THE KERZMAN-STEIN OPERATOR ON SMIRNOV SPACE AND AN ANALOGUE FOR DIRICHLET SPACE

by

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Abstract

When defined on the L^2 space of a smooth Jordan curve in $\mathbb{C}$, the skew-adjoint part of the Cauchy integral operator $\mathbf{C}$, called the Kerzman-Stein operator, is notable both for its compactness and role in the Kerzman-Stein formula relating $\mathbf{C}$ to the Szegő projection $\mathbf{P}$. We investigate various identities and properties of the Kerzman-Stein operator on the Smirnov space of harmonic half-order differentials, as well as introduce a natural analogue of the operator for the homogeneous Dirichlet space of harmonic functions, and show that it is related to the Grunsky operator.

Working over Ahlfors-regular Jordan domains in the Smirnov setting, we derive adjoint formulas for the Cauchy operators that allow us to generalize the Kerzman-Stein formula to this new analytic setting. Further, we provide sufficient $C^{n,\alpha}$ curve conditions for the Kerzman-Stein operator $\mathbf{A}$ to belong to the Schatten p-classes for all $p \geqslant 2$. An extension of $\mathbf{A}$ is also considered, which is an isomorphic map between the spaces on the complementary components of the curve. This mapping is related to $\mathbf{A}$ through overfare, and has the property of being unitary precisely for disks. Alongside this, we show that the Smirnov space Grunsky operator recently introduced by Kristel et al. [26] is in the Schatten class S_p precisely when $\mathbf{A}$ is in S_p. As a consequence of this new perspective, we apply our results to show that all $C^{1,1/2+\varepsilon}$ Jordan curves are Weil-Petersson class quasicircles.

Adjacent to this is our work in the Dirichlet space setting, where we define an analogue of $\mathbf{A}$, which we denote $\dot{\mathbf{A}}$, that shares many similar properties as its Smirnov space counterpart. In particular, we show that a Kerzman-Stein formula with $\dot{\mathbf{A}}$ holds for quasicircles, and we compute identities which closely relate these operators with overfare. From here, we show that the (classical) Grunsky operator is in S_p if and only if $\dot{\mathbf{A}}$ is in S_p. In particular, the independent results of Jones [23], Takhtajan and Teo [44], and Shen [42] imply that Hilbert-Schmidtness of $\dot{\mathbf{A}}$ is met precisely for Weil-Petersson class quasidisks.

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Introduction

Near the beginning of every student's journey into complex analysis, one learns that the Cauchy integral

$$(\mathbf{C}u)(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{u(\zeta)}{\zeta - z}\, d\zeta$$

is one of the most effective tools for studying holomorphic functions and boundary values. Beyond its foundational usage, the study of generalized Cauchy operators on Hilbert spaces has remained in the active research literature since the 19th century. In a famous paper, Kerzman and Stein [24] discovered a new way to think about the Cauchy integral operator, allowing for subsequent simplifications of the proofs of many classical theorems that form the basis of complex function theory. In this thesis, our goal is to shed additional light on the Kerzman-Stein perspective using modern advances in the theory of Smirnov spaces, as well as open the door for further investigations surrounding these ideas in the Dirichlet space setting, where the analogous operator had previously not been defined.

Setting aside analytic assumptions for now, let us set the stage for these contributions of Kerzman and Stein. Let Ω be a simply-connected domain in $\mathbb{C}$ bounded by a smooth Jordan curve Γ. For a holomorphic function $h : \Omega \to \mathbb{C}$ with sufficiently nice boundary values, it is natural to ask about the boundary behaviour of $\mathbf{C}h$. That is, does the nontangential limit of $(\mathbf{C}h)(z)$ as $z \in \Omega$ approaches some $z_0 \in \Gamma$ exist, and if so, does it define a nice extension of $\mathbf{C}h$? Subject to the boundary regularity, the answer is yes, with the limit given by the *Sokhotski-Plemelj* formula

$$(\mathbf{C}h)(z_0) = \frac{1}{2}h(z_0) + \text{P.V.}\frac{1}{2\pi i}\int_{\Gamma} \frac{h(\zeta)}{\zeta - z_0}\, d\zeta, \quad z_0 \in \Gamma$$

which in fact yields a sensible notion of a singular Cauchy integral on all of $L^2(\Gamma, |d\zeta|)$. With this version of the Cauchy operator, Kerzman and Stein [24] considered the difference of $\mathbf{C}$ and its L^2 adjoint, $\mathbf{C}^*$:

$$\mathbf{A} := \mathbf{C} - \mathbf{C}^*.$$

One may also recognize this expression as the skew-adjoint part of $\mathbf{C}$ (up to scalar). They were able to show that, remarkably, the singularity at $\zeta = z_0$ present in the Cauchy kernel is cancelled out in this subtraction, forming a nonsingular integral operator with smooth kernel function. In particular, this cancellation implies that the Cauchy kernel is "close" to being self-adjoint near its diagonal, and the operator $\mathbf{A}$, now eponymously called the *Kerzman-Stein* operator, is compact. This is a very striking result, given that $\mathbf{C}$ by itself is always singular and never compact. Further, consider the *Szegő projection* $\mathbf{P}$ on $L^2(\Gamma, |d\zeta|)$, which is the orthogonal projection that picks out those L^2 functions that are boundary values of holomorphic functions on Ω. With their new operator, they showed that $\mathbf{I}+\mathbf{A}$ is invertible on $L^2(\Gamma, |d\zeta|)$, and

$$\mathbf{P}(\mathbf{I}+\mathbf{A}) = \mathbf{C}. \tag{1.0.1}$$

This formula, called the *Kerzman-Stein* formula, says that the Szegő projection and the Cauchy operator are closely related, and it allows for functional-analytic properties of one to be passed to the other. In fact, there is a sense in which the failure of $\mathbf{P}$ and $\mathbf{C}$ to coincide measures the irregularity of Γ. As is demonstrated in the book of Bell [3], the connection between $\mathbf{P}$ and $\mathbf{C}$ allows for the development of function theory on smooth planar domains to be greatly simplified.

The background analytic setting associated to the Szegő projection is the *Smirnov space* $E^2(\Omega)$ (a generalization of the *Hardy space*), which is a Hilbert space of holomorphic functions with L^2 boundary values. On "nice" domains, the L^2 space of the boundary has the orthogonal decomposition:

$$L^2(\Gamma, |d\zeta|) = E^2(\Omega) \oplus \overline{T}\,\overline{E^2(\Omega)} \tag{1.0.2}$$

where $E^2(\Omega)$ has been identified with its boundary values for convenience, and T denotes the unit tangent function for Γ. The proof of this decomposition contained in Bell [3] is one of the key examples where the Kerzman-Stein approach simplifies the analysis.

Later, Barrett and Bolt [1] considered the Cauchy and Kerzman-Stein operators as mappings of *half-order differentials* associated to the Smirnov space, rather than functions. These are objects of the form $hdz^{1/2}$, for a nice function h (in our case, $h \in E^2(\Omega)$). While this can be made completely rigorous, for our work the symbol $dz^{1/2}$ is treated formally as the "square root" of the differential $dz = dx + idy$, which possesses many algebraic properties that reinforce this notational standard. Their approach is a natural progression of the theory, as elements of the Smirnov space transform in a similar way to how half-order differentials transform under pull-back (i.e. conformal reparameterization). Indeed, making the change of variables $\zeta = f(\tau)$ in the L^2-norm, we have that

$$\int_\Gamma |h(\zeta)|^2 \, |d\zeta| = \int_{f^{-1}(\Gamma)} |h \circ f(\tau)|^2 |f'(\tau)| \, |d\tau|$$

or equivalently

$$\|h\|_\Gamma = \left\| (h \circ f)\sqrt{f'} \right\|_{f^{-1}(\Gamma)}. \tag{1.0.3}$$

In particular, we have that $h \in E^2(\Omega)$ if and only if $(h \circ f)\sqrt{f'} \in E^2(f^{-1}(\Omega))$. Similarly, if we pull-back $h(\zeta)d\zeta^{1/2}$ by f, as the notation suggests, we get an expression of the form

$$(h \circ f)(\tau)(df(\tau))^{1/2} = (h \circ f)(\tau)\sqrt{f'(\tau)}d\tau^{1/2}.$$

Thus, half-order differentials are the geometrically "correct" objects for a conformally invariant definition of the Smirnov space. Very recently, Kristel et al. [26] took this perspective further by phrasing the classical Smirnov space theory surrounding the Cauchy integral in terms of harmonic half-order differentials on Ahlfors-regular Jordan domains. This particular class of domain is apt for studying the Cauchy operator as an integral operator over the boundary, as it is precisely the class for which it is bounded, per the work of David [11]. Therefore, it is the widest class of Jordan domains for which the definition of the Kerzman-Stein operator makes sense. However, we were unable to locate an investigation of the Kerzman-Stein operator in this analytic setting. The first part of this thesis is dedicated to proving results for this setting, as well as extending the theory using tools from Kristel et al. [26] .

Apart from the notational difference, one of the main insights of Kristel et al. [26] (in analogy with the work of the second two authors in the Bergman space) is the concept of *overfare*, which identifies elements of the Smirnov space on the "inside" and "outside" domains associated to a Jordan curve. For context, given a "nice" function on the boundary, a classical problem is finding the corresponding harmonic function with those boundary values. If one solves this problem on both complementary domains, then the identification of these solutions defines overfare. This operator appears implicitly in the classical jump formula for the Cauchy operators, and encodes geometric data of the complementary spaces. Using overfare, Kristel et al. [26] defined an analogous version of the *Grunsky operator* for Smirnov spaces. This operator is related to the Cauchy integral, and previously only had definitions associated with the Bergman and Dirichlet space settings, where it has long been studied for its deep connections to univalence and Teichmüller space theory. Briefly, the Bergman space contains those holomorphic functions with finite L^2-norm on their domain, and is most aptly modelled in terms of holomorphic L^2 one-forms, instead of functions. Closely related is the Dirichlet space, consisting of holomorphic functions whose first derivative is in the Bergman space. We may interpret this condition as identifying the holomorphic functions whose images have finite area (up to multiplicity). In the Dirichlet setting, there is a definition of the Cauchy integral operator that behaves in a similar, but slightly more nuanced way than in the Smirnov space. However, the ideas of Kerzman-Stein have yet to be implemented in this setting. Given the mileage of their viewpoint in the Smirnov space, the interesting questions it has generated, and the analogies that already exist between the spaces, it seems natural to investigate these ideas in the Dirichlet setting.

The interplay between the geometry of domains and the analytic properties of associated function spaces is at the heart of this thesis. In particular, we are most interested in the relationship between the jump problem, overfare, Cauchy integral operators, and how they all interact based on the Hilbert space setting and the regularity of the underlying domain. With the Kerzman-Stein operator being built from the Cauchy operator, our work on this matter seeks to highlight the ways in which it may be seen as an "improvement" of the Cauchy operator that also captures geometric data. We shall consider these operators and the corresponding circle of questions in two separate analytic settings – first, in the Smirnov space of harmonic half-order differentials in Chapter 3, and then in the homogeneous Dirichlet space of harmonic functions in Chapter 4.

Summarizing our original contributions in the Smirnov setting, we have:

(s1) Explicit adjoint formulas for the Cauchy integral operators on Ahlfors-regular Jordan domains (Section 3.5).

(s2) A proof of the Kerzman-Stein formula for Ahlfors-regular Jordan curves (Section 3.6 and 3.7).

(s3) A generalization of the Kerzman-Stein operator whose domain and target space differ, corresponding to the distinct complementary components of a Jordan curve (Section 3.6). Moreover, we show that this operator is an isomorphism, and is in fact unitary precisely for disks (Section 3.9).

(s4) Sufficient curve conditions for the Kerzman-Stein kernel to belong to $C^{n,\alpha}$, yielding sufficient conditions for the Kerzman-Stein operator to belong to the Schatten p-classes S_p for $p \geqslant 2$ and $p = 1$ (Section 3.8).

(s5) A proof that the Kerzman-Stein operator is in S_p if and only if the Grunsky operator for the Smirnov space is in S_p, from which we are able to conclude that for all $\varepsilon > 0$, the $C^{1,1/2+\varepsilon}$ Jordan curves belong to the Weil-Petersson class of quasicircles (Section 3.9).

(s6) A construction of the scattering matrix of overfare is given in terms of Cauchy blocks, and quadratic adjoint identities for the Cauchy operators that suggest this scattering matrix has a symplectic interpretation (Section 3.10).

Following this in the Dirichlet setting, we have:

(d1) An explicit adjoint formula for the homogeneous overfare associated to quasicircles, which we then interpret to mean that it is a "skew-symplectic" transformation (Section 4.5).

(d2) An analogue of the Kerzman-Stein operator and the generalization in (s3) for the homogeneous Dirichlet space of harmonic functions (Section 4.7).

(d3) A Kerzman-Stein formula on quasidisks (Section 4.7).

(d4) A proof that the Kerzman-Stein operator of (d3) is in S_p if and only if the (classical) Grunsky operator is in S_p. In particular, since the conditions for the latter have previously been determined, we have necessary and sufficient conditions for the former for all $1 \leqslant p \leqslant \infty$ in terms of regularity conditions on the domain (Section 4.7).

2

Preliminaries

2.1 Classical Results on the Riemann Sphere

Let $\mathbb{C}$ denote the complex plane. For a nonempty open set $U \subset \mathbb{C}$, a function $h : U \to \mathbb{C}$ is said to be *holomorphic* on U if h is complex differentiable at every point in U. A function $g : U \to \mathbb{C}$ is said to be *antiholomorphic* on U if $g = \overline{H}$ for some holomorphic function H on U.

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the *Riemann sphere*. We equip $\overline{\mathbb{C}}$ with the standard charts

$$\varphi_0(z) = z, \quad z \in \mathbb{C},$$
$$\varphi_\infty(z) = \frac{1}{z}, \quad z \in \overline{\mathbb{C}} \setminus \{0\},$$

using the convention that $1/\infty = 0$. Thus, if $\infty \in U$, then $h : U \to \mathbb{C}$ is holomorphic at ∞ if $h \circ \varphi_\infty = h(1/z)$ is holomorphic on a neighbourhood of 0. Antiholomorphicity at ∞ is defined correspondingly as holomorphicity.

An onto function $f : U \to V$ is a *biholomorphism* if it is both one-to-one and holomorphic. In this case, $f^{-1} : V \to U$ is also holomorphic. We say that $f : U \to \mathbb{C}$ is *conformal* if it is a biholomorphism onto its image. The set of all biholomorphisms between open sets U and V is denoted $\mathrm{Conf}(U,V)$. A particularly important example is the set of automorphisms of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which consists precisely of the Möbius transformations mapping the disk to itself:

$$\mathrm{Conf}(\mathbb{D},\mathbb{D}) = \left\{ M(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z} : a \in \mathbb{D}, 0 \leqslant \theta < 2\pi \right\}. \tag{2.1.1}$$

For arbitrary open connected sets (called *domains*), we have the following foundational result:

Theorem 2.1.1 (Riemann Mapping Theorem). *Let $D \subsetneq \overline{\mathbb{C}}$ be a proper simply-connected domain with at least two boundary points. Then* $\mathrm{Conf}(D,\mathbb{D}) \neq \varnothing$.

Elements of $\mathrm{Conf}(D,\mathbb{D})$ (called *Riemann maps*) are unique up to composition by Möbius transformation of the disk. That is, given $F \in \mathrm{Conf}(D,\mathbb{D})$, we have

$$\mathrm{Conf}(D,\mathbb{D}) = \{M \circ F : M \in \mathrm{Conf}(\mathbb{D},\mathbb{D})\}.$$

A set of points $\Gamma \subset \overline{\mathbb{C}}$ is called a *curve* if there is a continuous function $\varphi : [0,1] \to \mathbb{C}$ such that $\Gamma = \mathrm{image}(\varphi)$. The map φ is called a *parameterization* of Γ. Often, it is convenient to identify a parameterization with the curve it defines. We call Γ a *Jordan curve* if φ is injective on $[0,1)$, and $\varphi(0) = \varphi(1)$, or equivalently if Γ is a homeomorphic image of $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. A key property of such curves is that they split the sphere into two distinct regions:

Theorem 2.1.2 (Jordan Curve Theorem). *Let $\Gamma \subset \overline{\mathbb{C}}$ be a Jordan curve. Then $\overline{\mathbb{C}} \backslash \Gamma$ has exactly two connected components whose boundary is* Γ.

The resulting complementary components for a given Jordan curve are called *Jordan domains*. Throughout, these components are labelled Ω_1 and Ω_2, and whenever Γ is bounded, let us agree that Ω_1 is the bounded component. In much of the following sections, it will be particularly constructive to think of the sphere comprising of these three sets. Clearly, Ω_1 and Ω_2 are always proper simply-connected domains in the sphere. However, not every proper simply-connected domain is a Jordan domain (for example, take $D = \overline{\mathbb{C}} \backslash [-1,1]$). For a set $D \subset \overline{\mathbb{C}}$, let $\mathrm{cl}(D)$ denote its topological closure. Jordan domains are characterized by the following extension theorem for their Riemann maps.

Theorem 2.1.3 (Carathéodory's Theorem). *Let $D \subsetneq \overline{\mathbb{C}}$ be a proper simply-connected domain with boundary Γ, and let $f \in \mathrm{Conf}(\mathbb{D},D)$. Then f has a continuous and injective extension to* $\mathrm{cl}(\mathbb{D})$ *if and only if* Γ *is a Jordan curve.*

In particular, whenever Γ is a Jordan curve, the extension of f is a homeomorphism of $\mathbb{S}^1$ – meaning we can take the conformal parameterization $t \mapsto f(e^{2\pi i t})$ for Γ.

The *length* $\ell(\Gamma) \in [0,\infty]$ of a bounded curve Γ is defined as the supremum over all sums of the form

$$\sum_{n=1}^{N} |\varphi(t_n) - \varphi(t_{n-1})|.$$

where φ parameterizes Γ, and $0 \leqslant t_0 < t_1 < \cdots < t_N \leqslant 1$. Γ is called *rectifiable* if $\ell(\Gamma) < \infty$. In this case, φ is of bounded variation, and it is a well-known fact that such functions are differentiable almost everywhere (a.e.) on $[0,1]$. Thus, for rectifiable curves Γ, we define the *unit tangent function* $T : \Gamma \to \mathbb{S}^1$ a.e. by the formula

$$T(\varphi(t)) = \frac{\varphi'(t)}{|\varphi'(t)|}, \quad t \in [0,1].$$

Finally, Γ is said to be *positively oriented* if the direction of traversal of its parameterization keeps Ω_1 on the left-hand side. Otherwise, we say Γ is *negatively oriented*.

2.2 Complex Conjugation on Vector Spaces

Before moving forward, we briefly review complexification of real vector spaces as a means of defining a canonical notion of conjugation on complex vector spaces. This process is especially important in upcoming sections since many of the spaces will have a natural decomposition induced by the conjugate, and our techniques rely on the manipulation of conjugated operators.

Let $V_{\mathbb{R}}$ be an arbitrary real vector space. The $\mathbb{R}$-tensor product of $\mathbb{C}$ with $V_{\mathbb{R}}$ is called the *complexification* of $V_{\mathbb{R}}$, and is denoted

$$V_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}.$$

In our presentation, we drop the tensor notation for elements of $V_{\mathbb{C}}$. Scalar multiplication over $\mathbb{C}$ is then defined as $c'(cv) = (c'c)v$ for all $c' \in \mathbb{C}$ and $cv \in V_{\mathbb{C}}$, which makes $V_{\mathbb{C}}$ a complex vector space. Every vector $v \in V_{\mathbb{C}}$ can be written uniquely as $v = v_1 + iv_2$, with $v_1, v_2 \in V_{\mathbb{R}}$, and so $V_{\mathbb{C}}$ can be decomposed into real and imaginary parts:

$$V_{\mathbb{C}} = V_{\mathbb{R}} \oplus iV_{\mathbb{R}}.$$

A $\mathbb{C}$-basis for $V_{\mathbb{C}}$ consists of elements of the form $e_k + i0$, where $e_k \in B$ for some basis B of $V_{\mathbb{R}}$. If $V_{\mathbb{R}}$ comes equipped with an inner product $\langle \cdot, \cdot \rangle_{V_{\mathbb{R}}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$, then $V_{\mathbb{C}}$ is also an inner product space with $\langle \cdot, \cdot \rangle_{V_{\mathbb{C}}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ given by

$$\langle u_1 + iu_2, v_1 + iv_2 \rangle_{V_{\mathbb{C}}} := \langle u_1, v_1 \rangle_{V_{\mathbb{R}}} + \langle u_2, v_2 \rangle_{V_{\mathbb{R}}} + i \left[\langle u_2, v_1 \rangle_{V_{\mathbb{R}}} - \langle u_1, v_2 \rangle_{V_{\mathbb{R}}} \right]. \tag{2.2.1}$$

Thus, complexification can extend any real space to a complex space in a natural sense. Conversely, given any complex vector space V, it is always possible to find a real subspace W of V such that V is the complexification of W, so that $V = W \oplus iW$. A *conjugation* on V is a function $\overline{(\cdot)} : V \to V$ satisfying the following properties:

(a) (*Conjugate-linearity*) $\overline{(cv_1+v_2)} = \overline{c}\cdot\overline{v_1}+\overline{v_2}$ for all $v_1, v_2 \in V$ and $c \in \mathbb{C}$,

(b) (*Involutory*) $\overline{\overline{v}} = v$ for all $v \in V$.

For any subspace $U \subset V$, we call the pairing $(U, \overline{(\cdot)})$ a *conjugable* space, and write $\overline{U}$ to refer to the subspace of V consisting of elements of the form $\overline{u}$ for $u \in U$. The canonical conjugation is defined in terms of the complexification of a real subspace. Writing $V = W \oplus iW$ as before, define $\overline{(\cdot)} : V \to V$ for all $v = w_1 + iw_2 \in V$ by

$$\overline{v} := w_1 - iw_2.$$

If V is an inner product space, then we have the following formula for the inner product of conjugates:

$$\langle \overline{u}, \overline{v} \rangle_V = \overline{\langle u, v \rangle_V} = \langle v, u \rangle_V. \tag{2.2.2}$$

Let us now recall some familiar spaces and conjugations that will be relevant in upcoming sections.

Example 2.2.1. Let $V_{\mathbb{R}}(D)$ denote the space of (real) one-forms u on a domain $D \subset \overline{\mathbb{C}}$ that are *harmonic*, meaning that $dv = u$ for a harmonic function $v : D \to \mathbb{R}$. In local coordinates, we can write $u = adx + bdy$ for some C^1-smooth functions a and b. Define the $*$-operator in these local coordinates by $*u = -bdx + ady$. Harmonic one-forms may also be characterized as the smooth one-forms that are *closed* ($du = 0$) and *co-closed* ($d[*u] = 0$). Complexifying $V_{\mathbb{R}}(D)$ according to the procedure above, we obtain the space of *complex harmonic one-forms* on D

$$\Omega_{\text{harm}}(D) := V_{\mathbb{R}}(D) \oplus iV_{\mathbb{R}}(D).$$

Elements $u \in \Omega_{\text{harm}}(D)$ can be expressed in local coordinates as

$$u = (a_1 + ib_1)dx + (a_2 + ib_2)dy \tag{2.2.3}$$

where $a_1, b_1, a_2, b_2 \in V_{\mathbb{R}}(D)$. Now, set $dz := dx + idy$, and $d\bar{z} := dx - idy$. A one-form is called *holomorphic* if it it given by $h(z)dz$ in local coordinates for some function h holomorphic in z. Similarly, a one-form is called *antiholomorphic* if it can be written as $\overline{g(z)}d\bar{z}$ for some function $\overline{g}$ antiholomorphic in z. Using the Cauchy-Riemann equations, it can be shown that every $u \in \Omega_{\text{harm}}(D)$ can be written as the sum of holomorphic and antiholomorphic one-forms in local coordinates:

$$u = hdz + \overline{H}d\bar{z}.$$

Thus, if we denote the subspace of holomorphic one-forms hdz by $\Omega^{1,0}(D)$, and the space of antiholomorphic one-forms by $\Omega^{0,1}(D) := \overline{\Omega^{1,0}(D)}$, we have the decomposition

$$\Omega_{\text{harm}}(D) = \Omega^{1,0}(D) \oplus \Omega^{0,1}(D).$$

Expressed in this way, it can be shown that the canonical conjugation of u is given by

$$\overline{u} = Hdz + \overline{h}d\bar{z}.$$

Example 2.2.2. Let $\Gamma \subset \mathbb{C}$ be a bounded rectifiable Jordan curve. Denote by $L^2(\Gamma, |d\zeta|)$ the Hilbert space of measurable functions $u : \Gamma \to \mathbb{C}$ for which

$$\|u\|_\Gamma^2 = \int_\Gamma |u(\zeta)|^2 \, |d\zeta| < \infty.$$

The norm $\|\cdot\|_\Gamma$ is induced by the inner product

$$\langle u, v \rangle_\Gamma = \int_\Gamma u(\zeta)\overline{v(\zeta)} \, |d\zeta|.$$

Since $u \in L^2(\Gamma, |d\zeta|)$ if and only if $\operatorname{Re} u$, $\operatorname{Im} u \in L^2(\Gamma, |d\zeta|)$, we can view $L^2(\Gamma, |d\zeta|)$ as the complexification of the space of real measurable functions $\Gamma \to \mathbb{R}$ with finite L^2-norm. Thus, the canonical conjugation is simply conjugations of functions. Later when we think of $L^2(\Gamma, |d\zeta|)$ in terms of the boundary values of the Smirnov space, we will mention a different conjugation for the space.

Definition 2.2.3. Let $\mathbf{T} : V \to W$ be a linear operator between conjugable spaces. Define the conjugate operator $\overline{\mathbf{T}} : \overline{V} \to \overline{W}$ by

$$\overline{\mathbf{T}}\overline{v} = \overline{\mathbf{T}v}, \qquad \overline{v} \in \overline{V}.$$

We collect some basic algebraic facts regarding conjugate operators. Let V_1, V_2, V_3 be conjugable spaces, and let $\mathbf{T} : V_1 \to V_2$, $\mathbf{L} : V_1 \to V_2$, $\mathbf{S} : V_2 \to V_3$ be linear operators. Then the following are easily verified:

(a) $\overline{\overline{\mathbf{T}}} = \mathbf{T}$.

(b) $\overline{(c_1\mathbf{T} + c_2\mathbf{L})} = \bar{c_1}\overline{\mathbf{T}} + \bar{c_2}\overline{\mathbf{L}}$ for all $c_1, c_2 \in \mathbb{C}$.

(c) $\overline{(\mathbf{ST})} = \overline{\mathbf{S}} \circ \overline{\mathbf{T}}$.

(d) If $\mathbf{T}$ is invertible, then $\overline{\mathbf{T}}$ is invertible and $\left(\overline{\mathbf{T}}\right)^{-1} = \overline{(\mathbf{T}^{-1})}$.

(e) $\|\overline{\mathbf{T}}\| = \|\mathbf{T}\|$, where $\|\cdot\|$ denotes the operator norm.

For Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, denote the set of all bounded operators $\mathbf{B} : \mathcal{H}_1 \to \mathcal{H}_2$ by $B(\mathcal{H}_1, \mathcal{H}_2)$. When $\mathcal{H}_1 = \mathcal{H}_2$, we simply write $B(\mathcal{H}_1)$. The symbol $\mathbf{I}$ is used generically for the identity operator whenever the underlying space is clear from context. Recall that for every $\mathbf{B} \in B(\mathcal{H}_1, \mathcal{H}_2)$, the *adjoint* of $\mathbf{B}$ is the unique operator $\mathbf{B}^* \in B(\mathcal{H}_2, \mathcal{H}_1)$ defined for all $u_2 \in \mathcal{H}_2$ by $u_2 \mapsto \mathbf{B}^* u_2$ such that

$$\langle \mathbf{B}u_1, u_2 \rangle_{\mathcal{H}_2} = \langle u_1, \mathbf{B}^* u_2 \rangle_{\mathcal{H}_1} \text{ for all } u_1 \in \mathcal{H}_1. \tag{2.2.4}$$

The existence and uniqueness of such an operator is a consequence of the Riesz representation theorem. If we equip each of $\mathcal{H}_1$ and $\mathcal{H}_2$ with their canonical conjugation, then

$$\left\langle \overline{\mathbf{B}}\overline{u_1}, \overline{u_2} \right\rangle_{\mathcal{H}_2} = \overline{\langle \mathbf{B}u_1, u_2 \rangle}_{\mathcal{H}_2} = \overline{\langle u_1, \mathbf{B}^* u_2 \rangle}_{\mathcal{H}_1} = \left\langle \overline{u_1}, \overline{\mathbf{B}^* u_2} \right\rangle_{\mathcal{H}_1}$$

and hence

$$\left(\overline{\mathbf{B}}\right)^* = \overline{(\mathbf{B}^*)}. \tag{2.2.5}$$

3

Smirnov Spaces of Harmonic Half-Order Differentials

3.1 Generalizations of the Hardy Space

The Smirnov space associated to a planar domain is one of the natural generalizations of the Hardy space of the disk, and is a particularly nice setting to conduct function theory due to the regularity of the boundary values and pleasing blend of analysis with the geometry of curves. While many of the standard sources on the topic represent elements of the space in terms of functions, Barrett and Bolt [1] considered representing the corresponding Cauchy and Kerzman-Stein operators using half-order differentials, which naturally aligns with how elements of the space transform under pull-back of conformal mappings (see equation (1.0.3)). Very recently, Kristel et al. [26] followed up on this approach by recasting many of the classical objects and theorems in the literature with this newer framework, whilst also considering the relation between the spaces on the bounded and unbounded domains.

Our main goal for this chapter is to expand on aspects of this setting explored by Kristel et al. [26], whom we primarily follow in the following three sections – first, with the conformally invariant definition of the Smirnov space (Section 3.2), then the description of the boundary values (Section 3.2), and then with properties of the half-order Cauchy operator (Section 3.4). Following this in Section 3.5, we compute adjoint formulas for the half-order Cauchy operator on Ahlfors-regular Jordan domains using algebraic properties of the "overfare" operator, which identifies elements of the Smirnov spaces on either sides of a Jordan curve that share the same L^2 boundary values. Utilizing overfare proves to be effective throughout the chapter for deriving many other identities from the jump formula in the generality established by David [11]. Following the adjoint formulas, we consider the Kerzman-Stein operator in Section 3.6. Here, we prove that the Kerzman-

Stein formula holds in the context of Ahlfors-regular Jordan domains. We also define a related operator, which we call the d,c-Kerzman-Stein operator, that is an isomorphism between the Smirnov spaces on either side of the curve. In the following section, we use our previous results along with theorems of Privalov [36] and David [11] to describe the boundary values of the integral kernel of the Kerzman-Stein operator, and we show that the Kerzman-Stein formula holds on Ahlfors-regular Jordan curves. Section 3.8 deals with how properties of the Kerzman-Stein kernel function are intimately linked to the boundary regularity of the domain, and we include sufficient $C^{n,\alpha}$ curve conditions for the Kerzman-Stein operator to belong to the Schatten p-classes S_p for all $p \geqslant 2$. In Section 3.9, we cover the analogue of the Grunsky operator for the Smirnov space defined by Kristel et al. [26]. In particular, we show that it is related to the Kerzman-Stein operator in the sense that it belongs to S_p if and only if the Kerzman-Stein operator is in S_p. We also characterize disk domains in terms of vanishing of the Smirnov Grunsky operator, and properties of the Kerzman-Stein operator, including unitarity of the d,c-Kerzman-Stein operator. We end the chapter with a computation of the scattering matrix for the overfare operator in Section 3.10, which has previously only been done in the Bergman space setting by Schippers and Staubach [38], [41].

Before proceeding, we briefly review the standard function-space theory in the plane, following Kristel et al. [26]. The *Hardy space of the disk*, denoted $H^2(\mathbb{D})$, is the Hilbert space of holomorphic functions $h : \mathbb{D} \to \mathbb{C}$ satisfying

$$\|h\|^2_{H^2(\mathbb{D})} = \sup_{0<r<1} \int_0^{2\pi} |h(re^{i\theta})|^2 \, d\theta < \infty. \tag{3.1.1}$$

This growth condition on h ensures that it has nontangential boundary values almost everywhere on $\mathbb{S}^1$, and they in fact belong to $L^2(\mathbb{S}^1, |d\zeta|)$ (Theorem 2.2 in Duren [13]). The generalization of Hardy space theory to arbitrary domains in the plane can take different forms, since the set of functions $H^2(\mathbb{D})$ may be characterized in different ways that are not necessarily equivalent on more general domains. We review these before settling on the definition we will be using.

First, it is well-known that $h \in H^2(\mathbb{D})$ if and only if the subharmonic function $|h|^2$ has a harmonic majorant in $\mathbb{D}$ (Theorem 2.12 in Duren [13]). Thus, a natural extension of the Hardy space to an arbitrary simply-connected domain $\Omega \subset \mathbb{C}$ (with at least two boundary points) would be to require this condition for candidate holomorphic functions $h : \Omega \to \mathbb{C}$. In fact, this is equivalent to demanding that $h \circ f \in H^2(\mathbb{D})$ for some $f \in \mathrm{Conf}(\mathbb{D}, \Omega)$ (see Theorem 10.1 in Duren [13]).

Alternatively, one could ask for a limiting integral condition, akin to equation (3.1.1).

The *Smirnov space* on a simply-connected domain $\Omega \subset \mathbb{C}$, denoted $E^2(\Omega)$, is the space of holomorphic functions $h : \Omega \to \mathbb{C}$ with the property that there is a sequence of rectifiable Jordan curves $\Gamma_n \subset \Omega$ that eventually enclose every compact subdomain of Ω such that

$$\|h\|^2_{E^2(\Omega)} = \sup_{0<n<1} \int_{\Gamma_n} |h(\zeta)|^2 \, |d\zeta| < \infty. \tag{3.1.2}$$

In fact, for each $h \in E^2(\Omega)$, it suffices to consider level curves $\Gamma_r = f(|\zeta| = r)$ of a conformal map $f \in \mathrm{Conf}(\mathbb{D},\Omega)$ (Theorem 10.1, Duren [13]). The subtlety arising from (3.1.2) that is hidden in (3.1.1) is the influence of the boundary behaviour of the mapping function f. In particular, we have that $h \in E^2(\Omega)$ if and only if $(h \circ f)\sqrt{f'} \in H^2(\mathbb{D})$ (Theorem 10.1 in Duren [13]). Therefore, it is perhaps most clear in this instance that the geometry of the domain Ω has great effect on the space of functions $E^2(\Omega)$. In fact, $E^2(\Omega)$ coincides with the harmonic majorant extension of the Hardy space only for sufficiently smooth domains (see Theorem 10.2 in Duren [13]).

Another possibility is to define the Hardy space based on properties of the desired boundary functions. Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded rectifiable Jordan curve with complementary components Ω_1 and Ω_2. A function $u \in L^2(\Gamma, |d\zeta|)$ belongs to the *$L^2(\Gamma, |d\zeta|)$-closure of polynomials in z* if there is a sequence of polynomials $p_n(z)$ such that $\|u - p_n\|_\Gamma \to 0$ as $n \to \infty$. Making the identification of $H^2(\mathbb{D})$ with its boundary values, it can be shown that $H^2(\mathbb{D})$ is exactly the $L^2(\mathbb{S}^1, |d\zeta|)$-closure of the polynomials in $z = e^{i\theta}$ (Theorem 3.3 in Duren [13]). Thus, one can define the space $\mathcal{H}^2(\Omega_1)$ to be the $L^2(\Gamma, |d\zeta|)$-closure of the polynomials in z. For the unbounded component Ω_2, the space $\mathcal{H}^2(\Omega_2)$ is defined to be the $L^2(\Gamma, |d\zeta|)$ closure of the polynomials in z^{-1} that vanish at ∞.

