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Realization of Non-commutative Functions around a Matrix-Centre

by

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Abstract

This work is concerned with fundamental aspects of functions of several non-commutative (NC) variables, where the inputs are d -tuples of square matrices of the same but arbitrary sizes. We provide an introduction to NC function theory and discuss analytic properties of these functions in Chapters 1 and 2, following the work of Kaliuzhnyi-Verbovetskyi and Vinnikov [11]. In particular, we present the Taylor–Taylor formula, named after Brook Taylor, who established the classical Taylor series for commutative variables, and Joseph L. Taylor, who extended it to functions of several NC variables, thereby laying the foundation of NC function theory [23]. These Taylor–Taylor (TT) series take the form of power series in multilinear maps, capturing the analytic structure of NC functions. As shown by K.-Verbovetskyi and Vinnikov, one can find sufficient conditions (known as the Lost-Abbey conditions) under which a power series of multilinear maps defines an NC function.

Building on this foundation, we focus on the realization theory of more general NC functions. As shown in [1], analytic NC functions whose domains include the origin (that is, the zero d -tuple $(0, \dots, 0)$ of matrices), have operator realizations, meaning that they can be entirely described by a simple rational expression involving finitely many operators acting between Hilbert spaces. Moreover, the realization is unique up to a type of isomorphism, provided that certain minimality conditions are satisfied. Even if the realization is not minimal, one can apply the Kalman decomposition, which constructs a minimal realization for a given NC function from any realization by compressing the original realizations to a certain semi-invariant subspace.

A question that naturally arises is whether it would be feasible to extend the results explained above to the case where the realizations are “centred” around a non-zero d -tuple of matrices, Y , as was done by Porat and Vinnikov in [18, 19], for NC rational functions. Their result showed that matrix-centre realization of NC rational functions are necessarily finite dimensional. By considering general, infinite-dimensional matrix-centre realizations we extend the matrix-centre realization theory to a much broader

class of analytic NC functions. In addition to this, we prove that if a minimal realization around any matrix centre, Y , is translated to another realization about a different matrix-centre, X , then the minimality property is preserved by this operation.

Together, these results provide a better understanding of analytic NC functions and further extend realization theory. In particular, we prove that any uniformly analytic NC function admits a matrix-centre realization about an arbitrary point in its domain [13].

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Chapter 1

Introduction to Non-commutative Function Theory

In this chapter, we introduce some basic definitions, terminology and notations. First, we review the classical notion of realization and briefly explain how non-commutative systems motivate the study of realizations in the non-commutative case. Then, in Sections (1.1) and (1.2) we provide the formal definition of non-commutative sets, non-commutative functions and their properties. The rest of the chapter explains the difference-differential operators and studies analytic properties of non-commutative functions.

Let us first consider the classical discrete-time, i/o/s (input/ output/ state) linear system Σ of the following form:

$$\Sigma = \begin{cases} x(n+1) = Ax(n) + Bu(n) \\ y(n) = Cx(n) + Du(n) \end{cases}$$

Here, $x(n)$ is the state of the system, which takes values in $\mathcal{H}$. The input $u(n)$ takes the values in input space $\mathcal{U}$, and $y(n)$ is the output at time n taking values in output space $\mathcal{Y}$. All of the spaces $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{Y}$ are complex finite-dimensional vector spaces.

The connection matrix (or impulse-response matrix) of the system Σ is defined as:

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{U} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H} \\ \mathcal{U} \end{pmatrix},$$

such $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{U}, \mathcal{H})$, $C \in \mathcal{B}(\mathcal{H}, \mathcal{U})$ and $D \in \mathcal{B}(\mathcal{U})$ (they are bounded linear operators from one vector space to another). If we start at an initial value $x(0)$ at time 0 and feed this to the linear system Σ , we can construct the following formulas for the input and output of the system at time n :

$$\begin{aligned} x(n) &= A^n x(0) + \sum_{k=0}^{n-1} A^{n-1-k} B u(k) \\ y(n) &= C A^n x(0) + \sum_{k=0}^{n-1} C A^{n-1-k} B u(k) + D u(n) \end{aligned} \quad (1.1)$$

Next, we need to briefly define the Z-transform. This the main tool to convert a discrete-time signal into a frequency-domain. In other words, instead of time, we want to see how the input and output is distributed within different frequency bands. The unilateral Z-transform is defined by:

$$x_n \mapsto \sum_{n=0}^{\infty} x_n z^n,$$

in which $\{x_n\}_{n=0}^{\infty}$ is the underlying sequence (typically a discrete-time signal) and z is a complex variable. Applying the unilateral Z-transform to the Equations (1.1) we get the frequency-domain formulas:

$$\begin{aligned} \hat{x}(z) &= (I - zA)^{-1} x(0) + z(I - zA)^{-1} B \hat{u}(z), \\ \hat{y}(z) &= C(I - zA)^{-1} x(0) + T_{\Sigma}(z) \hat{u}(z), \end{aligned} \quad (1.2)$$

where,

$$T_{\Sigma}(z) = D + zC(I - zA)^{-1}B \quad (1.3)$$

is a rational function. For each z in a ball of radius $\|A\|^{-1}$ around the origin ($|z| < \|A\|^{-1}$), the geometric series $(I - zA)^{-1} = \sum_{i=0}^{\infty} (zA)^i$ converges in matrix norm, for each value of $z \in \mathbb{C}$ such that $(I - zA)^{-1}$ exists and $T_{\Sigma}(z) \in \mathcal{B}(\mathcal{U}, \mathcal{Y})$ is a finite-dimensional transformation with rational entries in z . We call this function a **transfer function** of the system Σ .

Given the linear system above one can ask a natural question of whether an arbitrary state $h \in \mathcal{H}$ is achievable if we initialize the system with the state $x(0) = 0$ and feed the system with a series of inputs $\{u(k)\}_{k=0}^N$ (for $N > 0$).

As noticed above, T_{Σ} can encode the necessary information about an input-output system. It can reveal the relation between the input and output of the system and also reveals the internal state of that system. One example, which is of particular interest, is when the input-state-output spaces are vector spaces consisting of matrices and instead of the variable z , we have multiple variables. We note that $T_{\Sigma}(z)$ is a rational function in z . Here, by **rational function** in single-variable z , we mean a function of the form $r(z) = \frac{p(z)}{q(z)}$, for univariate polynomials $p, q \in \mathbb{C}[z]$. Later, we will prove that any such function has a finite-dimensional realization $(A, B, C, D) \in \mathbb{C}^{n \times n} \times \mathbb{C}^n \times \mathbb{C}^{1 \times n} \times \mathbb{C}$. Conversely, one can prove that $T_{\Sigma}(z)$ in Equation (1.3), is rational by computing $(I - zA)^{-1}$ via Schur complements.

To obtain a rational function in d variables, we replace z with commuting variables $z_1, \dots, z_d$ and replace $A \in \mathbb{C}^{n \times n}$ with $A_1, \dots, A_d$, a d -tuple of matrices in $\mathbb{C}^{n \times n}$ and B with $B_1, \dots, B_d \in \mathbb{C}^n$. More accurately, we apply the following maps to the system:

$$\begin{aligned} zA &\longmapsto z_1A_1 + \dots + z_dA_d \\ zB &\longmapsto z_1B_1 + \dots + z_dB_d \end{aligned}$$

There exist some occasions in system theory where the evolution of the underlying system is along a free monoid (defined in the following sections) rather than $\mathbb{N} \cup \{0\}$. In this case, the non-commutativity of variables becomes a necessity. To get a non-commutative rational expression we replace $z_1, \dots, z_d$ with d formal non-commutative (NC) variables $\mathfrak{z}_1, \dots, \mathfrak{z}_d$. In this case, we have:

$$T_\Sigma(\mathfrak{z}_1, \dots, \mathfrak{z}_d) = D + C \left(I - \sum_{j=1}^d \mathfrak{z}_j A_j \right)^{-1} \sum_{j=1}^d \mathfrak{z}_j B_j. \quad (1.4)$$

Choosing $\mathcal{U} = \mathbb{C}$, we obtain $A \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times d}$, $B \in \mathbb{C}^d \otimes \mathbb{C}^n$, $C \in \mathbb{C}^{1 \times n}$ and $D \in \mathbb{C}$. We can also view this as a function on d -tuple $Z = (Z_1, \dots, Z_d)$ such that $Z_j \in \mathbb{C}^{m \times m}$ and $m \in \mathbb{N}$ by replacing:

$$\begin{aligned} \mathfrak{z}_j A_j &\longmapsto Z_j \otimes A_j \\ \mathfrak{z}_j B_j &\longmapsto Z_j \otimes B_j, \end{aligned}$$

so that the transfer function (1.4) turns into:

$$\underbrace{T_\Sigma(Z_1, \dots, Z_d)}_{\in \mathbb{C}^{m \times m}} = DI_m + I_m \otimes C \left(I_m \otimes I_n - \sum_{j=1}^d Z_j \otimes A_j \right)^{-1} \sum_{j=1}^d Z_j \otimes B_j, \quad (1.5)$$

for $(Z_1, \dots, Z_d) \in \mathbb{C}^{(m \times m) \cdot d}$ that are chosen such that the inverse $\left(I_m \otimes I_n - \sum_{j=1}^d Z_j \otimes A_j \right)^{-1}$ exists. This is an example of a **non-commutative (NC) rational function** which means roughly, that it is obtained by applying the operations, $+$, $-$, $\times$, and $(\cdot)^{-1}$ to the free algebra of all complex polynomials in d NC variables $\mathfrak{z}_1, \dots, \mathfrak{z}_d$. The technical definition and the proof of why this is in fact an NC rational function can be found in Chapter 3. In particular, Equations (1.4) is an NC rational function if and only if

$$(A, B, C, D) \in \left(\mathbb{C}^{(m \times m)} \otimes \mathbb{C}^{1 \times d} \right) \times \left(\mathbb{C}^m \otimes \mathbb{C}^d \right) \times \mathbb{C}^{1 \times m} \times \mathbb{C}$$

in which $m \in \mathbb{N}$ is finite. The quadruple (A, B, C, D) is called the **Fornasini-Marchesini (FM) realization**. In this thesis, we consider more general realizations,

$$(A, B, C, D) \in \mathcal{B}(\mathcal{H})^{1 \times d} \times \mathcal{H} \otimes \mathbb{C}^d \times \mathcal{H}^\dagger \times \mathbb{C}.$$

where $\mathcal{H}$ is a separable, complex, infinite dimensional Hilbert space and $\mathcal{H}^\dagger$ is the dual of the Hilbert space $\mathcal{H}$.

As pointed out earlier, in system theory, when we want to move from one dimensional to multidimensional systems, we replace the evolution along $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ by the evolution along $\mathbb{N}_0^d$ (which is a monoid with respect to the operation “+”). In order to move from the multidimensional commutative case to the multidimensional non-commutative case we replace $\mathbb{N}_0^d$ by the free monoid $\mathbb{F}_d^+$ (defined below). Moreover, if we have a non-commutative rational expression (defined in section 2) which is defined at zero, then choosing appropriate matrices A, B, C and D , we can write that non-commutative rational function in the form of Equation (1.4).

The function T_Σ defined above is called a transfer function in the context of (classical) control theory. For a non-commutative rational function $\mathfrak{r}$, such a representation is called **realization** of $\mathfrak{r}$ (centred at zero).

The theory of realizations was first developed by R.E. Kalman in [12]. The central question in that paper of Kalman was that if we are given a impulse response matrix of an experimentally observed system, how can we identify the linear dynamical system (in this case, an input-output system) that generated that matrix? Kalman proved that one can uniquely determine the input/ state/ output system by a finite realization (A, B, C, D) if and only if the realization is *minimal*, meaning that the state space dimension is minimum. We will define these terms formally in Chapter 3.

1.1 Noncommutative Sets

We start by defining the *non-commutative universe*, which contains all the non-commutative (NC) sets. These NC sets serve as the domain of NC functions.

Definition 1.1. 1. Let $\mathbb{C}$ be the set of complex numbers and $d \in \mathbb{N}$. We define the d -dimensional complex NC universe as:

$$\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} := \bigsqcup_{n=1}^{\infty} \mathbb{C}^{(n \times n) \cdot d}, \quad \mathbb{C}^{(n \times n) \cdot d} := \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times d}.$$

Here, $\bigsqcup$ denotes the disjoint union, and $\otimes$ is the tensor product of vector spaces, which in this context identifies

$$\mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times d} \cong (\mathbb{C}^{n \times n})^d,$$

the space of d -tuples of $n \times n$ complex matrices. An element X of $\mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times d}$ is a row d -tuple of matrices:

$$(X_1, \dots, X_d) \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times d}, \quad \text{with each } X_j \in \mathbb{C}^{n \times n}.$$

2. A subset Ω of the NC universe, is called an **NC set** if it is closed under direct sums in the sense that for all $X \in \Omega \cap \mathbb{C}^{(n \times n) \cdot d}$ and $Y \in \Omega \cap \mathbb{C}^{(m \times m) \cdot d}$ ($n, m \in \mathbb{N}$ are arbitrary), we have :

$$X \oplus Y := \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \Omega \cap \mathbb{C}^{((n+m) \times (n+m)) \cdot d}$$

We usually denote $\Omega_n = \Omega \cap \mathbb{C}^{(n \times n) \cdot d}$ and call it the **level n of the NC set Ω** .

The formal definition of NC universe stated above is defined over $\mathbb{C}^d$, however, it is a special case of the NC universe defined in [11], in which it is defined by using modules over unital commutative rings instead of $\mathbb{C}^d$. The definition of d -dimensional

NC universe, where we consider d -tuples of matrices, is taken from [1]. Throughout this thesis, unless stated otherwise, we work with this d -dimensional NC universe and simply call it the NC universe.

Example 1.1. Consider the set $\Omega = \text{Nilp}(\mathbb{C}) \subseteq \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ which consists of nilpotent matrices over $\mathbb{C}$. In this particular case since $\mathbb{C}$ is a field, this means that for $X \in \Omega$, there exists a T such that TXT^{-1} is strictly upper-triangular (i.e. 0's on and below the diagonal). This Ω is an NC set.

In this thesis, we equip the NC universe and NC sets, as defined above, with various natural topologies. All results in the theory of NC functions, including our original contributions, are presented using the uniform topology, defined below. When applicable, we explicitly note where results extend to other topologies.

Definition 1.2. Considering the Banach space $\mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^d, \mathbb{C}^n)$, we define the **row norm** of $Z \in \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^d, \mathbb{C}^n)$ as:

$$\|Z\|_{\text{row}} := \|(Z_1, \dots, Z_d)\|_{\mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^d, \mathbb{C}^n)} = \sqrt{\left\| \sum_{j=1}^d Z_j Z_j^* \right\|}.$$

Likewise, for the Banach space $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^d)$, we define the **column norm** as :

$$\|A\|_{\text{col}} := \left\| \begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} \right\|_{\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^d)} = \sqrt{\left\| \sum_{j=1}^d A_j^* A_j \right\|},$$

Consider a point in the NC universe at level s , namely, $Y \in \mathbb{C}^{(s \times s) \cdot d}$. We define:

$$r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y) := \bigcap_{m=1}^{\infty} r \cdot \mathbb{B}_{sm}^d(Y) = \bigcap_{m=1}^{\infty} \{X \in \mathbb{C}^{(sm \times sm) \cdot d} : \|X - I_m \otimes Y\|_{\text{row}} < r\} \quad (1.6)$$

Here, $I_m \otimes Y$ is Kronecker product of matrices, meaning that $I_m \otimes Y$ is a diagonal block matrix with the matrix Y on the each block entry of the diagonal. We call $r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$

the **uniform row-ball** of radius $r > 0$, centred at Y . By Proposition (7.12) of [11], for any $Z \in r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$ there exist a $\nu > 0$ such that $\nu \cdot \mathbb{B}_{s\mathbb{N}}^d(Z) \subseteq r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$. As a consequence, these NC balls form a basis for a topology on $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ which is known as the **uniform topology** on $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$. The open sets in the uniform topology are called **uniformly open NC sets**.

The uniform topology is not Hausdorff, since two points $I_m \otimes Y$ and $I_{m'} \otimes Y$ cannot be separated by the NC balls we defined above if $m \neq m'$. However, if we define an equivalence relation $\sim$ as follows:

$$X \sim X' \quad \text{if and only if} \quad X = I_m \otimes Y \text{ and } X' = I_{m'} \otimes Y \quad \text{for some } Y \in \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d},$$

then the quotient space $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} / \sim$, equipped with the quotient topology is Hausdorff.

For a set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$, the **radical of Ω** is the set:

$$\text{rad } \Omega := \left\{ Y \in \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} : \exists m \in \mathbb{N} \text{ such that } I_m \otimes Y \in \Omega \right\}.$$

We will say that $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is a **radical set** if $\text{rad } \Omega = \Omega$.

Remark 1.1. *rad Ω is not necessarily an NC set, even if Ω is an NC set. Indeed, suppose $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is an NC set which contains the matrices $0_{(2 \times 2)}$ and $I_{(2 \times 2)} \in \Omega_2$ and all their direct sums, in any possible order. Then the 1×1 matrices 0 and 1 belong to $\text{rad } \Omega$ while their direct sum does not.*

There is another topology that we can consider on $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$. Suppose we have a pseudo-metric defined as follows:

$$\mu_{\text{row}}(Z, W) := \|I_{t_1} \otimes Z - I_{t_2} \otimes W\|_{\text{row}},$$

in which $Z \in \mathbb{C}^{(n \times n) \cdot d}$ and $W \in \mathbb{C}^{(m \times m) \cdot d}$ with $n, m \in \mathbb{N}$ such that $\text{lcm}(n, m) = nt_1 = mt_2$. Here, $\text{lcm}(n, m)$ denotes the least common divisor of both n and m . The

corresponding topology on $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is induced by the NC balls of the form

$$\mathbb{B}_{\mu_{row}}(Y, \delta) := \{X \in \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} : \mu_{row}(X, Y) < \delta\},$$

where $\delta > 0$ and the NC ball is centred around a matrix point $Y \in \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$. These NC balls form a basis for the **pseudo-metric row topology** on the NC universe $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$.

We can obtain easily that the open balls in the pseudo-metric row topology are radical, therefore the open sets in this topology are radical as well. We shall call the open sets in this topology to be μ_{row} -open sets and denote by $\mathbb{B}_{\mu_{row}}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ the open balls in the pseudo-metric row topology.

We would like to compare the uniform topology and pseudo-metric topology.

Proposition 1.1 (See [11] for a proof). *1. For every $s \in \mathbb{N}$ and every fixed matrix point $Y \in \mathbb{C}^{(s \times s) \cdot d}$, and $\delta > 0$ we have :*

$$\delta \cdot \mathbb{B}_{\mathbb{N}}^d(Y) = \mathbb{B}_{\mu_{row}}(Y, \delta) \cap \bigcap_{m=1}^{\infty} \mathbb{C}^{(sm \times sm) \cdot d}$$

- 2. If $\Omega \in \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is a uniformly open set, then $\text{rad } \Omega$ is the smallest μ_{row} -open set containing Ω .*
- 3. The NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is a μ_{row} -open set if and only if Ω is uniformly open and radical.*

Considering the μ_{row} -equivalence relationship $X \sim_{\mu_{row}} X' \iff \mu_{row}(X, X') = 0$, we can prove that the quotient space of $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ with respect to this equivalence class is equal to the quotient space of $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ with respect to the equivalence class $\sim$, as sets.

One other natural way of defining a topology on $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is to consider the intersection of NC sets with the Euclidean open sets in each level. We call the NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ **finitely open** if for every $n \in \mathbb{N}$ the set $\Omega_n \subseteq \mathbb{C}^{(n \times n) \cdot d}$ is open in the

row matrix-norm, i.e. the intersection of Ω with any $\mathbb{C}^{(n \times n) \cdot d}$ is open in the Euclidean topology of the $\mathbb{C}^{(n \times n) \cdot d}$. The topology induced by these open sets is called the **finitely open topology** on $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$.

Definition 1.3. An NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is **right admissible** if for every $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathbb{C}^{n \times m}$ there exists a non-zero $r \in \mathbb{C}$ such that

$$\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Remark 1.2. If there is no risk of confusion, we will simply say that for every $X \in \Omega_n$, $Y \in \Omega_m$, and every **appropriate** $Z \in \mathbb{C}^{n \times m}$, we have

$$\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

What we mean by this is that there exists a nonzero $r \in \mathbb{C}$ as in the definition above, but we choose to write Z instead of rZ in order to avoid repetition.

Example 1.2. Any uniformly open NC set Ω is right admissible: if we take $X \in \Omega_m$ and $Y \in \Omega_n$ then:

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \Omega_{m+n}.$$

There is an $r > 0$ such that $r \cdot \mathbb{B}_{(n+m) \cdot \mathbb{N}}^d \left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right) \subseteq \Omega$. If $\delta > 0$ is small then

$$\begin{pmatrix} X & \delta Z \\ 0 & Y \end{pmatrix} \in r \cdot \mathbb{B}_{(n+m) \cdot \mathbb{N}}^d \left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right)$$

for small enough $\delta > 0$, because

$$\left\| \begin{pmatrix} X & \delta Z \\ 0 & Y \end{pmatrix} - \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\|_{\text{row}} = \left\| \begin{pmatrix} 0 & \delta Z \\ 0 & 0 \end{pmatrix} \right\|_{\text{row}} = \|\delta Z\|_{\text{row}} = \delta \|Z\|_{\text{row}}.$$

Suppose that we have an NC set Ω . We say that Ω is **invariant under similarities** if for every square invertible matrix $S \in \mathbb{C}^{n \times n}$, of appropriate size (n can vary and be arbitrary), SXS^{-1} is also in Ω .

Definition 1.4. We can construct an NC set from a given NC set in the NC universe that is invariant under similarities, and yet, preserves the essential properties of the original NC set. Suppose that $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is an NC set. Then

$$\tilde{\Omega} := \{SXS^{-1} : X \in \Omega_n, S \in \mathbb{C}^{n \times n} \text{ invertible}, n \in \mathbb{N}\}$$

is an NC set. (Note that we define $SXS^{-1} := (SX_1S^{-1}, \dots, SX_dS^{-1}) \in \Omega_n$.)

Clearly, $\tilde{\Omega}$ is the smallest NC set that contains Ω and that is invariant under similarities. We shall call $\tilde{\Omega}$ the **similarity invariant envelope** of the NC set Ω .

Now let $X \in \Omega_n$, $Y \in \Omega_m$, and let $S \in \mathbb{C}^{n \times n}$, $T \in \mathbb{C}^{m \times m}$ be invertible, $n, m \in \mathbb{N}$. Then $S \oplus T \in \mathbb{C}^{(n+m) \times (n+m)}$ is invertible, $X \oplus Y \in \Omega_{n+m}$, and

$$SXS^{-1} \oplus TYT^{-1} = (S \oplus T)(X \oplus Y)(S \oplus T)^{-1} \in \tilde{\Omega}_{n+m}.$$

Hence, $\tilde{\Omega}$ is closed under direct sums.

Proposition 1.2 ([11], Proposition A.2). If $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is a right admissible NC set, then so is its similarity invariant envelope $\tilde{\Omega}$. Moreover, for any $\tilde{X} \in \tilde{\Omega}_n$, $\tilde{Y} \in \tilde{\Omega}_m$, and $Z \in \mathbb{C}^{n \times m}$, one has

$$\begin{bmatrix} \tilde{X} & Z \\ 0 & \tilde{Y} \end{bmatrix} \in \tilde{\Omega}_{n+m},$$

and, respectively, for any $\tilde{X} \in \tilde{\Omega}_n$, $\tilde{Y} \in \tilde{\Omega}_m$, and $Z \in \mathbb{C}^{m \times n}$, one has

$$\begin{bmatrix} \tilde{X} & 0 \\ Z & \tilde{Y} \end{bmatrix} \in \tilde{\Omega}_{n+m}.$$

Proof. Let $\tilde{X} \in \tilde{\Omega}_n$, $\tilde{Y} \in \tilde{\Omega}_m$, and $Z \in \mathbb{C}^{n \times m}$. Then there exist $X \in \Omega_n$, $Y \in \Omega_m$, and

invertible $S \in \mathbb{C}^{n \times n}$, $T \in \mathbb{C}^{m \times m}$, and $r \in \mathbb{C}$ such that $\tilde{X} = SXS^{-1}$, $\tilde{Y} = TYT^{-1}$, and

$$\begin{bmatrix} X & rS^{-1}ZT \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Then

$$\begin{bmatrix} \tilde{X} & Z \\ 0 & \tilde{Y} \end{bmatrix} = \begin{bmatrix} r^{-1}S & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} X & rS^{-1}ZT \\ 0 & Y \end{bmatrix} \begin{bmatrix} r^{-1}S & 0 \\ 0 & T \end{bmatrix}^{-1} \in \tilde{\Omega}_{n+m}.$$

□

1.2 NC functions and their Differentials

The definition of an NC function is as follows:

Definition 1.5. Let $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ be an NC set. A mapping $f : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$, for $d, d' \in \mathbb{N}$ with $f(\Omega_n) \subseteq \mathbb{C}^{(n \times n) \cdot d'}$, $n \in \mathbb{N}$ (f respects grading), is called an **NC function** if f satisfies the following two conditions:

- f respects direct sums:

$$f\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} f(X) & 0 \\ 0 & f(Y) \end{bmatrix}.$$

- f respects similarities: if $X \in \Omega_n$ and $S \in \mathbb{C}^{n \times n}$ is invertible with $SXS^{-1} = (SX_1S^{-1}, \dots, SX_dS^{-1}) \in \Omega_n$, then

$$f(SXS^{-1}) = Sf(X)S^{-1}.$$

We can combine the two conditions here into a single condition which is originally due to J. L. Taylor in [23] .

Proposition 1.3. A mapping $f : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$, with $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ and $f(\Omega_n) \subseteq \mathbb{C}^{(n \times n) \cdot d'}$, $n \in \mathbb{N} \cup \{0\}$ respects direct sums and similarities if and only if it respects intertwinings: for any $X \in \Omega_n$, $Y \in \Omega_m$, and $T \in \mathbb{C}^{n \times m}$, one has

$$XT = TY \implies f(X)T = Tf(Y),$$

For $XT := (X_1T, \dots, X_dT)$ and $TY = (TY_1, \dots, TY_d)$.

Proof. Proposition 2.1 of [11]. □

Proposition 1.4 (Canonical extension of NC functions). Let $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ be an NC set, and let $f : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$ be an NC function. Then there exists a unique NC function $\tilde{f} : \tilde{\Omega} \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$ such that $\tilde{f}|_{\Omega} = f$.

Proof. For any $n \in \mathbb{N}$ and $\tilde{X} \in \tilde{\Omega}_n$, there exist $X \in \Omega_n$ and an invertible $S \in \mathbb{C}^{n \times n}$ such that $\tilde{X} = SXS^{-1}$, and we set $\tilde{f}(\tilde{X}) = Sf(X)S^{-1}$. This definition is correct, since $SXS^{-1} = TYT^{-1}$ for $X, Y \in \Omega_n$ and for invertible $S, T \in \mathbb{C}^{n \times n}$ implies that $Y = (T^{-1}S)X(T^{-1}S)^{-1}$, hence $f(Y) = (T^{-1}S)f(X)(T^{-1}S)^{-1}$ and $Tf(Y)T^{-1} = Sf(X)S^{-1}$. It is straightforward to check that $\tilde{f} : \tilde{\Omega} \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$ is an NC function and that $\tilde{f}|_{\Omega} = f$. If $\tilde{g} : \tilde{\Omega} \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$ is another NC function such that $\tilde{g}|_{\Omega} = f$, then for any $X \in \Omega_n$ and invertible $S \in \mathbb{C}^{n \times n}$ we have

$$\tilde{g}(SXS^{-1}) = S\tilde{g}(X)S^{-1} = Sf(X)S^{-1} = \tilde{f}(SXS^{-1}),$$

i.e., we have necessarily $\tilde{g} = \tilde{f}$. □

We will call the NC function $\tilde{f} : \tilde{\Omega} \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$ from Proposition (1.4) the **canonical extension of the NC function** $f : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$.

Example 1.3. For an arbitrary holomorphic function defined on an open domain in the complex plane, we can naturally construct an NC function: If f is holomorphic function in

an open $\Omega \subset \mathbb{C}$, define $\widehat{\Omega} \subseteq \mathbb{C}^{\mathbb{N} \times \mathbb{N}} := \bigsqcup_{n=1}^{\infty} \mathbb{C}^{n \times n}$ by

$$\widehat{\Omega}_n := \{X \in \mathbb{C}^{n \times n} : \sigma(X) \subseteq \Omega\},$$

where $\sigma(X)$ denotes the spectrum of X . It follows immediately that $\widehat{\Omega}$ is an NC set. f is an NC function on $\widehat{\Omega}$ by Riesz-Dunford holomorphic functional calculus (see [15], Chapter 9 for the precise definition), which is defined by:

$$f(X) := \frac{1}{2\pi i} \int_{\gamma} f(z)(zI - X)^{-1} dz.$$

where γ is a piecewise smooth ($\mathcal{C}^1$), simple counter-clockwise contour such that $\text{Ran } \gamma \subseteq \Omega$ and $\sigma(X)$ is contained in the interior of γ .

We can simply prove that f satisfies the conditions of an NC function: Let $X \in \mathbb{C}^{(n \times n) \cdot d}$ and $Y \in \mathbb{C}^{(m \times m) \cdot d}$. We have:

$$\begin{aligned} f \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} &= \frac{1}{2\pi i} \int_{\gamma} f(z) \left(\begin{pmatrix} zI_{(n \times n)} & 0 \\ 0 & zI_{(m \times m)} \end{pmatrix} - \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right)^{-1} dz \\ &= \frac{1}{2\pi i} \int_{\gamma} f(z) \left(\begin{pmatrix} zI_{(n \times n)} - X & 0 \\ 0 & zI_{(m \times m)} - Y \end{pmatrix} \right)^{-1} dz \\ &= \begin{pmatrix} \frac{1}{2\pi i} \int_{\gamma} f(z)(zI_{(n \times n)} - X)^{-1} dz & 0 \\ 0 & \frac{1}{2\pi i} \int_{\gamma} f(z)(zI_{(m \times m)} - Y)^{-1} dz \end{pmatrix} \\ &= \begin{pmatrix} f(X) & 0 \\ 0 & f(Y) \end{pmatrix}. \end{aligned}$$

Similarly, it is easy to show that f respects joint similarities and the grading sizes (meaning that f takes $n \times n$ matrices to $n \times n$ matrices). Hence f is an NC function.

Example 1.4. The NC unit ball $\Omega = 1 \cdot \mathbb{B}_{\mathbb{N}}^1(0_{1 \times 1})$ (in one variable) is an NC set. Its

similarity invariant envelope is

$$\tilde{\Omega} = \{Z \in \mathbb{C}^{(\mathbb{N} \times \mathbb{N})} : r_{\text{spec}}(Z) < 1\}.$$

where $r_{\text{spec}}(Z)$ is the spectral radius of the matrix $Z \in \mathbb{C}^{n \times n}$. We need to show that $Z = SXS^{-1}$ for some $X \in \mathbb{C}^{n \times n}$ with $\|X\| < 1$ and some invertible $S \in \mathbb{C}^{n \times n}$. It suffices to show this for Z a Jordan cell, i.e., for

$$Z = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

with $|\lambda| < 1$. Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - |\lambda|$, ensuring that $\|X\| < 1$, where

$$X = \begin{bmatrix} \lambda & \epsilon & & \\ & \ddots & \ddots & \\ & & \ddots & \epsilon \\ & & & \lambda \end{bmatrix}.$$

Then $Z = SXS^{-1}$ with $S = \text{diag}(\underbrace{\epsilon^{-1}, \dots, \epsilon^{-1}}_{n \text{ times}})$.

Clearly the NC function $f : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ defined by $f(Z) = (I - Z)^{-1}$ can be canonically extended to an NC function on $\tilde{\Omega}$ using the same formula.

1.2.1 The NC Difference-Differential Operator

The direct-sum property in Definition (1.5) is applied to block-diagonal 2×2 matrices, where all off-diagonal (in particular, upper-right) blocks are zero. The main idea in this section is to show the following : if one evaluates an NC function, $f : \Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$, on a block upper triangular matrix, $\begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}$, with $X \in \Omega_n$,

$Y \in \Omega_m$ and $Z \in \mathbb{C}^{n \times m} \otimes \mathbb{C}^{1 \times d}$, one can show that:

$$f \left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right) = \begin{bmatrix} f(X) & \Delta f(X, Y)(Z) \\ 0 & f(Y) \end{bmatrix}. \quad (1.7)$$

where $Z \mapsto \Delta f(X, Y)(Z)$ is a $\mathbb{C}$ -linear operator from $\mathbb{C}^{n \times m} \otimes \mathbb{C}^{1 \times d}$ to $\mathbb{C}^{n \times m}$. The formal definition of $\Delta f(X, Y)(Z)$ will be given in Definition (1.6). We follow [11] to show the properties of this map and prove that it acts as a “right NC difference operator” when $Y \neq X$, and as a “right NC differential operator” when $Y = X$.

