeworth, F. Y., (1925), Papers Relating to Political Economy, 3 vols., The Royal Economic Society, 1925.

Enthoven, A. C., and Arrow, K. J., (1956), "A Theorem on Expectations and the Stability of Equilibrium," Econo-Metrica, 24 (1956), 288-293.

Ezekiel, M., (1937), "The Cobweb Theorem," Quarterly Journal of Economics, 52 (1937), 255-280.

Fisher, F. M., (1970), "Quasi-Competitive Price Adjustment by Individual Firms: A Preliminary Paper," Journal of Economic Theory, 2 (1970), 195-206.

Fisher, F. M., (1972), "On Price Adjustment Without an Auctioneer," Review of Economic Studies, 39 (1972), 1-15.

Frisch, R., (1936), "On the Notion of Equilibrium and Dis-equilibrium," Review of Economic Studies, III (1936), 100-105.

Georgesku-Roegen, N., (1968), "Revisiting Marshal's Con-stancy of Marginal Utility of Money," Southern Economic Journal, 35 (1968), 176-181.

Gorman, W. M., (1964), "More Scope for Qualitative Economics," Review of Economic Studies, 31 (1964), 64-68.

Grossman, H. I., (1971), "Money, Interest and Prices in Market Disequilibrium," Journal of Political Economy, 71 (1971), 943-961.

Hahn, F. H., (1958), "Gross Substitutes and the Dynamic Stability of General Equilibrium," Econometrica, 26 (1958), 169-170.

Hahn, F. H., and Matthews, R. C. O., (1967), "The Theory of Economic Growth: A survey," in Surveys of Economic Theory: Volume II, London: MacMillan, 1967.

Hahn, F. H., and Negishi, T., (1962), "A Theorem on Non-Tatonnement Stability," Econometrica, 30 (1962), 463-469.

Herglotz, G., (1924), "Uber die Wurzelanzahl algebraischer Gleichungen innerhalb und auf dem Ein ehitskreis," Mathematische Zeitschrift, 19 (1924), 26-34.

Hicks, J. R., (1946), Value and Capital: An Inquiry into Some Fundamental Principles of Economic Theory. Second Edition, Oxford: Clarendon Press, 1946.

Hicks, J. R., (1950), A Contribution to the Theory of the Trade Cycle. Oxford: Oxford University Press, 1950.

Hicks, J. R., (1959), "A Value and Capital Growth Model," Review of Economic Studies, 26 (1959), 159-177.

Hicks, J. R., (1965), Capital and Growth. Oxford: Oxford University Press, 1965.

Higgins, G., and Liebhafsky, H., (1968), "Pareto and the Marshallian Constancy Assumption," Southern Economic Journal, 35 (1968), 167-175.

Jaffe, W., (1967), "Walras' Theory of Tatonnement: A Critique of Recent Interpretations," The Journal of Political Economy, 75 (1967), 1-19.

Kaldor, N., (1933-1934), "A Classificatory Note on the Determinateness of Equilibrium," Review of Economic Studies, 1 (1933-1934), 122-136.

Keynes, J. M., (1936), The General Theory of Employment, Interest and Money. London: Macmillan and Co., 1964.

Koopmans, T. C., (1957), Three Essays in the State of Economic Science. New York: MacGraw Hill Book Co., 1957.

Lancaster, K., (1962), "The Scope of Qualitative Economics," Review of Economic Studies, 29 (1962), 99-123.

Lancaster, K., (1964), "Partitionable Systems and Qualitative Economics," Review of Economic Studies, 31 (1964), 69-72.

Lancaster, K., (1968), Mathemential Economics. London: MacMillan, 1968.

Lange, O., (editor), (1942), Studies in Mathematical Economics and Econometrics. Chicago, 1942.

Liebhafsky, H., (1961), "Marshall and Slutsky on the Theory of Demand," Canadian Journal of Economics and Political Science, XXVII (1961), 176-191.

Marshall, A., (1920), Principles of Economics. Eighth edition, London: MacMillan, 1920.

McKenzie, L. W., (1960), "Stability of Equilibrium and the Value of Positive Excess Demand," Econometrica, 28 (1960),

Metzler, L. A., (1945), "Stability of Multiple Markets: the Hicks Conditions," Econometrica, 13 (1945), 266-292.

