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THE MAXIMAX RESPONSE OF LINEAR, STABLE STRUCTURES

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the University of Manitoba in partial fulfillment of the requirements
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TO THE MEMORY OF

MY FATHER

ABSTRACT

A general theory is developed to estimate the greatest maximum response of linear structures subjected to incompletely described loads. The estimation is established from readily available characteristics of the load and structure. The analysis deals with the simultaneous limitation in both time and frequency which is imposed often in measurements. The accuracy of the method is evaluated and compared with two more established procedures for fairly comprehensive circumstances. One of the methods is least accurate but the ranking of the other appears to depend upon the load information and, amongst all the structural parameters, the damping. Extension of the method to discrete multi degree-of-freedom systems is shown to produce reasonable estimates for both the maximax and maximax mean square response. Reasonable estimates of the largest feasible (maximax) response become increasingly valuable in assessing the "worst" design case. Practical applications are illustrated for both single and two degree-of-freedom structures. A particular example of growing practical importance, the constant tuned vibration neutralizer is investigated in greater detail to determine its optimum parameters.

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NOMENCLATURE

E	analogous to the terminology of signal analysis, energy is the time integral of the square of a function
$E_f(T_f)$	energy of the dynamic loading $f(t)$
E_{f_p}	energy of the load $f_p(t)$
E_g	energy of the response $g(t)$
E_{g_q}	energy of the response $g_q(t)$
E_{g_o}	energy of the optimum response
E_{g_B}	energy of a time and band limited response
E_h	total energy of the impulse response function $h(t)$
E'_h	energy of $h(t)$ in the interval $[T'_h, \infty]$
E'_{h_ϵ}	energy of $h(t)$ in the interval $[0, T'_h]$
E'_H	energy of $H(j\omega)$ in the interval $[-\Omega'_h, \Omega'_h]$
E'_{H_ϵ}	energy of $H(j\omega)$ in the interval $[\Omega'_h, \infty]$
$f(t)$	input dynamic load
$f_p(t)$	load applied upon the p th mass of a discrete multi degree-of-freedom system
$\langle f^2(t) \rangle$	mean square value of $f(t)$
$\langle f_p^2(t) \rangle$	mean square value of $f_p(t)$
$F(j\omega)$	Fourier transform of the load $f(t)$

$g(t)$	response (displacement, velocity, acceleration, jerk)
$g_{qp}(t)$	the response of the q th mass to $f_p(t)$
$g_q(t)$	the response of the q th mass for all applied loads
$g_B(t)$	time and band limited response
$\langle g^2(t) \rangle$	mean square response $g(t)$
$\langle g^2(t) \rangle_0$	maximum mean square response
$\langle g_q^2(t) \rangle$	mean square value of $g_q(t)$
$G(j\omega)$	Fourier transform of $g(t)$
$G^*(j\omega)$	complex conjugate of $G(j\omega)$
$G_B(j\omega)$	frequency limited $G(j\omega)$
$G_B^*(j\omega)$	complex conjugate of $G_B(j\omega)$
$h(t)$	impulse response function of the system
$H(j\omega)$	frequency response function, the Fourier transform of $h(t)$
$H_{qp}(j\omega)$	frequency response function of the q th mass due to $f_p(t)$
j	equal $\sqrt{-1}$
$J_{n+m}(h\zeta)$	spherical Bessel functions of order $n+m$
k	stiffness of a simple oscillator

k_1, k_2, k_3	stiffness of primary, attached secondary and tertiary system respectively
$L(t)$	inverse Fourier transform of $ H(j\omega) ^2$
m	mass of a simple oscillator
m_1, m_2, m_3	mass of primary, attached secondary and tertiary system respectively
$P(t)$	boxcar window in time, equals unity for $t \leq T'_h$ and zero for $t > T'_h$
$P_{n+m}^m(n)$	associated Legendre functions of order m and degree $n+m$
$P_n^0(t)$	associated Legendre functions of zero order and n degree
$Q(t)$	boxcar window in time equals zero for $t \leq T'_h$ and unity for $t > T'_h$
$R_f(0)$	autocorrelation function of the load at zero time delay, $\tau = 0$
$R_{g_0}(0)$	optimum autocorrelation function of the response at zero time delay, $\tau = 0$
$R_{0n}^{(1)}(c, t)$	radial prolate spheroidal function
$R(j\omega)$	analogous to the boxcar window $P(t)$ in the frequency domain
s	$j\omega$

$S(j\omega)$	analogous to the boxcar window $P(t)$ in the frequency domain
$S_f(\omega)$	power spectral density of the load
$S_{g_0}(c, t)$	optimum spectral density of the response
$S_{mL}(h, n)$	angular prolate spheroidal function
$S_{0n}(c, t)$	angular prolate spheroidal function ($m = 0$, $L = n$)
t	time
T	duration
T_f	some instant during the force's application
T_g	duration of the response
T_{g_q}	the duration of the response of the q th mass
T_h	duration of the impulse response function $h(t)$
T'_h	approximation of T_h
T_n	natural period of the oscillator
$\varepsilon(t)$	error due to limiting the bandwidth
ε_h	the ratio between the amplitude of the impulse response function at time T'_h and its maximum value
ε_H	the ratio between the amplitude of $ H(j\omega) $ at Ω'_h and its maximum value

ζ	damping ratio of the oscillator
$\zeta_{11}, \zeta_{22}, \zeta_{23}$	damping ratio of primary, attached secondary and tertiary system respectively
ζ_g	parameter depends upon the characteristics of a particular earthquake load
η	ratio of E_g/E_f , eigenvalue of a homogeneous Weiner-Hopf integral equation of the second kind
η_0	ratio of E_{g_0}/E_f , maximum eigenvalue of a homogeneous Weiner-Hopf integral equation of the second kind
λ	eigenvalue of the prolate spheroidal wave functions of zero order
λ_0	first eigenvalue of the prolate spheroidal wave functions of the zero order
μ	ratio of m_2/m_1
ν	ratio of E_{g_B}/E_f
π	3.141592653589793
σ_0	parameter depends upon the characteristics of a particular earthquake load
$\psi(t)$	prolate spheroidal wave functions of zero order
$\psi_0(t)$	first prolate spheroidal wave functions of zero order

ω_g	parameter depends upon the characteristics of a particular earthquake load
ω_n	undamped natural frequency of the oscillator
$\omega_{11}, \omega_{22}, \omega_{23}$	$\sqrt{k_1}/m_1, \sqrt{k_2}/m_2, \sqrt{k_3}/m_2$ respectively
Ω	bandwidth
Ω_{g_q}	bandwidth of the response of the qth mass
Ω_f	bandwidth of the load
Ω_g	bandwidth of the response
Ω_h	bandwidth of the impulse response function
Ω'_h	approximation for Ω_h
$ $	absolute value
$< >$	mean square value
$\approx$	approximately equal
$\sim$	asymptotically equal
$< \text{or} >$	inequality
$\neq$	unequal
$[]$	square matrix
$\{ \}$	column matrix
$\frac{d}{dt}$ or dot	derivative with respect to time, t
superscript	

CHAPTER 1

INTRODUCTION

The greatest maximum response of a structure is a significant if not the most critical parameter in predicting the damage potential of dynamic loads. If the characteristics of the load and the system are known, the response and hence its maximum amplitude may be determined by using analytical or experimental methods [1 - 16]. However details of the load are not always known and a comprehensive time history of the response is often not required. Such considerations motivated intensive research to determine with sufficient accuracy the greatest maximum (maximax) response from a limited knowledge of the loading and the structure.

An early method was developed originally by Tufts and Shnidman [17] for the analogous matched filter problem of radar theory and the "bang-bang" problem of optimal control. The method utilizes the energy[†] of both the load and structure rather than their detailed time and frequency characteristics. It was extended and applied to a large number of engineering problems in the areas of circuit theory [18,19], linear structures [20-22] and non-linear structures [23-25]. Although the method has attracted considerable attention, its accuracy is generally unsatisfactory. Shinozuka [26-28] and Yang [29] attempted to improve the estimation by using the envelopes of

[†]Analogous to the terminology of signal analysis, energy is the time integral of the square of a function.

the frequency and time characteristics. However these approaches suffer from two practical disadvantages. Firstly, a digital computer is needed and secondly, the envelopes are not usually available. A critical review of these methods will be presented in Chapter 3.

The aim of this work is to develop a practical but accurate technique to determine the maximax response of linear structures. The maximax response is considered to be the greatest maximum response for loads with the same energy. It is the same class of loads considered by Tufts, Shnidman Papoulis and Drenick [17,19,20]. However it is a larger class than that considered by Shinozuka [26] in which the maximax response is defined as the greatest maximum response for a class of loads with the same envelope of the Fourier transform. Such a class of loads can be considered as a subclass of the loads with same load energy because if the envelope of the Fourier transform is known the energy can be evaluated by using Parseval's theorem [36]. However if only the energy is known, the envelope of the Fourier transform can not be determined. A subsidiary but important objective is to obviate the need for a load's time history or explicit calculations of the entire response. Therefore highly variable details of disturbances like earthquakes, blasts from quarrying and sonic booms should not require specification.

The approach, developed in Chapter 2, using a modification of the work of Slepian, Landau and Pollak [30,36],

determines the largest amplitude in a class of response functions whose energies are identical to the maximum produced by a given energy of the loads. Such a class of response functions are both time and frequency limited. Although this simultaneous restriction violates the uncertainty principle [30], it is imposed often in measurements due to time and instrument limitations. Therefore the method is consistent with many practical applications.

The accuracy of the technique is evaluated in Chapter 3 for fairly comprehensive but ideal circumstances where the loads and hence the exact solutions are known fully. It is shown that the approach produces generally better accuracy and certainly requires less knowledge than comparable theories. Extension to more realistic situations which involve many degrees-of-freedom is outlined in Chapter 4. Practical applications are illustrated throughout and in references [31-33].

CHAPTER 2

A THEORY OF THE GREATEST MAXIMUM RESPONSE OF LINEAR STRUCTURES

2.1. Introduction

A structure's maximum response and energy are often critical parameters in predicting likely damage by deterministic or random loads. If the properties of the load and structure are known, the history of the response and, hence, the maxima may be evaluated explicitly by using methods such as the normal mode or Fourier transform methods [1-15]. In practice the form of any one loading (e.g., earthquake, sonic boom, wind gust) is seldom unique and the structure is prone during its lifetime to different combinations of loads. Moreover, identifying the parameters of a complex structure is usually difficult so that it is often important to compare the results of different methods. Such considerations were the primary motivation for Drenick [20], Shinozuka [26-28] and researchers in analogous filter problems [17-19] to determine the excitation amongst all possible excitations which could produce the greatest maximum (maximax) response. Unfortunately their estimation of the maximax response grows with increasing duration of both the external load and impulse response function. It is often inaccurate because the corresponding excitation is virtually unattainable especially in the usual situation of poorly damped structures. Although

some of the additional bounds presented subsequently by Liu [22] are more accurate they suffer from the disadvantage that the response has to be known *a priori*.

A novel theory will be proposed here which usually will produce better estimates and certainly will require less knowledge than all previous theories. The analysis is approached by deducing the maximum response as the maximum over all time of an optimum function with a certain energy contained in a particular bandwidth. The parameters in this function will be defined in terms of usually known load variables. Loads will be classified as having either short or long duration relative to the predominant natural period of the structure. A blast or explosion is a typical short duration load while an earthquake could be treated as a long duration load. The theory is general for any linear structure. However practical applications will be illustrated in this chapter for a structure idealized as a simple, viscously damped oscillator.

2.2. Background

With reference to Figure 2.1 a force, $f(t)$, excites a causal, stable, linear system with impulse response function, $h(t)$. The resultant response, $g(t)$, may be the displacement, velocity, acceleration or so forth. The energy of $f(t)$, well known in signal analysis, is defined as [34,35] (the

nomenclature is given on page vii)

$$E_f(T_f) = \int_0^{T_f} [f(t)]^2 dt, \quad (2.1)$$

where T_f is some instant during the force's application. Here, and in what follows, a subscript f will denote a variable associated with the load, $f(t)$; similarly g will refer to the response and h to the impulse response function. All functions under consideration are assumed to be of bounded variation; this means that the Fourier integrals exist [34]. When the force is time limited and structural damping exists,

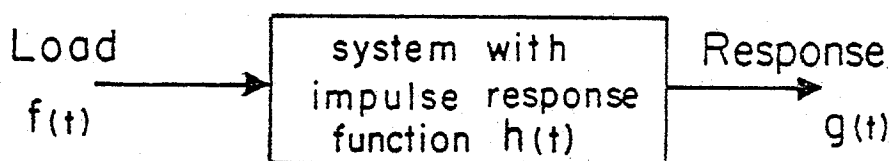


Figure 2.1 Linear Stable System.

the response vanishes after a finite time. Then by using Duhamel's integral, the response's duration, T_g , may be expressed explicitly in terms of the excitation time by

$$T_g = T_f + T_h. \quad (2.2)$$

Conversely when $E_f(T_f)$ grows rapidly with increasing T_f it may be more convenient to use the power, $dE_f(T_f)/dt$, or its mean square value [5]

$$\langle f^2(t) \rangle_f = E_f(T_f)/T_f, \quad (2.3)$$

rather than the energy.

Although a time limited function is not frequency limited too, high frequency components cannot be actually measured due to instrumentation limitations. This practical difficulty suggests that the response's energy may be reasonably assumed to be confined within a frequency band. It is shown in reference [30] that the concentration may be measured in either the time or frequency domain by using the Heisenberg uncertainty principle. Duration and bandwidth cannot be small simultaneously. The response's bandwidth, however, may be represented as a function of its excitation's counterpart by [22]

$$\Omega_g = \min(\Omega_f, \Omega_h). \quad (2.4)$$

2.3 The Maximum Response

The optimum function having the largest energy in the baseband $|\omega| \leq \Omega_g$ will be investigated. Let E_g be the total energy of the time limited response, $g(t)$, and E_{gB} be the energy of the band limited response, $g_B(t)$, which has significant values for frequencies $|\omega| \leq \Omega_g$. If

$$v = E_{gB}/E_g, \quad (2.5)$$

then v is never greater than unity. Following procedures similar to those used by Slepian and Pollak [36], the energy E_{gB} may be written as

$$E_{gB} = \int_{-\infty}^{\infty} [g_B(u)]^2 du, \quad (2.6)$$

where

$$g_B(u) = \frac{1}{2\pi} \int_{-\Omega_g}^{\Omega_g} G(j\omega) e^{j\omega u} d\omega = \frac{1}{2\pi} \int_{-\Omega_g}^{\Omega_g} e^{j\omega u} \int_0^T g(v) e^{-j\omega v} dv d\omega.$$

Interchanging the order of integration yields

$$g_B(u) = \frac{1}{2\pi} \int_0^T g(v) \int_{-\Omega_g}^{\Omega_g} e^{j\omega(u-v)} d\omega dv = \int_0^T g(v) \frac{\sin \Omega_g(u-v)}{\pi(u-v)} dv. \quad (2.7)$$

From equations (2.6) and (2.7), details are given in Appendix V

$$E_{gB} = \int_{-\infty}^{\infty} \int_0^T \int_0^T g(v) g(t) \frac{\sin \Omega_g(u-v)}{\pi(u-v)} \frac{\sin \Omega_g(t-u)}{\pi(t-u)} dv dt du.$$

By using Parseval's theorem, it can be shown that [36]

$$E_{gB} = \int_0^T \int_0^T g(v) g(t) \frac{\sin \Omega_g(t-v)}{\pi(t-v)} dv dt \quad (2.8)$$

and hence, from equation (2.5)

$$v = \frac{\int_0^{T_g} g(t) \int_0^{T_g} g(v) \frac{\sin \Omega_g(t-v)}{\pi(t-v)} dv dt}{\int_0^{T_g} [g(t)]^2 dt}. \quad (2.9)$$

The problem of finding an optimum response function which has most of its energy in the band Ω_g reduces to finding the maximum value of v . The time origin of $g(t)$ is shifted to the center of the duration T_g to make the limits of the integrals symmetrical. It is shown in reference [36] that the maximum value of v occurs when $g(t-T_g/2)$ is proportional to the first characteristic function $\psi_0(t)$ corresponding to the maximum characteristic value, λ_0 , of the integral equation

$$\lambda \psi(t) = \int_{-T_g/2}^{T_g/2} \psi(v) \frac{\sin \Omega_g(t-v)}{\pi(t-v)} dv \quad (2.10)$$

The kernel $\sin \Omega_g t / \pi t$ is positive definite so that the characteristic values of the above equation are all positive and infinite in number. The eigenfunctions $\psi_i(t)$ ($i = 0, 1, 2, \dots$) are the prolate spheroidal wave functions[†] which depend only upon the product of the response's duration and frequency. The optimum functions, $\psi_0(t)$, will have the greatest maximum amplitude so that the maximum response may

[†]Properties of these functions can be found in references [30] and [36].

be written as [36,37]

$$\max g_o(t) = \sqrt{E_{g_o}/\lambda_o} \max \psi_o(t + T_g/2, \Omega_g T_g/2). \quad (2.11)$$

Subscript o refers to optimum values and the quantity $\sqrt{E_{g_o}/\lambda_o}$ in equation (2.11) is obtained from the normalization of the eigenfunctions, $\psi_o(t)$. If E_g is known instead of E_{g_o} then equation (2.11) gives an estimate for $\max g(t)$ rather than $\max g_o(t)$. Substituting equation (2.10) into equation (2.9) gives v equal to λ , and hence its maximum value v_o is equal to λ_o . $\max g_o(t)$ will be shown to give a good estimate when T_g is finite. However in the case of long duration loads the mean square power should be used in the normalization. Then equation (2.11) may be rewritten as

$$\max g_o(t) = \sqrt{\langle g^2(t) \rangle_o / \lambda_o} \max \psi_o(t + \frac{1}{2}, \Omega_g/2) \quad (2.12)$$

where $\langle g^2(t) \rangle_o$ is the mean square power defined in equation (2.3).

Values of λ_o and $\max \psi_o(\Omega_g T_g/2)$ in the normalized form $[T_g/2]^{\frac{1}{2}} \max \psi_o(\Omega_g T_g/2)$ are presented in Appendix I for values of $\Omega_g T_g/2$ up to eight. However $\Omega_g T_g/2$ may be greater than eight in some practical situations. Then λ_o is virtually unity and $[T_g/2]^{\frac{1}{2}} \max \psi_o(\Omega_g T_g/2)$ is given by the asymptotic solution [38]

$$[T_g/2]^{\frac{1}{2}} \max \psi_o(\Omega_g T_g/2) = [(\frac{\Omega_g T_g}{2\pi})^{\frac{1}{2}} / \{1 + 0.0234 / (\frac{\Omega_g T_g}{2})^2\}]^{\frac{1}{2}}, \quad \frac{\Omega_g T_g}{2} \geq 6. \quad (2.13)$$

Results from the above equation coincide in the cross-over region $6 \leq \Omega_g T_g/2 \leq 8$ with those given in Appendix I. Numerical values of λ_0 and $[T_g/2]^{1/2} \max \psi_0(\Omega_g T_g/2)$ are shown in Figure 2.2 for $\Omega_g T_g/2$ varying from 0.1 to 1,000. The combined function $[T_g/2 \lambda_0]^{1/2} \max \psi_0(\Omega_g T_g/2)$ is also given to make prospective enumerations of formula (2.11) easier.

2.4. Relationship Between Input and Output Energy and Power

To make the equations more useful in practice, relations between the response parameters (e.g., energy, mean square power) and their corresponding parameters in the excitation will be established by using [34]

$$E_g = \int_0^{\infty} g^2(t) dt = \frac{1}{2\pi} \int_0^{\infty} |F(j\omega)H(j\omega)|^2 d\omega. \quad (2.14)$$

Whether the load is deterministic or not two cases will be considered, one with finite duration T_f and the other with long duration, $T_f \rightarrow \infty$.

2.4.1. Short Duration Loading

When T_f is short (e.g., impulsive loading) equation (2.14) may be written in the form

$$E_g = \frac{1}{2\pi} \int_0^{\infty} \int_0^{T_f} f(\tau) e^{-j\omega\tau} \int_0^{T_f} f(x) e^{j\omega x} |H(j\omega)|^2 d\omega dx d\tau. \quad (2.15)$$

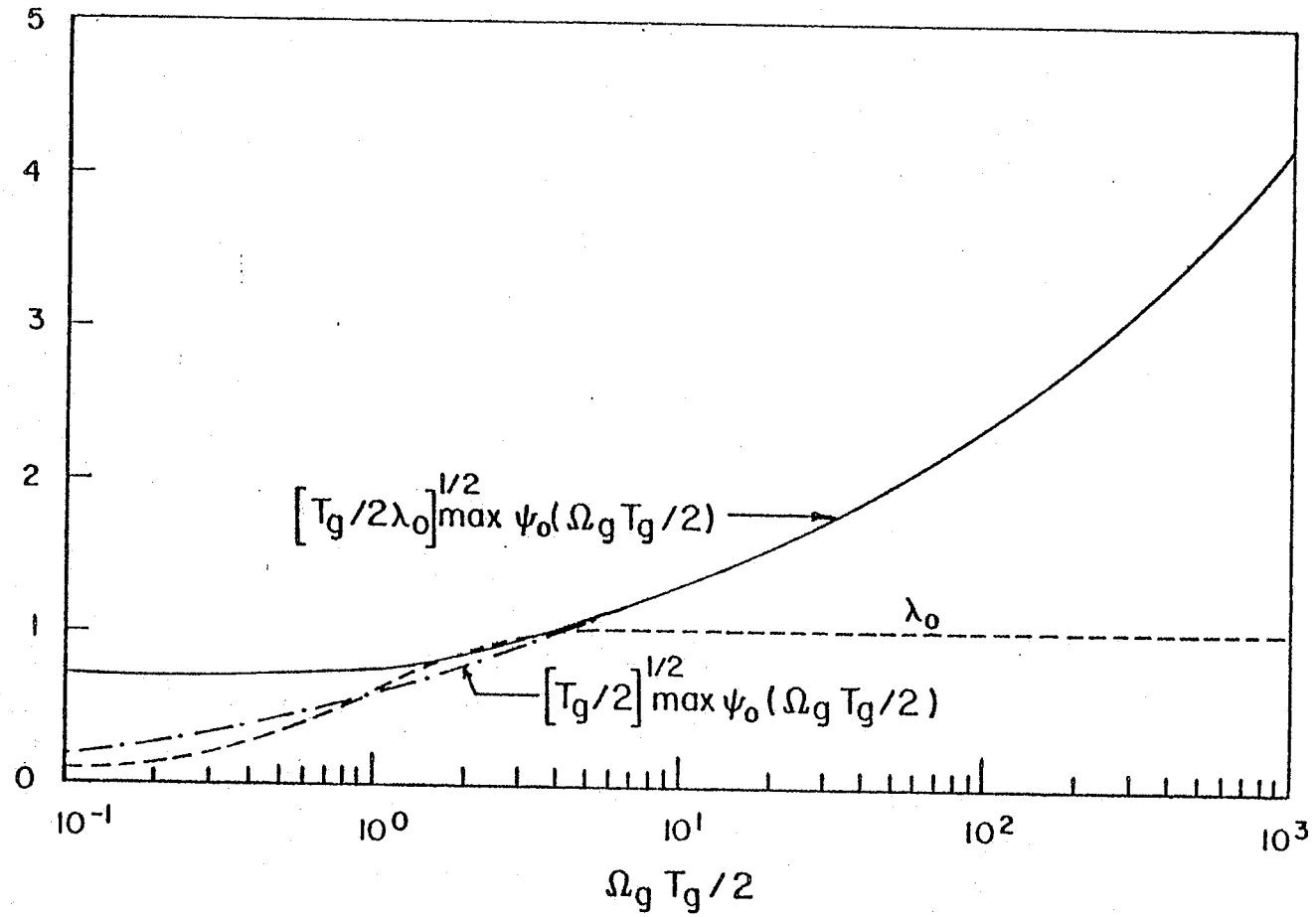


Figure 2.2. $(T_g / 2\lambda_0)^{1/2} \max \psi_0(\Omega_g T_g / 2)$, $(T_g / 2)^{1/2} \max \psi_0(\Omega_g T_g / 2)$ and λ_0 against $\Omega_g T_g / 2$.

