

Generalized amenability in C^* -algebras and quantum groups

by

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Abstract

Let A be an operator algebra and X be a dual Banach A -bimodule. A bounded linear map $D : A \rightarrow X$ is called a derivation if $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. The algebra is said to be amenable if for each derivation D from A into any dual Banach A -bimodule X , there is an $x \in X$ such that $D(a) = ax - xa$ for all $a \in A$. If for each derivation D , there is a net (or sequence) (x_i) in X such that $D(a) = \lim_i (ax_i - x_i a)$ for all $a \in A$, either in the norm or weak* topology of X , then A is called approximately (sequentially) amenable. Clearly, amenability implies approximate amenability; however, the converse does not hold in general. A question posed by Ghahramani, Zhang, and others asks whether this converse implication holds for C^* -algebras.

In this thesis, we explore generalized amenability in C^* -algebras, von Neumann algebras, quantum groups, and certain operator algebras arising in non-commutative function theory. We first show that, for certain locally compact quantum groups, approximate amenability implies amenability. This allows us to recover and extend some classical results of Ghahramani et al.

Secondly, we study the sequential form of Ghahramani et al.'s question. We show that every sequentially amenable C^* -algebra is amenable. We establish this by considering the enveloping von Neumann algebra of the given C^* -algebra and a generalized version of Connes amenability for von Neumann algebras. Our proof relies on the theories of Alain Connes, Tomita–Takesaki for von Neumann algebras, spatial normal tensor products (laid out in Appendix B), lifting derivations via Hilbert W^* -modules, and the weak* decomposition and Radon–Nikodym theorem for quantum expectations (developed in Appendix A).

Afterwards, we introduce a non-commutative function-theoretic approach to study approximate amenability. This point of view leads to generalizations of several classical results. In particular, we study the Cuntz–Toeplitz C^* -algebra $\mathcal{E}_d$, the universal C^* -algebra generated by d non-commuting isometries. We show that $\mathcal{E}_d$ is not a compact quantum group, while it possesses a compact quantum semigroup structure. However, we prove that it is an amenable Banach algebra.

Finally, in Appendix A, we study weak* continuity and weak* singularity for quantum expectations, and we introduce a weak* decomposition for them. Using the Grothendieck–Haagerup inequality together with the notion of KMS states, we show that our weak* decomposition agrees with that of Arveson–Gheondea–Kavruk. This enables us to compute the Radon–Nikodym derivative for quantum expectations, which is then used in Chapter 4 to prove the equivalence between sequential Connes amenability and Connes amenability for finite von Neumann algebras.

Dedication

To my family:
my wife Maryam,
my sons Rojman and Jivan,
my mother Fatemeh,
and to the memory of my father Seddiq

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Chapter 1

Introduction

1.1 Mean-amenability of groups

The roots of the concept of amenability can be traced back to Lebesgue, who introduced his integral and wondered whether the monotone convergence theorem could be deduced from his existing axioms for integration. This property relies crucially on the countable additivity of the measure, and it appears difficult to relax this requirement. In fact, Banach gave a negative answer to Lebesgue's question by constructing a finitely additive measure on $\mathbb{R}$ that is translation invariant and, interestingly, defined on all of its subsets. Later, Banach and Tarski used the axiom of choice and showed that one cannot construct a measure on $\mathbb{R}^n$, $n \geq 3$, such that it is finitely additive, invariant under solid motions of the space (translations, rotations, and reflections), and is defined on all subsets of $\mathbb{R}^n$. Otherwise, if such a measure existed, the Banach–Tarski paradox would imply that a solid object as small as a pea could be decomposed into finitely many pieces and reassembled, using only rigid motions, into an object as large as the Earth. Of course, to avoid such a paradox, one needs to restrict the domain of the measure to the so called measurable sets.

In 1929, von Neumann studied the Banach–Tarski's paradox, and noticed that their proof is based on two assumptions:

- The axiom of choice, and
- the fact that the free group $\mathbb{F}_2$ on two generators fails to possess a property which he called *amenability*.

Von Neumann called a discrete group G *amenable* if there exists a finitely additive probability measure m on all subsets of G such that

$$m(xE) = m(E), \quad \forall E \subseteq G, \forall x \in G,$$

and

$$m(G) = 1.$$

The first systematic study of amenability began with Mahlon Day's paper [22]. He noticed that integration against such a positive measure m gives rise to a positive linear functional μ on $\ell_\infty(G)$

such that

$$\begin{aligned}\mu(f) &= \int f \, dm, \quad \text{and} \\ \mu(1) &= \int 1 \, dm = m(G) = 1, \\ \|\mu\| &= 1\end{aligned}$$

Day called such functionals a *mean*, and he then placed their theory on a more functional analytic foundation. For the indicator function $f = \chi_E$ of the set E , we have

$$m(E) = \mu(\chi_E).$$

This formula builds a bridge between measure theory and functional analysis. Now, consider the left translation operator L_x defined on $\ell_\infty(G)$ by

$$L_x f(y) = f(xy), \quad \text{where } x, y \in G.$$

Then,

$$m(a.E) = \mu(\chi_{a.E}) = \mu(L_{a^{-1}}(\chi_E)).$$

So, the left translation-invariant property $m(a.E) = m(E)$ translates to

$$\mu(L_{a^{-1}}(\chi_E)) = \mu(\chi_E), \quad \text{for all } a \in G.$$

Since the characteristic functions are dense in L^p spaces, we can substitute χ_E by a general function f . However, when G is a semigroup, we must avoid the inverses; hence, the following definition is more appropriate:

Definition 1.1.1. A positive linear functional μ on $\ell_\infty(G)$ is called a mean if $\mu(1) = 1 = \|\mu\|$. A mean is called a left invariant mean (LIM) if $\mu(L_x f) = \mu(f)$.

M. Day then defines amenability as follows:

Definition 1.1.2. A discrete group G is *amenable* if there exists a left-invariant mean (LIM) on $\ell_\infty(G)$.

The decade 1960's saw a surge in abstract harmonic analysis, so one could generalize Definitions 1.1.1 and 1.1.2 for any locally compact group by using $L^\infty(G)$:

Definition 1.1.3. A locally compact group G is *amenable* if there exists a left-invariant mean (LIM) on $L^\infty(G)$.

Sometimes one can use a more intuitive approach, known as the Følner condition, to show that a group G is amenable and, in that case, to construct a LIM on it; see [59] and [76, P.54].

Theorem 1.1.4. A locally group G with Harr measure λ is amenable if and only if for each compact subset K of G and for any $\varepsilon > 0$, there is a Borel subset B of G with $0 < \lambda(B) < \infty$ such that

$$\frac{\lambda[(xB \setminus B) \cup (B \setminus xB)]}{\lambda(B)} < \varepsilon, \quad \forall x \in K.$$

It is known that any locally compact abelian group and any compact group are amenable, see [59] and [76]. We list some examples of amenable and non-amenable groups here.

Example 1.1.5. The following groups are all amenable:

- The additive groups of $\mathbb{Z}$, $\mathbb{R}$ are amenable. Also, the additive group of real numbers with discrete topology, denoted by $\mathbb{R}_d$, is amenable.
- The unit circle $\mathbb{T}$ under complex multiplication is a compact amenable group.
- $ax + b$ group: Consider the upper half of the complex plain $\mathbb{H}^+ = \mathbb{R} \times (0, \infty)$ with the semi-direct product

$$(b, a)(b', a') = (b + ab', aa').$$

Now, if we endow $\mathbb{H}^+$ with the Euclidean topology carried from $\mathbb{R}^2$, then it becomes a locally compact non-abelian group with the left Haar measure $d\lambda = \frac{dx dy}{y^2}$. This group is amenable.

- Every solvable group is amenable. In particular, the Heisenberg group

$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

is amenable, see [76, Chapter 1].

- The group of rotations in the 3 dimensional space, $SO(3)$, under the Euclidean topology is compact, and hence amenable. The same is true for $SO(n)$ in $\mathbb{R}^n$ when $n \geq 3$ with the usual Euclidean topology.

Remark 1.1.6. Before presenting the next example, we recall some facts from [76, Chapter 1]. Firstly, amenability is preserved under continuous onto homomorphisms. Secondly, amenability is inherited by closed subgroups (in the same topology as the ambient group). Thirdly, if we equip the group G with the discrete topology, denoted by G_d , then the amenability of G_d implies the amenability of G . However, the converse is not true.

Example 1.1.7. The followings are examples of non-amenable groups:

- The free group $\mathbb{F}_2$ on two generators, equipped with the discrete topology, is not amenable. Notably, $\mathbb{F}_2$ is an algebraic subgroup (though not closed) of the compact amenable group $SO(3)$. This illustrates that amenability is not necessarily hereditary in the algebraic sense—that is, when considering only the algebraic structure, changing the topology can significantly alter amenability properties. However, if we equip $SO(3)$ with the discrete topology, denoted by $SO(3)_d$, then it will no longer be amenable. In this case, $\mathbb{F}_2$ becomes a closed subgroup of $SO(3)_d$.
- $SL_2(\mathbb{R})$, the space of invertible 2×2 invertible matrices with determinant equal to 1, is a locally compact group under the matrix multiplication and Euclidean topology. It has a normal closed subgroup $N = \{\pm I\}$ such that the quotient $\frac{SL_2(\mathbb{R})}{N}$ contains a closed subgroup isomorphic and homeomorphic to $\mathbb{F}_2$, see [76, Chapter 1]. So, $\frac{SL_2(\mathbb{R})}{N}$ cannot be amenable. Hence, $SL_2(\mathbb{R})$ cannot be amenable. As $SL_2(\mathbb{R})$ embeds into $SL_n(\mathbb{R})$ and $GL_n(\mathbb{R})$ for $n \geq 2$, one deduces that none of the latter groups is amenable.

We would like to mention that, in addition to its geometric origins, amenability has numerous applications. For instance, in statistical hypothesis testing, the Hunt-Stein theorem guarantees the existence of an optimal test if a certain group is amenable [59, Chapter 4]. Amenability also finds applications in automata and cellular automata theory [8], and even in differential geometry [76, Preface].

1.2 Derivation-amenability of Banach algebras

Despite its many interesting examples and applications, the approach to amenability of a group G via a left invariant state on $L^\infty(G)$ has a notable limitation: it relies heavily on the translation operators L_x . In 1972, Johnson [42] introduced a more algebraic characterization of amenability in terms of the convolution algebra $L^1(G)$, employing cohomological techniques and the notion of inner derivations.

Recall that $L^1(G)$ is the Banach algebra of all Haar-integrable functions on G , equipped with the norm

$$\|f\|_1 = \int |f(x)| d\lambda(x), \quad (1.1)$$

where λ is a Haar measure on G , and with the convolution product

$$(f \star g)(x) = \int f(xy^{-1})g(y) d\lambda(y). \quad (1.2)$$

Let X be a dual $L^1(G)$ -module. A bounded operator $D : L^1(G) \rightarrow X$ with the Leibnitz property $D(f \star g) = D(f) \cdot g + f \cdot D(g)$ is called a *derivation*. Johnson proved that G is amenable if and only if every such D is inner, i.e., there is an $x \in X$ such that $D(f) = f \cdot x - x \cdot f$ for any $f \in L^1(G)$. Johnson's work was a landmark since it initiated the theory of amenability in Banach algebra. Johnson's ideas were followed by Ruan for the setting of non-commutative functional analysis (i.e., operator spaces), called operator amenability. Finally, in [33] and [34] approximate versions of Johnson's amenability were introduced, requiring that every derivation D be a pointwise limit of a net of inner derivations.

1.3 Thesis outline

This thesis is organized as follows:

- In Chapter 2, we present the necessary preliminaries and basic definitions, while also placing the main notions in their historical and mathematical context. In particular, we review the development from Johnson's amenability to approximate amenability, introduced by Ghahremani et al. as a limiting form of Johnson's notion. We next move to the dual Banach algebra framework, where Connes amenability plays the corresponding role, and this leads naturally to the notion of approximate Connes amenability for von Neumann algebras. Since much of our work involves sequences of approximating inner derivations, we refer to the corresponding sequential version as *sequential amenability* in Banach algebras.
- In Chapter 3, we investigate approximate amenability for quantum groups in the von Neumann algebraic setting. After reviewing the basic definitions, we prove that, for certain classes

of discrete quantum groups, approximate amenability implies the existence of an invariant mean. This allows us to recover and unify several classical results, proved by Ghahremani et al., relating approximate amenability of group algebras to amenability of the underlying group. For instance, our quantum group approach implies that, for a discrete group G , the amenability of G , the approximate amenability of $\ell^1(G)$, and the approximate amenability of $C_r^*(G)$ are all equivalent.

- In Chapters 4 and 5, we continue our study of approximate amenability, showing that sequential amenability and amenability coincide for C^* -algebras. We first establish that the sequential amenability of a C^* -algebra ensures the sequential Connes amenability of its double dual, which is a von Neumann algebra. We then prove that, for von Neumann algebras, sequential Connes amenability implies Connes amenability. The proof of the latter result depends on the following key tools developed in this thesis:
 - (i) Relating sequential Connes amenability of finite von Neumann algebras to the existence of certain normal functionals. Specifically, we show that if a finite von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is sequentially Connes amenable, then it possesses a faithful normal tracial state τ which also satisfies a weak tracial property on $\mathcal{B}(\mathcal{H})$. We refer to such traces as *ultra traces*. We should mention that such states are called *amenable traces* in [7, Section 6.2].
 - (ii) A theory of *weak* decomposition of quantum expectations* on C^* -algebras, together with a computation of their relative *Radon–Nikodym derivatives*, developed in Appendix A. In Section 4.3, this theory is applied to show that if a finite von Neumann algebra $\mathcal{M}$ is sequentially Connes amenable, then the Radon–Nikodym derivative can be computed with respect to its ultra trace τ . This produces a conditional expectation onto $\mathcal{M}$ and therefore proves that $\mathcal{M}$ is Connes amenable.
 - (iii) We develop a Hilbert W^* -module technique for *lifting derivations* from a corner of a von Neumann algebra to the ambient von Neumann algebra. This allows us to prove that a corner inherits sequential Connes amenability from the ambient algebra, thereby reducing the study of semifinite von Neumann algebras to their finite corners. The lifting process requires the spatial normal tensor product for dual Banach spaces and its balanced version. The key definitions are given in Section 5.1, while the proofs of the basic properties are collected in Appendix B.
 - (iv) For type III von Neumann algebras, we use their crossed-product decomposition and the associated W^* -dynamical systems to reduce the problem to the semifinite case.

With these tools and reductions in hand, we are able to extend Connes’ theorem and show that every sequentially Connes amenable von Neumann algebra is injective. Our theorem then implies that, for C^* -algebras, sequential amenability implies amenability. This establishes the sequential form of a question posed by Ghahramani, Zhang, and others; see [33, Question 9.4], [36, Section 5], [91, Questions 16 & 17], and [92, Question 3.5]. The corresponding conclusion for nets is discussed in Chapter 3, where the investigation is only for discrete Kac algebras.

- In Chapter 6, we investigate the amenability of certain non-commutative operator algebras that arise naturally in non-commutative function theory (NCFT). One may regard NCFT as a non-commutative analogue of multi-variable complex analysis. The algebras in NCFT are

generated by finitely many operators subject to the Cuntz–Toeplitz relations. We examine amenability-like properties of these algebras. In particular, by employing techniques from NCFT, we show that the non-commutative Hardy algebra $\mathbb{H}_d^\infty$ —the counterpart of the classical Hardy algebra H^∞ —fails to be approximately amenable. We also study compact quantum groups in the C^* -algebraic setting. However, the Cuntz–Toeplitz C^* -algebra $\mathcal{E}_d$ is only a compact quantum semigroup and therefore does not admit a left invariant mean. Consequently, although $\mathcal{E}_d$ is derivation amenable, it fails to be mean-amenable.

- In Appendix [A](#), we study quantum expectations (i.e., states) on C^* -algebras and establish a Radon–Nikodym theorem for them. Given a pair of quantum expectations (μ, λ) on a C^* -algebra A , we introduce the notions of weak*-continuity and weak*-singularity of μ with respect to λ . We then decompose μ with respect to λ into a weak*-continuous part and a weak*-singular part. If λ satisfies the KMS condition, we use the Grothendieck–Haagerup inequality to show that our weak*-decomposition coincides with the Arveson–Gheondea–Kavruk decomposition of quantum expectations. This enables us to compute the Radon–Nikodym derivative of μ with respect to λ , denoted by $D_\lambda(\mu)$. We use this derivative in Section [4.3](#) to construct a conditional expectation for a finite, sequentially Connes amenable von Neumann algebra.
- We develop the basic properties of spatial normal tensor products and balanced normal tensor products in Appendix [B](#). These tensor products are used in Sections [5.1](#) and [5.2](#) to construct the lifted modules that appear in the Hilbert W^* -module lifting of derivations from finite corners to ambient von Neumann algebras.

Chapter 2

Preliminaries

2.1 Amenability of Banach algebras

Definition 2.1.1. Let A be a Banach algebra, and X be a Banach space. We call X a *left Banach A -Module* if X is an algebraic module over A , and if the bilinear map

$$\begin{aligned} A \times X &\longrightarrow X \\ (a, x) &\longrightarrow a \cdot x \end{aligned}$$

is bounded, i.e., $\|a \cdot x\| \leq C\|a\|\|x\|$ for some positive number C and for all $a \in A$ and $x \in X$. *Right Banach A -modules* are defined similarly. X is called a *Banach A -bimodule*, if it is both left and right A -module and the associativity

$$a.(x \cdot b) = (a \cdot x) \cdot b, \quad (a, b \in A, x \in X)$$

holds between left and right actions. The module X is *unital* if A has a unit 1 and

$$1 \cdot x = x, \quad \forall x \in X.$$

Remark 2.1.2. When X is a left A -module, we can make X^* into a right A -module via

$$\begin{aligned} X^* \times A &\longrightarrow X^* \\ (f, a) &\longrightarrow f \cdot a, \quad \langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle. \end{aligned}$$

In this case, we call X^* a *right dual Banach A -module*. In addition, we can make X^{**} into a left A -module by defining

$$\begin{aligned} A \times X^{**} &\longrightarrow X^{**} \\ (a, F) &\longrightarrow a \cdot F, \quad \langle a \cdot F, f \rangle = \langle F, f \cdot a \rangle. \end{aligned}$$

Similarly, we can start with a right A -module X and proceed to define left and right actions on X^* and X^{**} respectively.

We can regard A as a Banach A -bimodule with module actions given by its multiplication.

Definition 2.1.3. Let A be Banach algebra and A^{**} be its second dual. For any $\Phi, \Psi \in A^{**}$, we denote the first Arens product of Φ and Ψ by $\Phi \square \Psi$ and define it by :

$$\begin{aligned}\langle \Phi \square \Psi, f \rangle &= \langle \Phi, \Psi \cdot f \rangle, \quad f \in A^* \\ \langle \Psi \cdot f, a \rangle &= \langle \Psi, f \cdot a \rangle, \quad a \in A \\ \langle f \cdot a, b \rangle &= \langle f, ab \rangle, \quad b \in A.\end{aligned}$$

We also have the second Arens product of Φ and Ψ , denoted by $\Phi \diamond \Psi$, defined by :

$$\begin{aligned}\langle \Phi \diamond \Psi, f \rangle &= \langle \Psi, f \cdot \Phi \rangle, \quad f \in A^* \\ \langle f \cdot \Phi, a \rangle &= \langle \Phi, a \cdot f \rangle, \quad a \in A \\ \langle a \cdot f, b \rangle &= \langle f, ba \rangle, \quad b \in A.\end{aligned}$$

Remark 2.1.4. 1. Consider the natural embedding $j : A \longrightarrow A^{**}$, and equip A^{**} with the first Arens product. For any $a, b \in A$, let $\Phi = j(a)$ and $\Psi = j(b)$. By the first Arens product, for each $f \in A^*$ we have

$$\langle j(a) \square j(b), f \rangle = \langle f, ab \rangle = \langle j(ab), f \rangle.$$

Hence, $j : A \longrightarrow (A^{**}, \square)$ is a homomorphism. Now, we can define the first and second Arens product using the natural embedding j . Recall that $\overline{j(A)}^{wk*} = A^{**}$ by the Goldstine theorem; here wk^* denotes the weak* topology $\sigma(A^{**}, A^*)$. So,

$$\Phi = w^*\lim_{\alpha} j(a_{\alpha}), \quad \Psi = w^*\lim_{\beta} j(b_{\beta}).$$

Thus, we can define

$$\begin{aligned}\Phi \square \Psi &= w^*\lim_{\alpha} w^*\lim_{\beta} j(a_{\alpha} b_{\beta}), \\ \Phi \diamond \Psi &= w^*\lim_{\beta} w^*\lim_{\alpha} j(a_{\alpha} b_{\beta}).\end{aligned}$$

2. The natural question arises as to whether the first and second Arens products on the second dual A^{**} of a Banach algebra A coincide. If they do, A is called *Arens regular*; otherwise, A is called *Arens irregular*. It is known that for a locally compact group G , the convolution Banach algebra $L^1(G)$ is Arens regular if and only if G is finite [76, Example 5.1.19]. By contrast, every operator algebra is Arens regular [4, Corollar 2.5.4].
3. Finally, we should mention that if a Banach algebra A is Arens regular, then the Arens product on A^{**} is separately weak* continuous [4, Remark 2.5.3].

Definition 2.1.5. Let A be a Banach algebra, and X be a Banach A -bimodule. A bounded linear map $D : A \longrightarrow X$ is called a *derivation* if it enjoys the Leibniz product rule

$$D(ab) = D(a) \cdot b + a \cdot D(b), \quad \forall a, b \in A.$$

The set of all derivations is denoted by $Z^1(A, X)$.

Example 2.1.6. Let A be a Banach algebra, X be a Banach A -bimodule, and $x \in X$. Define

$$\begin{aligned}ad_x : A &\longrightarrow X \\ ad_x(a) &= a \cdot x - x \cdot a\end{aligned}$$

Then, ad_x is a derivation, called an *inner derivation*. The set of all inner derivations is denoted by $B^1(A, X)$, which is a linear subspace of $Z^1(A, X)$.

Definition 2.1.7. Suppose that A is a Banach algebra and X is a Banach A -bimodule. Then, the quotient space

$$H^1(A, X) = \frac{Z^1(A, X)}{B^1(A, X)}$$

is called the *first cohomology group* of A with coefficients in X .

Definition 2.1.8. A Banach algebra A is called *amenable* if for any Banach A -bimodule X , we have $H^1(A, X^*) = 0$. That is, every derivation $D : A \rightarrow X^*$ is inner. Here, X^* is the dual module as defined in Remark 2.1.2.

Definition 2.1.9. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. For $\xi, \eta \in \mathcal{H}$, the linear functional

$$\begin{aligned} \omega_{\xi, \eta} : \mathcal{B}(\mathcal{H}) &\longrightarrow \mathbb{C} \\ \omega_{\xi, \eta}(T) &= \langle T\xi, \eta \rangle \end{aligned}$$

is called the *vector functional* associated with ξ and η . If $\xi = \eta$, then $\omega_{\xi, \xi}$ is a positive vector functional and is denoted by ω_ξ . If $\|\xi\| = 1$, then ω_ξ has norm one and is called a *vector state*.

- (i) The *weak operator topology*, denoted by WOT, is the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the seminorms

$$T \longmapsto |\omega_{\xi, \eta}(T)| = |\langle T\xi, \eta \rangle|, \quad (\xi, \eta \in \mathcal{H}).$$

- (ii) The *σ -weak operator topology*, also called the *ultraweak topology* or the weak* topology on $\mathcal{B}(\mathcal{H})$, denoted by $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})_*)$, is the locally convex topology generated by the seminorms

$$T \longmapsto \left| \sum_{n=1}^{\infty} \omega_{\xi_n, \eta_n}(T) \right|,$$

where $\{\xi_n\}_{n=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$ are sequences in $\mathcal{H}$ satisfying

$$\sum_{n=1}^{\infty} \|\xi_n\|^2 < +\infty, \quad \sum_{n=1}^{\infty} \|\eta_n\|^2 < +\infty.$$

- (iii) The *strong operator topology*, denoted by SOT, is the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the seminorms

$$T \longmapsto \omega_\xi(T^*T)^{1/2} = \|T\xi\|, \quad (\xi \in \mathcal{H}).$$

- (iv) The *σ -strong operator topology* is the locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by the seminorms

$$T \longmapsto \left(\sum_{n=1}^{\infty} \omega_{\xi_n}(T^*T)^{1/2} \right)^{1/2},$$

where $\{\xi_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{H}$ satisfying $\sum_{n=1}^{\infty} \|\xi_n\|^2 < +\infty$.