Negishi, T., (1958), "A Note on the Stability of an Economy Where All Goods Are Gross Substitutes," Econometrica, 26 (1958), 445-447.

Negishi, T., (1961), "On the Formation of Prices," International Economic Review, 2 (1961), 122-126.

Negishi, T., (1962), "The Stability of a Competitive Economy: A Survey Article," Econometrica, 30 (1962), 635-659.

Negishi, T., (1964), "The Stability of Exchange and Adaptive Expectations," International Economic Review, 5 (1964), 104-111.

Nerlove, M., (1958), "Adaptive Expectations and Cobweb Phenomena," Quarterly Journal of Economics, 72 (1958), 227-240.

Newman, P., (1959-1961), "Some Notes on Stability Conditions," Review of Economic Studies, 28-29 (1959-1961), 1-9.

Newman, P., (1961), "Approaches to Stability Analysis," Econometrica, 28 (1961), 12-29.

Newman, P., (1965), The Theory of Exchange. Prentice Hall, 1965.

Ogata, K., (1967), State Space Analysis of Control Systems. Englewood Cliffs, New Jersey, 1967.

Pareto, V., (1927), Manuel d' Economie Politique. Paris, 1927.

Patinkin, D., (1962), "The Limitations of Samuelson's Correspondence Principle," Metroeconomica, IV (1962), 37-43.

Patinkin, D., (1965), Money, Interest and Prices: An Integration of Monetary and Value Theory. Second Edition, New York: Harper and Row, 1965.

Ross, S. L., (1964), Differential Equations. Massachusetts: Ginn, Waltham, 1964.

Samuelson, P., (1939), "Interactions between the Multiplier Analysis and the Principle of Acceleration," The Review of Economics and Statistics, XXI (1939), 75-78.

Samuelson, P., (1941), "Conditions that the Roots of a Polynomial be less than Unity in Absolute Value," Annals of Mathematical Statistics, 12 (1941), 360-364.

Samuelson, P., (1941), "The Stability of Equilibrium; Comparative Statics and Dunamics," Econometrica, 9 (1941), 97-120.

Samuelson, P., (1944), "The Relation Between Hicksian Stability and True Dynamic Stability," Econometrica, 12 (1944), 256-257.

Samuelson, P., (1947), Foundations of Economic Analysis. Cambridge: Harvard University Press, 1947.

Scarf, H., (1960), "Some Examples of Global Instability of the Competitive Equilibrium" International Economic Review, 1 (1960), 157-172.

Schur, J., (1917), "Uber Potenzreihen, die im Innern des Einheitsskreis beschrankt sind," Journal fur Mathematik, 147 (1917), 205-232.

Smithies, A., (1939), "The Maximization of Profits Over Time With Changing Cost and Demand Functions," Econometrica, VII (1939), 312-318.

Solow, R., (1956), Book Review of Elements of Pure Economics, by Leon Walras, translated by William Jaffe, Homewood, Illinois: R. Irwin Inc., 1954, in Econometrica, 24 (1956), 87-89.

Takayama, A., (1974), Mathematical Economics. Hindsdale, Illinois: The Dryden Press, 1974.

Uzawa, H., (1961), "The Stability of Dynamic Processes," Econometrica, 29 (1961), 617-631.

Uzawa, H., (1962), "On the Stability of Edgeworth's Barter Process," International Economic Review, 3 (1962), 218-232.

Vind, K., (1965), "A Theorem on the Core of an Economy," Review of Economic Studies, 32 (1965), 47-48.

Walker, D., (1969), "Marshall's Theory of Competitive Exchange," Canadian Journal of Economics, II (1969), 589-598.

Walker, D., (1973), "Edgeworth's Theory of Recontract," Economic Journal, 83 (1973), 138-149.

Walker, D. A., (1973a),"Competitive Tatonnement Exchange Markets," Kyklos, 25 (1973), 345-363.

Walras, L., (1954), Elements of Pure Economics. Translated by W. Jaffe, Homewood, Illinois: Irwin Inc., 1954.