Analogously, one can define the left NC differential operator by considering the lower triangular matrices. We will only focus on the right NC differential operator and we will simply call it the NC differential operator. The definition and properties of the left NC differential operator can be derived by symmetric arguments from the right NC differential operators.

To see that the matrix on the right side of the Equation (1.7) is actually upper triangular, denote

$$f \left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right) = \begin{bmatrix} A & \Delta f(X, Y)(Z) \\ C & D \end{bmatrix}.$$

We have

$$\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \begin{bmatrix} I_n \\ 0 \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix} X.$$

By Proposition (1.3),

$$\begin{pmatrix} A \\ C \end{pmatrix} = f \left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right) \begin{bmatrix} I_n \\ 0 \end{bmatrix} = \begin{bmatrix} I_n \\ 0 \end{bmatrix} f(X) = \begin{pmatrix} f(X) \\ 0 \end{pmatrix}.$$

This implies that $A = f(X)$ and $C = 0$. We have also

$$Y \begin{bmatrix} 0 & I_m \end{bmatrix} = \begin{bmatrix} 0 & I_m \end{bmatrix} \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}.$$

By Proposition (1.3),

$$f(Y) \begin{bmatrix} 0 & I_m \end{bmatrix} = \begin{bmatrix} 0 & I_m \end{bmatrix} f \left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right),$$

which implies that $D = f(Y)$ and that again $C = 0$. Thus, (1.7) holds ([11], Proposition 2.2).

Given an NC set Ω , the assumption that

$$\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m} \quad \text{for all } X \in \Omega_n,$$

and $Y \in \Omega_m$, and $Z \in \mathbb{C}^{n \times m}$ would be too strong. We can put weaker assumptions on the NC set Ω . The key observation is the following property of $\Delta f(X, Y)(\cdot)$:

Let f be an NC function on an NC set Ω . Let $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathbb{C}^{(n \times m) \cdot d}$ be such that

$$\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Then (1.7) holds, where the off-diagonal block entry $\Delta f(X, Y)(Z)$ is determined uniquely by (1.7) and has the following property. If $r \in \mathbb{C}$ is such that

$$\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}, \tag{1.8}$$

then

$$\Delta f(X, Y)(rZ) = r \Delta f(X, Y)(Z); \tag{1.9}$$

for every $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathbb{C}^{n \times m}$, (See [11]. Proposition (2.2) for the proof of this homogeneity property).

Using this homogeneity property, Equation (1.9), we can define $\Delta f(X, Y)(\cdot) : \mathbb{C}^{(n \times m) \cdot d} \rightarrow \mathbb{C}^{n \times m}$, for any $X \in \Omega_n$ and $Y \in \Omega_m$, if Ω is any right-admissible NC set:

Definition 1.6. Let Ω be right admissible NC set, and let f be an NC function on Ω . Then for every $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathbb{C}^{n \times m}$, and for non-zero $r \in \mathbb{C}$ such that

$$\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$$

we define first $\Delta f(X, Y)(rZ)$ as the $(1, 2)$ block of the matrix $f \left(\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \right)$ and then define the **NC difference-differential operator** $\Delta f(X, Y)(\cdot) : \mathbb{C}^{n \times m} \otimes \mathbb{C}^{1 \times d} \rightarrow \mathbb{C}^{n \times m}$ by:

$$\Delta f(X, Y)(Z) = r^{-1} \Delta f(X, Y)(rZ). \quad (1.10)$$

Using the homogeneity property (1.9), and the fact that $\Delta f(X, Y)(Z)$ is uniquely determined by Equation (1.7), the right-hand side of (1.10) is defined unambiguously for all $Z \in \mathbb{C}^{(n \times m) \cdot d} := \mathbb{C}^{n \times m} \otimes \mathbb{C}^{1 \times d}$.

The NC -difference-differential operators has the following basic properties:

1. Let f be an NC function defined on an NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$. The mapping $Z \mapsto \Delta f(X, Y)(Z)$ is a $\mathbb{C}$ -linear mapping, meaning that:

$$\Delta f(X, Y)(sZ_1 + Z_2) = s \Delta f(X, Y)(Z_1) + \Delta f(X, Y)(Z_2),$$

for $s \in \mathbb{C}$ and every $X \in \Omega_n$, $Y \in \Omega_m$, and $Z_1, Z_2 \in \mathbb{C}^{(n \times m) \cdot d}$ (See Proposition (2.4) of [11]).

2. If $f : \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is a constant NC function, i.e., there is a $c \in \mathbb{C}$ such that

$f(X) = cI_n (= \text{diag}[c, \dots, c])$ for all $n \in \mathbb{N}$ and all $X \in \mathbb{C}^{n \times n}$, then

$$\Delta f(X, Y)(Z) = 0 \quad \text{for all } X, Y, Z \text{ of appropriate sizes.}$$

3. If $f, g : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$ are NC functions on a right admissible NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ and $a, b \in \mathbb{C}$, then

$$\Delta(af + bg)(X, Y)(\cdot) = a\Delta f(X, Y)(\cdot) + b\Delta g(X, Y)(\cdot),$$

for every X and Y of appropriate sizes.

4. Let $l : \mathbb{C} \rightarrow \mathbb{C}$ be an $\mathbb{C}$ -linear mapping. We extend l to a linear mapping from $\mathbb{C}^{n \times m}$ to $\mathbb{C}^{n \times m}$, $n, m \in \mathbb{N}$ by $l([\mu_{ij}]) = [l(\mu_{ij})]$. Then l is an NC function from $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ to $\mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$, and

$$\Delta l(X, Y)(Z) = l(Z) \quad \text{for all } X, Y, Z \text{ of appropriate sizes.}$$

In particular, if $l_j : \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$ is the j -th coordinate NC function, i.e.,

$$l_j(X) = l_j(X_1, \dots, X_d) = X_j,$$

then

$$\Delta l_j(X, Y)(Z) = Z_j,$$

for all d -tuples X, Y, Z of complex matrices of appropriate sizes.

5.(Product Rule) Let $f : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$ and $g : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$ be NC functions on a right admissible NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$. We extend the multiplication operation in complex numbers to matrices over $\mathbb{C}$ of appropriate sizes. It is easy to check that $f \cdot g : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$ is an NC function.

Then

$$\Delta(f \cdot g)(X, Y)(Z) = f(X) \cdot \Delta g(X, Y)(Z) + \Delta f(X, Y)(Z) \cdot g(Y)$$

for all $n, m \in \mathbb{N}$, $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathbb{C}^{(n \times m) \cdot d}$.

Proof 5. Without loss of generality, we can assume that Ω is similarity invariant, i.e., $\tilde{\Omega} = \Omega$; see Remark 2.5 of [11]. Then by Proposition (1.4), for arbitrary $X \in \Omega_n$, $Y \in \Omega_m$, and appropriate $Z \in \mathbb{C}^{n \times m}$ one has

$$\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Therefore we have:

$$(f \cdot g) \left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right) = \begin{bmatrix} (f \cdot g)(X) & \Delta(f \cdot g)(X, Y)(Z) \\ 0 & (f \cdot g)(Y) \end{bmatrix}. \quad (1.11)$$

On the other hand,

$$\begin{aligned} f \left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right) \cdot g \left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right) &= \begin{bmatrix} f(X) & \Delta f(X, Y)(Z) \\ 0 & f(Y) \end{bmatrix} \cdot \begin{bmatrix} g(X) & \Delta g(X, Y)(Z) \\ 0 & g(Y) \end{bmatrix} \\ &= \begin{bmatrix} f(X) \cdot g(X) & f(X) \cdot \Delta g(X, Y)(Z) + \Delta f(X, Y)(Z) \cdot g(Y) \\ 0 & f(Y) \cdot g(Y) \end{bmatrix}. \end{aligned} \quad (1.12)$$

Comparing the (1, 2) blocks on the right-hand sides of (1.11) and (1.12), we obtain the required formula for $\Delta(f \cdot g)(X, Y)(Z)$. $\square$

6. (Chain Rule) Let $f : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$ and $g : \Lambda \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$ be NC functions on right admissible NC sets $\Omega, \Lambda \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$, such that $f(\Omega) \subseteq \Lambda$. Then the composition $g \circ f :$

$\Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$ is an NC function, and

$$\Delta(g \circ f)(X, Y)(Z) = \Delta g(f(X), f(Y))(\Delta f(X, Y)(Z))$$

for all $n, m \in \mathbb{N}$, $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathbb{C}^{n \times m}$.

Proof 6. Without loss of generality, that Ω is similarity invariant, i.e., $\tilde{\Omega} = \Omega$; see Remark 2.5 of [11]. Then, for arbitrary $n, m \in \mathbb{N}$, $X \in \Omega_n$, $Y \in \Omega_m$, and $Z \in \mathbb{C}^{n \times m}$ one has

$$\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Therefore,

$$(g \circ f) \left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right) = \begin{bmatrix} (g \circ f)(X) & \Delta(g \circ f)(X, Y)(Z) \\ 0 & (g \circ f)(Y) \end{bmatrix}. \quad (1.13)$$

On the other hand,

$$\begin{aligned} g \left(f \left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \right) \right) &= g \left(\begin{bmatrix} f(X) & \Delta f(X, Y)(Z) \\ 0 & f(Y) \end{bmatrix} \right) \\ &= \begin{bmatrix} g(f(X)) & \Delta g(f(X), f(Y))(\Delta f(X, Y)(Z)) \\ 0 & g(f(Y)) \end{bmatrix}. \end{aligned} \quad (1.14)$$

Comparing the (1, 2) blocks on the right-hand sides of (1.13) and (1.14), we obtain the required formula for $\Delta(g \circ f)(X, Y)(Z)$. $\square$

Definition 1.7 (Invertibility domain of an NC function). *Let $f : \Omega \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N})}$ be an NC function on a right admissible NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$. Let*

$$\Omega^{inv} := \bigsqcup_{n=1}^{\infty} \{X \in \Omega_n : f(X) \text{ is invertible in } \mathbb{C}^{n \times n}\}.$$

*Then Ω^{inv} is a right admissible NC set and we call it the **invertibility domain** of f for*

which $f^{-1} : \Omega^{\text{inv}} \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ defined by $f^{-1}(X) := f(X)^{-1}$ is an NC function.

If $\text{Dom } f = \Omega$ is uniformly open, so is $\text{Dom } f^{-1} = \Omega^{\text{inv}}$ and if f is uniformly continuous on Ω then f^{-1} is also uniformly continuous on Ω^{inv} .

6. Let f be the NC function as in the definition above and let f^{-1} be its inverse. We have:

$$\Delta f^{-1}(X, Y)(Z) = -f(X)^{-1} \Delta f(X, Y)(Z) f(Y)^{-1}$$

for all $n, m \in \mathbb{N}$, $X \in \Omega_n^{\text{inv}}$, $Y \in \Omega_m^{\text{inv}}$, and $Z \in \mathbb{C}^{n \times m}$.

Proof. Note that Ω^{inv} is an NC set since f respects direct sums. We show next that if Ω is right admissible, then so is Ω^{inv} : Let $X \in \Omega_n^{\text{inv}} \subseteq \Omega_n$, $Y \in \Omega_m^{\text{inv}} \subseteq \Omega_m$, and $Z \in \mathbb{C}^{n \times m}$. Then there exists an invertible $r \in \mathbb{C}$ such that

$$\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

We have

$$f \left(\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \right) = \begin{bmatrix} f(X) & \Delta f(X, Y)(rZ) \\ 0 & f(Y) \end{bmatrix}.$$

Since $f(X)$ and $f(Y)$ are invertible, so is the right-hand side. Therefore,

$$\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}^{\text{inv}},$$

and Ω^{inv} is right admissible.

Since we have:

$$f^{-1} \left(\begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \right) = \begin{bmatrix} f(X)^{-1} & -f(X)^{-1} \Delta f(X, Y)(rZ) f(Y)^{-1} \\ 0 & f(Y)^{-1} \end{bmatrix},$$

we conclude that:

$$\Delta f^{-1}(X, Y)(rZ) = -f(X)^{-1} \Delta f(X, Y)(rZ) f(Y)^{-1},$$

which implies the required formula for $\Delta f^{-1}(X, Y)(Z)$ by linearity. $\square$

1.2.2 Finite difference-differential formulas

Theorem 1.1 ([11], Section 1.2). *Let $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ be an NC function on a right admissible NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$. Then for all $n, m \in \mathbb{N}$, and arbitrary $X \in \Omega_n$, $Y \in \Omega_m$, and $S \in \mathbb{C}^{m \times n}$,*

$$Sf(X) - f(Y)S = \Delta f(Y, X)(SX - YS). \quad (1.15)$$

Proof. We may assume, without loss of generality, that Ω is similarity invariant, i.e., $\tilde{\Omega} = \Omega$; see Remark 2.5 of [11]. For arbitrary $n, m \in \mathbb{N}$, $X \in \Omega_n$, $Y \in \Omega_m$, and appropriate $S \in \mathbb{C}^{m \times n}$ (in the sense of Remark (1.2)), since Ω is an NC set, one has:

$$\begin{bmatrix} Y & SX - YS \\ 0 & X \end{bmatrix} \in \Omega_{m+n}.$$

Clearly,

$$\begin{bmatrix} Y & SX - YS \\ 0 & X \end{bmatrix} \begin{bmatrix} S \\ I_n \end{bmatrix} = \begin{bmatrix} S \\ I_n \end{bmatrix} X.$$

Therefore, by the result of Proposition (1.3), we have:

$$f \left(\begin{bmatrix} Y & SX - YS \\ 0 & X \end{bmatrix} \right) \begin{bmatrix} S \\ I_n \end{bmatrix} = \begin{bmatrix} S \\ I_n \end{bmatrix} f(X),$$

i.e., by Equation (1.7):

$$\begin{bmatrix} f(Y) & \Delta f(Y, X)(SX - YS) \\ 0 & f(X) \end{bmatrix} \begin{bmatrix} S \\ I_n \end{bmatrix} = \begin{bmatrix} S \\ I_n \end{bmatrix} f(X).$$

Comparing the $(1, 1)$ block entries in the matrix products in the right-hand side and in the left-hand side, we obtain

$$f(Y)S + \Delta f(Y, X)(SX - YS) = Sf(X),$$

which yields (1.15). So far, the Equation (1.15) is obtained for appropriate $S \in \mathbb{C}^{m \times n}$. For arbitrary $S \in \mathbb{C}^{m \times n}$, one can multiply both sides of Equation (1.15) by $r \in \mathbb{C}$ and the desired result of the theorem can be obtained by considering linearity of difference-differential operator. $\square$

Naturally, one can define the Gâteaux derivative of an NC function f at the point $X \in \Omega_n$ in the direction of $H \in \mathbb{C}^{n \times n}$ as follows: Given the equality (1.15), one can specifically write it as

$$f(X + tH) - f(X) = \Delta f(X + tH, X)(tH) = t \cdot \Delta f(X + tH, X)(H).$$

If for every $n \in \mathbb{N}$, and every $X \in \Omega_n$ and $H \in \mathbb{C}^{n \times n}$, there exists a $\epsilon > 0$ such that $f(X + tH)$ is bounded for $|t| < \epsilon$, then, by dividing both sides by t , we have:

$$\frac{f(X + tH) - f(X)}{t} = \Delta f(X + tH, X)(H). \quad (1.16)$$

For the right-hand side the this equation, by definition, we can write:

$$f \begin{pmatrix} X + tH & H \\ 0 & X \end{pmatrix} = \begin{pmatrix} f(X + tH) & \Delta f(X + tH, X)(H) \\ 0 & f(X) \end{pmatrix}. \quad (1.17)$$

If $f|_{\text{Dom}(f)}$ is (row) matrix-norm continuous at each level (equivalently, if f is con-

tinuous in the finitely open topology), then the left-hand side of Equation (1.17) is a continuous function of t . This proves that :

$$\lim_{t \rightarrow 0} \Delta f(X + tH, X)(H) = \Delta f(X, X)(H).$$

Taking the limit from the left-hand side of Equation (1.16), gives us the **Gâteaux derivative (G-derivative)** of the NC function f at the matrix point X in the direction of H :

$$\partial_H f(X) := \lim_{t \rightarrow 0} \frac{f(X + tH) - f(X)}{t} = \Delta f(X, X)(H).$$

Here, we can prove a surprising proposition. We can prove that if an NC function is continuous, then it is holomorphic, meaning that the directional derivatives of the NC function exist.

Proposition 1.5. *Suppose $f : \text{Dom}(f) \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is an NC function with a uniformly open NC domain $\text{Dom}(f)$ (it can just be open in the finitely open topology i.e. each $\text{Dom}_n(f) := \text{Dom}(f) \cap \mathbb{C}^{(n \times n) \cdot d}$ is an open set in the Euclidean topology of $\mathbb{C}^{(n \times n) \cdot d}$), and assume that $f|_{\text{Dom}(f)}$ is (row) matrix-norm continuous at each level, i.e. f is continuous in the finitely open topology. Then all the directional derivatives of f exist at any point $X \in \text{Dom } f$.*

Remark 1.3. *Any uniformly open set is finitely open (open in finitely open topology). Hence, if f is continuous with respect to the uniformly open topology, it is continuous in the finitely open topology.*

Proof of Proposition (1.5). Choose $X \in \text{Dom}_n(f)$ and any direction $H \in \mathbb{C}^{(n \times n) \cdot d}$. consider

$$f \begin{pmatrix} X + tH & tH \\ 0 & X \end{pmatrix}$$

with $t \in \mathbb{C}$, small enough so that $\begin{pmatrix} X + tH & tH \\ 0 & X \end{pmatrix} \in \text{Dom}_n(f)$ is in the domain of f (this is

possible because $\text{Dom}(f)$ is assumed to be a uniformly open NC set). We also consider

$$S_t = \begin{pmatrix} I_n & -t^{-1}I_n \\ 0 & I_n \end{pmatrix}, \quad S_t^{-1} = \begin{pmatrix} I_n & t^{-1}I_n \\ 0 & I_n \end{pmatrix}.$$

Since f is an NC function, the equality $f(S_t X S_t^{-1}) = S_t f(X) S_t^{-1}$ must be true. Because of continuity, we know that

$$f \begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \lim_{t \rightarrow 0} f \begin{pmatrix} X + tH & H \\ 0 & X \end{pmatrix}.$$

So, we have

$$\begin{aligned} f \left(S_t \begin{pmatrix} X + tH & 0 \\ 0 & X \end{pmatrix} S_t^{-1} \right) &= f \begin{pmatrix} X + tH & H \\ 0 & X \end{pmatrix} \\ &= S_t f \begin{pmatrix} X + tH & 0 \\ 0 & X \end{pmatrix} S_t^{-1} = S_t \begin{pmatrix} f(X + tH) & 0 \\ 0 & f(X) \end{pmatrix} S_t^{-1} \\ &= \begin{pmatrix} f(X + tH) & \frac{f(X + tH) - f(X)}{t} \\ 0 & f(X) \end{pmatrix}. \end{aligned}$$

Therefore, we conclude that :

$$f \begin{pmatrix} X & H \\ 0 & X \end{pmatrix} = \lim_{t \rightarrow 0} \begin{pmatrix} f(X + tH) & \frac{f(X + tH) - f(X)}{t} \\ 0 & f(X) \end{pmatrix},$$

which means that the limit

$$\lim_{t \rightarrow 0} \frac{f(X + tH) - f(X)}{t} = \partial_H f(X),$$

exists. Since this is true for all directions H , we conclude that f is holomorphic. $\square$

With a bit more work, one can show that “local boundedness” implies holomorphic for an NC function.

Definition 1.8. Let $f : \text{Dom } f \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$ be an NC function. We say that f is **uniformly locally bounded** if given any $Y \in \text{Dom } f$, there exist a uniformly open neighbourhood of Y , say Λ , on which f is uniformly bounded in matrix-norm.

Lemma 1.1. (See for example [11], Theorem 7.23) Let $f : \text{Dom}(f) \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ be an NC function on the uniformly open NC set, $\text{Dom}(f)$. Then, if f is uniformly locally bounded, it is holomorphic, meaning that for any $X \in \text{Dom}_n(f)$ and any $H \in \mathbb{C}^{(n \times n) \cdot d}$, the derivative $\partial_H f(X)$ exists and is equal to $\Delta f(X, X)(H)$.

An NC function f has nice properties when we take its NC difference-differential with direct sums, similarities, and intertwining in the arguments of the derivative. These next propositions show how $\Delta f(X, Y)(Z)$ behaves with respect to direct sums and joint similarities.

Proposition 1.6. (See [11], Proposition 2.15) Let $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ be an NC function on a right admissible NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$. Then:

$$\Delta f(X' \oplus X'', Y) \left(\begin{bmatrix} Z' \\ Z'' \end{bmatrix} \right) = \begin{bmatrix} \Delta f(X', Y)(Z') \\ \Delta f(X'', Y)(Z'') \end{bmatrix}, \quad (1X)$$

for $n', n'', m \in \mathbb{N}$, $X' \in \Omega_{n'}$, $X'' \in \Omega_{n''}$, $Y \in \Omega_m$, $Z' \in \mathbb{C}^{n' \times m}$, $Z'' \in \mathbb{C}^{n'' \times m}$;

$$\Delta f(X, Y' \oplus Y'') \left(\begin{bmatrix} Z' & Z'' \end{bmatrix} \right) = \begin{bmatrix} \Delta f(X, Y')(Z') & \Delta f(X, Y'')(Z'') \end{bmatrix}, \quad (1Y)$$

for $n, m', m'' \in \mathbb{N}$, $X \in \Omega_n$, $Y' \in \Omega_{m'}$, $Y'' \in \Omega_{m''}$, $Z' \in \mathbb{C}^{n \times m'}$, $Z'' \in \mathbb{C}^{n \times m''}$;

$$\Delta f(TXT^{-1}, Y)(TZ) = T \Delta f(X, Y)(Z), \quad (2X)$$

for $n, m \in \mathbb{N}$, $X \in \Omega_n$, $Y \in \Omega_m$, $Z \in \mathbb{C}^{n \times m}$, and an invertible $T \in \mathbb{C}^{n \times n}$ such that

$$TXT^{-1} \in \Omega_n;$$

$$\Delta f(X, SYS^{-1})(ZS^{-1}) = \Delta f(X, Y)(Z)S^{-1}, \quad (2Y)$$

for $n, m \in \mathbb{N}$, $X \in \Omega_n$, $Y \in \Omega_m$, $Z \in \mathbb{C}^{n \times m}$, and an invertible $S \in \mathbb{C}^{m \times m}$ such that $SYS^{-1} \in \Omega_m$;

If $TX = \tilde{X}T$, then

$$T\Delta f(X, Y)(Z) = \Delta f(\tilde{X}, Y)(TZ), \quad (3X)$$

for $n, \tilde{n}, m \in \mathbb{N}$, $X \in \Omega_n$, $\tilde{X} \in \Omega_{\tilde{n}}$, $Y \in \Omega_m$, $Z \in \mathbb{C}^{n \times m}$, and $T \in \mathbb{C}^{\tilde{n} \times n}$;

If $YS = S\tilde{Y}$, then

$$\Delta f(X, Y)(Z)S = \Delta f(X, \tilde{Y})(ZS), \quad (3Y)$$

for $n, m, \tilde{m} \in \mathbb{N}$, $X \in \Omega_n$, $Y \in \Omega_m$, $\tilde{Y} \in \Omega_{\tilde{m}}$, $Z \in \mathbb{C}^{n \times m}$, and $S \in \mathbb{C}^{m \times \tilde{m}}$.

Combining the formulation in the proposition above, we get the following:

Proposition 1.7. [11] Let $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ be an NC function on a right admissible NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ for $d \in \mathbb{N}$. Then:

$$\Delta f(X_1 \oplus X_2, Y_1 \oplus Y_2) \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} \Delta f(X_1, Y_1)(Z_1) & \Delta f(X_1, Y_2)(Z_2) \\ \Delta f(X_2, Y_1)(Z_3) & \Delta f(X_2, Y_2)(Z_4) \end{bmatrix}, \quad (1.18)$$

for $n_1, m_1, n_2, m_2 \in \mathbb{N}$, $X_1 \in \Omega_{n_1}$, $X_2 \in \Omega_{n_2}$, $Y_1 \in \Omega_{m_1}$, $Y_2 \in \Omega_{m_2}$, and

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} \in \mathbb{C}^{((n_1+n_2) \times (m_1+m_2)) \cdot d},$$

with block entries of appropriate sizes. If either n_2 or m_2 is 0, the corresponding block entry is void. Furthermore,

$$\Delta f(TXT^{-1}, SYS^{-1})(TZS^{-1}) = T\Delta f(X, Y)(Z)S^{-1}, \quad (1.19)$$

for $n, m \in \mathbb{N}$, $X \in \Omega_n$, $Y \in \Omega_m$, $Z \in \mathbb{C}^{n \times m}$, and invertible $T \in \mathbb{C}^{n \times n}$, $S \in \mathbb{C}^{m \times m}$ such

that $TXT^{-1} \in \Omega_n$ and $SY S^{-1} \in \Omega_m$.

If $TX = \tilde{X}$ and $YS = \tilde{Y}$, then

$$T\Delta f(X, Y)(Z)S = \Delta f(\tilde{X}, \tilde{Y})(TZS). \quad (1.20)$$

Proposition 1.8 ([11], Proposition 3.1). *Let $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ be an NC function on a right admissible NC set $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ for $d \in \mathbb{N}$. If $n_1, n_2 \in \mathbb{N}$, $X^1 \in \Omega_{n_1}$, $X^2 \in \Omega_{n_2}$, and $Z^1 \in \mathbb{C}^{(n_1 \times n_2) \cdot d}$, then:*

(1) *If $TX^1 = \tilde{X}^1 T$, then:*

$$T\Delta f(X^1, X^2)(Z^1) = \Delta f(\tilde{X}^1, X^2)(TZ^1)$$

for $\tilde{n}_1 \in \mathbb{N}$, $\tilde{X}^1 \in \Omega_{\tilde{n}_1}$, and $T \in \mathbb{C}^{\tilde{n}_1 \times n_1}$.

(2) *If $X^2 T = T \tilde{X}^2$, then:*

$$\Delta f(X^1, X^2)(Z^1)T = \Delta f(X^1, \tilde{X}^2)(Z^1 T)$$

for $\tilde{n}_2 \in \mathbb{N}$, $\tilde{X}^2 \in \Omega_{\tilde{n}_2}$, and $T \in \mathbb{C}^{n_2 \times \tilde{n}_2}$.

Example 1.5. *Given the NC function $h(X_1, X_2) = (X_1 X_2 - X_2 X_1)^{-1}$, we want to compute $\Delta h(X, Y)(Z)$. Note that if we take $g(X) = X^{-1}$ and $f(X_1, X_2) = X_1 X_2 - X_2 X_1$, then using the chain rule in subsection (1.2.1), we can write:*

$$\Delta(g \circ f)(X, Y)(Z) = \Delta g(f(X), f(Y))(\Delta f(X, Y)(Z))$$

in which $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ and also $Z = (Z_1, Z_2)$. We can compute $\Delta g(X, Y)(Z)$ as follows:

$$g \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} = \begin{pmatrix} X^{-1} & -X^{-1} Z Y^{-1} \\ 0 & Y^{-1} \end{pmatrix}.$$

so it follows that $\Delta g(X, Y)(Z) = -X^{-1}ZY^{-1}$. We can also compute the first-order derivative, $\Delta f(X, Y)(Z)$ as follows:

$$\begin{aligned} f \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} &= \begin{pmatrix} X_1 & Z_1 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} X_2 & Z_2 \\ 0 & Y_2 \end{pmatrix} - \begin{pmatrix} X_2 & Z_2 \\ 0 & Y_2 \end{pmatrix} \begin{pmatrix} X_1 & Z_1 \\ 0 & Y_1 \end{pmatrix} \\ &= \begin{pmatrix} X_1X_2 - X_2X_1 & X_1Z_2 + Z_1Y_2 - X_2Z_1 - Z_2Y_1 \\ & Y_1Y_2 - Y_2Y_1 \end{pmatrix}. \end{aligned}$$

Therefore, we get the following as the first-order derivative of h :

$$\Delta h(X, Y)(Z) = -h(X) (X_1Z_2 + Z_1Y_2 - X_2Z_1 - Z_2Y_1) h(Y) =: s(X, Y)(Z).$$

We can check that $s(X, Y)(Z)$ satisfies the property of Proposition (1.7). First we want to show that:

$$s(X \oplus X', Y) \begin{pmatrix} Z \\ Z' \end{pmatrix} = \begin{pmatrix} s(X, Y)(Z) \\ s(X', Y)(Z') \end{pmatrix}.$$

The left hand side is equal to :

$$-h(X \oplus X') \left(\begin{pmatrix} X_1 & 0 \\ 0 & X'_1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z'_1 \end{pmatrix} + \begin{pmatrix} Z_1 \\ Z'_1 \end{pmatrix} Y_2 - \begin{pmatrix} X_2 & 0 \\ 0 & X'_2 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z'_1 \end{pmatrix} - \begin{pmatrix} Z_2 \\ Z'_2 \end{pmatrix} Y_1 \right) h(Y)$$

and after computation, it will easily follow that the equality above holds true. The same is true for the second argument, namely:

$$s(X, Y \oplus Y')(Z, Z') = (s(X, Y)(Z), s(X, Y')(Z')).$$

For testing the condition related to the similarities, it suffices to prove that :

$$s(TXT^{-1}, TYT^{-1})[TZT^{-1}] = T(s(X, Y)(Z))T^{-1}.$$

This holds because h is an NC function, which by definition respects similarities. Furthermore, each term in the expression $X_1Z_1 + Z_1Y_2 - X_2Z_1 - Z_2Y_1$ is an NC function in the sense that each respects direct sums and similarities in its arguments. Since $s(X, Y)(Z) = -h(X)(X_1Z_1 + Z_1Y_2 - X_2Z_1 - Z_2Y_1)h(Y)$ is linear in Z and composed of NC functions, it follows that $s(X, Y)(Z)$ is an NC function, as it satisfies linearity in Z and preserves direct sums and similarities in all arguments.

1.3 Higher order NC difference-differential operators

In this section, we study the notion of higher-order difference-differential operators for NC functions. These higher-order difference-differential operators also have natural invariance properties with respect to $\oplus$ and joint similarity. We also present finite difference formulas related to these higher-order difference-differential operators.

While this section is based on Chapter 3 of [11], our approach differs slightly. In [11], the theory of higher-order NC functions is developed first, followed by the introduction of the corresponding difference-differential operators. Here, we begin with the ℓ -th order difference-differential operator of an NC function, and state the finite difference-differential formulas of higher order (related to higher order derivation of NC functions). This will provide a minimal and sufficient foundation to introduce “Taylor-Taylor series” of NC functions in the next chapter.

Definition 1.9. Let $f : \Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \longrightarrow \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d'}$ be an NC function, with $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ a right admissible NC set.

Let $\ell \in \mathbb{N}$. For every $X^0 \in \Omega_{n_0}, \dots, X^\ell \in \Omega_{n_\ell}, Z^1 \in \mathbb{C}^{(n_0 \times n_1) \cdot d}, \dots, Z^\ell \in$

$\mathbb{C}^{(n_{\ell-1} \times n_{\ell}) \cdot d}$ such that:

$$\begin{bmatrix} X^0 & Z^1 & 0 & \cdots & 0 \\ 0 & X^1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & Z^{\ell-1} & 0 \\ 0 & \cdots & 0 & X^{\ell-1} & Z^{\ell} \\ 0 & \cdots & \cdots & 0 & X^{\ell} \end{bmatrix} \in \Omega_{n_0 + \cdots + n_{\ell}},$$

one has:

$$f \left(\begin{bmatrix} X^0 & Z^1 & 0 & \cdots & 0 \\ 0 & X^1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & Z^{\ell-1} & 0 \\ 0 & \cdots & 0 & X^{\ell-1} & Z^{\ell} \\ 0 & \cdots & \cdots & 0 & X^{\ell} \end{bmatrix} \right) = \quad (1.21)$$

$$\begin{bmatrix} f(X^0) & * & \cdots & \Delta^{\ell} f(X^0, \dots, X^{\ell})(Z^1, \dots, Z^{\ell}) \\ 0 & f(X^1) & \ddots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & f(X^{\ell}) \end{bmatrix},$$

in which $\Delta^{\ell} f(X^0, \dots, X^{\ell})(\cdot)$ is a mapping from $\mathbb{C}^{(n_0 \times n_1) \cdot d} \times \mathbb{C}^{(n_1 \times n_2) \cdot d} \times \cdots \times \mathbb{C}^{(n_{\ell-1} \times n_{\ell}) \cdot d}$ to $\mathbb{C}^{(n_0 \times n_{\ell}) \cdot d'}$. We call $\Delta^{\ell} f(X^0, \dots, X^{\ell})(\cdot)$, the **ℓ -th order difference-differential operator** of the NC function f .

