

The Hilbert Functions of Families of Fat Point Schemes

by

Samuel Adeshina

A Thesis submitted to the Faculty of Graduate Studies of
The University of Manitoba
in partial fulfilment of the requirements of the degree of

MASTER OF SCIENCE

Department of Mathematics
University of Manitoba
Winnipeg

Copyright © 2025 by Samuel Adeshina

Abstract

We determine the Hilbert functions of families of fat point schemes supported on special point sets called *partial intersections*. The partial intersection on which we focus is of type $p = (3, 2)$ and $q = (2\ell, b)$, where $\ell = 1, 2, 3$. We employ an algorithm known as the *greedy algorithm* to reduce the multiplicities of associated fat point schemes to zero. The algorithm produces a vector that can be used to determine an upper and a lower bound of the Hilbert function of any fat point scheme. For each family $\ell = 1, 2, 3$ that we investigate, the vector is special enough (called *GMS*) that the upper and lower bounds are equal, determining the Hilbert function in each case.

Acknowledgements

First, I give God the Almighty all the glory for his favor upon my life. He blessed me with sound health, without which I could not have completed this research. I deeply appreciate Dr. Susan Cooper for taking me in as her research student. Her support—through patience, encouragement and expertise—was immeasurable. I learned more under her supervision than in all my previous academic years. My gratitude extends to Professor Shaun Lui and Professor Adam Clay for being there for me when I needed them most. I will never forget their contributions to my academic journey. I would also like to thank the members of my thesis examination committee, Dr. Leo Butler and Professor Yang Zhang, for their time and thoughtful feedback during my defense. Also on my mind are all the wonderful instructors who taught me, especially Dr. Thomas Kucera and Dr. Andrii Arman, for a truly enriched learning experience. To the administrative personnel at the Department of Mathematics office— Kristina, Melanie, Kerry, Brittany, and Alison—thank you for always making my day whenever I visited. My deepest gratitude to my mother, Mrs. Lola Comfort Adeshina, for her prayers, sacrifices, and unwavering support. She remains my greatest inspiration. Finally, to every member of my family—Tobi, Sanya, Daniel, and dad—I am lucky to have you. Thank you for your support.

Contents

Abstract	i
Acknowledgements	ii
List of Tables	iii
List of Figures	iv
1 Introduction	1
2 Preliminaries	4
2.1 The Hilbert Function of a Set of Reduced Points	4
2.1.1 The 0-Sequence	9
2.1.2 Data Obtained from Hilbert Functions	16
3 Fat Point Schemes	28
3.1 Hilbert Functions of Fat Point Schemes: Definitions and Literature Review .	28
3.2 The Reduction Approach	32
3.2.1 The Greedy Algorithm	40
3.3 Fat Point Schemes Whose Support is a Partial Intersection	41
3.3.1 Hilbert Functions of Fat Points Supported on Partial Intersections . .	42
3.3.2 The Hilbert Function of a Fat Point Scheme Supported on a Partial Intersection of Type $p = (3, 2)$ and $q = (2\ell, b)$	45

List of Tables

2.1	3-binomial expansion of 18.	10
2.2	3-binomial expansion of 18.	11

List of Figures

2.1	The set of 8 points of $\mathbb{X}$ in $\mathbb{A}^2 \subset \mathbb{P}^2$	18
3.1	Scheme A with 3 points each of multiplicity 2 and 12 points of multiplicity 1.	33
3.2	Reduced point set with arrows showing $\text{diag}(\mathbf{v})$	35
3.3	Diagram showing $\text{diag}(5, 4, 3, 3, 3, 2, 1)$	37
3.4	Scheme A with lines l_1, l_2 & l_3	38
3.5	Diagram formed by $d = (7, 5, 3, 1)$	38
3.6	Reduced point set formed by $\mathbf{v} = (6, 6, 3, 1)$	39
3.7	Subscheme $A = 2p_1 + p_2 + p_3 + 2p_4 + p_5 + 2p_6$	40
3.8	Scheme A with three points of multiplicity 2.	40
3.9	Partial intersection with three levels of blocks.	41
3.10	Partial intersection of type $p = (5, 3, 2)$ and $q = (3, 2, 1)$	42
3.11	Partial intersection of type $p = (7, 5, 3)$ and $q = (2, 2, 2)$	42
3.12	Partial intersection of type $p = (3, 2)$ and $q = (4, 2)$	43
3.13	Diagram with $\mathbf{v} = (12, 12, 8, 6, 6, 4)$	43
3.14	Partial intersection of type $p = (5, 3)$ and $q = (2, 2)$	44
3.15	Reduced point set from $d = (15, 15, 10, 10, 9, 9, 6, 6, 5, 5, 3, 3)$	44
3.16	Partial intersection of type $p = (3, 2)$ and $q = (2, 1)$	45
3.17	Partial intersection of type $p = (3, 2)$ and $q = (2, b)$	46
3.18	Partial intersection of type $p = (3, 2)$ and $q = (4, b)$	48
3.19	Partial intersection of type $p = (3, 2)$ and $q = (4, 1)$	61
3.20	Partial intersection of type $p = (3, 2)$ and $q = (4, 2)$	61
3.21	Partial intersection of type $p = (3, 2)$ and $q = (4, 3)$	61
3.22	Partial intersection of type $p = (3, 2)$ and $q = (6, b)$	62
3.23	Partial intersection of type $p = (3, 2)$ and $q = (6, 1)$	86
3.24	Partial intersection of type $p = (3, 2)$ and $q = (6, 2)$	86
3.25	Partial intersection of type $p = (3, 2)$ and $q = (6, 3)$	87
3.26	Partial intersection of type $p = (3, 2)$ and $q = (6, 4)$	87

3.27	Partial intersection of type $p = (3, 2)$ and $q = (6, 5)$	88
------	--	----

Chapter 1

Introduction

Hilbert functions, a topic studied in algebraic geometry and commutative algebra, have played significant roles in helping mathematicians understand the geometry of points and the constraints imposed on those points in projective space. This thesis discusses how the Hilbert function has helped in: classifying points in projective space; determining the dimensions of graded rings; and in the studying of the linear conditions imposed on homogeneous polynomials vanishing on point sets. Interest in the Hilbert function was first pursued at the beginning of the twentieth century by Italian geometers when they studied the vector space of all homogeneous polynomials of a given degree, say t , which vanish at a given set of points [21]. Their work focused on counting the number of linearly independent forms of this degree. The number of these linearly independent forms is the codimension of the vector space. This codimension is a value of the Hilbert function of a graded ring at degree t . David Hilbert, who was one of the greatest mathematicians of his time, made significant contributions to the field while studying invariant theory. Chief among his contributions to this topic is his description of the nature of these functions. Hence, the function was named after him.

From the time of Hilbert, who described the Hilbert functions and gave it a framework, to that of Macaulay, who described the geometry of points and their constraints by using combinatorial techniques, to this day, the interest in Hilbert functions has continued to grow and attract the interest of many mathematicians. The interest of mathematicians in this topic stems partly from its many significant results and interesting unresolved problems and partly from its various applications in other branches of mathematics. In the Literature Review section of Chapter 3, we discuss some interesting results involving the Hilbert function and also mention some unsolved problems in the field. Some of the results we discuss are Macaulay's classification and description of the Hilbert function of reduced points in pro-

jective space, the equivalence of the Hilbert function of different fat point schemes, and the Betti numbers of graded rings. The open problems discussed have to do with the problem of extending known results in projective 2-space, $\mathbb{P}^2$, to higher dimensions, and the problem of exploring an arbitrary field in addition to the algebraically closed field in which most works are done today.

The goal of this thesis is to examine the Hilbert functions of families of fat points. Specifically, we are interested in classifying the Hilbert functions of fat points supported on special configurations known as *partial intersections*. The approach of this thesis is to first lay the groundwork for the Hilbert function of fat point schemes. Chapter 2 opens with a brief introduction of affine and projective space. What follows is a brief discussion on homogeneous polynomials and homogeneous ideals. These topics are relevant to discussions that revolve around graded structures. We give some important examples to enrich this discussion. Thereafter, we state the definition of the Hilbert function. What follows is an introduction to the 0-sequence and Macaulay's Theorem. Knowledge of the 0-sequence and Macaulay's Theorem are central to the understanding of the growth of the Hilbert function and the constraints on this growth. This is covered in depth in subsection 2.1.2. The constraints on the growth of the Hilbert function can describe the distribution of points on hypersurfaces and cones. In this section, we also state many important theorems and give relevant examples. Some of the examples treated show us how to determine the number of points from a given set that lie on a hypersurface.

In Chapter 3, we focus on the Hilbert function of fat point schemes. We start with a brief introduction to Hilbert functions of fat point schemes. Here, we state the definitions of some basic objects. Thereafter, we discuss some important results and give a list of open problems in the Literature Review section. Since fat point schemes deal with points with multiplicities ≥ 1 , our approach to investigating these schemes lies in systematically reducing the multiplicities of these points to zero. So, we discuss briefly the reduction approach used in [6] and also show how the reduction process gives what is known as the *reduction vector*. We state the properties of the reduction vector and also state the conditions that a *nice* reduction vector must satisfy for arising bounds of a Hilbert function to be equal. This is followed by an examination of some important theorems that play a pivotal role in establishing this concept. Next, we explore the Hilbert function of partial intersections.

The last section of Chapter 3 explores the Hilbert functions of families of fat point schemes supported on partial intersections. We explore three families. By reducing the multiplicities

of their fat point schemes to zero using the greedy algorithm, we get a reduction vector. We observe that we have a *nice* reduction vector for each family and conclude that the bounds of the Hilbert function for each family are equal. This determines the Hilbert functions for these families.

Chapter 2

Preliminaries

In this chapter, we lay important groundwork by stating the necessary definitions of the terminologies needed for this thesis. These concepts and terminologies are general knowledge in the fields of algebraic geometry and commutative algebra. In addition, we will give relevant examples to help clarify these concepts and facilitate understanding. We also state some important theorems and establish their relevance in the study of Hilbert functions. We begin with the definition of affine and projective space. This is important because of our interest in points in projective space.

2.1 The Hilbert Function of a Set of Reduced Points

In this section, we introduce the space in which we work and define the terms necessary for a discussion of the Hilbert function. In the end, we state the definition of the Hilbert function and give relevant examples.

Definition 2.1. **Affine n-space** over a field k , denoted $\mathbb{A}^n(k) = k^n$, is the set

$$k^n = \{(a_1, \dots, a_n) \mid a_1, \dots, a_n \in k\}.$$

In affine space, non-parallel lines meet in at least one point. This gives a solution of polynomial equations. However, parallel lines do not meet. This means that not all polynomial equations have solutions in affine space. Hence, the need for projective space.

Definition 2.2. An **n-dimensional projective space** over a field k , denoted as $\mathbb{P}^n(k)$, is the set of equivalence classes of $\sim$ on $k^{n+1} \setminus \{(0, \dots, 0)\}$ under the relation $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ if and only if there exists $\lambda \in k \setminus \{0\}$ such that $x_i = \lambda y_i$ for all $1 \leq i \leq n$. We write:

$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{(0, \dots, 0)\}) / \sim .$$

Given a point $(x_0, \dots, x_n) \in k^{n+1} \setminus \{(0, \dots, 0)\}$, its equivalence class $p \in \mathbb{P}^n(k)$ will be denoted by $p = (x_0 : \dots : x_n)$. We say $(x_0 : \dots : x_n)$ are **homogeneous coordinates** of p . So,

$$(x_0 : \dots : x_n) = (y_0 : \dots : y_n) \Leftrightarrow (x_0, \dots, x_n) = \lambda(y_0, \dots, y_n)$$

for some $\lambda \in k \setminus \{0\}$.

In this thesis, we assume that k is an algebraically closed field.

Example 2.3. The 3-dimensional projective space over the real field $\mathbb{R}$ is written as $\mathbb{P}^3(\mathbb{R})$. So, $\mathbb{P}^3(\mathbb{R}) = (\mathbb{R}^4 \setminus \{(0, 0, 0, 0)\})/\sim$.

Notice that $(1 : -1 : 2 : 1) \in \mathbb{P}^3(\mathbb{R})$, and that

$$(2 : -2 : 4 : 2) = (1 : -1 : 2 : 1) \text{ since } (2, -2, 4, 2) = 2(1, -1, 2, 1).$$

As we have already seen, homogeneous coordinates are not unique. For example,

$$(1 : 1 : 1) = (2 : 2 : 2) = (\pi : \pi : \pi) = (\sqrt{2} : \sqrt{2} : \sqrt{2}) \in \mathbb{P}^2(\mathbb{R}).$$

since they represent the same point in projective space, $\mathbb{P}^2(\mathbb{R})$.

Projective space is affine space taken with a point at infinity. In projective space, all parallel lines meet at one point—at infinity. Hence, all systems of polynomial equations have solutions in projective space. If we homogenize points and equations in affine space, we get points and polynomial equations in projective space. These equations always have solutions in projective space. Hence, projective space has an advantage over affine space. The following example clarifies this concept.

Example 2.4. Consider the polynomial equation $f(x, y) = x - y + 1$ in the affine space $\mathbb{R}^2$. The point $(0, 1) \in \mathbb{R}^2$ satisfies the equation, but the points $\lambda(0, 1)$ do not satisfy the equation for $\lambda \in \mathbb{R} \setminus \{0, 1\}$.

If we homogenize the above equation, we get $F(x_0, x_1, x_2) = x_1 - x_2 + x_0$. Notice that not only does the homogenized point $(1 : 0 : 1) \in \mathbb{P}^2(\mathbb{R})$ satisfy the equation, but also do the points $(\lambda : 0 : \lambda)$ for $\lambda \neq 0$.

The polynomial $F(x_0, x_1, x_2) = x_1 - x_2 + x_0$ in the above example is a **homogeneous polynomial**. Next, we state the definition of a homogeneous polynomial.

The set of all polynomials in $x_0, \dots, x_n$ with coefficients in k is denoted $k[x_0, \dots, x_n]$. For example, $f = 2x^3y^2z + \frac{3}{2}y^3z^3 - 3xyz + y^2$ is a polynomial in $\mathbb{Q}[x, y, z]$.

Definition 2.5. A **homogeneous polynomial** in $R = k[x_0, \dots, x_n]$ is a linear combination of monomials of the same total degree.

By total degree, we mean the sum of the powers of the variables of a monomial. That is, if x^α is a monomial, where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and α_i is a non-negative integer for each $i \in \{1, 2, \dots, n\}$, the total degree of x^α is $\alpha_1 + \cdots + \alpha_n$.

Example 2.6. Let $f = 2x^3 + 3x^2y + 5xy^2 + y^3$. We see that f is a homogeneous polynomial with a total degree of 3 (the total degree of each monomial is 3).

Next, we discuss homogeneous ideals and the ideal of a set of points.

Definition 2.7. A **homogeneous ideal** is an ideal I in $k[x_0, \dots, x_n]$ that is generated by homogeneous polynomials.

Definition 2.8. The **ideal of a point set**, $I(X)$, of a set of points $X \in \mathbb{P}^n(k)$ is the set of polynomials that vanish at every point of X . That is,

$$I(X) = \{f \in k[x_0, \dots, x_n] \mid f((a_0 : \cdots : a_n)) = 0 \text{ for all } (a_0 : \cdots : a_n) \in X\}.$$

Notice that if k is an infinite field, then $I(X)$ must be a homogeneous ideal. In such a field, if $f(x_0, \dots, x_n) \in I(X)$, we have

$$g_i(\lambda a_0, \dots, \lambda a_n) = \lambda^d g_i(a_0, \dots, a_n)$$

where d is the degree of f , g_i is any homogeneous component of f and $(a_0 : \cdots : a_n) \in X$. Thus, $g_i \in I(X)$.

Note that the ideal of a set of projective points is a radical ideal (i.e., equal to its radical), where if J is any ideal of a commutative ring S , then the **radical** of J is

$$\sqrt{J} := \{f \in S \mid f^\ell \in J \text{ for some } \ell > 0\}.$$

Next, we state the definition of the Hilbert function. For $i \geq 0$, we set R_i to be the k -vector space generated by all the monomials of $R = k[x_0, \dots, x_n]$ of degree i . Then $R = k[x_0, \dots, x_n] = \bigoplus_{i \geq 0} R_i$. We also let $I \subseteq R$ be a homogeneous ideal and $I_i = I \cap R_i$ for $i \geq 0$. If $A = R/I$, then $A = \bigoplus_{i \geq 0} R_i/I_i = \bigoplus_{i \geq 0} A_i$. The dimension of $A_i = R_i/I_i$ is known as the **Hilbert function** for the graded ring A at degree i . It is denoted as $H(A, i)$ or $h_A(i)$.

Definition 2.9. [8, 9, 11] With $A = R/I$ as above, we define the **Hilbert function** of $A = R/I$ to be the function:

$$H(A, -) : \mathbb{N}_{\geq 0} \rightarrow \mathbb{N}_{\geq 0}$$

described by:

$$H(A, i) = \dim A_i = \dim(R/I)_i = \dim R_i - \dim I_i.$$

When X is a finite set of points in $\mathbb{P}^n(k)$, we call the Hilbert function of X the Hilbert function of $A = R/I(X)$. In other words, the Hilbert function counts the number of independent polynomials of each degree that vanish at a finite set of points of X in $\mathbb{P}^n(k)$.

We know the Hilbert function of the graded ring $A = R/I$ at degree i is the dimension of R_i/I_i . Therefore,

$$H(A, i) = \dim A_i = \dim R_i - \dim I_i = \binom{n+i}{i} - \dim I_i$$

for every degree i .

Fix X to be a finite set of points in $\mathbb{P}^n(k)$, none lying on $x_0 = 0$, and $R = k[x_0, \dots, x_n]$. Let $A = R/I(X) = \bigoplus_{t \geq 0} A_t$ and $\bar{x}_0$ denote the image of x_0 in A . Then, $B = A/\langle \bar{x}_0 \rangle$ is a 0-dimensional ring. Now,

$$B \cong \frac{\langle R/I \rangle}{\langle x_0, I \rangle / I} \cong \frac{R}{\langle x_0, I \rangle} \cong \frac{k[x_0, \dots, x_n]}{\langle x_0, I \rangle}.$$

We see that $\langle x_0, I \rangle = \langle x_0, J \rangle$, where J is an ideal in $k[x_1, \dots, x_n]$ and $x_0 = 0$ in all elements of I . So, $B \cong \frac{k[x_1, \dots, x_n]}{J} = \bigoplus_{t \geq 1} B_t$. It turns out that J is a radical ideal, where $\sqrt{J} = \langle x_1, \dots, x_n \rangle$.

Here are some facts about B (from [8, 9, 11]):

- (1) $B_t = 0$ for all $t \gg 0$;
- (2) if $B_t = 0$, then $B_{t+1} = 0$.

Now, $\bar{x}_0 \in A$ defines a linear transformation:

$$\bar{x}_0 : A_i \rightarrow A_{i+1}$$

given by multiplication. Since $\bar{x}_0$ is not a zero divisor, this linear transformation is one-to-one. This knowledge helps us appreciate the following.

Proposition 2.10. [8, 9, 11] If X is a set of s distinct points in $\mathbb{P}^n(k)$, then, $H(X, s-1) = s$.

Corollary 2.11. [8, 9, 11] There is an integer $e > 0$ such that

(i) $H(X, t) = e$ for all $t \gg 0$.

(ii) If $H(X, t) = e$, then $H(X, t+1) = e$ (i.e., the Hilbert function is non-decreasing).

(iii) If $H(X, t) = c$ and $H(X, t+1) = c$, then $c = e$.

Proof. (i) Take $|X| = e$.

(ii) We know

$$\dim B_{t+1} = \dim A_{t+1} - \dim A_t.$$

That is,

$$0 = \dim A_{t+1} - e$$

($B_{t+1} = 0$ since $B_t = 0$ for all $t \gg 0$). Therefore, $\dim A_{t+1} = e$. That is,

$$H(X, t+1) = e.$$

(iii) Since

$$\overline{x_0} : A_t \rightarrow A_{t+1}$$

is one-to-one, $H(X, t)$ and $H(X, t+1)$ are unique. Since $H(X, t) = e$ and $H(X, t) = c$, then $c = e$. The same argument follows for $H(X, t+1)$. □

Proposition 2.12. [8, 9, 11] If X consists of exactly s distinct points, then $H(X, t) \leq s$ for all t .

Proof. The polynomials vanishing at the points of X give s homogeneous equations. This means the number of independent solutions, which is the rank of the coefficient matrix formed from the homogeneous equations, is $\leq s$. Therefore, $H(X, t) \leq s$. □

Now, we demonstrate some examples with $R = k[x_0, x_1, x_2]$ and $\mathbb{P}^2(k)$.

Example 2.13. [8, 9, 11] Let $X \subseteq \mathbb{P}^2(k)$ be 6 points configured as

$$X = \left\{ \begin{pmatrix} o & o & o \\ o & o & \\ o & & \end{pmatrix} \right\}.$$

We have:

$$H(X, 0) = 1 - 0 = 1;$$

$$H(X, 1) = 3 - 0 = 3;$$

$$H(X, 2) = 6 - 0 = 6.$$

Since $|X| = 6$ and the Hilbert function is non-decreasing, the sequence of the Hilbert function values is $1, 3, 6, 6, 6, \dots$ (Recall that if $|X| = s$, then $H(X, t) \leq s$).

Example 2.14. [8, 9, 11] Consider the following set $X \subset \mathbb{P}^2(k)$ of 6 points

$$X = \left\{ \begin{matrix} o & o & o \\ o & o & o \end{matrix} \right\}.$$

We have:

$$H(X, 0) = 1 - 0 = 1;$$

$$H(X, 1) = 3 - 0 = 3;$$

$$H(X, 2) = 6 - 1 = 5.$$

Since $|X| = 6$ and the Hilbert function is non-decreasing, either $H(X, 3) = 5$ or 6. If $H(X, 3) = 5$, then $H(X, t) = 5$ for all t . However, we know that $H(X, 5) = 6$ (since $H(X, s - 1) = s$ if $|X| = s$). This is a contradiction. Hence, $H(X, 3) = 6$. Again, since the Hilbert function is non-decreasing, we have $H(X) = (1, 3, 5, 6, 6, \dots)$.

2.1.1 The 0-Sequence

Next, we discuss the 0-sequence. The 0-sequence helps in determining how fast the Hilbert function grows and in studying the linear constraints imposed on point sets. This not only helps us understand the geometry of points, but also helps us to classify them. To study the 0-sequence, we employ the combinatorial techniques in determining the maximum dimension of the linear forms that vanish at a set of points X at a fixed degree. We start with a little introduction to this technique, and show some interesting results that help to understand the nature of the Hilbert function.

Definition 2.15. [8] Let $h, i \in \mathbb{Z}^+$. The i -**binomial expansion** of h is the unique expression

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \dots + \binom{m_j}{j},$$

where

$$m_i > m_{i-1} > \cdots > m_j \geq j \geq 1.$$

Given this expansion, we define

$$h^{(i)} = \binom{m_i + 1}{i + 1} + \binom{m_{i-1} + 1}{i} + \cdots + \binom{m_j + 1}{j + 1},$$

where $0^{(i)} := 0$.

The fact that such an expression exists for h can be found in [8]. The following examples will help us appreciate this technique.

