MATRIX SOLUTION APPROACH TO PROJECT SCHEDULING UNDER CRISP AND FUZZY ENVIRONMENT

BY

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A Thesis submitted to
the Faculty of Graduate Studies
In Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

Department of Mechanical and Industrial Engineering
University of Manitoba
Winnipeg, Manitoba

THE UNIVERSITY OF MANITOBA

FACULTY OF GRADUATE STUDIES

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ABSTRACT

In the present thesis, we formulate and solve a Critical Path problem using simple Matrix algebra techniques. We use our developed technique to further solve and analyze the Critical Path problem with imprecise or fuzzy, activity durations. Furthermore, we model our technique to solve a production-scheduling problem under Group Technology (GT) setup. Significance of this model can be seen in the light of widespread usage of Critical Path method in the field of Project Management and Manufacturing Engineering under uncertainty.

The methods presented in this thesis are computationally simple and very useful in classroom teaching of Critical Path problem.

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CHAPTER 1

INTRODUCTION

The solution to a majority of problems in industry and the business world exists in the better control and use of existing resources. This involves a critical examination, analysis and planning of all the activities that lead to the consumption of these resources in every project.

A project is defined by a set of activities, in a specific sequence, aimed at achieving a specific and relatively short-term objective. A project could be a one-time occurrence or repetitive, as in case of production of parts from raw material, within a work-cell or a company as a whole. Good project management means a good management of these activities for better or optimum use of the resources. The resources for a company usually consist of labor, capital, time and technology. One of these most valuable resource, time, that usually determines the failure or success of a project, is often neglected while managing projects. It is a common belief that projects hardly ever finish on time. Though there is an inherent uncertainty in a project, yet it can still be finished in time with better control of its activities. However, acceptance of better project management techniques has generally not been easier.

Project management can be defined as the examination, planning, directing and controlling of the consumption of resources, within certain constraints, to obtain the desired outcome. The principles of project management can be applied to as diverse activities as Research & Development, construction, operations management, design of production cells, creation of a movie etc.

In the present thesis, we seek to introduce, both under a crisp and a fuzzy environment, a simple computational procedure to plan a project's completion in the shortest possible time, given the sequence of activities to complete the project. We consider projects with different activities and objectives and model our technique based on the well known Djikstra's Algorithm and principle of recursive equation.

In the present thesis, we provide alternative solution techniques to two of the well-known time oriented network techniques, Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT). Both of these techniques are often referred to as arrow diagramming methods, or activity-on-arrow networks. Both techniques though very similar, have some differentiating features as follows:

- PERT uses three time estimates (optimistic, most likely, and pessimistic), whereas CPM uses one time estimate that represents the normal time.
- PERT is probabilistic in nature, based on a beta distribution for each activity time and a normal distribution for expected time duration, allowing the user to calculate the risk in completing a project. CPM, on the other hand, is deterministic in nature as it is based on a single time estimate.

The principle advantages of CPM or PERT techniques are to,

- provide management with the ability to plan for best possible use of resources to achieve a given goal within time and cost limitations,
- help management handle the uncertainties involved in programs by determining the influence of time delays in certain elements on the project completion and facilitate "what if" exercises,

- help management determine the existence of slack between elements, if any,
- identify the longest or critical path and the elements that are crucial to meet the completion deadline,
- provide management with a means for evaluating alternatives,
- provide a method to determine manpower, material, and capital requirements as well as providing a means for checking progress by utilizing time network analysis technique,
- reveal interdependencies of various activities in the project.

1.1. Nomenclature

Both the techniques (CPM and PERT) use similar nomenclature and terminology as represented below. We intend to use similar nomenclature in the present thesis also.

1.1.1. Dummy Activities

Dummy activities are superficial activities that do not require time or any other resource. They are used to represent precedence requirements accurately and to uniquely identify activities. They are usually represented by a dotted line on the network diagram. For example, the home-builder can have the foundation laid and order materials concurrently. Though, building the home requires completion of both activities, therefore a dummy activity is used to show this precedence relationship.

1.1.2. Critical Path

Critical path is that sequence of activities and events, whose accomplishment will require the longest expected time. It represents the minimum time that is required to finish the project. Any delays on the critical path would result in the delay of the completion of the whole project. Hence critical path is vital for resource scheduling and allocation as the project manager can focus on timely completion of critical path activities, provided that the critical path does not change.

1.1.3. Slack Time

As there is only one longest path length (although there could be multiple paths with the longest path length) through the network, the other paths must be either equal in length to or shorter than that path. Therefore, there must exist events and activities that can be completed before the time when they are actually needed. The time differential between the scheduled completion date and the required date to meet critical path is referred to as the slack time.

Slack can be defined as the difference between the latest allowable date and the earliest expected data based on the nomenclature below:

T_E = the earliest time on which an event can be expected to take place

T_L= the latest time on which an event can take place without extending the completion date of the project

Slack time = $T_L - T_E$

Optimum use of slack times provides for a better balance of resources throughout the project, and may possibly reduce project costs by eliminating idle or waiting time.

There are two types of slack, total slack and free slack. Free slack is the amount of time that completion of an activity can slip without delaying the start of any subsequent activity. Total slack is the amount of time that completion of an activity can slip without delaying the completion of project from schedule, provided all the other activities are finished on schedule. Total slack may include free slack and slack shared with another activity.

Monitoring the slack time can provide timely warnings to the project manager. As an example, if the total slack time available begins to decrease from one reporting period to the next, that could indicate that work is taking longer than anticipated or that more highly skilled labor is needed. A new critical path could also be forming.

1.1.4. Earliest Start Time (ES)

The earliest start time of an activity is the sum of all the activity times on the longest path to that activity. It is the earliest time an activity can begin, provided all preceding activities on this path begin as early as possible. To calculate the earliest starting times, we must make a forward pass through the network (i.e., left to right). The earliest starting time of a successor activity is the latest of the earliest finish dates of the predecessors.

1.1.5. Earliest Finish Time (EF)

The earliest finish time of an activity is equal to its earliest start time plus the activity duration.

1.1.6. Latest Start Time (LS)

The latest start time for an activity is the latest time it can be started without delaying the completion of the project. LS of an activity is equal to the scheduled project completion time minus the time requirements of the longest path from the end of that activity to the completion of the project and the activity duration. It can also be evaluated by subtracting activity duration from Latest Finish Time (LF) of an activity.

1.1.7. Latest Finish Time (LF)

LF of an activity is equal to the scheduled project completion time minus the time requirements of the longest path from the end of that activity to the completion of that project.

1.2. Solution Approaches

The traditional approaches for solving CPM and PERT problems will be discussed in detail in the literature survey in the next chapter. Along with these approaches, some modern concepts presented by various authors will also be discussed in the next chapter.

The traditional techniques of CPM and PERT are generally criticized for the time consuming and labor intensive effort required in the calculations. In a lot of practical applications, there exists a lack of historical data for time estimates. Also, there exists a lack of functional ownership in estimates. However, in the present thesis, we have used a matrix approach for solving the CPM and PERT problem. To obtain the solution for the

CPM Problem from a source node to destination node, we use the philosophy of the recursive equation and the principle of optimality used in dynamic programming.

In a lot of real world project planning problems, the activity durations cannot be crisply defined. This can happen because sometimes it is simply not possible to obtain precise data, or the cost of obtaining precise data is too high. Under such circumstances, the problem becomes that of modeling with imprecise data. We have analyzed our problems by means of a fuzzy logic approach when some sort of ambiguity in activity time is involved. Fuzzy set theory is a tool that gives reasonable analysis of complex systems without making the process of analysis too complex.

In what follows, we briefly explain the fundamentals of recursive equation and the principle of optimality and give a brief introduction to matrix algebra and fuzzy set theory.

1.2.1. Principle of Optimality

Bellman (1957) introduced a principle of optimality and a recursive equation to develop an optimization approach, called dynamic programming (DP), that optimizes n–stage sequential decision problems on a step-by-step basis using information from the preceding steps. However, in such an approach each single step contains information that identifies a segment of the optimal solution. Thus, in a dynamic program we obtain, in general, an optimal solution to an optimization problem through a series of steps and tableaus.

The principle of optimality states "An optimal set of decision rules has the property that, regardless of the i^{th} decision, the remaining decisions must be optimal with respect to the outcome that results from the i^{th} decision".

1.2.2. Recursive Equation

In developing our technique use the following recursive equation with the principle of optimality, as defined by Bellman (1957).

Using the terminology of Bellman, we have

$$\text{Cumulative resource for Stage } j = (\text{Direct resource for Stage } j) \;\Theta\; (\text{Cumulative resource for Stage } (j-1)) \qquad j = 1, 2, \ldots, N$$

where,

(i) the Θ stands for a binary operation such as ordinary addition, multiplication, or division of real numbers

(ii) it is assumed that for $j = 1$,

the Cumulative resource for Stage 1 = Direct resource for Stage 1

Based on the above philosophy of the recursive equation and the principle of optimality, we develop a simple matrix representation for solving the CPM problem for a given network.

1.2.3. Fuzzy Logic and Fuzzy Set Theory

As explained by Liberatore (2002), fuzzy logic is not fuzzy thinking. Zadeh (1965) introduced fuzzy sets to represent knowledge that is vague or imprecise, that is, "fuzzy." In a classical set theory, an element either is or is not a member of a set. In contrast to the sharp or "crisp" boundaries of classical sets, fuzzy sets allow degrees of membership in a set, as expressed by a number between 0 and 1.

In the following section, we introduce some of the basic terminology of fuzzy set theory. Theory of fuzzy sets is basically a theory of graded concepts (Zimmerman, 1991).

- **Fuzzy Set**

Let X be a classical set of objects, called the universe, whose generic elements are denoted by x. The membership in a crisp subset of X is viewed as a characteristic function μ_A from X to [0, 1] such that:

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

where [0, 1] is called a valuation set (Lai and Hwang, (1992)).

If the valuation set is allowed to be the real interval [0, 1], A is called a fuzzy set proposed by Zadeh (1996). $\mu_A(x)$ is the degree of membership of x in A. The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A. Therefore, A is completely characterized by the set of ordered pairs:

$$A = \{(x, \mu_A(x)) / x \in X\}$$

where μ_A (x) maps X to the membership space [0, 1]. Elements with zero degree of membership are usually not listed. If Sup μ (x) = 1, $\forall$ x $\in$ R, then the fuzzy set A is called a normal fuzzy set in R. A fuzzy set that is not normal is called subnormal fuzzy set.

- **α – Level Set or α – Cut**

One of the most important concepts of fuzzy sets is the concept of an α-cut or α-level set. An α-cut denoted by A_α is the crisp set of elements x in R whose degree of belongings to the fuzzy set A is at least $\alpha \in [0, 1]$. This means

$$A_\alpha = \{x \in R \mid \mu_A(x) \geq \alpha, \alpha \in [0, 1]\}$$

that is, the α-cut or α-level set of a fuzzy set is the crisp set A_α that contains all elements of the universal set $X \in R$ whose membership grades in A are greater than or equal to the specified value of α, $\alpha \in [0, 1]$.

- **Support of a Fuzzy Set**

The support of a fuzzy set A is a set S(A) such that $x \in S(A) \Leftrightarrow \mu_A(x) > 0$. If $\mu_A(x)$ is constant over S(A), then A is non-fuzzy.

- **Algebraic Operations on Fuzzy Sets**

In addition to the set theoretic operations, we can also define a number of other ways of forming combinations of fuzzy sets and relating them to one another. Here we present some more important operations among those:

1. The algebraic sum of A and B is A + B whose membership function is defined as

$$\mu_{(A+B)}(x) = \mu_A(x)\ (+)\ \mu_B(x), \quad \forall\ x \in X$$

provided $\mu_A(x)\ (+)\ \mu_B(x) \leq 1, \quad \forall\ x \in X$

2. The absolute difference $|A - B|$, of A and B is given by

$$\mu_{|A-B|}(x) = \left|\mu_A(x)\ (-)\ \mu_B(x)\right| \quad x \in X$$

- **Convexity of Fuzzy Set**

The notion of convexity can be extended to fuzzy sets in such a way as to preserve many of the properties that it has in case of crisp sets. In what follows, we assume that the set X is the n-dimensional space R^n. We now have the following two equivalent definitions of convexity of a fuzzy set.

A fuzzy set A is convex if and only if every set $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ for all $\alpha \in [0, 1]$ is a convex set.

The second definition of convexity of a fuzzy set is as follows:

A fuzzy set A is said to be a convex set if

$$\mu\ (\lambda x_1 + (1-\lambda)x_2) \geq \min\ (\mu(x_1), \mu(x_2)), \quad x_1, x_2 \in X,\ \ \lambda \in [0, 1].$$

- **Fuzzy Number**

The first definition of a fuzzy set allows us to extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts.

An ordinary number ‘a’ can be characterized by using the notation of membership function as,

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

A fuzzy number A is a fuzzy set on the real line R, which possesses the following properties:

- A is a normal, convex fuzzy set on R
- The α-level set A_α must be a closed interval for every $\alpha \in [0, 1]$

The support of A, $S(A) = \{x \mid \mu_A(x) > 0\}$, must be bounded.

- **Fuzzy Arithmetic**

  Fuzzy arithmetic is based on two properties of fuzzy numbers:

1. Each fuzzy set and thus, each fuzzy number can be fully and uniquely represented by its α-level sets.
2. α-level sets of each fuzzy numbers are closed intervals of real numbers for all $\alpha \in [0, 1]$

These properties enable us to define an arithmetic operation on fuzzy numbers in terms of arithmetic operations on their α-level sets (i.e. arithmetic operations on closed intervals).

- **Fuzzy Arithmetic Based On Operations On Closed Intervals**

  A fuzzy number can be characterized by an interval of confidence at level α,

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$$

which has the property

$$\alpha \leq \alpha' \Rightarrow A_{\alpha'} \subset A_\alpha$$

According to Kaufmann and Gupta (1985, 1988), let A = [a, b] ∈ R and B = [c, d] ∈ R be two fuzzy numbers, then the arithmetic operations on them as follows:

Addition	$A + B = [a + c, b + d]$
Subtraction	$A - B = [a - d, b - c]$
Minimum ($\wedge$)	$A \wedge B = [a \wedge c, b \wedge d]$
Maximum ($\vee$)	$A \vee B = [a \vee c, b \vee d]$

Let A and B be two fuzzy numbers, $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ be the α-level set of A, and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-level set of B.

Let * denote any of the arithmetic operations +, –, $\wedge$ and $\vee$ on fuzzy numbers.

Then, we define a fuzzy set A * B in R, by defining its α-level sets $(A * B)_\alpha$ as

$$(A * B)_\alpha = A_\alpha * B_\alpha \text{ for any } \alpha \in [0, 1]$$

Since $(A * B)_\alpha$ is a closed interval for each $\alpha \in [0, 1]$ and A and B are fuzzy numbers, A * B is also a fuzzy number.

- **Triangular Fuzzy Number**

A triangular fuzzy number (T.F.N.), A, is denoted by the triplet (a_1, a_2, a_3) and its membership function is written as

$$\mu_A(x) = \begin{cases} 0 & x \le a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & a_2 \le x \le a_3 \\ 0 & x \ge a_3 \end{cases}$$

The α-level set of a triangular fuzzy number is

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3] \qquad \forall\, \alpha \in [0, 1]$$

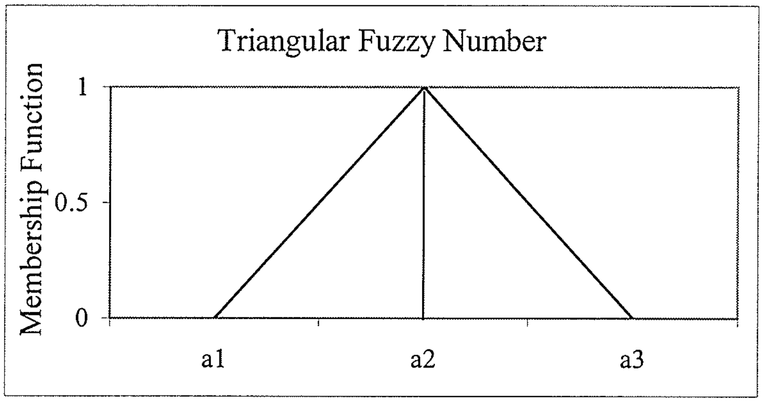


Figure 1.1. Graphic representation of a Triangular Fuzzy Number

Algebraic Operations on T.F.N.

Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be two T.F.Ns then,

- Addition $\qquad A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
- Subtraction $\qquad A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

- **Ranking of Fuzzy Numbers**

Ranking of fuzzy numbers is required in several applications. But, since the fuzzy numbers do not form a natural linear order like crisp numbers, it becomes a challenge to rank them. Several researchers have developed different methods to rank and compare fuzzy numbers. Each method has its own merits and de-merits and cannot be easily contrasted to other.

Let $A_i = (a_{1i}, a_{2i}, a_{3i})$ for I = 1, 2, 3, …, n, represent the triangular fuzzy numbers to be ranked. Researchers, as we will show later, have developed a real number as an $(\text{Index})_i$ for each A_i as the ordering or ranking value of A_i. The $(\text{Index})_i$ is treated as a fuzzy measure of A_i. Then the fuzzy numbers are ranked as per the order of their corresponding indices.

We briefly present some of the indices developed by various researchers.

a) Chang's Ranking Index (Komolananij, 1995)

$$(\text{Index})_i = \frac{(a_{3i} - a_{1i})(a_{1i} + a_{2i} + a_{3i})}{6} \quad \text{for } i = 1, 2, \ldots, n$$

b) Chiu and Park Ranking Index (Chiu et al, 1994)

Let w_{i1} and w_{i2} be the weights associated with the fuzzy numbers A_i, to be ranked. Then,

$$(\text{Index})_i = \frac{(a_{1i} + a_{2i} + a_{3i})w_{i1}}{3} + w_{i2}a_{2i} \quad \text{for } i = 1, 2, \ldots, n$$

Chiu and Park suggest that the appropriate values for $w_{i1} = 1$ and w_{i2} should be between [0.1 and 0.3].

c) Kaufmann and Gupta Ranking Index (Kaufmann and Gupta, 1985)

$$(\text{Index})_i = \frac{(a_{1i} + 2a_{2i} + a_{3i})}{4} \quad \text{for } i = 1, 2, \ldots, n$$

Kaufmann and Gupta suggest that if we have a set of fuzzy numbers with same indices, then the number with the largest $[a_{2i}]$ will be given the highest rank. Also, if the indices as well as the $[a_{2i}]$ match for a set of fuzzy numbers, then the range of the numbers, $[a_{1i} - a_{3i}]$, is examined. The number with the largest range, $[a_{1i} - a_{3i}]$, is ranked the highest.

d) Fuzzy Weighted Methods (Bortolan et al, 1985)

Let w_{i1} and w_{i2} be the weights associated with the fuzzy numbers A_i, to be ranked.

Then,

$$(\text{Index})_i = \frac{(a_{1i} + a_{3i})w_{i1}}{2} + w_{i2}a_{3i} \qquad \text{for } i = 1, 2, \ldots, n$$

Bortolan and Degani suggest that the appropriate values for $w_{i2} = 1$ and $w_{i1} = 0.5$.

1.2.4. Matrix Algebra

The relationship between two or more variables of data can be represented by putting data in rows and columns. The number of Rows and the number of Columns define the size of the matrix, as a block. A matrix that has m rows and n columns is said to be a **(m x n) matrix** (pronounced as m-by-n matrix). For example, we show a (2x3) matrix as below:

$$\text{MONTH} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

The above matrix could represent the expenses of two families A and B (represented in rows) on food, utilities and health for a particular month.

When the numbers of rows and columns are equal, we call the matrix a **square matrix**. A square matrix of **order n** is a (n x n) matrix.

Shown below are some of the matrix operations that we will use during our calculations.

- **Addition of Matrices**

In order to add two matrices, we add the entries one by one. Note: Matrices involved in the addition operation must have the same size.

For example, we have

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} + \begin{pmatrix} \alpha & \beta & \gamma \\ \theta & \nu & \mu \end{pmatrix} = \begin{pmatrix} a + \alpha & b + \beta & c + \gamma \\ d + \theta & e + \nu & f + \mu \end{pmatrix}$$

- **Multiplication of a Matrix by a Number**

In order to multiply a matrix by a number, you multiply every entry by the given number. For example, for any number λ, we will have

$$\lambda \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b & \lambda c \\ \lambda d & \lambda e & \lambda f \end{pmatrix}$$

- **Subtraction of two matrices**

Subtraction is a combination of the above shown two rules. Indeed, if M and N are two matrices, then we will write

$$M - N = M + (-1)N$$

So first, you multiply the matrix N by -1, and then add the result to the matrix M.

In case of multiple operations on the same matrices (such as $J + F - 3M$), the calculations may be performed in any order. This is called **associativity of the operations**.

- **Other properties involving addition**

Let A, B, and C are m x n matrices. We have

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$

- **Transpose of a Matrix**

Considering a matrix, A

$$A = \begin{pmatrix} 0 & 1 & -1 & 3 \\ 0 & 2 & 3 & 1 \\ -1 & 0 & 2 & -3 \end{pmatrix}$$

The rows of matrix, A are given as

$$(0 \quad 1 \quad -1 \quad 3), (0 \quad 2 \quad 3 \quad 1), (-1 \quad 0 \quad 2 \quad -3)$$

The columns of matrix, A are given as

$$\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$$

The transpose operation interchanges the rows and the columns of a matrix. Now, the matrix transpose of A can be written as

$$A^{T} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 2 & 0 \\ -1 & 3 & 2 \\ 3 & 1 & -3 \end{pmatrix}$$

As seen above, the rows of matrix A^T are shown as

$$(0 \quad 0 \quad -1), (1 \quad 2 \quad 0), (-1 \quad 3 \quad 2), (3 \quad 1 \quad -3)$$

As we can see, the transpose of the columns of A are the rows of A^T.

1.3. Organization of the Thesis

In the present thesis, we model two problems, from different yet similar areas, using a matrix solution approach under crisp and fuzzy environments. In addition, we discuss the methods to obtain their solutions and interpretation to the solutions.

Chapter 1 provided an introduction to the concepts and problems considered in this thesis. Chapter 2 deals with the literature review of the problems considered with an objective to recognize the work done by other researchers. In Chapter 3, a matrix solution approach to the Critical Path problem is presented under crisp environment. In Chapter 4, the matrix solution approach to Critical Path problem is extended in the presence of fuzzy activity durations and the presented approach is compared to traditional PERT method using an example. In Chapter 5, we have modified the matrix solution technique to address the production-scheduling problem under GT setup. In Chapter 6, we have applied the technique presented in Chapter 5 to a real-world scenario as a case study. Finally, the conclusion and the discussion on the contributions made by the thesis, along with some recommendations for further research, are given in Chapter 6.

CHAPTER 2

LITERATURE SURVEY

This chapter provides a survey of the literature dealing with Project Scheduling problem, Production Scheduling problem under Group Technology setup, and other concepts considered in this thesis. The purpose of this chapter is to review the developments, and to identify the status of existing literature in this area.

2.1 Review of Literature on Project Scheduling Problem

Several methods have been developed in the past, to schedule and manage the projects of different dimensions. Most of these techniques retain their own usefulness in some form or the other. These methods as described by Kerzner (2001) may be classified as follows:

2.1.1. Preliminary Techniques

Preliminary techniques such as Gantt or Bar charts, Milestone charts etc. Henry Gantt introduced Gantt or bar charts in early 1900s. These charts represent the duration of each activity on a bar chart, with time usually represented on X-axis. An activity represents the amount of work required to proceed from one point to the next. The major de-merit of this type of charts is their inability to show the interdependencies between activities and events. These interdependencies must be identified in order to develop a master plan, that provides an easily understandable and current picture of operations at all the times.

Examples of Gantt chart and Milestone chart are shown below:

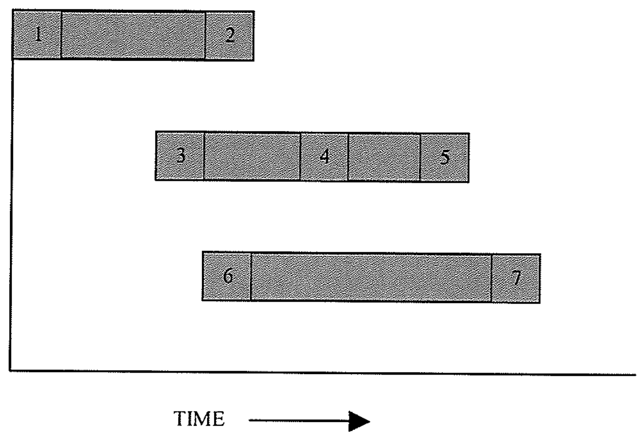


Figure 2.1. Gantt chart

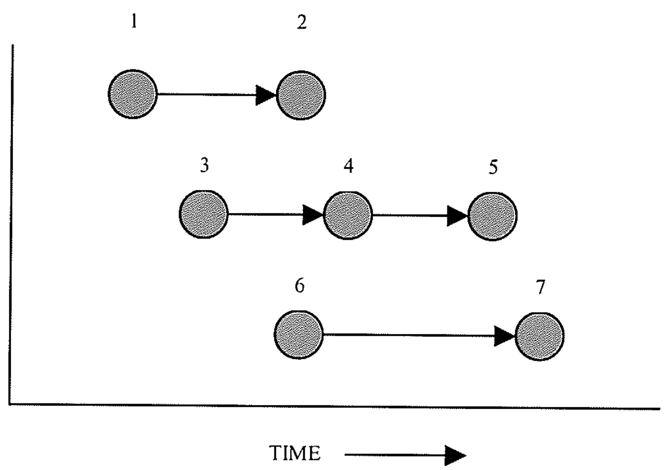


Figure 2.2. Milestone chart

2.1.2. Advanced Techniques

Advanced techniques such as Critical Path method (CPM) (also called the Arrow Diagram method (ADM)) or Program Evaluation and Review Technique (PERT) are used to show these interdependencies. This task is accomplished through the construction of networks. Networks are composed of events and activities. An event is defined as the starting or ending point for a group of activities. A circular node on the network diagram usually represents an event. An activity is the work required to proceed from one event or point in time to next. An arrow on the network diagram usually represents an activity. Network analysis provides valuable information for planning, integration of plans, time studies, scheduling, and resource management. Network planning eliminates the need for crisis management by providing a pictorial representation of the total program. The following management information can be obtained from such a representation:

- Interdependencies of activities
- Project completion time
- Impact of late starts
- Impact of early starts
- Trade-offs between resources a time
- "What if" exercises
- Cost of a crash program
- Slippages in planning or performance
- Evaluation of performance

A typical network diagram is shown below:

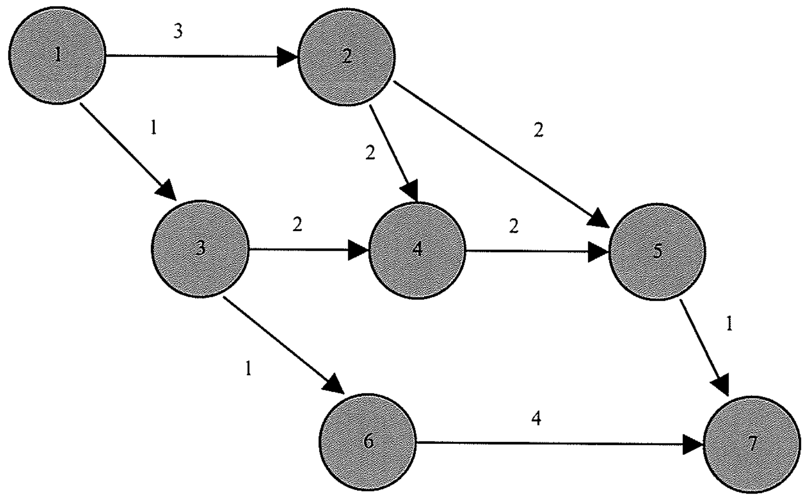


Figure 2.3. Network Diagram Example

PERT was originally developed in 1958-59 by the special projects office of the U.S. Navy in collaboration with the management-consulting firm of Booz, Allen, and Hamilton. It was successfully introduced on Navy's Polaris weapon system in 1958. Since that time, PERT has spread rapidly throughout almost all industries. At about the same time the Navy was developing PERT, the DuPont Company initiated a similar technique known as the critical path method (CPM), which also has spread widely, and is particularly concentrated in the construction and process industries.

CPM or PERT is basically a management planning and control tool. These can be considered as a road map for a particular program or project in which all of the major elements (events) have been completely identified together with their corresponding interrelations. CPM or PERT charts are often constructed from back to front because, for many projects, the end date is fixed and the contractor has front-end flexibility. Major purpose of constructing the CPM or PERT chart is to determine how much time is needed to complete the project. Both the techniques use time as a common denominator to analyze the elements that directly influence the success of the project, namely, time, cost, and performance. The construction of the network requires two inputs. First, a selection must be made as to whether the events represent the start or the completion of an activity. Event completions are generally preferred. The next step is to define the sequence of events, which relates each event to its immediate predecessor. Answering the following three questions help in the construction of CPM or PERT network:

- What job immediately precedes this job?
- What job immediately follows this job?
- What jobs can be run concurrently?

2.1.3. Recent Advances

Several new advanced techniques have gained popularity in recent days such as Graphical Evaluation and Review Technique (GERT), Precedence Diagram method (PDM) and Critical Chain Technique (CCS).

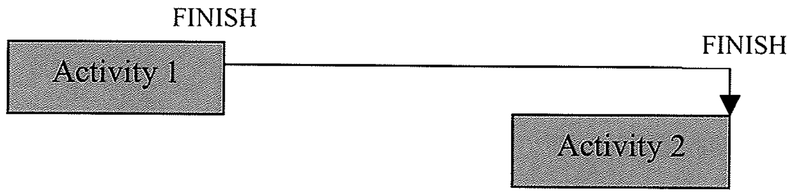
GERT is similar to PERT but has the distinct advantages of allowing for looping, branching, and multiple project end results. GERT enables management to select one of several other different branches to continue the project, in case a test fails.

Recently several software packages introduced in the market use the technique of Precedence Diagram Method (PDM). This technique allows the use of constraint between activities. Some typical PDM constraints and their explanation is as follows:

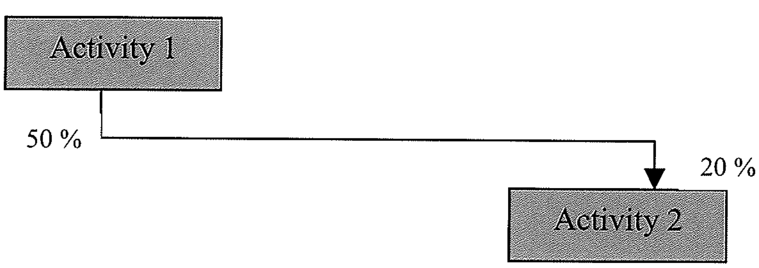
1. Finish-To-Start constraint implies that next activity, Activity 2, can start no earlier than the completion of previous activity, Activity 1.

2. Start-To-Start constraint implies that Activity 2 cannot start prior to Activity 1.

3. Finish-To-Finish constraint implies that Activity 2 cannot finish until Activity 1 finishes.



4. Percent Complete constraint implies that the last 20 percent of Activity 2 cannot be started until 50 percent of Activity 1 has been completed.



Critical Chain Scheduling (CCS) (Willy Herrolin et al., 2002) has gained a lot of attention recently among practitioners as well as philosophers. There still exists a lot of controversy regarding the merits and de-merits of this approach. The fundamentals of CCS are summarized in the following lines.

CCS builds a baseline schedule using activity duration estimates based on a 50% confidence level. Activity due dates and project milestones are eliminated, and multitasking is to be avoided. To minimize work in progress, a precedence-feasible schedule is constructed by timing activities at their latest start dates based on critical path calculations. If conflicts among resources common to certain activities occur, they are resolved by staggering those activities. The critical chain then is defined as that chain of precedence and resource dependent activities that determines the overall duration of a project. If there is more than one critical chain, just select one.