We now present some examples and well-known theorems on the relationship between the amenability of a locally compact group G and various Banach algebras associated to it. Before doing so, we recall some standard definitions from abstract harmonic analysis. Let m be a fixed left Haar measure on G and Δ be the modular function of G , which is a continuous group homomorphism from G to the set of positive real numbers under scalar multiplication such that $m(Ex) = \Delta(x)m(E)$ for all $x \in G$ and all Borel subset $E \subseteq G$; see [31, Section 2.4] for more information. Then, the group algebra $L^1(G)$ is a Banach $*$ -algebra under the convolution product (1.2) and the involution

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}, \quad \forall x \in G. \quad (2.1)$$

Recall that a unitary representation

$$\pi : G \longrightarrow \mathcal{U}(\mathcal{H}_\pi)$$

is a group homomorphism from G to the group of unitary operators $\mathcal{U}(\mathcal{H}_\pi)$ on a Hilbert space $\mathcal{H}_\pi$, which is continuous when $\mathcal{U}(\mathcal{H}_\pi)$ is equipped with the strong operator topology. Any such representation π can be lifted to a non-degenerate $*$ -representation of $L^1(G)$, denoted by $\tilde{\pi}$, i.e.,

$$\begin{aligned} \tilde{\pi} : L^1(G) &\longrightarrow \mathcal{B}(\mathcal{H}_\pi) \\ f &\longmapsto \tilde{\pi}(f), \end{aligned}$$

such that for all $\xi, \eta \in \mathcal{H}_\pi$

$$\langle \tilde{\pi}(f)\xi, \eta \rangle = \int f(x) \langle \pi(x)\xi, \eta \rangle dm(x).$$

Sometime $\tilde{\pi}$ is still denoted by π . By [31, Section 3.2], there is a one-to-one correspondence between the non-degenerate $*$ -representations of $L^1(G)$ and unitary representations of G via this lifting process. We define $C^*(G)$, the group C^* -algebra of G , to be the completion of $L^1(G)$ with respect to the norm

$$\|f\|_* = \sup \|\tilde{\pi}(f)\|,$$

where the supremum is taken over all non-degenerate $*$ -representations $\tilde{\pi}$ of $L^1(G)$. Denote the set of all continuous positive type function on G by $P(G)$. These are functions $\phi \in C_b(G)$, which define positive linear functionals on $L^1(G)$ in the sense that for each $f \in L^1(G)$

$$\langle \phi, f^* \star f \rangle = \int (f^* \star f)\phi(x) dm(x) \geq 0.$$

We define $B(G)$ to be the linear span of $P(G)$. If $\pi : G \longrightarrow \mathcal{U}(\mathcal{H}_\pi)$ is a unitary representation of G , then for any $\xi, \eta \in \mathcal{H}_\pi$, the *matrix element* of π with respect to ξ and η , denoted by $\pi_{\xi, \eta}$, is the continuous function on G determined by the vector functional $\omega_{\xi, \eta}$; that is,

$$\begin{aligned} \pi_{\xi, \eta} : G &\longrightarrow \mathbb{C} \\ \pi_{\xi, \eta}(x) &= \langle \pi(x)\xi, \eta \rangle. \end{aligned}$$

By [30], $B(G)$ is generated by all matrix elements of all representations of G . Moreover, $B(G)$ is a Banach algebra under the pointwise multiplication and the norm defined by

$$\|\phi\| = \sup \{ |\langle \phi, f \rangle| : f \in L^1(G), \|f\|_* \leq 1 \}, \quad \forall \phi \in B(G); \quad (2.2)$$

where $\langle \phi, f \rangle = \int_G f(t) \phi(t) dm(t)$. This algebra is called the Fourier-Stieltjes algebra of G . It is known that $B(G)$ is the dual Banach space of $C^*(G)$ with duality given by the mapping

$$\begin{aligned} J : B(G) &\longrightarrow C^*(G)^* \\ \phi &\longmapsto J_\phi, \end{aligned} \tag{2.3}$$

which sends the matrix element $\phi = \pi_{\xi, \eta}$ to the vector functional $J_\phi = \omega_{\xi, \eta}$, i.e.,

$$\langle f, J_\phi \rangle = \langle \tilde{\pi}(f), \omega_{\xi, \eta} \rangle, \quad \forall f \in L^1(G).$$

By [30], the duality (2.3) is also given by

$$\langle f, \phi \rangle = \int f(t) \phi(t) dm(t), \quad \forall f \in L^1(G), \forall \phi \in B(G).$$

So, the norm 2.2 is in fact the norm determined by the Hahn-Banach dual pairing, see [74, Theorem 4.3]. Let $\mathcal{B}(L^2(G))$ be the set of all bounded operators on the Hilbert space $L^2(G)$ and

$$\begin{aligned} \lambda : G &\longrightarrow \mathcal{B}(L^2(G)) \\ x &\longrightarrow \lambda(x), \quad \lambda(x)h(y) = h(x^{-1}y) \quad \forall h \in L^2(G) \end{aligned}$$

be the left regular representation of G . By [31, P. 79], the lifted representation, still denoted by λ , is given by

$$\begin{aligned} \lambda : L^1(G) &\longrightarrow \mathcal{B}(L^2(G)) \\ f &\longrightarrow \lambda(f), \quad \lambda(f)h = f * h \quad \forall h \in L^2(G). \end{aligned}$$

The group von Neumann algebra generated by λ , $VN(G)$, is the closure of $\lambda(L^1(G))$ in the weak operator topology of $\mathcal{B}(L^2(G))$. By [31, Theorem 3.12], $VN(G)$ coincides with the closure of the linear span of $\lambda(G)$ in the weak operator topology.

Let $C_c(G)$ denote the set of all continuous functions on G with compact support. The Fourier algebra of G , denoted by $A(G)$, is defined as the closure, with respect to the norm in (2.2), of the linear span of $P(G) \cap C_c(G)$. It is known that $A(G)$ is the set of all matrix elements of the left regular representation, while $B(G)$ is the set of all matrix elements of all continuous unitary representations of G , [30]. It is well-known that $A(G) = B(G)$ when G is compact. In general, $A(G)$ is a closed ideal in $B(G)$ (see [30] for details). One can show that $A(G)$ is the pre-dual space of the von Neumann algebra $VN(G)$ with the following duality map:

$$\begin{aligned} E : VN(G) &\longrightarrow A(G)^* \\ T &\longrightarrow E_T \end{aligned}$$

where E_T acts on each matrix element $\lambda_{\xi, \eta} \in A(G)$ by evaluation on (or the dual pairing with) the vector functional $\omega_{\xi, \eta}$, i.e.,

$$\langle E_T, \lambda_{\xi, \eta} \rangle = \langle T, \omega_{\xi, \eta} \rangle.$$

When G is an abelian group, the above-mentioned objects take familiar forms. Recall that in this case, the dual of G , denoted by $\hat{G}$, is the set of all continuous group homomorphisms from G to the unit circle $\mathbb{T}$, equipped with the group operation of pointwise multiplication. The dual group $\hat{G}$ can be identified with the spectrum of the Banach algebra $L^1(G)$ and, under the weak* topology, it

becomes a locally compact abelian group; see [31, Chapter 4]. The identifications in the following table can be found in [24], [30], and [31].

General group	Abelian group
$A(G)$	$L^1(\hat{G})$
$B(G)$	$M(\hat{G})$
$C^*(G)$	$C_0(\hat{G})$
$VN(G)$	$L^\infty(\hat{G})$
$A(G) \trianglelefteq B(G)$	$L^1(\hat{G}) \trianglelefteq M(\hat{G})$
$B(G) \cong C^*(G)^*$	$M(\hat{G}) \cong C_0(\hat{G})^*$
$VN(G) \cong A(G)^*$	$L^\infty(\hat{G}) \cong L^1(\hat{G})^*$

Table 2.1: Standard identifications under Pontryagin duality (when G is abelian).

One of the most famous theorems in the theory of amenability is the following result due to Johnson [42].

Theorem 2.1.10. *A locally compact group G is amenable in the sense of Definition 1.1.2 (in terms of left invariant means for groups) if and only if $L^1(G)$ is amenable in the sense of Definition 2.1.8 (in terms of inner derivations for Banach algebras).*

Since $A(G) \cong L^1(\hat{G})$ in the abelian case, one might assume that the amenability of the Fourier algebra $A(G)$ preserves the same properties characterized by the above theorem. However, this is untrue. In fact, we have the following characterization:

Theorem 2.1.11. [76, Theorem 3.2.10] *The Fourier algebra $A(G)$ is amenable if and only if G has an abelian subgroup of finite index.*

Thus, even though the group $G = SO(3)$ is amenable, the above theorem implies that $A(G)$ is not amenable [43]. To overcome this asymmetry, one needs the notion of *operator amenability* for the Banach algebra $A(G)$ —that is, amenability of the algebra at all matricial levels when it acts on operator spaces. In this general setting, the following fundamental theorem due to Ruan [72] holds.

Theorem 2.1.12. [76, Theorem 8.2.1] *A locally compact group G is amenable if and only if the Fourier algebra $A(G)$ is operator amenable.*

2.2 Connes amenability of dual Banach algebras and approximate amenability of Banach algebras

We have the following theorems regarding the Johnson’s amenability of Banach algebras $B(G)$ and $M(G)$.

Theorem 2.2.1. [76, Theorem 3.2.11] *The Fourier-Stieltjes algebra $B(G)$ is amenable if and only if G has an abelian compact subgroup of finite index.*

Theorem 2.2.2. [76, Theorem 3.1.1] *The measure algebra $M(G)$ is amenable if and only if G is discrete and amenable.*

Here, the asymmetry in results is due to the fact that Johnson's amenability does not take into account the *dual space structure* of $B(G)$ and $M(G)$. Bringing the dual space structure into account is precisely what Connes' amenability is designed to address. Indeed, many well-known Banach algebras such as $L^\infty(G)$, $M(G)$, $B(G)$, and $VN(G)$ are dual Banach spaces. These algebras serve as primary examples of the so-called *dual Banach algebras*, a concept first systematically studied in [44].

Definition 2.2.3. A *dual Banach algebra* is a pair (A, A_*) of Banach spaces such that :

- A is a Banach algebra,
- $A = (A_*)^*$,
- the multiplication of A is weak*-separately continuous.

For dual Banach algebras, we may consider dual Banach modules defined as follows.

Definition 2.2.4. Let A be a dual Banach algebra, and let X be a dual Banach space.

- (i) We say that X is a *dual Banach right A -module* if X is a right Banach A -module and, for each $a \in A$, the right multiplication operator

$$R_a : X \longrightarrow X, \quad x \longmapsto x \cdot a,$$

is weak*-weak* continuous.

- (ii) We say that X is a *normal right A -module* if it is a dual Banach right A -module and, for each $x \in X$, the right orbital map

$$r_x : A \longrightarrow X, \quad a \longmapsto x \cdot a,$$

is weak*-weak* continuous.

- (iii) Similarly, we say that X is a *dual Banach left A -module* if X is a left Banach A -module and, for each $a \in A$, the left multiplication operator

$$L_a : X \longrightarrow X, \quad x \longmapsto a \cdot x,$$

is weak*-weak* continuous.

- (iv) We say that X is a *normal left A -module* if it is a dual Banach left A -module and, for each $x \in X$, the left orbital map

$$\ell_x : A \longrightarrow X, \quad a \longmapsto a \cdot x,$$

is weak*-weak* continuous.

Definition 2.2.5. Let A be a dual Banach algebra, and let X be a dual Banach space.

- (i) X is called a *dual Banach A -bimodule* if it is both a dual Banach left A -module and a dual Banach right A -module in the sense of Definition 2.2.4.
- (ii) X is called a *normal A -bimodule* if it is both a normal left A -module and a normal right A -module in the sense of Definition 2.2.4. Equivalently, the module actions are separately weak*-weak* continuous; that is, the maps

$$A \times X \longrightarrow X, \quad (a, x) \longmapsto a \cdot x,$$

and

$$X \times A \longrightarrow X, \quad (x, a) \longmapsto x \cdot a,$$

are weak*-weak* continuous in each variable separately.

- (iii) For a dual Banach algebra A and a normal A -bimodule X , a bounded derivation

$$D : A \longrightarrow X$$

is called a *normal derivation* if it is weak*-weak* continuous.

Remark 2.2.6. Assume that A is a dual Banach algebra.

- (i) Let X be a normal left A -module. Then, for each $x \in X$ and $f \in X_*$, the vector functional

$$\begin{aligned} x \odot f : A &\longrightarrow \mathbb{C} \\ x \odot f(a) &= \langle f, a \cdot x \rangle \end{aligned}$$

is weak* continuous; equivalently, $x \odot f \in A_*$.

- (ii) Let X be a normal right A -module. Then, for each $x \in X$ and $f \in X_*$, the vector functional

$$\begin{aligned} f \odot x : A &\longrightarrow \mathbb{C} \\ f \odot x(a) &= \langle f, x \cdot a \rangle \end{aligned}$$

is weak* continuous; equivalently, $f \odot x \in A_*$.

- (iii) In particular, if X is a normal A -bimodule, then for each $x \in X$ and $f \in X_*$, both vector functionals $x \odot f$ and $f \odot x$ are weak* continuous; equivalently, $x \odot f, f \odot x \in A_*$.

Definition 2.2.7. A dual Banach algebra A is called *Connes-amenable*, if for any normal dual Banach A -bimodule X , every normal derivation $D : A \longrightarrow X$ is inner.

By taking the dual Banach algebra structure into account, Connes amenability restores the expected symmetry, as shown in the following theorem [76, Theorem 5.1.27]:

Theorem 2.2.8. *For a locally compact group G the following are equivalent:*

- (i) G is amenable
- (ii) $M(G)$ is Connes-amenable
- (iii) $VN(G)$ is Connes-amenable

Suppose that A is a Banach algebra, X is an A -bimodule, and the derivation $D : A \longrightarrow X^*$ is not inner. A natural question is whether D can be approximated by inner derivations. This idea was first explored in [38], [33], [34], and [36], and it led to the generalized notion of amenability.

Definition 2.2.9. A Banach algebra A is called

- (i) *approximately amenable* if for every Banach A -bimodule X and every bounded derivation $D : A \longrightarrow X^*$ there is a net $(f_i)_{i \in I}$ in X^* such that

$$D(a) = \|\cdot\|_X - \lim_i (a \cdot f_i - f_i \cdot a), \quad \forall a \in A.$$

If the implementing net is a sequence, A is called *sequentially amenable*.

- (ii) *weak* approximately amenable* if for every Banach A -bimodule X and every bounded derivation $D : A \longrightarrow X^*$ there is a net $(f_i)_{i \in I}$ in X^* such that

$$D(a) = w^* - \lim_i (a \cdot f_i - f_i \cdot a), \quad \forall a \in A.$$

If the implementing net is a sequence, A is called *weak* sequentially amenable*.

We should remark that, by [38, Proposition 1], in the preceding definition one may always assume that the implementing net (sequence) is unbounded.

Example 2.2.10. It follows immediately from the definition that every amenable Banach algebra is sequentially amenable. Also, every sequentially amenable Banach algebra is approximately amenable. The converses, however, fail in general. We record some standard examples.

- (i) Consider the Heisenberg group $\mathbb{H}$ introduced in Example 1.1.5. By [76, Example 4.4.4], the Fourier algebra $A(\mathbb{H})$ is approximately amenable but not amenable.
- (ii) There are approximately amenable Banach algebras which are neither sequentially amenable nor amenable. Let Λ be an uncountable well-ordered set, and define a semigroup product on Λ by

$$\alpha \vee \beta = \max\{\alpha, \beta\}, \quad (\alpha, \beta \in \Lambda).$$

We denote this semigroup by $\Lambda_\vee$. By [9, Theorem 6.1 & Theorem 6.4], the convolution algebra $\ell^1(\Lambda_\vee)$ is approximately amenable. However, it is neither sequentially amenable nor amenable.

- (iii) There are also Banach algebras which are sequentially amenable but not amenable. Let $\mathbb{N}_\wedge$ be the set of natural numbers with the product defined by

$$m \wedge n = \min\{m, n\}, \quad (m, n \in \mathbb{N}).$$

This makes $\mathbb{N}_\wedge$ into a semigroup. By [34, Example 4.6], the convolution algebra $\ell^1(\mathbb{N}_\wedge)$ is sequentially amenable, but it is not amenable. Also, [33, Example 6.1] provides another example of a sequentially amenable Banach algebra which is not amenable.

The following result is established in [34, Theorem 2.1]:

Theorem 2.2.11 ([34]). *A Banach algebra A is approximately amenable if and only if it is weak* approximately amenable.*

Remark 2.2.12. The above theorem shows that, as long as one works with nets, weak* and norm approximations coincide. However, as noted in [34], this equivalence may fail when nets are replaced by sequences. Therefore, when discussing sequential versions of amenability, it is important to specify the type of convergence being assumed. This also implies that approximate amenability is weaker than sequential amenability.

2.3 Von Neumann algebras

Recall that a *von Neumann algebra* is a unital *-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ that is closed in the strong (or, equivalently, weak) operator topology. The von Neumann commutant $\mathcal{M}'$ is the set of all those operators $T \in \mathcal{B}(\mathcal{H})$ that commute with all members of $\mathcal{M}$. Alternatively, $\mathcal{M}$ can be characterized as equal to its bicommutant $\mathcal{M}''$; see [56, Chapter 4]. The following definition is taken from [76], [81], and [82].

Definition 2.3.1. Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{M}^+$ be its cone of positive elements. A function $\phi : \mathcal{M}^+ \rightarrow [0, +\infty]$ is called a *weight* if:

- it is additive, i.e., $\phi(x + y) = \phi(x) + \phi(y)$ for $x, y \in \mathcal{M}^+$,
- it is positive homogenous, i.e., $\phi(\lambda x) = \lambda \phi(x)$ for $\lambda \geq 0$ and $x \in \mathcal{M}^+$.

Here, we accept the convention that $0(\infty) = 0$. We define some important sets determined by ϕ , which are important to us later:

- $\mathfrak{p}_\phi = \{x \in \mathcal{M}^+ : \phi(x) < \infty\}$, and let $\mathfrak{m}_\phi$ be the linear span of $\mathfrak{p}_\phi$.
- $\mathfrak{n}_\phi = \{x \in \mathcal{M} : \phi(x^*x) < \infty\}$, which is a left ideal in $\mathcal{M}$ by [82, Lemma VII-2.1].
- In fact, $\mathfrak{m}_\phi = \mathfrak{n}_\phi^* \mathfrak{n}_\phi = \{\sum_{j=1}^n y_j^* x_j : n \in \mathbb{N}, x_j, y_j \in \mathfrak{n}_\phi\}$, and it is a hereditary *-subalgebra of $\mathcal{M}$ such that $\mathfrak{m}_\phi \cap \mathcal{M}^+ = \mathfrak{p}_\phi$ by [82, Lemma VII-2.1] and [53, P. 842]. $\mathfrak{m}_\phi$ is called the *definition subalgebra* of ϕ .
- $N_\phi = \{x \in \mathcal{M} : \phi(x^*x) = 0\}$, which is the left ideal of zero length elements in the GNS construction.

We can extend the weight ϕ to a linear map on $\mathfrak{m}_\phi$. The weight ϕ is called

- *finite* if $\mathfrak{p}_\phi = \mathcal{M}^+$. This is equivalent to $\phi(1) < \infty$.
- *semifinite* if $\mathfrak{m}_\phi$ is weak* dense in $\mathcal{M}$. In other words, every nonzero $x \in \mathcal{M}^+$ majorizes some $y > 0$, $y \in \mathcal{M}$, with $\phi(y) < \infty$.
- *faithful* if $\phi(x) = 0$ for any $x \in \mathcal{M}^+$ implies $x = 0$.
- *normal* if it has the monotone convergence property for the bounded increasing nets, i.e., for any bounded increasing net (x_i) in $\mathcal{M}^+$ one must have $\phi(\sup_i x_i) = \sup_i \phi(x_i)$.
- *tracial* if it satisfies the trace property $\phi(x^*x) = \phi(xx^*)$ for all $x \in \mathcal{M}$.

Remark 2.3.2. Since $\mathcal{M}^+$ spans $\mathcal{M}$, any finite weight ϕ can be extended to a bounded linear functional on $\mathcal{M}$. Additionally, for a linear functional defined on all of $\mathcal{M}$, normality is equivalent to weak*-continuity. However, when $\mathfrak{m}_\phi$ is only a proper subspace of $\mathcal{M}$, the notion of weak*-continuity is not applicable. It is also worth noting that the concept of normality originates from the monotone convergence theorem in measure and integration theory.

Remark 2.3.3. By [82, Theorem VII.2.7], every von Neumann algebra $\mathcal{M}$ admits a normal, faithful, semifinite weight ϕ . If, in addition, ϕ is tracial, then $\mathcal{M}$ is called a *semifinite* von Neumann algebra [81, Theorem V.2.15]. In this case, the tracial property of ϕ ensures that $\mathfrak{m}_\phi$ is a (two-sided) ideal of $\mathcal{M}$ [81, p. 319], referred to as the *definition ideal* of ϕ . Moreover, the formula

$$\|x\|_1 = \phi(|x|), \quad (x \in \mathfrak{m}_\phi) \quad (2.4)$$

defines a norm on $\mathfrak{m}_\phi$. The completion of $\mathfrak{m}_\phi$ with respect to this norm is denoted by $L^1(\mathcal{M}, \phi)$, and it is isometrically isomorphic to the predual $\mathcal{M}_*$ of $\mathcal{M}$ [81, p. 320]. This isomorphism is given by

$$\begin{aligned} L^1(\mathcal{M}, \phi) &\cong \mathcal{M}_*, \\ h &\mapsto \phi_h, \quad \phi_h(z) = \phi(zh), \quad z \in \mathcal{M}. \end{aligned}$$

In addition, $L^1(\mathcal{M}, \phi)$ is a Banach $\mathcal{M}$ -bimodule since for all $a \in \mathcal{M}$ and $\xi \in L^1(\mathcal{M}, \phi)$

$$\begin{aligned} \|a\xi\|_1 &\leq \|a\| \|\xi\|_1, \\ \|\xi a\|_1 &\leq \|\xi\|_1 \|a\|. \end{aligned}$$

In this case, the von Neumann algebra $\mathcal{M}$ may be denoted by $L^\infty(\mathcal{M}, \phi)$ and the norm of $a \in \mathcal{M}$ by $\|a\|_\infty$. In the ideal $\mathfrak{n}_\phi$, we define a sesquilinear form

$$\langle x | y \rangle_\phi = \phi(y^*x), \quad (x, y \in \mathfrak{n}_\phi).$$

By the positivity and faithfulness of ϕ , this sesquilinear form is an inner product in $\mathfrak{n}_\phi$, and we write the norm of $x \in \mathfrak{n}_\phi$ by

$$\|x\|_2 = \phi(x^*x)^{1/2}. \quad (2.5)$$

We denote by $L^2(\mathcal{M}, \phi)$ the completion of $\mathfrak{n}_\phi$ under $\|\cdot\|_2$. It is also a Banach $\mathcal{M}$ -bimodule, and for all $a \in \mathcal{M}$, $x \in L^2(\mathcal{M}, \phi)$, we have

$$\begin{aligned} \|ax\|_2 &\leq \|a\|_\infty \|x\|_2, \\ \|xa\|_2 &\leq \|a\|_\infty \|x\|_2. \end{aligned}$$

The $*$ -operation in $\mathfrak{n}_\phi$ is an isometry with respect to the $\|\cdot\|_2$ -norm; hence, it is extended to a conjugate linear isometry

$$\begin{aligned} J : L^2(\mathcal{M}, \phi) &\rightarrow L^2(\mathcal{M}, \phi) \\ J(x) &= x^* \end{aligned}$$

with period two.

As an example, let $\mathcal{H}$ be a separable Hilbert space, and let ϕ denote the canonical trace on $\mathcal{M} = \mathcal{B}(\mathcal{H})$. Then,

$$\begin{aligned} L^1(\mathcal{M}, \phi) &= \mathcal{B}^1(\mathcal{H}), \\ L^2(\mathcal{M}, \phi) &= \mathcal{B}^2(\mathcal{H}). \end{aligned}$$

Here, $\mathcal{B}^1(\mathcal{H})$ is ideal of all trace class operators, and $\mathcal{B}^2(\mathcal{H})$ denotes the ideal of all Hilbert–Schmidt operators. Moreover, by [56, Theorem 4.2.3], every weak*-continuous functional ω on $\mathcal{M}$ is given by the trace and a trace class operator x , i.e.,

$$\omega(y) = \phi(yx) = \phi_x(y), \quad (y \in \mathcal{M}).$$

Definition 2.3.4. Let $\mathcal{M}$ be a von Neumann algebra contained in some $\mathcal{B}(\mathcal{H})$. Denote by $\mathcal{P}(\mathcal{M})$ the set of all projections in $\mathcal{M}$, and let $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ be the center of $\mathcal{M}$. Recall that the center is always an abelian von Neumann algebra, which contains $\mathbb{C}I$.

- $\mathcal{M}$ is called a *factor* if it has a trivial center, i.e., $\mathcal{Z}(\mathcal{M}) = \mathbb{C}I$.
- A projection $p \in \mathcal{P}(\mathcal{M})$ is called *central* if $p \in \mathcal{Z}(\mathcal{M})$. We denote the set of all central projections by $\mathcal{ZP}(\mathcal{M})$.
- For two projections $e, f \in \mathcal{P}(\mathcal{M})$, we say that e is a sub-projection of f (or f majorizes e), denoted by $e \leq f$ (or $f \geq e$), if and only if $ef = e$, or equivalently by taking the involution, $fe = e$. If we set $f^\perp = 1 - f$, where $1 = 1_{\mathcal{M}}$ is the identity element of $\mathcal{M}$, then obviously, $f \geq e$, if and only if $f^\perp \leq e^\perp$.
- For two projections $e, f \in \mathcal{P}(\mathcal{M})$, their joint maximum, denoted by $e \vee f$, is the projection onto the closed subspace generated by the union of ranges of e and f . Their joint minimum, denoted by $e \wedge f$, is the projection onto the intersection of the ranges of e and f . One can even generalize these operations to a family of projections in an obvious way. The set $\mathcal{P}(\mathcal{M})$ is closed under these operations. With $\vee$ and $\wedge$, $\mathcal{P}(\mathcal{M})$ is a *complete lattice*, see [81, Proposition V.1.1].
- For two projections $e, f \in \mathcal{P}(\mathcal{M})$, we say that e is orthogonal to f , denoted by $e \perp f$, if and only if $ef = 0$, or equivalently, by taking the involution, $fe = 0$.
- A projection $p \in \mathcal{P}(\mathcal{M})$ is called *minimal* if it does not majorize any non-zero projection (except itself). This is equivalent to $p\mathcal{M}p = \mathbb{C}p$.
- A projection $p \in \mathcal{P}(\mathcal{M})$ is called *abelian* if the von Neumann algebra $p\mathcal{M}p$ is abelian. We denote the set of abelian projections by $\mathcal{AP}(\mathcal{M})$.
- Two projections $p, q \in \mathcal{P}(\mathcal{M})$ are called *Murray–von Neumann equivalent* (or *simply equivalent*), denoted by $p \sim q$, if there is a partial isometry $\nu \in \mathcal{M}$ such that $p = \nu^*\nu$ and $q = \nu\nu^*$. The partial isometry ν sets up an isometry between the ranges of p and q ; hence, equivalent projections have the same Hilbert dimension for their ranges. We remark that for any partial isometry ν , the projections $\nu^*\nu$ and $\nu\nu^*$ are trivially equivalent by the definition.
- A projection $p \in \mathcal{P}(\mathcal{M})$ is *sub-equivalent* to a projection $q \in \mathcal{P}(\mathcal{M})$, denoted by $p \preceq q$, if there is a sub-projection $q_0 \leq q$ such that $p \sim q_0$. This is a partial order on the projections, which makes $\mathcal{P}(\mathcal{M})$ into a partially directed set, see [81, Proposition V.1.3].
- A projection $q \in \mathcal{P}(\mathcal{M})$ is called *finite* if $\{p \in \mathcal{P}(\mathcal{M}) : p \leq q, p \sim q\} = \{q\}$. That is the only sub-projection of q , which is equivalent to q , is q itself. We denote the set of all finite projections by $\mathcal{FP}(\mathcal{M})$, and the set of all central finite projections by $\mathcal{ZFP}(\mathcal{M})$.