Example 2.16. [8] If $h = 18$ and $i = 3$, then we have the 3-binomial expansion of 18:

$$18 = \binom{5}{3} + \binom{4}{2} + \binom{2}{1} = 10 + 6 + 2.$$

So,

$$18^{(3)} = \binom{6}{4} + \binom{5}{3} + \binom{3}{2} = 15 + 10 + 3 = 28.$$

Alternatively, we can write the i -binomial expansion of h by looking at values in Pascal's Triangle. Let's re-arrange the values of Pascal's Triangle into vertical columns in Table 2.1.

	0	1	2	3	4 ...
0	1	1	1	1	1 ...
1	1	2	3	4	5 ...
2	1	3	6	10	15 ...
3	1	4	10	20	35 ...
4	1	5	15	35	70 ...
⋮					

Table 2.1: 3-binomial expansion of 18.

We label the rows (and columns) of the table as $0, 1, 2, \dots$. Since $h = 18$ and $i = 3$, we get the 3-binomial expansion of 18 by writing the highest value ≤ 18 in the column labeled '3'. This value is 10. So, we have $18 - 10 = 8$ left to expand. We repeat this process by writing the highest value ≤ 8 in the next lower column (in this case, in the column labeled '2'). This value is 6. So, we have $10 + 6 = 16$ now. Since we need 2 to make 18, we will repeat the process one more time by writing the highest value ≤ 2 in the next lower column (in this case, in the column labeled '1'.) We have 2 in this column. So, the 3-binomial expansion of 18 is:

$$18 = 10 + 6 + 2.$$

The values are shown in bold in Table 2.1.

To get $h^{(i)}$, we simply write the value to the right of each bold value in the table and add them together. For example, we have 15 to the right of 10, 10 to the right of 6, and 3 to the right of 2. So, we have:

$$18^{(3)} = 15 + 10 + 3 = 28.$$

Example 2.17. The 4-binomial expansion of 11 is

$$11 = \binom{5}{4} + \binom{4}{3} + \binom{2}{2} + \binom{1}{1} = 5 + 4 + 1 + 1$$

and

$$11^{(4)} = \binom{6}{5} + \binom{5}{4} + \binom{3}{3} + \binom{2}{2} = 6 + 5 + 1 + 1 = 13.$$

Alternatively, we can look these values up on the table. We get the 4-binomial expansion of 11 by first writing the highest value ≤ 11 in the fourth column. This value is 5. We have $11 - 5 = 6$. Next, we write the value ≤ 6 in the third column. This value is 4 in the third column. We have $6 - 4 = 2$ left. We repeat this process by writing the value ≤ 2 in the second column. This value is 1 in the second column. We have $2 - 1 = 1$ left. Lastly, we write 1 since 1 is in the first column. These values are in bold in Table 2.2.

	0	1	2	3	4	...
0	1	1	1	1	1	...
1	1	2	3	4	5	...
2	1	3	6	10	15	...
3	1	4	10	20	35	...
4	1	5	15	35	70	...
$\vdots$						

Table 2.2: 3-binomial expansion of 18.

So, we have

$$11 = 5 + 4 + 1 + 1.$$

We determine $11^{(4)}$ by summing the values to the right of the values in bold in the table. These values are 1, 1, 5 and 6. So,

$$11^{(4)} = 6 + 5 + 1 + 1 = 13.$$

The combinatorial techniques we just discussed lay the groundwork for the discussion of 0-sequences. Now, we discuss 0-sequences and some important theorems related to them. We start by stating the definition of an 0-sequence.

Definition 2.18. [8] A sequence of nonnegative integers $\{c_i\}_{i \geq 0}$, is called an **0-sequence** if $c_0 = 1$ and $c_{i+1} \leq c_i^{\langle i \rangle}$ for all $i \geq 1$.

Notice that each column of the table above is an 0-sequence.

Example 2.19. Consider the sequence $\{c_i\}_{i \geq 0} = \{1, 3, 6, 10, 13, 13, \dots\}$.

Now, $c_0 = 1$. We also have

$$c_1 = 3 = \binom{3}{1},$$

Hence,

$$c_1^{\langle 1 \rangle} = \binom{4}{2} = 6 = c_2,$$

$$c_2^{\langle 2 \rangle} = \binom{5}{3} = 10 = c_3,$$

and

$$c_4 = 13 < c_3^{\langle 3 \rangle} = \binom{6}{4} = 15.$$

One can continue in this way to verify that the sequence is indeed an 0-sequence.

Example 2.20. Consider the sequence $\{c_i\}_{i \geq 0} = \{1, 4, 10, 24, 28, 36, 36 \dots\}$.

Now, $c_0 = 1$. Also,

$$c_1 = 4 = \binom{4}{1},$$

$$c_1^{\langle 1 \rangle} = \binom{5}{2} = 10 = c_2,$$

$$c_2^{\langle 2 \rangle} = \binom{6}{3} = 20.$$

Now, $c_3 = 24 > c_2^{\langle 2 \rangle}$. Hence, the sequence is not an 0-sequence.

Example 2.21. Consider the sequence $\{c_i\}_{i \geq 0} = \{1, 3, 5, 7, 9, 10, 10 \dots\}$.

Now, $c_0 = 1$. We calculate

$$c_1 = 3 = \binom{3}{1},$$

$$c_1^{\langle 1 \rangle} = \binom{4}{2} = 6 > 5 = c_2,$$

$$\begin{aligned}
c_2 &= 5 = \binom{3}{2} + \binom{2}{1}, \\
c_2^{\langle 2 \rangle} &= \binom{4}{3} + \binom{3}{2} = 4 + 3 = 7 = c_3, \\
c_3^{\langle 3 \rangle} &= \binom{5}{4} + \binom{4}{3} = 5 + 4 = 9 = c_4, \\
c_4^{\langle 4 \rangle} &= \binom{6}{5} + \binom{5}{4} = 6 + 5 = 11 > 10 = c_5.
\end{aligned}$$

From this point, it is easy to continue verifying that the sequence is an 0-sequence.

The following theorems are important as they provide the necessary foundation upon which geometric data can be obtained in the study of points in projective space.

Theorem 2.22. (*Macaulay's Theorem*) [1] *The following are equivalent:*

- (a) $\{c_i\}_{i \geq 0}$ is an 0-sequence;
- (b) $\{c_i\}_{i \geq 0}$ is the Hilbert function of a standard k -algebra A ,

where a standard graded k -algebra is a quotient of the form $A = \bigoplus_{i \geq 0} A_i = \bigoplus_{i \geq 0} R_i/I_i = R/I$ with $I \subseteq R = k[x_0, \dots, x_n]$ a homogeneous ideal.

This theorem helps determine whether a given sequence is a Hilbert function.

The following properties are helpful in working with 0-sequences.

Theorem 2.23. [8]

- (a) Let α, β, i be positive integers with $\alpha < \beta$. Then, $\alpha^{\langle i \rangle} \leq \beta^{\langle i \rangle}$.
- (b) Let $\{b_i\}_{i \geq 0}$ be an 0-sequence with $b_1 = m$. Then, $b_{i+1} \leq \binom{m+i}{i+1}$ for every $i \geq 0$.
- (c) Let $h = \binom{(n-1)+i}{i} - 1$. Then, $h^{\langle i \rangle} = \binom{n+i}{i+1} - n$.

Next, we discuss *differentiable 0-sequences*, which are 0-sequences whose first difference is also an 0-sequence. According to Macaulay's Theorem [1], the Hilbert function is an 0-sequence. So, if the first difference of a Hilbert function is also an 0-sequence, then the Hilbert function is a differentiable 0-sequence. This topic is important because if the Hilbert function is a differentiable 0-sequence, we will be able to predict the geometry of points on a hyperplane by studying its first difference. It is interesting to note that not all Hilbert functions are differentiable 0-sequences. According to the paper [8], a Hilbert function that

is a differentiable 0-sequence arises from a reduced graded ring $A = R/I$, where $I \subset R = k[x_0, \dots, x_n]$ is a radical ideal. In fact, the paper also shows that the converse is true. We start by stating a formal definition of a differentiable 0-sequence, and stating some theorems relevant to differentiable 0-sequences.

Definition 2.24. Let $\{b_i\}_{i \geq 0}$ be an 0-sequence. We say that $\{b_i\}$ is **differentiable** if the difference sequence $\Delta\{b_i\}_{i \geq 0} = \{c_i\}_{i \geq 0}$ defined by $c_i = b_i - b_{i-1}$ is again an 0-sequence ($b_{-1} = 0$).

Notice from $c_i = b_i - b_{i-1}$ that $c_0 = b_0$ if we take $b_{-1} = 0$.

Example 2.25. Consider the sequence $\mathcal{H} = \{1, 3, 6, 10, 15, 15, \dots\}$ once again. Note that it is an 0-sequence. We examine whether it is a differentiable 0-sequence.

Observe that the first difference of the given sequence is $\mathcal{H}' = \{1, 2, 3, 4, 5, 0, 0, \dots\}$. This is an 0-sequence. Therefore, the given sequence $\mathcal{H}$ is a differentiable 0-sequence. Also, observe that the given sequence is the second column in Table 2.2 and its first difference is the entries of the first column of the table. This means that both sequences are 0-sequences.

Definition 2.26. [8] The 0-sequence $\{b_i\}_{i \geq 0}$ has **dimension** d if there is a polynomial $f(x)$ with rational coefficients of degree d with the property that for all $t \gg 0$, $f(t) = b_t$.

To show that a differentiable 0-sequence arises from a reduced graded ring, the paper [8] constructed two differentiable 0-sequences from an existing one. Each of these sequences is the Hilbert function of a reduced variety. The union of these varieties give a reduced variety whose Hilbert function is the union of the sequences constructed. This shows that there is a reduced variety for a differentiable 0-sequence. We construct the sequences below and also state some important theorems.

Let $S = \{b_i\}_{i \geq 0}$ be a differentiable 0-sequence with $b_1 = n + 1$, where $n \in \mathbb{Z}^+$. Also, let $d_i = \binom{n+i-1}{i}$ be the Hilbert function of a polynomial ring with n variables at degree $i \geq 0$, and $c_i = b_{i+1} - d_{i+1}$.

- (1) Assume $c_i \leq c_{i+1}$ for all $i \geq 0$. Set $S_1 = \{c_i\}_{i \geq 0}$.
- (2) Assume there is some $h \geq 0$ such that $1 = c_0 \leq c_1 \leq c_2 \leq \dots \leq c_{h-1}$ and $c_{h-1} > c_h$. Set $S_2 = \{c_0, c_1, \dots, c_{h-1}, c_{h-1}, \dots\}$. That is, we are interested in the sequence up to the point before it decreases.

If we define $h = \infty$ in Case I, then $\{i \mid i \geq \infty\} = \emptyset$. We now define $S'_1 = \{c'_i\}_{i \geq 0}$, where

$$c'_i = \begin{cases} d_i, & i \leq h \\ b_i - c_{h-1}, & i \geq h. \end{cases}$$

Observe that both definitions agree when $i = h$.

Example 2.27. [8] Consider the sequence

$$1 \quad 3 \quad 6 \quad 10 \quad 14 \quad 17 \quad 20 \quad 21 \quad 22 \quad 22 \quad 22 \dots$$

Observe that this is an 0-sequence. Also, its first difference is

$$1 \quad 2 \quad 3 \quad 4 \quad 4 \quad 3 \quad 3 \quad 1 \quad 1 \quad 0 \quad 0 \dots$$

which is also an 0-sequence. This means the given sequence is a differentiable 0-sequence.

We construct sequences S_1 and S'_1 from the given sequence as follows.

Since $b_1 = 3$ in the given sequence, we have $n = 2$. So, $d_i = 1 + i$. Now, $c_i = b_{i+1} - d_{i+1}$. Therefore, $c_0 = 1, c_1 = 3, c_2 = 6, c_3 = 9, c_4 = 11, c_5 = 13, c_6 = 13, c_7 = 13$ and $c_8 = 12$. So, sequence S_1 is $\{1, 3, 6, 9, 11, 13, 13, 13, \dots\}$.

Next, we construct S'_1 . We have $c'_0 = 1, c'_1 = 2, c'_2 = 3, c'_3 = 4, c'_4 = 5, c'_5 = 6, c'_6 = 7, c'_7 = 8, c'_8 = 9, c'_9 = 9, c'_{10} = 9 \dots$. So, the sequence S'_1 is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 9, 9 \dots\}$.

Therefore, the sequences are:

$$S_1 : 1 \quad 3 \quad 6 \quad 9 \quad 11 \quad 13 \quad 13 \quad 13 \dots$$

and

$$S'_1 : 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 9 \dots$$

Observe that both S_1 and S'_1 are differentiable 0-sequences. We have the following theorem.

Theorem 2.28. [8] Let $S = \{b_i\}_{i \geq 0}$ with $b_1 = n + 1$, be a differentiable 0-sequence. Then, the sequences S_1 and S'_1 constructed above are both differentiable 0-sequences.

The paper [8] proves the following theorem by decomposing the differentiable 0-sequence S into S_1 and S'_1 .

Theorem 2.29. [8] Let k be an infinite field and let $S = \{b_i\}_{i \geq 0}$ be a differentiable 0-sequence with $b_1 = n + 1$. There is a radical ideal I in $k[x_0, \dots, x_n]$ such that S is the Hilbert function of $k[x_0, \dots, x_n]/I$.

Corollary 2.30. [8] *Let k be an infinite field and let $S = \{b_i\}_{i \geq 0}$ be a d -dimensional differentiable 0-sequence with $b_1 = n + 1$. There is a d -dimensional variety $V \subset \mathbb{P}^n(k)$ which is the union of a finite number of sub-varieties of dimension $\leq d$, having Hilbert function S .*

Example 2.31. Let S be the zero-dimensional differentiable 0-sequence

$$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 11 \quad 11 \quad 11 \rightarrow$$

That is, S is the Hilbert function of 11 points in $\mathbb{P}^2$. The first difference function is

$$1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 0 \quad 0 \rightarrow$$

By the procedure outlined after Definition 2.26, we have $c_0 = 1, c_1 = 2, c_2 = 3, c_3 = 4, c_4 = 5$, and $c_5 = 4$. Hence, $h = 5$. Therefore,

$$S_1 = \{1, 2, 3, 4, 5, 5, 5, \dots\}.$$

Now, $c'_0 = 1, c'_1 = 2, c'_2 = 3, c'_3 = 4, c'_4 = 5, c'_5 = 6, c'_6 = 6, c'_7 = 6$, and so on. Therefore,

$$S'_1 = \{1, 2, 3, 4, 5, 6, 6, 6, \dots\}.$$

Observe that the Hilbert function S'_1 stabilizes at 6. This corresponds to 6 points on a line ℓ_1 in $\mathbb{P}^2$.

We repeat the process with sequence $S_1 = \{1, 2, 3, 4, 5, 5, 5, \dots\}$. Notice, however, that $c_0 = 0$. Hence, we cannot split S_1 further. Since S_1 stabilizes at 5, then we have 5 points on a line ℓ_2 in $\mathbb{P}^2$.

Therefore, S_1 and S'_1 are the Hilbert functions of 5 and 6 points, respectively, whose union gives 11 points with Hilbert function S .

2.1.2 Data Obtained from Hilbert Functions

From Macaulay's Theorem, we see that the 0-sequence determines the growth of a Hilbert function. The growth can sometimes reveal useful geometric information. In this section, we briefly demonstrate the potential utility of the Hilbert function.

We continue fixing $R = k[x_0, \dots, x_n]$ with k an algebraically closed field. Let $I \subset R$ be a homogeneous ideal of R . Suppose $H(A, d) = h_d$ is the Hilbert function of A at degree d , where $A = R/I = \bigoplus_{i \geq 0} A_i = \bigoplus_{i \geq 0} R_i/I_i$. Now,

$$h_d = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_j}{j}$$

where $m_d > m_{d-1} > \cdots > m_j \geq j \geq 1$. If $H(A, d+1) = h_{d+1}$, then

$$h_{d+1} \leq \binom{m_d+1}{d+1} + \binom{m_{d-1}+1}{d} + \cdots + \binom{m_j+1}{j+1} = h_d^{\langle d \rangle}.$$

That is, $H(A, d+1) \leq H(A, d)^{\langle d \rangle}$. This means that the 0-sequence determines the growth of Hilbert function from degree d to $d+1$. We say A has **maximal growth** in degree d if $H(A, d+1) = H(A, d)^{\langle d \rangle}$.

We first explore what happens when $A = R/I$ has maximal growth, and we look at quotients. The following definitions are needed.

Definition 2.32. An **associated prime ideal** of R/I is a prime ideal that is the annihilator of a nonzero element of R/I .

Definition 2.33. [1] The ideal $\mathfrak{m} = \langle x_0, \dots, x_n \rangle \subset R = k[x_0, \dots, x_n]$ is the **irrelevant maximal ideal** of R . If I is a homogeneous ideal and $A = R/I$, then the image of $\mathfrak{m}$ in A is $\overline{\mathfrak{m}}$. This is called the irrelevant maximal ideal of A .

Definition 2.34. [1] The **saturation** of a homogeneous ideal $I \subset R$, denoted as I^{sat} , is defined by

$$I^{\text{sat}} := \{f \in R/I : f\overline{\mathfrak{m}}^l \subseteq I \text{ for some integer } l\}.$$

The ideal I is **saturated** if $I = I^{\text{sat}}$.

Now, we discuss the consequences of maximal growth of the Hilbert functions of saturated ideals by looking at the following results.

Lemma 2.35. [1] *Let I be a saturated ideal in the polynomial ring $R = k[x_0, \dots, x_n]$ and let L be a linear form in R which is not a zero divisor on the ring $A = R/I$. Let J be the ideal $I/LI \simeq \langle I, L \rangle / \langle I \rangle$ in the polynomial ring $S = R/\langle L \rangle$ and set $B = S/J$. If A has maximal growth in degree d , then B also has maximal growth in degree d .*

The converse of this lemma does not necessarily hold. Also, a Hilbert function with maximal growth may not have an associated saturated ideal. Consider the following examples.

Example 2.36. [1] Consider the set of 8 points of $\mathbb{X}$ in $\mathbb{A}^2 \subset \mathbb{P}^2$, configured as follows:

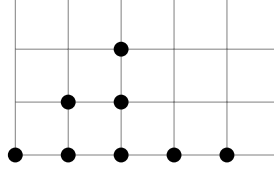


Figure 2.1: The set of 8 points of $\mathbb{X}$ in $\mathbb{A}^2 \subset \mathbb{P}^2$.

The ideal $I = I_{\mathbb{X}} \subseteq R = k[x_0, x_1, x_2]$ of these points is a saturated ideal and $A = R/I$ has Hilbert function

$$1 \quad 3 \quad 6 \quad 7 \quad 8 \quad 8 \cdots.$$

Let the equation of a line in $\mathbb{P}^2$ which misses the points of $\mathbb{X}$ be L . Then L is not a zero divisor in A . So, the first difference of the Hilbert function of A is $B = A/\overline{L}A$, given as

$$1 \quad 2 \quad 3 \quad 1 \quad 1 \quad 0 \cdots.$$

Let $H(B) = \{c_i\}_{i \geq 0}$. Then

$$c_3 = 1 = \binom{3}{3}$$

and

$$c_3^{(3)} = \binom{4}{4} = 1 = c_4.$$

That is, the passage from c_3 to c_4 is the maximum permitted by Macaulay's Theorem. This means B has maximal growth in degree 3.

Let $H(A) = \{c'_i\}_{i \geq 0}$. We see that

$$c'_3 = 7 = \binom{4}{3} + \binom{3}{2}$$

and

$$(c'_3)^{(3)} = \binom{5}{4} + \binom{4}{3} = 5 + 4 = 9.$$

However, $c'_4 = 8 < 9$. Hence, A does not have maximal growth in degree 3.

This example shows that the converse of Lemma 2.35 does not hold.

Example 2.37. [1] Suppose there are ideals in R whose Hilbert function starts with

$$1 \quad 4 \quad 6.$$

We can find many saturated ideals $I \subseteq R = k[x_0, x_1, x_2, x_3]$ that satisfy this property.

Let c'_i be the values of this Hilbert function. Then

$$c'_2 = 6 = \binom{4}{2}.$$

Hence,

$$(c'_2)^{\langle 2 \rangle} = \binom{5}{3} = 10.$$

This means that the Hilbert function has maximal growth in degree 2 if we let $c'_3 = 10$. So, there are ideals I' in R whose Hilbert function starts with

$$1 \quad 4 \quad 6 \quad 10.$$

That is, with first difference Hilbert function which starts with

$$\{c_i\}_{i \geq 0} = 1 \quad 3 \quad 2 \quad 4.$$

We claim that I' is not a saturated ideal. To see this, assume I' is saturated. Then,

$$c_2 = 2 = \binom{2}{2} + \binom{1}{1}.$$

So,

$$c_2^{\langle 2 \rangle} = \binom{3}{3} + \binom{2}{2} = 2.$$

However, $c_3 = 4 > c_2^{\langle 2 \rangle}$. Hence, the first difference is not an 0-sequence, which is impossible. This means we cannot find a saturated ideal with this Hilbert function.

Next, we will look at a lemma that tells us when we can have a saturated ideal.

Lemma 2.38. *[1] Let $I \subset R = k[x_0, \dots, x_n]$ be a saturated ideal and let $L \in R_1$ be a general linear form. Let*

$$J = I/\langle LI \rangle \simeq \langle I, L \rangle / \langle L \rangle \subset R/\langle L \rangle = S.$$

Suppose S/J has maximal growth in degree d . Let $\bar{I} = \langle I_{\leq d} \rangle$ and let $\tilde{I} = \bar{I}^{sat}$ be the saturation of the ideal $\bar{I}$. Then $\bar{I} = \tilde{I}$. That is, $\bar{I}$ is a saturated ideal.

Example 2.39. Let $\mathbb{X}$ be a set of 16 points in $\mathbb{P}^3$ with Hilbert function

$$1 \quad 4 \quad 9 \quad 16 \quad 16 \dots.$$

The first difference Hilbert function is

$$1 \quad 3 \quad 5 \quad 7 \quad 0 \quad 0 \cdots.$$

Let c_i denote the values of the first difference function and $\bar{I} = \bar{I}(\mathbb{X})$. Then

$$c_2 = 5 = \binom{3}{2} + \binom{2}{1}.$$

So,

$$c_2^{(2)} = \binom{4}{3} + \binom{3}{2} = 7 = c_3.$$

This means there is maximal growth at degree 2. Hence, $\bar{I} = \langle I_{\leq 2} \rangle$ is a saturated ideal.

Example 2.40. Let $\mathbb{X}$ be a set of 45 points in $\mathbb{P}^4$ with Hilbert function

$$1 \quad 5 \quad 13 \quad 25 \quad 42 \quad 45 \quad 45 \cdots.$$

and first difference Hilbert function

$$1 \quad 4 \quad 8 \quad 12 \quad 17 \quad 3 \quad 0 \quad 0 \cdots.$$

Let c_i denote the values of the first difference function. Then

$$c_3 = 12 = \binom{5}{3} + \binom{2}{2} + \binom{1}{1}.$$

So,

$$c_3^{(3)} = \binom{6}{4} + \binom{3}{3} + \binom{2}{2} = 17 = c_4.$$

This means there is maximal growth at degree 3. Hence, $\bar{I} = \langle I_{\leq 3} \rangle$ is a saturated ideal.