Wicksteed, P., (1933), The Common Sense of Political Economy. London: MacMillan and Co., 1933.

Wilson, E., (1935), "Generalization of Pareto's Demand Theorem," Quarterly Journal of Economics, XLIX (1935), 715-717.

Wilson, E., (1939), "Pareto Versus Marshall," Quarterly Journal of Economics, LIII (1939), 645-649.

APPENDICES

APPENDIX TO CHAPTER III

STABILITY: CONCEPTS AND METHODS OF ANALYSIS

The brief discussion in this Appendix is based in part on Newman[1]. First, local stability is discussed and then the second method of Lyapounov, both very briefly.

An equilibrium is locally, asymptotically stable if a sufficiently small displacement of the system from equilibrium results to a tendency of the system to approach equilibrium, given sufficient time. Consider the dynamic system

$$\dot{P}_i = K_i(E_i^*) \qquad (i=1,2,..n) \qquad (1.11)$$

discussed in Chapter I. Assume that there exists a price vector at which all excess demands become zero, and linearize the system in the neighbourhood of this equilibrium price vector to obtain

$$\dot{P}=KJP \qquad (3.A.1)$$

Where $\dot{P}$ is the nx1 dimensional vector of time derivatives of prices, K is the nxn diagonal matrix of speeds of adjustment, J the nxn Jacobian of excess demands and P the nx1 dimensional price vector. Modification of units of measurement to make all elements in the main diagonal of K equal to unity transforms the system to

---

[1](1961).

$$\dot{P} = JP \qquad (3.A.2)$$

Global stability of this linear system implies only local stability of the original system since the Jacobian J has been evaluated in the neighbourhood of equilibrium. For stability of the linear system (3.A.2) it is well-known that the real parts of all the characteristic roots of J must be negative.

Global stability necessitates consideration of the original, nonlinear system. In this respect, Lyapounov's second method can be used. One must be able to claim that the system tends to return to equilibrium after a displacement, irrespective of magnitude.

Lyapounov's method utilizes, most of the time, some measure of "distance" from the equilibrium. The aim is to show that such a measure decreases as time passes.

One possible definition for a Lyapounov function may now be given. Consider the system (1.11), now written as follows:

$$\dot{P} = f(P) \qquad (1.11a)$$

where $\dot{P}$ and P are nx1 dimensional vectors and the assumption is that equilibrium is at the origin[1]. Suppose that there exists a scalar function (from n-dimensional space to single

[1] A harmless assumption, since it involves re-definition of P to measure derivations from any non-zero equilibrium.

dimensional one), with continuous first partial derivatives, such that

(a) $V(o)=0$ (where o is the null vector)

(b) $V(P)$ positive for all non-zero vectors P

(c) $\dot{V}(P)$ negative for all non-zero P, along the solution of (1.11a)

(d) $V(P)$ tends to infinity as the norm of P does so.

Then, the system is asymptotically stable in the large, i.e., globally stable.

Distance functions satisfy (a), (b), and (d) of the above requirements. It thus remains, in order to prove stability by this method, to show that (c) above holds.

As has been already mentioned in Section 3.1 of this Chapter, this method or variations of it have been extensively used to prove global stability in tatonnement and non-tatonnement analyses.

APPENDIX TO CHAPTER IV

(A) GLOBAL STABILITY ANALYSIS OF THE MODEL WITH NON-TATONNEMENT IN PRODUCTION (SECTION 4.2)

Consider the non-linear fundamental equation of the system of Section 4.2.c, solved explicitly for P:

$$P_t = H^{-1}[F(P_{t-1})] = f(P_{t-1}) \qquad \text{(A4.11a)}$$

and transform the axes so that the equilibrium price is zero, that is,

$$f(0)=0 \qquad \text{(A4.14).}$$

The equilibrium origin of the dynamic system (4.11a) is then uniformly asymptotically stable in the large if a function V(P) can be found such that[1]

(a) $V(P_t)>0$ for $P_t \neq 0$

(b) $\Delta V(P_t)<0$ for $P_t \neq 0$

where $\Delta V(P_t)=V(P_t)-V(P_{t-1})=V[f(P_{t-1})]-V(P_{t-1})$

(c) $V(0)=0$

(d) $V(P_t)\to\infty$ as $\| P_t \| \to \infty$.