A similar argument can be made, as in Definition (1.6), about why we are considering right admissible NC set as our domain of definition.

The definition above gives us the ℓ -th order difference-differential operator as the upper-right corner entry of the right-hand side matrix of Equation (1.21). The remaining block entries of the right-hand side of Equation (1.21) can be completed as follows:

Lemma 1.2. *With the same conditions as in the Definition (1.9), the matrix on the right-hand side of Equation (1.21) is:*

$$\begin{bmatrix} f(X^0) & \Delta f(X^0, X^1)(Z^1) & \Delta^2 f(X^1, X^2, X^3)(Z^1, Z^2) & \cdots & \Delta^\ell f(X^0, \dots, X^\ell)(Z^1, \dots, Z^\ell) \\ 0 & f(X^1) & \Delta f(X^1, X^2)(Z^2) & \cdots & \Delta^{\ell-1} f(X^1, \dots, X^\ell)(Z^2, \dots, Z^\ell) \\ \vdots & \ddots & & \ddots & \vdots \\ & & & f(X^{\ell-1}) & \Delta f(X^{\ell-1}, X^\ell)(Z^\ell) \\ 0 & \cdots & & 0 & f(X^\ell) \end{bmatrix}$$

Outline of the Proof. Suppose we are given the following matrix:

$$D = \begin{pmatrix} X & G & 0 \\ 0 & Y & H \\ 0 & 0 & Z \end{pmatrix}$$

and we are asked to find $f(D)$ for an NC function f . We break this 3×3 matrix into a smaller block matrices as follows:

$$X' = \begin{pmatrix} X & G \\ 0 & Y \end{pmatrix}, \quad H' = \begin{pmatrix} 0 \\ H \end{pmatrix}$$

so that:

$$f \begin{pmatrix} X' & H' \\ 0 & Z \end{pmatrix} = \begin{pmatrix} f(X') & \Delta f(X', Z)[H'] \\ 0 & f(Z) \end{pmatrix}.$$

We know that

$$f(X') = \begin{pmatrix} f(X) & \Delta f(X, Y)[G] \\ 0 & f(Y) \end{pmatrix},$$

so to find $\Delta f(X', Z)[H']$, we consider:

$$\Delta f(X', Z)[H'] = \begin{pmatrix} A \\ B \end{pmatrix}.$$

Note that:

$$\begin{pmatrix} 0 & I_n \end{pmatrix} \begin{pmatrix} X & G \\ 0 & Y \end{pmatrix} = Y \begin{pmatrix} 0 & I_n \end{pmatrix},$$

and using (1) in Proposition (1.8), we have:

$$\begin{pmatrix} 0 & I_n \end{pmatrix} \Delta f(X', Z) \begin{bmatrix} 0 \\ H \end{bmatrix} = \begin{pmatrix} 0 & I_n \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \Delta f(Y, Z)[H].$$

On the other hand, we can take:

$$Y' = \begin{pmatrix} Y & H \\ 0 & Z \end{pmatrix}, \quad G' = [G \quad 0]$$

using Proposition (1.6), we have:

$$\Delta f(X, \begin{pmatrix} Y & H \\ 0 & Z \end{pmatrix})[G \quad 0] = [\Delta f(X, Y)[G] \quad A]$$

in which $A = \Delta^2 f(X, Y, Z)[G, H]$ is the second order derivative of f . In conclusion we have:

$$f \begin{pmatrix} X & G & 0 \\ 0 & Y & H \\ 0 & 0 & Z \end{pmatrix} = \begin{pmatrix} f(X) & \Delta f(X, Y)[G] & \Delta^2 f(X, Y, Z)[G, H] \\ 0 & f(Y) & \Delta f(Y, Z)[H] \\ 0 & 0 & f(Z) \end{pmatrix}$$

The key take-away from the preceding calculation is this: for an $\ell \times \ell$ block upper-triangular matrix $\tilde{D}$, the super-diagonal block entries (those above the main diagonal) of $f(\tilde{D})$ are precisely the (first-order) difference-differential operators of f , provided that they exist. This can be computed systematically by sliding a 2×2 window along the diagonal blocks of $\tilde{D}$ and applying the first derivative formula at each position.

This observation naturally extends to higher derivatives. Consider now sliding a 3×3 window along the diagonal of $\tilde{D}$ and evaluating f on each such block. The calculations illustrated above reveal that the entries on the *second super-diagonal* (two positions

above the main diagonal) encode the second-order difference-differential operators of f .

More generally, this pattern continues: the k -th super-diagonal of $f(\tilde{D})$ contains the k -th order directional derivatives of f , which can be extracted by examining $(k+1) \times (k+1)$ diagonal blocks. This process will ultimately give us the matrix that contains all the derivatives of f up until the order ℓ . $\square$

In the proposition below, we figure out how the ℓ -th difference-differential is uniquely determined and also it reveals that $\Delta^\ell f$ is calculated iteratively by evaluating NC functions of increasing orders on 2×2 block upper triangular matrices at each step.

Proposition 1.9. *Let $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ be an NC set. Let $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$, and for a $k \in \mathbb{N}$, let $X^0 \in \Omega_{n_0}, \dots, X^{k-1} \in \Omega_{n_{k-1}}, X^k, X^{k'} \in \Omega_{n_k}, Z^1 \in \mathbb{C}^{n_0 \times n_1}, \dots, Z^{k-1} \in \mathbb{C}^{n_{k-2} \times n_{k-1}}, Z^k \in \mathbb{C}^{n_{k-1} \times n_k}$, and $Z^{k''} \in \mathbb{C}^{n_{k-1} \times n_{k''}}$. Let $Z \in \mathbb{C}^{n_k \times n_{k''}}$ be such that*

$$\begin{bmatrix} X^{k'} & Z \\ 0 & X^{k''} \end{bmatrix} \in \Omega_{n_k + n_{k''}}.$$

Then

$$\begin{aligned} & \Delta^k f \left(X^0, \dots, X^{k-1}, \begin{bmatrix} X^{k'} & Z \\ 0 & X^{k''} \end{bmatrix} \right) (Z^1, \dots, Z^{k-1}, (Z^{k'} \quad Z^{k''})) \quad (1.22) \\ &= \left[\begin{array}{c} \Delta^k f(X^0, \dots, X^{k-1}, X^{k'}) (Z^1, \dots, Z^{k-1}, Z^{k'}) \\ \Delta^{k+1} f(X^0, \dots, X^{k-1}, X^{k'}, X^{k''}) (Z^1, \dots, Z^{k-1}, Z^{k'}, Z) \\ + \Delta^k f(X^0, \dots, X^{k-1}, X^{k''}) (Z^1, \dots, Z^{k-1}, Z^{k''}) \end{array} \right]^T \end{aligned}$$

Here $\Delta^{k+1} f(X^0, \dots, X^{k-1}, X^{k'}, X^{k''}) (Z^1, \dots, Z^{k-1}, Z^{k'}, Z)$ is determined uniquely by

(1.22), is independent of $Z^{k''}$, and has the following property. If $r \in \mathbb{C}$ is such that

$$\begin{bmatrix} X^{k'} & rZ \\ 0 & X^{k''} \end{bmatrix} \in \Omega_{n_k+n_{k''}},$$

then

$$\begin{aligned} \Delta^{k+1} f(X^0, \dots, X^{k-1}, X^{k'}, X^{k''})(Z^1, \dots, Z^{k-1}, Z^{k'}, rZ) \\ = r \Delta^{k+1} f(X^0, \dots, X^{k-1}, X^{k'}, X^{k''})(Z^1, \dots, Z^{k-1}, Z^{k'}, Z). \end{aligned} \quad (1.23)$$

For $k = 1$, the matrices $Z^1, \dots, Z^{k-1}$ do not show up in (1.22) and (1.23).

Theorem 1.2 (Multi-linearity property of difference-differential operator Δ). *Let $f : \Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ be an NC function, with $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ being a right admissible NC set. For every $X^0 \in \Omega_{n_0}, \dots, X^k \in \Omega_{n_k}, X^{k+1} \in \Omega_{n_{k+1}}$, the mapping*

$$(Z^1, \dots, Z^{k+1}) \mapsto \Delta^{k+1} f(X^0, \dots, X^{k+1})(Z^1, \dots, Z^{k+1}),$$

where each $Z^i = (Z_1^i, \dots, Z_d^i)$ is a d -tuple with $Z_j^i \in \mathbb{C}^{n_{i-1} \times n_i}$ for $i = 1, \dots, k+1$, from $\mathbb{C}^{(n_0 \times n_1) \cdot d} \times \dots \times \mathbb{C}^{(n_k \times n_{k+1}) \cdot d}$ to $\mathbb{C}^{n_0 \times n_{k+1}}$, is $(k+1)$ -linear over $\mathbb{C}$. That is, for every $X^0 \in \Omega_{n_0}, \dots, X^k \in \Omega_{n_k}, X^{k+1} \in \Omega_{n_{k+1}}, Z_I, Z_{II} \in \mathbb{C}^{(n_0 \times n_1) \cdot d} \times \dots \times \mathbb{C}^{(n_k \times n_{k+1}) \cdot d}$, and $c \in \mathbb{C}$, the following holds:

$$\begin{aligned} \Delta^{k+1} f(X^0, \dots, X^{k+1})(cZ_I + Z_{II}) \\ = c \Delta^{k+1} f(X^0, \dots, X^{k+1})(Z_I) + \Delta^{k+1} f(X^0, \dots, X^{k+1})(Z_{II}). \end{aligned}$$

1.4 Finite difference formulas of Higher Order

We present an analogue of Equations (1.15) for higher order NC functions. Theorems (1.3) and (1.4), are special cases of Theorems 3.19 and 3.20 of [11]:

Theorem 1.3. Let $f : \Omega \longrightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$, where $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is a right admissible NC set. Then for all $n_0, \dots, n_k \in \mathbb{N}$ and $k \in \mathbb{N}$, and arbitrary $X^0 \in \Omega_{n_0}, \dots, X^{k-1} \in \Omega_{n_{k-1}}, X, Y \in \Omega_n, Z^1 \in \mathbb{C}^{n_0 \times n_1}, \dots, Z^k \in \mathbb{C}^{n_{k-1} \times n_k}$, we have:

$$\begin{aligned} \Delta^k f(X^0, \dots, X^{k-1}, X)(Z^1, \dots, Z^k) - \Delta^k f(X^0, \dots, X^{k-1}, Y)(Z^1, \dots, Z^k) \\ = \Delta^{k+1} f(X^0, \dots, X^{k-1}, Y, X)(Z^1, \dots, Z^k, X - Y), \end{aligned} \quad (1.24)$$

and,

$$\begin{aligned} \Delta^k f(X^0, \dots, X^{k-1}, X)(Z^1, \dots, Z^k) - \Delta^k f(X^0, \dots, X^{k-1}, Y)(Z^1, \dots, Z^k) \\ = \Delta^{k+1} f(X^0, \dots, X^{k-1}, X, Y)(Z^1, \dots, Z^k, X - Y). \end{aligned} \quad (1.25)$$

Theorem (1.3) is a consequence of the following more general result.

Theorem 1.4. Let $f : \Omega \longrightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$, where $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ is a right admissible NC set. Then for all $n_0, \dots, n_k \in \mathbb{N}$ and $k \in \mathbb{N}$, and arbitrary $X^0 \in \Omega_{n_0}, \dots, X^{k-1} \in \Omega_{n_{k-1}}, X \in \Omega_n, Y \in \Omega_m, Z^1 \in \mathbb{C}^{n_0 \times n_1}, \dots, Z^k \in \mathbb{C}^{n_{k-1} \times m}, S \in \mathbb{C}^{m \times n}$, we have:

$$\begin{aligned} \Delta^k f(X^0, \dots, X^{k-1}, X)(Z^1, \dots, Z^k S) - \Delta^k f(X^0, \dots, X^{k-1}, Y)(Z^1, \dots, Z^k S) \\ = \Delta^{k+1} f(X^0, \dots, X^{k-1}, Y, X)(Z^1, \dots, Z^k, SX - YS). \end{aligned} \quad (1.26)$$

Remark 1.4. Setting $X = Y + tZ$ (with $t \in \mathbb{C}$) in Theorem (1.3), we obtain:

$$\begin{aligned} \Delta^k f(X^0, \dots, X^{k-1}, Y + tZ)(Z^1, \dots, Z^k) - \Delta^k f(X^0, \dots, X^{k-1}, Y)(Z^1, \dots, Z^k) \\ = t \Delta^{k+1} f(X^0, \dots, X^{k-1}, Y, Y + tZ)(Z^1, \dots, Z^k, Z). \end{aligned} \quad (1.27)$$

Using an argument similar to the one before Proposition (1.5), it follows that:

$$\Delta^{k+1} f(X^0, \dots, X^{k-1}, Y, Y)(Z^1, \dots, Z^k, Z)$$

is the directional derivative of

$$\Delta^k f(X^0, \dots, X^{k-1}, \cdot)(Z^1, \dots, Z^k)$$

at Y in the direction Z , i.e.,

$$\Delta^{k+1} f(X^0, \dots, X^{k-1}, Y, Y)(Z^1, \dots, Z^k, \cdot)$$

is the G -derivative of

$$\Delta^k f(X^0, \dots, X^{k-1}, \cdot)(Z^1, \dots, Z^k)$$

at Y .

Chapter 2

Analytic Properties of Non-commutative Functions

In this chapter, we aim to present and review the basics of analytical properties of NC functions as stated in the textbook of Vinnikov and Kaliuzhnyi-Verbovetskyi (Chapters 4, 7, and 8 of [11]). We begin with the definition and properties of Taylor-Taylor formulas and continue to prove theorems on how local boundedness and analyticity are related in this setting. We then discuss the “radius of convergence” of Taylor-Taylor series of NC functions.

2.1 Basics of Taylor-Taylor Expansion of NC Functions

First, we would like to present the way the Taylor-Taylor (TT) expansion of an NC function is constructed and discuss the basic properties.

Theorem 2.1. *Let f be an NC function defined on the right admissible NC set Ω . Let $n \in \mathbb{N}$, and $Y \in \Omega_n$. Then for each $N \in \mathbb{N}$ and arbitrary $X \in \Omega_n$,*

$$f(X) = \sum_{\ell=0}^N \frac{1}{\ell!} (\partial_{X-Y}^{(\ell)} f)(Y) + \Delta^{N+1} f(\underbrace{Y, \dots, Y}_{N+1 \text{ times}}, X) (\underbrace{X - Y, \dots, X - Y}_{N+1 \text{ times}}). \quad (2.1)$$

The notation for derivatives in the Theorem above is defined in (2.3).

Proof. Using Equation (1.15) and Theorem (1.3), we obtain

$$\begin{aligned} f(X) &= f(Y) + \Delta^{(1)} f(Y, X)(X - Y) \\ &= f(Y) + (\partial_{X-Y}^{(1)} f)((Y, Y)) + \Delta^{(2)} f(Y, Y, X)(X - Y, X - Y) \\ &\vdots \\ &= f(Y) + (\partial_{X-Y}^{(1)} f)((Y, Y)) + \dots + \frac{1}{N!} (\partial_{X-Y}^{(N)} f)(\underbrace{Y, \dots, Y}_{N+1 \text{ times}}) + \\ &\quad \Delta^{(N+1)} f(\underbrace{Y, \dots, Y}_{N+1 \text{ times}}, X) (\underbrace{X - Y, \dots, X - Y}_{N+1 \text{ times}}). \end{aligned}$$

which is the same as the series above. $\square$

The following corollary can be derived directly from Theorem (2.1), concerning the case where X and Y in the series expansion, need not to be of the same size.

Corollary 2.1. Suppose $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is an NC function with $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ a right admissible NC set, and $Y \in \Omega_s$ (level s of the NC set Ω). Then for each $N \in \mathbb{N}$ and arbitrary $m \in \mathbb{N}$ and $X \in \Omega_{ms}$,

$$\begin{aligned} f(X) &= \sum_{\ell=0}^N \frac{1}{\ell!} \left(\partial_{X-I_m \otimes Y}^{(\ell)} f \right) (\underbrace{I_m \otimes Y, \dots, I_m \otimes Y}_{\ell+1 \text{ times}}) \\ &\quad + \left(\Delta^{N+1} f \right) (\underbrace{I_m \otimes Y, \dots, I_m \otimes Y}_{N+1 \text{ times}}, X) (\underbrace{X - I_m \otimes Y, \dots, X - I_m \otimes Y}_{N+1 \text{ times}}) \end{aligned} \quad (2.2)$$

Theorem 2.2 (Theorem 7.2 of [11]). *Let an NC function $f : \Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ be locally bounded in the finitely-open topology, where $\Omega = \text{Dom } f$ is finitely-open NC set. Then*

1. *f is G-differentiable.*
2. *For every $n \in \mathbb{N}$, $Y \in \Omega_n$, $Z \in \mathbb{C}^{(n \times n) \cdot d}$, and each $N \in \mathbb{N}$,*

$$\left. \frac{1}{N!} \frac{d^N}{dt^N} f(Y + tZ) \right|_{t=0} = \Delta^N f(Y, \dots, Y)(Z, \dots, Z),$$

where Y appears $N + 1$ times and Z appears N times.

We also use the following notation more frequently, as it is more consistent with our previous difference-differential operator notation:

$$\frac{1}{N!} \partial_Z^{(N)} f(Y) := \Delta^N f(Y, \dots, Y)(Z, \dots, Z). \quad (2.3)$$

2.2 Operator Spaces

In this section, we investigate the convergence of the Taylor–Taylor (TT) expansion of an NC function around a matrix centre. Our goal is to determine when the remainder term in the finite expansion vanishes in norm, ensuring the existence of an infinite series representation.

In other words, we are interested in conditions under which

$$\left\| \left(\Delta^{N+1} f \right) \left(\underbrace{I_m \otimes Y, \dots, I_m \otimes Y}_{N+1 \text{ times}}, X \right) \left(\underbrace{X - I_m \otimes Y, \dots, X - I_m \otimes Y}_{N+1 \text{ times}} \right) \right\| \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

so that

$$f(X) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \partial_{X - I_m \otimes Y}^{(\ell)} f(I_m \otimes Y) \quad (2.4)$$

holds as a norm-convergent series.

Each coefficient in (2.4) can be viewed as an ℓ -linear map

$$\widehat{f}_\ell(H^{(1)}, \dots, H^{(\ell)}) := \partial_{H^{(1)}, \dots, H^{(\ell)}}^{(\ell)} f(I_m \otimes Y), \quad \widehat{f}_\ell : (\mathbb{C}^{(n \times n) \cdot d})^\ell \rightarrow \mathbb{C}^{n \times n}.$$

By identifying $\widehat{f}_\ell$ with the corresponding linear map

$$f_\ell : \underbrace{\mathbb{C}^{(n \times n) \cdot d} \otimes \dots \otimes \mathbb{C}^{(n \times n) \cdot d}}_{\ell\text{-fold tensor product}} \rightarrow \mathbb{C}^{n \times n},$$

we may, by Paulsen's Theorem [16], interpret f_ℓ as a completely bounded multilinear map in the sense of operator space theory. This observation naturally leads to an NC analogue of the Cauchy–Hadamard formula, where the *radius of convergence* of the TT series (2.4) is expressed in terms of the completely bounded norms of the coefficient maps $f^{(\ell)}$.

Definition 2.1 (Concrete operator space, cf. [4, 15]). *Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the bounded linear operators on $\mathcal{H}$ with the operator norm. A linear subspace*

$$\mathcal{E} \subseteq \mathcal{B}(\mathcal{H})$$

*that is closed in the operator norm is called a **concrete operator space**. For each $n \in \mathbb{N}$ we equip the matrix space $M_n(\mathcal{E})$ with the norm inherited from $\mathcal{B}(\mathcal{H}^{\oplus n})$ by the identification*

$$M_n(\mathcal{E}) \ni [x_{ij}] \mapsto \left(\xi_1 \oplus \dots \oplus \xi_n \mapsto \sum_{j=1}^n x_{ij} \xi_j \right) \in \mathcal{B}(\mathcal{H}^{\oplus n}).$$

Definition 2.2 (Abstract operator space, Ruan's axioms [4, Section 2.1]). *An **abstract operator space** is a complex vector space E together with a family of norms*

$$\{\|\cdot\|_{M_n(E)} : M_n(E) \rightarrow [0, \infty)\}_{n=1}^\infty$$

satisfying the following axioms (Ruan's axioms):

(R1) (**Matrix direct sum**) For all $m, n \in \mathbb{N}$ and all $x \in M_n(E)$, $y \in M_m(E)$,

$$\|x \oplus y\|_{M_{n+m}(E)} = \max\{\|x\|_{M_n(E)}, \|y\|_{M_m(E)}\}.$$

(R2) (**Compatibility with scalar multiplication**) For every $n, m \in \mathbb{N}$, $x \in M_m(E)$, and $\alpha \in M_{n,m}(\mathbb{C})$, $\beta \in M_{m,n}(\mathbb{C})$,

$$\|\alpha x \beta\|_{M_n(E)} \leq \|\alpha\|_{M_{n,m}} \|x\|_{M_m(E)} \|\beta\|_{M_{m,n}}.$$

Definition 2.3 (Ampliations, cf. [15]). If $\varphi : E \rightarrow F$ is a linear map between operator spaces, then for each $n \in \mathbb{N}$ the n -th **ampliation** $\varphi_n : M_n(E) \rightarrow M_n(F)$ is defined entrywise by

$$\varphi_n([x_{ij}]) := [\varphi(x_{ij})].$$

Definition 2.4 (Completely bounded maps and cb-norm [15], and [4, Section 2.2]). A linear map $\varphi : E \rightarrow F$ is called **completely bounded (CB)** if

$$\|\varphi\|_{cb} := \sup_{n \in \mathbb{N}} \|\varphi_n\| < \infty.$$

Here $\|\varphi_n\|$ denotes the operator norm of φ_n between the Banach spaces $M_n(E)$ and $M_n(F)$. If $\|\varphi\|_{cb} \leq 1$, then φ is called **completely contractive**.

Definition 2.5 (Completely isometric maps). A linear map $\varphi : E \rightarrow F$ is a **complete isometry** if for every $n \geq 1$ and all $x \in M_n(E)$,

$$\|\varphi_n(x)\|_{M_n(F)} = \|x\|_{M_n(E)}.$$

The notion of complete boundedness generalizes boundedness in the category of operator spaces: instead of just controlling the norm of φ , one requires uniform control over all matrix ampliations. This reflects the fact that the correct morphisms in operator space theory are completely bounded maps (rather than merely bounded ones).

The following theorem below shows that the abstract axioms faithfully capture the concrete setting. In practice, this guarantees that one may always work concretely inside $\mathcal{B}(\mathcal{H})$ without loss of generality.

Theorem 2.3 (Ruan’s theorem [4, Thm. 2.3.5]). *Every abstract operator space is completely isometrically isomorphic to a concrete operator space. More precisely: if E is an abstract operator space, then there exist a Hilbert space $\mathcal{H}$ and a linear complete isometry*

$$\mathcal{J} : E \longrightarrow \mathcal{B}(\mathcal{H}),$$

so that $\mathcal{J}(E)$ is a concrete operator space and the matrix norms on E coincide with those inherited from $\mathcal{J}$.

2.2.1 Complete Boundedness for ℓ -linear maps

Let E and F be an operator spaces. Given $P \in M_{i,j}(E)$ and $Q \in M_{j,k}(E)$, their **Haagerup tensor product** is

$$P \odot Q := \left(\sum_{\gamma=1}^j P_{\alpha\gamma} \otimes Q_{\gamma\beta} \right)_{\substack{1 \leq \alpha \leq i \\ 1 \leq \beta \leq k}} \in M_{i,k}(E \otimes E),$$

and for $C \in M_{i,k}(E \otimes E)$, its **Haagerup tensor norm** is

$$\|C\|_h := \inf \left\{ \|P\|_E \cdot \|Q\|_E \mid P \in M_{i,j}(E), Q \in M_{j,k}(E), P \odot Q = C \right\}.$$

The algebraic tensor product $E \otimes E$ equipped with $\|\cdot\|_h$ will then be an operator space, which is usually denoted by $E \otimes_h E$. More generally, $E^{\otimes \ell} = \underbrace{E \otimes \cdots \otimes E}_{\ell \text{ times}}$ equipped with the Haagerup tensor norm is an operator space. Here, given $C \in$

$M_{i,k}(E^{\otimes \ell})$, setting $m_0 := i$ and $m_\ell := k$,

$$\|C\|_h = \inf \left\{ \prod_{s=1}^{\ell} \|A^{(s)}\|_E \mid A^{(s)} \in E^{m_{s-1} \times m_s}, A^{(1)} \odot A^{(2)} \odot \dots \odot A^{(\ell)} = C \right\}.$$

Suppose we are given a multilinear map:

$$g : \underbrace{E \times \dots \times E}_{\ell \text{ times}} \longrightarrow F.$$

Next, consider the linear map:

$$g^{(\ell)} : \underbrace{E \otimes_h \dots \otimes_h E}_{\ell \text{ times}} \longrightarrow F,$$

which is given by

$$g^{(\ell)}(H^{(1)} \otimes H^{(2)} \otimes \dots \otimes H^{(\ell)}) := g(H^{(1)}, H^{(2)}, \dots, H^{(\ell)}), \quad H^{(i)} \in E.$$

We then say that g is a **completely bounded ℓ -linear map** if $g^{(\ell)}$ is completely bounded, and one sets $\|g\|_{\text{CB}} = \|g^{(\ell)}\|_{\text{CB}}$. More generally, in this way one gets a one-to-one correspondence between completely bounded multi-linear maps and completely bounded linear maps on tensor products of operator spaces, where corresponding maps are often treated as one and the same. In particular, whenever $H^{(1)}, \dots, H^{(\ell)} \in M_m(E)$, we can extend the definition of the multi-linear map g to such tuples of matrices over E via

$$g(H^{(1)}, \dots, H^{(\ell)}) := g^{(\ell)}(H^{(1)} \odot \dots \odot H^{(\ell)}).$$

In our case, $g_\ell : \underbrace{\mathbb{C}^{(s \times s) \cdot d} \times \dots \times \mathbb{C}^{(s \times s) \cdot d}}_{\ell \text{ times}} \rightarrow \mathbb{C}^{s \times s}$ is a ℓ -linear. Viewing $\mathbb{C}^{(s \times s) \cdot d} \simeq \mathcal{B}(\mathbb{C}^s \otimes \mathbb{C}^d, \mathbb{C}^s)$, and so we can view g_ℓ as a completely bounded linear map on the operator space $\mathbb{C}^{(s \times s) \cdot d} \otimes \dots \otimes \mathbb{C}^{(s \times s) \cdot d}$ (ℓ times), equipped with the Haagerup tensor norm, taking values in $\mathbb{C}^{s \times s}$.

It is convenient to introduce the Haagerup tensor product adapted to our setting, where $E = \mathbb{C}^{s \times s}$ and $M_m(E) = (\mathbb{C}^{s \times s})^{m \times m}$, as follows: If $P = [P_{ij}]_{1 \leq i, j \leq m} \in (\mathbb{C}^{s \times s})^{m \times m}$ and $Q = [Q_{ij}]_{1 \leq i, j \leq m} \in (\mathbb{C}^{s \times s})^{m \times m}$, then we use the notation

$$P \odot_s Q := \left[\sum_{k=1}^m P_{ik} \otimes Q_{kj} \right]_{1 \leq i, j \leq m} \quad \text{in which } P \otimes Q = [P_{ij}Q]_{1 \leq i, j \leq m}$$

for the **Haagerup tensor product** of P and Q , viewed as $m \times m$ matrices over the tensor algebra of $\mathbb{C}^{s \times s}$.

This notation requires an explanation: Suppose that we are given a ℓ -linear map,

$$\hat{g}^{(\ell)} : \mathbb{C}^{(n \times n) \cdot d} \times \mathbb{C}^{(n \times n) \cdot d} \times \dots \times \mathbb{C}^{(n \times n) \cdot d} \longrightarrow \mathbb{C}^{n \times n}.$$

(As an example of such map that we can consider is:

$$\hat{g}^{(\ell)}(\underbrace{(\cdot, \dots, \cdot)}_{\ell \text{ times}}) = \partial^{(\ell)}(\underbrace{(\cdot, \dots, \cdot)}_{\ell \text{ times}}) f(Y),$$

where f is a uniformly analytic NC function in a uniformly open row ball around $Y \in \mathbb{C}^{(n \times n) \cdot d}$. See Definition (2.6) for a definition of uniformly analytic NC function).

One can identify $\hat{g}^\ell$ with the linear map:

$$G^{(\ell)} : \mathbb{C}^{(n \times n) \cdot d} \otimes \mathbb{C}^{(n \times n) \cdot d} \otimes \dots \otimes \mathbb{C}^{(n \times n) \cdot d} \longrightarrow \mathbb{C}^{n \times n},$$

defined by:

$$G^{(\ell)}(X^{(1)} \otimes \dots \otimes X^{(\ell)}) := \hat{g}^{(\ell)}(X^{(1)}, \dots, X^{(\ell)}).$$

The question that arises naturally is: how does $G^{(\ell)}$ act on $X^{(1)} \odot \dots \odot X^{(\ell)}$, where each $X^{(j)} \in \mathbb{C}^{(mn \times mn) \cdot d}$?

Here we take $\ell = 2$ and $m = 2$ to illuminate the situation: Given $X \otimes Z$ where $X, Z \in$

$\mathbb{C}^{(2n \times 2n) \cdot d}$ and we consider $X \odot Z$ with $\mathbb{C}^{(n \times n) \cdot d}$ denoted as E , so that $X, Z \in E^{2 \times 2}$,

$$X = \begin{pmatrix} X^{(1,1)} & X^{(1,2)} \\ X^{(2,1)} & X^{(2,2)} \end{pmatrix}, \quad \text{with each } X^{(i,j)} \in \mathbb{C}^{(n \times n) \cdot d}.$$

The Haagerup product is therefore:

$$\begin{aligned} X \odot Z &= \begin{pmatrix} X^{(1,1)} & X^{(1,2)} \\ X^{(2,1)} & X^{(2,2)} \end{pmatrix} \odot \begin{pmatrix} Z^{(1,1)} & Z^{(1,2)} \\ Z^{(2,1)} & Z^{(2,2)} \end{pmatrix} \\ &= \begin{pmatrix} X^{(1,1)} \otimes Z^{(1,1)} + X^{(1,2)} \otimes Z^{(2,1)} & X^{(1,1)} \otimes Z^{(1,2)} + X^{(1,2)} \otimes Z^{(2,2)} \\ X^{(2,1)} \otimes Z^{(1,1)} + X^{(2,2)} \otimes Z^{(2,1)} & X^{(2,1)} \otimes Z^{(1,2)} + X^{(2,2)} \otimes Z^{(2,2)} \end{pmatrix}. \end{aligned}$$

In this case, let $\widehat{g}^{(2)}$ be defined as follows:

$$\widehat{g}^{(2)} : \mathbb{C}^{(n \times n) \cdot d} \times \mathbb{C}^{(n \times n) \cdot d} \rightarrow \mathbb{C}^{n \times n}$$

$$\widehat{g}^{(2)}(\cdot, \cdot) := \partial_{(\cdot, \cdot)}^{(2)} f(Y).$$

for f being a uniformly analytic NC function and for $X, Z \in \mathbb{C}^{(2n \times 2n) \cdot d}$, then $\widehat{g}^{(2)}(X, Z)$ can be computed, by what has been said above, in the following way:

$$\begin{aligned} \widehat{g}^{(2)}(X, Z) &= \left(\text{id}_2 \otimes \partial_{(\cdot, \cdot)}^{(2)} f(Y) \right) \circ (X \odot Z) \\ &= \left(\text{id}_2 \otimes \partial_{(\cdot, \cdot)}^{(2)} f(Y) \right) \circ \begin{pmatrix} X^{(1,1)} \otimes Z^{(1,1)} + X^{(1,2)} \otimes Z^{(2,1)} & \dots \\ \dots & \dots \end{pmatrix} \\ &= \begin{pmatrix} \partial_{(X^{(1,1)}, Z^{(1,1)})}^{(2)} f(Y) + \partial_{(X^{(1,2)}, Z^{(2,1)})}^{(2)} f(Y) & \dots \\ \dots & \dots \end{pmatrix} \end{aligned}$$

and the rest of the matrix is computed in the same way. In the equation above, $X \odot Z$ is a 2×2 block matrix, and $\text{id}_2 \otimes \partial_{(\cdot, \cdot)}^{(2)} f(Y)$ is the second ampliation, meaning that we

apply the mapping $\partial_{(\cdot, \cdot)}^{(2)} f(Y)$ to each block of the 2×2 block matrix $X \odot Z$.