We now consider ideals of points. If $I = I(X)$ for $X \subset \mathbb{P}^n$, then once the maximal growth is reached for $H(X)$, the set X can sometimes be reduced to two disjoint sets, where one of the sets lies in $\mathbb{P}^m \subset \mathbb{P}^n$. The theorem below explains this more concretely.

Theorem 2.41. [1] *Let Y be a closed, reduced subscheme of $\mathbb{P}^{r+1}$ of dimension ≥ 0 , with the first difference of the Hilbert function satisfying $\Delta H(Y, d) = \binom{d+m-1}{d}$ and $\Delta H(Y, d+1) = \binom{d+m}{d+1}$. Then*

- (a) *Y is the disjoint union of a scheme Y_1 and a scheme Y_2 , where Y_1 lies in a $\mathbb{P}^m = \Lambda$ and Y_2 is a finite set of points.*

(b) The first difference of the Hilbert function of Y_1 is

$$\Delta H(Y_1, t) = \begin{cases} \Delta H(\Lambda, t) = \binom{t+m-1}{t}, & \text{if } t \leq d+1; \\ \Delta H(Y, t), & \text{if } t \geq d. \end{cases}$$

(c) $(I_{Y_1})_t = (I_Y + I_\Lambda)_t$ for all $t \geq d-1$.

Let us consider the following corollary.

Corollary 2.42. [1] Let $Z \subset \mathbb{P}^{r+1}$ be a reduced set of points with $\Delta H(Z, d) = \binom{d+m-1}{d}$ and $\Delta H(Z, d+1) = \binom{d+m}{d+1}$. Then Z contains a subset Z_1 lying on a $\mathbb{P}^m = \Lambda$, and the first difference of the Hilbert function of Z_1 is the sequence

$$1 \quad m \quad \binom{m+1}{2} \cdots \binom{d+m-1}{d} \quad \binom{d+m}{d+1} \quad \Delta H(Z, d+2) \cdots$$

(In particular, Z contains at least $\binom{m+d+1}{m}$ points on a $\mathbb{P}^m$).

We will consider some points in the following examples.

Example 2.43. [1] Let $\mathbb{X}$ be a set of 30 points in $\mathbb{P}^5$ with Hilbert function

$$1 \quad 6 \quad 21 \quad 25 \quad 30 \quad 30 \cdots$$

We have $\Delta H(\mathbb{X})$ is

$$1 \quad 5 \quad 15 \quad 4 \quad 5 \quad 0 \quad 0 \cdots$$

Let c_i denote the values of the difference function. Observe that

$$c_3 = 4 = \binom{4}{3}$$

and

$$c_3^{(3)} = \binom{5}{4} = 5 = c_4.$$

Hence, the passage from c_3 to c_4 is the maximum permitted by Macaulay's Theorem. That is, there is maximal growth at degree 3.

Now, from Corollary 2.42, we have

$$\Delta H(\mathbb{X}, 3) = \binom{3+m-1}{3} = \binom{2+m}{3} = 4.$$

So, $m = 2$.

Observe that

$$\Delta H(\mathbb{X}, 4) = \binom{2+m+1}{3+1} = \binom{5}{4} = 5.$$

So, $\mathbb{X}$ is reduced into two subschemes, $\mathbb{X}_1$ and $\mathbb{X}_2$, where $\mathbb{X}_1$ lies in a $\mathbb{P}^2 = \Lambda$ (since $m = 2$).

The first difference of the Hilbert function of $\mathbb{X}_1$ is

$$\Delta H(\mathbb{X}_1, t) = \begin{cases} \Delta H(\Lambda, t) = \binom{t+1}{t}, & \text{if } t \leq 4; \\ \Delta H(\mathbb{X}, t), & \text{if } t \geq 3. \end{cases}$$

From Corollary 2.42, we see that $\mathbb{X}$ contains at least $\binom{m+d+1}{m}$ points on a $\mathbb{P}^m$. Now,

$$\binom{2+3+1}{2} = \binom{6}{2} = 15.$$

So, $\mathbb{X}_1$ has at least 15 points on a $\mathbb{P}^2$.

Example 2.44. Let $\mathbb{X}$ be a set of 46 points in $\mathbb{P}^5$ with Hilbert function

$$1 \quad 6 \quad 16 \quad 36 \quad 44 \quad 46 \quad 46 \cdots$$

The first difference function $\Delta H(\mathbb{X})$ is

$$1 \quad 5 \quad 10 \quad 20 \quad 8 \quad 2 \quad 0 \quad 0 \cdots$$

Let c_i denote the values of the difference function. Observe that

$$c_2 = 10 = \binom{5}{2}$$

and

$$c_2^{\langle 2 \rangle} = \binom{6}{3} = 20 = c_3.$$

Hence, the passage from c_2 to c_3 is the maximum permitted by Macaulay's Theorem. That is, there is maximal growth at degree 2.

Now, from Corollary 2.42, we have

$$\Delta H(\mathbb{X}, 2) = \binom{2+m-1}{2} = \binom{1+m}{2} = 10.$$

So, $m = 4$.

Observe that

$$\Delta H(\mathbb{X}, 3) = \binom{1+4+1}{2+1} = \binom{6}{3} = 20.$$

So, $\mathbb{X}$ is reduced into two subschemes, $\mathbb{X}_1$ and $\mathbb{X}_2$, where $\mathbb{X}_1$ lies in a $\mathbb{P}^4 = \Lambda$ (since $m = 4$).

The first difference of the Hilbert function of $\mathbb{X}_1$ is

$$\Delta H(\mathbb{X}_1, t) = \begin{cases} \Delta H(\Lambda, t) = \binom{t+3}{t}, & \text{if } t \leq 3; \\ \Delta H(\mathbb{X}, t), & \text{if } t \geq 2. \end{cases}$$

From Corollary 2.42, we see that $\mathbb{X}$ contains at least $\binom{m+d+1}{m}$ points on a $\mathbb{P}^m$. Now,

$$\binom{4+2+1}{4} = \binom{7}{4} = 35.$$

So, $\mathbb{X}_1$ has at least 35 points on a $\mathbb{P}^4$.

Example 2.45. Let $\mathbb{X}$ be a set of 62 points in $\mathbb{P}^5$ with Hilbert function

$$1 \quad 4 \quad 10 \quad 20 \quad 35 \quad 56 \quad 60 \quad 62 \quad 62 \cdots.$$

Then $\Delta H(\mathbb{X})$ is

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 4 \quad 2 \quad 0 \quad 0 \cdots.$$

Let c_i denote the values of $\Delta H(\mathbb{X})$. Observe that

$$c_4 = 15 = \binom{6}{4}$$

and

$$c_4^{(4)} = \binom{7}{5} = 21 = c_5.$$

Hence, the passage from c_4 to c_5 is the maximum permitted by Macaulay's Theorem. That is, there is maximal growth at degree 4.

Now, from Corollary 2.42, we have

$$\Delta H(\mathbb{X}, 4) = \binom{4+m-1}{4} = \binom{3+m}{4} = 15.$$

So, $m = 3$.

Observe that

$$\Delta H(\mathbb{X}, 5) = \binom{4+m}{4+1} = \binom{7}{5} = 21.$$

So, $\mathbb{X}$ is reduced into two subschemes, $\mathbb{X}_1$ and $\mathbb{X}_2$, where $\mathbb{X}_1$ lies in a $\mathbb{P}^3 = \Lambda$ (since $m = 3$).

The first difference of the Hilbert function of $\mathbb{X}_1$ is

$$\Delta H(\mathbb{X}_1, t) = \begin{cases} \Delta H(\Lambda, t) = \binom{t+2}{t}, & \text{if } t \leq 5; \\ \Delta H(\mathbb{X}, t), & \text{if } t \geq 4. \end{cases}$$

From Corollary 2.42, we see that $\mathbb{X}$ contains at least $\binom{m+d+1}{m}$ points on a $\mathbb{P}^m$. Now,

$$\binom{3+4+1}{3} = \binom{8}{3} = 56.$$

So, $\mathbb{X}_1$ has at least 56 points on a $\mathbb{P}^3$.

Next, we will look at another theorem which is helpful in describing the geometry of the set of points that are different from the ones we have already seen.

Theorem 2.46. [1] *Let $Y \subset \mathbb{P}^{r+1}$ be a reduced scheme of any dimension. Assume that for some d , $\Delta H(Y, d) = \Delta H(Y, d+1) = s$, where $d \geq s$. Then*

(a) $\dim Y \leq 1$.

(b) $\langle (I_Y)_{\leq d} \rangle$ is the saturated ideal of a curve V of degree s (not necessarily unmixed). Furthermore, V is reduced.

Let C be the unmixed one-dimensional component of the reduced curve in (b). Let Y_1 be the subvariety of Y on C and Y_2 the “residual” subvariety.

(c) $\langle (I_Y)_{\leq d} \rangle = I_C$.

(d) $\dim Y_2 = 0$ and $H(Y_1, t) = H(Y, t) - |Y_2|$ for all $t \geq d-1$.

(e)

$$\Delta H(Y_1, t) = \begin{cases} \Delta H(C, t) & \text{for } t \leq d+1; \\ \Delta H(Y, t), & \text{for } t \geq d. \end{cases}$$

In particular, $\Delta H(Y_1, t) = s$ for all $s \leq t \leq d+1$.

Next, we consider the following examples where we will apply this theorem.

Example 2.47. [1] Let $\mathbb{X}$ be a set of 21 points in $\mathbb{P}^2$ with Hilbert function

$$1 \quad 3 \quad 6 \quad 10 \quad 14 \quad 16 \quad 18 \quad 20 \quad 21 \quad 21 \dots$$

We calculate

$$1 \quad 2 \quad 3 \quad 4 \quad 4 \quad 2 \quad 2 \quad 2 \quad 1 \quad 0 \quad 0 \dots$$

Since $\Delta H(\mathbb{X}, 5) = H(\mathbb{X}, 6) = H(\mathbb{X}, 7) = 2$, we choose $d = 5$ and $s = 2$. This satisfies the requirement $d \geq s$ of Theorem 2.46. Hence, a subset $\mathbb{X}_1 \subset \mathbb{X}$ lies on a conic in $\mathbb{P}^2$ (since $s = 2$). By condition (e) of Theorem 2.46, we have

$$\Delta H(\mathbb{X}_1, t) = \begin{cases} \Delta H(C, t), & \text{for } t \leq 6; \\ \Delta H(\mathbb{X}, t), & \text{for } t \geq 5, \end{cases}$$

which simplifies to

$$1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 1 \quad 0 \rightarrow$$

This means $|\mathbb{X}_1| = 16$. Therefore, $\mathbb{X}_1$ has 16 points on a conic in $\mathbb{P}^2$.

Lastly, if we denote c_i as the values of the first difference, where $i \geq 0$, we see that

$$c_5 = 2 = \binom{5}{5} + \binom{4}{4}$$

and

$$c_5^{(5)} = \binom{6}{6} + \binom{5}{5} = 2 = c_6.$$

This means the passage from c_5 to c_6 is the maximum permitted by Macaulay's Theorem.

Example 2.48. [1] Let $\mathbb{X}$ be a set of points in $\mathbb{P}^3$ with Hilbert function

$$1 \quad 4 \quad 10 \quad 15 \quad 16 \quad 17 \quad 18 \quad 18 \dots$$

Then $\Delta H(\mathbb{X})$ is

$$1 \quad 3 \quad 6 \quad 5 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \dots$$

Using c_i to denote the terms of $\Delta H(\mathbb{X})$, observe that

$$c_4 = 1 = \binom{4}{4}$$

and

$$c_4^{\langle 4 \rangle} = \binom{5}{5} = 1 = c_5.$$

That is, the passage from c_4 to c_5 is the maximum permitted by Macaulay's Theorem.

Now, observe that $\Delta H(\mathbb{X}, 4) = H(\mathbb{X}, 5) = H(\mathbb{X}, 6) = 1$. So, we choose $d = 4$ and $s = 1$. This satisfies the requirement $d \geq s$ of Theorem 2.46. Hence, a subset $\mathbb{X}_1 \subset \mathbb{X}$ lies on a $\mathbb{P}^1$ (since $s = 1$).

By condition (e) of Theorem 2.46, we have

$$\Delta H(\mathbb{X}_1, t) = \begin{cases} \Delta H(C, t), & \text{for } t \leq 5; \\ \Delta H(\mathbb{X}, t), & \text{for } t \geq 4, \end{cases}$$

which simplifies to $\Delta H(\mathbb{X}_1, t) =$

$$1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \rightarrow$$

So, $\mathbb{X}_1$ has 7 points lying on a $\mathbb{P}^1$.

Example 2.49. Let $\mathbb{X}$ be a set of points in $\mathbb{P}^2$ with Hilbert function

$$1 \quad 4 \quad 10 \quad 17 \quad 21 \quad 23 \quad 25 \quad 27 \quad 27 \cdots,$$

We calculate $\Delta H(\mathbb{X})$ to be

$$1 \quad 3 \quad 6 \quad 7 \quad 4 \quad 2 \quad 2 \quad 2 \quad 0 \quad 0 \rightarrow$$

Using c_i to denote the terms of the difference sequence, observe that

$$c_5 = 2 = \binom{5}{5} + \binom{4}{4}$$

and

$$c_5^{\langle 5 \rangle} = \binom{6}{6} + \binom{5}{5} = 2 = c_6.$$

That is, the passage from c_5 to c_6 is the maximum permitted by Macaulay's Theorem.

Now, observe that $\Delta H(\mathbb{X}, 5) = H(\mathbb{X}, 6) = H(\mathbb{X}, 7) = 2$. So, we choose $d = 5$ and $s = 2$. This satisfies the requirement $d \geq s$ of Theorem 2.46. Hence, a subset $\mathbb{X}_1 \subset \mathbb{X}$ lies on a conic in $\mathbb{P}^2$ (since $s = 2$).

By condition (e) of Theorem 2.46, we have

$$\Delta H(\mathbb{X}_1, t) = \begin{cases} \Delta H(C, t), & \text{for } t \leq 6; \\ \Delta H(\mathbb{X}, t), & \text{for } t \geq 5, \end{cases}$$

which simplifies to $\Delta H(\mathbb{X}_1) =$

$$1 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 0 \quad 0 \rightarrow$$

So, $\mathbb{X}_1$ has 15 points lying on a conic in $\mathbb{P}^2$.

Chapter 3

Fat Point Schemes

There has been a great deal of interest in the Hilbert function of a set of points with multiplicities greater than 1. These sets are known as fat point schemes. The study of fat point schemes have had applications in algebraic geometry, commutative algebra, and number theory to name a few. For example, conjectures have been made about ideals of fat point subschemes in $\mathbb{P}^2$; upper and lower bounds for the Hilbert function of any fat point subschemes in $\mathbb{P}^n$ have been determined (by using a combinatorial approach); and there is an ongoing effort to relate non-reduced fat point schemes to a special type of reduced point set called a *partial intersection*.

In this thesis, we seek to further understand the Hilbert function of the homogeneous fat point scheme by studying the Hilbert function of some families of fat point schemes. In this Chapter, we will state some basic definitions and highlight some important theorems that will be central to our understanding of this topic.

Throughout this section, let $R = k[x_0, \dots, x_n]$ be a graded polynomial ring and k be an algebraically closed field.

3.1 Hilbert Functions of Fat Point Schemes: Definitions and Literature Review

We start by stating the definition of a fat point scheme and its support. In addition, we discuss some important results and open problems in the study of the Hilbert function.

Definition 3.1. Let $p_1, \dots, p_r \in \mathbb{P}^n$ and $m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}$. A fat point scheme $\mathbb{Y} \subset \mathbb{P}^n$ is defined by the ideal $I = I_{\mathbb{Y}} = \bigcap_i I_{p_i}^{m_i} \subset R$, where I_{p_i} is the ideal generated by all homogeneous

polynomials (i.e., forms) that vanish at p_i . The **fat point scheme** $\mathbb{Y} \subset \mathbb{P}^n$ is denoted by $\mathbb{Y} = m_1 p_1 + \cdots + m_r p_r$ and I by $I_{\mathbb{Y}}$.

If $\mathbb{X} = p_1 + \cdots + p_r$ and $m_1 = \cdots = m_r = m$, then we write $\mathbb{Y} = m\mathbb{X}$ and say that $\mathbb{Y}$ is **homogeneous**. The **support** of $\mathbb{Y}$ is the set of points p_i with $m_i > 0$.

Definition 3.2. The **Hilbert function** of a fat point scheme $\mathbb{Y}$ is defined as the sequence $H(\mathbb{Y}, t) = \dim_k R_t - \dim_k (I_{\mathbb{Y}})_t$.

Related to the Hilbert function are graded Betti numbers. We let $R(-j)$ be a shift by degree j in the grading of R .

Definition 3.3. [5] Given the exact sequence

$$0 \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow I_{\mathbb{Y}} \rightarrow 0$$

where $F_i = \bigoplus_{j \geq 0} R(-j)^{\sigma_{i,j}(\mathbb{Y})}$, the **graded Betti numbers** of $I_{\mathbb{Y}}$ are the numbers $\sigma_{i,j}$ of factors of $R(-j)$ in each sum.

Macaulay made significant contributions to the knowledge of the Hilbert function by describing the Hilbert function of a homogeneous ideal I in a polynomial ring R [10]. In this section, we will review some significant results in the study of Hilbert functions and also discuss some of the important open problems in the area.

We start by discussing the results of the classification of the Hilbert function of reduced 0-dimensional subschemes of projective space (i.e., fat point schemes with all multiplicities equal to one). All possible Hilbert functions of reduced 0-dimensional subschemes of projective space have already been classified [8]. As previously discussed, the paper [8] proved the following important theorem.

Theorem 3.4. [8] *Let k be an infinite field and let $S = \{b_i\}_{i \geq 0}$, be a differentiable 0-sequence with $b_1 = n + 1$. There is a radical ideal I in $k[x_0, \dots, x_n]$ such that S is the Hilbert function of $k[x_0, \dots, x_n]/I$.*

The theorem above describes the Hilbert function of reduced point sets $X \subset \mathbb{P}^n$. In addition, the paper provides an algorithm for constructing any desirable reduced point set. The paper also shows how we can split a differentiable 0-sequence S into two sequences S_1 and S_2 , and proved the following useful theorem.

Theorem 3.5. [8] *Let $S = \{b_i\}_{i \geq 0}$ with $b_1 = n + 1$, be a differentiable 0-sequence. Then the sequences S_1 and S_2 , constructed from S using the procedure outlined after Definition 2.26, are differentiable 0-sequences.*

The proof of Theorem 3.5 follows by showing that S_1 is the Hilbert function of the subvariety of $X \subset \mathbb{P}^{b_1-1}$ and S_2 is the Hilbert function of the subvariety $V \subset H$, where H is a hyperplane not containing any irreducible component of X . The argument concludes that S is the Hilbert function of $X \cup V$.

Apart from the classification results, there are many results in the description of the nature of the Hilbert function of a fat point scheme. Let $\mathbb{Y}$ be a fat point subscheme, given as $\mathbb{Y} = m_1p_1 + \cdots + m_rp_r$, where $p_1, \dots, p_r \subset \mathbb{P}^n$ and m_i is the multiplicity of each p_i . Let $H(\mathbb{Y}, t)$ be the Hilbert function of the fat point subscheme at degree t . In [10], we see that $H(\mathbb{Y}, t)$ is nondecreasing and satisfies $H(\mathbb{Y}, 0) = 1$. Furthermore, the Hilbert function stabilizes for $t \geq \deg(\mathbb{Y}) - 1$. That is, $H(\mathbb{Y}, t) = \deg(\mathbb{Y})$ for $t \geq \deg(\mathbb{Y}) - 1$.

Also of interest is the result about the existence of a maximum Hilbert function discussed in [12]. The paper proves the existence of a maximum Hilbert function, $h^{\max}$, for a double-point scheme, i.e., for $\mathbb{Y} = 2p_1 + \cdots + 2p_r$. In addition, it determines the graded Betti numbers of $I(\mathbb{Y})$ for $\mathbb{Y} = m_1p_1 + \cdots + m_rp_r$ if either $r \leq 6$ or the points lie in a conic.

Another important result is the *Hilbert function equivalence* for ordered sets of points in $\mathbb{P}^2$, which is discussed in [10]. The Hilbert function equivalence helps in the classification of fat point schemes. We state the definition of the Hilbert function equivalence as follows.

Definition 3.6. [10] Let $p_1, \dots, p_r$ and $p'_1, \dots, p'_r$ be ordered sets of distinct points of $\mathbb{P}^2$. We say these sets are **Hilbert function equivalent** if $H(\mathbb{Y}, t) = H(\mathbb{Y}', t)$ for every choice of nonnegative integers m_i , where $\mathbb{Y} = m_1p_1 + \cdots + m_rp_r$ and $\mathbb{Y}' = m_1p'_1 + \cdots + m_rp'_r$. We refer to a Hilbert function equivalence class as a **configuration type**.

Given r , there are many known results in counting the number of configuration types of fat point schemes as seen in [10]. From the papers [18] and [19], we know that there are finitely many configuration types of fat point schemes $\mathbb{Y} = m_1p_1 + \cdots + m_rp_r$ given r and $m_1, \dots, m_r$. Example 5.9 of [10] shows that there are infinitely many configuration types among sets of r points, for each $r \geq 9$. From [10], we see that there are finitely many configuration types for $r \leq 8$. We can find the case of $r \leq 6$ points in [14]. In [10], we find that the case of $r = 7$ points gives 29 types, and that of $r = 8$ points gives 143 types. We also know that if the given points lie on a conic, then there are finitely many configuration types [3], [14], and [19].

In [12], the authors investigate the Hilbert function and the Betti numbers of symbolic powers $I^{(m)}$ of the ideal I of a finite set of reduced points in $\mathbb{P}^2$ when $m = 2$. The case of $I^{(m)}$ for a larger m when I defines a reduced subscheme of projective space whose support is close to a complete intersection is investigated in [15] and [16].

Macaulay gave necessary and sufficient conditions for a nondecreasing integer-valued function to be the Hilbert function of a graded k -algebra. Furthermore, the algebraic and geometric consequences that arise from the extremal cases of Macaulay's Theorem are investigated in [1]. The paper [2] investigates how the Hilbert function constrains the graded Betti numbers in the case where the height of an ideal $I \subset R$ is 2. The height of an ideal I is $\inf_{\mathfrak{p} \supseteq I} \text{ht}(\mathfrak{p})$, where $\text{ht}(\mathfrak{p})$ is the height of a prime ideal $\mathfrak{p} \subset R$. The height of a prime ideal $\mathfrak{p}$ is the supremum of all integers ℓ such that we can find a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_\ell = \mathfrak{p}$. The authors of [7] studied graded Betti numbers for graded modules M over R .

Although there has been significant progress in the study of Hilbert functions as many difficult problems have been put to rest and new concepts developed, there are still many unsolved problems in this area. This makes the study of Hilbert functions interesting and growing. We discuss some of the unsolved problems in the investigation of Hilbert function here.

First, we discuss the determination of the minimum Hilbert function for a double-point scheme. Although the existence of a maximal Hilbert function, $h^{\max}$, is known for a double-point scheme, the paper [12] raises the question of whether the minimum Hilbert function $h^{\min}$ always exists for a double-point scheme. This question remains open.