The final candidate considered for an extension of the Hardy space is, in loose sense, one that picks out the functions in $L^2(\Gamma, |d\zeta|)$ that are orthogonal to antiholomorphic functions (this is related to the decomposition in equation (1.0.2), which we discuss again in a later section). The *generalized Hardy space* $H^2(\Omega_1)$ is defined to be the (closed) subspace consisting of elements $u \in L^2(\Gamma, |d\zeta|)$ such that for all $n \geqslant 0$, one has

$$\int_\Gamma u(\zeta)\zeta^n \, d\zeta = 0.$$

Similarly, the generalized Hardy space of the unbounded component $H^2(\Omega_2)$ is defined to be the (closed) subspace consisting of elements $u \in L^2(\Gamma, |d\zeta|)$ such that for all $n \geqslant 1$

$$\int_\Gamma u(\zeta)\zeta^{-n} \, d\zeta = 0.$$

For sufficiently "regular" domains, different characterizations may coincide – the most general class being *Smirnov domains*.

Definition 3.1.1. A bounded rectifiable Jordan domain $\Omega \subset \mathbb{C}$ is called a Smirnov domain if for some $f \in \mathrm{Conf}(\mathbb{D}, \Omega)$, the harmonic function $\log|f'(z)|$ can be represented by the Poisson integral of its nontangential boundary values:

$$\log|\varphi'(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\log|f'(e^{it})|\,dt, \quad re^{i\theta} \in \mathbb{D}$$

where $P(r, \theta)$ is the Poisson kernel function

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

As the name suggests, Smirnov domains are well-suited for hosting Smirnov spaces. Indeed, by Theorem 10.6 Duren [13], the boundary values of $E^2(\Omega)$ (which we simply identify with $E^2(\Omega)$ for convenience) coincide with $\mathcal{H}^2(\Omega)$ if and only if Ω is a Smirnov domain. From here, further increasing the regularity can allow for more characterizations to agree.

Definition 3.1.2. A bounded rectifiable Jordan curve $\Gamma \subset \overline{\mathbb{C}}$ is said to be Ahlfors-regular if there is a constant $M > 0$ with the property that

$$\ell(B(z;r) \cap \Gamma) \leqslant rM$$

for all $z \in \Gamma$ and $r > 0$. We call the complementary components of Γ Ahlfors-regular Jordan domains.

Zinsmeister [47] showed that Ahlfors-regular Jordan domains are Smirnov, and they likely remain the most general class of Smirnov domains known (Pritsker [35]). Moreover, for a bounded Ahlfors-regular Jordan curve $\Gamma \subset \overline{\mathbb{C}}$ with complementary components Ω_1 and Ω_2, we have that

$$H^2(\Omega_d) = E^2(\Omega_d) = \mathcal{H}^2(\Omega_d) = \hat{H}(\Omega_d), \quad d \in \{1,2\}.$$

For a proof of this fact, see Theorem 8 in Section 12 of Meyer and Coifman [29]. Furthermore, Ahlfors-regular Jordan domains are precisely those for which the Cauchy operator on $E^2(\Omega)$ is bounded and the jump decomposition holds (see Sections 3.4 and 3.7). For these reasons, we shall mainly focus on this class for the upcoming sections.

3.2 Half-Order Differentials

Following Kristel et al. [26], we now state and discuss basic definitions and objects associated to the Smirnov space in a differential-geometric language that makes the space conformally invariant.

Let $D_1, D_2 \subsetneq \overline{\mathbb{C}}$ be proper simply-connected domains in the sphere. For any conformal map $f \in \mathrm{Conf}(D_1, D_2)$, there are exactly two holomorphic functions $g : D_1 \to \mathbb{C}\backslash\{0\}$ with the property that $g^2 = f'$. Define the collection

$$\widehat{\mathrm{Conf}}(D_1, D_2) := \left\{(f, g) : f \in \mathrm{Conf}(D_1, D_2),\, g^2 = f'\right\}.$$

We write elements of $\widehat{\mathrm{Conf}}(D_1, D_2)$ as $\hat{f} = (f, \sqrt{f'})$, where $\sqrt{f'}$ denotes a definite choice of branch of square root of f'.

We now give an informal description of half-order differentials in terms of their algebraic properties and how they transform between domains. See the appendix in Kristel et al. [26] for an explicit construction from a differential-geometric viewpoint. The vector space of *holomorphic half-order differentials* on $D \subsetneq \mathbb{C}$, denoted by $\Omega^{\frac{1}{2},0}(D)$, consists of elements $hdz^{1/2}$, where $h : D \to \mathbb{C}$ is a holomorphic function (called the *coefficient* of $hdz^{1/2}$). We equip this space with a bilinear pairing

$$\left(hdz^{1/2},\, gdz^{1/2}\right) \mapsto hgdz \in \Omega^{1,0}(D).$$

For our purposes, the symbol $dz^{1/2}$ is treated formally, reinforcing the above pairing and the transformation law

$$\Omega^{\frac{1}{2},0}(D_2) \to \Omega^{\frac{1}{2},0}(D_1)$$

given by pull-back under $\hat{f}$:

$$\hat{f}^*\left(hdz^{1/2}\right) := (h \circ f)\sqrt{f'}dz^{1/2} \in \Omega^{\frac{1}{2},0}(D_1). \tag{3.2.1}$$

The transformation law allows us to extend the definition of $\Omega^{\frac{1}{2},0}(D)$ to arbitrary proper simply-connected domains in the sphere. However, observe that if D_2 contains ∞, then for all $hdz^{1/2} \in \Omega^{\frac{1}{2},0}(D_2)$, necessarily $h(1/z)/z$ must be holomorphic at 0 since if $\iota(z) = 1/z$ denotes the meromorphic reciprocal function, then

$$\iota^*\left(h(z)dz^{1/2}\right) = \frac{i}{z}h\left(\frac{1}{z}\right)dz^{1/2}.$$

In particular, $h(z)$ must have a zero at ∞. Thus, holomorphicity at ∞ for half-order differentials is stricter than the usual condition of holomorphicity at ∞ for functions.

Similarly, we define the vector space of *antiholomorphic half-order differentials* on $D \subsetneq \overline{\mathbb{C}}$, denoted by $\Omega^{0,\frac{1}{2}}(D)$, as the conjugation of the corresponding holomorphic space:

$$\Omega^{0,\frac{1}{2}}(D) := \overline{\Omega^{\frac{1}{2},0}(D)}.$$

Elements of $\Omega^{0,\frac{1}{2}}(D)$ are of the form $\overline{H}d\bar{z}^{1/2}$, where $\overline{H} : D \to \mathbb{C}$ is an antiholomorphic function. The bilinear pairing of the holomorphic space is inherited as expected:

$$\left(\overline{H}d\bar{z}^{1/2}, \overline{G}d\bar{z}^{1/2}\right) \mapsto \overline{HG}d\bar{z} \in \Omega^{0,1}(D).$$

Similarly, we have the transformation law for $\Omega^{0,\frac{1}{2}}(D_2) \to \Omega^{0,\frac{1}{2}}(D_1)$ given by

$$\hat{f}^* \left(\overline{H}d\bar{z}^{1/2}\right) = \overline{(H \circ f)}\overline{\sqrt{f'}}d\bar{z}^{1/2} \in \Omega^{0,\frac{1}{2}}(D_1). \tag{3.2.2}$$

We further extend the bilinear pairing to elements from both spaces simultaneously:

$$\left(hdz^{1/2}, \overline{H}d\bar{z}^{1/2}\right) \mapsto h\overline{H}|dz|$$

where $|dz|$ denotes the arc-length differential.

Next, we refine the vector spaces of half-order differentials by imposing growth conditions on the elements near the boundary. Let $D \subsetneq \overline{\mathbb{C}}$ be a proper simply-connected domain, and let $g_D(z,q)$ denote Green's function of D with singularity at $q \in D$. For each $0 < r < 1$, let $\Gamma_{q,r}$ denote the level curve

$$\Gamma_{q,r} := \{z \in \Omega : g_D(z,q) = -\log r\}. \tag{3.2.3}$$

Using conformal invariance of Green's functions and the fact that

$$g_{\mathbb{D}}(z,q) = -\log\left|\frac{z-q}{1-\bar{q}z}\right|$$

it follows that for any $f \in \mathrm{Conf}(\mathbb{D}, D)$ with $f(0) = q$, we have

$$\Gamma_{q,r} = f(|z| = r). \tag{3.2.4}$$

In particular, (3.2.3) describes the set of points associated to the real analytic Jordan curve $t \mapsto f(re^{2\pi it})$ (see Remark 3.8.15 for further comment). For each $q \in D$, define a norm

$\|\cdot\|_{D,q}$ on $\Omega^{\frac{1}{2},0}(D)$ by the formula

$$\|hdz^{1/2}\|_{D,q}^2 := \lim_{r\nearrow 1}\int_{\Gamma_{q,r}} |h(z)|^2\,|dz|. \tag{3.2.5}$$

For any two $p,q \in D$, it can be shown that we have equality of the norms with respect to both p and q (see Proposition 2.3, Kristel et al. [26]):

$$\|hdz^{1/2}\|_{D,q} = \|hdz^{1/2}\|_{D,p}.$$

Therefore, we drop the notation for $q \in D$, and agree that a choice of singularity has been implicitly made:

$$\|hdz^{1/2}\|_{D}^2 = \lim_{r\nearrow 1}\int_{\Gamma_{r}} |h(z)|^2\,|dz|. \tag{3.2.6}$$

Definition 3.2.1. Let $D \subsetneq \overline{\mathbb{C}}$ be a proper simply-connected domain. The Smirnov space of half-order differentials, denoted $\mathcal{A}^{1/2}(D)$, is the Hilbert space of holomorphic half-order differentials $hdz^{1/2} \in \Omega^{\frac{1}{2},0}(D)$ with the property that $\|hdz^{1/2}\|_D < \infty$.

In other words, $\mathcal{A}^{1/2}(D)$ consists of holomorphic half-order differentials $hdz^{1/2}$ whose coefficient h is an element of the Smirnov space $E^2(D)$ defined in the introduction of the section. We equip $\mathcal{A}^{1/2}(D)$ with the compatible inner product to the norm in (3.2.5):

$$\left\langle hdz^{1/2}, gdz^{1/2}\right\rangle_{D,q} := \lim_{r\nearrow 1}\int_{\Gamma_{q,r}} h(z)dz^{1/2}\overline{g(z)}d\bar{z}^{1/2}. \tag{3.2.7}$$

The integral on the right-hand side is interpreted using the bilinear pairing associated to the differentials:

$$\lim_{r\nearrow 1}\int_{\Gamma_{q,r}} h(z)dz^{1/2}\overline{g(z)}d\bar{z}^{1/2} = \lim_{r\nearrow 1}\int_{\Gamma_{q,r}} h(z)\overline{g(z)}\,|dz|.$$

Using (3.2.6) and the polarization identity, it follows that the inner product is unchanged upon varying $q \in D$, so we again assume that a choice of singularity for Green's function has been made, and simply write

$$\left\langle hdz^{1/2}, gdz^{1/2}\right\rangle_{D} = \lim_{r\nearrow 1}\int_{\Gamma_{r}} h(z)\overline{g(z)}\,|dz|.$$

We define the corresponding antiholomorphic space $\overline{\mathcal{A}^{1/2}(D)}$ by conjugation, which inherits the norm and inner product as described in Section 2.2.

Definition 3.2.2. Let $D \subsetneq \overline{\mathbb{C}}$ be a proper simply-connected domain. The Smirnov space of harmonic half-order differentials on D is defined to be the direct sum

$$\mathcal{A}^{1/2}_{\text{harm}}(D) = \mathcal{A}^{1/2}(D) \oplus \overline{\mathcal{A}^{1/2}(D)}.$$

We equip this larger space with the extension of the inner product (3.2.7) that makes the holomorphic and antiholomorphic subspaces orthogonal to each other. That is, for $u, v \in \mathcal{A}^{1/2}_{\text{harm}}(D)$ with

$$u = h dz^{1/2} + \overline{H} d\bar{z}^{1/2},\ v = g dz^{1/2} + \overline{G} d\bar{z}^{1/2}$$

we have

$$\langle u, v \rangle_D = \left\langle h dz^{1/2}, g dz^{1/2} \right\rangle_D + \left\langle \overline{H} d\bar{z}^{1/2}, \overline{G} d\bar{z}^{1/2} \right\rangle_D.$$

Note that the extension of the inner product is both consistent with the Cauchy-Goursat theorem and the bilinear product, since

$$\lim_{r \nearrow 1} \int_{\Gamma_r} h(\zeta) G(\zeta)\, dz = 0 = \left\langle h dz^{1/2}, \overline{G} d\bar{z}^{1/2} \right\rangle_D$$

and

$$\lim_{r \nearrow 1} \int_{\Gamma_r} \overline{H(\zeta) g(\zeta)}\, d\bar{z} = 0 = \left\langle \overline{H} d\bar{z}^{1/2}, g dz^{1/2} \right\rangle_D.$$

Moving forward, we generically refer to $\mathcal{A}^{1/2}_{\text{harm}}(D)$ as the Smirnov space. The following result can be interpreted as the saying that the Smirnov space, when modelled using half-order differentials, is *conformally invariant*.

Proposition 3.2.3. *Let $D_1, D_2 \subsetneq \overline{\mathbb{C}}$ be proper simply-connected domains. For any $\hat{f} \in \widehat{\text{Conf}}(D_2, D_1)$, the map*

$$\hat{f}^* : \mathcal{A}^{1/2}(D_1) \to \mathcal{A}^{1/2}(D_2)$$

is unitary.

Proof. This is shown in steps throughout Section 2 of Kristel et al. [26]. □

Remark 3.2.4. Similarly, we have that $\hat{f}^* : \overline{\mathcal{A}^{1/2}(D_1)} \to \overline{\mathcal{A}^{1/2}(D_2)}$ is unitary. Moreover, if we extend both notions of pull-back linearly to all of $\mathcal{A}^{1/2}_{\text{harm}}(D_1)$, then it follows that $\overline{\hat{f}^*} = \hat{f}^*$.

Often, it will be useful to analyze the restriction of linear operators to the principal orthogonal subspaces in Definition 3.2.2. For an operator $\mathbf{B} : \mathcal{A}^{1/2}_{\text{harm}}(D_1) \to \mathcal{A}^{1/2}_{\text{harm}}(D_2)$, we denote these restrictions as

$$\mathbf{B}^h := \mathbf{B}|_{\mathcal{A}^{1/2}(D_1)} : \mathcal{A}^{1/2}(D_1) \to \mathcal{A}^{1/2}_{\text{harm}}(D_2),$$
$$\mathbf{B}^a := \mathbf{B}|_{\overline{\mathcal{A}^{1/2}(D_1)}} : \overline{\mathcal{A}^{1/2}(D_1)} \to \mathcal{A}^{1/2}_{\text{harm}}(D_2).$$

Concerning the identity operator $\mathbf{I}$, we sometime use the above restriction notation for book-keeping in larger expressions. Similarly, we shall keep track of the underlying domains using subscripts on the operator (eg. $\mathbf{B}_{1,2}$).

Example 3.2.5. The Smirnov space of the disk has a particularly nice characterization. Let $h : \mathbb{D} \to \mathbb{C}$ be a holomorphic function. Then h has a power series representation centred at 0, say $h(z) = \sum_{n=0}^{\infty} a_n z^n$. The set $\{z^n dz^{1/2} : n \geqslant 0\}$ forms an orthonormal basis for $\mathcal{A}^{1/2}(\mathbb{D})$, and so we have

$$\|hdz^{1/2}\|_{\mathbb{D}}^2 = \sum_{n=0}^{\infty} \left\langle a_n z^n dz^{1/2}, a_n z^n dz^{1/2} \right\rangle_{\mathbb{D}} = \sum_{n=0}^{\infty} |a_n|^2.$$

Therefore, $hdz^{1/2} \in \mathcal{A}^{1/2}(\mathbb{D})$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ (compare with the holomorphic Dirichlet space of the disk in Example 4.2.5).

Definition 3.2.6. The Szegő projection is the orthogonal projection

$$\mathbf{P} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega) \to \mathcal{A}^{1/2}(\Omega)$$

defined for all $u = \alpha + \overline{\beta} \in \mathcal{A}^{1/2}(\Omega) \oplus \overline{\mathcal{A}^{1/2}(\Omega)}$ by $\mathbf{P}u = \alpha$.

Remark 3.2.7. The complementary orthogonal projection $\mathbf{I} - \mathbf{P} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega) \to \overline{\mathcal{A}^{1/2}(\Omega)}$ is denoted by the symbol $\overline{\mathbf{P}}$. This notation agrees with the definition of the conjugate operator since

$$\overline{\mathbf{P}}u = \overline{\mathbf{P}\overline{u}} = \overline{\beta} = (\mathbf{I} - \mathbf{P})u.$$

It is easy to check that the Szegő projection commutes with pull-back under $\hat{f} \in \widehat{\text{Conf}}(D_1, D_2)$, and by Remark 3.2.4, so does $\overline{\mathbf{P}}$:

$$\hat{f}^* \mathbf{P}_2 = \mathbf{P}_1 \hat{f}^*,$$
$$\hat{f}^* \overline{\mathbf{P}}_2 = \overline{\mathbf{P}}_1 \hat{f}^*.$$

The associated *Szegő kernel* is a well-studied integral kernel that reproduces holomorphic boundary values in $L^2(\Gamma, |d\zeta|)$, which can be analogously defined for $\mathcal{A}^{1/2}_{\text{harm}}(\Omega)$. Recall that by the Riesz-representation theorem, there is a unique integral kernel $S_{\Omega,z}(\zeta) \in$

$\mathcal{A}^{1/2}_{\mathrm{harm}}(\Omega)$ with the property that for all $u \in \mathcal{A}^{1/2}_{\mathrm{harm}}(\Omega)$

$$(\mathbf{P}u)(z) = \left\langle u, \overline{S_{\Omega,z} dz^{1/2}} \right\rangle_{\Omega}, \quad z \in \Omega. \tag{3.2.8}$$

Starting in the unit disk $\mathbb{D}$, the Szegő kernel is given by

$$S_{\mathbb{D},z}(\zeta) dz^{1/2} = S_{\mathbb{D}}(z,\zeta) d\bar{\zeta}^{1/2} dz^{1/2} = \frac{1}{2\pi} \frac{d\bar{\zeta}^{1/2} dz^{1/2}}{1 - \bar{\zeta} z}, \quad z, \zeta \in \mathbb{D}.$$

For arbitrary proper simply-connected domains $\Omega \subsetneq \overline{\mathbb{C}}$, the Szegő kernel can be computed via pull-back from that for the disk. That is, if $\hat{F} \in \widehat{\mathrm{Conf}}(\Omega, \mathbb{D})$, then

$$\begin{aligned} S_{\Omega,z}(\zeta) dz^{1/2} = S_{\Omega}(z,\zeta) d\bar{\zeta}^{1/2} dz^{1/2} &= (\hat{F} \times \hat{F})^* \left(S_{\mathbb{D}}(z,\zeta) d\bar{\zeta}^{1/2} dz^{1/2} \right) \\ &= \frac{1}{2\pi} \frac{\overline{\sqrt{F'(\zeta)}} \sqrt{F'(z)}}{1 - \overline{F(\zeta)} F(z)} d\bar{\zeta}^{1/2} dz^{1/2}. \end{aligned}$$

It can be shown that the above expression is Möbius invariant, and thus well-defined. The fact that $S_{\Omega}(z,\zeta) d\bar{\zeta}^{1/2} dz^{1/2}$ is the integral kernel for $\mathbf{P}$ as claimed in (3.2.8) is classical. See Proposition 4.1 in Kristel et al. [26] for a proof in this analytic setting.

Similarly for $\overline{\mathbf{P}}$, we have the *Garabedian kernel*, denoted $L_{\Omega,z}(\zeta)$, which has the property that

$$\left(\overline{\mathbf{P}} u \right)(z) = \left\langle u, -i L_{\Omega,z} dz^{1/2} \right\rangle_{\Omega}, \quad z \in \Omega. \tag{3.2.9}$$

For the unit disk $\mathbb{D}$, the Garabedian kernel is given by

$$L_{\mathbb{D},z}(\zeta) dz^{1/2} = L_{\mathbb{D}}(\zeta, z) d\zeta^{1/2} dz^{1/2} = \frac{1}{2\pi} \frac{d\zeta^{1/2} dz^{1/2}}{\zeta - z} \tag{3.2.10}$$

and for general Ω, the kernel computed via pull-back as before:

$$\begin{aligned} L_{\Omega,z}(\zeta) dz^{1/2} = L_{\Omega}(\zeta, z) d\zeta^{1/2} dz^{1/2} &= (\hat{F} \times \hat{F})^* (L_{\mathbb{D}}(\zeta, z) d\zeta^{1/2} dz^{1/2}) \\ &= \frac{1}{2\pi} \frac{\sqrt{F'(\zeta)} \sqrt{F'(z)}}{F(\zeta) - F(z)} d\zeta^{1/2} dz^{1/2}. \end{aligned}$$

Again, the above is Möbius invariant, and hence the kernel is well-defined. By expansion into a power series in $\zeta \in \Omega$, one can show that the function $L_{\Omega}(\zeta, z)$ has a simple pole at $\zeta = z$ with residue $1/2\pi$. Therefore, the expression in equation (3.2.9) is a slight abuse of notation as the kernel is not actually an element of the Smirnov space. Thus, we interpret the inner product in the equation in terms of the limiting-integral expression in equation (3.2.7). By construction, both kernels are *conformally invariant* – that is, both kernels transform via pull-back by $\hat{g} \in \widehat{\mathrm{Conf}}(\Omega_1, \Omega_2)$:

$$(\hat{g} \times \hat{g})^* \left(S_{\Omega_2}(z,\zeta) dz^{1/2} d\bar{\zeta}^{1/2} \right) = S_{\Omega_1}(g(z), g(\zeta)) \sqrt{g'(z)} \overline{\sqrt{g'(\zeta)}} dz^{1/2} d\bar{\zeta}^{1/2},$$
$$(\hat{g} \times \hat{g})^* \left(L_{\Omega_2}(\zeta, z) d\zeta^{1/2} dz^{1/2} \right) = L_{\Omega_1}(g(\zeta), g(z)) \sqrt{g'(\zeta)} \sqrt{g'(z)} dz^{1/2} d\zeta^{1/2}.$$

3.3 Boundary Values and Overfare

Much of the classical theory of operators on Smirnov spaces is phrased in terms of the corresponding boundary space $L^2(\Gamma, |d\zeta|)$. In this section, we discuss the identification of the two spaces using nontangential boundary values of functions. The upshot of this is two-fold. First, all of the classical function-theoretic results become available in the new setting. Second, it gives us a way of relating the Smirnov spaces on either side of a rectifiable Jordan curve, using *overfare*. Passing between the boundary space and either side of the curve is a surprisingly powerful technique that will be used repeatedly in upcoming sections.

We begin with the notion of boundary values of functions on the disk, and then extend to more general domains using conformal reparameterization. Let $g : \mathbb{D} \to \mathbb{C}$ be a function. Then g is said to have a *nontangential* limit $l \in \mathbb{C}$ at a point $z_0 \in \mathbb{S}^1$ if for every $1 < M < \infty$, we have

$$\lim_{\substack{z \to z_0 \\ z \in W(z_0;M)}} g(z) = l$$

where z lies in the *nontangential wedge*

$$W(z_0; M) := \{ z \in \mathbb{D} : |z_0 - z| < M(1 - |z|) \}.$$

Loosely, the restriction on how z may approach z_0 ensures that the path taken does not approach the tangent line to $\mathbb{S}^1$ at z_0, along which irregular boundary behaviour could affect the convergence of the limit.

Now, consider a general Jordan domain $\Omega \subsetneq \overline{\mathbb{C}}$ with boundary Γ. We say that a function $g : \Omega \to \overline{\mathbb{C}}$ has a *conformally nontangential* limit (or *CNT* limit) $l \in \mathbb{C}$ at $z_0 \in \Gamma$ if there is $f \in \mathrm{Conf}(\mathbb{D}, \Omega)$ such that $g \circ f : \mathbb{D} \to \Omega$ has a nontangential limit of l at $f^{-1}(z_0) \in \mathbb{S}^1$. It can be shown that this definition is indeed well-defined – that is, if the limit exists, then it is identical for any choice of conformal map (see Lemma 2.29 in Schippers and Staubach [39], for example). The following theorem says that elements of the Smirnov space have particularly well-behaved CNT limits for rectifiable Jordan domains.

Theorem 3.3.1. *Let $\Omega \subset \overline{\mathbb{C}}$ be a rectifiable Jordan domain with boundary Γ. For each $hdz^{1/2} \in \mathcal{A}^{1/2}(\Omega)$, the function h has a conformally nontangential limit $h_\Gamma(\zeta)$ for almost every $\zeta \in \Gamma$. The resulting boundary function $h_\Gamma : \Gamma \to \mathbb{C}$ belongs to $L^2(\Gamma, |d\zeta|)$, and cannot vanish on set of positive measure unless h is identically zero. Finally,*

$$\|hdz^{1/2}\|_\Omega^2 = \int_\Gamma |h_\Gamma(\zeta)|^2 \, |d\zeta| < \infty.$$

Proof. For bounded domains, this result is classical (see Theorem 10.3 in Duren [13]). If Ω is unbounded, then $\infty \notin \Gamma$ since Γ is rectifiable. Choose a point $q \notin \Omega \cup \Gamma$ and define $f(z) = (z-q)^{-1}$ for all $z \in \Omega$. Then $f \in \mathrm{Conf}(\Omega, \Omega_q)$, where $\Omega_q = f(\Omega)$ is a bounded domain. Now, let $F = f^{-1}$, and choose a branch of square root of F' to extend F to an element $\hat{F} \in \widehat{\mathrm{Conf}}(\Omega_q, \Omega)$. For any $\alpha \in \mathcal{A}^{1/2}(\Omega)$, the result holds for $\hat{F}^*\alpha \in \mathcal{A}^{1/2}(\Omega_q)$, so by conformal invariance of the Smirnov space and unitarity of $\hat{F}^*$, the proof is complete. □

By conjugating, it is clear that the same properties hold for the elements of $\overline{\mathcal{A}^{1/2}(\Omega)}$, and thus the entire space $\mathcal{A}^{1/2}_{\mathrm{harm}}(\Omega)$. Moreover, by an application of the polarization identity, we also have an expression for the inner product in terms of the boundary values.

Corollary 3.3.2. *Let $\Omega \subset \overline{\mathbb{C}}$ be a rectifiable Jordan domain with boundary Γ. For all $hdz^{1/2}, gdz^{1/2} \in \mathcal{A}^{1/2}(\Omega)$, we have*

$$\left\langle hdz^{1/2}, gdz^{1/2} \right\rangle_\Omega = \int_\Gamma h_\Gamma(\zeta)\overline{g_\Gamma(\zeta)} \, |d\zeta|.$$

Similarly, for all $\overline{H}d\bar{z}^{1/2}, \overline{G}d\bar{z}^{1/2} \in \overline{\mathcal{A}^{1/2}(\Omega)}$, we have

$$\left\langle \overline{H}d\bar{z}^{1/2}, \overline{G}d\bar{z}^{1/2} \right\rangle_\Omega = \int_\Gamma \overline{H_\Gamma(\zeta)} G_\Gamma(\zeta) \, |d\zeta|.$$

Theorem 3.3.1 says that, in particular, the boundary values of elements in $\mathcal{A}^{1/2}_{\mathrm{harm}}(\Omega)$ belong to $L^2(\Gamma, |dz|)$ (defined in Example 2.2.2). The converse is also true, as we will note in the upcoming theorem. Denote the set of "boundary values" of $\mathcal{A}^{1/2}(\Omega)$ as

$$\mathcal{A}(\Omega, \Gamma) = \left\{ h_\Gamma \in L^2(\Gamma, |d\zeta|) : hdz^{1/2} \in \mathcal{A}^{1/2}(\Omega) \right\} \tag{3.3.1}$$

and similarly for $\overline{\mathcal{A}^{1/2}(\Omega)}$:

$$\overline{\mathcal{A}(\Omega, \Gamma)} = \left\{ \overline{H}_\Gamma \in L^2(\Gamma, |d\zeta|) : \overline{H}d\bar{z}^{1/2} \in \overline{\mathcal{A}^{1/2}(\Omega)} \right\}.$$

Theorem 3.3.3. *Let $\Omega \subset \overline{\mathbb{C}}$ be a rectifiable Jordan domain with boundary Γ and unit tangent function T. Then the L^2 space of the boundary Γ has the following orthogonal decomposition:*

$$L^2(\Gamma, |d\zeta|) = \mathcal{A}(\Gamma) \oplus \overline{T}\,\overline{\mathcal{A}(\Gamma)}.$$

Furthermore, the corresponding operator $\mathbf{b} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega) \to L^2(\Gamma, |d\zeta|)$ defined for all $u = h dz^{1/2} + \overline{H} d\bar{z}^{1/2}$ by

$$\mathbf{b}u = h_\Gamma + \overline{TH}_\Gamma.$$

is unitary.

Proof. This appears in the C^∞-smooth setting in Bell [3]. See Theorem 2.16 of Kristel et al. [26] for the generalization to this setting. □

Remark 3.3.4. Concretely, the second part of the theorem says that **b** preserves the inner product, so that

$$\langle \mathbf{b}u, \mathbf{b}v \rangle_\Gamma = \langle u, v \rangle_\Omega$$

or equivalently, $\mathbf{b}^* = \mathbf{b}^{-1}$. Thus, Hilbert space properties are shared between $\mathcal{A}^{1/2}_{\text{harm}}(\Omega)$ and $L^2(\Gamma, |d\zeta|)$.

The appearance of the unit tangent function in the orthogonal decomposition can be understood as follows. Recall that the arc-length differential $|dz|$ is related to the differential dz by the formula

$$dz = T|dz|$$

and thus $d\bar{z} = \overline{T}|dz|$. Using the bilinear pairing from the previous section, we have $|dz| = dz^{1/2} d\bar{z}^{1/2}$. Now, formally manipulating the half-order differentials as the notation suggests, we find that

$$\begin{aligned} \overline{T} dz^{1/2} &= \overline{T} T^{1/2} |dz|^{1/2} \\ &= \overline{T}^{1/2} |dz|^{1/2} \\ &= d\bar{z}^{1/2}. \end{aligned}$$

Next, we compute an essential property of the operator **b**. With $u \in \mathcal{A}^{1/2}_{\text{harm}}(\Omega)$ as in the statement of Theorem 3.3.3, we have that

$$\begin{aligned}\overline{\mathbf{b}}u &= \overline{\mathbf{b}\left(\overline{h}d\bar{z}^{1/2} + Hdz^{1/2}\right)} \\ &= Th_\Gamma + \overline{H}_\Gamma \\ &= T\left[h_\Gamma + \overline{TH}_\Gamma\right] \\ &= (T\mathbf{b})\,u.\end{aligned}$$

using the fact that complex conjugation is a continuous operation, and $|T| = 1$. In particular, we have shown that

$$\overline{\mathbf{b}} = T\mathbf{b}. \tag{3.3.2}$$

With this property, we are lead to an alternative notion of conjugation for L^2 induced by the conjugation in the associated Smirnov space:

$$h_\Gamma + \overline{TH}_\Gamma \mapsto \overline{Th_\Gamma} + H_\Gamma. \tag{3.3.3}$$

Moreover, the operator $\overline{\mathbf{b}} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega) \to L^2(\Gamma, |d\zeta|)$ induces an alternate orthogonal decomposition:

$$L^2(\Gamma, |d\zeta|) = T\mathcal{A}(\Omega,\Gamma) \oplus \overline{\mathcal{A}(\Omega,\Gamma)}.$$

Unless otherwise stated, we implicitly use the decomposition in Theorem 3.3.3 to agree with the literature. As an example of the utility of both decompositions, we prove a well-known relation between the boundary values of the Szegő and Garabedian kernels.

Example 3.3.5. For a rectifiable Jordan domain $\Omega \subset \overline{\mathbb{C}}$ with boundary Γ, identify the Szegő and Garabedian kernel functions $S_z(\zeta)$ and $L_z(\zeta)$ with their boundary values. Using the kernel representation of $\overline{\mathbf{P}}u$ and Remark 3.3.4, for any $u \in \mathcal{A}^{1/2}_{\text{harm}}(\Omega)$ we may write

$$\left(\overline{\mathbf{P}}u\right)(z) = \left\langle \mathbf{b}u, -iL_z dz^{1/2} \right\rangle_\Gamma.$$

Now, consider the Riesz representation of $\mathbf{P}\overline{u}$. If we instead push to the boundary using $\overline{\mathbf{b}}$, we find that

$$\begin{aligned}(\mathbf{P}\overline{u})(z) &= \left\langle \overline{\mathbf{b}}\overline{u}, \overline{\mathbf{b}}\left(\overline{S_z(\zeta)d\bar{\zeta}^{1/2}}\right) d\bar{z}^{1/2} \right\rangle_\Gamma \\ &= \left\langle \overline{\mathbf{b}u}, T\overline{S_z}d\bar{z}^{1/2} \right\rangle_\Gamma\end{aligned}$$

whence

$$\overline{(\mathbf{P}\overline{u})(z)} = \left\langle \mathbf{b}u, \overline{T}S_z dz^{1/2} \right\rangle_\Gamma.$$

Therefore, since $\overline{\mathbf{P}}u = \overline{\mathbf{P}\overline{u}}$, we conclude that

$$S_z(\zeta)dz^{1/2} = \frac{1}{i}T(\zeta)L_z(\zeta)dz^{1/2}, \quad \zeta \in \Gamma, z \in \Omega.$$

In the context of Jordan curves, we always have exactly two complementary components on either side of the curve, labelled Ω_1 and Ω_2. Thus, we need additional notation to communicate which domain the boundary values have been taken with respect to. Define the "standard" Szegő projection $\mathbf{P}_d^\Gamma : L^2(\Gamma, |d\zeta|) \to \mathcal{A}(\Omega_d, \Gamma)$ by conjugating $\mathbf{P}_d$ by $\mathbf{b}_d : \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d) \to L^2(\Gamma, |d\zeta|)$:

$$\mathbf{P}_d^\Gamma = \mathbf{b}_d \mathbf{P}_d \mathbf{b}_d^{-1}. \tag{3.3.4}$$

Similarly, the complementary orthogonal projection $\overline{\mathbf{P}}_d^\Gamma : L^2(\Gamma, |d\zeta|) \to \overline{T}\,\overline{\mathcal{A}(\Omega_d, \Gamma)}$ can be obtained by applying the alternative conjugation operation of equation (3.3.3) to (3.3.4), and

$$\overline{\mathbf{P}}_d^\Gamma = \mathbf{b}_d \overline{\mathbf{P}}_d \mathbf{b}_d^{-1}.$$

Next, we turn our attention to identifying the Smirnov spaces on either side of the curve.

Definition 3.3.6. Let $\Gamma \subset \overline{\mathbb{C}}$ be a rectifiable Jordan curve with complementary components Ω_1 and Ω_2. For distinct $d, c \in \{1, 2\}$, the d,c-overfare $\mathbf{O}_{d,c} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d) \to \mathcal{A}^{1/2}_{\text{harm}}(\Omega_c)$ is the operator $\mathbf{O}_{d,c} = \mathbf{b}_c^{-1}\mathbf{b}_d$.