By Proposition (3.3) in [11], we use the following notations, all of which have the same meaning:

$$\begin{aligned} \left(\partial_{X - I_m \otimes Y}^{(\ell)} f \right) (I_m \otimes Y) &= \left(\text{id}_m \otimes \partial_{(\cdot)}^{(\ell)} f(Y) \right) \circ (X - I_m \otimes Y)^{\odot_s^\ell} \\ &= \ell! \Delta^\ell f(\underbrace{Y, \dots, Y}_{\ell+1 \text{ times}}) (X - I_m \otimes Y)^{\odot_s^\ell}, \quad Y \in \mathbb{C}^{(s \times s) \cdot d} \end{aligned} \quad (2.5)$$

As a final result to this section, we would like to state a result from Paulsen, as in [16].

Theorem 2.4. *Let E be an operator space and let $\varphi : \mathbb{C}^{n \times n} \rightarrow E$ be a linear map. Then:*

$$\|\varphi\|_{cb} \leq n\sqrt{n} \|\varphi\|.$$

This theorem shows that if the domain of the linear map is finite dimensional, then boundedness of the linear map φ , automatically gives us the complete boundedness of the φ .

2.3 Taylor-Taylor Series

The Taylor-Taylor formula (2.2), gives rise to analytic properties of NC functions. To extend the summation above to an infinite sum, it is required to discuss particular subsets in the NC universe.

Let $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ be an open NC set. Recall that an NC function $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is called **uniformly locally bounded** if for every $n \in \mathbb{N}$ the function $f|_\Omega$ is locally bounded on a uniformly open neighbourhood of Y , i.e., for every $Y \in \Omega_n$ there exists $\delta > 0$ such that f is bounded on uniform ball $\delta \cdot \mathbb{B}_{n\mathbb{N}}^d(Y)$.

Remark 2.1. *Clearly, a uniformly locally bounded NC function is locally bounded in the*

finitely-open topology.

Definition 2.6. Let $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ be an open NC set. An NC function $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is called **Fréchet (F-) differentiable** if for every $n \in \mathbb{N}$ the function $f|_{\Omega_n}$ is F-differentiable, i.e., $f|_{\Omega_n}$ is G-differentiable and for any $X \in \Omega_n$ the linear operator $\delta f(X) : \mathbb{C}^{(n \times n) \cdot d} \rightarrow \mathbb{C}^{n \times n}$, defined as $\delta f(X) = \Delta f(X, X)$ is bounded. In this case

$$\lim_{\|Z\| \rightarrow 0} \frac{\|f(X + Z) - f(X) - \delta f(X)(Z)\|}{\|Z\|_{\text{row}}} = 0$$

for all $X \in \Omega_n$ and $\|\cdot\|_{n\text{-row}}$ being the matrix norm of level $n \in \mathbb{N}$ for a d -tuple of matrices Z . An NC function $f : \Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is called **analytic** if for every $n \in \mathbb{N}$ the function $f|_{\Omega_n}$ is analytic, i.e., $f|_{\Omega_n}$ is locally bounded and G-differentiable. The NC function is called **uniformly analytic** if it is uniformly locally bounded and G-differentiable.

By [7], Theorem 3.17.1, if the NC function $f : \Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is uniformly locally bounded, then $f|_{\Omega_n}$ is also uniformly continuous and F-differentiable. It follows that the NC function $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is continuous in the finitely-open topology (i.e., all its restrictions $f|_{\Omega_n}$ are continuous) and F-differentiable.

Proposition 2.1. Let $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ be a uniformly open NC set. If an NC function $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is continuous with respect to the uniform topologies on $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ and $\mathbb{C}^{\mathbb{N} \times \mathbb{N}}$, then f is uniformly locally bounded.

Proof. Given $s \in \mathbb{N}$ and $Y \in \Omega_s$, it follows from continuity that there exists an NC ball $r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y) \subseteq \Omega$ such that $\|f(X) - f(I_m \otimes Y)\|_{sm} < 1$ for all $m \in \mathbb{N}$ and $X \in r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)_{sm}$. Then

$$\|f(X)\|_{sm} \leq \|f(I_m \otimes Y)\|_{sm} + 1 = \|I_m \otimes f(Y)\|_{sm} + 1 \leq \|f(Y)\|_s + 1.$$

□

Proposition 2.2. *Suppose that we are given a uniformly analytic NC function:*

$$f : \text{Dom}(f) \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \longrightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}.$$

Let $Y \in \text{Dom}_s(f)$. Then there exists a radius $r > 0$ such that for every $X \in r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y) \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$, we have:

$$f(X) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \partial_{X-I_m \otimes Y}^{(\ell)} f(Y). \quad (2.6)$$

Proof. By definition of f being a uniformly analytic NC function, there exists $r > 1$ such that for every $X \in r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$, the map

$$t \longmapsto f(Y + t(X - Y))$$

is analytic in the complex variable t on $r \cdot \mathbb{D}(0)$, with $\mathbb{D}(0)$ being the open unit disk around the origin $\mathbb{D}(0) := \{s \in \mathbb{C} : |s| < 1\}$. This analyticity follows from the existence of all Gâteaux derivatives of f .

Hence, for such X , the matrix-valued function:

$$\widehat{f}(t) := f(I_m \otimes Y + t(X - I_m \otimes Y))$$

admits the usual Taylor expansion at $t = 0$:

$$f(I_m \otimes Y + t(X - I_m \otimes Y)) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{d^\ell}{dt^\ell} f(I_m \otimes Y + t(X - I_m \otimes Y)) \Big|_{t=0} t^\ell.$$

By Theorem (2.2), each derivative term can be expressed as

$$\frac{d^\ell}{dt^\ell} f(I_m \otimes Y + t(X - I_m \otimes Y)) \Big|_{t=0} = \partial_{X-I_m \otimes Y}^{(\ell)} f(Y),$$

so that

$$f(I_m \otimes Y + t(X - I_m \otimes Y)) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (\partial_{X-I_m \otimes Y}^{(\ell)} f(Y)) t^\ell.$$

Finally, setting $t = 1$ gives

$$f(X) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \partial_{X-I_m \otimes Y}^{(\ell)} f(Y),$$

and the series converges absolutely on the uniformly open row ball $r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$, as desired. $\square$

Theorem 2.5. ([11], Theorem 7.21) *Let an NC function $f : \Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ be uniformly locally bounded on a uniformly open NC set, Ω . Let $s \in \mathbb{N}$, $Y \in \Omega_s$, and let $r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$ be an NC ball about Y such that $r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y) \subseteq \Omega$ and f is bounded on $r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$. Then, for every $0 < r' < r$, $m \in \mathbb{N}$, and $X \in r' \cdot \mathbb{B}_{sm}^d(Y)$, we have:*

$$f(X) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\partial_{X-I_m \otimes Y}^{(\ell)} f \right) (Y), \quad (2.7)$$

where the series converges absolutely and uniformly on $r' \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$. in the sense that:

$$\sum_{\ell=0}^{\infty} \sup_{m \in \mathbb{N}, X \in r' \cdot \mathbb{B}_{sm}^d(Y)} \left\| \frac{1}{\ell!} \left(\partial_{X-I_m \otimes Y}^{(\ell)} f \right) \underbrace{(Y, \dots, Y)}_{\ell+1 \text{ times}} \right\|_{sm} < \infty. \quad (2.8)$$

Proof. By assumption, we have $\|X - Y\|_{row} < r' < r$. Let $X(\lambda) = Y + \lambda(X - Y)$. Then:

$$\|X(\lambda) - Y\|_{row} = |\lambda| \|X - Y\|_{row} < |\lambda| \cdot r'.$$

So $X(\lambda) \in r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$ for all $|\lambda| < \frac{r}{r'}$ such that $\frac{r}{r'} > 1$. In particular, there exist a $\epsilon > 0$ such that:

$$X(\lambda) \in r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y) : \forall |\lambda| \leq 1 + \epsilon.$$

Now, for any $Z \in r \cdot \mathbb{B}_{s\mathbb{N}}^d(0)$, the function $f(Y + tZ)$ is matrix-valued analytic function of t in $\{t \in \mathbb{C} : |t| \leq 1\}$, bounded above in the row norm by a constant $M \geq 0$. Using

the Cauchy estimate, we have :

$$\frac{1}{\ell!} \left\| \partial_Z^{(\ell)} f(Y) \right\|_{sm} \leq M.$$

So, taking $Z = X(1 + \epsilon) - Y \in r \cdot \mathbb{B}_{s\mathbb{N}}^d(0)$, then:

$$\frac{1}{\ell!} \left\| \partial_{X(1+\epsilon)-Y}^{(\ell)} f(Y) \right\|_{sm} \leq M.$$

Note that $X(1 + \epsilon) - Y = (1 + \epsilon)(X - Y)$.

By ℓ -linearity, $\partial_{(1+\epsilon)(X-Y)}^{(\ell)} f(Y) = (1 + \epsilon)^\ell \partial_{X-Y}^\ell f(Y)$ and therefore,

$$\frac{1}{\ell!} \left\| \partial_{X-Y}^{(\ell)} f(Y) \right\|_{sm} \leq \frac{M}{(1 + \epsilon)^\ell}.$$

Therefore, Equation (2.8) holds and the Taylor-Taylor series (2.7) converges absolutely and uniformly on $r \cdot \mathbb{B}_{sm}^d(Y)$. $\square$

Corollary 2.2. ([11], Corollary 7.28) *Let $\Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ be a uniformly open NC set. Then an NC function $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ is uniformly locally bounded if and only if f is continuous with respect to the uniformly-open topologies on $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ and $\mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ if and only if f is uniformly analytic.*

Theorem 2.6. (See [11], Theorem (7.9)) *Suppose that $Y \in \Omega_s \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ and $r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$, a uniform open ball of radius $r > 0$, which is the largest so that $r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y) \subseteq \Omega$. Let $f : \Omega \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ be a uniformly analytic NC function, and suppose that:*

$$f(X) = \sum_{\ell=0}^{\infty} f_\ell \left(\underbrace{X - I_m \otimes Y, \dots, X - I_m \otimes Y}_{\ell \text{ - times}} \right), \quad (2.9)$$

for $X \in r \cdot \mathbb{B}_{sm}^d(Y)$, and $f_\ell : (\mathbb{C}^{(s \times s) \cdot d})^\ell \rightarrow \mathbb{C}^{s \times s}$ is a ℓ -linear mapping. Then

$$\ell! f_\ell(H) = \left(\partial_H^{(\ell)} f \right) (Y),$$

for every $\ell \in \mathbb{N} \cup \{0\}$.

Before the proof of the this theorem, we provide an explanation about how the theorem holds for term $\ell = 2$ and when $d = 2$:

Let X be defined as:

$$X := \begin{pmatrix} Y & Z^1 & 0 \\ 0 & Y & Z^2 \\ 0 & 0 & Y \end{pmatrix}$$

so that:

$$X - I_3 \otimes Y = \begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix},$$

then:

$$\begin{aligned} f(X) &= \sum_{\ell=0}^{\infty} f_{\ell} \circ \begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix}^{\odot \ell} = \\ &f_0 \circ \begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix}^{\odot 0} + f_1 \circ \begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix}^{\odot 1} + f_2 \circ \begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix}^{\odot 2} + \text{higher order terms.} \end{aligned}$$

Note that :

$$\begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix}^{\odot 0} = \begin{pmatrix} I_3 & 0 & 0 \\ 0 & I_3 & 0 \\ 0 & 0 & I_3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & Z^1 \otimes Z^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\text{id}_3 \otimes f_2 \begin{pmatrix} 0 & 0 & Z^1 \otimes Z^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & f_2(Z^1, Z^2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that the series can be written as:

$$\begin{aligned} f(X) &= \sum_{\ell=0}^{\infty} f_{\ell} \circ \begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix}^{\odot \ell} \\ &= \begin{pmatrix} f_0 & 0 & 0 \\ 0 & f_0 & 0 \\ 0 & 0 & f_0 \end{pmatrix} + \begin{pmatrix} 0 & f_1(Z^1) & 0 \\ 0 & 0 & f_1(Z^2) \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & f_2(Z^1, Z^2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The 4th term in the series,

$$f_3 \circ \begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix}^{\odot_s^3} = f_3 \circ \left(\begin{pmatrix} 0 & 0 & Z^1 \otimes Z^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & Z^1 & 0 \\ 0 & 0 & Z^2 \\ 0 & 0 & 0 \end{pmatrix} \right) = 0$$

Comparing the (1, 3)-entry of the matrix :

$$\begin{pmatrix} f_0 & f_1(Z^1) & f_2(Z^1, Z^2) \\ 0 & f_0 & f_1(Z^2) \\ 0 & 0 & f_0 \end{pmatrix}$$

with Equation (1.21), we can deduce that $f_2(Z^1, Z^2) = \Delta f(Y)(Z^1, Z^2)$. Moreover, we can use Lemma (1.2), and find $f_0 = f(Y)$, $f_1(Z^j) = \Delta f(Y)(Z^j)$ for $j = 1, 2$.

Given the series (2.9), with f_{ℓ} being defined as ℓ -linear mapping from ℓ copies of $\mathbb{C}^{(sm \times sm) \cdot d}$ to $\mathbb{C}^{s \times s}$, the problem is that the input of f_{ℓ} may be a $(\mathbb{C}^{s \times s})^{m \times m}$ matrix. Then, just like what we have done in Section (2.2.1), we amplify the mapping f_{ℓ} to all the entries of the $m \times m$ matrix. Namely, we take:

$$X := \begin{pmatrix} Y & Z^1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & Z^{\ell-1} & 0 \\ & & & \ddots & Z^{\ell} \\ 0 & \dots & \dots & \dots & Y \end{pmatrix},$$

and would like to compute:

$$f(X) = \sum_{\ell=0}^{\infty} f_{\ell}(X - I_m \otimes Y)^{\odot \ell} = \sum_{\ell=0}^{\infty} f_{\ell} \circ \begin{pmatrix} 0 & Z^1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & Z^{\ell-1} & 0 \\ 0 & \dots & \dots & \ddots & Z^{\ell} \\ & & & & 0 \end{pmatrix}^{\odot \ell}$$

the first term $f_0 = f(Y)$ would be

$$\text{id}_{\ell+1} \otimes f_0 \begin{pmatrix} Y & 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ & & Y & 0 & 0 \\ & & & \ddots & 0 \\ 0 & \dots & \dots & \dots & Y \end{pmatrix} = \begin{pmatrix} f_0 & 0 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & f_0 & 0 & 0 \\ & & & \ddots & 0 \\ 0 & \dots & \dots & \dots & f_0 \end{pmatrix}.$$

Similarly, for $f_1(X - I_m \otimes Y)$ we can compute:

$$\text{id}_{\ell+1} \otimes f_1 \begin{pmatrix} 0 & Z^1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & Z^{\ell-1} & 0 \\ & & & \ddots & Z^{\ell} \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & f_1(Z^1) & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & f_1(Z^{\ell-1}) & 0 \\ & & & \ddots & f_1(Z^{\ell}) \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}.$$

Using

$$\begin{pmatrix} 0 & Z^1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & Z^{\ell-1} & 0 \\ & & & \ddots & Z^{\ell} \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & Z^1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & Z^{\ell-1} & 0 \\ & & & \ddots & Z^{\ell} \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & Z^1 \otimes Z^2 & 0 & \dots & 0 \\ 0 & 0 & 0 & Z^2 \otimes Z^3 & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & 0 & Z^{\ell-1} \otimes Z^{\ell} \\ & & & & 0 & 0 \\ 0 & \dots & & & & 0 \end{pmatrix}.$$

we have for the second order term:

$$\text{id}_{\ell+1} \otimes f_2 \begin{pmatrix} 0 & 0 & Z^1 \otimes Z^2 & 0 & \dots & 0 \\ & 0 & 0 & Z^2 \otimes Z^3 & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & 0 & Z^{\ell-1} \otimes Z^\ell \\ & & & & 0 & 0 \\ 0 & \dots & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & f_2(Z^1, Z^2) & 0 & \dots & 0 \\ & 0 & 0 & f_2(Z^2, Z^3) & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & 0 & f_2(Z^{\ell-1}, Z^\ell) \\ & & & & 0 & 0 \\ 0 & \dots & & & & 0 \end{pmatrix}$$

Continuing this process, one can compute as a standard linear algebra fact, that we have:

$$(X - I_m \otimes Y)^{\odot_{s^\ell}} = \begin{pmatrix} 0 & 0 & 0 & \dots & Z^1 \otimes \dots \otimes Z^\ell \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

so that

$$\text{id}_{\ell+1} \otimes f_\ell \begin{pmatrix} 0 & 0 & 0 & \dots & Z^1 \otimes \dots \otimes Z^\ell \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \dots & f_\ell(Z^1 \otimes \dots \otimes Z^\ell) \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix},$$

and again by direct computation, one can deduce that $(X - I_m \otimes Y)^{\odot_{s^n}}$ will be a zero

matrix for any $n > \ell$. In short, we can write:

$$f \left(\begin{bmatrix} Y & Z^1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & Z^{\ell-1} & 0 \\ & & & \ddots & Z^\ell \\ 0 & \cdots & \cdots & \cdots & Y \end{bmatrix} \right) = \begin{bmatrix} f_0 & f_1(Z^1) & f_2(Z^1, Z^2) & \cdots & f_\ell(Z^1, \dots, Z^\ell) \\ 0 & f_0 & f_1(Z^2) & \cdots & f_{\ell-1}(Z^2, \dots, Z^\ell) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & f_0 & f_1(Z^\ell) \\ 0 & \cdots & \cdots & 0 & f_0 \end{bmatrix}.$$

This will be in particular useful to us in the following proof:

Proof of Theorem (2.6). Using the explanation prior to this proof, for every $Z^1, \dots, Z^\ell \in \mathbb{C}^{(s \times s) \cdot d}$ there exists a real $t > 0$ such that

$$f \left(\begin{bmatrix} Y & tZ^1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & tZ^{\ell-1} & 0 \\ & & & \ddots & tZ^\ell \\ \underbrace{0 \quad \cdots \quad \cdots \quad \cdots \quad Y}_{=: X(t)} \end{bmatrix} \right) = \begin{bmatrix} f_0 & f_1(tZ^1) & \cdots & \cdots & f_\ell(tZ^1, \dots, tZ^\ell) \\ 0 & f_0 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & f_0 & f(tZ^\ell) \\ 0 & \cdots & \cdots & 0 & f_0 \end{bmatrix}.$$

Using the properties of NC functions and given that f_ℓ is multi-linear mapping, we see that $f_\ell(tZ^1, \dots, tZ^\ell) = t^\ell f_\ell(Z^1, \dots, Z^\ell)$, and thus

$$f_\ell(Z^1, \dots, Z^\ell) = t^{-\ell} f_\ell(tZ^1, \dots, tZ^\ell),$$

is uniquely determined by $f(X(t))$. On the other hand, if $t > 0$ is sufficiently small, we have $X(t) \in r \cdot \mathbb{B}_{s\mathbb{N}}(Y)$ and the series (2.9) converges—see Theorem (2.5). Then we have by Equation (1.21) that:

$$f_\ell(Z^1, \dots, Z^\ell) = \Delta^\ell f(Y, \dots, Y)(Z^1, \dots, Z^\ell) = \frac{1}{\ell!} \left(\partial_{Z^1, \dots, Z^\ell}^{(\ell)} f(Y) \right)$$

□

Remark 2.2. The Equation (2.9) above, is an infinite power series expansion around a matrix centre $Y \in \Omega_s$. We can consider this as a uniqueness theorem for convergent NC power series expansions of an NC function, about the matrix centre $Y \in \mathbb{C}^{(s \times s) \cdot d}$.

In what follows, we will expand the Taylor-Taylor series even further, using “partial difference-differential operators” to obtain a power series of multi-linear maps indexed by the free monoid—the set of all words in d letters:

Definition 2.7. The **free monoid**, $\mathbb{F}_d^+$, consists of the **words**, $\omega = i_1 \dots i_n$, composed of **letters** i_j , chosen from the **alphabet**, $\{1, \dots, d\}$. The product on the free monoid is given by concatenation of words with the empty word $\emptyset$ being the unit of the monoid. The **length** of the word is also denoted by $|\omega|$, which is equal to n in our case here.

The free monomial $\mathfrak{z}^\omega$, is defined in the obvious way: if $\omega = i_1 \dots i_n$ then $\mathfrak{z}^\omega = \mathfrak{z}^{i_1} \mathfrak{z}^{i_2} \dots \mathfrak{z}^{i_n}$ and $\mathfrak{z}^\emptyset = 1$. The **free algebra** of all NC polynomials in the variable $\mathfrak{z}$ is denoted as $\mathbb{C}\langle \mathfrak{z} \rangle$, and the ring of all formal power series (FPS) with complex coefficients will be denoted by $\mathbb{C}\langle\langle \mathfrak{z} \rangle\rangle$. Moreover, if $\omega = i_1 i_2 \dots i_\ell$ then ω^t is defined as $\omega^t := i_\ell \dots i_1$.

Similar to the definition above, if $X = (X_1, \dots, X_d) \in (\mathbb{C}^{sm \times sm})^d$ and $\omega = i_1 \dots i_\ell$, then

$$X^{\odot_{s\omega}} := X_{i_1} \odot_s \dots \odot_s X_{i_\ell}.$$

With the definition above, we can extend the result of this chapter further to the case of power series of multi-linear maps indexed by the free monoid.

Definition 2.8 (See [11, Section 3.5 & Corollary 4.4]). Let $\{e_i\}_{i=1}^\ell$, the standard orthonormal basis for $\mathbb{C}^\ell$ and let $A = (A^1, \dots, A^\ell) \in \left(\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}\right)^\ell$, meaning that each A^i is an element of $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ for $i = 1, \dots, \ell$. So, for example, $A^1 e_{i_1}$ is the i_1 -element of the ℓ -tuple of matrices A . Consider a word $\omega = i_1 i_2 \dots i_\ell$. In Equation (1.21), if we put $Z^1 = A^1 e_{i_1}, \dots, Z^\ell = A^\ell e_{i_\ell}$, we get the **partial difference-differential operator** of order $|\omega|$, of an NC function f at $(Y^0, \dots, Y^\ell) \in \left(\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}\right)^{\ell+1}$. In other words, we have:

$$f \left(\begin{bmatrix} Y^0 & A^1 e_{i_1} & 0 & \cdots & 0 \\ 0 & Y^1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & Y^{\ell-1} & A^\ell e_{i_\ell} \\ 0 & \cdots & \cdots & 0 & Y^\ell \end{bmatrix} \right) \quad (2.10)$$

$$= \begin{bmatrix} f(Y^0) & \Delta f(Y^0, Y^1)(A^1 e_{i_1}) & \cdots & \Delta f(Y^0, \dots, Y^\ell)(A^1 e_{i_1}, \dots, A^\ell e_{i_\ell}) \\ 0 & f(Y^1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Delta f(Y^{\ell-1}, Y^\ell)(A^\ell e_{i_\ell}) \\ 0 & \cdots & 0 & f(Y^\ell) \end{bmatrix}$$

and the partial difference-differential operator of f at $Y = (Y^0, \dots, Y^\ell) \in (\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d})^{\ell+1}$ of order $|\omega|$ is $\Delta f(Y^0, \dots, Y^\ell)(A^1 e_{i_1}, \dots, A^\ell e_{i_\ell})$ denoted by $\Delta_{(\cdot)}^{\omega^t} f(Y)$:

$$\Delta_{(A^1, \dots, A^\ell)}^{\omega^t} f(Y) := \Delta f(Y^0, \dots, Y^\ell)(A^1 e_{i_1}, \dots, A^\ell e_{i_\ell})$$

where the notation ω^t is defined in Definition (2.7). For any $n \in \mathbb{N}$, $\Delta_{(\cdot), (\cdot), \dots, (\cdot)}^{\omega^t} f(Y) = \Delta_{(\cdot), (\cdot), \dots, (\cdot)}^{\omega^t} f((Y_n^0, \dots, Y_n^\ell) : \mathbb{C}^{(n \times n) \cdot d} \otimes \mathbb{C}^{|\omega|} \rightarrow \mathbb{C}^{n \times n})$ is a multi-linear map from $|\omega|$ copies of $\mathbb{C}^{(n \times n) \cdot d}$ into $\mathbb{C}^{n \times n}$.

We have written $\Delta_X^{\omega^t} f(Y)$ as a short-form notation for $\Delta_{X, \dots, X}^{\omega^t} f(Y, \dots, Y)$, where $X \in \mathbb{C}^{(n \times n) \cdot d}$ is repeated $|\omega|$ times and $Y \in \mathbb{C}^{(n \times n) \cdot d}$ is repeated $|\omega| + 1$ times. That is, $\Delta_{A^{i_1}, \dots, A^{i_{|\omega|}}}^{\omega^t} f(Y)$ is multilinear in the ‘directional arguments’, $A^{i_j} \in \mathbb{C}^{(n \times n) \cdot d}$, $1 \leq j \leq |\omega|$.

Recall the Taylor-Taylor series of an NC function $f : \Omega \subseteq \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ around

a matrix centre $Y \in \Omega_s$:

$$f(X) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\partial_{X-I_m \otimes Y}^{(\ell)} f \right) (Y) : \quad X \in r' \cdot \mathbb{B}_{sm}^d(Y).$$

Also, recall from notation that we set right after Theorem (2.2), that we defined, $\frac{1}{\ell!} \partial_{X-I_m \otimes Y}^{(\ell)} f = \Delta^{(\ell)} f(Y, \dots, Y)(X - I_m \otimes Y, \dots, X - I_m \otimes Y)$. Let us for a moment denote $Z = X - I_m \otimes Y$. Here Z is a d -tuple of $s \times s$ complex matrices. Therefore, we have

$$Z = \sum_{j=1}^d Z e_j : \quad \{e_j\}_{j=1}^d \text{ is the basis of } \mathbb{C}^d.$$

So we can write :

$$\begin{aligned} \frac{1}{\ell!} \partial_{X-I_m \otimes Y}^{(\ell)} f &= \Delta^{(\ell)} f(Y, \dots, Y)(Z, \dots, Z) \\ &= \Delta^{(\ell)} f(Y, \dots, Y) \left(\sum_{j_1=1}^d Z e_{j_1}, \dots, \sum_{j_\ell=1}^d Z e_{j_\ell} \right) \\ &= \sum_{j_1=1}^d \sum_{j_2=1}^d \dots \sum_{j_\ell=1}^d \Delta^{(\ell)} f(Y, \dots, Y)(Z e_{j_1}, \dots, Z e_{j_\ell}) \\ &= \sum_{|\omega|=\ell} \Delta_Z^{\omega^t} f(Y) \end{aligned} \tag{2.11}$$

This observation guides us to write the another form of the Taylor-Taylor series of an NC function f , around matrix-centre $Y \in \mathbb{C}^{s \times s}$, indexed by the free monoid $\mathbb{F}_+^d$, given that the NC function satisfies the assumptions of Theorem (2.5):

$$f(X) = \sum_{\ell=0}^{\infty} \sum_{|\omega|=\ell} \Delta_{X-I_m \otimes Y}^{\omega^t} f(Y) = \sum_{\omega \in \mathbb{F}_+^d} \Delta_{X-I_m \otimes Y}^{\omega^t} f(Y) \tag{2.12}$$

For this case, the convergence formulas similar to Theorem (2.5), can be obtain similarly as above by the relation in (2.11). Moreover, similar to the Remark (2.2), if f is uniformly analytic in a uniformly open neighbourhood of $0 = (0, \dots, 0) \in \mathbb{C}^{1 \times d}$,

given any $\omega \in \mathbb{F}_+^d$, one can compute that [1]

$$\Delta_X^{\omega^t} f(0) = \widehat{f}_\omega X^\omega, \quad \widehat{f}_\omega \in \mathbb{C},$$

so that this gives a free power series expansion of $f(X)$, and any such f can be identified with a free formal power series ([11], Corollary 4.4). Meaning that at $Y = 0$ we have:

$$f(X) = \sum_{\omega \in \mathbb{F}_+^d} \widehat{f}_\omega X^\omega.$$

So, the conclusion is that using the convergence theorems stated so far, the free formal power series converges absolutely and uniformly in $r \cdot \mathbb{B}_{\mathbb{N}}^d$ for a $0 < r < R_f$, where R_f can be computed using by Popescu's generalization of the Hadamard formula [1],

$$\frac{1}{R_f} = \limsup_{n \rightarrow \infty} \sqrt[n]{\sum_{|\omega|=n} |\widehat{f}_\omega|^2}. \quad (2.13)$$

which is similar to radius of convergence defined in ([11]), and $|\widehat{f}_\omega|$ is the modulus of the coefficient of the free power series above.

Example 2.1. We would like to present an example of the partial difference-differential of a simple NC function. Consider $f(Y_1, Y_2) = Y_1 Y_2 - Y_2 Y_1$. We aim to compute the partial difference-differential operator $\Delta_H^{(21)} f(Y, Y, Y)$ in which $Y = (Y_1, Y_2)$ and $H = (H_1, H_2)$.

First, to compute $\Delta_H^{(1)} f(Y, Y)$, we do as follows:

$$\begin{aligned} f \left(\begin{bmatrix} Y & H e_1 \\ 0 & Y \end{bmatrix} \right) &= \begin{pmatrix} Y_1 & H_1 \\ 0 & Y_1 \end{pmatrix} \begin{pmatrix} Y_2 & 0 \\ 0 & Y_2 \end{pmatrix} - \begin{pmatrix} Y_2 & 0 \\ 0 & Y_2 \end{pmatrix} \begin{pmatrix} Y_1 & H_1 \\ 0 & Y_1 \end{pmatrix} \\ &= \begin{pmatrix} Y_1 Y_2 - Y_2 Y_1 & H_1 Y_2 - Y_2 H_1 \\ 0 & Y_1 Y_2 - Y_2 Y_1 \end{pmatrix} = \begin{pmatrix} f(Y) & \Delta_H^{(1)} f(Y, Y) \\ 0 & f(Y) \end{pmatrix} \end{aligned}$$

Therefore, we get $\Delta_H^{(1)} f(Y, Y) = H_1 Y_2 - Y_2 H_1$, which is compatible with the limit defini-

tion of the derivative in the direction of $(H_1, 0)$:

$$\lim_{t \rightarrow 0} \frac{f(Y_1 + tH_1, Y_2) - f(Y_1, Y_2)}{t} = H_1Y_2 - Y_2H_1.$$

For the partial difference-differential operator $\Delta_H^{(21)} f(Y, Y, Y)$, we carry a similar computation:

$$\begin{aligned} f \begin{pmatrix} Y & He_1 \\ 0 & Y & He_2 \\ 0 & 0 & Y \end{pmatrix} &= \begin{pmatrix} Y_1 & H_1 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & Y_1 \end{pmatrix} \begin{pmatrix} Y_2 & 0 & 0 \\ 0 & Y_2 & H_2 \\ 0 & 0 & Y_2 \end{pmatrix} - \begin{pmatrix} Y_2 & 0 & 0 \\ 0 & Y_2 & H_2 \\ 0 & 0 & Y_2 \end{pmatrix} \begin{pmatrix} Y_1 & H_1 & 0 \\ 0 & Y_1 & 0 \\ 0 & 0 & Y_1 \end{pmatrix} \\ &= \begin{pmatrix} f(Y) & * & H_1H_2 \\ 0 & f(Y) & * \\ 0 & 0 & f(Y) \end{pmatrix}, \quad \text{in which } Y = (Y_1, Y_2). \end{aligned}$$

Hence, this shows that $\Delta_H^{(21)} f(Y, Y, Y) = H_1H_2$. We would also like to show here how this partial differentiation of f is different from G -differentiation of f . For example, we can compute $\partial_{He_2} \partial_{He_1} f(Y)$ as:

$$\partial_{He_1} f(Y) = H_1Y_2 - Y_2H_1.$$

Considering the limit definition of the G -derivative, if we take the NC function $g(Y_2)[H_1] := H_1Y_2 - Y_2H_1$, we would have:

$$\partial_{He_2} \partial_{He_1} f(Y) = \lim_{t \rightarrow 0} \frac{g(Y_2 + tH_2) - g(Y_2)}{t} = H_1H_2 - H_2H_1 \neq \Delta_H^{(21)} f(Y, Y, Y).$$

2.4 Lost-Abbey Conditions

So far, we have assumed that the function f is an NC function, and then we asked the question of whether we can expand this NC function to a Taylor-Taylor series. Here we

ask the converse question: Given a series

$$\sum_{\ell=0}^{\infty} f_{\ell}(\underbrace{X - I_m \otimes Y, \dots, X - I_m \otimes Y}_{\ell \text{ times}}), \quad (2.14)$$

with $Y \in \mathbb{C}^{(s \times s) \cdot d}$ and $f_{\ell} : (\mathbb{C}^{(s \times s) \cdot d})^{\ell} \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$, for $\ell \in \mathbb{N} \cup \{0\}$, being ℓ -linear maps and $X \in \mathbb{C}^{(sm \times sm) \cdot d}$, what conditions makes this series an NC function? The following conditions are called **Lost- Abbey conditions** for NC functions [11]:

Proposition 2.3. (See [11], Remark 4.3) *The infinite series (2.14) is an NC function if and only if the following conditions holds true:*

$$Sf_0 - f_0S = f_1(SY - YS), \quad (2.15)$$

and for $\ell \in \mathbb{N} \cup \{0\}$,

$$Sf_{\ell}(Z^1, \dots, Z^{\ell}) - f_{\ell}(SZ^1, Z^2, \dots, Z^{\ell}) = f_{\ell+1}(SY - YS, Z^1, \dots, Z^{\ell}), \quad (2.16)$$

$$\begin{aligned} & f_{\ell}(Z^1, \dots, Z^{j-1}, Z^j S, Z^{j+1}, \dots, Z^{\ell}) - f_{\ell}(Z^1, \dots, Z^j, SZ^{j+1}, Z^{j+2}, \dots, Z^{\ell}) \\ &= f_{\ell+1}(Z^1, \dots, Z^j, SY - YS, Z^{j+1}, \dots, Z^{\ell}), \end{aligned} \quad (2.17)$$

$$f_{\ell}(Z^1, \dots, Z^{\ell-1}, Z^{\ell} S) - f_{\ell}(Z^1, \dots, Z^{\ell}) S = f_{\ell+1}(Z^1, \dots, Z^{\ell}, SY - YS), \quad (2.18)$$

for every $S \in \mathbb{C}^{s \times s}$.