- A projection $p \in \mathcal{P}(\mathcal{M})$ is called *semi-finite* if there exists a family of pairwise orthogonal projections q_i in $\mathcal{FP}(\mathcal{M})$ such that $p = \sum_i q_i$ in strong operator topology.
- A projection $q \in \mathcal{P}(\mathcal{M})$ is called *infinite* if it is not finite.
- A projection $q \in \mathcal{P}(\mathcal{M})$ is called *purely infinite* if $\{p \in \mathcal{FP}(\mathcal{M}) : p \leq q\} = \{0\}$.
- A projection $q \in \mathcal{P}(\mathcal{M})$ is called *properly infinite* if $\{p \in \mathcal{ZFP}(\mathcal{M}) : p \leq q\} = \{0\}$.
- A projection $p \in \mathcal{P}(\mathcal{M})$ is called countably decomposable if every family of mutually orthogonal nonzero subprojections of p is countable.

Remark 2.3.5. For a projection $p \in \mathcal{P}(\mathcal{M})$ the following implications hold:

- minimal $\Rightarrow$ abelian $\Rightarrow$ finite $\Rightarrow$ semi-finite
- purely infinite $\Rightarrow$ properly infinite
- a semi-finite projection cannot be purely infinite.

Definition 2.3.6. A von Neumann algebra $\mathcal{M}$ is called *finite*, *semi-finite*, *infinite*, *properly infinite*, *purely infinite*, or *countably decomposable* if its maximal projection, i.e., $1 = 1_{\mathcal{M}}$, has the corresponding property respectively. Also, it is worth mentioning that $\mathcal{M}$ is called *atomic* if every non-zero projection in $\mathcal{M}$ majorizes a non-zero minimal projection in $\mathcal{M}$. Atomic von Neumann algebras are non-commutative generalizations of ℓ^∞ spaces, see [81, P. 155].

Definition 2.3.7. A von Neumann algebra $\mathcal{M}$ is called of

- type I if for any non-zero projection $p \in \mathcal{ZP}(\mathcal{M})$ there is a nonzero projection $q \in \mathcal{AP}(\mathcal{M})$ such that $q \leq p$.
 - Let n be a finite or infinite cardinal. A type I von Neumann algebra $\mathcal{M}$ is said to be of type I_n if $1_{\mathcal{M}}$ is the sum of n mutually orthogonal equivalent abelian projections.
- type II if $\mathcal{AP}(\mathcal{M}) = \{0\}$ and if for any non-zero projection $p \in \mathcal{ZP}(\mathcal{M})$ there is a nonzero projection $q \in \mathcal{FP}(\mathcal{M})$ such that $q \leq p$. A type II von Neumann algebra is called
 - type II_1 if 1 is finite.
 - type II_∞ if 1 is properly infinite.
- type III if $\mathcal{FP}(\mathcal{M}) = \{0\}$

Remark 2.3.8. The type classification of von Neumann algebras can be understood in terms of the possible “sizes” of projections. For example, if $\mathcal{M}$ is a countably decomposable factor, then, by [3, Theorem III.7.1.9], there is a dimension function on Murray–von Neumann equivalence classes of projections,

$$d : \mathcal{P}(\mathcal{M}) / \sim \longrightarrow [0, \infty],$$

which is unique up to normalization, such that

$$p \preceq q \iff d([p]) \leq d([q]), \quad p, q \in \mathcal{P}(\mathcal{M}).$$

In this case, the range of d is

$$\begin{cases} \{0, 1, \dots, n\}, & \text{if } \mathcal{M} \text{ is of type } I_n, \\ \{0, 1, 2, \dots, \infty\}, & \text{if } \mathcal{M} \text{ is of type } I_\infty, \\ [0, 1], & \text{if } \mathcal{M} \text{ is of type } II_1, \\ [0, \infty], & \text{if } \mathcal{M} \text{ is of type } II_\infty, \\ \{0, \infty\}, & \text{if } \mathcal{M} \text{ is of type III.} \end{cases}$$

Example 2.3.9. Here, we list some important examples of various types of von Neumann algebras:

- (i) Let n be a finite or infinite cardinal. For any Hilbert space $\mathcal{H}$ of dimension n , the von Neumann algebra $\mathcal{B}(\mathcal{H})$ is of type I_n . See [3, III.1.5.12], and [81, Section V.1].
- (ii) $VN(\mathbb{F}_2)$, the group von Neumann algebra of $\mathbb{F}_2$, is of type II_1 , see [3, Proposition III.3.3.7].
- (iii) The tensor product of a type II_1 von Neumann algebra with a type I_∞ von Neumann algebra is a type II_∞ von Neumann algebra, see [48, Theorem 6.7.10] and [81, Theorem V.2.30]. We study tensor products in Section 2.4.
- (iv) Any type III von Neumann algebra can be realized as a crossed product $II_\infty \overline{\rtimes} \mathbb{R}$; see [80]. The notation and definition of the crossed product are given in Definition 5.3.2.
- (v) The type decomposition of von Neumann algebras, [81, Theorem V.1.19], states that for any von Neumann algebra $\mathcal{M}$, one can find the pairwise orthogonal and unique central projections $p_t \in \mathcal{ZP}(\mathcal{M})$, $t \in \{I, II_1, II_\infty, III\}$, such that $\mathcal{M}p_t$ is of type t , and

$$\mathcal{M} = \mathcal{M}p_I \oplus \mathcal{M}p_{II_1} \oplus \mathcal{M}p_{II_\infty} \oplus \mathcal{M}p_{III}.$$

- (vi) Semi-finite von Neumann algebras are characterized by $p_{III} = 0$ in the type decomposition. That is, a von Neumann algebra is semi-finite if and only if it does not have a summand of type III. Thus, semi-finite von Neumann algebras are of either type I or type II.
- (vii) By [12] and [3, pp. 253-255], for any semi-finite von Neumann algebra $\mathcal{M}$, there is an increasing net of finite projections p_α such that $p_\alpha \xrightarrow{\text{tot}} 1$, and each corner algebra $\mathcal{M}_\alpha = p_\alpha \mathcal{M} p_\alpha$ is a finite von Neumann algebra. Moreover, $\overline{\bigcup_\alpha \mathcal{M}_\alpha}^{\text{-weak}^*} = \mathcal{M}$.
- (viii) A von Neumann algebra $\mathcal{M}$ is of type t if and only if its commutant $\mathcal{M}'$ is of type t . As before, $t \in \{I, II_1, II_\infty, III\}$, see [3, III.5.1.20].
- (ix) By [48, Proposition 6.3.7], for any von Neumann algebra $\mathcal{M}$, one can find a unique central projection $p \in \mathcal{ZP}(\mathcal{M})$ such that the von Neumann algebra $\mathcal{M}p$ is finite, the von Neumann algebra $\mathcal{M}p^\perp$ is properly infinite, and $\mathcal{M} = \mathcal{M}p \oplus \mathcal{M}p^\perp$.
- (x) Let $\mathcal{M}$ be a von Neumann algebra inside some $\mathcal{B}(\mathcal{H})$, and consider a nonzero $z \in \mathcal{ZP}(\mathcal{M})$. If $\mathcal{M}$ is of type $t \in \{I, II_1, II_\infty, III\}$, then $\mathcal{M}z$ is of the same type, see [48, P. 422]. If $\mathcal{M}$ is finite and z is finite, then $z\mathcal{M}z$ is finite as well, [48, Proposition 6.3.2].

Remark 2.3.10. For a von Neumann algebra $\mathcal{M}$, we have the following important observations.

- (i) By [82, Theorem VII. 2. 7], $\mathcal{M}$ has a normal faithful semi-finite weight.
- (ii) $\mathcal{M}$ is **semi-finite** if and only if it has normal faithful semi-finite **tracial** weight, see [81, Theorem V. 2. 15].
- (iii) $\mathcal{M}$ is **finite** if and only if it has sufficiently many normal **finite traces** that can **separate** each strictly positive element from zero, see [81, Theorem V. 2. 4].
- (iv) If $\mathcal{M}$ is a finite factor, then [81, Theorem V.2.6] implies that $\mathcal{M}$ has a **faithful** normal tracial **state** τ . For example, $VN(\mathbb{F}_2)$ has such a faithful state given by the vector state at identity, see [17, Chapter VII]. However, in the non-factor case, the faithfulness of τ is not guaranteed.

Definition 2.3.11. Let A be a C*-algebra, and B be a C*-subalgebra of it. A linear mapping $\mathcal{E} : A \rightarrow B$ is called a *conditional expectation* if it is a norm one onto projection.

Definition 2.3.12. Let $\mathcal{M}$ be a von Neumann algebra inside $\mathcal{B}(\mathcal{H})$ and $\mathcal{M}'$ be its commutant.

1. $\mathcal{M}$ is called *semi-discrete* if there exists a net of unital normal and completely positive maps $S_\alpha : \mathcal{M} \rightarrow \mathcal{M}$ of finite rank such that $S_\alpha a \rightarrow a$ for each $a \in \mathcal{M}$ in the weak*-topology. Completely positive means the ampliation maps on different matrix levels are all positive, i.e., $(S_\alpha)_n([x_{i,j}]_{n \times n}) = [S_\alpha(x_{i,j})]_{n \times n}$ is positive for each $n \in \mathbb{N}$.
2. $\mathcal{M}$ is called *hyperfinite* if there exists an increasing net of finite dimensional *-subalgebras $(\mathcal{M}_\alpha)$ in $\mathcal{M}$ such that the union $\cup_\alpha \mathcal{M}_\alpha$ is weak*-dense in $\mathcal{M}$.
3. $\mathcal{M}$ is called *injective* if there exists a conditional expectation $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}'$.

Remark 2.3.13. For a von Neumann algebra $\mathcal{M}$ inside some $\mathcal{B}(\mathcal{H})$ the following are equivalent:

- (i) $\mathcal{M}$ is semi-discrete,
- (ii) $\mathcal{M}$ is hyper-finite,
- (iii) $\mathcal{M}$ is injective,
- (iv) $\mathcal{M}'$ is injective.

See [3, P. 367] and [76, Chapter 5] for the details.

2.4 Tensor products

From time to time, we will need to work with tensor products. While one always starts from the algebraic tensor product, the essential distinction arises when they attempt to equip the algebraic construction with a suitable norm. The choice of norm significantly affects the resulting topological tensor product and leads to different functional-analytic properties.

Let A and B be two Banach spaces (or algebras), and $\mathcal{L}^2(A^* \times B^*, \mathbb{C})$ be the space of all bounded bilinear maps on $A^* \times B^*$. For each $a \in A$ and $b \in B$, we can specify a member of $\mathcal{L}^2(A^* \times B^*, \mathbb{C})$ by declaring:

$$\begin{aligned} a \otimes b : A^* \times B^* &\longrightarrow \mathbb{C} \\ a \otimes b(f, g) &= f(a)g(b), \quad \forall f \in A^*, \forall g \in B^*. \end{aligned}$$

We define the algebraic tensor product of A and B , denoted by $A \otimes B$, to be the space of all linear spans of all $a \otimes b$ with $a \in A$ and $b \in B$. By definition, $A \otimes B \subseteq \mathcal{L}^2(A^* \times B^*, \mathbb{C})$. One can show that given any $u \in A \otimes B$, there always exist two linearly independent sets $\{a_i\}$ and $\{b_i\}$ such that $u = \sum_{i=1}^n a_i \otimes b_i$. However, this representation is not unique. The mapping

$$\begin{aligned} \gamma : A \otimes B &\longrightarrow [0, \infty) \\ \gamma(u) = \|u\|_\gamma &= \inf \left\{ \sum_{i=1}^n \|a_i\| \|b_i\| : u = \sum_{i=1}^n a_i \otimes b_i, n \in \mathbb{N} \right\}, \end{aligned}$$

defines a norm on $A \otimes B$, which is called the *projective tensor norm*. This norm is a *cross norm*, i.e., $\|a \otimes b\| = \|a\| \|b\|$. If A and B are $*$ -algebras, then $A \otimes B$ is an $*$ -algebra with the product and involution specified by

$$(a \otimes b)(c \otimes d) = ac \otimes bd, \quad (a \otimes b)^* = a^* \otimes b^*.$$

Definition 2.4.1. The completion of $A \otimes B$ under the projective tensor norm is called the *projective tensor product* of A and B and is denoted by $A \otimes_\gamma B$. It is worth noting that each $u \in A \otimes_\gamma B$ has a (non-unique) representation of the form $u = \sum_{i=1}^\infty a_i \otimes b_i$ such that $\|u\|_\gamma \leq \sum_{i=1}^\infty \|a_i\| \|b_i\| < \infty$. So,

$$\|u\|_\gamma = \inf \left\{ \sum_{i=1}^\infty \|a_i\| \|b_i\| : u = \sum_{i=1}^\infty a_i \otimes b_i \right\}.$$

Remark 2.4.2. Projective tensor norm is the largest possible cross tensor norm that one can impose on an algebraic tensor product. There is another important cross tensor norm, called injective tensor norm which is defined by:

$$\begin{aligned} \lambda : A \otimes B &\longrightarrow [0, \infty) \\ \lambda(u) = \|u\|_\lambda &= \sup \{ |(\phi \otimes \psi)(u)| : \phi \in A^*, \psi \in B^*, \|\phi\| = \|\psi\| = 1 \}. \end{aligned}$$

This is the smallest possible cross tensor norm on the algebraic tensor product $A \otimes B$ of Banach spaces (or algebras). The completion of $A \otimes B$ under λ is called the *injective tensor product* and is denoted by $A \otimes_\lambda B$.

Example 2.4.3. (i) By [81, Theorem IV.2.3], we have the canonical isometric identification

$$(A \widehat{\otimes}_\gamma B)^* \cong \mathcal{B}(A, B^*). \quad (2.6)$$

For a description of the dual space $(A \widehat{\otimes}_\lambda B)^*$, see [77, Proposition 3.14].

(ii) For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we denote the canonical Hilbert space tensor product of them by $\mathcal{H} \otimes_2 \mathcal{K}$, which is the completion of the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ under the canonical inner product

$$\langle a \otimes b, c \otimes d \rangle := \langle a, c \rangle \langle b, d \rangle.$$

Denote the trace class operators by $\mathcal{B}^1(\mathcal{H})$, the Hilbert-Schmidt operators by $\mathcal{B}^2(\mathcal{H})$, and the compact operators by $\mathcal{K}(\mathcal{H})$. Then, by [81, Theorem IV.2.5],

$$\mathcal{H} \otimes_\gamma \mathcal{H} \cong \mathcal{B}^1(\mathcal{H}), \quad \mathcal{H} \otimes_2 \mathcal{H} \cong \mathcal{B}^2(\mathcal{H}), \quad \mathcal{H} \otimes_\lambda \mathcal{H} \cong \mathcal{K}(\mathcal{H}). \quad (2.7)$$

Remark 2.4.4. We should note that the projective and injective tensor products of two Banach $*$ -algebras are still Banach $*$ -algebras. However, when forming the tensor product of two C^* -algebras (or von Neumann algebras), we aim to equip the algebraic tensor product with a suitable tensor norm so that its completion becomes a C^* -algebra (or a von Neumann algebra).

Definition 2.4.5. Let A and B be two C^* -algebras with faithful universal representations $\phi : A \rightarrow \mathcal{B}(\mathcal{H}_\phi)$ and $\psi : B \rightarrow \mathcal{B}(\mathcal{H}_\psi)$. Consider the algebraic tensor product $A \otimes B$, and the representation $\phi \otimes \psi : A \otimes B \rightarrow \mathcal{B}(\mathcal{H}_\phi \otimes_2 \mathcal{H}_\psi)$ defined by $\phi \otimes \psi(a \otimes b) = \phi(a) \otimes \psi(b)$, see [56, Theorem 6.3.]. Following [81, Chapter 4] or [56, Chapter 6], the *minimal* or *spatial tensor norm* is defined by:

$$\begin{aligned} \|\cdot\|_{\min} : A \otimes B &\longrightarrow [0, \infty) \\ \|u\|_{\min} &= \|\phi \otimes \psi(u)\|. \end{aligned}$$

This norm satisfies the C^* -equation $\|uu^*\| = \|u\|^2$. So, the completion of $A \otimes B$ under it is a C^* -algebra, called the *minimal* or *spatial tensor product* of A and B , and is denoted by $A \otimes_{\min} B$.

Definition 2.4.6. Let A and B be two C^* -algebras. Consider the algebraic tensor product $A \otimes B$. Following [81, Chapter 4] or [56, Chapter 6], the *maximal* tensor norm on $A \otimes B$ is defined by:

$$\begin{aligned} \|\cdot\|_{\max} : A \otimes B &\longrightarrow [0, \infty) \\ \|u\|_{\max} &= \sup \{ \|\pi(u)\| : \pi \text{ is a } * \text{-representation of } A \otimes B \}. \end{aligned}$$

$\|\cdot\|_{\max}$ also satisfies the C^* -equation $\|uu^*\| = \|u\|^2$. So, the completion of $A \otimes B$ under it is a C^* -algebra, called the *maximal tensor product* of A and B , and is denoted by $A \otimes_{\max} B$.

Remark 2.4.7. By definition, $\|\cdot\|_{\min} \leq \|\cdot\|_{\max}$ and in general, $A \otimes_{\max} B$ is quite larger than $A \otimes_{\min} B$. Note that these tensor products are independent of the Hilbert space representations; see [81]. By [81, Proposition IV.2.2], we have

$$\|\cdot\|_{\lambda} \leq \|\cdot\|_{\min} \leq \|\cdot\|_{\max} \leq \|\cdot\|_{\gamma}.$$

Definition 2.4.8. A C^* -algebra A is called *nuclear* if for every C^* -algebra B , we have

$$A \otimes_{\min} B \cong A \otimes_{\max} B,$$

isometrically $*$ -isomorphism of C^* -algebras, i.e., there is only one possible C^* -tensor norm on the algebraic tensor product $A \otimes B$.

Example 2.4.9. All abelian C^* -algebras are nuclear [56, Theorem 6.4.15], whereas the operator algebra $\mathcal{B}(\mathcal{H})$ on an infinite-dimensional Hilbert space $\mathcal{H}$ is not [65].

The following theorem characterizes nuclear C^* -algebras in terms of amenability. The forward implication was proved by A. Connes [12], and the converse was established by U. Haagerup [39].

Theorem 2.4.10. *A C^* -algebra is amenable if and only if it is nuclear.*

Definition 2.4.11. Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras inside $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively. Similar to Definition 2.4.5, we can embed the algebraic tensor product $\mathcal{M} \otimes \mathcal{N}$ into

$\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})$ as a unital $*$ -algebra. The von Neumann algebraic tensor product of $\mathcal{M}$ and $\mathcal{N}$ is defined as the double commutant of $\mathcal{M} \otimes \mathcal{N}$ inside $\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})$, i.e.,

$$\mathcal{M} \overline{\otimes} \mathcal{N} := (\mathcal{M} \otimes \mathcal{N})''.$$

One can show that the von Neumann algebraic tensor product is independent of the representations of $\mathcal{M}$ and $\mathcal{N}$ that we begin with [81, Section IV.5]. If $\mathcal{M}_*$ and $\mathcal{N}_*$ are pre-duals of $\mathcal{M}$ and $\mathcal{N}$ respectively, then by [76, E.4.5], we have:

$$(\mathcal{M}_* \otimes_\gamma \mathcal{N}_*)^* = \mathcal{M} \overline{\otimes} \mathcal{N}.$$

Here, $\otimes_\gamma$ denotes the (operator space) projective tensor product.

2.5 Approximate and virtual diagonals

When A is a Banach algebra, the projective tensor product $A \otimes_\gamma A$ is an A -bimodule under the canonical actions:

$$a \cdot (b \otimes c) = ab \otimes c, \text{ and } (b \otimes c) \cdot a = b \otimes ca, \quad \forall a, b, c \in A. \quad (2.8)$$

The *multiplication operator* on $A \otimes_\gamma A$ is defined by :

$$\begin{aligned} \pi : A \otimes_\gamma A &\longrightarrow A \\ \pi(a \otimes b) &= ab. \end{aligned} \quad (2.9)$$

One can see that under canonical module actions 2.8, π is a module homomorphism. The first and the second dual of π are denoted by $\pi^* : A^* \longrightarrow (A \otimes_\gamma A)^*$ and $\pi^{**} : (A \otimes_\gamma A)^{**} \longrightarrow A^{**}$.

Theorem 2.5.1. *For a Banach algebra A the followings are equivalent ([42]):*

- (i) A is amenable.
- (ii) A has a bounded approximate diagonal, i.e., there is a bounded net (u_i) in $A \otimes_\gamma A$ such that for any $a \in A$, we have $au_i - u_i a \longrightarrow 0$ and $\pi(u_i)a \longrightarrow a$.
- (iii) A has a virtual diagonal, i.e., there is an $M \in (A \otimes_\gamma A)^{**}$ such that $aM - Ma = 0$ and $\pi^{**}(M)a = a$ for all $a \in A$.

Definition 2.5.2. Let A be a Banach algebra and $c \geq 0$. We say that A is c -amenable if it admits a bounded approximate diagonal of norm at most c . The *amenability constant* of A is then defined by

$$AM(A) := \inf \{ c \geq 0 : A \text{ is } c\text{-amenable} \},$$

with the usual convention that $\inf \emptyset = +\infty$. By Theorem 2.5.1, A is amenable if and only if $AM(A) < \infty$.

Definition 2.5.3 ([36]). A Banach algebra A is said to be *pseudo-amenable* if there exists a net (u_i) in $A \otimes_\gamma A$ such that, for every $a \in A$,

$$au_i - u_i a \longrightarrow 0 \quad \text{and} \quad \pi(u_i)a \longrightarrow a.$$

The net (u_i) is not required to be bounded, and is called an *approximate diagonal* for A .

Remark 2.5.4. Approximate amenability (Definition 2.2.9) and pseudo-amenability (Definition 2.5.3) are collectively referred to as the *generalized notions of amenability*. Clearly, every amenable Banach algebra is both approximately amenable and pseudo-amenable; however, the converse fails in general, see [33, Example 6.1], [91] and [76, Example 4.4.4]:

- By Example 2.2.10, the Fourier algebra $A(\mathbb{H})$ of the Heisenberg group is approximately amenable but not amenable.
- The Fourier algebra $A(\mathbb{F}_2)$ of the free group $\mathbb{F}_2$ is pseudo-amenable but neither amenable nor approximately amenable, see [35] and [76, Example 4.4.4].

Thus, the generalized notions of amenability are strictly broader than amenability and also differ from each other. It was conjectured that approximate amenability implies pseudo-amenability. This implication has been proved when the Banach algebra has a bounded approximate identity; however, the general case remains unproved [91].

Let A be a Banach algebra. We can always associate to it a unital Banach algebra, denoted by $A^\#$. Explicitly, $A^\#$ is the set $A \times \mathbb{C}$ equipped with the operations

$$\begin{aligned} (a, \lambda) + (b, \mu) &= (a + b, \lambda + \mu), \\ \mu(a, \lambda) &= (\mu a, \mu \lambda), \\ (a, \lambda)(b, \mu) &= (ab + \mu a + \lambda b, \lambda \mu), \\ \|(a, \lambda)\| &= \|a\| + |\lambda|. \end{aligned}$$

The Banach algebra $A^\#$ is called the *unitization* of A with the unit $u = (0, 1)$. We note that $\|u\| = 1$. It is sometimes convenient to write $A \oplus \mathbb{C}$ instead of $A \times \mathbb{C}$. We remark that the embedding $a \mapsto (a, 0)$ realizes A as a closed ideal of codimension 1 in $A^\#$.

Theorem 2.5.5. [36, Theorem 3.1] *A Banach algebra A is approximately amenable if and only if $A^\#$ is pseudo-amenable.*

Remark 2.5.6. By [56, Theorem 3.1.1], every C^* -algebra admits a bounded approximate identity. Hence, by [36, Proposition 3.2], any approximately amenable C^* -algebra is pseudo-amenable. On the other hand, approximate amenability and pseudo-amenability coincide for unital Banach algebras [92, Proposition 10]. Therefore, by Theorem 2.2.11, we obtain the following equivalence for unital C^* -algebras.

Theorem 2.5.7. *For a unital C^* -algebra A the followings are equivalent:*

- *A is approximately amenable.*
- *A is weak* approximately amenable.*
- *A is pseudo amenable.*

2.6 Hilbert C^* - and W^* -Modules

Hilbert modules generalize the concept of Hilbert spaces by allowing the inner product to take values in a C^* -algebra or a von Neumann algebra. They play a fundamental role in the theory of operator

algebras, K -theory, and non-commutative geometry. Hilbert W^* -modules provide the natural framework for Morita equivalence and tensor product constructions. We use them in Section 5.2 to lift derivations from finite corners of a von Neumann algebra to the ambient algebra. The materials in this section are directly taken from [3, Section II.7.1] and [4, Chapter 8].