Remark 3.3.7. Observe that

$$\left(\mathbf{O}_{d,c}\right)^{-1} = \mathbf{O}_{c,d}$$

and so we may refer to the operators together simply as *the overfare for* Γ.

In words, for each $u_d \in \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d)$, the overfare of u_d is the unique element in the Smirnov space of $\mathcal{A}^{1/2}_{\text{harm}}(\Omega_c)$ with the same boundary values as u_d (up to a set of measure zero). By Theorem 3.3.3, it follows that overfare is a unitary operator:

$$\left(\mathbf{O}_{d,c}\right)^* = \mathbf{O}_{c,d}. \tag{3.3.5}$$

Now, observe that

$$\left(\overline{\mathbf{b}}\right)^{-1} = (T\mathbf{b})^{-1} = \mathbf{b}^{-1}\overline{T}. \tag{3.3.6}$$

Therefore, it follows that overfare is invariant under conjugation:

$$\overline{\mathbf{O}}_{d,c} = \mathbf{O}_{d,c}. \tag{3.3.7}$$

Consequently, the choice of orthogonal decomposition of $L^2(\Gamma, |d\zeta|)$ does not change overfare.

In the setting of the unit circle, this operator is particularly easy to understand as illustrated by the following example.

Example 3.3.8. Let $\Gamma = \mathbb{S}^1$, so that $\Omega_1 = \mathbb{D}$ and $\Omega_2 = \mathbb{D}^- = \{z \in \overline{\mathbb{C}} : |z| > 1\}$. Then it is not to hard to see that the formula for the overfare of $h(z)dz^{1/2} \in \mathcal{A}^{1/2}(\mathbb{D})$ is given by

$$\mathbf{O}_{1,2}\left(h(z)dz^{1/2}\right)(w) = \frac{1}{\bar{w}}h\left(\frac{1}{\bar{w}}\right)d\bar{w}^{1/2}, \quad w \in \mathbb{D}^-.$$

Hence, overfare in the setting of the unit circle takes holomorphic elements to antiholomorphic elements. A similar formula holds for antiholomorphic elements (and in fact follows from the one above immediately from the general property in equation (3.3.7)).

While overfare behaves in a similarly straightforward manner for all disk domains (this is observed later in Theorem 3.9.11), it remains much more subtle in the case of general curves. Since the decomposition of $L^2(\Gamma, |d\zeta|)$ in Theorem 3.3.3 depends on the side of the curve in which the nontangential limits are taken, we cannot yet give any deeper description of how overfare transforms elements of the Smirnov space. In the following section, we will start gathering the tools to do so.

3.4 The Cauchy Operator and Jump Decomposition

With much of the essential structural properties of $\mathcal{A}^{1/2}_{\text{harm}}(\Omega)$ stated, we now turn to defining the Cauchy operator on the space. In contrast with the Dirichlet space case (see Section 4.4), this is a relatively straightforward task due to the identification of $\mathcal{A}^{1/2}_{\text{harm}}(\Omega)$ and $L^2(\Omega, |d\zeta|)$ from the previous section. Moreover, we make the assumption that all of our boundary curves are bounded in this setting.

Definition 3.4.1. Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded rectifiable Jordan curve with complimentary components Ω_1 and Ω_2, and assume that Γ is oriented positively with respect to Ω_1. For fixed $d, c \in \{1, 2\}$, the (half-order) d,c-Cauchy operator $\mathbf{J}_{d,c}$ is defined for all $u \in \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d)$ by

$$(\mathbf{J}_{d,c}u)(z) = \lim_{r \nearrow 1} \int_{\zeta \in \Gamma_{p,r}} u(\zeta)\left(\frac{1}{2\pi i}\frac{d\zeta^{1/2}dz^{1/2}}{\zeta - z}\right), \qquad z \in \Omega_c$$

where $\Gamma_{p,r} \subset \Omega_d$ is defined for all $0 < r < 1$ as in equation (3.2.4), and shares the orientation of Γ.

Remark 3.4.2. The decision to fix the orientation of the curve in this definition was made to simplify the presentation and computations, and to not stray too far from the source literature. Ideally, a *Möbius invariant* definition of the Cauchy kernel would be implemented, which would require that the direction be allowed to change. This has already been done in the Dirichlet setting, which we consider in Section 4.4. The added advantage of this generalization would be the extension of the associated theorems to Jordan curves which pass through ∞.

Using the boundary values of elements in $\mathcal{A}^{1/2}_{\text{harm}}(\Omega_d)$, we can express $\mathbf{J}_{d,c}$ by the standard Cauchy integral. Write $u \in \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d)$ according to its orthogonal decomposition, say

$$u = h dz^{1/2} + \overline{H} d\bar{z}^{1/2}.$$

Then for $0 \leqslant |z| < r < 1$, we have that

$$\int_{\zeta \in \Gamma_{p,r}} \frac{u(\zeta) d\zeta^{1/2} dz^{1/2}}{\zeta - z} = \int_{\zeta \in \Gamma_{p,r}} \frac{h(\zeta) d\zeta + \overline{H(\zeta)} |d\zeta|}{\zeta - z}.$$

For a choice of $f \in \mathrm{Conf}(\mathbb{D}, \Omega_d)$, the integral on the right-hand side is equal to

$$\int_0^{2\pi} \frac{r i e^{i\theta} \left[h \circ f(re^{i\theta})\right] f'(re^{i\theta})\, d\theta + r \left[\overline{H \circ f(re^{i\theta})}\right] |f'(re^{i\theta})|\, |d\theta|}{re^{i\theta} - z}.$$

Now, the functions h, $\overline{H}$, and f' have conformally nontangential limits at almost every boundary point, so taking the limit as $r \nearrow 1$ yields:

$$\begin{aligned} &\int_0^{2\pi} \frac{\left[h_\Gamma \circ f(e^{i\theta})\right] f'(e^{i\theta}) i e^{i\theta} d\theta + \left[\overline{H_\Gamma \circ f(e^{i\theta})}\right] |f'(e^{i\theta})| |d\theta|}{e^{i\theta} - z} \\ &= \int_{\zeta \in \Gamma} \frac{\left[h_\Gamma(\zeta) d\zeta + \overline{H_\Gamma(\zeta) T(\zeta)}\right]}{\zeta - z} d\zeta. \end{aligned}$$

Therefore, we have

$$(\mathbf{J}_{d,c} u)(z) = \frac{1}{2\pi i} \int_{\zeta \in \Gamma} \frac{(\mathbf{b}_d u)(\zeta)}{\zeta - z} d\zeta \cdot dz^{1/2} \tag{3.4.1}$$

$$= \frac{1}{2\pi i} \int_{\zeta \in \Gamma} \frac{h_\Gamma(\zeta)}{\zeta - z} d\zeta \cdot dz^{1/2} + \frac{1}{2\pi i} \int_{\zeta \in \Gamma} \frac{\overline{H_\Gamma(\zeta)}}{\zeta - z} |d\zeta| \cdot dz^{1/2}. \tag{3.4.2}$$

Since $\mathbf{b}_d : \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d) \to L^2(\Gamma, |d\zeta|)$ is an isomorphism, we see that the integral expression in (3.4.1) coincides with the standard Cauchy integral operator on $L^2(\Gamma, |d\zeta|)$. For $d = 1$, we have the following classical theorem:

Theorem 3.4.3 (Cauchy Integral Formula I)**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded rectifiable Jordan curve with complementary components Ω_1 and Ω_2. Then for all $\alpha \in \mathcal{A}^{1/2}(\Omega_1)$:*

$$\mathbf{J}_{1,c}\alpha = \begin{cases} \alpha, & c = 1 \\ 0, & c = 2. \end{cases}$$

Proof. This follows from the above observations and Theorem 10.4 in Duren [13]. □

Later in this section, we derive the corresponding formula for $\mathbf{J}_{2,c}\mathbf{P}_2$. Next, we restrict our attention to Ahlfors-regular Jordan domains (see Definition 3.1.2), with which there are many very strong results surrounding the Cauchy operators. The first of which is the jump decomposition, or *jump formula*, which has been studied since the 19th century, beginning with Sokhotski. The theorem statement below is a slight rephrasing of the result due to David [11], which is a version of the jump decomposition on $L^2(\Gamma, |d\zeta|)$ in term of the Smirnov spaces on Ω_1 and Ω_2. A proof of this theorem in the function space setting can also be found in the book by Meyer and Coifman [29].

Theorem 3.4.4 (Jump Decomposition, David [11])**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded rectifiable Jordan curve with complementary components Ω_1 and Ω_2. Then we have a decomposition*

$$L^2(\Gamma, |d\zeta|) = \mathcal{A}(\Omega_1, \Gamma) \oplus \mathcal{A}(\Omega_2, \Gamma)$$

if and only if Γ is Ahlfors-regular. When this holds, the Cauchy operators are bounded linear operators $\mathbf{J}_{1,c} : \mathcal{A}^{1/2}_{\mathrm{harm}}(\Omega_1) \to \mathcal{A}^{1/2}(\Omega_c)$, and the decomposition is given explicitly for $u_\Gamma \in L^2(\Gamma)$ by

$$u_\Gamma = \mathbf{b}_1\mathbf{J}_{1,1}u - \mathbf{b}_2\mathbf{J}_{1,2}u$$

where $u = \mathbf{b}_1^{-1}u_\Gamma \in \mathcal{A}^{1/2}_{\mathrm{harm}}(\Omega_1)$.

Remark 3.4.5. A further discussion of boundedness of Cauchy integrals in connection with Ahlfors-regularity of the curve can be found in Pritsker [35].

We shall write the above decomposition as the operator identity

$$\mathbf{J}_{1,1} - \mathbf{O}_{2,1}\mathbf{J}_{1,2} = \mathbf{I} \tag{3.4.3}$$

which holds on all of $\mathcal{A}^{1/2}_{\mathrm{harm}}(\Omega_1)$. Equation (3.4.3) is what we refer to as the *jump formula*, and it will be used extensively in the remain sections to derive further relations between overfare, the Cauchy operators, and the Smirnov spaces of the inside and outside domains.

From the jump formula, the following important fact may be deduced:

Theorem 3.4.6. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2. Then the restricted Cauchy operator*

$$\mathbf{J}^a_{1,2} : \overline{\mathcal{A}^{1/2}(\Omega_1)} \to \mathcal{A}^{1/2}(\Omega_2)$$

is a bounded isomorphism.

Proof. See Corollary 3.4 in Kristel et al. [26]. □

Remark 3.4.7. This is closely related to a result of Bilalov and Najafov [6].

With the Cauchy integral formula and the above isomorphism, we can start extracting information about the operators in the jump formula. Restricting equation (3.4.3) to $\overline{\mathcal{A}^{1/2}(\Omega_1)}$ and taking projections, we find that

$$\begin{aligned} \mathbf{P}_1\mathbf{O}_{2,1}\mathbf{J}^a_{1,2} &= \mathbf{J}^a_{1,1}, \\ -\overline{\mathbf{P}}_1\mathbf{O}_{2,1}\mathbf{J}^a_{1,2} &= \mathbf{I}^a. \end{aligned} \tag{3.4.4}$$

In particular, the second equation says that we can express the inverse of $\mathbf{J}^a_{1,2}$ as

$$\left(\mathbf{J}^a_{1,2}\right)^{-1} = -\overline{\mathbf{P}}_1\mathbf{O}^h_{2,1}. \tag{3.4.5}$$

Example 3.4.8. Returning to the remarks immediately following Example 3.3.8, we are now equipped with tools to further understand the 2,1-overfare of holomorphic elements. Let $\alpha \in \mathcal{A}^{1/2}(\Omega_2)$. Then, there exists a unique $\overline{\beta} \in \overline{\mathcal{A}^{1/2}(\Omega_1)}$ with the property that $\alpha = \mathbf{J}^a_{1,2}\overline{\beta}$, and so by the the jump formula

$$\mathbf{O}_{2,1}\alpha = \mathbf{J}_{1,1}\overline{\beta} - \overline{\beta}.$$

In particular, this means that

$$\text{image}(\mathbf{O}^h_{2,1}) = \text{image}\left(\mathbf{J}^a_{1,1} - \mathbf{I}^a\right) \tag{3.4.6}$$

and thus in general, holomorphic elements overfare do not overfare exactly to antiholomorphic elements as they do in the setting of the unit circle. However, there is a sense in which the image of holomorphic overfare is "dominated" by its antiholomorphic part, with the holomorphic part being "small" (see Section 3.8 for a discussion of compactness and related properties of $\mathbf{J}^a_{1,1}$).

Additionally, equation (3.4.6) leads to a useful description of the null space of $\mathbf{J}_{1,1}$. Using the fact that the Cauchy operator is a projection operator:

$$(\mathbf{J}_{1,1})^2 = \mathbf{J}_{1,1}$$

we deduce from (3.4.6) that $\mathbf{J}_{1,1}\mathbf{O}^h_{2,1}$ is identically zero. Conversely, if $u \in \operatorname{null}\mathbf{J}_{1,1}$, then

$$\mathbf{J}^a_{1,1}\overline{\mathbf{P}}_1 u = -\mathbf{P}_1 u$$

so

$$\mathbf{O}_{2,1}\mathbf{J}_{1,2}\overline{\mathbf{P}}_1(-u) = -\mathbf{J}^a_{1,1}\overline{\mathbf{P}}_1 u + \overline{\mathbf{P}}_1 u = u.$$

Thus, we conclude that

$$\operatorname{null}(\mathbf{J}_{1,1}) = \operatorname{image}(\mathbf{O}^h_{2,1}). \tag{3.4.7}$$

Next, we wish to extend the previous theorem to the Smirnov space of the unbounded domain. In equation (3.4.1), we saw that the (half-order) Cauchy operators can be viewed as acting on the boundary space $L^2(\Gamma, |d\zeta|)$. Thus, given fixed boundary values, the Cauchy operator is the same no matter the side of the curve in which the limit is taken:

$$\mathbf{J}_{d,d} = \mathbf{J}_{c,d}\mathbf{O}_{d,c}, \quad d \neq c. \tag{3.4.8}$$

Since $(\mathbf{O}_{d,c})^{-1} = \mathbf{O}_{c,d}$, one could also write the above identity as $\mathbf{J}_{c,d} = \mathbf{J}_{d,d}\mathbf{O}_{c,d}$. We immediately apply these observations in the following proof of the Cauchy integral formula for unbounded domains.

Corollary 3.4.9 (Cauchy Integral Formula II). *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2. Then for all $\alpha \in \mathcal{A}^{1/2}(\Omega_2)$:*

$$\mathbf{J}_{2,c}\alpha = \begin{cases} 0, & c = 1 \\ -\alpha, & c = 2. \end{cases}$$

Proof. Let $\overline{\beta} \in \overline{\mathcal{A}^{1/2}(\Omega_1)}$ be the unique element such that $\alpha = \mathbf{J}^a_{1,2}\overline{\beta}$. Then by equation (3.4.5), $-\overline{\mathbf{P}}\mathbf{O}_{2,1}\alpha = \overline{\beta}$, and so by (3.4.8) and Theorem 3.4.3

$$\mathbf{J}_{2,2}\alpha = \mathbf{J}_{1,2}\overline{\mathbf{P}}_1\mathbf{O}_{2,1}\alpha = -\mathbf{J}^a_{1,2}\overline{\beta} = -\alpha$$

as desired. Now, if we rearrange equation (3.4.8), we have $\mathbf{J}_{2,1}\alpha = \mathbf{J}_{1,1}\mathbf{O}_{2,1}\alpha$, which vanishes by equation (3.4.7). □

Remark 3.4.10. Certainly, one could employ a much more standard technique to prove a result like this. For our purposes however, the technique in the above proof serves to highlight simple algebraic consequences that may be derived from Theorem 3.4.4.

Remark 3.4.11. On first glance, Corollary 3.4.9 appears to contradict the standard Cauchy integral formula for functions on the unbounded domain (see Gakhov [18], for example). That is, if $h : \Omega_2 \to \mathbb{C}$ is holomorphic on a smooth unbounded domain Ω_2 with sufficiently nice boundary values, then

$$\frac{1}{2\pi i}\int_{\zeta\in\Gamma}\frac{h_\Gamma(\zeta)}{\zeta - z}\,d\zeta = \begin{cases} -h(z) + h(\infty), & z \in \Omega_2 \\ h(\infty), & z \in \Omega_1. \end{cases}$$

However, by the transformation law for elements of the Smirnov space (equation (3.2.1)), if $h(z)dz^{1/2} \in \mathcal{A}^{1/2}(\Omega_2)$, then $h(1/z)/z$ must be holomorphic on a neighbourhood of 0, whence $h(\infty) = 0$.

Next, we use identity (3.4.8) to derive a version of the jump formula for the Cauchy operators on the unbounded domain. Observe that:

$$\begin{aligned} \mathbf{J}_{2,2} - \mathbf{O}_{1,2}\mathbf{J}_{2,1} &= \mathbf{J}_{1,2}\mathbf{O}_{2,1} - \mathbf{O}_{1,2}\mathbf{J}_{1,1}\mathbf{O}_{2,1} \\ &= \mathbf{O}_{1,2}(\mathbf{O}_{2,1}\mathbf{J}_{1,2} - \mathbf{J}_{1,1})\mathbf{O}_{2,1} \\ &= -\mathbf{I}. \end{aligned}$$

Thus, a more general jump formula can be packaged as follows:

$$\mathbf{J}_{d,d} - \mathbf{O}_{c,d}\mathbf{J}_{d,c} = (-1)^{d+1}\mathbf{I}, \quad d \neq c. \tag{3.4.9}$$

The argument of Kristel et al. [26] for invertibility of $\mathbf{J}^a_{1,2}$ now goes through when applied to $\mathbf{J}^a_{2,1}$, yielding:

Corollary 3.4.12. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2. Then the restricted Cauchy operator*

$$\mathbf{J}^a_{2,1} : \overline{\mathcal{A}^{1/2}(\Omega_2)} \to \mathcal{A}^{1/2}(\Omega_1)$$

is a bounded isomorphism.

Example 3.4.13. The previous corollary allows us to derive a particularly simple representation of the inverse of $\mathbf{J}^a_{1,2}$ in the setting of the unit circle. First, using the (generalized) jump formula of equation (3.4.9), we compute that

$$\left(\mathbf{J}^a_{2,1}\right)^{-1} = \overline{\mathbf{P}}_2\mathbf{O}^h_{1,2}. \tag{3.4.10}$$

Combining the conjugate of the above with identity (3.4.5) for the inverse of $\mathbf{J}^a_{1,2}$, we have

$$(\overline{\mathbf{J}}^h_{2,1})^{-1}\left(\mathbf{J}^a_{1,2}\right)^{-1} = -\mathbf{P}_2\mathbf{O}_{1,2}\overline{\mathbf{P}}_1\mathbf{O}^h_{2,1}.$$

Now, set $\Omega_1 = \mathbb{D}$ and $\Omega_2 = \mathbb{D}^{-1}$. By Example 3.3.8, overfare in this setting takes holomorphic to antiholomorphic and vice versa, thus $\mathbf{P}_2\mathbf{O}_{1,2}\overline{\mathbf{P}}_1\mathbf{O}^h_{2,1} = \mathbf{I}^h$, and so it follow that when $\Gamma = \mathbb{S}^1$:

$$\left(\mathbf{J}^a_{1,2}\right)^{-1} = -\overline{\mathbf{J}}^h_{2,1}. \tag{3.4.11}$$

3.5 Adjoint Formulas for the Cauchy Operators

In this section, we derive adjoint formulas for the Cauchy operators in the setting of Ahlfors-regular Jordan domains. In the setting of Lipschitz boundary curves, the adjoint for the corresponding linear operator on $L^2(\Gamma, |d\zeta|)$ has been known for quite some time (see Muskhelishvili [30], for example). Having an explicit form of the adjoint is necessary for the study of the Kerzman-Stein operator, which is the main focus of the next section. The approach to our computation goes through the jump formula and properties of the overfare operator, which allow us to bypass a standalone analytic argument.

Theorem 3.5.1. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2, and let $d, c \in \{1,2\}$. Then*

$$\left(\mathbf{J}_{d,c}\right)^* = \begin{cases} (-1)^{d+1}\mathbf{I} - \overline{\mathbf{J}}_{d,d}, & d = c \\ -\overline{\mathbf{J}}_{c,d}, & d \neq c. \end{cases}$$

Remark 3.5.2. By combining the Cauchy integral formulas and conjugating, it follows that

$$\overline{\mathbf{J}}_{d,c} = \begin{cases} (-1)^{d+1}\overline{\mathbf{P}}_d + \overline{\mathbf{J}}_{d,d}\mathbf{P}_d, & d = c \\ \overline{\mathbf{J}}_{d,c}\mathbf{P}_c, & d \neq c. \end{cases}$$

and so these adjoint formulas for the Cauchy operators can instead be expressed as

$$\left(\mathbf{J}_{d,c}\right)^* = \begin{cases} (-1)^{d+1}\mathbf{P}_d - \overline{\mathbf{J}}_{d,d}\mathbf{P}_d, & d = c \\ -\overline{\mathbf{J}}_{c,d}\mathbf{P}_c, & d \neq c. \end{cases}$$

Proof. Restricting the jump formula to $\overline{\mathcal{A}^{1/2}(\Omega_d)}$ and rearranging, we can express the holomorphic overfare as the integral equation

$$\mathbf{O}^h_{d,c} = \left[\mathbf{J}^a_{c,c} + (-1)^{d+1}\mathbf{I}^a\right]\left(\mathbf{J}^a_{c,d}\right)^{-1}. \tag{3.5.1}$$

By the conjugation rule for the overfare in equation (3.3.7), it follows that $\overline{\mathbf{O}_{d,c}\mathbf{P}_d} = \mathbf{O}_{d,c}\overline{\mathbf{P}}_d$, and so

$$\mathbf{O}^a_{d,c} = \left[\overline{\mathbf{J}}^h_{c,c} + (-1)^{d+1}\mathbf{I}^h\right]\left(\overline{\mathbf{J}}^h_{c,d}\right)^{-1}. \tag{3.5.2}$$

Putting these together yields the following integral equation representation for the general overfare:

$$\mathbf{O}_{d,c} = \left[\mathbf{J}^a_{c,c} + (-1)^{d+1}\mathbf{I}^a\right]\left(\mathbf{J}^a_{c,d}\right)^{-1}\mathbf{P}_d + \left[\overline{\mathbf{J}}^h_{c,c} + (-1)^{d+1}\mathbf{I}^h\right]\left(\overline{\mathbf{J}}^h_{c,d}\right)^{-1}\overline{\mathbf{P}}_d. \tag{3.5.3}$$

Now, using the fact that $\mathbf{O}_{d,c} = \left(\mathbf{O}_{c,d}\right)^*$, we obtain

$$\begin{aligned} &\left[\mathbf{J}^a_{c,c} + (-1)^{d+1}\mathbf{I}^a\right]\left(\mathbf{J}^a_{c,d}\right)^{-1}\mathbf{P}_d + \left[\overline{\mathbf{J}}^h_{c,c} + (-1)^{d+1}\mathbf{I}^h\right]\left(\overline{\mathbf{J}}^h_{c,d}\right)^{-1}\overline{\mathbf{P}}_d \\ =& \left[\left(\mathbf{J}^a_{d,c}\right)^{-1}\right]^*\left[\left(\mathbf{J}^a_{d,d}\right)^*\mathbf{P}_d + (-1)^{c+1}\overline{\mathbf{P}}_d\right] \\ &+ \left[\left(\overline{\mathbf{J}}^h_{d,c}\right)^{-1}\right]^*\left[\left(\overline{\mathbf{J}}^h_{d,d}\right)^*\overline{\mathbf{P}}_d + (-1)^{c+1}\mathbf{P}_d\right]. \end{aligned} \tag{3.5.4}$$

By right-composing the above with $\overline{\mathbf{P}}_d$ and left-composing with $\mathbf{P}_c$, we get that

$$\left(\overline{\mathbf{J}}^h_{c,d}\right)^{-1} = -\left[\left(\mathbf{J}^a_{d,c}\right)^{-1}\right]^*.$$

Finally, by interchanging the inverse and adjoint sign on the right-hand side and then inverting both sides, we arrive at

$$\left(\mathbf{J}^a_{d,c}\right)^* = -\overline{\mathbf{J}}^h_{c,d}, \quad d \neq c.$$

In particular, since $\mathbf{J}_{d,c}$ vanishes on $\mathcal{A}^{1/2}(\Omega_d)$ by the Cauchy integral formula, we may extend the adjoint formula by zero to the whole space to get

$$\left(\mathbf{J}_{d,c}\right)^* = -\overline{\mathbf{J}}_{c,d}, \quad d \neq c. \tag{3.5.5}$$

From here, it is easy to deduce $(\mathbf{J}_{d,d})^*$. Indeed, taking the adjoint of the expression for $\mathbf{J}_{d,d}$ in equation (3.4.8):

$$\begin{aligned}\left(\mathbf{J}_{d,d}\right)^* = \left(\mathbf{J}^a_{c,d}\overline{\mathbf{P}}_c\mathbf{O}_{d,c}\right)^* &= -\mathbf{O}_{c,d}\overline{\mathbf{J}}^h_{d,c}\mathbf{P}_d \\ &= -\mathbf{O}_{c,d}\overline{\mathbf{J}}_{d,c}.\end{aligned}$$

Rearranging the jump formula and taking the complex conjugate yields

$$-\mathbf{O}_{c,d}\overline{\mathbf{J}}_{d,c} = (-1)^{d+1}\mathbf{I} - \overline{\mathbf{J}}_{d,d}.$$

and so combined we get that

$$\left(\mathbf{J}_{d,d}\right)^* = (-1)^{d+1}\mathbf{I} - \overline{\mathbf{J}}_{d,d}, \quad d \in \{1,2\}.$$

□

Remark 3.5.3. There is another interesting identity that appears out of equation (3.5.4). Indeed, right-composing the expression with $\mathbf{P}_c$ and left-composing with $\mathbf{P}_d$, it follows from the Cauchy adjoint formulas that

$$\mathbf{J}^a_{d,d}\left(\mathbf{J}^a_{d,c}\right)^{-1} = \left(\overline{\mathbf{J}}^h_{d,c}\right)^{-1}\overline{\mathbf{J}}^h_{c,c}, \quad d \neq c. \tag{3.5.6}$$

3.6 An Extension of the Kerzman-Stein Operator

In this section, we consider the (half-order) Kerzman-Stein operator on both the bounded and unbounded component. Further, we introduce a natural extension of the operator that maps between the spaces $\mathcal{A}^{1/2}_{\text{harm}}(\Omega_1)$ and $\mathcal{A}^{1/2}_{\text{harm}}(\Omega_2)$. Phrasing the Kerzman-Stein operator in terms of half-order differentials was first done by Barrett and Bolt [1] for C^∞-smooth curves. With explicit adjoint formulas for the Cauchy operators on Ahlfors-regular Jordan domains, we are able to derive many properties and identities for the operator, including the Kerzman-Stein formula, for this wider class of domains.

Let $\Gamma \subset \mathbb{C}$ be a bounded Ahlfors-regular Jordan curve with bounded complementary component Ω_1. The (half-order) *Kerzman-Stein operator* $\mathbf{A}_{1,1} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1) \to \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1)$

is defined to be

$$\mathbf{A}_{1,1} := \mathbf{J}_{1,1} - (\mathbf{J}_{1,1})^*. \tag{3.6.1}$$

With our adjoint formula for $\mathbf{J}_{1,1}$, we can express this operator as $\mathbf{A}_{1,1} = \mathbf{J}_{1,1} + \overline{\mathbf{J}}_{1,1} - \mathbf{I}$, and so we have that

$$\mathbf{P}_1(\mathbf{I} + \mathbf{A}_{1,1}) = \mathbf{J}_{1,1}. \tag{3.6.2}$$

Equation (3.6.2) is called the *Kerzman-Stein formula* (compare with equation (1.0.1)). By Lemma A.1.1 in the Appendix, $\mathbf{I} + \mathbf{A}_{1,1} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1) \to \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1)$ is a bounded isomorphism, so we can rewrite the above formula as

$$\mathbf{P}_1 = \mathbf{J}_{1,1}(\mathbf{I} + \mathbf{A}_{1,1})^{-1}. \tag{3.6.3}$$

This relationship between the Szegő projection and the Cauchy operator was one of key insights of Kerzman and Stein [24], as it allows for properties of these operator to be obtained from each other. As pointed out by Lanzani [27], the significance of this relationship when formulated on the boundary space $L^2(\Gamma, |d\zeta|)$ is that the Cauchy operator is an explicit integral operator, where as the Szegő projection is implicit. Conversely, boundedness of Szegő projection is trivial, whereas boundedness of the Cauchy operator is not. The book of Bell [3] uses these insights to prove many classical theorems in function theory and conformal mapping very efficiently on C^∞ domains, however the author notes that the smoothness requirements may be weakened in many instances.

Another key property of the classical Kerzman-Stein operator is its compactness for sufficiently smooth boundaries. This is surprising since neither the standard Cauchy operator nor its adjoint are compact on $L^2(\Gamma, |d\zeta|)$. We postpone further discussion regarding compactness of $\mathbf{A}_{1,1}$ until Section 3.8. Next, consider the following:

Definition 3.6.1. Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2. For fixed $d, c \in \{1, 2\}$, define the d,c-Kerzman-Stein operator

$$\mathbf{A}_{d,c} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d) \to \mathcal{A}^{1/2}_{\text{harm}}(\Omega_c)$$

by the formula

$$\mathbf{A}_{d,c} = \mathbf{J}_{d,c} - \left(\mathbf{J}_{c,d}\right)^*.$$

For the remainder of this section, we derive some basic algebraic properties of these operators. First, we see that

$$\left(\mathbf{A}_{d,c}\right)^* = -\mathbf{A}_{c,d}.$$

meaning that $\mathbf{A}_{d,c}$ is skew-adjoint for the case $d = c$. Now, using the adjoint formulas from

Remark 3.5.2, it follows that

$$\mathbf{A}_{d,c} = \mathbf{J}^{a}_{d,c}\overline{\mathbf{P}}_d + \overline{\mathbf{J}}^{h}_{d,c}\mathbf{P}_d, \quad d,c \in \{1,2\} \tag{3.6.4}$$

from which we have

$$\overline{\mathbf{A}}_{d,c} = \mathbf{A}_{d,c}. \tag{3.6.5}$$

Now, we restrict our attention specifically to the operators $\mathbf{A}_{d,d}$. The 2,2-Kerzman-Stein operator $\mathbf{A}_{2,2}$ is simply the analogue of the standard operator $\mathbf{A}_{1,1}$ for unbounded domains. The formula in (3.6.3) has an analogue for the unbounded domain, which we record momentarily. First, define the $*$-operator for $u = hdz^{1/2} + \overline{H}d\bar{z}^{1/2}$ by

$$*u := -ihdz^{1/2} + i\overline{H}d\bar{z}^{1/2}.$$

We wish to note that we could not find reference in the literature for this operator in this setting, but it is the direct analogue of the usual complexification of the Hodge-$*$ operator for smooth one-forms, and appears to behave similarly. On the Smirnov space, we can express this operator as $* = -i\mathbf{P} + i\overline{\mathbf{P}}$.

Theorem 3.6.2 (Kerzman-Stein Formula)**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan domain with complementary components Ω_1 and Ω_2. Then for each $d \in \{1,2\}$, the operator*

$$\mathbf{I} + \mathbf{A}_{d,d} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d) \to \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d)$$

is a bounded isomorphism, and

$$\mathbf{P}_d = \begin{cases} \mathbf{J}_{1,1}\left(\mathbf{I} + \mathbf{A}_{1,1}\right)^{-1}, & d = 1 \\ -i\mathbf{J}_{2,2} *_2 \left(\mathbf{I} + \mathbf{A}_{2,2}\right)^{-1}, & d = 2. \end{cases}$$

Proof. The case when $d = 1$ was handled previously. For $d = 2$, the fact that $\mathbf{I} + \mathbf{A}_{2,2}$ is a bounded isomorphism follows from Lemma A.1.1. Now, observe that $\mathbf{J}_{2,2} - (\mathbf{J}_{2,2})^* = \mathbf{J}_{2,2} + \mathbf{I} + \overline{\mathbf{J}}_{2,2}$, and so

$$\begin{aligned} \mathbf{P}_2\left(\mathbf{I} + \mathbf{A}_{2,2}\right) = 2\mathbf{P}_2 + \mathbf{J}_{2,2} &= \mathbf{P}_2 + \mathbf{J}_{2,2}\overline{\mathbf{P}}_2 \\ &= -i\mathbf{J}_{2,2}*_2 \end{aligned}$$

as required. □

Remark 3.6.3. In the next section, we will "push" this formula to the boundary (See equation (3.7.4)).

As is shown next, the 1,1 and 2,2-Kerzman-Stein operators are in fact equivalent up to conjugation by overfare. Indeed, using the jump formula in (3.4.9), we can express the d,d-Cauchy operator as $\mathbf{J}_{d,d} = \mathbf{O}_{c,d}\mathbf{J}_{d,c} + (-1)^{d+1}\mathbf{I}$, and so the adjoint can be written

$$\left(\mathbf{J}_{d,d}\right)^* = \left(\mathbf{O}_{c,d}\mathbf{J}_{d,c}\right)^* + (-1)^{d+1}\mathbf{I}.$$

Subtracting these expressions, we have

$$\mathbf{A}_{d,d} = \mathbf{O}_{c,d}\mathbf{J}_{d,c} - \left(\mathbf{O}_{c,d}\mathbf{J}_{d,c}\right)^*. \tag{3.6.6}$$

Along with property (3.6.5), we use this representation to compute:

$$\begin{aligned}\mathbf{A}_{1,1} &= \mathbf{O}_{2,1}\mathbf{J}_{1,2} + \overline{\mathbf{J}}_{2,1}\mathbf{O}_{1,2} \\ &= \mathbf{O}_{2,1}\left(\mathbf{O}_{1,2}\overline{\mathbf{J}}_{2,1} + \mathbf{J}_{1,2}\mathbf{O}_{2,1}\right)\mathbf{O}_{1,2} \\ &= \mathbf{O}_{2,1}\mathbf{A}_{2,2}\mathbf{O}_{1,2}\end{aligned}$$

or equivalently

$$\mathbf{O}_{1,2}\mathbf{A}_{1,1} = \mathbf{A}_{2,2}\mathbf{O}_{1,2}. \tag{3.6.7}$$

Next, let us consider the operators $\mathbf{A}_{d,c}$ for when $d \neq c$. Similar to overfare, the d,c-Kerzman-Stein operators are isomorphisms of the "inside" and "outside" Smirnov spaces.