The safety time that is eliminated from the critical chain activity durations by selecting aggressive duration estimates is shifted to the end of the critical chain in the form of a project buffer (PB). This PB should protect the project due date promised to the customer from variability in the critical chain activities.

Feeding buffers (FB) are inserted whenever a non-critical chain activity joins the critical chain. Their aim is to protect the critical chain from disruptions on the activities feeding it and to allow critical chain activities to start early in case things go well. Although more detailed methods can be used for sizing the buffers, the default procedure is to use the 50% buffer sizing rule, i.e., to use a PB of half the project duration and to set the size of a FB to half the duration of the longest non-critical chain path leading into it.

Resource buffers (RB), usually in the form of an advance warning, are placed whenever a resource has to perform an activity on the critical chain, and the previous critical chain activity is done by a different resource.

The diagram as given in, Eliyahu M. Goldratt (1997), illustrates this method:

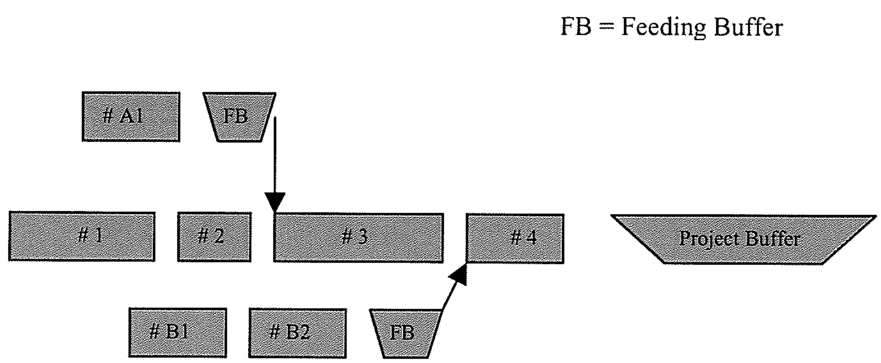


Figure 2.4. Critical Chain Scheduling

During project execution, both the critical chain and the baseline schedule should be fixed. Project activities are executed according to the roadrunner mentality using an un-buffered projected schedule. This schedule is early start-based, except for the gating tasks (activities without predecessors), which are to be started at their baseline scheduled start times. Early completion of activities must be reported, and activities should be started as soon as work becomes available. The execution of the project is managed by using the buffers as a proactive warning mechanism.

As activities are completed, project management should keep track of buffer consumption. As long as a predetermined portion of the buffer remains [buffer consumption is only in the green (O.K.) zone], everything is assumed to go well. If buffer consumption moves beyond a certain point, a warning is raised (the yellow watch-and-plan zone). If it deteriorates past a critical point (the red act zone), corrective action must be taken.

2.1.4. Other Approaches to Project Scheduling Problem

Chanas et al (1981) developed a method called FPERT, where the activity duration times were modeled as triangular fuzzy variables. The project completion time was obtained using the critical path method.

Chang et al (1995) proposed a methodology for calculating the completion time and the degree of criticality for each path on the project, using triangular fuzzy numbers (TFN) as activity durations. Fuzzy Delphi method, as developed by Kaufmann et al (1988) was used to estimate a reliable time interval (fuzzy interval) of each method.

Chanas et al (2001) presented an approach to calculate the criticality degree of a path in the presence of fuzzy activity times. The approach is further treated as a classical case of linear programming problem under certain assumptions.

Ozdamar et al (2001) proposed a heuristic method for modeling uncertainty in software projects using 6-points LR fuzzy numbers as task durations. The proposed method takes into account the learning rate curve of the persons assigned to the project.

Ghomi et al (2001) proposed a formula to compute the path critical index (PCI) and activity critical index (ACI) for the PERT network with any structure. The criticality index of the path (PCI) is defined as the probability that the duration of the path is greater than or equal to the duration of every other path on the network. The criticality index of an activity (ACI) is defined as the probability that any particular activity is on the said critical path of the network.

Velsky et al (2002) developed a polynomial algorithm as a generalization of Dijkstra's algorithm for scheduling of projects using 'And' and 'Or' nodes in the graphs. For any operation whose starting point is an 'And' node, its execution can begin only after all of its predecessors have been completed, whereas an 'Or' node represents the start of operation that can be executed when just one of its immediate predecessors has been completed. Traditional methods like CPM/PERT lack the capability of defining 'Or' nodes.

Liberatore (2002) used a fuzzy logic approach to model the uncertainty involved in project scheduling. The approach is further compared against the probabilistic approach used by PERT method.

Dubois et al (2003) proposed a fuzzy set based approach for scheduling of projects. Two distinct uses of fuzzy sets were depicted: representing preference profiles and modeling uncertainty distributions. The first setting produced to a valued, non-compensatory generalization of constraint-directed scheduling. The other setting yielded a possibility-theoretic counterpart of PERT, where probability distributions of activity durations got changed into possibility distributions, for the purpose of modeling incomplete information.

Dubois et al (2003) proposed a heuristic technique for determining latest starting dates and slack times in the presence of fuzzy activity durations. The approach of interval-valued durations was extended to fuzzy intervals to obtain the solution.

2.2 Review of Literature on Production Scheduling Problem under Group Technology (GT) setup

The group technology approach originally proposed by Mitrafanov and Burbidge is a philosophy that exploits the proximity among the attributes of given objects. Cellular manufacturing (CM) is an application of GT in manufacturing. The machining cells are formed to capture the inherent advantages of GT like reduced setup times, reduced in-process inventories, improved product quality, shorter lead time, reduced tool requirements, improved productivity, better overall control of operations, etc. (Singh and Rajamani (1996)).

Gupta (1991) proposed a dynamic algorithm for application of Critical Path method to scheduling of continuous production systems under GT setup, using actual times rather than predicted values. The algorithm attempts to minimize the sum of manufacturing and inventory storage cost.

Jamshidi et al (1993) developed a methodology for integrating scheduling and sequencing of jobs, with common cycles of production, under GT setup. A number of variables were considered namely, demand patterns, coefficients of variation, cost ratios, lot sizing rules, and a sequencing rule. Based on the results of changes in each variable, appropriate tables were developed to assist the decision maker in making decisions concerning the quantity and timing of orders.

Bector et al (1995) developed a simplified computational technique for solving a process-planning problem under GT setup. A multiple part and multiple operations problem was solved, using 'Fordyce and Webster (1984)' approach.

2.3 Summary of the Thesis

The results and methods proved in this thesis are contained in Chapter 3, Chapter 4, and Chapter 5. We summarize them as follows:

Chapter 3 A Matrix Solution Approach to the Critical Path Problem

The purpose of the present chapter is to provide a simple and effective computational technique to the Critical Path problem. The traditional CPM method is considerably lengthy and requires lot of mathematical computations. Therefore, method proposed in this chapter, is an attempt to provide an easy tool to address this kind of problems. The technique is further modeled into a computerized solution using MS Excel that can be very suitably applied to classroom teaching of the method.

Chapter 4 A Comparison of Matrix Solution of Critical Path Problem under Fuzzy Environment with Traditional PERT Method

In this chapter, we extend the Matrix Solution approach of solving a Critical Path Problem, when the activity duration times are fuzzy variables. The proposed method can be very useful for project scheduling in the field of research & development, software development etc. where there is a lack of history for activity durations or a fast learning curve is involved. The proposed approach is compared against the traditional PERT method of modeling uncertainty in projects. Advantage of using fuzzy mathematics is that it gives decision-maker flexibility and quantifies the uncertainty involved in the problem in question.

Chapter 5 A Matrix Solution Approach to Production Scheduling Problem under GT Setup under Fuzzy Environment

In this chapter, the matrix solution approach is adapted to solve a continuous - production scheduling problem under GT setup under fuzzy environment. In the present chapter, problem of continuous -production scheduling under GT setup is solved where the processing times are considered as fuzzy variables to model the inherent variability in processing times due to different operator speeds.

Chapter 6 Case Study - A Matrix Solution Approach to Production Scheduling Problem (under GT Setup) under Fuzzy Environment at XYZ Industries Ltd.

In this chapter, we apply the technique developed in Chapter 5 to a scheduling problem in a real world case.

Chapter 7 Conclusion, Contribution and Recommendations

In this chapter, we present the conclusions and contributions, along with some recommendations for further research on the problems considered in this thesis.

CHAPTER 3

CRITICAL PATH METHOD USING A SIMPLE MATRIX TECHNIQUE

In the present chapter, we develop a simple solution technique for the well-known Critical Path (CPM) Problem using basic matrix algebra. However, the technique presented here, provides an optimal solution and alternative optima if any, in two matrices (initial and the final matrix) and in three simple steps. Furthermore, we compute the early start, late start, early finish, late finish, free slack and total slack using basic matrix algebraic techniques in two additional but simple steps.

3.1. Introduction

A network diagram is the starting point for the CPM problem. A network diagram represents a sequential flow of activities (represented by arcs), ending in different nodes. The traditional approach of solving a critical path problem involves defining each possible path from the beginning to the termination of the project, then calculating the length of each path, and, finally, determining the longest path length. But the technique presented in this chapter, will give a solution to our problem in three simple steps, using basic matrix algebra. We demonstrate, by solving a numerical example, how all the calculations can be very easily performed using MS Excel. The given technique can also be suitably applied for classroom teaching of a Critical Path Problem and its applications.

3.2. General Formulation Of Critical Path Problem (CPM)

Suppose we are given a network that has n number of nodes, of which the source node is named as Node 1 and the destination node as Node n and i, j , . ., k , . . be the other nodes on the network. The n nodes are connected through arcs (i, j) and r_{ij} is the resource, time required for completion of an activity in this case, associated with each arc (i, j), where r_{ij} is not feasible for i $\geq$ j, and arcs providing a one way direction only.

3.2.1. Notations

Let,

M = Initial Resource matrix,

r_{ij} = Elements of Matrix M,

C = Critical Path matrix,

c_{ij} = Elements of Matrix C,

d_k = Longest path of Node k from the source Node 1,

(d_k, i) = label of node k of Matrix C, where Node i is the immediately preceding node to the Node k, on the longest path of the Node k from the source Node 1,

D_n^*: is the length of critical path of Node n from Node 1.

A = Activity matrix,

a_{ij} = Elements of Matrix A, or activity from node i to node j,

M^T = Transpose of Matrix M (initial resource matrix),

R_{ij} = Elements of Matrix M^T,

L = Late Start matrix,

ℓ_{ij} = Elements of Matrix L,

ℓ_k = Minimum Element in Column k of Matrix L,

(ℓ_k, i) = label of Node k of Matrix L, where Node i is the immediately preceding node to the Node k, on the shortest path of the Node k from the source Node 1,

L^T = Transpose of Matrix L,

L_{ij} = Elements of Matrix L^T,

T = Total Slack matrix,

t_{ij} = Elements of Matrix T.

3.2.2. Obtaining the Initial Resource Matrix M and the Activity Matrix A

The Initial resource Matrix M having n rows and n columns is obtained as follows:

a) Assign the resource r_{ij} corresponding to the arc (i, j) in the network to the cell corresponding to, the row i and the column j of the Matrix M, where, i, j = 1, 2, . . ., n and j > i.

b) Assign a resource of 0 to those cells of the Matrix M that correspond to the nodes in the network connected by dummy activities.

Obtain the Activity Matrix A having n rows and n columns by assigning the activity a_{ij} corresponding to the arc (i, j) in the network, to the cell corresponding to, the row i and the column j of the Matrix A, where i, j = 1, 2, . . . n.

3.2.3. Obtaining the Critical Path Matrix, C

a) (i) Obtain the 1st row of Matrix C by copying the 1st row of Matrix M. Declare Node 1 as labeled in C by putting an asterisk on Node 1, in row 1.

(ii) Label Node 2 in Matrix C as $(r_{12}, 1)$ by putting $(r_{12}, 1)$ above the column of Node 2. Highlight Node 2 column, in both Matrix M and Matrix C.

b) (i) In Matrix C, identify the maximum element among the highlighted elements in the column(s) of the labeled node(s). Let this maximum element be at r_{ik} and its value is d_k. If there is more than one maximum element available in different columns of Matrix C, we break the tie arbitrarily.

(ii) Circle the maximum element $r_{ik} = d_k$ in Matrix C. Label Node k, on the matrix by putting (d_k, i) above the column of Node k in Matrix C. If there are more than one maximum elements available in the same column k, but for different values of i = p, q say, this indicates, that there is an alternative critical path available from the source Node 1 to purpose Node n. In this case, we circle both the elements r_{pk} and r_{qk}, and the labels of Node 'k' will be (d_k, p) and (d_k, q).

(iii) Highlight Column k, in both Matrix M and Matrix C.

(iv) Obtain row of the Node k of Matrix C by adding d_k to each elements of the row of Node k of the initial resource Matrix M. Declare Node k as labeled by putting an asterisk on Node k in row k of Matrix C.

c) (i) Repeat Step (b) until all the nodes in Matrix C are labeled.

3.2.4. Identifying the Critical Path and the length of the Critical Path

Looking at the label of column of Node n in the Matrix C, say (d_n, p). Then, d_n gives us the required length (D_n^*) of Critical Path from source Node 1 to destination Node n.

In addition, label (d_n, p) of Matrix C means that, one segment of the Critical Path from the source Node 1 to the destination Node n is "from Node p to Node n".

Next, go to the column of Node p in the Matrix C. Suppose its label is (d_p, q). This implies that another segment of the Critical Path from the source Node 1 to the destination Node n is "from Node q to Node p". Continue in the same manner until a Node m with label (d_m, 1) in Matrix C is reached. This indicates that a critical path from the source Node 1 to the destination Node n has been obtained. Thus obtained critical path from the source Node 1 to the destination Node n is as follows:

"Start at Node 1, go to Node m, ..., go to Node q, go to Node p, go to Node n", or written as (1, m, ..., q, p, n) with the overall critical path length equal to D_n^*.

3.2.5 Alternative Optimal Solutions

As already pointed out in Step 2 (b)–(ii), if there are more than one maximum element available in the same column k, but for different values of i = p, q say, this indicates that there is an alternative critical path available from the source Node 1 to destination Node n. In this case, we circle both the elements r_{pk} and r_{qk}, and the labels of Node k will be (d_k, p) and (d_k, q).

3.2.6. Early Start time, Early Finish time and Free Slack time of an activity

- **Early Start time**

Looking at the originating Node, say Node k, of the activity in the network diagram, go to the column of Node k in Matrix C. Let the label of Node k be (d_k, p), then,

d_k^* = Early Start time of the activity, a_{kj}.

- **Early Finish time**

Early Finish time of the activity is obtained by comparing the elements of Activity Matrix A with the corresponding elements of Matrix C. The elements, c_{ij} of Matrix C give the Early Finish time of corresponding activities, a_{ij} of Activity Matrix A, i.e.,

Early Finish time of activity $[a_{ij}]$ = $[c_{ij}]$

- **Free Slack time**

To find the Free Slack time of an activity in Column k of Activity Matrix A, go to the column of Node k in Matrix C. Let the label of Node k be (d_k, p), then,

Free Slack time of activity a_{ik} is obtained by subtracting the corresponding element c_{ik} of Matrix C from d_k, i.e.,

Free slack of activity, $a_{ik} = d_k - c_{ik}$

3.2.7. Late Start time, Late Finish time and Total Slack time of an activity

- **Late Start matrix**

Late Start Matrix L, having n rows and n columns is obtained as a starting point for determining the Late Start time, Late Finish time and Total Slack time of an activity, using the approach given below:

a) (i) Write the transpose of Matrix M (written in actual calculations in Section 4.2.7. as a mirror image for calculation purposes) and call it Matrix M^T, having elements R_{ij}, corresponding to the Row i and the Column j of the matrix, where, i, j = 1, 2, . . ., n, such that, $[R_{ij}] = [r_{ji}]$

(ii) The first Element ℓ_{11} of Matrix L is equal to critical path length, i.e., D_N*. Node 1 of Matrix L is declared as labeled by putting (ℓ_1, 1) above the column of Node 1 and an asterisk in the row of Node 1. Highlight Node 1 column, in both matrices, i.e., Matrix L and Matrix M^T.

b) (i) Identify the minimum element among the highlighted column elements of Matrix L. Let the minimum element be in Column k and its value be ℓ_k. Circle the minimum element $\ell_{ik} = \ell_k$ in Matrix L and label Node k in the Matrix L as (ℓ_k, i) above the column of Node k. Highlight Column k, in both Matrix L and Matrix M^T.

(ii) Obtain row of the Node k of Matrix L by subtracting each element of the row of Node k of Matrix M^T from ℓ_k. Declare Node k, as labeled by putting an asterisk on Node k, in Row k of Matrix L.

c) (i) Repeat Step (b), until all the nodes in Matrix L are labeled.

- **Transpose of Late Start Matrix L^T**

Transpose of Matrix L is written as L^T, having elements L_{ij} corresponding to the Row i and the Column j of the matrix, where, i, j = 1, 2, . . ., n, such that,

$$[L_{ij}] = [\ell_{ji}]$$

- **Late Start time**

Late Start time of an activity is obtained by comparing the elements of Activity Matrix A with the corresponding elements of matrix L^T. The elements L_{ij} of matrix L^T give us the Late Start time of corresponding activities a_{ij} of Activity Matrix A, i.e.,

$$\text{Late Start time of activity } [a_{ij}] = [L_{ij}]$$

- **Late Finish time**

Identify the minimum element, say $L_{ij_{min}}$, in each row of Matrix L^T and label the node of each row with the minimum element.

Identify the destination node, say Node k of the activity, of which we wish to find the Late Finish Time. Go to the row of Node k, in matrix L^T. Let its label be $L_{ik_{min}}$, then,

$L_{ik_{min}}$ = Late Finish time of all the activities with destination Node as Node k.

- **Total Slack time**

Another matrix called Total Slack Matrix T, having elements t_{ij} is obtained using the following relationship:

$$[t_{ij}] = [L_{ij}] - [c_{ij}] + [r_{ij}]$$

Total Slack time of an activity is obtained by comparing the elements of Activity Matrix A with the corresponding elements of above defined Total Slack Matrix T. The elements t_{ij} of Matrix T give us the Total Slack time of corresponding activities a_{ij} of Activity Matrix A, i.e.,

$$\text{Total Slack time of activity } [a_{ij}] = [t_{ij}]$$

3.3. Numerical Example

We will solve the following example (source Production and Operations Management, pp 517, Author Fogarty et al.) to illustrate our method.

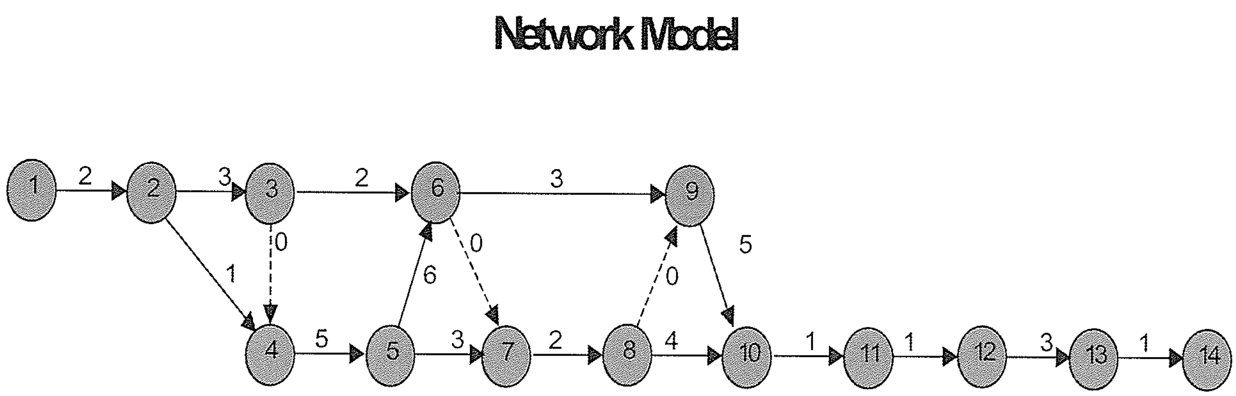


Figure 3.1. Network Diagram (given)

In the above given network diagram, numbers inside the circles represent the i_{th} node, and the numerical values above the arrows represent the duration of the activity, a_{ij}.

3.3.2. Obtaining Initial Resource Matrix M and Activity Matrix A

Following the technique given in Section 3.3.1., we obtain the Initial Resource Matrix M and the Activity Matrix A from the given network as follows:

Matrix M

Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	2	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	3	1	_	_	_	_	_	_	_	_	_	_
3	_	_	_	0	_	2	_	_	_	_	_	_	_	_
4	_	_	_	_	5	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	6	3	_	_	_	_	_	_	_
6	_	_	_	_	_	_	0	_	3	_	_	_	_	_
7	_	_	_	_	_	_	_	2	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	0	4	_	_	_	_
9	_	_	_	_	_	_	_	_	_	5	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	1	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	1	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	3	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	1
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Matrix A

Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	a_{12}	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	a_{23}	a_{24}	_	_	_	_	_	_	_	_	_	_
3	_	_	_	a_{34}	_	a_{36}	_	_	_	_	_	_	_	_
4	_	_	_	_	a_{45}	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	a_{56}	a_{57}	_	_	_	_	_	_	_
6	_	_	_	_	_	_	a_{67}	_	a_{69}	_	_	_	_	_
7	_	_	_	_	_	_	_	a_{78}	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	a_{89}	$a_{8,10}$	_	_	_	_
9	_	_	_	_	_	_	_	_	_	$a_{9,10}$	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	$a_{10,11}$	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	$a_{11,12}$	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	$a_{12,13}$	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	$a_{13,14}$
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

3.3.3. Obtaining the Critical Path Matrix C

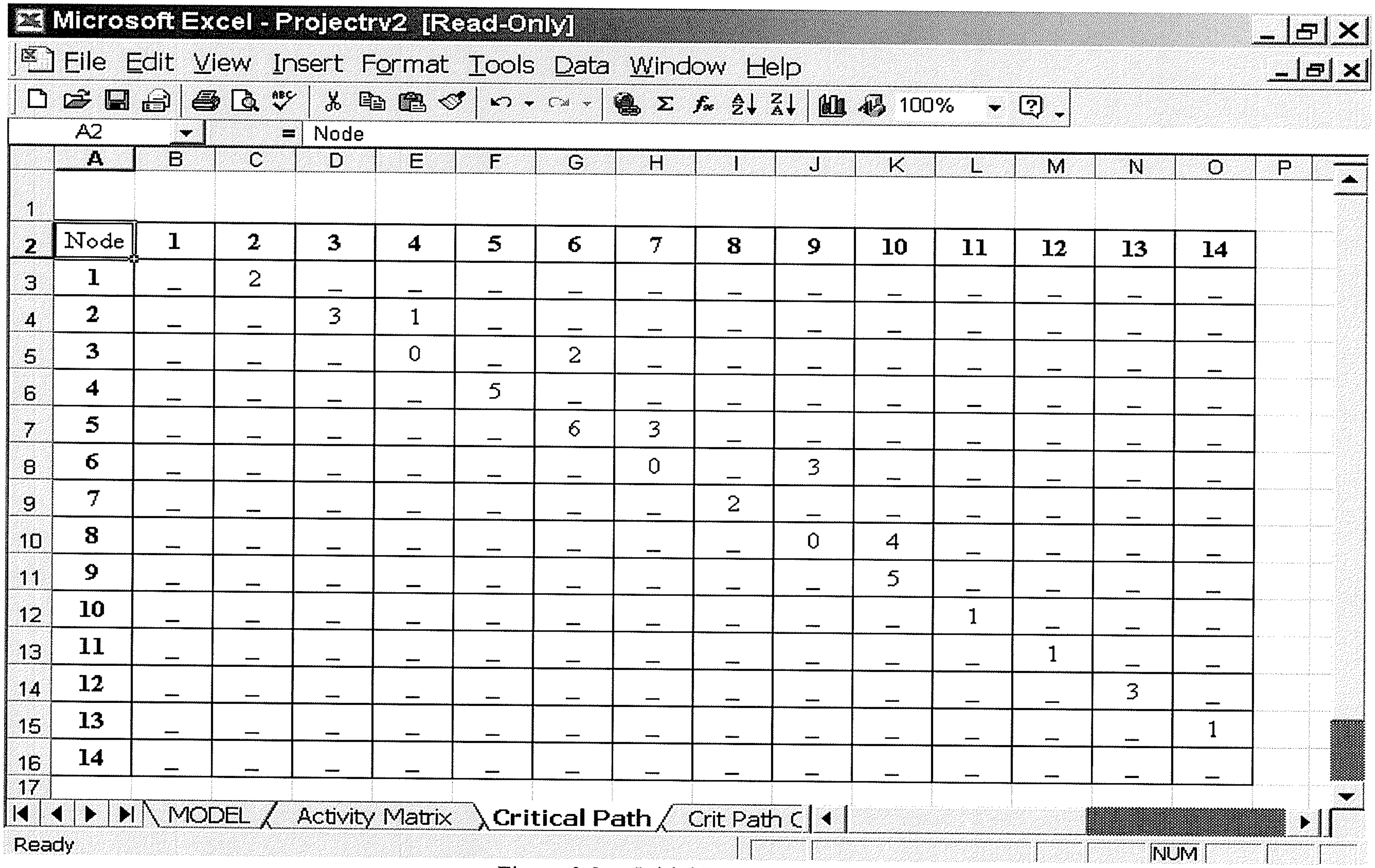
Following the steps outlined in Section 3.2.3., we obtain Matrix C as shown below. The highlighted elements (in red color) represent the nodes that lie on the Critical Path, determined after the matrix is complete.

Matrix C

		(2, 1)	(5, 2)	(5, 3)	(10, 4)	(16, 5)	(16, 6)	(18, 7)	(19, 6)	(24, 9)	(25,10)	(26,11)	(29,12)	(30,13)
Node	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
1*	_	**2**	_	_	_	_	_	_	_	_	_	_	_	_
2*	_	_	**5**	3	_	_	_	_	_	_	_	_	_	_
3*	_	_	_	**5**	_	7	_	_	_	_	_	_	_	_
4*	_	_	_	_	**10**	_	_	_	_	_	_	_	_	_
5*	_	_	_	_	_	**16**	13	_	_	_	_	_	_	_
6*	_	_	_	_	_	_	**16**	_	**19**	_	_	_	_	_
7*	_	_	_	_	_	_	_	**18**	_	_	_	_	_	_
8*	_	_	_	_	_	_	_	_	18	22	_	_	_	_
9*	_	_	_	_	_	_	_	_	_	**24**	_	_	_	_
10*	_	_	_	_	_	_	_	_	_	_	**25**	_	_	_
11*	_	_	_	_	_	_	_	_	_	_	_	**26**	_	_
12*	_	_	_	_	_	_	_	_	_	_	_	_	**29**	_
13*	_	_	_	_	_	_	_	_	_	_	_	_	_	**30**
14*	_	_	_	_	_	_	_	_	_	_	_	_	_	_

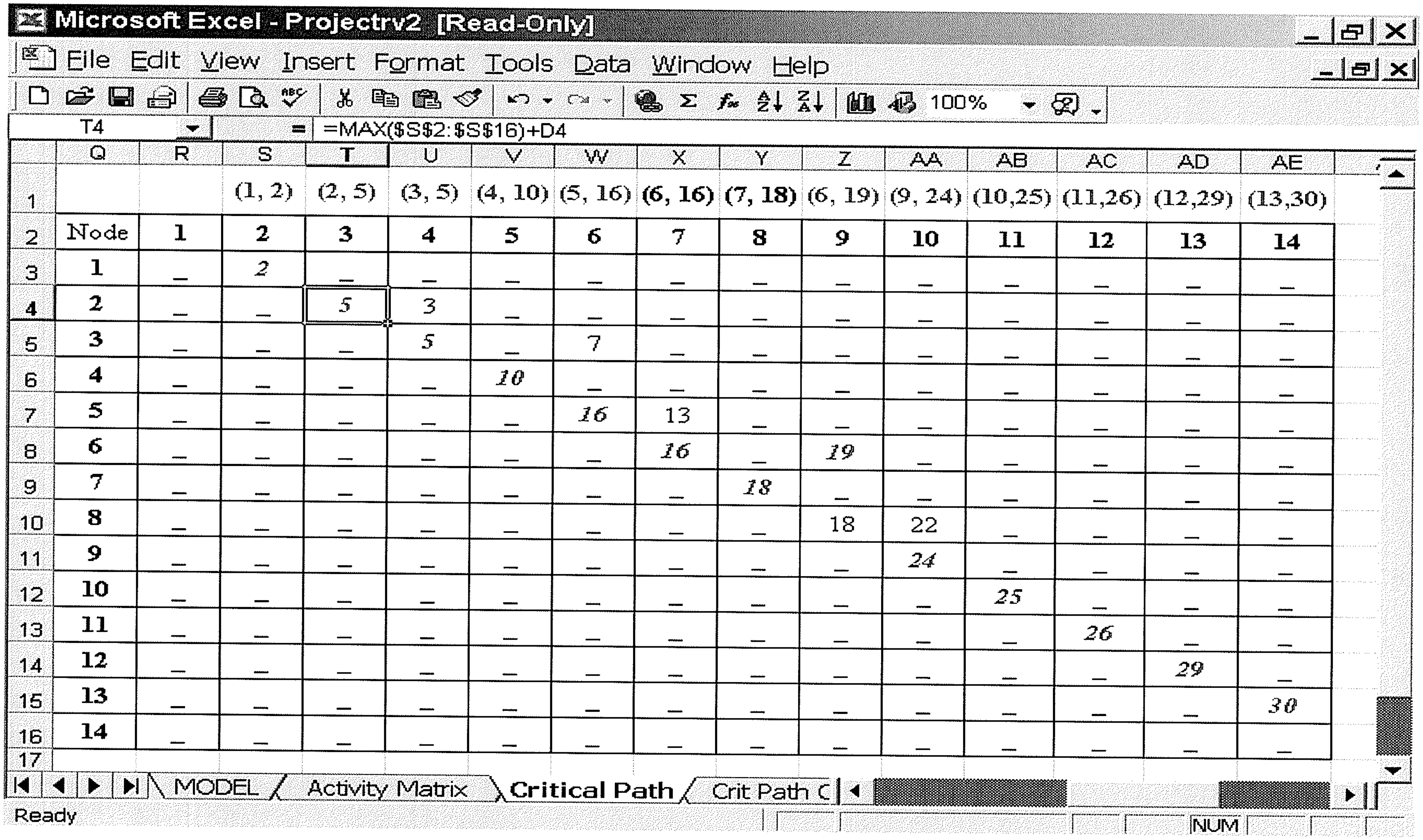
Figure 3.2. and Figure 3.3. below shows how the above calculations can be easily performed using MS Excel. The key formulae used in different cells in Figure 3.3. are shown below:

CELLS	KEY FORMULAE
T5	=MAX($S$3:$S$17)+D5
COPIED TO	T5 : AE5
U6	=MAX($T$3:$T$17)+E6
COPIED TO	U6 : AE6
AND SO ON	



Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	2	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	3	1	_	_	_	_	_	_	_	_	_	_
3	_	_	_	0	_	2	_	_	_	_	_	_	_	_
4	_	_	_	_	5	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	6	3	_	_	_	_	_	_	_
6	_	_	_	_	_	_	0	_	3	_	_	_	_	_
7	_	_	_	_	_	_	_	2	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	0	4	_	_	_	_
9	_	_	_	_	_	_	_	_	_	5	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	1	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	1	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	3	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	1
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.2. – Initial Resource Matrix



Microsoft Excel - Projectrv2 [Read-Only]

File Edit View Insert Format Tools Data Window Help

T4 = =MAX($S$2:$S$16)+D4

	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE
1			(1, 2)	(2, 5)	(3, 5)	(4, 10)	(5, 16)	**(6, 16)**	**(7, 18)**	(6, 19)	(9, 24)	(10,25)	(11,26)	(12,29)	(13,30)
2	Node	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
3	**1**	_	*2*	_	_	_	_	_	_	_	_	_	_	_	_
4	**2**	_	_	*5*	3	_	_	_	_	_	_	_	_	_	_
5	**3**	_	_	_	*5*	_	7	_	_	_	_	_	_	_	_
6	**4**	_	_	_	_	*10*	_	_	_	_	_	_	_	_	_
7	**5**	_	_	_	_	_	*16*	13	_	_	_	_	_	_	_
8	**6**	_	_	_	_	_	_	*16*	_	*19*	_	_	_	_	_
9	**7**	_	_	_	_	_	_	_	*18*	_	_	_	_	_	_
10	**8**	_	_	_	_	_	_	_	_	18	22	_	_	_	_
11	**9**	_	_	_	_	_	_	_	_	_	*24*	_	_	_	_
12	**10**	_	_	_	_	_	_	_	_	_	_	*25*	_	_	_
13	**11**	_	_	_	_	_	_	_	_	_	_	_	*26*	_	_
14	**12**	_	_	_	_	_	_	_	_	_	_	_	_	*29*	_
15	**13**	_	_	_	_	_	_	_	_	_	_	_	_	_	*30*
16	**14**	_	_	_	_	_	_	_	_	_	_	_	_	_	_
17															

MODEL / Activity Matrix / **Critical Path** / Crit Path C

Ready NUM

Figure 3.3. – Critical Path Matrix

3.3.4. Identifying Critical Path and determining Critical Path length

Following the technique outlined in Section 3.2.4., the Critical Path for the given project from Node 1 to Node 14, as determined from the Critical Path Matrix C is as shown below:

Node 1 to Node 2 to Node 3 to Node 4 to Node 5 to Node 6 to Node 9 to Node 10 to Node 11 to Node 12 to Node 13 to Node 14,

Also,

Critical Path length $D_n^* = 30$

Shown below is the given network with Critical Path highlighted in red color:

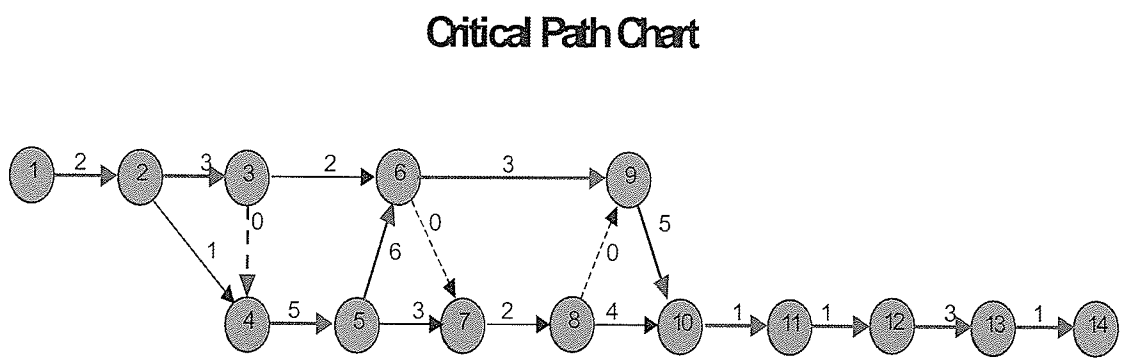


Figure 3.4. – Network Diagram (outlining Critical Path)

3.3.5. Alternative Optimal solutions

As explained in Section 3.2.5., alternative optimal solution exists in case of an occurrence of more than one maximum elements in the same column. As we did not have any such tie in our solution, hence the obtained solution is unique to the given network.