Next is the unresolved problems that relate to the classification of configuration types. The paper [10] classifies the configuration types of the cases of $r = 7$ and $r = 8$ points. However, it shows that each $r \geq 9$ has infinitely many configuration types. Hence, this case has not been fully classified. Also, according to [4], [13], [17], [20], and [22], the Hilbert functions of the generic points for multiplicities ≥ 2 do not have a general formula.

Another important unresolved problem in this area relates to graded Betti numbers. From the paper [10], we see that the graded Betti numbers are known for $r \leq 6$ and for points on a conic. However, there is no known classification for the graded Betti numbers for $r \geq 7$. Hence, this remains an open problem.

The authors in [12] produce results on the Hilbert functions of fat point subschemes of the form $2p_1 + \cdots + 2p_r$ for any r distinct points in $\mathbb{P}^2$, while the authors in [10] investigate r points in $\mathbb{P}^2$, where $r \leq 8$ or the points lie on a conic. In other words, most of the results in this area focus on distinct points in $\mathbb{P}^2$.

Lastly, of interest is the fact that most work produced have only considered cases of algebraically closed field. There has not been any significant work done on an arbitrary field. Hence, the problem of r points in $\mathbb{P}^2$ or higher dimension over an arbitrary field is still open.

3.2 The Reduction Approach

Central to the remainder of this thesis is the determination of the Hilbert function of a fat point scheme by a reduction process. We briefly discuss the reduction approach used and illustrate it with relevant examples.

Definition 3.7. [5] Let $A = a_1p_1 + a_2p_2 + \cdots + a_rp_r$ be a fat point scheme in $\mathbb{P}^n$. Fix a sequence $L_1, \dots, L_{n+1}$ of hyperplanes in $\mathbb{P}^n$, not necessarily distinct.

- (a) We define fat point schemes $A_0, \dots, A_{n+1}$ by $A_0 = A$ and $A_j = A_{j-1} : L_j$ for $1 \leq j \leq n+1$. As a matter of notation, we write $A : L_1, \dots, L_j = (((A : L_1) : L_2) : \cdots : L_j)$, so we have $A_j = A : L_1, \dots, L_j$.
- (b) We will say the sequence $L_1, \dots, L_{n+1}$ **totally reduces** A if $A_{n+1} = \emptyset$ is the empty scheme, or equivalently, if for each point p_i , the hyperplane L_j passes through p_i for at least a_i of the indices j .
- (c) We associate to A and the sequence $L_1, \dots, L_{n+1}$ an integer vector

$$\mathbf{d} = \mathbf{d}(A; L_1, \dots, L_{n+1}) = (d_1, \dots, d_{n+1}),$$

where d_j is the $\deg(L_j \cap A_{j-1})$ of the scheme theoretic intersection of L_j with A_{j-1} . We refer to $\mathbf{d}$ as the **reduction vector** for A induced by the sequence $L_1, \dots, L_{n+1}$. We say that $\mathbf{d}$ is a **full** reduction vector for A if $L_1, \dots, L_{n+1}$ totally reduces A .

Definition 3.8. [5] Let $\mathbf{v} = (v_1, \dots, v_r)$ be a vector. We say $\mathbf{v}$ is **non-negative**, **positive**, **non-increasing** or **strictly increasing** if its components are. If $\mathbf{v}$ is a non-negative, non-increasing integer vector, then $\mathbf{v}$ is said to be **GMS** if $v_i - v_j \geq j - i - 1$ for each $1 \leq i < j \leq r$.

Definition 3.9. Let $\mathbf{v} = (v_1, \dots, v_r)$ be a reduction vector. Then, $\Delta(v_r, \dots, v_1) = (u_r, \dots, u_1)$, where $u_r = v_r$ and $u_i = v_i - v_{i+1}$ for $1 \leq i \leq r-1$.

Proposition 3.10. [5] Let $\mathbf{v} = (v_1, \dots, v_r)$ be a non-negative, non-increasing integer vector. Then the following are equivalent:

- (i) $\mathbf{v}$ is *GMS*;
- (ii) between any two zero entries of $\Delta(v_r, \dots, v_1)$ there is an entry strictly bigger than 1;
- (iii) $\mathbf{v}$ does not contain a sequence of consecutive entries of the form (a, a, a) , or of the form $(a_i, \dots, a_{i+j+1})$ for $j > 1$ where $a_i = a_{i+1}$, $a_{i+j} = a_{i+j+1}$, and $a_{i+1}, \dots, a_{i+j}$ are consecutive integers.

Having introduced the reduction approach and the reduction vector, we now define bounds for the Hilbert function arising from reduction and provide relevant examples on how to determine these bounds.

Definition 3.11. [5] Let $\mathbf{v} = (v_1, \dots, v_{n+1})$ be an integer vector. We define the functions $f_{\mathbf{v}}$ and $F_{\mathbf{v}}$ by setting

$$f_{\mathbf{v}}(t) = \sum_{i=0}^n \binom{\min(t-i+1, \mathbf{v}_{i+1})}{1}$$

and

$$F_{\mathbf{v}}(t) = \min_{0 \leq i \leq n+1} \left(\binom{t+2}{2} - \binom{t-i+2}{2} + \sum_{i+1 \leq j \leq n+1} \mathbf{v}_j \right)$$

for $t \geq 0$.

The functions $f_{\mathbf{v}}$ and $F_{\mathbf{v}}$ are the lower and upper bounds of the Hilbert function of a fat point scheme. Moreover, the bounds are equal when d is GMS.

Theorem 3.12. [5] Let $l_1, \dots, l_n$ be a totally reducing sequence of lines for the fat point scheme $\mathbb{Y}$ with the associated reduction vector $\mathbf{v}$, then the Hilbert function $H(\mathbb{Y})$ is bounded by

$$f_{\mathbf{v}} \leq H(\mathbb{Y}) \leq F_{\mathbf{v}}.$$

Furthermore, if $\mathbf{v}$ is GMS, then $f_{\mathbf{v}} = H(\mathbb{Y}) = F_{\mathbf{v}}$.

Thus, the condition of a reduction vector $\mathbf{v}$ being GMS guarantees that the Hilbert function of a fat point scheme will be uniquely determined by $\mathbf{v}$.

Example 3.13. [5] Consider the scheme $A \subset \mathbb{P}^2$ consisting of three points each of multiplicity 2 and twelve points of multiplicity 1 arranged as shown in Figure 3.1.

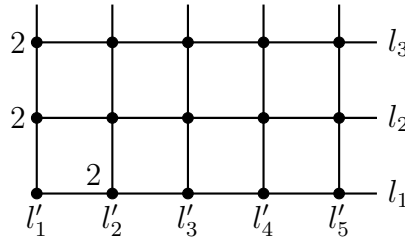


Figure 3.1: Scheme A with 3 points each of multiplicity 2 and 12 points of multiplicity 1.

If we make greedy choices by passing a line through A a line with points whose sum of multiplicities is the highest (i.e., through l_3, l_2, l_1, l'_1 , and l'_2 , respectively), we will obtain the reduction vector $\mathbf{v}$ given by

$$\mathbf{v} = (6, 6, 6, 2, 1).$$

The method we used here is called the *greedy algorithm*, and is discussed fully in Section 3.2.1. However, if we use the lines $l'_1, \dots, l'_5, l'_1$ and l'_2 , respectively, we will obtain the reduction vector $\mathbf{v}'$ given as

$$\mathbf{v}' = (5, 4, 3, 3, 3, 2, 1).$$

Notice that we did not obtain $\mathbf{v}'$ by the greedy algorithm. Also, observe that both $\mathbf{v}$ and $\mathbf{v}'$ are not GMS since they do not satisfy the conditions in Proposition 3.10.

First, we try to obtain the Hilbert function of the scheme A by using the reduction vector $\mathbf{v}$. Since $\mathbf{v}$ is not GMS, we expect to have $f_{\mathbf{v}} \neq F_{\mathbf{v}}$. Hence, $f_{\mathbf{v}}$ and $F_{\mathbf{v}}$ will give us lower and upper bounds of the Hilbert function of A , respectively. That is, we will have $f_{\mathbf{v}}(t) \leq H(A, t) \leq F_{\mathbf{v}}(t)$.

(1) When $\mathbf{v} = (6, 6, 6, 2, 1)$, we can determine $f_{\mathbf{v}}$ by using the formula

$$f_{\mathbf{v}}(t) = \sum_{i=0}^n \binom{\min(t - i + 1, \mathbf{v}_{i+1})}{1}.$$

We see that $n = 4$ (since $\mathbf{v}$ has 5 components).

Now, when $t - i + 1 \leq 0$, we simply write 0. We calculate

$$\begin{aligned} f_{\mathbf{v}}(0) &= \min(1, 6) + \min(0, 6) + \min(0, 6) + \min(0, 2) + \min(0, 1) = 1, \\ f_{\mathbf{v}}(1) &= \min(2, 6) + \min(1, 6) + \min(0, 6) + \min(0, 2) + \min(0, 1) = 3, \\ f_{\mathbf{v}}(2) &= \min(3, 6) + \min(2, 6) + \min(1, 6) + \min(0, 2) + \min(0, 1) = 6, \\ f_{\mathbf{v}}(3) &= \min(4, 6) + \min(3, 6) + \min(2, 6) + \min(1, 2) + \min(0, 1) = 10, \\ f_{\mathbf{v}}(4) &= \min(5, 6) + \min(4, 6) + \min(3, 6) + \min(2, 2) + \min(1, 1) = 15, \\ f_{\mathbf{v}}(5) &= \min(6, 6) + \min(5, 6) + \min(4, 6) + \min(3, 2) + \min(2, 1) = 18, \\ f_{\mathbf{v}}(6) &= \min(7, 6) + \min(6, 6) + \min(5, 6) + \min(4, 2) + \min(3, 1) = 20, \\ f_{\mathbf{v}}(7) &= \min(8, 6) + \min(7, 6) + \min(6, 6) + \min(5, 2) + \min(4, 1) = 21. \end{aligned}$$

Since the total degree of the given fat point scheme is the sum of the components of $\mathbf{v}$, $f_{\mathbf{v}}$ will stabilize here. That is, since

$$\sum_{i=0}^4 \mathbf{v}_{i+1} = 6 + 6 + 6 + 2 + 1 = 21,$$

$f_{\mathbf{v}}$ stabilizes at 21. That is, $f_{\mathbf{v}}(t) = 21$ for $t \geq 7$. Therefore,

$$f_{\mathbf{v}} = (1, 3, 6, 10, 15, 18, 20, 21, 21, \dots).$$

Alternatively, we may determine $f_{\mathbf{v}}$ by using the components of $\mathbf{v}$ to draw a reduced point set as shown in Figure 3.2. As explained in [5], we use the diagonal count vector. Counting the points along the diagonal lines of slope -1 in the diagram in Figure 3.2, we have

$$\text{diag}(\mathbf{v}) = (1, 2, 3, 4, 5, 3, 2, 1, 0, 0, \dots).$$

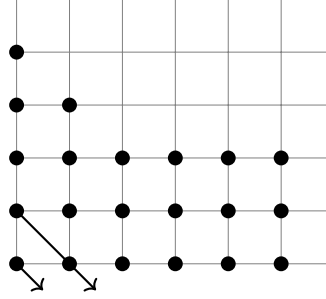


Figure 3.2: Reduced point set with arrows showing $\text{diag}(\mathbf{v})$.

Now, $\text{diag}(\mathbf{v})$ is the first difference of $f_{\mathbf{v}}$. So, we can easily get $f_{\mathbf{v}}$ as

$$f_{\mathbf{v}} = \sum \text{diag}(\mathbf{v}) = (1, 3, 6, 10, 15, 18, 20, 21, 21, \dots).$$

Next, let us determine $F_{\mathbf{v}}$. We will do so by using the formula given in Definition 3.11. We calculate

$$F_{\mathbf{v}}(t) = \min_{0 \leq i \leq 5} \left(\binom{t+2}{2} - \binom{t-i+2}{2} + \sum_{i+1 \leq j \leq 5} \mathbf{v}_j \right).$$

When $t - i + 2 \leq 1$, we take $\binom{t-i+2}{2} = 0$. We find, for various values of t :

(i) For $t = 0$ and $i = 0, \dots, 5$, respectively, we have

$$\binom{2}{2} - \binom{2}{2} + 21 = 21, \binom{2}{2} - 0 + 15 = 16, \binom{2}{2} - 0 + 9 = 10,$$

$$\binom{2}{2} - 0 + 3 = 4, \binom{2}{2} - 0 + 1 = 2, \binom{2}{2} - 0 + 0 = 1.$$

The minimum is 1. So, $F_{\mathbf{v}}(0) = 1$.

(ii) For $t = 1$ and $i = 0, \dots, 5$, respectively, we have:

$$\binom{3}{2} - \binom{3}{2} + 21 = 21, \binom{3}{2} - \binom{2}{2} + 15 = 18, \binom{3}{2} - 0 + 9 = 12,$$

$$\binom{3}{2} - 0 + 3 = 6, \binom{3}{2} - 0 + 1 = 4, \binom{3}{2} - 0 + 0 = 3.$$

The minimum is 3. So, $F_v(1) = 3$.

(iii) For $t = 2$ and $0 \leq i \leq 5$, we have:

$$\binom{4}{2} - \binom{4}{2} + 21 = 21, \binom{4}{2} - \binom{3}{2} + 15 = 18, \binom{4}{2} - \binom{2}{2} + 9 = 14,$$

$$\binom{4}{2} - 0 + 3 = 9, \binom{4}{2} - 0 + 1 = 7, \binom{4}{2} - 0 + 0 = 6.$$

The minimum is 6. So, $F_v(2) = 6$.

(iv) For $t = 3$ and $0 \leq i \leq 5$, we have:

$$\binom{5}{2} - \binom{5}{2} + 21 = 21, \binom{5}{2} - \binom{4}{2} + 16 = 20, \binom{5}{2} - \binom{3}{2} + 12 = 19,$$

$$\binom{5}{2} - \binom{2}{2} + 9 = 18, \binom{5}{2} - 0 + 6 = 16, \binom{5}{2} - 0 + 3 = 13.$$

$$\binom{5}{2} - 0 + 1 = 11, \binom{5}{2} - 0 + 0 = 10.$$

The minimum is 10. So, $F_v(3) = 10$.

(v) For $t = 4$ and $0 \leq i \leq 5$, we have:

$$\binom{6}{2} - \binom{6}{2} + 21 = 21, \binom{6}{2} - \binom{5}{2} + 15 = 20, \binom{6}{2} - \binom{4}{2} + 9 = 18,$$

$$\binom{6}{2} - \binom{3}{2} + 3 = 15, \binom{6}{2} - \binom{2}{2} + 1 = 15, \binom{6}{2} - 0 + 0 = 15.$$

The minimum is 15. So, $F_v(4) = 15$.

(vi) For $t = 5$ and $0 \leq i \leq 5$, we calculate:

$$\binom{7}{2} - \binom{7}{2} + 21 = 21, \binom{7}{2} - \binom{6}{2} + 15 = 21, \binom{7}{2} - \binom{5}{2} + 9 = 20,$$

$$\binom{7}{2} - \binom{4}{2} + 3 = 18, \binom{7}{2} - \binom{3}{2} + 1 = 19, \binom{7}{2} - \binom{2}{2} + 0 = 20.$$

The minimum is 18. So, $F_{\mathbf{v}}(5) = 18$.

(vii) For $t = 6$ and $0 \leq i \leq 5$, we calculate:

$$\begin{aligned} \binom{8}{2} - \binom{8}{2} + 21 &= 21, \binom{8}{2} - \binom{7}{2} + 15 = 22, \binom{8}{2} - \binom{6}{2} + 9 = 22, \\ \binom{8}{2} - \binom{5}{2} + 3 &= 21, \binom{8}{2} - \binom{4}{2} + 1 = 23, \binom{8}{2} - \binom{3}{2} + 0 = 25. \end{aligned}$$

The minimum is 21. So, $F_{\mathbf{v}}(6) = 21$.

Now, $F_{\mathbf{v}}(t) = 21$ for $t \geq 6$ (since $\sum_{j=1}^5 \mathbf{v}_j = 21$). So,

$$F_{\mathbf{v}}(t) = (1, 3, 6, 10, 15, 18, 21, 21, \dots).$$

Observe that $f_{\mathbf{v}}(t) = F_{\mathbf{v}}(t)$ for $0 \leq t \leq 5$, and $f_{\mathbf{v}}(t) \leq F_{\mathbf{v}}(t)$ for all $t > 5$. Therefore, $f_{\mathbf{v}} \leq F_{\mathbf{v}}$ for all t . Hence, $f_{\mathbf{v}}(t) \leq H(A, t) \leq F_{\mathbf{v}}(t)$.

Again, observe that $\mathbf{v}$ is not GMS since it does not satisfy Proposition 3.10. Hence, $f_{\mathbf{v}}(t) \neq F_{\mathbf{v}}(t)$.

- (2) When $\mathbf{v}' = (5, 4, 3, 3, 3, 2, 1)$, we will determine $f_{\mathbf{v}'}$ using Figure 3.3, where the entries in $\mathbf{v}'$ give the row counts for the dots.

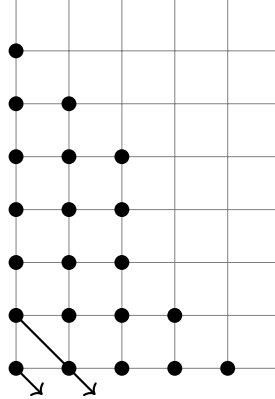


Figure 3.3: Diagram showing $\text{diag}(5, 4, 3, 3, 3, 2, 1)$.

From Figure 3.3, we have

$$\Delta f_{\mathbf{v}'} = \text{diag}(\mathbf{v}') = (1, 2, 3, 4, 5, 3, 3, 0, 0, \dots).$$

Therefore,

$$f_{\mathbf{v}'} = (1, 3, 6, 10, 15, 21, 21, \dots).$$

Next, we will determine $F_{\mathbf{v}'}$ by using the formula in Definition 3.11. We have $n = 6$ (since $\mathbf{v}'$ has 7 components), and

$$F_{\mathbf{v}'} = (1, 3, 6, 10, 15, 21, 21, \dots).$$

By comparing $f_{\mathbf{v}'}$ with $F_{\mathbf{v}'}$, we see that $f_{\mathbf{v}'} \leq F_{\mathbf{v}'}$ for all t . Hence, $f_{\mathbf{v}'} \leq H(A, t) \leq F_{\mathbf{v}'}$.

Again, observe that $20 = f_{\mathbf{v}}(6) < f_{\mathbf{v}'}(6) = 21$ and that the total degree of the fat point scheme is 21. This means $\mathbf{v}'$ gives a strictly better lower bound than $\mathbf{v}$, which is obtained by applying the greedy algorithm.

Example 3.14. Consider the scheme A in $\mathbb{P}^2$ shown in Figure 3.4.

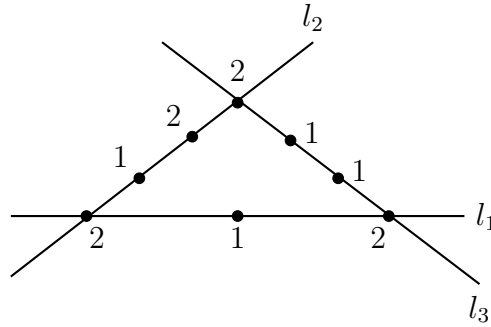


Figure 3.4: Scheme A with lines l_1, l_2 & l_3 .

Reducing with the lines l_2, l_3, l_1 , and l_2 , respectively, we obtain the reduction vector $d = (7, 5, 3, 1)$. However, if we reduce the points along lines l_3, l_2, l_1 , and l_2 , respectively, we obtain the reduction vector $\mathbf{v} = (6, 6, 3, 1)$. Observe that both d and $\mathbf{v}$ are GMS as they each satisfy the conditions of Proposition 3.10. We show that for both d and $\mathbf{v}$, the upper and lower bounds of the Hilbert function coincide.

(1) First, we determine f_d for $d = (7, 5, 3, 1)$ using the diagram in Figure 3.5.

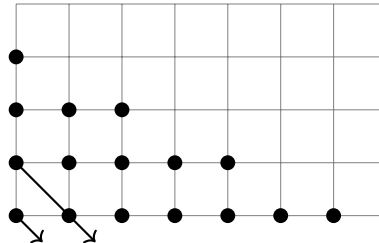


Figure 3.5: Diagram formed by $d = (7, 5, 3, 1)$.

From Figure 3.5, we see that

$$\Delta f_d = \text{diag}(d) = (1, 2, 3, 4, 3, 2, 1, 0, 0, \dots).$$

Hence,

$$f_d = (1, 3, 6, 10, 13, 15, 16, 16, \dots).$$

Next, we determine F_d using Definition 3.11. So, we have

$$F_d = (1, 3, 6, 10, 13, 15, 16, \dots).$$

We see that $f_d = F_d$ as expected. Hence, $H(A, t) = f_d = F_d$.

(2) Now, we determine $f_{\mathbf{v}}$ for $\mathbf{v} = (6, 6, 3, 1)$ by the diagram in Figure 3.6.

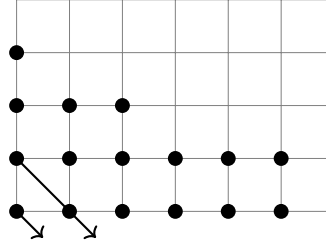


Figure 3.6: Reduced point set formed by $\mathbf{v} = (6, 6, 3, 1)$.

From Figure 3.6, we see that

$$\Delta f_{\mathbf{v}} = \text{diag}(\mathbf{v}) = (1, 2, 3, 4, 3, 2, 1, 0, 0, \dots).$$

This gives

$$f_{\mathbf{v}} = (1, 3, 6, 10, 13, 15, 16, 16, \dots).$$

When we apply the formula in Definition 3.11, we calculate

$$F_{\mathbf{v}} = (1, 3, 6, 10, 13, 15, 16, 16, \dots).$$

We see that $H(A, t) = f_{\mathbf{v}} = F_{\mathbf{v}}$ as expected (since v is GMS).

Notice that $f_{\mathbf{v}} = F_{\mathbf{v}} = f_d = F_d$. That is, since d and $\mathbf{v}$ are GMS, we obtain the same result.

3.2.1 The Greedy Algorithm

Let $\mathbb{Y} \subset \mathbb{P}^n$ be a fat point scheme defined by $\mathbb{Y} = m_1 p_1 + \cdots + m_r p_r$, where $p_1, \dots, p_r \in \mathbb{P}^n$ and $m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}$. We systematically reduce the multiplicities of the points to zero by repeatedly passing a hyperplane through the line that contains points with the highest sum of multiplicities. This method of reduction is referred to as the “greedy algorithm”. This gives a reduction vector d . If d satisfies Proposition 3.10, we say d is GMS.

Example 3.15. Consider the subscheme $A = 2p_1 + p_2 + p_3 + 2p_4 + p_5 + 2p_6 \subset \mathbb{P}^2$ shown in Figure 3.7.

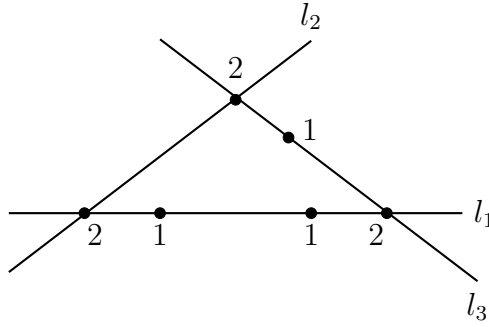


Figure 3.7: Subscheme $A = 2p_1 + p_2 + p_3 + 2p_4 + p_5 + 2p_6$.