Theorem 3.6.4. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular domain with complementary components Ω_1 and Ω_2. Then for distinct $d,c \in \{1,2\}$, the d,c-Kerzman-Stein operator is a bounded isomorphism, and*

$$\left(\mathbf{A}_{d,c}\right)^{-1} = \left(\mathbf{J}^a_{d,c}\right)^{-1}\mathbf{P}_c + \left(\overline{\mathbf{J}}^h_{d,c}\right)^{-1}\overline{\mathbf{P}}_c.$$

Proof. From Theorems 3.4.6 and 3.4.12, we have that $\mathbf{J}^a_{d,c} : \overline{\mathcal{A}^{1/2}(\Omega_d)} \to \mathcal{A}^{1/2}(\Omega_c)$ is an isomorphism, and hence so is its conjugation $\overline{\mathbf{J}}^h_{d,c} : \mathcal{A}^{1/2}(\Omega_d) \to \overline{\mathcal{A}^{1/2}(\Omega_c)}$. Therefore by equation (3.6.4) and the orthogonal decomposition defining the harmonic Smirnov space (Definition 3.2.2), it follows that $\mathbf{A}_{d,c} = \mathbf{J}^a_{d,c}\overline{\mathbf{P}}_d + \overline{\mathbf{J}}^h_{d,c}\mathbf{P}_d$ is an isomorphism with the described inverse formula. □

Remark 3.6.5. By our formulas for the inverse of the restricted Cauchy operators (equation 3.4.5 and equation 3.4.10), it follows that we can write

$$\left(\mathbf{A}_{d,c}\right)^{-1} = (-1)^d\left(\overline{\mathbf{P}}_d\mathbf{O}_{c,d}\mathbf{P}_c + \mathbf{P}_d\mathbf{O}_{d,c}\overline{\mathbf{P}}_c\right).$$

Example 3.6.6. Set $\Gamma = \mathbb{S}^1$, so that $\Omega_1 = \mathbb{D}$ and $\Omega_2 = \mathbb{D}^- = \{z \in \mathbb{C} : |z| > 1\}$. By our work in Example 3.4.13, it is immediate from the above inverse formula that $(\mathbf{A}_{1,2})^{-1} = -\mathbf{A}_{2,1}$. That is, $\mathbf{A}_{1,2}$ is unitary in the setting of the unit circle.

Next, we want to see how the d,c-Kerzman-Stein operator interacts with overfare. By applying $\mathbf{O}_{c,d}$ to the expression in (3.6.4), observe that

$$\mathbf{O}_{c,d}\mathbf{A}_{d,c} = \mathbf{O}_{c,d}\mathbf{J}_{d,c}\overline{\mathbf{P}}_d + \mathbf{O}_{c,d}\overline{\mathbf{J}}_{d,c}\mathbf{P}_d.$$

Substituting each term on the right-hand side using the jump formula yields

$$\begin{aligned}\mathbf{O}_{c,d}\mathbf{J}_{d,c}\overline{\mathbf{P}}_d + \mathbf{O}_{c,d}\overline{\mathbf{J}}_{d,c}\mathbf{P}_d &= \mathbf{J}_{d,d}\overline{\mathbf{P}}_d + (-1)^{d+1}\overline{\mathbf{P}}_d + \overline{\mathbf{J}}_{d,d}\mathbf{P}_d + (-1)^{d+1}\mathbf{P}_d \\ &= \mathbf{A}_{d,d} + (-1)^{d+1}\mathbf{I}\end{aligned}$$

and so we arrive at a "jump formula" for the d,c-Kerzman-Stein operators:

$$\mathbf{A}_{d,d} - \mathbf{O}_{c,d}\mathbf{A}_{d,c} = (-1)^{d+1}\mathbf{I}, \quad d \neq c. \tag{3.6.8}$$

From this, we can extract a relation between the 1,2 and 2,1-Kerzman-Stein operators. Observe that

$$\begin{aligned}\mathbf{O}_{1,2}\mathbf{A}_{1,2}\mathbf{O}_{2,1} &= (\mathbf{A}_{1,1} - \mathbf{I}_1)\mathbf{O}_{2,1} \\ &= \mathbf{O}_{2,1}(\mathbf{A}_{2,2} - \mathbf{I}_2) \\ &= \mathbf{A}_{2,1} - 2\mathbf{O}_{2,1}\end{aligned}$$

which rearranges to the identity

$$\mathbf{O}_{1,2}\mathbf{A}_{2,1} - \mathbf{A}_{1,2}\mathbf{O}_{2,1} = -2\mathbf{I}.$$

3.7 Boundary Values of the Cauchy Kernel

The main goal of this section is to compute the boundary values of the Kerzman-Stein kernel function for Ahlfors-regular Jordan curves. This will allow us to translate between properties of $\mathbf{A}_{1,1}$, and the "standard" integral operator on $L^2(\Gamma, |d\zeta|)$ from the introductory chapter. We will follow the computation done in Bell [3] for C^∞-smooth curves, which generalizes nicely due to theorems of Privalov [36] and David [11].

First, we begin with the boundary values of the Cauchy operator. To integrate the Cauchy kernel when both variables lie on the curve, one needs the notion of *principal* *value*. For a bounded Jordan curve Γ and a point $z_0 \in \Gamma$, set

$$\Gamma(z_0;r) := \Gamma \backslash B(z_0;r), \quad r > 0$$

where $B(z_0;r)$ denotes the open disk of radius r centred at z_0. Using a topological argument, it can be shown that if r is made sufficiently small, then $B(z_0;r)$ contains exactly one connected component of Γ, so that $\Gamma(z_0;r)$ is an open curve. This curve is used in defining the analogue of the Cauchy operator for inputs on the curve:

Definition 3.7.1. Let $\Gamma \subset \mathbb{C}$ be a bounded rectifiable Jordan curve with bounded complementary domain Ω_1. For all $u \in \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1)$, define the singular Cauchy operator $\mathbf{H}$ by

$$(\mathbf{H}u)(z_0) = \lim_{r \searrow 0} \frac{1}{2\pi i} \int_{\Gamma(z_0;r)} \frac{u_\Gamma(\zeta)}{\zeta - z_0}\, d\zeta, \qquad z_0 \in \Gamma.$$

The integral on the right-hand side is called the *principal value of the singular Cauchy integral*, and is commonly expressed with the notation

$$\text{P.V.} \frac{1}{2\pi i} \int_{\Gamma} \frac{u_\Gamma(\zeta)}{\zeta - z_0}\, d\zeta := \lim_{r \searrow 0} \frac{1}{2\pi i} \int_{\Gamma(z_0;r)} \frac{u_\Gamma(\zeta)}{\zeta - z_0}\, d\zeta.$$

The following theorem characterizes precisely when the classical *Sokhotski-Plemelj formula* for the boundary values of the Cauchy operator holds.

Theorem 3.7.2 (Privalov [36])**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded rectifiable Jordan curve with complementary components Ω_1 and Ω_2. The nontangential boundary values of the Cauchy operators $\mathbf{J}_{1,1}$ and $\mathbf{J}_{1,2}$ exist almost everywhere on Γ if and only if the principal value Cauchy integral operator $\mathbf{H}$ is well-defined. In the affirmative case, $\mathbf{H}$ maps into $L^2(\Gamma, |d\zeta|)$ and*

$$\mathbf{b}_c \mathbf{J}_{1,c} = (-1)^{c+1} \frac{1}{2} \mathbf{b}_c + \mathbf{H}, \quad c \in \{1,2\}.$$

The problem of characterizing the boundedness of $\mathbf{H}$ is solved by considering Privalov's result in conjunction with David's result in Theorem 3.4.4.

Theorem 3.7.3 (David [11])**.** *Let $\Gamma \subset \mathbb{C}$ be a bounded rectifiable Jordan curve. Then $\mathbf{H}$ is a bounded operator into $L^2(\Gamma, |d\zeta|)$ if and only if Γ is Ahlfors-regular.*

A more complete discussion around these results and their relation to the Cauchy operator can be found in Pritsker [35]. Armed with Theorems 3.5.1 and 3.7.2 , we can easily compute the boundary values of the Kerzman-Stein operator using a technique displayed in Bell [3]. First, we compute the adjoint of $\mathbf{H}$:

$$\begin{aligned}\mathbf{H}^* &= \left(\mathbf{I} - \overline{\mathbf{J}}_{1,1}\right)\mathbf{b}_1^{-1} - \frac{1}{2}\mathbf{b}_1^{-1} \\ &= \frac{1}{2}\mathbf{b}_1^{-1} - \overline{\left(\frac{1}{2}\mathbf{I} + \mathbf{b}_1^{-1}\mathbf{H}\right)}\mathbf{b}_1^{-1} \\ &= -\left(\overline{\mathbf{b}}_1\right)^{-1}\overline{\mathbf{H}}\mathbf{b}_1^{-1} \\ &= -\mathbf{b}_1^{-1}\left(T\overline{\mathbf{H}}\right)\mathbf{b}_1^{-1}\end{aligned}$$

where in the last equality we used equation 3.3.6. Taking the adjoint of the Sokhotski-Plemelj formula and rearranging, we then have that

$$(\mathbf{J}_{1,1})^* = \frac{1}{2}\mathbf{I} - \mathbf{b}_1^{-1}\left(T\overline{\mathbf{H}}\right).$$

Therefore, $\mathbf{A}_{1,1}$ as an operator on the boundary Γ can be expressed in terms of Cauchy principal value integrals:

$$\mathbf{b}_1\mathbf{A}_{1,1} = \mathbf{H} + \overline{T\overline{\mathbf{H}}}. \tag{3.7.1}$$

Equivalently, if we define $\mathbf{A}_\Gamma := \mathbf{b}_1\mathbf{A}_{1,1}\mathbf{b}_1^{-1}$ and $\mathbf{H}_\Gamma := \mathbf{H}\mathbf{b}_1^{-1}$, then (3.7.1) becomes

$$\begin{aligned}\mathbf{A}_\Gamma u &= \mathbf{H}_\Gamma u + \overline{T\overline{\mathbf{H}}_\Gamma u} \\ &= \text{P.V.}\int_\Gamma A(\zeta,\cdot)u(\zeta)\,|d\zeta|\end{aligned} \tag{3.7.2}$$

where $u \in L^2(\Gamma, |d\zeta|)$, and

$$A(\zeta,z) = \frac{1}{2\pi i}\left[\frac{T(\zeta)}{\zeta - z} - \frac{\overline{T(z)}}{\overline{\zeta} - \overline{z}}\right], \quad \zeta, z \in \Gamma. \tag{3.7.3}$$

Equation (3.7.2) is the usual formulation of the Kerzman-Stein operator in the literature. As we will see in the next section, working directly with this kernel function pairs well with more standard integral operator techniques.

We may now write the Kerzman-Stein formula (3.6.2) in terms of the corresponding linear operators on L^2. Let $\mathbf{J}_\Gamma := \mathbf{b}_1\mathbf{J}_{1,1}\mathbf{b}_1^{-1}$ denote the "standard" Cauchy integral operator on $L^2(\Gamma, |d\zeta|)$. Then $\mathbf{A}_\Gamma = \mathbf{J}_\Gamma - (\mathbf{J}_\Gamma)^*$, and

$$\mathbf{P}_1^\Gamma(\mathbf{I} + \mathbf{A}_\Gamma) = \mathbf{J}_\Gamma.$$

Moreover, $\mathbf{I} + \mathbf{A}_\Gamma : L^2(\Gamma, |d\zeta|) \to L^2(\Gamma, |d\zeta|)$ is a bounded isomorphism by Lemma A.1.1, and so

$$\mathbf{P}_1^\Gamma = \mathbf{J}_\Gamma(\mathbf{I}+\mathbf{A}_\Gamma)^{-1}. \tag{3.7.4}$$

Remark 3.7.4. The weakest conditions we could locate in the literature for which the Kerzman-Stein formula holds is for Lipschitz domains, which was proven by Lanzani [28]. In her paper, she expressed that it might be possible to extend to the case of (not necessarily Jordan) Ahlfors-regular domains, with one major obstacle being the characterization of the Hardy space for such domains. Since Ahlfors-regular Jordan domains yield a nice characterization of the Hardy space (as well as David's theorem), we were able to circumvent these technicalities in proving (3.7.4).

3.8 Regularity of the Kerzman-Stein Kernel

In this section, we emphasize the interplay between the analytic properties of the Kerzman-Stein kernel function $A(\zeta,z)$ and the regularity of the boundary curve. One of the key properties of the Kerzman-Stein operator is that it is compact on a wide class of smooth curves. This was first proved by Kerzman and Stein [24] for C^∞ curves, by way of the stronger result that $A(\zeta,z)$ is C^∞-smooth in $(\zeta,z)\in\Gamma\times\Gamma$. Later on, Lanzani [28] showed that compactness also held for C^1 curves. Moreover, she provided an example of a piecewise-C^1 curve for which the operator was not compact. Subsequently, Bolt and Raich [8] showed that $\mathbf{A}_{1,1}$ is never compact for piecewise-C^1 curves with a corner. Here, we shall consider curves belonging to the continuously-differentiable classes $C^{n,\alpha}$, and provide sufficient conditions for $\mathbf{A}_\Gamma$ (and hence $\mathbf{A}_{d,d}$) to belong to the Schatten p-classes for all $p\geqslant 2$.

Recall that a function is of class C^n ($n\geqslant 0$) if all of its nth partial derivatives exist and are continuous on the domain of definition. If, moreover, all of the partial derivatives are α-Hölder continuous ($0\leqslant\alpha\leqslant 1$), then we say that the function is of class $C^{n,\alpha}$. When $\alpha=1$, this is known as the Lipschitz condition.

Definition 3.8.1. A bounded Jordan curve $\Gamma\subset\mathbb{C}$ is said to be of class C^n (for some fixed $n\geqslant 1$), written $\Gamma\in C^n$, if there exists a parameterization function $\varphi:[0,1]\to\Gamma$ that is n-times continuously differentiable, satisfies $\varphi^{(k)}(0)=\varphi^{(k)}(1)$ for all $0\leqslant k\leqslant n$, and $\varphi'(t)\neq 0$ for all $t\in[0,1]$. Moreover, Γ is said to be of class $C^{n,\alpha}$, where $0<\alpha\leqslant 1$ if $\Gamma\in C^n$ and the nth derivative of φ is α-Hölder continuous. If $\Gamma\in C^n$ for all $n\in\mathbb{N}$, then we say that Γ is of class C^∞.

Remark 3.8.2. C^1 Jordan curves are in particular Ahlfors-regular.

For convenience, set $C^{n,0} := C^n$. It is clear that the $C^{n,\alpha}$ classes obey the following chain of inclusions, which we call the *continuously differentiable hierarchy*:

$$C^n \supsetneq C^{n,\alpha} \supsetneq C^{n+1} \supsetneq C^\infty, \qquad n \geqslant 1, 0 < \alpha \leqslant 1. \tag{3.8.1}$$

Next, we identify some of the relevant collections of bounded operators for our work.

Definition 3.8.3. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n : n \geqslant 1\}$. The trace of a positive-semidefinite operator $\mathbf{B} \in B(\mathcal{H})$ is the quantity $\mathrm{Tr}(\mathbf{B}) \in [0, \infty]$ defined by the formula

$$\mathrm{Tr}(\mathbf{B}) = \sum_{n=1}^{\infty} \langle \mathbf{B} e_k, e_k \rangle.$$

Remark 3.8.4. This definition can be shown to be independent of choice of orthonormal basis.

Now, let $\mathbf{B} \in B(\mathcal{H}_1, \mathcal{H}_2)$ be a bounded linear operator between separable Hilbert spaces. Then $\mathbf{B}^*\mathbf{B} \in B(\mathcal{H}_1)$ is a positive-semidefinite bounded operator, and so for all $n \geqslant 1$, there exists $(\mathbf{B}^*\mathbf{B})^{1/n} \in B(\mathcal{H}_1)$ with the property that $\left[(\mathbf{B}^*\mathbf{B})^{1/n}\right]^n = \mathbf{B}^*\mathbf{B}$ (see Corollary 12.45 in Einsiedler and Ward [15], for example). For the case that $n = 2$, write $|\mathbf{B}| := (\mathbf{B}^*\mathbf{B})^{1/2}$. We say that $\mathbf{B}$ is *trace class* if $\mathrm{Tr}(|\mathbf{B}|) < \infty$, and the set of trace class operators, denoted by $S_1(\mathcal{H}_1, \mathcal{H}_2)$, forms a Banach space with norm $\|\cdot\|_{S_1} := \mathrm{Tr}(|\cdot|)$. It can be shown that all trace class operators are compact.

Definition 3.8.5. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces. An operator $\mathbf{B} \in B(\mathcal{H}_1, \mathcal{H}_2)$ is said to be Hilbert-Schmidt if there is an orthonormal basis $\{e_n : n \geqslant 1\}$ for $\mathcal{H}_1$ such that

$$\|\mathbf{B}\|_{S_2}^2 := \sum_{n=0}^{\infty} \|\mathbf{B} e_n\|_{\mathcal{H}_2}^2 < \infty.$$

It is well-known that the the set of Hilbert-Schmidt operators, denoted $S_2(\mathcal{H}_1, \mathcal{H}_2)$, form a Hilbert space with norm $\|\cdot\|_{S_2}$, and are again examples of compact operators. Additionally, we have that $\mathbf{B}$ is Hilbert-Schmidt if and only if $|\mathbf{B}|^2 = \mathbf{B}^*\mathbf{B}$ has finite trace. Hilbert-Schmidt integral operators are easily characterized, as observed by the following classical result.

Proposition 3.8.6. *Let $\Gamma \subset \mathbb{C}$ be a bounded rectifiable Jordan curve, and let $K : \Gamma \times \Gamma \to \mathbb{C}$ be a measurable function. Define an integral operator $\mathbf{K} : L^2(\Gamma, |d\zeta|) \to L^2(\Gamma, |d\zeta|)$ for all $u \in L^2(\Gamma, |d\zeta|)$ by*

$$(\mathbf{K}u)(z_0) = \int_{\zeta \in \Gamma} K(\zeta, z_0) u(\zeta)\, |d\zeta|, \quad z_0 \in \Gamma.$$

Then $\mathbf{K}$ *is Hilbert-Schmidt if and only if* $K(\zeta, z) \in L^2(\Gamma \times \Gamma, |d\zeta| \otimes |dz|)$.

Proof. For example, see Chapter III Section 9 of Gohberg and Kreĭn [22]. □

Remark 3.8.7. In particular, a simple sufficient condition for Hilbert-Schmidtness of $\mathbf{K}$ is that its kernel function K is both measurable and essentially bounded.

Next, we consider the *Schatten classes*, which are a sort of analogue of L^p spaces for operators. For this, we briefly follow the presentation of Delgado and Ruzhansky [12]. See Chapter 7 of Weidmann [45] for an introduction to the Schatten class where many of the well-known properties we state are proven.

Definition 3.8.8. Let $1 \leqslant p < \infty$. A compact operator $\mathbf{B} \in (\mathcal{H}_1, \mathcal{H}_2)$ is said to belong to the Schatten p-class if

$$\|\mathbf{B}\|_{S_p} = [\mathrm{Tr}(|\mathbf{B}|^p)]^{1/p} < \infty.$$

The class of Schatten p-class operators is denoted by $S_p(\mathcal{H}_1, \mathcal{H}_2)$, and forms a Banach space with norm $\|\cdot\|_{S_p}$. The particular cases $p = 1$ and $p = 2$ correspond to the trace class and the Hilbert-Schmidt class, respectively. For $p = \infty$, $S_\infty(\mathcal{H}_1, \mathcal{H}_2)$ is defined to be the space of compact operators, and the corresponding Schatten norm is simply the operator norm. Each class forms a separate two-sided ideal in the space $B(\mathcal{H}_1, \mathcal{H}_2)$. It is also possible to define analogous Schatten classes for $0 < p < 1$, however they are not considered in this work. The Schatten classes obey the following inclusion property:

$$S_p(\mathcal{H}_1, \mathcal{H}_2) \subset S_q(\mathcal{H}_1, \mathcal{H}_2), \quad 1 \leqslant p < q \leqslant \infty.$$

When $p \geqslant 2$, there are sufficient integrability conditions on the kernel of an integral operator that yield a finite Schatten p-norm, similar in spirit to that of Lemma 3.8.6. We discuss this, as well as the failure of classifying integral operators in S_p for $p < 2$ in Section A.2 of Appendix A.

Let us make note of the equivalence of the Schatten-ness of the Kerzman-Stein operators and the restricted Cauchy operators.

Proposition 3.8.9. *Let* $\Gamma \subset \overline{\mathbb{C}}$ *be a bounded Ahlfors-regular Jordan curve with complementary components* Ω_1 *and* Ω_2*, and fix* $d \in \{1, 2\}$*. Then the following are equivalent:*

(a) $\mathbf{J}^a_{d,d} \in S_p(\overline{\mathcal{A}^{1/2}(\Omega_d)}, \mathcal{A}^{1/2}(\Omega_d))$

(b) $\mathbf{A}_{d,d} \in S_p(\mathcal{A}^{1/2}_{\mathrm{harm}}(\Omega_d))$.

Proof. By equation (3.6.4), we can write $\mathbf{A}_{d,d} = \mathbf{J}^a_{d,d}\overline{\mathbf{P}}_d + \overline{\mathbf{J}^a_{d,d}\overline{\mathbf{P}}_d}$. Since the operations of extending by zero, restricting to a subspace, projecting onto an orthogonal subspace, and conjugating all preserve the Schatten classes (see Theorem 7.8 in Weidmann [45]), the claim follows. □

Next, we record the compactness result of the Kerzman-Stein operator due to Lanzani [28] in terms of the half-order operators on the bounded and unbounded domains.

Theorem 3.8.10 (Lanzani [28])**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded C^1 Jordan curve with complementary components Ω_1 and Ω_2. Then for each $d \in \{1,2\}$, $\mathbf{A}_{d,d}$ is compact.*

Proof. By Theorem 3.2 in Lanzani [28], $\mathbf{A}_\Gamma$ is compact, and hence so is $\mathbf{A}_{1,1} = \mathbf{b}_1^{-1}\mathbf{A}_\Gamma\mathbf{b}_1$ since $\mathbf{b}_1$ is a unitary operator. Compactness of $\mathbf{A}_{2,2}$ then follows from that of $\mathbf{A}_{1,1}$ as a result of the overfare conjugation identity in equation (3.6.7). □

Remark 3.8.11. By contrast, $\mathbf{J}^a_{d,c}$ (and hence $\mathbf{A}_{d,c}$) is never compact for $d \neq c$ by Theorem 3.4.6.

Lanzani's proof relies on a local approximation of $\mathbf{A}_\Gamma$ in the operator norm by a sequence of related integral operators with bounded kernel. The technique doesn't immediately generalize to a proof of Hilbert-Schmidtness due to the well-known fact that S_2 is not closed in the operator norm for infinite-dimensional spaces. Due to the work of Bolt and Raich [8], her compactness result is sharp for this class of curves, since if Γ is only piecewise-C^1 curve (i.e. has C^1 segments with a finite number of corners), then necessarily $\mathbf{A}_\Gamma$ won't be compact.

For the remainder of the section, we consider classes of curves with greater regularity than C^1 in conjunction with the kernel function of the Kerzman-Stein operator. Let Ω be a simply-connected domain, and recall the Garabedian kernel $L_\Omega(\zeta,z)d\zeta^{1/2}dz^{1/2}$ defined in Section 3.2. Since $L_\Omega(\zeta,z)$ has a simple pole at $\zeta = z$ with residue $1/2\pi$, if we define

$$L^{\text{hol}}_\Omega(\zeta,z)d\zeta^{1/2}dz^{1/2} := \frac{d\zeta^{1/2}dz^{1/2}}{\zeta - z} - 2\pi L_\Omega(\zeta,z)d\zeta^{1/2}dz^{1/2}, \quad \zeta,z \in \Omega \tag{3.8.2}$$

then the singularities of both terms on the right-hand side cancel out, and hence $L^{\text{hol}}_\Omega(\zeta,z)$ is a holomorphic function in both variables simultaneously. We call $L^{\text{hol}}_\Omega(\zeta,z)d\zeta^{1/2}dz^{1/2}$ the *desingularized Garabedian kernel*, and it can be used it to write the restricted Cauchy operator $\mathbf{J}^a_{1,1}$ with respect to a holomorphic integral kernel:

Proposition 3.8.12. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with bounded complementary component Ω_1. For any $\overline{\beta} = \overline{H} d\bar{z}^{1/2} \in \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1)$, we have*

$$\left(\mathbf{J}^a_{1,1}\overline{\beta}\right)(z) = \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{\zeta \in \Gamma_{p,r}} L^{\text{hol}}_{\Omega_1}(\zeta, z) d\zeta^{1/2} dz^{1/2} \overline{H(\zeta)} d\bar{\zeta}^{1/2}, \quad z \in \Omega_1.$$

Proof. This is shown during the proof of Corollary 4.6 in Kristel et al. [26]. □

Recall that a function of (possibly several) real variables is called *real analytic* if it is locally equal to its Taylor series expansion. The class of real analytic functions is denoted C^ω. We say that a function $K : \Gamma \times \Gamma \to \mathbb{C}$ is C^ω if $K(\varphi(t), \varphi(s))$ is C^ω, where $\varphi : [0,1] \to \Gamma$ is a parameterization of Γ.

Definition 3.8.13. Let $\Gamma \subset \mathbb{C}$ be a bounded Jordan curve. We say that Γ is real analytic if it can be parameterized by a function $\varphi : [0,1] \to \Gamma$, which when extended periodically to the real line, has the property that $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ are both real analytic.

Remark 3.8.14. The class of real analytic curves is (strictly) contained in C^∞.

Remark 3.8.15. Using the generalized Schwarz reflection principle, it can be shown that Γ is real analytic if and only if every $f \in \text{Conf}(\mathbb{D}, \Omega_1)$ extends to a one-to-one holomorphic function on a neighbourhood of $\Omega_1 \cup \Gamma$ (see Section 11 in Bell [3], for example).

For real analytic curves, we have the following strengthening of Proposition 3.8.12. Let $\Omega_1 \subset \mathbb{C}$ be a bounded real analytic Jordan domain, and express the Garabedian kernel for Ω_1 with respect to some fixed $\hat{f} \in \widehat{\text{Conf}}(\mathbb{D}, \Omega_1)$:

$$L_{\Omega_1}(\zeta, z) d\zeta^{1/2} dz^{1/2} = \frac{1}{2\pi} \frac{\sqrt{f'(\zeta)}\sqrt{f'(z)}}{f(\zeta) - f(z)} d\zeta^{1/2} dz^{1/2}.$$

Since f extends to a homeomorphism $\mathbb{S}^1 \to \Gamma$ by Carathéodory's Theorem, and both $\mathbb{D}$ and Ω_1 have real analytic boundary, it follows by the generalized Schwarz reflection principle that f extends to a holomorphic function on a neighbourhood of Ω_1. Thus, since $L_{\Omega_1}(\zeta, z)$ has a simple pole with residue $1/2\pi$ at each $\zeta = z \in \text{cl}(\Omega_1)$, the desingularized function $L^{\text{hol}}_{\Omega_1}(\zeta, z)$ is holomorphic on a neighbourhood of $\text{cl}(\Omega_1) \times \text{cl}(\Omega_1)$. In this case, we may write

$$\left(\mathbf{J}^a_{1,1}\overline{\beta}\right)(z) = \frac{1}{2\pi i} \int_{\zeta \in \Gamma} L^{\text{hol}}_{\Omega_1}(\zeta, z) \overline{H(\zeta)}\, |d\zeta| \cdot dz^{1/2}, \quad z \in \Omega_1. \tag{3.8.3}$$

Burbea [9] used the technique of desingularizing the Cauchy kernel in a sketch of a proof that the Kerzman-Stein operator is compact for a class of curves Γ called *Dini-smooth*, which lies between $C^{1,1}$ and C^2 (see Pommerenke [33], for example). The difficulty with extending this technique, as pointed out by Barrett and Edholm [2], is in gaining enough control of $L_{\Omega_1}(\zeta, z)$ for ζ near z in $\mathrm{cl}(\Omega_1)$, and so the technique of Burbea appears to have been abandoned in the context of the Kerzman-Stein operator. Thus, we shall focus our attention instead to analyzing $A(\zeta, z)$. The book of Bell [3] contains a slick proof that $A(\zeta, z)$ is of class C^∞ for C^∞ curves. Here, we generalize this technique to much of the remaining continuously differentiable hierarchy. To analyze the kernel function $A(\zeta, z)$ in the context of the continuously differentiable classes, we make use of the following result (cf. page 19, Bell [3]).

Lemma 3.8.16. *Let $I \subsetneq \mathbb{R}$ be an open interval, and suppose that $K(x,y) : I \times I \to \mathbb{C}$ is a C^1 function with the property that $K(x,x) = 0$ for all $x \in I$. Then there exists a continuous function $K_0(x,y) : I \times I \to \mathbb{C}$ with the property that*

$$K(x,y) = (x-y)K_0(x,y), \quad (x,y) \in I \times I.$$

Additionally:

(a) *If $K(x,y) \in C^\omega$, then $K_0(x,y) \in C^\omega$.*

(b) *If $K(x,y) \in C^{n,\alpha}$ for some $n \geqslant 1$, $0 \leqslant \alpha \leqslant 1$, then $K_0(x,y) \in C^{n-1,\alpha}$.*

Proof. Since $K(x,y) \in C^1$, by definition its first-order partial derivatives are continuous, and so by the fundamental theorem of calculus we can write

$$K(t,y) = \int_y^t \frac{\partial K}{\partial x}(x,y)\,dx.$$

If we make the change of variables $u = (x-y)/(t-y)$, then $(t-y)du = dx$ and

$$K(t,y) = (t-y)\int_0^1 \frac{\partial K}{\partial x}(y+u(t-y),y)\,du.$$

Now, set

$$K_0(t,y) := \int_0^1 \frac{\partial K}{\partial x}(y+u(t-y),y)\,du. \tag{3.8.4}$$

Then $K_0(t,y)$ is indeed a continuous function. From here, we consider two cases. First, if $K(x,y) \in C^\omega$, then all of the partial derivatives of $K(t,y)$ are in the C^ω-class, and hence $K_0(t,y)$ is as well. Now, assume that $K(t,y) \in C^{n,\alpha}$. Differentiating (3.8.4) under the integral sign $n-1$ times with respect to t yields

$$\frac{\partial^{n-1}K_0}{\partial t^{n-1}}(t,y) = \int_0^1 \frac{\partial^n K}{\partial x^n}(y+u(t-y),y)u^{n-1}\,du. \tag{3.8.5}$$

For any (t,y), $(t_0,y_0) \in I \times I$, we use (3.8.5) to estimate that

$$\begin{aligned}&\left|\frac{\partial^{n-1}K_0}{\partial t^{n-1}}(t,y) - \frac{\partial^{n-1}K_0}{\partial t^{n-1}}(t_0,y_0)\right| \\ &\leqslant \sup_{u\in[0,1]}\left|\frac{\partial^n K}{\partial x^n}(y+u(t-y),y) - \frac{\partial^n K}{\partial x^n}(y_0+u(t_0-y_0),y_0)\right|\end{aligned}$$

so choose $u' \in [0,1]$ for which

$$\begin{aligned}&\left|\frac{\partial^{n-1}K_0}{\partial t^{n-1}}(t,y) - \frac{\partial^{n-1}K_0}{\partial t^{n-1}}(t_0,y_0)\right| \\ &\leqslant \left|\frac{\partial^n K}{\partial x^n}(y+u'(t-y),y) - \frac{\partial^n K}{\partial x^n}(y_0+u'(t_0-y_0),y_0)\right|.\end{aligned} \tag{3.8.6}$$

Since the nth-partial derivatives of $K(t,y)$ are α-Hölder continuous, and the mapping

$$(t,y) \mapsto (y+u'(t-y),y)$$

is differentiable (and in particular Lipschitz continuous), it follows that the composition of the two is α-Hölder continuous. Therefore, there exists a constant $M > 0$ such that

$$\left|\frac{\partial^n K}{\partial x^n}(y+u'(t-y),y) - \frac{\partial^n K}{\partial x^n}(y_0+u'(t_0-y_0),y_0)\right| \leqslant M\|(y,t)-(y_0,t_0)\|^{\alpha}. \tag{3.8.7}$$

The inequalities (3.8.6) and (3.8.7) together imply that $K_0(t,y)$ is of class $C^{n-1,\alpha}$. □

With this technical result, the geometric properties of $C^{n,\alpha}$ curves can be directly translated into analytic properties of the Kerzman-Stein kernel. In particular, we show next that the apparent singularities of $A(\zeta,z)$ along the diagonal $\zeta = z$ cancel out for sufficiently smooth classes of curves, and are at the very least controllable for some less regular classes.

Theorem 3.8.17. *Let $\Gamma \subset \mathbb{C}$ be a bounded Jordan curve.*

(a) *If Γ is real analytic, then $A(\zeta,z)$ has a C^{ω} extension to $\Gamma \times \Gamma$.*

(b) *If $\Gamma \in C^{n,\alpha}$ for some $n \geqslant 2$ and $0 \leqslant \alpha \leqslant 1$, then $A(\zeta,z)$ has a $C^{n-2,\alpha}$ extension to $\Gamma \times \Gamma$.*

(c) *If $\Gamma \in C^{1,1}$, then $A(\zeta,z)$ is essentially bounded.*

(d) *If $\Gamma \in C^{1,\alpha}$ for some $0 < \alpha < 1$, then $A(\zeta,z) \in L^p(\Gamma \times \Gamma, |d\zeta| \otimes |dz|)$ for every $p < \frac{1}{1-\alpha}$.*

Proof. Let $\varphi : [0,1] \to \Gamma$ be a C^1 parameterization of Γ by arc-length. Then the unit tangent function of Γ can be expressed as $T(\varphi(s)) = \varphi'(s)$. From (3.7.3), we may then write the kernel of the Kerzman-Stein operator $\mathbf{A}_\Gamma$ as

$$A(\varphi(s),\varphi(t)) = \frac{1}{2\pi i}\left[\frac{\varphi'(s)}{\varphi(s)-\varphi(t)} - \frac{\overline{\varphi'(t)}}{\overline{\varphi(s)}-\overline{\varphi(t)}}\right].$$

Now set $K(s,t) = \varphi(s)-\varphi(t)$. Then $K(s,t)$ satisfies the assumptions of the lemma, so there is a continuous function $K_0(s,t)$ such that

$$\varphi(s)-\varphi(t) = (s-t)K_0(s,t).$$

Observe that $K_0(t,t) = \varphi'(t) \neq 0$, so there is $M_0 > 0$ such that $M_0^{-1} \leqslant |K_0(s,t)| \leqslant M_0$ by continuity of φ' and injectivity of φ. In particular, we can write

$$(s-t)A(\varphi(s),\varphi(t)) = \frac{1}{2\pi i}\left[\frac{\varphi'(s)}{K_0(s,t)} - \frac{\overline{\varphi'(t)}}{\overline{K_0(s,t)}}\right]. \tag{3.8.8}$$

From here, we split off into cases. First, to prove (a) and (b), let us assume that Γ is at least C^2. In this case, both $\varphi'(s)$ and $K_0(s,t)$ are C^1-smooth, so the right-hand side of (3.8.8) is C^1 and vanishes along the diagonal. Applying the lemma again yields a continuous function $K_1(s,t)$ with the property that

$$\frac{1}{2\pi i}\left[\frac{\varphi'(s)}{K_0(s,t)} - \frac{\overline{\varphi'(t)}}{\overline{K_0(s,t)}}\right] = (s-t)K_1(s,t)$$

whence

$$A(\varphi(s),\varphi(t)) = K_1(s,t), \quad s \neq t. \tag{3.8.9}$$

If additionally Γ is real analytic, then both $\varphi(s)$, $K_0(s,t) \in C^\omega$, and so $K_1(s,t) \in C^\omega$. Similarly, if $\Gamma \in C^{n,\alpha}$ for some $n \geqslant 2$ and $0 \leqslant \alpha \leqslant 1$, then $\varphi(s) \in C^{n,\alpha}$ by assumption, and hence $K(s,t) \in C^{n,\alpha}$. It is then clear in this case that $K(s,t) \in C^{n,\alpha}$, and thus $K_0(s,t) \in C^{n-1,\alpha}$. Again, one can check that the right-hand side of (3.8.8) is a $C^{n-1,\alpha}$ function in (s,t), and so $K_1(s,t) \in C^{n-2,\alpha}$ by the lemma. This completes the cases (a) and (b).