3.3.6. Early Start time, Early Finish time and Free Slack time of an activity

Following the technique outlined in Section 3.2.5., Early Start time, Early Finish time and Free Slack time of an activity are obtained as below:

- **Early Start time**

Following the technique explained in Section 3.2.6., we construct a table (Table 3.1.) showing the Early Start time for all the activities in the Network Diagram.

Table 3.1.

Node	Originating Activity (s)	Early Start time = d_k (from labels in Matrix C)
1	a_{12}	0
2	a_{23}, a_{24}	2
3	a_{34}, a_{36}	5
4	a_{45}	5
5	a_{56}, a_{57}	10
6	a_{67}, a_{69}	16
7	a_{78}	16
8	a_{89}, $a_{8,10}$	18
9	$a_{9,10}$	19
10	$a_{10,11}$	24
11	$a_{11,12}$	25
12	$a_{12,13}$	26
13	$a_{13,14}$	29

- **Early Finish time**

As explained earlier in Section 3.2.6., Early Finish time of an activity can be obtained by comparing elements of Activity Matrix A with the corresponding elements of Critical Path Matrix C. We construct a table (Table 3.2.) showing Early Finish time of all the activities on the Network Diagram, following the above technique.

Table 3.2.

Activity [a_{ij}]	Early Finish time = [c_{ij}]
a_{12}	2
a_{23}	5
a_{24}	3
a_{34}	5
a_{36}	7
a_{45}	10
a_{56}	16
a_{57}	13
a_{67}	16
a_{69}	19
a_{78}	18
a_{89}	18
$a_{8,10}$	22
$a_{9,10}$	24
$a_{10,11}$	25
$a_{11,12}$	26
$a_{12,13}$	29
$a_{13,14}$	30

- **Free Slack time**

We construct another table (Table 3.3.) showing the Free Slack time for all the activities on the Network Diagram, following the technique outlined in Section 3.2.6.

Table 3.3.

Activity $[a_{ik}]$	d_k (label of column k in Matrix C)	$[c_{ik}]$ (Matrix C)	Free Slack time = $[d_k] - [c_{ik}]$
a_{12}	2	2	0
a_{23}	5	5	0
a_{24}	5	3	2
a_{34}	5	5	0
a_{45}	10	10	0
a_{36}	16	7	9
a_{56}	16	16	0
a_{57}	16	13	3
a_{67}	16	16	0
a_{78}	18	18	0
a_{69}	19	19	0
a_{89}	19	18	1
$a_{8,10}$	24	22	2
$a_{9,10}$	24	24	0
$a_{10,11}$	25	25	0
$a_{11,12}$	26	26	0
$a_{12,13}$	29	29	0
$a_{13,14}$	30	30	0

3.3.7. Late Start time, Late Finish time and Total Slack time of an Activity

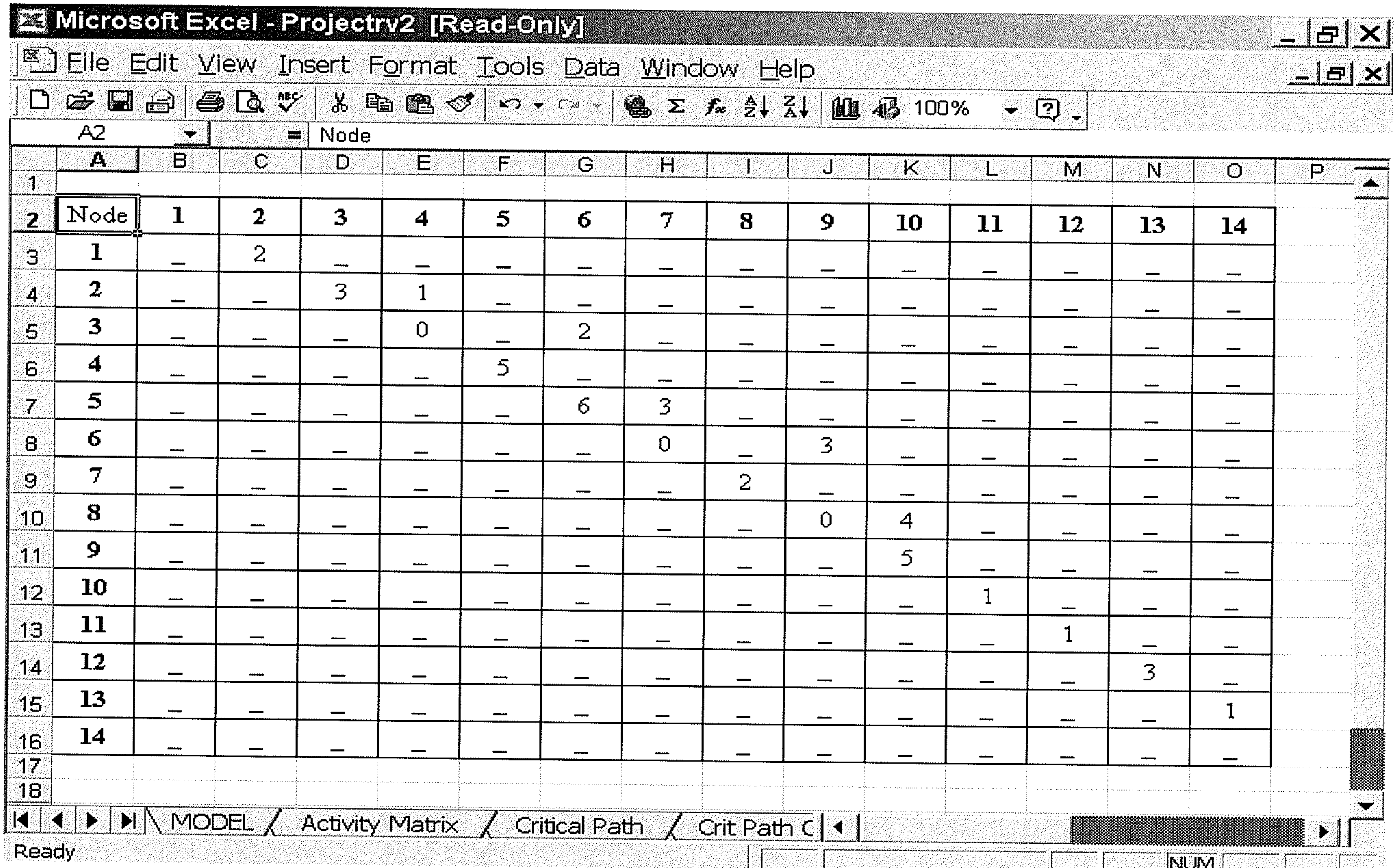
- **Late Start Matrix**

As described in Section 3.2.7, we obtain the Transpose of Matrix M and call it Matrix M^T as follows:

Matrix M^T

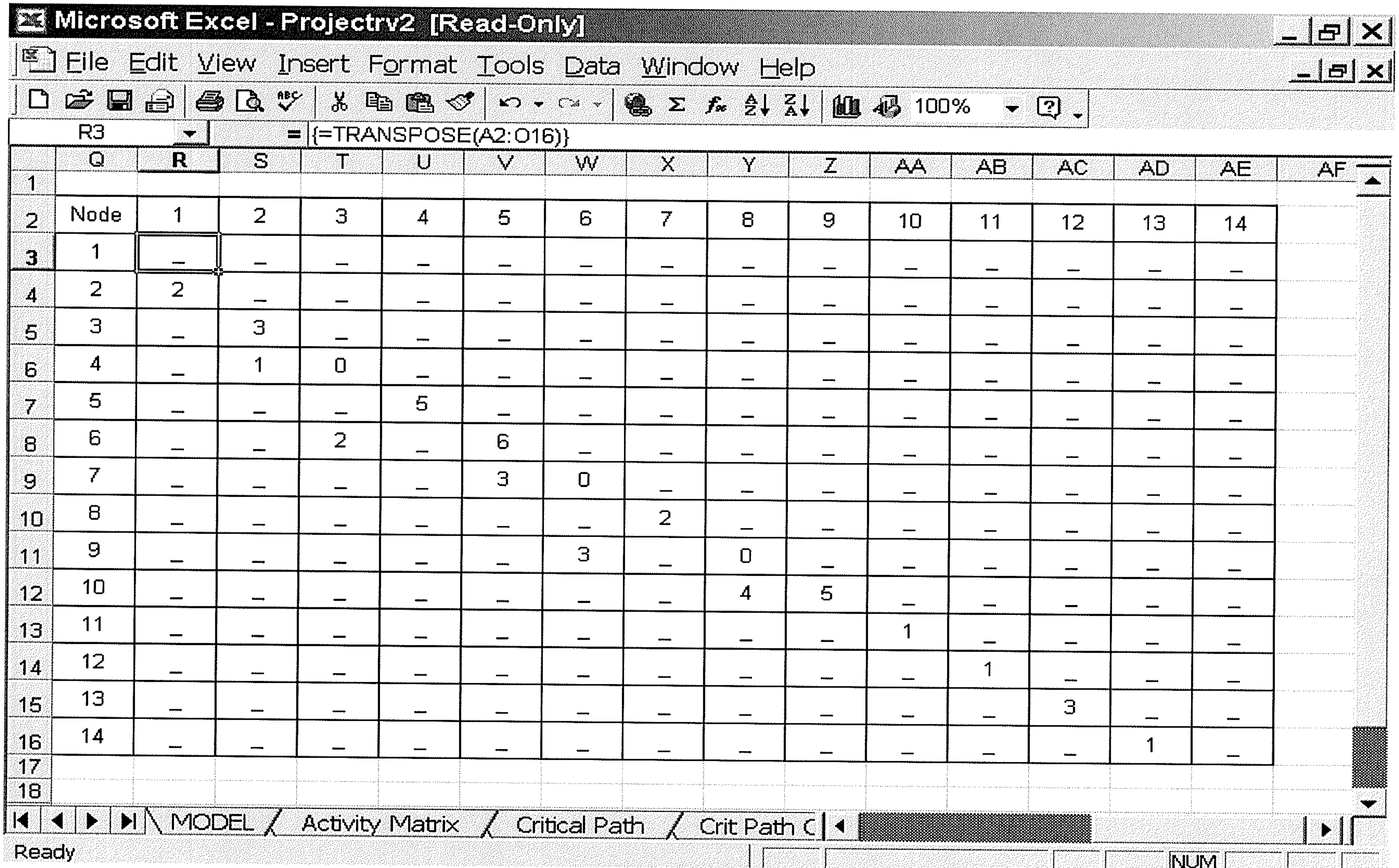
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	1	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	3	_	_	_	_	_	_	_	_	_	_	_
3	_	_	_	1	_	_	_	_	_	_	_	_	_	_
4	_	_	_	_	1	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	5	4	_	_	_	_	_	_	_
6	_	_	_	_	_	_	0	_	3	_	_	_	_	_
7	_	_	_	_	_	_	_	2	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	0	3	_	_	_	_
9	_	_	_	_	_	_	_	_	_	6	_	2	_	_
10	_	_	_	_	_	_	_	_	_	_	5	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	0	1	_
12	_	_	_	_	_	_	_	_	_	_	_	_	3	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	2
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.5. and Figure 3.6. below, show how the above transpose can be easily obtained using MS Excel, along with the key formulae used.



Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	2	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	3	1	_	_	_	_	_	_	_	_	_	_
3	_	_	_	0	_	2	_	_	_	_	_	_	_	_
4	_	_	_	_	5	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	6	3	_	_	_	_	_	_	_
6	_	_	_	_	_	_	0	_	3	_	_	_	_	_
7	_	_	_	_	_	_	_	2	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	0	4	_	_	_	_
9	_	_	_	_	_	_	_	_	_	5	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	1	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	1	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	3	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	1
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.5. – Matrix M



Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	_	_	_	_	_	_	_	_	_	_	_	_	_
2	2	_	_	_	_	_	_	_	_	_	_	_	_	_
3	_	3	_	_	_	_	_	_	_	_	_	_	_	_
4	_	1	0	_	_	_	_	_	_	_	_	_	_	_
5	_	_	_	5	_	_	_	_	_	_	_	_	_	_
6	_	_	2	_	6	_	_	_	_	_	_	_	_	_
7	_	_	_	_	3	0	_	_	_	_	_	_	_	_
8	_	_	_	_	_	_	2	_	_	_	_	_	_	_
9	_	_	_	_	_	3	_	0	_	_	_	_	_	_
10	_	_	_	_	_	_	_	4	5	_	_	_	_	_
11	_	_	_	_	_	_	_	_	_	1	_	_	_	_
12	_	_	_	_	_	_	_	_	_	_	1	_	_	_
13	_	_	_	_	_	_	_	_	_	_	_	3	_	_
14	_	_	_	_	_	_	_	_	_	_	_	_	1	_

Figure 3.6. - Matrix M^T

The key formulae used in Figure 3.6. shown above are as follows:

CELLS	KEY FORMULAE
Q2 : AE16	{=TRANSPOSE(A2:O16)}

When an array formula is entered, Microsoft Excel automatically inserts the formula between { } (braces). The technique to enter array formula is

- If the array formula will return one result, click the cell in which you want to enter the array formula. If the array formula will return multiple results, select the range of cells in which you want to enter the array formula.
- Type the array formula and press CTRL+SHIFT+ENTER.

Late Start Matrix L is obtained following the steps outlined in Section 3.2.7., as shown below. We highlight the minimum elements of each column in red color.

Matrix L

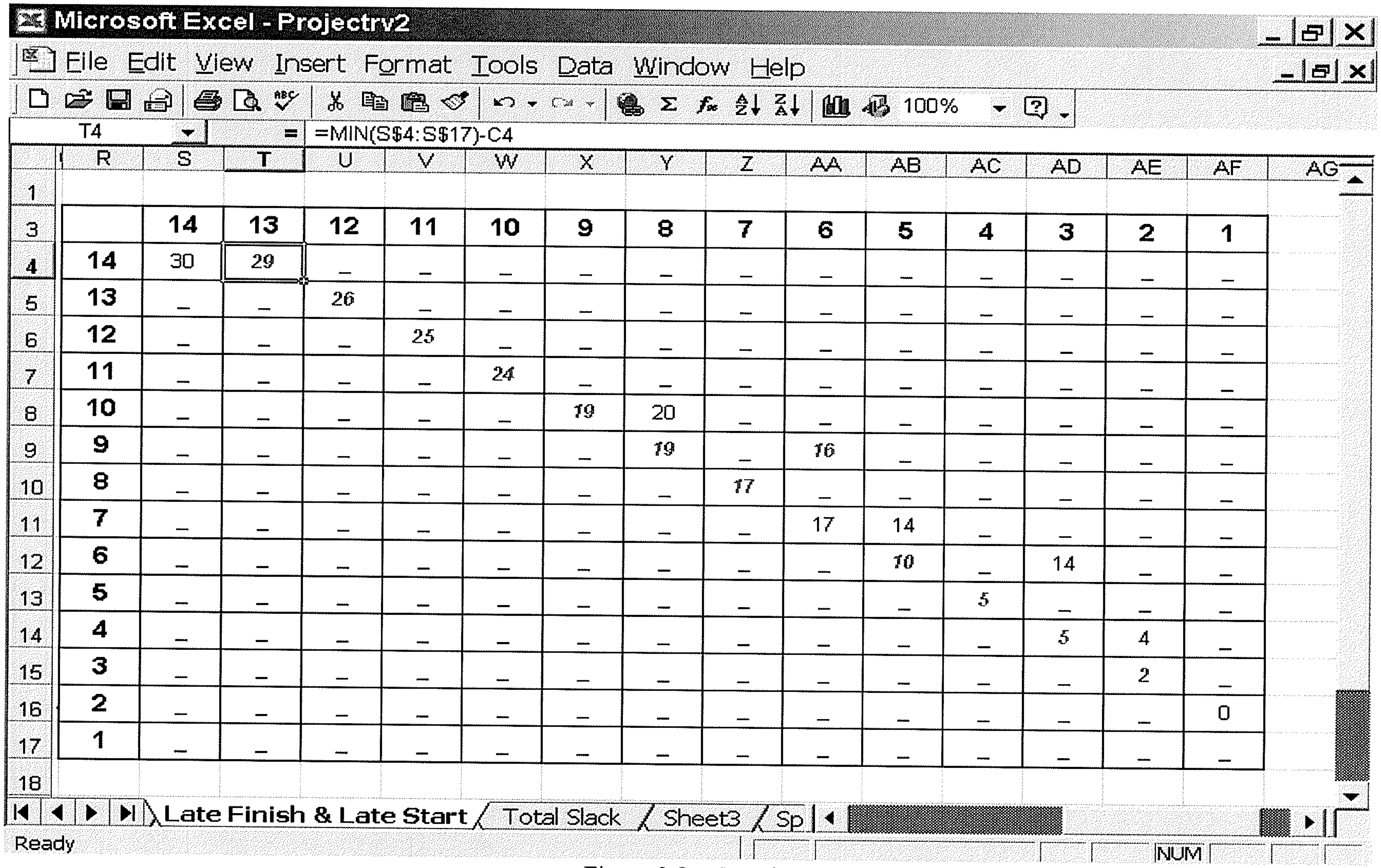
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	30	29	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	26	_	_	_	_	_	_	_	_	_	_	_
3	_	_	_	25	_	_	_	_	_	_	_	_	_	_
4	_	_	_	_	24	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	19	20	_	_	_	_	_	_	_
6	_	_	_	_	_	_	19	_	16	_	_	_	_	_
7	_	_	_	_	_	_	_	17	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	17	14	_	_	_	_
9	_	_	_	_	_	_	_	_	_	10	_	14	_	_
10	_	_	_	_	_	_	_	_	_	_	5	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	5	4	_
12	_	_	_	_	_	_	_	_	_	_	_	_	2	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	0
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.7. and Figure 3.8. below, show how the above calculations can be easily performed using MS Excel. The key formulae used in different cells in Figure 3.8. are as shown below:

CELLS	KEY FORMULAE
T4	=MIN(S$4:S$17)-C4
COPIED TO	T5 : AF4
U5	=MIN(T$4:T$17)-D5
COPIED TO	U5 : AF5
AND SO ON	

Node	14	13	12	11	10	9	8	7	6	5	4	3	2	1
14	_	1	_	_	_	_	_	_	_	_	_	_	_	_
13	_	_	3	_	_	_	_	_	_	_	_	_	_	_
12	_	_	_	1	_	_	_	_	_	_	_	_	_	_
11	_	_	_	_	1	_	_	_	_	_	_	_	_	_
10	_	_	_	_	_	5	4	_	_	_	_	_	_	_
9	_	_	_	_	_	_	0	_	3	_	_	_	_	_
8	_	_	_	_	_	_	_	2	_	_	_	_	_	_
7	_	_	_	_	_	_	_	_	0	3	_	_	_	_
6	_	_	_	_	_	_	_	_	_	6	_	2	_	_
5	_	_	_	_	_	_	_	_	_	_	5	_	_	_
4	_	_	_	_	_	_	_	_	_	_	_	0	1	_
3	_	_	_	_	_	_	_	_	_	_	_	_	3	_
2	_	_	_	_	_	_	_	_	_	_	_	_	_	2
1	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.7. – Mirror image of Matrix M^T



	14	13	12	11	10	9	8	7	6	5	4	3	2	1
14	30	29	_	_	_	_	_	_	_	_	_	_	_	_
13	_	_	26	_	_	_	_	_	_	_	_	_	_	_
12	_	_	_	25	_	_	_	_	_	_	_	_	_	_
11	_	_	_	_	24	_	_	_	_	_	_	_	_	_
10	_	_	_	_	_	19	20	_	_	_	_	_	_	_
9	_	_	_	_	_	_	19	_	16	_	_	_	_	_
8	_	_	_	_	_	_	_	17	_	_	_	_	_	_
7	_	_	_	_	_	_	_	_	17	14	_	_	_	_
6	_	_	_	_	_	_	_	_	_	10	_	14	_	_
5	_	_	_	_	_	_	_	_	_	_	5	_	_	_
4	_	_	_	_	_	_	_	_	_	_	_	5	4	_
3	_	_	_	_	_	_	_	_	_	_	_	_	2	_
2	_	_	_	_	_	_	_	_	_	_	_	_	_	0
1	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.8. – Matrix L

- **Transpose of Matrix L, as Matrix L^T**

Following the steps outlined in Section 3.2.7., we write the transpose of Matrix L as follows:

Matrix L^T

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
1	_	0	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	2	4	_	_	_	_	_	_	_	_	_	_
3	_	_	_	5	_	14	_	_	_	_	_	_	_	_
4	_	_	_	_	5	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	10	14	_	_	_	_	_	_	_
6	_	_	_	_	_	_	17	_	16	_	_	_	_	_
7	_	_	_	_	_	_	_	17	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	19	20	_	_	_	_
9	_	_	_	_	_	_	_	_	_	19	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	24	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	25	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	26	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	29
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

- **Late Start Time**

Following the steps outlined in Section 3.2.7., we construct a table (Table 3.4.) to show the Late Start time for all the activities on the Network Diagram.

Table 3.4.

Activity $[a_{ij}]$	Late Start time = $[L_{ij}]$
a_{12}	0
a_{23}	2
a_{24}	4
a_{34}	5
a_{36}	14
a_{45}	5
a_{56}	10
a_{57}	14
a_{67}	17
a_{69}	16
a_{78}	17
a_{89}	19
$a_{8,10}$	20
$a_{9,10}$	19
$a_{10,11}$	24
$a_{11,12}$	25
$a_{12,13}$	25
$a_{13,14}$	29

- **Late Finish Time**

We prepare a table (Table 3.5.) showing the Late Finish time of all the activities on the Network Diagram based on the technique given in Section 3.2.7., as follows:

Table 3.5.

Node	Terminating Activity (s)	Late Finish time = $L_{ik_{min}}$ (from row 'k' of matrix 'L^T')
2	a_{12}	2
3	a_{23}	5
4	a_{34}, a_{24}	5
5	a_{45}	10
6	a_{56}, a_{36}	16
7	a_{67}, a_{57}	17
8	a_{78}	19
9	a_{89}, a_{69}	19
10	$a_{9,10}$, $a_{8,10}$	24
11	$a_{10,11}$	25
12	$a_{11,12}$	26
13	$a_{12,13}$	29
14	$a_{13,14}$	30

- **Total Slack Time**

As described in Section 3.2.7., we obtain another Matrix T called the Total Slack matrix having elements t_{ij} as shown below, where,

$[t_{ij}] = [L_{ij}] - [c_{ij}] + [r_{ij}]$, Matrix T

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	0	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	0	2	_	_	_	_	_	_	_	_	_	_
3	_	_	_	0	_	9	_	_	_	_	_	_	_	_
4	_	_	_	_	0	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	0	4	_	_	_	_	_	_	_
6	_	_	_	_	_	_	1	_	0	_	_	_	_	_
7	_	_	_	_	_	_	_	1	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	1	2	_	_	_	_
9	_	_	_	_	_	_	_	_	_	0	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	0	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	0	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	0	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	0
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.9. to Figure 3.12. below, show how the above shown Matrix T can be easily obtained using MS Excel. The key formulae used in different cells are as shown below:

CELLS	KEY FORMULAE
S18	=S2+C18-C2
COPIED TO	S18 : AE30

Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	2	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	5	3	_	_	_	_	_	_	_	_	_	_
3	_	_	_	5	_	7	_	_	_	_	_	_	_	_
4	_	_	_	_	10	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	16	13	_	_	_	_	_	_	_
6	_	_	_	_	_	_	16	_	19	_	_	_	_	_
7	_	_	_	_	_	_	_	18	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	18	22	_	_	_	_
9	_	_	_	_	_	_	_	_	_	24	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	25	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	26	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	29	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	30
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

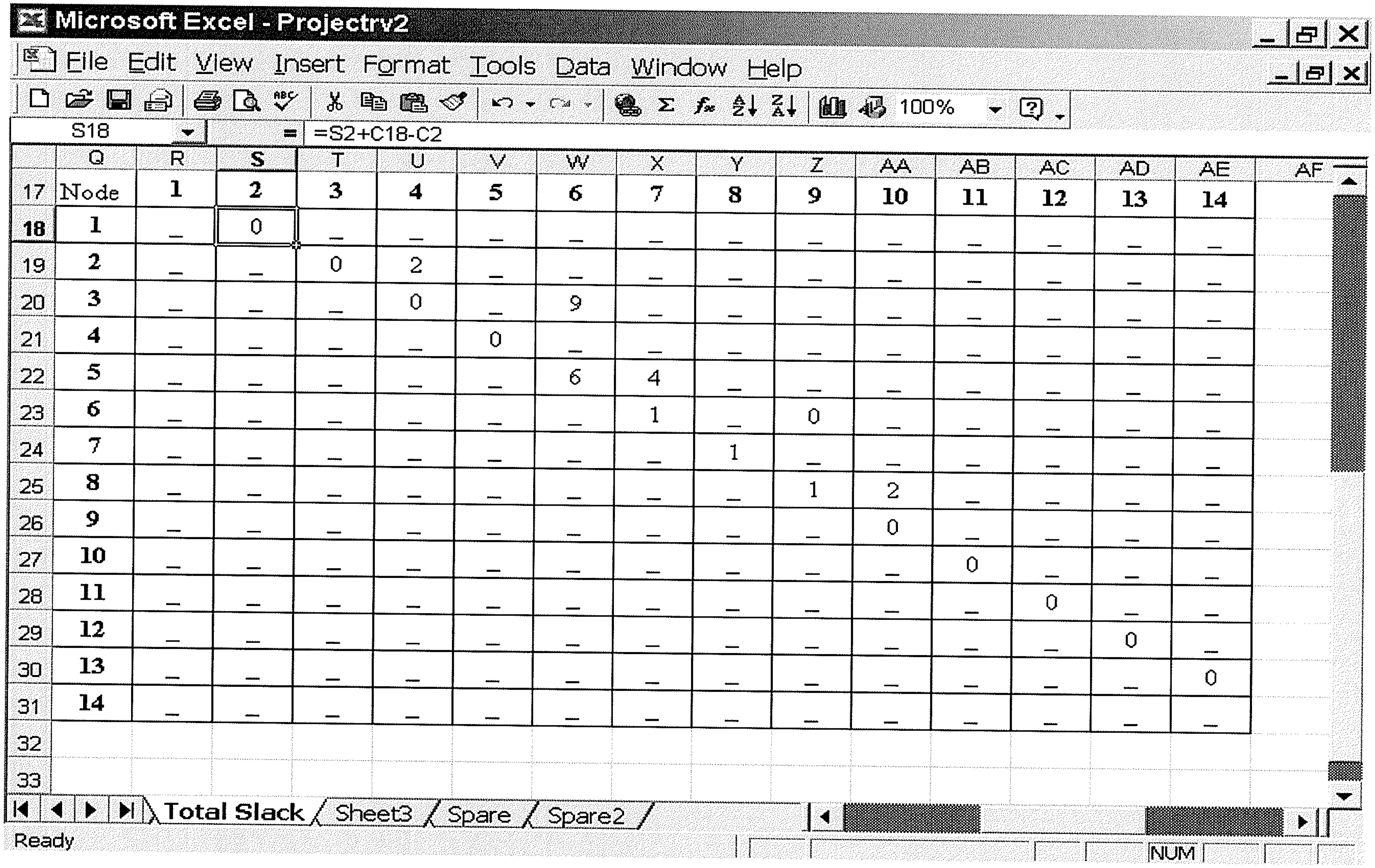
Figure 3.9. – Matrix C

Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	0	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	2	4	_	_	_	_	_	_	_	_	_	_
3	_	_	_	5	_	14	_	_	_	_	_	_	_	_
4	_	_	_	_	5	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	16	14	_	_	_	_	_	_	_
6	_	_	_	_	_	_	17	_	16	_	_	_	_	_
7	_	_	_	_	_	_	_	17	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	19	20	_	_	_	_
9	_	_	_	_	_	_	_	_	_	19	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	24	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	25	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	26	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	29
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.10. – Matrix L

Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	2	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	3	1	_	_	_	_	_	_	_	_	_	_
3	_	_	_	0	_	2	_	_	_	_	_	_	_	_
4	_	_	_	_	5	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	6	3	_	_	_	_	_	_	_
6	_	_	_	_	_	_	0	_	3	_	_	_	_	_
7	_	_	_	_	_	_	_	2	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	0	4	_	_	_	_
9	_	_	_	_	_	_	_	_	_	5	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	1	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	1	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	3	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	1
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.11. – Matrix M



Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	0	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	0	2	_	_	_	_	_	_	_	_	_	_
3	_	_	_	0	_	9	_	_	_	_	_	_	_	_
4	_	_	_	_	0	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	6	4	_	_	_	_	_	_	_
6	_	_	_	_	_	_	1	_	0	_	_	_	_	_
7	_	_	_	_	_	_	_	1	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	1	2	_	_	_	_
9	_	_	_	_	_	_	_	_	_	0	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	0	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	0	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	0	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	0
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

Figure 3.12. – Matrix T

We construct a table (Table 3.6.) that shows the Total Slack time for all the activities on the Network Diagram, following the technique described in Section 3.2.7., as shown below:

Table 3.6.