Let $V(P_t)=P_t^2$, which obviously satisfies (a), (c) and (d), be the Lyapounov function for the system. For condition (b) to be satisfied so that the system is stable as defined we must have

$$V[f(P_{t-1})]-V(P_{t-1})=f(P_{t-1})^2-P_{t-1}^2<0 \qquad \text{(A4.15)}$$

[1] See Chapter III, Appendix, for the continuous time variant of this method. For discrete time formulations see Takayama, (1974), and Ogata, (1967), pp.486ff.

or the equivalent condition

$$|f(P_t)| < |P_t| \qquad \text{(A4.16)}$$

The condition (4.15) is represented graphically in Figure A.4.2, which shows the graph of $f(P_t)$ along with two 45° lines. As long as the graph of the function lies in the shaded area between the two 45° lines the system is stable as defined above.

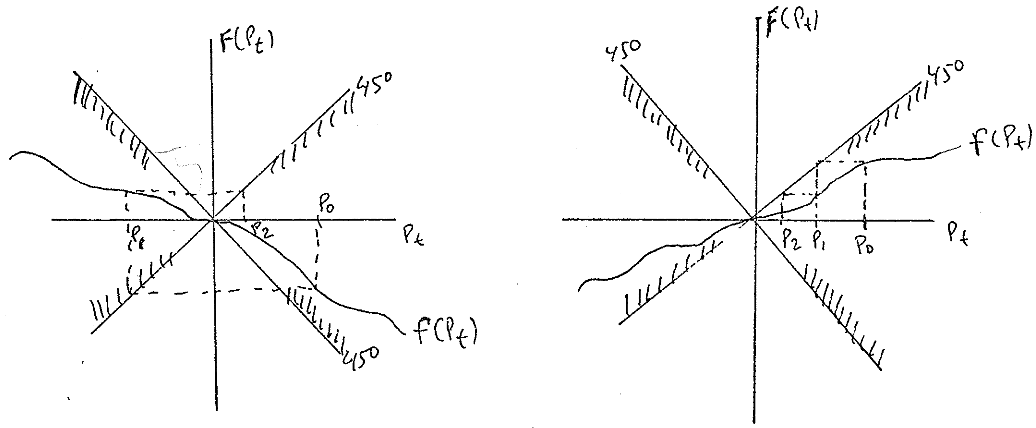


Figure A.4.2

It may be useful to consider the economic meaning of the stability condition (4.10) before discussion of the cases presented in Figure A.4.2 is undertaken. Closer examination of equation (4.11a) shows that $f(P_t)$ denotes the demand price necessary for market clearance of the commodity supplied, given the price of production P_t. In the light of this interpretation, the stability condition (4.16) says that the demand price of each amount produced must always be less in absolute value than the price on the basis of which production proceeded in period t. Thus, the price at which the market clears in period t+1 must always be less in absolute value than the price of period t on the basis of which production became available. In this way, the "energy" of the system becomes constantly dissipated and the equilibration process tends to the origin with time.

An examination of the two cases of Figure A.4.2 is now in order. Case (a) depicts f(P) negatively sloped, indicating that supply and demand responses are such that a fall in price requires a rise in the demand price necessary to clear the market. This will happen when the supply response to a change in price is opposite in sign to the demand response. For example, if the quantity supplied rises and the quantity demanded falls with a rise in price, a fall in the price will bring forward less supply and will thus require a rise in the demand price necessary for flow equilibrium in the market.

Suppose, for example, that the initial price is P_o in the Figure. Given the quantity produced on the basis of this price, the demand price is P_1. On the basis of this price there is again production, the clearance of which necessitates a higher demand price (P_2) than before. Thus, a fall in the price from P_o to P_1 raises the demand price from P_1 to P_2. The system is seen to cycle around the origin and approach it in the process. The case is equivalent to that of case (a) of Figure 4.1 of the text but the rather restrictive condition on the slopes of the curves is now replaced by the more general one of equation (4.16). The generality of this latter condition will be discussed below.