For (c) and (d), we return to the expression in equation (3.8.8) for C^1 curves, and assume further that $\Gamma \in C^{1,\alpha}$ for some $0 < \alpha \leqslant 1$. Note that

$$\begin{aligned} \left| \frac{\varphi'(s)}{K_0(s,t)} - \frac{\overline{\varphi'(t)}}{\overline{K_0(s,t)}} \right| &= \left| \frac{\varphi'(s)\overline{K_0(s,t)} - \overline{\varphi'(t)}K_0(s,t)}{K_0(s,t)\overline{K_0(s,t)}} \right| \\ &\leqslant M_0^2 \left| \varphi'(s)\overline{K_0(s,t)} - \overline{\varphi'(t)}K_0(s,t) \right|. \end{aligned} \tag{3.8.10}$$

Define a function $R : [0,1]^4 \to \mathbb{C}$ by

$$R(x_1,x_2,x_3,x_4) := K_0(x_1,x_2)\overline{K_0(x_3,x_4)}.$$

Since $K_0(s,t) \in C^{0,\alpha}$ for these cases, it follows that R is a $C^{0,\alpha}$ function on $[0,1]^4$. In particular, there exists $M_1 > 0$ such that for any $s,t \in [0,1]$

$$\begin{aligned} \left| \varphi'(s)\overline{K_0(s,t)} - \overline{\varphi'(t)}K_0(s,t) \right| &= |R(s,s,s,t) - R(s,t,t,t)| \\ &\leqslant M_1 \|(0, s-t, s-t, 0)\|^\alpha \\ &= 2^{\alpha/2} M_1 |s-t|^\alpha. \end{aligned}$$

Combining this with (3.8.10), we obtain

$$\left| \frac{\varphi'(s)}{K_0(s,t)} - \frac{\overline{\varphi'(t)}}{\overline{K_0(s,t)}} \right| \leqslant 2^{\alpha/2} M_0^2 M_1 |s-t|^\alpha.$$

Putting everything together so far for the case $\Gamma \in C^{1,\alpha}$, we obtain the bound

$$|A(\varphi(t),\varphi(s))| \leqslant \frac{M}{|s-t|^{1-\alpha}}, \quad s \neq t \tag{3.8.11}$$

for a constant $M > 0$. If $\alpha = 1$, then the function $A(\varphi(t),\varphi(s))$ is bounded for $s \neq t$, proving (c). If $0 < \alpha < 1$, by Lemma A.2.1 the function $(s,t) \mapsto (s-t)^{(\alpha-1)}$ is in $L^p([0,1]^2, dA)$ for all $0 < p < (1-\alpha)^{-1}$. Combining this fact with the Fubini-Tonelli theorem, we have

$$\begin{aligned} \iint_{\Gamma\times\Gamma} |A(\zeta,z)|^p \, (|d\zeta| \otimes |dz|) &= \int_0^1 \int_0^1 |A(\varphi(s),\varphi(t))|^p \, ds dt \\ &\leqslant M \int_0^1 \int_0^1 \frac{1}{|s-t|^{p(1-\alpha)}} \, ds dt \\ &< \infty \end{aligned}$$

as required for (d). □

Remark 3.8.18. Note that the diagonal of $\Gamma \times \Gamma$ is a measure zero set, and so by equation 3.8.9, we can simply take $K_1(s,t)$ to be the kernel of $\mathbf{A}_\Gamma$ in practice for $\Gamma \in C^2$.

With control over the kernel function, we can easily deduce some useful qualitative facts of the Kerzman-Stein operator – first of which is a sufficient condition for removing the principal value sign from $\mathbf{A}_\Gamma$. Indeed, for all $\Gamma \in C^{1,\alpha}$ where $\alpha > 0$, we have that $A_z(\zeta) := A(\zeta, z) \in L^1(\Gamma, |d\zeta|)$ for each fixed $z \in \Gamma$. Thus, it follows from the dominated convergence theorem that the principal value integral in equation (3.7.2) coincides with the standard Lebesgue integral over Γ, so we may write

$$(\mathbf{A}_\Gamma u)(z) = \int_{\zeta \in \Gamma} A(\zeta, z) u(\zeta) \, |d\zeta|, \quad z \in \Gamma. \tag{3.8.12}$$

Next, we record the sufficient conditions for membership of the Kerzman-Stein operator to various operator classes.

Corollary 3.8.19. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded C^4 curve with complementary components Ω_1 and Ω_2. Then for each $d \in \{1,2\}$, $\mathbf{A}_{d,d}$ is trace class.*

Proof. By Proposition 3.8.17 (b), the kernel function of $\mathbf{A}_\Gamma$ is C^2, so by Proposition 3.5 in Sugiura [43], $\mathbf{A}_\Gamma$ must be trace class. Thus, since $\mathbf{b}_1 : \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1) \to L^2(\Gamma, |d\zeta|)$ is unitary, it follows that $\mathbf{A}_{1,1} = \mathbf{b}_1^{-1} \mathbf{A}_\Gamma \mathbf{b}_1$ is trace class. By equation (3.6.7), we then have that $\mathbf{A}_{2,2}$ is also trace class. □

Perhaps the most significant consequence of Theorem 3.8.17 is that the Kerzman-Stein operator is Hilbert-Schmidt for a relatively wide class of continuously differentiable curves.

Corollary 3.8.20. *For any $\alpha > \frac{1}{2}$ and bounded $C^{1,\alpha}$ Jordan curve $\Gamma \subset \overline{\mathbb{C}}$ with complementary components Ω_1 and Ω_2, $\mathbf{A}_{d,d}$ is Hilbert-Schmidt for each $d \in \{1,2\}$.*

Proof. By Proposition 3.8.17 (d), the kernel function of $\mathbf{A}_\Gamma$ is square-integrable, so by Proposition 3.8.6, the integral operator $\mathbf{A}_\Gamma$ is Hilbert-Schmidt, and the result follows. □

Due to the significance of this operator class, it is natural to pose the following problem concerning the limits of Hilbert-Schmidtness of $\mathbf{A}_{1,1}$.

Problem 3.8.21. *Determine all Hölder indices α for which $\mathbf{A}_{1,1} : \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1) \to \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1)$ remains Hilbert-Schmidt for every $\Omega_1 \in C^{1,\alpha}$.*

Jumping ahead to the topics in the next section, this can be viewed as a special case of an open problem (Problem 3.9.5) regarding the Hilbert-Schmidtness of the Grunsky operator for the Smirnov space. For now, we deal with the remaining cases $0 < \alpha \leqslant 1/2$ by considering the Schatten classes.

Corollary 3.8.22. *Fix some* $0 < \alpha \leqslant \frac{1}{2}$*, and suppose that* $\Gamma \subset \overline{\mathbb{C}}$ *is a bounded* $C^{1,\alpha}$ *Jordan curve with complementary components* Ω_1 *and* Ω_2*. Then* $\mathbf{A}_{d,d} \in S_p\left(\mathcal{A}^{1/2}_{\text{harm}}(\Omega_d)\right)$ *for all* $\frac{1}{\alpha} < p < \infty$.

Proof. It suffices to show that $\mathbf{A}_\Gamma$ belongs to the required classes. If $p > 1/\alpha \geqslant 2$ and $q \in \mathbb{R}$ satisfies $1/p + 1/q = 1$, then $1 < q < (1-\alpha)^{-1} < 2$. Applying Theorem A.2.3 using the fact that $\overline{A(\zeta,z)} = -A(z,\zeta)$, we get

$$\|\mathbf{A}_\Gamma\|_{S_p} \leqslant \|A\|_{q,p}$$

where

$$\|A\|^p_{q,p} = \int_\Gamma \left[\int_\Gamma |A(\zeta,z)|^q \, |d\zeta|\right]^{\frac{p}{q}} |dz|.$$

Now, using equation (3.8.11), we have for some $M > 0$ the bound

$$\int_\Gamma \left[\int_\Gamma |A(\zeta,z)|^q \, |d\zeta|\right]^{\frac{p}{q}} |dz| \leqslant M \int_0^1 \left[\int_0^1 \frac{ds}{|s-t|^{q(1-\alpha)}}\right]^{\frac{p}{q}} dt$$

By a computation similar to that in the proof of Lemma A.2.1, the right-hand side is finite for the given range of p and q. Therefore, $\|\mathbf{A}_\Gamma\|_{S_p} < \infty$ for all $p > \alpha^{-1}$. □

3.9 The Grunsky Operator on Smirnov Space

In its various forms, the Grunsky operator has long been studied in the context of complex function theory, specifically in relation to the Bergman and Dirichlet spaces (see Section 4.6). Associated to the Grunsky operator are the Grunsky coefficients, which appear in Grunsky's famous characterization for univalence of holomorphic functions on $\mathbb{D}^-$. Recently, Kristel et al. [26] introduced an analogue of the Grunsky operator for the half-order Smirnov space of the disk using their work on the Cauchy operator and overfare. In this section, we follow their presentation for defining the operator and briefly exploring its properties. We use the operator to supplement a geometric characterization for the Kerzman-Stein operator on Ahlfors-regular Jordan domains (acting both as a slight generalization and extension of a theorem of Kerzman and Stein [24]). We also discuss the problem of characterizing Hilbert-Schmidtness of the Smirnov space Grunsky operator, which was explored by Kristel et al. [26]. We show that their operator belongs to the Schatten p-class S_p if and only if $\mathbf{A}_{1,1} \in S_p$. By our work in the previous chapter, this yields sufficient conditions for Schatten-ness of the Smirnov space Grunsky operator for all $p \geqslant 2$.

Definition 3.9.1. Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2. For $f \in \mathrm{Conf}(\mathbb{D}, \Omega_1)$, fix a choice of branch of $\sqrt{f'}$ to obtain $\hat{f} \in \widehat{\mathrm{Conf}}(\mathbb{D}, \Omega_1)$. The Grunsky operator associated to f is the operator $\mathbf{Gr}_f : \overline{\mathcal{A}^{1/2}(\mathbb{D})} \to \mathcal{A}^{1/2}(\mathbb{D})$ defined by

$$\mathbf{Gr}_f = -\hat{f}^* \mathbf{P}_1 \mathbf{O}_{2,1} \mathbf{J}^a_{1,2} (\hat{f}^{-1})^*.$$

Remark 3.9.2. The choice of branch of $\sqrt{f'}$ does not change the output of the operator.

There are two important integral representations of $\mathbf{Gr}_f$. Recall from equation (3.4.4) the identity $\mathbf{P}_1 \mathbf{O}_{2,1} \mathbf{J}^a_{1,2} = \mathbf{J}^a_{1,1}$. With this, we may also write

$$\mathbf{Gr}_f = -\hat{f}^* \mathbf{J}^a_{1,1} (\hat{f}^{-1})^*. \tag{3.9.1}$$

Moreover, the kernel function in (3.9.1) can be desingularized with the Garabedian kernel by changing variables in the expression (3.8.3). That is, for any $\overline{\beta} = \overline{H} d\bar{z}^{1/2} \in \overline{\mathcal{A}^{1/2}(\mathbb{D})}$, we have

$$\left(\mathbf{Gr}_f \overline{\beta}\right)(z) = -\frac{1}{2\pi i} \int_{\zeta \in \mathbb{S}^1} \overline{H(\zeta)} d\bar{\zeta}^{1/2} \left(\frac{\sqrt{f'(\zeta)}\sqrt{f'(z)}}{f(\zeta) - f(z)} - \frac{1}{\zeta - z} \right) d\zeta^{1/2} dz^{1/2}. \tag{3.9.2}$$

Recall from Proposition 3.2.3 that pull-back is a unitary operation on the Smirnov spaces. Thus, by equation (3.9.1) and Theorem 3.5.1, we have the adjoint formula

$$\left(\mathbf{Gr}_f\right)^* = -\overline{\mathbf{Gr}_f}.$$

Next, we focus on comparing the Grunsky and Kerzman-Stein operators. We have the following result relating the two operators in terms of the Schatten classes.

Theorem 3.9.3. *Let $\Gamma \subset \mathbb{C}$ be a bounded Ahlfors-regular Jordan curve with bounded complementary component Ω_1. For all $1 \leqslant p \leqslant \infty$, the following are equivalent:*

(a) $\mathbf{A}_{1,1} \in S_p\left(\mathcal{A}^{1/2}_{\mathrm{harm}}(\Omega_1)\right)$.

(b) $\mathbf{Gr}_f \in S_p\left(\overline{\mathcal{A}^{1/2}(\mathbb{D})}, \mathcal{A}^{1/2}(\mathbb{D})\right)$ *for all* $f \in \mathrm{Conf}(\mathbb{D}, \Omega_1)$.

It is sufficient for (b) *to hold for at least one such mapping f.*

Proof. Since pull-back by $\hat{f}$ and $\hat{f}^{-1}$ are both bounded isomorphisms on the Smirnov spaces, and the Schatten classes form a two-sided ideal in the space of bounded operators (see Theorem 7.8 (c) in Weidmann [45], for example), it follows that $\mathbf{Gr}_f \in S_p$ if and only if $\mathbf{J}^a_{1,1} \in S_p$ by equation (3.9.1). Hence we may apply Proposition 3.8.9 to prove the claim. □

Remark 3.9.4. In particular, the results of the previous section imply that:

(a) $\Omega_1 \in C^4 \implies \mathbf{Gr}_f$ is trace class.

(b) $\Omega_1 \in C^{1,1/2+\varepsilon} \implies \mathbf{Gr}_f$ is Hilbert-Schmidt.

(c) $\Omega_1 \in C^1 \implies \mathbf{Gr}_f$ is compact.

With this observation, the problem of characterizing Schatten-ness of $\mathbf{A}_{1,1}$ is connected to the following open problem.

Problem 3.9.5 (Kristel et al. [26]). *Determine necessary and sufficient conditions on both Ω and the map $f \in \mathrm{Conf}(\mathbb{D}, \Omega)$ that guarantee that $\mathbf{Gr}_f$ is a Hilbert-Schmidt operator.*

The authors provided a necessary condition for Hilbert-Schmidtness. To state it, we first need a definition.

Definition 3.9.6. Let $\Omega \subset \overline{\mathbb{C}}$ be a Jordan domain. We say that a map $f \in \mathrm{Conf}(\mathbb{D}, \Omega)$ is Weil-Petersson class if $f''/f' \in L^2(\mathbb{D}, dA)$, where $dA = (d\bar{z} \wedge dz)/2i$ denotes the Lebesgue area measure. In this case, we also say that Ω and Γ are Weil-Petersson class.

Remark 3.9.7. By the Kellogg-Warschawski theorem (see Theorem 3.6 in Pommerenke [33]), if Ω is a C^2 domain, then f' and f'' extend continuously to $\mathrm{cl}(\mathbb{D})$ and $f' \neq 0$ on $\overline{\mathbb{D}}$. Hence, such f are immediately in the Weil-Petersson class. However, there are Weil-Petersson curves which are not C^1 (see Bishop [7], for example).

Their result can now be stated as follows.

Theorem 3.9.8 (Kristel et al. [26]). *If $\mathbf{Gr}_f$ is Hilbert-Schmidt, then f is Weil-Petersson class.*

Characterizing Hilbert-Schmidtness of $\mathbf{Gr}_f$ is of interest partially due to the ramifications of this property in the Bergman space setting in connection to the universal Teichmüller space. By Theorem 3.9.3, we can phrase the above result in terms of the Kerzman-Stein operator.

Corollary 3.9.9. *If $\mathbf{A}_{d,d}$ is Hilbert-Schmidt, then Ω_d is Weil-Petersson class.*

With this, our sufficient conditions for Hilbert-Schmidtness from Theorem 3.8.20 find immediate application.

Theorem 3.9.10. *If $\Gamma \subset \mathbb{C}$ is a bounded Jordan curve of class $C^{1,1/2+\varepsilon}$ for some $\varepsilon > 0$, then Γ is Weil-Petersson class.*

Continuing with the theme of operator properties distinguishing classes of domains, the upcoming theorem provides a characterization of disk domains in the plane in terms of the properties of Kerzman-Stein operator and its extension from Section 3.6. This originally stems from the work of Kerzman and Stein [24], who showed that for a C^{∞}-smooth curve Γ, the corresponding Kerman-Stein operator $\mathbf{A}_{\Gamma}$ vanishes identically precisely when Γ is a circle (or equivalently, when the Cauchy and Szegő kernel coincide). Their argument involves an explicit geometric computation with the kernel function, which we bypass using a property of $\mathbf{Gr}_f$ for the general case of Ahlfors-regular Jordan curves.

Theorem 3.9.11. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2. Then the following are equivalent:*

(a) *Γ is a circle.*

(b) *For every $f \in \mathrm{Conf}(\mathbb{D}, \Omega_1)$, $\mathbf{Gr}_f = 0$.*

(c) $\mathbf{P}_1 = \mathbf{J}_{1,1}$.

(d) $\mathbf{A}_{1,1} = 0$.

(e) $\mathbf{A}_{1,2} = -\mathbf{O}_{1,2}$.

(f) $\mathrm{image}\left(\mathbf{O}^h_{1,2}\right) = \overline{\mathcal{A}^{1/2}(\Omega_2)}$.

(g) $\mathbf{A}_{1,2}$ *is unitary.*

It is sufficient that (b) *holds for at least one such mapping f.*

Proof. First, we prove the equivalence of (a) and (b). Note that $\mathbf{Gr}_f = 0$ if and only if the desingularized kernel in (3.9.2) vanishes identically, i.e.

$$\frac{\sqrt{f'(\zeta)}\sqrt{f'(z)}}{f(\zeta) - f(z)} - \frac{1}{\zeta - z} \equiv 0. \tag{3.9.3}$$

If Γ is a circle, then every $f \in \mathrm{Conf}(\mathbb{D}, \Omega_1)$ must be a Möbius transformation, and hence (3.9.3) holds by a simple but tedious computation. Conversely, if $\mathbf{Gr}_f = 0$, then from (3.9.3) it follows that

$$\frac{f'(\zeta)f'(z)}{(f(\zeta) - f(z))^2} - \frac{1}{(\zeta - z)^2} \equiv 0. \tag{3.9.4}$$

The left-hand side of (3.9.4) is the desingularized Bergman kernel, which is well-known to vanish only if f is a Möbius transform (see Proposition 4.6.2). Hence, Γ must be a circle.

Next, by the expression in (3.9.1) relating $\mathbf{Gr}_f$ and $\mathbf{J}^a_{1,1}$, we have that $\mathbf{J}^a_{1,1} = 0$ if and only if $\mathbf{Gr}_f = 0$. From the Cauchy integral formula, this yields (b) $\Longleftrightarrow$ (c), and by the Kerzman-Stein formula (Theorem 3.6.2), it follows that (c) $\Longleftrightarrow$ (d). By the formula in equation (3.6.8), we have (d) $\Longleftrightarrow$ (e). Now, if $\mathbf{A}_{1,2} = -\mathbf{O}_{1,2}$, then by precomposing with $\mathbf{P}_1$, we have that $\mathbf{O}^h_{1,2} = -\overline{\mathbf{J}}^h_{1,2}$. Since $\overline{\mathbf{J}}^h_{1,2} : \mathcal{A}^{1/2}(\Omega_1) \to \overline{\mathcal{A}^{1/2}(\Omega_2)}$ is in particular surjective (Theorem 3.4.6), this gives us (e) $\Longrightarrow$ (f). Conversely, assuming (f), by the integral representation for the overfare in equation (3.5.1), it must be the case that $\mathbf{J}^a_{2,2} = 0$, or equivalently $\mathbf{A}_{2,2} = 0$. Thus since $\mathbf{O}_{2,1}\mathbf{A}_{2,2} = \mathbf{A}_{1,1}\mathbf{O}_{1,2}$, we have $\mathbf{A}_{1,1} = 0$, so (f) $\Longrightarrow$ (d). Clearly, (e) $\Longrightarrow$ (g) by equation (3.3.5). Finally, if $\mathbf{A}_{1,2}$ is unitary, then $\mathbf{O}_{2,1}\mathbf{A}_{1,2} = \mathbf{I} - \mathbf{A}_{1,1}$ is also unitary (this is equation (3.6.8)). By definition, this implies that $(\mathbf{I} + \mathbf{A}_{1,1})(\mathbf{I} - \mathbf{A}_{1,1}) = \mathbf{I}$, or equivalently $(\mathbf{A}_{1,1})^2 = 0$. Thus for any $u \in \mathcal{A}^{1/2}_{\text{harm}}(\Omega_1)$, we have

$$\|\mathbf{A}_{1,1}u\|^2 = \langle \mathbf{A}_{1,1}u, \mathbf{A}_{1,1}u\rangle = -\langle u, (\mathbf{A}_{1,1})^2 u\rangle = 0$$

whence (g) $\Longrightarrow$ (d). □

3.10 A Scattering Matrix for Overfare

In our final section dedicated to the Smirnov space, we derive quadratic adjoint identities for the Cauchy operator, and compute the so-called "scattering matrix" of the overfare operator $\mathbf{O}_{2,1}$. This is analogous to the work of Schippers and Staubach [38], [41] in the Bergman space setting, which is connected to problems in scattering theory of the potential (see Section 4.5). This greater perspective is out of scope for our work here, however the identities produced in the Smirnov setting are interesting in their own right, and set the groundwork for a future investigation into the matter.

Theorem 3.10.1 (Quadratic Adjoint Identities). *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2. Then*

$$\begin{aligned} -\mathbf{I}^a_1 &= \left(\mathbf{J}^a_{1,1}\right)^* \mathbf{J}^a_{1,1} - \left(\mathbf{J}^a_{1,2}\right)^* \mathbf{J}^a_{1,2}, \\ \mathbf{I}^a_2 &= \left(\mathbf{J}^a_{2,1}\right)^* \mathbf{J}^a_{2,1} - \left(\mathbf{J}^a_{2,2}\right)^* \mathbf{J}^a_{2,2}, \\ 0 &= \left(\mathbf{J}^a_{1,1}\right)^* \mathbf{J}^a_{2,1} - \left(\mathbf{J}^a_{1,2}\right)^* \mathbf{J}^a_{2,2}, \\ 0 &= \left(\mathbf{J}^a_{2,2}\right)^* \mathbf{J}^a_{1,2} - \left(\mathbf{J}^a_{2,1}\right)^* \mathbf{J}^a_{1,1}. \end{aligned}$$

Proof. To begin, we demonstrate the first identity – the second identity is computed similarly. With the adjoint formulas of Section 3.5 in hand, write

$$\left(\mathbf{J}^a_{1,1}\right)^* \mathbf{J}^a_{1,1} - \left(\mathbf{J}^a_{1,2}\right)^* \mathbf{J}^a_{1,2} = -\overline{\mathbf{J}}^h_{1,1}\mathbf{J}^a_{1,1} + \overline{\mathbf{J}}^h_{2,1}\mathbf{J}^a_{1,2}.$$

Now, rewrite identity (3.4.8) as $\mathbf{J}_{1,1}\mathbf{O}_{2,1} = \mathbf{J}_{2,1}$. Then

$$-\overline{\mathbf{J}}^h_{1,1}\mathbf{J}^a_{1,1} + \overline{\mathbf{J}}^h_{2,1}\mathbf{J}^a_{1,2} = -\overline{\mathbf{J}}^h_{1,1}\mathbf{J}^a_{1,1} + \overline{\mathbf{J}}_{1,1}\mathbf{O}_{2,1}\mathbf{J}^a_{1,2}$$

By the jump formula and Cauchy integral formula, the right-hand side simplifies to

$$\begin{aligned} -\overline{\mathbf{J}}^h_{1,1}\mathbf{J}^a_{1,1} + \overline{\mathbf{J}}_{1,1}\mathbf{O}_{2,1}\mathbf{J}^a_{1,2} &= -\overline{\mathbf{J}}^h_{1,1}\mathbf{J}^a_{1,1} + \overline{\mathbf{J}}_{1,1}\left(\mathbf{J}^a_{1,1} - \mathbf{I}^a_1\right) \\ &= -\overline{\mathbf{J}}^a_{1,1} \\ &= -\mathbf{I}^a_1 \end{aligned}$$

as required. Next, we compute the third identity – the fourth identity follows by taking its adjoint. Using the identity in (3.4.8) again, we have

$$\begin{aligned} \left(\mathbf{J}^a_{1,1}\right)^* \mathbf{J}^a_{2,1} - \left(\mathbf{J}^a_{1,2}\right)^* \mathbf{J}^a_{2,2} &= \left(\mathbf{J}^a_{1,1}\right)^* \mathbf{J}_{1,1}\mathbf{O}^a_{2,1} - \left(\mathbf{J}^a_{1,2}\right)^* \mathbf{J}_{1,2}\mathbf{O}^a_{2,1} \\ &= \left[(\mathbf{J}^a_{1,1})^*\mathbf{J}_{1,1} - \left(\mathbf{J}^a_{1,2}\right)^* \mathbf{J}_{1,2}\right]\mathbf{O}^a_{1,2}. \end{aligned}$$

By the first identity we computed and the Cauchy integral formula, the right-hand side becomes

$$\begin{aligned} \left[(\mathbf{J}^a_{1,1})^*\mathbf{J}_{1,1} - (\mathbf{J}^a_{1,2})^*\mathbf{J}_{1,2}\right]\mathbf{O}^a_{1,2} &= -\overline{\mathbf{P}}_1\mathbf{O}^a_{2,1} + \left[-\overline{\mathbf{J}}_{1,1}\mathbf{J}_{1,1} + \overline{\mathbf{J}}_{2,1}\mathbf{J}_{1,2}\right]\mathbf{P}_1\mathbf{O}^a_{2,1} \\ &= -\overline{\mathbf{P}}_1\mathbf{O}^a_{2,1} - \overline{\mathbf{J}}_{1,1}\mathbf{P}_1\mathbf{O}^a_{2,1} \\ &= -\overline{\mathbf{J}}_{1,1}\mathbf{O}^a_{2,1}. \end{aligned}$$

The composition in the final equality vanishes identically by the conjugation of (3.4.7), and so the proof is complete. □

Remark 3.10.2. By conjugating each of the identities in Theorem 3.10.1, we also obtain:

$$\begin{aligned} -\mathbf{I}^h_1 &= \mathbf{J}^a_{1,1}\left(\mathbf{J}^a_{1,1}\right)^* - \mathbf{J}^a_{2,1}\left(\mathbf{J}^a_{2,1}\right)^*, \\ \mathbf{I}^h_2 &= \mathbf{J}^a_{1,2}\left(\mathbf{J}^a_{1,2}\right)^* - \mathbf{J}^a_{2,2}\left(\mathbf{J}^a_{2,2}\right)^*, \\ 0 &= \mathbf{J}^a_{1,2}\left(\mathbf{J}^a_{1,1}\right)^* - \mathbf{J}^a_{2,1}\left(\mathbf{J}^a_{2,2}\right)^*, \\ 0 &= \mathbf{J}^a_{2,2}\left(\mathbf{J}^a_{2,1}\right)^* - \mathbf{J}^a_{1,2}\left(\mathbf{J}^a_{1,1}\right)^*. \end{aligned}$$

An interpretation of the first two identities in Theorem 3.10.1 can be made in terms of operator norms. Indeed, for any $\overline{\beta}_1 \in \overline{\mathcal{A}^{1/2}(\Omega_1)}$ we have that

$$\begin{aligned}\left\langle \mathbf{J}_{1,2}\overline{\beta}_1, \mathbf{J}_{1,2}\overline{\beta}_1 \right\rangle_{\Omega_2} - \left\langle \mathbf{J}_{1,1}\overline{\beta}_1, \mathbf{J}_{1,1}\overline{\beta}_1 \right\rangle_{\Omega_1} &= \left\langle \overline{\beta}_1, \left[\left(\mathbf{J}^a_{1,2}\right)^* \mathbf{J}_{1,2} - \left(\mathbf{J}^a_{1,1}\right)^* \mathbf{J}_{1,1} \right] \overline{\beta}_1 \right\rangle_{\Omega_1} \\ &= \left\langle \overline{\beta}_1, \overline{\beta}_1 \right\rangle_{\Omega_1}\end{aligned}$$

which implies that

$$\|\mathbf{J}^a_{1,2}\|^2 - \|\mathbf{J}^a_{1,1}\|^2 = 1. \tag{3.10.1}$$

Similarly, for any $\overline{\beta}_2 \in \overline{\mathcal{A}^{1/2}(\Omega_2)}$, we have that

$$\begin{aligned}\left\langle \mathbf{J}_{2,1}\overline{\beta}_2, \mathbf{J}_{2,1}\overline{\beta}_2 \right\rangle_{\Omega_1} - \left\langle \mathbf{J}_{2,2}\overline{\beta}_2, \mathbf{J}_{2,2}\overline{\beta}_2 \right\rangle_{\Omega_2} &= \left\langle \overline{\beta}_2, \left[\left(\mathbf{J}^a_{2,1}\right)^* \mathbf{J}_{2,1} - \left(\mathbf{J}^a_{2,2}\right)^* \mathbf{J}_{2,2} \right] \overline{\beta}_2 \right\rangle_{\Omega_2} \\ &= \left\langle \overline{\beta}_2, \overline{\beta}_2 \right\rangle_{\Omega_2}\end{aligned}$$

and so

$$\left\|\mathbf{J}^a_{2,1}\right\|^2 - \left\|\mathbf{J}^a_{2,2}\right\|^2 = 1. \tag{3.10.2}$$

Further, we can use these quadratic norm identities to relate the norms of the Cauchy operators on either side of the curve. Note that since $\overline{(\mathbf{J}^a_{1,2})^*} = -\mathbf{J}^a_{2,1}$, we could have previously observed that

$$\|\mathbf{J}^a_{1,2}\| = \|\mathbf{J}^a_{2,1}\|.$$

Subtracting equation (3.10.1) from (3.10.2) and using the above substitution, it then follows that

$$\|\mathbf{J}^a_{1,1}\| = \|\mathbf{J}^a_{2,2}\|. \tag{3.10.3}$$

Let us also note the following, which says that the extension of the Cauchy operators in Theorem 3.10.1 to the rest of the space yields quadratic adjoint identities related to the d,c-Kerzman-Stein operators.

Corollary 3.10.3 (Extended Quadratic Adjoint Identities)**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2. Then*

$$\begin{aligned} i\left[*_1\left(\mathbf{I}+\mathbf{A}_{1,1}\right)\right] &= \left(\mathbf{J}_{1,1}\right)^* \mathbf{J}_{1,1} - \left(\mathbf{J}_{1,2}\right)^* \mathbf{J}_{1,2}, \\ -i\left[*_2\left(\mathbf{I}-\mathbf{A}_{2,2}\right)\right] &= \left(\mathbf{J}_{2,1}\right)^* \mathbf{J}_{2,1} - \left(\mathbf{J}_{2,2}\right)^* \mathbf{J}_{2,2}, \\ \mathbf{A}_{2,1} &= \left(\mathbf{J}_{1,1}\right)^* \mathbf{J}_{2,1} - \left(\mathbf{J}_{1,2}\right)^* \mathbf{J}_{2,2}, \\ \mathbf{A}_{1,2} &= \left(\mathbf{J}_{2,2}\right)^* \mathbf{J}_{1,2} - \left(\mathbf{J}_{2,1}\right)^* \mathbf{J}_{1,1}. \end{aligned}$$

Proof. Observe that

$$\begin{aligned}(\mathbf{J}_{1,1})^*\mathbf{J}_{1,1} - (\mathbf{J}_{1,2})^*\mathbf{J}_{1,2} &= \left[\mathbf{P}_1 + \left(\mathbf{J}^a_{1,1}\right)^*\mathbf{P}_1\right]\left[\mathbf{P}_1 + \mathbf{J}_{1,1}\overline{\mathbf{P}}_1\right] - \left(\mathbf{J}^a_{1,2}\right)^*\mathbf{J}^a_{1,2}\overline{\mathbf{P}}_1 \\ &= \mathbf{P}_1 + \mathbf{J}_{1,1}\overline{\mathbf{P}}_1 - \overline{\mathbf{J}}_{1,1}\mathbf{P}_1 + \left[\left(\mathbf{J}^a_{1,1}\right)^*\mathbf{J}^a_{1,1} - \left(\mathbf{J}^a_{1,2}\right)^*\mathbf{J}^a_{1,2}\right]\overline{\mathbf{P}}_1\end{aligned}$$

and using the first equation in Theorem 3.10.1

$$\mathbf{P}_1 + \mathbf{J}_{1,1}\overline{\mathbf{P}}_1 - \overline{\mathbf{J}}_{1,1}\mathbf{P}_1 + \left[\left(\mathbf{J}^a_{1,1}\right)^*\mathbf{J}^a_{1,1} - \left(\mathbf{J}^a_{1,2}\right)^*\mathbf{J}^a_{1,2}\right]\overline{\mathbf{P}}_1 = \mathbf{P}_1 - \overline{\mathbf{P}}_1 + \mathbf{J}_{1,1}\overline{\mathbf{P}}_1 - \overline{\mathbf{J}}_{1,1}\mathbf{P}_1$$

which proves our first identity. Next, we have

$$\begin{aligned}(\mathbf{J}_{2,2})^*\mathbf{J}_{1,2} - (\mathbf{J}_{2,1})^*\mathbf{J}_{1,1} &= \left[-\mathbf{P}_2 + \left(\mathbf{J}^a_{2,2}\right)^*\mathbf{P}_2\right]\mathbf{J}_{1,2}\overline{\mathbf{P}}_1 - \left(\mathbf{J}^a_{2,1}\right)^*\left[\mathbf{P}_1 + \mathbf{J}_{1,1}\overline{\mathbf{P}}_1\right] \\ &= \mathbf{J}_{1,2}\overline{\mathbf{P}}_1 + \overline{\mathbf{J}}_{1,2}\mathbf{P}_1 + \left[\left(\mathbf{J}^a_{2,2}\right)^*\mathbf{J}^a_{1,2} - \left(\mathbf{J}^a_{2,1}\right)^*\mathbf{J}^a_{1,1}\right]\overline{\mathbf{P}}_1.\end{aligned}$$

Now, using the fourth equation in Theorem 3.10.1, we find that

$$\mathbf{J}_{1,2}\overline{\mathbf{P}}_1 + \overline{\mathbf{J}}_{1,2}\mathbf{P}_1 + \left[\left(\mathbf{J}^a_{2,2}\right)^*\mathbf{J}^a_{1,2} - \left(\mathbf{J}^a_{2,1}\right)^*\mathbf{J}^a_{1,1}\right]\overline{\mathbf{P}}_1 = \mathbf{J}_{1,2}\overline{\mathbf{P}}_1 + \overline{\mathbf{J}}_{1,2}\mathbf{P}_1$$

proving our fourth identity. The second identity can be computed in a similar fashion as the first, and the third can be obtained from taking the adjoint of the fourth. □

Next, we shall compute an object called the *scattering matrix* for the overfare operator $\mathbf{O}_{2,1}$, which is related to the quadratic adjoint identities. From our point of view, the essential property of the scattering matrix is that it tells us exactly how the holomorphic and antiholomorphic parts of the overfare equation $\mathbf{O}_{2,1}u_2 = u_1$ interact. As mentioned previously, this is the analogue of the work of Schippers and Staubach [38], [41], who showed that in the Bergman space setting, the scattering matrix is unitary and consists of adjoint-Schiffer blocks. We briefly discuss their result in Section 4.5 in the context of the Dirichlet space. In the Smirnov setting, it turns out that the scattering matrix consists of adjoint Cauchy-blocks (up to a sign change).