Activity [a_{ij}]	Total Slack time = [t_{ij}]
a_{12}	0
a_{23}	0
a_{24}	2
a_{34}	0
a_{36}	9
a_{45}	0
a_{56}	0
a_{57}	4
a_{67}	1
a_{69}	0
a_{78}	1
a_{89}	1
$a_{8,10}$	2
$a_{9,10}$	0
$a_{10,11}$	0
$a_{11,12}$	0
$a_{12,13}$	0
$a_{13,14}$	0

3.4. Conclusion

In this chapter, we have described a simple step-by-step computational technique for solving a Critical Path problem, which, we believe, is easier to use than traditional methods. At the same time, we have shown how MS Excel can be used to perform so easily all the calculations for us. Also, the suggested algorithms that yield Early Start time, Early Finish time, Late Start time, Late Finish time, Total Slack time and Free Slack time of every activity on the Network Diagram, are very easy to follow and give the required solution at a glance.

CHAPTER 4

PROJECT SCHEDULING UNDER FUZZY ENVIRONMENT USING MATRIX ALGEBRA TECHNIQUE

The uncertainty in the activity time estimates has always troubled project managers and has been an immensely interesting topic both for academics and researchers. A lot of different probabilistic approaches have been presented in the past to tackle the issue of uncertainty in the activity time estimates. In the present chapter, we analyze the probabilistic approach of Program Evaluation and Review Technique (PERT) method. We also propose an alternate approach to probability theory for quantifying uncertainty related to activity duration. We compare and contrast the proposed method with the PERT method using an example.

4.1. Introduction

Matthew J. Liberatore (2002) presented a method for assessing effects of uncertainty on project scheduling using fuzzy logic. At the same time, the author states "improved computational procedures are needed so that fuzzy logic can be applied to projects of arbitrary size". We illustrate a simple computational technique for solving the project-scheduling problem using the fuzzy logic approach.

PERT has been an accepted traditional approach, for addressing project schedule uncertainty. PERT, as explained earlier in Chapter 2, is based on critical path analysis and applies probability theory to quantify the uncertainty related to the time required to complete project activities. However, there is an alternative approach for measuring schedule uncertainty based on fuzzy logic. Fuzzy logic is an approach for measuring imprecision or vagueness in estimation and may be preferred to probability theory in capturing activity duration uncertainty in some situations.

Fuzzy Logic vs. Probability (Liberatore (2002))

Probability theory often is used to model the uncertain duration of a project activity. We might say that the probability that a given activity will take seven days to complete is 0.8 and the probability that it will take eight days is 0.2. Because these two probabilities sum to 1.0, no other activity durations are possible in this example. The interpretation of these probability statements is that after repeating this activity many times, the activity time was seven days in 80% of the cases, and the activity time was eight days in 20% of the cases. In essence, there is an 80% chance it will take seven days and a 20% chance it will require eight days. This example shows that randomness describes the uncertainty of event occurrence in the future. The event "activity time takes seven days" does or does not occur, and the event "activity time takes eight days" does or does not occur. After the activity is completed, randomness dissipates because the activity is known to have taken a specific amount of time; that is, one of these events occurred.

Fuzziness is concerned with event ambiguity. It measures the degree to which an event occurs, not whether it occurs. Using fuzzy logic, we might say that the given activity takes "approximately seven days" to complete. This statement reflects the imprecision or vagueness of our time estimate. Here, we define "approximately seven days" as a particular type of fuzzy set called a fuzzy quantity. The membership of seven within this set is 1.0, because seven certainly is a member. Other possible times such as eight are assigned membership values, also called beliefs, based on our meaning of the vague notion of "approximately seven days." Note that the beliefs are not required to sum to 1.0. Fuzziness remains after the event occurs because the actual time is one of the possible times in our fuzzy set and still is "approximately seven days."

4.2. PERT Method in Detail (Fogarty et al (1989))

PERT achieves a probabilistic estimate of project completion by obtaining three estimates for each activity, describing the statistical distribution of possible times for each activity, and determining the standard deviation of each activity time and of the total project completion time.

According to PERT, the three time estimates for each activity are:

- Optimistic time (a)
- Pessimistic time (b)
- Most Likely time (m)

The a and b times are estimated on the basis that the probability of an actual time falling outside the range is about 1%. The expected activity time and the variance calculations assume that the activity times follow beta (β) distribution (Fogarty et al., 1989).

4.2.1. Notations

Let,

t_{ei} = Expected Activity time of i^{th} Activity,

σ^2_{ti} = Variability of i^{th} Activity time,

i = j for Activities on Critical Path,

k = Number of Activities on the Critical Path,

T_E = Expected time required to complete project,

$\sigma^2{}_T$ = Variability of Expected project completion time.

4.2.2. General Solution

The estimates of the Expected Activity time and its variability measures for any particular activity are as follows:

$$t_{ei} = \frac{a + 4m + b}{6}$$

$$\sigma^2{}_{ti} = \left(\frac{b-a}{6}\right)^2$$

Critical Path length is the sum of the Expected times of all the activities on the Critical Path, i.e.,

$$T_E = \sum_{j=1}^{k} t_{ei}$$

The variance of the Expected project completion time is equal to the sum of the variances of all the activities on the Critical Path, i.e.,

$$\sigma^2{}_T = \sum_{j=1}^{k} \sigma^2{}_{ti}$$

4.2.3. Numerical Example (Fogarty et al. (1989), pp. 524-25)

We reproduce the example solved by Fogarty et al (1989) to illustrate the technique. The Network Diagram used in this case is the same as the one shown in Chapter 3, Section 3.3.1. The activity times given in the Network Diagram are considered as Most Likely times (m) for those activities.

We recreate table 15-4 (pp 524) as Table 4.1., showing the three time estimates, Expected activity time, and its variability for all the activities.

Table 4.1.

Activity	'a'	'm'	'b'	t_{ei}	σ^2_{ti}
a_{12}	1.6	2.0	2.4	2	0.018
a_{23}	2.0	3.0	4.6	3.1	0.188
a_{24}	0.9	1.0	2.0	1.15	0.034
a_{45}	3.0	5.0	7.0	5.0	0.444
a_{36}	0.6	2.0	2.8	1.9	0.134
a_{56}	4.6	6.0	7.4	6.0	0.218
a_{57}	2.5	3.0	3.5	3.0	0.028
a_{78}	2.0	2.0	3.0	2.17	0.028
a_{69}	2.0	3.0	5.0	3.17	0.250
$a_{8,10}$	2.0	4.0	6.0	4.0	0.444
$a_{9,10}$	4.0	5.0	8.0	5.33	0.444
$a_{11,12}$	1.0	1.0	1.0	1.0	0.000
$a_{10,11}$	0.8	1.0	2.0	1.13	0.040
$a_{12,13}$	2.8	3.0	3.6	3.07	0.018
$a_{13,14}$	1.0	1.0	3.0	1.33	0.111

In this example, the critical path consists of Activities, a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{69}, $a_{9,10}$, $a_{10,11}$, $a_{11,12}$, $a_{12,13}$, and $a_{13,14}$. Using the data from the above table, we have,

$$T_E = 2.0 + 3.1 + 0 + 5.0 + 6.0 + 3.17 + 5.33 + 1.13 + 1.0 + 3.07 + 1.33$$

$$= 31.13$$

$$\sigma^2_T = .018 + .188 + 0 + .444 + .218 + .25 + .444 + .04 + 0 + .018 + .111$$

$$= 1.731$$

4.3. Fuzzy Logic Approach to Project Scheduling

Fuzzy Logic can be used to model the uncertainty present in the activity times. We will use the technique described in Chapter 3, Section 3.2., to obtain the solution to our problem.

We define t_{ij} as time required to finish the task between the source Node i and destination Node j where i = 1, 2, . . . n; j = 1, 2, . . . n and j > i, meaning that an activity providing one way direction only.

Further, t_{ij} is defined as a triangular fuzzy number of the type:

$$t_{ij} = [t_{ij1}, t_{ij2}, t_{ij3}]$$

The different values of t_{ij} can be obtained from the experts who share the same information but different opinion.

Each fuzzy number t_{ij} and its membership function are determined on the line of 'Kaufman and Gupta (1985, 1988)' by using the interval of confidence for t_{ij} at α - level.

$$t_{ij}^{\alpha} = \{(t_{ij2} - t_{ij1})\alpha + t_{ij1}, (t_{ij2} - t_{ij3})\alpha + t_{ij3}\} \ \forall \quad \alpha \in [0,1] \qquad (1)$$

where,

- Setting $\alpha = 0$, we get the end points t_{ij1} and t_{ij3} of the fuzzy number t_{ij}
- Setting $\alpha = 1$ gives the middle point t_{ij2} of t_{ij}

Kaufmann and Gupta Ranking Index (Kaufmann and Gupta (1985)), as described in Chapter 1, Section 1.3.3., is used for ranking or ordering of fuzzy numbers. 'Ranking Index', also called 'Associated Ordinary Number' (A.O.N.) is written and compared for each fuzzy number to establish the ranking. For example, associated ordinary number for T_{ij} is:

$$\text{A.O.N. } [T_{ij}] = \frac{T_{ij1} + 2(T_{ij2}) + T_{ij3}}{4} \qquad (2)$$

The membership function is obtained from the interval of confidence by setting separately, each of the function, equal to x and solving each of those equations.

4.3.1. Numerical Example

In this section on next page, we consider a similar numerical example as in Section 4.2.3. in which we use the Network Diagram used in Chapter 3, Section 3.3.1. However, here we use the activity time t_{ij} in terms of a triangular fuzzy number,

$t_{ij} = [t_{ij1}, t_{ij2}, t_{ij3}]$, where $t_{ij1} = a$, $t_{ij2} = m$, and $t_{ij3} = b$.

Matrix M

Node	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	_	(1.6,2,2.4)	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	(2,3,4.6)	(0.9,1,2)	_	_	_	_	_	_	_	_	_	_
3	_	_	_	(0,0,0)	_	(0.6,2,2.8)	_	_	_	_	_	_	_	_
4	_	_	_	_	(3,5,7)	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	(4.6,6,7.4)	(2.5,3,3.5)	_	_	_	_	_	_	_
6	_	_	_	_	_	_	(0,0,0)	_	(2,3,5)	_	_	_	_	_
7	_	_	_	_	_	_	_	(2,2,3)	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	(0,0,0)	(2,4,6)	_	_	_	_
9	_	_	_	_	_	_	_	_	_	(4,5,8)	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	(0.8,1,2)	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	(1,1,1)	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	(2.8,3,3.6)	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	(1,1,3)
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

4.3.2. Intervals of Confidence, at α - level, for t_{ij}'s

Based on the definition given in Chapter 1, Section 1.3.3., and from the equation (1) shown above, the intervals of confidence for elements of Matrix M are written as below:

$$t_{12}^{\alpha} = [0.4\alpha + 1.6, (-0.4)\alpha + 2.4]$$

$$t_{23}^{\alpha} = [1\alpha + 2.0, -1.6\alpha + 4.6]$$

$$t_{24}^{\alpha} = [0.1\alpha + 0.9, (-1)\alpha + 2.0]$$

$$t_{34}^{\alpha} = [0\alpha + 0, 0\alpha + 0]$$

$$t_{36}^{\alpha} = [1.4\alpha + 0.6, -0.8\alpha + 2.8]$$

$$t_{45}^{\alpha} = [2\alpha + 3, -2\alpha + 7]$$

$$t_{56}^{\alpha} = [1.4\alpha + 4.6, (-1.4)\alpha + 7.4]$$

$$t_{57}^{\alpha} = [0.5\alpha + 2.5, (-0.5)\alpha + 3.5]$$

$$t_{67}^{\alpha} = [0\alpha + 0, 0\alpha + 0]$$

$$t_{69}^{\alpha} = [1\alpha + 3, -2\alpha + 5]$$

$$t_{78}^{\alpha} = [0\alpha + 2.0, (-1)\alpha + 3.0]$$

$$t_{89}^{\alpha} = [0\alpha + 0, 0\alpha + 0]$$

$$t_{8,10}^{\alpha} = [2\alpha + 2, -2\alpha + 6]$$

$$t_{9,10}^{\alpha} = [1\alpha + 4, -3\alpha + 8]$$

$$t_{10,11}^{\alpha} = [0.2\alpha + 1.0, (-1)\alpha + 2.0]$$

$$t_{11,12}^{\alpha} = [0\alpha + 1, 0\alpha + 1]$$

$$t_{12,13}^{\alpha} = [0.2\alpha + 2.8, -0.6\alpha + 3.6]$$

$$t_{13,14}^{\alpha} = [0\alpha + 1.0, (-2)\alpha + 3.0]$$

4.3.3. Obtaining Critical Path Matrix C having elements $[T_{ij}]$

Following the technique outlined in Chapter 3, Section 3.2.3., we obtain the Critical Path Matrix C as shown below:

a) (i) First row of Matrix C is obtained by copying the first row element(s) of Matrix M. First row of Matrix C is considered as labeled by putting an asterisk on Node 1 in Row 1.

(ii) Label Node 2 Column of Matrix C by putting ([1.6,2,2.4], 1) above Column of Node 2. Highlight Node 2 column in both, Matrix M and Matrix C.

Hence, we have, $T_{12} = (1.6, 2, 2.4)$

Now, the membership function of T_{12} is obtained using the definition given in Chapter 1, Section 1.3.3., for the membership function of a triangular fuzzy number, as shown below:

$$T_{12}^{\alpha} = [0.4\alpha + 1.6, (-0.4)\alpha + 2.4]$$

We now set

$$0.4\alpha + 1.6 = x \text{ and } (-0.4)\alpha + 2.4 = x$$

This yields,

$$0.4\alpha + 1.6 - x = 0 \qquad \text{(i)}$$

$$(-0.4)\alpha + 2.4 - x = 0 \qquad \text{(ii)}$$

In (i) setting $\alpha = 0$ we get, $x = 1.6$

In (ii) setting $\alpha = 0$ we get, $x = 2.4$

Setting $\alpha = 1$ in either we get, $x = 2$

Solving $0.4\alpha + 1.6 - x = 0$ for α we obtain

$$\alpha = \frac{x - 1.6}{0.4} \qquad \text{for } 1.6 \leq x \leq 2$$

and solving $(-0.4)\alpha + 2.4 - x = 0$ we obtain

$$\alpha = \frac{x - 2.4}{-0.4} \qquad \text{for } 2 \leq x \leq 2.4$$

Thus, the membership function for $T_{12}^{\alpha} = [0.4\alpha + 1.6, (-0.4)\alpha + 2.4]$ is

$$\mu_{T_{12}}(x) = \begin{cases} 0 & x \leq 1.6 \\ \dfrac{x - 1.6}{0.4} & 1.6 \leq x \leq 2 \\ \dfrac{x - 2.4}{-0.4} & 2 \leq x \leq 2.4 \\ 0 & x \geq 2.4 \end{cases}$$

b) Since there is only one element T_{12} in highlighted columns of Matrix C, it is added to the elements in Row 2 of Matrix M to obtain the corresponding elements of Row 2 of Matrix C, i.e.,

$$T_{23} = T_{12} + t_{23} \qquad \text{and}$$

$$T_{24} = T_{12} + t_{24}$$

T_{23} is obtained using the relationship, $\quad T_{23}^{\alpha} = T_{12}^{\alpha} + t_{23}^{\alpha}$

$$T_{12}^{\alpha} = [0.4\alpha + 1.6, (-0.4)\alpha + 2.4]$$

$$t_{23}^{\alpha} = [1\alpha + 2.0, -1.6\alpha + 4.6]$$

$$T_{23}^{\alpha} = [0.4\alpha + 1.6, (-0.4)\alpha + 2.4] + [1\alpha + 2.0, -1.6\alpha + 4.6]$$

$$= [1.4\alpha + 3.6, -2.0\alpha + 7.0]$$

We now set

$1.4\alpha + 3.6 = x$ and $-2.0\alpha + 7.0 = x$

This yields,

$$1.4\alpha + 3.6 - x = 0 \qquad \text{(iii)}$$

and $$-2.0\alpha + 7.0 - x = 0 \qquad \text{(iv)}$$

In (iii) setting $\alpha = 0$ we get, $x = 3.6$

In (iv) setting $\alpha = 0$ we get, $x = 7$

Setting $\alpha = 1$ in either we get, $x = 5$

Therefore, $T_{23} = (3.6, 5, 7)$

Similarly, we can calculate T_{24}

$T_{24} = (2.5, 3, 4.4)$

Node 2 Row is labeled by putting an asterisk beside Node 2 in Row 2 of Matrix C. Column of Node 3 is labeled with maximum element in the column, i.e., ([3.6, 5, 7], 2). Node 3 column is highlighted in both, Matrix M and Matrix C.

c) Following exactly the above step (b), we obtain,

$T_{34} = (3.6, 5, 7)$ and

$T_{36} = (4.2, 7, 9.8)$

We label Node 3 Row by putting an asterisk beside it. As we notice in Node 4 column, there are two elements, T_{24} and T_{34}. The maximum of these two numbers will be determined using 'Kaufmann and Gupta Ranking Index', as described above on pp 61-61.

Using equation (2), we find the A.O.N. for both the fuzzy numbers as below:

$$\text{A.O.N. } [T_{24}] = \frac{2.5 + 2 \times 3 + 4.4}{4} = 3.225$$

$$\text{A.O.N. } [T_{34}] = \frac{3.6 + 2 \times 5 + 7}{4} = 5.15$$

Comparing the above two numbers, we see that T_{34} is greater than T_{24}. Therefore, we label the Node 4 Column in Matrix C with ([3.6, 5, 7], 3). Henceforth, we obtain fourth row of Matrix C by adding T_{34} in the fourth row elements of the Matrix M.

d) Similarly, we obtain the rest of the elements of Matrix C.

Matrix C

		([1.6,2,2.4], 1)	([3.6,5,7], 2)	([3.6,5,7], 3)	([6.6,10,14], 4)	([11.2, 16, 21.4], 5)	([11.2, 16,	([13.2, 18,	([13.2, 19,	([17.2, 24,	([18, 25, 36.4],	([19, 26, 37.4],	([21.8, 29, 41],	([22.8, 30, 44],
Node	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
1	_	(1.6, 2, 2.4)	_	_	_	_	_	_	_	_	_	_	_	_
2	_	_	(3.6, 5, 7)	(2.5, 3, 4.4)	_	_	_	_	_	_	_	_	_	_
3	_	_	_	(3.6, 5, 7)	_	(4.2, 7, 9.8)	_	_	_	_	_	_	_	_
4	_	_	_	_	(6.6, 10, 14)	_	_	_	_	_	_	_	_	_
5	_	_	_	_	_	(11.2, 16, 21.4)	(9.1, 13, 17.5)	_	_	_	_	_	_	_
6	_	_	_	_	_	_	(11.2, 16, 21.4)	_	(13.2, 19, 26.4)	_	_	_	_	_
7	_	_	_	_	_	_	_	(13.2, 18, 24.4)	_	_	_	_	_	_
8	_	_	_	_	_	_	_	_	(13.2, 18, 24.4)	(15.2, 22, 30.4)	_	_	_	_
9	_	_	_	_	_	_	_	_	_	(17.2, 24, 34.4)	_	_	_	_
10	_	_	_	_	_	_	_	_	_	_	(18, 25, 36.4)	_	_	_
11	_	_	_	_	_	_	_	_	_	_	_	(19, 26, 37.4)	_	_
12	_	_	_	_	_	_	_	_	_	_	_	_	(21.8, 29, 41)	_
13	_	_	_	_	_	_	_	_	_	_	_	_	_	(22.8, 30, 44)
14	_	_	_	_	_	_	_	_	_	_	_	_	_	_

with the membership functions given below:

Membership function for T_{23} = (3.6, 5, 7) is

$$\mu_{T_{23}}(x) = \begin{cases} 0 & x \le 3.6 \\ \dfrac{x-3.6}{1.4} & 3.6 \le x \le 5 \\ \dfrac{x-7}{-2} & 5 \le x \le 7 \\ 0 & x \ge 7 \end{cases}$$

Membership function for T_{24} = (2.5, 3, 4.4),

$$\mu_{T_{24}}(x) = \begin{cases} 0 & x \le 2.5 \\ \dfrac{x-2.5}{0.5} & 2.5 \le x \le 3 \\ \dfrac{x-4.4}{-1.4} & 3 \le x \le 4.4 \\ 0 & x \ge 4.4 \end{cases}$$

Membership function for T_{34} = (3.6, 5, 7),

$$\mu_{T_{23}}(x) = \begin{cases} 0 & x \le 3.6 \\ \dfrac{x-3.6}{1.4} & 3.6 \le x \le 5 \\ \dfrac{x-7}{-2} & 5 \le x \le 7 \\ 0 & x \ge 7 \end{cases}$$

Membership function for T_{36} = (4.2, 7, 9.8),

$$\mu_{T_{36}}(x) = \begin{cases} 0 & x \le 4.2 \\ \dfrac{x-4.2}{2.8} & 4.2 \le x \le 7 \\ \dfrac{x-9.8}{-2.8} & 7 \le x \le 9.8 \\ 0 & x \ge 9.8 \end{cases}$$

Membership function for $T_{45} = (6.6, 10, 14)$,

$$\mu_{T_{45}}(x) = \begin{cases} 0 & x \le 6.6 \\ \dfrac{x-6.6}{3.4} & 6.6 \le x \le 10 \\ \dfrac{x-14}{-4} & 10 \le x \le 14 \\ 0 & x \ge 14 \end{cases}$$

Membership function for $T_{56} = (11.2, 16, 21.4)$,

$$\mu_{T_{56}}(x) = \begin{cases} 0 & x \le 11.2 \\ \dfrac{x-11.2}{4.8} & 11.2 \le x \le 16 \\ \dfrac{x-21.4}{-5.4} & 16 \le x \le 21.4 \\ 0 & x \ge 21.4 \end{cases}$$

Membership function for $T_{57} = (9.1, 13, 17.5)$,

$$\mu_{T_{57}}(x) = \begin{cases} 0 & x \le 9.1 \\ \dfrac{x-9.1}{3.9} & 9.1 \le x \le 13 \\ \dfrac{x-17.5}{-4.5} & 13 \le x \le 17.5 \\ 0 & x \ge 17.5 \end{cases}$$

Membership function for $T_{67} = (11.2, 16, 21.4)$,

$$\mu_{T_{67}}(x) = \begin{cases} 0 & x \le 11.2 \\ \dfrac{x-11.2}{4.8} & 11.2 \le x \le 16 \\ \dfrac{x-21.4}{-5.4} & 16 \le x \le 21.4 \\ 0 & x \ge 21.4 \end{cases}$$

Membership function for $T_{69} = (13.2, 19, 26.4)$,

$$\mu_{T_{69}}(x) = \begin{cases} 0 & x \leq 13.2 \\ \dfrac{x-13.2}{5.8} & 13.2 \leq x \leq 19 \\ \dfrac{x-26.4}{-7.4} & 19 \leq x \leq 26.4 \\ 0 & x \geq 26.4 \end{cases}$$

Membership function for $T_{78} = (13.2, 18, 24.4)$,

$$\mu_{T_{78}}(x) = \begin{cases} 0 & x \leq 13.2 \\ \dfrac{x-13.2}{4.8} & 13.2 \leq x \leq 18 \\ \dfrac{x-24.4}{-6.4} & 18 \leq x \leq 24.4 \\ 0 & x \geq 24.4 \end{cases}$$

Membership function for $T_{89} = (13.2, 18, 24.4)$,

$$\mu_{T_{89}}(x) = \begin{cases} 0 & x \leq 13.2 \\ \dfrac{x-13.2}{4.8} & 13.2 \leq x \leq 18 \\ \dfrac{x-24.4}{-6.4} & 18 \leq x \leq 24.4 \\ 0 & x \geq 24.4 \end{cases}$$

Membership function for $T_{8,10} = (15.2, 22, 30.4)$,

$$\mu_{T_{8,10}}(x) = \begin{cases} 0 & x \leq 15.2 \\ \dfrac{x-15.2}{6.8} & 15.2 \leq x \leq 22 \\ \dfrac{x-30.4}{-8.4} & 22 \leq x \leq 30.4 \\ 0 & x \geq 30.4 \end{cases}$$

Membership function for $T_{10,11} = (18, 25, 36.4)$,

$$\mu_{T_{10,11}}(x) = \begin{cases} 0 & x \leq 18 \\ \dfrac{x-18}{7} & 18 \leq x \leq 25 \\ \dfrac{x-36.4}{-11.4} & 25 \leq x \leq 36.4 \\ 0 & x \geq 36.4 \end{cases}$$

Membership function for $T_{11,12} = (19, 26, 37.4)$,

$$\mu_{T_{11,12}}(x) = \begin{cases} 0 & x \leq 19 \\ \dfrac{x-19}{7} & 19 \leq x \leq 26 \\ \dfrac{x-37.4}{-11.4} & 26 \leq x \leq 37.4 \\ 0 & x \geq 37.4 \end{cases}$$

Membership function for $T_{12,13} = (21.8, 29, 41)$,

$$\mu_{T_{12,13}}(x) = \begin{cases} 0 & x \leq 21.8 \\ \dfrac{x-21.8}{7.2} & 21.8 \leq x \leq 29 \\ \dfrac{x-41}{-12} & 29 \leq x \leq 41 \\ 0 & x \geq 41 \end{cases}$$

Membership function for $T_{13,14} = (22.8, 30, 44)$,

$$\mu_{T_{13,14}}(x) = \begin{cases} 0 & x \leq 22.8 \\ \dfrac{x-22.8}{7.2} & 22.8 \leq x \leq 30 \\ \dfrac{x-44}{-14} & 30 \leq x \leq 44 \\ 0 & x \geq 44 \end{cases}$$

4.3.4. Identifying Critical Path and determining Critical Path length

Following the technique outlined in Chapter 3, Section 3.2.4., the Critical Path for the given project from Node 1 to Node 14, as determined from the Critical Path Matrix C is as shown below:

Node 1 to Node 2 to Node 3 to Node 4 to Node 5 to Node 6 to Node 9 to Node 10 to Node 11 to Node 12 to Node 13 to Node 14,

Also,

Critical Path length $D_n^* = [22.8, 30, 44]$

Membership function for $D_n^* = (22.8, 30, 44)$,

$$\mu_{D^*_n}(x) = \begin{cases} 0 & x \leq 22.8 \\ \dfrac{x - 22.8}{7.2} & 22.8 \leq x \leq 30 \\ \dfrac{x - 44}{-14} & 30 \leq x \leq 44 \\ 0 & x \geq 44 \end{cases}$$

4.4. Interpretation of the Results

It is evident, from two different solutions to the numerical problem, probability method in PERT neglects the activity duration uncertainty while calculating the Critical path length. Whereas, fuzzy logic approach takes into account the uncertainty of activity duration and provides better results with missing decision points.

In Appendix 1, we calculate the various values of T_{ij}, when α lies between 0 and 1 and in Appendix 2, we plot the membership function graphs for different values of T_{ij}.

For example, the final length of Critical path is calculated as (22.8, 30, 44). As we can see in graph D_n, when $22.8 \leq D_n \leq 30$ the membership function increases monotonically to the left and goes to its maximum value of 1 at an interior point $D_n = 30$, and when $30 \leq D_n \leq 44$ the membership function decreases monotonically to the right and goes to 0 at a right end point $D_n = 44$, starting from 1 at $D_n = 30$. Similarly we can see the membership function graphs for rest of T_{ij}'s along with the values of x. Most of the data available in project scheduling problems in the industry is in the form of fuzzy estimates. Fuzzy set theory permits the partial belonging of an element to a fuzzy set characterized by a membership function that takes values in the interval [0, 1]. Thus, fuzzy approach yields a relatively "more satisfactory and flexible solution" within a pre-specified intervals. Another advantage of the present approach under fuzzy environments is that, it also provided us a range of project completion time showing the lower and upper bounds of the possible solution. Also, fuzzy logic approach may be suitable in those situations where past data is either unavailable or not relevant, the definition of the activity itself is somewhat unclear, or the notion of the activity's completion is vague, as it covers both ambiguous and unpredictable situation.

CHAPTER 5

MATRIX SOLUTION TECHNIQUE APPLIED TO GROUP TECHNOLOGY UNDER FUZZY ENVIRONMENT

In the present chapter, we develop a method, on the lines of the method developed in Chapter 4, to solve multiple parts scheduling problem, in a group technology (GT) based method of processing. We tackle the problem in the presence of fuzzy data.

5.1. Introduction

Modern industrial production differs in the nature and use of different equipment, end products and the type of industry. Overall, different technologies provide different alternatives for converting raw materials to finished product. This provides different options for part processing routes where each alternative has different characteristics associated with it such as degree of automation, capital investment, maintenance cost, labor cost etc. The problem is to identify the best route among these options based on some goal such as minimum processing time, minimum cost, maximum productivity, maximum profit etc. Also, the processing times for a sequence are likely to vary depending on machine speed or simply the speed of the operator. For example, a time standard has built into it an allowance for rest to overcome fatigue, an allowance for unavoidable delay etc. (Fogarty et. al., 1991). The variability caused by factors such as the above mentioned allowances in the processing times can be modeled using fuzzy numbers.

Fierce competition in the manufacturing industry has brought to the forefront several new factors, such as, very short delivery times, high product customization etc., that predominantly determine the competitive advantage of manufacturing enterprises. With the whole supply chain getting tuned to JIT production, the pressure for shrinking delivery lead times is mounting every day. Along with that, the customer demands lower volumes of a high variety of products. In this environment, many companies are facing horrendous problems, such as, huge set-up times, high work-in-process inventories, in-efficient material handling etc., leading to very high cost, poor quality and late deliveries.

Group Technology (GT) has gained popularity recently because it provides answer to the above stated concerns. GT is essentially a production management philosophy of exploiting the similarities between parts, tasks, and ideas. GT essentially provides the flexibility of a job-shop with high productivity of a flow shop. Some of the benefits offered by GT are:

- Minimized part travel time and distance, accounting for efficient material handling
- Increase in throughput
- Reduction in set-up time
- Reduction in batch size or small lot production
- Lower work-in-process inventories

5.2. General Formulation

In this section, we analyze a method to solve a process-planning problem with the goal of attaining minimum processing time. Various alternate work routes considered here include the ones for group processing in which parts with common attributes are grouped together to be processed on the same sequence of machines using universal fixtures. The routes for group processing can be appended to a conventional work flow diagram. Hence the problem is to search for an optimum route among various alternative routes. The method can be applied under different cases, for example, single part or multiple parts considering single operation or multiple operations.

5.2.1. Obtaining Initial Matrix

Initial Matrix M having n rows and n columns is obtained by assigning the resource r_{ij} corresponding to the arc (i, j) in the network to the cell corresponding to the Row i and the Column j of the Matrix M, where, i, j = 1, 2, . . ., n and j > i. The cells corresponding to the non-connected nodes can be left blank.

5.2.2. Obtaining the Optimal Matrix 'C'

Optimal Matrix C having n rows and n columns is obtained using the technique shown below:

a) (i) Row 1 of Matrix C is obtained by duplicating Row 1 of Matrix M.

b) To obtain Row 2 (Node 2 Row) of Optimal Matrix C

   (i) Identify the minimum element in Column 2 of Matrix C. Let the minimum element be v_2.