Interchanging the order of integration leads to

$$E_g = \int_0^{T_f} f(\tau) \int_0^{T_f} f(x) L(x - \tau) dx d\tau, \quad (2.16)$$

where $L(t)$ is the Fourier transform of $|H(j\omega)|^2$ and, as such, is unique. The maximum values of E_g occur when $f(\tau)$ equals the characteristic functions, $f_0(\tau)$, corresponding to the largest characteristic values η_0 of the homogeneous Wiener-Hopf integral equation [34,37]

$$\eta f(\tau) = \int_0^{T_f} f(x) L(\tau - x) dx. \quad (2.17)$$

$f_0(\tau)$ is the excitation which maximizes the energy content of the structure's response. Substituting equation (2.17) into equation (2.16) gives

$$E_{g0} = \eta_0(T_f) E_f. \quad (2.18)$$

The maximum response can be found immediately in terms of the excitation's energy by combining the above equation with equation (2.11) so that

$$\max g_0(t) = \sqrt{\eta_0(T_f) E_f / \lambda_0} \max \psi_0(t + T_g/2, \Omega_g T_g/2). \quad (2.19)$$

In many applications finding the greatest maximum (maximax) response is a major concern. This maximax will correspond to the maximum value of $\eta_0(T_f)$ obtained from the

Fourier transform of equation (2.17); that is,

$$\max \eta_0 = \max |H(j\omega)|^2 \quad (2.20)$$

and

$$\max_{\max} g_0(t) = \sqrt{\max |H(j\omega)|^2 E_f / \lambda_0} \max \psi_0(t + T_g/2, \Omega_g T_g/2). \quad (2.21)$$

Equation (2.19) estimates the maximum response for any input load with known energy while equation (2.21) provides the maximax response for all excitations with the same input energy E_f *regardless of the loading pattern $f(t)$* . Application of these equations will be presented in Chapter 3.

2.4.2. Long Duration Loading

Long duration loads can be treated in a fashion similar to that for short duration loads by letting T_f tend to infinity in equation (2.16). Then

$$E_{g_0} = \eta_0 E_f \quad (2.22)$$

and η_0 no longer depends upon the duration T_f .

Taking the derivative of equation (2.22) relates the input and output power in the form

$$dE_{g_0}/dt = \eta_0 dE_f/dt. \quad (2.23)$$

For stationary and ergodic random processes neither energy nor power are known entirely but are defined usually in statistical terms. The mean square power can be deduced

from equations (2.3) and (2.22) as

$$\langle g^2(t) \rangle = \eta_0 \langle f^2(t) \rangle, \quad (2.24)$$

which, when combined with equation (2.12), leads to

$$\max g_0(t) = \sqrt{\eta_0 \langle f^2(t) \rangle / \lambda_0} \max \psi_0(t + \frac{1}{2}, \Omega_g/2). \quad (2.25)$$

The autocorrelation and spectral density functions are related to the mean square response by [5]

$$P_f = R_f(0) = \int_0^\infty S(\omega) d\omega. \quad (2.26)$$

It follows that

$$R_{g_0}(0) = \eta_0 R_f(0) \quad (2.27)$$

and

$$\int_0^\infty S_{g_0}(\omega) d\omega = \eta_0 \int_0^\infty S_f(\omega) d\omega. \quad (2.28)$$

2.5. A Viscously Damped Oscillator

The application of the theory is illustrated by considering the particular example of a structure idealized as a single degree-of-freedom, viscously damped system. Values of η_0 were evaluated by solving the eigenvalue integral equation (2.16), by using the numerical technique of Youla [39]. Details of this method are given for both short and long duration loads in Appendix II, for the (displacement)

frequency response function [2]

$$|H(s)|^2 = (1/k^2)/[(s/\omega_n)^4 + (2 - 4\zeta^2)(s/\omega_n)^2 + 1], \quad s = j\omega.$$

Here ω_n is the undamped natural frequency of the system and ζ is the damping ratio. Similar procedures can be implemented in the case of velocity or acceleration frequency response functions.

For short duration loading, values of $k^2\eta_0$ are plotted in Figure 2.3 against $T_f/T_n^\dagger$ for damping ratios ζ between 0.02 and 0.30, which encompass most practical situations. It is found that for values of T_f/T_n smaller than 1.0, damping has much less importance in controlling $k^2\eta_0$. For this reason only the undamped response to impulsive loading has been frequently considered [2]. When the ratio T_f/T_n becomes greater than nine, $k^2\eta_0$ approaches asymptotically its maximum value. Therefore the eigenvalues for long duration loads can be found directly from the asymptotic values. Alternatively

$$\begin{aligned} \max \eta_0 &= \max |H(j\omega)|^2 = 1/(4k^2 \zeta^2(1 - \zeta^2)), \quad \zeta < 1/\sqrt{2} \\ &= 1/k^2, \quad 1/\sqrt{2} \leq \zeta < 1. \end{aligned} \quad (2.29)$$

Values of $\max k^2\eta_0$ are plotted against the damping ratio in Figure 2.4. For finite duration loads the maximax displacement, obtained by inserting the values of $\max \eta_0$ into equation (2.21), is given by

$^\dagger T_n$ is the natural period of the oscillator.

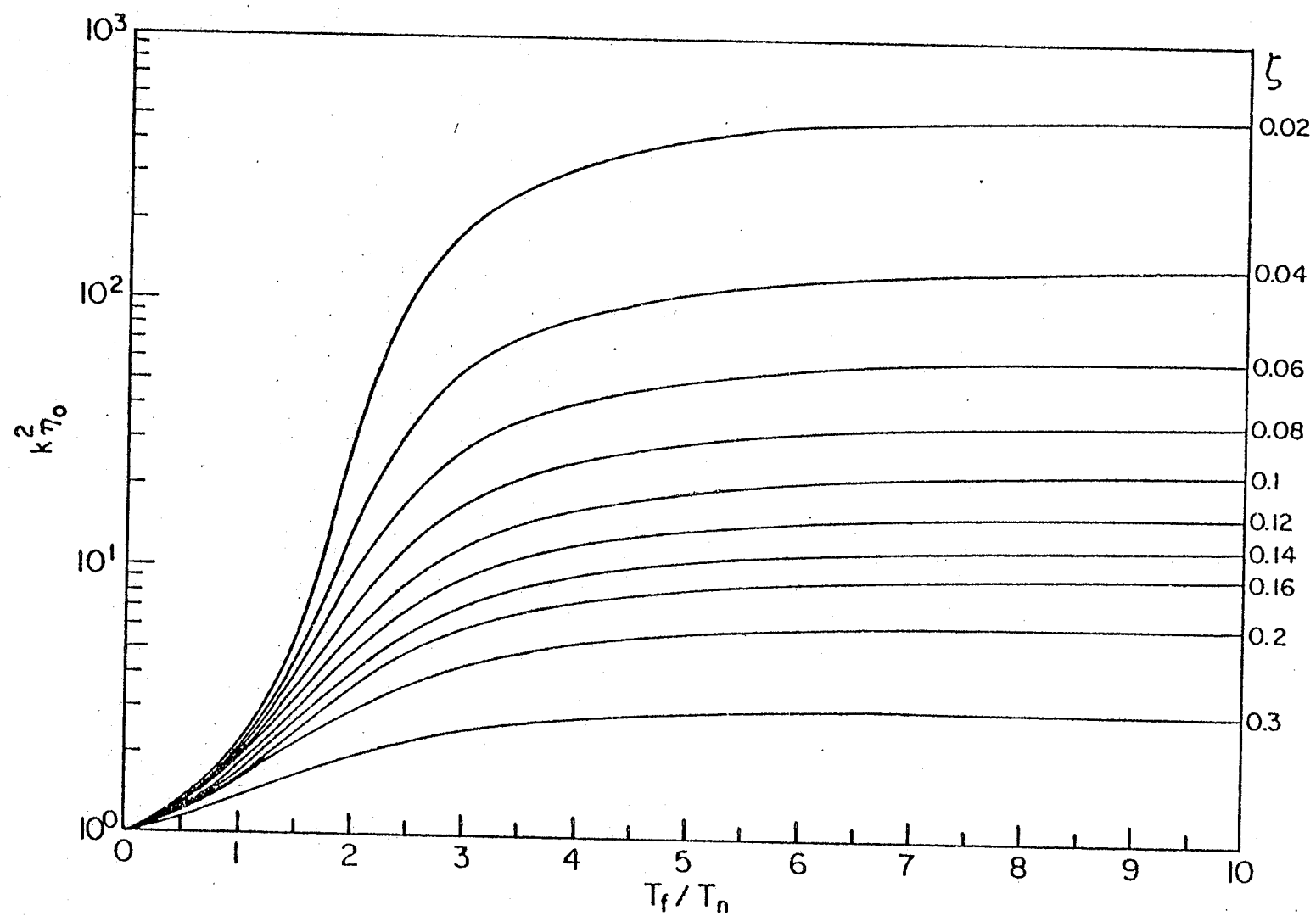


Figure 2.3. $k^2_{\eta_0}$ against T_f/T_n for different viscous damping ratios ζ .

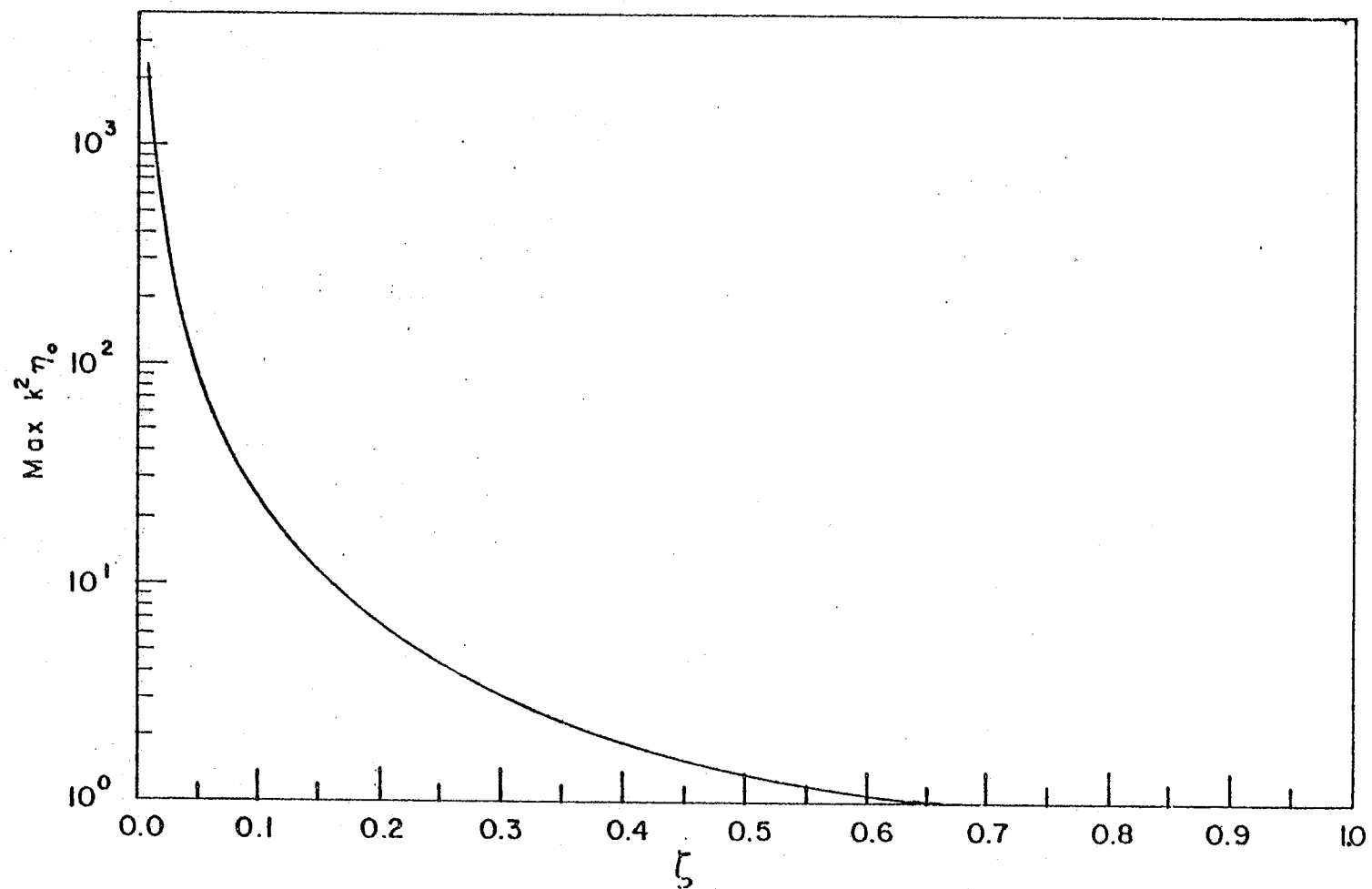


Figure 2.4. Maximum $k^2 \eta_0$ against viscous damping ratio ζ .

$$\text{maximax } g_0(t) = [2k\zeta\sqrt{(1-\zeta^2)}]^{-1} \sqrt{E_f/\lambda_0} \max \psi_0(t+T_g/2, \Omega_g T_g/2), \quad \zeta < 1/\sqrt{2},$$

$$\text{maximax } g_0(t) = (k)^{-1} \sqrt{E_f/\lambda_0} \max \psi_0(t+T_g/2, \Omega_g T_g/2), \quad 1/\sqrt{2} \leq \zeta < 1. \quad (2.30)$$

2.6. Maximum Stochastic Response

The maximum mean square response to a stationary random load having an arbitrary random power spectrum can be evaluated readily by using the foregoing equations. However, it is interesting to compare the specific result for η_0 to that given in references [2] and [15] for a lightly damped oscillator excited by white noise. Equation (2.28) can be rewritten as

$$\int_0^{\infty} [S_g(\omega) - \eta S_f(\omega)] d\omega = 0. \quad (2.31)$$

The input and output power spectral density functions are related through [2,15]

$$S_g(\omega) = |H(j\omega)|^2 S_f(\omega),$$

so that

$$\int_0^{\infty} [|H(j\omega)|^2 - \eta] S_f(\omega) d\omega = 0. \quad (2.32)$$

Upon assuming that $S_f(\omega)$ is twice continuously differentiable with respect to ω and using the fundamental lemma of

calculus of variations, it follows that [37]

$$\eta = |H(j\omega)|^2.$$

Therefore

$$\eta_0 = \max |H(j\omega)|^2 = 1/(4k^2 \zeta^2(1 - \zeta^2)).$$

Agreement can be shown by substituting unity for $(1 - \zeta^2)$ because the system is lightly damped ($\zeta \ll 1$).

2.7. Estimation of the Response's Bandwidth and Duration

The vexatious task of forecasting Ω_g and T_g given complete details of the oscillator and knowing the load's duration T_f , and energy or the bounds of $|F(j\omega)|$ is well recognised in analogous problems in circuit theory [35,40]. Difficulties arise because the estimation will depend upon the particular criterion selected and, hence, will not be unique. From a practical standpoint however any easily implemented criterion which produced inconsequential errors should be satisfactory. Consequently simple guidelines to obtain fair estimates will be developed by using criteria selected somewhat arbitrarily from those known to be reasonable for electrical circuits.

Knowing T_f , the period of the oscillator, T_h , should be strictly chosen so that the response's period of interest, T_g , has to satisfy equation (2.2) to be fully compatible with

Duhamel's integral

$$g(t) = \lim_{T \rightarrow \infty} \int_0^T f(\tau) h(t-\tau) d\tau = \frac{1}{2\pi} \lim_{T \rightarrow \infty} \int_{-\Omega}^{\Omega} F(j\omega) H(j\omega) e^{j\omega t} d\omega \quad (2.33)$$

The duration T_h is often approximated in circuit theory by T'_h obtained from

$$h(T'_h) = \epsilon_h \max h(t). \quad (2.34)$$

Therefore T'_h will depend upon ϵ_h which is normally selected such that the response to an individual load hardly changes relative to its maximum value for all time t greater than $(T_f + T'_h)$ [41]. Obviously the maximax rather than the maximum response is a more fitting reference for several loads with similar energy and duration or Fourier transforms. Then an easily instituted guideline for selecting ϵ_h appropriately may be generated in a manner similar to the derivation of the upper bound given in references [18,19]. Duhamel's integral is rewritten first as

$$g(t) = \lim_{T \rightarrow \infty} \left[\int_0^T f(\tau) P(t-\tau) h(t-\tau) d\tau + \int_0^T f(\tau) Q(t-\tau) h(t-\tau) d\tau \right] \quad (2.35)$$

where

$$P(t) = \begin{cases} 1 & t \leq T'_h \\ 0 & t > T'_h \end{cases}$$

and

$$Q(t) = \begin{cases} 0 & t \leq T'_h \\ 1 & t > T'_h \end{cases} \quad (2.36)$$

The first integral in equation (2.35) is the response up to the time $(T_f + T'_h)$ associated with a given ϵ_h and the second represents the subsequent response. Hence, applying Schwartz's inequality to the second integral yields

$$g(t) - \int_0^{T_f+T'_h} f(\tau)h(t-\tau)d\tau = \lim_{T \rightarrow \infty} \int_0^T f(\tau)Q(t-\tau)h(t-\tau)d\tau \leq [E_f E'_{h\epsilon}]^{\frac{1}{2}} \quad (2.37)$$

where, because of $Q(t-\tau)$,

$$E'_{h\epsilon} = \lim_{T \rightarrow \infty} \int_{T'_h}^T h^2(t)dt. \quad (2.38)$$

Similarly, applying Schwartz's inequality to the time integral of equation (2.33) gives the maximax estimation of references [17-20] as

$$\text{maximax } g(t) = [E_f E_h]^{\frac{1}{2}} \quad (2.39)$$

where

$$E_f = \lim_{T \rightarrow \infty} \int_0^T f^2(t)dt$$

and

$$E_h = \lim_{T \rightarrow \infty} \int_0^T h^2(t)dt \quad (2.40)$$

Combining formula (2.37) and (2.39) gives

$$\frac{g(t) - \int_0^{T_f + T'_h} f(\tau)h(t-\tau)d\tau}{\text{maximax } g(t)} \leq \left[\frac{E'_{h\epsilon}}{E_h} \right]^{\frac{1}{2}} \leq \left[\frac{E'_h}{E_h} \right]^{\frac{1}{2}} \quad (2.41)$$

because E'_h , the energy time integral truncated at $T'_h < T$, cannot be greater than E_h . This inequality implies that if ϵ_h is chosen so that, irrespective of the load, the ratio $[E'_{h\epsilon}/E'_h]^{\frac{1}{2}}$ or $[E'_{h\epsilon}/E_h]^{\frac{1}{2}}$ is small then the inaccuracy in the corresponding maximax response caused solely by error in T_h and, hence, T_g will be no larger.

The frequency bandwidth of the response given in equation (2.4), Ω_g , is recognised in circuit theory to be the smaller of the load and oscillator's bandwidths. Therefore Ω_g is independent of the load when Ω_h is less than Ω_f so that it may be approximated fairly accurately then in a comparable manner to T_h . Greater uncertainty might be introduced in Ω_g when Ω_f is less than Ω_h because generally less detail is presumed for the load than for the oscillator. The bandwidth Ω_f and, hence, Ω_g can be assessed straightforwardly if $|F(j\omega)|$ is assumed to be known but some difficulty is experienced if only E_f and T_f are known. The only recourse in the latter case is to employ the experimental strategy of judging the cut-off frequency by experience with kindred loads. For example, the cut-off frequency $2.5/T_f$

which is a reasonable approximation for half-sine, rectangular and triangular impulsive loads [42] could be used for similarly shaped impulses. Such a procedure however is viable only if the estimation of the maximax response is insensitive to large differences in Ω_f when Ω_f equals Ω_g . Some idea of this sensitivity may be gauged from formula (2.21) which is probably the most critical from a theoretical standpoint because it depends not only upon Ω_g but T_g too. Functions of Ω_g and T_g in this formula are combined and plotted in Figure 2.2 in the normalized form $[T_g/2\lambda_o]^{\frac{1}{2}} \max \psi_o(\Omega_g T_g/2)$. It may be seen that $[T_g/2\lambda_o]^{\frac{1}{2}} \max \psi_o(\Omega_g T_g/2)$ is almost constant up to $\Omega_g T_g/2$ around one and approximately doubles for subsequent tenfold increases in $\Omega_g T_g/2$. The error which is introduced by limiting the bandwidth of a time limited response is evaluated in Appendix III and shown to be negligible for $\Omega_g T_g/2$ greater than 4. Therefore the maximax response appears fairly tolerant of errors in the product $\Omega_g T_g$ with slightly less latitude allowed for T_g because of the normalization factor $T_g^{\frac{1}{2}}$. Consequently formula (2.21) permits reasonable inaccuracy in Ω_f or, for that matter, Ω_h and is just more sensitive to the fortunately surer error in T_g (or T_h given T_f).