Theorem 3.10.4 (Scattering Matrix of the Smirnov Overfare)**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Ahlfors-regular Jordan curve with complementary components Ω_1 and Ω_2, and let*

$$\mathbf{Q} : \mathcal{A}^{1/2}(\Omega_1) \oplus \mathcal{A}^{1/2}(\Omega_2) \to \overline{\mathcal{A}^{1/2}(\Omega_1)} \oplus \overline{\mathcal{A}^{1/2}(\Omega_2)}$$

be the linear operator with matrix representation

$$\mathbf{Q} = \begin{pmatrix} -\overline{\mathbf{J}}^h_{1,1} & \overline{\mathbf{J}}^h_{2,1} \\ -\overline{\mathbf{J}}^h_{1,2} & \overline{\mathbf{J}}^h_{2,2} \end{pmatrix}.$$

Then $\mathbf{Q}$ *is the scattering matrix of* $\mathbf{O}_{2,1}$ *in the sense that for all* $\alpha_d + \bar{\beta}_d \in \mathcal{A}^{1/2}_{\text{harm}}(\Omega_d)$ *satisfying* $\mathbf{O}_{2,1}\left(\alpha_2 + \bar{\beta}_2\right) = \alpha_1 + \bar{\beta}_1$*, we have*

$$\begin{pmatrix} \bar{\beta}_1 \\ \bar{\beta}_2 \end{pmatrix} = \begin{pmatrix} -\overline{\mathbf{J}}^h_{1,1} & \overline{\mathbf{J}}^h_{2,1} \\ -\overline{\mathbf{J}}^h_{1,2} & \overline{\mathbf{J}}^h_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Proof. Using equation (3.4.8), we have that

$$\overline{\mathbf{J}}_{1,2}\left(\alpha_1 + \overline{\beta_1}\right) = \overline{\mathbf{J}}_{2,2}\left(\alpha_2 + \overline{\beta_2}\right)$$

which, using the Cauchy integral formula, reduces to the identity

$$\overline{\mathbf{J}}_{1,2}\alpha_1 - \overline{\mathbf{J}}_{2,2}\alpha_2 = -\overline{\beta_2}.$$

A similar application of equation (3.4.8) yields

$$\overline{\mathbf{J}}_{1,1}\left(\alpha_1 + \overline{\beta_1}\right) = \overline{\mathbf{J}}_{2,1}\left(\alpha_2 + \overline{\beta_2}\right)$$

which reduces to

$$-\overline{\mathbf{J}}_{1,1}\alpha_1 + \overline{\mathbf{J}}_{2,1}\alpha_2 = \overline{\beta_1}.$$

□

To end the section, we remark on the implications of the quadratic adjoint identities as they relate to the scattering matrix. Define a linear operator $\mathbf{S} : \mathcal{A}^{1/2}(\Omega_1) \oplus \mathcal{A}^{1/2}(\Omega_2) \to \mathcal{A}^{1/2}(\Omega_1) \oplus \mathcal{A}^{1/2}(\Omega_2)$ by the matrix

$$\mathbf{S} = \begin{pmatrix} -i\mathbf{I}_1 & 0 \\ 0 & i\mathbf{I}_2 \end{pmatrix}.$$

Then by Theorem 3.10.1, it follows immediately that $\mathbf{Q}^*\overline{\mathbf{S}}\mathbf{Q} = \mathbf{S}$, where the adjoint of $\mathbf{Q}$ is taken with respect to the direct sum inner product. This identity suggests that the matrix $\mathbf{Q}$ may be a symplectic transformation in some sense. Our speculation here is further supported by the observation that $\mathbf{O}_{2,1}$ is a unitary operator, while the overfare for the Dirichlet space is skew-symplectic (we prove this in Section 4.5), and has a unitary scattering matrix. Thus, it is quite possible that the properties "interchange" between the settings (after accounting for the differing standard for curve orientation).

Problem 3.10.5. *Find a symplectic form* ω *which yields a symplectic interpretation of the matrix* $\mathbf{Q}$*, say of the form* $\omega(\mathbf{Q}u, \mathbf{Q}v) = \omega(u, v)$*, that informs the identity* $\mathbf{Q}^*\overline{\mathbf{S}}\mathbf{Q} = \mathbf{S}$.

4

Homogeneous Dirichlet Spaces of Harmonic Functions

4.1 Motivation and Analogies

The *Dirichlet space* is a one of the classical function spaces in complex analysis. It consists of the holomorphic functions having finite Dirichlet integral (or *energy*) on a fixed domain. This integral appears in Dirichlet's principle, which is a method for solving Laplace's equation with (continuous) boundary condition:

$$\begin{aligned} \text{find:} \quad & u : \mathrm{cl}(D) \to \mathbb{C} \\ \text{subject to:} \quad & \Delta u = 0 \text{ in } D, \\ & u = g \text{ on } \partial D. \end{aligned}$$

Riemann's approach to this problem involves finding the minimizer of the Dirichlet energy over all C^2 functions with boundary values g, which turns out to yield the (unique) solution to the problem.

The Dirichlet space finds applications in potential theory and geometric function theory, and is perhaps the simplest conformally invariant and quasi-invariant function space. By differentiating its elements, we obtain the closely related *Bergman space*, which is characterized as the space of holomorphic functions with finite L^2 norm, and has its own rich theory and applications in hyperbolic geometry and PDEs. In this chapter, we focus on the algebraic properties of operators defined on an extension of the classical Dirichlet space to harmonic functions, leveraging its relation to the Bergman space for certain key analytic results. More specifically, we consider analogues of those operators previously studied in the Smirnov space, while highlighting the similarities and differences between the two settings. This perspective culminates in an analogue of the Kerzman-Stein operator for the Dirichlet setting, which is currently absent from the literature.

In Section 4.2 and 4.3, we give a basic description of the Bergman and Dirichlet space on simply-connected domains in the sphere, followed by a short exposition on boundary values, overfare, and the significance of quasicircles for operator theory on Dirichlet space. Section 4.4 is used to discuss a "Cauchy-like" operator on the Dirichlet space of quasicircles. Our hesitance with the terminology is due to the fact that quasicircles are generally not rectifiable, and thus there is no way to directly formulate integrals over the boundary. This problem is circumvented by considering limits of Cauchy integrals along curves that approach the boundary. As we will see, many of the properties we hope for, such as a jump formula, hold under this definition for quasicircles, and thus simply calling it the Cauchy operator is justified. In each of these three sections, we follow the presentation of Schippers and Staubach [39]. Section 4.5 contains adjoint formulas for the Cauchy operator. Using these, we employ a similar trick as in Section 3.5 to show that overfare in this setting is a skew-symplectic transformation. In Section 4.6, we cover the definitions of the classical Grunsky operator as formulated on the Dirichlet space, Bergman space, and $\ell^2(\mathbb{C})$. Finally, in Section 4.7, we introduce a version of the Kerzman-Stein operator for the Dirichlet space, and prove that a "Kerzman-Stein-like" formula holds in the setting of quasicircles in $\overline{\mathbb{C}}$. Moreover, we show that this operator is in the Schatten p-class S_p if and only if the Grunsky operator is in S_p, which by the work of Jones [23], Shen [42], and Takhtajan and Teo [44], completely characterizes Schatten-ness of this operator in terms properties of the Riemann map for the underlying domain.

As many of the operators in the previous chapter have analogues in the Dirichlet and Bergman space settings, we retain much of the previously used notation (after appropriately redefining it) both for simplicity and to highlight the similarities and differences between the settings. We shall also carry over many of the conventions we set, unless otherwise specified.

4.2 Bergman and Dirichlet Spaces

To begin, we discuss a model for the Bergman space in terms of harmonic one-forms (see Example 2.2.1), which is justified by the simple fact that elements of the usual function space transform like one-forms under conformal reparameterization. There are many standard references on the Bergman space of functions, for example see Duren and Schuster [14], Zhu [46], or Bell [3].

Let $D \subset \overline{\mathbb{C}}$ be an open connected set. For all (smooth) one-forms $u = h dz + H d\bar{z}$ on D, define the $*$-operator by

$$*u := -ihdz + iHd\bar{z}$$

and define a function $\|\cdot\|_D$ (called the *Bergman norm*) taking values in $[0,\infty]$ by

$$\|u\|_D^2 = \frac{1}{2\pi}\iint_D u \wedge *\overline{u}. \tag{4.2.1}$$

In local coordinates, the integral on the right-hand side is given by

$$\|u\|_D^2 = \frac{1}{\pi}\iint_{D\setminus\{\infty\}} \left(|h(z)|^2 + |H(z)|^2\right)\, dA \tag{4.2.2}$$

where $dA = (d\bar{z} \wedge dz)/2i$ is the Lebesgue area measure. The point at infinity can be removed from the domain of integration in (4.2.2) since it neither affects convergence nor the value of the integral – for if $\infty \in D$, then there is a neighbourhood of ∞, say $B(\infty;R)$, whose closure is contained in D. Therefore, making the change of variables $w = 1/z$ on $B(\infty;R)$, we can write

$$\pi\|u\|_\Omega^2 = \iint_{D\setminus \mathrm{cl}(B(\infty;R))} \left(|h(z)|^2 + |H(z)|^2\right)\, dA + \iint_{cl(B(0;1/R))} \frac{\left(|h(1/w)|^2 + |H(1/w)|^2\right)}{|w|^2}\, dA.$$

Since both $h(1/w)/w^2$ and $H(1/w)/w^2$ are continuous on $\mathrm{cl}(B(0;1/R))$, the second integral on the right is always finite, and approaches 0 in the limit $R \to \infty$.

Definition 4.2.1. Let $D \subsetneq \overline{\mathbb{C}}$ be a proper simply-connected domain in the sphere. The Bergman space of harmonic one-forms on D is the Hilbert space with elements

$$\mathscr{A}_{\mathrm{harm}}(D) = \{u \in \Omega_{\mathrm{harm}}(D) : \|u\|_D < \infty\}$$

equipped with inner product

$$\langle u, v\rangle_D = \frac{1}{2\pi}\iint_D u \wedge *\overline{v}.$$

Clearly, the inner product $\langle\cdot,\cdot\rangle_D$ induces the norm defined in (4.2.1). With this structure, $\mathscr{A}_{\mathrm{harm}}(D)$ inherits the decomposition from the space of harmonic one-forms. Indeed, if we denote the subspaces containing only holomorphic and antiholomorphic one-forms (respectively) by

$$\mathscr{A}(D) := \mathscr{A}_{\text{harm}}(D) \cap \Omega^{1,0}(D),$$
$$\overline{\mathscr{A}(D)} := \mathscr{A}_{\text{harm}}(D) \cap \Omega^{0,1}(D)$$

then we have the orthogonal decomposition

$$\mathscr{A}_{\text{harm}}(D) = \mathscr{A}(D) \oplus \overline{\mathscr{A}(D)}.$$

Thus in local coordinates, we can express the inner product of $\mathscr{A}_{\text{harm}}(D)$ as

$$\left\langle hdz + \overline{H}d\bar{z},\, gdz + \overline{G}d\bar{z} \right\rangle_D = \frac{1}{\pi} \iint\limits_{D\setminus\{\infty\}} \left[h(z)\overline{g(z)} + \overline{H(z)}G(z) \right] dA.$$

When modelled in terms of functions, the Bergman space is a "larger" space than the Smirnov space for sufficiently smooth domains (See page 26 in Bell [3]). For example, if $D' \subset \mathbb{C}$ is a bounded C^∞-smooth domain, then there exists a constant $M > 0$ such that for all $u \in E^2(D')$, we have that

$$\iint\limits_{D'} |u(z)|^2 \, dA \leqslant M \|u\|^2_{E^2(D')}.$$

Next, we turn to the definition of the Dirichlet space on simply-connected domains. For a full introduction to the Dirichlet space on the unit disk, see El-Fallah et al. [16]. Recall that a C^2 function u is *harmonic* if

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

On simply-connected domains, this condition is met if and only if there exists holomorphic and antiholomorphic functions h and $\overline{H}$ (respectfully) such that $u = h + \overline{H}$. When in this form, we note the following special properties of u concerning the exterior derivative d:

(a) $du = \frac{\partial h}{\partial z} dz + \frac{\partial \overline{H}}{\partial \bar{z}} d\bar{z}$.

(b) $\overline{d}u = du$.

For non simply-connected domains, a holomorphic-antiholomorphic decomposition may not exist for general u (see Section 14 of Bell [3], for example).

On a simply-connected domain $D \subset \overline{\mathbb{C}}$, the *Dirichlet energy* of a C^∞-smooth function

$u : D \to \mathbb{C}$ is defined to be the quantity

$$\mathfrak{D}(u) = \|du\|_D^2. \tag{4.2.3}$$

Written in local coordinates, Dirichlet energy can be expressed as

$$\mathfrak{D}(u) = \frac{1}{\pi} \iint\limits_{D\setminus\{\infty\}} \left(\left| \frac{\partial h(z)}{\partial z} \right|^2 + \left| \frac{\partial \overline{H}(z)}{\partial \bar{z}} \right|^2 \right) dA. \tag{4.2.4}$$

Whenever u is holomorphic, we see from this representation that the Dirichlet energy of u measures the area of the image of u, counting multiplicity.

Definition 4.2.2. Let $D \subsetneq \overline{\mathbb{C}}$ be a proper simply-connected domain in the sphere. The Dirichlet space of harmonic functions on D is the set

$$\mathfrak{D}_{\text{harm}}(D) = \{u : D \to \mathbb{C} \text{ harmonic} : \mathfrak{D}(u) < \infty\}.$$

We denote the corresponding holomorphic and antiholomorphic Dirichlet spaces (respectfully) by $\mathcal{D}(D)$ and $\overline{\mathcal{D}(D)}$. Turning the set $\mathcal{D}_{\text{harm}}(D)$ into a Hilbert space with a unique holomorphic-antiholomorphic decomposition is more subtle than in previous settings. The germ of the problem is that nonzero constant functions have zero Dirichlet energy (which follows directly from the definition in (4.2.3)), and so the map $u \mapsto \sqrt{\mathfrak{D}(u)}$ only defines a semi-norm on $\mathcal{D}_{\text{harm}}(D)$. Moreover, constants belong to both $\mathcal{D}(D)$ and $\overline{\mathcal{D}(D)}$, meaning that $\mathcal{D}_{\text{harm}}(D) = \mathcal{D}(D) + \overline{\mathcal{D}(D)}$ is not a unique decomposition. The fix we shall implement is to mod out by constants.

Definition 4.2.3. Let $D \subsetneq \overline{\mathbb{C}}$ be a proper simply-connected domain in the sphere. The homogeneous Dirichlet space of harmonic functions on D is the set of equivalence classes

$$\dot{\mathcal{D}}_{\text{harm}}(D) = \mathcal{D}_{\text{harm}}(D)/\sim \tag{4.2.5}$$

where $u_1 \sim u_2$ if and only if $u_1 - u_2 : D \to \mathbb{C}$ is constant.

Remark 4.2.4. When it is clear from context, we shall refer to $\dot{\mathcal{D}}_{\text{harm}}(D)$ simply as the Dirichlet space of D.

On this set, the mapping $u \mapsto \|u\|_{\dot{\mathfrak{D}}_{\text{harm}}(D)} := \sqrt{\mathfrak{D}(u)}$ is a norm induced by the inner product

$$\langle u, v\rangle_{\dot{\mathcal{D}}_{\text{harm}}(D)} = \langle du, dv\rangle_{\mathscr{A}_{\text{harm}}(D)}. \tag{4.2.6}$$

When it is clear from context, we drop the reference to the spaces in the subscript, and simply write $\|u\|_D$ or $\langle du, dv\rangle_D$. We now have the desired orthogonal decomposition

$$\dot{\mathcal{D}}_{\text{harm}}(D) = \dot{\mathcal{D}}(D) \oplus \overline{\dot{\mathcal{D}}(D)} \tag{4.2.7}$$

and we denote the corresponding orthogonal projection onto the holomorphic subspace by

$$\begin{aligned} \dot{\mathbf{P}} : \dot{\mathcal{D}}_{\text{harm}}(D) &\to \dot{\mathcal{D}}(D), \\ u = h + \overline{H} &\mapsto h. \end{aligned}$$

Similarly, we have the associated orthogonal projection $\overline{\dot{\mathbf{P}}}u = \left(\mathbf{I} - \dot{\mathbf{P}}\right)u = \overline{H}$. Note that the choice to mod out by constants is not the only option to turn $\mathcal{D}_{\text{harm}}(D)$ into a Hilbert space (see Schippers and Staubach [39]).

By construction, the exterior derivative

$$d : \dot{\mathcal{D}}_{\text{harm}}(D) \to \mathscr{A}_{\text{harm}}(D) \tag{4.2.8}$$

is a unitary mapping of the homogeneous Dirichlet space of functions to the Bergman space of one-forms. Moreover, if we restrict to the orthogonal subspaces, then we have that

$$d\left(\dot{\mathcal{D}}(D)\right) = \mathscr{A}(D), \quad d\left(\overline{\dot{\mathcal{D}}(D)}\right) = \overline{\mathscr{A}(D)}.$$

While our main focus in the upcoming sections will be on deriving analogies between the Dirichlet and Smirnov settings, the identification (4.2.8) will be used frequently to translate analytic results from the Bergman space.

Example 4.2.5. In the case of the unit disk $D = \mathbb{D}$, we have relatively simple descriptions of the Dirichlet and Bergman spaces (compare with Example 3.2.5). By integrating in polar coordinates, one can check that the set $\{\sqrt{n}z^{n-1}dz : n \geqslant 1\}$ forms an orthonormal basis for the holomorphic Bergman space $\mathscr{A}(\mathbb{D})$. Computing as we did in Example 3.2.5, it follows that

$$h(z)dz = \sum_{n=0}^{\infty} a_n z^n dz \in \mathscr{A}(\mathbb{D}) \iff \sum_{n=1}^{\infty} \frac{|a_{n-1}|^2}{n} < \infty.$$

Furthermore, if we denote the space of square-integrable sequences by

$$\ell^2(\mathbb{C}) = \left\{ (\lambda_n)_{n=1}^{\infty} \subset \mathbb{C} : \sum_{n=1}^{\infty} |\lambda_n|^2 < \infty \right\}$$

and equip it with the standard inner product $\langle \lambda, \lambda' \rangle_{\ell^2} = \sum_{n=1}^{\infty} \lambda_n \overline{\lambda_n'}$, then the mapping

$\ell^2(\mathbb{C}) \to \mathscr{A}(\mathbb{D})$ given by

$$(\lambda_n)_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} \lambda_n \sqrt{n} z^{n-1} dz \tag{4.2.9}$$

is unitary. Similarly, the set $\{z^n/\sqrt{n} : n \geqslant 1\}$ forms an orthonormal basis for the holomorphic Dirichlet space $\dot{\mathcal{D}}(\mathbb{D})$, and

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \in \dot{\mathcal{D}}(\mathbb{D}) \Longleftrightarrow \sum_{n=1}^{\infty} n|a_n|^2 < \infty.$$

Moreover, the mapping $\ell^2(\mathbb{C}) \to \dot{\mathcal{D}}(\mathbb{D})$ given by

$$(\lambda_n)_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} \frac{\lambda_k}{\sqrt{n}} z^n \tag{4.2.10}$$

is unitary. By conjugating, we can also produce similar mappings $\ell^2(\mathbb{C}) \to \overline{\mathscr{A}(\mathbb{D})}$ and $\ell^2(\mathbb{C}) \to \overline{\dot{\mathcal{D}}(\mathbb{D})}$.

For $f \in \mathrm{Conf}(D_2, D_1)$, the pull-back of the Dirichlet space by f is the unitary mapping given by

$$\begin{aligned} f^* : \dot{\mathcal{D}}_{\text{harm}}(D_1) &\to \dot{\mathcal{D}}_{\text{harm}}(D_2), \\ u &\mapsto u \circ f. \end{aligned}$$

In particular, the Dirichlet space is conformally invariant. Similarly, the pull-back of the Bergman space of holomorphic one-forms by f is the unitary mapping given by

$$\begin{aligned} f^* : \mathscr{A}(D_1) &\to \mathscr{A}(D_2), \\ h(z)dz &\mapsto [h \circ f(z)] f'(z) dz. \end{aligned}$$

Pull-back of the antiholomorphic subspace is defined similarly.

Example 4.2.6. Let $\iota(z) = 1/z$ for all $z \in \overline{\mathbb{C}}$. Then pull-back of $\mathscr{A}(\mathbb{D}^-)$ by ι is the map $\iota^* : \mathscr{A}(\mathbb{D}^-) \to \mathscr{A}(\mathbb{D})$ defined for all $h(z)dz \in \mathscr{A}(\mathbb{D}^-)$ by

$$\iota^*(h(z)dz) = -\frac{1}{z^2} h\left(\frac{1}{z}\right) dz.$$

In particular, since the coefficient on the right-hand side must be holomorphic in $\mathbb{D}$, we see that $h(z)$ must have a zero of order at least two at ∞.

4.3 Quasicircles and Boundary Values

In this section, we give a brief overview of quasicircles and some of their defining properties in connection to the harmonic Dirichlet space. Quasicircles are Jordan curves that are, in some sense, the "right" class of curves for studying the Dirichlet space, as in many cases they precisely characterize behaviour of the operators defined on them. This is somewhat similar to the relationship between Ahlfors-regularity and the generalizations of the Hardy space discussed in Chapter 3. In this case however, there is a much stronger case to be made for quasicircles and Dirichlet space. For brevity, our goal is to state some fundamental properties in connection to the boundary space of $\mathcal{D}_{\text{harm}}(\Omega)$ so that we have a well-defined notion of overfare at our disposal. The work of Gehring and Hag [20] contains extensive background on quasicircles and their pervasiveness in geometric function theory. One may also wish to consult the book of Pommerenke [33] for an introduction to these topics. We start with the *bounding turning* definition for quasicircles in the plane.

Definition 4.3.1. A bounded Jordan curve $\Gamma \subset \mathbb{C}$ is called a quasicircle if there exists $M > 0$ such that for each pair of points $\zeta, \tau \in \Gamma$, if $\Gamma_{\zeta,\tau}$ denotes the smaller of the two open curves in $\Gamma \backslash \{\zeta, \tau\}$ in terms of set diameter, then

$$\operatorname{diam} \Gamma_{\zeta,\tau} \leqslant M|\zeta - \tau|.$$

If Γ is a quasicircle, then we call the complementary components Ω_1 and Ω_2 quasidisks.

Remark 4.3.2. Using this characterization, it can be shown that a piecewise C^1 Jordan curve is a quasicircle if and only if it contains no cusps (see Chapter 5 in Pommerenke [33], for example).

Remark 4.3.3. The above definition can be extended to the sphere as follows: A Jordan curve $\Gamma \subset \overline{\mathbb{C}}$ passing through ∞ is a *quasicircle* if there exists a Möbius transformation T such that $T(\Gamma) \subset \mathbb{C}$ is a quasicircle.

There are numerous equivalent ways of defining quasicircles, with entire expository works dedicated to equivalent definitions (for example, see Gehring [19]). One such example is that a Jordan curve Γ in the sphere is a quasicircle if and only if there is a quasiconformal map $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with the property that $\Gamma = \Phi(\mathbb{S}^1)$. Under this formulation, if we think of conformal mappings as the transformations that locally preserve angle, then quasiconformal mappings allow for bounded distortion of angles.

Before moving on to the boundary values of the Dirichlet space, let us touch on the relevant class of negligible subsets of the boundary which do not affect the specification of elements in $\dot{\mathcal{D}}_{\text{harm}}(\Omega)$. In the Smirnov space setting, recall that these are the subsets of Γ with Lebesgue measure zero. For the Dirichlet space, the correct class are the *null sets*, which are "smaller" than measure zero. Formally, a Borel subset $I \subset \Gamma$ is said to be *null* with respect to Ω if there exists a map $f \in \text{Conf}(\mathbb{D}, \Omega)$ such that the set $f^{-1}(I)$ has logarithmic capacity zero, and a property is said to hold *quasi-everywhere* if it holds everywhere except possibly on a subset of logarithmic capacity zero (see Conway [10] for details on logarithmic capacity and its applications in potential theory). Note that Carathéodory's Theorem (Theorem 2.1.3) is used implicitly in this definition to extend f to Γ. The need to specify a complementary component in the definition of null sets is a subtly arising from the fact that the homeomorphic extension of the mapping function does not necessarily preserve measure zero subsets of $\mathbb{S}^1$. Thus for general Jordan curves, it need not be the case that a null set with respect to one complementary domain be null with respect to the other. In the case of quasicircles however, the null sets with respect to both components are identical, which derives from the fact that quasiconformal maps of the sphere preserve subsets of logarithmic capacity zero. This consequence will be recorded again in the upcoming *Transmission Theorem*.

Next, we review results regarding the boundary values of the Dirichlet space. Note that by Example 3.2.5 and 4.2.5, it follows that $\mathcal{D}(\mathbb{D}) \subset E^2(\mathbb{D})$. Thus, since elements of the Smirnov space of functions on the disk have nontangential limits to the boundary almost everywhere, the elements of the Dirichlet space of arbitrary Jordan domains automatically have conformally nontangential limits to the boundary (defined in Section 3.3). With the greater analytic requirements for functions in the Dirichlet space, a stronger result exists:

Theorem 4.3.4. *Let $\Omega \subset \overline{\mathbb{C}}$ be a Jordan domain with boundary Γ. Then every function $u \in \mathcal{D}_{harm}(\Omega)$ has conformally nontangential limits at every point in Γ except possibly on a null set of with respect to Ω. Moreover, if the boundary values of u vanish outside of a null set with respect to Ω, then u is identically zero.*

Proof. See Theorem 2.32 in Schippers and Staubach [39]. □

Now, let $\mathcal{B}(\Gamma, \Omega)$ denote the set of equivalence classes consisting of functions that agree up to a null set of Γ with respect to Ω. By the above theorem, each class may contain at most one boundary function associated to an element in $\mathcal{D}_{\text{harm}}(\Omega)$. The *Osborn space* of Γ with respect to Ω, denoted $\mathscr{H}(\Gamma, \Omega)$, is the set of functions $u_\Gamma \in \mathcal{B}(\Gamma, \Omega)$ that are the boundary values of elements $u \in \mathcal{D}_{\text{harm}}(\Omega)$. The corresponding map identifying the two spaces is called the *trace operator*, and is denoted by

$$\mathbf{b} : \mathcal{D}_{\text{harm}}(\Omega) \to \mathscr{H}(\Gamma, \Omega)$$

with the inverse $\mathbf{b}^{-1} : \mathscr{H}(\Gamma,\Omega) \to \mathcal{D}_{\text{harm}}(\Omega)$ called the *extension operator*. Similar to the situation with null sets, the Osborn spaces $\mathscr{H}(\Gamma,\Omega_1)$ and $\mathscr{H}(\Gamma,\Omega_2)$ may not agree in general. The next theorem rectifies the situation.

Theorem 4.3.5 (Transmission Theorem, Schippers and Staubach [39])**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a Jordan curve with complementary components Ω_1 and Ω_2, and fix distinct $d,c \in \{1,2\}$. Then* (a) *and* (b) *below are equivalent.*

(a) *Γ is a quasicircle.*

(b) (i) *If $I \subset \Gamma$ is null with respect to Ω_d, then it is also null with respect to Ω_c,*

(ii) $\mathscr{H}(\Gamma,\Omega_d) \subset \mathscr{H}(\Gamma,\Omega_c)$,

(iii) $\mathbf{b}_c^{-1}\mathbf{b}_d : \mathcal{D}_{harm}(\Omega_d) \to \mathcal{D}_{harm}(\Omega_c)$ *is bounded with respect to Dirichlet energy.*

With this, we are equipped to define a notion of *overfare* for the Dirichlet space. For our purposes, we only consider the homogeneous version.

Definition 4.3.6. Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary domains Ω_1 and Ω_2. For distinct $d,c \in \{1,2\}$, the bounded operator $\dot{\mathbf{O}}_{d,c} = \dot{\mathbf{b}}_c^{-1}\dot{\mathbf{b}}_d : \dot{\mathcal{D}}_{\text{harm}}(\Omega_d) \to \dot{\mathcal{D}}_{\text{harm}}(\Omega_c)$ is called the homogenous d,c-overfare.

From definition, it is clear that

$$\left(\dot{\mathbf{O}}_{d,c}\right)^{-1} = \dot{\mathbf{O}}_{c,d}$$

and so we may consider both the d,c and c,d-operators together as the (homogeneous) overfare for Γ. Now, since $\dot{\mathbf{b}}_d$ is defined in terms of conformally nontangential limits of functions, it follows that that $\overline{\dot{\mathbf{b}}}_d = \dot{\mathbf{b}}_d$, and so

$$\overline{\dot{\mathbf{O}}}_{d,c} = \dot{\mathbf{O}}_{d,c}. \tag{4.3.1}$$

4.4 The Schiffer and Cauchy Operators

Now that we are equipped with the relevant structural properties of the Dirichlet space, we are ready to define the other main operator for our study — the Cauchy operator $\dot{\mathbf{J}}_{d,c}$. Barring some of the technicalities, $\dot{\mathbf{J}}_{d,c}$ is the analogue of the Cauchy integral operator studied previous in Chapter 3, and so we shall come to have many analogous theorems and identities at our disposal. To obtain these, we translate properties of the *Schiffer operator* on Bergman space, which itself is a well-studied singular integral operator.

Earlier, the complementary components of a Jordan curve $\Gamma \subset \overline{\mathbb{C}}$ were denoted Ω_1 and Ω_2, with the convention that $\infty \in \Omega_2$ whenever Γ is bounded. While the distinction for bounded curves was necessary for our presentation of the Smirnov setting, the corresponding operators in the Dirichlet setting have all been formulated for general Jordan curves in $\overline{\mathbb{C}}$. Therefore, the subscripts on the complementary domains now represent an arbitrary, but definitive choice of labelling.

Definition 4.4.1. Let $\Gamma \subset \overline{\mathbb{C}}$ be a Jordan curve with complementary components Ω_1 and Ω_2. For each $d, c \in \{1,2\}$, the d,c-Schiffer operator $\mathbf{T}_{d,c}$ is defined for all $\overline{\beta} = \overline{H} d\bar{z} \in \overline{\mathscr{A}(\Omega_d)}$ by

$$\left(\mathbf{T}_{d,c}\overline{\beta}\right)(z) = \text{P.V.}\,\frac{1}{\pi} \iint_{w\in\Omega_d} \frac{\overline{H(w)}}{(w-z)^2}\, dA \cdot dz, \qquad z \in \Omega_c.$$

Remark 4.4.2. In the case that $d \neq c$, we can immediately remove the principal value sign. When $d = c$, the integral kernel of $\mathbf{T}_{d,c}$ must first be desingularized using the *Schiffer kernel*.

While the Schiffer operator is interesting in its own right, its inclusion in this work is for extracting properties about the Cauchy operator, which we define now.

Definition 4.4.3. Let $\Gamma \subset \overline{\mathbb{C}}$ be a bounded Jordan curve with complementary components Ω_1 and Ω_2. Fix $d, c \in \{1,2\}$, and orient Γ positively with respect to Ω_d. The d,c-Cauchy operator $\mathbf{J}^q_{d,c}$ with normalization at $q \notin \Gamma$ is defined for all $u \in \mathcal{D}_{\text{harm}}(\Omega_d)$ by

$$\left(\mathbf{J}^q_{d,c}u\right)(z) = \lim_{r\nearrow 1} \frac{1}{2\pi i} \int_{\zeta\in\Gamma_{p,r}} u(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - q}\right) d\zeta, \qquad z \in \Omega_c$$

where $\Gamma_{p,r} \subset \Omega_d$ is the level curve defined in equation (3.2.4), and shares the orientation of Γ.

Remark 4.4.4. It can be shown that the limit is independent of the choice of normalization p associated to $\Gamma_{p,r}$.

In the case that Γ is a bounded curve and the normalization is made at $q = \infty$, then

$$\left(\mathbf{J}^q_{d,c}u\right)(z) = \lim_{r\nearrow 1} \frac{1}{2\pi i} \int_{\zeta\in\Gamma_{p,r}} \frac{u(\zeta)}{\zeta - z}\, d\zeta$$

which is nearly the "standard" Cauchy integral operator. Notice that this kernel function has a pole at $\zeta = z$ and at $\zeta = \infty$ (the latter case becomes more apparent by making the change of variables $\tau = 1/\zeta$ in the above integral). The conventional choice of putting the pole at $q = \infty$ is arbitrary, and in fact conceals the Möbius invariance of the Cauchy kernel, which we demonstrate for the q-normalized kernel now. For any Möbius transformation M and $q \notin \Gamma$, a calculation shows that

$$M'(\zeta)\left(\frac{1}{M(\zeta)-M(z)} - \frac{1}{M(\zeta)-M(q)}\right) = \frac{1}{\zeta - z} - \frac{1}{\zeta - q}$$

and so using this in conjunction with conformal invariance of the Dirichlet space and a change of variables in the integral expression of $\mathbf{J}^q_{d,c}u$, we have that

$$\mathbf{J}^q_{d,c}u = \left(\mathbf{J}^{M(q)}_{M(\Omega_d),M(\Omega_c)} u \circ M^{-1}\right) \circ M.$$

With this property, we may now extend the definition of the Cauchy operator to curves Γ containing ∞ by conjugating by a transformation M taking Γ to the finite plane.

As defined, $\mathbf{J}^q_{d,c}$ is not quite the integral operator that we might expect. In the Smirnov space setting, the Cauchy operator was also defined using a limit of Cauchy integrals on level curves, however we restricted ourselves to a class of rectifiable domains, on which the limit indeed coincides with the standard Cauchy integral. We were justified in our restriction to this class due to its characterizing properties of the Hardy space (Section 3.1) and the Cauchy integral operator (Theorem 3.4.4). Now, in the Dirichlet space setting, we are most interested in quasicircles, which are not rectifiable in general. Thus, we do not have any way to recover a sensible interpretation of $\mathbf{J}^q_{d,c}$ as a Lebesgue contour integral over the boundary. We will see shortly that $\mathbf{J}^q_{d,c}$ (and its homogeneous counterpart) behaves much in the same way that the Cauchy integral operator does, and so we are justified in our naming convention. The problem of finding an appropriate principal value interpretation for $\mathbf{J}^q_{d,c}$ on the boundary of quasicircles which produces a Plemelj-Sokhotski-like formula is posed and discussed in Schippers and Staubach [39].

Since we are most interested in the homogeneous Dirichlet space, we consider the corresponding "homogeneous" Cauchy operators, denoted by

$$\dot{\mathbf{J}}_{d,c} : \dot{\mathcal{D}}_{\text{harm}}(\Omega_d) \to \dot{\mathcal{D}}(\Omega_c), \quad d,c \in \{1,2\}.$$

We take for granted the fact that these are well-defined bounded operators mapping into the homogeneous holomorphic Dirichlet space. Note that since we have modded out by constants in z in both the domain and image spaces, the operator is unchanged by the choice of normalization at $q \notin \Gamma$, justifying its removal from the notation. Henceforth, we implicitly assume that a fixed choice of normalization has been made.

For the remainder of this section, we record some of the known properties and identities surrounding these operators. As we have already covered analogous results in the Smirnov case, we shall keep our exposition here to a minimum. Denote the *Wirtinger derivatives* on the Riemann sphere by ∂ and $\overline{\partial}$. In local $z = x + iy$ coordinates, these are given by

$$\partial := \frac{\partial}{\partial z} dz = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) dz,$$
$$\overline{\partial} := \frac{\partial}{\partial \bar{z}} d\bar{z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) d\bar{z}.$$

If h is a holomorphic function, then by the Cauchy-Riemann equations we have $\partial h = dh$. In fact, on simply-connected domains, for each harmonic function $u = h + \overline{H}$ we have that $du = \partial u + \overline{\partial} u = \partial h + \overline{\partial}\overline{H}$. In particular, when restricting to the standard orthogonal subspaces of $\dot{\mathcal{D}}_{\text{harm}}(\Omega)$, the Wirtinger derivatives restrict to unitary maps:

$$\partial^h := d|_{\dot{\mathcal{D}}(\Omega)} : \dot{\mathcal{D}}(\Omega) \to \mathscr{A}(\Omega),$$
$$\overline{\partial}^a := d|_{\overline{\dot{\mathcal{D}}(\Omega)}} : \overline{\dot{\mathcal{D}}(\Omega)} \to \overline{\mathscr{A}(\Omega)}.$$

The first theorem we present directly relates the Cauchy operator to the Schiffer operator via the Wirtinger derivatives.