Here is a brief calculation as an evidence why the proposition is true: We want to prove that $f(X) = \sum_{\ell=0}^{\infty} f_{\ell}(X - I_m \otimes Y)$, is an NC function. For the invariance under direct sum, we have

$$f \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} = \sum_{\ell=0}^{\infty} f_{\ell} \left(\begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} - I_m \otimes Y, \dots, \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} - I_m \otimes Y \right) = \begin{pmatrix} f(X) & 0 \\ 0 & f(X') \end{pmatrix}$$

for $X \in \mathbb{C}^{(ms \times ms) \cdot d}$ and $X' \in \mathbb{C}^{(sm' \times sm') \cdot d}$, mainly because each f_{ℓ} is an NC function itself.

Next, we need to show that if $X' = SXS^{-1}$, we have $Sf(X) = f(X')S$. First, using Equation (2.15), we have:

$$\begin{aligned} Sf_0 &= f_0S + f_1(SY - YS), \\ Sf_1(X - Y) &= f_1(S(X - Y)) + f_2(SY - YS, X - Y) \end{aligned}$$

and therefore,

$$Sf_0 + Sf_1(X - Y) = f_0S + f_1(SY) - f_1(YS) + f_1(\underbrace{SX}_{=X'S}) - f_1(SY) + f_2(SY - YS, X - Y)$$

by Equation (2.18):

$$\begin{aligned} f_1(X'S) &= f_1(X')S + f_2(X', YS - SY) \\ f_1(YS) &= f_1(Y)S + f_2(Y, YS - SY) \end{aligned}$$

and therefore:

$$Sf_0 + Sf_1(X - Y) = f_0S + f_1(X' - Y)S + f_2(SY - YS, X - Y) + f_2(X' - Y, SY - YS).$$

We can take one more step: if we add Equation (2.16), to both sides of the last equation above, we get:

$$\begin{aligned} &Sf_0 + Sf_1(X - Y) + Sf_2(X - Y, X - Y) \\ &= f_0S + f_1(X' - Y)S + f_2(SY - YS, X - Y) + f_2(X' - Y, SY - YS) + \\ &\quad (f_2(S(X - Y), X - Y) + f_3(SY - YS, X - Y, X - Y)) \\ &= f_0S + f_1(X' - Y)S + f_2(X' - Y, SY - YS) + f_2(X'S - YS, X - Y) \\ &\quad + f_3(X' - Y, SY - YS, X - Y) \end{aligned}$$

Now, using Equation (2.17) on the term $f_2(X'S - YS, X - Y)$, we get:

$$\begin{aligned}
&= Sf_0 + Sf_1(X - Y) + Sf_2(X - Y, X - Y) \\
&= f_0S + f_1(X' - Y)S + f_2(X' - Y, X' - Y)S \\
&+ f_3(SY - YS, X - Y, X - Y) + f_3(X' - Y, SY - YS, X - Y) \\
&+ f_3(X' - Y, X' - Y, SY - YS).
\end{aligned}$$

Continuing this process, if one can show that this iterative process converges, it would follow that $Sf(X) = f(X')S$ is indeed true.

2.5 Convergence theorems for NC Power Series

An interesting question that can arise within the context of power series defined up until here, is that when does a power series converge to an NC function on its domain of definition? More accurately, given that following power series satisfies the Lost-Abbey conditions around a matrix centre $Y \in \mathbb{C}^{(s \times s) \cdot d}$:

$$f(X) := \sum_{\ell=0}^{\infty} f_{\ell}(\underbrace{X - I_m \otimes Y, \dots, X - I_m \otimes Y}_{\ell \text{ times}}) \quad (2.19)$$

we want to see if it actually converges to an NC function (or when does it have a radius of convergence), given the ℓ -linear mappings $f_{\ell} : (\mathbb{C}^{(s \times s) \cdot d})^{\ell} \longrightarrow \mathbb{C}^{s \times s}$, $\ell = 0, 1, \dots$ and $X \in \mathbb{C}^{(sm \times sm) \cdot d}$ $m = 0, 1, \dots$; the power series centred around the matrix point $Y \in \mathbb{C}^{(s \times s) \cdot d}$.

Furthermore, let

$$\mu_{\text{cb}} = \limsup_{\ell \rightarrow \infty} \sqrt[\ell]{\|f_{\ell}\|_{\text{cb}}}, \quad \rho_{\text{cb}} = \mu_{\text{cb}}^{-1}. \quad (2.20)$$

The next theorem shows that ρ_{cb} acts as the radius of convergence for the series (2.2).

Theorem 2.7. ([11], Theorem 8.11) 1. The series (2.19) converges absolutely and uni-

formly on every NC ball $\delta \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$ with $\delta < \rho_{\text{cb}}$, in the following sense:

$$\sum_{\ell=0}^{\infty} \sup_{W \in \delta \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)} \|f_{\ell}(W - I_{m_W} \otimes Y)\|_{sm_W} < \infty, \quad Y \in \mathbb{C}^{(s \times s) \cdot d}. \quad (2.21)$$

Here m_W is the size of a block matrix W with $s \times s$ blocks. Furthermore, the series fails to converge uniformly on any $\delta \cdot \mathbb{B}_{s\mathbb{N}}^d$ with $\delta > \rho_{\text{cb}}$.

2. For any uniformly-open NC set $\Gamma \subseteq \rho_{\text{cb}} \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$, $f|_{\Gamma}$ is a uniformly analytic NC function.

Proof of part 1. When $0 < \delta < \rho_{\text{cb}}$, then we have:

$$\mu_{\text{cb}} = \limsup_{\ell \rightarrow \infty} \sqrt[\ell]{\|f_{\ell}\|_{\text{cb}}} < \infty.$$

and by choosing $\epsilon > 0$ such that $\epsilon < \delta^{-1} - \mu_{\text{cb}}$, there exist a $C > 0$ (see [11, page. 137]):

$$\|f_{\ell}\|_{\text{cb}} < C(\mu_{\text{cb}} + \epsilon)^{\ell}, \quad \text{for } \ell \in \mathbb{N} \cup \{0\},$$

and therefore, for $W \in \delta \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$ with $\delta < \rho_{\text{cb}}$:

$$\begin{aligned} \|f(W)\| &= \left\| \sum_{\ell=0}^{\infty} f_{\ell}(W - I_{m_W} \otimes Y)^{\odot \ell} \right\| \leq \sum_{\ell=0}^{\infty} \sup_{W \in \delta \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)} \|f_{\ell}(W - I_{m_W} \otimes Y)^{\odot \ell}\|_{sm_W} \\ &= \sum_{\ell=0}^{\infty} \sup_{\|W - I_{m_W} \otimes Y\|_{sm_W} = \delta} \|f_{\ell}(W - I_{m_W} \otimes Y)^{\odot \ell}\|_{sm_W} \\ &= \sum_{\ell=0}^{\infty} \delta^{\ell} \sup_{\|W - I_{m_W} \otimes Y\|_{sm_W} = 1} \|f_{\ell}(W - I_{m_W} \otimes Y)^{\odot \ell}\|_{sm_W} \end{aligned} \quad (2.22)$$

in which m_W is chosen so that W and $I_{m_W} \otimes Y$ are of the same size. We know that:

$$\|f_{\ell}(W - I_{m_W} \otimes Y)^{\odot \ell}\| \leq \|f_{\ell}\|_{\text{cb}} \|(W - I_{m_W} \otimes Y)^{\odot \ell}\|_h \leq \|f_{\ell}\|_{\text{cb}} \|W - I_{m_W} \otimes Y\|_h^{\ell}$$

So, knowing that $\delta(\mu_{\text{cb}} + \epsilon) < 1$, we have:

$$\begin{aligned} & \sum_{\ell=0}^{\infty} \delta^{\ell} \sup_{\|W - I_{m_W} \otimes Y\|_{sm_W} = 1} \left\| f_{\ell} (W - I_{m_W} \otimes Y)^{\odot \ell} \right\|_{sm_W} \\ & \leq \sum_{\ell=0}^{\infty} \delta^{\ell} \|f_{\ell}\|_{\text{cb}} \leq C \sum_{\ell=0}^{\infty} \delta^{\ell} (\mu_{\text{cb}} + \epsilon)^{\ell} < \infty. \end{aligned} \quad (2.23)$$

Now suppose $\delta > \rho_{\text{cb}}$. Then for any $\epsilon : 0 < \epsilon < \mu_{\text{cb}} - \delta^{-1}$ and $L > 0$ there exists $\ell > L$ such that $\|f_{\ell}\|_{\text{cb}} > (\mu_{\text{cb}} - \epsilon)^{\ell}$. For any such ℓ there exists $W \in \delta \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$ with $\|W - I_{m_W} \otimes Y\| = (\mu_{\text{cb}} - \epsilon)^{-1} < \delta$ such that

$$\|f_{\ell} (W - I_{m_W} \otimes Y)^{\odot \ell}\| > (\mu_{\text{cb}} - \epsilon)^{\ell} \|W - I_{m_W} \otimes Y\|^{\ell} = 1.$$

Hence the sequence $\|f_{\ell} (W - I_{m_W} \otimes Y)^{\odot \ell}\|$, $\ell \in \mathbb{N} \cup \{0\}$, does not converge to 0 and the series does not converge uniformly on $\delta \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$. $\square$

Proof of part 2. Note that we assumed at the beginning of this section that the general term in the series (2.19), obeys the Lost- Abbey condition. It follows from Part 1, that f is uniformly locally bounded: For $W \in \delta \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$ with $\delta < \rho_{\text{cb}}$:

$$\|f(W)\| = \left\| \sum_{\ell=0}^{\infty} f_{\ell} (W - I_{m_W} \otimes Y)^{\odot \ell} \right\| \leq \sum_{\ell=0}^{\infty} \sup_{W \in \delta \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)} \left\| f_{\ell} (W - I_{m_W} \otimes Y)^{\odot \ell} \right\| < \infty. \quad (2.24)$$

So the series at right-hand side of Inequality (2.24), converges and this shows that $f(W)$ is uniformly bounded in $\delta \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$ for any $\delta < \rho_{\text{cb}}$. Therefore, by Lemma (1.1) and Definition (2.6), f is uniformly analytic in $\rho_{\text{cb}} \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$.

In conclusion, $f|_{\Gamma}$ is uniformly analytic NC function. $\square$

Chapter 3

Realization Theory of Noncommutative Functions

In this section, we present some basic theorems for the “realization” of NC functions, also known as the linearization trick in the free probability literature. (See for example, [6].) First, we discuss basic definitions and properties of realizations and their corresponding NC functions and characterize them. In particular, we review the results which prove that an NC function has a finite-dimensional realization if and only if it is a noncommutative rational NC function centred around 0. We set the necessary preliminaries and present the conditions for minimality of the realization and Kalman decomposition around the origin.

Next, in the second section, based on [1], we build a realization of an NC function around a matrix centre given the assumption that the NC function has a realization around 0. Also, we present the analytical properties (e.g. the existence and explicit form of the associated power series) of the realization corresponding to a function and present conditions on the realization itself to make it a realization corresponding to an NC function.

We aim to set the stage here in the first two sections to provide some results in the last section regarding the realization of an NC function around an arbitrary matrix

point.

This chapter is closely aligned with the papers [18, 19] as well as the paper [1] for results regarding infinite-dimensional operator realizations.

3.1 Introduction to Realizations: Definitions and Basic Properties

A one-dimensional *realization* is a triple, $(A, b, c) \in \mathcal{B}(\mathcal{H}) \times \mathcal{H} \times \mathcal{H}$ where $\mathcal{H}$ is a complex, separable or finite-dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the Banach space of bounded linear operators on $\mathcal{H}$. Any realization defines an analytic function in an open neighbourhood of 0 by the realization formula,

$$h(z) := b^*(I - zA)^{-1}c,$$

and we write $h \sim (A, b, c)$ if h is defined by the realization in this way. Here, if $|z| < \|A\|^{-1}$, then the inverse of the *linear pencil*, $L_A(z)^{-1} := (I - zA)^{-1}$ can be expanded as a convergent geometric series so that

$$h(z) = \sum_{n=0}^{\infty} (b^* A^n c) z^n, \quad |z| < \|A\|^{-1},$$

is an absolutely convergent Taylor series. Conversely, we present an argument from ([1]) in this chapter, that shows a one-variable formal power series, $h \in \mathbb{C}[[z]]$, is the Taylor series of an analytic function in an open neighbourhood of 0 then it has a realization, $h \sim (A, b, c)$, in which case we say that h is *familiar* ([1], Lemma 3.1).

Definition 3.1. Let $A \in \mathcal{B}(\mathcal{H})^d$. The **linear pencil** L_A is defined as an affine linear

function on $\mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$:

$$L_A(Z) := I_n \otimes I_{\mathcal{H}} - \sum_{j=1}^d Z_j \otimes A_j =: I_n \otimes I - Z \otimes A; \quad Z \in \mathbb{C}^{(n \times n) \cdot d}.$$

Definition 3.2. A **descriptor realization** is a triple, (A, b, c) , with $A \in \mathcal{B}(\mathcal{H})^d := \mathcal{B}(\mathcal{H}) \otimes \mathbb{C}^d$, $b, c \in \mathcal{H}$. Here, note that $A \in \mathcal{B}(\mathcal{H})^d$ is viewed as a column d -tuple,

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} \in \mathcal{B}(\mathcal{H})^d.$$

The **transfer function**, h , of the descriptor realization, (A, b, c) , can be defined as the NC function with the domain of invertibility of $L_A(Z)$, given as:

$$\text{Dom } h := \bigsqcup_{n=1}^{\infty} \text{Dom}_n h; \quad \text{Dom}_n h := \{Z \in \mathbb{C}^{(n \times n) \cdot d} \mid L_A(Z) \text{ is invertible}\},$$

via the formula:

$$h(Z) := I_n \otimes b^* L_A(Z)^{-1} I_n \otimes c.$$

We will write $h \sim (A, b, c)$ to denote that h is the transfer-function of this realization.

Definition 3.3. The **invertibility domain of the realization** (A, b, c) defined as:

$$\mathcal{D}(A) = \{Z \in \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d} : L_A(Z) \text{ is invertible}\}.$$

The following results on the inversion map on $\mathcal{B}(\mathcal{H})$ are well known:

Lemma 3.1. Let $T \in \mathcal{B}(\mathcal{H})$, in which $\mathcal{H}$ is a Hilbert space and T is in the space of bounded linear operators on $\mathcal{H}$. Suppose that $\|T\| < 1$. Then the series $\sum_{n=0}^{\infty} T^n$ converges in operator norm to $(I - T)^{-1}$.

Using this theorem it follows that the inverse operation, sending each operator to its inverse, is in fact continuous:

Proposition 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. The map $T \mapsto T^{-1}$ is continuous.*

Proof. Let $S \in \mathcal{B}(\mathcal{H})$ satisfy:

$$\|S - T\| < \frac{1}{\|T^{-1}\|}.$$

Set $E = T - S$, so that $\|E\| < \|T^{-1}\|^{-1}$. Then

$$T^{-1}S = I - T^{-1}E, \quad \|T^{-1}E\| \leq \|T^{-1}\| \cdot \|E\| < 1.$$

Since $\|T^{-1}E\| < 1$, the operator $I - T^{-1}E$ is invertible by the Neumann series lemma, and

$$\|(I - T^{-1}E)^{-1}\| \leq \frac{1}{1 - \|T^{-1}E\|} \leq \frac{1}{1 - \|T^{-1}\| \cdot \|S - T\|}.$$

As $S = T(I - T^{-1}E)$, we obtain

$$S^{-1} = (I - T^{-1}E)^{-1}T^{-1},$$

and therefore

$$\|S^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|T^{-1}\| \cdot \|S - T\|}.$$

Now consider the difference:

$$S^{-1} - T^{-1} = S^{-1}(S - T)T^{-1} = S^{-1}ET^{-1}.$$

Taking norms yields

$$\|S^{-1} - T^{-1}\| \leq \|S^{-1}\| \cdot \|S - T\| \cdot \|T^{-1}\|.$$

Substituting the upper bound for $\|S^{-1}\|$, we see that as $\|S - T\| \rightarrow 0$, the right-hand side tends to 0. Hence $S^{-1} \rightarrow T^{-1}$ in the operator norm.

This shows that the inversion map is continuous at every invertible operator T . $\square$

Corollary 3.1. *The set of all invertible elements in $\mathcal{B}(\mathcal{H})$ is open in the operator norm topology. In particular, if $A \in \mathcal{B}(\mathcal{H})$ is invertible, then $B \in \mathcal{B}(\mathcal{H})$ is invertible if*

$$\|A - B\| < \frac{1}{\|A^{-1}\|}. \quad (3.1)$$

Lemma 3.2. *The invertibility domain of the realization (A, b, c) is a joint similarity invariant uniformly open NC set.*

Proof. It is straightforward to check that

$$L_A \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} = \begin{pmatrix} L_A(Z) & 0 \\ 0 & L_A(W) \end{pmatrix}.$$

so that $\mathcal{D}(A)$ is indeed an NC set. One can also check that

$$\begin{aligned} L_A(SZS^{-1}) &= I \otimes I_{\mathcal{H}} - \sum_{j=1}^d SZ_j S^{-1} \otimes A_j \\ &= SS^{-1} \otimes I_{\mathcal{H}} - \sum_{j=1}^d SZ_j S^{-1} \otimes A_j \\ &= S \left(I \otimes I_{\mathcal{H}} - \sum_{j=1}^d Z_j \otimes A_j \right) S^{-1} \\ &= (S \otimes I_{\mathcal{H}}) L_A(Z) (S^{-1} \otimes I_{\mathcal{H}}). \end{aligned}$$

so $L_A(SZS^{-1})$ is invertible if and only if $L_A(Z)$ is invertible. That is $Z \in \mathcal{D}(A)$ if and only if $SZS^{-1} \in \mathcal{D}(A)$. Therefore, $\mathcal{D}(A)$ is joint similarity invariant.

It remains to show that $\mathcal{D}(A)$ is uniformly open: We show that given $Z \in \mathcal{D}(A)$, we can find a uniformly open ball containing Z which is inside $\mathcal{D}(A)$.

Let us denote by $\mathcal{D}_s(A)$, the level s of the NC set $\mathcal{D}(A)$, defined as

$$\mathcal{D}_s(A) := \{Z \in \mathbb{C}^{(s \times s) \cdot d} : L_A(Z) \text{ is invertible}\}.$$

Given $Z \in \mathcal{D}_s(A)$, we consider $\|Z^{\oplus m} - W\|_{\text{row}}$ where $W \in \mathbb{C}^{(sm \times sm) \cdot d}$, and we know that $L_A(Z)$ is invertible. We can determine when the inverse of $L_A(W)$ exist. Observe that:

$$\|L_A(Z^{\oplus m}) - L_A(W)\| = \|(Z^{\oplus m} - W) \otimes A\|_{\text{row}} \leq \|(Z^{\oplus m} - W)\|_{\text{row}} \|A\|_{\text{col}}.$$

We can take $\|Z^{\oplus m} - W\|_{\text{row}} < \frac{1}{\|A\|} \frac{1}{\|L_A(Z^{\oplus m})^{-1}\|}$, then we will have:

$$\|L_A(Z^{\oplus m}) - L_A(W)\| < \frac{1}{\|L_A(Z^{\oplus m})^{-1}\|}$$

so the condition (3.1) is satisfied. By Corollary (3.1), we conclude that $L_A(W)$ is also invertible. This proves that a uniform open NC ball of sufficiently small radius, is contained entirely in $\mathcal{D}(A)$. $\square$

We want to show that in a uniformly open neighbourhood of $0 \in \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$, any realization defines a holomorphic NC function and vice versa. Here, we consider the multi-variable case. First, note that for $Z \in \mathbb{C}^{(n \times n) \cdot d}$ we have:

$$\begin{aligned} \|Z \otimes A\| &= \|(Z \otimes I_{\mathcal{H}})(I_n \otimes A)\|, & \text{Note: } \boxed{(A \otimes B)(C \otimes D) = AC \otimes BD} \\ &= \left\| \sum_{j=1}^d Z_j \otimes A_j \right\|_{\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}^{n \times n}} \\ &\leq \underbrace{\|(Z_1 \otimes I_{\mathcal{H}}, \dots, Z_d \otimes I_{\mathcal{H}})\|}_{\in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H} \otimes \mathbb{C}^{1 \times d}, \mathbb{C}^n \otimes \mathcal{H})} \underbrace{\left\| \begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} \right\|}_{\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^d)} = \|Z\|_{\text{row}} \|A\|_{\text{col}}. \end{aligned}$$

So if $\|Z\|_{\text{row}} < \|A\|_{\text{col}}^{-1}$ then $\|Z \otimes A\| < 1$. Therefore,

$$h(Z) = I_n \otimes b^* \sum_{j=0}^{\infty} (Z \otimes A)^j I_n \otimes c = \sum_{\omega \in \mathbb{F}_+^d} (b^* A^\omega c) Z^\omega. \quad (3.2)$$

Note that as an operator on $\mathcal{H} \otimes \mathbb{C}^n$,

$$\left\| \sum_{j=1}^d Z_j \otimes A_j \right\|_{\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}^{n \times n}} \leq \|Z\|_{\text{row}} \|A\|_{\text{col}},$$

so that the operator $L_A(Z)$ is invertible in $\|A\|_{\text{col}}^{-1} \cdot \mathbb{B}_{\mathbb{N}}^d(0)$. Therefore, $\|A\|_{\text{col}}^{-1} \cdot \mathbb{B}_{\mathbb{N}}^d(0) \subseteq \text{Dom } h$, and

$$\|h(Z)\| \leq \frac{\|b\| \cdot \|c\|}{1 - r \|A\|_{\text{col}}^{-1}}; \quad \forall Z \in r \cdot \mathbb{B}_{\mathbb{N}}^d(0) \text{ for } 0 < r < \|A\|_{\text{col}}^{-1}.$$

As a consequence, h will be uniformly bounded on any NC row-ball of radius $\|A\|_{\text{col}}^{-1} > r > 0$, hence, using Corollary (2.2), uniformly analytic on this uniformly open NC row ball $\|A\|_{\text{col}}^{-1} \cdot \mathbb{B}_{\mathbb{N}}^d(0)$. Since $\mathcal{D}(A)$ is uniformly open, we conclude that h is uniformly analytic on all of $\mathcal{D}(A)$.

By evaluating h on jointly nilpotent points Z in $r \cdot \mathbb{B}_{\mathbb{N}}^d(0)$ (See for example [11], Chapter 5), we see that any h that admits a realization (A, b, c) can be identified with the formal power series, $h \in \mathbb{C}\langle\mathfrak{z}\rangle$,

$$h(\mathfrak{z}) = \sum_{\omega \in \mathbb{F}_+^d} \hat{h}_\omega \mathfrak{z}^\omega, \quad \text{where} \quad \hat{h}_\omega = b^* A^\omega c.$$

Indeed, this also follows readily from expansion of $L_A(Z)^{-1}$ as a convergent geometric series for any Z in $r \cdot \mathbb{B}_{\mathbb{N}}^d(0)$, $r = \|A\|_{\text{col}}^{-1}$, as in Equation (3.2).

In the case of one variable functions defined in a unit disk in the complex plane $\mathbb{C}$, the **Hardy space** is defined as the Hilbert space of square-summable Taylor series in

the complex unit disk, denoted by:

$$H^2(\mathbb{D}) := \{f \in \mathcal{O}(\mathbb{D}) : \sum_{j=0}^{\infty} |f_j|^2 < \infty \text{ if } f(z) = \sum_{j=0}^{\infty} f_j z^j\} \quad (3.3)$$

in which $\mathcal{O}(\mathbb{D})$ is the set of holomorphic functions defined on the unit disk in the complex plane.

Equip $H^2(\mathbb{D})$ with the inner product and norm given by the ℓ^2 -norm of Taylor coefficients:

$$\langle f, g \rangle_{H^2} := \sum_{n=0}^{\infty} a_n \overline{b_n}, \quad \|f\|_{H^2} := \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2},$$

where $f(z) = \sum_{n \geq 0} a_n z^n$ and $g(z) = \sum_{n \geq 0} b_n z^n$. Under this inner product $H^2(\mathbb{D})$ is isometrically isomorphic to $\ell^2(\mathbb{N}_0)$ via the map $f \mapsto (a_0, a_1, \dots)$ [1, 20]. One can easily check that $(z^n)_{n=0}^{\infty}$ is an orthonormal basis of $H^2(\mathbb{D})$.

There are three common and equivalent viewpoints for $H^2(\mathbb{D})$: 1) the *coefficient* viewpoint (Taylor coefficients in ℓ^2); 2) the *disk* viewpoint (holomorphic functions on $\mathbb{D}$ whose Taylor coefficients are square summable); 3) the *boundary viewpoint* (See [8]): the radial limits $f(e^{it}) = \lim_{r \nearrow 1} f(re^{it})$ exist for a.e. t , and the boundary function lies in $L^2(\partial\mathbb{D})$. Moreover, the map $V : H^2(\mathbb{D}) \rightarrow L^2(\partial\mathbb{D})$ which identifies every $h \in H^2(\mathbb{D})$ with its radial boundary limits is an isometry [8].

These viewpoints can be summarized into the identities below:

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 dt.$$

This identity (coefficient ℓ^2 norm = boundary L^2 norm) is the Parseval (or Plancherel) identity for $H^2(\mathbb{D})$, for a certain choice of the orthonormal basis of $H^2(\mathbb{D})$.

Also consider the **Hardy algebra**, which consists of all bounded holomorphic func-

tion in $\mathbb{D}$, explicitly defined as:

$$H^\infty(\mathbb{D}) = \{f \in \mathcal{O}(\mathbb{D}) : \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < +\infty\}. \quad (3.4)$$

Lemma 3.3. *We have $H^\infty(\mathbb{D}) \subseteq H^2(\mathbb{D})$.*

Proof. (See [8]) Let $f \in H^\infty(\mathbb{D})$, so $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| = M < \infty$. For each fixed $r \in [0, 1)$, apply the maximum modulus principle to f on the closed disk $\overline{\mathbb{D}}_r = \{z \in \mathbb{C} : |z| \leq r\}$. This yields

$$\sup_{|z|=r} |f(z)| = \max_{z \in \overline{\mathbb{D}}_r} |f(z)| \leq M.$$

Thus, $|f(re^{it})| \leq M$ for all $t \in [0, 2\pi]$, and

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt \leq \frac{1}{2\pi} \int_0^{2\pi} M^2 dt = M^2.$$

It follows that

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt \leq M^2 < \infty.$$

which implies that $f \in H^2(\mathbb{D})$ by definition.

Moreover, the means $\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt$ are nondecreasing in r (as $|f|^2$ is subharmonic), so the limit

$$\|f\|_{H^2}^2 = \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt \leq M^2 < \infty$$

exists and is finite. By the boundary viewpoint (and Fatou's radial theorem [21] for bounded holomorphic functions), the radial limits $f(e^{it}) = \lim_{r \rightarrow 1^-} f(re^{it})$ exist for a.e. $t \in [0, 2\pi]$, and the Parseval identity gives

$$\|f\|_{H^2}^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 dt \leq M^2 < \infty,$$

so f lies in $L^2(\partial\mathbb{D})$ as required. Therefore, $H^\infty(\mathbb{D}) \subseteq H^2(\mathbb{D})$. $\square$

Let h be holomorphic map defined on an open neighbourhood of the origin so that there exists a real number $r > 0$ such that $h(rz) \in H^\infty(\mathbb{D}) \subseteq H^2(\mathbb{D})$. In $H^2(\mathbb{D})$ we denote by :

$$S : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}), \quad (Sf)(z) := zf(z),$$

the unilateral shift operator, and by

$$1 \in H^2(\mathbb{D})$$

the constant function 1, i.e. the basis element z^0 . Considering the orthonormal basis $(z^n)_{n=0}^\infty$ we can write the following expansion for h :

$$h_r(z) := h(rz) = \sum_{n=0}^{\infty} \hat{h}_{r,n} z^n : \quad \text{ONB expansion}$$

so that one can obtain the coefficients by $\hat{h}_{r,n} = \langle z^n, h_r \rangle = \langle S^n 1, h_r \rangle = \langle 1, S^{*n} h_r \rangle$. Therefore, we deduce that $(S^*, 1, h_r)$ is a realization for h_r . Thus, for the function h with the expansion

$$h(z) = \sum_{n=0}^{\infty} \hat{h}_n z^n$$

which is not necessarily in the Hardy space, we can write

$$h(rz) = \sum_{n=0}^{\infty} \hat{h}_n r^n z^n$$

so that $\hat{h}_{n,r} = \hat{h}_n r^n = \langle 1, S^{*n} h_r \rangle$, meaning that $\hat{h}_n = \langle 1, (\frac{1}{r} S^*)^n h_r \rangle_{H^2}$. Hence, $(r^{-1} S^*, 1, h_r)$ is a realization for h around the origin.

In conclusion, we have proven the following one-variable case of the following result from [1]:

Lemma 3.4. *An NC function, h , is uniformly analytic in a uniformly open neighbourhood*

of $0 \in \mathbb{C}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$ if and only if it is familiar, i.e. if and only if it has an operator realization.

It will also be convenient to consider *Fornasini–Marchesini* (FM) realizations.

Definition 3.4. An *Fornasini–Marchesini (FM) realization* is a quadruple (A, B, C, D) , where

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} \in \mathcal{B}(\mathcal{H})^d, \quad B = \begin{pmatrix} B_1 \\ \vdots \\ B_d \end{pmatrix} \in \mathcal{H} \otimes \mathbb{C}^d, \quad C \in \mathcal{H}^*, \quad \text{and} \quad D \in \mathbb{C}.$$

An FM realization is often written in the block operator–colligation form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathbb{C} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H} \otimes \mathbb{C}^d \\ \mathbb{C} \end{pmatrix}.$$

As before, we then define the transfer function, h , of the FM realization (A, B, C, D) as a function with domain

$$\text{Dom } h = \bigsqcup_n \text{Dom}_n h; \quad \text{Dom}_n h = \{Z \in \mathbb{C}^{(n \times n) \cdot d} \mid L_A(Z)^{-1} \text{ exists}\},$$

via the formula:

$$h(Z) := DI_n + I_n \otimes CL_A(Z)^{-1} \sum_{j=1}^d (Z_j \otimes B_j) =: D + I_n \otimes CL_A(Z)^{-1} Z \otimes B.$$

In particular, as discussed above, there is an $r > 0$ so that $r \cdot \mathbb{B}_{\mathbb{N}}^d(0) \subseteq \text{Dom } h$, and h will be uniformly bounded on any NC row-ball of radius $\|A\|_{\text{col}}^{-1} > r > 0$, hence it is uniformly analytic on this NC domain. We will write $h \sim (A, B, C, D)$ to denote that h is the transfer function of this FM realization.