   (ii) Obtain Row 2 of Optimal Matrix C by adding v_2 to each element of the Row 2 of Matrix M.

c) In general, we obtain Row k, where k = 1,2,…,n ; of Optimal Matrix C as follows:

   (i) Identify the minimum element among the Column k elements of Matrix C. Let the value of the minimum element be v_k

(ii) Obtain Row of the Node k of Matrix C by adding v_k to each element of the Row of Node k of Matrix M.

5.2.3. Identifying the Optimal Solution

Looking at the Column of Node n in the Matrix C, identify the minimum element v_n^* in it. Then, v_n^* gives us the minimum overall processing time from source Node 1 to destination Node n. Suppose, the position of v_n^* is in Row p, $p < n$, of Optimal Matrix C. This means that, one segment of the Optimal path from the source Node 1 to the destination Node n is "from Node p to Node n".

Next, go to the column of Node p in the Matrix C. Identify the minimum element in the Column of Node p as v_p^*. Suppose, the position of v_p^* is in Row q, $q < p$, of Optimal Matrix C. This implies that another segment of the shortest route from the source Node 1 to the destination Node n is "from Node q to Node p".

Continue in the same manner until, we identify a minimum element say v_m^* in Optimal Matrix C, such that v_m^* lies in Row 1. This indicates that an optimal path from the source Node 1 to the destination Node n has been obtained. Thus obtained critical path from the source Node 1 to the destination Node n is as follows:

"Start at Node 1, go to Node m, ..., go to Node q, go to Node p, go to Node n", or written as (1, m, ..., q, p, n) with the overall optimal path length equal to v_n^*.

5.2.4. Alternative Optimal Solutions

In Section 5.2.3., if the minimum element available in the same Column k but in more than one Row, this indicates that there is an alternative critical path available from the source Node 1 to destination Node n, and in this case we choose any of the Rows arbitrarily.

5.3. Numerical Example under Fuzzy Environment

We consider in a numerical example, a case of multiple parts and multiple operations. The network representation (Figure 5.1.) of the problem including Group processing for two operations for completing three parts is taken from Ham et al (1985). We consider the processing times as Triangular Fuzzy numbers (T.F.N.'s).

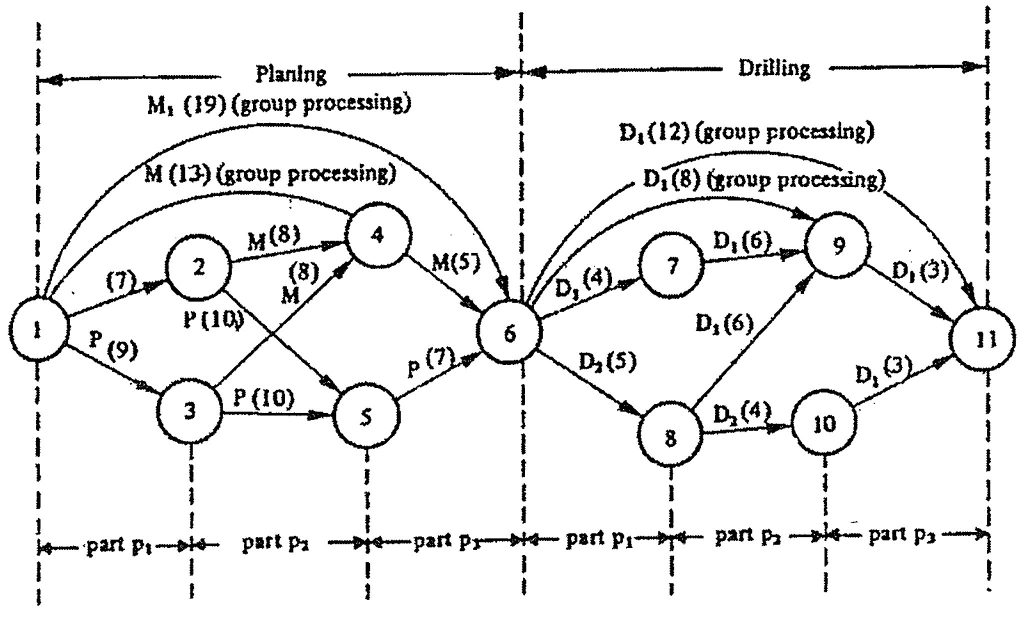


Figure 5.1. – Network Diagram (given)

5.3.1. Notations

In Figure 5.1.

M: Milling Machine, P: Planer, D_1: Drilling Machine 1, D_2: Drilling Machine 2,

and $r_{ij} = [r_{ij1}, r_{ij2}, r_{ij3}]$ is a T.F.N. representing the processing times (in hrs.) in Figure 5.1, where r_{ij1}, r_{ij2} and r_{ij3} are selected with expert opinion.

5.3.2. Obtaining Initial matrix M

Following the technique outlined in Section 5.2.1., we obtain Matrix M from Figure 5.1., as follows:

Matrix M

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	–	(5, 7, 8)	(8, 9, 12)	(10, 13, 14)	–	(15, 19, 22)	–	–	–	–	–
2	–	–	–	(6, 8, 9)	(8, 10, 13)	–	–	–	–	–	–
3	–	–	–	(7, 8, 10)	(9, 10, 12)	–	–	–	–	–	–
4	–	–	–	–	–	(4, 5, 8)	–	–	–	–	–
5	–	–	–	–	–	(6, 7, 9)	–	–	–	–	–
6	–	–	–	–	–	–	(3, 4, 6)	(4, 5, 7)	(5, 8, 10)	–	(11, 12, 15)
7	–	–	–	–	–	–	–	–	(4,6,7)	–	–
8	–	–	–	–	–	–	–	–	(5, 6, 8)	(3, 4, 7)	–
9	–	–	–	–	–	–	–	–	–	–	(2, 3, 5)
10	–	–	–	–	–	–	–	–	–	–	(2, 3, 6)
11	–	–	–	–	–	–	–	–	–	–	–

5.3.3 Intervals of Confidence, at α – level, for t_{ij}'s

$t_{12}{}^{\alpha} = [2\alpha + 5, -1\alpha + 8]$

$t_{13}{}^{\alpha} = [1\alpha + 8, 3\alpha + 12]$

$t_{14}{}^{\alpha} = [3\alpha + 10, -1\alpha + 14]$

$t_{16}{}^{\alpha} = [4\alpha + 15, -3\alpha + 22]$

$t_{24}{}^{\alpha} = [2\alpha + 6, -1\alpha + 9]$

$t_{25}{}^{\alpha} = [2\alpha + 8, -3\alpha + 13]$

$t_{34}{}^{\alpha} = [1\alpha + 7, -2\alpha + 10]$

$t_{35}{}^{\alpha} = [1\alpha + 9, -2\alpha + 12]$

$t_{46}{}^{\alpha} = [1\alpha + 4, -3\alpha + 8]$

$t_{56}{}^{\alpha} = [1\alpha + 6, -2\alpha + 9]$

$t_{67}{}^{\alpha} = [1\alpha + 3, -2\alpha + 6]$

$t_{68}{}^{\alpha} = [1\alpha + 4, -2\alpha + 7]$

$t_{69}{}^{\alpha} = [3\alpha + 5, -2\alpha + 10]$

$t_{6,11}{}^{\alpha} = [1\alpha + 11, -3\alpha + 15]$

$t_{79}{}^{\alpha} = [2\alpha + 4, -1\alpha + 7]$

$t_{89}{}^{\alpha} = [1\alpha + 5, -2\alpha + 8]$

$t_{8,10}{}^{\alpha} = [1\alpha + 3, -3\alpha + 7]$

$t_{9,11}{}^{\alpha} = [1\alpha + 2, -2\alpha + 5]$

$t_{10,11}{}^{\alpha} = [1\alpha + 2, -3\alpha + 6]$

5.3.4. Obtaining the Optimal Matrix C having Elements c_{ij}

Kaufmann and Gupta Ranking Index (Kaufmann and Gupta (1985)), as described in Chapter 1, Section 1.3.3., is used for ranking or ordering of fuzzy numbers. 'Ranking Index', also called 'Associated Ordinary Number' (A.O.N.) is written and compared for each fuzzy number to establish the ranking. For example, associated ordinary number for c_{ij} is:

$$\text{A.O.N. } [c_{ij}] \quad = \frac{c_{ij1} + 2(c_{ij2}) + c_{ij3}}{4} \qquad (1)$$

The membership function for all c_{ij}'s is obtained based on the definition given in Section 1.3.3., from the intervals of confidence by setting separately, each of the function, equal to x and solving each of those equations.

a) (i) Row 1 of Matrix C is obtained by duplicating Row 1 of Matrix M.

b) Since there is only one element (5, 7, 8) in Column 2 of Matrix C, we obtain Row 2 of Optimal Matrix C by adding (5, 7, 8) to each element of the Row 2 of Matrix M, as shown below:

To obtain c_{24} and c_{25}, we use the relationship,

$$c_{24}^{\alpha} = c_{12}^{\alpha} + t_{24}^{\alpha}$$

$$c_{25}^{\alpha} = c_{12}^{\alpha} + t_{25}^{\alpha}$$

Now,

$$c_{12}^{\alpha} = [2\alpha + 5, -1\alpha + 8] \qquad \text{and}$$

$$t_{24}^{\alpha} = [2\alpha + 6, -1\alpha + 9]$$

$$c_{24}^{\alpha} = [2\alpha + 5, -1\alpha + 8] + [2\alpha + 6, -1\alpha + 9]$$

$$c_{24}^{\alpha} = [4\alpha + 11, -2\alpha + 17]$$

We now set, $4\alpha + 11 = x$ and $-2\alpha + 17 = x$

This yields, $4\alpha + 11 - x = 0$, and $-2\alpha + 17 - x = 0$

Setting $\alpha = 0$ in $4\alpha + 11 - x = 0$, and $-2\alpha + 17 - x = 0$ we get,

$x = 11$, and $x = 17$, respectively.

Setting $\alpha = 1$ in either we get, $x = 15$

Therefore, $c_{24} = (11, 15, 17)$

Similarly, we can calculate c_{25}

$$c_{25} = (13, 17, 21)$$

c) Similarly, we obtain elements of Row 3 of matrix 'C' as

$$c_{34} = (15, 17, 22), \text{ and } c_{35} = (17, 19, 24)$$

d) In Column 4 of Optimal Matrix C, there are 3 different elements. The minimum element among the Column 4 elements is determined by locating the minimum of Associated Ordinary Numbers (A.O.N.'s) for these fuzzy numbers, as shown below. Using equation (1) given above,

$$\text{A.O.N. } [c_{14}] = \frac{10 + 2 \times 13 + 14}{4} = 12.5$$

Similarly, A.O.N. $[c_{24}] = 14.5$, and A.O.N. $[c_{34}] = 17.75$.

Comparing the above three numbers, we see that, c_{14} is smallest.

Hence, we obtain fourth row of Matrix C by adding c_{14} in the fourth row elements of the Initial Matrix M.

$$c_{46} = (14, 18, 22)$$

Similarly, we obtain all the other elements of Optimal Matrix C, which looks like follows:

Matrix C

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	–	(5, 7, 8)	(8, 9, 12)	(10, 13, 14)	–	(15, 19, 22)	–	–	–	–	–
2	–	–	–	(11, 15, 17)	(13, 17, 21)	–	–	–	–	–	–
3	–	–	–	(15, 17, 22)	(17, 19, 24)	–	–	–	–	–	–
4	–	–	–	–	–	(14, 18, 22)	–	–	–	–	–
5	–	–	–	–	–	(19, 24, 30)	–	–	–	–	–
6	–	–	–	–	–	–	(17, 22, 25)	(18, 23, 29)	(19, 26, 32)	–	(25, 30, 37)
7	–	–	–	–	–	–	–	–	(21, 28, 32)	–	–
8	–	–	–	–	–	–	–	–	(23, 29, 37)	(21, 27, 36)	–
9	–	–	–	–	–	–	–	–	–	–	(21, 29, 37)
10	–	–	–	–	–	–	–	–	–	–	(23, 30, 42)
11	–	–	–	–	–	–	–	–	–	–	–

With the membership functions given below:

Membership function for c_{12} = (5, 7, 8) is

$$\mu_{c_{12}}(x) = \begin{cases} 0 & x \le 5 \\ \frac{x-5}{2} & 5 \le x \le 7 \\ \frac{x-8}{-1} & 7 \le x \le 8 \\ 0 & x \ge 8 \end{cases}$$

Membership function for c_{13} = (8, 9, 12) is

$$\mu_{c_{13}}(x) = \begin{cases} 0 & x \le 8 \\ \frac{x-8}{1} & 8 \le x \le 9 \\ \frac{x-9}{-3} & 9 \le x \le 12 \\ 0 & x \ge 12 \end{cases}$$

Membership function for c_{14} = (10, 13, 14) is

$$\mu_{c_{14}}(x) = \begin{cases} 0 & x \le 10 \\ \frac{x-10}{3} & 10 \le x \le 13 \\ \frac{x-14}{-1} & 13 \le x \le 14 \\ 0 & x \ge 14 \end{cases}$$

Membership function for c_{16} = (15, 19, 22) is

$$\mu_{c_{16}}(x) = \begin{cases} 0 & x \le 15 \\ \frac{x-15}{4} & 15 \le x \le 19 \\ \frac{x-22}{-3} & 19 \le x \le 22 \\ 0 & x \ge 22 \end{cases}$$

Membership function for c_{24} = (11, 15, 17) is

$$\mu_{c_{24}}(x) = \begin{cases} 0 & x \le 11 \\ \dfrac{x-11}{4} & 11 \le x \le 15 \\ \dfrac{x-17}{-2} & 15 \le x \le 17 \\ 0 & x \ge 17 \end{cases}$$

Membership function for c_{25} = (13, 17, 21) is

$$\mu_{c_{25}}(x) = \begin{cases} 0 & x \le 13 \\ \dfrac{x-13}{4} & 13 \le x \le 17 \\ \dfrac{x-21}{-4} & 17 \le x \le 21 \\ 0 & x \ge 21 \end{cases}$$

Membership function for c_{34} = (15, 17, 22) is

$$\mu_{c_{34}}(x) = \begin{cases} 0 & x \le 15 \\ \dfrac{x-15}{2} & 15 \le x \le 17 \\ \dfrac{x-22}{-5} & 17 \le x \le 22 \\ 0 & x \ge 22 \end{cases}$$

Membership function for c_{35} = (17, 19, 24) is

$$\mu_{c_{35}}(x) = \begin{cases} 0 & x \le 17 \\ \dfrac{x-17}{2} & 17 \le x \le 19 \\ \dfrac{x-24}{-5} & 19 \le x \le 24 \\ 0 & x \ge 24 \end{cases}$$

Membership function for c_{46} = (14, 18, 22) is

$$\mu_{c_{46}}(x) = \begin{cases} 0 & x \leq 14 \\ \frac{x-14}{4} & 14 \leq x \leq 18 \\ \frac{x-22}{-4} & 18 \leq x \leq 22 \\ 0 & x \geq 22 \end{cases}$$

Membership function for c_{56} = (19, 24, 30) is

$$\mu_{c_{56}}(x) = \begin{cases} 0 & x \leq 19 \\ \frac{x-19}{5} & 19 \leq x \leq 24 \\ \frac{x-30}{-6} & 24 \leq x \leq 30 \\ 0 & x \geq 30 \end{cases}$$

Membership function for c_{67} = (17, 22, 25) is

$$\mu_{c_{67}}(x) = \begin{cases} 0 & x \leq 17 \\ \frac{x-17}{5} & 17 \leq x \leq 22 \\ \frac{x-25}{-3} & 22 \leq x \leq 25 \\ 0 & x \geq 25 \end{cases}$$

Membership function for c_{68} = (18, 23, 29) is

$$\mu_{c_{68}}(x) = \begin{cases} 0 & x \leq 18 \\ \frac{x-18}{5} & 18 \leq x \leq 23 \\ \frac{x-29}{-6} & 23 \leq x \leq 29 \\ 0 & x \geq 29 \end{cases}$$

Membership function for c_{69} = (19, 26, 32) is

$$\mu_{c_{69}}(x) = \begin{cases} 0 & x \le 19 \\ \dfrac{x-19}{7} & 19 \le x \le 26 \\ \dfrac{x-32}{-6} & 26 \le x \le 32 \\ 0 & x \ge 32 \end{cases}$$

Membership function for $c_{6,11}$ = (25, 30, 37) is

$$\mu_{c_{6,11}}(x) = \begin{cases} 0 & x \le 25 \\ \dfrac{x-25}{5} & 25 \le x \le 30 \\ \dfrac{x-37}{-7} & 30 \le x \le 37 \\ 0 & x \ge 37 \end{cases}$$

Membership function for c_{79} = (21, 28, 32) is

$$\mu_{c_{79}}(x) = \begin{cases} 0 & x \le 21 \\ \dfrac{x-21}{7} & 21 \le x \le 28 \\ \dfrac{x-32}{-4} & 28 \le x \le 32 \\ 0 & x \ge 32 \end{cases}$$

Membership function for c_{89} = (23, 29, 37) is

$$\mu_{c_{89}}(x) = \begin{cases} 0 & x \le 23 \\ \dfrac{x-23}{6} & 23 \le x \le 29 \\ \dfrac{x-37}{-8} & 29 \le x \le 37 \\ 0 & x \ge 37 \end{cases}$$

Membership function for $c_{8,10}$ = (21, 27, 36) is

$$\mu_{c_{8,10}}(x) = \begin{cases} 0 & x \le 21 \\ \dfrac{x-21}{6} & 21 \le x \le 27 \\ \dfrac{x-36}{-9} & 27 \le x \le 36 \\ 0 & x \ge 36 \end{cases}$$

Membership function for $c_{9,11}$ = (21, 29, 37) is

$$\mu_{c_{9,11}}(x) = \begin{cases} 0 & x \le 21 \\ \dfrac{x-21}{8} & 23 \le x \le 29 \\ \dfrac{x-37}{-8} & 29 \le x \le 37 \\ 0 & x \ge 37 \end{cases}$$

Membership function for $c_{10,11}$ = (23, 30, 42) is

$$\mu_{c_{10,11}}(x) = \begin{cases} 0 & x \le 23 \\ \dfrac{x-23}{7} & 23 \le x \le 30 \\ \dfrac{x-42}{-12} & 30 \le x \le 42 \\ 0 & x \ge 42 \end{cases}$$

5.3.5. Identifying the Optimal solution

As seen in the last column (Column n) of Matrix C, the minimum element is given as, $c_{9,11}$ = (21, 29, 37)

So, the minimum processing time required to complete the process, v_n^* = (21, 29, 37)

Also, the Optimal path is determined using the technique outlined in Section 5.2.3., as follows:

Node1 → Node 4 → Node 6 → Node 9 → Node 11

5.4. Interpretation of Results

In the present chapter, we use the computational technique developed in Chapter 4, to solve a process-planning problem, with GT setup, considering fuzzy data. Using fuzzy numbers to model uncertainty, enables us to perform micro-analysis on the effect of variability in processing times on the ultimate result.

In Appendix 3, we calculate the various values of x, when α lies between 0 and 1 and in Appendix 4, we plot the membership function graphs for different values of c_{ij}. For example, the final length of optimal path is calculated as (21, 29, 37). As we can see in graph v_n, when $21 \leq v_n \leq 29$ the membership function increases monotonically to the left and goes to its maximum value of 1 at an interior point $v_n = 29$, and when $29 \leq v_n \leq 37$ the membership function decreases monotonically to the right and goes to 0 at a right end point $v_n = 37$, starting from 1 at $v_n = 29$. Similarly we can see the membership function graphs for rest of c_{ij}'s along with the values of x. Most of the times, there is an inherent variability in the processing times for a part. Fuzzy set theory permits the partial belonging of an element to a fuzzy set characterized by a membership function that takes values in the interval [0, 1]. Thus, fuzzy approach yields a relatively "more satisfactory and flexible solution" within a pre-specified intervals. Another advantage of the present approach under fuzzy environments is that, it also provided us a range of processing times showing the lower and upper bounds of the possible solution. As pointed out in Chapter 4, fuzzy logic approach may also be suitable in those situations where past data is either unavailable or not relevant, the definition of the activity itself is somewhat unclear, or the notion of the activity's completion is vague.

CHAPTER 6

CASE STUDY – SCHEDULING PROBLEM (UNDER GT SETUP) SOLVED FOR XYZ INDUSTRIES LTD.

XYZ Industries Ltd., Winnipeg is a leading producer of hydraulic cylinders in North America. The various models of hydraulic cylinders manufactured at XYZ vary from 2" to 5" bore, and 6" to 48" stroke length with all the combinations between bore size and stroke length. The cylinders with 3" bore size are the most popular size. The 3" rod caps used in 3" bore size cylinders are available in different configurations, but have some common attributes making them suitable for producing in the same batch. In the present chapter, we solve a scheduling problem for the rod cap cell under GT setup, under fuzzy environment.

6.1. Problem

Generally, there are three operations performed on a rod cap. Operation 10 can be performed on either a CNC lathe or a manual lathe (ML) where the raw casting is faced and grooved. The semi-finished part is then drilled on a manual multi-spindle drill press (MS) to finish operation 20. The part can be further processed on either a Vertical Machining Center (VMC) or a Radial drill (RD), where a port is drilled and threaded to finish operation 30. Each time the machine shop sets up for making a rod cap, it takes approximately half an hour to change the fixtures/tools for each machine and approximately another half an hour to reprogram the CNC lathe and the Vertical Machining Center (including the program prove out).

Next week orders indicate that there is a requirement for 100 pcs. each of two different kinds of 3" rod caps. Based on different possible options for part processing routes, including GT options, we show below a network diagram. The estimated processing times are not always accurate and depend on the operator speed in most cases. Hence, we take the processing times to be triangular fuzzy numbers in our problem. Figure 6.1., shown below depicts the network diagram under a GT setup.

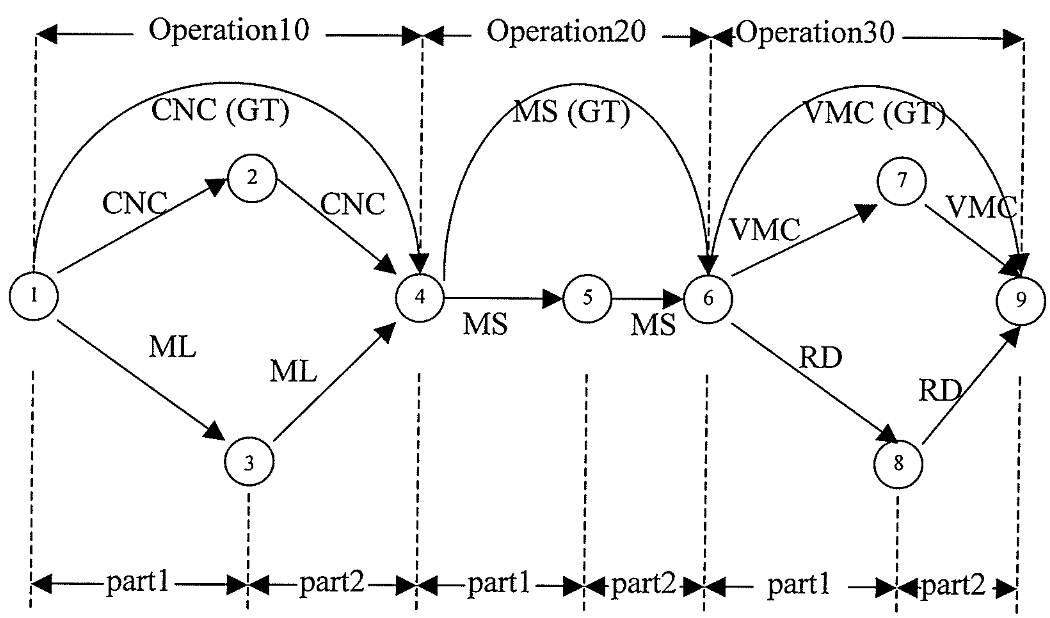


Figure 6.1. - Network Diagram

Given below is the table (Table 6.1.) of most likely setup and processing times for each machine. From the past data, it is observed that the actual processing times for each machine can vary between 75% to 110% of estimated times.

Table 6.1.

	Set-up Time	Processing Time (Part1or Part2)
CNC Lathe (CNC)	1 hrs	2 hrs
Manual Lathe (ML)	0.5 hrs	3 hrs
Multi Spindle Drill Press (MS)	0.5 hrs	1 hrs
Vertical Machining Center (VMC)	1 hrs	1 hrs
Radial Drill (RD)	0.5 hrs	1 hrs

6.2. Application of the Matrix Solution Technique

Based on the above network diagram and estimated processing times, the manufacturing engineer writes the initial Matrix M as described in Chapter 5.

Matrix M

Node	1	2	3	4	5	6	7	8	9
1	_	(2.7,3,3.8)	(3,3.5,4.4)	(4.5,5,6.3)	_	_	_	_	_
2	_	_	_	(2.7,3,3.8)	_	_	_	_	_
3	_	_	_	(3,3.5,4.4)	_	_	_	_	_
4	_	_	_	_	(1.4,1.5,2)	(2.3,2.5,3)	_	_	_
5	_	_	_	_	_	(1.4,1.5,2)	_	_	_
6	_	_	_	_	_	_	(1.8,2,2.5)	(1.4,1.5,2)	(2.7,3,3.8)
7	_	_	_	_	_	_	_	_	(1.8,2,2.5)
8	_	_	_	_	_	_	_	_	(1.4,1.5,2)
9	_	_	_	_	_	_	_	_	_

Upon solving the problem using the technique given in Chapter 5, we obtain our solution Matrix C as follows:

Matrix C

Node	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	_	(2.7,3,3.8)	(3,3.5,4.4)	(4.5,5,6.3)	_	_	_	_	_
2	_	_	_	(5.4,6,7.6)	–	_	_	_	_
3	_	_	_	(6,7,8.8)	_	_	_	_	_
4	_	_	_	_	(5.9,6.5,8.3)	(6.8,7.5,9.3)	_	_	_
5	_	_	_	_	_	(7.3,8,10.3)	_	_	_
6	_	_	_	_	_	_	(8.6, 9.5,11.8)	(8.2,9,11.3)	(9.5, 10.5, 13.1)
7	_	_	_	_	_	_	_	_	(10.4, 11.5,14.3)
8	_	_	_	_	_	_	_	_	(9.6, 10.5, 13.3)
9	_	_	_	_	_	_	_	_	_

6.3. Membership Functions for Elements of Matrix C

As explained in Section 5.3.4. in Chapter 5, we obtain the membership functions for the elements of Matrix C as below:

Membership function for c_{12} = (2.7, 3, 3.8) is

$$\mu_{c_{12}}(x) = \begin{cases} 0 & x \le 2.7 \\ \dfrac{x - 2.7}{0.3} & 2.7 \le x \le 3 \\ \dfrac{x - 3.8}{-0.8} & 3 \le x \le 3.8 \\ 0 & x \ge 3.8 \end{cases}$$

Membership function for c_{13} = (3, 3.5, 4.4) is

$$\mu_{c_{13}}(x) = \begin{cases} 0 & x \le 3 \\ \dfrac{x - 3}{0.5} & 3 \le x \le 3.5 \\ \dfrac{x - 4.4}{-0.9} & 3.5 \le x \le 4.4 \\ 0 & x \ge 4.4 \end{cases}$$

Membership function for c_{14} = (4.5, 5, 6.3) is

$$\mu_{c_{14}}(x) = \begin{cases} 0 & x \le 4.5 \\ \dfrac{x - 4.5}{0.5} & 4.5 \le x \le 5 \\ \dfrac{x - 6.3}{-1.3} & 5 \le x \le 6.3 \\ 0 & x \ge 6.3 \end{cases}$$

Membership function for c_{24} = (5.4, 6, 7.6) is

$$\mu_{c_{24}}(x) = \begin{cases} 0 & x \leq 5.4 \\ \dfrac{x - 5.4}{0.6} & 5.4 \leq x \leq 6 \\ \dfrac{x - 7.6}{-1.6} & 6 \leq x \leq 7.6 \\ 0 & x \geq 7.6 \end{cases}$$

Membership function for c_{34} = (6, 7, 8.8) is

$$\mu_{c_{34}}(x) = \begin{cases} 0 & x \leq 6 \\ \dfrac{x - 6}{1} & 6 \leq x \leq 7 \\ \dfrac{x - 8.8}{-1.8} & 7 \leq x \leq 8.8 \\ 0 & x \geq 8.8 \end{cases}$$

Membership function for c_{45} = (5.9, 6.5, 8.3) is

$$\mu_{c_{45}}(x) = \begin{cases} 0 & x \leq 5.9 \\ \dfrac{x - 5.9}{0.6} & 5.9 \leq x \leq 6.5 \\ \dfrac{x - 8.3}{-1.8} & 6.5 \leq x \leq 8.3 \\ 0 & x \geq 8.3 \end{cases}$$

Membership function for c_{46} = (6.8, 7.5, 9.3) is

$$\mu_{c_{46}}(x) = \begin{cases} 0 & x \leq 6.8 \\ \dfrac{x - 6.8}{0.7} & 6.8 \leq x \leq 7.5 \\ \dfrac{x - 9.3}{-1.8} & 7.5 \leq x \leq 9.3 \\ 0 & x \geq 9.3 \end{cases}$$

Membership function for c_{56} = (7.3, 8, 10.3) is

$$\mu_{c_{56}}(x) = \begin{cases} 0 & x \le 7.3 \\ \dfrac{x-7.3}{0.7} & 7.3 \le x \le 8 \\ \dfrac{x-10.3}{-2.3} & 8 \le x \le 10.3 \\ 0 & x \ge 10.3 \end{cases}$$

Membership function for c_{67} = (8.6, 9.5, 11.8) is

$$\mu_{c_{67}}(x) = \begin{cases} 0 & x \le 8.6 \\ \dfrac{x-8.6}{0.9} & 8.6 \le x \le 9.5 \\ \dfrac{x-11.8}{-2.3} & 9.5 \le x \le 11.8 \\ 0 & x \ge 11.8 \end{cases}$$

Membership function for c_{68} = (8.2, 9, 11.3) is

$$\mu_{c_{68}}(x) = \begin{cases} 0 & x \le 8.2 \\ \dfrac{x-8.2}{0.8} & 8.2 \le x \le 9 \\ \dfrac{x-11.3}{-2.3} & 9 \le x \le 11.3 \\ 0 & x \ge 11.3 \end{cases}$$

Membership function for c_{69} = (9.5, 10.5, 13.1) is

$$\mu_{c_{69}}(x) = \begin{cases} 0 & x \le 9.5 \\ \dfrac{x-9.5}{1} & 9.5 \le x \le 10.5 \\ \dfrac{x-13.1}{-2.6} & 10.5 \le x \le 13.1 \\ 0 & x \ge 13.1 \end{cases}$$

Membership function for c_{79} = (10.4, 11.5, 14.3) is

$$\mu_{c_{79}}(x) = \begin{cases} 0 & x \le 10.4 \\ \dfrac{x-10.4}{1.1} & 10.4 \le x \le 11.5 \\ \dfrac{x-14.3}{-2.8} & 11.5 \le x \le 14.3 \\ 0 & x \ge 14.3 \end{cases}$$

Membership function for c_{89} = (9.6, 10.5, 13.3) is

$$\mu_{c_{89}}(x) = \begin{cases} 0 & x \le 9.6 \\ \dfrac{x-9.6}{0.9} & 9.6 \le x \le 10.5 \\ \dfrac{x-13.3}{-2.8} & 10.5 \le x \le 13.3 \\ 0 & x \ge 13.3 \end{cases}$$

As the Matrix C indicates,

the minimum processing time required to complete the process,

$v_n^* = (9.5, 10.5, 13.1)$,

and the optimal path is determined as,

Node 1 → Node 4 → Node 6 → Node 9

In the absence of Group Technology options,

the minimum processing time required to complete the process is,

$v_n^* = (11, 12, 15.6)$,

and the optimal path is determined as,

Node 1 → Node 2 → Node 4 → Node 5 → Node 6 → Node 8 → Node 9

6.4. Interpretation of Results

Comparing the above two minimum processing times, it is evident that using Group Technology manufacturing technique under fuzzy environments gives better results as compared to the traditional Job Shop manufacturing technique. The given solution technique will enable the manufacturing engineer to compare and choose the appropriate manufacturing technique.