Definition 2.6.1. Let B be a C^* -algebra and $\mathcal{E}$ be a right B -module. A B -valued inner product on $\mathcal{E}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow B$ with the following properties for $\xi, \eta, \zeta \in \mathcal{E}$, $b \in B$, $\lambda \in \mathbb{C}$:

- (i) $\langle \xi, \eta + \zeta \rangle = \langle \xi, \eta \rangle + \langle \xi, \zeta \rangle$ and $\langle \xi, \lambda \eta \rangle = \lambda \langle \xi, \eta \rangle$,
- (ii) $\langle \xi, \eta b \rangle = \langle \xi, \eta \rangle b$,
- (iii) $\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$,
- (iv) $\langle \xi, \xi \rangle \geq 0$ (as an element of B).
- (v) $\langle \xi, \xi \rangle = 0$ implies $\xi = 0$

A right B -module with a B -valued inner product is called a (right) *Hilbert B -module*. By a *Hilbert B -module* we always mean a Hilbert right B -module.

Remark 2.6.2. If $\mathcal{E}$ is a left A -module, there is an analogous definition of an A -valued inner product with (i) replaced by linearity in the first variable and (ii) by the condition $\langle a\xi, \eta \rangle = a\langle \xi, \eta \rangle$. Conditions (iii) and (iv) remain the same. A left A -module with an A -valued inner product in this sense is called a *left Hilbert A -module*. It follows from (ii) and (iii) that

$$\langle \xi b, \eta \rangle = b^* \langle \xi, \eta \rangle,$$

i.e., $\langle \cdot, \cdot \rangle$ is “conjugate-linear” in the first variable. The inner product in a left Hilbert module is “conjugate-linear” in the second variable. If condition (v) does not hold, one can mod out $\mathcal{E}$ by the closed sub-module $N_{\mathcal{E}} = \{\xi : \langle \xi, \xi \rangle = 0\}$, and work with the Hilbert B -module $\frac{\mathcal{E}}{N_{\mathcal{E}}}$ instead.

Just as in the scalar case, there is a CBS inequality and an induced norm

$$\|\xi\|^2 = \|\langle \xi, \xi \rangle\| \tag{2.10}$$

on a Hilbert C^* -module.

Proposition 2.6.3. Let B be a C^* -algebra, $\mathcal{E}$ be a Hilbert B -module. Assume $\xi, \eta \in \mathcal{E}$ and $b \in B$. Then, for the norm defined by Equation (2.10), we have

- (i) $\langle \xi, \eta \rangle^* \langle \xi, \eta \rangle \leq \|\xi\|^2 \|\eta\|^2$ [CBS Inequality as elements of B]
- (ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$,
- (iii) $\|\xi b\| \leq \|\xi\| \|b\|$.

Example 2.6.4. (i) Every Hilbert space H is a Hilbert C^* -module over $\mathbb{C}$.

(ii) Any C^* -algebra B is a Hilbert B -module with the inner product $\langle x, y \rangle = x^*y$.

(iii) If τ is a positive linear functional on a C^* -algebra B , and $\mathcal{E}$ is a Hilbert B -module. Then, by [56, Theorem 3.3.7], we can endow $\mathcal{E}$ with an inner product defined by

$$\langle \xi, \eta \rangle_\tau := \tau(\langle \eta, \xi \rangle), \quad (\xi, \eta \in \mathcal{E}).$$

This makes $\mathcal{E}$ into a pre-Hilbert space.

Definition 2.6.5 (Hilbert W^* -module). Let $\mathcal{M}$ be a von Neumann algebra. A (right) Hilbert W^* -module over $\mathcal{M}$ is a Hilbert C^* -module $\mathcal{E}$ over $\mathcal{M}$ that is *self-dual*, i.e. every bounded $\mathcal{M}$ -module map $\phi : \mathcal{E} \rightarrow \mathcal{M}$ is given by $\phi(x) = \langle z, x \rangle$ for some $z \in \mathcal{E}$.

Remark 2.6.6. In the above definition, the element z is unique for the functional ϕ . For if $\langle z, x \rangle = \langle z', x \rangle$, then $\langle z - z', x \rangle = 0$. Now take $x = z - z'$. By Definition 2.6.1-v or the CBS inequality, we have $z = z'$.

The following is [4, Lemma 8.5.4]:

Theorem 2.6.7. Let $\mathcal{E}$ be a Hilbert C^* -module over the von Neumann algebra $\mathcal{M}$.

1. $\mathcal{E}$ is a Hilbert W^* -module over $\mathcal{M}$ if and only if $\mathcal{E}$ is a dual Banach space such that the $\mathcal{M}$ -valued inner product $\langle \cdot, \cdot \rangle$ is weak*-separately continuous.
2. If $\mathcal{E}$ is a Hilbert W^* -module over $\mathcal{M}$, then the pre-dual of $\mathcal{E}$ is unique.

Chapter 3

Generalized amenability of locally compact quantum groups

In this chapter, after reviewing quantum groups in the von Neumann algebraic setting, we establish several results concerning their approximate amenability. Quantum groups are C^* -algebras or von Neumann algebras endowed with additional structure that parallels the classical function spaces $C(G)$ (continuous functions on a compact group) and $L^\infty(G)$ (essentially bounded measurable functions on a group). According to [88, Chapter 6], quantum groups overcome the limitations of classical groups, which are inadequate to fully capture the symmetries arising in quantum mechanics. Moreover, quantum groups provide a natural framework for extending Pontryagin duality to a much broader setting.

3.1 locally compact quantum groups

Definition 3.1.1. A von Neumann algebra $\mathcal{M}$ is called a Hopf-von Neumann algebra if it has a co-multiplication $\Delta : \mathcal{M} \rightarrow \mathcal{M} \overline{\otimes} \mathcal{M}$, which is a normal, unital and injective $*$ -homomorphism such that Δ is co-associative, i.e., $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$. Here, id is the identity map on $\mathcal{M}$. We will refer to the pair $(\mathcal{M}, \Delta)$ as a Hopf-von Neumann algebra. Co-associativity of Δ means that the following diagram commutes

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\Delta} & \mathcal{M} \overline{\otimes} \mathcal{M} \\ \Delta \downarrow & & \downarrow \Delta \otimes \text{id} \\ \mathcal{M} \overline{\otimes} \mathcal{M} & \xrightarrow{\text{id} \otimes \Delta} & \mathcal{M} \overline{\otimes} \mathcal{M} \overline{\otimes} \mathcal{M} \end{array}$$

Figure 3.1: Co-associativity of co-multiplication.

Example 3.1.2. For a locally compact group G , $L^\infty(G)$ is a Hopf von Neumann algebra with the co-multiplication given by $\Delta(F)(s, t) = F(st)$. The co-associativity of Δ reflects the associativity of the group multiplication. Similarly, $VN(G)$ is a Hopf von Neumann algebra under the assignment $\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s)$, where λ is the left regular representation of G .

Remark 3.1.3. Consider a Hopf-von Neumann algebra $(\mathcal{M}, \Delta)$, where $\mathcal{M}$ is represented faithfully on some Hilbert space $\mathcal{H}$; that is, there exists a faithful $*$ -representation $\Phi : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H})$. We can identify $\mathcal{M}$ with $\Phi(\mathcal{M})$. Denote the operator norm on $\mathcal{M}$ by $\|\cdot\|_\infty$, and the norm on the predual $\mathcal{M}_*$ by $\|\cdot\|_1$. Following [15], we may define a (completely contractive) Banach algebra structure on $\mathcal{M}^*$ via the convolution product

$$\begin{aligned} \star : \mathcal{M}^* \times \mathcal{M}^* &\longrightarrow \mathcal{M}^* \\ (\mu, \omega) &\mapsto \mu \star \omega \\ \langle \mu \star \omega, m \rangle &= \langle \mu \otimes \omega, \Delta(m) \rangle, \quad \forall m \in \mathcal{M}. \end{aligned} \quad (3.1)$$

Here, $\mu \otimes \omega$ acts on elementary tensors via $\mu \otimes \omega(m \otimes n) = \mu(m)\omega(n)$. Recall that $(\mathcal{M}_*, \|\cdot\|_1)$ is the unique predual of $\mathcal{M}$, embedded isometrically into $\mathcal{M}^*$ through the canonical map

$$\begin{aligned} \wedge : \mathcal{M}_* &\longrightarrow \mathcal{M}^* \\ f &\longmapsto \hat{f}. \end{aligned}$$

Using this embedding, we may endow $\mathcal{M}_*$ with a Banach algebra structure inherited from $\mathcal{M}^*$ via the convolution product $\star$. Alternatively, this can be defined directly using the preadjoint of the normal coproduct Δ , which is a completely contractive mapping $\Delta_* : \mathcal{M}_* \otimes_\gamma \mathcal{M}_* \rightarrow \mathcal{M}_*$. We define

$$\begin{aligned} \star : \mathcal{M}_* \times \mathcal{M}_* &\longrightarrow \mathcal{M}_* \\ (f, g) &\mapsto f \star g \\ \langle f \star g, m \rangle &= \langle \Delta_*(f \otimes g), m \rangle, \quad \forall m \in \mathcal{M}. \end{aligned} \quad (3.2)$$

Since $\langle \Delta_*(f \otimes g), m \rangle = \langle f \otimes g, \Delta(m) \rangle$, Equations (3.1) and (3.2) along with the canonical embedding $\wedge$ show that both approaches yield the same algebra structure on $\mathcal{M}_*$. For simplicity, we will henceforth omit the notation $\wedge$ and treat elements of $\mathcal{M}_*$ as elements of $\mathcal{M}^*$.

Let $1 = 1_{\mathcal{M}}$ denote the identity of $\mathcal{M}$. We have $\Delta(1) = 1 \otimes 1$. Then, for any $f, g \in \mathcal{M}_*$, we obtain

$$\langle f \star g, 1 \rangle = \langle f \otimes g, \Delta(1) \rangle = \langle f \otimes g, 1 \otimes 1 \rangle = \langle f, 1 \rangle \langle g, 1 \rangle; \quad (3.3)$$

showing that the identity element of $\mathcal{M}$ is multiplicative on $(\mathcal{M}_*, \star)$. Finally, we observe that the multiplication $\star$ of the Banach algebra $\mathcal{M}_*$ endows $\mathcal{M}$ and $\mathcal{M}^*$ with the structure of dual Banach $\mathcal{M}_*$ -modules [76, E.5.5]. Specifically, for $f, g \in \mathcal{M}_*$ and $F \in \mathcal{M}$, the module actions of $\mathcal{M}_*$ on $\mathcal{M}$ are defined via:

$$\begin{aligned} \langle f \cdot F, g \rangle &= \langle F, g \star f \rangle, \\ \langle F \cdot f, g \rangle &= \langle F, f \star g \rangle. \end{aligned} \quad (3.4)$$

Also, by Equations (3.1), (3.2), and the canonical isometry $\wedge$, the module actions of $f, g \in \mathcal{M}_*$ on $\mu \in \mathcal{M}^*$ are given by

$$\begin{aligned} f \cdot \mu &= f \star \mu, \\ \mu \cdot g &= \mu \star g. \end{aligned}$$

Definition 3.1.4. A Hopf-von Neumann algebra $(\mathcal{M}, \Delta)$ is called *discrete* if $\mathcal{M}_*$ is unital.

Definition 3.1.5. Let $(\mathcal{M}, \Delta)$ be a Hopf-von Neumann algebra. A state $\mu \in \mathcal{M}^*$ is called

- a *left invariant mean (LIM)* (also referred to as a *left invariant quantum mean (LIQM)*) if

$$\mu((\omega \otimes \text{id})\Delta(F)) = \omega(1)\mu(F) \quad \forall F \in \mathcal{M}, \omega \in \mathcal{M}_*,$$

or equivalently,

$$\omega \star \mu = \omega(1)\mu, \quad \forall \omega \in \mathcal{M}_*.$$

- a *right invariant mean (RIM)* (also referred to as a *right invariant quantum mean (RIQM)*) if

$$\mu((\text{id} \otimes \omega)\Delta(F)) = \mu(F)\omega(1), \quad \forall F \in \mathcal{M}, \omega \in \mathcal{M}_*,$$

or equivalently,

$$\mu \star \omega = \omega(1)\mu, \quad \forall \omega \in \mathcal{M}_*.$$

These notions generalize the classical left and right invariant means on $L^\infty(G)$ for a locally compact group G ; see [23] and [75] for further details.

Recall that for a von Neumann algebra $\mathcal{M}$, the flip map $\chi : \mathcal{M} \overline{\otimes} \mathcal{M} \rightarrow \mathcal{M} \overline{\otimes} \mathcal{M}$ is defined by $\chi(a \otimes b) = b \otimes a$. The following definition is taken from [28, Section 1.2].

Definition 3.1.6. A *co-involutive Hopf-von Neumann algebra* is a triple $(\mathcal{M}, \Delta, \kappa)$, in which $(\mathcal{M}, \Delta)$ is Hopf-von Neumann algebra, and κ is a co-involution on $\mathcal{M}$, i.e., κ is a unital involutive normal anti-automorphism of order 2 such that

$$(\kappa \otimes \kappa) \circ \Delta = \chi \circ \Delta \circ \kappa.$$

That is, the following diagram commutes

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\kappa} & \mathcal{M} \\ \Delta \downarrow & & \downarrow \Delta \\ \mathcal{M} \overline{\otimes} \mathcal{M} & & \mathcal{M} \overline{\otimes} \mathcal{M} \\ & \searrow \kappa \otimes \kappa \quad \swarrow \chi & \\ & \mathcal{M} \overline{\otimes} \mathcal{M} & \end{array}$$

Figure 3.2: Commutativity of co-involution with co-multiplication.

Since κ is normal, we can make $\mathcal{M}_*$ into an involutive Banach algebra by defining

$$\langle \omega^*, m \rangle := \overline{\langle \omega, \kappa(m)^* \rangle}, \quad \forall \omega \in \mathcal{M}_*, \forall m \in \mathcal{M},$$

where $\kappa(m)^*$ is the usual adjoint of the operator $\kappa(m) \in \mathcal{M}$.

Note that κ is normal, linear, and compatible with the co-multiplication. In general, κ is distinct from the natural adjoint of the von Neumann algebra.

Example 3.1.7. For a unimodular locally compact group G , consider the Hopf-von Neumann algebra $L^\infty(G)$. Define $\kappa(F)(t) = F(t^{-1})$ for $F \in L^\infty(G)$ and $t \in G$. Then, by [28, Example 1.2.9], κ is a co-involution on $L^\infty(G)$. We have $\omega^*(t) = \overline{\omega(t^{-1})}$ for $\omega \in L^1(G)$ and $t \in G$.

Since locally compact groups possess Haar measures, we expect Hopf-von Neumann algebras to have a similar property. It turns out that the theory of normal weights on von Neumann algebras serves this purpose. The following definition is adapted from [28, Definition 2.2.1].

Definition 3.1.8. A quadruple $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi)$ is called a *Kac algebra* if $(\mathcal{M}, \Delta, \kappa)$ is a co-involutive Hopf-von Neumann algebra, and ϕ is a normal, faithful, semi-finite weight on $\mathcal{M}^+$ with the following *left invariance property*

$$(\text{id} \otimes \phi)\Delta(x) = \phi(x)1, \quad \forall x \in \mathfrak{p}_\phi,$$

where $\mathfrak{p}_\phi = \{x \in \mathcal{M}^+ : \phi(x) < \infty\}$ as in Definition 2.3.1. Following [28, Chapter 6], we say that a Kac algebra $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi)$ is *discrete* if $\mathcal{M}_*$ is unital. Also, a Kac algebra is called *unimodular* if ϕ is invariant under κ , i.e., $\phi \circ \kappa(x) = \phi(x)$ for any $x \in \mathfrak{p}_\phi$.

Remark 3.1.9. The left invariance equation $(\text{id} \otimes \phi)\Delta(x) = \phi(x)1$ agrees with the left invariance of Haar measures in harmonic analysis, according to the following argument which is well-known in quantum group theory (see [84, P. 40] for instance). Take a locally compact group G and a left Haar measure ϕ on it, then for $f \in L^1(G) \cap L^\infty(G)$, we define $\phi(f) := \int f(y) d\phi(y)$. Now, for any $x \in G$, define $F(x) = \int f(xy) d\phi(y)$. By the left invariance of ϕ , we see that $F(x) = \phi(L_{x^{-1}}(f)) = \phi(f)$, so $F = \phi(f)1$ where 1 is the constant function in the von Neumann algebra $L^\infty(G)$. Suppose that $\Delta(f)$ can be written as a sum of elementary tensors $f_1 \otimes f_2$ defined by $f_1 \otimes f_2(x, y) = f_1(x)f_2(y)$. We have

$$f(xy) = \Delta(f)(x, y) = \left[\sum_i f_{1,i} \otimes f_{2,i} \right](x, y) = \sum_i f_{1,i}(x)f_{2,i}(y).$$

Substituting this into the equation $F(x) = \int f(xy) d\phi(y)$ yields

$$\begin{aligned} F(x) &= \int \left(\sum_i f_{1,i}(x) f_{2,i}(y) \right) d\phi(y) \\ &= \sum_i f_{1,i}(x) \int f_{2,i}(y) d\phi(y) \\ &= \sum_i f_{1,i}(x) \phi(f_{2,i}) \\ &= (\text{id} \otimes \phi) \left(\sum_i f_{1,i} \otimes f_{2,i} \right)(x) \\ &= (\text{id} \otimes \phi)(\Delta f)(x). \end{aligned}$$

Thus, $F = (\text{id} \otimes \phi)\Delta(f)$, and the left invariance formula $F = \phi(f)1$ becomes

$$(\text{id} \otimes \phi)\Delta(f) = \phi(f)1.$$

We note that left invariance property can be characterized by either equations:

$$(\text{id} \otimes \phi)\Delta(x) = \phi(x)1, \quad \forall x \in \mathfrak{p}_\phi,$$

or

$$\phi((\omega \otimes \text{id})\Delta(x)) = \phi(x)\omega(1), \quad \forall x \in \mathfrak{p}_\phi, \omega \in \mathcal{M}_*^+.$$

These equations are equivalent since, by [54], we have

$$\phi((\omega \otimes \text{id})\Delta(x)) = \omega((\text{id} \otimes \phi)\Delta(x)).$$

We now consider the definition of a locally compact quantum group in the von Neumann algebraic setting, see [55, P. 71].

Definition 3.1.10. A *locally compact quantum group* is a Hopf-von Neumann algebra $(\mathcal{M}, \Delta)$ that satisfies the following additional conditions:

- (i) there is a normal faithful semi-finite weight ϕ on $\mathcal{M}^+$ which is left invariant, i.e., $\phi[(\omega \otimes \text{id})\Delta(x)] = \omega(1)\phi(x)$ for all $\omega \in \mathcal{M}_*^+$ and $x \in \mathfrak{p}_\phi$. We call ϕ a *quantum left Haar measure*.
- (ii) there is a normal faithful semi-finite weight ψ on $\mathcal{M}^+$ which is right invariant, i.e., $\psi[(\text{id} \otimes \omega)\Delta(x)] = \omega(1)\psi(x)$ for all $\omega \in \mathcal{M}_*^+$ and $x \in \mathfrak{p}_\psi$. We call ψ a *quantum right Haar measure*.

Note that the quantum Haar measures on locally compact quantum groups are only densely defined. We use Ruan's notations for a locally compact quantum group $(\mathcal{M}, \Delta, \phi, \psi)$, e.g., we denote $\mathbb{G} = (\mathcal{M}, \Delta, \phi, \psi)$, $\mathcal{M} = L^\infty(\mathbb{G})$, $\mathcal{M}_* = L^1(\mathbb{G})$, and $L^2(\phi)$ for the GNS Hilbert space induced by ϕ . For a discrete quantum group, we use the lower case notations $\ell^\infty(\mathbb{G})$, $\ell^1(\mathbb{G})$, and $\ell^2(\phi)$ respectively.

Example 3.1.11. For a locally compact group G , the classical $L^\infty(G)$ and the group von Neumann algebra $VN(G)$ are examples of locally compact quantum groups. Moreover, by [28, Chapter 6] and [54], the Kac algebra $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi)$ is also a locally compact quantum group. The Kac algebra played a significant role in the formulation of locally compact quantum groups in [54].

Remark 3.1.12. One can also formulate a locally compact quantum group in the C*-algebraic framework. However, as shown in [55] and [16], this approach is equivalent to the von Neumann algebraic formulation. We adopt the von Neumann algebraic definition for locally compact quantum groups, whereas for compact quantum groups, we prefer the C*-algebraic approach. As explained in [16, pp. 1-2], when dealing with locally compact quantum groups, it is more convenient to use the von Neumann algebraic approach.

Following [75] and [85], we adopt the following terminology:

Definition 3.1.13. Let $(L^\infty(\mathbb{G}), \Delta, \phi, \psi)$ be a locally compact quantum group. We say that

- $(L^\infty(\mathbb{G}), \Delta)$ is compact if its left Haar weight ϕ is bounded, i.e., ϕ is a LIM (left invariant mean) defined on the whole $L^\infty(\mathbb{G})$.
- $(L^\infty(\mathbb{G}), \Delta)$ is discrete if $L^1(\mathbb{G})$ is unital.

3.2 The GNS representation of a locally compact quantum group

For any locally compact quantum group $\mathbb{G} = (\mathcal{M}, \Delta, \phi, \psi)$, we construct its *canonical GNS representation* and *left regular representation* as defined in [16], [54] [55]. The GNS construction is based on the theory of weights on von Neumann algebras, and it does not need any quantum group theory, see [82, Chapter VII]. From Definition 2.3.1, recall that $\mathfrak{n}_\phi = \{x \in \mathcal{M} : \phi(x^*x) < \infty\}$, and $N_\phi = \{x \in \mathcal{M} : \phi(x^*x) = 0\}$ is a left ideal of $\mathcal{M}$ such that $N_\phi \subseteq \mathfrak{n}_\phi$. On the modulus space $\frac{\mathfrak{n}_\phi}{N_\phi}$, we can define an inner product by $\langle x + N_\phi, y + N_\phi \rangle = \phi(y^*x)$ for all $x, y \in \mathfrak{n}_\phi$, then we complete it to obtain a Hilbert space, denoted by $L^2(\mathbb{G})$ or $L^2(\phi)$. In the theory of locally compact quantum groups, one writes

$$L^\infty(\mathbb{G}) = \mathcal{M}, \quad L^1(\mathbb{G}) = \mathcal{M}_*.$$

Here, $L^1(\mathbb{G})$ denotes the predual of $L^\infty(\mathbb{G})$. We also define

$$\begin{aligned}\Lambda_\phi : \mathfrak{n}_\phi &\longrightarrow \frac{\mathfrak{n}_\phi}{N_\phi} \\ a &\longrightarrow a + N_\phi.\end{aligned}$$

Now, the GNS representation is given by

$$\begin{aligned}\pi_\phi : L^\infty(\mathbb{G}) &\longrightarrow \mathcal{B}(L^2(\phi)) \\ \pi_\phi(a)(b + N_\phi) &= ab + N_\phi.\end{aligned}$$

Since ϕ is faithful in our case, $N_\phi = 0$, and hence N_ϕ can be dropped in the above formulas. By [82, Proposition VII-1.4], π_ϕ is a faithful, normal $*$ -representation. The representation π_ϕ is called the *standard representation* of $\mathcal{M}$ since as far as the von Neumann algebraic structures are concerned, one has $\mathcal{M} \cong \pi_\phi(\mathcal{M})$. By [55, Theorem 1.2], there is a unitary operator

$$W : L^2(\phi) \otimes_2 L^2(\phi) \longrightarrow L^2(\phi) \otimes_2 L^2(\phi)$$

such that

$$W^*[\Lambda_\phi(x) \otimes \Lambda_\phi(y)] = (\Lambda_\phi \otimes \Lambda_\phi)[\Delta(y)(x \otimes 1)], \quad \forall x, y \in \mathfrak{n}_\phi.$$

Note that $W \in \mathcal{M} \otimes \mathcal{B}(L^2(\phi))$. We call W the *multiplicative unitary* or the *Kac-Takesaki unitary operator* of the quantum group $\mathbb{G}$. For $\mathcal{M} = L^\infty(G)$, where G is a locally compact group, we have

$$\begin{aligned}W : L^2(G) \otimes_2 L^2(G) &\longrightarrow L^2(G) \otimes_2 L^2(G) = L^2(G \times G) \\ W[\xi \otimes \eta](s, t) &= \xi \otimes \eta(s, s^{-1}t) = \xi(s)\eta(s^{-1}t),\end{aligned}$$

for all $\xi, \eta \in L^2(G)$ and all $s, t \in G$. By [55, P. 71], we obtain $\Delta(x) = W^*(1 \otimes x)W$ for all $x \in \mathfrak{n}_\phi$. Moreover, by [55, P. 74], one can define the injective algebra homomorphism

$$\begin{aligned}\lambda : L^1(\mathbb{G}) &\longrightarrow \mathcal{B}(L^2(\phi)) \\ \lambda(\omega) &= (\omega \otimes \text{id})(W),\end{aligned}$$

where $\omega \otimes \text{id} : \mathcal{M} \otimes \mathcal{B}(L^2(\phi)) \longrightarrow \mathcal{B}(L^2(\phi))$ is the slicing map defined by $\omega \otimes \text{id}(a \otimes T) = \omega(a)T$ for all $a \in \mathcal{M}$ and $T \in \mathcal{B}(L^2(\phi))$. The homomorphism λ plays the role of the left regular representation in harmonic analysis, see [55, P. 74]. The von Neumann algebra generated by the range of λ is denoted by $\hat{\mathbb{G}}$ or $VN(\mathbb{G})$. One of the most remarkable results in the quantum group theory is the generalized Pontryagin duality property, i.e., $L^\infty(\mathbb{G})$ and $VN(\mathbb{G})$ are dual to each other in some sense similar to the Pontryagin duality in harmonic analysis. We refer to [55, P. 74] and [75, Section 2] for more details. By the symbolism of quantum groups, we may write $L^\infty(\hat{\mathbb{G}})$ for $VN(\mathbb{G})$ and $A(\mathbb{G})$ for $L^1(\hat{\mathbb{G}}) = L^\infty(\hat{\mathbb{G}})_*$ [75].