Minimal Realizations

The other notion that we consider in this chapter is *minimality*. In a finite-dimensional realization ($\mathcal{H}$ being a finite-dimensional Hilbert space), we say that a realization $A \in \mathbb{C}^{(n \times n) \cdot d}$ is minimal if n is as small as possible. In [1], it is shown how we can generalize this notion to infinite-dimensional Hilbert spaces, as follows:

Definition 3.5. Suppose (A, b, c) is a realization. We define the controllable subspace as

$$\mathcal{C}_{A,c} := \bigvee_{\omega \in \mathbb{R}_+^d} A^\omega c, \quad (\text{closed linear span}) \quad (3.5)$$

and the observable subspace as:

$$\mathcal{O}_{A^*,b} := \bigvee_{\omega \in \mathbb{R}_+^d} A^{*\omega} b, \quad (3.6)$$

and the realization is said to be controllable if $\mathcal{C}_{A,c} = \mathcal{H}$ and observable if $\mathcal{O}_{A^*,b} = \mathcal{H}$. We say that a realization (A, b, c) is **minimal** if it is both controllable and observable.

The definition of minimality above, matches with the finite-dimensional case where it is a theorem that a finite-dimensional realization is minimal if and only if it is both observable and controllable. Minimal realizations are unique up to a natural notion of isomorphism [6], as shall be discussed in the following sections.

We can also define minimality for FM realizations as follows:

Definition 3.6. As before, we will say that the FM realization, (A, B, C, D) , for h , is minimal, if

$$\mathcal{H} = \bigvee_{\substack{\omega \in \mathbb{R}_+^d, \\ 1 \leq j \leq d}} A^\omega B_j =: \mathcal{C}_{A,B}, \quad (3.7)$$

and

$$\mathcal{H} = \bigvee_{\omega \in \mathbb{F}_d^+} A^{*\omega} C^* =: \mathcal{O}_{A^*, C}. \quad (3.8)$$

The subspaces $\mathcal{C}_{A,B}$ and $\mathcal{O}_{A^*,C}$ of $\mathcal{H}$ defined above are called the *controllable subspace* and the *observable subspace* of the FM realization and we say the FM realization is *controllable* if $\mathcal{H} = \mathcal{C}_{A,B}$ and *observable* if $\mathcal{H} = \mathcal{O}_{A^*,C}$. As before, (A, B, C, D) is said to be *minimal* if it is both controllable and observable.

3.1.1 From Descriptor Realizations to FM Realization and vice versa

Remark 3.1. If we have an NC function that admits an FM realization, one can show that it also admits a descriptor realization and vice versa. If h has descriptor realization (A, b, c) then one can define

$$\mathcal{H}' := \bigvee_{\omega \neq \emptyset} A^\omega c, \quad A' := A|_{\mathcal{H}'}, \quad B_j := A_j c, \quad C := (P_{\mathcal{H}'} b)^*, \quad \text{and} \quad D := h(0),$$

where $P_{\mathcal{H}'}$ is the projection onto $\mathcal{H}'$. It follows that (A', B, C, D) is an FM realization of h . Moreover, it is readily checked that if (A, b, c) is a minimal descriptor realization then (A', B, C, D) is a minimal FM realization.

Conversely, if (A, B, C, D) is an FM realization of h on $\mathcal{H}$, one can construct a descriptor realization, $(\hat{A}, b, c)$ on $\hat{\mathcal{H}} := \mathcal{H} \oplus \mathbb{C}$ by setting $\hat{A}_j|_{\mathcal{H}} := A_j$ and defining $\hat{A}_j 0 \oplus 1 := B_j \in \mathcal{H}$. That is, with respect to the direct sum decomposition $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathbb{C}$,

$$\hat{A}_j = \begin{pmatrix} A_j & B_j \langle 0 \oplus 1, \cdot \rangle_{\hat{\mathcal{H}}} \\ 0 & 0 \end{pmatrix}.$$

Next, one defines $b := C^* \oplus \overline{D}1$ and $c := 0 \oplus 1$. It is easy to verify that $(\hat{A}, b, c)$ is then a descriptor realization of h . If (A, B, C, D) is a minimal FM realization, then $(\hat{A}, b, c)$ is controllable, but it may not be observable. However, even if it is not observable, its

observable subspace has codimension at most one in $\widehat{\mathcal{H}}$.

Proof of Remark 3.1. Suppose that we are given an NC function h and its descriptor realization $h \sim (A, b, c)$. So, by definition and Equation (3.2), we have:

$$h(Z) = I_n \otimes b^* \left(\sum_{\omega \in \mathbb{F}_d^+} (Z \otimes A)^\omega \right) I_n \otimes c = \sum_{\omega \in \mathbb{F}_d^+} (b^* A^\omega c) Z^\omega.$$

We notice that $A_j \mathcal{H}' \subseteq \mathcal{H}'$, because for $\omega \neq \emptyset$, we have $A_j(A^\omega c) = A^{j\omega} c \in \mathcal{H}'$. Moreover, for every $\omega \in \mathbb{F}_d^+$:

$$b^* A^\omega c = \langle A^\omega c, b \rangle = \langle A'^\omega c, P|_{\mathcal{H}'} b \rangle.$$

and for $\omega = \emptyset$, we get the first term in the series expansion which is $h(0) = I_n \otimes b^* c$. so that we can write the series expansion above as :

$$\begin{aligned} h(Z) &= h(0) + \sum_{\omega \neq \emptyset} \langle A'^\omega c, P|_{\mathcal{H}'} b \rangle Z^\omega = h(0) + \sum_{\omega \in \mathbb{F}_d^+} \langle A'^\omega B_j, P|_{\mathcal{H}'} b \rangle Z^\omega = h(0) + \sum_{\substack{\omega \in \mathbb{F}_d^+ \\ j=1, \dots, d}} C A'^\omega B_j Z^{\omega j}. \\ &= D I_n + \sum_{\substack{\omega \in \mathbb{F}_d^+ \\ j \in \mathbb{N}}} I_n \otimes C (Z \otimes A')^\omega Z^j \otimes B_j = D I_n + I_n \otimes C \sum_{\omega \in \mathbb{F}_d^+} (Z \otimes A')^\omega \sum_{j=1, \dots, d} Z_j \otimes B_j. \\ &= D I_n + I_n \otimes C (I \otimes I_{\mathcal{H}'} - Z \otimes A)^{-1} \sum_{j=1}^d Z \otimes B \end{aligned}$$

We can also prove easily that the realization (A', B, C, D) is both controllable and observable, given that the initial realization (A, b, c) is controllable and observable. For the controllability we have:

$$\bigvee_{\substack{\omega \in \mathbb{F}_d^+ \\ j=1, \dots, d}} A'^\omega B_j = \bigvee_{\substack{\omega \in \mathbb{F}_d^+ \\ j=1, \dots, d}} A^\omega |_{\mathcal{H}'} A_j c = \bigvee_{\omega \in \mathbb{F}_d^+} A^\omega c = \mathcal{H}'.$$

And for the observability condition, we have:

$$\bigvee_{\omega \in \mathbb{F}_d^+} A'^{*}{}^\omega C^* = \bigvee_{\omega \in \mathbb{F}_d^+} (A^*|_{\mathcal{H}})^\omega P_{\mathcal{H}'} b = P_{\mathcal{H}'} \bigvee_{\omega \in \mathbb{F}_d^+} A^{*\omega} b = \mathcal{H}'.$$

This shows that if (A, b, c) is minimal, then (A', B, C, D) is also minimal.

Conversely, suppose $h \sim (A, B, C, D)$ is an FM realization, and for $x \oplus \alpha \in \widehat{\mathcal{H}}$ we have $\widehat{A}_j(x \oplus \alpha) = A_j x + B_j \alpha \oplus 0 = (A_j x + B_j \alpha, 0)$ and $c = 0 \oplus 1$. We observe that:

$$\widehat{A}_j c = \widehat{A}_j 0 \oplus 1 = B_j \oplus 0.$$

Thus, for $\omega = i_1 \cdots i_n \in \mathbb{F}_d^+$, we can write:

$$\widehat{A}^\omega c = A^{i_1} \cdots A^{i_{n-1}} B_{i_n} \oplus 0 \in \mathcal{H} \oplus \mathbb{C}.$$

Moreover, for $\omega \neq \emptyset$, we have:

$$\begin{aligned} CA^\omega B_{i_n} &= CA^{i_1} \cdots A^{i_{n-1}} B_{i_n} = \langle A^{i_1} \cdots A^{i_{n-1}} B_{i_n}, C^* \rangle_{\mathcal{H}} \\ &= \langle A^{i_1} \cdots A^{i_{n-1}} B_{i_n} \oplus 0, C^* \oplus \overline{D}1 \rangle_{\mathcal{H}'} = \langle \widehat{A}^\omega c, b \rangle. \end{aligned}$$

Considering the first term $D = h(0) = b^* c$ corresponding to $\omega = \emptyset$, we can write:

$$h(Z) = \sum_{\omega \in \mathbb{F}_d^+} b^* \widehat{A}^\omega c Z^\omega = b^* L_{\widehat{A}}(Z)^{-1} c.$$

Now, with the assumption that (A, B, C, D) is controllable and observable, we can prove that $(\widehat{A}, b, c)$ is controllable, because:

$$\bigvee_{\omega \in \mathbb{F}_d^+} \widehat{A}^\omega c = \bigvee_{\omega \in \mathbb{F}_d^+} (A^{i_1} \cdots A^{i_{|\omega|-1}} B_{i_{|\omega|}} \oplus 0) \oplus (0 \oplus \mathbb{C}) = \mathcal{H} \oplus \mathbb{C} = \widehat{\mathcal{H}}.$$

However, the observability may not happen because:

$$\bigvee_{\omega \in \mathbb{F}_d^+} \widehat{A}^{*\omega} b = \bigvee_{\omega \in \mathbb{F}_d^+} A_{i_1} \cdots A_{i_{|\omega|}} (C^* \oplus \overline{D}1),$$

and the span on the right-hand side may not contain the subspace $0 \oplus \mathbb{C}$. $\square$

As stated in [6, 18] and also in a more general setting, in [1], we can build realizations of NC functions by synthesis, meaning that we can start with simple realizations and construct other realizations of sums, products and inverses of NC functions. The algorithm is similar to the one stated in references mentioned above.

3.1.2 How to construct Realizations of Sums, Products and Inverses?

Given two realizations $(A, b, c) \sim g$ and $(A', b', c') \sim h$ with $A \in \mathcal{B}(\mathcal{H})$ and $A' \in \mathcal{B}(\mathcal{H}')$, there is an algorithm for constructing descriptor realizations of sums, products and inverses as follows [1]. A descriptor realization for $g + h$ is given by (A^+, b^+, c^+) where,

$$A_j^+ := A_j \oplus A'_j, \quad b^+ := \begin{pmatrix} b \\ b' \end{pmatrix}, \quad \text{and} \quad c^+ := \begin{pmatrix} c \\ c' \end{pmatrix}. \quad (3.9)$$

A descriptor realization for $g \cdot h$ is given by $(A^\times, b^\times, c^\times)$, where

$$A_j^\times := \begin{pmatrix} A_j & A_j c b'^* \\ 0 & A' \end{pmatrix}, \quad b^\times := \begin{pmatrix} b \\ c^* b' \end{pmatrix}, \quad \text{and} \quad c^\times := \begin{pmatrix} 0 \\ c' \end{pmatrix}. \quad (3.10)$$

And a descriptor realization for g^{-1} , assuming that $g(0) = b^* c \neq 0$, on the Hilbert space $\hat{\mathcal{H}} := \mathcal{H} \oplus \mathbb{C}$, is given by $(A^{(-1)}, b^{(-1)}, c^{(-1)})$, where $A_j^{(-1)} \in \mathcal{B}(\hat{\mathcal{H}})$, $1 \leq j \leq d$,

$b^{(-1)}, c^{(-1)} \in \hat{\mathcal{H}}$ and,

$$A_j^{(-1)} := \begin{pmatrix} A_j - \frac{1}{b^*c} A_j c b^* & \frac{1}{b^*c} A_j c (0 \oplus 1_{\mathbb{C}}) \\ 0 & 0 \end{pmatrix}, \quad b^{(-1)} := \frac{1}{c^*b} (-b \oplus 1), \quad \text{and} \quad c^{(-1)} := 0 \oplus 1. \quad (3.11)$$

We can also use the same argument to obtain the sum, product and inverse of a Fornasini-Marchesini realization. Let $(A, B, C, D) \sim g$ and $(A', B', C', D') \sim h$ be FM operator realizations on separable, complex Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively. An FM operator realization for $g + h$ is given by (A^+, B^+, C^+, D^+) where,

$$A_j^+ := A_j \oplus A'_j, \quad B_j^+ := \begin{pmatrix} B_j \\ B'_j \end{pmatrix}, \quad C_j^+ := (C_j, C'_j), \quad \text{and} \quad D^+ := D + D'. \quad (3.9)$$

An FM realization for $g \cdot h$ is given by $(A^\times, B^\times, C^\times, D^\times)$, where

$$A_j^\times := \begin{pmatrix} A_j & B_j C'_j \\ 0 & A'_j \end{pmatrix}, \quad B_j^\times := \begin{pmatrix} B_j D' \\ B'_j \end{pmatrix}, \quad C^\times := (C, D C'), \quad \text{and} \quad D^\times := D \cdot D'. \quad (3.10)$$

An FM operator realization for g^{-1} , assuming that $g(0) = D \neq 0$, is given by $(A^{(-1)}, B^{(-1)}, C^{(-1)}, D^{(-1)})$, where

$$A_j^{(-1)} := A_j - \frac{1}{D} B_j C, \quad B_j^{(-1)} := -\frac{1}{D} B_j, \quad C^{(-1)} := \frac{1}{D} C, \quad \text{and} \quad D^{(-1)} := \frac{1}{D}. \quad (3.11)$$

It is well known that these formulas give realizations for sums, products and inverses in the case of finite-dimensional realizations. Here we calculate the inverse realization of the realization of $g \sim (A, B, C, D)$. The fact that Equation (3.11) above gives an FM realization for g^{-1} , for example, can be checked using the Woodbury identity [24], for FM realization:

Woodbury matrix identity: $(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$ given that A^{-1} , C^{-1} , and $(C^{-1} + VA^{-1}U)^{-1}$ exist.

Applying this to the FM realization:

$$\begin{aligned}
g(Z)^{-1} &= \left(D + CL_A(Z)^{-1}Z \otimes B \right)^{-1} \\
&= D^{-1} - D^{-1}C \left(L_A(Z) + Z \otimes BD^{-1}C \right)^{-1} Z \otimes BD^{-1} \\
&= D^{-1} - D^{-1}C \left(I - Z \otimes A + Z \otimes BD^{-1}C \right)^{-1} Z \otimes BD^{-1} \\
&= \frac{1}{D}I_n - \frac{1}{D}C \left(I - \sum_{j=1}^d Z_j \otimes (A_j - \frac{1}{D}B_jC) \right)^{-1} \sum_{j=1}^d \frac{1}{D}Z_j \otimes B_j.
\end{aligned}$$

which, by comparison gives us the Equations (3.11).

Remark 3.2. If (A, B, C, D) is a minimal FM realization so that $D \neq 0$, then it is easy to check that $(A^{(-1)}, B^{(-1)}, C^{(-1)}, D^{(-1)})$ is also minimal, as in [18], section 5.2. Indeed, assuming (A, B, C, D) is minimal, we have:

$$\bigvee_{\substack{\omega \in \mathbb{F}_+^d \\ 1 \leq j \leq d}} A^{(-1)\omega} B_j^{(-1)} \supseteq \bigvee A^\omega B_j = \mathcal{H}, \quad \text{and} \quad \bigvee_{\omega \in \mathbb{F}_+^d} A^{(-1)*\omega} C^{(-1)*} \supseteq \bigvee A^{*\omega} C^* = \mathcal{H}.$$

Definition 3.7. An NC rational expression $\mathfrak{r}$, is any expression by applying $+$, $-$, $\cdot$ and $(\cdot)^{-1}$ to the elements of $\mathbb{C}\langle \mathfrak{z} \rangle$, the ring of free polynomials. The domains of valid NC rational expression is $\text{Dom } \mathfrak{r} = \{X : \mathfrak{r}(X) \text{ is defined}\}$. We consider the equivalence relation defined by $\mathfrak{r}_1 \sim \mathfrak{r}_2$ if $\mathfrak{r}_1(X) = \mathfrak{r}_2(X)$ for all $X \in \text{Dom}(\mathfrak{r}_1) \cap \text{Dom}(\mathfrak{r}_2)$ ($\mathfrak{r}$ is valid if $\text{Dom}(\mathfrak{r}) \neq \emptyset$). An NC rational function is an equivalence class of valid NC rational expressions with respect to this relation.

We can prove here, as an example, that a finite-dimensional realization (A, b, c) (meaning that A is being finite-dimensional), always corresponds to a rational NC function, the equivalence classes of rational expressions with common domain. If $f(Z)$ is an

NC rational function, it means that it is in fact a rational expression applied to d -tuple of matrices Z . By using either of the algorithms above, we can easily show that any NC rational function has a finite-dimensional realization. This is mainly because of how we construct the realization of sums, products and inverses, as represented in at the beginning of this section and it is known as **synthesis** which is introduced in [18]. We know that the FM realization for 1 is given by $(1, 0, 0, 0)$ and the FM realization for Z_k is given by $(0, \delta_{k,j}, 1, 0)$ (or equivalently, the descriptor realization is given by $(0, Z, e_k)$, in which $Z = (Z_1, \dots, Z_d)$ and e_k is the k -th element of the standard orthonormal basis of $\mathbb{C}^d$). This shows us that these realizations are finite dimensional. Moreover, we saw earlier in this section, how we can construct the realizations of sums, products and inverses. Note that if start with 1 or a monomial Z_k , for a rational expression, we should only apply a finite number of these rules to these monomials and therefore, we can conclude that the final result, namely a realization for a rational expression, will be finite dimensional.

Conversely, if (A, b, c) is a finite-dimensional realization of size $m \in \mathbb{N}$, for an NC function f it means that:

$$f(Z) = b^*(I \otimes I_m - Z \otimes A)^{-1} c$$

for such realization with $A \in \mathbb{C}^{(n \times n) \cdot d}$. We can sketch a proof of why this is in fact an NC rational function. As a sketch, we take $n = 2$ and also $d = 2$. Given $Z \in \mathbb{C}^{(m \times m) \cdot d}$, we assume that the equality above holds. Note that here in this case (because $n = d = 2$), we have:

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

in which

$$A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \text{ and } A_2 = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}$$

so that $L_A(Z) = I_m \otimes I_2 - Z_1 \otimes A_1 - Z_2 \otimes A_2$, which is equal to

$$\begin{pmatrix} I_m - Z_1 a_{11} - Z_2 a'_{11} & I_m - Z_1 a_{12} - Z_2 a'_{12} \\ I_m - Z_1 a_{21} - Z_2 a'_{21} & I_m - Z_1 a_{22} - Z_2 a'_{22} \end{pmatrix} \in \mathbb{C}^{m \times m} \otimes \mathbb{C}^{2 \times 2}.$$

Now, using the following Schur complement formula, we want to compute the inverse of the block matrix of $L_A(Z)$:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightarrow M^{-1} = \begin{pmatrix} (M/D)^{-1} & -(M/D)^{-1}BD^{-1} \\ -D^{-1}C(M/D)^{-1} & D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1} \end{pmatrix}$$

in which we assume that the inverse of D and $M/D := A - BD^{-1}C$ exist. Here, we observe that the entries of inverse of the block matrix of $L_A(Z)$ will be a rational functions of Z_1 and Z_2 . By calculating the inverse formula for the other values of n and d , although with more calculations, we see that just similar to this case here, the realizations in the form of $b^*(I \otimes I_m - Z \otimes A)^{-1}c$, fit into the definition of NC rational functions.

Example 3.1. *We would like to provide an example here, to demonstrate that if the realization is finite dimensional, then the corresponding NC function is indeed rational.*

We can take for example:

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So the linear pencil would be:

$$L_A(Z) = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} - \begin{pmatrix} Z_1 & Z_1 \\ 0 & Z_1 \end{pmatrix} - \begin{pmatrix} 0 & Z_2 \\ Z_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} I_n - Z_1 & -(Z_1 + Z_2) \\ -Z_2 & I_n - Z_1 \end{pmatrix}.$$

Now using the Schur complement formula, we have:

$$L_A(Z)^{-1} = \begin{pmatrix} (M/D)^{-1} & \boxed{-(M/D)^{-1}(-Z_1 - Z_2)(I_n - Z_1)^{-1}} \\ -(I_n - Z_1)(-Z_2)(M/D)^{-1} & * \end{pmatrix},$$

in which (M/D) is defined as:

$$(M/D) = (I_n - Z_1) - (Z_1 + Z_2)(I_n - Z_1)^{-1}Z_2.$$

Therefore, the NC function corresponding to the realization would be:

$$h(Z) = I_n \otimes (1, 0)L_A(Z)^{-1} \begin{pmatrix} 0 \\ I_n \end{pmatrix} = -(M/D)^{-1}(-Z_1 - Z_2)(I_n - Z_1)^{-1}.$$

This clearly demonstrates that the NC function h is a rational NC function.

3.1.3 Uniqueness of Minimal Realizations

With respect to specific transformations, minimal realizations can be considered to be unique.

Definition 3.8 (Closed and closeable operators). Recall that a linear map $L : \text{Dom } L \subseteq \mathcal{H} \rightarrow \mathcal{J}$, with dense linear domain, $\text{Dom } L$, in a Hilbert space, $\mathcal{H}$, and range in a Hilbert space, $\mathcal{J}$, is said to be **closed** if its graph is a closed subspace of $\mathcal{H} \oplus \mathcal{J}$. Such a linear map is **closeable** if it has a closed extension, $\widehat{L}$, i.e. $\text{Dom } L \subseteq \text{Dom } \widehat{L} \subseteq \mathcal{H}$ and $\widehat{L}|_{\text{Dom } L} = L$.

If L is closeable, its closure, $\bar{L}$, is defined as its minimal closed extension, which can be obtained by taking the closure of the graph of L in $\mathcal{H} \oplus \mathcal{J}$.

A subset, $\mathcal{D} \subseteq \text{Dom } L$ of the domain of a closed linear operator, L , is called a **core** for L if $(\mathcal{D}, L\mathcal{D})$ is dense in the graph of L . In particular, if L is closeable with closure $\bar{L}$, then $\text{Dom } L$ is a core for $\bar{L}$ (See [20], Section VI.2).

Remark 3.3. In the definition above, we write $L \subseteq \hat{L}$ and say that L is a restriction of $\hat{L}$. In particular, L is closeable if and only if the closure of its graph in $\mathcal{H} \oplus \mathcal{J}$ does not contain any vectors of the form, $0 \oplus y$ for $y \neq 0$. Equivalently, L is closeable if and only if whenever $x_n \in \text{Dom } L$, $x_n \rightarrow 0$ and $y_n = Lx_n \rightarrow y \in \mathcal{J}$, then necessarily $y = 0$. One can prove that L^* is densely-defined if and only if L is closeable, in which case $\bar{L} = L^{**}$, i.e. the bi-adjoint is the minimal closed extension of L .

Definition 3.9. Let $X : \mathcal{H} \rightarrow \mathcal{J}$ be a closed, densely-defined linear map on $\text{Dom } X \subseteq \mathcal{H}$. We say that X is a **pseudo-affinity**, if X is injective and has a dense range. Suppose that $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{J})$ are bounded linear maps such that $A \text{Dom } X \subseteq \text{Dom } X$ and $XA \subseteq BX$, then we say that A and B are pseudo-similar.

The following uniqueness theorem is stated in [1] and we refer to that paper for the proof. A similar statement can be proved for FM realizations.

Theorem 3.1 (Uniqueness of minimal realizations). *Minimal realizations are unique up to unique pseudo-similarity. That is, if (A, b, c) and (A', b', c') are two minimal realizations defined on $\mathcal{H}$ and $\mathcal{H}'$, respectively, for the same NC function, h , then there is a closed, injective and densely-defined linear transformation, $S : \text{Dom } S \subseteq \mathcal{H} \rightarrow \mathcal{H}'$, with dense range so that for all $p, q \in \mathbb{C}\langle \mathfrak{z} \rangle$,*

- (i) *The non-closed linear span of $(\mathbb{C}\langle A \rangle_c, S\mathbb{C}\langle A \rangle_c)$ is dense in the graph of S and $S(p(A)c) = p(A')c$.*
- (ii) *The non-closed linear span of $(\mathbb{C}\langle A'^* \rangle_c, S\mathbb{C}\langle A'^* \rangle_c)$ is dense in the graph of S^* and $S^*q(A'^*)b' = q(A^*)b$.*

Given an arbitrary realization (A, b, c) of an NC function h defined on $\mathcal{H}$, we can always find the minimal realization associated to h by compressing to a certain “semi-invariant subspace” [1].

Suppose $\mathcal{J}$ is a Hilbert space and Σ is a semigroup of operators on $\mathcal{J}$. If $\mathcal{H}$ is a subspace of $\mathcal{J}$ and Q is the orthogonal projection onto $\mathcal{H}$, then $\mathcal{H}$ is called **semi-invariant under** Σ provided $QW_1QW_2Q = QW_1W_2Q$ for all W_1 and W_2 in Σ . When this happens, the family of operators $\{QW|_{\mathcal{H}} : W \in \Sigma\}$ is a semigroup—called the *projection* of Σ onto $\mathcal{H}$ —and the natural map of Σ onto its projection is a homomorphism [22].

Let $(A, b, c) \sim h$ be any descriptor realization of h on $\mathcal{H}$. Define the **minimal space** of (A, b, c) as the subspace

$$\mathcal{M}_{A,b,c} := \mathcal{C}_{A,c} \ominus \mathcal{O}_{A^*,b}^\perp \cap \mathcal{C}_{A,c}.$$

Then, $\mathcal{M}_{A,b,c}$ is the direct difference of the nested, A -invariant subspaces, $\mathcal{C}_{A,c}$ and $\mathcal{O}_{A^*,b}^\perp \cap \mathcal{C}_{A,c}$, and is hence *semi-invariant* for A . That is, if $Q_0 : \mathcal{H} \rightarrow \mathcal{M}_{A,b,c}$ denotes the orthogonal projection, then

$$Q_0 A^\omega Q_0 = (Q_0 A Q_0)^\omega, \quad \omega \in \mathbb{F}_d^+,$$

so that the compression to $\mathcal{M}_{A,b,c}$ is a surjective, unital homomorphism of the unital algebra generated by A , $\text{Alg}\{I, A_1, \dots, A_d\}$, onto $\text{Alg}\{I, A_1^{(0)}, \dots, A_d^{(0)}\}$ where $A^{(0)} := Q_0 A|_{\mathcal{M}_{A,b,c}}$.

Theorem 3.2 (Kalman decomposition). *Let (A, b, c) be any descriptor realization of h on $\mathcal{H}$, and define the minimal space, $\mathcal{M}_{A,b,c}$, as in Equation (3.8) above, with projector Q_0 . Then setting $A^{(0)} := Q_0 A|_{\mathcal{M}_{A,b,c}}$, $b_0 := Q_0 b$ and $c := Q_0 c$, $(A^{(0)}, b_0, c_0)$ is a minimal realization of h .*

Chapter 4

Operator Realization around a matrix-centre

We have seen before that an NC function f is uniformly analytic in a uniformly open neighbourhood of $0 := (0, \dots, 0) \in \mathbb{C}^{1 \times d}$ if and only if it has a realization $f \sim (A, b, c)$. For an NC function f , with the realization $f \sim (A, b, c)$, the **moments** of the realization are $b^* A^\omega c = \widehat{f}_\omega$ with $\widehat{f}_\omega$ being the Taylor-Taylor coefficients of f at 0. Hence, our motivation in this section is the question of whether an analytic NC function f about a matrix centre $Y \in \mathbb{C}^{(m \times m) \cdot d}$, has a *realization centred at Y* , which somehow determines the Taylor-Taylor series coefficients of f at Y .

In the case of functions with one variable, we can see that if f is holomorphic in an open neighbourhood of $y \in \mathbb{C}$, it has a “realization centred at y ”, $f \sim_y (A, b, c)$ in the sense that

$$\begin{aligned} f(z) &= b^* (I - (z - y)A)^{-1} c \\ &= \sum_{j=0}^{\infty} b^* A^j c (z - y)^j \end{aligned}$$

in which, the term $\widehat{f}_j := b^* A^j c$ is the Taylor coefficient of the function f at the point

y . We can prove this as follows: let $g(z) := f(z + y)$ so that g would be a holomorphic function in a disk centred at the origin of a given non-zero radius. Therefore, by the results of the previous section, g has a realization, lets say $g \sim (A, b, c)$. So, we can write:

$$\begin{aligned} f(z) &= g(z - y) = b^* (I - (z - y)A)^{-1} c \\ &= b^* (L_A(y)(I - zL_A(y)^{-1}A))^{-1} c, \quad \boxed{L_A(y) := I - yA} \\ &= b^* (I - zL_A(y)^{-1}A)^{-1} L_A(y)^{-1} c. \end{aligned}$$

Therefore, we can consider the realization of f at y , a translation of the realization of $g \sim (A, b, c)$ explicitly written as $f \sim_y (L_A(y)^{-1}A, b, L_A(y)^{-1}c)$.

Similar to the case of one complex variable above, if an NC function $f \sim (A, b, c)$, with $A \in \mathcal{B}(\mathcal{H})^d$ and $Y \in \mathbb{C}^{(m \times m) \cdot d}$, $Y \in \mathcal{D}_m(A)$, we can construct a matrix centre realization of f about y , $f \sim_Y (\hat{A}, \hat{b}, \hat{c})$ using a similar “translation process”.

Definition 4.1 (Completely bounded norm for column d -tuples). *For a d -tuple of completely bounded linear maps:*

$$\begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} : \mathbb{C}^{m \times m} \otimes \mathbb{C}^{1 \times d} \longrightarrow \mathcal{B}(\mathcal{H}).$$

i.e.

$$\begin{pmatrix} A_1 \\ \vdots \\ A_d \end{pmatrix} \circ (Z_1, \dots, Z_d) = \sum_{j=1}^d A_j(Z_j) =: A(Z).$$

We denote by $\|(A_1, \dots, A_d)^T\|_{\text{CB}}$, the column cb-norm of this linear map, which is defined as in Definition (2.4).

Definition 4.2. Let $Y \in \mathbb{C}^{(m \times m) \cdot d}$. A **descriptor matrix-centre realization** of Y , is a

triple $(A, b, c)_Y$, where $A = (A_1, \dots, A_d)$ is a d -tuple of linear maps,

$$A_j \in \mathcal{B}(\mathbb{C}^{m \times m}, \mathcal{B}(\mathcal{H})),$$

with $\mathcal{H}$ being a complex, separable Hilbert space and $b, c \in \mathcal{B}(\mathbb{C}^m, \mathcal{H})$. We define the **linear pencil** as a function $L_A(X)$ given as:

$$L_A(X) := I_{\mathcal{H}} - \sum_{j=1}^d A_j(X_j).$$

The **invertibility domain** of the realization is defined as:

$$\mathcal{D}^Y(A) := \bigsqcup_{n=1}^{\infty} \mathcal{D}_n^Y(A).$$

such that

$$\mathcal{D}_n^Y(A) := \{X \in \mathbb{C}^{(mn \times mn) \cdot d} \mid L_A(X - I_n \otimes Y)^{-1} \text{ exists} \}$$

Definition 4.3. If for any $X \in \mathcal{D}_s^Y(A)$,

$$f(X) = I_s \otimes b^* L_A(X - I_s \otimes Y)^{-1} I_s \otimes c.$$

We say that f is a **quantized transfer function** of $(A, b, c)_Y$ and write $f \sim_Y (A, b, c)$.

Remark 4.1. Note that the quantized transfer function may not be an NC function as it may not obey the joint similarity condition [13]. It is however, always graded and respects the direct sum. However, quantized transfer function is uniformly bounded, as it is a convergent geometric series with a positive radius of convergence.

Remark 4.2. It is a result of Paulsen that for operators $\hat{A}_j : \mathbb{C}^{n \times n} \rightarrow \mathcal{B}(\mathcal{H})$, from a finite-dimensional vector space to $\mathcal{B}(\mathcal{H})$, boundedness implies complete boundedness [See Theorem (2.4)].