In Appendix 5, we calculate the various values of x, when α lies between 0 and 1 and in Appendix 6, we plot the membership function graphs for different values of c_{ij}. For example, the final length of optimal path is calculated as (9.5, 10.5, 13.1). As we can see in graph v_n (or c_{69}), when $9.5 \leq v_n \leq 10.5$ the membership function increases monotonically to the left and goes to its maximum value of 1 at an interior point $v_n = 10.5$, and when $10.5 \leq v_n \leq 13.1$ the membership function decreases monotonically to the right and goes to 0 at a right end point $v_n = 13.1$, starting from 1 at $v_n = 10.5$. Similarly, we can interpret the membership function graphs for rest of c_{ij}'s along with the values of x.

CHAPTER 7

CONCLUSION, CONTRIBUTION AND RECOMMENDATIONS

In the present chapter, we state the contributions and conclusions of this thesis. Finally, we give some recommendations for further research on the problems considered in this thesis.

7.1 Conclusion and Contribution

In the present dissertation, an important problem in the field of industrial engineering i.e. critical path method of project scheduling (originally proposed by DuPont Industries, 1958) has been revisited. We have modeled our solution technique as a matrix approach to critical path problem, based on the philosophy of Principle of Optimality and Recursive Equation. Earlier this approach has been used for solution of transportation type problems. It is suggested that the methods presented in this thesis are computationally simple and useful for determining the satisfactory solution to project scheduling problems. Also, this thesis shows how the method can be implemented using a simple and widely known software, MS Excel.

In general, the activity duration in the critical path method is assumed to be deterministically known. But, in the case of innovative projects, activity duration is usually forecasted and forecasts rarely-if-ever turn out to be crisply correct. Traditionally such a problem of uncertainty in activity duration is dealt using PERT technique. However, in the present thesis, we deal with this problem under fuzzy environment. In Chapter 4, we model Critical Path problem as a matrix solution approach under fuzzy data and compare it with traditional PERT, and we observe that the results obtained using fuzzy logic approach provide more flexibility in the form of range of estimates/values of the activity durations. Assuming that the data for activity durations is known in the form of triangular fuzzy numbers (T.F.N.'s), we find the critical path length in the form of a triangular fuzzy number (T.F.N.).

In Chapter 5 of the thesis, we apply the same matrix solution technique for solving a manufacturing engineering problem in a Group Technology (GT) setup under fuzzy environment. Assuming that the data for processing times of the parts is known in the form of T.F.N.'s, we find the minimum processing time in the form of a triangular fuzzy number (T.F.N.) and also determine the optimal routings for the parts under consideration.

7.2 Recommendations for Future Research

It is believed that a number of extensions are possible to the project scheduling problems. The results of Chapter 3 and Chapter 4 can be extended to the case when the estimates for activity durations are provided in the form of trapezoidal fuzzy numbers (Tr.F.N.'s). Similarly, the results of Chapter 5 can be extended to the case when the estimates of setup time per lot and run time per piece are provided in the form of Tr.F.N.'s.

Further research is possible involving the application of the technique presented in this thesis to,

- GERT method, where the probabilistic time estimates can be modeled as Triangular Fuzzy Numbers,

- CCS method, where feeding buffers and project buffer can be modeled as Triangular Fuzzy Numbers,
- PDM method, where the starting times and/or finishing times of activities can be modeled as Triangular Fuzzy Numbers.

In brief, in most of the scheduling problems, the basic goal is to identify the least time consuming way of reaching the objective, given a variety of working and business constraints and parameters. Therefore, for a scheduling problem with multiple activities and multiple available alternative routes, where instinct and experience are not able to cope up with the size and complexity of the problem, use of fuzzy logic and fuzzy sets is an aid to be examined.

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APPENDIX 1

Interval of Confidence Tables

for T_{ij}'s from Chapter 4

for $0 \leq \alpha \leq 1$

(page 122 - 129)

Interval of Confidence for $0 \le \alpha \le 1$

Corresponding to T_{12}

α	$x = 0.4\,\alpha + 1.6$	α	$x = -0.4\,\alpha + 2.4$
0	1.60	1	2.00
0.1	1.64	0.9	2.04
0.2	1.68	0.8	2.08
0.3	1.72	0.7	2.12
0.4	1.76	0.6	2.16
0.5	1.80	0.5	2.20
0.6	1.84	0.4	2.24
0.7	1.88	0.3	2.28
0.8	1.92	0.2	2.32
0.9	1.96	0.1	2.36
1	2.00	0	2.40

Corresponding to T_{23}

α	$x = 1.4\,\alpha + 3.6$	α	$x = -2.0\,\alpha + 7.0$
0	3.60	1	5.00
0.1	3.74	0.9	5.20
0.2	3.88	0.8	5.40
0.3	4.02	0.7	5.60
0.4	4.16	0.6	5.80
0.5	4.30	0.5	6.00
0.6	4.44	0.4	6.20
0.7	4.58	0.3	6.40
0.8	4.72	0.2	6.60
0.9	4.86	0.1	6.80
1	5.00	0	7.00

Corresponding to T_{24}

α	$x = .5\,\alpha + 2.5$	α	$x = -1.4\,\alpha + 4.4$
0	2.50	1	3.00
0.1	2.55	0.9	3.14
0.2	2.60	0.8	3.28
0.3	2.65	0.7	3.42
0.4	2.70	0.6	3.56
0.5	2.75	0.5	3.70
0.6	2.80	0.4	3.84
0.7	2.85	0.3	3.98
0.8	2.90	0.2	4.12
0.9	2.95	0.1	4.26
1	3.00	0	4.40

Corresponding to T_{34}			
α	$x = 1.4\alpha + 3.6$	α	$x = -2.0\alpha + 7.0$
0	3.60	1	5.00
0.1	3.74	0.9	5.20
0.2	3.88	0.8	5.40
0.3	4.02	0.7	5.60
0.4	4.16	0.6	5.80
0.5	4.30	0.5	6.00
0.6	4.44	0.4	6.20
0.7	4.58	0.3	6.40
0.8	4.72	0.2	6.60
0.9	4.86	0.1	6.80
1	5.00	0	7.00

Corresponding to T_{36}			
α	$x = 2.8\alpha + 4.2$	α	$x = -2.8\alpha + 9.8$
0	4.20	1	7.00
0.1	4.48	0.9	7.28
0.2	4.76	0.8	7.56
0.3	5.04	0.7	7.84
0.4	5.32	0.6	8.12
0.5	5.60	0.5	8.40
0.6	5.88	0.4	8.68
0.7	6.16	0.3	8.96
0.8	6.44	0.2	9.24
0.9	6.72	0.1	9.52
1	7.00	0	9.80

Corresponding to T_{45}			
α	$x = 3.4\alpha + 6.6$	α	$x = -4.0\alpha + 14.0$
0	6.60	1	10.00
0.1	6.94	0.9	10.40
0.2	7.28	0.8	10.80
0.3	7.62	0.7	11.20
0.4	7.96	0.6	11.60
0.5	8.30	0.5	12.00
0.6	8.64	0.4	12.40
0.7	8.98	0.3	12.80
0.8	9.32	0.2	13.20
0.9	9.66	0.1	13.60
1	10.00	0	14.00

Corresponding to T_{56}			
α	x = 4.8 α + 11.2	α	x = − 5.4 α + 21.4
0	11.20	1	16.00
0.1	11.68	0.9	16.54
0.2	12.16	0.8	17.08
0.3	12.64	0.7	17.62
0.4	13.12	0.6	18.16
0.5	13.60	0.5	18.70
0.6	14.08	0.4	19.24
0.7	14.56	0.3	19.78
0.8	15.04	0.2	20.32
0.9	15.52	0.1	20.86
1	16.00	0	21.40

Corresponding to T_{57}			
α	x = 3.9 α + 9.1	α	x = −4.5 α + 17.5
0	9.10	1	13.00
0.1	9.49	0.9	13.45
0.2	9.88	0.8	13.90
0.3	10.27	0.7	14.35
0.4	10.66	0.6	14.80
0.5	11.05	0.5	15.25
0.6	11.44	0.4	15.70
0.7	11.83	0.3	16.15
0.8	12.22	0.2	16.60
0.9	12.61	0.1	17.05
1	13.00	0	17.50

Corresponding to T_{67}			
α	x = 4.8 α + 11.2	α	x = − 5.4 α + 21.4
0	11.20	1	16.00
0.1	11.68	0.9	16.54
0.2	12.16	0.8	17.08
0.3	12.64	0.7	17.62
0.4	13.12	0.6	18.16
0.5	13.60	0.5	18.70
0.6	14.08	0.4	19.24
0.7	14.56	0.3	19.78
0.8	15.04	0.2	20.32
0.9	15.52	0.1	20.86
1	16.00	0	21.40

Corresponding to T_{69}			
α	$x = 5.8\,\alpha + 13.2$	α	$x = -7.4\,\alpha + 26.4$
0	13.20	1	19.00
0.1	13.78	0.9	19.74
0.2	14.36	0.8	20.48
0.3	14.94	0.7	21.22
0.4	15.52	0.6	21.96
0.5	16.10	0.5	22.70
0.6	16.68	0.4	23.44
0.7	17.26	0.3	24.18
0.8	17.84	0.2	24.92
0.9	18.42	0.1	25.66
1	19.00	0	26.40

Corresponding to T_{78}			
α	$x = 4.8\,\alpha + 13.2$	α	$x = -6.4\,\alpha + 24.4$
0	13.20	1	18.00
0.1	13.68	0.9	18.64
0.2	14.16	0.8	19.28
0.3	14.64	0.7	19.92
0.4	15.12	0.6	20.56
0.5	15.60	0.5	21.20
0.6	16.08	0.4	21.84
0.7	16.56	0.3	22.48
0.8	17.04	0.2	23.12
0.9	17.52	0.1	23.76
1	18.00	0	24.40

Corresponding to T_{89}			
α	$x = 4.8\,\alpha + 13.2$	α	$x = -6.4\,\alpha + 24.4$
0	13.20	1	18.00
0.1	13.68	0.9	18.64
0.2	14.16	0.8	19.28
0.3	14.64	0.7	19.92
0.4	15.12	0.6	20.56
0.5	15.60	0.5	21.20
0.6	16.08	0.4	21.84
0.7	16.56	0.3	22.48
0.8	17.04	0.2	23.12
0.9	17.52	0.1	23.76
1	18.00	0	24.40

Corresponding to $T_{8,10}$

α	$x = 6.8\alpha + 15.2$	α	$x = -8.4\alpha + 30.4$
0	15.20	1	22.00
0.1	15.88	0.9	22.84
0.2	16.56	0.8	23.68
0.3	17.24	0.7	24.52
0.4	17.92	0.6	25.36
0.5	18.60	0.5	26.20
0.6	19.28	0.4	27.04
0.7	19.96	0.3	27.88
0.8	20.64	0.2	28.72
0.9	21.32	0.1	29.56
1	22.00	0	30.40

Corresponding to $T_{9,10}$

α	$x = 6.8\alpha + 17.2$	α	$x = -10.4\alpha + 34.4$
0	17.20	1	24.00
0.1	17.88	0.9	25.04
0.2	18.56	0.8	26.08
0.3	19.24	0.7	27.12
0.4	19.92	0.6	28.16
0.5	20.60	0.5	29.20
0.6	21.28	0.4	30.24
0.7	21.96	0.3	31.28
0.8	22.64	0.2	32.32
0.9	23.32	0.1	33.36
1	24.00	0	34.40

Corresponding to $T_{10,11}$

α	$x = 7.0\alpha + 18$	α	$x = -11.4\alpha + 36.4$
0	18.00	1	25.00
0.1	18.70	0.9	26.14
0.2	19.40	0.8	27.28
0.3	20.10	0.7	28.42
0.4	20.80	0.6	29.56
0.5	21.50	0.5	30.70
0.6	22.20	0.4	31.84
0.7	22.90	0.3	32.98
0.8	23.60	0.2	34.12
0.9	24.30	0.1	35.26
1	25.00	0	36.40

Corresponding to $T_{11,12}$

α	$x = 7.0\,\alpha + 19$	α	$x = -11.4\,\alpha + 37.4$
0	19.00	1	26.00
0.1	19.70	0.9	27.14
0.2	20.40	0.8	28.28
0.3	21.10	0.7	29.42
0.4	21.80	0.6	30.56
0.5	22.50	0.5	31.70
0.6	23.20	0.4	32.84
0.7	23.90	0.3	33.98
0.8	24.60	0.2	35.12
0.9	25.30	0.1	36.26
1	26.00	0	37.40

Corresponding to $T_{12,13}$

α	$x = 7.2\,\alpha + 21.8$	α	$x = -12\,\alpha + 41$
0	21.80	1	29.00
0.1	22.52	0.9	30.20
0.2	23.24	0.8	31.40
0.3	23.96	0.7	32.60
0.4	24.68	0.6	33.80
0.5	25.40	0.5	35.00
0.6	26.12	0.4	36.20
0.7	26.84	0.3	37.40
0.8	27.56	0.2	38.60
0.9	28.28	0.1	39.80
1	29.00	0	41.00

Corresponding to $T_{13,14}$

α	$x = 7.2\,\alpha + 22.8$	α	$x = -14\,\alpha + 44$
0	22.80	1	30.00
0.1	23.52	0.9	31.40
0.2	24.24	0.8	32.80
0.3	24.96	0.7	34.20
0.4	25.68	0.6	35.60
0.5	26.40	0.5	37.00
0.6	27.12	0.4	38.40
0.7	27.84	0.3	39.80
0.8	28.56	0.2	41.20
0.9	29.28	0.1	42.60
1	30.00	0	44.00

Corresponding to D_{n*}			
α	x = 7.2 α + 22.8	α	x = − 14 α + 44
0	22.80	1	30.00
0.1	23.52	0.9	31.40
0.2	24.24	0.8	32.80
0.3	24.96	0.7	34.20
0.4	25.68	0.6	35.60
0.5	26.40	0.5	37.00
0.6	27.12	0.4	38.40
0.7	27.84	0.3	39.80
0.8	28.56	0.2	41.20
0.9	29.28	0.1	42.60
1	30.00	0	44.00

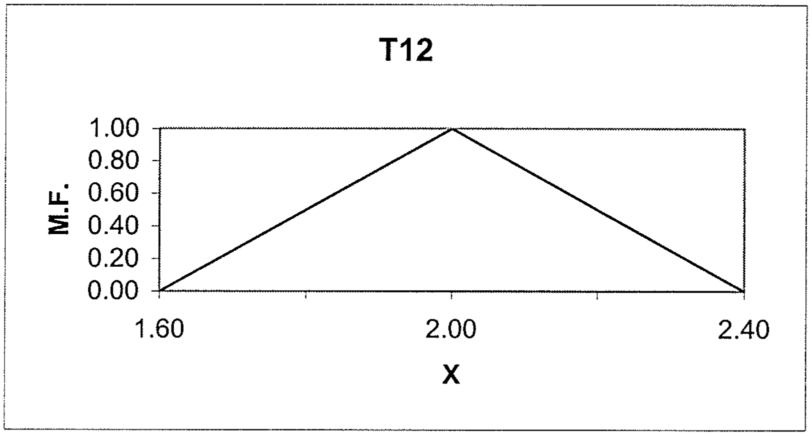
APPENDIX 2

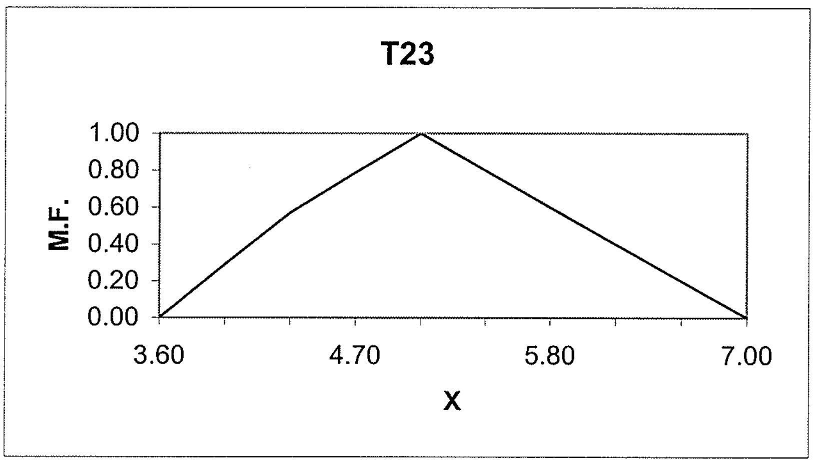
Graph of Membership Functions

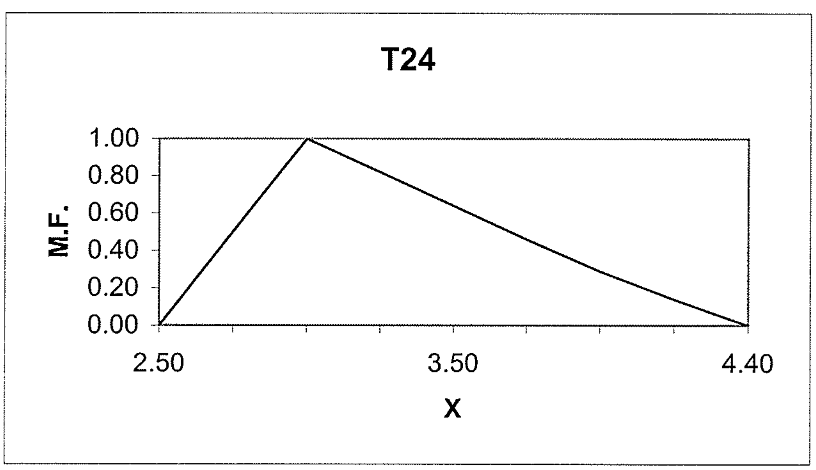
for T_{ij}'s from Chapter 4

(page 130 - 137)

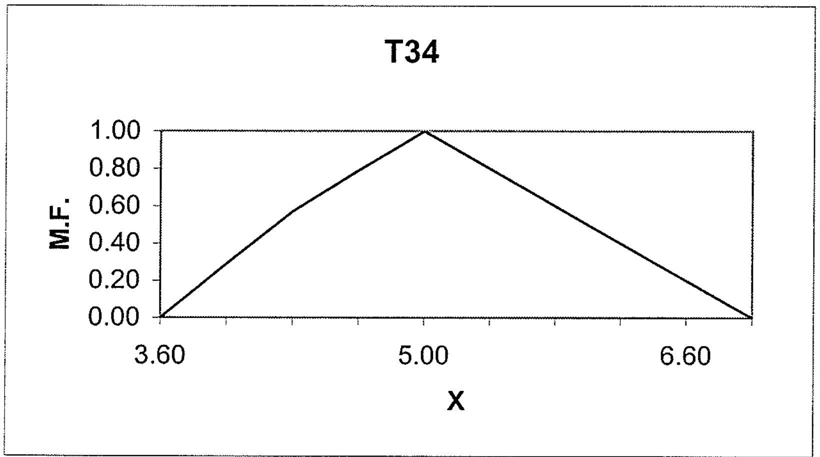
Graph of Membership Functions for Tij 's

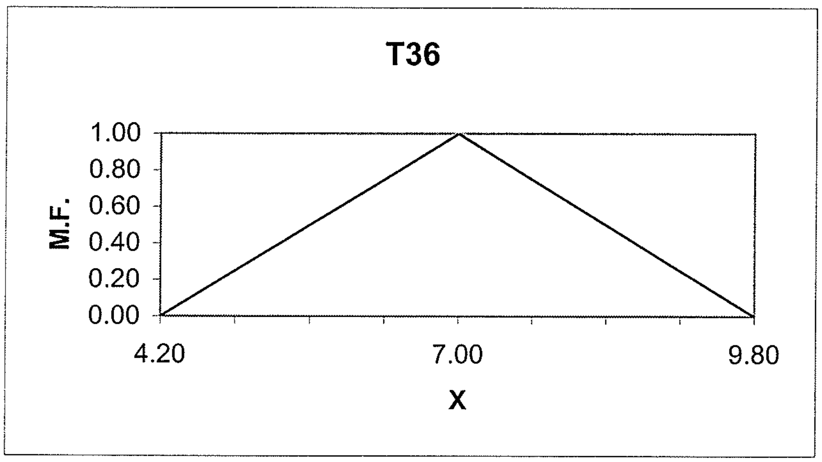


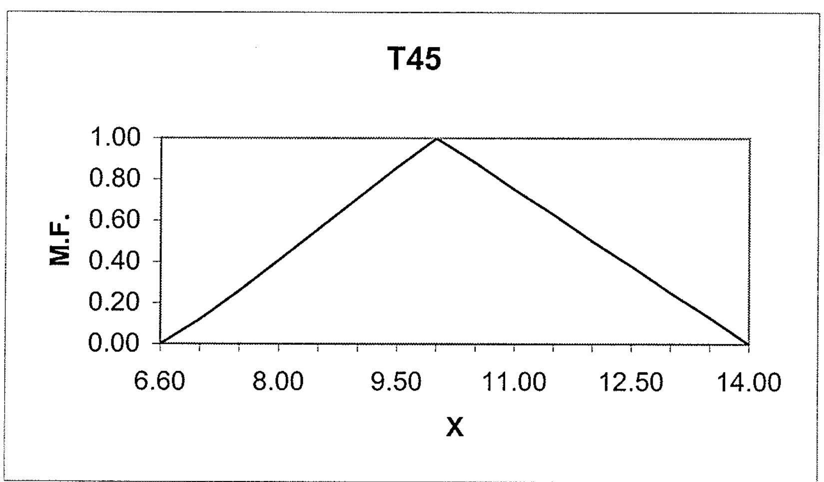




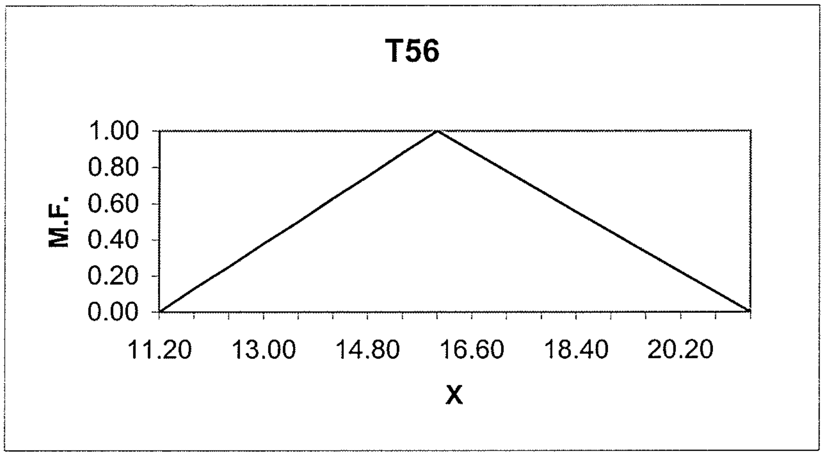
Graph of Membership Functions for Tij 's

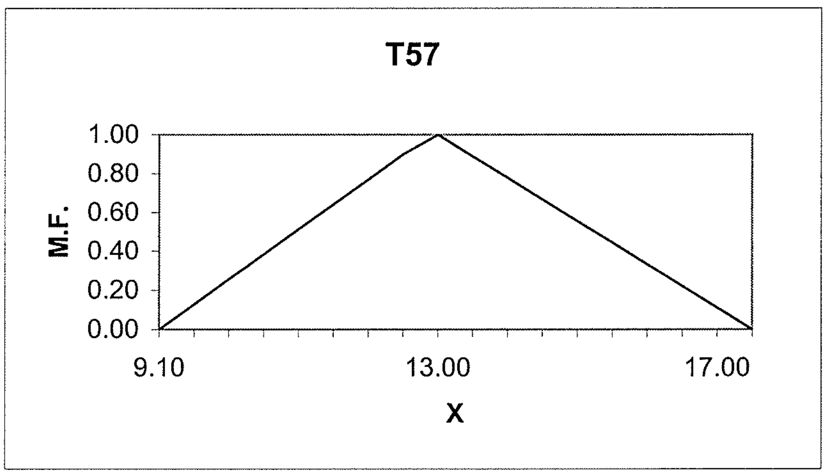


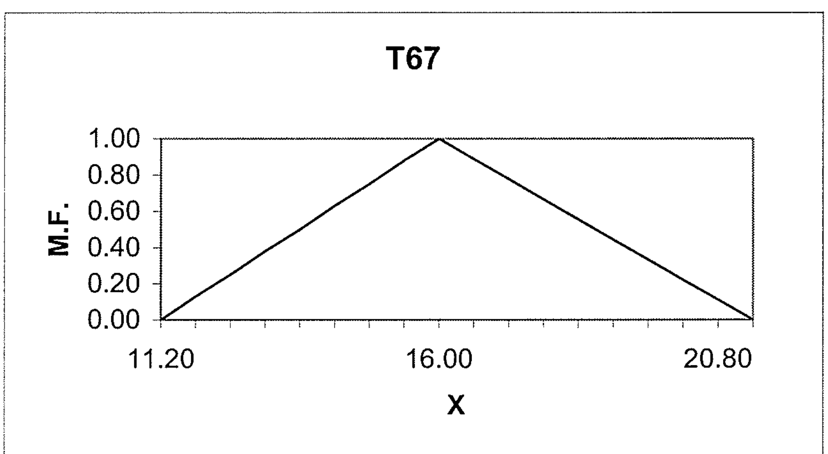




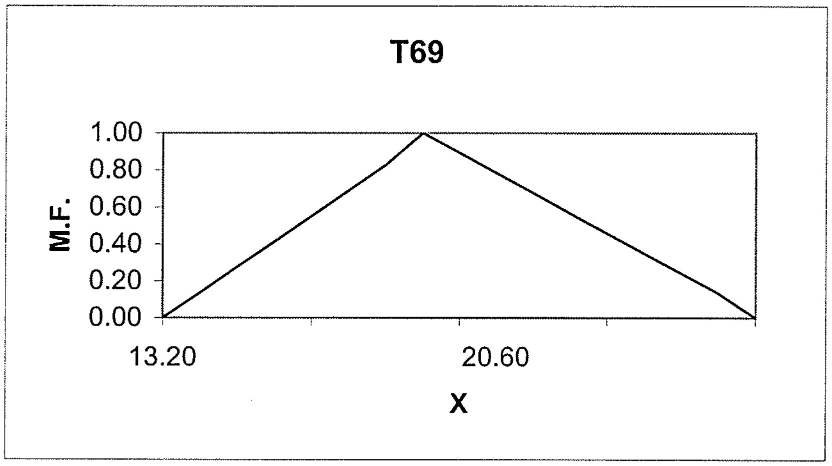
Graph of Membership Functions for Tij 's

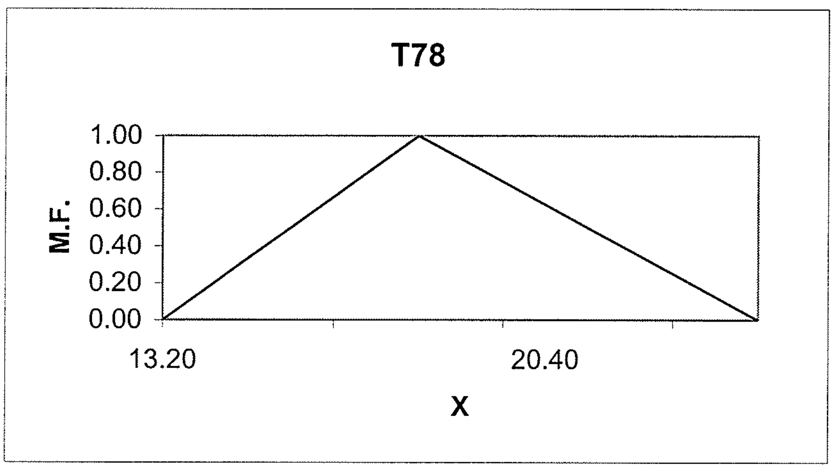


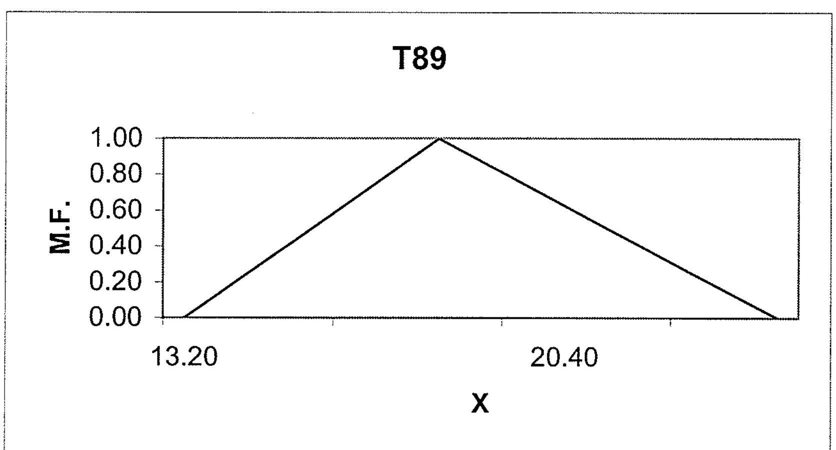




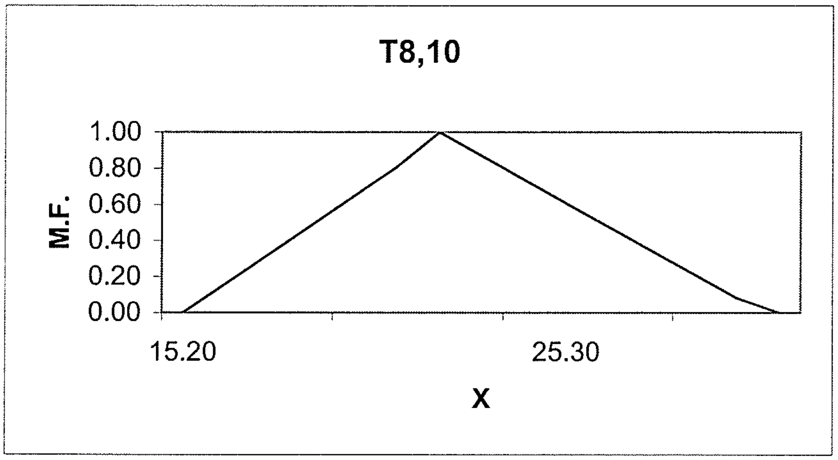
Graph of Membership Functions for Tij 's