3.3 Kac intertwined discrete systems

When studying discrete quantum groups, the symbol 1 may denote different units, such as the unit of the predual, the unit of the von Neumann algebra, or the usual multiplicative scalar unit. The intended meaning will always be clear from the context.

Lemma 3.3.1. *Let $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi)$ be a discrete Kac algebra. Then, $\mathbb{K}$ is a unimodular Kac algebra, and $\mathcal{M}_*$ is a Banach $*$ -algebra under its involution induced by κ . Furthermore, if $\omega \in \mathcal{M}_*$ is a positive linear functional on $\mathcal{M}$ and is unitary in $\mathcal{M}_*$, that is, $\omega \star \omega^* = \omega^* \star \omega = 1$, then ω defines a normal state on $\mathcal{M}$; equivalently, $\|\omega\| = \omega(1) = 1$.*

Proof. By the structure theorem for discrete Kac algebras, [28, Corollary 6.3.4], $\mathbb{K}$ is a unimodular Kac algebra, and by [28, PP. 15 - 16], $\mathcal{M}_*$ is a Banach $*$ -algebra. We therefore only need to show that $\|\omega\| = \omega(1) = 1$. From the definition of convolution on $\mathcal{M}_*$, it follows that $1 \in \mathcal{M}$ acts as a multiplicative linear functional on $\mathcal{M}_*$:

$$\langle f \star g, 1_{\mathcal{M}} \rangle = \langle f \otimes g, \Delta(1) \rangle = \langle f \otimes g, 1 \otimes 1 \rangle = \langle f, 1 \rangle \langle g, 1 \rangle, \quad \forall f, g \in \mathcal{M}_*. \quad (3.5)$$

We claim that

$$1_{\mathcal{M}}(1_{\mathcal{M}_*}) = \langle 1_{\mathcal{M}_*}, 1_{\mathcal{M}} \rangle \neq 0.$$

By the Identity (3.5) and the fact that $1 \star f = f$ for all $f \in \mathcal{M}_*$, we obtain

$$\langle f, 1 \rangle = \langle 1 \star f, 1 \rangle = \langle 1, 1 \rangle \langle f, 1 \rangle. \quad (3.6)$$

Substituting $f = 1$ yields

$$\langle 1, 1 \rangle = \langle 1, 1 \rangle^2.$$

If $\langle 1, 1 \rangle = 0$, then $1_{\mathcal{M}}$ would be the zero functional on $\mathcal{M}_*$ by Equation (3.6). Since $\mathcal{M}_*$ separates points of $\mathcal{M}$, this would imply $1_{\mathcal{M}} = 0$, a contradiction. Therefore, $\langle 1, 1 \rangle = 1$. Using the unitarity of ω , we compute

$$1 = \langle 1, 1 \rangle = \langle \omega \star \omega^*, 1 \rangle = \omega(1) \overline{\omega(1)} = |\omega(1)|^2.$$

Since $\omega(1) \geq 0$ by assumption, we obtain that $\omega(1) = 1$. However, by [56, Corollary 3.3.4], for the positive linear functional ω on $\mathcal{M}$, we have $\|\omega\| = \omega(1)$. Thus, $\|\omega\| = 1$. As members of $\mathcal{M}_*$ are normal functionals, ω is a normal state on $\mathcal{M}$. $\square$

Remark 3.3.2. For a discrete Kac algebra $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi)$, we use the following notations:

- $\mathcal{U}(\mathcal{M}_*)_+$: the set of unitary elements of $\mathcal{M}_*$ which are positive linear functionals on $\mathcal{M}$;
- $\mathcal{S}(\mathcal{M}_*)_+$: the set of normal states on $\mathcal{M}$, that is, positive, normal linear functionals on $\mathcal{M}$ of norm one.

By Lemma 3.3.1, we observe that $\mathcal{U}(\mathcal{M}_*)_+ \subseteq \mathcal{S}(\mathcal{M}_*)_+$.

Definition 3.3.3. A Kac Left Intertwined Discrete System (KLIDS) is a six-tuple $(\mathcal{M}, \Delta, \kappa, \phi, \pi_\phi, \lambda)$ such that:

- $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi)$ is a discrete Kac algebra,
- π_ϕ and λ are the representations defined in Section 3.2 and λ is a unital $*$ -homomorphism,
- the set of normal states $\mathcal{S}(\mathcal{M}_*)_+$ is the closed convex hull of $\mathcal{U}(\mathcal{M}_*)_+$ in $\mathcal{M}_*$,
- the following “intertwining relationships” holds for the left module action of all elements $\omega \in \mathcal{U}(\mathcal{M}_*)_+$ on $F \in \mathcal{M}$:

$$\pi_\phi(\omega \cdot F) = \lambda(\omega) \pi_\phi(F) \lambda(\omega)^*.$$

A *Kac Right Intertwined Discrete System (KRIDS)* is defined analogously, with conditions (i-iii) the same as above, and the intertwining condition (iv) replaced by

$$\pi_\phi(F \cdot \omega) = \lambda(\omega)^* \pi_\phi(F) \lambda(\omega), \quad F \in \mathcal{M}, \omega \in \mathcal{U}(\mathcal{M}_*)_+.$$

Here, we recall that the module actions $\omega \cdot F$ and $F \cdot \omega$ are given by Equations (3.4).

We now provide an example of a KRIDS.

Example 3.3.4. For a discrete group G , let $\mathcal{M} = \ell^\infty(G)$ and $\mathcal{M}_* = \ell^1(G)$. Consider the co-multiplication $\Delta : \ell^\infty(G) \rightarrow \ell^\infty(G) \bar{\otimes} \ell^\infty(G)$ defined by

$$\Delta(f)(s, t) = f(st), \quad s, t \in G,$$

the co-involution $\kappa : \ell^\infty(G) \rightarrow \ell^\infty(G)$ defined by

$$\kappa(f)(t) = f(t^{-1}), \quad t \in G,$$

and the regular counting measure ϕ on G . The co-involution κ induces the classical involution

$$\omega^*(s) = \overline{\omega(s^{-1})}$$

on $\ell^1(G)$ and makes it into a Banach $*$ -algebra with convolution product

$$(\omega * \eta)(s) = \sum_{t \in G} \omega(t) \eta(t^{-1}s).$$

In this setting, $\pi_\phi : \ell^\infty(G) \rightarrow \mathcal{B}(\ell^2(G))$ is the representation of $\ell^\infty(G)$ by multiplication operators, and λ is the unital $*$ -homomorphism induced by the left regular representation of G , i.e.,

$$\begin{aligned} \lambda : \ell^1(G) &\rightarrow \mathcal{B}(\ell^2(G)) \\ (\lambda(\omega)\xi)(s) &= \sum_{t \in G} \omega(t) \xi(t^{-1}s) = (\omega * \xi)(s), \quad (\omega \in \ell^1(G), \xi \in \ell^2(G), s \in G). \end{aligned}$$

(i) We first claim that

$$\mathcal{U}(\ell^1(G))_+ = \{\delta_g : g \in G\}.$$

For each $g \in G$, δ_g is unitary in $\ell^1(G)$ since

$$\delta_g^* * \delta_g = \delta_{g^{-1}} * \delta_g = \delta_{g^{-1}g} = \delta_e, \quad \delta_g * \delta_g^* = \delta_{gg^{-1}} = \delta_e.$$

Moreover, viewed as a normal linear functional on $\ell^\infty(G)$ via the canonical pairing

$$\langle \omega, F \rangle = \sum_{s \in G} \omega(s) F(s), \quad \omega \in \ell^1(G), F \in \ell^\infty(G),$$

we observe that

$$\langle \delta_g, F \rangle = \sum_{s \in G} \delta_g(s) F(s) = F(g) \geq 0, \tag{3.7}$$

for $F \geq 0$ in $\ell^\infty(G)$. So, δ_g is a positive normal linear functional on $\ell^\infty(G)$, and hence $\delta_g \in \mathcal{U}(\ell^1(G))_+$ by Lemma 3.3.1.

Conversely, suppose $\omega \in \mathcal{U}(\ell^1(G))_+$. As in equation (3.7), positivity yields

$$\omega(\delta_s) = \omega(s) \geq 0.$$

By Lemma 3.3.1, we have $\|\omega\| = 1$. Since the natural injection $j : \ell^1(G) \hookrightarrow (\ell^\infty(G))^*$ is isometric, it follows that

$$\sum_{s \in G} \omega(s) = 1.$$

Unitarity means $\omega^* * \omega = \delta_e$, and the $*$ -representation $\lambda : \ell^1(G) \rightarrow \mathcal{B}(\ell^2(G))$ gives

$$\lambda(\omega)^* \lambda(\omega) = I,$$

so $\lambda(\omega)$ is a unitary on $\ell^2(G)$. Since

$$(\lambda(\omega)\delta_e)(x) = \sum_{t \in G} \omega(t)\delta_e(t^{-1}x) = \omega(x),$$

we have $\lambda(\omega)\delta_e = \omega$ as a vector in $\ell^2(G)$. Similarly, for any $s \in G$

$$(\lambda(\omega)\delta_s)(x) = \omega(xs^{-1}).$$

As $\lambda(\omega)$ is unitary, we see that

$$0 = \langle \delta_s, \delta_e \rangle = \langle \lambda(\omega)\delta_s, \lambda(\omega)\delta_e \rangle = \sum_{x \in G} \omega(xs^{-1})\omega(x) \quad (s \neq e).$$

However, we recall that $\omega(s) \geq 0$ for all $s \in G$; hence,

$$\omega(xs^{-1})\omega(x) = 0 \quad \forall x \in G, s \neq e.$$

Now pick any two distinct group elements $y, z \in G$. Let $s = z^{-1}y \neq e$. Setting $x = y$ in the last relation gives:

$$\omega(y)s^{-1})\omega(y) = \omega(z)\omega(y) = 0.$$

Thus for any distinct $y \neq z$,

$$\omega(y)\omega(z) = 0.$$

Since $\omega \geq 0$, this means that no two distinct points can both have positive values under ω . The equality $\sum_s \omega(s) = 1$ then implies that exactly one point has mass 1, and all others have mass 0. That is,

$$\omega = \delta_g, \quad \text{for some } g \in G.$$

Therefore,

$$\mathcal{U}(\ell^1(G))_+ = \{\delta_g : g \in G\}.$$

(ii) Now, we show that the closed convex hull of $\mathcal{U}(\ell^1(G))_+$ is $\mathcal{S}(\ell^1(G))_+$.

Every finite convex combination

$$\sum_{j=1}^n \alpha_j \delta_{g_j}, \quad \alpha_j \geq 0, \quad \sum_{j=1}^n \alpha_j = 1,$$

is positive and of norm one, hence lies in $\mathcal{S}(\ell^1(G))_+$. Thus

$$\overline{\text{co}}(\mathcal{U}(\ell^1(G))_+) \subseteq \mathcal{S}(\ell^1(G))_+.$$

Conversely, let $\omega \in \mathcal{S}(\ell^1(G))_+$. Then, $\omega(s) \geq 0$ and $\sum_s \omega(s) = 1$ by the natural injection $:\ell^1(G) \longrightarrow [\ell^1(G)]^{**} = [\ell^\infty(G)]^*$. The equation

$$\|\omega\|_1 = \sum_{s \in G} |\omega(s)| = \sup \left\{ \sum_{s \in F} |\omega(s)| : F \subseteq G \text{ finite} \right\} < \infty,$$

immediately implies that for each $\varepsilon > 0$, there exists a finite set $F \subset G$ such that

$$\sum_{s \notin F} \omega(s) < \varepsilon.$$

Define

$$\nu(s) = \begin{cases} \omega(s), & s \in F, \\ 0, & \text{o.w.} \end{cases}$$

Then,

$$\|\omega - \nu\|_1 = \sum_{s \notin F} \omega(s) < \varepsilon.$$

Set $a = \|\nu\|_1$, and define $\tilde{\nu} = \nu/a$. We estimate the difference between ω and $\tilde{\nu}$ by writing

$$\|\omega - \tilde{\nu}\|_1 \leq \|\omega - \nu\|_1 + \|\nu - \tilde{\nu}\|_1.$$

The first term is less than ε by the construction of ν . To estimate the second term, note that

$$\tilde{\nu} = \frac{\nu}{a}, \quad a = \|\nu\|_1 = \sum_{s \in F} \omega(s) \geq 1 - \varepsilon.$$

Thus,

$$\tilde{\nu} = \sum_{s \in F} \frac{\omega(s)}{a} \delta_s \in \text{co}(\{\delta_s : s \in F\})$$

and

$$\|\nu - \tilde{\nu}\|_1 = \left| 1 - \frac{1}{a} \right| \|\nu\|_1 = |1 - a| < \varepsilon.$$

Combining the two estimates gives

$$\|\omega - \tilde{\nu}\|_1 \leq \|\omega - \nu\|_1 + \|\nu - \tilde{\nu}\|_1 < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, ω is in the closed convex hull of $\{\delta_g : g \in G\}$. Consequently,

$$\overline{\text{co}}(\mathcal{U}(\ell^1(G))_+) = \mathcal{S}(\ell^1(G))_+.$$

- (iii) For $F \in \ell^\infty(G)$ and $g \in G$, we show that the right module action of the point mass $\delta_g \in \ell^1(G)$ at F is given by

$$(F \cdot \delta_g)(s) = F(gs), \quad s \in G.$$

By the definition of the right module action,

$$\langle F \cdot \delta_g, \mu \rangle = \langle F, \delta_g \star \mu \rangle, \quad \forall \mu \in \ell^1(G).$$

However,

$$(\delta_g \star \mu)(s) = \sum_{t \in G} \delta_g(t) \mu(t^{-1}s) = \mu(g^{-1}s).$$

Hence,

$$\begin{aligned} \langle F \cdot \delta_g, \mu \rangle &= \langle F, \delta_g \star \mu \rangle \\ &= \sum_{s \in G} F(s) [\delta_g \star \mu](s) \\ &= \sum_{s \in G} F(s) \mu(g^{-1}s) \quad [r = g^{-1}s, s = gr] \\ &= \sum_{r \in G} F(gr) \mu(r) \\ &= \langle F(g \cdot), \mu \rangle \end{aligned}$$

Since this holds for all $\mu \in \ell^1(G)$, it follows that

$$(F \cdot \delta_g)(s) = F(gs).$$

- (iv) For $F \in \ell^\infty(G)$, $\xi \in \ell^2(G)$, and $g, s \in G$, we have

$$(\pi_\phi(F)\xi)(s) = F(s)\xi(s), \quad (\lambda(\delta_g)\xi)(s) = \xi(g^{-1}s), \quad (\lambda(\delta_g)^*\xi)(s) = \xi(gs).$$

So,

$$\begin{aligned} [\lambda(\delta_g)^* \pi_\phi(F) \lambda(\delta_g) \xi](s) &= [\pi_\phi(F) \lambda(\delta_g) \xi](gs) \\ &= [F(\cdot) \lambda(\delta_g) \xi](gs) \\ &= F(gs) (\lambda(\delta_g) \xi)(gs) \\ &= F(gs) \xi(g^{-1}gs) \\ &= F(gs) \xi(s) \\ &= [F \cdot \delta_g](s) \xi(s) \\ &= [\pi_\phi(F \cdot \delta_g) \xi](s). \end{aligned}$$

Since this holds for all $\xi \in \ell^2(G)$, we conclude that the intertwining relationship through the right action holds.

Combining parts (i), (ii), and (iii) above, we conclude that $(\ell^\infty(G), \Delta, \kappa, \phi, \pi_\phi, \lambda)$ is a Kac Right Intertwined Discrete System (KRIDS).

3.4 Approximate amenability of discrete quantum groups

In this section, we aim to prove that, for certain discrete quantum groups, approximate amenability implies amenability. As a consequence, we recover some classical results of Ghahramani et al.

Remark 3.4.1. Let $\mathcal{M}$ be a von Neumann algebra with the pre-dual $\mathcal{M}_*$. We recall the following facts from [81, Sections III.3 and III.4].

- (i) Since the multiplication in $\mathcal{M}$ is weak* separately continuous, $\mathcal{M}_*$ can be viewed as an $\mathcal{M}$ -bimodule with the actions given by

$$\begin{aligned} \mathcal{M} \times \mathcal{M}_* &\longrightarrow \mathcal{M}_* \\ (a, \mu) &\longrightarrow a \cdot \mu, \quad \langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle, \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_* \times \mathcal{M} &\longrightarrow \mathcal{M}_* \\ (\mu, a) &\longrightarrow \mu \cdot a, \quad \langle \mu \cdot a, b \rangle = \langle \mu, ab \rangle. \end{aligned}$$

- (ii) Let $\omega \in \mathcal{M}_*^+$ be a positive normal linear functional on $\mathcal{M}$. Then, there exists a unique non-zero projection $p \in \mathcal{M}$ such that

$$p \cdot \omega = \omega \cdot p = p \cdot \omega \cdot p = \omega,$$

and ω is faithful on the reduced von Neumann algebra $p\mathcal{M}p$. The projection p is called the *support projection* of ω and is denoted by $s(\omega)$.

- (iii) For an arbitrary normal linear functional $\Omega \in \mathcal{M}_*$, there exists a unique partial isometry $v \in \mathcal{M}$ and a unique positive normal functional $\omega \in \mathcal{M}_*^+$ such that

$$\Omega = v \cdot \omega, \quad v^*v = s(\omega).$$

This representation is called the *polar decomposition* of Ω , and ω is called the *absolute value* of Ω and is denoted by $\omega = |\Omega|$. We have $\|\Omega\| = \|\omega\|$.

Lemma 3.4.2. Let $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi, \pi_\phi, \lambda)$ be a KLIDS, $\omega \in \mathcal{U}(\mathcal{M}_*)_+$, and $\eta \in \mathcal{M}_*$. Then,

$$|\eta \star \omega| = |\eta| \star \omega,$$

and

$$\| |\eta| \star \omega - |\eta| \| \leq 2 \|\eta\|^{1/2} \|\eta \star \omega - \eta\|^{1/2}. \quad (3.8)$$

An analogous statement holds also for a KRIDS, with the orders of η and ω interchanged in the above formulas.

Proof. Fix $\omega \in \mathcal{U}(\mathcal{M}_*)_+$, and define

$$\begin{aligned} \Phi_\omega : \mathcal{M} &\longrightarrow \mathcal{M} \\ \Phi_\omega(F) &:= (\text{id} \otimes \omega)\Delta(F) \end{aligned}$$

Since Δ and the slicing map $\text{id} \otimes \omega$ are normal, Φ_ω is normal as well. By definition, for any $\eta \in \mathcal{M}_*$, $F \in \mathcal{M}$, we have

$$\langle \eta, \omega \cdot F \rangle = \langle \eta \star \omega, F \rangle = \langle \eta \otimes \omega, \Delta(F) \rangle. \quad (3.9)$$

Applying the slicing map $\text{id} \otimes \omega$ on simple tensors $\Delta(F) = F_1 \otimes F_2 \in \mathcal{M} \otimes \mathcal{M}$ shows that

$$\langle \eta, (\text{id} \otimes \omega) \Delta(F) \rangle = \langle \eta \otimes \omega, \Delta(F) \rangle. \quad (3.10)$$

Combining Equations (3.9) and (3.10), we deduce that

$$\Phi_\omega(F) = (\text{id} \otimes \omega) \Delta(F) = \omega \cdot F. \quad (3.11)$$

Plugging (3.11) into the intertwining relation of Definition 3.3.3 (iv) yields that for every $F \in \mathcal{M}$

$$\pi_\phi(\Phi_\omega(F)) = \pi_\phi(\omega \cdot F) = \lambda(\omega) \pi_\phi(F) \lambda(\omega)^*.$$

Since $\omega \in \mathcal{U}(\mathcal{M}_*)_+$, the operator $\lambda(\omega)$ is unitary; hence, $X \mapsto \lambda(\omega) X \lambda(\omega)^*$ is a normal $*$ -automorphism of the von Neumann algebra $\pi_\phi(\mathcal{M})$. As π_ϕ is faithful, after identifying $\mathcal{M}$ with $\pi_\phi(\mathcal{M})$, we deduce that Φ_ω is a normal unital $*$ -automorphism of $\mathcal{M}$. Now, consider the pre-adjoint of Φ_ω defined by

$$\begin{aligned} \Phi_{\omega*} : \mathcal{M}_* &\longrightarrow \mathcal{M}_* \\ \langle \Phi_{\omega*}(\eta), F \rangle &= \langle \eta, \Phi_\omega(F) \rangle, \quad \eta \in \mathcal{M}_*, F \in \mathcal{M}. \end{aligned}$$

It is readily seen that

$$\Phi_{\omega*}(\eta) = \eta \star \omega, \quad \forall \eta \in \mathcal{M}_*. \quad (3.12)$$

Also, $\langle \eta \circ \Phi_\omega, F \rangle = \langle \eta, \Phi_\omega(F) \rangle = \langle \eta \star \omega, F \rangle$ yields

$$\eta \circ \Phi_\omega = \eta \star \omega. \quad (3.13)$$

We claim that for every $\eta \in \mathcal{M}_*$,

$$|\eta \circ \Phi_\omega| = |\eta| \circ \Phi_\omega. \quad (3.14)$$

Assume the polar decomposition $\eta = v \cdot |\eta|$ with $v^* v = s(|\eta|)$. Since Φ_ω is a $*$ -automorphism, set $v_\omega := \Phi_\omega^{-1}(v)$ and $\rho_\omega := |\eta| \circ \Phi_\omega$. Then, v_ω is a partial isometry and $\rho_\omega \in \mathcal{M}_*^+$. Now, for all $F \in \mathcal{M}$

$$\begin{aligned} (\eta \circ \Phi_\omega)(F) &= \eta(\Phi_\omega(F)) \\ &= |\eta|(\Phi_\omega(F) v) \\ &= |\eta|(\Phi_\omega(F) \Phi_\omega(\Phi_\omega^{-1}(v))) \\ &= (|\eta| \circ \Phi_\omega)(F \Phi_\omega^{-1}(v)) \\ &= \rho_\omega(F v_\omega) \\ &= v_\omega \cdot \rho_\omega(F). \end{aligned}$$

So, $\eta \circ \Phi_\omega = v_\omega \cdot \rho_\omega$. By the uniqueness of the polar decomposition, we obtain (3.14). Combining Equations (3.14) and (3.13), we have

$$|\eta \star \omega| = |\eta| \star \omega. \quad (3.15)$$

By the main theorem of [52], for all $\eta, \theta \in \mathcal{M}_*$, we have

$$\| |\theta| - |\eta| \|^2 \leq 2 \|\theta + \eta\| \|\theta - \eta\|. \quad (3.16)$$

For $\theta = \eta \star \omega$, we note that $\|\eta \star \omega\| \leq \|\eta\|$ and $\|\eta + \theta\| \leq 2\|\eta\|$. Substituting these in (3.16) and using (3.15), we obtain

$$\| |\eta| \star \omega - |\eta| \| \leq 2 \|\eta\|^{1/2} \|\eta \star \omega - \eta\|^{1/2},$$

which is exactly (3.8). For a KRIDS, we need to define $\Phi_\omega(F) = F \cdot \omega$ and proceed similarly. $\square$

Theorem 3.4.3. *Let $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi, \pi_\phi, \lambda)$ be a KLIDS (respectively, KRIDS), and denote by $C_r^*(\mathbb{K})$ and $VN(\mathbb{K})$ the C^* -algebra and the von Neumann algebra generated by the range of λ . Suppose that $VN(\mathbb{K})$ admits a normal faithful tracial state τ and that $C_r^*(\mathbb{K})$ is approximately amenable. Then $\mathcal{M}$ admits a right (respectively, left) invariant mean.*

Proof. Since $C_r^*(\mathbb{K})$ is unital, it is pseudo amenable. Suppose the net $d_\alpha = \sum_{n=1}^\infty a_n^\alpha \otimes b_n^\alpha \in C_r^*(\mathbb{K}) \otimes_\gamma C_r^*(\mathbb{K})$ is an approximate diagonal for $C_r^*(\mathbb{K})$ and $\pi : C_r^*(\mathbb{K}) \otimes_\gamma C_r^*(\mathbb{K}) \rightarrow C_r^*(\mathbb{K})$ is the multiplication map. Since $C_r^*(\mathbb{K})$ is unital, we may suppose that $\pi(d_\alpha) = \sum_{n=1}^\infty a_n^\alpha b_n^\alpha = 1 \in C_r^*(\mathbb{K})$. Moreover, we can assume that each d_α has a finite sum representation; i.e., $d_\alpha = \sum_{n=1}^{r_\alpha} a_n^\alpha \otimes b_n^\alpha$ with $r_\alpha < \infty$. Note that the identity element of $C_r^*(\mathbb{K})$ is the same of that of $\mathcal{B}(\mathcal{H})$ by construction; where $\mathcal{H} = \mathcal{H}_\phi$ as in Section 3.2. Let Π be the multiplication map of the projective tensor product $\mathcal{B}(\mathcal{H}) \otimes_\gamma \mathcal{B}(\mathcal{H})$. Since for the projective norms we have $\|\cdot\|_{\gamma, C_r^*(\mathbb{K})} \geq \|\cdot\|_{\gamma, \mathcal{B}(\mathcal{H})}$, we deduce that $d_\alpha \in \mathcal{B}(\mathcal{H}) \otimes_\gamma \mathcal{B}(\mathcal{H})$. So, we have $\Pi(d_\alpha) = \sum_{n=1}^\infty a_n^\alpha b_n^\alpha = 1 \in \mathcal{B}(\mathcal{H})$.