Case (b) of Figure A.4.2 depicts a situation where a fall in price necessitates a fall in the demand price for flow equilibrium. This will happen when the demand and supply responses to price are of the same sign. If for example, a fall in price reduces both quantity supplied and demanded, a lower price will bring forward less supply and this will in turn require a lower price for market clearance. Consider P_o as the initial price: the demand price necessary for market clearance is P_1 from Figure A.4.2b. Given this price, the amount supplied in the next period is such that the demand price for market clearance is P_2. Thus, a fall of the price from P_o to P_1 necessitates a similar fall in the demand price from P_1 to P_2. The case is similar to (b) in Figure 4.1 of the text, and the approach to equilibrium is from one side, that of the initial price.

The stability condition of equation (4.16), and its graphical representation in Figure A.4.2 are more general than the condition derived from the linearized version of the dynamic equation, i.e., from equation (4.12) in the text. The stability condition from that method was that the ratio of price responsiveness of the two functions, F_p/H_p, be less than unity in absolute value. The implication was that the own price elasticity of demand at each and every point must be greater than the elasticity of supply with respect to the product price at each and every price-quantity combination.[2] As may be seen from inspection of Figure A.4.2, however, this need not be the case for stability of the system to obtain: it is conceivable that the slope conditions are violated in some range but stability may still obtain. To see this more clearly, it may be desirable to linearize the function f around a random point P*. The linearized form then becomes:

$$f(P_t) = f(P_t^*) + F_p/H_p \; (P_t - P_t^*) \qquad \text{(A4.17)}$$

the slope of which is F_p/H_p. It should be clear that the stability condition (4.16) may be satisfied even if for some price ranges the slope of the above function is greater than unity in absolute value, since (4.16) only requires that P_t be greater than $f(P_t)$ in absolute value.[3]

Thus, in the case of non-linear specification of functions, it may be the case that the typical stability conditions derived from the standard treatment of the problem are violated and yet the system is stable. An illustration

---

[2]The own price elasticity of demand may be written $-H_pP/Q$ and the elasticity of supply F_pP/Q, where Q is quantity. The requirement that the absolute value of F_p is less than the absolute value of H_p thus means that the elasticities must bear the relationship mentioned in the text.

[3]The discussion of course implies that f is nonlinear, and consequently F_p, H_p are functions of P.

of this point is in Figure A4.2b; stability is present throughout, but nevertheless the slope of f at, e.g., point A is greater than unity in absolute value.

(B) SOLUTION OF THE MONOPOLIST'S PROBLEM (SECTION 4.4)

The problem is to maximize (assuming the discount rate is zero)

$$Z=\int_0^A z(t)\,dt=\int_0^A [R(t)-C(t)]\,dt \qquad \text{(A4.40)}$$

with cost and demand functions given by

$$C(t)=aQ^o(t)^2+b[I(t)-I^*]^2 \qquad \text{(A4.41)}$$

$$Q^s(t) = c-dP(t) \qquad \text{(A4.42)}$$

and inventory at time t given by

$$I(t)=\int_0^t [Q^o(x)-Q^s(x)]\,dx \qquad \text{(A4.49)}$$

Use of the following transformations:

$$q^o(t)=\int_0^t Q^o(x)\,dx,\quad q^s(t)=\int_0^t Q^s(x)\,dx \qquad \text{(A4.50)}$$

enables one to write

$$Z=\int_0^A [R(t)-C(t)]\,dt=\int_0^A \left[\frac{c\dot{q}^s-(\dot{q}^s)^2}{d}-a\dot{q}^o-b(q^o-q^s+I(0)-I^*)^2\right]dt$$

that is,

$$Z=\int_0^A F(q^s,\dot{q}^s,q^o,\dot{q}^o)\,dt \qquad \text{(A4.51)}$$

Conditions necessary for the maximum of Z are[4]

$$F_{q^s}-d/dt(F_{\dot{q}^s})=0 \text{ and } F_{q^o}-d/dt(F_{\dot{q}^o})=0 \qquad \text{(A4.52)}$$

Calculation of the derivatives and substitution in (A4.52) yields the optimal rules given in the text, equations (4.43) and (4.44).

---

[4]See A. Danese, (1965).