Our emphasis on complete boundedness here, stems from the need to analyze the Taylor–Taylor expansion of the realization and, in particular, to control its Hadamard radius

of convergence (2.13). When studying realizations of the NC functions, each coefficient in the corresponding Taylor–Taylor series involves multilinear maps acting on matrix tuples of the form $b^* A^\omega c$. Ensuring that these maps are completely bounded guarantees uniform control of their norms at all matrix levels as discussed following Definition (2.5).

Remark 4.3. Since each $A_j \in \mathcal{B}(\mathbb{C}^{n \times n}, \mathcal{B}(\mathcal{H}))$ is a linear map on a finite-dimensional operator space $\mathbb{C}^{n \times n}$, each A_j is completely bounded [16]. Therefore, one can consider A as a completely bounded linear map from the column operator space (with d components) over $\mathbb{C}^{n \times n}$ into $\mathcal{B}(\mathcal{H})$, $A \in \text{CB}(\mathbb{C}^{(n \times n) \cdot d}, \mathcal{B}(\mathcal{H}))$ and define :

$$\|A\|_{\text{CB}} := \|(A_1, \dots, A_d)^T\|_{\text{CB}}.$$

Proposition 4.1. The set $\mathcal{D}^Y(A)$ is a uniformly open NC set.

Proof. First, it is clear that $\mathcal{D}^Y(A)$ is closed under ‘ $\oplus$ ’ and therefore by definition it is an NC set. Suppose that $Z \in \mathcal{D}^Y(A)$, which means that $L_A(Z - I_n \otimes Y)^{-1}$ exists. If W is in an open uniform ball of radius $\delta > 0$ around Z , we have :

$$\|(Z - I_n \otimes Y) - (W - I_n \otimes Y)\|_{\text{row}} = \|Z - W\|_{\text{row}} < \delta.$$

Furthermore, this means that :

$$\|L_A(Z - I_n \otimes Y) - L_A(W - I_n \otimes Y)\| < \delta \cdot \|A\|_{\text{CB}}$$

$$\implies \|I - L_A(Z - I_n \otimes Y)^{-1} L_A(W - I_n \otimes Y)\| < \delta \cdot \|A\|_{\text{CB}} \cdot \|L_A(Z - I_n \otimes Y)^{-1}\|$$

If $\delta > 0$ is taken to be sufficiently small, i.e. $\delta < \frac{1}{\|A\|_{\text{CB}} \cdot \|L_A(Z - I_n \otimes Y)^{-1}\|}$, then we have:

$$\|I - L_A(Z - I_n \otimes Y)^{-1} L_A(W - I_n \otimes Y)\| < 1.$$

This implies that:

$$I - (I - L_A(Z - I_n \otimes Y)^{-1} L_A(W - I_n \otimes Y)) = L_A(Z - I_n \otimes Y)^{-1} L_A(W - I_n \otimes Y)$$

is invertible. Therefore $L_A(W - I_n \otimes Y)$ is invertible (See Lemma (3.1)), and therefore $W \in \mathcal{D}^Y(A)$. This shows that any uniformly open NC row ball of sufficiently small radius around any point in $\mathcal{D}^Y(A)$, is contained in $\mathcal{D}^Y(A)$. $\square$

Theorem 4.1. *For every X in the NC open row ball centred at Y , of radius of $\|(A_1, \dots, A_d)^T\|_{\text{CB}}^{-1}$ (or in other words, for any X with $\|X - I_s \otimes Y\| < \|(A_1, \dots, A_d)^T\|_{\text{CB}}^{-1}$):*

$$\begin{aligned} L_A(X - I_s \otimes Y)^{-1} &= \sum_{\ell=0}^{\infty} A(X - I_s \otimes Y)^\ell \\ &= \sum_{\omega \in \mathbb{F}_d^+} A^{(\omega)}(X - I_s \otimes Y), \end{aligned}$$

where $A^{(\omega)}(X - I_s \otimes Y) = A_{i_1}(X_{i_1} - I_s \otimes Y_{i_1}) \dots A_{i_d}(X_{i_d} - I_s \otimes Y_{i_d})$ if $\omega = i_1 \dots i_d \in \mathbb{F}_d^+$, $i_j \in \{1, \dots, d\}$.

Proof. Expanding the linear pencil $L_A(X - I_s \otimes Y)$, we have:

$$L_A(X - I_s \otimes Y) = I_m \otimes I_{\mathcal{H}} - A(X - I_s \otimes Y) = I_m \otimes I_{\mathcal{H}} - \sum_{j=0}^d A_j(X_j - I_s \otimes Y_j).$$

The convergence criteria of the series

$$L_A(X - I_s \otimes Y)^{-1} = \sum_{\ell=0}^{\infty} A(X - I_s \otimes Y)^\ell$$

is that

$$\|A(X - I_s \otimes Y)\| \leq \|(A_1, \dots, A_d)^T\|_{\text{CB}} \|X - I_s \otimes Y\|_{\text{row}} < 1.$$

If $\|X - I_s \otimes Y\|_{\text{row}} < \frac{1}{\|(A_1, \dots, A_d)^T\|_{\text{CB}}} \text{ for } A = (A_1, \dots, A_d) \in \text{CB}(\mathbb{C}^{m \times m} \otimes \mathbb{C}^{1 \times d}, \mathcal{B}(\mathcal{H}))$,

the series above converges and therefore the inverse of the linear pencil exists.

It is readily checked that:

$$A(X - I_s \otimes Y)^\ell = \sum_{|\omega|=\ell} A^\omega(X - I_s \otimes Y), \quad \text{such that } \omega \in \mathbb{F}_+^d,$$

where if $\omega = i_1 i_2 \dots i_k$:

$$A^\omega(X - I_s \otimes Y) := A_{i_1}(X_{i_1} - I_s \otimes Y_{i_1}) \dots A_{i_k}(X_{i_k} - I_s \otimes Y_{i_k})$$

we can write the expanded geometric series of the linear pencil as :

$$L_A(X - I_s \otimes Y)^{-1} = \sum_{\omega \in \mathbb{F}_+^d} A^\omega(X - I_s \otimes Y).$$

The same criteria is also sufficient for the existence of the inverse of $L_A(X - I_s \otimes Y)$ and the convergence of the expanded geometric series above. $\square$

Given the theorem above, we can now connect the terms of the Taylor-Taylor series of an NC function at a given point in its domain of definition, $Y \in \mathbb{C}^{m \times m}$. Note that:

$$f(X) = I_m \otimes b^* L_A(X)^{-1} I_m \otimes c = \sum_{\omega \in \mathbb{F}_+^d} b^* A^\omega(X - I_s \otimes Y) c, \quad (4.1)$$

and that $b^* A^\omega(X - I_s \otimes Y) c$ are $|\omega|$ -linear maps acting on $\underbrace{(X - I_s \otimes Y, \dots, X - I_s \otimes Y)}_{d\text{-times}}$.

By the uniqueness theorem in Chapter 2, Theorem (2.6) and its version for series indexed by free monoid (See page 64), and Equation (2.12), we have that the series (4.1), is equal to the Taylor - Taylor expansion of f at a matrix centre Y . In other words, we obtained the following:

Lemma 4.1. *Let $Y \in \text{Dom}_m f$, where f is an NC function. If $f \sim_Y (A, b, c)$, then the Taylor–Taylor series of f at Y is given by the multilinear partial difference-differential maps,*

$$\Delta_{X-I_s \otimes Y}^{\omega^t} f(Y) = I_s \otimes b^* A^\omega (X - I_s \otimes Y) I_s \otimes c. \quad (4.2)$$

where $\omega = i_1 \cdots i_k \in \mathbb{F}_+^d$ and $X - I_s \otimes Y \in \mathbb{C}^{(sm \times sm) \cdot d}$.

That is, if the Taylor–Taylor series of $f \sim_Y (A, b, c)$ centred at $Y \in \text{Dom}_m f$ converges at $X \in \mathbb{C}^{(sm \times sm) \cdot d}$, then

$$f(X) = \sum_{\omega \in \mathbb{F}_+^d} \Delta_{X-I_s \otimes Y}^{\omega^t} f(Y),$$

where the terms of this series are given by Equation (4.2).

In the next theorem, we consider the relation between Lost-Abbey conditions and realizations. Note that in all of the results presented so far, the assumption is that the NC function f is defined at the origin, and then we build realization with this assumption. Recall that the quantized transfer function of a matrix-centre realization at Y need not to be an NC function. It will be an NC function if and only if it obeys the Lost-Abbey conditions:

Theorem 4.2. *Consider the series:*

$$f(X) := \sum_{\ell=0}^{\infty} f_\ell(\underbrace{X - I_m \otimes Y, \dots, X - I_m \otimes Y}_{\ell \text{ times}}), \quad (4.3)$$

with $Y \in \mathbb{C}^{(s \times s) \cdot d}$ and $f_\ell : (\mathbb{C}^{(s \times s) \cdot d})^\ell \rightarrow \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$, for $\ell \in \mathbb{N} \cup \{0\}$, being ℓ -linear maps and $X \in \mathbb{C}^{(sm \times sm) \cdot d}$. Then $f \sim_Y (A, b, c)$ is a uniformly analytic NC function on the uniformly open NC set $\mathcal{D}^Y(A)$, if and only if the $|\omega|$ -linear maps:

$$f_\ell(\cdot, \dots, \cdot) = \sum_{|\omega|=\ell} b^* A^\omega(\cdot, \dots, \cdot) c, \quad \ell \in \mathbb{N} \cup \{0\},$$

obey the Lost-Abbey conditions.

Proof. If we assume that f is a uniformly analytic NC function, using the facts that the linear maps $b^* A^\omega(\cdot, \dots, \cdot) c$, are the terms of the Taylor-Taylor series expansion of the NC function f around Y (because of Lemma (4.1)), we can deduce from Proposition (2.3), that the $|\omega|$ -linear maps f_ℓ obeys the Lost-Abbey conditions.

If $f \sim_Y (A, b, c)$ obey Lost-Abbey conditions at matrix point Y , then it is uniformly bounded on a $r \cdot \mathbb{B}_{s\mathbb{N}}^d(Y)$ for any $r < \|A\|_{\text{CB}}$. Hence, since uniform boundedness implies uniform analytic property, it follows that all the partial derivatives of f exists and hence the series $\sum_{\ell=0}^{\infty} f_\ell(\underbrace{X - I_m \otimes Y, \dots, X - I_m \otimes Y}_{\ell \text{ times}})$ defines a unique NC function f which is uniformly analytic on its domain.

□

4.0.1 Minimality of Descriptor Realizations around a Matrix Centre

For a realization around a matrix centre, controllability and observability can be defined in a similar way to the definitions presented in the first section of this Chapter, both for descriptor and FM realizations. Moreover, the translation of realizations here is derived from the arguments in [1] for the finite-dimensional matrix-centred realizations of NC rational functions, in which the same propositions are stated for FM realizations.

Notation: If $\omega = i_1 \cdots i_{|\omega|}$, let $A^\omega : \mathbb{C}^{m \times m} \otimes \mathbb{C}^{1 \times |\omega|} \rightarrow \mathcal{B}(\mathcal{H})$ be the $|\omega|$ -multilinear map defined as:

$$A^\omega(G_1, \dots, G_{|\omega|}) := A_{i_1}(G_1)A_{i_2}(G_2) \cdots A_{i_{|\omega|}}(G_{|\omega|}) \in \mathcal{B}(\mathcal{H}), \quad G_i \in \mathbb{C}^{n \times n}.$$

Also define the $|\omega|$ -linear map, $A^{\omega^\dagger}$ as:

$$A^{\omega^\dagger}[\vec{G}] = \left(A^\omega[\vec{G}^*] \right)^*, \quad \vec{G} := (G_1, \dots, G_{|\omega|}); G_i \in \mathbb{C}^{n \times n},$$

where $\vec{G}^*$ denotes component-wise adjoint.

Definition 4.4. [13] Given any realization, $(A, b, c)_Y$, around a matrix centre $Y \in \mathbb{C}^{(n \times n) \cdot d}$, we define the **controllable subspace**,

$$\mathcal{C}_{A,c} := \bigvee_{\omega \in \mathbb{F}_d^+} \text{Ran } A^\omega c,$$

where, if $\omega = i_1 \cdots i_{|\omega|}$, then $A^\omega : \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times |\omega|} \rightarrow \mathcal{B}(\mathcal{H})$ is the $|\omega|$ -multilinear completely bounded map defined as before:

$$A^\omega(G_1, \dots, G_{|\omega|}) := A_{i_1}(G_1)A_{i_2}(G_2) \cdots A_{i_{|\omega|}}(G_{|\omega|}) \in \mathcal{B}(\mathcal{H}), \quad G_i \in \mathbb{C}^{n \times n}.$$

The realization, $(A, b, c)_Y$, is said to be **controllable** if $\mathcal{H} = \mathcal{C}_{A,c}$. Similarly, we define the **observable subspace**,

$$\mathcal{O}_{A^\dagger, b} := \bigvee_{\omega \in \mathbb{F}_d^+} \text{Ran } A^{\omega\dagger} b,$$

where, if $\omega = i_1 \cdots i_{|\omega|}$ and $G_i \in \mathbb{C}^{n \times n}$ then

$$A^{\omega\dagger}(G_1, \dots, G_{|\omega|}) := A_{i_{|\omega|}}(G_{|\omega|}^*)^* A_{i_{|\omega|-1}}(G_{|\omega|-1}^*)^* \cdots A_{i_1}(G_1^*)^* \in \mathcal{B}(\mathcal{H}).$$

As before, $(A, b, c)_Y$ is said to be **observable**, if $\mathcal{O}_{A^\dagger, b} = \mathcal{H}$.

Note that here, we define A^ω as a $|\omega|$ -linear map on $|\omega|$ -tuples of $n \times n$ matrices, whereas earlier in our discussion of Taylor–Taylor series of a function with a matrix-centre realization, we defined a similar $|\omega|$ -linear map on $|\omega|$ -tuples of elements $H^{(i)}$ of $\mathbb{C}^{(n \times n) \cdot d}$, $\hat{A}^\omega$, which is related to A^ω by

$$\hat{A}^\omega[H^{(1)}, \dots, H^{(\omega)}] = A^\omega[H_{i_1}^{(1)}, \dots, H_{i_{|\omega|}}^{(|\omega|)}].$$

Hence,

$$\text{Ran } A^\omega c = \bigvee_{\substack{G_i \in \mathbb{C}^{n \times n}, \\ v \in \mathbb{C}^n}} A_{i_1}(G_1) \cdots A_{i_{|\omega|}}(G_{|\omega|}) c(v).$$

Definition 4.5 (Minimal realization around a matrix centre). *A realization $(A, b, c)_Y$ is said to be **minimal**, if it is both controllable and observable.*

4.1 New Results on Realization theory around a matrix centre

In this section, we present several results that extend the realization theory of NC functions and allow us to build matrix-centre realizations of more general NC functions. Namely, we can build realizations of an NC rational function $\mathfrak{r}$ in m NC variables composed with m -tuples of NC functions $g_1, \dots, g_m$ that have realizations at 0. Our results in this section provides a way of building realizations for such functions, $f := \mathfrak{r} \circ (g_1, \dots, g_m)$, even if f is not defined at 0. Recall from Definition (2.7) that the set of all NC polynomials in d non-commuting variables $\mathfrak{z} = (\mathfrak{z}_1, \dots, \mathfrak{z}_d)$ is called **free algebra** and is denoted by $\mathbb{C}\langle \mathfrak{z} \rangle$.

Definition 4.6 ([1, 13]). *Given any $p \in \mathbb{C}\langle \mathfrak{z} \rangle$,*

$$p = \sum_{|\omega| \leq N} \hat{p}_\omega \mathfrak{z}^\omega,$$

we can view $p(A)$ as a multi-affine linear map in $n_p := \sum_{|\omega| \leq N; \hat{p}_\omega \neq 0} |\omega|$ – arguments in $\mathbb{C}^{n \times n}$ into $\mathcal{B}(\mathcal{H})$.

Namely, given two words, $\alpha, \omega \in \mathbb{F}_d^+$ and $c \in \mathbb{C}$, we define:

$$A^\alpha + cA^\omega : \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times (|\alpha| + |\omega|)} \longrightarrow \mathcal{B}(\mathcal{H}),$$

in the obvious way:

$$(A^\alpha + cA^\omega)(G_1, \dots, G_{|\alpha|}, H_1, \dots, H_{|\omega|}) := A^\alpha(G) + cA^\omega(H),$$

for $G = (G_1, \dots, G_{|\alpha|})$ and $H = (H_1, \dots, H_{|\omega|})$, $G_i, H_j \in \mathbb{C}^{n \times n}$.

For simplicity of notation, when we view $p(A)$ for $p \in \mathbb{C}\langle \mathfrak{z} \rangle$ as a multi-affine linear map in n_p copies of $\mathbb{C}^{n \times n}$, we will often simply write

$$p(A)[\vec{G}]$$

for the image of $\vec{G} := (G_1, \dots, G_{n_p}) \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times n_p}$ under $p(A)$.

Definition 4.7. Suppose we have an NC polynomial of the form:

$$p(A) = \sum_{|\omega| \leq N} \hat{p}_\omega A^\omega,$$

The **conjugate** of this NC polynomial is defined as:

$$\bar{p}(A) = \sum_{|\omega| \leq N} \overline{\hat{p}_\omega} A^\omega,$$

where $\overline{\hat{p}_\omega}$ is the complex conjugate of the complex number $\hat{p}_\omega$.

Similar to the theorems stated in the previous section of this chapter, we aim to extend Theorems (3.2) and (3.1) to matrix-centre realizations. We start with an analogue of Theorem (3.1) for this case.

Definition 4.8 (Analytic equivalence realizations around matrix point Y ; See [1]). Let $A \in (A_1, \dots, A_d)^T$ and $A' \in (A'_1, \dots, A'_d)^T$ be two column d -tuples of linear maps $A_j : \mathbb{C}^{n \times n} \rightarrow \mathcal{B}(\mathcal{H})$ and $A'_j : \mathbb{C}^{n \times n} \rightarrow \mathcal{B}(\mathcal{H}')$. We say that A and A' are **analytically equivalent around Y** and denote it by $(A, b, c)_Y \sim (A', b', c')_Y$, if there exist $b, c \in \mathcal{B}(\mathbb{C}^n, \mathcal{H})$ and $b', c' \in \mathcal{B}(\mathbb{C}^n, \mathcal{H}')$ so that the realizations $(A, b, c)_Y$ and $(A', b', c')_Y$ are both minimal and define the same quantized transfer function in a uniformly open neighbourhood of $Y \in \mathbb{C}^{m \times m}$. That is, A and A' are analytically equivalent if there exists

an $r > 0$ so that

$$b^* L_A (Z - I_s \otimes Y)^{-1} c = b'^* L_{A'} (Z - I_s \otimes Y)^{-1} c', \quad \forall Z \in r \cdot \mathbb{B}^{(s\mathbb{N} \times s\mathbb{N}) \cdot d},$$

or equivalently (See [13], Lemma (2.7) for a proof),

$$b^* A^\omega c = b'^* A'^\omega c' \quad \forall \omega \in \mathbb{F}^d.$$

If $(A, b, c)_Y \sim (A', b', c')_Y$, where $A' \in \text{CB}(\mathbb{C}^{n \times n}, \mathcal{B}(\mathcal{H}'))^d$, we observe that :

$$\mathcal{D}^Y(A) \cap \mathcal{D}^Y(A') \supseteq r \cdot \mathbb{B}_{mN}^d(Y), \text{ where } r := \min \left\{ \|A\|_{\text{CB}; \text{col}}^{-1}, \|A'\|_{\text{CB}; \text{col}}^{-1} \right\},$$

$$\|A\|_{\text{CB}; \text{col}} := \|(A_1, \dots, A_d)^T\|_{\text{CB}}.$$

Recall that the definitions for the following theorem are stated at the beginning of Section (3.1.1):

Theorem 4.3. *Suppose that $(A, b, c)_Y \sim (A', b', c')_Y$ are minimal analytically equivalent realizations at Y . Then there is a unique pseudo-similarity, $S : \text{Dom } S \subset \mathcal{H} \rightarrow \mathcal{H}'$ so that for all $p, q \in \mathbb{C}\langle \mathfrak{z} \rangle$:*

- (i) $\text{Ran } \mathbb{C}\langle A \rangle c$ is a core for S and $Sp(A)[\vec{G}]c(v) = p(A')[\vec{G}']c'(v)$ for all $\vec{G} \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times n_p}$ and $v \in \mathbb{C}^n$.
- (ii) $\text{Ran } \mathbb{C}\langle A'^\dagger \rangle b'$ is a core for S^* and $S^*q(A)^\dagger[\vec{G}]b'(u) = q(A)^\dagger b(u)$ for all $\vec{G} = (G_1, \dots, G_{n_q}), G_i \in \mathbb{C}^{n \times n}$ and $u \in \mathbb{C}^n$.

Proof. We need to first check that S is a well-defined linear map. Namely, if $p(A)[\vec{G}]c(v) = 0 \in \mathcal{H}$, we need to show that $p(A')[\vec{G}]c'(v) = 0 \in \mathcal{H}'$. For any $q \in \mathbb{C}\langle z \rangle$, and any $\vec{H} = (H_1, \dots, H_{n_q}), H_j \in \mathbb{C}^{m \times m}$ and $u \in \mathbb{C}^m$, since $(A, b, c)_Y \sim (A', b', c')_Y$, we have :

$$\begin{aligned} 0 &= b(u)^* q(A)[\vec{H}^*]p(A)[\vec{G}]c(v) = \langle q(A)^\dagger[\vec{H}]b(u), p(A)[\vec{G}]c(v) \rangle_{\mathcal{H}} \\ &= b'(u)^* q(A')[\vec{H}^*]p(A')[\vec{G}]c'(v) = \langle q(A')^\dagger[\vec{H}]b'(u), p(A')[\vec{G}]c'(v) \rangle_{\mathcal{H}'}. \end{aligned}$$

However, since $(A', b', c')_Y$ is assumed to be minimal,

$$\mathcal{H}' = \mathcal{O}_{A'^\dagger, b'} = \bigvee_{q \in \mathbb{C}\langle z \rangle} \text{Ran } q(A')^\dagger b',$$

and we conclude that $p(A')[\vec{G}]c'(v) = 0$, as required.

We want to prove that S is closeable. We need to show that $p_n(A)[\vec{G}]c(v) \xrightarrow{n \rightarrow \infty} 0$ and $Sp_n(A)[\vec{G}]c(v) \xrightarrow{n \rightarrow \infty} y'$ imply $y' = 0$.

Since the linear transformation $Lv = q(A')[\vec{H}]v$ is defined everywhere, $\bar{q}(A')[\vec{H}]b'(u)$ (See Definition (4.7) and (4.4), for the notation $q(A')^\dagger$) has the property that:

$$\begin{aligned} \langle q(A'^\dagger)[\vec{H}]b'(u), y' \rangle &= \lim_{n \rightarrow \infty} \langle b'(u), \bar{q}(A')[\vec{H}^*]p_n(A')[\vec{G}]c(v) \rangle_{\mathcal{H}'} \quad \boxed{\vec{H}^* := (H_1^*, \dots, H_{n_q}^*)} \\ &= \lim_{n \rightarrow \infty} \langle b(u), \bar{q}(A)[\vec{H}^*]p_n(A)[\vec{G}]c(v) \rangle_{\mathcal{H}} \\ &= \lim_{n \rightarrow \infty} b^*(u) \bar{q}(A)[\vec{H}^*]p_n(A)[\vec{G}]c(v) = 0 \end{aligned}$$

and because of controllability of (A', b', c') it implies that $y' = 0$. This suffices to prove that S is closeable.

Next we want to show that S is injective. First, let us prove that $Sp(A)[\vec{H}]c(v) = 0$ then $p(A)[\vec{H}]c(v) = 0$. Because (A, b, c) is controllable, we can conclude that for every $h \in \mathcal{H}$ if $Sh = 0$ implies $h = 0$. So, our assumption is that

$$Sp(A)[\vec{H}]c(v) = p(A')[\vec{H}]c'(v) = 0.$$

On the other hand, by equivalence of realizations, the following equality is true:

$$b^* A^\omega [\vec{W}]c = b'^* A'^\omega [\vec{W}]c', \quad \forall \vec{W} = (W_1, \dots, W_{|\omega|}) \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times |\omega|},$$

so we have for every $q \in \mathbb{C}\langle \mathfrak{z} \rangle$

$$\begin{aligned} p(A')[\vec{H}]c'(v) = 0 &\implies b^*q(A')[\vec{G}]p(A')[\vec{H}]c'(v) = 0 \\ &\iff \langle q(A'^\dagger)[\vec{G}]b, p(A')[\vec{H}]c'(v) \rangle_{\mathcal{H}'} = 0 \\ &\iff \langle q(A^\dagger)[\vec{G}]b, p(A)[\vec{H}]c(v) \rangle_{\mathcal{H}} = 0 \end{aligned}$$

which implies that $p(A)[\vec{H}]c(v) = 0$.

Additionally, we claim that $\mathbb{C}\langle (A')^\dagger \rangle b'(u) \subseteq \text{Dom } S^*$ and that $S^*q(A'^\dagger)[\vec{G}]b'(u) = q(A)[\vec{G}]b(u)$ for any $q \in \mathbb{C}\langle \mathfrak{z} \rangle$. For every $p \in \mathbb{C}\langle \mathfrak{z} \rangle$:

$$\begin{aligned} \langle q(A'^\dagger)[\vec{G}]b'(u), Sp(A')[\vec{H}]c'(v) \rangle &= \langle b'(u), \bar{q}(A')[\vec{G}]Sp(A')[\vec{H}]c'(v) \rangle_{\mathcal{H}'} \\ &= \langle b(u), \bar{q}(A)[\vec{G}]Sp(A)[\vec{H}]c(v) \rangle_{\mathcal{H}} \\ &= \langle q(A^\dagger)[\vec{G}]b(u), Sp(A)[\vec{H}]c(v) \rangle_{\mathcal{H}}. \end{aligned}$$

Therefore, by the definition of the Hilbert space adjoint of a densely-defined linear map $S^*q(A'^\dagger)[\vec{G}]b'(u) = q(A^\dagger)[\vec{G}]b(u)$. By a symmetric argument, it follows that $\mathbb{C}\langle A'^\dagger \rangle b'$ is a core for S^* . $\square$

Theorem 4.4 (Kalman decomposition [13]). *Let $f \sim (A, b, c)_Y$ be a realization centred at $Y \in \mathbb{C}^{(m \times m) \cdot d}$ and let Q_0 be the orthogonal projection onto $\mathcal{M}_{A,b,c} \subseteq \mathcal{H}$. Then $(A^{(0)}, b_0, c_0)_Y$ is minimal realization of f .*

Proof. Given the realization $(A, b, c)_Y$ around a matrix centre $Y \in \mathbb{C}^{(m \times m) \cdot d}$, we define the minimal subspace $\mathcal{M}_{A,b,c}$ as

$$\mathcal{C}_{A,c} \ominus (\mathcal{C}_{A,c} \cap \mathcal{O}_{A^\dagger, b}^\perp).$$

Let $A'(G) = A(G)|_{\mathcal{C}_{A,c}} \in \mathcal{B}(\mathcal{C}_{A,c})^d$ be the restriction of A to the controllable subspace $\mathcal{C}_{A,c}$ for any fixed $G \in \mathbb{C}^{(m \times m)} \otimes \mathbb{C}^{1 \times |\omega|}$, which is a bounded linear map. If

we take P' to be the orthogonal projection onto $\mathcal{C}_{A,c}$, defining $b'(u) = P'b(u)$ and $c'(v) = c(v) \in \mathcal{C}_{A,c}$ as bounded linear operators from $\mathbb{C}^m$ to $\mathcal{H}$. Indeed, $(A', b', c')_Y$ is a controllable realization on $\mathcal{H}' = \mathcal{C}_{A,c}$ because:

$$\begin{aligned}\mathcal{C}_{A',c'} &= \bigvee_{\omega \in \mathbb{F}_d^+} \text{Ran } A'^{\omega} c' = \bigvee_{\substack{\vec{G} \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times |\omega|} \\ v \in \mathbb{C}^n}} A'^{\omega} [\vec{G}] c'(v) \\ &= \bigvee (P' A P')^{\omega} [\vec{G}] c'(v) = \bigvee A^{\omega} [\vec{G}] \underbrace{P' c(v)}_{=c(v)} = \mathcal{C}_{A,c} = \mathcal{H}',\end{aligned}$$

using the A -invariance of P' . Since

$$\begin{aligned}b'^*(u) A'^{\omega} [\vec{G}] c'(v) &= (P' b(u))^* (P' A [\vec{G}] P')^{\omega} c(v) = b^*(u) A^{\omega} [\vec{G}] P' c(v) \\ &= b^*(u) A^{\omega} [\vec{G}] c(v) = u^* \hat{f}^{\omega} [\vec{G}] u,\end{aligned}$$

we can see that $(A', b', c')_Y$ is a controllable realization of the NC function f around the matrix centre Y , which is analytically equivalent to $(A, b, c)_Y$. We define the subspace $\mathcal{M} := \bigvee \text{Ran } A'^{\omega \dagger} b' \subseteq \mathcal{C}_{A,c}$ with orthogonal projection $Q_0 \leq \mathcal{C}_{A,c}$. If we consider the space $\mathcal{C}_{A,c} \ominus \mathcal{M}$, we can show that:

$$\begin{aligned}x \in \mathcal{C}_{A,c} \ominus \mathcal{M} &\iff 0 = \langle A'^{\omega \dagger} [\vec{G}] b'(u), x \rangle \\ &\iff 0 = \langle A^{\omega \dagger} [\vec{G}] b(u), x \rangle \\ &\iff x \in \mathcal{C}_{A,c} \ominus (\mathcal{C}_{A,c} \cap \mathcal{O}_{A^\dagger, b}^\perp) = \mathcal{M}_{A,b,c}.\end{aligned}$$

We denote $A^{(0)}[G] = Q_0 A[G]|_{\mathcal{M}_{A,b,c}}$, $b_0(u) = b(u)$ and $c_0(v) = Q_0 c(v)$ and similar to what we have above, we can write:

$$\begin{aligned}
b_0(u)^* A^{(0)\omega}[\vec{G}]c_0(v) &= b'(u)^*(Q_0 A' Q_0)^\omega[\vec{G}]Q_0 c'(v) \\
&= b'(u)^* Q_0 A'^\omega[\vec{G}]c(v) \quad (Q_0 \text{ being co-invariant for } A') \\
&= (P'b(u))^* A'^\omega[\vec{G}]P'c(v) \quad (b'(u) \in \text{Ran } Q_0 = \mathcal{M}_{A,b,c}) \\
&= b(u)^* A^\omega[\vec{G}]c(v) = u^* \hat{f}_\omega[\vec{G}]v
\end{aligned}$$

for every $\vec{G} = (G_1, \dots, G_{|\omega|}) \in \mathbb{C}^{(m \times m)} \otimes \mathbb{C}^{1 \times |\omega|}$. This proves that $(A^{(0)}, b_0, c_0)$ is a realization for f .