Since τ is a normal state on von Neumann algebra $VN(\mathbb{K})$, by [48, Theorem 7.1.12] τ is a countable sum of vector states, i.e., $\tau = \sum_{n=1}^\infty \omega_{\xi_n}|_{VN(\mathbb{K})}$, where $\omega_{\xi_n}(T) = \langle T\xi_n, \xi_n \rangle$ and $(\xi_n)_n$ is an orthogonal family in $\mathcal{H}$ such that $\sum_{n=1}^\infty \|\xi_n\|^2 = 1$. Then, $\tilde{\tau} = \sum_{n=1}^\infty \omega_{\xi_n}$ is a normal state on $\mathcal{B}(\mathcal{H})$, which extends τ . So, $\tilde{\tau}|_{VN(\mathbb{K})} = \tau$ and $\|\tilde{\tau}\| = \|\tau\| = \tau(1) = 1$. Let χ be the flip map defined by

$$\begin{aligned} \chi : \mathcal{B}(\mathcal{H}) \otimes_\gamma \mathcal{B}(\mathcal{H}) &\longrightarrow \mathcal{B}(\mathcal{H}) \otimes_\gamma \mathcal{B}(\mathcal{H}) \\ \chi(a \otimes b) &= b \otimes a. \end{aligned} \quad (3.17)$$

Recall that $\|\Pi\| = 1$, $\|\pi_\phi\| = 1$, and $\|\chi\| = 1$. Now, we define a family of linear functionals on $\mathcal{M}$ with the aid of the approximate diagonal of $C_r^*(\mathbb{K})$:

$$\begin{aligned} \Omega_\alpha : \mathcal{M} &\longrightarrow \mathbb{C} \\ \langle \Omega_\alpha, F \rangle &= \langle \tilde{\tau}, \Pi \circ \chi(\pi_\phi(F) \cdot d_\alpha) \rangle. \end{aligned}$$

We compute that

$$\begin{aligned} |\Omega_\alpha(F)| &\leq \|\tilde{\tau}\| \|\Pi\| \|\pi_\phi\| \|F\| \|d_\alpha\| \\ &\leq \|F\| \|d_\alpha\|. \end{aligned}$$

Also,

$$\begin{aligned}
\|\Omega_\alpha\| &\geq |\Omega_\alpha(1)| = \langle \tilde{\tau}, \Pi \circ \chi(d_\alpha) \rangle \\
&= \left\langle \tau, \sum_{n=1}^{r_\alpha} b_n^\alpha a_n^\alpha \right\rangle \\
&= \left\langle \tau, \sum_{n=1}^{r_\alpha} a_n^\alpha b_n^\alpha \right\rangle \quad (\tau \text{ is tracial on } VN(\mathbb{K})) \\
&= \langle \tau, 1 \rangle = 1.
\end{aligned}$$

So, $1 \leq \|\Omega_\alpha\| \leq \|d_\alpha\| < \infty$, which shows that each Ω_α is non-trivial and bounded. Now, by the property of the flip map, for any $T \in \mathcal{B}(\mathcal{H})$ and $c \in C_r^*(\mathbb{K})$, we obtain

$$\begin{aligned}
\Pi \circ \chi(T \cdot d_\alpha \cdot c) &= \Pi \circ \chi\left(\sum T a_n^\alpha \otimes b_n^\alpha c\right) \\
&= \Pi\left(\sum b_n^\alpha c \otimes T a_n^\alpha\right) \\
&= \sum b_n^\alpha [c T a_n^\alpha] \\
&= \Pi \circ \chi\left(\sum [c T] a_n^\alpha \otimes b_n^\alpha\right) \\
&= \Pi \circ \chi(c T \cdot d_\alpha).
\end{aligned}$$

Thus,

$$\Pi \circ \chi(c T \cdot d_\alpha) = \Pi \circ \chi(T \cdot d_\alpha \cdot c). \quad (3.18)$$

We now split the remainder of the proof according to the two possible cases of the intertwining relation.

(i) Assume that $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi, \pi_\phi, \lambda)$ is a KLIDS, i.e.,

$$\pi_\phi(\omega \cdot F) = \lambda(\omega) \pi_\phi(F) \lambda(\omega)^*, \quad \forall F \in \mathcal{M} \quad \forall \omega \in \mathcal{U}(\mathcal{M}_*)_+.$$

Setting $c = \lambda(\omega)$ and $T = \pi_\phi(F) \lambda(\omega)^*$ in Equation (3.18), we obtain

$$\Pi \circ \chi(\lambda(\omega) \pi_\phi(F) \lambda(\omega)^* \cdot d_\alpha) = \Pi \circ \chi(\pi_\phi(F) \lambda(\omega)^* \cdot d_\alpha \cdot \lambda(\omega)). \quad (3.19)$$

Applying the intertwining relationship for $\omega \cdot F$, we compute

$$\begin{aligned}
(\Omega_\alpha \star \omega - \Omega_\alpha)(F) &= \Omega_\alpha(\omega \cdot F) - \Omega_\alpha(F) \\
&= \langle \Omega_\alpha, \omega \cdot F - F \rangle \\
&= \langle \tilde{\tau}, \Pi \circ \chi(\pi_\phi(\omega \cdot F - F) \cdot d_\alpha) \rangle \\
&= \langle \tilde{\tau}, \Pi \circ \chi(\lambda(\omega) \pi_\phi(F) \lambda(\omega)^* - \pi_\phi(F)) \cdot d_\alpha \rangle \\
&= \langle \tilde{\tau}, \Pi \circ \chi(\lambda(\omega) \pi_\phi(F) \lambda(\omega)^* \cdot d_\alpha - \pi_\phi(F) \cdot d_\alpha) \rangle.
\end{aligned}$$

From Equation (3.19) and the fact that $\lambda(\omega)^* \lambda(\omega) = 1$, we have

$$\begin{aligned}
(\Omega_\alpha \star \omega - \Omega_\alpha)(F) &= \langle \tilde{\tau}, \Pi \circ \chi(\pi_\phi(F) \lambda(\omega)^* \cdot d_\alpha \cdot \lambda(\omega) - \pi_\phi(F) \cdot d_\alpha) \rangle \\
&= \langle \tilde{\tau}, \Pi \circ \chi(\pi_\phi(F) \lambda(\omega)^* \cdot d_\alpha \cdot \lambda(\omega) - \pi_\phi(F) \lambda(\omega)^* \lambda(\omega) \cdot d_\alpha) \rangle \\
&= \langle \tilde{\tau}, \Pi \circ \chi(\pi_\phi(F) \lambda(\omega)^* \cdot [d_\alpha \cdot \lambda(\omega) - \lambda(\omega) \cdot d_\alpha]) \rangle.
\end{aligned}$$

This yields

$$\|\Omega_\alpha \star \omega - \Omega_\alpha\| \leq \|\lambda(\omega)^*\| \|d_\alpha \cdot \lambda(\omega) - \lambda(\omega) \cdot d_\alpha\|,$$

and since $\lambda(\omega)$ is a unitary operator, we have

$$\|\Omega_\alpha \star \omega - \Omega_\alpha\| \leq \|d_\alpha \cdot \lambda(\omega) - \lambda(\omega) \cdot d_\alpha\|, \quad \forall \omega \in \mathcal{U}(\mathcal{M}_*)_+ . \quad (3.20)$$

However, d_α is an approximate diagonal for $C_r^*(\mathbb{K})$ and $\lambda(\omega) \in C_r^*(\mathbb{K})$. So,

$$\|d_\alpha \cdot \lambda(\omega) - \lambda(\omega) \cdot d_\alpha\| \longrightarrow 0$$

This implies

$$\|\Omega_\alpha \star \omega - \Omega_\alpha\| \longrightarrow 0, \quad \forall \omega \in \mathcal{U}(\mathcal{M}_*)_+ . \quad (3.21)$$

Since $1 \leq \|\Omega_\alpha\| \leq \|d_\alpha\| < \infty$, we may replace Ω_α with $\frac{\Omega_\alpha}{\|\Omega_\alpha\|}$ in (3.21). Therefore, without loss of generality, we may assume that $\|\Omega_\alpha\| = 1$. Consider the polar decomposition

$$\Omega_\alpha = v_\alpha \cdot \mu_\alpha, \quad 1 = \|\Omega_\alpha\| = \|\mu_\alpha\| = \mu_\alpha(1).$$

By Lemma 3.4.2, we have

$$\|\mu_\alpha \star \omega - \mu_\alpha\| \leq 2 \|\Omega_\alpha \star \omega - \Omega_\alpha\|^{1/2}. \quad (3.22)$$

Using (3.21), we deduce that

$$\|\mu_\alpha \star \omega - \mu_\alpha\| \longrightarrow 0, \quad \forall \omega \in \mathcal{U}(\mathcal{M}_*)_+ . \quad (3.23)$$

Since each μ_α is positive and $\|\mu_\alpha\| = 1 = \mu_\alpha(1)$, the net $(\mu_\alpha)_\alpha$ lies in the unit ball of $\mathcal{M}^*$, which is weak*-compact. Hence, it admits a weak* cluster point $\mu \in \mathcal{M}_+^*$. Passing to a subnet if necessary, we can assume

$$\mu_\alpha \xrightarrow{w^*} \mu.$$

Now, fix any $\omega \in \mathcal{U}(\mathcal{M}_*)_+$ and any $F \in \mathcal{M}$. Using the equation $\langle \mu_\alpha \star \omega, F \rangle = \langle \mu_\alpha, \omega \cdot F \rangle$, we obtain

$$\langle \mu_\alpha \star \omega, F \rangle \longrightarrow \langle \mu \star \omega, F \rangle, \quad \langle \mu_\alpha, F \rangle \longrightarrow \langle \mu, F \rangle.$$

So, by (3.23), we have

$$\langle \mu \star \omega - \mu, F \rangle = \lim_\alpha \langle \mu_\alpha \star \omega - \mu_\alpha, F \rangle = 0.$$

As F was arbitrary, we conclude that $\mu \star \omega = \mu$ for all $\omega \in \mathcal{U}(\mathcal{M}_*)_+$. Since $\omega(1) = 1$, we have

$$\mu \star \omega = \omega(1)\mu, \quad \forall \omega \in \mathcal{U}(\mathcal{M}_*)_+ . \quad (3.24)$$

This equation remains valid if we replace $\omega \in \mathcal{U}(\mathcal{M}_*)_+$ by any convex combination of such elements, and it continues to hold in the limit of them as well. Since $\overline{\text{co}}[\mathcal{U}(\mathcal{M}_*)_+] = \mathcal{S}(\mathcal{M}_*)_+$ from the definition of KLIDS, we have

$$\mu \star \omega = \omega(1)\mu, \quad \forall \omega \in \mathcal{S}(\mathcal{M}_*)_+ . \quad (3.25)$$

However, by [49, Corollary 4.3.7], each linear functional on $\mathcal{M}$ is the sum of four states. Each state is uniquely decomposed into a normal and singular parts by [81, P. 127]. This implies that each member of $\mathcal{M}_*$ is the sum of at most four members of $\mathcal{S}(\mathcal{M}_*)_+$. So,

$$\mu \star \omega = \omega(1)\mu, \quad \forall \omega \in \mathcal{M}_* . \quad (3.26)$$

Consequently, by Definition (3.1.5), μ is a right invariant mean.

(ii) Suppose that $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi, \pi_\phi, \lambda)$ is a KRIDS, i.e.,

$$\pi_\phi(F \cdot \omega) = \lambda(\omega)^* \pi_\phi(F) \lambda(\omega) \quad \forall F \in \mathcal{M} \quad \forall \omega \in \mathcal{U}(\mathcal{M}_*)_+.$$

Setting $c = \lambda(\omega)^*$, $T = \pi_\phi(F) \lambda(\omega)$ in Equation (3.18), and proceeding by a computation analogous to the one in part (i), we obtain

$$(\omega \star \Omega_\alpha - \Omega_\alpha)(F) = \langle \tilde{\tau}, \Pi \circ \chi(\pi_\phi(F) \lambda(\omega) \cdot [d_\alpha \cdot \lambda(\omega)^* - \lambda(\omega)^* \cdot d_\alpha]) \rangle,$$

and consequently,

$$\|\omega \star \Omega_\alpha - \Omega_\alpha\| \leq \|d_\alpha \cdot \lambda(\omega)^* - \lambda(\omega)^* \cdot d_\alpha\|. \quad (3.27)$$

We now claim that

$$\|\omega \star \mu_\alpha - \mu_\alpha\| \leq 2\|\omega \star \Omega_\alpha - \Omega_\alpha\|^{1/2}. \quad (3.28)$$

Recall that $\mathcal{M}_*$ is a Banach $*$ -algebra under the involution given by Definition 3.1.6. In Inequality (3.22), we consider $\Omega_\alpha^*, \mu_\alpha^*, \omega^* \in \mathcal{M}_*$ instead of $\Omega_\alpha, \mu_\alpha, \omega$. Hence,

$$\|\mu_\alpha^* \star \omega^* - \mu_\alpha^*\| \leq \|\Omega_\alpha^* \star \omega^* - \Omega_\alpha^*\|. \quad (3.29)$$

Then,

$$\begin{aligned} \|\omega \star \mu_\alpha - \mu_\alpha\| &= \|(\mu_\alpha^* \star \omega^* - \mu_\alpha^*)^*\| \\ &= \|\mu_\alpha^* \star \omega^* - \mu_\alpha^*\| \\ &\leq 2\|\Omega_\alpha^* \star \omega^* - \Omega_\alpha^*\|^{1/2} \\ &= 2\|(\omega \star \Omega_\alpha - \Omega_\alpha)^*\|^{1/2} \\ &= 2\|\omega \star \Omega_\alpha - \Omega_\alpha\|^{1/2}, \end{aligned}$$

which proves (3.28). By Inequality (3.27), we obtain

$$\|\omega \star \mu_\alpha - \mu_\alpha\| \longrightarrow 0, \quad \forall \omega \in \mathcal{U}(\mathcal{M}_*)_+. \quad (3.30)$$

As in part (i), we then obtain a state $\mu \in \mathcal{M}^*$ such that

$$\omega \star \mu = \omega(1)\mu, \quad \forall \omega \in \mathcal{M}_*. \quad (3.31)$$

So, by Definition (3.1.5), μ is a left invariant mean.

□

With Theorem 3.4.3 at hand, we can give alternative proofs for some results of [9]. For instance, the following result was proved in [9, Corollary 7.3]; here, we provide another proof using quantum group techniques.

Corollary 3.4.4. *Let G be a discrete group. If $C_r^*(G)$ is approximately amenable, then G is amenable.*

Proof. By Example 3.3.4, the system $(\ell^\infty(G), \Delta, \kappa, \phi, \pi_\phi, \lambda)$ is a KRIDS. By [48, Example 8.1.4], the assignment $\tau(A) = \langle A\delta_e, \delta_e \rangle$, where δ_e is the characteristic function at the identity of G , defines a faithful normal tracial state on $VN(\mathbb{K}) = VN(G)$. Now, we can apply Theorem 3.4.3 to deduce that $\ell^\infty(G)$ has a LIM. Thus, by [76, Proposition 1.1.9], G must be amenable. □

Corollary 3.4.5. $C_r^*(\mathbb{F}_2)$ is not approximately amenable.

Proof. If $C_r^*(\mathbb{F}_2)$ were approximately amenable, then by Corollary 3.4.4, the discrete group $\mathbb{F}_2$ would be amenable, which is absurd. $\square$

Corollary 3.4.6. The full group C^* -algebra $C^*(\mathbb{F}_2)$ is not approximately amenable.

Proof. By [17, Theorem VII.2.5], there is an onto homomorphism $\lambda : C^*(\mathbb{F}_2) \rightarrow C_r^*(\mathbb{F}_2)$ which is continuous since $\|\lambda(f)\| \leq \|f\|_*$. If $C^*(\mathbb{F}_2)$ is approximately amenable, then by [33, Proposition 2.2], $C_r^*(\mathbb{F}_2)$ must be approximately amenable, which is not the case by Corollary 3.4.5. $\square$

Corollary 3.4.7. For any discrete group G , the followings are equivalent:

- (i) G is amenable
- (ii) $\ell^1(G)$ is approximately amenable.
- (iii) $C_r^*(G)$ is approximately amenable

Proof. Equivalence $i \iff ii$ is [33, Theorem 3.2].

The implication $iii \implies i$ is Corollary 3.4.4. For the converse, see [59, P. 44]. $\square$

Definition 3.4.8. A locally compact quantum group $\mathbb{G} = (\mathcal{M}, \Delta, \phi, \psi)$ is said to be

- Day–Von Neumann (DV) amenable (or *mean amenable*) if $\mathcal{M} = L^\infty(\mathbb{G})$ has a LIM or RIM.
- Johnson (J) amenable (or *derivation amenable*) if $\mathcal{M}_* = L^1(\mathbb{G})$ is an amenable Banach algebra.

We now generalize the idea of the proof of Ghahramani et al., in [36] and [76, Theorem 4.4.19]. This allows us to obtain the following theorem:

Theorem 3.4.9. Let $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi, \pi_\phi, \lambda)$ be a KLIDS. If $\mathcal{M}_*$ is approximately amenable, then $\mathcal{M}$ has a RIM. In other words, if $\mathbb{K}$ is J approximately amenable, then it is DV amenable.

Proof. Since $\mathcal{M}_*$ is unital, it is pseudo-amenable. Suppose that (d_α) with $d_\alpha = \sum_{n=1}^\infty a_n^\alpha \otimes_\gamma b_n^\alpha \in \mathcal{M}_* \otimes \mathcal{M}_*$ forms an approximate diagonal for $\mathcal{M}_*$ such that $\pi(d_\alpha) = 1_{\mathcal{M}_*}$. From Equation (3.3), $1 \in \mathcal{M}$ is a multiplicative linear functional on $\mathcal{M}_*$. Define

$$f_\alpha = (\text{id} \otimes 1)d_\alpha = \sum_{n=1}^\infty a_n^\alpha \langle 1, b_n^\alpha \rangle \in \mathcal{M}_*.$$

Let $\omega \in \mathcal{U}(\mathcal{M}_*)_+$. Recall that $\mathcal{M}_* \hat{\otimes} \mathcal{M}_*$ is an $\mathcal{M}_*$ -bimodule, and

$$\begin{aligned} \omega \cdot d_\alpha &= \sum_{n=1}^\infty \omega \star a_n^\alpha \otimes b_n^\alpha, \\ d_\alpha \cdot \omega &= \sum_{n=1}^\infty a_n^\alpha \otimes b_n^\alpha \star \omega. \end{aligned}$$

Thus,

$$(\text{id} \otimes 1)(\omega \cdot d_\alpha) = \sum_{n=1}^{\infty} \omega \star a_n^\alpha \langle 1, b_n^\alpha \rangle = \omega \star f_\alpha,$$

and

$$(\text{id} \otimes 1)(d_\alpha \cdot \omega) = \sum_{n=1}^{\infty} a_n^\alpha \langle 1, b_n^\alpha \star \omega \rangle = \sum_{n=1}^{\infty} a_n^\alpha \langle 1, b_n^\alpha \rangle \langle 1, \omega \rangle = f_\alpha \langle 1, \omega \rangle = f_\alpha.$$

Hence,

$$\omega \star f_\alpha - f_\alpha = (\text{id} \otimes 1)[\omega \cdot d_\alpha - d_\alpha \cdot \omega].$$

We then derive

$$\|\omega \star f_\alpha - f_\alpha\|_1 = \|(\text{id} \otimes 1)(\omega \cdot d_\alpha - d_\alpha \cdot \omega)\| \leq \|\omega \cdot d_\alpha - d_\alpha \cdot \omega\| \longrightarrow 0 \quad (3.32)$$

since (d_α) is an approximate diagonal. From $\pi(d_\alpha) = 1$, we obtain

$$\langle 1, f_\alpha \rangle = \langle 1, \sum_{n=1}^{\infty} a_n^\alpha \langle 1, b_n^\alpha \rangle \rangle = \sum_{n=1}^{\infty} \langle 1, a_n^\alpha \rangle \langle 1, b_n^\alpha \rangle = \sum_{n=1}^{\infty} \langle 1, a_n^\alpha \star b_n^\alpha \rangle = \langle 1, \pi(d_\alpha) \rangle = 1_{\mathcal{M}_*}.$$

Thus, $\|f_\alpha\|_1 \geq 1$ for all α . By [81, Theorem III.4.2], let $|f_\alpha|$ be the absolute value functional of f_α in its polar decomposition, and note that $\| |f_\alpha| \|_1 = \|f_\alpha\|_1$ by [81, Proposition III.4.6]. Then, $\omega_\alpha = \frac{|f_\alpha|}{\|f_\alpha\|_1} \in \mathcal{M}_*^+$ has norm one. By Lemma 3.4.2, we have

$$\|\omega \star \omega_\alpha - \omega_\alpha\|_1 \leq 2 \|\omega \star f_\alpha - f_\alpha\|_1^{1/2}.$$

So, by (3.32) we obtain

$$\|\omega \star \omega_\alpha - \omega_\alpha\| \longrightarrow 0.$$

As in the proof of Theorem 3.4.3, we conclude that every weak* cluster point of $(\omega_\alpha)_\alpha$ is a right invariant mean of $\mathcal{M}$. $\square$

Corollary 3.4.10. *For a KLIDS $\mathbb{K} = (\mathcal{M}, \Delta, \kappa, \phi, \pi_\phi, \lambda)$, the notions of approximate J amenability, J amenability, and DV amenability are the same.*

Proof. When $\mathbb{K}$ is discrete, its pre-dual has a unit. Now, combine Theorem 3.4.9 with the main theorem of [73, Theorem 4.5]. $\square$

Chapter 4

Generalized amenability of C^* -algebras and finite von Neumann algebras

By definition, any amenable Banach algebra is approximately amenable. The converse, in general, is not true; see [33, Example 6.1]. We now ask whether the converse holds if restricting to unital C^* -algebras. In this chapter and the next, we prove that sequential amenability and amenability coincide for unital C^* -algebras.

4.1 Defect in approximate amenability of Certain C^* -algebras

Recall that pseudo-amenability and approximate amenability coincide for unital C^* -algebras. Consequently, whenever a unital C^* -algebra is approximately amenable, we may work with approximate diagonals.

Remark 4.1.1. Recall that the spectrum of a C^* -algebra A , denoted by $\hat{A}$, is the set of all unitary equivalence classes of non-zero irreducible representations of A . This set contains all GNS representations associated with pure states of A . We also denote by $\text{Prim}(A)$ the set of all primitive ideals of A , equipped with the hull-kernel topology (see [17, Section VII.3] and [56, Chapter 5]). There is a natural (though not necessarily injective) map from $\hat{A}$ to $\text{Prim}(A)$, which sends each irreducible representation π to its kernel, $\ker(\pi)$. Thus, one can pull back the hull-kernel topology of $\text{Prim}(A)$ onto $\hat{A}$. This mapping is bijective for GCR C^* -algebras (see [56, Theorem 5.6.4] and [56, Theorem 5.4.9]), i.e., for those C^* -algebras such that for each irreducible representation π , the image $\pi(A)$ contains the ideal of compact operators. For a simple C^* -algebra A , $\text{Prim}(A)$ contains only two points. Therefore, $\hat{A}$ can be much bigger than $\text{Prim}(A)$ in general. An example of such an algebra is the reduced group C^* -algebra $C_r^*(\mathbb{F}_2)$ of the free group $\mathbb{F}_2$ on two generators (see [17, Theorem VII.7.5]).

Recall that there are two Arens products on the double dual (or bidual) of any Banach algebra (see Definition 2.1.3). However, these two products are the same for C^* -algebras and operator algebras (see Remark 2.1.4). We always assume that the bidual is equipped with the Arens product.

Proposition 4.1.2. *Suppose that A is C^* -algebra and one of the following conditions holds.*

- *A is separable and the spectrum $\hat{A}$ of A is discrete;*

- A is a closed two sided ideal in its bidual A^{**} under its Arens product.

Then, A is amenable.

Proof. If the first condition holds, by [24, Items 4.7.3 and 10.10.6], one can write A as the c_0 -direct sum of the algebras of compact operators on separable Hilbert spaces, i.e., $A = c_0 - \bigoplus_i \mathcal{K}(\mathcal{H}_i)$. If the second condition holds, by [81, p. 157, Exercises 3.a-c and 4.e], we have $A = c_0 - \bigoplus_i \mathcal{K}(\mathcal{H}_i)$. It is well known that the amenability constant of each $\mathcal{K}(\mathcal{H}_i)$ is one [39, Theorem 3.1]. By [76, Proposition 2.3.19], A must be amenable in either cases. $\square$

Example 4.1.3. Suppose that G is a second countable locally compact group. The spectrum of the full group C*-algebra $C^*(G)$ is denoted by $\hat{G}$, and is called the dual of G . Also, the spectrum of the reduced group C*-algebra $C_r^*(G)$ is denoted by $\hat{G}_r$ and is called the reduced dual of G . Now if $\hat{G}$ (resp. $\hat{G}_r$) is discrete, then by the above proposition $C^*(G)$ (resp. $C_r^*(G)$) must be amenable. Since for $G = \mathbb{F}_2$, the free group on two generators, neither the full nor the reduced group C*-algebra is amenable, it follows that neither of the duals is discrete.

By a result of Kadison [47, Theorem 4], if a C*-algebra A is represented in some $B(H)$, then every derivation $D : A \longrightarrow A$ is of the form

$$D(a) = aT - Ta, \quad \text{for some } T \in B(H). \quad (4.1)$$

He calls such derivations *spatial*. If, in addition, A is a von Neumann algebra, by a result of Sakai [78], D is inner, i.e., $T \in A$. Suppose that A is merely a C*-algebra and a derivation $D : A \longrightarrow A$ is given. Can we still conclude that D is implemented by an element $T \in A$ in Equation (4.1)? The answer is negative! The next example confirms this significant difference between a von Neumann algebra and a C*-algebra.