A priori error bounds can be assessed more systematically and closely when Ω_g equals Ω_h rather than Ω_f . An approximation to Ω_h , Ω_h' , can be determined by analogy with

T_h' in equation (2.34) from

$$|H(j\Omega_h')| = \epsilon_H \max |H(j\omega)|. \quad (2.42)$$

Selection of ϵ_H is based again on the requirement that the response scarcely changes relative to the maximum value but for frequencies ω greater than Ω_h' . Differences in the response arising from the neglect of frequencies greater than Ω_h' can be derived from equation (2.33) as

$$g(t) - g_B(t) = \frac{1}{2\pi} \text{Lt}_{\Omega \rightarrow \infty} \int_{-\Omega}^{\Omega} F(j\omega)H(j\omega)e^{j\omega t} d\omega - \frac{1}{2\pi} \int_{-\Omega_h'}^{\Omega_h'} F(j\omega)H(j\omega)e^{j\omega t} d\omega \quad (2.43)$$

where $g_B(t)$ is the frequency limited response. Alternatively

$$g(t) - g_B(t) = \frac{1}{2\pi} \text{Lt}_{\Omega \rightarrow \infty} \left[\int_{-\Omega}^{\Omega} F(j\omega)H(j\omega)R(j\omega)e^{j\omega t} d\omega + \int_{-\Omega}^{\Omega} F(j\omega)H(j\omega)S(j\omega)e^{j\omega t} d\omega \right] \quad (2.44)$$

where

$$R(j\omega) = \begin{cases} 1 & \omega \leq -\Omega_h' \\ 0 & \omega > -\Omega_h' \end{cases}$$

and

$$S(j\omega) = \begin{cases} 0 & \omega < \Omega_h' \\ 1 & \omega \geq \Omega_h' \end{cases} \quad (2.45)$$

By employing symmetry about ω zero and the procedure used to arrive at inequality (2.37), the modulus of equation (2.44) can be shown to be bounded by

$$|g(t) - g_B(t)| \leq 2[E_f E_{H_\epsilon}']^{\frac{1}{2}} \quad (2.46)$$

where

$$E_{H_\epsilon}' = \frac{1}{2\pi} \text{Lt}_{\Omega \rightarrow \infty} \int_{-\Omega}^{\Omega} [|H(j\omega)| |S(j\omega)|]^2 d\omega. \quad (2.47)$$

Therefore the difference in $g(t)$ relative to the maximax response of formula (2.39) is

$$\frac{g(t) - g_B(t)}{\text{maximax } g(t)} \leq 2 \left[\frac{E_{H_\epsilon}'}{E_h'} \right]^{\frac{1}{2}} \leq 2 \left[\frac{E_{H_\epsilon}'}{E_h'} \right]^{\frac{1}{2}} \quad (2.48)$$

if the same reasoning needed to formulate inequality (2.41) is applied. E_h' is the energy integral in the frequency rather than the time domain which is terminated at $\pm \Omega_h'$ i.e.

$$E_h' = \frac{1}{2\pi} \int_{-\Omega_h'}^{\Omega_h'} |H(j\omega)|^2 d\omega. \quad (2.49)$$

Hence ϵ_H chosen so that the corresponding Ω_h' leads to a small ratio of $2[E_{H_\epsilon}'/E_h']^{\frac{1}{2}}$ will produce, by itself, no greater error in the maximax response. Of course, a standard analysis based as appropriate on inequalities (2.41) and (2.48) should be performed when errors occur in both frequency and time.

All relevant detail has been generated now to apply formula (2.21). The accuracy will be compared in Chapter 3 with alternative methods by using idealized loads which typify practical examples.

2.8. Conclusions

The greatest maximum response for linear structures can be predicted from generally available knowledge. The load's pattern and duration are eliminated in finding the maximax response. However, when the duration is known a better estimation can be obtained. The results are applicable to all linear structures and to deterministic and random loadings.

CHAPTER 3

APPLICATION AND COMPARISON OF MAXIMAX RESPONSE ESTIMATES

3.1. Introduction

Details of nominally identical shock loads, unlike their analytical counterparts, often vary in practice due to uncontrolled fluctuations in turbulence, packaging of dynamite, etc., yet their general forms remain similar. Therefore considerable attention has been shown recently in predicting the maximum response and energy of structures excited by grossly described loads [17-29]. Tufts and Shnidman [17] and later Drenick [21] estimated the greatest maximum response of linear, stable (damped) structures from knowledge of the energies of the load and the impulse response function of the structure. Shinozuka [28] subsequently improved the estimation by employing comparable information in the frequency rather than the time domain. This last approach however suffers from the practical disadvantage that a computer is needed. An alternative procedure selectively using overall time and, from an experimental viewpoint, more readily available frequency characteristics of the load and structure has been outlined in Chapter 2. The theory has the additional attraction of estimating the maximum energy as well as the greatest maximum of the response but requires further elaboration to cover realistic problems.

A critical review of the three approaches is presented with an assessment of their relative accuracies for diverse but typical shock and earthquake loads. The loads are applied to structures idealized as in the previously mentioned references by simple, viscously damped oscillators. Values of the damping and resonant frequency are varied to encompass the range of most practical applications.

3.2. Review

The response, $g(t)$, of a linear, stable, causal oscillator initially at rest is obtained from Duhamel's integral as [2]

$$g(t) = \lim_{T \rightarrow \infty} \int_0^T f(\tau) h(t-\tau) d\tau = \frac{1}{2\pi} \lim_{\Omega \rightarrow \infty} \int_{-\Omega}^{\Omega} F(j\omega) H(j\omega) e^{j\omega t} d\omega. \quad (3.1)$$

A detailed description of $h(t)$ and $H(j\omega)$ will depend partly upon the stiffness k , natural period T_n and damping ratio ζ of the oscillator and partly upon the response variable (displacement, velocity or acceleration etc.) of interest. Even if either $h(t)$ or $H(j\omega)$ is known, $g(t)$ can be evaluated from equation (3.1) only if the appropriate time or frequency characteristics of the load are determined completely. This is often a difficult if not impossible practical task which is compounded by the possibility of a structure experiencing greatly different types of loads over its life-span. In many practical situations however the maximum

response to a particular load or the largest maximum (maximax) generated by similar loads is often of greater concern than the time history of $g(t)$. Then the problem becomes more tractable because these maxima may be estimated in terms of the energy of the load, E_f , which is less changeable than $f(t)$ or $F(j\omega)$.

The earliest method of approximating the maximax response was to apply Schwartz's inequality to the time integral of equation (3.1) [17-21]. The result

$$\text{maximax } g_0(t) = [E_f E_h]^{1/2} \quad (3.2)$$

is usually an overestimate and valid only if the energies E_f and E_h are finite. In principle these last requirements do not necessarily infer that $f(t)$ and $h(t)$ are time limited but this is often implied in practice. If, for example, the functions do not have simple analytical descriptions then digital computation of the integrals would be truncated at some conveniently large but finite T . Although formula (3.2) is exact when $f(t)$ and $h(t)$ are proportional, it is generally less precise than the upper bound [28]

$$\text{maximax } g_0(t) = \frac{1}{2\pi} L t \int_{-\Omega}^{\Omega} |F(j\omega)| |H(j\omega)| d\omega \quad (3.3)$$

providing $|H(j\omega)|$ and especially $|F(j\omega)|$, the envelopes of $H(j\omega)$ and the similar $F(j\omega)$, are accurate. The bound is derived by taking the modulus of the frequency integral

(3.1) which is never greater than the integral of the modulus. However formula (3.3) still requires evaluation usually by terminating the limits again at a finite value, Ω_g , found by inspection. Even then, the computation is seldom a simple task.

A different approach presented in Chapter 2 determines the largest amplitude in a class of response functions whose energies are identical to the maximum produced by a given energy of the loads. All functions belonging to this class are both time and frequency limited. Although this simultaneous restriction violates the uncertainty principle [30], it is imposed commonly but often unknowingly in measurements due to time and instrument limitations. Therefore this approach would appear to favour truly experimental applications. If the theoretical violation is accepted pragmatically then the function having the greatest energy and amplitude in time T_g and the frequency range $[-\Omega_g, \Omega_g]$ is the first prolate spheroidal wave function of zero order, $\psi_0(\Omega_g T_g/2)$ [36]. Therefore the largest response, subject to the corresponding energy being maximum, is approximated in Chapter 2 by

$$\max g_0(t) = [\eta_0 E_f / \lambda_0]^{1/2} \max \psi_0(\Omega_g T_g/2) \quad (3.4)$$

where $[\eta_0 E_f / \lambda_0]^{1/2}$ is basically a normalization factor. λ_0 is a function only of the product $\Omega_g T_g/2$ whereas η_0 depends generally upon the characteristics of the oscillator

and the duration of the load. However the maximum value of η_0 is given by

$$\max \eta_0 = \max |H(j\omega)|^2 \quad (3.5)$$

which is independent of the load. An approximation to the maximax response may be derived by substituting the last equation into the maximum of formula (3.4) to produce

$$\text{maximax } g_0(t) = [\max |H(j\omega)|^2 E_f/\lambda_0]^{1/2} \max \psi_0(\Omega_g T_g/2). \quad (3.6)$$

Even a cursory inspection of formula (3.6) will reveal that the still unknown response variables Ω_g and T_g have to be evaluated reliably from *a priori* knowledge. Formula (3.2) and (3.3) on the other hand appear to need only one of these variables to effectively truncate their integrals. This might be the case indeed in theoretical calculations but often Ω_g and T_g are both restricted in experimental measurements. For example the time history $h(t)$ cannot be practically acquired in perpetuity yet the digital equivalent of its Fourier transform cannot go faithfully beyond the Nyquist frequency [43]. Consequently the measured unlike the theoretical response of a commonly occurring lightly damped structure, where $h(t)$ decays very slowly, will likely be time and frequency bounded. Therefore all formulae could usefully employ guidelines for appraising both Ω_g and T_g *a priori* in experimental situations to avoid extraneous deviations from theoretical predictions.

3.3. Typical Examples

Particular emphasis will be placed on the application of the relatively unknown formula (3.6) whose estimates form the basis for comparing the other two procedures. Absolute accuracies are determined wherever possible by contrasting approximate against exact solutions for a variety of loads. The loads circumscribe pulse modulated sinusoids, the most frequently occurring mechanical shocks and representative earthquakes. They all independently excite a common structure having the idealized impulse response function [2]

$$h(t) = \frac{\omega_n}{k(1-\zeta^2)^{1/2}} e^{-\zeta\omega_n t} \sin [(1-\zeta^2)^{1/2} \omega_n t] \quad (3.7)$$

and the frequency response function

$$|H(j\omega)| = \frac{1}{k[(\frac{\omega}{\omega_n})^4 + 2(2\zeta^2-1)(\frac{\omega}{\omega_n})^2 + 1]^{1/2}} \quad (3.8)$$

for its displacement. The structural characteristics T_h and Ω_h may be estimated by first deriving $\max h(t)$ and $\max |H(j\omega)|$ from the above and then substituting these relations into equation (2.34) and (2.42). It can be shown by using calculus of variations that

$$\max h(t) = \frac{\omega_n}{k\zeta\gamma} \sin (\tan^{-1} \gamma) e^{(-\gamma^{-1} \tan^{-1} \gamma)} \quad (3.9)$$

where

$$\gamma = (1 - \zeta^2)^{1/2} / \zeta \quad (3.10)$$

and

$$\max |H(j\omega)| = 1/[2 k \zeta (1 - \zeta^2)^{1/2}] \quad , \quad \zeta \leq \frac{1}{\sqrt{2}} \quad (3.11)$$

Hence

$$T'_h = \frac{1}{\omega_n} [(1 - \zeta^2)^{1/2} \tan^{-1} \gamma - \zeta^{-1} \ln\{\epsilon_h \sin(\tan^{-1} \gamma)\}] \quad (3.12)$$

and

$$\Omega'_h = \omega_n [(1 - 2\zeta^2)\{1 + 2\zeta \frac{(1 - \zeta^2)^{1/2}}{(1 - 2\zeta^2)} (\epsilon_H^{-2} - 1)^{1/2}\}] \quad (3.13)$$

providing

$$\epsilon_H \leq 2\zeta(1 - \zeta^2)^{1/2} \quad \text{and} \quad \zeta < 1/\sqrt{2} \quad (3.14)$$

Values of T'_h/T_n (or $\omega_n T'_h/2\pi$) and Ω'_h/ω_n computed from the above equations for various ζ less than 0.5 and both ϵ_h and ϵ_H equal to 0.025 are plotted in Figures 3.1 and 3.2. The ϵ_h and ϵ_H were selected somewhat arbitrarily but such that the bounds $[E'_h/E_h]^{1/2}$ and $2[E'_H/E_H]^{1/2}$ in inequalities (2.41) and (2.48) respectively are less than a normally sufficiently small 0.06. The curves in the figures agree with the intuitive concept that as the damping of a given structure increases, T_h decreases with Ω_h behaving almost reciprocally. They also provide a convenient way of approximating T_h and Ω_h quickly.

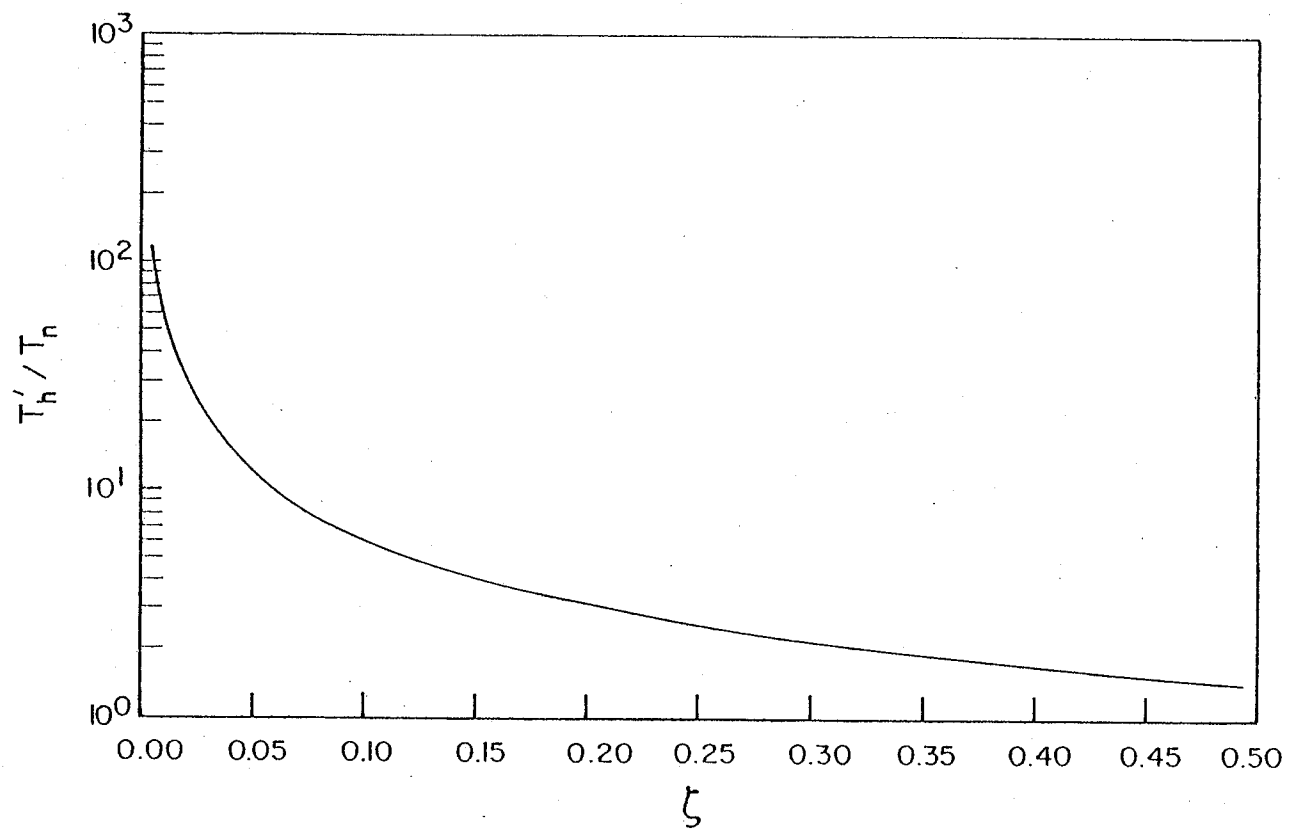


Figure 3.1. T'_h / T_n for different viscous damping ratios ζ .

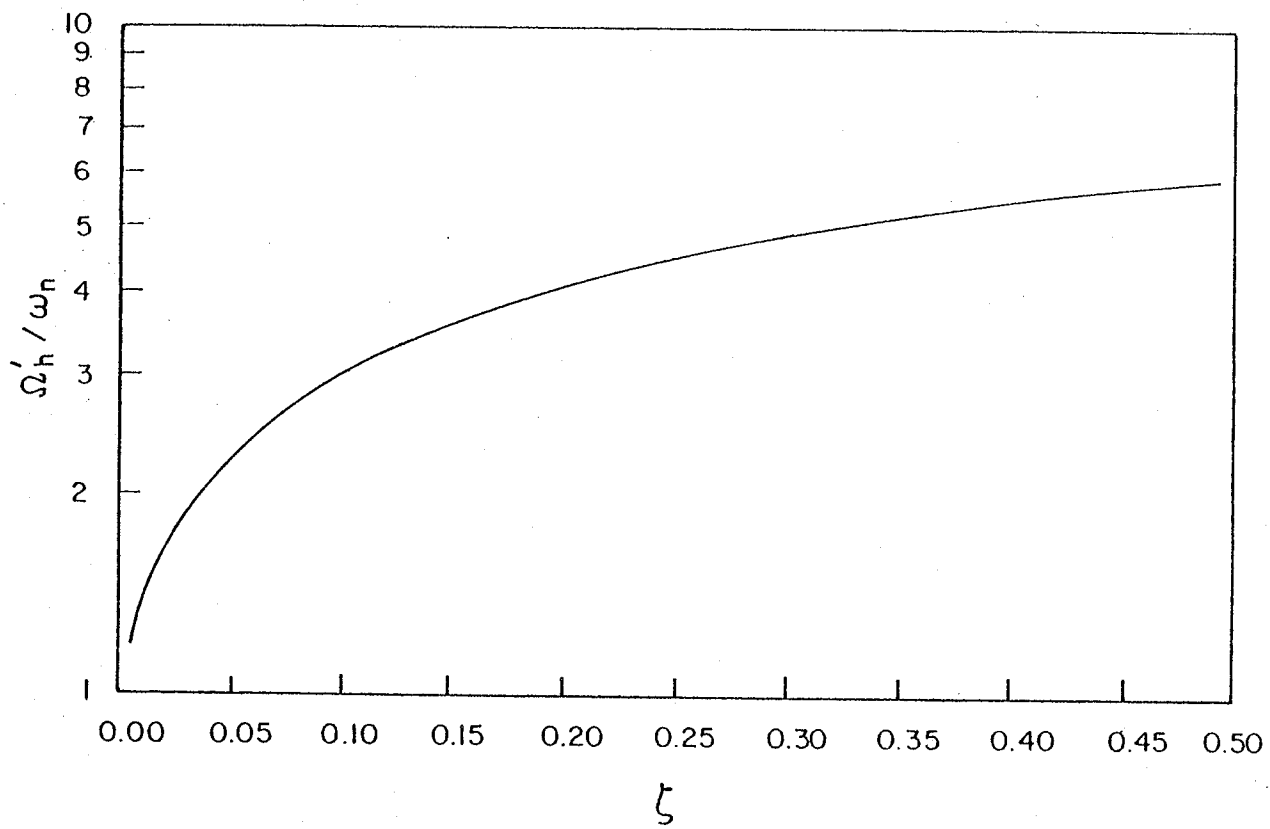


Figure 3.2. Ω'_h / ω_n for different viscous damping ratios ζ .

The only outstanding variable related to the structure is the energy of its impulse response function, E_h , which is wanted in formula (3.2). By substituting equation (3.7) into (2.40), it may be shown analytically that

$$E_h = \frac{\pi}{2k^2 \zeta T_n} \quad (3.15)$$

Further information required pertains to the properties of the type of load under consideration. Although the loads will be appraised separately, their time history or Fourier transform will be assumed in each example. This detailed knowledge, if consistent and available in practice, would make the need for estimates somewhat superfluous. However it is necessary to evaluate the exact solutions and, by comparison, test the accuracy of the three formulae.

3.3.1. Pulse Modulated Sinewave Excitations

The initially stationary oscillator will have the exact maximax displacement [2]

$$\text{maximax } g(t) = \frac{p}{k} \cdot \frac{1}{2\zeta (1 - \zeta^2)^{\frac{1}{2}}} \quad (3.16)$$

when excited by the pulse modulated sinewave $p_{T_f}(t) \cdot \sin \omega_f t$ shown in Figure 3.3 under the conditions

$$\omega_f = \omega_n (1 - 2\zeta^2)^{\frac{1}{2}} \quad \text{and} \quad T_f \gg T_n \quad (3.17)$$

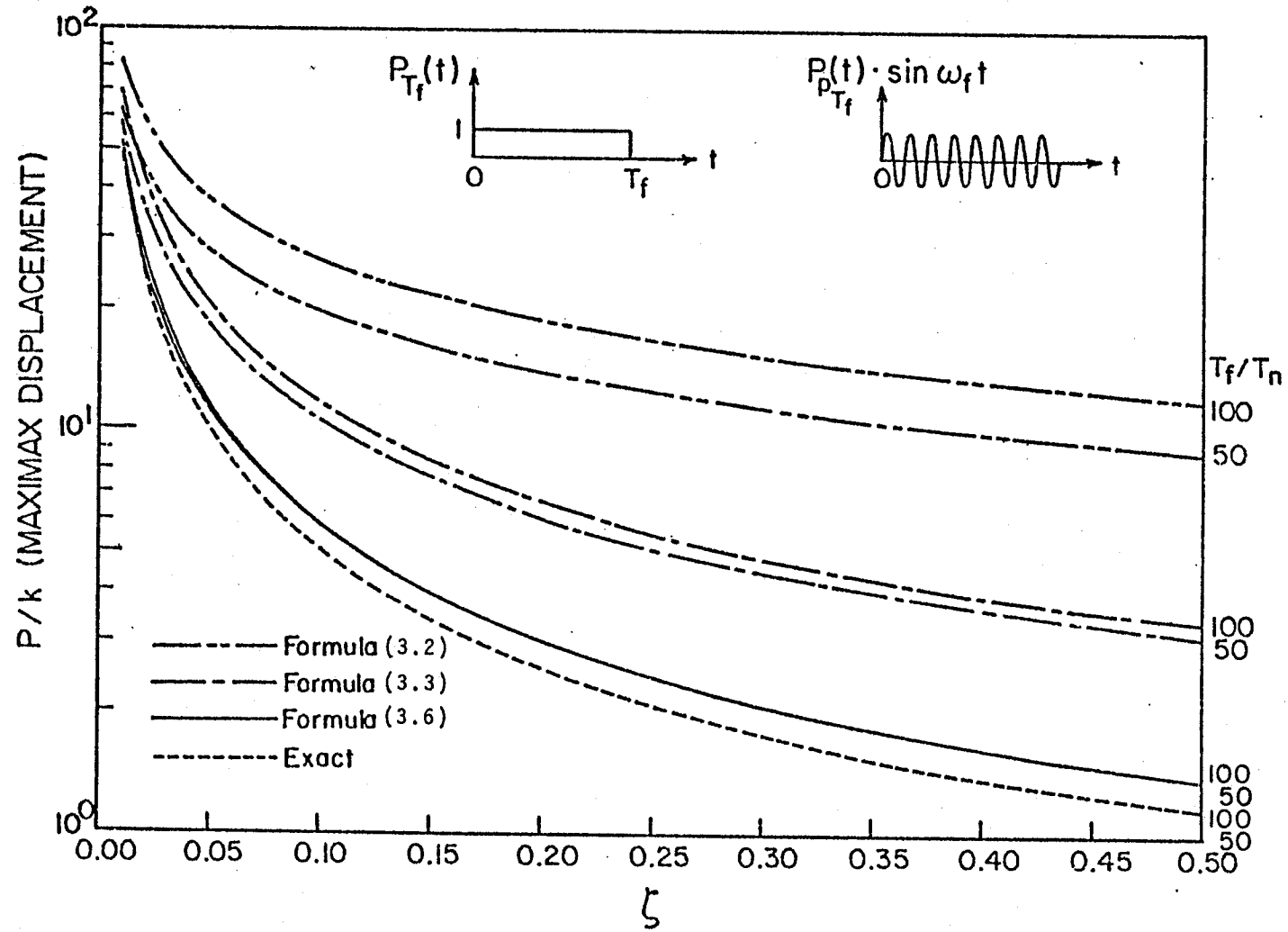


Figure 3.3. Comparison of estimates for the pulse modulated sinewave.