Theorem 4.4.5. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a Jordan curve with complementary components Ω_1 and Ω_2, and let $d, c \in \{1,2\}$ be distinct. Then*

$$\partial \mathbf{J}^q_{d,c} = \begin{cases} \partial + \mathbf{T}_{d,d}\overline{\partial}, & d = c \\ \mathbf{T}_{d,c}\overline{\partial}, & d \neq c. \end{cases}$$

Proof. See Theorem 5.9 in Schippers and Staubach [39]. □

Next, we have a precise characterization for when the restricted Cauchy operator $\dot{\mathbf{J}}^a_{d,c}$ has a (bounded) inverse.

Theorem 4.4.6 (Napalkov Jr. and Yulmukhametov [31], [32])**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a Jordan curve with complementary components Ω_1 and Ω_2, and let $d, c \in \{1,2\}$ be distinct. Then the following are equivalent:*

(a) *Γ is a quasicircle.*

(b) *$\dot{\mathbf{J}}^a_{d,c} : \overline{\dot{\mathcal{D}}(\Omega_d)} \to \dot{\mathcal{D}}(\Omega_c)$ is a bounded isomorphism.*

(c) *$\mathbf{T}_{d,c} : \overline{\mathscr{A}(\Omega_d)} \to \mathscr{A}(\Omega_c)$ is a bounded isomorphism.*

Proof. This was first proven by Napalkov Jr. and Yulmukhametov [31], [32]. A different proof can be found in the survey paper of Schippers and Staubach [39]. □

The next identity says that for quasicircles, the Cauchy operators agree no matter the side of the curve the limit is taken (up to a change of sign to account for a change in the orientation of the boundary curve).

Theorem 4.4.7. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2. Then for distinct $d, c \in \{1,2\}$, we have*

$$\dot{\mathbf{J}}_{d,d} = -\dot{\mathbf{J}}_{c,d}\dot{\mathbf{O}}_{d,c}.$$

Proof. See Theorem 5.11 in Schippers and Staubach [39]. □

Remark 4.4.8. The proof of the corresponding result (3.4.8) in the Smirnov case comes from the simple observation that the overfare identifies elements sharing boundary values, and the (half-order) Cauchy operator can be identified with the standard Cauchy integral over the shared boundary. One could argue in a similar manner here, provided we impose further regularity on Γ. At the generality of quasicircles, the Cauchy operator has no interpretation on the boundary, and thus this result is not obvious. One of the key steps towards the proof is the agreement of the Osborn spaces in the Transmission Theorem (Theorem 4.3.5). In Section A.3, we discuss the relationship between this identity and the (upcoming) jump formula.

The next two theorems are reformulations of classical results.

Theorem 4.4.9 (Cauchy Integral Formula)**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a Jordan curve with complementary components Ω_1 and Ω_2, and let $d, c \in \{1,2\}$. Then for all $h \in \dot{\mathcal{D}}(\Omega_d)$, we have*

$$\dot{\mathbf{J}}_{d,c}h = \begin{cases} h, & d = c \\ 0, & d \neq c. \end{cases}$$

Theorem 4.4.10 (Homogeneous Jump Formula)**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2, and let $d, c \in \{1,2\}$ be distinct. Then the following identity holds on all of $\dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_d)$:*

$$\dot{\mathbf{J}}_{d,d} - \dot{\mathbf{O}}_{c,d}\dot{\mathbf{J}}_{d,c} = \mathbf{I}.$$

Proof. See Theorem 5.16 in Schippers and Staubach [39]. □

Remark 4.4.11. By the same argument used in the Smirnov case (see equation (3.4.5)), it follows that

$$\left(\dot{\mathbf{J}}^a_{d,c}\right)^{-1} = -\overline{\dot{\mathbf{P}}}_d \dot{\mathbf{O}}^h_{c,d}.$$

4.5 Adjoint Formulas and the Scattering Matrix

Continuing with the operators defined thus far, we now consider their adjoints. First, we record the adjoint formulas on quasidisks for the homogeneous Cauchy operators using those for the Schiffer operators. With this, we deduce the explicit adjoint of the homogeneous overfare operator using the jump formula, and provide a symplectic interpretation for it. Finally, we record the scattering matrix of the homogeneous overfare using the corresponding results of Schippers and Staubach [38], [41] in the Bergman setting (compare with Section 3.10).

Theorem 4.5.1. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2. Then for $d, c \in \{1, 2\}$, we have*

$$\left(\mathbf{T}_{d,c}\right)^* = \overline{\mathbf{T}}_{c,d}.$$

Proof. This is a special case of Theorem 3.12 in Schippers and Staubach [40]. □

With this result, one can easily deduce adjoint formulas for $\dot{\mathbf{J}}_{d,c}$. To state these, we first define the $*$-operator for the Dirichlet space. For $u = h + \overline{H} \in \dot{\mathcal{D}}_{\text{harm}}(\Omega)$, let

$$*u := -ih + i\overline{H}.$$

Note by linearity of the exterior derivative, we have that $d(*u) = *(du)$.

Theorem 4.5.2. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2. Then for $d, c \in \{1, 2\}$, we have*

$$\left(\dot{\mathbf{J}}_{d,c}\right)^* = \begin{cases} \mathbf{I} + i\overline{\dot{\mathbf{J}}}_{d,d} *_d, & d = c \\ \overline{\dot{\mathbf{J}}}_{c,d}, & d \neq c. \end{cases}$$

Remark 4.5.3. Alternatively, we could simplify the above expression using the Cauchy integral formula to obtain:

$$\left(\dot{\mathbf{J}}_{d,c}\right)^* = \begin{cases} \dot{\mathbf{P}}_d + \overline{\dot{\mathbf{J}}}_{d,d} \dot{\mathbf{P}}_d, & d = c \\ \overline{\dot{\mathbf{J}}}_{c,d} \overline{\dot{\mathbf{P}}}_c, & d \neq c. \end{cases}$$

Proof. By Theorem 4.4.5 (a), we may write

$$\dot{\mathbf{J}}^a_{d,c} = (\partial^h)^{-1}\mathbf{T}_{d,c}\overline{\partial}^a.$$

Taking the adjoint and using the result of Theorem 4.5.1, we then have

$$\left(\dot{\mathbf{J}}^a_{d,c}\right)^* = \left(\overline{\partial}^a\right)^*(\mathbf{T}_{d,c})^*\left[(\partial^h)^{-1}\right]^* = \left(\overline{\partial}^a\right)^{-1}\overline{\mathbf{T}}_{d,c}\partial^h = \overline{\dot{\mathbf{J}}}^h_{d,c}.$$

To expand this formula to the whole space, we consider the cases separately. When $d = c$, the Cauchy integral formula says that $\overline{\dot{\mathbf{J}}}^a_{d,d}\overline{\dot{\mathbf{P}}}_d = \overline{\dot{\mathbf{P}}}_d$, and so extending $\dot{\mathbf{J}}^a_{d,c}$ by zero allows us to write

$$\begin{aligned}\left(\dot{\mathbf{J}}_{d,d}\right)^* = \dot{\mathbf{P}}_d + \left(\dot{\mathbf{J}}^a_{d,d}\right)^*\dot{\mathbf{P}}_d &= \dot{\mathbf{P}}_d + \overline{\dot{\mathbf{P}}}_d + \overline{\dot{\mathbf{J}}}_{d,d}\left(\dot{\mathbf{P}}_d - \overline{\dot{\mathbf{P}}}_d\right)\\ &= \mathbf{I} + i\overline{\dot{\mathbf{J}}}_{d,d} *_d .\end{aligned}$$

Similarly, when $d \neq c$ the Cauchy integral formula implies that $\dot{\mathbf{J}}^h_{d,c} = 0$ and $\overline{\dot{\mathbf{J}}}^a_{c,d} = 0$, thus

$$\left(\dot{\mathbf{J}}_{d,c}\right)^* = \left(\dot{\mathbf{J}}^a_{d,c}\right)^*\dot{\mathbf{P}}_c = \overline{\dot{\mathbf{J}}}^h_{c,d}\dot{\mathbf{P}}_c = \overline{\dot{\mathbf{J}}}_{c,d}$$

completing the proof. □

Next, we compute the adjoint of $\dot{\mathbf{O}}_{d,c}$ using a similar algebraic method as in Section 3.5 for the Smirnov space overfare.

Theorem 4.5.4. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2, and let $d, c \in \{1,2\}$ be distinct. Then*

$$\left(\dot{\mathbf{O}}_{d,c}\right)^* = *_d\dot{\mathbf{O}}_{c,d} *_c .$$

Proof. Since $\dot{\mathbf{J}}^a_{c,d} : \overline{\dot{\mathcal{D}}(\Omega_c)} \to \dot{\mathcal{D}}(\Omega_d)$ is an isomorphism, we can rearrange the jump formula to obtain

$$\dot{\mathbf{O}}^h_{d,c} = \left(\dot{\mathbf{J}}^a_{c,c} - \mathbf{I}^a\right)\left(\dot{\mathbf{J}}^a_{c,d}\right)^{-1}. \tag{4.5.1}$$

Now, equation (4.3.1) implies that $\overline{\dot{\mathbf{O}}_{2,1}\dot{\mathbf{P}}_2} = \dot{\mathbf{O}}_{2,1}\overline{\dot{\mathbf{P}}}_2$, and so the conjugation of (4.5.1) is

$$\dot{\mathbf{O}}^a_{d,c} = \left(\overline{\dot{\mathbf{J}}}^h_{c,c} - \mathbf{I}^h\right)\left(\overline{\dot{\mathbf{J}}}^h_{c,d}\right)^{-1}. \tag{4.5.2}$$

Putting these two expressions together yields the following integral equation representation of overfare:

$$\dot{\mathbf{O}}_{d,c} = \left(\dot{\mathbf{J}}^a_{c,c} - \mathbf{I}^a\right)\left(\dot{\mathbf{J}}^a_{c,d}\right)^{-1}\dot{\mathbf{P}}_d + \left(\overline{\dot{\mathbf{J}}}^h_{c,c} - \mathbf{I}^h\right)\left(\overline{\dot{\mathbf{J}}}^h_{c,d}\right)^{-1}\overline{\dot{\mathbf{P}}}_d. \qquad (4.5.3)$$

Thus, the adjoint of overfare can be expressed as

$$\left(\dot{\mathbf{O}}_{d,c}\right)^* = \left(\overline{\dot{\mathbf{J}}}^h_{d,c}\right)^{-1}\left(\overline{\dot{\mathbf{J}}}^h_{c,c}\dot{\mathbf{P}}_c - \overline{\dot{\mathbf{P}}}_c\right) + \left(\dot{\mathbf{J}}^a_{d,c}\right)^{-1}\left(\dot{\mathbf{J}}^a_{c,c}\overline{\dot{\mathbf{P}}}_c - \dot{\mathbf{P}}_c\right). \qquad (4.5.4)$$

Now, by the Cauchy integral formula and Theorem 4.4.7, we may write

$$\dot{\mathbf{J}}^a_{c,c} = -\dot{\mathbf{J}}_{d,c}\overline{\dot{\mathbf{P}}}_d\dot{\mathbf{O}}^a_{c,d}$$

Substituting (4.5.2) into this expression and rearranging, we get

$$\left(\dot{\mathbf{J}}^a_{d,c}\right)^{-1}\dot{\mathbf{J}}^a_{c,c} = -\overline{\dot{\mathbf{J}}}^h_{d,d}\left(\overline{\dot{\mathbf{J}}}^h_{d,c}\right)^{-1}.$$

Using both this and its conjugate, we rewrite equation (4.5.4) as

$$\left(\dot{\mathbf{O}}_{d,c}\right)^* = -\left(\dot{\mathbf{J}}^a_{d,d} + \mathbf{I}^a\right)\left(\dot{\mathbf{J}}^a_{d,c}\right)^{-1}\dot{\mathbf{P}}_c - \left(\overline{\dot{\mathbf{J}}}^h_{d,d} + \mathbf{I}^h\right)\left(\overline{\dot{\mathbf{J}}}^h_{d,c}\right)^{-1}\overline{\dot{\mathbf{P}}}_c.$$

Taking projections and switching the indices in (4.5.3), we obtain the following four identities:

$$\begin{aligned}
\overline{\dot{\mathbf{P}}}_d\left(\dot{\mathbf{O}}_{d,c}\right)^*\dot{\mathbf{P}}_c &= \overline{\dot{\mathbf{P}}}_d\dot{\mathbf{O}}_{c,d}\dot{\mathbf{P}}_c,\\
\overline{\dot{\mathbf{P}}}_d\left(\dot{\mathbf{O}}_{d,c}\right)^*\overline{\dot{\mathbf{P}}}_c &= -\overline{\dot{\mathbf{P}}}_d\dot{\mathbf{O}}_{c,d}\overline{\dot{\mathbf{P}}}_c,\\
\dot{\mathbf{P}}_d\left(\dot{\mathbf{O}}_{d,c}\right)^*\dot{\mathbf{P}}_c &= -\dot{\mathbf{P}}_d\dot{\mathbf{O}}_{c,d}\dot{\mathbf{P}}_c,\\
\dot{\mathbf{P}}_d\left(\dot{\mathbf{O}}_{d,c}\right)^*\overline{\dot{\mathbf{P}}}_c &= \dot{\mathbf{P}}_d\dot{\mathbf{O}}_{c,d}\overline{\dot{\mathbf{P}}}_c.
\end{aligned}$$

Adding all of these equations together, we have

$$\left(\dot{\mathbf{O}}_{d,c}\right)^* = -\left(\overline{\dot{\mathbf{P}}}_d - \dot{\mathbf{P}}_d\right)\dot{\mathbf{O}}_{c,d}\left(\overline{\dot{\mathbf{P}}}_c - \dot{\mathbf{P}}_c\right) = *_d\dot{\mathbf{O}}_{c,d}*_c$$

as desired. □

Remark 4.5.5. The key inputs for this technique are the jump formula, the conjugation property of overfare, and one of the adjoint formulas (in this case, those for the Cauchy operators). If we had started with the adjoint formula for $\dot{\mathbf{O}}_{d,c}$ instead, then we could follow a similar procedure to that in Section 3.5 to deduce the adjoint formulas for $\dot{\mathbf{J}}_{d,c}$.

Next, we discuss a natural symplectic form on the Dirichlet space. Following Example 6.11 in Kristel and Schippers [25], define $\omega(\cdot,\cdot) : \dot{\mathcal{D}}_{\text{harm}}(\Omega) \times \dot{\mathcal{D}}_{\text{harm}}(\Omega) \to \mathbb{C}$ by the formula

$$\omega(u,v) = \langle u, -(*\overline{v})\rangle.$$

The function ω is a *strong symplectic form* – that is, a continuous skew-symmetric bilinear form with the property that the map $\varphi : \dot{\mathcal{D}}_{\text{harm}}(\Omega) \to \dot{\mathcal{D}}_{\text{harm}}(\Omega)^*$ given by $\varphi(v) = \omega(v,\cdot)$ is an isomorphism. One could recognize $(\langle\cdot,\cdot\rangle, *, \omega(\cdot,\cdot))$ as a *compatible triple*, where the inner product and symplectic form are related via the *complex structure* $u \mapsto -(*\overline{u})$. The function ω has existed in the literature for a long time in association to the Riemann bilinear relations (see Farkas and Kra [17], for example). The relation to the inner product allows us to express ω with the formula

$$\omega(u,v) = \frac{1}{2\pi}\iint_{\Omega} du \wedge dv.$$

We note the following basic algebraic properties related to $(\langle\cdot,\cdot\rangle, *, \omega(\cdot,\cdot))$:

(a) $\overline{*}u = *u$ and $*^2 u = -u$.

(b) $\langle *u, v\rangle = \langle u, -(*v)\rangle$.

(c) $\omega(*u, *v) = \omega(u,v)$.

(d) $\langle *u, *v\rangle = \langle u, v\rangle$.

We are almost ready to provide a symplectic interpretation for overfare. An operator $\mathbf{B}$ is called a *symplectic transformation* if $\omega(\mathbf{B}u, \mathbf{B}v) = \omega(u,v)$ for all $u,v \in \dot{\mathcal{D}}_{\text{harm}}(\Omega)$. Similarly, we shall call $\mathbf{B}$ a *skew-symplectic* if instead $\omega(\mathbf{B}u, \mathbf{B}v) = -\omega(u,v)$. We claim that $\dot{\mathbf{O}}_{d,c}$ is a skew-symplectic transformation. For $d \in \{1,2\}$, let $u_d, v_d \in \dot{\mathcal{D}}_{\text{harm}}(\Omega_d)$ have the property that $u_2 = \dot{\mathbf{O}}_{1,2}u_1$ and $v_2 = \dot{\mathbf{O}}_{1,2}v_1$. Given nice analytic conditions, our claim follows from two applications of Stokes' Theorem. That is, assuming that Γ is smooth, and u_d, v_d have smooth extensions to the boundary, then

$$\begin{aligned}\iint_{\Omega_1} du_1 \wedge dv_1 = \int_{\Gamma} \left(\dot{\mathbf{b}}_1 u_1\right) d\left(\dot{\mathbf{b}}_1 v_1\right) &= -\int_{-\Gamma} \left(\dot{\mathbf{b}}_2 u_2\right) d\left(\dot{\mathbf{b}}_2 v_2\right) \\ &= -\iint_{\Omega_2} du_2 \wedge dv_2.\end{aligned}$$

Now if $\Gamma \subset \overline{\mathbb{C}}$ is an arbitrary quasicircle, and we make no added assumptions on the boundary values of u_d and v_d, we are not necessarily able to argue in this way. However, we can employ Theorem 4.5.4 to obtain

$$\omega\left(\dot{\mathbf{O}}_{2,1}u_2, v_1\right)_{\Omega_1} = \left\langle \dot{\mathbf{O}}_{2,1}u_2, -(*_1\overline{v_1})\right\rangle_{\Omega_1} = \left\langle u_2, *_2\left(\overline{\dot{\mathbf{O}}_{1,2}v_1}\right)\right\rangle_{\Omega_2}$$
$$= -\omega\left(u_2, \dot{\mathbf{O}}_{1,2}v_1\right)_{\Omega_1}$$

and so

$$\omega(u_1, v_1)_{\Omega_1} = -\omega(u_2, v_2)_{\Omega_2}.$$

To finish the section, we present the scattering matrix of $\dot{\mathbf{O}}_{d,c}$, which was first explored in the Bergman space setting by Schippers and Staubach [38] in connection to scattering of the potential. Recall that in Section 3.10, we derived the analogous result in the Smirnov space setting. Here, we rephrase the results of Schippers and Staubach in terms of the Dirichlet space as a point of comparison, beginning with the following identities.

Theorem 4.5.6 (Schippers and Staubach [38], [41], Quadratic Adjoint Identities)**.** *Let* $\Gamma \subset \overline{\mathbb{C}}$ *be a quasicircle with complementary components* Ω_1 *and* Ω_2*. Then*

$$\mathbf{I}_1^a = \left(\dot{\mathbf{J}}_{1,1}^a\right)^* \dot{\mathbf{J}}_{1,1}^a + \left(\dot{\mathbf{J}}_{1,2}^a\right)^* \dot{\mathbf{J}}_{1,2}^a,$$
$$\mathbf{I}_2^a = \left(\dot{\mathbf{J}}_{2,1}^a\right)^* \dot{\mathbf{J}}_{2,1}^a + \left(\dot{\mathbf{J}}_{2,2}^a\right)^* \dot{\mathbf{J}}_{2,2}^a,$$
$$0 = \left(\dot{\mathbf{J}}_{1,1}^a\right)^* \dot{\mathbf{J}}_{2,1}^a + \left(\dot{\mathbf{J}}_{1,2}^a\right)^* \dot{\mathbf{J}}_{2,2}^a,$$
$$0 = \left(\dot{\mathbf{J}}_{2,2}^a\right)^* \dot{\mathbf{J}}_{1,2}^a + \left(\dot{\mathbf{J}}_{2,1}^a\right)^* \dot{\mathbf{J}}_{1,1}^a.$$

Proof. Since $\dot{\mathbf{J}}_{d,c}^a = (\partial^h)^{-1}\mathbf{T}_{d,c}\overline{\partial}^a$, this result is simply a restatement of Theorem 4.23 in Schippers and Staubach [38]. Alternatively, one could compute as in Section 3.10. □

Remark 4.5.7. By conjugating each of the identities above, one can also derive the following:

$$\mathbf{I}_1^h = \dot{\mathbf{J}}_{1,1}^a \left(\dot{\mathbf{J}}_{1,1}^a\right)^* + \dot{\mathbf{J}}_{2,1}^a \left(\dot{\mathbf{J}}_{2,1}^a\right)^*,$$
$$\mathbf{I}_2^h = \dot{\mathbf{J}}_{1,2}^a \left(\dot{\mathbf{J}}_{1,2}^a\right)^* + \dot{\mathbf{J}}_{2,2}^a \left(\dot{\mathbf{J}}_{2,2}^a\right)^*,$$
$$0 = \dot{\mathbf{J}}_{1,2}^a \left(\dot{\mathbf{J}}_{1,1}^a\right)^* + \dot{\mathbf{J}}_{2,1}^a \left(\dot{\mathbf{J}}_{2,2}^a\right)^*,$$
$$0 = \dot{\mathbf{J}}_{2,2}^a \left(\dot{\mathbf{J}}_{2,1}^a\right)^* + \dot{\mathbf{J}}_{1,2}^a \left(\dot{\mathbf{J}}_{1,1}^a\right)^*.$$

Computing as we did in Section 3.10, the quadratic adjoint identities inform the following operator norm identities:

$$\begin{aligned} \left\| \dot{\mathbf{J}}^a_{1,2} \right\|^2 + \left\| \dot{\mathbf{J}}^a_{1,1} \right\|^2 &= 1, \\ \left\| \dot{\mathbf{J}}^a_{2,1} \right\|^2 + \left\| \dot{\mathbf{J}}^a_{2,2} \right\|^2 &= 1. \end{aligned} \tag{4.5.5}$$

Now, since $\overline{\left(\dot{\mathbf{J}}^a_{1,2}\right)^*} = \bar{\dot{\mathbf{J}}}^a_{2,1}$, it follows that

$$\left\| \dot{\mathbf{J}}^a_{1,2} \right\| = \left\| \dot{\mathbf{J}}^a_{2,1} \right\|$$

and so from (4.5.5) we deduce

$$\left\| \dot{\mathbf{J}}^a_{1,1} \right\| = \left\| \dot{\mathbf{J}}^a_{2,2} \right\|. \tag{4.5.6}$$

Theorem 4.5.8 (Schippers and Staubach [38], [41], Scattering Matrix of the Homogeneous Overfare)**.** *Let* $\dot{\mathbf{Q}} : \dot{\mathcal{D}}(\Omega_1) \oplus \dot{\mathcal{D}}(\Omega_2) \to \overline{\dot{\mathcal{D}}(\Omega_1)} \oplus \overline{\dot{\mathcal{D}}(\Omega_2)}$ *be the linear operator with matrix representation*

$$\begin{pmatrix} -\bar{\dot{\mathbf{J}}}^h_{1,1} & -\bar{\dot{\mathbf{J}}}^h_{2,1} \\ -\bar{\dot{\mathbf{J}}}^h_{1,2} & -\bar{\dot{\mathbf{J}}}^h_{2,2} \end{pmatrix}.$$

Then $\dot{\mathbf{Q}}$ *is the scattering matrix of* $\dot{\mathbf{O}}_{2,1}$ *in the sense that for all* $\alpha_d + \bar{\beta}_d \in \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_d)$ *satisfying* $\dot{\mathbf{O}}_{2,1}(\alpha_2 + \bar{\beta}_2) = \alpha_1 + \bar{\beta}_1$*, we have*

$$\begin{pmatrix} \bar{\beta}_1 \\ \bar{\beta}_2 \end{pmatrix} = \begin{pmatrix} -\bar{\dot{\mathbf{J}}}^h_{1,1} & -\bar{\dot{\mathbf{J}}}^h_{2,1} \\ -\bar{\dot{\mathbf{J}}}^h_{1,2} & -\bar{\dot{\mathbf{J}}}^h_{2,2} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Moreover, $\dot{\mathbf{Q}}$ *is unitary with respect to the direct sum inner product.*

Proof. Unitarity of the scattering matrix follows directly from the quadratic adjoint identities in Theorem 4.5.6 and Remark 4.5.7. The remainder of the statement follows from the corresponding result in the Bergman space (Theorem 8.8 of Schippers and Staubach [38]) and the fact that $\mathbf{J}^a_{d,c} = (\partial^h)^{-1}\mathbf{T}_{d,c}\overline{\partial}^a$ and overfare on the Bergman space is defined as $\mathbf{O}'_{d,c} d := d\dot{\mathbf{O}}_{d,c}$. □

4.6 The Classical Grunsky Operator

We now give a brief description of the (classical) Grunsky operator. In particular, we are most interested in deriving its properties based on its construction as an integral operator on the Bergman space, which dates back to the work of Bergman and Schiffer [5].

This operator has a storied history in geometric function theory through its connection to univalence, Faber series, and the universal Teichmüller space. An introduction to these ideas can be found in Pommerenke [34]. First, we define the Grunsky operator in terms of the Cauchy and overfare operators following the survey paper of Schippers and Staubach [39]. Subsequently, we discuss the construction of the Grunsky operator via generating function following Shen [42], and its relation to the corresponding integral operator on the Bergman space.

To define the Grunsky operator on the Dirichlet space, it is convenient to first specify how it acts on the space of nonconstant-polynomials in the variable $\bar{z}$, which we denote by $\mathbb{C}_0[\bar{z}]$. For the details, we refer the reader back to Schippers and Staubach [39]. Let $\Gamma \subset \overline{\mathbb{C}}$ be a Jordan curve with complementary components Ω_1 and Ω, and fix distinct $d, c \in \{1,2\}$. For $f \in \mathrm{Conf}(\mathbb{D}, \Omega_d)$ and $q \in \Omega_c$, the *Grunsky operator associated to f on polynomials* is defined

$$\mathbf{Gr}_f := -f^*\mathbf{P}_d\mathbf{O}_{c,d}\mathbf{J}^q_{d,c}(f^{-1})^* : \mathbb{C}_0[\bar{z}] \to \mathcal{D}(\mathbb{D}). \tag{4.6.1}$$

It can be shown that this expression extends by density to a norm-bounded operator on all of $\overline{\mathcal{D}(\mathbb{D})}$ that is independent of q. Henceforth, we use the symbol $\mathbf{Gr}_f$ to refer to this extension. Now, further assume that Γ is a quasicircle. In this case, the formula in (4.6.1) holds on all of $\overline{\dot{\mathcal{D}}(\mathbb{D})}$. Moreover, if we consider the corresponding homogeneous operator $\dot{\mathbf{G}}\mathbf{r} : \overline{\dot{\mathcal{D}}(\mathbb{D})} \to \dot{\mathcal{D}}(\mathbb{D})$, then since the jump formula restricted to antiholomorphic function is $\dot{\mathbf{O}}_{c,d}\dot{\mathbf{J}}^a_{d,c} = \dot{\mathbf{J}}^a_{d,d} - \mathbf{I}^a$, we can write

$$\dot{\mathbf{G}}\mathbf{r}_f = -f^*\dot{\mathbf{J}}^a_{d,d}(f^{-1})^* : \overline{\dot{\mathcal{D}}(\mathbb{D})} \to \dot{\mathcal{D}}(\mathbb{D}). \tag{4.6.2}$$

Remark 4.6.1. By unitarity of pull-back and Theorem 4.5.2, it follows from the above representation that

$$\left(\dot{\mathbf{G}}\mathbf{r}_f\right)^* = \overline{\dot{\mathbf{G}}\mathbf{r}}_f.$$

Next, let us consider the corresponding operator on the Bergman space. For a Jordan curve Γ and $f \in \mathrm{Conf}(\mathbb{D}, \Omega_d)$, define $\mathbf{Gr}'_f : \overline{\mathscr{A}(\mathbb{D})} \to \mathscr{A}(\mathbb{D})$ by $\mathbf{Gr}'_f\bar{\partial} := \partial\mathbf{Gr}_f$. Then it can be shown that

$$\begin{aligned}\left(\mathbf{Gr}'_f\overline{\beta}\right)(z) &= \left[f^*\mathbf{T}_{d,d}(f^{-1})^*\overline{\beta}\right](z)\\ &= \frac{1}{2\pi i}\iint\limits_{w\in\mathbb{D}}\left[\frac{f'(w)f'(z)}{(f(w)-f(z))^2} - \frac{1}{(w-z)^2}\right]\overline{\beta(w)} \wedge dw \cdot dz, \quad z \in \mathbb{D}.\end{aligned} \tag{4.6.3}$$

This nonsingular integral expression for the Grunsky operator is what we want the upcoming computations. Recall that the *Schwarzian derivative* of a locally injective holomorphic function f is defined to be

$$\mathcal{S}f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

One of the notable characteristics of the Schwarzian derivative is the fact that $\mathcal{S}f = 0$ if and only if f is a Möbius transformation. For a general map $f \in \mathrm{Conf}(\mathbb{D}, \Omega)$, we have the following relation between the Schwarzian and the desingularized Schiffer kernel:

$$(\mathcal{S}f)(z) = 6\left[\frac{f'(w)f'(z)}{(f(w)-f(z))^2} - \frac{1}{(w-z)^2}\right]_{w=z}. \tag{4.6.4}$$

With this, we can prove the following well-known result characterizing the vanishing of the Grunsky operator:

Proposition 4.6.2. *Let $\Omega \subset \overline{\mathbb{C}}$ be a Jordan domain, and let $f \in \mathrm{Conf}(\mathbb{D}, \Omega)$. Then $\mathbf{Gr}'_f = 0$ if and only if f is a Möbius transformation.*

Proof. It suffices to show that the integral kernel in (4.6.3) vanishes identically in $\mathbb{D} \times \mathbb{D}$ and only if f is a Möbius transformation. The crucial fact we need to use here is Möbius invariance of the Grunsky operator. That is, for any Möbius transformation M, we have that

$$\mathbf{Gr}'_{M\circ f} = \mathbf{Gr}'_f \tag{4.6.5}$$

and the same holds for $\mathbf{Gr}_f$ (see Corollary 6.10 in Schippers and Staubach [39], for example). Now, if f is a Möbius transformation, then so is f^{-1}, and hence $\mathbf{Gr}'_f = \mathbf{Gr}'_{\mathrm{id}}$, where id denotes the identity function on $\mathbb{D}$. But then we have that $\mathbf{Gr}'_{\mathrm{id}} = 0$ by (4.6.3). Conversely, if $\mathbf{Gr}'_f = 0$, then the kernel function in (4.6.3) vanishes outside a set of (area) measure zero. However, this kernel is holomorphic in (w, z), and so we must have identical vanishing, i.e.

$$\frac{f'(w)f'(z)}{(f(w)-f(z))^2} - \frac{1}{(w-z)^2} \equiv 0. \tag{4.6.6}$$

Therefore by equation (4.6.4), $\mathcal{S}f = 0$, whence f is a Möbius transformation. □

There are many different ways of saying what the Grunsky operator is. Another common variation is defined on square-summable sequences, which we now define following the presentation of Shen [42]. Recall that a function f is called *univalent* if it is both holomorphic and injective. Let $g : \mathbb{D}^- \to \overline{\mathbb{C}}$ be a meromorphic function of the form $g(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}$. For such functions, one can construct the following analytic function in two variables:

$$\log\left(\frac{g(w)-g(z)}{w-z}\right) = -\sum_{m,n=1}^{\infty} b_{m,n}(g)w^{-m}z^{-n}, \quad z,w \in \mathbb{D}^-.$$

The coefficients $b_{mn}(g)$ are called the *Grunsky coefficients* of g, and they appear in a characterizing of univalence as well as in the *Grunsky inequalities* (see Pommerenke [34] for example). Now, if a function g of the above form is also univalent, we say that $g \in \Sigma_0$. For any such $g \in \Sigma_0$, define the sequence map

$$G(g) : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C}),$$
$$(\lambda_n)_{n=1}^{\infty} \mapsto \left(\sum_{m=1}^{\infty} \sqrt{mn}\, b_{m,n}(g)\lambda_m\right)_{n=1}^{\infty}.$$

We call $G(g)$ the ℓ^2-*Grunsky operator* associated to g. The following theorem demonstrates the relationship between $G(g)$ and the integral operator in equation (4.6.3).

Theorem 4.6.3. *Let $\Omega \subset \overline{\mathbb{C}}$ be a Jordan domain, and let $f \in \mathrm{Conf}(\mathbb{D},\Omega)$. Then*

$$\mathbf{Gr}'_f = -S[G(\iota \circ f \circ \iota)]\overline{S}^{-1}$$

where $\iota : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the meromorphic reciprocal function $z \mapsto 1/z$, and $S : \ell^2(\mathbb{C}) \to \mathscr{A}(\mathbb{D})$ denotes the unitary mapping defined in Example 4.2.5.