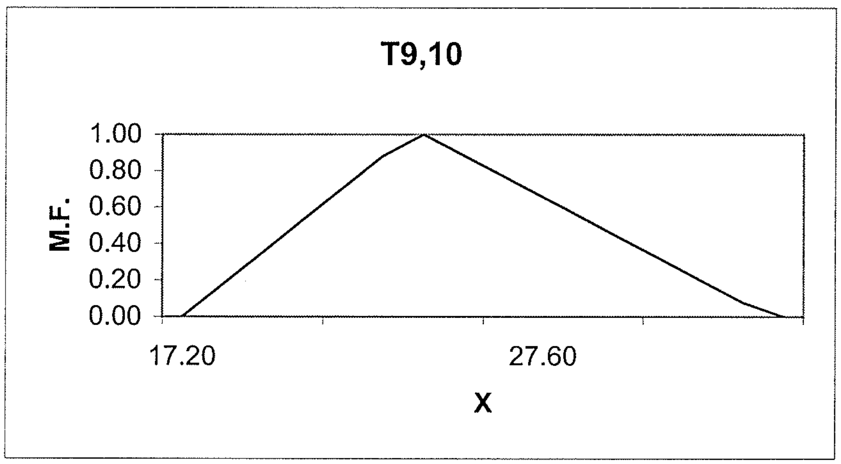


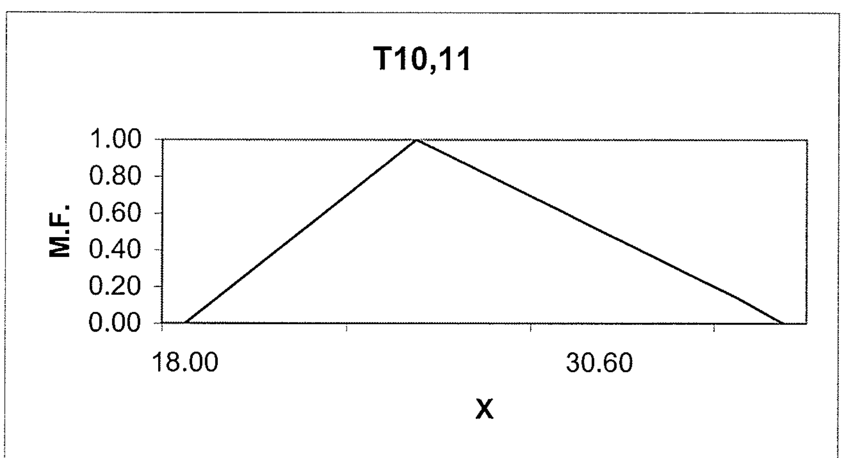




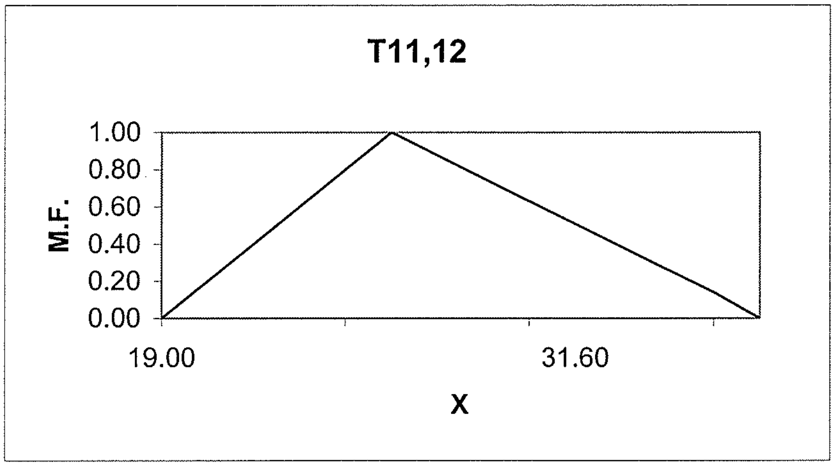
Graph of Membership Functions for Tij 's

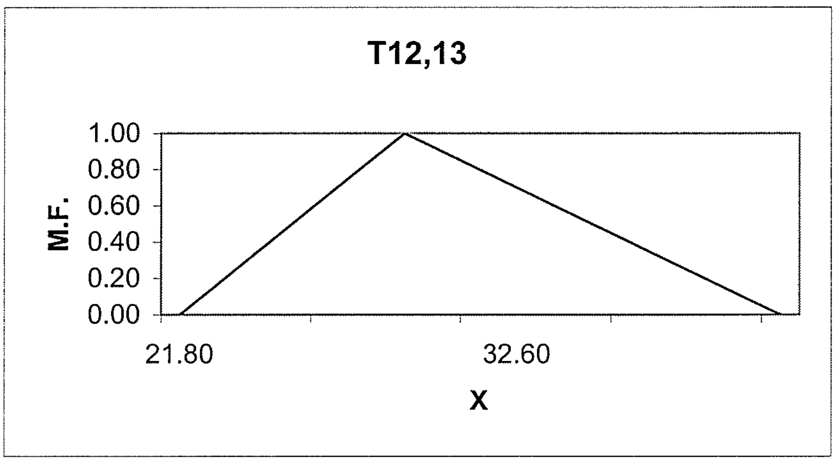


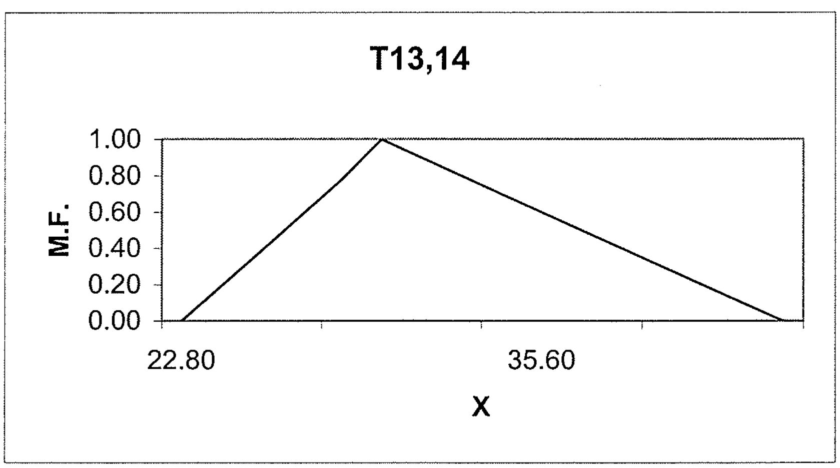




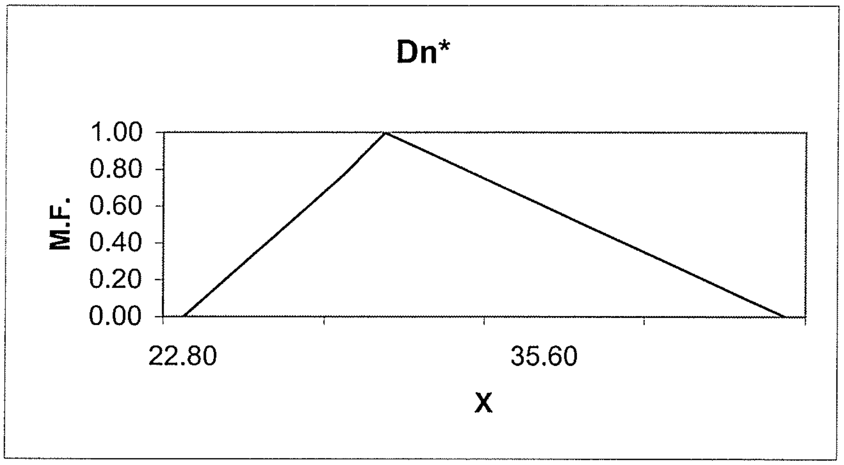
Graph of Membership Functions for Tij 's







Graph of Membership Functions for Tij 's



APPENDIX 3

Interval of Confidence Tables

for c_{ij}'s from Chapter 5

for $0 \leq \alpha \leq 1$

(page 138 - 145)

Interval of Confidence for $0 \leq \alpha \leq 1$

Corresponding to c_{12}

α	$x = 2.0\ \alpha + 5.0$	α	$x = -1.0\ \alpha + 8.0$
0	5.00	1	7.00
0.1	5.20	0.9	7.10
0.2	5.40	0.8	7.20
0.3	5.60	0.7	7.30
0.4	5.80	0.6	7.40
0.5	6.00	0.5	7.50
0.6	6.20	0.4	7.60
0.7	6.40	0.3	7.70
0.8	6.60	0.2	7.80
0.9	6.80	0.1	7.90
1	7.00	0	8.00

Corresponding to c_{13}

α	$x = 1.0\ \alpha + 8.0$	α	$x = -3.0\ \alpha + 12.0$
0	8.00	1	9.00
0.1	8.10	0.9	9.30
0.2	8.20	0.8	9.60
0.3	8.30	0.7	9.90
0.4	8.40	0.6	10.20
0.5	8.50	0.5	10.50
0.6	8.60	0.4	10.80
0.7	8.70	0.3	11.10
0.8	8.80	0.2	11.40
0.9	8.90	0.1	11.70
1	9.00	0	12.00

Corresponding to c_{14}

α	$x = 3.0\ \alpha + 10.0$	α	$x = -1.0\ \alpha + 14.0$
0	10.00	1	13.00
0.1	10.30	0.9	13.10
0.2	10.60	0.8	13.20
0.3	10.90	0.7	13.30
0.4	11.20	0.6	13.40
0.5	11.50	0.5	13.50
0.6	11.80	0.4	13.60
0.7	12.10	0.3	13.70
0.8	12.40	0.2	13.80
0.9	12.70	0.1	13.90
1	13.00	0	14.00

Corresponding to c_{16}			
α	$x = 4.0\ \alpha + 15.0$	α	$x = -3.0\ \alpha + 22.0$
0	15.00	1	19.00
0.1	15.40	0.9	19.30
0.2	15.80	0.8	19.60
0.3	16.20	0.7	19.90
0.4	16.60	0.6	20.20
0.5	17.00	0.5	20.50
0.6	17.40	0.4	20.80
0.7	17.80	0.3	21.10
0.8	18.20	0.2	21.40
0.9	18.60	0.1	21.70
1	19.00	0	22.00

Corresponding to c_{24}			
α	$x = 4.0\ \alpha + 11.0$	α	$x = -2.0\ \alpha + 17.0$
0	11.00	1	15.00
0.1	11.40	0.9	15.20
0.2	11.80	0.8	15.40
0.3	12.20	0.7	15.60
0.4	12.60	0.6	15.80
0.5	13.00	0.5	16.00
0.6	13.40	0.4	16.20
0.7	13.80	0.3	16.40
0.8	14.20	0.2	16.60
0.9	14.60	0.1	16.80
1	15.00	0	17.00

Corresponding to c_{25}			
α	$x = 4.0\ \alpha + 13.0$	α	$x = -4.0\ \alpha + 21.0$
0	13.00	1	17.00
0.1	13.40	0.9	17.40
0.2	13.80	0.8	17.80
0.3	14.20	0.7	18.20
0.4	14.60	0.6	18.60
0.5	15.00	0.5	19.00
0.6	15.40	0.4	19.40
0.7	15.80	0.3	19.80
0.8	16.20	0.2	20.20
0.9	16.60	0.1	20.60
1	17.00	0	21.00

Corresponding to c_{34}

α	x = 2.0 α + 15.0	α	x = − 5.0 α + 22.0
0	15.00	1	17.00
0.1	15.20	0.9	17.50
0.2	15.40	0.8	18.00
0.3	15.60	0.7	18.50
0.4	15.80	0.6	19.00
0.5	16.00	0.5	19.50
0.6	16.20	0.4	20.00
0.7	16.40	0.3	20.50
0.8	16.60	0.2	21.00
0.9	16.80	0.1	21.50
1	17.00	0	22.00

Corresponding to c_{35}

α	x = 2.0 α + 17.0	α	x = −5.0 α + 24.0
0	17.00	1	19.00
0.1	17.20	0.9	19.50
0.2	17.40	0.8	20.00
0.3	17.60	0.7	20.50
0.4	17.80	0.6	21.00
0.5	18.00	0.5	21.50
0.6	18.20	0.4	22.00
0.7	18.40	0.3	22.50
0.8	18.60	0.2	23.00
0.9	18.80	0.1	23.50
1	19.00	0	24.00

Corresponding to c_{46}

α	x = 4.0 α + 18.0	α	x = − 4.0 α + 22.0
0	18.00	1	18.00
0.1	18.40	0.9	18.40
0.2	18.80	0.8	18.80
0.3	19.20	0.7	19.20
0.4	19.60	0.6	19.60
0.5	20.00	0.5	20.00
0.6	20.40	0.4	20.40
0.7	20.80	0.3	20.80
0.8	21.20	0.2	21.20
0.9	21.60	0.1	21.60
1	22.00	0	22.00

Corresponding to c_{56}			
α	$x = 5.0\,\alpha + 19.0$	α	$x = -6.0\,\alpha + 30.0$
0	19.00	1	24.00
0.1	19.50	0.9	24.60
0.2	20.00	0.8	25.20
0.3	20.50	0.7	25.80
0.4	21.00	0.6	26.40
0.5	21.50	0.5	27.00
0.6	22.00	0.4	27.60
0.7	22.50	0.3	28.20
0.8	23.00	0.2	28.80
0.9	23.50	0.1	29.40
1	24.00	0	30.00

Corresponding to c_{67}			
α	$x = 5.0\,\alpha + 17.0$	α	$x = -3.0\,\alpha + 25.0$
0	17.00	1	22.00
0.1	17.50	0.9	22.30
0.2	18.00	0.8	22.60
0.3	18.50	0.7	22.90
0.4	19.00	0.6	23.20
0.5	19.50	0.5	23.50
0.6	20.00	0.4	23.80
0.7	20.50	0.3	24.10
0.8	21.00	0.2	24.40
0.9	21.50	0.1	24.70
1	22.00	0	25.00

Corresponding to c_{68}			
α	$x = 5.0\,\alpha + 18.0$	α	$x = -6.0\,\alpha + 29.0$
0	18.00	1	23.00
0.1	18.50	0.9	23.60
0.2	19.00	0.8	24.20
0.3	19.50	0.7	24.80
0.4	20.00	0.6	25.40
0.5	20.50	0.5	26.00
0.6	21.00	0.4	26.60
0.7	21.50	0.3	27.20
0.8	22.00	0.2	27.80
0.9	22.50	0.1	28.40
1	23.00	0	29.00

Corresponding to c_{69}			
α	x = 7.0 α + 19.0	α	x = – 6.0 α + 32.0
0	19.00	1	26.00
0.1	19.70	0.9	26.60
0.2	20.40	0.8	27.20
0.3	21.10	0.7	27.80
0.4	21.80	0.6	28.40
0.5	22.50	0.5	29.00
0.6	23.20	0.4	29.60
0.7	23.90	0.3	30.20
0.8	24.60	0.2	30.80
0.9	25.30	0.1	31.40
1	26.00	0	32.00

Corresponding to $c_{6,11}$			
α	x = 5.0 α + 25.0	α	x = – 7.0 α + 37.0
0	25.00	1	30.00
0.1	25.50	0.9	30.70
0.2	26.00	0.8	31.40
0.3	26.50	0.7	32.10
0.4	27.00	0.6	32.80
0.5	27.50	0.5	33.50
0.6	28.00	0.4	34.20
0.7	28.50	0.3	34.90
0.8	29.00	0.2	35.60
0.9	29.50	0.1	36.30
1	30.00	0	37.00

Corresponding to c_{79}			
α	x = 7.0 α + 21.0	α	x = – 4.0 α + 32.0
0	21.00	1	28.00
0.1	21.70	0.9	28.40
0.2	22.40	0.8	28.80
0.3	23.10	0.7	29.20
0.4	23.80	0.6	29.60
0.5	24.50	0.5	30.00
0.6	25.20	0.4	30.40
0.7	25.90	0.3	30.80
0.8	26.60	0.2	31.20
0.9	27.30	0.1	31.60
1	28.00	0	32.00

Corresponding to c_{89}

α	$x = 6.0\,\alpha + 23.0$	α	$x = -8.0\,\alpha + 37.0$
0	23.00	1	29.00
0.1	23.60	0.9	29.80
0.2	24.20	0.8	30.60
0.3	24.80	0.7	31.40
0.4	25.40	0.6	32.20
0.5	26.00	0.5	33.00
0.6	26.60	0.4	33.80
0.7	27.20	0.3	34.60
0.8	27.80	0.2	35.40
0.9	28.40	0.1	36.20
1	29.00	0	37.00

Corresponding to $c_{8,10}$

α	$x = 6.0\,\alpha + 21.0$	α	$x = -9.0\,\alpha + 36.0$
0	21.00	1	27.00
0.1	21.60	0.9	27.90
0.2	22.20	0.8	28.80
0.3	22.80	0.7	29.70
0.4	23.40	0.6	30.60
0.5	24.00	0.5	31.50
0.6	24.60	0.4	32.40
0.7	25.20	0.3	33.30
0.8	25.80	0.2	34.20
0.9	26.40	0.1	35.10
1	27.00	0	36.00

Corresponding to $c_{9,11}$

α	$x = 8.0\,\alpha + 21.0$	α	$x = -8.0\,\alpha + 37.0$
0	21.00	1	29.00
0.1	21.80	0.9	29.80
0.2	22.60	0.8	30.60
0.3	23.40	0.7	31.40
0.4	24.20	0.6	32.20
0.5	25.00	0.5	33.00
0.6	25.80	0.4	33.80
0.7	26.60	0.3	34.60
0.8	27.40	0.2	35.40
0.9	28.20	0.1	36.20
1	29.00	0	37.00

Corresponding to $c_{10,11}$			
α	$x = 7.0\,\alpha + 23.0$	α	$x = -12.0\,\alpha + 42.0$
0	23.00	1	30.00
0.1	23.70	0.9	31.20
0.2	24.40	0.8	32.40
0.3	25.10	0.7	33.60
0.4	25.80	0.6	34.80
0.5	26.50	0.5	36.00
0.6	27.20	0.4	37.20
0.7	27.90	0.3	38.40
0.8	28.60	0.2	39.60
0.9	29.30	0.1	40.80
1	30.00	0	42.00

Corresponding to v_{n*}			
α	$x = 8.0\,\alpha + 21.0$	α	$x = -8.0\,\alpha + 37.0$
0	21.00	1	29.00
0.1	21.80	0.9	29.80
0.2	22.60	0.8	30.60
0.3	23.40	0.7	31.40
0.4	24.20	0.6	32.20
0.5	25.00	0.5	33.00
0.6	25.80	0.4	33.80
0.7	26.60	0.3	34.60
0.8	27.40	0.2	35.40
0.9	28.20	0.1	36.20
1	29.00	0	37.00

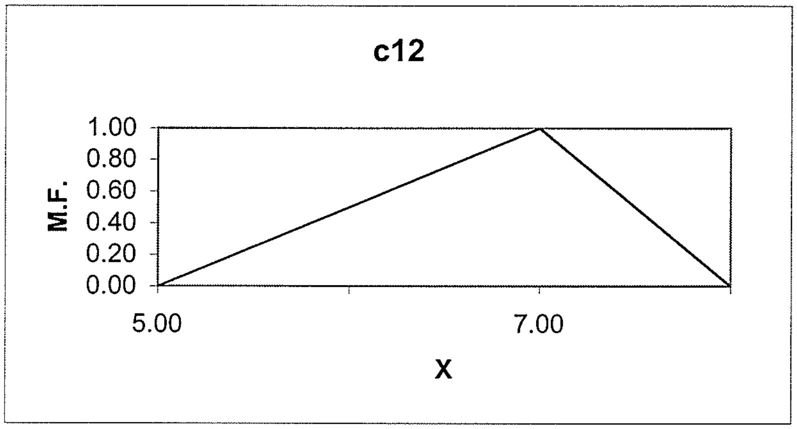
APPENDIX 4

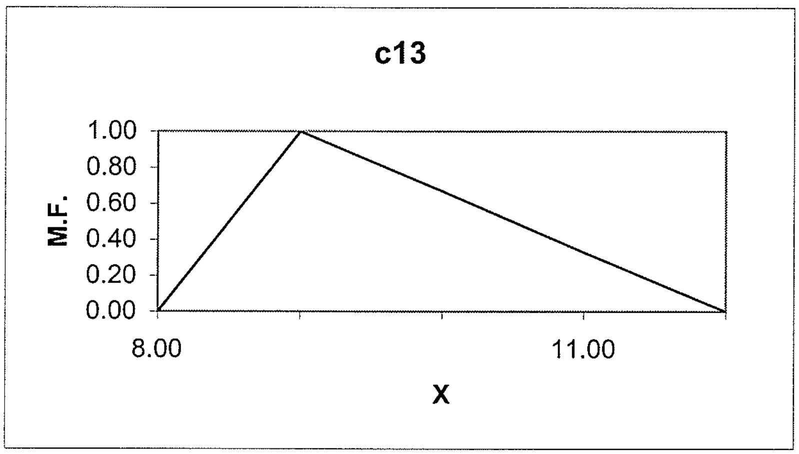
Graph of Membership Functions

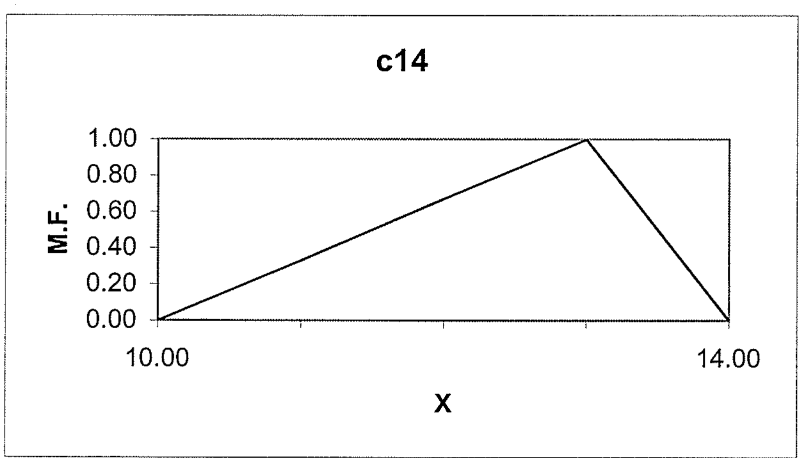
for c_{ij}'s from Chapter 5

(page 146 - 153)

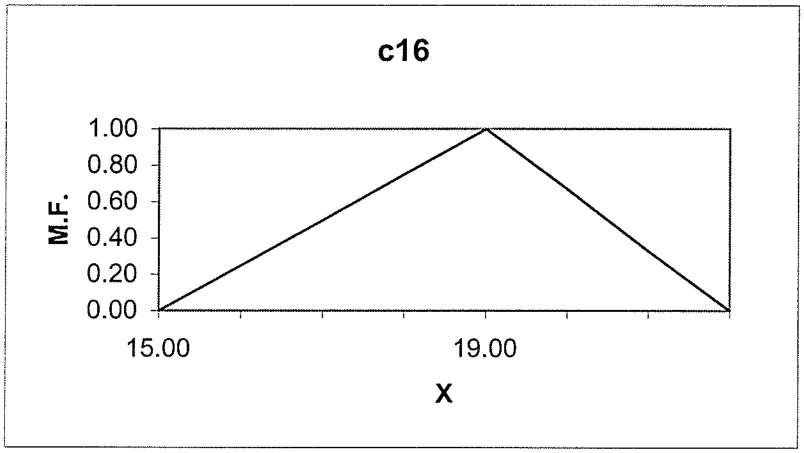
Graph of Membership Functions for cij 's

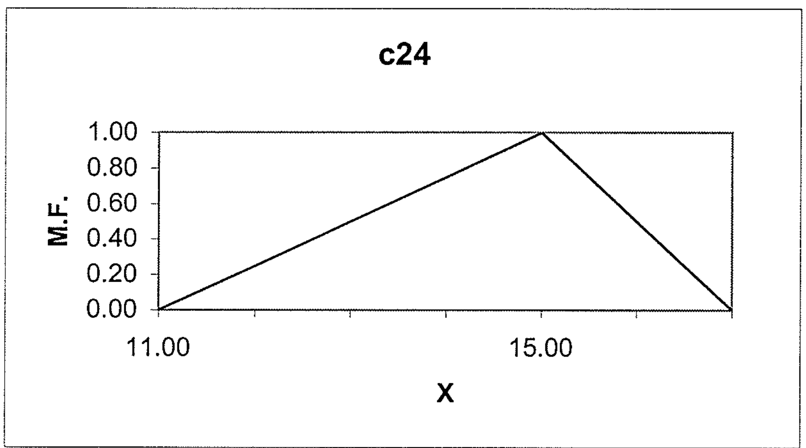


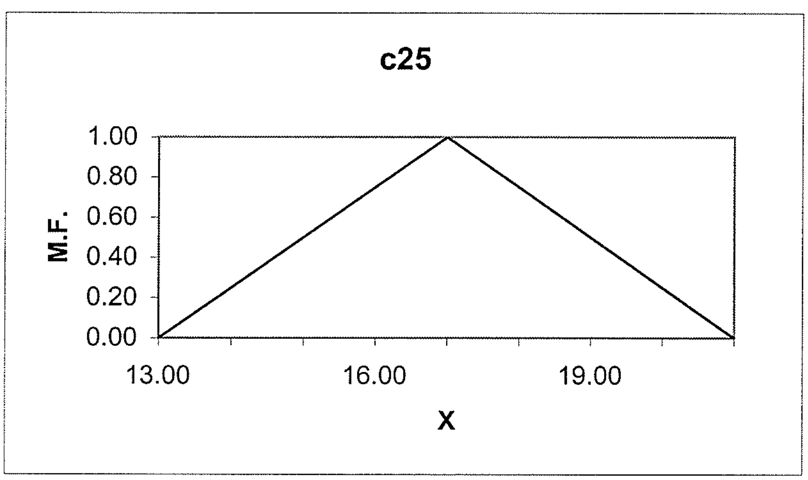




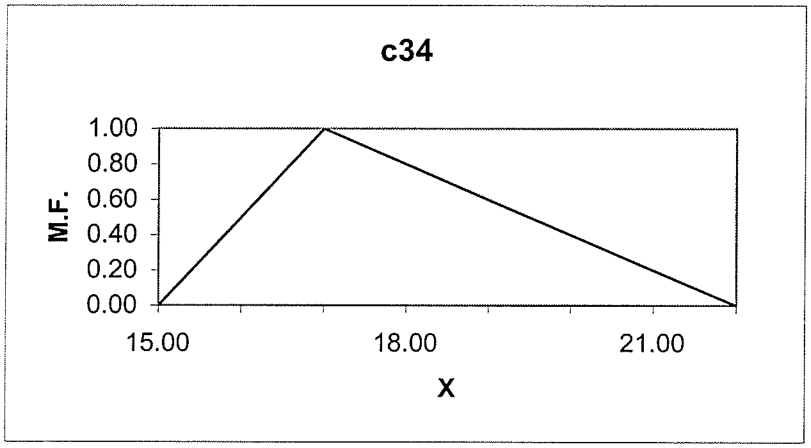
Graph of Membership Functions for cij 's

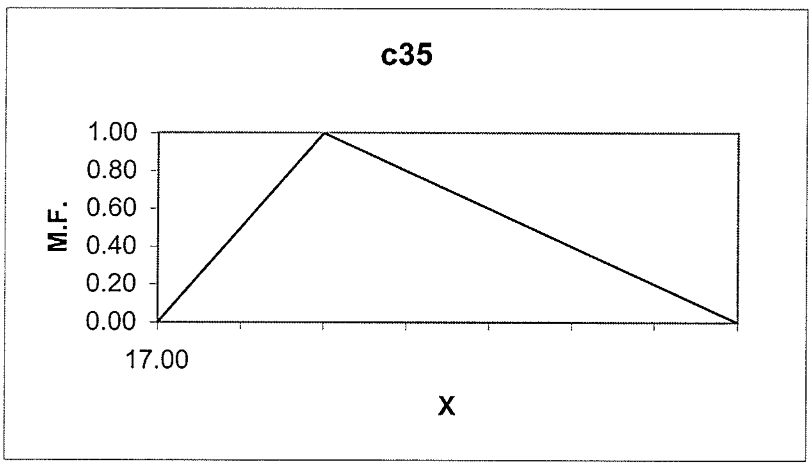


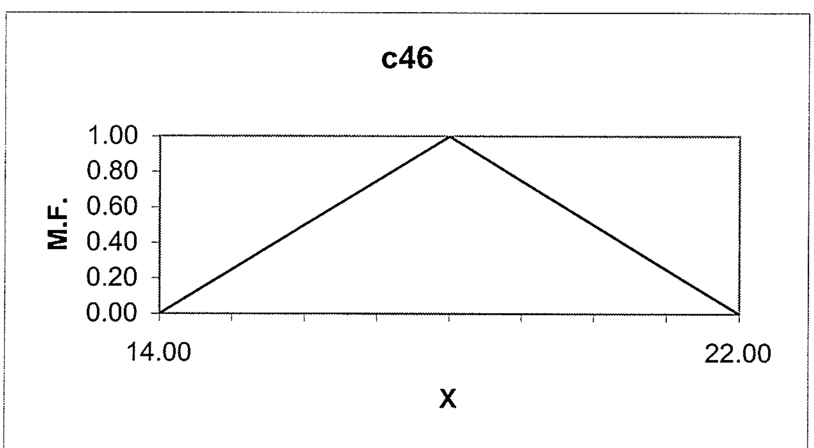




Graph of Membership Functions for cij 's







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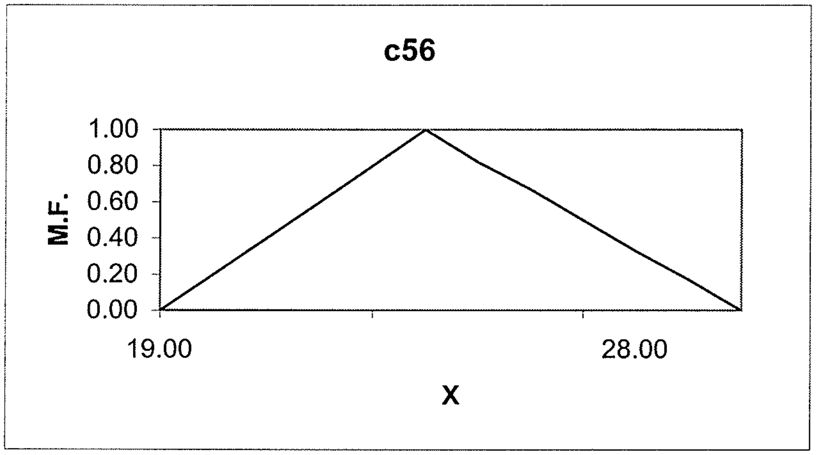
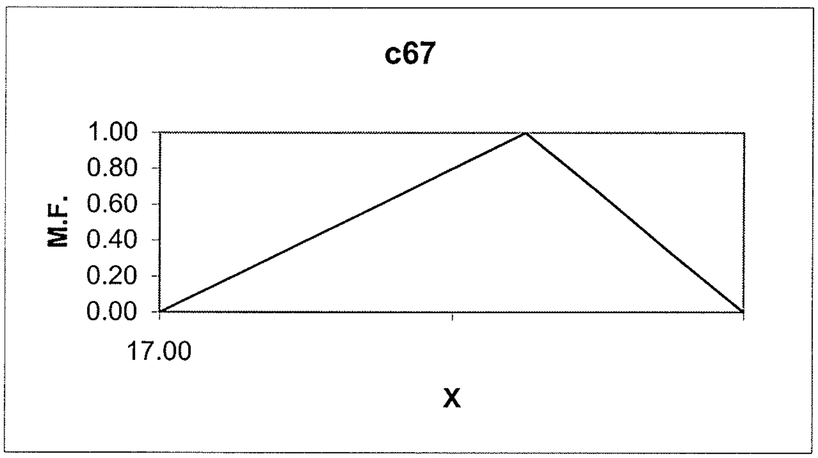
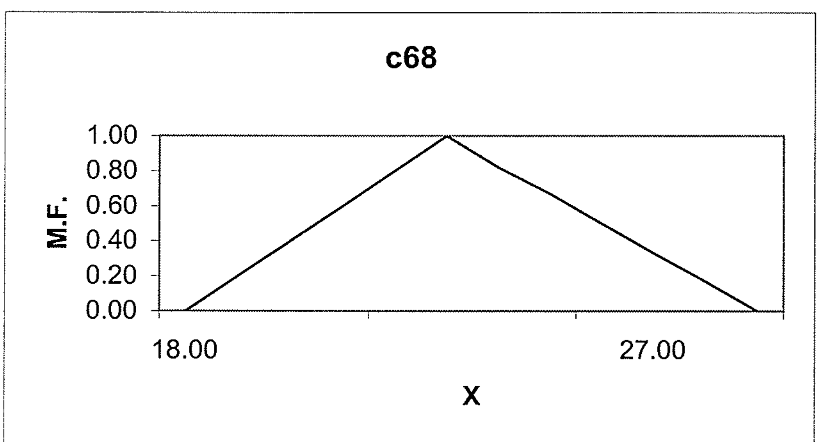
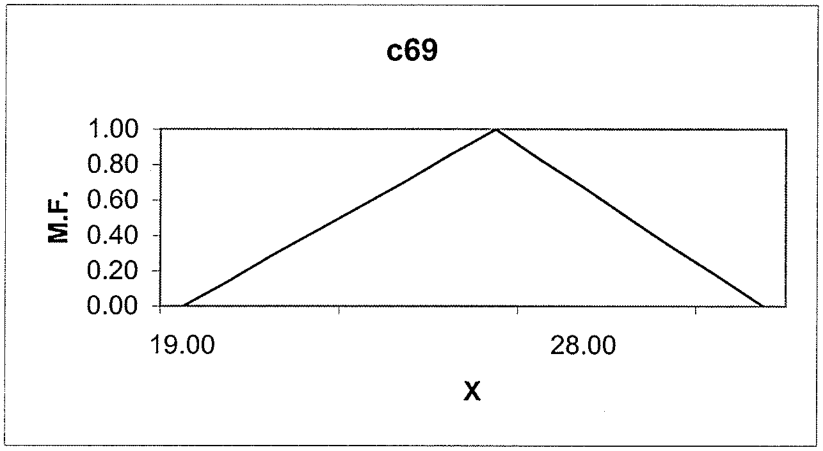
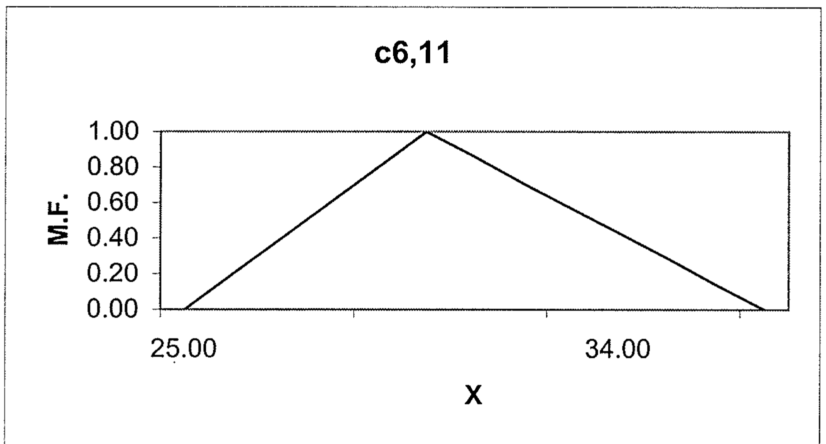