All formulae require the evaluation of at least E_f or $|F(j\omega)|$ from the load's given time history. By using equation (2.40) and condition (3.17),

$$E_f = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \{P_{pT_f}(t) \sin \omega_f t\}^2 dt = p^2 \int_0^{T_f} \sin^2 \omega_f t dt \doteq p^2 T_f / 2 \quad (3.18)$$

to a sufficient degree of accuracy. This relationship can be substituted together with equation (3.15) into formula (3.2) to give the approximation

$$\text{maximax } g(t) = \frac{p}{2k} \left[\frac{\pi T_f}{\zeta T_n} \right]^{\frac{1}{2}} \quad (3.19)$$

$|F(j\omega)|$ can be derived likewise from the time history as

$$|F(j\omega)| = \frac{p}{\omega_f \left[\left(\frac{\omega}{\omega_f} \right)^2 - 1 \right]} \cdot \left[1 + \cos^2 \omega_f T_f + \left(\frac{\omega}{\omega_f} \right)^2 \sin^2 \omega_f T_f - 2 \cos \omega_f T_f \cos \omega T_f - 2 \left(\frac{\omega}{\omega_f} \right) \sin \omega_f T_f \sin \omega T_f \right]^{\frac{1}{2}} \quad (3.20)$$

by using standard procedures. However ω_f must satisfy equation (3.17) again. Equations (3.8) and (3.20) when combined with formula (3.3) lead to a complicated integral expression for the estimate of $\text{maximax } g(t)$ which is more easily ascertained numerically.

Formula (3.6) demands Ω_f in addition to E_f and T_f to enable the factor $\lambda_0^{-\frac{1}{2}} \max \psi_0(\Omega_g T_g / 2)$ to be calculated.

Now Ω_f can be approximated by utilizing either experience or the bandwidth of the major lobe of the pulse modulated sinewave's Fourier transform, $4\pi/T_f$ [34]. The latter approximation is adopted here. It is compared with the Ω'_h determined for a given ζ and ω_n from Figure 3.2 in order to estimate Ω_g in equation (2.4). Then T_g is appraised by putting the known T_f and T'_h , which is assessed similarly to Ω'_h but from Figure 3.1, into equation (2.2). Subsequently the product $\Omega_g T_g/2$ is approximated so that, having figured T_g , $\lambda_0^{-1/2} \max \psi_0(\Omega_g T_g/2)$ can be estimated by using Figure 2.2. Finally this factor is substituted along with equation (3.11) and (3.18) into formula (3.6).

The three estimates originating from formula (3.2), (3.3) and (3.6) are non-dimensionalized and compared with the exact solution (3.16) in Figure 3.3 for different ζ and T_f/T_n satisfying (3.17). Numerical values were obtained following the procedures outlined. The integral inherent to formula (3.3) was computed digitally by using a readily available numerical quadrature algorithm which utilizes a cautious Romberg extrapolation to find an acceptable estimate [44]. This technique coped successfully with the integrand's singularity. Integration was terminated somewhat arbitrarily at ω/ω_n of a thousand because a tenfold increase in this limit did not affect the solution's first two digits. Two digit accuracy was considered sufficient for plotting purposes.

Figure 3.3 shows that for this example, formula (3.6) is most accurate followed by formula (3.3) and then (3.2). The exact maximax displacement is independent of T_f/T_n whereas formula (3.6) is slightly, formula (3.3) more so and formula (3.2) most sensitive to T_f/T_n . Inaccuracies always grow at a rate depending upon the sensitivity with increasing T_f/T_n . For a given T_f/T_n , the absolute difference between the exact value and any one of the estimates appears to decrease with increasing ζ .

3.3.2. Mechanical Shocks

Mechanical shock testing typically utilizes half-sine, rectangular or triangular-like pulses [9]. By assuming, often unrealistically, that such shocks do not deviate from a simple idealized shape, the resulting maximum or maximax response can be predicted "exactly" from equation (3.1). Maxima computed for different T_f/T_n and possibly ζ are presented most conveniently in the graphical format called a shock spectrum. Customary growths in the oscillator's maximax displacement created by decreasing ζ are illustrated by the exact spectra for the half-sine pulse shown as dashed curves in Figure 3.4. These curves indicate that the frequent neglect of small ζ to avoid computational complexities leads to normally slightly large maxima. Consequently comparable spectra for the rectangular and triangular pulses should differ little from the ζ

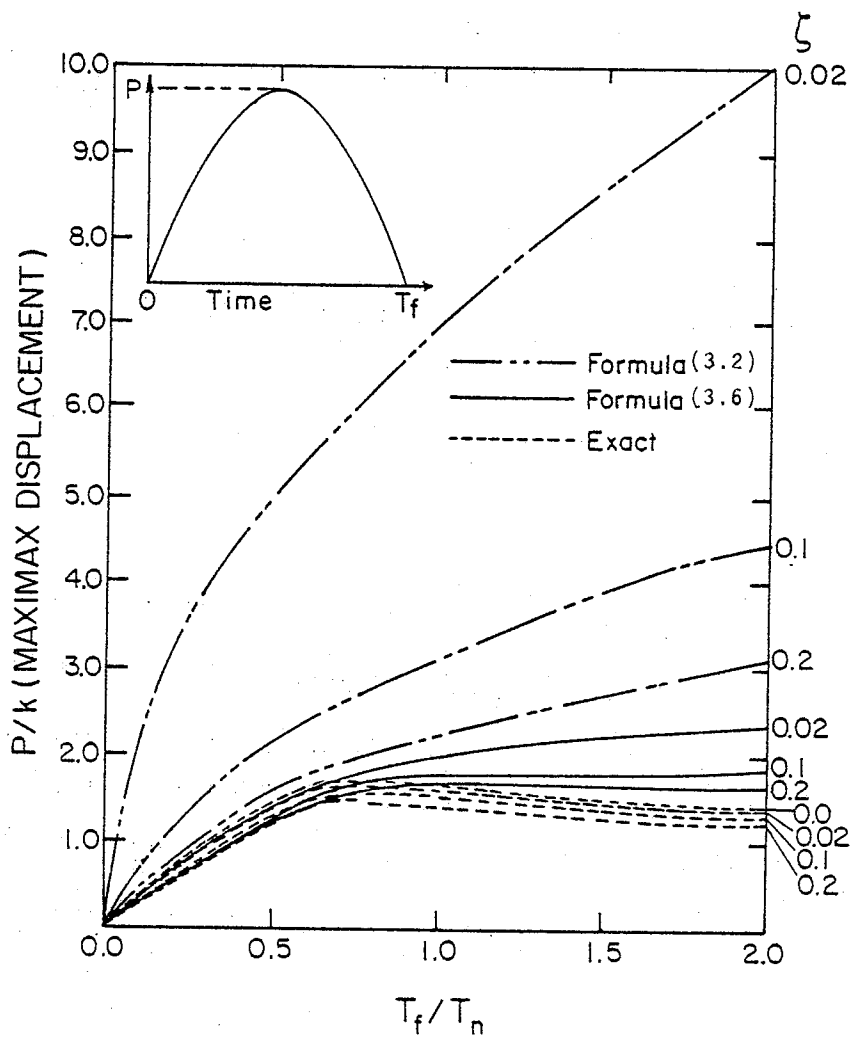


Figure 3.4. Contrasting formula (3.2) and (3.6) with the half sine wave's exact shock spectra.



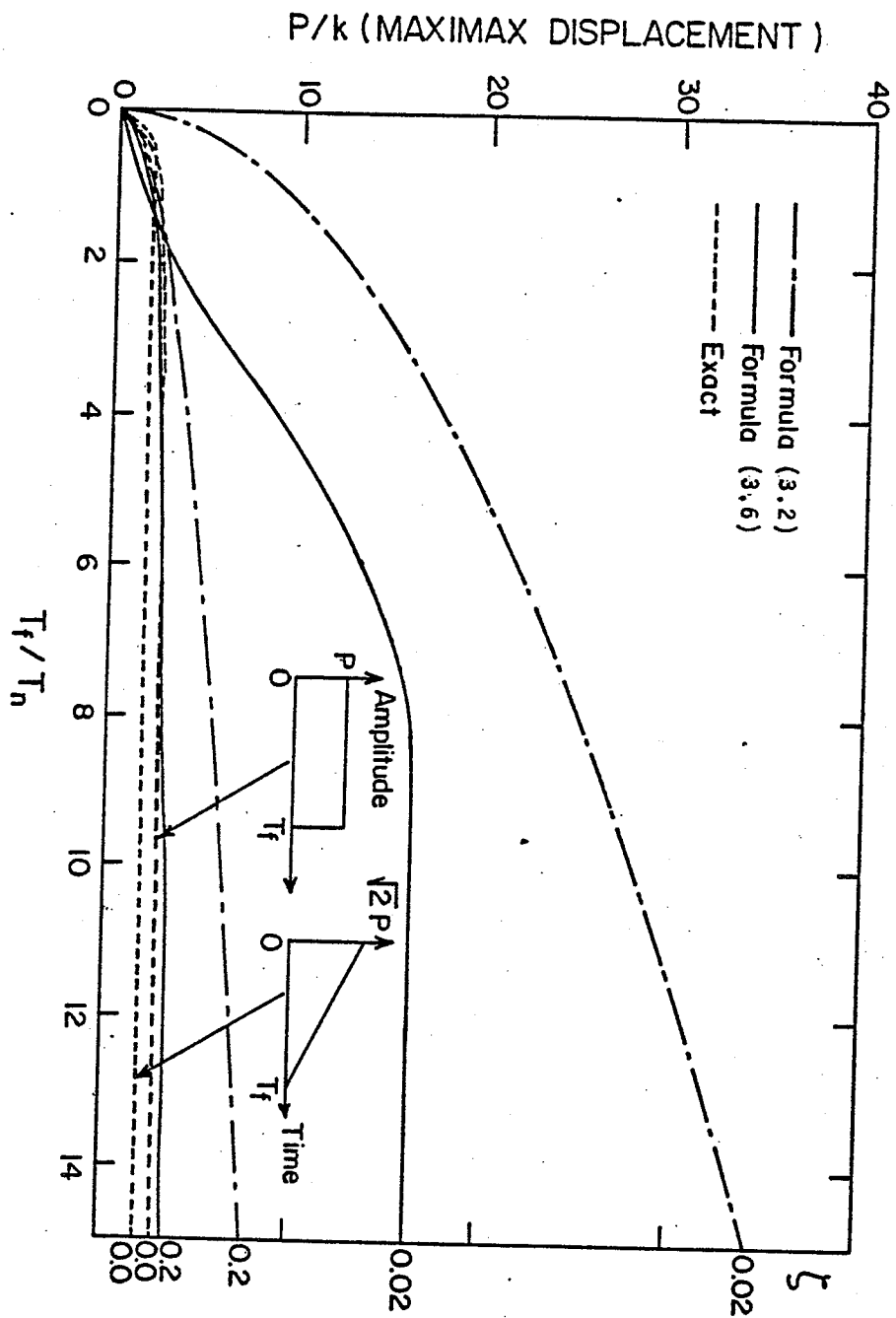


Figure 3.5. Contrasting formula (3.2) and (3.6) with the shock spectra of the rectangular and triangular pulses.

zero curves selectively presented in Figure 3.5 to improve its clarity. The peak amplitude of the triangular pulse in the latter figure is $\sqrt{2}P$ rather than P to make the energies and durations of all three loads identical.

The corresponding estimates from formula (3.2) and (3.6) are also shown in the same figures. Estimates were computed by using identical procedures to those in the previous example but Ω_f equals $2.5/T_f$ rather than $4\pi/T_f$. Both Figure 3.4 and 3.5 demonstrate that again formula (3.6) is invariably more accurate than (3.2). The formulae also behave like before in comparison with the exact spectra for varying ζ and T_f/T_n . However formula (3.6) increases more rapidly at low T_f/T_n but ultimately tends, as previously, to be constant when T_f/T_n is large.

3.3.3. Earthquake Loads

Earthquake loads are often presented in the form of frequency spectra rather than time histories. For example the frequency envelopes

$$|F_1(j\omega)| = \sigma_0 \left(\frac{1 + 4\zeta_g^2 (\omega/\omega_g)^2}{[1 - (\omega/\omega_g)^2]^2 + 4\zeta_g^2 (\omega/\omega_g)^2} \right)^{\frac{1}{2}}, \quad (3.21)$$

$$|F_2(j\omega)| = \sigma_0 \left(\frac{1}{[1 - (\omega/\omega_g)^2]^2 + 4\zeta_g^2 (\omega/\omega_g)^2} \right)^{\frac{1}{2}}, \quad (3.22)$$

and

$$|F_3(j\omega)| = \sigma_0 (\omega/\omega_g)^2 e^{-(\omega/\omega_g)^2} \quad (3.23)$$

have been suggested as reasonable representations of certain trends observed from records of strong-motion earthquakes [26, 45]. Parameters ζ_g and ω_g depend upon the characteristics of a particular earthquake but typical values, 0.5 and 10π rad/sec respectively, are often assumed [26]. Variable σ_0 with dimensions of feet per second is dictated by the total energy of the earthquake at a specific location. Shinozuka [26] used all three load spectra to assess formula (3.3) with (3.2) as a reference. His results, evaluated as the ratio of the estimates from the two formulae, are displayed as the dashed curves in Figure 3.6 through 3.8. Accepting this precedent, similar ratios were derived from formula (3.6).

By dividing formula (3.6) by (3.2) and using equation (3.11) and (3.15), the approximation can be rewritten as

$$\frac{\text{maximax } g(t)}{[E_f E_h]^{1/2}} = [\zeta(1 - \zeta^2) \omega_n]^{-1/2} \cdot \lambda_0^{-1/2} \max \psi_0(\Omega_g T_g/2) \quad (3.24)$$

where $\omega_n = 2\pi/T_n$. Given ζ and ω_g (10π rad/sec), the first term on the righthand side can be evaluated straightforwardly in the range $0 < \omega_n/\omega_g \leq 2$ considered in reference [26]. Shinozuka demonstrated by simply plotting the $|F_i(j\omega)|$ ($i = 1, 2, 3$) and $|H(j\omega)|$ for various ω/ω_g

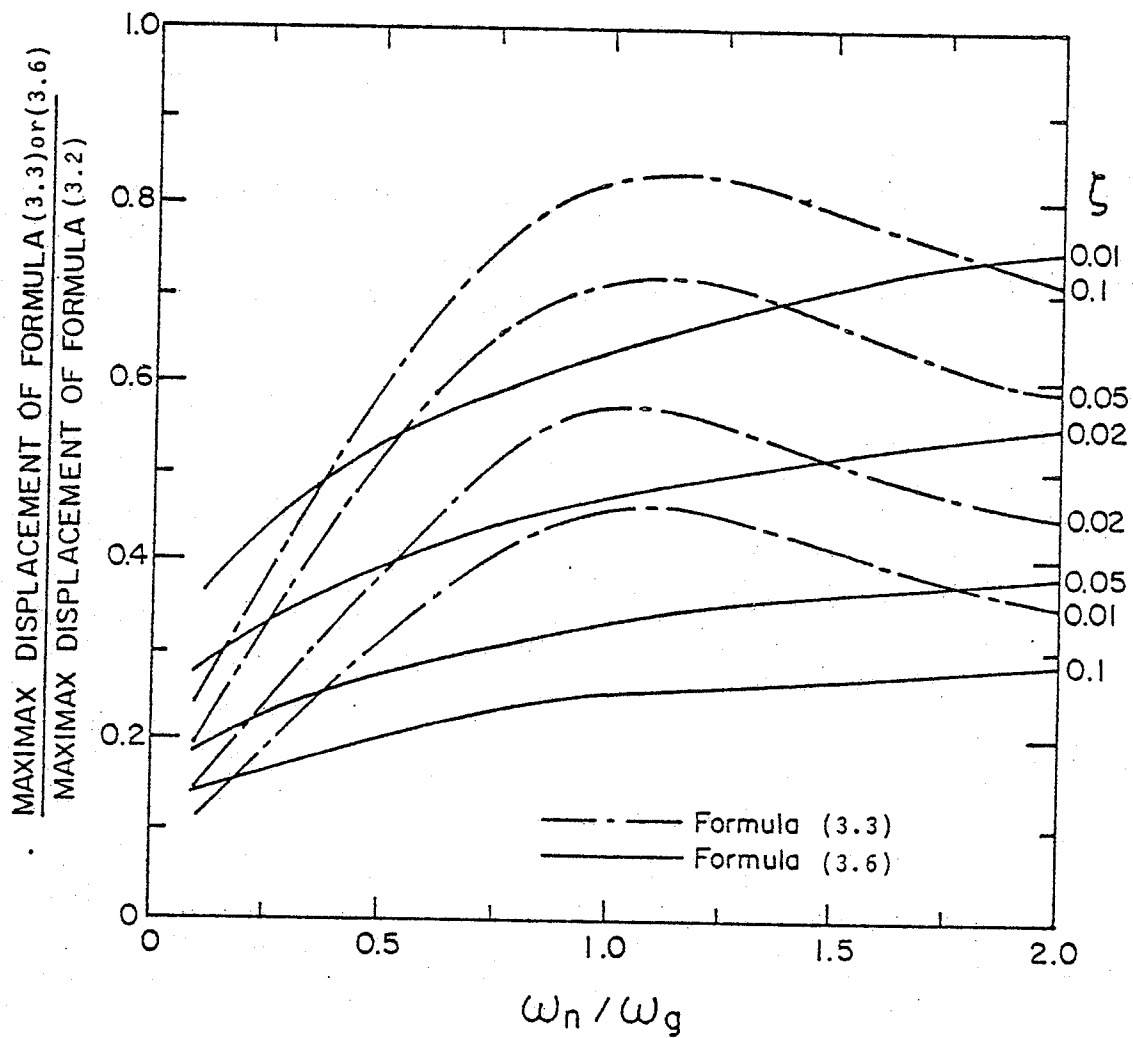


Figure 3.6. Comparison of estimates for the earthquake spectrum given by equation (3.21).

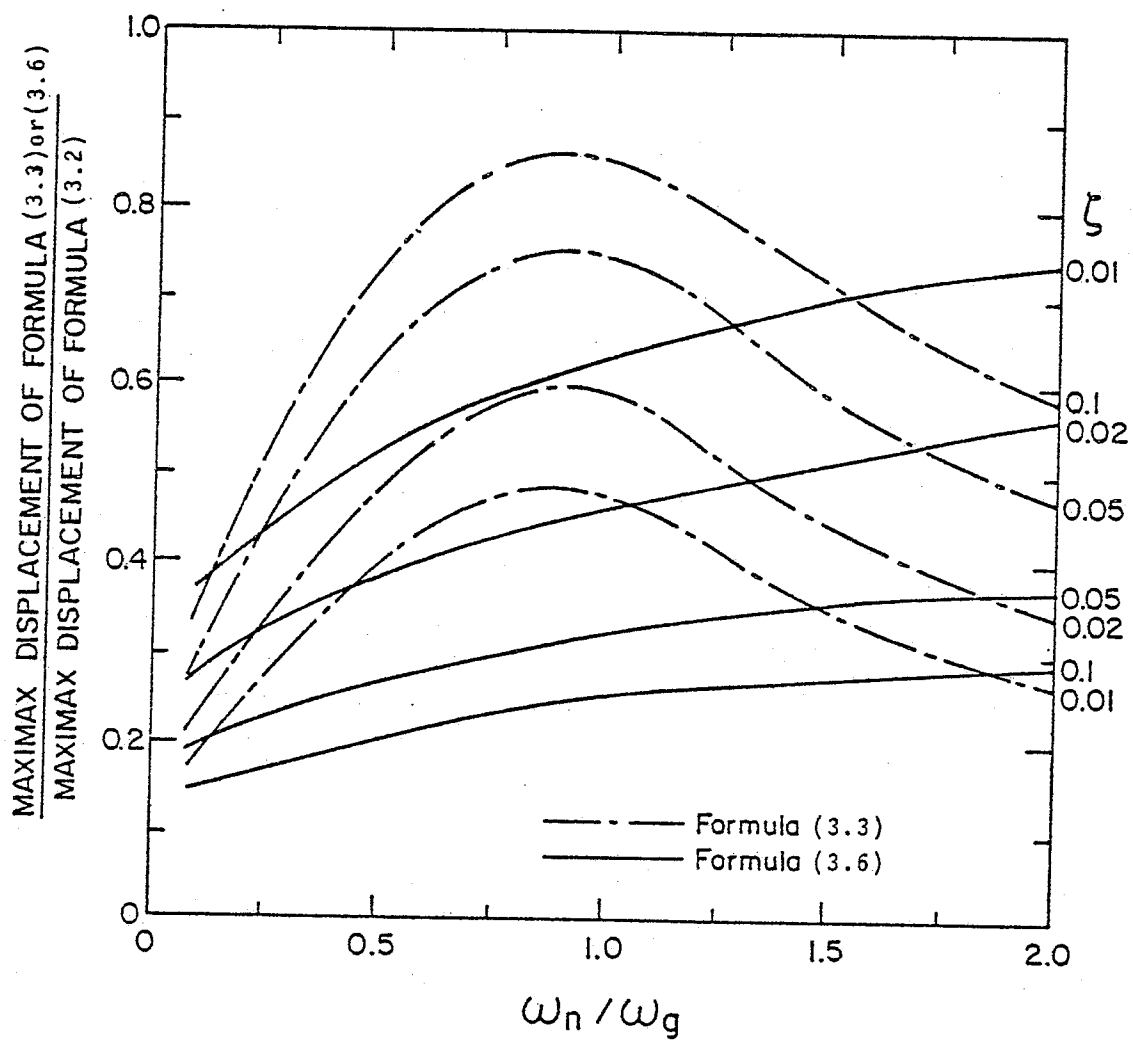


Figure 3.7. Comparison of estimates for the earthquake spectrum given by equation (3.22).