Proposition 4.1.4. *There exists an amenable C*-algebra A and a derivation $D : A \longrightarrow A$ which is not inner by elements of A .*

Proof. Let H^2 be the classical Hardy space, $A = K(H^2)$ be the ideal of compact operators on it, and $\mathcal{T}(H^2)$ be the Cuntz-Toeplitz C*-algebra generated by the forward shift s . It is well-known that $\mathcal{T}(H^2)$ contains $K(H^2)$ as an ideal, and the commutator ideal of $\mathcal{T}(H^2)$ is $K(H^2)$ [56, Theorem 3.5.10]. So, $[a, s] = as - sa \in K(H^2)$ for any $a \in \mathcal{T}(H^2)$. Now, consider the following derivation on $A = K(H^2)$:

$$\begin{aligned} D : K(H^2) &\longrightarrow K(H^2) \\ D(a) &= as - sa. \end{aligned}$$

We know that $K(H^2)$ is amenable; hence, it is approximately amenable. So, by [34, Theorem 2.1] any derivation with domain $K(H^2)$ into any ordinary module (not necessarily a dual module) must be approximately inner. Thus, there is a net of compact operators (K_i) in $K(H^2)$ such that $as - sa = D(a) = \lim_i aK_i - K_ia$. So, D is approximately inner. Note that $s^*s = I$, so $s \notin K(H^2)$. We claim that D cannot be an inner derivation implemented by any member of $K(H^2)$, i.e., D is not inner. For if it were inner, there would be a $k \in K(H^2)$ such that $D(a) = ak - ka$, then $as - sa = ak - ka$ for all $a \in K(H^2)$. Hence, $(s - k)a = a(s - k)$ for all $a \in K(H^2)$, so $s - k$ must be in the commutant $K(H^2)'$ of $K(H^2)$. On the other hand, it is well known that $K(H^2)' = \mathbb{C}I$, see for example [56, Theorem 4.1.12]. So, there must be a $\lambda \in \mathbb{C}$ such that $s = k + \lambda I$.

It is known that s is a Fredholm operator with index equal to -1 , and the index of λI is zero. However, index is invariant under a compact perturbation [56, Theorem 1.4.18]. So, we would have $-1 = \text{ind}(s) = \text{ind}(k + \lambda I) = \text{ind}(\lambda I) = 0$, a contradiction. Consequently, the derivation D cannot be inner. $\square$

Remark 4.1.5. It was brought to our attention by Prof. Nico Spronk that the above proof can be made more general. Indeed, one may choose any $s \in \mathcal{B}(\mathcal{H}) \setminus (\mathcal{K}(\mathcal{H}) + \mathbb{C}I)$. Since $\mathcal{K}(\mathcal{H})$ is strictly weak*-dense in $\mathcal{B}(\mathcal{H})$, there is a net (k_i) in $\mathcal{K}(\mathcal{H})$ converging to s in weak* topology. Hence,

$$ak_i - k_ia \xrightarrow{w^*} as - sa = D(a), \quad (a \in \mathcal{K}(\mathcal{H})).$$

So, by [34, Theorem 2.1], D is approximately inner. If D were inner, implemented by $k \in \mathcal{K}(\mathcal{H})$, then

$$(s - k)a = a(s - k) \quad (a \in \mathcal{K}(\mathcal{H})).$$

Since $\mathcal{K}(\mathcal{H})' = \mathbb{C}I$, we would have $s - k \in \mathbb{C}I$, contradicting $s \notin \mathcal{K}(\mathcal{H}) + \mathbb{C}I$.

Remark 4.1.6. For an infinite-dimensional Hilbert space H , one can infer from Ozawa's argument in [58] that $\mathcal{B}(H)$ is not approximately amenable. In the sequel, we present an alternative proof of this in Theorem 4.1.8. We should also mention that there is a related classical obstruction at the level of Johnson amenability for $\mathcal{B}(H)$. Wassermann [89, Corollary 1.9] showed that a von Neumann algebra which is nuclear as a C*-algebra must be subhomogeneous. A von Neumann algebra $\mathcal{M}$ is called **subhomogeneous** if there exists an $N \in \mathbb{N}$ such that for every irreducible *-representation $\pi : \mathcal{M} \rightarrow B(H_\pi)$ one has $\dim(H_\pi) \leq N$. Since nuclearity and Johnson amenability coincide for C*-algebras, one can deduce that $\mathcal{B}(\mathcal{H})$ fails to be amenable for an infinite-dimensional Hilbert space $\mathcal{H}$. However, Theorem 4.1.8 gives the stronger conclusion that $\mathcal{B}(\mathcal{H})$ is not even approximately amenable.

Denote by $\mathcal{P}$ the set of all prime numbers, and note that for any $p \in \mathcal{P}$, the set $\mathbb{Z}_p \cong \frac{\mathbb{Z}}{p\mathbb{Z}}$ is a finite field. By defining module operations modulo p , $\mathbb{Z}_p^3$ becomes a $\mathbb{Z}_p$ -module, which can also be regarded as a vector space over $\mathbb{Z}_p$. For any two non-zero $x, y \in \mathbb{Z}_p^3$, define $x \sim y$ if and only if there is a non-zero $\lambda \in \mathbb{Z}_p$ such that $x = \lambda y$. Thus, $\sim$ is an equivalence relationship on $\mathbb{Z}_p^3$ for which the modulus space is the *projective plane* over $\mathbb{Z}_p$, and we will denote it by $\mathbb{P}_p$. The group $SL(3, \mathbb{Z})$ of 3×3 matrices with entries in $\mathbb{Z}$ and determinant one can act on $\mathbb{Z}_p^3$ through matrix multiplication. This induces an action of $SL(3, \mathbb{Z})$ on $\mathbb{P}_p$, and hence, a unitary representation π_p of $SL(3, \mathbb{Z})$ on $\ell^2(\mathbb{P}_p)$. Let $x_1, \dots, x_m$ be the algebraic generators of $SL(3, \mathbb{Z})$, and let $(e_\times)_{\times \in \mathbb{P}_p}$ be the canonical basis of $\ell^2(\mathbb{P}_p)$. In addition, we define $\mathbb{P}$ to be the disjoint union of all $\mathbb{P}_p$ as p runs through prime numbers. The following lemma can be found in [76, Lemma 3.4.5].

Lemma 4.1.7. *It is impossible to find for any $\epsilon > 0$, a number $r \in \mathbb{N}$ such that for each $p \in \mathcal{P}$, there are $\xi_{1,p}, \eta_{1,p}, \dots, \xi_{r,p}, \eta_{r,p} \in \ell^2(\mathbb{P}_p)$ such that $\sum_{k=1}^r \xi_{k,p} \otimes \eta_{k,p} \neq 0$ and for any $1 \leq j \leq m$*

$$\left\| \sum_{k=1}^r [\xi_{k,p} \otimes \eta_{k,p} - (\pi_p(x_j) \otimes \pi_p(x_j))(\xi_{k,p} \otimes \eta_{k,p})] \right\| \leq \epsilon \left\| \sum_{k=1}^r \xi_{k,p} \otimes \eta_{k,p} \right\|,$$

where the norms are the projective tensor norm of $\ell^2(\mathbb{P}_p) \otimes \ell^2(\mathbb{P}_p)$.

Theorem 4.1.8. *The C*-algebra $\mathcal{B}(\mathcal{H})$ is not approximately amenable for any infinite dimensional separable Hilbert space $\mathcal{H}$.*

Proof. We follow the argument of [76, Theorem 3.4.13]. Suppose to the contrary that $\mathcal{B}(\mathcal{H})$ is approximately amenable, and Let I denote the identity operator on $\mathcal{H}$. Since $\mathcal{B}(\mathcal{H})$ is unital, it is pseudo-amenable by [33]. Consequently, there exists an approximate diagonal $(d_\alpha) \subseteq \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ such that $\pi(d_\alpha) \rightarrow I$. Since $\|\pi(d_\alpha) - I\| \rightarrow 0$, we can replace d_α by:

$$D_\alpha = d_\alpha + (I - \pi(d_\alpha)) \otimes I.$$

Then, $(D_\alpha) \subseteq \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ is an approximate diagonal with $\pi(D_\alpha) = I$. In addition, each D_α is a finite sum of simple tensors. Thus, without loss of generality, we may assume that d_α is of the form:

$$d_\alpha = \sum_{k=1}^r a_k^\alpha \otimes b_k^\alpha,$$

and $I = \sum_{k=1}^r a_k^\alpha b_k^\alpha$. By fixing a particular α , we can drop the superscript α and write:

$$I = \sum_{k=1}^r a_k b_k, \quad \text{and} \quad d_\alpha = \sum_{k=1}^r a_k \otimes b_k.$$

Without loss of generality we can also suppose that $\mathcal{H}$ is $\ell^2(\mathbb{P}) = \ell^2 - \bigoplus_{p \in \mathcal{P}} \ell^2(\mathbb{P}_p)$. Thus, $\ell^\infty - \bigoplus_{p \in \mathcal{P}} \mathcal{B}(\ell^2(\mathbb{P}_p))$ can be viewed as the sub-algebra of block diagonal matrices in $\mathcal{B}(\ell^2(\mathbb{P}))$. Let $x_1, \dots, x_j$ be the generators of $SL(3, \mathbb{Z})$, and consider the operators $\pi_p(x_1), \dots, \pi_p(x_m)$ on $\ell^2(\mathbb{P}_p)$, so $(\pi_p(x_j))_{p \in \mathcal{P}}$ can be regarded as an element of $\mathcal{B}(\ell^2(\mathbb{P}))$ for each j . Now, let $\epsilon > 0$, $I_{\ell^2(\mathbb{P})} = \sum_{k=1}^r a_k b_k$, and $T = \sum_{k=1}^r a_k \otimes b_k = d_\alpha \in \mathcal{B}(\ell^2(\mathbb{P})) \otimes^\gamma \mathcal{B}(\ell^2(\mathbb{P}))$. By the definition of the approximate diagonal, for all $1 \leq j \leq m$ we may assume that

$$\|(\pi_p(x_j))_{p \in \mathcal{P}} T - T (\pi_p(x_j))_{p \in \mathcal{P}}\| < \frac{\epsilon}{m+1}.$$

For each $p \in \mathcal{P}$, let $P_p : \ell^2(\mathbb{P}) \rightarrow \ell^2(\mathbb{P}_p)$ be the canonical projection, and define

$$\begin{aligned} T_p : \mathbb{P} &\longrightarrow \ell^2(\mathbb{P}_p) \otimes^\gamma \ell^2(\mathbb{P}_p) \\ T_p(\times) &:= \sum_{k=1}^r P_p a_k(e_\times) \otimes P_p^* b_k^*(e_\times). \end{aligned}$$

As in the proof of [76, Theorem 3.4.13], for all $1 \leq j \leq m$ we obtain that

$$\left\| \sum_{k=1}^r T_p(\times) - (\pi_p(x_j) \otimes \pi_p(x_j)) (T_p(\times)) \right\| \leq \epsilon \|T_p(\times)\|.$$

Now, plug in $\xi_{k,p} := P_p a_k(e_\times)$ and $\eta_{k,p} := P_p^* b_k^*(e_\times)$ in the above inequality. Then, for $\epsilon > 0$, we have got a number $r \in \mathbb{N}$ such that for each $p \in \mathcal{P}$, there are $\xi_{1,p}, \eta_{1,p}, \dots, \xi_{r,p}, \eta_{r,p} \in \ell^2(\mathbb{P}_p)$ such that $\sum_{k=1}^r \xi_{k,p} \otimes \eta_{k,p} \neq 0$ and for any $1 \leq j \leq m$,

$$\left\| \sum_{k=1}^r [\xi_{k,p} \otimes \eta_{k,p} - (\pi_p(x_j) \otimes \pi_p(x_j)) (\xi_{k,p} \otimes \eta_{k,p})] \right\| \leq \epsilon \left\| \sum_{k=1}^r \xi_{k,p} \otimes \eta_{k,p} \right\|.$$

However, Lemma 4.1.7 assures that such a result is *impossible*. Thus, $\mathcal{B}(\mathcal{H})$ cannot be pseudo-amenable, and consequently it cannot be approximately amenable. $\square$

4.2 Connes's approximate amenability of von Neumann algebras

The following definition is motivated by [29]; however, in contrast to [29], we consider weak* convergence instead of norm convergence.

Definition 4.2.1. A von Neumann algebra $\mathcal{M}$ is called *approximately Connes' amenable* if for any normal dual Banach $\mathcal{M}$ -bimodule X , every normal derivation $D : \mathcal{M} \rightarrow X$ is weak*-approximately inner, i.e., $D(a) = w^* - \lim_{\alpha} a \cdot x_{\alpha} - x_{\alpha} \cdot a$. We say that $\mathcal{M}$ is *sequentially Connes' amenable* if the implementing net is a sequence, i.e., $D(a) = w^* - \lim_{k \rightarrow \infty} a \cdot x_k - x_k \cdot a$.

To avoid possible confusion, we will refer to the classical (approximately) amenable Banach algebras as (approximately) Johnson amenable Banach algebras.

Remark 4.2.2. Obviously, Connes amenability implies approximate Connes amenability; however, the converse is false for dual Banach algebras as the following example shows. The natural numbers $\mathbb{N}$ is a commutative unital semigroup under the joint maximum $m \vee n = \max\{m, n\}$. We denote this semigroup by $\mathbb{N}_{\vee}$. By [29, Example 2.1], the dual Banach algebra $\ell^1(\mathbb{N}_{\vee}) = c_0(\mathbb{N}_{\vee})^*$ is approximately Connes amenable but not Connes amenable.

We also note that [29, Theorem 4.2] continues to hold when the relevant notions are interpreted in the sense of Definition 4.2.1. Moreover, by using the structural properties of von Neumann algebras, the corresponding weak* sequential version of [29, Theorem 4.2] remains valid. Although the proofs require some modifications, we omit the details in order to keep the focus on the main results of this work.

For von Neumann algebras, a natural question is under what conditions approximate Connes amenability implies Connes amenability. In general for C*-algebras, when approximate Johnson's amenability implies Johnson's amenability? In the remainder of this chapter, and throughout the next chapter, we investigate these questions.

Proposition 4.2.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neuman algebras, and let $\phi : \mathcal{M} \rightarrow \mathcal{N}$ be an onto normal *-homomorphism. If $\mathcal{M}$ is approximately Connes amenable, then so is $\mathcal{N}$.*

Proof. Let X be a normal dual Banach $\mathcal{N}$ bi-module, and $\delta : \mathcal{N} \rightarrow X$ be a normal derivation. We can make X into a normal dual $\mathcal{M}$ -bi-module by defining the module actions as $m \cdot x := \phi(m) \cdot x$ and $x \cdot m := x \cdot \phi(m)$. Since ϕ is a homomorphism and δ is a derivation, $D = \delta \circ \phi : \mathcal{M} \rightarrow X$ is also a derivation. By weak*-continuity assumption on ϕ , we see that D is a normal derivation on $\mathcal{M}$. So, it is approximately inner by our assumption; i.e., we can find a net (x_{α}) in X such that $D(a) = w^* - \lim_{\alpha} a \cdot x_{\alpha} - x_{\alpha} \cdot a$. By our definition of the action of $\mathcal{M}$ on X , this means $\delta(\phi(a)) = w^* - \lim_{\alpha} \phi(a) \cdot x_{\alpha} - x_{\alpha} \cdot \phi(a)$. Since ϕ is onto, we deduce that $\delta(n) = w^* - \lim_{\alpha} n \cdot x_{\alpha} - x_{\alpha} \cdot n$ for any $n \in \mathcal{N}$. $\square$

Recall that every type I von Neumann algebra is a direct sum of von Neumann algebras of the form $A_{\alpha} \overline{\otimes} B(H_{\alpha})$ with A_{α} being an abelian von Neumann algebra.

Definition 4.2.4. A type I von Neumann algebra $\mathcal{M}$ is said to be of:

1. **type I_f** if it can be written as a finite sum $\mathcal{M} \cong \ell^{\infty} - \bigoplus_{\alpha=1}^n A_{\alpha} \overline{\otimes} B(H_{\alpha})$,

2. **type I_{uf}** if $\mathcal{M} \cong \ell^\infty\text{-}\bigoplus_{i \in I} A_i \bar{\otimes} \mathbb{M}_{n_i}$, where I is an arbitrary index set, each A_i is an abelian von Neumann algebra, $\mathbb{M}_{n_i}$ is the algebra of $n_i \times n_i$ complex matrices, and $N = \sup\{n_i : i \in I\} < \infty$.

Theorem 4.2.5. *Let $\mathcal{M}$ be a type I_f von Neumann algebra. If $\mathcal{M}$ is approximately amenable, then it is amenable. In particular, $\mathcal{M}$ is of type I_{uf}.*

Proof. Consider the coordinate homomorphism $\phi : \mathcal{M} \longrightarrow A_\alpha \bar{\otimes} B(H_\alpha)$, which is bounded and onto. Hence, by [33, Proposition 2.2] $A_\alpha \bar{\otimes} B(H_\alpha)$ is approximately amenable. Since A_α is a commutative von Neumann algebra, we can choose a non-zero multiplicative linear functional τ from its Gelfand spectrum, see [31, Chapter 1]. Now, construct the following slicing map which is a homomorphism :

$$\begin{aligned} \psi = \tau \otimes id : A_\alpha \bar{\otimes} B(H_\alpha) &\longrightarrow B(H_\alpha) \\ a \otimes T &\longrightarrow \tau(a)T, \end{aligned}$$

and then expand by linearity. By construction, ψ is an algebra homomorphism. Since

$$\|\tau(a)T\| = |\tau(a)|\|T\| \leq \|a\|\|T\| = \|a \otimes T\|,$$

we see that ψ is bounded. Obviously, ψ is onto. Hence, by [33, Proposition 2.2], $B(H_\alpha)$ is approximately amenable. However, by Theorem 4.1.8, H_α cannot be infinite dimensional. Hence, $B(H_\alpha) = \mathbb{M}_{n,\alpha}$ is a finite dimensional von Neumann algebra, so it must be amenable (see [76, Example 2.2.4, or Theorem 2.3.25] for example). Let $A_{\alpha,*}$ denote the predual of the von Neumann algebra A_α , and obviously the predual of $\mathbb{M}_{n,\alpha}$ is itself. Now, By Effross-Ruan von Neumann tensor product formula [27, Theorem 3.2], we have

$$A_\alpha \bar{\otimes} \mathbb{M}_{n,\alpha} \cong (A_{\alpha,*} \hat{\otimes} \mathbb{M}_{n,\alpha})^*.$$

However, by standard operator theory discussions (see [4, Section 1.5]), we have

$$(A_{\alpha,*} \hat{\otimes} \mathbb{M}_{n,\alpha})^* \cong B(A_{\alpha,*}, \mathbb{M}_{n,\alpha}) = \mathbb{M}_{n,\alpha}(B(A_{\alpha,*}, \mathbb{C})) = \mathbb{M}_{n,\alpha}(A_\alpha).$$

Thus, $A_\alpha \bar{\otimes} \mathbb{M}_{n,\alpha} \cong \mathbb{M}_{n,\alpha}(A_\alpha)$, so $A_\alpha \bar{\otimes} B(H_\alpha) = \mathbb{M}_{n,\alpha}(A_\alpha)$ is amenable by [76, Theorem 7.1.9], and as a C*-algebra the amenability constant of $A_\alpha \bar{\otimes} B(H_\alpha)$ is equal to 1 (see [33, Section 6]). Consequently, the finite direct sum of $A_\alpha \bar{\otimes} B(H_\alpha)$, which is $\mathcal{M}$, must be amenable by [76, Proposition 2.3.19]. $\square$

Remark 4.2.6. The proof of Theorem 4.1.8 demonstrates that the finite sum condition in Theorem 4.2.5 cannot be weakened. For example, $\mathcal{M} \cong \ell^\infty\text{-}\bigoplus_{i=1}^\infty A_i \bar{\otimes} \mathbb{M}_{n_i}$ is not amenable; where each A_i is a non-zero abelian von Neumann algebra, $\mathbb{M}_{n_i}$ is the algebra of $n_i \times n_i$ complex matrices, and $\sup\{n_i : i \in \mathbb{N}\} = \infty$. We later observed that the type I_{uf} von Neumann algebras in Definition 4.2.4 are precisely the subhomogeneous von Neumann algebras. Indeed, for $I_n = \{i \in I : n_i = n\}$ and $B_n = \ell^\infty\text{-}\bigoplus_{i \in I_n} A_i$, we see that each B_n is again an abelian von Neumann algebra, and

$$\mathcal{M} \cong \bigoplus_{n=1}^N B_n \bar{\otimes} \mathbb{M}_n.$$

Thus, the uniformly bounded matrix-size condition in Definition 4.2.4 is exactly the usual subhomogeneity condition discussed in Remark 4.1.6, and subhomogeneous von Neumann algebras are amenable by [89, Corollary 1.9].

The following proposition is proved in [44, Corollary 6.5]; however, we can give a simpler argument based on [76, Section 7.2].

Proposition 4.2.7. *Let $\mathcal{M}$ be a type I von Neumann algebra. Then, $\mathcal{M}$ is Connes amenable.*

Proof. By [81, Theorem V.1.27], every type I von Neumann algebra is a direct sum of von Neumann algebras of the form $A_\alpha \overline{\otimes} B(H_\alpha)$ with A_α being an abelian von Neumann algebra. By [76, Example 2.3.4], the abelian von Neumann algebra A_α is Connes amenable. By Connes' theorem [12], the full operator algebra $B(H_\alpha) = K(H_\alpha)^{**}$ is Connes amenable. By [76, P. 297], the tensor product $A_\alpha \overline{\otimes} B(H_\alpha)$ is Connes amenable. So, $A_\alpha \overline{\otimes} B(H_\alpha)$ is injective by [76, Corollary 7.2.9]. The direct sum of injective von Neumann algebras is still injective [76, P. 297]. Thus, $\mathcal{M}$ is injective. Hence, by Johnson- Kadison- Ringrose Theorem [44, Corollary 6.4], $\mathcal{M}$ must be Connes amenable. $\square$

Let A be a C^* -algebra, and let $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ denote its universal representation. Then, $\mathcal{M} = \pi(A)''$ is the *universal enveloping von Neumann algebra* of A . Our goal is to relate this object to A^{**} . Recall from Definition 2.1.3 and Remark 2.1.4 that A^{**} carries a unique Arens product. The next proposition is a well-known result. A version of it, stated in older terminology, can be found in [11, Theorem 7.1]. We include a proof in order to make explicit the realization of A^{**} as the enveloping von Neumann algebra, which will be used throughout the sequel.

Proposition 4.2.8. *For any C^* -algebra A , the double dual A^{**} under the Arens product has the structure of a von Neumann algebra, which is isomorphic to the enveloping von Neumann algebra of A .*

Proof. Recall that by [56, Theorem 4.2.4], the weak* and weak operator topology coincide on bounded balls of $\mathcal{B}(\mathcal{H})$. Hence, by Krein-Smulian theorem [13, Theorem V.12.1], every von Neumann algebra is closed in its weak*-topology. By Sherman-Takeda theorem [81, Theorem III.2.4], A^{**} and $\mathcal{M} = \pi(A)''$ are isometrically isomorphic as Banach spaces, and we can use the transpose map of the universal representation of A to set up a isometric linear isomorphism $\tilde{\pi}$ from A^{**} onto $\mathcal{M}$. If we denote the natural embedding of A into A^{**} by j , then $\pi(a) = \tilde{\pi} \circ j(a)$. The mapping $\tilde{\pi}$ is also a homeomorphism with respect to the weak* topology of A^{**} induced by A^* and the weak* topology of $\mathcal{M}$ inherited from $\mathcal{B}(\mathcal{H})$. The Arens product is unique and separately weak* continuous on A^{**} [4, Remark 2.5.3]. By [4, Corollaries 2.5.6 and 2.7.14], we see that $\tilde{\pi}$ is indeed an algebra homomorphism which is also weak*-weak* continuous. Thus, $A^{**} = \overline{A}^{wk*} \cong \mathcal{M}$ is a von Neumann algebra. $\square$

By the preceding proposition, we can identify A^{**} with the universal enveloping von Neumann algebra of the C^* -algebra A . For the following notions, we refer to [81] and [82].

Definition 4.2.9. A C^* -algebra A is called of type t if its universal enveloping von Neumann algebra A^{**} is of type t , where $t \in \{I, II_1, II_\infty, III\}$.

It is well-known that type I C^* -algebras are nuclear and hence amenable. Here, we can prove this with the aid of Proposition 4.2.7.

Corollary 4.2.10. *Let A be a type I C^* -algebra. Then, A is amenable. In particular, for type I C^* -algebras, amenability and approximate amenability are the same.*

Proof. By assumption, the enveloping von Neumann algebra $\mathcal{M}$ of A is a type I von Neumann algebra. By Corollary 4.2.7, $\mathcal{M}$ is Connes amenable. So, by Connes' Theorem [12, Theorem 1], $\mathcal{M}$ is injective. By [10, Theorem 3], A must be amenable. $\square$

Corollary 4.2.11. *Let p and q be two projections on a Hilbert space H , and A be the C^* -algebra generated by p and q . Then, A is amenable.*

Proof. Let $\mathcal{M}$ be the enveloping von Neumann algebra of A . By [81, Theorem V.1.41], $\mathcal{M}$ must be of type I. From Corollary 4.2.10, A is amenable. $\square$

Remark 4.2.12. We thank Prof. Nico Spronk for suggesting the following observation which provides an alternative proof for Corollary 4.2.11. Let

$$D = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle u, v : u^2 = v^2 = 1 \rangle$$

be the infinite discrete dihedral group. Let δ_s denote the point mass at $s \in D$, and define

$$p = \frac{\delta_1 - \delta_u}{2}, \quad q = \frac{\delta_1 - \delta_v}{2}$$

in $\ell^1(D)$. Since $\delta_u^2 = \delta_v^2 = \delta_1$, both p and q are projections. Moreover, $\delta_u = \delta_1 - 2p$ and $\delta_v = \delta_1 - 2q$. As u and v generate D , it follows that $\ell^1(D)$ is the unital Banach algebra generated by δ_1, p , and q . Consider the character

$$\varepsilon : \ell^1(D) \longrightarrow \mathbb{C}, \quad \varepsilon(f) = \sum_{s \in D} f(s),$$

and its kernel $\ell_0^1(D) = \text{Ker}(\varepsilon)$ which is a closed ideal of $\ell^1(D)$. Since $\varepsilon(p) = \varepsilon(q) = 0$, the closed non-unital algebra generated by p and q is precisely $\ell_0^1(D)$. The group D is amenable, so are $\ell^1(D)$ and its closed ideal $\ell_0^1(D)$. If $\mathcal{A}$ is a C^* -algebra generated by projections P and Q , one can construct a continuous homomorphism

$$\ell_0^1(D) \longrightarrow \mathcal{A} : \quad p \longmapsto P, \quad q \longmapsto Q$$

with dense range. Thus, the amenability of the two-projection case is already visible at the level of the group algebra of the amenable group $\mathbb{Z}_2 * \mathbb{Z}_2$. However, the situation changes when one passes from two to three or more projections. In fact, for higher free multiplications like $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, the subgroup consisting of words of even length has index 2 and is isomorphic to the free group of the free group $\mathbb{F}_2$. Consequently, the C^* -algebra generated by three or more projections is non-amenable.