To prove the minimality, we first observe that :

$$\bigvee_{\omega \in \mathbb{F}_d^+} A^{(0)\omega^\dagger}[\vec{G}]b_0(u) = \bigvee_{\omega \in \mathbb{F}_d^+} A'^{\omega^\dagger}[\vec{G}] \underbrace{Q_0 b'(u)}_{b'(u)} = \mathcal{M}_{A,b,c}$$

for fixed $\vec{G} \in \mathbb{C}^{(m \times m)} \otimes \mathbb{C}^{1 \times |\omega|}$ and $u \in \mathbb{C}^m$. Moreover,

$$\begin{aligned}
\bigvee_{\omega \in \mathbb{F}_d^+} A^{(0)\omega}[\vec{G}]c_0(v) &= \bigvee_{\omega \in \mathbb{F}_d^+} Q_0 A'^\omega[\vec{G}]Q_0 c(v) \\
&= Q_0 \bigvee_{\omega \in \mathbb{F}_d^+} A'^\omega[\vec{G}]c(v), \quad \text{since } Q_0 \text{ is } A' \text{--co-invariant,} \\
&= Q_0 \bigvee_{\omega \in \mathbb{F}_d^+} A^\omega[\vec{G}]c(v), \quad \text{since } A' = A|_{\mathcal{C}_{A,c}}, \\
&= Q_0 \mathcal{C}_{A,c} = \mathcal{M}_{A,b,c},
\end{aligned}$$

since $\mathcal{M}_{A,b,c} \subseteq \mathcal{C}_{A,c}$. This proves that the realization $(A^{(0)}, b_0, c_0)$ on $\mathcal{M}_{A,b,c}$ is minimal. □

4.1.1 Translation of Realizations

Given a matrix-centre realization $f \sim (A, b, c)_Y$, with $Y \in \mathbb{C}^{(m \times m) \cdot d}$ and $X, Z \in \mathcal{D}_n^Y(A)$, we have that

$$\begin{aligned}
f(Z) &= I_n \otimes b^* L_A (Z - I_n \otimes Y)^{-1} I_n \otimes c \\
&= I_n \otimes b^* (I_n \otimes I_{\mathcal{H}} - A(Z - I_n \otimes Y))^{-1} I_n \otimes c \\
&= I_n \otimes b^* (I_n \otimes I - (I_n \otimes A)(Z - X) - (\text{id}_n \otimes A)(X - I_n \otimes Y))^{-1} I_n \otimes c \\
&= I_n \otimes b^* \left(I_n \otimes I - L_A (X - I_n \otimes Y)^{-1} (\text{id}_n \otimes A)(Z - X) \right)^{-1} L_A (X - I_n \otimes Y)^{-1} I_n \otimes c \\
&=: b'^* L_{A'} (Z - X)^{-1} c',
\end{aligned}$$

where $b', c' \in \mathcal{B}(\mathbb{C}^{mn}, \mathcal{H}')$, $A'_j \in \text{CB}(\mathbb{C}^{mn \times mn}, \mathcal{B}(\mathcal{H}'))$, $\mathcal{H}' := \mathbb{C}^n \otimes \mathcal{H}$, and for any $G \in \mathbb{C}^{mn \times mn}$,

$$A'_j(G) := L_A (X - I_n \otimes Y)^{-1} \text{id}_n \otimes A_j(G), \quad b' = I_n \otimes b, \quad \text{and} \quad c' = L_A (X - I_n \otimes Y)^{-1} I_n \otimes c.$$

and id_n denotes the identity map on $n \times n$ matrices. This is essentially the same calculation as in [13], Section 5.2. Hence we have shown that $f \sim (A', b', c')_X$.

Next, we would like to show that the minimality property of a realization is preserved through translations.

First, we set the stage for one-dimensional realizations: If $f \sim (A, b, c) \in \mathcal{B}(\mathcal{H}) \times \mathcal{H} \times \mathcal{H}$ is a minimal realization at $0 \in \mathbb{C}$, we would like to show that $f \sim_{z'} (A', b', c')$ is also a minimal realization of f at z' ([13]). Suppose $f \sim_0 (A, b, c) \in \mathcal{B}(\mathcal{H}) \times \mathcal{H} \times \mathcal{H}$ is a one-dimensional realization centred at $0 \in \mathbb{C}$. We will simply write $(A, b, c) = (A, b, c)_0$ in this case. Then, for any $z' \in \mathcal{D}(A)$, we can construct a realization of f centred at z' , $f \sim_{z'} (A', b', c')_{z'}$:

$$A' := (I - z'A)^{-1}A, \quad b' = b, \quad \text{and} \quad c' = (I - z'A)^{-1}c.$$

The controllability subspace of (A', b', c') is

$$\mathcal{C}_{A',c'} = \bigvee_{j=0}^{\infty} A'^j c' = \bigvee_{j=0}^{\infty} A^j (I - z'A)^{-j-1} c.$$

Hence, $(I - z'A)^{-1} c \in \mathcal{C}_{A',c'}$, and

$$z'A(I - z'A)^{-2} c = (I - z'A)^{-2} c - (I - z'A)^{-1} c \in \mathcal{C}_{A',c'}.$$

In particular it follows that $\mathcal{C}_{A',c'}$ contains the linear span of $(I - z'A)^{-2} c$ and $(I - z'A)^{-1} c$. Iterating this argument yields

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{j=1}^{\infty} (I - z'A)^{-j} c.$$

Set $T := (I - z'A)^{-1}$, a bounded, invertible operator and choose any $\lambda \in \mathbb{C}$ so that $|\lambda| > \|T\|$. Then, $\lambda I - T$ is invertible and

$$(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n,$$

is a convergent geometric series since $\|\lambda^{-1} T\| = |\lambda|^{-1} \|T\| < 1$. Hence,

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{|\lambda| > \|T\|} (\lambda I - T)^{-1} T c.$$

Since T is invertible, $0 \notin \sigma(T)$, so that the functions $e_{-n}(\lambda) := 1/\lambda^n$, $n \in \mathbb{N}$ are analytic in an open set containing the spectrum of T , $e_{-n} \in \mathcal{O}(\sigma(T))$. Hence, by the Riesz-Dunford functional calculus (see [15], Chapter 9 for the precise definition),

$$T^{-n} = \frac{1}{2\pi i} \oint_{r \cdot \partial \mathbb{D}} \lambda^{-n} (\lambda I - T)^{-1} d\lambda, \quad r > \|T\|.$$

so that

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{n \in \mathbb{N}} T^{-n} T c = \bigvee_{n=0}^{\infty} T^{-n} c = \bigvee_{n=0}^{\infty} (I - z' A)^n c.$$

Hence, $c \in \mathcal{C}_{A',c'}$ and

$$(I - z' A)c = c - z' A c \in \mathcal{C}_{A',c'},$$

so that $Ac \in \mathcal{C}_{A',c'}$. Continuing in this manner,

$$(I - z' A)c = c - 2z' A c + z'^2 A^2 c \in \mathcal{C}_{A',c'},$$

so that $A^2 c \in \mathcal{C}_{A',c'}$, and we conclude that

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{n=0}^{\infty} A^n c = \mathcal{C}_{A,c} = \mathcal{H},$$

so that (A', b', c') is controllable. The proof for observability is analogous to the proof above.

Theorem 4.5 (Minimal matrix translations). *If $f \sim (A, b, c)_Y$ is a minimal realization at $Y \in \mathbb{C}^{(m \times m) \cdot d}$, and $X \in \mathcal{D}_n^Y(A)$ is arbitrary, then $f \sim (A', b', c')_X$ is also a minimal realization of f at X .*

Proof. The controllable subspace of $(A', b', c')_Y$ is

$$\mathcal{C}_{A',c'} = \bigvee_{\substack{\omega \in \mathbb{F}_d^+; v \in \mathbb{C}^{mn} \\ \vec{G} \in \mathbb{C}^{(m \times m) \cdot n} \otimes \mathbb{C}^{1 \times |\omega|}}} (L_A(X - I_n \otimes Y)^{-1} A)^{\omega} [\vec{G}] L_A(X - I_n \otimes Y)^{-1} (I_n \otimes c)(v),$$

and we want to prove that this is equal to $\mathbb{C}^n \otimes \mathcal{H}$. Setting $T := A(X - I_n \otimes Y)$,

$L := (I - T) = L_A(X - I_n \otimes Y)$ and $\Lambda := L^{-1}$, we have that

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{\substack{\omega \in \mathbb{F}_d^+; v \in \mathbb{C}^{mn} \\ \vec{G} \in \mathbb{C}^{(m \times m) \cdot n} \otimes \mathbb{C}^{1 \times |\omega|} \\ 1 \leq j \leq d}} (\Lambda A)^\omega[\vec{G}] \Lambda A_j (X_j - I_n \otimes Y_j) \Lambda(I_n \otimes c)(v),$$

and hence, by summing over j ,

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{\substack{\omega \in \mathbb{F}_d^+; v \in \mathbb{C}^{mn} \\ \vec{G} \in \mathbb{C}^{(m \times m) \cdot n} \otimes \mathbb{C}^{1 \times |\omega|}}} (\Lambda A)^\omega[\vec{G}] \Lambda \underbrace{A(X - I_n \otimes Y)}_{=T} \Lambda(I_n \otimes c)(v).$$

Iterating this argument yields that

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{\substack{\omega \in \mathbb{F}_d^+; v \in \mathbb{C}^{mn} \\ \vec{G} \in \mathbb{C}^{(m \times m) \cdot n} \otimes \mathbb{C}^{1 \times |\omega|} \\ j \in \mathbb{N} \cup \{0\}}} (\Lambda A)^\omega[\vec{G}] T^j \Lambda^{j+1} (I_n \otimes c)(v) = \bigvee_{\substack{\omega \in \mathbb{F}_d^+; v \in \mathbb{C}^{mn} \\ \vec{G} \in \mathbb{C}^{(m \times m) \cdot n} \otimes \mathbb{C}^{1 \times |\omega|} \\ j \in \mathbb{N}}} (\Lambda A)^\omega[\vec{G}] \Lambda^j (I_n \otimes c)(v),$$

since T and $\Lambda = (I - T)^{-1}$ commute and $T\Lambda = T(I - T)^{-1} = (I - T)^{-1} - I = \Lambda - I$.

In particular, it follows as before that

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{\substack{\omega \in \mathbb{F}_d^+; v \in \mathbb{C}^{mn} \\ \vec{G} \in \mathbb{C}^{(m \times m) \cdot n} \otimes \mathbb{C}^{1 \times |\omega|} \\ |\lambda| > \|\Lambda\|}} (\Lambda A)^\omega[\vec{G}] (\lambda I - \Lambda)^{-1} \Lambda(I_n \otimes c)(v),$$

since for any $\lambda \in \mathbb{C}$ so that $|\lambda| > \|\Lambda\|$,

$$(\lambda I - \Lambda)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} \Lambda^n,$$

converges in operator norm. Again, by the Riesz–Dunford functional calculus, since Λ is invertible, $0 \notin \sigma(\Lambda)$ and $1/\lambda^j$ is holomorphic in an open neighbourhood of the

spectrum of Λ so that

$$\begin{aligned}
\mathcal{C}_{A',c'} &\supseteq \bigvee_{h \in \mathcal{O}(\sigma(\Lambda))} (\Lambda A)^\omega[\vec{G}]h(\Lambda)\Lambda(I_n \otimes c)(v) \\
&\supseteq \bigvee_{j=1}^{\infty} (\Lambda A)^\omega[\vec{G}]\Lambda^{-j}\Lambda(I_n \otimes c)(v) \\
&= \bigvee_{j=0}^{\infty} (\Lambda A)^\omega[\vec{G}]\Lambda^{-j}(I_n \otimes c)(v) \\
&\supseteq \bigvee (\Lambda A)^\omega[\vec{G}](I_n \otimes c)(v) \supseteq \bigvee (I_n \otimes c)(v).
\end{aligned}$$

We now iterate the argument as before.

$$\begin{aligned}
\mathcal{C}_{A',c'} &\supseteq \bigvee (\Lambda A)^\omega[\vec{G}](I_n \otimes c)(v) \\
&\supseteq \bigvee_{\substack{1 \leq j \leq d; v \in \mathbb{C}^{mn} \\ G_j \in \mathbb{C}^{(m \times m) \cdot n}}} (\Lambda A)^\omega[\vec{G}]\Lambda(id_n \otimes A_j)(G_j)(I_n \otimes c)(v) \\
&\supseteq \bigvee_{\substack{1 \leq j \leq d; v \in \mathbb{C}^{mn} \\ G \in \mathbb{C}^{mn \times mn}}} (\Lambda A)^\omega[\vec{G}]\Lambda(id_n \otimes A_j)(X_j - I_m \otimes Y_j)\Lambda(id_n \otimes A_k)(G)(I_n \otimes c)(v) \\
&\supseteq \bigvee (\Lambda A)^\omega[\vec{G}]\Lambda T \Lambda(id_n \otimes A_k)(G)(I_n \otimes c)(v),
\end{aligned}$$

where the last formula is obtained by summing over $1 \leq j \leq d$. Iterating this argument then yields:

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{s=0}^{\infty} (\Lambda A)^\omega[\vec{G}](\Lambda T)^s \Lambda(id_m \otimes A_k)(G)(I_n \otimes c)(v).$$

Again, we now use that Λ, T commute and that $\Lambda T = \Lambda - I$ to conclude that

$$\mathcal{C}_{A',c'} \supseteq \bigvee_{s=1}^{\infty} (\Lambda A)^\omega[\vec{G}]\Lambda^s(id_m \otimes A_k)(G)(I_n \otimes c)(v),$$

and hence as before,

$$\begin{aligned}
\mathcal{C}_{A',c'} &\supseteq \bigvee_{|\lambda| > \|\Lambda\|} (\Lambda A)^\omega[\vec{G}](\lambda I - \Lambda)^{-1} \Lambda(id_m \otimes A_k)(G_k)(I_n \otimes c)(v) \\
&\supseteq \bigvee_{n=0}^{\infty} (\Lambda A)^\omega[\vec{G}] \Lambda^{-n}(id_m \otimes A_k)(G_k)(I_n \otimes c)(v) \\
&\supseteq \bigvee (\Lambda A)^\omega[\vec{G}](id_m \otimes A_k)(G_k)(I_n \otimes c)(v).
\end{aligned}$$

In particular,

$$\mathbb{C}^n \otimes \bigvee_{G_k \in \mathbb{C}^{m \times m}} A_k(G_k)c(v) + \mathbb{C}^n \otimes \bigvee_{v \in \mathbb{C}^m} c(v) \subseteq \mathcal{C}_{A',c'},$$

and iterating this argument shows that

$$\mathbb{C}^n \otimes \mathcal{H} = \mathbb{C}^n \otimes \mathcal{C}_{A,c} = \mathbb{C}^n \otimes \bigvee_{\substack{\omega \in \mathbb{F}_d^+; v \in \mathbb{C}^m \\ \vec{G} \in \mathbb{C}^{(m \times m) \cdot n} \otimes \mathbb{C}^{1 \times |\omega|}}} A^\omega[\vec{G}]c(v) \subseteq \mathcal{C}_{A',c'},$$

so that $(A', b', c')_Y$ is controllable. The proof for observability is analogous to the proof above.

The final statement will follow from our construction of $(A', b', c')_X$. If $Z \in \mathcal{D}_{km}(A)$, then, since $X \in \mathcal{D}_m(A)$, by assumption,

$$\begin{aligned}
L_A(Z - I \otimes Y)^{-1} &= (I \otimes I - A(X - I \otimes Y) - A(Z - I \otimes X))^{-1} \\
&= (I \otimes I - L_A(X - I \otimes Y)^{-1} A(Z - I \otimes X))^{-1} L_A(X - I \otimes Y)^{-1} \\
&= L_{A'}(Z - I \otimes X)^{-1} L_A(X - I \otimes Y)^{-1}.
\end{aligned}$$

That is, invertibility of $L_A(Z - I \otimes Y)$ implies invertibility of $L_{A'}(Z - I \otimes X)$. This proves that $\mathcal{D}_{km}^Y(A) \subseteq \mathcal{D}_{km}^X(A')$, and the reverse inclusion also holds by symmetry. $\square$

4.1.2 FM Realization about a Matrix-centre and FM Algorithm

We can define the FM operator realization centred at a matrix point Y in a similar fashion to what is said in Chapter 3. The main objective of this section is to present formally the FM algorithm (also defined in Chapter 3 as synthesis).

Definition 4.9 (FM realizations around Y). *An FM realization at a matrix-point $Y \in \mathbb{C}^{(n \times n) \cdot d}$ is a quadruple $(A, B, C, D)_Y$, where, as before, $A = (A_1, \dots, A_d) : \mathbb{C}^{(n \times n) \cdot d} \rightarrow$*

$\mathcal{B}(\mathcal{H})$ is a row d -tuple of linear maps, $B = \begin{pmatrix} B_1 \\ \vdots \\ B_d \end{pmatrix} \in \mathcal{B}(\mathbb{C}^{n \times n}, \mathcal{B}(\mathbb{C}^n, \mathcal{H})) \otimes \mathbb{C}^d$, $C \in$

$\mathcal{B}(\mathcal{H}, \mathbb{C}^n)$, and $D \in \mathbb{C}^{n \times n}$. That is, each $B_j : \mathbb{C}^{n \times n} \rightarrow \mathcal{B}(\mathbb{C}^n, \mathcal{H})$ is a bounded linear map.

The linear pencil of A , L_A , is defined as before, and its invertibility domain (at Y), $\mathcal{D}^Y(A) \subseteq \mathbb{C}^{(\mathbb{N}n \times \mathbb{N}n) \cdot d}$ is defined as

$$\mathcal{D}^Y(A) := \bigcup_{m=1}^{\infty} \mathcal{D}_m^Y(A) = \{Z \in \mathbb{C}^{(mn \times mn) \cdot d} \mid L_A(Z - I_m \otimes Y) \text{ is invertible}\}.$$

The FM realization $(A, B, C, D)_Y$ defines a quantized transfer function, h on $\mathcal{D}_m^Y(A)$ by the formula:

$$h(X) = I_m \otimes D + I_m \otimes C \underbrace{L_A(X - I_m \otimes Y)^{-1} \sum_{j=1}^d B_j(X_j - I_m \otimes Y_j)}_{=: B(X - I_m \otimes Y)}; \quad X \in \mathcal{D}_m^Y(A) \subseteq \mathbb{C}^{(mn \times mn) \cdot d}.$$

The notions of controllability, observability and minimality can be defined similarly:

Definition 4.10. *If $(A, B, C, D)_Y$ is a matrix-centre realization at $Y \in \mathbb{C}^{(n \times n) \cdot d}$, we define, as before, the **controllable subspace**,*

$$\mathcal{C}_{A,B} := \bigvee_{\substack{\omega \in \mathbb{F}_d^+ \\ 1 \leq j \leq d}} \text{Ran } A^\omega B_j,$$

and the **observable subspace**,

$$\mathcal{O}_{A^\dagger, C^*} := \bigvee_{\omega \in \mathbb{F}_d^+} \text{Ran } A^{\omega; \dagger} C^*,$$

Again, as before, we say that $(A, B, C, D)_Y$ is **minimal** if it is both controllable and observable. (Both Theorems (4.4) and (4.3) as well as all of the results of previous sections are readily adapted to FM realizations.)

As described in Chapter 3, Remark (3.1), (for the case of realizations at 0) one can construct an FM realization from a descriptor one and vice versa. Namely, let $(A, b, c)_Y$ be a descriptor realization at $Y \in \mathbb{C}^{(n \times n) \cdot d}$. Then define the A -invariant subspace

$$\mathcal{H}' := \bigvee_{\omega \neq \emptyset} \text{Ran } A^\omega c,$$

with orthogonal projection P' ,

$$A'_j[G] := A_j[G]|_{\mathcal{H}'}, \quad B_j[G] := A_j[G]c, \quad C := b^* P', \quad \text{and} \quad D := b^* c \in \mathbb{C}^{n \times n},$$

where $G \in \mathbb{C}^{n \times n}$. It is easy to check that minimality of $(A, b, c)_Y$ implies minimality of $(A', B, C, D)_Y$ and both of these realizations define the same quantized transfer function in a uniform row-ball of positive radius centred at Y .

Conversely, if $(A, B, C, D)_Y$ is an FM realization about $Y \in \mathbb{C}^{(n \times n) \cdot d}$, let $\widehat{\mathcal{H}} := \mathcal{H} \oplus \mathbb{C}^n$ and define

$$\widehat{A}_j[G] := \begin{pmatrix} A_j[G] & B_j[G] \\ 0 & 0 \end{pmatrix}, \quad b := C^* \oplus D^*, \quad c := 0 \oplus I_n.$$

Then $(\widehat{A}, b, c)_Y$ is a descriptor realization centred at Y , which defines the same quantized transfer function in a uniformly open neighbourhood of Y . If $(A, B, C, D)_Y$ is

controllable, then

$$\mathcal{C}_{\hat{A},c} = \bigvee_{\omega \in \mathbb{F}_d^+} \hat{A}^\omega [\tilde{G}] c(v) = \bigvee_{\substack{G \in \mathbb{C}^{n \times n}, v \in \mathbb{C}^n \\ 1 \leq j \leq d}} B_j[G](v) \oplus \bigvee_{\substack{\omega = \omega' j \in \mathbb{F}_d^+ \\ \omega' \in \mathbb{F}_d^+, 1 \leq j \leq d}} A^{\omega'} [\tilde{G}] B_j[G](v) = \mathcal{C}_{A,B}.$$

so that $(\hat{A}, b, c)_Y$ is also controllable. If $(A, B, C, D)_Y$ is observable, then $(\hat{A}, b, c)_Y$ need not be observable, but its observable subspace has codimension at most n .

4.1.3 FM Algorithm (Synthesis)

Similar to what is preseted in Section (3.1.2), and also as in [18, Theorem 2.4] and [1, Subsection 3.4], if $f \sim_Y (A, B, C, D)$ and $g \sim_Y (A', B', C', D')$, then $f + g$, $f \cdot g$, and, assuming $f(Y)$ is invertible, f^{-1} , all have matrix-centre realizations about Y given by a natural extension of the Fornasini–Marchesini algorithm for realizations at 0. Namely, a matrix-centre realization for $f + g$ is given by (A^+, B^+, C^+, D^+) where

$$A_j^+ := A_j \oplus A'_j, \quad B_j^+ := \begin{pmatrix} B_j \\ B'_j \end{pmatrix}, \quad C^+ := (C, C'), \quad D^+ := D + D'.$$

A matrix-centre realization for $f \cdot g$ is given by $(A^\times, B^\times, C^\times, D^\times)$, where

$$A_j^\times := \begin{pmatrix} A_j & B_j(\cdot)C' \\ A'_j & \end{pmatrix}, \quad B^\times := \begin{pmatrix} B_j(\cdot)D' \\ B'_j \end{pmatrix}, \quad C^\times := (C, DC'), \quad D^\times := DD'.$$

Finally, a matrix-centre realization for f^{-1} , assuming that $D = f(Y)$ is invertible, is given by $(A^{(-1)}, B^{(-1)}, C^{(-1)}, D^{(-1)})$, where

$$A_j^{(-1)} := A_j - B_j(\cdot)D^{-1}C, \quad B_j^{(-1)} := -B_j(\cdot)D^{-1}, \quad C^{(-1)} := D^{-1}C, \quad D^{(-1)} := D^{-1}.$$

Verification of these formulas is a straightforward computation, and we refer the

reader to [18, Theorem 2.4] for their proofs (in the NC rational/finite-dimensional setting).

Given a rational function, one can use the above realizations for $f + g$, $f \cdot g$, and f^{-1} (if it exists), to build a realization for that rational function. This algorithm is known as **FM algorithm**.

Remark 4.4. *An analogous algorithm exists for descriptor realizations centred at $Y \in \mathbb{C}^{(n \times n) \cdot d}$. In our opinion the formulas for the FM algorithm are simpler and easier to work with in many calculations. Another advantage of the FM algorithm over the descriptor realization algorithm is that if $(A, B, C, D)_Y$ is minimal (and $D \in \mathbb{C}^{n \times n}$ is invertible) then the realization $(A^{(-1)}, B^{(-1)}, C^{(-1)}, D^{(-1)})_Y$ is also minimal. (This is not the case for the corresponding descriptor algorithm formula for the realization of the inverse of the quantized transfer function.)*

4.1.4 Applications:

The purpose of the theorems stated above is to extend the theory of realizations to a larger class of NC functions.

By Lemma (3.4), an NC function, f is uniformly analytic in a uniformly open neighbourhood of 0 if and only if f has a realization, i.e. $f \sim (A, b, c)$. Let $\mathcal{B} := \mathcal{B}(\mathcal{H})^d$, and let $\mathcal{O}_0^{\mathcal{B}}$ denote the set of all NC functions that have realizations. Then $\mathcal{O}_0^{\mathcal{B}}$ is a ring, using Section (3.1.2). One can construct a field of NC functions, $\mathfrak{M}_0^{\mathcal{B}}$, called the **field of fractions of $\mathcal{O}_0^{\mathcal{B}}$** as follows:

Consider the set of all rational expressions in $\mathfrak{z}$, denoted by $\mathbb{C} \langle \mathfrak{z} \rangle$, and an NC rational expressions, $\mathfrak{r} \in \mathbb{C} \langle \mathfrak{z} \rangle$, composed with elements of $\mathcal{O}_0^{\mathcal{B}}$, that is $\mathfrak{r} \circ (f_1, \dots, f_m)$, $m \in \mathbb{N}$. We say an expression, $\mathfrak{r} \circ (f_1, \dots, f_m) \in \mathbb{C} \langle \mathfrak{z} \rangle \circ \mathcal{O}_0^{\mathcal{B}}$ is **valid**, if its domain is not empty. We say

$$\mathfrak{r}_1 \circ (f_1, \dots, f_m) \sim \mathfrak{r}_2 \circ (g_1, \dots, g_n), \quad \mathfrak{r}_1, \mathfrak{r}_2: \text{NC rational}, \quad n \in \mathbb{N},$$

in which $(g_1, \dots, g_n) \in \mathcal{O}_0^{\mathcal{B}}$, if $\mathfrak{r}_1(f_1, \dots, f_m) = \mathfrak{r}_2(g_1, \dots, g_n)$ for every $X \in \text{Dom}(\mathfrak{r}_1 \circ (f_1, \dots, f_m)) \cap \text{Dom}(\mathfrak{r}_2 \circ (g_1, \dots, g_n)) \cap r \cdot \mathbb{B}^{(\mathbb{N} \times \mathbb{N}) \cdot d}$, for every $r > 0$. One can show that $\sim$ is an equivalence relation and the set $\mathfrak{M}_0^{\mathcal{B}}$, of all equivalence classes is a skew field (See [14, Section (6.2)]) . Elements of $\mathfrak{M}_0^{\mathcal{B}}$ can be viewed as **meromorphic NC germs** in a uniformly open neighbourhood of 0.

Suppose $f \in \mathfrak{M}_0^{\mathcal{B}}$, such that $f(x) = \mathfrak{r}(g_1, \dots, g_n)(x)$, in which $\mathfrak{r}^0$ is an NC rational function with each $g_j \in \mathcal{O}_0^{\mathcal{B}}$ and $g_j \sim (A'^j, b'_j, c'_j)$. Suppose $Y \in \text{Dom}_s(f)$ so that $Y \in \text{Dom}_s g_j$, for $1 \leq j \leq n$. Therefore , by translation results we have $g_j \sim_Y (A^{(j)}, b_j, c_j)$ and by the FM algorithm we have $f \sim_Y (A, b, c)$ (see Section (4.1.3)). We conclude that any NC function that is an NC rational expression composed with functions that have realizations at 0 will have a matrix-centred realization about any matrix-tuple in its domain.

Theorem 4.6. *Let $\mathcal{B} := \mathcal{B}(\mathcal{H})^d$ and let $\mathcal{O}_0^{\mathcal{B}}$ denote the ring of all NC functions that admit realizations centred at 0. Let $\mathfrak{M}_0^{\mathcal{B}}$ be its skew field of fractions, i.e. the field of meromorphic NC germs in a uniformly open neighbourhood of 0, as described above. Then every $f \in \mathfrak{M}_0^{\mathcal{B}}$ admits a matrix-centred realization about any point $Y \in \text{Dom}_n(f) \subseteq \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times d}$. In particular, if*

$$f(x) = \mathfrak{r}(g_1, \dots, g_n)(x),$$

with $\mathfrak{r}$ an NC rational function and each $g_j \in \mathcal{O}_0^{\mathcal{B}}$, then for every $Y \in \text{Dom}_s(f)$ there exists a matrix-centre realization, centred at Y :

$$f \sim_Y (A, b, c), \quad A_j : \mathbb{C}^{s \times s} \longrightarrow \mathcal{B}(\mathcal{H}).$$

Proof. Let $\mathbb{C} \langle \mathfrak{z}_1, \dots, \mathfrak{z}_k \rangle_k$ denote the set of all rational expressions on k NC variables. By construction (see [1, Subsection 5.2] and [14, Section 6]), every element $f \in \mathfrak{M}_0^{\mathcal{B}}$ can be represented as an NC rational expression composed with functions in $\mathcal{O}_0^{\mathcal{B}}$. That is, there exists $k \in \mathbb{N}$, an NC rational function $\mathfrak{r} \in \mathbb{C} \langle \mathfrak{z}_1, \dots, \mathfrak{z}_k \rangle_k$, and functions

$g_1, \dots, g_k \in \mathcal{O}_0^{\mathcal{B}}$ such that

$$f = \mathfrak{r}(g_1, \dots, g_k).$$

The domain of f is then the largest NC set on which this composition is well-defined.

By Lemma (3.4), each $g_j \in \mathcal{O}_0^{\mathcal{B}}$ admits a realization at 0, say $g_j \sim_0 (A^{(j)}, b_j, c_j)$ with $A^{(j)} \in \mathcal{B}$. If $Y \in \text{Dom}_s(f)$, then in particular $Y \in \text{Dom}_s(g_j)$ for each $1 \leq j \leq k$. By the translation results (Section 4 and Theorem (4.5)), the realizations of g_j can be translated from 0 to Y , yielding

$$g_j \sim_Y (\tilde{A}^{(j)}, \tilde{b}_j, \tilde{c}_j), \quad \tilde{A}_i^{(j)} : \mathbb{C}^{s \times s} \longrightarrow \mathcal{B}(\mathcal{H}).$$

Applying the FM (or descriptor) realization algorithm to the composition $\mathfrak{r}(g_1, \dots, g_k)$ then produces a matrix-centred realization of f about Y . Hence,

$$f \sim_Y (A, b, c), \quad A_j : \mathbb{C}^{s \times s} \longrightarrow \mathcal{B}(\mathcal{H}),$$

as required. □

Chapter 5

Future Work and Directions

The skew field $\mathfrak{M}_d^{\mathbb{C}}$. Let $\mathcal{O}_d^{\mathbb{C}}$ denotes the ring, consisting of all uniformly analytic NC functions f that admit realizations of the form:

$$f \sim_0 (A, b, c), \quad (A, b, c) \in \mathcal{C}(\mathcal{H})^{1 \times d} \times \mathcal{H} \times \mathcal{H},$$

where $\mathcal{C}(\mathcal{H})$ is the closed two-sided ideal of compact operators on $\mathcal{H}$. As shown in [1, Corollary (6.3), Theorem (6.10)], the ring $\mathcal{O}_d^{\mathbb{C}}$ is a semi-free ideal ring in the sense of P.M. Cohn [2], and therefore admits a universal skew field of fractions, denoted $\mathfrak{M}_d^{\mathbb{C}}$.

Elements of $\mathfrak{M}_d^{\mathbb{C}}$ can be defined as equivalence classes of NC rational expressions with coefficients in $\mathcal{O}_d^{\mathbb{C}}$. Concretely, given $f_1, \dots, f_k \in \mathcal{O}_d^{\mathbb{C}}$ and an NC rational function $\mathfrak{r} \in \mathbb{C} \langle \mathfrak{z} \rangle_k$, we obtain an element $\mathfrak{r}(f_1, \dots, f_k) \in \mathfrak{M}_d^{\mathbb{C}}$ whenever the expression is valid (i.e. its domain of evaluation is nonempty). Two such expressions are identified if they agree on the intersection of their domains inside every uniformly open NC neighbourhood.

Thus $\mathfrak{M}_d^{\mathbb{C}}$ is the **universal skew field of fractions of $\mathcal{O}_d^{\mathbb{C}}$** . Its elements may be viewed as *uniformly meromorphic NC germs whose realizations take values in compact operators*. In this sense, $\mathfrak{M}_d^{\mathbb{C}}$ provides a natural multivariate noncommutative analogue of the classical field of meromorphic functions on $\mathbb{C}$.

Theorem (4.2) guarantees that any NC function f which is uniformly analytic in a uniformly open neighbourhood of a matrix point $Y \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^d$ admits a matrix-centred realization at Y . However, this does not imply that f belongs to the skew field $\mathfrak{M}_d^{\mathcal{B}}$ of uniformly meromorphic germs at 0. Indeed, [14, Theorem 7.2] shows that for a matrix point Y , one can construct a uniformly analytic NC function defined near Y whose evaluations take values in nilpotent matrices of arbitrarily prescribed order. Such an f cannot belong to any skew field of fractions, because a skew field has no nonzero nilpotent elements. On the other hand, if an NC function admits a finite-dimensional matrix-centred realization, then it must be an NC rational function.

This leads to the following open problem, originally posed by V. Vinnikov: suppose f has a matrix-centred realization

$$f \sim_Y (A, b, c), \quad A_j : \mathbb{C}^{n \times n} \longrightarrow \mathcal{C}(\mathcal{H}),$$

with values in compact operators on $\mathcal{H}$. Is it then necessarily true that $f \in \mathfrak{M}_d^{\mathcal{C}}$?

As explained in [1], the field $\mathfrak{M}_d^{\mathcal{C}}$ is a natural noncommutative and multivariate analogue of the classical field of meromorphic functions on $\mathbb{C}$. The resolution of this question would clarify the precise scope of compact-valued matrix-centred realizations within the framework of NC meromorphic function theory.

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