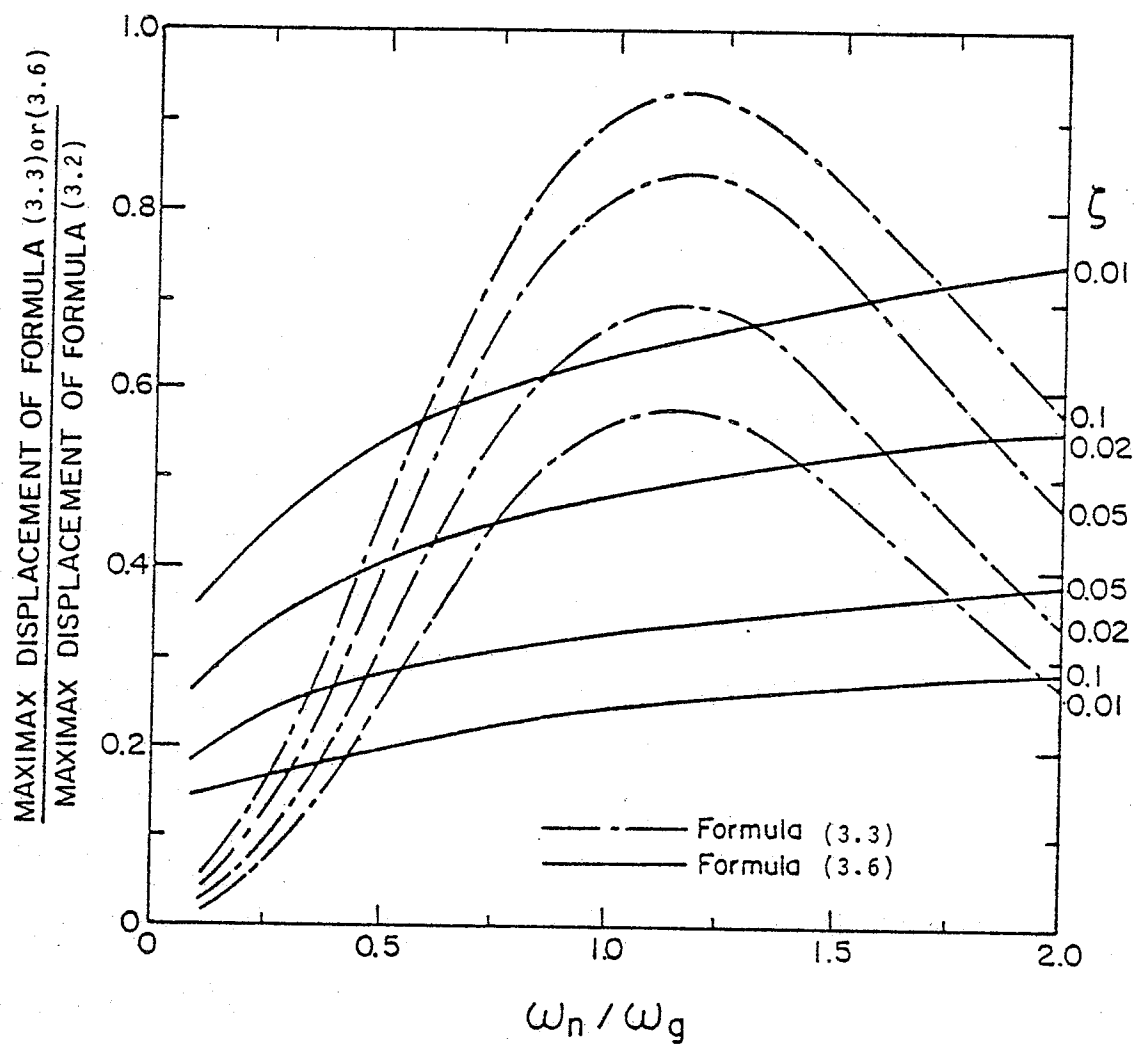


Figure 3.8. Comparison of estimates for earthquake spectrum given by equation (3.23).

that Ω_h is always less than any of the Ω_f when $\omega_n/\omega_g \leq 2$. Consequently equation (2.4) indicates that Ω_g is identical to Ω_h which can be appraised from Figure 3.2. T_g can be approximated as before by employing equation (2.2), T'_h from Figure 3.1 and the duration T_f of 25 seconds for the typical strong-motion earthquake [45,46]. Having evaluated Ω_g and T_g , the second term $\lambda_0^{-1/2} \max \psi_0(\Omega_g T_g/2)$ in formula (3.24) can be evaluated from Figure 2.2. Numerical values obtained by following this procedure are shown as the continuous curves in Figure 3.6 through 3.8.

The figures indicate clearly that all the ratios are invariably less than unity so that formula (3.3) and (3.6) are evidently more accurate than formula (3.2) again. Formula (3.6) is inferior to (3.3) when ζ equals 0.01 but its relative performance improves progressively until generally it is apparently superior for ζ greater than 0.05. This change is particularly noticeable around ω_n/ω_g of one due to the seemingly greater sensitivity of formula (3.3) to ω_n/ω_g .

3.4 Conclusions

Three distinct formulae can be used to predict the maximax response of a given linear structure when it is excited by partially specified loads. The loads are described generally either by their energy and possibly duration or by the

upper envelope of their Fourier transforms. The form of information available for the loads will largely determine the ease of applying any one formula. For example, if only the overall time behaviour is known then the necessary calculation of the Fourier transforms in formula (3.3) would impose a heavy additional penalty. Also, care should be exercised to distinguish between theoretical and experimental applications. The crucial difference is that the response's frequency bandwidth, Ω_g , will be dictated by the characteristics of the measuring instrumentation in experiments but not in theory. These characteristics are chosen usually by following informally the guidelines developed in Section 2.7 but further comparative trials may be undertaken for finer "tuning". If Ω_g is thus ascertained, the computational complexities associated with the theoretical evaluation of formula (3.6) and, to a minor extent, formula (3.3) will be reduced. This potential improvement has been ignored.

The formulae are ranked purely on their theoretical accuracies for several transient loads occurring typically in practice. Formula (3.2) is certainly inferior on this basis. However the relative rating of the other two formulae is not so clear. Formula (3.6) is more accurate than formula (3.3) for the pulse modulated sinewave but the balance apparently tips for low structural damping likely in earthquake designs.

CHAPTER 4

THE MAXIMAX RESPONSE OF DISCRETE MULTI DEGREE-OF-FREEDOM
SYSTEMS4.1 Introduction

A technique has been developed in Chapter 2 to estimate directly the maximax response of a linear, stable oscillator to all possible loads having the same energy or mean square value. Comparisons with two more established procedures for a fair range of realistic loads, in Chapter 3, suggest that the technique produces accuracies better than one and comparable, at worst, to the other. However the representation of complex structures by a simple oscillator is somewhat limited so that the technique will be extended to discrete systems having many degrees-of-freedom. Specific examples chosen to illustrate the application and accuracy of the technique have theoretical solutions which assume a detailed knowledge of the loads. Analogous solutions cannot be determined explicitly when, as often occurs in practice, these details cannot be specified completely. Then reasonable estimates of the largest feasible (maximax) response become increasingly valuable in assessing the "worst" design case [28].

4.2 The Maximax and Maximax Mean Square of the Response

Formulae have been developed in Chapter 2 for estimating the maximax response characteristics of single degree-of-freedom, structural systems to incompletely described loads. The theory will be extended to more realistic systems having many degrees-of-freedom. Structures are still linear, causal and stable but they are idealized now by n separate masses coupled by weightless springs and viscous dashpots as in Figure 4.1. The typical p th mass, at rest initially, is loaded at time $t = 0$ by $f_p(t)$ which can be specified only grossly by either its energy

$$E_{f_p}^+ = \lim_{T \rightarrow \infty} \int_0^T f_p^2(t) dt \quad (4.1)$$

and duration T_f , if finite, or the mean square value

$$\langle f_p^2(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_p^2(t) dt. \quad (4.2)$$

Application of the above formulae requires E_{f_p} or $\langle f_p^2(t) \rangle$ to be bounded. Similar limitations apply to the loads acting simultaneously on the other masses. Given this partial information and the system's complete properties, the object is to

[†] Subscript f will denote a variable associated with a load to be consistent with the previous notation. Similarly g will refer to a response and h to an impulse response function. A single sub-subscript indicates the particular component under consideration.

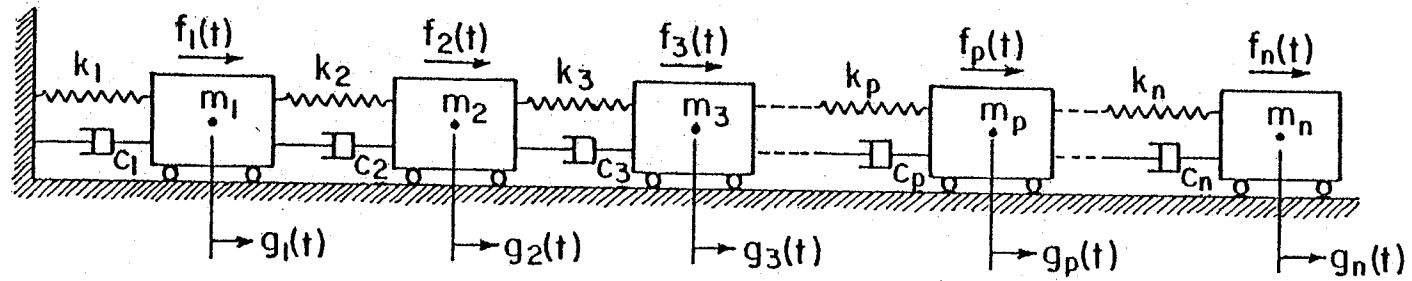


Figure 4.1. A discrete multi degree-of-freedom system.

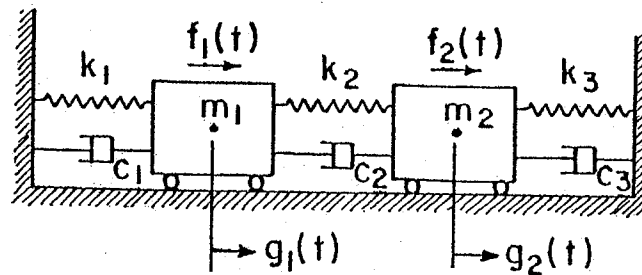


Figure 4.2. A two degree-of-freedom system.

estimate accurately the maximax or maximax mean square response of each mass without determining their entire time histories. This objective can be accomplished by first taking advantage of the system's linearity to determine the energies or the mean squares of the responses by using the principle of superposition. Then the greatest maximum energy or mean square of each response is assessed for *all* loads which have the same energy or mean square as a corresponding given load. This procedure is repeated for all the known loads so that the overall maximax energy or maximax mean square can be found for the response of each mass. Finally each response's maximax value can be ascertained from its maximax energy in an identical fashion to that detailed in Chapter 2.

The total response of the q th mass, say, to the n applied loads can be written as

$$g_q(t) = \sum_{p=1}^n g_{qp}(t) \quad q = 1, 2, 3, \dots, n \quad (4.3)$$

by employing superposition. Typically $g_{qp}(t)$ is the response of the q th mass to the load $f_p(t)$ applied independently at the p th mass. Of course $g_{qp}(t)$ and, hence, $g_q(t)$ can be enumerated straightforwardly only for the trivial case of $f_p(t) \equiv 0$ because otherwise the loads are not specified fully. However a tractable expression for the energy E_{g_q} corresponding to $g_q(t)$ can be developed when the energies of the loads are known. Now

$$E_{g_q} = \lim_{T \rightarrow \infty} \int_0^T g_q^2(t) dt = \lim_{T \rightarrow \infty} \int_0^T \left(\sum_{p=1}^n g_{qp}(t) \right)^2 dt, \quad q = 1, 2, \dots, n \quad (4.4)$$

by using an analogous definition to (4.1) and equation (4.3).

By manipulating terms

$$E_{g_q} = \sum_{p=1}^n \left(\lim_{T \rightarrow \infty} \int_0^T g_{qp}^2(t) dt \right) + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n \left(\lim_{T \rightarrow \infty} \int_0^T g_{qi}(t) \cdot g_{qj}(t) dt \right) \quad q = 1, 2, 3, \dots, n$$

or

$$E_{g_q} = \sum_{p=1}^n E_{g_{qp}} + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n \left(\lim_{T \rightarrow \infty} \int_0^T g_{qi}(t) g_{qj}(t) dt \right), \quad (4.5)$$

$q = 1, 2, 3, \dots, n.$

Schwartz's inequality [34]

$$\begin{aligned} \lim_{T \rightarrow \infty} \int_0^T g_{qi}(t) g_{qj}(t) dt &\leq \left(\lim_{T \rightarrow \infty} \int_0^T g_{qi}^2(t) dt \cdot \lim_{T \rightarrow \infty} \int_0^T g_{qj}^2(t) dt \right)^{\frac{1}{2}} \\ &= (E_{g_{qi}} E_{g_{qj}})^{\frac{1}{2}} \end{aligned}$$

has to be applied generally to formulate the extreme right of equation (4.5) in terms of appropriate energies. Then

$$E_{g_q} \leq \sum_{p=1}^n E_{g_{qp}} + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n (E_{g_{qi}} \cdot E_{g_{qj}})^{\frac{1}{2}}, \quad q = 1, 2, 3, \dots, n. \quad (4.6)$$

The equality holds only in the rare case when $g_{qi}(t)$ is proportional to $g_{qj}(t)$ so that inequality (4.6) will

normally produce an upper bound for E_{g_q} . When the loads are specified by their mean squares rather than their energies, the mean square response of the q th mass can be derived similarly to be

$$\langle g_q^2(t) \rangle \leq \sum_{p=1}^n \langle g_{qp}^2(t) \rangle + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n [\langle g_{qi}^2(t) \rangle \cdot \langle g_{qj}^2(t) \rangle]^{\frac{1}{2}}, \quad q = 1, 2, \dots, n. \quad (4.7)$$

Relation (4.6) certainly shows that E_{g_q} can never be greater than its right-hand side. Consequently

$$\begin{aligned} \text{maximax } E_{g_q}^+ &\leq \sum_{p=1}^n E_{g_{qp}} + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n (E_{g_{qi}} \cdot E_{g_{qj}})^{\frac{1}{2}} \leq \sum_{p=1}^n \text{maximax } E_{g_p} \\ &+ \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n (\text{maximax } E_{g_{qi}} \cdot \text{maximax } E_{g_{qj}})^{\frac{1}{2}}, \quad q = 1, 2, \dots, n. \quad (4.8) \end{aligned}$$

The representative term $\text{maximax } E_{g_{qp}}$ corresponds to the maximum energy of the response of the q th mass produced independently by *any* load which, regardless of its history, has the same energy as $f_p(t)$. By analogy to the single oscillator in Chapter 2 where^{††}

$$\text{maximax } E_g = \max |H(j\omega)|^2 E_f \quad (4.9)$$

[†] $\text{Maximax } E_{g_q}$ could be written here as $E_{g_{q_0}}$ for consistency with the notation of Chapter 2. The present convention however is simpler and less confusing.

^{††} The sub-subscript 1 is eliminated to be consistent with the notation of the single oscillator in Chapters 2 and 3.

it may be shown similarly that

$$\text{maximax } E_{g_{qp}} = \max |H_{qp}(j\omega)|^2 E_{f_p} . \quad (4.10)$$

$H_{qp}(j\omega)$, the generalization of the standard frequency response function $H(j\omega)$, is the ratio of the steady-state displacement of mass q to unit sinusoidal loading of mass p . More detailed descriptions of these functions together with their relationships to the corresponding impulse response functions, $h_{qp}(t)$ and $h(t)$, are given by Lin [15]. Henceforth it is assumed implicitly that of all the response variables only the displacements are of interest. If velocities or accelerations are preferred then the major change involves the replacement of $\max |H_{qp}(j\omega)|^2$ in formula (4.10) by $\max (\omega^2 |H_{qp}(j\omega)|^2)$ or $\max (\omega^4 |H_{qp}(j\omega)|^2)$, respectively. Substituting formulae like (4.10) into equality (4.8) leads to

$$\begin{aligned} \text{maximax } E_{g_q} \leq & \sum_{p=1}^n \max |H_{qp}(j\omega)|^2 E_{f_p} \\ & + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n (\max |H_{qi}(j\omega)|^2 \max |H_{qj}(j\omega)|^2 E_{f_i} E_{f_j})^{\frac{1}{2}}, \quad q = 1, 2, \dots, n \end{aligned} \quad (4.11)$$

for the displacement energy of mass q . Similarly

$$\begin{aligned} \text{maximax } \langle g_q^2(t) \rangle \leq & \sum_{p=1}^n \max |H_{qp}(j\omega)|^2 \cdot \langle f_p^2(t) \rangle + \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n (\max |H_{qi}(j\omega)|^2 \cdot \\ & \max |H_{qj}(j\omega)|^2 \cdot \langle f_i^2(t) \rangle \cdot \langle f_j^2(t) \rangle)^{\frac{1}{2}} \quad q = 1, 2, \dots, n \end{aligned} \quad (4.12)$$

when the loads are given by their mean squares. Finally the maximax displacement of the q th mass can be estimated by employing inequality (4.11) in conjunction with

$$\begin{aligned} \text{maximax } g_q(t) &= (\text{maximax } E_{g_q})^{\frac{1}{2}} \lambda_{o_q}^{-\frac{1}{2}} \max \psi_{o_q}(\Omega_{g_q} T_{g_q}/2), \\ q &= 1, 2, \dots, n \end{aligned} \quad (4.13)$$

generalized straightforwardly from the analogous approximation for the single oscillator in Chapter 2. Both λ_{o_q} and $\max \psi_{o_q}(\Omega_{g_q} T_{g_q}/2)$ are functions only of the product of the frequency and time bandwidth, $\Omega_{g_q} T_{g_q}$, of the displacement of the q th mass. Their behaviour was examined in reference 36 and 38 and it is shown without subscript q in the normalized form $[T_g/2 \lambda_o]^{\frac{1}{2}} \max \psi_o(\Omega_g T_g/2)$ presented in Figure 2.2.

The major task in applying any derivative of formula (4.13) is the determination of Ω_{g_q} and T_{g_q} from the *a priori* knowledge. A similar difficulty was encountered for the oscillator. Approximations to the displacement's frequency and duration bandwidths, Ω_g and T_g , were based there upon

$$\Omega_g = \min(\Omega_f, \Omega_g) \quad (4.14)$$

$$T_g = T_f + T_h \quad (4.15)$$

and gave reasonable accuracy as illustrated in Chapter 3. The frequency and time bandwidths of the oscillator, Ω_h and T_h , were found from the decay of $|H(j\omega)|$ and $h(t)$. On the other hand, the less certain effective bandwidth of the

loads, Ω_f , had to be gauged from experience or the characteristics of the measuring instrumentation. The duration of the loads, T_f , was assumed again to be known. Extension of these formulae to multi degree-of-freedom systems is, in general, not immediately apparent and requires further investigation. The situation becomes tractable however in those special cases where the character of the responses is obviously dictated by the loads. One such example is the pulse modulated sinewave which is important in the identification of systems [43]. The duration of the pulse, T_f , is typically so long that the system's natural periods are overshadowed to attain the steady-state response. Then the frequency bandwidth of the loads, described fairly in Chapter 3 by $4\pi/T_f$, becomes so narrow that the system's frequency bandwidth is relatively unimportant. This type of load will be used to examine the potential usefulness of formula (4.13) combined with (4.11). On the other hand, formula (4.12) is assessed more appropriately by considering white noise excitations whose mean square values are more likely than their energies to be finite.

Formulae will be applied to the relatively simple two degree-of-freedom system of Figure 4.2 for which exact solutions are more readily available. The particular example of growing practical importance, the constant tuned vibration neutralizer, will be investigated in greater detail to determine its optimum parameters.

4.3 Application to a Two Degree-of-Freedom System

The typical term $\max |H_{qp}(j\omega)|^2$ is common to formula (4.11) and (4.12). It is found by formulating the equations of motion for the system in Figure 4.2, determining $H_{qp}(j\omega)$ analytically and then evaluating appropriate maxima numerically. Equations of motion can be written by using Lagrange's equation as

$$\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2(\zeta_{11}\omega_{11} + \mu\zeta_{22}\omega_{22}) & -2\mu\zeta_{22}\omega_{22} \\ -2\zeta_{22}\omega_{22} & 2(\zeta_{22}\omega_{22} + \zeta_{23}\omega_{23}) \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} \omega_{11}^2 + \mu\omega_{22}^2 & -\mu\omega_{22}^2 \\ -\omega_{22}^2 & \omega_{22}^2 + \omega_{23}^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \frac{f_1(t)}{m_1} \\ \frac{f_2(t)}{m_2} \end{Bmatrix} \quad (4.16)$$

where a dot denotes differentiation with respect to time, t .

The non-dimensional parameters

$$\mu = m_2/m_1, \quad \omega_{11}^2 = \frac{k_1}{m_1}, \quad \omega_{22}^2 = \frac{k_2}{m_2}, \quad \omega_{23}^2 = \frac{k_3}{m_2}$$

and

$$\zeta_{11} = \frac{c_1}{2\sqrt{k_1 m_1}}, \quad \zeta_{22} = \frac{c_2}{2\sqrt{k_2 m_2}}, \quad \zeta_{23} = \frac{c_3}{2\sqrt{k_3 m_2}} \quad (4.17)$$

are defined conventionally. Equation (4.16) can be re-organized more conveniently in terms of the two first order,

ordinary differential equations

$$\{\dot{y}(t)\} + [C]\{y(t)\} + [K]\{g(t)\} = \{f(t)\} \quad (4.18)$$

and

$$\begin{Bmatrix} \dot{g}_1 \\ \dot{g}_2 \end{Bmatrix} \equiv \{\dot{g}(t)\} = \{y(t)\} . \quad (4.19)$$

The matrices of equation (4.16) and (4.18) essentially correspond one to one from left to right. Time variations of the variables in equation (4.18) and (4.19) have been inserted explicitly for emphasis. The last two equations can be rewritten in terms of the one matrix equation

$$\begin{Bmatrix} \dot{y}(t) \\ \dot{g}(t) \end{Bmatrix} = \begin{bmatrix} -[C] & -[K] \\ [I] & [0] \end{bmatrix} \begin{Bmatrix} \{y(t)\} \\ \{g(t)\} \end{Bmatrix} + \begin{Bmatrix} \{f(t)\} \\ \{0\} \end{Bmatrix} \quad (4.20)$$

where $[I]$ and $[0]$ are identity and null matrices, respectively. This last equation can be put in an abbreviated first rather than the original second order differential equation form as

$$\{\dot{X}(t)\} = [A]\{X(t)\} + \{F(t)\} \quad (4.21)$$

where $\{X(t)\} \equiv \begin{Bmatrix} y(t) \\ g(t) \end{Bmatrix}$ and the other matrices correspond again from left to right. The Laplace transform of equation (4.21) is simply

$$s\{X(s)\} = [A]\{X(s)\} + \{F(s)\} \quad (4.22)$$

because the system is at rest initially. Now

$$\{X(s)\} = [H(s)]\{F(s)\} \quad (4.23)$$

and $[H(s)]$ is the frequency response matrix with typical element $H_{qp}(s)$ when s equals $j\omega$. Combining equation (4.22) with (4.23) gives

$$H(s) = (s[I] - [A])^{-1}, \quad s = j\omega \quad (4.24)$$

providing the inverse exists. By substituting the components of $[A]$ in equation (4.24) appropriate to the two degree-of-freedom system, the sub-matrix of $[H(s = j\omega)]$ associated only with the displacement vector $\{g(t)\}$ can be shown to have elements

$$H_{11}(j\omega) = (\omega_{22}^2 + \omega_{23}^2 + 2j\omega(\zeta_{22}\omega_{22} + \zeta_{23}\omega_{23}) - \omega^2)/\Delta$$

$$H_{12}(j\omega) = \frac{1}{\mu} H_{21}(j\omega) = -(2j\zeta_{22}\omega_{22}\omega + \omega_{22}^2)/\Delta$$

$$H_{22}(j\omega) = (\omega_{11}^2 + \mu\omega_{22}^2 + 2j\omega(\zeta_{11}\omega_{11} + \mu\zeta_{22}\omega_{22}) - \omega^2)/\Delta$$

where

$$\Delta = (H_{11}(j\omega) H_{22}(j\omega) + \frac{1}{\mu} H_{12}^2(j\omega))^{-1}. \quad (4.25)$$

$\text{Max } |H_{qp}(j\omega)|^2$ can be determined from the largest value obtained by substituting the real roots of the equations stemming from

$$\frac{\partial}{\partial \omega} |H_{qp}(j\omega)|^2 \equiv \frac{\partial}{\partial \omega} (H_{qp}(j\omega) H_{qp}^*(j\omega)) = 0^\dagger \quad (4.26)$$

[†] A star superscript denotes the complex conjugate.

into $|H_{qp}(j\omega)|^2$, q and $p = 1, 2$. Analytical expressions derived by combining equation (4.25) and (4.26) are given in Appendix IV. Mathematical complexity was avoided by computing the roots of the resulting equations digitally by using Laguerre's method [47]. Numerical values of the roots were checked against the analytical formula for zero damping throughout and were found accurate to more than eight significant figures. The $\max |H_{qp}(j\omega)|^2$, q and $p = 1, 2$, were determined by inspection.