From the diagram, we see that if

- (1) we apply the greedy algorithm, then the reduction vector d will be $d = (6, 4, 2)$, which is GMS;
- (2) we do not apply the greedy algorithm but choose to reduce the multiplicities of the points by passing the lines l_3, l_1 , and l_2 , through the points, respectively, we will have the reduction vector $\mathbf{v} = (5, 5, 2)$, which is GMS.

Example 3.16. Consider the scheme A in $\mathbb{P}^2$ consisting of three points each of multiplicity 2 and 9 points each of multiplicity 1 shown in Figure 3.8.

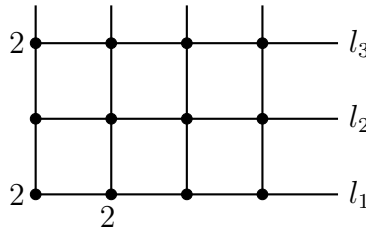


Figure 3.8: Scheme A with three points of multiplicity 2.

Observe that if

- (1) we apply the greedy algorithm by using the lines l_1, l_3, l_2, l_1 , and l_3 , respectively, we will have the reduction vector $d = (6, 5, 4, 2, 1)$, which is GMS;
- (2) we reduce using the vertical lines from left to right and we repeat, we will have the reduction vector $\mathbf{v} = (5, 4, 3, 3, 2, 1)$, which is GMS.

3.3 Fat Point Schemes Whose Support is a Partial Intersection

The rest of this thesis focuses on fat point subschemes in $\mathbb{P}^2$ with multiplicity m and supported on partial intersections. For visual purposes, we describe these support points as being in the intersection of lines that form a grid. However, these points are not found at every intersection of the lines. In fact, the points form levels of blocks. Hence, we say that these fat point schemes are supported on partial intersections. First, we state the definition of partial intersection. Then, we show how we can determine the Hilbert function of a partial intersection. Finally, we consider a family of fat point schemes supported on a special partial intersection and show that this family always has a reduction vector that is GMS.

Definition 3.17. [6] Let $p = (p_1, \dots, p_r)$ and $q = (q_1, \dots, q_r)$ be two r -tuples of positive integers with $p_1 > \dots > p_r > 0$ and set $a = q_1 + \dots + q_r$. Let $\{H_i\}$ be the set of points on “horizontal lines” for each $i \in \{1, \dots, a\}$ and $\{V_j\}$ be the set of points on “vertical lines” for each $j \in \{1, \dots, p_1\}$ and set $P_{i,j} = H_i \cap V_j$. Further, let $r(i) = \inf\{s \in \mathbb{N} \mid \sum_{j=1}^s q_j \geq i\}$ for $i \in \{1, \dots, a\}$. With this notation, we define the **partial intersection of type (p, q)** , denoted $\mathbb{Y}_{stairs}^{p,q}$, as the set of points:

$$\{P_{i,j(i)} \mid i = 1, \dots, a, j(i) = 1, \dots, p_{r(i)}\} \subseteq \mathbb{P}^2.$$

We can visualize a partial intersection as a grid with a collection of points that form blocks. The points are at some of the intersections of the vertical and horizontal lines which form the grid. The diagram in Figure 3.9 is a partial intersection with 3 levels of blocks.

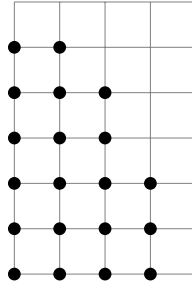


Figure 3.9: Partial intersection with three levels of blocks.

Example 3.18. Figure 3.10 depicts a partial intersection of type $p = (5, 3, 2)$ and $q = (3, 2, 1)$.

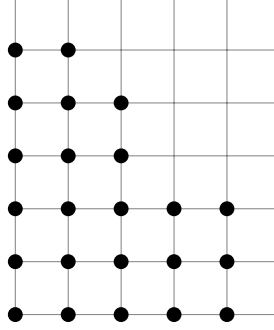


Figure 3.10: Partial intersection of type $p = (5, 3, 2)$ and $q = (3, 2, 1)$.

Example 3.19. A partial intersection of type $p = (7, 5, 3)$ and $q = (2, 2, 2)$ is shown in Figure 3.11.

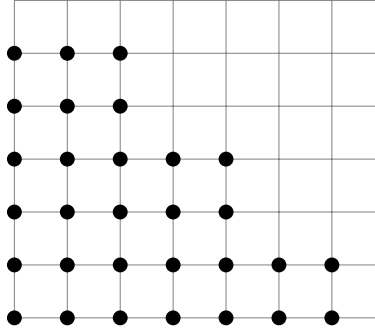


Figure 3.11: Partial intersection of type $p = (7, 5, 3)$ and $q = (2, 2, 2)$.

3.3.1 Hilbert Functions of Fat Points Supported on Partial Intersections

Given a fat point scheme supported on a partial intersection where each point has multiplicity $m \geq 1$, we perform a reduction process in which we pass lines (vertical, horizontal, or diagonal lines) containing points. This gives us a reduction vector $\mathbf{v}$. As demonstrated in the previous section, with the reduction vector we can form a diagram. The sequence $\sum \text{diag}(\mathbf{v})$ gives the lower bound of the Hilbert function of the fat point scheme. If $\mathbf{v}$ is GMS, then the Hilbert function of this fat point scheme will be $f_{\mathbf{v}}$.

The following examples describe the process.

Example 3.20. Consider the partial intersection of type $p = (3, 2)$ and $q = (4, 2)$ where each point has a multiplicity of 2 (see Figure 3.12).

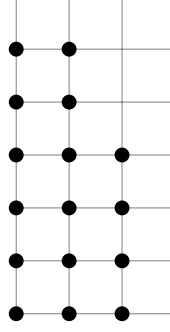


Figure 3.12: Partial intersection of type $p = (3, 2)$ and $q = (4, 2)$.

By applying the greedy algorithm using only the vertical lines, we totally reduce the fat point scheme. This will give the reduction vector $\mathbf{v} = (12, 12, 8, 6, 6, 4)$. This gives us the diagram in Figure 3.13, where the number of dots in each row is given by the corresponding entry in $\mathbf{v}$.

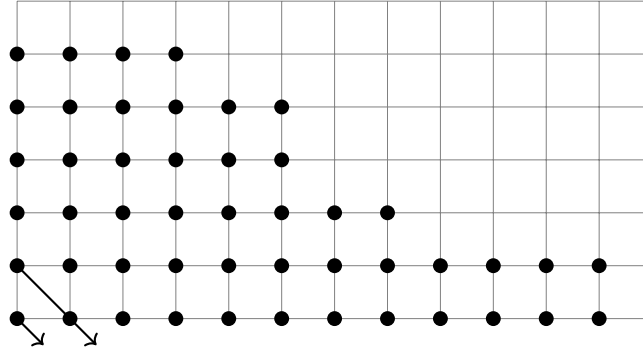


Figure 3.13: Diagram with $\mathbf{v} = (12, 12, 8, 6, 6, 4)$.

By drawing diagonal lines of slope -1 through the points, as shown in Figure 3.13, we have

$$\Delta f_{\mathbf{v}} = \text{diag}(\mathbf{v}) = (1, 2, 3, 4, 5, 6, 6, 6, 6, 4, 2, 2, 1, 0, 0, 0, \dots).$$

Hence,

$$f_{\mathbf{v}} = \sum \text{diag}(\mathbf{v}) = (1, 3, 6, 10, 15, 21, 27, 33, 39, 43, 45, 47, 48, 0, 0, 0, \dots).$$

Observe that $\mathbf{v} = (12, 12, 8, 6, 6, 4)$ is GMS since it satisfies all the conditions in Proposition 3.10. Hence the Hilbert function of the given fat point scheme supported on the partial intersection is $f_{\mathbf{v}}$.

Note: For the rest of this thesis, d stands for a reduction vector and d' stands for its reverse. For example, if $d = (5, 3, 1)$, then $d' = (1, 3, 5)$.

Example 3.21. Figure 3.14 depicts a partial intersection of type $p = (5, 3)$ and $q = (2, 2)$. Assign each point multiplicity 3.

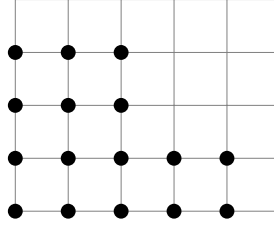


Figure 3.14: Partial intersection of type $p = (5, 3)$ and $q = (2, 2)$.

By applying the greedy algorithm, we totally reduce the points by passing only horizontal lines containing the points. We obtain the reduction vector $d = (15, 15, 10, 10, 9, 9, 6, 6, 5, 5, 3, 3)$. Observe that by reversing the order of the entries of d , we get

$$d' = (3, 3, 5, 5, 6, 6, 9, 9, 10, 10, 15, 15).$$

Hence,

$$\Delta d' = (3, 0, 2, 0, 1, 0, 3, 0, 1, 0, 5, 0).$$

We see that d is not GMS since there is at least two zero entries with only the number 1 in between. This means f_d will only be a lower bound of the Hilbert function for this fat point scheme. We now determine this lower bound.

Figure 3.15 is the diagram we obtain from d .

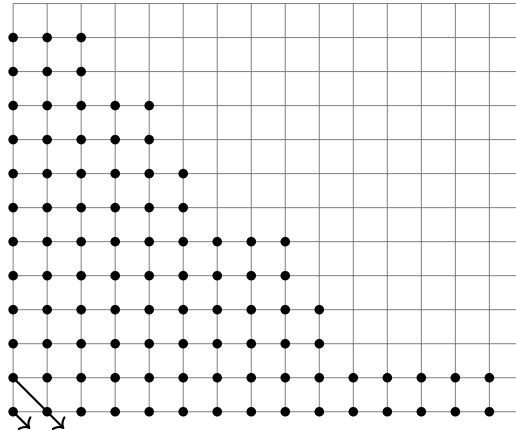


Figure 3.15: Reduced point set from $d = (15, 15, 10, 10, 9, 9, 6, 6, 5, 5, 3, 3)$.

From Figure 3.15, we see that

$$\Delta f_d = \text{diag}(d) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 10, 5, 2, 1, 0, 0, 0, \dots).$$

Hence,

$$f_d = (1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 88, 93, 95, 96, 96, \dots),$$

which is the lower bound for the Hilbert function for the given fat point scheme.

3.3.2 The Hilbert Function of a Fat Point Scheme Supported on a Partial Intersection of Type $p = (3, 2)$ and $q = (2\ell, b)$.

In this section, we investigate the Hilbert function of a fat point scheme supported on a partial intersection of type $p = (3, 2)$ and $q = (2\ell, b)$, where ℓ is a positive integer. We determine the reduction vector for three families (namely $\ell = 1, 2, 3$) and show that these reduction vectors are GMS. This yields the exact Hilbert function of every member of these families.

We begin our exploration with $\ell = 1$.

Theorem 3.22. *Let b and m be positive integers. Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (2, b)$ and $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$m \leq \frac{\text{lcm}(2+b, 2)}{2} - 1,$$

then d is GMS.

Proof. we have three cases to consider. Each case yields a reduction vector which is GMS. Let $d = (d_1, \dots, d_{3m})$ be the reduction vector and $d' = (d_{3m}, d_{3m-1}, \dots, d_2, d_1)$.

Case I: Assume that $b = 1$. The partial intersection $\mathbb{X}$ is visualized in Figure 3.16. By assumption $1 \leq m \leq 2$.

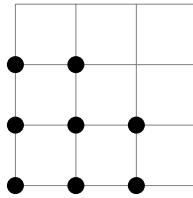


Figure 3.16: Partial intersection of type $p = (3, 2)$ and $q = (2, 1)$.

We consider the two values of m separately.

- (i) For $m = 1$, the resulting reduction vector is $d = (3, 3, 2)$.
- (ii) For $m = 2$, the resulting reduction vector is $d = (6, 6, 4, 3, 3, 2)$.

Clearly, d is GMS for cases (i) and (ii) above.

For cases II and III, the partial intersection $\mathbb{X}$ is depicted in Figure 3.17.

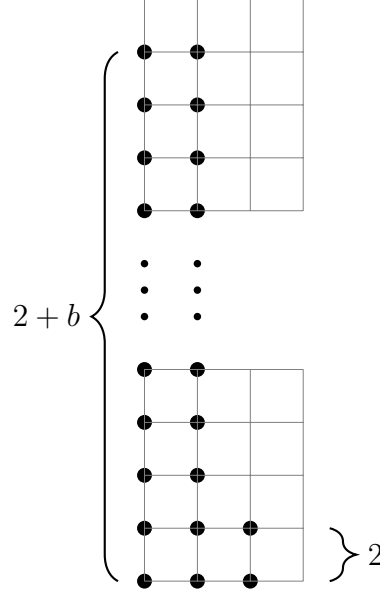


Figure 3.17: Partial intersection of type $p = (3, 2)$ and $q = (2, b)$.

Case II: Assume that $b > 1$ is odd, say $b = 2n + 1$ for some positive integer n . Now, by assumption, $1 \leq m \leq \frac{\text{lcm}(2+b, 2)}{2} - 1 = \frac{\text{lcm}(2n+3, 2)}{2} - 1 = 2(n + 1)$.

We consider two ranges of values for m and show that each of these yields a GMS reduction vector.

- (i) For $1 \leq m \leq n + 1$, the reduction vector is of the form $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$, where

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(2 + b), & \text{for } c = 0, \dots, m - 1; \\ d_{2m+c+1} = 2(m - c), & \text{for } c = 0, \dots, m - 1. \end{cases}$$

Observe that when checking the GMS criterion, the only zero entries of $\Delta d'$ of concern come from the terms of the form $d_{2c+1} = d_{2c+2}$. Now, for $0 \leq c \leq m - 2$, we have $d_{2c+2} = (m - c)(2 + b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m - (c + 1))(2 + b)$. So,

$$d_{2c+2} - d_{2c+3} = (m - c)(2 + b) - (m - (c + 1))(2 + b) = 2 + b \geq 5.$$

Thus, between any two zero entries of $\Delta d'$, there is an entry strictly greater than 1. We conclude that d is GMS (by Proposition 3.10).

(ii) For $n + 1 < m \leq 2(n + 1)$, the reduction vector is given by

$$d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m}),$$

where

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(2 + b), & \text{for } c = 0, \dots, m - 2; \\ d_{2m+c-1} = 2(m - c), & \text{for } c = 0, \dots, m - (n + 2); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(2 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = 2(m - c), & \text{for } c = m - (n + 1), \dots, m - 1. \end{cases}$$

Observe that the reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m - c)(2 + b)$ for $0 \leq c \leq m - 2$. Now, for $0 \leq c \leq m - 3$, we have $d_{2c+2} = (m - c)(2 + b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m - (c + 1))(2 + b)$. So,

$$d_{2c+2} - d_{2c+3} = (m - c)(2 + b) - (m - (c + 1))(2 + b) = 2 + b \geq 5.$$

In addition, we have

$$d_{2(m-2)+2} = 2(2 + b) \quad \text{and} \quad d_{2(m-2)+3} = d_{2(m-2)+2+1} = d_{2(m-1)+1} = d_{2m-1} = 2m.$$

Therefore, since $m \leq 2(n + 1)$, we have

$$\begin{aligned} d_{2(m-2)+2} - d_{2(m-2)+3} &= 2(2 + b) - 2m \\ &= 2(2 + 2n + 1) - 2m \\ &= 4 + 4n + 2 - 2m \\ &\geq 4 + 4n + 2 - 2(2n + 2) \\ &= 4 + 4n + 2 - 4n - 4 \\ &= 2. \end{aligned}$$

That is, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

Case III: Assume that $b \geq 2$ is even, say $b = 2n$ where n is a positive integer.

By assumption, $1 \leq m \leq \frac{\text{lcm}(2+b, 2)}{2} - 1 = \frac{\text{lcm}(2+2n, 2)}{2} - 1 = n$. We show that this range of m yields a GMS reduction vector.

The reduction vector for this range is of the form $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$, where

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(2+b), & \text{for } c = 0, \dots, m-1; \\ d_{2m+c+1} = 2(m-c), & \text{for } c = 0, \dots, m-1. \end{cases}$$

Note that the reduction vector d is the same as in (i) of Case II, and so d is GMS. □

We now consider the family of partial intersections of type $p = (3, 2)$ and $q = (2\ell, b)$ with $\ell = 2$. This family is handled via a series of propositions. Figure 3.18 depicts the partial intersection $\mathbb{X}$ of type $p = (3, 2)$ and $q = (4, b)$.

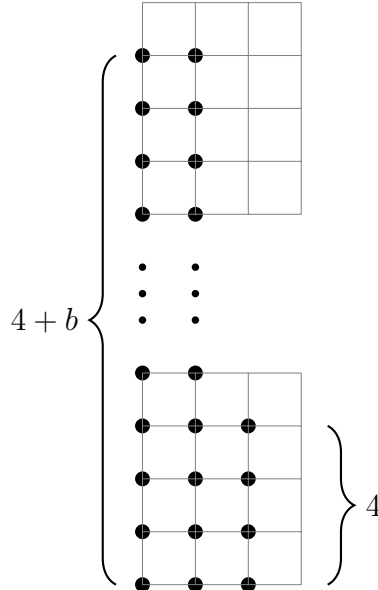


Figure 3.18: Partial intersection of type $p = (3, 2)$ and $q = (4, b)$.

Proposition 3.23. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (4, b)$ where $b = 4n$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(4+b, 4)}{4} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(4+4n, 4)}{4} - 1 = n$. Using Figure 3.18, we observe that the reduction vector d is of the form $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$ where

$$\begin{cases} d_{2c+1} = d_{2c+2} = 4(m-c)(1+n), & \text{for } c = 0, \dots, m-1; \\ d_{2m+c+1} = 4(m-c), & \text{for } c = 0, \dots, m-1. \end{cases}$$

Observe that when checking the GMS criterion, the only zero entries of $\Delta d'$ of concern come from the terms of the form $d_{2c+1} = d_{2c+2}$. Now, for $0 \leq c \leq m-2$, we have $d_{2c+2} = 4(m-c)(1+n)$ and $d_{2c+3} = d_{2(c+1)+1} = 4(m-(c+1))(1+n)$. So,

$$d_{2c+2} - d_{2c+3} = 4(m-c)(1+n) - 4(m-(c+1))(1+n) = 4(1+n) \geq 8.$$

Certainly, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Therefore, d is GMS (by Proposition 3.10). □

Proposition 3.24. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (4, b)$ where $b = 4n + 1$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(4+b, 4)}{4} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(4+b, 4)}{4} - 1 = \frac{\text{lcm}(4+4n+1, 4)}{4} - 1 = 4(1+n)$. We consider four ranges of values for m and show that each of these ranges yields a GMS reduction vector $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$.

(i) For $1 \leq m \leq 1+n$, Figure 3.18 can be used to see that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(4+b), & \text{for } c = 0, \dots, m-1; \\ d_{2m+c+1} = 4(m-c), & \text{for } c = 0, \dots, m-1. \end{cases}$$

Observe that when checking the GMS criterion for this range, the only zero entries of $\Delta d'$ of concern come from the terms of the form $d_{2c+1} = d_{2c+2}$. Now, for $0 \leq c \leq m-2$, we have $d_{2c+2} = (m-c)(4+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(4+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(4+b) - (m-(c+1))(4+b) = 4+b \geq 9.$$

Thus, between any two zero entries of $\Delta d'$, there is an entry strictly greater than 1. We conclude that d is GMS (by Proposition 3.10).

(ii) For the range $1 + n < m \leq 2(1 + n)$, the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(4 + b), & \text{for } c = 0, \dots, m - 2; \\ d_{2m+c-1} = 4(m - c), & \text{for } c = 0, \dots, m - ((n + 1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(4 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = 4(m - c), & \text{for } c = m - (n + 1), \dots, m - 1. \end{cases}$$

The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m - c)(4 + b)$ for $0 \leq c \leq m - 2$. Now, for $0 \leq c \leq m - 3$, we have $d_{2c+2} = (m - c)(4 + b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m - (c + 1))(4 + b)$. So,

$$d_{2c+2} - d_{2c+3} = (m - c)(4 + b) - (m - (c + 1))(4 + b) = 4 + b \geq 9.$$

In addition, we have

$$d_{2(m-2)+2} = 2(4 + b) \text{ and } d_{2(m-2)+3} = d_{2(m-2)+2+1} = d_{2(m-1)+1} = d_{2m-1} = 4m.$$

Therefore, since $m \leq 2(n + 1)$, we have

$$\begin{aligned} d_{2(m-2)+2} - d_{2(m-2)+3} &= 2(4 + b) - 4m \\ &= 2(4 + 4n + 1) - 4m \\ &= 8 + 8n + 2 - 4m \\ &\geq 8 + 8n + 2 - 4(2n + 2) \\ &= 8 + 8n + 2 - 8n - 8 \\ &= 2. \end{aligned}$$

That is, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

(iii) For the range $2(n + 1) < m \leq 3(n + 1)$, we can use Figure 3.18 to deduce that the

entries of the reduction vector d are of the form

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(4+b), & \text{for } c = 0, \dots, m-3; \\ d_{2m+c-3} = 4(m-c), & \text{for } c = 0, \dots, m-(2(n+1)+1); \\ d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(4+b), & \text{for } c = m-2; \\ d_{2m+c-1} = 4(m-c), & \text{for } c = m-2(n+1), \dots, m-((n+1)+1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(4+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 4(m-c), & \text{for } c = m-(n+1), \dots, m-1. \end{cases}$$

We have three situations, each of which yields zero entries of $\Delta d'$. We consider each separately.

Firstly, the reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(4+b)$ for $0 \leq c \leq m-3$. Now, for $0 \leq c \leq m-4$, we have $d_{2c+2} = (m-c)(4+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(4+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(4+b) - (m-(c+1))(4+b) = 4+b \geq 9.$$

Secondly, we have

$$d_{2(m-3)+2} = (m-(m-3))(4+b) = 3(4+b)$$

and

$$d_{2(m-3)+3} = d_{2(m-3)+2+1} = d_{2(m-2)+1} = d_{2m-3} = 4m.$$

Therefore, since $m \leq 3(n+1)$, we have

$$\begin{aligned} d_{2(m-3)+2} - d_{2(m-3)+3} &= 3(4+b) - 4m \\ &= 3(4+4n+1) - 4m \\ &= 12+12n+3-4m \\ &\geq 12+12n+3-4(3n+3) \\ &= 12+12n+3-12n-12 \\ &= 3. \end{aligned}$$

Thirdly, the last term of the terms of the form $d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(4+b)$ is when $c = m-2$. When $c = m-2$, we have

$$d_{2m-2n+c-2} = d_{2m-2n+m-2-2} = d_{3m-2n-4} = [m - (m - 2)](4 + b) = 2(4 + b).$$

The first term of the terms of the form $d_{2m+c-1} = 4(m - c)$ is when $c = m - 2(n + 1)$.