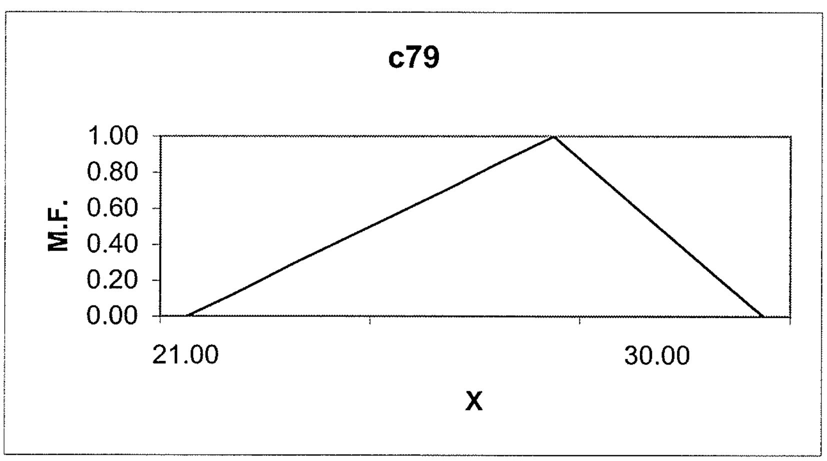




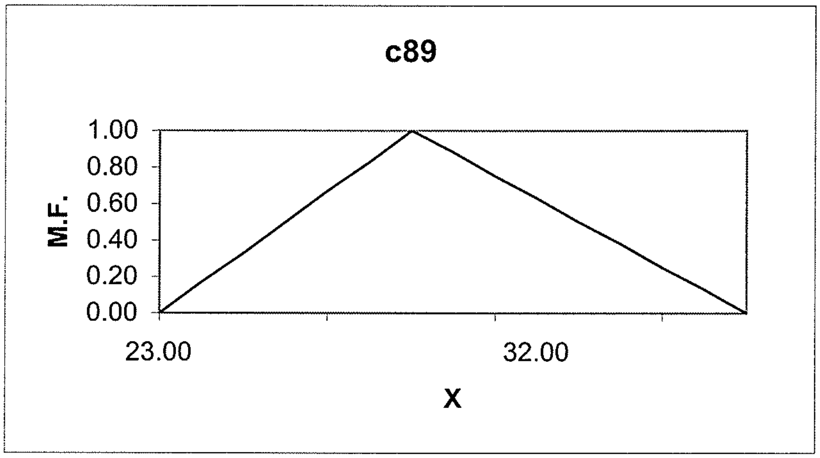
Graph of Membership Functions for cij 's

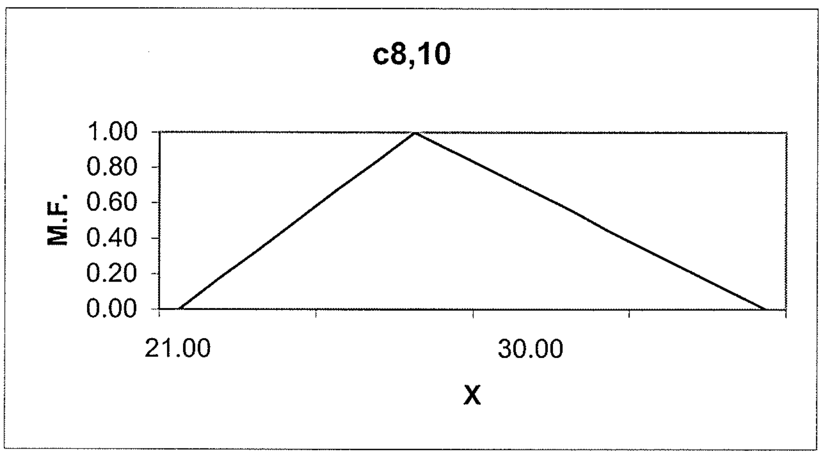


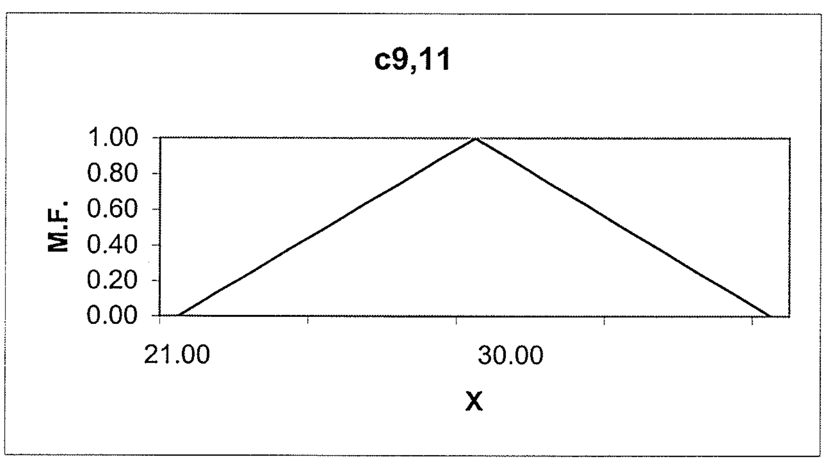




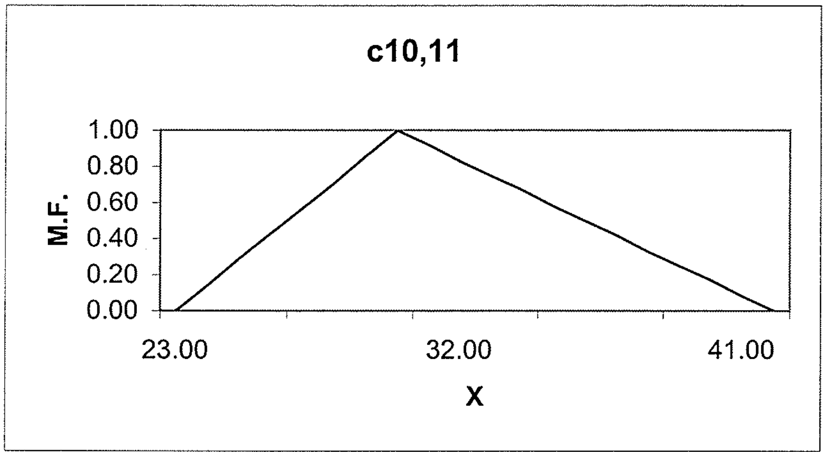
Graph of Membership Functions for cij 's

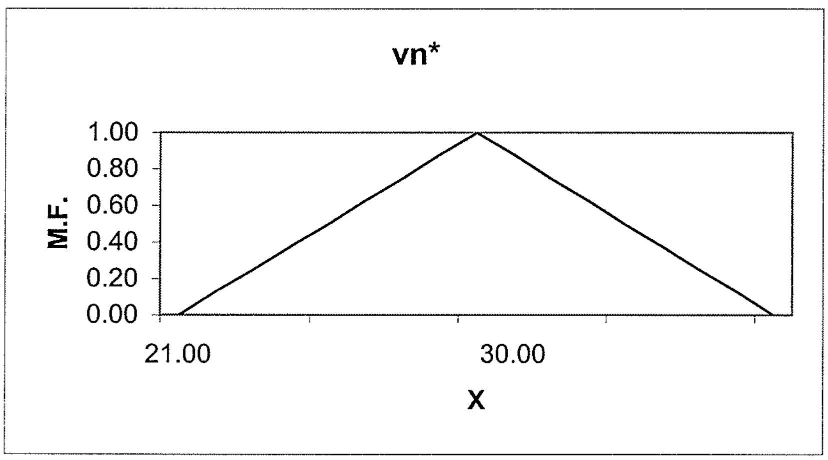






Graph of Membership Functions for cij 's





APPENDIX 5

Interval of Confidence Tables

for c_{ij}'s from Chapter 6

for $0 \leq \alpha \leq 1$

(page 154 - 159)

Interval of Confidence for $0 \le \alpha \le 1$

Corresponding to c_{12}

α	$x = 0.3\alpha + 2.7$	α	$x = -0.8\alpha + 3.8$
0	2.70	1	3.00
0.1	2.73	0.9	3.08
0.2	2.76	0.8	3.16
0.3	2.79	0.7	3.24
0.4	2.82	0.6	3.32
0.5	2.85	0.5	3.40
0.6	2.88	0.4	3.48
0.7	2.91	0.3	3.56
0.8	2.94	0.2	3.64
0.9	2.97	0.1	3.72
1	3.00	0	3.80

Corresponding to c_{13}

α	$x = 0.5\alpha + 3.0$	α	$x = -0.9\alpha + 4.4$
0	3.00	1	3.50
0.1	3.05	0.9	3.59
0.2	3.10	0.8	3.68
0.3	3.15	0.7	3.77
0.4	3.20	0.6	3.86
0.5	3.25	0.5	3.95
0.6	3.30	0.4	4.04
0.7	3.35	0.3	4.13
0.8	3.40	0.2	4.22
0.9	3.45	0.1	4.31
1	3.50	0	4.40

Corresponding to c_{14}

α	$x = 0.5\alpha + 4.5$	α	$x = -1.3\alpha + 6.3$
0	4.50	1	5.00
0.1	4.55	0.9	5.13
0.2	4.60	0.8	5.26
0.3	4.65	0.7	5.39
0.4	4.70	0.6	5.52
0.5	4.75	0.5	5.65
0.6	4.80	0.4	5.78
0.7	4.85	0.3	5.91
0.8	4.90	0.2	6.04
0.9	4.95	0.1	6.17
1	5.00	0	6.30

Corresponding to c_{24}			
α	x = 0.6 α + 5.4	α	x = − 1.6 α + 7.6
0	5.40	1	6.00
0.1	5.46	0.9	6.16
0.2	5.52	0.8	6.32
0.3	5.58	0.7	6.48
0.4	5.64	0.6	6.64
0.5	5.70	0.5	6.80
0.6	5.76	0.4	6.96
0.7	5.82	0.3	7.12
0.8	5.88	0.2	7.28
0.9	5.94	0.1	7.44
1	6.00	0	7.60

Corresponding to c_{34}			
α	x = 1.0 α + 6.0	α	x = − 1.8 α + 8.8
0	6.00	1	7.00
0.1	6.10	0.9	7.18
0.2	6.20	0.8	7.36
0.3	6.30	0.7	7.54
0.4	6.40	0.6	7.72
0.5	6.50	0.5	7.90
0.6	6.60	0.4	8.08
0.7	6.70	0.3	8.26
0.8	6.80	0.2	8.44
0.9	6.90	0.1	8.62
1	7.00	0	8.80

Corresponding to c_{45}			
α	x = 0.6 α + 5.9	α	x = −1.8 α + 8.3
0	5.90	1	6.50
0.1	5.96	0.9	6.68
0.2	6.02	0.8	6.86
0.3	6.08	0.7	7.04
0.4	6.14	0.6	7.22
0.5	6.20	0.5	7.40
0.6	6.26	0.4	7.58
0.7	6.32	0.3	7.76
0.8	6.38	0.2	7.94
0.9	6.44	0.1	8.12
1	6.50	0	8.30

Corresponding to c_{46}			
α	$x = 0.7\,\alpha + 6.8$	α	$x = -1.8\,\alpha + 9.3$
0	6.80	1	7.50
0.1	6.87	0.9	7.68
0.2	6.94	0.8	7.86
0.3	7.01	0.7	8.04
0.4	7.08	0.6	8.22
0.5	7.15	0.5	8.40
0.6	7.22	0.4	8.58
0.7	7.29	0.3	8.76
0.8	7.36	0.2	8.94
0.9	7.43	0.1	9.12
1	7.50	0	9.30

Corresponding to c_{56}			
α	$x = 0.7\,\alpha + 7.3$	α	$x = -2.3\,\alpha + 10.3$
0	7.30	1	8.00
0.1	7.37	0.9	8.23
0.2	7.44	0.8	8.46
0.3	7.51	0.7	8.69
0.4	7.58	0.6	8.92
0.5	7.65	0.5	9.15
0.6	7.72	0.4	9.38
0.7	7.79	0.3	9.61
0.8	7.86	0.2	9.84
0.9	7.93	0.1	10.07
1	8.00	0	10.30

Corresponding to c_{67}			
α	$x = 0.9\,\alpha + 8.6$	α	$x = -2.3\,\alpha + 11.8$
0	8.60	1	9.50
0.1	8.69	0.9	9.73
0.2	8.78	0.8	9.96
0.3	8.87	0.7	10.19
0.4	8.96	0.6	10.42
0.5	9.05	0.5	10.65
0.6	9.14	0.4	10.88
0.7	9.23	0.3	11.11
0.8	9.32	0.2	11.34
0.9	9.41	0.1	11.57
1	9.50	0	11.80

Corresponding to c_{68}			
α	$x = 0.8\,\alpha + 8.2$	α	$x = -2.3\,\alpha + 11.3$
0	8.20	1	9.00
0.1	8.28	0.9	9.23
0.2	8.36	0.8	9.46
0.3	8.44	0.7	9.69
0.4	8.52	0.6	9.92
0.5	8.60	0.5	10.15
0.6	8.68	0.4	10.38
0.7	8.76	0.3	10.61
0.8	8.84	0.2	10.84
0.9	8.92	0.1	11.07
1	9.00	0	11.30

Corresponding to c_{69}			
α	$x = 1.0\,\alpha + 9.5$	α	$x = -2.6\,\alpha + 13.1$
0	9.50	1	10.50
0.1	9.60	0.9	10.76
0.2	9.70	0.8	11.02
0.3	9.80	0.7	11.28
0.4	9.90	0.6	11.54
0.5	10.00	0.5	11.80
0.6	10.10	0.4	12.06
0.7	10.20	0.3	12.32
0.8	10.30	0.2	12.58
0.9	10.40	0.1	12.84
1	10.50	0	13.10

Corresponding to c_{79}			
α	$x = 1.1\,\alpha + 10.4$	α	$x = -2.8\,\alpha + 14.3$
0	10.40	1	11.50
0.1	10.51	0.9	11.78
0.2	10.62	0.8	12.06
0.3	10.73	0.7	12.34
0.4	10.84	0.6	12.62
0.5	10.95	0.5	12.90
0.6	11.06	0.4	13.18
0.7	11.17	0.3	13.46
0.8	11.28	0.2	13.74
0.9	11.39	0.1	14.02
1	11.50	0	14.30

Corresponding to c_{89}			
α	$x = 0.9\,\alpha + 9.6$	α	$x = -2.8\,\alpha + 13.3$
0	9.60	1	10.50
0.1	9.69	0.9	10.78
0.2	9.78	0.8	11.06
0.3	9.87	0.7	11.34
0.4	9.96	0.6	11.62
0.5	10.05	0.5	11.90
0.6	10.14	0.4	12.18
0.7	10.23	0.3	12.46
0.8	10.32	0.2	12.74
0.9	10.41	0.1	13.02
1	10.50	0	13.30

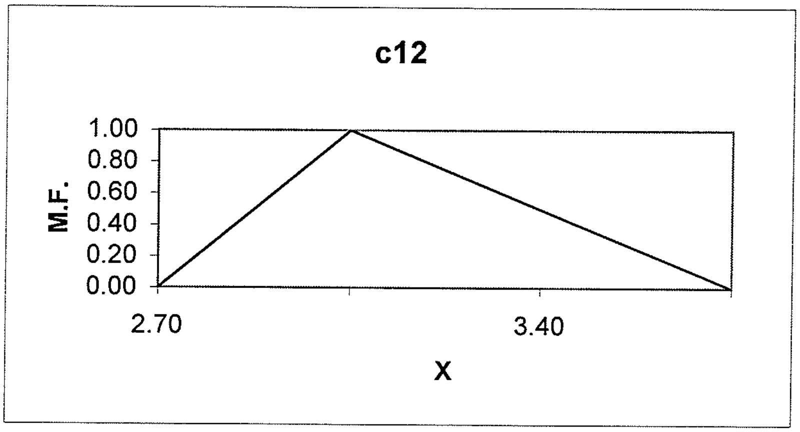
APPENDIX 6

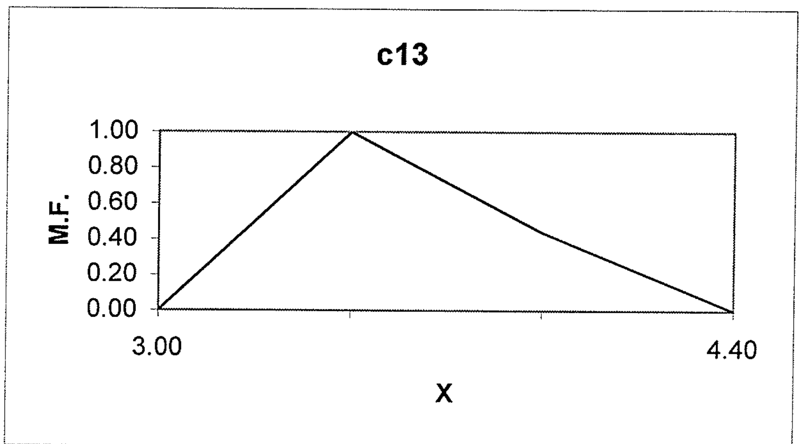
Graph of Membership Functions

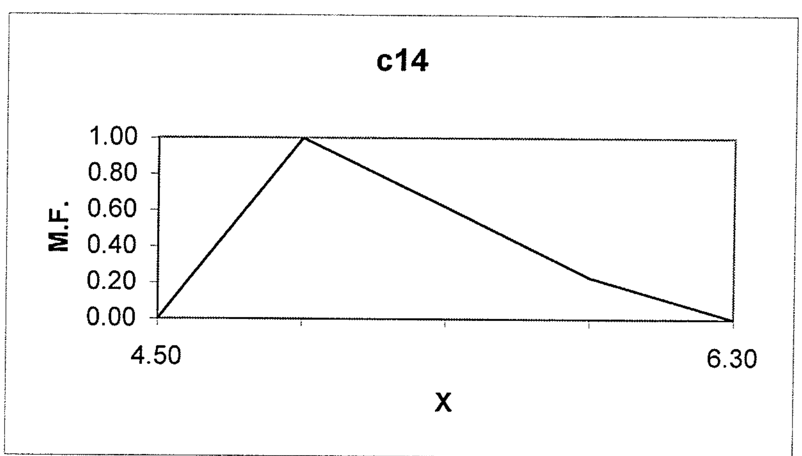
for c_{ij}'s from Chapter 6

(page 160 - 165)

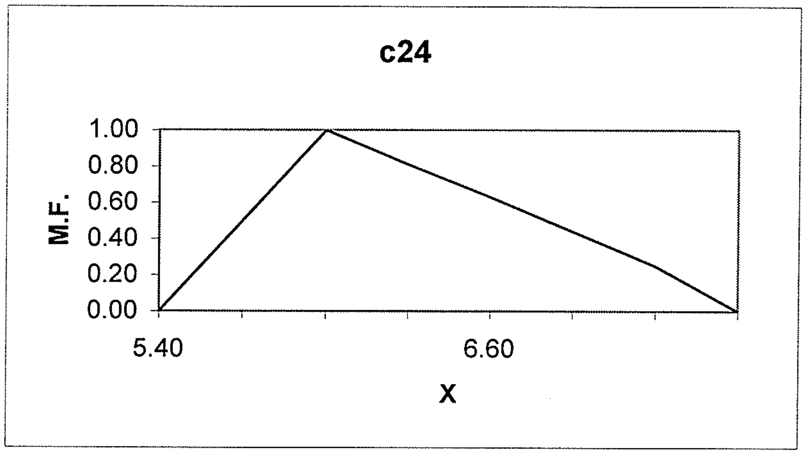
Graph of Membership Functions for cij 's

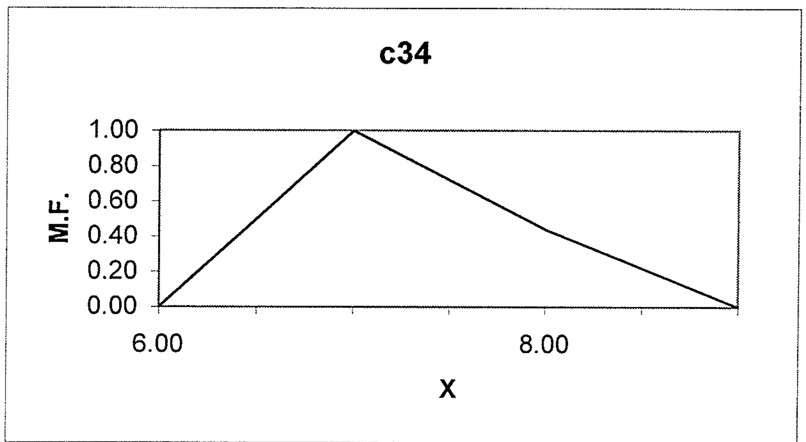


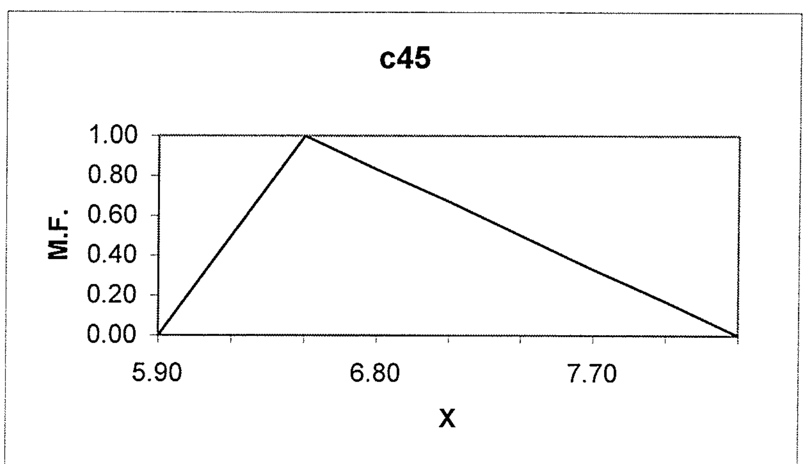




Graph of Membership Functions for cij 's







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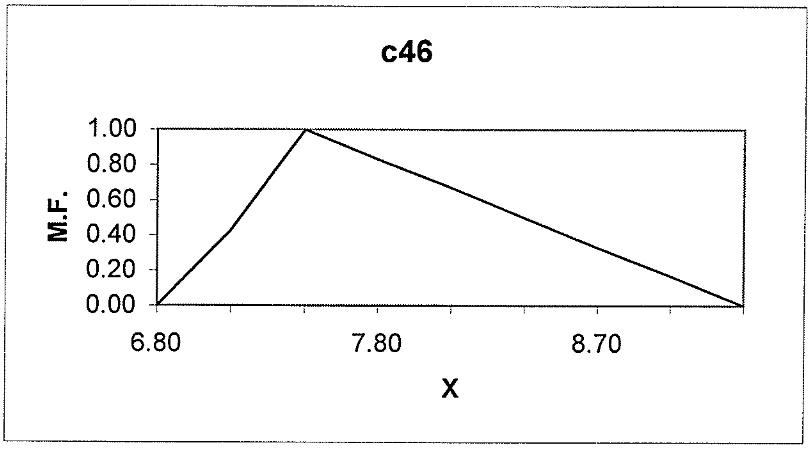

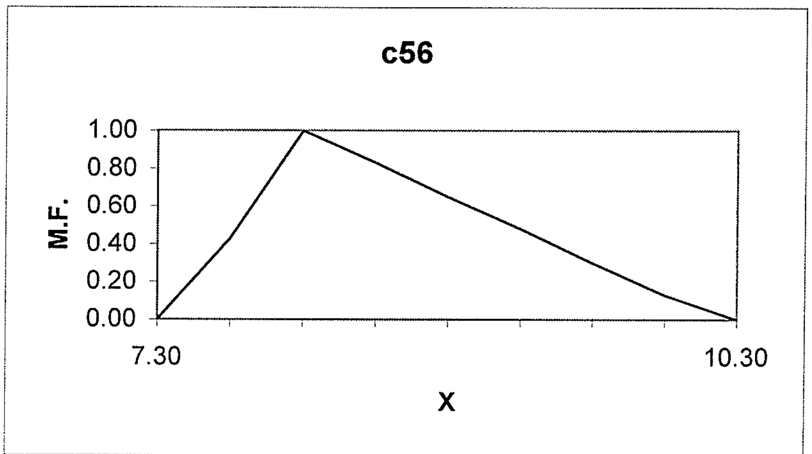

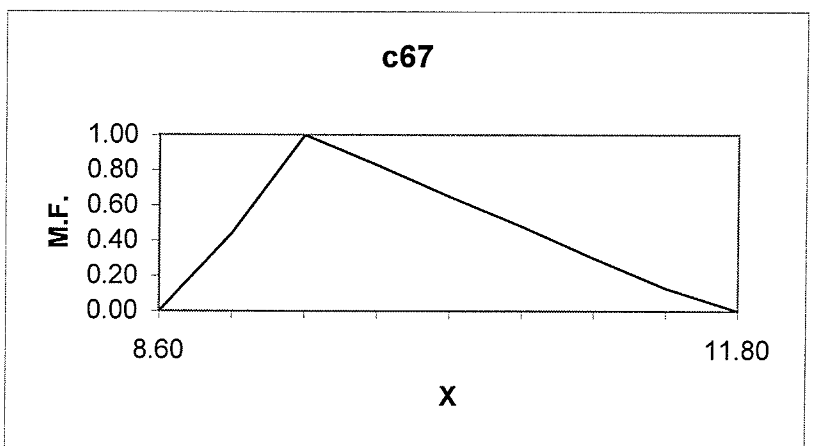

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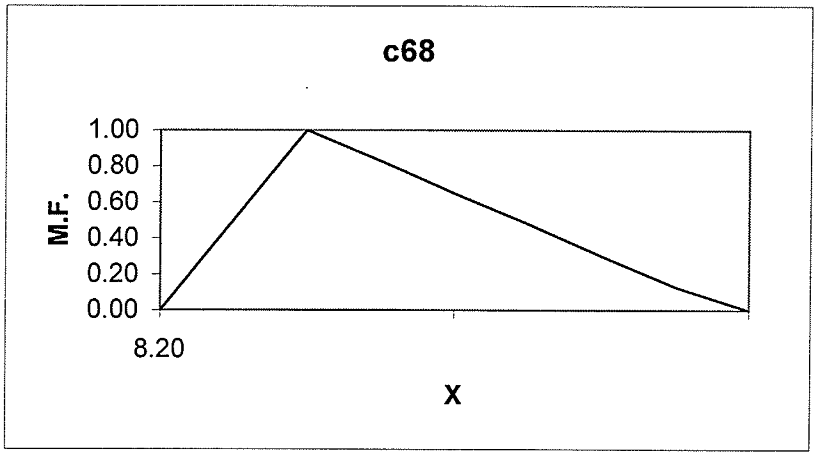

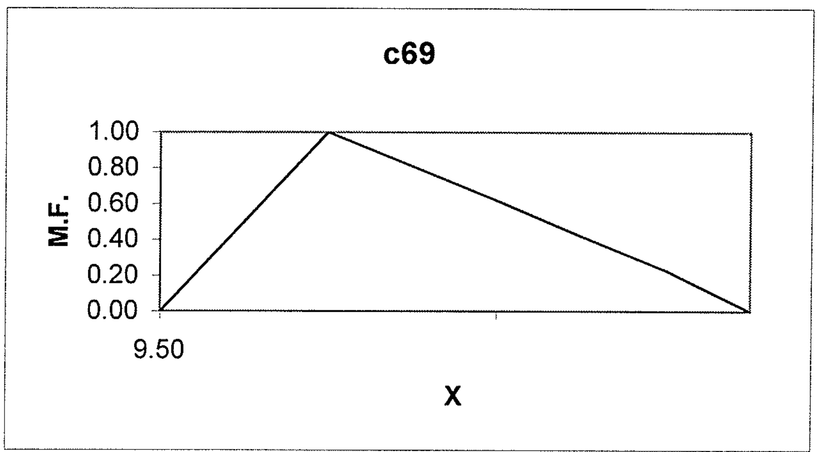

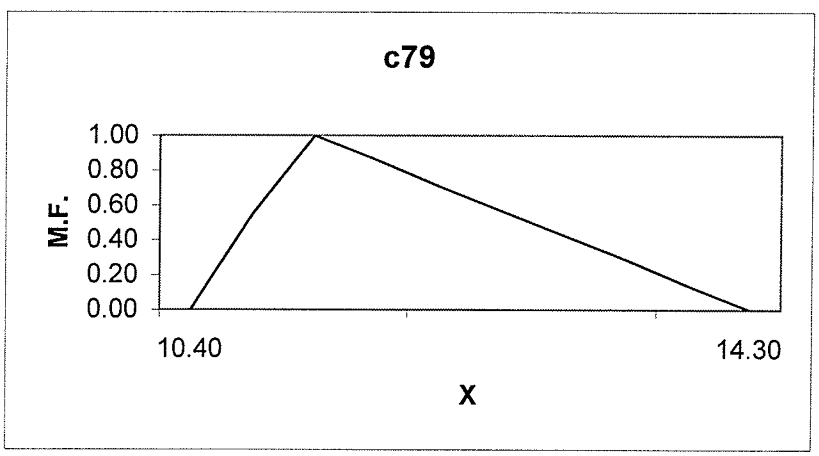

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