If the force in the coupling spring, $k_2(g_2(t) - g_1(t))$, is of interest then a different frequency response matrix $[H'(s = j\omega)]$ can be formulated by replacing $g_2(t)$ with $(g_2(t) - g_1(t))$. It may be demonstrated that the appropriate elements of $[H'(s = j\omega)]$ are

$$H'_{11}(j\omega) = H_{11}(j\omega), \quad H'_{12}(j\omega) = H_{12}(j\omega)$$

and

$$H'_{21}(j\omega) = H_{21}(j\omega) - H_{11}(j\omega)$$

$$H'_{22}(j\omega) = H_{22}(j\omega) - H_{12}(j\omega). \quad (4.27)$$

The same procedures as before can be adopted to enumerate the differing quantities $\max |H'_{2p}(j\omega)|^2$, $p = 1, 2$.

Selected illustrations of $|H_{qp}(j\omega)|$ and $|H'_{qp}(j\omega)|$ are given in Figures 4.3 and 4.4. More comprehensive examples of the maxima of their squares, normalized with respect to

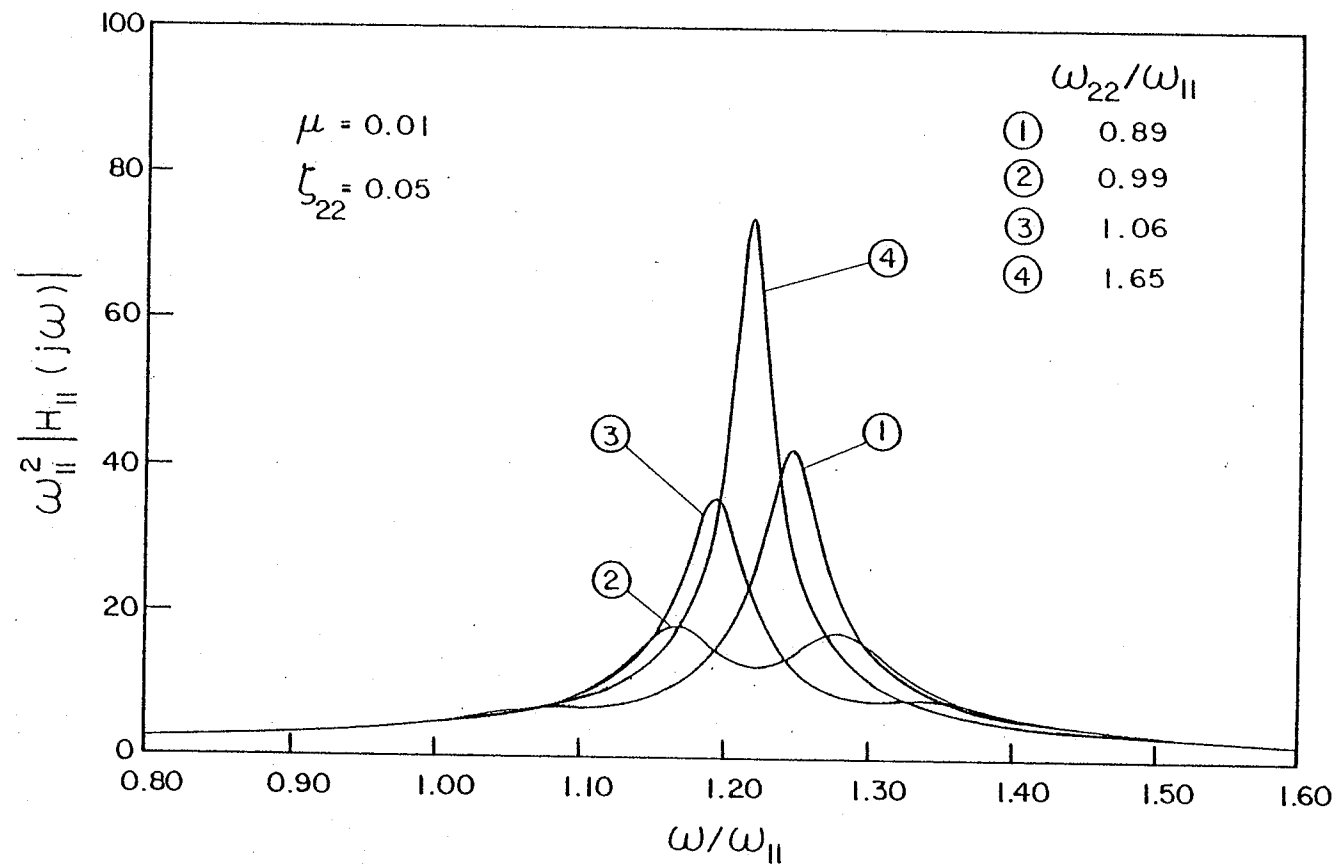


Figure 4.3. Plots of $\omega_{11}^2 |H_{11}(j\omega)|$ against ω/ω_{11} for various ratios of ω_{22}/ω_{11} .

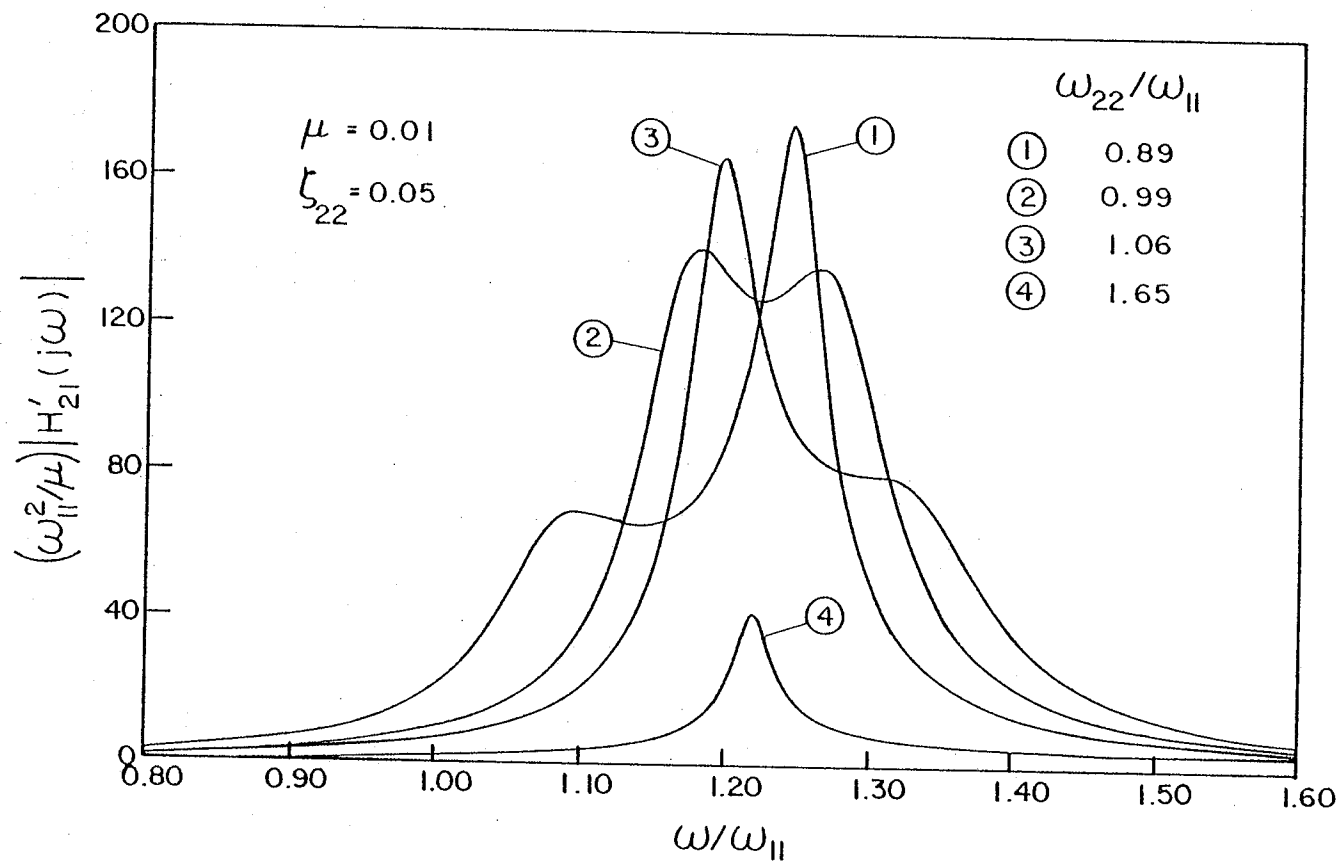


Figure 4.4. Plots of $(\omega_{11}^2/\mu)|H'_{21}(j\omega)|$ against ω/ω_{11} for various ratios of ω_{22}/ω_{11} .

$\max |H(j\omega)|^2$, are shown in Figure 4.5 through 4.8. The $H(j\omega)$ is simply the frequency response function of the single oscillator so that[†]

$$\max |H(j\omega)|^2 = \frac{1}{4 \omega_{11}^2 \zeta_{11}^2 (1 - \zeta_{11}^2)} \quad (4.28)$$

because ζ_{11} is less than $1/\sqrt{2}$ and μ is zero. Invariant parameters $\zeta_{11} = 0.01$, $\omega_{11}^2 = 1.51 \text{ (rad/sec)}^2$ and $\zeta_{23} = 0 = \omega_{23}$ not indicated in these figures coincide with the typical values used in reference [48]. Numerical values of the remaining parameters, μ , ω_{22} and ζ_{22} , are as presented.

Having determined the $\max |H_{qp}(j\omega)|^2$ or $\max |H'_{qp}(j\omega)|^2$ and knowing the E_{f_p} or $\langle f_p^2(t) \rangle$, formula (4.11) or, as applicable, (4.12) can be evaluated straightforwardly. The enumeration of formula (4.13) in association with (4.11) can be more onerous. Typical applications are presented in order to weigh the various formulae's estimates against exact solutions.

4.3.1. Maximax Mean Square Displacements to White Noise

Two degree-of-freedom systems have been employed recently to represent the action of an interior vibration neutralizer on the vibrations of buildings exposed to wind

[†]Here $H(j\omega) = G(j\omega)/(F(j\omega)/m_1)$ to be consistent with the conventional multi degree-of-freedom notation. However this $H(j\omega)$ is different from the $H(j\omega)$ defined in Chapter 2 as $G(j\omega)/F(j\omega)$. Therefore the present value of $\max |H(j\omega)|^2$ is multiplied by m_1 .

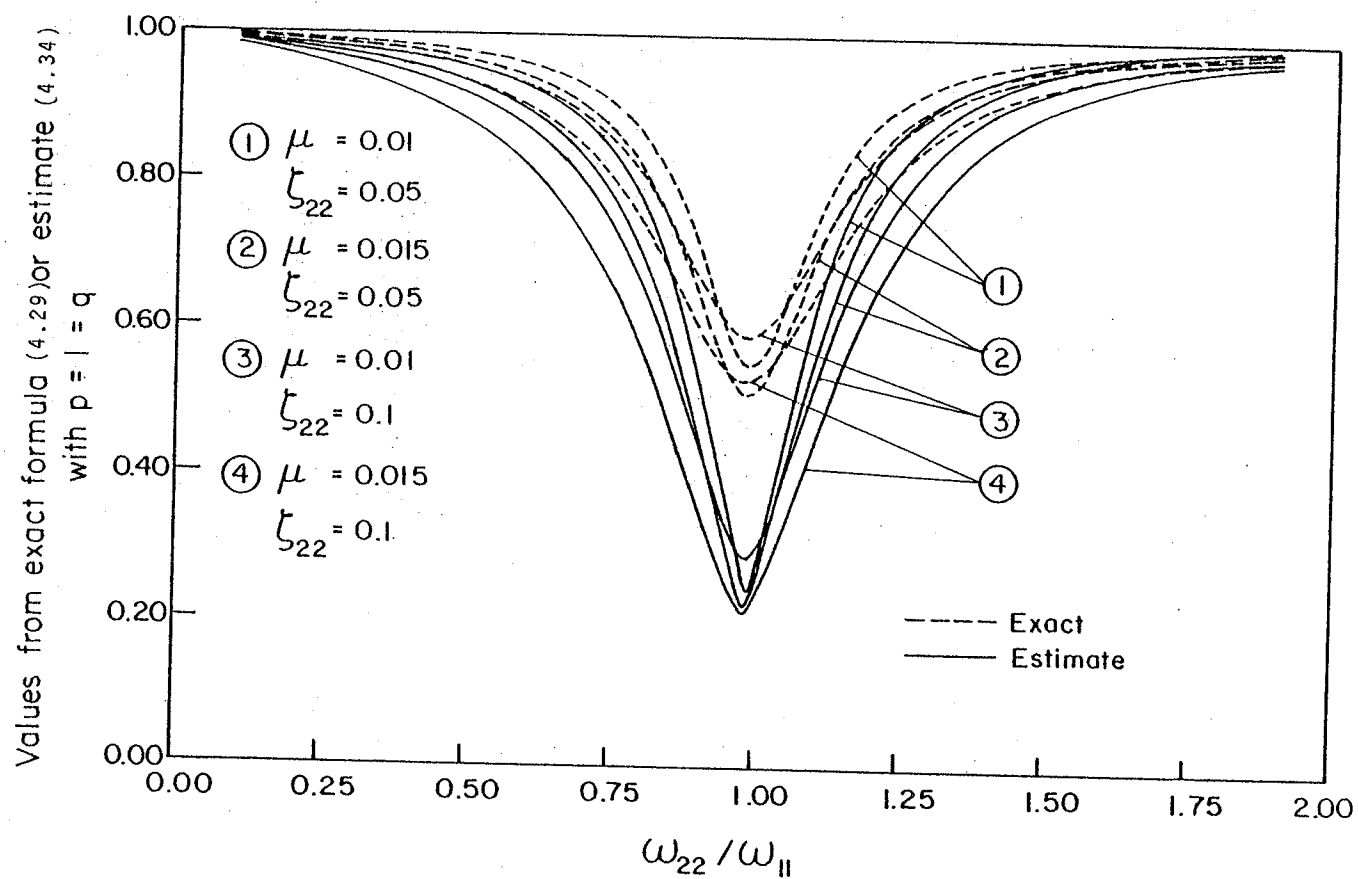


Figure 4.5. Comparison of exact $[\langle g_{11}^2(t) \rangle / \langle g^2(t) \rangle]^{1/2}$ and estimates.

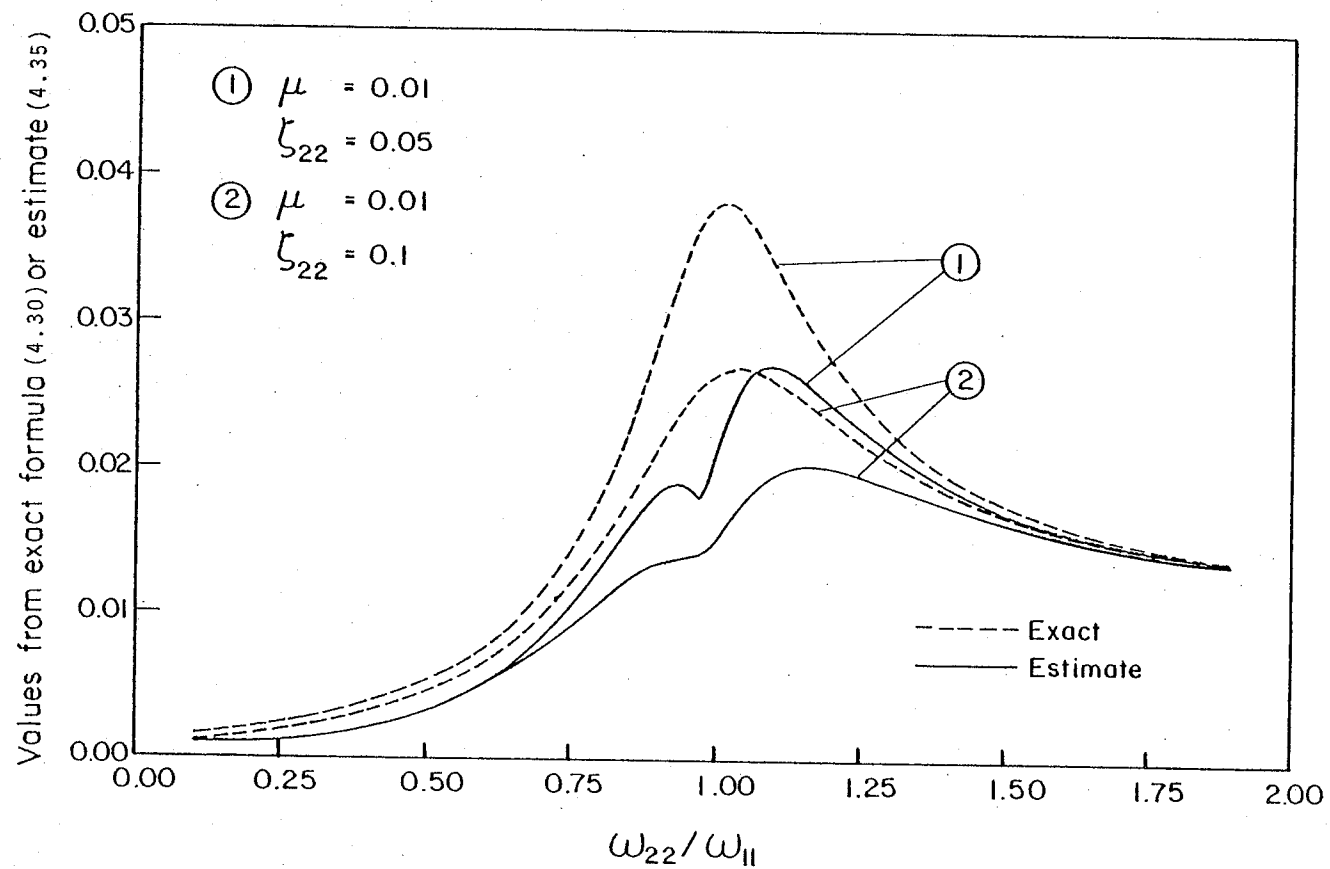


Figure 4.6. Comparison of exact values and estimates of $[\langle [g_{21}(t) - g_{11}(t)]^2 \rangle / \langle g^2(t) \rangle]^{1/2}$.

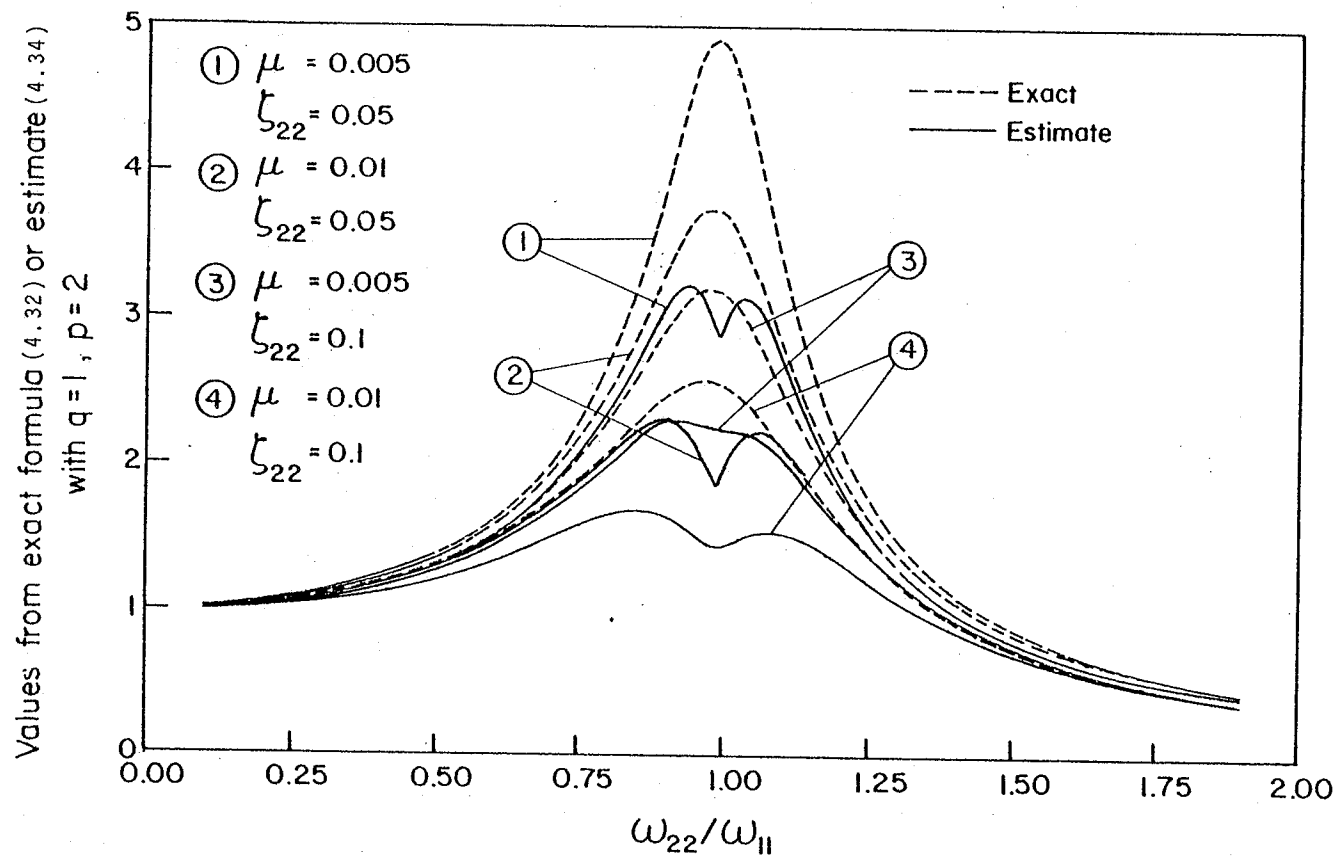


Figure 4.7. Comparison of exact $[\langle g_{12}^2(t) \rangle / \langle g^2(t) \rangle]^{1/2}$ and estimates.