Proof. By Möbius invariance of the Grunsky operator (equation (4.6.5)), we may assume that f has the normalization $f(0) = 0$ and $f'(0) = 1$. Let $g : \mathbb{D}^- \to \overline{\mathbb{C}}$ be the conformal map defined by

$$g(z) = \iota \circ f \circ \iota(z) = \frac{1}{f\left(\frac{1}{z}\right)}, \quad z \in \mathbb{D}^-.$$

Then with a simple power series manipulation, one can show that $g \in \Sigma_0$. Now, the Grunsky coefficients of g are given via the generating function

$$\log\left(\frac{g(w)-g(z)}{w-z}\right) = -\sum_{m,n=1}^{\infty} b_{m,n}(g)w^{-m}z^{-n}.$$

Observe that for all $w,z \in \mathbb{D}$

$$\log\left(\frac{wz}{f(w)f(z)}\frac{f(w)-f(z)}{w-z}\right) = \log\left(\frac{g\left(\frac{1}{w}\right)-g\left(\frac{1}{w}\right)}{\frac{1}{w}-\frac{1}{z}}\right) = -\sum_{m,n=1}^{\infty} b_{m,n}(g)w^m z^n$$

and differentiating the left-hand side with respect to each variable yields

$$\frac{\partial^2}{\partial z \partial w} \log\left(\frac{wz}{f(w)f(z)} \frac{f(w)-f(z)}{w-z}\right) = \frac{f'(w)f'(z)}{(f(w)-f(z))^2} - \frac{1}{(w-z)^2}.$$

Therefore, the desingularized Schiffer kernel function for Ω can be expressed as the double series

$$\frac{f'(w)f'(z)}{(f(w)-f(z))^2} - \frac{1}{(w-z)^2} = -\sum_{m,n=1}^{\infty} mnb_{m,n}(g)w^{m-1}z^{n-1}, \quad w,z \in \mathbb{D}. \tag{4.6.7}$$

Next, recall from Example 4.2.5 that $\{\sqrt{k}\bar{w}^{k-1}d\bar{w} : k \geqslant 1\}$ forms an orthonormal basis for $\overline{\mathscr{A}(\mathbb{D})}$. Since $\mathbf{Gr}'_f$ is bounded, if we let $\overline{\beta(w)} = \sum_{k=1}^{\infty} \lambda_k \sqrt{k}\bar{w}^{k-1}d\bar{w} \in \overline{\mathscr{A}(\mathbb{D})}$, then

$$\left(\mathbf{Gr}'_f\overline{\beta}\right)(z) = \sum_{k=1}^{\infty} \lambda_k \mathbf{Gr}'_f\left(\sqrt{k}\bar{w}^{k-1}d\bar{w}\right)(z). \tag{4.6.8}$$

For each $k \geqslant 1$, using the Fubini-Tonelli theorem we may write

$$\begin{aligned}
\mathbf{Gr}'_f\left(\sqrt{k}\bar{w}^{k-1}d\bar{w}\right)(z) &= -\frac{1}{2\pi i}\iint_{w\in\mathbb{D}}\left[\sum_{m,n=1}^{\infty} mnb_{m,n}(g)w^{m-1}z^{n-1}\right]\left(\sqrt{k}\bar{w}^{k-1}\right) d\bar{w}\wedge dw\cdot dz \\
&= -\sum_{m,n=1}^{\infty} n\sqrt{m}b_{m,n}(g)z^{n-1}\left[\frac{1}{\pi}\iint_{w\in\mathbb{D}}\left(\sqrt{m}w^{m-1}\right)\left(\sqrt{k}\bar{w}^{k-1}\right)dA\right]dz \\
&= -\sum_{n=1}^{\infty} n\sqrt{k}b_{k,n}(g)z^{n-1}dz.
\end{aligned}$$

Combining this with (4.6.8), we obtain

$$\left(\mathbf{Gr}'_f\overline{\beta}\right)(z) = -\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\sqrt{kn}b_{k,n}(g)\lambda_k\right)\sqrt{n}z^{n-1}dz, \quad z\in\mathbb{D}$$

and so under the identification $S : \ell^2(\mathbb{C}) \to \mathscr{A}(\mathbb{D})$ defined in equation 4.2.9, we have

$$-\left[S^{-1}\mathbf{Gr}'_f\overline{S}\right](\lambda_n)_{n=1}^{\infty} = \left(\sum_{k=1}^{\infty}\sqrt{kn}b_{k,n}(g)\lambda_k\right)_{n=1}^{\infty} = [G(g)](\lambda_n)_{n=1}^{\infty}$$

as desired. □

4.7 A Kerzman-Stein Operator for Dirichlet Space

In this final section of the chapter, we take steps to extend the Kerzman-Stein viewpoint to the Dirichlet and Bergman space setting. Specifically, we define the analogue of the d,c-Kerzman-Stein operator of Section 3.6 for the homogeneous Dirichlet space of harmonic functions. A previous discussion point was that the operator $\dot{\mathbf{J}}_{d,c}$ is nearly a standard Cauchy integral, with the issue of boundary irregularity preventing the identification. Nonetheless, we saw that it still behaves in much of the same as a Cauchy integral. Now, since the Kerzman-Stein operator is the skew-adjoint part of the Cauchy integral operator in the Smirnov setting, it is natural to define the corresponding operator for the Dirichlet space as the skew-adjoint part of $\dot{\mathbf{J}}_{d,d}$, and extend as we did in Definition 3.6.1. Let us recount the two most significant properties of the Kerzman-Stein operator for the Smirnov space:

(a) The Kerzman-Stein formula provides a link between the Szegő projection and the Cauchy integral (Theorem 3.6.3).

(b) For a wide class of curves, the Kerzman-Stein operator is compact, with increased curve regularity corresponding to increased operator regularity (Theorem 3.8.10).

The goal of this section is to verify that the analogous properties are met with the definition made in this new setting, as well as those explored with the extension of the standard operator in Sections 3.6, 3.8, and 3.9.

Definition 4.7.1. Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2. For $d,c \in \{1,2\}$, we define the (homogeneous) d,c-Kerzman-Stein operator

$$\dot{\mathbf{A}}_{d,c} : \dot{\mathcal{D}}_{\text{harm}}(\Omega_d) \to \dot{\mathcal{D}}_{\text{harm}}(\Omega_c)$$

by the formula

$$\dot{\mathbf{A}}_{d,c} = \dot{\mathbf{J}}_{d,c} - \left(\dot{\mathbf{J}}_{c,d}\right)^*.$$

Remark 4.7.2. One could also define Kerzman-Stein operators for the Bergman space of harmonic one-forms by conjugating $\dot{\mathbf{A}}_{d,c}$ by the exterior derivative.

Algebraically, $\dot{\mathbf{A}}_{d,c}$ behaves in a similar manner to its Smirnov space counterpart. First, we have that

$$\left(\dot{\mathbf{A}}_{d,c}\right)^* = -\dot{\mathbf{A}}_{c,d}$$

and so $\dot{\mathbf{A}}_{d,d}$ is skew-adjoint. By the Cauchy adjoint formulas in Section 4.5, we can express $\dot{\mathbf{A}}_{d,c}$ as

$$\dot{\mathbf{A}}_{d,c} = \dot{\mathbf{J}}^a_{d,c}\overline{\dot{\mathbf{P}}}_d - \overline{\dot{\mathbf{J}}}^h_{d,c}\dot{\mathbf{P}}_d. \tag{4.7.1}$$

from which it is clear that

$$\overline{\dot{\mathbf{A}}}_{d,c} = -\dot{\mathbf{A}}_{d,c}. \tag{4.7.2}$$

and

$$*_c\dot{\mathbf{A}}_{d,c} = -\dot{\mathbf{A}}_{d,c}*_d \tag{4.7.3}$$

The representation in (4.7.1) allows us to easily derive the following.

Theorem 4.7.3 (Kerzman-Stein Formula)**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2. For each $d \in \{1,2\}$, we have that the operator*

$$\mathbf{I} + \dot{\mathbf{A}}_{d,d} : \dot{\mathcal{D}}_{\text{harm}}(\Omega_d) \to \dot{\mathcal{D}}_{\text{harm}}(\Omega_d)$$

is a bounded isomorphism, and

$$\dot{\mathbf{P}}_d = \dot{\mathbf{J}}_{d,d}\left(\mathbf{I} + \dot{\mathbf{A}}_{d,d}\right)^{-1}.$$

Proof. Since we can write $\mathbf{I} + \dot{\mathbf{A}}_{d,d} = \dot{\mathbf{J}}_{d,d} - i\overline{\dot{\mathbf{J}}}_{d,d}*_d$, we immediately have that

$$\dot{\mathbf{P}}_d\left(\mathbf{I} + \dot{\mathbf{A}}_{d,d}\right) = \dot{\mathbf{J}}_{d,d} \tag{4.7.4}$$

and $\mathbf{I} + \dot{\mathbf{A}}_{d,d} : \dot{\mathcal{D}}_{\text{harm}}(\Omega_d) \to \dot{\mathcal{D}}_{\text{harm}}(\Omega_d)$ being a bounded isomorphism is a consequence of Lemma A.1.1. □

Next, we derive various identities relating these operators to overfare. By rearranging the jump formula, we have that $\dot{\mathbf{J}}_{d,d} = \dot{\mathbf{O}}_{c,d}\dot{\mathbf{J}}_{d,c} + \mathbf{I}$, and so the adjoint of $\dot{\mathbf{J}}_{d,c}$ can be expressed as

$$\left(\dot{\mathbf{J}}_{d,d}\right)^* = \left(\dot{\mathbf{O}}_{c,d}\dot{\mathbf{J}}_{d,c}\right)^* + \mathbf{I}.$$

Subtracting these two expressions yields

$$\dot{\mathbf{A}}_{d,d} = \dot{\mathbf{O}}_{c,d}\dot{\mathbf{J}}_{d,c} - \left(\dot{\mathbf{O}}_{c,d}\dot{\mathbf{J}}_{d,c}\right)^*, \quad d \neq c.$$

Expanding the right-hand side when $d = 1$, we find

$$\begin{aligned}
\dot{\mathbf{A}}_{1,1} &= \dot{\mathbf{O}}_{2,1}\dot{\mathbf{J}}_{1,2} - \overline{\dot{\mathbf{J}}}_{2,1}\left(*_2\dot{\mathbf{O}}_{1,2}*_1\right) \\
&= -\dot{\mathbf{O}}_{2,1}\left[\dot{\mathbf{O}}_{1,2}\overline{\dot{\mathbf{J}}}_{2,1} - \dot{\mathbf{J}}_{1,2}\left(*_1\dot{\mathbf{O}}_{2,1}*_2\right)\right]\left(*_2\dot{\mathbf{O}}_{1,2}*_1\right) \\
&= -\dot{\mathbf{O}}_{2,1}\overline{\dot{\mathbf{A}}}_{2,2}\left(*_2\dot{\mathbf{O}}_{1,2}*_1\right)
\end{aligned}$$

which, using property (4.7.2), rearranges to

$$\dot{\mathbf{O}}_{1,2}\left(\dot{\mathbf{A}}_{1,1*1}\right) = -\left(\dot{\mathbf{A}}_{2,2*2}\right)\dot{\mathbf{O}}_{1,2}. \tag{4.7.5}$$

Theorem 4.7.4. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2. Then for distinct $d,c \in \{1,2\}$, the d,c-Kerzman-Stein operator $\dot{\mathbf{A}}_{d,c}$ is a bounded isomorphism with*

$$\left(\dot{\mathbf{A}}_{d,c}\right)^{-1} = \left(\dot{\mathbf{J}}^{a}_{d,c}\right)^{-1}\dot{\mathbf{P}}_c - \left(\overline{\dot{\mathbf{J}}}^{h}_{d,c}\right)^{-1}\overline{\dot{\mathbf{P}}}_c.$$

Proof. By Theorem 4.4.6, $\dot{\mathbf{J}}^{a}_{d,c} : \overline{\dot{\mathcal{D}}(\Omega_d)} \to \dot{\mathcal{D}}(\Omega_c)$ is a bounded isomorphism, and hence so is its conjugation $\overline{\dot{\mathbf{J}}}^{h}_{d,c} : \dot{\mathcal{D}}(\Omega_d) \to \overline{\dot{\mathcal{D}}(\Omega_c)}$. Using equation (4.7.1) and the orthogonal decomposition (4.2.7), it follows that $\dot{\mathbf{A}}_{d,c} = \dot{\mathbf{J}}^{a}_{d,c}\overline{\dot{\mathbf{P}}}_d - \overline{\dot{\mathbf{J}}}^{h}_{d,c}\dot{\mathbf{P}}_d : \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_d) \to \dot{\mathcal{D}}_{\mathrm{harm}}(\Omega_c)$ is a bounded isomorphism with its inverse given as in the statement of the theorem. □

Remark 4.7.5. Substituting in the inverse formula for $\mathbf{J}^{a}_{d,c}$ from Remark 4.4.11, we can also express the inverse of $\dot{\mathbf{A}}_{d,c}$ as

$$\left(\dot{\mathbf{A}}_{d,c}\right)^{-1} = -\overline{\dot{\mathbf{P}}}_d\dot{\mathbf{O}}_{c,d}\dot{\mathbf{P}}_c + \dot{\mathbf{P}}_d\dot{\mathbf{O}}_{c,d}\overline{\dot{\mathbf{P}}}_c.$$

Next, consider the overfare of $\dot{\mathbf{A}}_{d,c}$. Applying the jump formula twice:

$$\begin{aligned}
\dot{\mathbf{O}}_{c,d}\dot{\mathbf{A}}_{d,c} &= \dot{\mathbf{O}}_{c,d}\dot{\mathbf{J}}_{d,c}\overline{\dot{\mathbf{P}}}_d - \dot{\mathbf{O}}_{c,d}\overline{\dot{\mathbf{J}}}_{d,c}\dot{\mathbf{P}}_d \\
&= \dot{\mathbf{J}}_{d,d}\overline{\dot{\mathbf{P}}}_d - \overline{\dot{\mathbf{P}}}_d - \left(\overline{\dot{\mathbf{J}}}_{d,d}\dot{\mathbf{P}}_d - \dot{\mathbf{P}}_d\right) \\
&= \dot{\mathbf{A}}_{d,d} + \dot{\mathbf{P}}_d - \overline{\dot{\mathbf{P}}}_d.
\end{aligned}$$

Rearranging this equation, we can write $\dot{\mathbf{A}}_{d,d} - \dot{\mathbf{O}}_{c,d}\dot{\mathbf{A}}_{d,c} = -i*_d$, or equivalently

$$\dot{\mathbf{A}}_{d,d}\,{*_d} - \dot{\mathbf{O}}_{c,d}\dot{\mathbf{A}}_{d,c}\,{*_d} = i\mathbf{I}, \quad d \neq c \tag{4.7.6}$$

which could be interpreted as a jump formula for the Kerzman-Stein operators. Using this formula in conjunction with equation (4.7.5), we compute

$$\begin{aligned}
\dot{\mathbf{O}}_{2,1}\left(\dot{\mathbf{A}}_{1,2}\,{*_1}\right)\dot{\mathbf{O}}_{2,1} &= \left[\left(\dot{\mathbf{A}}_{1,1}\,{*_1}\right) - i\mathbf{I}_1\right]\dot{\mathbf{O}}_{2,1} \\
&= -\dot{\mathbf{O}}_{2,1}\left[\left(\dot{\mathbf{A}}_{2,2}\,{*_2}\right) + i\mathbf{I}_2\right] \\
&= -\left(\dot{\mathbf{A}}_{2,1}\,{*_2}\right) - 2i\dot{\mathbf{O}}_{2,1}
\end{aligned}$$

and so

$$\dot{\mathbf{O}}_{2,1}\left(\dot{\mathbf{A}}_{1,2}{}^{*1}\right)+\left(\dot{\mathbf{A}}_{2,1}{}^{*2}\right)\dot{\mathbf{O}}_{1,2}=-2i\mathbf{I}.$$

Remark 4.7.6. By a nearly identical computation to that in the proof of Theorem 3.10.3, we can extend the quadratic adjoint identities from Theorem 4.5.6 to obtain the following expressions:

$$\begin{aligned}\mathbf{I}-i\dot{\mathbf{A}}_{d,d}{}^{*d}&=\left(\dot{\mathbf{J}}_{d,d}\right)^{*}\dot{\mathbf{J}}_{d,d}+\left(\dot{\mathbf{J}}_{d,c}\right)^{*}\dot{\mathbf{J}}_{d,c},\\ -i\dot{\mathbf{A}}_{d,c}{}^{*d}&=\left(\dot{\mathbf{J}}_{c,c}\right)^{*}\dot{\mathbf{J}}_{d,c}+\left(\dot{\mathbf{J}}_{c,d}\right)^{*}\dot{\mathbf{J}}_{d,d},\end{aligned}\qquad d\neq c.$$

Next, we provide a characterization for the homogeneous Kerzman-Stein operators in terms of *generalized circles* in $\overline{\mathbb{C}}$, which is the class of curves consisting of planar circles and lines which include the point at infinity (compare with Theorem 3.9.11).

Theorem 4.7.7. *Let $\Gamma\subset\overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2, and fix distinct $d,c\in\{1,2\}$. Then the following are equivalent:*

(a) Γ *is a generalized circle.*

(b) *For every $f\in\mathrm{Conf}(\mathbb{D},\Omega_d)$, $\dot{\mathbf{Gr}}_f=0$.*

(c) $\dot{\mathbf{P}}_d=\dot{\mathbf{J}}_{d,d}$.

(d) $\dot{\mathbf{A}}_{d,d}=0$.

(e) $\dot{\mathbf{A}}_{d,c}=i\dot{\mathbf{O}}_{d,c}{}^{*d}$.

(f) image $\left(\dot{\mathbf{O}}^{h}_{d,c}\right)=\overline{\dot{\mathcal{D}}(\Omega_c)}$.

(g) $\dot{\mathbf{A}}_{d,c}$ *is a symplectic transformation with respect to ω.*

It is sufficient for (b) *to hold for at least one such mapping f.*

Proof. The equivalence of (a) and (b) follows immediately from Proposition 4.6.2, as Möbius transformations take generalized circles to generalized circles. Since we can write $\dot{\mathbf{Gr}}_f=-f^{*}\dot{\mathbf{J}}^{a}_{d,d}(f^{-1})^{*}$, it follows from the Cauchy integral formula that (b)$\Longleftrightarrow$(c). By the Kerzman-Stein formula (Theorem 4.7.3), we have (c)$\Longleftrightarrow$(d). The statement (d)$\Longleftrightarrow$(e) is simply a consequence of identity (4.7.6). Now, if the formula in (e) holds for $\dot{\mathbf{A}}_{d,c}$, then by precomposing with $\dot{\mathbf{P}}_d$ we get that $\dot{\mathbf{O}}^{h}_{d,c}=-\overline{\dot{\mathbf{J}}}^{h}_{d,c}$, implying (f) by Theorem 4.4.6. Conversely, given (f), it follows from the integral expression of $\dot{\mathbf{O}}^{h}_{d,c}$ in (4.5.1) that $\dot{\mathbf{J}}^{a}_{c,c}=0$, or equivalently $\dot{\mathbf{A}}_{c,c}=0$. By the relation between Kerzman-Stein operators in (4.7.5), it follows in this case that $\dot{\mathbf{A}}_{d,d} = 0$, which is exactly (d). Finally, if (e) is true, then since $\dot{\mathbf{O}}_{d,c}$ is a skew-symplectic transformation, we have for all $u, v \in \dot{\mathcal{D}}_{\text{harm}}(\Omega_d)$ that

$$\omega\left(\dot{\mathbf{A}}_{d,c}u, \dot{\mathbf{A}}_{d,c}v\right)_{\Omega_c} = \omega\left(i\dot{\mathbf{O}}_{d,c} *_d u, i\dot{\mathbf{O}}_{d,c} *_d v\right)_{\Omega_c} = \omega(u,v)_{\Omega_d}$$

so in this case (g) is true. Conversely, if $\dot{\mathbf{A}}_{d,c}$ is a symplectic transformation, then the operator $\dot{\mathbf{A}}_{d,d} *_d - i\mathbf{I} = \dot{\mathbf{O}}_{c,d}\dot{\mathbf{A}}_{d,c} *_d$ is skew-symplectic. That is, for all $u, v \in \dot{\mathcal{D}}_{\text{harm}}(\Omega_d)$ we have that

$$-\left\langle \left(\dot{\mathbf{A}}_{d,d} *_d - i\mathbf{I}\right)u, *_d\overline{\left(\dot{\mathbf{A}}_{d,d} *_d - i\mathbf{I}\right)\overline{v}}\right\rangle_{\Omega_d} = \langle u, *_d\overline{v}\rangle_{\Omega_d}$$

In particular, this implies that

$$-\left(*_d\dot{\mathbf{A}}_{d,d} + i\mathbf{I}\right) *_d \overline{\left(\dot{\mathbf{A}}_{d,d} *_d - i\mathbf{I}\right)} = *_d.$$

Simplifying this expression and using identity (4.7.3), the above reduces to $\dot{\mathbf{A}}_{d,d}^2 = 0$. Therefore, for any $u \in \dot{\mathcal{D}}_{\text{harm}}(\Omega_d)$ we have

$$\left\|\dot{\mathbf{A}}_{d,d}u\right\|_{\Omega_d}^2 = \left\langle \dot{\mathbf{A}}_{d,d}u, \dot{\mathbf{A}}_{d,d}u\right\rangle_{\Omega_d} = -\left\langle u, \dot{\mathbf{A}}_{d,d}^2u\right\rangle_{\Omega_d} = 0$$

and so $\dot{\mathbf{A}}_{d,d} = 0$, which completes the proof. □

For the remainder of the section, we explore connections between the Grunsky and Kerzman-Stein operator. The crucial observation is, roughly, that both $\dot{\mathbf{Gr}}_f$ and $\dot{\mathbf{A}}_{d,d}$ belong to all of the same two-sided ideals in the corresponding spaces of bounded operators. In particular, we have the following characterization in terms of the Schatten classes.

Theorem 4.7.8. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2, and fix $d \in \{1,2\}$. For all $1 \leqslant p \leqslant \infty$, the following are equivalent:*

(a) $\dot{\mathbf{A}}_{d,d} \in S_p\left(\dot{\mathcal{D}}_{\text{harm}}(\Omega_d)\right)$.

(b) $\dot{\mathbf{Gr}}_f \in S_p\left(\overline{\dot{\mathcal{D}}(\mathbb{D})}, \dot{\mathcal{D}}(\mathbb{D})\right)$ *for all* $f \in \text{Conf}(\mathbb{D}, \Omega_d)$.

It is sufficient for (b) *to hold for at least one such mapping* f.

Proof. Since the Schatten-p class forms a two-sided ideal in the space of bounded operators, it follows from the expression in equation (4.7.1) that $\dot{\mathbf{A}}_{d,d} \in S_p$ if and only if $\dot{\mathbf{J}}_{d,d}^a \in S_p$. Now, as noted in Section 4.6, we can write $\dot{\mathbf{Gr}}_f = -f^*\dot{\mathbf{J}}_{d,d}^a(f^{-1})^*$ for quasidisks. Since pull-back by f is a bounded isomorphism, we have that $\dot{\mathbf{Gr}}_f \in S_p$ if and only if $\dot{\mathbf{J}}_{d,d}^a \in S_p$, completing the proof. □

The criterion for the Grunsky operator to belong to Schatten-p classes for $1 \leqslant p < \infty$ was discovered by Jones [23], who showed that $\mathbf{Gr}'_f$ (or equivalently $\dot{\mathbf{Gr}}_f$) is in the pth Schatten class if and only if $g = \log(f')$ is in the pth *analytic Besov space*, which for $p = 1$ means that $g'' \in L^1(\mathbb{D}, dA)$, and for $1 < p < \infty$ that

$$(1 - |z|^2)^{1-2/p} g'(z) \in L^p(\mathbb{D}, dA).$$

Later, the case $p = \infty$ was investigated by in the independent work of Takhtajan and Teo [44] and Shen [42], who also rediscovered equivalent conditions for the case $p = 2$. We emphasize these special cases in relation to the operator $\dot{\mathbf{A}}$ shortly. First, a definition:

Definition 4.7.9. Let $\Omega \subset \overline{\mathbb{C}}$ be a Jordan domain. We say that a map $f \in \mathrm{Conf}(\mathbb{D}, \Omega)$ is asymptotically conformal if

$$\lim_{|z| \to 1^-} (1 - |z|^2) \left(\frac{f''(z)}{f'(z)} \right) = 0.$$

In this case, we also call Ω and its boundary Γ asymptotically conformal.

The class of asymptotically conformal mappings can be defined in various equivalent ways, and the associated domains Ω are indeed examples of quasidisks (see Pommerenke [33], for example).

Theorem 4.7.10 (Takhtajan and Teo [44], Shen [42])**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2, and fix $d \in \{1, 2\}$. Then the following are equivalent:*

(a) *f is asymptotically conformal.*

(b) $\dot{\mathbf{Gr}}_f : \overline{\dot{\mathcal{D}}(\mathbb{D})} \to \dot{\mathcal{D}}(\mathbb{D})$ *is compact for every* $f \in \mathrm{Conf}(\mathbb{D}, \Omega_d)$.

It is sufficient for (b) *to hold for at least one such mapping f.*

Proof. Using the identification $\dot{\mathbf{Gr}}_f = (\partial^h)^{-1} \mathbf{Gr}'_f \overline{\partial}^a$ and Theorem 4.6.3, this is precisely Theorem 2 in Shen [42]. □

Combining their result with Theorem 4.7.8, we obtain the following characterization for compactness of the homogeneous Kerzman-Stein operator.

Corollary 4.7.11. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2, and fix $d \in \{1, 2\}$. Then the following are equivalent:*

(a) *Γ is asymptotically conformal.*

(b) $\dot{\mathbf{A}}_{d,d} : \dot{\mathcal{D}}_{\text{harm}}(\Omega_d) \to \dot{\mathcal{D}}_{\text{harm}}(\Omega_d)$ *is compact.*

Let us now consider the case $p = 2$, which corresponds to a particularly important subclass of asymptotically conformal mappings called the *Weil-Petersson* class. This class is often studied in relation to the *universal Teichmüller space*, which is a moduli space of Riemann surfaces. We previously defined the Weil-Petersson class in Section 3.9 as the mappings $f \in \text{Conf}(\mathbb{D}, \Omega)$ for which $f''/f' \in L^2(\mathbb{D}, dA)$. Notice that this condition on f may now be phrased in terms of the Dirichlet space as requiring that $\log(f') \in \mathcal{D}(\mathbb{D})$.

Theorem 4.7.12 (Jones [23], Takhtajan and Teo [44], Shen [42])**.** *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2, and fix $d \in \{1, 2\}$. Then the following are equivalent:*

(a) *f is Weil-Petersson class.*

(b) $\dot{\mathbf{Gr}}_f : \overline{\dot{\mathcal{D}}(\mathbb{D})} \to \dot{\mathcal{D}}(\mathbb{D})$ *is Hilbert-Schmidt for every* $f \in \text{Conf}(\mathbb{D}, \Omega_d)$.

It is sufficient for (b) *to hold for at least one such mapping f.*

Proof. Using the same justification as in the proof of Theorem 4.7.10, this is simply a restatement of Theorem 3 in Shen [42]. □

Similar to the previous situation, their result yields a characterization for Hilbert-Schmidtness of $\dot{\mathbf{A}}_{d,d}$ by Theorem 4.7.8.

Corollary 4.7.13. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle with complementary components Ω_1 and Ω_2, and fix $d \in \{1, 2\}$. Then the following are equivalent:*

(a) *Γ is Weil-Petersson class.*

(b) $\dot{\mathbf{A}}_{d,d} : \dot{\mathcal{D}}_{\text{harm}}(\Omega_d) \to \dot{\mathcal{D}}_{\text{harm}}(\Omega_d)$ *is Hilbert-Schmidt.*

We wish to emphasize that the previous two corollaries were an easy consequence of the characterization of the Schatten-ness of the Grunsky operator, as well as the formulation based on the restricted Cauchy operator. The upshot of our work is the new perspective, which includes the bridge built between these operators. Indeed, we observed in the Smirnov space setting that the "ordinary" Kerzman-Stein operator and the analogue of the Grunsky operator are equivalent from a certain functional-analytic viewpoint (Theorem 3.9.3). Following this, we defined an analogue of the Kerzman-Stein operator for the Dirichlet space, which both shares the defining characteristics of the original operator, and relates to the Grunsky operator in this setting (Theorem 4.7.8). A potentially interesting line of research questions based on these observations could be formulated in terms of deeper relations between these two operators, and the usefulness of the Kerzman-Stein operator and formula on Dirichlet spaces of certain regular curves.

Appendix A

Supplementary Results and Observations

This appendix consists of a few short sections, each of which is dedicated to filling in technical steps left out of the main part of the thesis, and providing additional observations.

A.1 Invertibility Property for the Skew-Adjoint Part

To begin, we record a generalization of a proof and result of Lanzani [28] which says that the linear operator

$$\mathbf{I} + \mathbf{J}_\Gamma - (\mathbf{J}_\Gamma)^* : L^2(\Gamma, |d\zeta|) \to L^2(\Gamma, |d\zeta|)$$

is invertible whenever the Cauchy integral operator $\mathbf{J}_\Gamma$ is bounded. This observation appears in a similar form in the paper of Kerzman and Stein [24] to relate the Szegő projection and the Cauchy integral operator. The need for a more general result is to prove a Kerzman-Stein formula on both complementary components in the Smirnov and Dirichlet space settings.

Lemma A.1.1. *Let $\mathcal{H}$ be a Hilbert space, and let $\mathbf{B} \in B(\mathcal{H})$. Then $\mathbf{I} + \mathbf{B} - \mathbf{B}^* \in B(\mathcal{H})$ is an isomorphism.*

Proof. To simplify notation, set $\mathbf{C} := \mathbf{I} + \mathbf{B} - \mathbf{B}^*$. Then $\mathbf{C}^* = \mathbf{I} - (\mathbf{B} - \mathbf{B}^*)$, and it is straightforward to check that the following identities hold:

$$\begin{aligned} \mathbf{C} + \mathbf{C}^* &= 2\mathbf{I}, \\ \mathbf{C}\mathbf{C}^* &= \mathbf{C}^*\mathbf{C}. \end{aligned}$$

Now, for any $u \in \mathcal{H}$ we have

$$\langle \mathbf{C}u, \mathbf{C}u \rangle = \langle u, \mathbf{C}^*\mathbf{C}u \rangle = \langle u, \mathbf{C}\mathbf{C}^*u \rangle = \langle \mathbf{C}^*u, \mathbf{C}^*u \rangle$$

or equivalently

$$\|\mathbf{C}u\| = \|\mathbf{C}^*u\|.$$

Using these computation, we find that

$$\|u\| = \frac{1}{2}\|(\mathbf{C}+\mathbf{C}^*)u\| \leqslant \|\mathbf{C}u\|, \|\mathbf{C}^*u\| \tag{A.1.1}$$

implying that $\mathbf{C}$ and $\mathbf{C}^*$ are both injective. Next, consider the following elementary decomposition of $\mathcal{H}$:

$$\mathcal{H} = \operatorname{null}\mathbf{C}^* \oplus \overline{\operatorname{image}\mathbf{C}}.$$

By the inequalities in equation (A.1.1) and boundedness of $\mathbf{C}$, it follows that the image of $\mathbf{C}$ is closed, so injectivity of $\mathbf{C}^*$ implies surjectivity of $\mathbf{C}$. Thus, $\mathbf{C} : \mathcal{H} \to \mathcal{H}$ is both bijective and bounded, and hence an isomorphism by the bounded inverse theorem. □

Remark A.1.2. One could also make sense of the statement of this lemma using spectral theory. Indeed, since $\mathbf{B} - \mathbf{B}^* : \mathcal{H} \to \mathcal{H}$ is skew-adjoint, its spectrum is purely imaginary, and so $-1 \neq \sigma(\mathbf{B} - \mathbf{B}^*)$.

A.2 Kernel Function Estimates

Next, we turn our attention to the analysis of the Kerzman-Stein integral kernel from Section 3.8. The upcoming lemma is used in the proof of Theorem 3.8.17 to make an estimate on the Kerzman-Stein kernel function $A(\zeta, z)$ when ζ is near z.

Lemma A.2.1. *For all* $0 < p < 1$*, we have*

$$\int_0^1 \int_0^1 \frac{1}{|x-y|^p}\, dxdy = \frac{2}{(1-p)(2-p)}.$$

In particular, $|x-y|^{-1} \in L^p([0,1]^2, dA)$.

Proof. By symmetry, it follows that

$$\int_0^1 \int_0^1 \frac{1}{|x-y|^p}\, dxdy = 2\int_0^1 \int_0^x \frac{1}{(x-y)^p}\, dydx$$

and we may directly compute the iterated integrals on the right-hand side. Indeed,

$$\int_0^x \frac{1}{(x-y)^p}\,dy = \frac{-1}{1-p}\cdot\frac{1}{(x-y)^{p-1}}\Big|_{y=0}^{y=x} = \frac{1}{1-p}\cdot\frac{1}{x^{p-1}}$$

and so

$$\frac{2}{1-p}\int_0^1 \frac{1}{x^{p-1}}\,dx = \frac{2}{1-p}\cdot\frac{1}{2-p}\,x^{2-p}\Big|_{x=0}^{x=1} = \frac{2}{(1-p)(2-p)}$$

as required. The second claim now follows from the Fubini-Tonelli theorem. □

Remark A.2.2. Note that when $p \geqslant 1$, the integral in Lemma A.2.1 diverges. Consequently, this limits the viability of our method for proving Hilbert-Schmidtness of the Kerzman-Stein operator for curve classes larger than $C^{1,1/2+\varepsilon}$, should the property still hold.

Related to the above computation in the context of Section 3.8, let us discuss sufficient conditions for an integral operator to belong to the Schatten p-class when $2 < p < \infty$. For $1 \leqslant p,q < \infty$, define the mixed (p,q)-Lebesgue norm of a measurable function $K : X \times X \to \mathbb{C}$ by

$$\|K\|_{p,q} = \left(\int_{x\in X}\left(\int_{y\in X}|K(x,y)|^p\,d\mu(x)\right)^{\frac{q}{p}} d\mu(y)\right)^{\frac{1}{q}}. \tag{A.2.1}$$

This is indeed a norm for the mixed-Lebesgue space, which is studied in the paper of Benedek and Panzone [4]. Define the Hermitian conjugate of K to be the function $K^*(x,y) = \overline{K(y,x)}$. Then we have the following:

Theorem A.2.3 (Russo [37], Goffeng [21]). *Let (X,μ) be a σ-finite measure space, and suppose that $\mathbf{K} \in B(L^2(X,\mu))$ is an integral operator with kernel $K : X \times X \to \mathbb{C}$, i.e.*

$$\mathbf{K}u = \int_X K(\cdot,y)u(y)d\mu(y), \quad u \in L^2(X,\mu).$$

Then for each $2 < p < \infty$, we have that

$$\|\mathbf{K}\|_{S_p} \leqslant \left(\|K\|_{q,p}\|K^*\|_{q,p}\right)^{\frac{1}{2}}$$

where q is the Hölder conjugate of p defined by the equation $\frac{1}{p} + \frac{1}{q} = 1$. In particular, if both $\|K\|_{q,p}$ and $\|K^*\|_{q,p}$ are finite, then $\mathbf{K} \in S_p(L^2(X,\mu))$.

This result is originally due to Russo [37] with the additional assumption that $K \in L^2(X \times X, \mu \otimes \mu)$. It was Goffeng [21] who later observed that this condition may be omitted. Paired with Lemma 3.8.6, we have nice tests for when integral operators belong to the Schatten classes for $2 \leqslant p < \infty$. When $p < 2$, no such simple criteria involving integrability of the kernel function exists. For a discussion on this matter, as well as alternative sufficient conditions, see Delgado and Ruzhansky [12].

A.3 The Adjoint of the Jump Formula

We now shift our attention to the Dirichlet space case. Recall that in Section 4.4, we had the identity

$$\dot{\mathbf{J}}_{d,d} = -\dot{\mathbf{J}}_{c,d}\dot{\mathbf{O}}_{d,c}$$

relating the limits of Cauchy integrals on each side of the curve, as well as the homogeneous jump formula

$$\dot{\mathbf{J}}_{d,d} - \dot{\mathbf{O}}_{c,d}\dot{\mathbf{J}}_{d,c} = \mathbf{I}.$$

Using our adjoint formulas from Section 4.5, it turns out these identities are equivalent in a sense. Indeed, taking the adjoint of the jump formula, we get that

$$\mathbf{I} = \mathbf{I} + i\overline{\dot{\mathbf{J}}}_{d,d} *_d - \overline{\dot{\mathbf{J}}}_{c,d}\left(*_c \dot{\mathbf{O}}_{d,c} *_d\right)$$

and hence

$$\begin{aligned} \overline{\dot{\mathbf{J}}}_{d,d} = -i\overline{\dot{\mathbf{J}}}_{c,d}\left(*_c \dot{\mathbf{O}}_{d,c}\right) &= -\overline{\dot{\mathbf{J}}}_{c,d}\left(\dot{\mathbf{P}}_c - \overline{\dot{\mathbf{P}}}_c\right)\dot{\mathbf{O}}_{d,c} \\ &= -\overline{\dot{\mathbf{J}}}_{c,d}\dot{\mathbf{O}}_{d,c}. \end{aligned}$$

Taking the complex conjugate now yields the desired identity. As it stands, this is merely a curiosity, since our proof of the overfare adjoint formula required the use of both of these identities. If the overfare adjoint identity could be proven independently (eg. mimic the Stokes' Theorem argument on smooth curves from Section 4.5), then this computation could be used to derive one identity from the other, of course provided further circular reasoning is ruled out.

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