We see that, for $c = m - 2(n + 1)$,

$$\begin{aligned} d_{2m+c-1} &= d_{2m+m-2(n+1)-1} = d_{3m-2n-3} \\ &= 4(m - (m - 2(n + 1))) \\ &= 4(2(n + 1)) \\ &= 8(n + 1). \end{aligned}$$

Therefore,

$$\begin{aligned} d_{3m-2n-4} - d_{3m-2n-3} &= 2(4 + b) - 8(n + 1) \\ &= 2(4 + 4n + 1) - 8(n + 1) \\ &= 8n + 10 - 8n - 8 \\ &= 2. \end{aligned}$$

The above arguments show that there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

- (iv) For $3(n + 1) < m \leq 4(n + 1)$, Figure 3.18 can be used to see that the reduction vector d satisfies

$$\left\{ \begin{array}{ll} d_{2c+1} = d_{2c+2} = (m - c)(4 + b), & \text{for } c = 0, \dots, m - 4; \\ d_{2m+c-5} = 4(m - c), & \text{for } c = 0, \dots, m - (3(n + 1) + 1); \\ d_{2m-3n+c-5} = d_{2m-3n+c-4} = (m - c)(4 + b), & \text{for } c = m - 3; \\ d_{2m+c-3} = 4(m - c), & \text{for } c = m - 3(n + 1), \dots, m - (2(n + 1) + 1); \\ d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m - c)(4 + b), & \text{for } c = m - 2; \\ d_{2m+c-1} = 4(m - c), & \text{for } c = m - 2(n + 1), \dots, m - ((n + 1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(4 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = (m - c)(4 + b), & \text{for } c = m - (n + 1), \dots, m - 1. \end{array} \right.$$

In this range, to verify that d is GMS, we need to consider four situations involving zero entries in $\Delta d'$.

Firstly, the reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m - c)(4 + b)$

for $0 \leq c \leq m - 4$. Now, for $0 \leq c \leq m - 5$, we have $d_{2c+2} = (m - c)(4 + b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m - (c + 1))(4 + b)$. So,

$$d_{2c+2} - d_{2c+3} = (m - c)(4 + b) - (m - (c + 1))(4 + b) = 4 + b \geq 9.$$

Secondly, we have

$$d_{2(m-4)+2} = 4(4 + b) \quad \text{and} \quad d_{2(m-4)+3} = d_{2(m-4)+2+1} = d_{2(m-3)+1} = d_{2m-5} = 4m.$$

Therefore, since $m \leq 4(n + 1)$, we have

$$\begin{aligned} d_{2(m-4)+2} - d_{2(m-4)+3} &= 4(4 + b) - 4m \\ &= 4(4 + 4n + 1) - 4m \\ &= 16 + 16n + 4 - 4m \\ &\geq 16 + 16n + 4 - 4(4n + 4) \\ &= 16 + 16n + 4 - 16n - 16 \\ &= 4. \end{aligned}$$

Thirdly, the last term of terms of the form $d_{2m-3n+c-5} = d_{2m-3n+c-4} = (m - c)(4 + b)$ is when $c = m - 3$. So, when $c = m - 3$, we have $d_{3m-3n-7} = 3(4 + b)$.

The first term of terms of the form $d_{2m+c-3} = 4(m - c)$ is when $c = m - 3(n + 1)$. So, when $c = m - 3(n + 1)$, we have

$$d_{2m+c-3} = d_{2m+m-3(n+1)-3} = d_{3m-3n-6} = 4(m - (m - 3(n + 1))) = 4(3(n + 1)) = 12(n + 1).$$

Therefore,

$$\begin{aligned} d_{3m-3n-7} - d_{3m-3n-6} &= 3(4 + b) - 12(n + 1) \\ &= 3(4 + 4n + 1) - 12(n + 1) \\ &= 12 + 12n + 3 - 12n - 12 \\ &= 3. \end{aligned}$$

And finally, the last term of terms of the form $d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m - c)(4 + b)$

is when $c = m - 2$. So, for $c = m - 2$, the term is

$$d_{2m-2n+c-2} = d_{2m-2n+m-2-2} = d_{3m-2n-4} = (m - (m - 2))(4 + b) = 2(4 + b).$$

The first term of terms of the form $d_{2m+c-1} = 4(m - c)$ is when $c = m - 2(n + 1)$. So, when $c = m - 2(n + 1)$, we have

$$d_{2m+c-1} = d_{2m+m-2(n+1)-1} = d_{3m-2n-3} = 4(m - (m - 2(n + 1))) = 4(2(n + 1)) = 8(n + 1).$$

Observe that

$$\begin{aligned} d_{3m-2n-4} - d_{3m-2n-3} &= 2(4 + b) - 8(n + 1) \\ &= 2(4 + 4n + 1) - 8(n + 1) \\ &= 8 + 8n + 2 - 8n - 8 \\ &= 2. \end{aligned}$$

We therefore conclude that there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

□

Proposition 3.25. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (4, b)$ where $b = 4n + 2$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(4 + b, 4)}{4} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(4+b, 4)}{4} - 1 = \frac{\text{lcm}(4+4n+2, 4)}{4} - 1 = 2(n + 1)$. We consider two ranges of values for m and show that the reduction vector $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$ is GMS for each of these ranges.

(i) For $1 \leq m \leq n + 1$, we can see from Figure 3.18 that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(4 + b), & \text{for } c = 0, \dots, m - 1; \\ d_{2m+c+1} = 4(m - c), & \text{for } c = 0, \dots, m - 1. \end{cases}$$

Note that the reduction vector d is the same as in Case (i) of Proposition 3.24. Hence, d is GMS.

- (ii) For the range $n + 1 < m \leq 2(n + 1)$, we can use Figure 3.18 to see that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(4 + b), & \text{for } c = 0, \dots, m - 2; \\ d_{2m+c-1} = 4(m - c), & \text{for } c = 0, \dots, m - (n + 2); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(4 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = 4(m - c), & \text{for } c = m - (n + 1), \dots, m - 1. \end{cases}$$

The reduction vector d is the same as in Case (ii) of Proposition 3.24. Hence, d is GMS.

□

Proposition 3.26. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (4, b)$ where $b = 4n + 3$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(4 + b, 4)}{4} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(4+b, 4)}{4} - 1 = \frac{\text{lcm}(4+4n+3, 4)}{4} - 1 = 4n + 6$. We consider four ranges of m and show that each range yields a GMS reduction vector d of the form $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$.

- (i) For $1 \leq m \leq n + 1$, Figure 3.18 can be used to see that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(4 + b), & \text{for } c = 0, \dots, m - 1; \\ d_{2m+c+1} = 4(m - c), & \text{for } c = 0, \dots, m - 1. \end{cases}$$

Observe that the reduction vector d is the same as in Case (i) of Proposition 3.24. Hence, d is GMS.

(ii) For $n + 1 < m \leq 2(n + 1) + 1$, the terms of the reduction vector d satisfy

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(4 + b), & \text{for } c = 0, \dots, m - 2; \\ d_{2m+c-1} = 4(m - c), & \text{for } c = 0, \dots, m - ((n + 1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(4 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = 4(m - c), & \text{for } c = m - (n + 1), \dots, m - 1. \end{cases}$$

The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m - c)(4 + b)$ for $0 \leq c \leq m - 2$. Now, for $0 \leq c \leq m - 3$, we have $d_{2c+2} = (m - c)(4 + b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m - (c + 1))(4 + b)$. So,

$$d_{2c+2} - d_{2c+3} = (m - c)(4 + b) - (m - (c + 1))(4 + b) = 4 + b \geq 11.$$

In addition, we have

$$d_{2(m-2)+2} = (m - (m - 2))(4 + b) = 2(4 + b) \quad \text{and}$$

$$d_{2(m-2)+3} = d_{2(m-2)+2+1} = d_{2(m-1)+1} = d_{2m-1} = 4m.$$

Therefore, since $m \leq 2(n + 1) + 1$, we have

$$\begin{aligned} d_{2(m-2)+2} - d_{2(m-2)+3} &= 2(4 + b) - 4m \\ &= 2(4 + 4n + 3) - 4m \\ &= 8 + 8n + 6 - 4m \\ &\geq 8 + 8n + 6 - 4(2(n + 1) + 1) \\ &= 8 + 8n + 6 - 8n - 12 \\ &= 2. \end{aligned}$$

(iii) For $2(n + 1) + 1 < m \leq 3(n + 1) + 2$, Figure 3.18 can be used to observe that the

reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(4+b), & \text{for } c = 0, \dots, m-3; \\ d_{2m+c-3} = 4(m-c), & \text{for } c = 0, \dots, m-(2(n+1)+2); \\ d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(4+b), & \text{for } c = m-2; \\ d_{2m+c-1} = 4(m-c), & \text{for } c = m-(2(n+1)+1), \dots, m-((n+1)+1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(4+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 4(m-c), & \text{for } c = m-(n+1), \dots, m-1. \end{cases}$$

In verifying the GMS conditions, we see three situations that give rise to entries equal to zero in $\Delta d'$.

Firstly, the reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(4+b)$ for $0 \leq c \leq m-3$. Now, for $0 \leq c \leq m-4$, we have $d_{2c+2} = (m-c)(4+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(4+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(4+b) - (m-(c+1))(4+b) = 4+b \geq 11.$$

Secondly, we have

$$\begin{aligned} d_{2m+m-(2n+4)-3} - d_{2m-2n+(m-2)-4} &= d_{3m-2n-7} - d_{3m-2n-6} \\ &= 4(m - (m - (2n+4))) - (m - (m-2))(4+b) \\ &= 4(2n+4) - 2(4+b) \\ &= 8n+16 - 2(4+4n+3) \\ &= 8n+16 - 8 - 8n - 6 \\ &= 2. \end{aligned}$$

Lastly, the last term of terms of the form $d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(4+b)$ is where $c = m-2$. So, when $c = m-2$, we have

$$d_{2m-2n+c-3} = d_{2m-2n+m-2-3} = d_{3m-2n-5} = (m-(m-2))(4+b) = 2(4+b).$$

The first term of terms of the form $d_{2m+c-1} = 4(m-c)$ is when $c = m-(2(n+1)+1)$.

When $c = m - (2(n + 1) + 1)$, we have

$$\begin{aligned}
d_{2m+c-1} &= d_{2m+m-(2(n+1)+1)-1} = d_{3m-2n-4} \\
&= 4(m - (m - (2(n + 1) + 1))) \\
&= 4(2(n + 1) + 1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
d_{3m-2n-5} - d_{3m-2n-4} &= 2(4 + b) - 4(2(n + 1) + 1) \\
&= 2(4 + 4n + 3) - 4(2n + 2 + 1) \\
&= 8 + 8n + 6 - 8n - 8 - 4 \\
&= 2.
\end{aligned}$$

Thus, the arguments above show that there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

- (iv) For $3(n + 1) + 2 < m \leq 4(n + 1) + 2$, Figure 3.18 demonstrates that the reduction vector d satisfies

$$\left\{ \begin{array}{ll} d_{2c+1} = d_{2c+2} = (m - c)(4 + b), & \text{for } c = 0, \dots, m - 4; \\ d_{2m+c-5} = 4(m - c), & \text{for } c = 0, \dots, m - (3(n + 1) + 3); \\ d_{2m-3n+c-7} = d_{2m-3n+c-6} = (m - c)(4 + b), & \text{for } c = m - 3; \\ d_{2m+c-3} = 4(m - c), & \text{for } c = m - (3(n + 1) + 2), \dots, m - (2(n + 1) + 2); \\ d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m - c)(4 + b), & \text{for } c = m - 2; \\ d_{2m+c-1} = 4(m - c), & \text{for } c = m - (2(n + 1) + 1), \dots, m - ((n + 1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(4 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = (m - c)(4 + b), & \text{for } c = m - (n + 1), \dots, m - 1. \end{array} \right.$$

We need to consider the four situations yielding zero entries in $\Delta d'$.

First, the reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m - c)(4 + b)$ for $0 \leq c \leq m - 4$. Now, for $0 \leq c \leq m - 5$, we have $d_{2c+2} = (m - c)(4 + b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m - (c + 1))(4 + b)$. So,

$$d_{2c+2} - d_{2c+3} = (m - c)(4 + b) - (m - (c + 1))(4 + b) = 4 + b \geq 11.$$

Second, we have

$$d_{2(m-4)+2} = 4(4+b) \quad \text{and} \quad d_{2(m-4)+3} = d_{2(m-4)+2+1} = d_{2(m-3)+1} = d_{2m-5} = 4m.$$

Therefore, since $m \leq 4(n+1) + 2$, we have

$$\begin{aligned} d_{2(m-4)+2} - d_{2(m-4)+3} &= 4(4+b) - 4m \\ &= 4(4+4n+3) - 4m \\ &= 16 + 16n + 12 - 4m \\ &\geq 16 + 16n + 12 - 4(4(n+1) + 2) \\ &= 16 + 16n + 12 - 16n - 24 \\ &= 4. \end{aligned}$$

Third, the last term of the terms of the form $d_{2m+c-3} = 4(m-c)$ is when $c = m - (2n+4)$. So, when $c = m - (2n+4)$, we have

$$d_{2m+c-3} = d_{2m+m-(2n+4)-3} = d_{3m-2n-7} = 4(m - (m - (2n+4))) = 4(2n+4).$$

The first term of $d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(4+b)$ is when $c = m - 2$. So, when $c = m - 2$, then

$$d_{2m-2n+c-4} = d_{2m-2n+(m-2)-4} = d_{3m-2n-6} = (m - (m - 2))(4+b) = 2(4+b).$$

Therefore,

$$\begin{aligned} d_{2m+c-3} - d_{2m-2n+c-4} &= d_{2m+m-(2n+4)-3} - d_{2m-2n+(m-2)-4} \\ &= d_{3m-2n-7} - d_{3m-2n-6} \\ &= 4(2n+4) - 2(4+b) \\ &= 8n + 16 - 2(4+4n+3) \\ &= 8n + 16 - 8 - 8n - 6 \\ &= 2. \end{aligned}$$

Finally, the last term of terms of the form $d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(4+b)$ is where $c = m - 2$. So, when $c = m - 2$,

$$d_{2m-2n+c-3} = d_{2m-2n+m-2-3} = d_{3m-2n-5} = (m - (m - 2))(4+b) = 2(4+b).$$

The first term of terms of the form $d_{2m+c-1} = 4(m-c)$ is when $c = m - (2(n+1) + 1)$. So, when $c = m - (2(n+1) + 1)$, we have

$$\begin{aligned} d_{2m+c-1} &= d_{2m+m-(2(n+1)+1)-1} = d_{3m-2n-4} \\ &= 4(m - (m - (2(n+1) + 1))) \\ &= 4(2(n+1) + 1). \end{aligned}$$

Therefore,

$$\begin{aligned} d_{3m-2n-5} - d_{3m-2n-4} &= 2(4+b) - 4(2(n+1) + 1) \\ &= 2(4 + 4n + 3) - 4(2n + 2 + 1) \\ &= 8 + 8n + 6 - 8n - 8 - 4 \\ &= 2. \end{aligned}$$

From the above arguments, we see that there is at least one entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10). □

Theorem 3.27. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (4, b)$ where b is a positive integer. Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(4+b, 4)}{4} - 1,$$

then d is GMS.

Proof. We first verify the cases for $b = 1, 2, 3$ by straight-forward calculations.

For $b = 1$, we assume that $1 \leq m \leq 4$. We consider each value of m separately, using Figure 3.19 to determine d .

- (i) For $m = 1$, we have $d = (5, 5, 4)$.
- (ii) For $m = 2$, we have $d = (10, 10, 8, 5, 5, 4)$.
- (iii) For $m = 3$, we have $d = (15, 15, 12, 10, 10, 8, 5, 5, 4)$.
- (iv) For $m = 4$, we have $d = (20, 20, 16, 15, 15, 12, 10, 10, 8, 5, 5, 4)$.

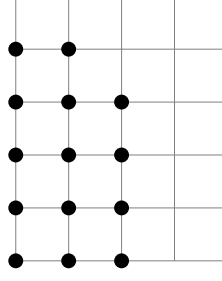


Figure 3.19: Partial intersection of type $p = (3, 2)$ and $q = (4, 1)$.

For $b = 2$, we assume that $1 \leq m \leq 2$. Considering each value of m separately, using Figure 3.20, we determine d as follows.

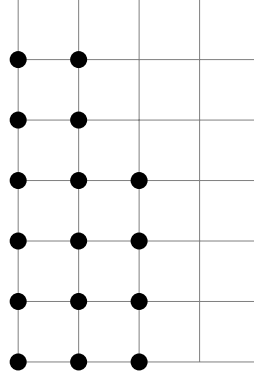


Figure 3.20: Partial intersection of type $p = (3, 2)$ and $q = (4, 2)$.

- (i) For $m = 1$, we have $d = (6, 6, 4)$,
- (ii) For $m = 2$, we have $d = (12, 12, 8, 6, 6, 4)$.

For $b = 3$, we assume that $1 \leq m \leq 6$. Using Figure 3.21, we consider each value of m separately and determine d as follows.

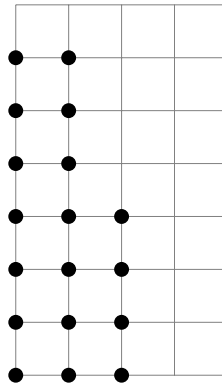


Figure 3.21: Partial intersection of type $p = (3, 2)$ and $q = (4, 3)$.

- (i) For $m = 1$, we have $d = (7, 7, 4)$.
- (ii) For $m = 2$, we have $d = (14, 14, 8, 7, 7, 4)$.
- (iii) For $m = 3$, we have $d = (21, 21, 14, 14, 12, 8, 7, 7, 4)$.
- (iv) For $m = 4$, we have $d = (28, 28, 21, 21, 16, 14, 14, 12, 8, 7, 7, 4)$.
- (v) For $m = 5$, we have $d = (35, 35, 28, 28, 21, 21, 20, 16, 14, 14, 12, 8, 7, 7, 4)$.
- (vi) For $m = 6$, we have $d = (42, 42, 35, 35, 28, 28, 24, 21, 21, 20, 16, 14, 14, 12, 8, 7, 7, 4)$.

The cases $b \geq 4$ follow from Propositions 3.24, 3.25, and 3.26.

□

We now consider the family of partial intersections of type $p = (3, 2)$ and $q = (2\ell, b)$ with $\ell = 3$. This family is handled via a series of propositions. Figure 3.22 depicts the partial intersection $\mathbb{X}$ of type $p = (3, 2)$ and $q = (6, b)$.

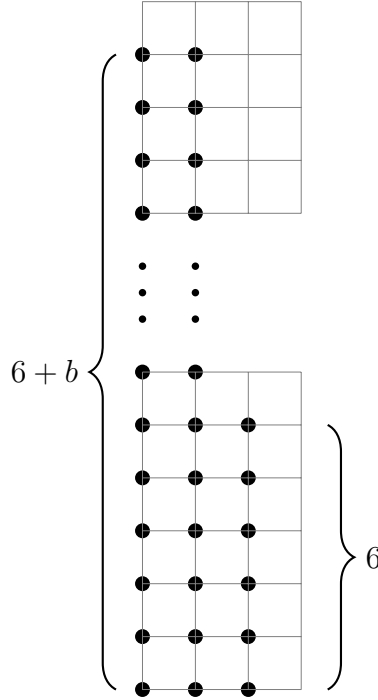


Figure 3.22: Partial intersection of type $p = (3, 2)$ and $q = (6, b)$.

Proposition 3.28. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (6, b)$ where $b = 6n$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$*

supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If

$$1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(6+6n, 6)}{6} - 1 = n$. We show that the reduction vector $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$ is GMS. Using Figure 3.22, we see that d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = 0, \dots, m-1. \end{cases}$$

Observe that when checking the GMS criterion, the only zero entries of $\Delta d'$ of concern come from the terms of the form $d_{2c+1} = d_{2c+2}$. Now, for $0 \leq c \leq m-2$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 12.$$

Certainly, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Therefore, d is GMS (by Proposition 3.10). □

Proposition 3.29. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (6, b)$ where $b = 6n + 1$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1 = \frac{\text{lcm}(6+6n+1, 6)}{6} - 1 = 6(1+n)$. We consider six ranges of values for m and show that each of these yields a GMS reduction vector $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$.

(i) For $1 \leq m \leq n+1$, Figure 3.22 can be used to see that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = 0, \dots, m-1. \end{cases}$$

Observe that when checking the GMS criterion, the only zero entries of $\Delta d'$ of concern come from the terms of the form $d_{2c+1} = d_{2c+2}$. Now, for $0 \leq c \leq m-2$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 13.$$

Thus, between any two zero entries of $\Delta d'$, there is an entry strictly greater than 1. We conclude that d is GMS (by Proposition 3.10).

(ii) For $1+n < m \leq 2(1+n)$, the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-2; \\ d_{2m+c-1} = 6(m-c), & \text{for } c = 0, \dots, m - ((n+1)+1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = m - (n+1), \dots, m-1. \end{cases}$$

The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(6+b)$ for $0 \leq c \leq m-2$. Now, for $0 \leq c \leq m-3$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 13.$$

In addition, we have

$$d_{2(m-2)+2} = 2(6+b) \text{ and } d_{2(m-2)+3} = d_{2(m-2)+2+1} = d_{2(m-1)+1} = d_{2m-1} = 6m.$$

Therefore, since $m \leq 2(n+1)$, we have

$$\begin{aligned} d_{2(m-2)+2} - d_{2(m-2)+3} &= 2(6+b) - 6m \\ &= 2(6+6n+1) - 6m \\ &= 12 + 12n + 2 - 6m \\ &\geq 12 + 12n + 2 - 6(2n+2) \\ &= 12 + 12n + 2 - 12n - 12 \\ &= 2. \end{aligned}$$

That is, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$.

Hence, we conclude that d is GMS (by Proposition 3.10).

- (iii) For $2(n+1) < m \leq 3(n+1)$, Figure 3.22 can be used to see that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-3; \\ d_{2m+c-3} = 6(m-c), & \text{for } c = 0, \dots, m - (2(n+1) + 1); \\ d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(6+b), & \text{for } c = m-2; \\ d_{2m+c-1} = 6(m-c), & \text{for } c = m-2(n+1), \dots, m - ((n+1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = m - (n+1), \dots, m-1. \end{cases}$$

In the following cases, we look at three situations where $\Delta d'$ has zero entries to verify that d is GMS.