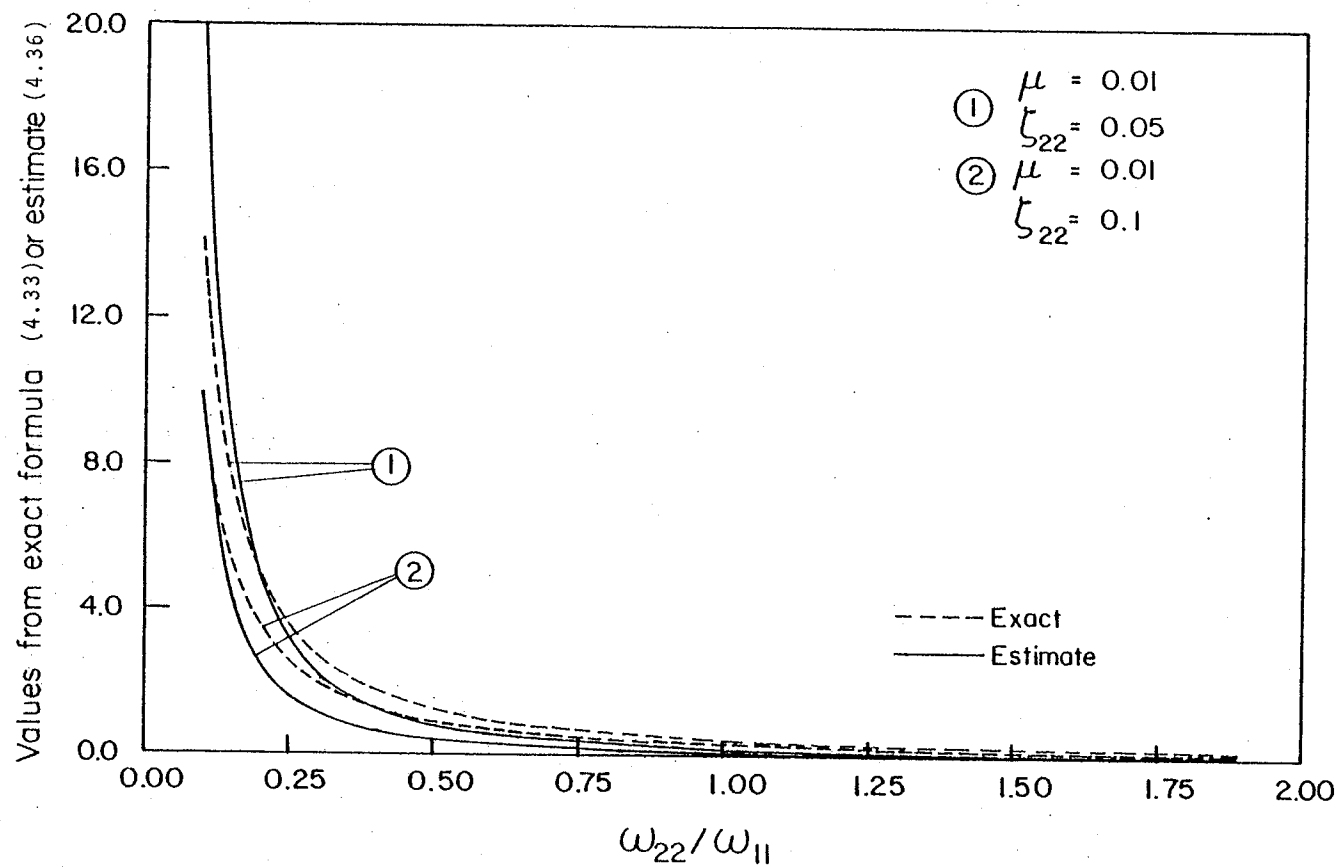


Figure 4.8. Comparison of exact $[\langle g_{22}(t) - g_{12}(t) \rangle^2 / \langle g^2(t) \rangle]^{1/2}$ and estimates.

loads [48]. The wind loads are idealized typically as a white noise, stationary excitation with constant spectral density. They only act directly on the building or mass 1 in Figure 4.2. The neutralizer's effectiveness is appraised usually from the exact result

$$\left(\frac{\langle g_{11}^2(t) \rangle}{\langle g^2(t) \rangle} \right)^{\frac{1}{2}} = \lim_{\Omega \rightarrow \infty} \left(\frac{\int_{-\Omega}^{\Omega} |H_{11}(j\omega)|^2 d\omega}{\int_{-\Omega}^{\Omega} |H(j\omega)|^2 d\omega} \right)^{\frac{1}{2}} \quad (4.29)$$

and the true mean square force in the coupling spring is deduced from

$$\left(\frac{\langle [g_{21}(t) - g_{11}(t)]^2 \rangle}{\langle g^2(t) \rangle} \right)^{\frac{1}{2}} = \lim_{\Omega \rightarrow \infty} \left(\frac{\int_{-\Omega}^{\Omega} |H'_{21}(j\omega)|^2 d\omega}{\int_{-\Omega}^{\Omega} |H(j\omega)|^2 d\omega} \right)^{\frac{1}{2}} \quad (4.30)$$

Denominator $\langle g^2(t) \rangle$ corresponds to the mean square displacement of mass 1 without the neutralizer. It is given by the analytical expression [14]

$$\lim_{\Omega \rightarrow \infty} \int_{-\Omega}^{\Omega} |H(j\omega)|^2 d\omega = \frac{\pi}{2 \zeta_{11} \omega_{11}^3} \quad (4.31)$$

Exact ratios evaluated from equation (4.29) and (4.30) by McNamara [48] are shown as the dashed curves in Figure 4.5 and 4.6. A similar procedure can be followed if internal

random disturbances are considered independently. Then corresponding to equation (4.29) and (4.30)

$$\left(\frac{\langle g_{q2}^2(t) \rangle}{\langle g^2(t) \rangle} \right)^{\frac{1}{2}} = \lim_{\Omega \rightarrow \infty} \left(\frac{\int_{-\Omega}^{\Omega} |H_{q2}(j\omega)|^2 d\omega}{\int_{-\Omega}^{\Omega} |H(j\omega)|^2 d\omega} \right)^{\frac{1}{2}}, \quad q = 1, 2. \quad (4.32)$$

and

$$\left(\frac{\langle [g_{22}(t) - g_2(t)]^2 \rangle}{\langle g^2(t) \rangle} \right)^{\frac{1}{2}} = \lim_{\Omega \rightarrow \infty} \left(\frac{\int_{-\Omega}^{\Omega} |H'_{22}(j\omega)|^2 d\omega}{\int_{-\Omega}^{\Omega} |H(j\omega)|^2 d\omega} \right)^{\frac{1}{2}} \quad (4.33)$$

The integrals in the above numerators, combined with equation (4.25) and its complex conjugate, were computed to three significant figures by using an algorithm for numerical quadrature which finds an acceptable estimate by cautious Romberg extrapolation [44]. Resulting, virtually exact ratios are presented as the dashed curves in Figure 4.7 and 4.8.

Bounds for the analogous maximax estimates can be based on inequality (4.12). Non-dimensionalizing these estimates to obtain ratios compatible with equation (4.29), (4.30), (4.32) and (4.33) leads to the approximations

$$\left(\frac{\text{maximax } \langle g_q^2(t) \rangle}{\text{maximax } \langle g^2(t) \rangle} \right)^{\frac{1}{2}} \left(\frac{\text{maximax } \langle g_{qp}^2(t) \rangle}{\text{maximax } \langle g^2(t) \rangle} \right)^{\frac{1}{2}} = \left(\frac{\max |H_{qp}(j\omega)|^2}{\max |H(j\omega)|^2} \right)^{\frac{1}{2}},$$

$p = 1, 2 \text{ with } q = 1. \quad (4.34)$

The identity occurs as a consequence of separate loading of the masses. Also

$$\left(\frac{\text{maximax } \langle [g_{21}(t) - g_{11}(t)]^2 \rangle}{\text{maximax } \langle g^2(t) \rangle} \right)^{\frac{1}{2}} = \left(\frac{\max |H'_{21}(j\omega)|^2}{\max |H(j\omega)|^2} \right)^{\frac{1}{2}} \quad (4.35)$$

and

$$\left(\frac{\text{maximax } \langle [g_{22}(t) - g_{12}(t)]^2 \rangle}{\text{maximax } \langle g^2(t) \rangle} \right)^{\frac{1}{2}} = \left(\frac{\max |H'_{22}(j\omega)|^2}{\max |H(j\omega)|^2} \right)^{\frac{1}{2}} \quad (4.36)$$

Therefore the estimates from relation (4.34), (4.35) and (4.36) correspond to the continuous curves in Figure 4.5 to 4.8, the details of which were outlined in the previous section. Differences between the appropriate continuous and dashed curves indicate the inaccuracy of the estimated ratios.

Figure 4.5 through 4.8 intimate that the approximate behave like the exact ratios with increasing ω_{22}/ω_{11} although they are normally lower. The greatest discrepancy generally occurs at ω_{22}/ω_{11} around one. Discrepancies stem typically from variations between $\max |H_{qp}(j\omega)|^2$ and $\text{Lt}_{\Omega \rightarrow \infty} \int_{-\Omega}^{\Omega} |H_{qp}(j\omega)|^2 d\omega$ because the corresponding and usually larger difference for the simple oscillator is essentially constant. Intuition suggests that this variation should grow as the domination of the maximum peak of $|H_{qp}(j\omega)|^2$ diminishes in the integration. This impression is confirmed by both Figure 4.3 and 4.4 where a single peak can be seen to prevail except at the frequency

ratio ω_{22}/ω_{11} of 0.99. These figures also indicate that small changes from this value of ω_{22}/ω_{11} will often reduce significantly the amplitude of the secondary peak and, hence, improve noticeably the integral's representation by the primary or larger peak. The sensitivity to perturbations about $\omega_{22}/\omega_{11} = 0.99$ is demonstrated most clearly in Figure 4.5.

Inequality (4.12)(and, of course, inequality (4.11)) can be applied more generally to the simultaneous loading of both masses. Inaccuracies will always increase, basically additively in the simplest case when mean square forces are identical. A more practical aspect relates to the possibility of improving the performance of a constant tuned neutralizer, say, where ω_{22}/ω_{11} always equals one [6]. This objective is achieved by adjusting ζ_{22} and μ under the constraint $\omega_{22}/\omega_{11} = 1$ so that equation (4.29) or the dashed curves in Figure 4.6 are as low as possible. Computational effort is reduced significantly however if the maximax estimates rather than the exact values are considered. This substitution seems justifiable because the estimates largely preserve the ranking of the exact values for different ζ_{22} and μ even though they may be not so accurate at $\omega_{22}/\omega_{11} = 1$. A necessary condition for the approximate ratio, given by relation (4.34) with $q = 1 = p$, to be minimum is

$$\frac{\partial}{\partial \zeta_{22}} (\max |H_{11}(j\omega)|^2)^{\frac{1}{2}} = 0 = \frac{\partial}{\partial \mu} (\max |H_{11}(j\omega)|^2)^{\frac{1}{2}} \quad (4.37)$$

because $H(j\omega)$ is invariably assumed fixed. Equation (4.37) can be rewritten as

$$\frac{\partial}{\partial \zeta_{22}} (\max |H_{11}(j\omega)|) = 0 = \frac{\partial}{\partial \mu} (\max |H_{11}(j\omega)|) \quad (4.38)$$

which is identical to the condition for minimizing the maximum steady state displacement of mass 1 when it is loaded sinusoidally. The same condition can be derived similarly from formula (4.11) and (4.13) for loads specified by their energy. Consequently requirement (4.38) appears to be reasonable for any type of loading.

Den Hartog [6] and Brock [49] in their now conventional analysis determined the "optimum" damping ζ_{22} assuming constant μ and ζ_{11} , in addition to ζ_{23} and ω_{23} , to be zero. Analytical equations were obtained for the "optimum" ζ_{22} and the resulting $\max |H_{11}(j\omega)|$ from the degenerative case

$$\frac{\partial}{\partial \zeta_{22}} (\max |H_{11}(j\omega)|) = 0 \quad (4.39)$$

of equation (4.38). Values computed from these equations and the consequent expressions for $\max |H_{21}(j\omega)|$ and $\max |H'_{21}(j\omega)|$ are shown as the dashed curves in Figure 4.9 and 4.10. However the question arises as to whether the ζ_{22} is truly optimum in the sense that $\max |H_{11}(j\omega)|$ really is as small as possible if μ as well as ζ_{22} is variable. To answer this question, different μ were substituted into $\max |H_{11}(j\omega)|$

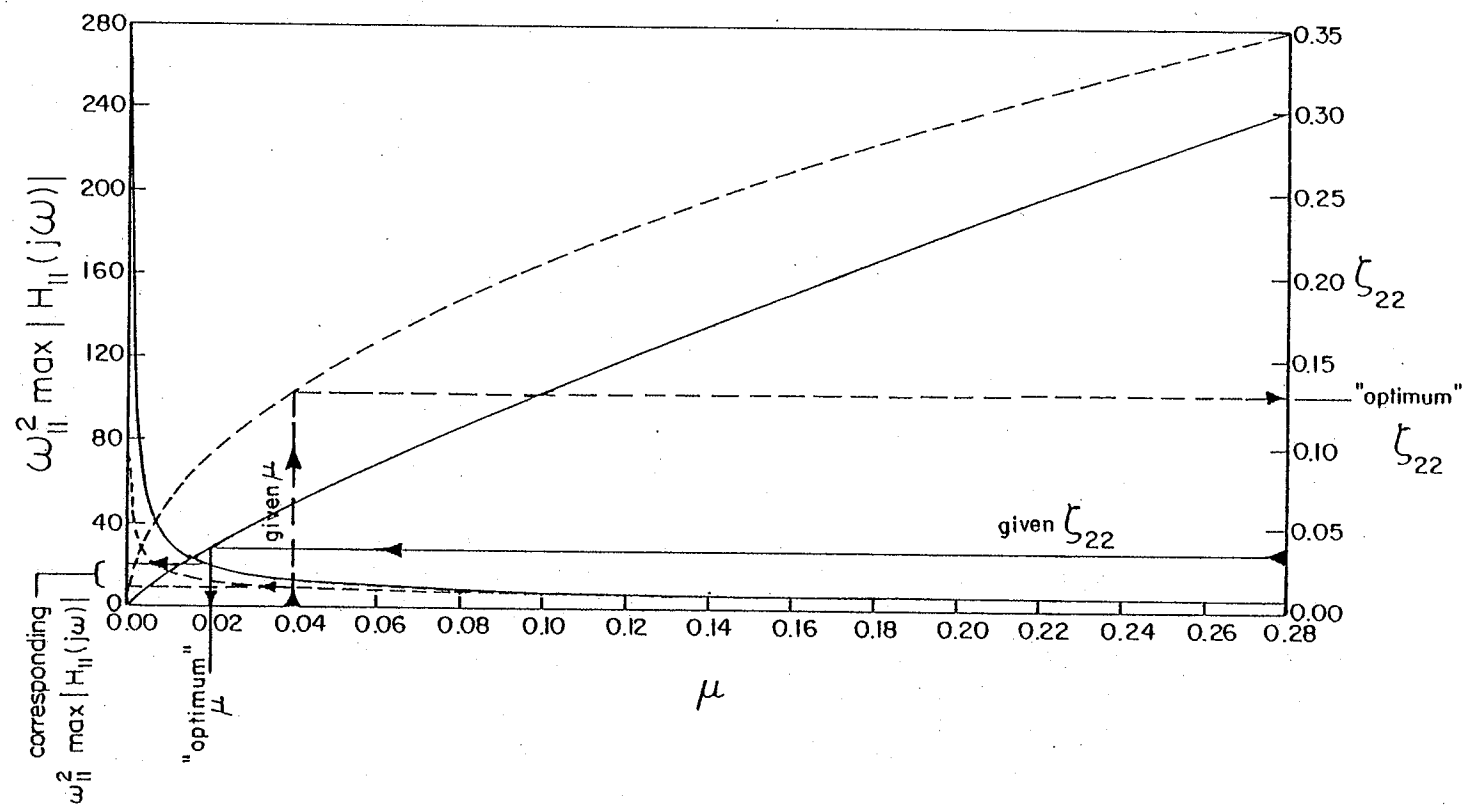


Figure 4.9 Properties of the constant tuned vibration neutralizer for either given μ and "optimum" ζ_{22} or given ζ_{22} and "optimum" μ .

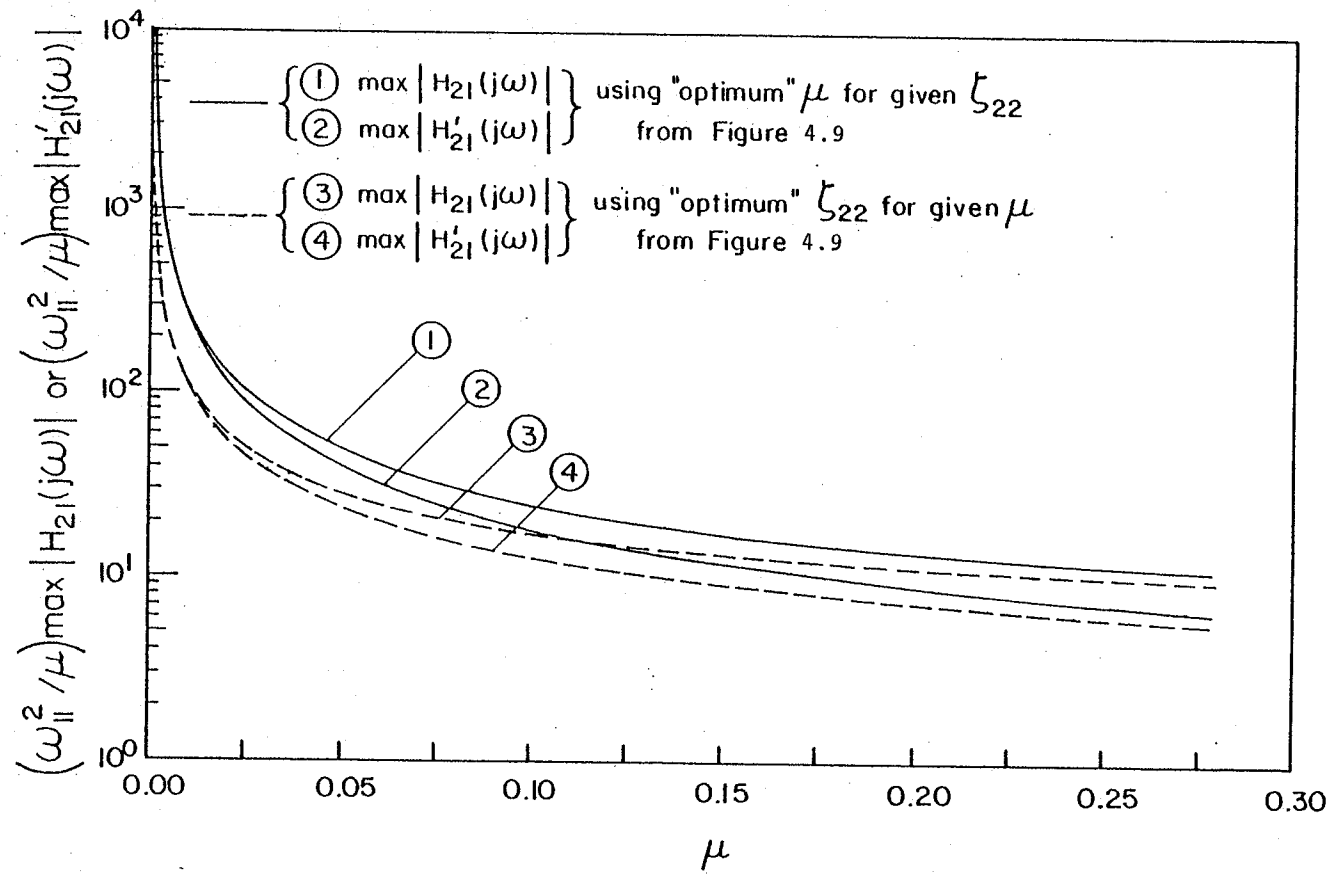


Figure 4.10. Behaviour of mass 2 or the interconnecting spring for the properties given in Figure 4.9.

for a given ζ_{22} , other variables as before, and the value of μ corresponding to the minimum of $\max |H_{11}(j\omega)|$ was found by iteration. This algorithm is equivalent to determining the μ which satisfies

$$\frac{\partial}{\partial \mu} (\max |H_{11}(j\omega)|) = 0 \quad (4.40)$$

for constant ζ_{22} and $\zeta_{11} = \zeta_{23} = \omega_{23} = 0$. Then the $\max |H_{qp}(j\omega)|$ and $\max |H'_{qp}(j\omega)|$ of interest were enumerated straightforwardly by using the known ζ_{22} and computed μ . The procedure was repeated for various ζ_{22} between 0.0 and 0.3 and the results are presented as the continuous curves in Figure 4.9 and 4.10. Of course, the true optimum must fulfill equation (4.39) and (4.40) simultaneously. This can occur only at the intersection of the $\max |H_{11}(j\omega)|$ generated separately by these two equations. Figure 4.9 indicates however that no such intersection and, hence, no unique optimum solution exists. Classical solutions based upon equation (4.39) invariably produce not only smaller $\max |H_{11}(j\omega)|$ but lower $\max |H_{21}(j\omega)|$ and $\max |H'_{21}(j\omega)|$ too so that they are superior.