- (a) The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(6+b)$ for $0 \leq c \leq m-3$. Now, for $0 \leq c \leq m-4$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 13.$$

- (b) In addition, we have

$$d_{2(m-3)+2} = 3(6+b) \quad \text{and} \quad d_{2(m-3)+3} = d_{2(m-3)+2+1} = d_{2(m-2)+1} = d_{2m-3} = 6m.$$

Therefore, since $m \leq 3(n+1)$, we have

$$\begin{aligned} d_{2(m-3)+2} - d_{2(m-3)+3} &= 3(6+b) - 6m \\ &= 3(6+6n+1) - 6m \\ &= 18 + 18n + 3 - 6m \\ &\geq 18 + 18n + 3 - 6(3n+3) \\ &= 18 + 18n + 3 - 18n - 18 \\ &= 3. \end{aligned}$$

- (c) The last term of terms of the form $d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(6+b)$ is

where $c = m - 2$. So, when $c = m - 2$, we have

$$d_{2m-2n+c-2} = d_{2m-2n+m-2-2} = d_{3m-2n-4} = (m - (m - 2))(6 + b) = 2(6 + b).$$

The first term of terms of the form $d_{2m+c-1} = 6(m - c)$ is when $c = m - 2(n + 1)$. So, when $c = m - 2(n + 1)$, we calculate

$$d_{2m+c-1} = d_{2m+m-2(n+1)-1} = d_{3m-2n-3} = 6(m - (m - 2(n + 1))) = 12(n + 1).$$

Therefore,

$$\begin{aligned} d_{3m-2n-4} - d_{3m-2n-3} &= 2(6 + b) - 12(n + 1) \\ &= 2(6 + 6n + 1) - 12(n + 1) \\ &= 12 + 12n + 2 - 12n - 12 \\ &= 2. \end{aligned}$$

That is, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

(iv) For $3(n + 1) < m \leq 4(n + 1)$, the reduction vector d satisfies

$$\left\{ \begin{array}{ll} d_{2c+1} = d_{2c+2} = (m - c)(6 + b), & \text{for } c = 0, \dots, m - 4; \\ d_{2m+c-5} = 6(m - c), & \text{for } c = 0, \dots, m - (3(n + 1) + 1); \\ d_{2m-3n+c-5} = d_{2m-3n+c-4} = (m - c)(6 + b), & \text{for } c = m - 3; \\ d_{2m+c-3} = 6(m - c), & \text{for } c = m - 3(n + 1), \dots, m - (2(n + 1) + 1); \\ d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m - c)(6 + b), & \text{for } c = m - 2; \\ d_{2m+c-1} = 6(m - c), & \text{for } c = m - 2(n + 1), \dots, m - ((n + 1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(6 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = (m - c)(6 + b), & \text{for } c = m - (n + 1), \dots, m - 1. \end{array} \right.$$

In the following cases, we consider four situations where $\Delta d'$ has zero entries.

(a) The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m - c)(6 + b)$ for $0 \leq c \leq m - 4$. Now, for $0 \leq c \leq m - 5$, we have $d_{2c+2} = (m - c)(6 + b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m - (c + 1))(6 + b)$. So,

$$d_{2c+2} - d_{2c+3} = (m - c)(6 + b) - (m - (c + 1))(6 + b) = 6 + b \geq 13.$$

(b) In addition, we have

$$d_{2(m-4)+2} = 4(6+b) \quad \text{and} \quad d_{2(m-4)+3} = d_{2(m-4)+2+1} = d_{2(m-3)+1} = d_{2m-5} = 6m.$$

Therefore, since $m \leq 4(n+1)$, we have

$$\begin{aligned} d_{2(m-4)+2} - d_{2(m-4)+3} &= 4(6+b) - 6m \\ &= 4(6+6n+1) - 6m \\ &= 24 + 24n + 4 - 6m \\ &\geq 24 + 24n + 4 - 6(4n+4) \\ &= 24 + 24n + 4 - 24n - 24 \\ &= 4. \end{aligned}$$

(c) The last term of terms of the form $d_{2m-3n+c-5} = d_{2m-3n+c-4} = (m-c)(6+b)$ is where $c = m-3$. So, for $c = m-3$, we have

$$d_{2m-3n+c-4} = d_{2m-3n+m-3-4} = d_{3m-3n-7} = [m - (m-3)](6+b) = 3(6+b).$$

The first term of terms of the form $d_{2m+c-3} = 6(m-c)$ is when $c = m-3(n+1)$. So, when $c = m-3(n+1)$, we calculate

$$d_{2m+c-3} = d_{2m+m-3(n+1)-3} = d_{3m-3n-6} = 6(m - (m-3(n+1))) = 18(n+1).$$

Therefore,

$$\begin{aligned} d_{3m-3n-7} - d_{3m-3n-6} &= 3(6+b) - 18(n+1) \\ &= 3(6+6n+1) - 18(n+1) \\ &= 18 + 18n + 3 - 18n - 18 \\ &= 3. \end{aligned}$$

(d) The last term of terms of the form $d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(6+b)$ is when $c = m-2$, and the first term of terms of the form $d_{2m+c-1} = 6(m-c)$ is when $c = m-2(n+1)$. So, we will have the same result as in case (c) of (iii) of Proposition 3.29.

We conclude that there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

- (v) For $4(n+1) < m \leq 5(n+1)$, Figure 3.22 can be used to see that the reduction vector d satisfies

$$\left\{ \begin{array}{ll} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-5; \\ d_{2m+c-7} = 6(m-c), & \text{for } c = 0, \dots, m - (4(n+1) + 1); \\ d_{2m-4n+c-7} = d_{2m-4n+c-6} = (m-c)(6+b), & \text{for } c = m-4; \\ d_{2m+c-5} = 6(m-c), & \text{for } c = m-4(n+1), \dots, m - (3(n+1) + 1); \\ d_{2m-3n+c-5} = d_{2m-3n+c-4} = (m-c)(6+b), & \text{for } c = m-3; \\ d_{2m+c-3} = 6(m-c), & \text{for } c = m-3(n+1), \dots, m - (2(n+1) + 1); \\ d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(6+b), & \text{for } c = m-2; \\ d_{2m+c-1} = 6(m-c), & \text{for } c = m-2(n+1), \dots, m - ((n+1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = m - (n+1), \dots, m-1. \end{array} \right.$$

We need to consider five cases involving zero entries of $\Delta d'$.

- (a) The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(6+b)$ for $0 \leq c \leq m-5$. Now, for $0 \leq c \leq m-6$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 13.$$

- (b) In addition, we have

$$d_{2(m-5)+2} = [m - (m-5)](6+b) = 5(6+b) \text{ and}$$

$$d_{2(m-5)+3} = d_{2(m-5)+2+1} = d_{2(m-4)+1} = d_{2m-7} = 6m.$$

Therefore, since $m \leq 5(n+1)$, we have

$$\begin{aligned} d_{2(m-5)+2} - d_{2(m-5)+3} &= 5(6+b) - 6m \\ &= 5(6+6n+1) - 6m \\ &= 30 + 30n + 5 - 6m \\ &\geq 30 + 30n + 5 - 6(5n+5) \\ &= 30 + 30n + 5 - 30n - 30 \\ &= 5. \end{aligned}$$

- (c) The last term of terms of the form $d_{2m-4n+c-7} = d_{2m-4n+c-6} = (m-c)(6+b)$ is where $c = m - 4$. So, when $c = m - 4$, we see that

$$d_{2m-4n+c-6} = d_{2m-4n+m-4-6} = d_{3m-4n-10} = (m - (m - 4))(6 + b) = 4(6 + b).$$

The first term of terms of the form $d_{2m+c-5} = 6(m-c)$ is when $c = m - 4(n+1)$. So, when $c = m - 4(n+1)$, we calculate

$$d_{2m+c-5} = d_{2m+m-4(n+1)-5} = d_{3m-4n-9} = 6(m - (m - 4(n+1))) = 24(n+1).$$

Therefore,

$$\begin{aligned} d_{3m-4n-10} - d_{3m-4n-9} &= 4(6+b) - 24(n+1) \\ &= 4(6+6n+1) - 24(n+1) \\ &= 24 + 24n + 4 - 24n - 24 \\ &= 4. \end{aligned}$$

- (d) The last term of terms of the form $d_{2m-3n+c-5} = d_{2m-3n+c-4} = (m-c)(6+b)$ is where $c = m - 3$, and the first term of terms of the form $d_{2m+c-3} = 6(m-c)$ is when $c = m - 3(n+1)$. So, we will have the same result as in case (c) of (iv).
- (e) The last term of terms of the form $d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(6+b)$ is when $c = m - 2$, and the first term of terms of the form $d_{2m+c-1} = 6(m-c)$ is when $c = m - 2(n+1)$. So, we will have the same result as in case (c) of (iii).

The above arguments (a)-(e) show that there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

- (vi) For $5(n+1) < m \leq 6(n+1)$, Figure 3.22 can be used to see that the reduction vector

d satisfies

$$\left\{ \begin{array}{ll} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-6; \\ d_{2m+c-9} = 6(m-c), & \text{for } c = 0, \dots, m - (5(n+1) + 1); \\ d_{2m-5n+c-9} = d_{2m-5n+c-8} = (m-c)(6+b), & \text{for } c = m-5; \\ d_{2m+c-7} = 6(m-c), & \text{for } c = m-5(n+1), \dots, m - (4(n+1) + 1); \\ d_{2m-4n+c-7} = d_{2m-4n+c-6} = (m-c)(6+b), & \text{for } c = m-4; \\ d_{2m+c-5} = 6(m-c), & \text{for } c = m-4(n+1), \dots, m - (3(n+1) + 1); \\ d_{2m-3n+c-5} = d_{2m-3n+c-4} = (m-c)(6+b), & \text{for } c = m-3; \\ d_{2m+c-3} = 6(m-c), & \text{for } c = m-3(n+1), \dots, m - (2(n+1) + 1); \\ d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(6+b), & \text{for } c = m-2; \\ d_{2m+c-1} = 6(m-c), & \text{for } c = m-2(n+1), \dots, m - ((n+1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = m - (n+1), \dots, m-1. \end{array} \right.$$

In the following cases, we look at the six solutions involving zero entries of $\Delta d'$.

- (a) The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(6+b)$ for $0 \leq c \leq m-6$. Now, for $0 \leq c \leq m-7$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 13.$$

- (b) In addition, we have

$$d_{2(m-6)+2} = (m - (m-6))(6+b) = 6(6+b) \text{ and}$$

$$d_{2(m-6)+3} = d_{2(m-6)+2+1} = d_{2(m-5)+1} = d_{2m-9} = 6m.$$

Therefore, since $m \leq 6(n+1)$, we have

$$\begin{aligned}
d_{2(m-6)+2} - d_{2(m-5)+1} &= 6(6+b) - 6m \\
&= 6(6+6n+1) - 6m \\
&= 36 + 36n + 6 - 6m \\
&\geq 36 + 36n + 6 - 6(6n+6) \\
&= 36 + 36n + 6 - 36n - 36 \\
&= 6.
\end{aligned}$$

- (c) The last term of terms of the form $d_{2m-5n+c-9} = d_{2m-5n+c-8} = (m-c)(6+b)$ is where $c = m-5$. So, when $c = m-5$, we have

$$d_{2m-5n+c-8} = d_{2m-5n+m-5-8} = d_{3m-5n-13} = (m-(m-5))(6+b) = 5(6+b).$$

The first term of terms of the form $d_{2m+c-7} = 6(m-c)$ is when $c = m-5(n+1)$. So, when $c = m-5(n+1)$, we find

$$d_{2m+c-7} = d_{2m+m-5(n+1)-7} = d_{3m-5n-12} = 6(m-(m-5(n+1))) = 30(n+1).$$

Therefore,

$$\begin{aligned}
d_{3m-5n-13} - d_{3m-5n-12} &= 5(6+b) - 30(n+1) \\
&= 5(6+6n+1) - 30(n+1) \\
&= 30 + 30n + 5 - 30n - 30 \\
&= 5.
\end{aligned}$$

- (d) The last term of terms of the form $d_{2m-4n+c-7} = d_{2m-4n+c-6} = (m-c)(6+b)$ is where $c = m-4$, and the first term of terms of the form $d_{2m+c-5} = 6(m-c)$ is when $c = m-4(n+1)$. So, we have the same result as in case (c) of (v).
- (e) The last term of terms of the form $d_{2m-3n+c-5} = d_{2m-3n+c-4} = (m-c)(6+b)$ is where $c = m-3$, and the first term of terms of the form $d_{2m+c-3} = 6(m-c)$ is where $c = m-3(n+1)$. So, we will have the same result as in case (c) of (iv).
- (f) The last term of terms of the form $d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(6+b)$ is when $c = m-2$, and the first term of terms of the form $d_{2m+c-1} = 6(m-c)$ is when $c = m-2(n+1)$. So, we will have the same result as in case (c) of (iii).

The above arguments (a)-(f) show that there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

□

Proposition 3.30. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (6, b)$ where $b = 6n + 2$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1 = \frac{\text{lcm}(6+6n+2, 6)}{6} - 1 = 3(1+n)$. We consider three ranges of values for m and show that each of these yields a GMS reduction vector $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$.

(i) For $1 \leq m \leq n+1$, Figure 3.22 can be used to see that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = 0, \dots, m-1. \end{cases}$$

Observe that the reduction vector d is the same as in Case (i) of Proposition 3.29. Hence, d is GMS.

(ii) For $1+n < m \leq 2(1+n)$, the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-2; \\ d_{2m+c-1} = 6(m-c), & \text{for } c = 0, \dots, m - ((n+1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = m - (n+1), \dots, m-1. \end{cases}$$

Observe that the reduction vector d is the same as in Case (ii) of Proposition 3.29. Hence, d is GMS.

(iii) For $2(n+1) < m \leq 3(n+1)$, the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-3; \\ d_{2m+c-3} = 6(m-c), & \text{for } c = 0, \dots, m - (2(n+1) + 1); \\ d_{2m-2n+c-3} = d_{2m-2n+c-2} = (m-c)(6+b), & \text{for } c = m-2; \\ d_{2m+c-1} = 6(m-c), & \text{for } c = m-2(n+1), \dots, m - ((n+1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = m - (n+1), \dots, m-1. \end{cases}$$

Observe that the reduction vector d is the same as in Case (iii) of Proposition 3.29. Hence, d is GMS.

□

Proposition 3.31. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (6, b)$ where $b = 6n + 3$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1 = \frac{\text{lcm}(6+6n+3, 6)}{6} - 1 = 2(n+1)$. We consider two ranges of values for m and show that each of these yields a GMS reduction vector $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$.

(i) For $1 \leq m \leq n+1$, Figure 3.22 can be used to see that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = 0, \dots, m-1. \end{cases}$$

Observe that the reduction vector d is the same as in Case (i) of Proposition 3.29. Hence, d is GMS.

(ii) For $1 + n < m \leq 2(n + 1)$, the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(6 + b), & \text{for } c = 0, \dots, m - 2; \\ d_{2m+c-1} = 6(m - c), & \text{for } c = 0, \dots, m - ((n + 1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(6 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = 6(m - c), & \text{for } c = m - (n + 1), \dots, m - 1. \end{cases}$$

Observe that the reduction vector d is the same as in Case (ii) of Proposition 3.29. Hence, d is GMS.

□

Proposition 3.32. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (6, b)$ where $b = 6n + 4$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(6 + b, 6)}{6} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1 = \frac{\text{lcm}(6+6n+4, 6)}{6} - 1 = 3(n + 1) + 1$. We consider three ranges of values for m and show that each of these yields a GMS reduction vector $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$.

(i) For $1 \leq m \leq n + 1$, Figure 3.22 can be used to calculate that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(6 + b), & \text{for } c = 0, \dots, m - 1; \\ d_{2m+c+1} = 6(m - c), & \text{for } c = 0, \dots, m - 1. \end{cases}$$

Observe that the reduction vector d is the same as in Case (i) of Proposition 3.29. Hence, d is GMS.

(ii) For $1 + n < m \leq 2(1 + n) + 1$, the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m - c)(6 + b), & \text{for } c = 0, \dots, m - 2; \\ d_{2m+c-1} = 6(m - c), & \text{for } c = 0, \dots, m - ((n + 1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(6 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = 6(m - c), & \text{for } c = m - (n + 1), \dots, m - 1. \end{cases}$$

The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m - c)(6 + b)$ for $0 \leq c \leq m - 2$. Now, for $0 \leq c \leq m - 3$, we have $d_{2c+2} = (m - c)(6 + b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m - (c + 1))(6 + b)$. So,

$$d_{2c+2} - d_{2c+3} = (m - c)(6 + b) - (m - (c + 1))(6 + b) = 6 + b \geq 16.$$

In addition, we have

$$d_{2(m-2)+2} = (m - (m - 2))(6 + b) = 2(6 + b) \text{ and}$$

$$d_{2(m-2)+3} = d_{2(m-2)+2+1} = d_{2(m-1)+1} = d_{2m-1} = 6m.$$

Therefore, since $m \leq 2(n + 1) + 1$, we have

$$\begin{aligned} d_{2(m-2)+2} - d_{2(m-2)+3} &= 2(6 + b) - 6m \\ &= 2(6 + 6n + 4) - 6m \\ &= 12 + 12n + 8 - 6m \\ &\geq 12 + 12n + 8 - 6(2(n + 1) + 1) \\ &= 12 + 12n + 8 - 6(2n + 2 + 1) \\ &= 12 + 12n + 8 - 12n - 12 - 6 \\ &= 2. \end{aligned}$$

That is, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

- (iii) For $2(n + 1) + 1 < m \leq 3(n + 1) + 1$, Figure 3.22 can be used to see that the reduction vector d satisfies

$$\left\{ \begin{array}{ll} d_{2c+1} = d_{2c+2} = (m - c)(6 + b), & \text{for } c = 0, \dots, m - 3; \\ d_{2m+c-3} = 6(m - c), & \text{for } c = 0, \dots, m - (2(n + 1) + 2); \\ d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m - c)(6 + b), & \text{for } c = m - 2; \\ d_{2m+c-1} = 6(m - c), & \text{for } c = m - (2(n + 1) + 1), \dots, m - ((n + 1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(6 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = 6(m - c), & \text{for } c = m - (n + 1), \dots, m - 1. \end{array} \right.$$

In checking the criteria for GMS, we consider three situations of 0-entries in $\Delta d'$.

- (a) The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(6+b)$ for $0 \leq c \leq m-3$. Now, for $0 \leq c \leq m-4$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 16.$$

- (b) In addition, we have

$$d_{2(m-3)+2} = (m-(m-3))(6+b) = 3(6+b) \text{ and}$$

$$d_{2(m-3)+3} = d_{2(m-3)+2+1} = d_{2(m-2)+1} = d_{2m-3} = 6m.$$

Therefore, since $m \leq 3(n+1) + 1$, we have

$$\begin{aligned} d_{2(m-3)+2} - d_{2(m-3)+3} &= 3(6+b) - 6m \\ &= 3(6+6n+4) - 6m \\ &= 18+18n+12-6m \\ &\geq 18+18n+12-6(3(n+1)+1) \\ &= 18+18n+12-6(3n+3+1) \\ &= 18+18n+12-18n-18-6 \\ &= 6. \end{aligned}$$

- (c) Finally, the last term of terms of the form $d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(6+b)$ is where $c = m-2$. So, when $c = m-2$, we have

$$d_{2m-2n+c-3} = d_{2m-2n+m-2-3} = d_{3m-2n-5} = (m-(m-2))(6+b) = 2(6+b).$$

The first term of terms of the form d_{2m+c-1} is when $c = m - (2(n+1) + 1)$. So, when $c = m - (2(n+1) + 1)$, we calculate

$$d_{2m+c-1} = d_{2m+m-(2(n+1)+1)-1} = d_{3m-2n-4} = 6(m-(m-(2(n+1)+1))) = 6(2(n+1)+1).$$

Therefore,

$$\begin{aligned}
d_{3m-2n-5} - d_{3m-2n-4} &= 2(6+b) - 6(2(n+1)+1) \\
&= 2(6+6n+4) - 6(2n+2+1) \\
&= 12+12n+8-12n-12-6 \\
&= 2.
\end{aligned}$$

That is, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

□

Proposition 3.33. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (6, b)$ where $b = 6n + 5$ for some positive integer n . Let $\mathbb{Y}$ be a homogeneous fat point scheme $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1,$$

then d is GMS.

Proof. By assumption, $1 \leq m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1 = \frac{\text{lcm}(6+6n+5, 6)}{6} - 1 = 6(n+1) + 4$. We consider six ranges of values for m and show that each of these yields a GMS reduction vector $d = (d_1, \dots, d_{2m}, d_{2m+1}, \dots, d_{3m})$.

(i) For $1 \leq m \leq n+1$, Figure 3.22 can be used to see that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = 0, \dots, m-1. \end{cases}$$

Observe that the reduction vector d is the same as in Case (i) of Proposition 3.29. Hence, d is GMS.

(ii) For $1+n < m \leq 2(1+n)+1$, Figure 3.22 we observe that the reduction vector d

satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-2; \\ d_{2m+c-1} = 6(m-c), & \text{for } c = 0, \dots, m - ((n+1)+1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = m - (n+1), \dots, m-1. \end{cases}$$

Observe that the reduction vector d is the same as in Case (ii) of Proposition 3.32. Hence, d is GMS.

- (iii) For $2(n+1)+1 < m \leq 3(n+1)+2$, Figure 3.22 demonstrates that the reduction vector d satisfies

$$\begin{cases} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-3; \\ d_{2m+c-3} = 6(m-c), & \text{for } c = 0, \dots, m - (2(n+1)+2); \\ d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(6+b), & \text{for } c = m-2; \\ d_{2m+c-1} = 6(m-c), & \text{for } c = m - (2(n+1)+1), \dots, m - ((n+1)+1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = m - (n+1), \dots, m-1. \end{cases}$$

In the following cases, we consider three solutions involving zero entries of $\Delta d'$.

- (a) The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(6+b)$ for $0 \leq c \leq m-3$. Now, for $0 \leq c \leq m-4$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 17.$$

- (b) In addition, we have

$$d_{2(m-3)+2} = (m - (m-3))(6+b) = 3(6+b) \text{ and}$$

$$d_{2(m-3)+3} = d_{2(m-3)+2+1} = d_{2(m-2)+1} = d_{2m-3} = 6m.$$

Therefore, since $m \leq 3(n+1) + 2$, we have

$$\begin{aligned}
d_{2(m-3)+2} - d_{2(m-3)+3} &= 3(6+b) - 6m \\
&= 3(6+6n+5) - 6m \\
&= 18 + 18n + 15 - 6m \\
&\geq 18 + 18n + 15 - 6(3(n+1) + 2) \\
&= 18 + 18n + 15 - 6(3n+3+2) \\
&= 18 + 18n + 15 - 18n - 18 - 12 \\
&= 3.
\end{aligned}$$

(c) Finally, the last term of terms of the form $d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(6+b)$ is where $c = m - 2$. So, when $c = m - 2$, we have

$$d_{2m-2n+c-3} = d_{2m-2n+m-2-3} = d_{3m-2n-5} = (m - (m-2))(6+b) = 2(6+b).$$

The first term of terms of the form d_{2m+c-1} is when $c = m - (2(n+1) + 1)$. So, when $c = m - (2(n+1) + 1)$, we calculate

$$d_{2m+c-1} = d_{2m+m-(2(n+1)+1)-1} = d_{3m-2n-4} = 6(m - (m - (2(n+1)+1))) = 6(2(n+1)+1).$$

Therefore,

$$\begin{aligned}
d_{3m-2n-5} - d_{3m-2n-4} &= 2(6+b) - 6(2(n+1) + 1) \\
&= 2(6+6n+5) - 6(2n+2+1) \\
&= 12 + 12n + 10 - 12n - 12 - 6 \\
&= 4.
\end{aligned}$$

That is, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

(iv) For $3(n+1) + 2 < m \leq 4(n+1) + 3$, the reduction vector d satisfies

$$\left\{ \begin{array}{ll} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-4; \\ d_{2m+c-5} = 6(m-c), & \text{for } c = 0, \dots, m-(3(n+1)+3); \\ d_{2m-3n+c-7} = d_{2m-3n+c-6} = (m-c)(6+b), & \text{for } c = m-3; \\ d_{2m+c-3} = 6(m-c), & \text{for } c = m-(3(n+1)+2), \dots, m-(2(n+1)+2); \\ d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(6+b), & \text{for } c = m-2; \\ d_{2m+c-1} = 6(m-c), & \text{for } c = m-(2(n+1)+1), \dots, m-((n+1)+1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = (m-c)(6+b), & \text{for } c = m-(n+1), \dots, m-1. \end{array} \right.$$

We consider four situations where $\Delta d'$ has zero entries while verifying the GMS condition.