4.3.2 Maximax Displacements Created by Pulse Modulated Sinewaves

Estimations of the maximax displacements to pulse modulated sinewaves applied simultaneously to both masses initially at rest will be weighed against exact solutions. The amplitude P , frequency ω_f , and duration T_f are taken

conveniently to be the same for both loads. T_f is assumed much more than twice the system's fundamental natural period so that the duration and bandwidth of the response are approximately T_f and $4\pi/T_f$. Then the energy of a load is nearly $P^2 T_f / 2$ [32]. Substituting these values into inequality (4.11) produces

$$\text{maximax } E_{g_1} \leq \frac{P^2 T_f}{2} ((\max |H_{11}(j\omega)|^2)^{\frac{1}{2}} + (\max |H_{12}(j\omega)|^2)^{\frac{1}{2}})^2$$

and

$$\text{maximax } E_{g_2} \leq \frac{P^2 T_f}{2} ((\max |H_{22}(j\omega)|^2)^{\frac{1}{2}} + (\max |H_{21}(j\omega)|^2)^{\frac{1}{2}})^2 \quad (4.41)$$

Assuming the equalities to be reasonable approximations, the above can be combined with formula (4.13) to give the estimates

$$\text{maximax } g_1(t) = 2^{\frac{1}{4}} P ((\max |H_{11}(j\omega)|^2)^{\frac{1}{2}} + (\max |H_{12}(j\omega)|^2)^{\frac{1}{2}})$$

and

$$\text{maximax } g_2(t) = 2^{\frac{1}{4}} P ((\max |H_{22}(j\omega)|^2)^{\frac{1}{2}} + (\max |H_{21}(j\omega)|^2)^{\frac{1}{2}})$$

because

$$\left(\frac{T_{g_q}}{2\lambda_{o_q}} \right)^{\frac{1}{2}} \max \psi_{o_q} (\Omega_{g_q} T_{g_q} / 2) \approx \left(\frac{T_f}{2\lambda_{o_q}} \right)^{\frac{1}{2}} \max \psi_{o_q} (\Omega_f T_f / 2) \approx \left(\frac{T_f}{2\lambda_{o_q}} \right)^{\frac{1}{2}} \max \psi_{o_q} (2\pi) \approx 2^{\frac{1}{4}},$$

(4.43)

using Figure 2.2. The exact solutions [50] have the factor one rather than $2^{\frac{1}{4}}$ and, hence, are about 19% smaller than these estimates.

4.4 Conclusions

A technique for estimating the maximax response to random or deterministic loads which are specified incompletely has been extended to discrete multi degree-of-freedom systems. Promising results obtained for the relatively simple but practically important two degree-of-freedom system encourage further research. Examples in which the characteristics of the system rather than the loading predominantly determine the response should be especially emphasised.

Detailed considerations of a special two degree-of-freedom system, the constant tuned vibration neutralizer, indicate that an optimum solution does not exist when μ and ζ_{22} vary simultaneously. However the conventional approach of fixing μ and finding the ζ_{22} which minimises $\max |H_{11}(j\omega)|$ appears reasonable.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

A novel theory has been developed to estimate the greatest maximum response of linear structures subjected to incompletely described loads. The maximum response energy has been determined for a class of loads with identical energies. This maximum energy has been shaped into a function which has the largest peak amplitude between all functions with similar energies. This function has been proved to be the first prolate spheroidal wave function of zero order so that it depends explicitly upon the product of the response's bandwidth and duration. The analysis has dealt attentively with simultaneous time and frequency concentration of the response which is a common experience in measurements. The accuracy of the proposed procedure has been evaluated for fairly comprehensive circumstances. It has been shown that the method produces generally better estimates than previous theories.

The analysis has been extended to complex linear structures with more than one degree-of-freedom, subjected to any type of loading. In particular, the maximax and the maximax mean square response of discrete multi-degree-of-freedom systems has been considered. Practical applications have been illustrated for structures idealized as one or two

degrees-of-freedom subjected to a wide variety of shock and earthquake loads. Therefore the method may prove invaluable for the design of structures in earthquake-prone regions. It may assist in establishing the possibility of damage to residences close to quarries and to estimate the peak sound level from impact machinery like drop forges.

Further research is required to investigate the effective duration and bandwidth of complex multi degree-of-freedom structures; to develop the method for distributed linear structures; and to extend the analysis to certain classes of non-linear structures.

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APPENDIX I

COMPUTATION OF MAXIMUM VALUES OF ψ_0 AND λ_0

The values λ_0 and $\max \psi_0$ are evaluated by solving the integral equation

$$\lambda f(t) = \int_{-T/2}^{T/2} \frac{\sin \Omega(\tau - t)}{\pi(\tau - t)} f(\tau) d\tau. \quad (A1)$$

An approximate method for obtaining the first characteristic function has been given by Chalk [51]. However, the solution depends upon the number of terms included in an infinite series. For better accuracy, the analytic solution given in reference [36] is employed. The solution involves the following relation [52,53],

$$\frac{2c}{\pi} [R_{0n}^{(1)}(c,1)]^2 S_{0n}(c,t) = \int_{-1}^1 \frac{\sin c(t-s)}{\pi(t-s)} S_{0n}(c,s) ds, \quad n=0,1,2,\dots (A2)$$

The functions $S_{0n}(c,t)$ and $R_{0n}(c,t)$ are angular prolate and radial prolate spheroidal functions, respectively [52-54]. $S_{0n}(c,t)$ are the exact eigensolutions of equation (A2) corresponding to the eigenvalues

$$\lambda_n(c) = (2c/\pi) [R_{0n}^{(1)}(c,1)]^2. \quad (A3)$$

The solutions of integral equation (A1) are

$$\sqrt{T/2} \psi_n(c,t) = [\sqrt{\lambda_n(c)}/u_n(c)] S_{0n}(c,2t/T), \quad (A4)$$

where the $u_n(c)$ are normalization factors given by

$$[u_n(c)]^2 = \int_{-1}^1 [S_{0n}(c,t)]^2 dt. \quad (A5)$$

The functions S_{0n} and R_{0n} may be calculated by using the expansions [52]

$$S_{mL}(h,n) = \sum_n d_n(h/mL) P_{n+m}^m(n), \quad R_{mL}(h,1) = \sum_n a_n(h/mL) J_{n+m}(h\zeta),$$

$$m, L = 0. \quad (A6)$$

The $P_{n+m}^m(n)$ are the associated Legendre functions of order m and degree $n + m$ while the $J_{n+m}(h\zeta)$ are spherical Bessel functions of order $n + m$.

The coefficients $d_n(h/mL)$ and $a_n(h/mL)$ can be taken from reference [52] or [54]. Since the $\psi_0(t)$ are even functions with maxima when t equals zero, only the values of $S_{00}(c,0)$ are calculated in evaluating $\max \psi_0(t)$. The normalization factors are obtained by substituting S_{0n} from equations (A6) into equation (A5). Then

$$[u_n(c)]^2 = \int_{-1}^1 \left(\sum_n d_n P_n^0(t) \right)^2 dt. \quad (A7)$$

The orthogonality property of the associated Legendre functions,

$$\int_{-1}^1 P_n^0(t) P_\ell^0(t) dt = \begin{cases} 0, & \ell \neq n, \\ 1/(n + \frac{1}{2}), & \ell = n, \end{cases} \quad (A8)$$

may be used in equation (A7) to give

$$[u_n(c)]^2 = \sum_n d_n^2 / (n + \frac{1}{2}), \quad n = 1, 2, 3, \dots \quad (A9)$$

The associated Legendre functions were computed to six decimal places by using an orthogonal polynomial for which the coefficients are given in reference [55]. However, spherical Bessel functions were calculated to six places from the recurrence relations of reference [55]. The accuracy of the solution depends upon the accuracy of the evaluation of a_n and d_n which were obtained from reference [52]. The data was checked for a number of points and found reliable to at least five decimal places.

APPENDIX II
SOLUTION OF THE HOMOGENEOUS WIENER-HOPF INTEGRAL EQUATION
OF THE SECOND KIND

AII.1. Solution of the Finite Duration Loading Integral Equation

The equation is

$$f(t) = \int_0^{T_f} f(\tau) L(t - \tau) d\tau, \quad 0 \leq t \leq T_f. \quad (A10)$$

An elementary solution given by Youla [39] can be used because the Fourier transform of the kernel, $L(t)$, satisfies the requirements for successful application of the method. Let $k^2 |H(s^2)| = N(s^2)/D(s^2)$, where $N(s^2)$ and $D(s^2)$ are polynomials in s^2 . If the roots of $D(s^2)$ are $\pm \mu_n$, $n = 1, 2$, then

$$\mu_1 = \omega[-(1 - 2\zeta^2) + 2\zeta\sqrt{\zeta^2 - 1}]^{\frac{1}{2}}, \quad \mu_2 = \omega[-(1 - 2\zeta^2) - 2\zeta\sqrt{\zeta^2 - 1}]^{\frac{1}{2}}. \quad (A11)$$

The roots of $[D(s^2) - \chi N(s^2)]$ are $\pm \omega_n(\chi)$, $n = 1, 2$, where

$$\begin{aligned} \omega_1(\chi) &= \omega[-(1 - 2\zeta^2) + [4\zeta^2(\zeta^2 - 1) + \chi]^{\frac{1}{2}}]^{\frac{1}{2}}, \\ \omega_2(\chi) &= \omega[-(1 - 2\zeta^2) - [4\zeta^2(\zeta^2 - 1) + \chi]^{\frac{1}{2}}]^{\frac{1}{2}}, \end{aligned} \quad (A12)$$

and $\chi = k^2 \eta$. The following relations between the eigenvalues are taken from Youla [39]

$$\omega_2 \tanh \theta_2 = \omega_1 \tanh \theta_1, \quad (\text{A13})$$

$$\omega_2 \coth \theta_2 = \omega_1 \coth \theta_1, \quad (\text{A14})$$

in which

$$\begin{aligned} \tanh \theta_1 = \{ [1 + (\omega_1/\omega)^2] \tanh (\omega_1 T_f/2) + 2\zeta \omega_1/\omega \} / \{ [1 + (\omega_1/\omega)^2] \\ + 2\zeta (\omega_1/\omega) \tanh (\omega_1 T_f/2) \}, \end{aligned} \quad (\text{A15})$$

$$\begin{aligned} \tanh \theta_2 = \{ [1 + (\omega_2/\omega)^2] \tanh (\omega_2 T_f/2) + 2\zeta \omega_2/\omega \} / \{ [1 + (\omega_2/\omega)^2] \\ + 2\zeta (\omega_2/\omega) \tanh (\omega_2 T_f/2) \}. \end{aligned} \quad (\text{A16})$$

Using equations (A12-A16), specifying different values of ζ , ω and T_f and iterating for values of χ yields the eigenvalues n of equation (A10). The calculation can be made as accurate as required by iterating for values of χ with arbitrarily small intervals. The computed eigenvalues satisfy equations (A13,A14) to four significant figures.

AII.2. Solution of the Semi-Infinite Loading Integral Equation

It is shown in reference [39] that the eigenvalues of the semi-infinite range (i.e., T_f tends to infinity) of equation (A10) can be determined by the requirement of convergence for the integral. This requirement leads to

$$\text{Re } \omega_n(\chi) < \text{Re } \mu_n, \quad n = 1, 2.$$

Therefore, substituting from equations (A11) and (A12) gives

$$\operatorname{Re}\{\omega[-(1 - 2\zeta^2) \pm [4\zeta^2(\zeta^2 - 1) + \chi]^{1/2}]^{1/2}\} < \operatorname{Re}\{\omega[-(1 - 2\zeta^2) \pm 2\zeta\sqrt{\zeta^2 - 1}]^{1/2}\}, \quad (\text{A17})$$

and hence

$$\chi = \begin{cases} 1/4\zeta^2(1 - \zeta^2), & \zeta < 1/\sqrt{2}, \\ 1, & 1/\sqrt{2} \leq \zeta < 1. \end{cases}$$

APPENDIX III ERROR ANALYSIS

Let $\epsilon(t)$ be the error in the maximum response estimation due to limiting the bandwidth: i.e.,

$$\epsilon(t) = g(t) - g_B(t). \quad (A18)$$

Assume that the Fourier transform $G(j\omega)$ of $g(t)$ has significant values only when $\omega \leq \Omega_L$ and likewise $G_B(j\omega)$ of $g_B(t)$ vanishes for all $\omega > \Omega_g$, $\Omega_g \leq \Omega_L$. Therefore

$$\epsilon(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [G(j\omega) - G_B(j\omega)] e^{j\omega t} d\omega. \quad (A19)$$

Applying Schwarz's inequality [34] to expression (A19) gives

$$|\epsilon(t)|^2 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega) - G_B(j\omega)|^2 d\omega,$$

or

$$|\epsilon(t)|^2 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} [G(j\omega) G^*(j\omega) + G_B(j\omega) G_B^*(j\omega) - G(j\omega) G_B^*(j\omega) - G^*(j\omega) G_B(j\omega)] d\omega.$$

Since $\Omega_g \leq \Omega_L$, then

$$G(j\omega) G_B^*(j\omega) = G^*(j\omega) G_B(j\omega) = G_B(j\omega) G_B^*(j\omega),$$

so that

$$|\epsilon(t)|^2 \leq (E_g - E_{gB}). \quad (A20)$$

Combining equation (A20) with equation (2.5) leads to

$$|\epsilon(t)|^2 \leq E_{g_0}(1 - \lambda_0), \quad (\text{A21})$$

because $g_0(t - T_g/2)$ is proportional to the optimum function $\psi_0(t)$ and $v_0 \equiv \lambda_0$. It may be seen from Figure 2.2 that the error, $\epsilon(t)$, is negligible when $\Omega_g T_g/2$ is greater than four because λ_0 is then almost unity.

APPENDIX IV

EQUATIONS NEEDED TO DETERMINE $\max |H_{qp}(j\omega)|^2$ AND $\max |H'_{qp}(j\omega)|^2$
FOR TWO DEGREE-OF-FREEDOM SYSTEMS

Analytical equations derived from equation (4.26) or,
for non-trivial cases $|H_{qp}(j\omega)| \neq 0$ and $|H'_{qp}(j\omega)| \neq 0$,

$$\frac{\partial}{\partial \omega} |H'_{qp}(j\omega)| = \frac{\partial}{\partial \omega} |H_{qp}(j\omega)| = 0 \quad q = 1 \text{ and } p = 1,2$$

and

$$\frac{\partial}{\partial \omega} |H'_{qp}(j\omega)| = 0 \quad q = 2 \text{ and } p = 1,2 \quad (A22)$$

can be written in the shorthand form

$$\sum_{r=0}^5 (A_{pq})_r \cdot (-\omega^2)^r = 0 \quad (A23)$$

when p equals q and

$$\sum_{r=0}^4 (A_{pq})_r \cdot (-\omega^2)^r = 0 \quad (A24)$$

otherwise. The various expressions for the coefficients,
 $(A_{pq})_r$, can be simplified by using the following relations

$$a_1 = 2(\zeta_{11}\omega_{11} + \mu \zeta_{22}\omega_{22})$$

$$b_1 = \omega_{11}^2 + \mu \omega_{22}^2$$

$$c_1 = 2\mu \zeta_{22}\omega_{22}$$

$$d_1 = \mu \omega_{22}^2$$

$$a_2 = 2(\zeta_{22}\omega_{22} + \zeta_{23}\omega_{23})$$

$$b_2 = \omega_{22}^2 + \omega_{23}^2$$

$$c_2 = 2\zeta_{22}\omega_{22}$$

$$d_2 = \omega_{22}^2$$

and

$$e = \omega_{11}^2 (1 - 2\zeta_{11}^2) \quad (A25)$$

Then

$$(A_{11})_0 = -\frac{1}{2} (2b_2 - a_2^2)(b_1b_2 - d_1d_2)^2 + b_2^2(b_1b_2 - d_1d_2)(b_1 + b_2 + a_1a_2 - c_1c_2) - \frac{1}{2} (a_1b_2 + b_1a_2 - c_1d_2 - c_2d_1)^2$$

$$(A_{11})_1 = -(b_1b_2 - d_1d_2)^2 + 2b_2^2(b_1b_2 - d_1d_2) - 2b_2^2(a_1 + a_2)(a_1b_2 + b_1a_2 - c_1d_2 - d_1c_2) + b_2^2(b_1 + b_2 + a_1a_2 - c_1c_2)^2$$

$$(A_{11})_2 = (b_1b_2 - d_1d_2)(b_1 + b_2 + a_1a_2 - c_1c_2) - \frac{1}{2} (a_1b_2 + b_1a_2 - c_1d_2 - c_2d_1)^2 + 3b_2^2(b_1 + b_2 + a_1a_2 - c_1c_2) - \frac{3}{2} (a_1 + a_2)^2 b_2^2 - (2b_1 - a_1^2) b_2^2 - (2b_2 - a_2^2) b_1^2 + (c_2^2 d_1^2 + c_1^2 d_2^2) + 2d_1d_2(b_1 + b_2 + a_1a_2) - 2(c_1d_2 + d_1c_2)(a_1b_2 + a_2b_1) + 2b_1b_2 c_1c_2 + \frac{1}{2} (2b_2 - a_2^2)[2(b_1b_2 - d_1d_2) - 2(a_1 + a_2)(a_1b_2 + b_1a_2 - c_1d_2 - d_1c_2) + (b_1 + b_2 + a_1a_2 - c_1c_2)^2]$$

$$(A_{11})_3 = (2b_2 - a_2^2)[2(b_1 + b_2 + a_1a_2 - c_1c_2) - (a_1 + a_2)^2] + 2b_2^2$$

$$(A_{11})_4 = [2(2b_2 - a_2^2) + \frac{1}{2}(2b_1 - a_1^2) - c_1c_2]$$

$$(A_{11})_5 = 1$$

$$(A_{12})_0 = -2 c_2^2 (b_1^2 b_2^2 + d_1^2 d_2^2 - 2 d_1 d_2 b_1 b_2)$$

$$-2 d_2^2 [2(b_1 b_2 - d_1 d_2)(b_1 + b_2 + a_1 a_2 - c_1 c_2)$$

$$+ (a_1 b_2 + b_1 a_2 - c_1 d_2 - c_2 d_1)^2]$$

$$(A_{12})_1 = -2 d_2^2 [4(b_1 b_2 - d_1 d_2) - 4(a_1 + a_2)(a_1 b_2 + b_1 a_2 - c_1 d_2 - d_1 c_2)$$

$$+ 2(b_1 + b_2 + a_1 a_2 - c_1 c_2)^2]$$

$$(A_{12})_2 = -6 d_2^2 [2(b_1 + b_2 + a_1 a_2 - c_1 c_2) - (a_1 + a_2)^2]$$

$$+ 2 c_2^2 [(b_1^2 + b_2^2) + (2 b_1 - a_1^2)(2 b_2 - a_2^2) + c_1^2 c_2^2$$

$$+ 2(a_1 + a_2)(c_1 d_2 + d_1 c_2) - 2 d_1 d_2 - 2 c_1 c_2 (a_1 a_2 + b_1 + b_2)]$$

$$(A_{12})_3 = 4 [2(b_1 + b_2) - (a_1^2 + a_2^2) - 2 c_1 c_2] c_2^2 - 8 d_2^2$$

$$(A_{12})_4 = 6 c_2^2$$

$$(A_{21})_0 = -2 (b_1^2 b_2^2 + d_1^2 d_2^2 - 2 d_1 d_2 b_1 b_2)$$

$$(A_{21})_1 = - (2b_1 - a_1^2) b_2^2 - (2 b_2 - a_2^2) b_1^2 + (c_2^2 d_1^2 + c_1^2 d_2^2)$$

$$+ 2 d_1 d_2 (a_1 a_2 + b_1 + b_2) - 2 (c_1 d_2 + d_1 c_2)(a_1 b_2 + a_2 b_1)$$

$$- 2 b_1 b_2 c_1 c_2$$

$$(A_{21})_2 = 0$$

$$(A_{21})_3 = 2(b_1 + b_2) - (a_1^2 + a_2^2) - 2 c_1 c_2$$

$$(A_{21})_4 = 2$$

$$\begin{aligned}
 (A_{22})_0 = & -2e (b_1^2 b_2^2 + d_1^2 d_2^2 - 2 d_1 d_2 b_1 b_2) \\
 & + 2 b_2^2 [2(b_1 b_2 - d_1 d_2)(b_1 + b_2 + a_1 a_2 - c_1 c_2) \\
 & - (a_1 b_2 + b_1 a_2 - c_1 d_2 - c_2 d_1)^2]
 \end{aligned}$$

$$\begin{aligned}
 (A_{22})_1 = & 2 b_2^2 [4 (b_1 b_2 - d_1 d_2) - 4 (a_1 + a_2)(a_1 b_2 + b_1 a_2 \\
 & - c_1 d_2 - d_1 c_2) + 2 (b_1 + b_2 + a_1 a_2 - c_1 c_2)^2] \\
 & - 4 (b_1 b_2 - d_1 d_2)^2
 \end{aligned}$$

$$\begin{aligned}
 (A_{22})_2 = & -2 [2(b_1 b_2 - d_1 d_2)(b_1 + b_2 + a_1 a_2 - c_1 c_2) \\
 & - (a_1 b_2 + b_1 a_2 - c_1 d_2 - c_2 d_1)^2] \\
 & + 2e [2 (b_1 b_2 - d_1 d_2) - 2(a_1 + a_2)] \cdot (a_1 b_2 + b_1 a_2 \\
 & - c_1 d_2 - d_1 c_2) + (b_1 + b_2 + a_1 a_2 - c_1 c_2)^2] \\
 & + 6 b_2^2 [2 (b_1 + b_2 + a_1 a_2 - c_1 c_2) - (a_1 + a_2)^2]
 \end{aligned}$$

$$(A_{22})_3 = 4e [2 (b_1 + b_2) - (a_1^2 + a_2^2) - 2 c_1 c_2] + 8 b_2^2$$

$$(A_{22})_4 = 2 [2 (b_1 + b_2) - (a_1^2 + a_2^2) - 2 c_1 c_2] + 6e$$

and

$$(A_{22})_5 = 4$$

(A26)

APPENDIX V

MATHEMATICAL DETAILS FOR DEVELOPING EQUATION (2.8) FROM EQUATION (2.7)

In order to derive equation (2.8), it must be proved that

$$\int_{-\infty}^{\infty} \frac{\sin(\Omega_g(u-v))}{\pi(u-v)} \cdot \frac{\sin(\Omega_g(t-u))}{\pi(t-u)} du = \frac{\sin(\Omega_g(t-v))}{\pi(t-v)} \quad (A27)$$

Parseval's theorem states that [36]

$$\int_{-\infty}^{\infty} f(u) g(u) du = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) G(\omega) d\omega \quad (A28)$$

If $F(\omega)$ or $G(\omega)$ are frequency limited in the bandwidth $(-\Omega_g, \Omega_g)$, equation (A28) can be rewritten as

$$\int_{-\infty}^{\infty} f(u) g(u) du = \frac{1}{2\pi} \int_{-\Omega_g}^{\Omega_g} F(\omega) G(\omega) d\omega \quad (A29)$$

Now $\sin(\Omega_g \zeta)/\pi \zeta$ is an even function which can be obtained from

$$\frac{\sin(\Omega_g \zeta)}{\pi \zeta} = \frac{1}{2\pi} \int_{-\Omega_g}^{\Omega_g} e^{j\omega \zeta} d\omega \quad (A30)$$

Applying equation (A29) and (A30) to the integral expression of (A27) gives

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\sin(\Omega_g(u-v))}{\pi(u-v)} \cdot \frac{\sin(\Omega_g(t-u))}{\pi(t-u)} du &= \frac{1}{2\pi} \int_{-\Omega_g}^{\Omega_g} e^{j\omega(t-u)} e^{j\omega(u-v)} d\omega \\ &= \frac{1}{2\pi} \int_{-\Omega_g}^{\Omega_g} e^{j\omega(t-v)} d\omega = \frac{\sin(\Omega_g(t-v))}{\pi(t-v)} \end{aligned} \quad (A31)$$

which is the required relationship.