- (a) The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(6+b)$ for $0 \leq c \leq m-4$. Now, for $0 \leq c \leq m-5$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 17.$$

- (b) In addition, we have

$$d_{2(m-4)+2} = (m-(m-4))(6+b) = 4(6+b) \text{ and}$$

$$d_{2(m-4)+3} = d_{2(m-4)+2+1} = d_{2(m-3)+1} = d_{2m-5} = 6m.$$

Therefore, since $m \leq 4(n+1)+3$, we have

$$\begin{aligned} d_{2(m-4)+2} - d_{2(m-4)+3} &= 4(6+b) - 6m \\ &= 4(6+6n+5) - 6m \\ &= 24 + 24n + 20 - 6m \\ &\geq 24 + 24n + 20 - 6(4(n+1)+3) \\ &= 24 + 24n + 20 - 6(4n+4+3) \\ &= 24 + 24n + 20 - 24n - 24 - 18 \\ &= 2. \end{aligned}$$

- (c) The last term of terms of the form $d_{2m-3n+c-7} = d_{2m-3n+c-6} = (m-c)(6+b)$ is

where $c = m - 3$. So, when $c = m - 3$, we have

$$d_{2m-3n+c-6} = d_{2m-3n+m-3-6} = d_{3m-3n-9} = (m - (m - 3))(6 + b) = 3(6 + b).$$

The first term of terms of the form $d_{2m+c-3} = 6(m-c)$ is when $c = m - (3(n+1)+2)$. So, when $c = m - (3(n+1)+2)$, we calculate

$$\begin{aligned} d_{2m+c-3} &= d_{2m+m-(3(n+1)+2)-3} = d_{3m-3n-8} \\ &= 6(m - (m - (3(n+1)+2))) \\ &= 6(3(n+1)+2). \end{aligned}$$

Therefore,

$$\begin{aligned} d_{3m-3n-9} - d_{3m-3n-8} &= 3(6 + b) - 6(3(n+1)+2) \\ &= 3(6 + 6n + 5) - 6(3n + 3 + 2) \\ &= 18 + 18n + 15 - 18n - 18 - 12 \\ &= 3. \end{aligned}$$

- (d) Finally, the last term of terms of the form $d_{2m-2n+c-4} = d_{2m-3n+c-3} = (m-c)(6+b)$ is where $c = m - 2$, and the first term of terms of the form $d_{2m+c-1} = 6(m-c)$ is when $c = m - (2(n+1)+1)$. The same work as in case (c) of (iii) of Proposition 3.32 shows that d is GMS.

Therefore, there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

(v) For $4(n+1) + 3 < m \leq 5(n+1) + 4$, the reduction vector d satisfies

$$\left\{ \begin{array}{ll} d_{2c+1} = d_{2c+2} = (m-c)(6+b), & \text{for } c = 0, \dots, m-5; \\ d_{2m+c-7} = 6(m-c), & \text{for } c = 0, \dots, m - (4(n+1) + 4); \\ d_{2m-4n+c-10} = d_{2m-4n+c-9} = (m-c)(6+b), & \text{for } c = m-4; \\ d_{2m+c-5} = 6(m-c), & \text{for } c = m - (4(n+1) + 3), \dots, m - (3(n+1) + 3); \\ d_{2m-3n+c-7} = d_{2m-3n+c-6} = (m-c)(6+b), & \text{for } c = m-3; \\ d_{2m+c-3} = 6(m-c), & \text{for } c = m - (3(n+1) + 2), \dots, m - (2(n+1) + 2); \\ d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(6+b), & \text{for } c = m-2; \\ d_{2m+c-1} = (m-c)(6+b), & \text{for } c = m - (2(n+1) + 1), \dots, m - ((n+1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m-c)(6+b), & \text{for } c = m-1; \\ d_{2m+c+1} = 6(m-c), & \text{for } c = m - (n+1), \dots, m-1. \end{array} \right.$$

We look at five cases yielding zero entries of $\Delta d'$.

- (a) The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m-c)(6+b)$ for $0 \leq c \leq m-5$. Now, for $0 \leq c \leq m-6$, we have $d_{2c+2} = (m-c)(6+b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m-(c+1))(6+b)$. So,

$$d_{2c+2} - d_{2c+3} = (m-c)(6+b) - (m-(c+1))(6+b) = 6+b \geq 17.$$

- (b) The last term of the terms of the form $d_{2m+c-7} = 6(m-c)$ is when $c = m - (4n+8)$. So, when $c = m - (4n+8)$,

$$d_{2m+c-7} = d_{2m+m-(4n+8)-7} = d_{3m-4n-15} = 6(m - (m - (4n+8))).$$

The first term of $d_{2m-4n+c-10} = d_{2m-4n+c-9} = (m-c)(6+b)$ is when $c = m-4$. So, when $c = m-4$, we will have

$$d_{2m-4n+c-10} = d_{2m-4n+(m-4)-10} = d_{3m-4n-14} = (m - (m-4))(6+b).$$

Therefore,

$$\begin{aligned}
d_{2m+c-7} - d_{2m-4n+c-10} &= d_{2m+m-(4n+8)-7} - d_{2m-4n+(m-4)-10} \\
&= d_{3m-4n-15} - d_{3m-4n-14} \\
&= 6(m - (m - (4n + 8))) - (m - (m - 4))(6 + b) \\
&= 6(4n + 8) - 4(6 + b) \\
&= 24n + 48 - 4(6 + 6n + 5) \\
&= 24n + 48 - 24 - 24n - 20 \\
&= 4.
\end{aligned}$$

- (c) The last term of terms of the form $d_{2m-4n+c-10} = d_{2m-4n+c-9} = (m - c)(6 + b)$ is where $c = m - 4$. So, when $c = m - 4$, we calculate

$$d_{2m-4n+c-9} = d_{2m-4n+m-4-9} = d_{3m-4n-13} = (m - (m - 4))(6 + b) = 4(6 + b).$$

The first term of terms of the form $d_{2m+c-5} = 6(m - c)$ is when $c = m - (4(n + 1) + 3)$. So, when $c = m - (4(n + 1) + 3)$, we have

$$\begin{aligned}
d_{2m+c-5} &= d_{2m+m-(4(n+1)+3)-5} = d_{3m-4n-12} \\
&= 6(m - (m - (4(n + 1) + 3))) \\
&= 6(4(n + 1) + 3) \\
&= 24n + 42.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d_{3m-4n-13} - d_{3m-4n-12} &= 4(6 + b) - (24n + 42) \\
&= 4(6 + 6n + 5) - (24n + 42) \\
&= 24 + 24n + 20 - 24n - 42 \\
&= 2.
\end{aligned}$$

- (d) The last term of terms of the form $d_{2m-3n+c-7} = d_{2m-3n+c-6} = (m - c)(6 + b)$ is where $c = m - 3$, and the first term of terms of the form $d_{2m+c-3} = 6(m - c)$ is when $c = m - (3(n + 1) + 2)$. So, we will have the same calculations as in case (c) of (iv).
- (e) The last term of terms of the form $d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m - c)(6 + b)$ is

where $c = m - 2$, and the first term of terms of the form $d_{2m+c-1} = 6(m - c)$ is when $c = m - (2(n + 1) + 1)$. We have the same calculations as in case (c) of (iii) of Proposition 3.32.

The above arguments show that there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

(vi) For $5(n + 1) + 4 < m \leq 6(n + 1) + 4$, Figure 3.22 can be used to see that the reduction vector d satisfies

$$\left\{ \begin{array}{ll} d_{2c+1} = d_{2c+2} = (m - c)(6 + b), & \text{for } c = 0, \dots, m - 6; \\ d_{2m+c-9} = 6(m - c), & \text{for } c = 0, \dots, m - (5(n + 1) + 5); \\ d_{2m-5n+c-13} = d_{2m-5n+c-12} = (m - c)(6 + b), & \text{for } c = m - 5; \\ d_{2m+c-7} = 6(m - c), & \text{for } c = m - (5(n + 1) + 4), \dots, m - (4(n + 1) + 4); \\ d_{2m-4n+c-10} = d_{2m-4n+c-9} = (m - c)(6 + b), & \text{for } c = m - 4; \\ d_{2m+c-5} = 6(m - c), & \text{for } c = m - (4(n + 1) + 3), \dots, m - (3(n + 1) + 3); \\ d_{2m-3n+c-7} = d_{2m-3n+c-6} = (m - c)(6 + b), & \text{for } c = m - 3; \\ d_{2m+c-3} = (m - c)(6 + b), & \text{for } c = m - (3(n + 1) + 2), \dots, m - (2(n + 1) + 2); \\ d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m - c)(6 + b), & \text{for } c = m - 2; \\ d_{2m+c-1} = 6(m - c), & \text{for } c = m - (2(n + 1) + 1), \dots, m - ((n + 1) + 1); \\ d_{2m-n+c-1} = d_{2m-n+c} = (m - c)(6 + b), & \text{for } c = m - 1; \\ d_{2m+c+1} = 6(m - c), & \text{for } c = m - (n + 1), \dots, m - 1. \end{array} \right.$$

We consider six situations that involve zero entries of $\Delta d'$.

(a) The reduction vector d has terms of the form $d_{2c+1} = d_{2c+2} = (m - c)(6 + b)$ for $0 \leq c \leq m - 6$. Now, for $0 \leq c \leq m - 7$, we have $d_{2c+2} = (m - c)(6 + b)$ and $d_{2c+3} = d_{2(c+1)+1} = (m - (c + 1))(6 + b)$. So,

$$d_{2c+2} - d_{2c+3} = (m - c)(6 + b) - (m - (c + 1))(6 + b) = 6 + b \geq 17.$$

(b) In addition, we have

$$d_{2(m-6)+2} = [m - (m - 6)](6 + b) = 6(6 + b) \text{ and}$$

$$d_{2(m-6)+3} = d_{2(m-6)+2+1} = d_{2(m-5)+1} = d_{2m-9} = 6m.$$

Therefore, since $m \leq 6(n+1) + 4$, we have

$$\begin{aligned}
d_{2(m-6)+2} - d_{2(m-6)+3} &= 6(6+b) - 6m \\
&= 6(6+6n+5) - 6m \\
&= 36 + 36n + 30 - 6m \\
&\geq 36 + 36n + 30 - 6(6(n+1) + 4) \\
&= 36 + 36n + 30 - 6(6n+6+4) \\
&= 36 + 36n + 30 - 36n - 36 - 24 \\
&= 6.
\end{aligned}$$

- (c) The last term of the terms of the form $d_{2m+c-7} = 6(m-c)$ is when $c = m - (4n+8)$, and the first term of $d_{2m-4n+c-10} = d_{2m-4n+c-9} = (m-c)(6+b)$ is when $c = m - 4$. So, we will have the same calculations as in case (b) of (v).
- (d) The last term of terms of the form $d_{2m-4n+c-10} = d_{2m-4n+c-9} = (m-c)(6+b)$ is where $c = m - 4$, and the first term of terms of the form $d_{2m+c-5} = 6(m-c)$ is when $c = m - (4(n+1) + 3)$. So, we will have the same calculations as in case (c) of (v).
- (e) The last term of terms of the form $d_{2m-3n+c-7} = d_{2m-3n+c-6} = (m-c)(6+b)$ is where $c = m - 3$, and the first term of terms of the form $d_{2m+c-3} = 6(m-c)$ is when $c = m - (3(n+1) + 2)$. So, we will have the same calculations as in case (c) of (iv) of Proposition 3.33.
- (f) The last term of terms of the form $d_{2m-2n+c-4} = d_{2m-2n+c-3} = (m-c)(6+b)$ is where $c = m - 2$, and the first term of terms of the form $d_{2m+c-1} = 6(m-c)$ is when $c = m - (2(n+1) + 1)$. So, we will have the same result as in case (c) of (iii) of Proposition 3.32.

Arguments (a)-(f) show that there is an entry strictly greater than 1 between any two zero entries of $\Delta d'$. Hence, we conclude that d is GMS (by Proposition 3.10).

□

Theorem 3.34. *Let $\mathbb{X}$ be a partial intersection of type $p = (3, 2)$ and $q = (6, b)$ where $b \in \mathbb{Z}^+$. Let $\mathbb{Y}$ be a homogeneous fat point schemes $\mathbb{Y} = m\mathbb{X}$ supported on $\mathbb{X}$ where $m \in \mathbb{Z}^+$. Apply the Greedy Algorithm to $\mathbb{Y}$ using only vertical lines to obtain the reduction vector d . If*

$$m \leq \frac{\text{lcm}(6+b, 6)}{6} - 1,$$

then d is *GMS*.

Proof. By applying the greedy algorithm on $\mathbb{Y}$, we have the following cases. Each case yields a reduction vector that is *GMS*.

1. For $b = 1$, we assume that $1 \leq m \leq 6$. Using Figure 3.23, we calculate the following reduction vectors.

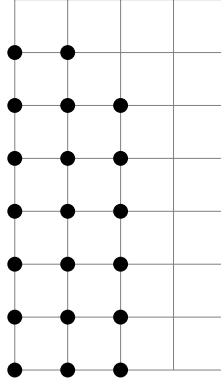


Figure 3.23: Partial intersection of type $p = (3, 2)$ and $q = (6, 1)$.

- (i) For $m = 1$, we have $d = (7, 7, 6)$.
 - (ii) For $m = 2$, we have $d = (14, 14, 12, 7, 7, 6)$.
 - (iii) For $m = 3$, we have $d = (21, 21, 18, 14, 14, 12, 7, 7, 6)$.
 - (iv) For $m = 4$, we have $d = (28, 28, 24, 21, 21, 18, 14, 14, 12, 7, 7, 6)$.
 - (v) For $m = 5$, we have $d = (35, 35, 30, 28, 28, 24, 21, 21, 18, 14, 14, 12, 7, 7, 6)$.
 - (vi) For $m = 6$, we have $d = (42, 42, 36, 35, 35, 30, 28, 28, 24, 21, 21, 18, 14, 14, 12, 7, 7, 6)$.
2. For $b = 2$, we assume that $1 \leq m \leq 3$. We use Figure 3.24 for this case to find the possible reduction vectors.

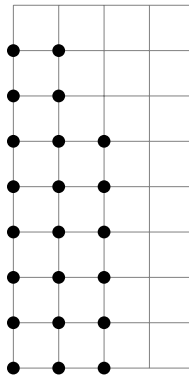


Figure 3.24: Partial intersection of type $p = (3, 2)$ and $q = (6, 2)$.

- (i) For $m = 1$, we have $d = (8, 8, 6)$.
 - (ii) For $m = 2$, we have $d = (16, 16, 12, 8, 8, 6)$.
 - (iii) For $m = 3$, we have $d = (24, 24, 18, 16, 16, 12, 8, 8, 6)$.
3. For $b = 3$, we assume that $1 \leq m \leq 2$ and use Figure 3.25.

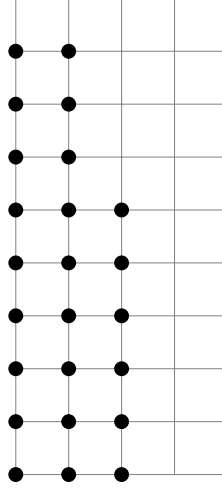


Figure 3.25: Partial intersection of type $p = (3, 2)$ and $q = (6, 3)$.

- (i) For $m = 1$, we have $d = (9, 9, 6)$.
 - (ii) For $m = 2$, we have $d = (18, 18, 12, 9, 9, 6)$.
4. For $b = 4$, we assume that $1 \leq m \leq 4$ and consider Figure 3.26.

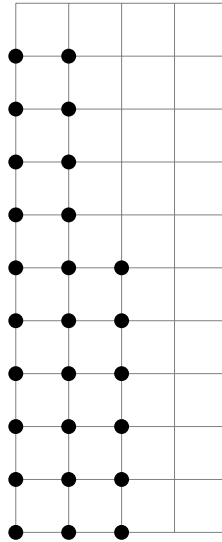


Figure 3.26: Partial intersection of type $p = (3, 2)$ and $q = (6, 4)$.

- (i) For $m = 1$, we have $d = (10, 10, 6)$.
 - (ii) For $m = 2$, we have $d = (20, 20, 12, 10, 10, 6)$.
 - (iii) For $m = 3$, we have $d = (30, 30, 20, 20, 18, 12, 10, 10, 6)$.
 - (iv) For $m = 4$, we have $d = (40, 40, 30, 30, 24, 20, 20, 18, 12, 10, 10, 6)$
5. For $b = 5$, we assume that $1 \leq m \leq 10$. Using Figure 3.27, we assume the necessary reduction vectors as below.

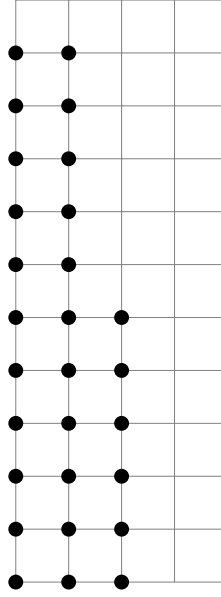


Figure 3.27: Partial intersection of type $p = (3, 2)$ and $q = (6, 5)$.

- (i) For $m = 1$, we have $d = (11, 11, 6)$.
- (ii) For $m = 2$, we have $d = (22, 22, 12, 11, 11, 6)$.
- (iii) For $m = 3$, we have $d = (33, 33, 22, 22, 18, 12, 11, 11, 6)$.
- (iv) For $m = 4$, we have $d = (44, 44, 33, 33, 24, 22, 22, 18, 12, 11, 11, 6)$
- (v) For $m = 5$, we have $d = (55, 55, 44, 44, 33, 33, 30, 24, 22, 22, 18, 12, 11, 11, 6)$
- (vi) For $m = 6$, we have

$$d = (66, 66, 55, 55, 44, 44, 36, 33, 33, 30, 24, 22, 22, 18, 12, 11, 11, 6)$$

- (vii) For $m = 7$, we have

$$d = (77, 77, 66, 66, 55, 55, 44, 44, 42, 36, 33, 33, 30, 24, 22, 22, 18, 12, 11, 11, 6).$$

(viii) For $m = 8$,

$$d = (88, 88, 77, 77, 66, 66, 55, 55, 48, 44, 44, 42, 36, 33, 33, \\ 30, 24, 22, 22, 18, 12, 11, 11, 6).$$

(ix) For $m = 9$,

$$d = (99, 99, 88, 88, 77, 77, 66, 66, 55, 55, 54, 48, \\ 44, 44, 42, 36, 33, 33, 30, 24, 22, 22, 18, 12, 11, 11, 6).$$

(x) For $m = 10$,

$$d = (110, 110, 99, 99, 88, 88, 77, 77, 66, 66, 60, 55, 55, 54, 48, 44, 44, 42, 36, \\ 33, 33, 30, 24, 22, 22, 18, 12, 11, 11, 6).$$

For each of the above cases, we easily see that d is GMS. The cases for $b \geq 5$ follow immediately from Propositions 3.28, 3.29, 3.30, 3.31, 3.32, and 3.33.

□

Bibliography

- [1] Anna Bigatti, Anthony V Geramita, and Juan C Migliore. Geometric Consequences of Extremal Behavior in a Theorem of Macaulay. *Transactions of the American Mathematical Society*, 346(1):203–235, 1994.
- [2] Giulio Campanella. Standard Bases of Perfect Homogeneous Polynomial Ideals of Height 2. *Journal of Algebra*, 101(1):47–60, 1986.
- [3] Maria Virginia Catalisano. Fat Points on a Conic. *Communications in Algebra*, 19(8):2153–2168, 1991.
- [4] Ciro Ciliberto. Geometric Aspects of Polynomial Interpolation in More Variables and of Waring’s Problem. In *European Congress of Mathematics: Barcelona, July 10–14, 2000, Volume I*, pages 289–316. Springer, 2001.
- [5] Susan Cooper, Brian Harbourne, and Zach Teitler. Combinatorial Bounds on Hilbert Functions of Fat Points in Projective Space. *Journal of Pure and Applied Algebra*, 215(9):2165–2179, 2011.
- [6] Susan M. Cooper and Elena Guardo. Fat Points, Partial Intersections and Hamming Distance. *Journal of Algebra and Its Applications*, 19(04):2050071, 2020.
- [7] E. Graham Evans Jr and Benjamin P Richert. Possible Resolutions for a given Hilbert Function. *Communications in Algebra*, 30(2):897–906, 2002.
- [8] Anthony V. Geramita, P. Maroscia, and L.G. Roberts. The Hilbert Function of a Reduced k -Algebra. *Journal of the London Mathematical Society*, 2(3):443–452, 1983.
- [9] Anthony V Geramita and Paolo Maroscia. The Ideal of Forms Vanishing at a Finite Set of Points in $\mathbb{P}^n$. *J. Algebra*, 90(2):528–555, 1984.
- [10] Anthony Vito Geramita, Brian Harbourne, and Juan Migliore. Classifying Hilbert Functions of Fat Point Subschemes in $\mathbb{P}^2$. *Collectanea mathematica*, 60:159–192, 2009.

- [11] A.V. Geramita. *Seminar Notes*. Curves Seminar at Queen's University, unpublished.
- [12] A.V. Geramita, J. Migliore, and L. Sabourin. On the First Infinitesimal Neighborhood of a Linear Configuration of Points in $\mathbb{P}^2$. *Journal of Algebra*, 298(2):563–611, 2006.
- [13] Alessandro Gimigliano. On Linear Systems of Plane Curves. *Thesis, Queen's University, Kingston*, 1987.
- [14] Elena Guardo and Brian Harbourne. Resolutions of Ideals of any Six Fat Points in $\mathbb{P}^2$. *Journal of Algebra*, 318(2):619–640, 2007.
- [15] Elena Guardo and Adam Van Tuyl. Powers of Complete Intersections: Graded Betti Numbers and Applications. *Illinois Journal of Mathematics*, 49(1):265–279, 2005.
- [16] Elena Guardo and Adam Van Tuyl. Some Results on Fat Points whose Support is a Complete Intersection Minus a Point. *Projective varieties with unexpected properties*, pages 257–266, 2005.
- [17] Brian Harbourne. The Geometry of Rational Surfaces and Hilbert Functions of Points in the Plane. In *Proceedings of the 1984 Vancouver conference in algebraic geometry*, volume 6, pages 95–111, 1986.
- [18] Brian Harbourne. Rational Surfaces with $k^2 > 0$. *Proceedings of the American Mathematical Society*, 124(3):727–733, 1996.
- [19] Brian Harbourne. Free Resolutions of Fat Point Ideals on $\mathbb{P}^2$. *Journal of Pure and Applied Algebra*, 125(1-3):213–234, 1998.
- [20] André Hirschowitz. Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1989(397):208–213, 1989.
- [21] Martin Kreuzer and Lorenzo Robbiano. *Computational Commutative Algebra*, volume 1. Springer, 2000.
- [22] Joaquim Roé. On the Existence of Plane Curves with Imposed Multiple Points. *Journal of Pure and Applied Algebra*, 156(1):115–126, 2001.