Remark 4.2.13. Let A be a unital C^* -algebra and B be a C^* -sub-algebra of A containing the unit element of A . By [56, Corollary 3.3.4], any positive linear functional on a unital C^* -algebra attains its norm at the identity element. Thus, we have the following observation:

- Suppose that $\mu : A \longrightarrow \mathbb{C}$ is a positive linear functional on A , and $\mu' : B \longrightarrow \mathbb{C}$ is the restriction of μ to B . Then, $\|\mu\| = \|\mu'\|$.

Remark 4.2.14. By [81, Theorem III.2.14], the dual space $\mathcal{M}^*$ admits an ℓ^1 -direct sum decomposition

$$\mathcal{M}^* = \mathcal{M}_* \oplus_1 \mathcal{M}_*^\perp.$$

Here, $\mathcal{M}_*^\perp$ denotes the *topological complement* of $\mathcal{M}_*$ in $\mathcal{M}^*$ (see [13, p. 94]). The elements of $\mathcal{M}_*$ are the normal functionals, while those in $\mathcal{M}_*^\perp$ are *singular functionals*. Consequently, every functional $\omega \in \mathcal{M}^*$ can be uniquely written as

$$\omega = \omega_n + \omega_s, \quad \omega_n \in \mathcal{M}_*, \quad \omega_s \in \mathcal{M}_*^\perp,$$

with the norm identity

$$\|\omega\| = \|\omega_n\| + \|\omega_s\|.$$

The following facts are important to us:

- (i) A functional ω on $\mathcal{M}$ is called Hermitian if $\omega(a^*) = \overline{\omega(a)}$ for all $a \in \mathcal{M}$. By the Jordan decomposition theorem [63, Theorem 3.2.5], there exist unique positive functionals ω^+ and ω^- on $\mathcal{M}$ such that

$$\omega = \omega^+ - \omega^-, \quad \|\omega\| = \|\omega^+\| + \|\omega^-\|.$$

If, in addition, ω is normal, then [81, Theorem III.4.2] ensures that both ω^+ and ω^- are normal, and their support projections $s(\omega^+)$ and $s(\omega^-)$ are orthogonal.

- (ii) For positive linear functionals, singularity means that it does not majorize any positive normal functional, see [3, Definition III.2.11]. Another equivalent characterization is given in [81, Theorem III.3.8], which implies that a positive linear functional ω is singular if and only if any non-zero projection has a non-zero sub-projection at which ω vanishes.
- (iii) A functional is singular if and only if it is a linear combination of positive singular ones, see [48, Proposition 10.1.17].
- (iv) Both $\mathcal{M}_*$ and $\mathcal{M}_*^\perp$ are closed linear subspace of $\mathcal{M}^*$.
- (v) According to [81, Theorem III.5.1], $\mathcal{M}_*$ is sequentially complete in the weak* topology of $\mathcal{M}^*$. Similarly, $\mathcal{M}_*^\perp$ is also weak* sequentially complete, as established in [81, Proposition III.5.8].
- (vi) Let ν be a normal positive linear functional on a von Neumann algebra $\mathcal{M}$. Then, the set of ν -zero length vectors, $N_\nu = \{ a \in \mathcal{M} : \nu(a^*a) = 0 \}$, is a weak* closed left ideal. If ν is tracial, then N_ν is a two sided weak* closed ideal. By [81, Proposition II.3.12], there is a central projection $f \in \mathcal{M}$ such that $N_\nu = \mathcal{M}(1 - f)$. It follows from [81, Lemma III.3.6 and the proof of Theorem V.2.4] that the support of ν is $s(\nu) = f$, and that ν is faithful on $f\mathcal{M}f$. That is, the functional $\nu_f : f\mathcal{M}f \rightarrow \mathbb{C}$ defined by $\nu_f(fmf) = \nu(fmf) = \nu(m)$ is a normal faithful positive linear functional on $f\mathcal{M}f$. Clearly, ν_f is the restriction of ν to $f\mathcal{M}f$, i.e., $\nu_f = \nu|_{f\mathcal{M}f}$.

So far, by using the enveloping von Neumann algebra, we have shown that for type I C^* -algebras, amenability and approximate amenability coincide. We now investigate approximate amenability of other types of C^* -algebras. It turns out once again that this links to Connes' approximate amenability of the enveloping von Neumann algebra.

Theorem 4.2.15. *Let A be a C^* -algebra. If A is sequentially Johnson's amenable, then the enveloping von Neumann algebra A^{**} of A is sequentially Connes' amenable.*

Proof. Since A is a C^* -algebra, it is Arens regular by [4, Corollary 2.5.4]. Let π be the universal representation of A and $\tilde{\pi} : A^{**} \rightarrow \pi(A)''$ be the Sherman-Takeda isomorphism as in Proposition 4.2.8 and [81, Theorem III.2.4]. Let $\mathcal{M} = A^{**}$ and $\mathcal{M}_* = A^*$. Recall that $\pi = \tilde{\pi} \circ j$. Suppose that X is a normal dual Banach $\mathcal{M}$ bi-module, and $\delta : \mathcal{M} \rightarrow X$ be a normal derivation. The action of $\mathcal{M} = A^{**}$ on X induces the left and right actions of A on X given by $a \cdot x = j(a) \cdot x$ and $x \cdot a = x \cdot j(a)$. These actions make X into a dual Banach A bi-module. Since j is a homomorphism and δ is a derivation, $D = \delta \circ j : A \rightarrow X$ is also a derivation. Thus, it is sequentially inner by our assumption on A ; i.e., there is a sequence (x_k) in X such that for all $a \in A$

$$D(a) = \lim_{k \rightarrow \infty} a \cdot x_k - x_k \cdot a. \quad (4.2)$$

So, we have

$$D(a) = \lim_{k \rightarrow \infty} j(a) \cdot x_k - x_k \cdot j(a), \quad (4.3)$$

in the norm topology of X . For each $k \in \mathbb{N}$, the derivation $D_k(a) = a \cdot x_k - x_k \cdot a$ is the restriction to A of the derivation $\delta_k : A^{**} \rightarrow X$ defined by $\delta_k(m) = m \cdot x_k - x_k \cdot m$. By [4, Lemma A.2.2], D_k has a unique extension $\tilde{D}_k$ to A^{**} such that $\|\tilde{D}_k\| = \|D_k\|$. So, $\tilde{D}_k = \delta_k$ and $\|\delta_k\| = \|D_k\|$. Since D_k converges pointwise to D , by principle of uniform boundedness, the sequence (D_k) is uniformly bounded. Hence, the sequence (δ_k) is uniformly bounded. Now, we claim that for any $m \in A^{**}$, the sequence $(\delta_k(m))$ converges to $\delta(m)$ in the weak* topology. If this is not the case, then there is an $m \in A^{**}$ and a weak* continuous functional $f \in X_*$ such that the sequence $(f \circ \delta_k(m))$ does not converge to $f \circ \delta(m)$. However, each functional $f \circ \delta_k$ and the functional $f \circ \delta$ are normal on the von Neumann algebra A^{**} . In addition, the sequence $(f \circ \delta_k)$ is a uniformly bounded sequence of normal functionals; hence, it has a weak* cluster point ϕ by Banach-Alaoglu theorem. The set $C = \{f \circ \delta_k : k \in \mathbb{N}\}$ is a bounded subset of $\mathcal{M}_*$. By the Eberlein-Smulian theorem [13, Theorem V.13.1], the weak closure of C is weakly sequentially compact. However, the weak topology of C coincides with its weak* topology induced by $\mathcal{M}^*$. Thus, we can find a subsequence of $(f \circ \delta_k)$ converging to ϕ in the weak* topology. By [81, Theorem III.5.1] or Remark 4.2.14-v, the functional ϕ must be normal. Thus, ϕ and $f \circ \delta$ are in the pre-dual $\mathcal{M}_*$ of the von Neumann algebra $\mathcal{M}$. However, by our assumption $\phi(m) \neq f \circ \delta(m)$, so the normal functional ϕ and $f \circ \delta$ cannot be equal, i.e., $\phi \neq f \circ \delta$ as members of A^* . Consequently, there is an $a \in A$ such that $\phi(a) \neq f \circ \delta(a)$. However, for any $a \in A$, we have $f \circ \delta_k(a) = f \circ D_k(a)$ and $f \circ \delta(a) = f \circ D(a)$. Moreover, since $\delta_k(a) \xrightarrow{wk^*} \delta(a)$, we deduce that $f \circ \delta_k(a) \rightarrow f \circ \delta(a)$. Thus, $\phi(a) = f \circ \delta(a)$, but this contradicts $\phi(a) \neq f \circ \delta(a)$. Therefore, for any m , the sequence $\delta_k(m)$ converges to the functional $\delta(m)$ in the weak* topology. This proves that the normal derivation δ is wk* sequentially inner on $\mathcal{M}$, i.e., $\mathcal{M}$ is sequentially Connes amenable. $\square$

Remark 4.2.16. It is natural to ask whether the sequentiality assumption in Theorem 4.2.15 can be removed; namely, whether approximate Johnson amenability of a C^* -algebra A implies approximate Connes amenability of A^{**} . An affirmative answer would allow the arguments and results developed in the sequel to be modified so as to settle completely the question of Ghahramani et al. The proof above, however, uses sequentiality in an essential way and therefore does not immediately extend to arbitrary nets.

4.3 Connes sequential amenability of finite von Neumann algebras

In this section, we aim to prove the counter part of Proposition 4.2.7 for type II_1 von Neumann algebras. Then, by combining it with Theorem 4.2.15, we show that amenability and sequential amenability are the same for C^* -algebras whose enveloping von Neumann algebra is finite. To do so, we need to review Connes' original work in [12, Theorem 1]. The same result for other types of C^* -algebras will be obtained in the next chapter.

Proposition 4.3.1. *Let τ_1 and τ_2 be topologies on a set X such that $\tau_1 \subseteq \tau_2$. Then, for any subset $A \subseteq X$, we have $\overline{A}^{\tau_1} \supseteq \overline{A}^{\tau_2}$.*

Proof. Let $x \in \overline{A}^{\tau_2}$. By definition, for every $U \in \tau_2$ with $x \in U$, we have $U \cap A \neq \emptyset$. Let $V \in \tau_1$ and $x \in V$. Since $\tau_1 \subseteq \tau_2$, we have $V \in \tau_2$, and so $V \cap A \neq \emptyset$. Hence, $x \in \overline{A}^{\tau_1}$. $\square$

Proposition 4.3.2. *Let $\mathcal{H}$ be any Hilbert space. The ideal of trace class operators $\mathcal{B}^1(\mathcal{H})$ is dense in $\mathcal{B}(\mathcal{H})$ in both weak operator and weak* topologies.*

Proof. Note that $\mathcal{B}^1(\mathcal{H})$ contains the set of all finite rank operators $\mathcal{F}(\mathcal{H})$; see [56, Theorem 2.4.17]. Denote by $\mathcal{K}(\mathcal{H})$ the ideal of compact operators. By Proposition 4.3.1 and the fact that taking topological closure is monotone, we have

$$\mathcal{K}(\mathcal{H}) = \overline{\mathcal{F}(\mathcal{H})}^{\|\cdot\|} \subseteq \overline{\mathcal{F}(\mathcal{H})}^{\text{wot}} \subseteq \overline{\mathcal{B}^1(\mathcal{H})}^{\text{wot}}.$$

By taking another wot-closure, we see that

$$\overline{\mathcal{K}(\mathcal{H})}^{\text{wot}} \subseteq \overline{\mathcal{B}^1(\mathcal{H})}^{\text{wot}}. \quad (4.4)$$

On the other hand, $\mathcal{K}(\mathcal{H})' = \mathbb{C}I$, see [56, Remark 4.2.2]. So, by [56, Theorem 4.1.12] $\overline{\mathcal{K}(\mathcal{H})}^{\text{tot}} = \mathcal{B}(\mathcal{H})$. However, by [56, Theorem 4.2.7] $\overline{\mathcal{K}(\mathcal{H})}^{\text{tot}} = \overline{\mathcal{K}(\mathcal{H})}^{\text{wot}}$. Putting these together and using Equation (4.4), we deduce that

$$\mathcal{B}(\mathcal{H}) \subseteq \overline{\mathcal{B}^1(\mathcal{H})}^{\text{wot}}.$$

The reverse inclusion follows from the Goldstine theorem, or from [81, Theorem II.2.6-iv]. $\square$

Remark 4.3.3. Recall that the duality between $\mathcal{H}$ and $\mathcal{H}^*$ is conjugate linear. Thus, we can regard $\mathcal{H}^*$ as the same $\mathcal{H}$, but with the conjugate linear scalar product defined by $\lambda \cdot \xi := \overline{\lambda}\xi$. As in [12], we use the notation $\mathcal{H}^c$ instead of $\mathcal{H}^*$. Now, any operator $T \in \mathcal{B}(\mathcal{H})$ has a counterpart $T^c \in \mathcal{B}(\mathcal{H}^c)$ satisfying $T^c(\lambda \cdot \xi) = T(\overline{\lambda}\xi) = \overline{\lambda}T(\xi)$ and $\|T\| = \|T^c\|$.

Remark 4.3.4. Every von Neumann algebra $\mathcal{M}$ admits a (so-called) *standard representation*. By [82, Theorem VII.2.7], there exists a normal faithful semifinite weight φ on $\mathcal{M}$. By Section 3.2, $\Lambda_\varphi(\mathcal{N}_\varphi^* \cap \mathcal{N}_\varphi)$ is a pre-Hilbert space under the inner product $\langle x + N_\varphi, y + N_\varphi \rangle = \varphi(y^*x)$. Its completion, $\mathcal{H}_\varphi$, is a Hilbert space. As in [82, Proposition VII.1.4], there is a normal faithful non-degenerate $*$ -representation π_φ of $\mathcal{M}$ on $\mathcal{H}_\varphi$. The representation $(\pi_\varphi, \mathcal{H}_\varphi)$ is called the *standard representation* of $\mathcal{M}$ with respect to φ . When $\mathcal{M}$ has an everywhere defined normal faithful tracial state τ , this process recovers the classical cyclic GNS representation $(\pi_\tau, H_\tau, \xi_\tau)$. Here, H_τ is the completion of $\frac{\mathcal{M}}{N_\tau}$ under $\langle x + N_\tau, y + N_\tau \rangle = \tau(y^*x)$, and $\xi_\tau = 1 + N_\tau$. Note that $N_\tau = 0$ since τ

is faithful. So, H_τ can be considered as the completion of $\mathcal{M}$ under the norm $\|x\|_2 = \sqrt{\tau(x^*x)}$. Sometimes, we may use $L^2(\tau)$ instead of H_τ . Let's write $\|x\|_\infty$ for the operator norm of x . By the inequality $|\tau(a)| \leq \|a\|_\infty$, it should be clear that $\|x\|_2 \leq \|x\|_\infty$. Also, note that $\xi_\tau = 1$ is the cyclic vector for the GNS representation, see [56, Theorem 5.1.1]. By [82, Proposition VII.1.4] the GNS representation $(\pi_\tau, L^2(\tau), 1)$ is a faithful normal representation of $\mathcal{M}$. So, the representation of $\mathcal{M}$ inside $\mathcal{B}(L^2(\tau))$,

$$\begin{aligned} \pi_\tau : \mathcal{M} &\longrightarrow \mathcal{B}(L^2(\tau)) \\ x &\longrightarrow \pi_\tau(x), \quad \pi_\tau(x)(y + N_\tau) = xy + N_\tau \end{aligned}$$

is a von Neumann algebra isomorphism from $\mathcal{M}$ onto its range and $\tau(x) = \langle \pi_\tau(x)\xi_\tau, \xi_\tau \rangle$. Since $\{\pi_\tau(\mathcal{M})h : h \in L^2(\tau)\}$ is dense in $L^2(\tau)$, the action of $\pi_\tau(\mathcal{M})$ on $L^2(\mathcal{M})$ is non-degenerate, i.e., for any vector $h \in L^2(\tau)$ there is an $x \in \mathcal{M}$ such that $\pi_\tau(x)h \neq 0$. When $\mathcal{M}$ is in its standard representation, we can write elements of $\mathcal{M}$ acting on elements of $L^2(\tau)$ without mentioning π_τ , e.g., $x^*\xi_\tau = \pi_\tau(x^*)\xi_\tau$, or $\tau(x) = \langle x\xi_\tau, \xi_\tau \rangle$. Now, consider the conjugate linear isometry

$$\begin{aligned} S : \mathcal{M}\xi_\tau &\longrightarrow \mathcal{M}\xi_\tau \\ S(m\xi_\tau) &= m^*\xi_\tau, \end{aligned}$$

which is only densely defined on $L^2(\tau)$. By [82, Section VI.1], one can find a polar decomposition $S = J|S|$, where J is a conjugate linear isometry on $L^2(\tau)$. Now, by Tomita Takesaki theory (see [82, Theorem VI.1.19]), we have $J\mathcal{M}J = \mathcal{M}'$. In addition, by [3, Theorem III.4.3.2], one can see that $J(\xi_\tau) = \xi_\tau$.

Lemma 4.3.5. *Let $\mathcal{M}$ be a von Neumann algebra inside $\mathcal{B}(\mathcal{H})$ and $T \in \mathcal{M}$. Using the same notation as in Remark 4.3.3, we have that $T^c \in \mathcal{M}'$.*

Proof. By Tomita-Takesaki theory and also by [82, Chapters VI - VIII], for any von Neumann algebra $\mathcal{M}$ inside $\mathcal{B}(\mathcal{H})$, there is a conjugate linear isometric isomorphism $J : \mathcal{H} \longrightarrow \mathcal{H}$ such that $J = J^* = J^{-1}$ and $J\mathcal{M}J = \mathcal{M}'$. Evidently, $J : \mathcal{H} \longrightarrow \mathcal{H}^c$ is a linear isometric isomorphism. On the other hand, it is easy to check that $JTJ = T^c$ for any $T \in \mathcal{B}(\mathcal{H})$, see also [12, P. 252]. Thus, $T^c \in \mathcal{M}'$ for $T \in \mathcal{M}$. $\square$

We have observed the importance of traces in studying Connes' amenability for von Neumann algebras. We define the following notion.

Definition 4.3.6. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A positive linear functional $\theta : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ is called an *ultra-trace* for $\mathcal{M}$ if its restriction to $\mathcal{M}$ is nonzero and it satisfies the tracial property

$$\theta(xT) = \theta(Tx) \quad (x \in \mathcal{M}, T \in \mathcal{B}(\mathcal{H})).$$

Equivalently, we say that θ is an ultra-trace for $(\mathcal{M}, \mathcal{B}(\mathcal{H}))$.

Remark 4.3.7. The notion introduced in Definition 4.3.6 is another manifestation of the notion of an *amenable trace* in Brown–Ozawa [7, Definition 6.2.1]. More precisely, Brown and Ozawa call a trace τ , on a concretely represented unital C*-algebra $A \subseteq \mathcal{B}(\mathcal{H})$, *amenable* if there exists a state φ on $\mathcal{B}(\mathcal{H})$ such that $\varphi|_A = \tau$ and $\varphi(uTu^*) = \varphi(T)$ for all $T \in \mathcal{B}(\mathcal{H})$ and all unitary elements $u \in A$. Since the terminology *ultra-trace* is adapted to the von Neumann algebraic setting considered here, we shall continue to use it in the sequel.

Definition 4.3.8. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra with a faithful normal tracial state τ . Given an ultra-trace θ for $(\mathcal{M}, \mathcal{B}(\mathcal{H}))$, we say that θ *majorizes* τ if there exists $\lambda > 0$ such that

$$\tau(x) \leq \lambda \theta(x) \quad (x \in \mathcal{M}_+).$$

Remark 4.3.9. Suppose that θ is an ultra trace for $(\mathcal{M}, \mathcal{B}(\mathcal{H}))$ as in Definition 4.3.6.

(i) The functional θ need not be normal or faithful on $\mathcal{B}(\mathcal{H})$. However, its normality on $\mathcal{M}$ is essential in applications. If θ is not normal, we may replace it by its normal part, provided that this part is nonzero on $\mathcal{M}$. Hence, without loss of generality, we may assume that θ

- is a positive linear functional on $\mathcal{B}(\mathcal{H})$,
- satisfies the ultra trace property $\theta(xT) = \theta(Tx)$ for all $x \in \mathcal{M}$ and $T \in \mathcal{B}(\mathcal{H})$,
- is a normal state on $\mathcal{M}$.

(ii) In applications of Definition 4.3.8, we may always assume $\lambda = 1$ by replacing θ with $\frac{1}{\lambda}\theta$. Moreover, θ is faithful on $\mathcal{M}$: if $\theta(a^*a) = 0$ for some $a \in \mathcal{M}$, then $\tau(a^*a) = 0$, and since τ is faithful, it follows that $a = 0$. It may also occur that θ coincides with τ on $\mathcal{M}$, i.e., θ extends τ . In this case, θ is a faithful normal tracial state on $\mathcal{M}$.

The following facts about von Neumann algebras are well known. We include a proof here for completeness.

Proposition 4.3.10. *Let $\mathcal{M}$ be a von Neumann algebra.*

- (i) *If $\mathcal{M}$ admits a faithful tracial state τ , then it is finite.*
- (ii) *If $\mathcal{M}$ is finite, then every isometry is a unitary.*

Proof. (i) Let q be a sub-projection of 1 with $q \sim 1$. By definition, there is a partial isometry v such that $v^*v = 1$ and $vv^* = q$. Since τ is tracial, $\tau(q) = \tau(1)$. Hence, $\tau(1 - q) = 0$. As $1 - q \geq 0$ and τ is faithful, we obtain $1 - q = 0$, i.e. $q = 1$. Thus, 1 has no proper equivalent sub-projection. So, $\mathcal{M}$ is finite.

(ii) For an isometry v , we have $v^*v = 1$. Any isometry is trivially a partial isometry, so $p = vv^*$ is a projection by [56, Theorem 2.3.3]. Since v^* is a contraction, we see that $p \leq 1$ by [1, Proposition 1.12]. However, the identity is a finite projection, and by definition no proper sub-projection of a finite projection is equivalent to it; hence, $p = 1$, which proves that v is unitary. □

Remark 4.3.11. In [12, Theorem 1], Connes proved that every Johnson amenable C^* -algebra A is nuclear. To establish this, he showed that the enveloping von Neumann algebra $\mathcal{M} = A^{**}$ is Connes amenable and that every Connes amenable von Neumann algebra is injective. Connes' proof of the latter relies on the existence of a majorizing ultra-trace on a finite von Neumann algebra equipped with a faithful normal tracial state. Following [12], there is a normal faithful tracial weight τ on a semi-finite von Neumann algebra $\mathcal{M}$. By [3, Corollary IV.2.6.2] and [12], one can find a net of finite projections $\{e_\alpha\}$ which increases to identity in strong operator topology such that:

- For each projection $e \in \{e_\alpha\}$, the corner algebra $e\mathcal{M}e = \mathcal{M}_e$ is a finite von Neumann algebra with the normal faithful tracial state $\tau_e(exe) = \frac{\tau(exe)}{\tau(e)}$ on the von Neumann algebra $\mathcal{M}_e$. Fixing the projection e , we use the notation τ_1 for τ_e in the sequel.
- the union of all $\mathcal{M}_e$'s is weak* dense in $\mathcal{M}$.
- Each τ_1 is a non-zero normal faithful tracial state on $\mathcal{M}_e$. Connes [12] proved that if $\mathcal{M}$ is Connes amenable, then there is a *non-zero ultra-trace* ψ_1^+ for $\mathcal{M}_e$ majorizing τ_1 .

From the above properties, Connes showed that if $\mathcal{M}$ is Connes amenable, then it is injective. We will show that if a finite von Neumann algebra is sequentially Connes amenable, then it has a faithful ultra trace. We will further prove that any finite von Neumann algebra with a faithful ultra trace is injective.

The proof of the next theorem is different from Connes' argument in [12]. Here, our proof is direct and is based on the weak* decomposition of positive linear functionals developed in the Appendix A. The proof is inspired by a method used in [76, Lemma 7.2.6]. We note that the result itself can also be deduced from [7, Theorem 6.2.7]. However, we include the proof below because it is formulated in our terminology and fits naturally with the framework developed in this thesis.

Theorem 4.3.12. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state τ . Consider the standard representation $\pi_\tau : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H})$, where $\mathcal{H} = L^2(\tau)$ as in Remark 4.3.4. Suppose that θ is an ultra trace for $(\mathcal{M}, \mathcal{B}(\mathcal{H}))$.*

(i) *If θ majorizes τ , then $\mathcal{M}$ is injective and hence Connes amenable.*

(ii) *In particular, if the ultra-trace θ is normal and faithful on $\mathcal{M}$, then $\mathcal{M}$ is Connes amenable.*

Proof. (i) By Remark 4.3.9, θ is also faithful on $\mathcal{M}$; hence $N_\tau = N_\theta = \{0\}$. By Remark 4.2.13, θ is a bounded linear functional with the norm $\|\theta\| = \theta(1)$. Considering θ as a positive linear functional on $\mathcal{M}$, we can apply the weak* decomposition stated in Remark 4.2.14 to decompose θ against τ into normal and singular parts. Thus, $\theta = \theta_n + \theta_s$, where θ_n is a positive linear and normal functional, while θ_s a positive linear and singular functional. By [48, Corollary 10.1.16], for any $T \in \mathcal{B}(\mathcal{H})$, the functionals $T \cdot \theta_n$ and $T \cdot \theta_s$ are normal and singular respectively. Similarly, $\theta_n \cdot T$ and $\theta_s \cdot T$ are normal and singular functionals respectively. Since the decomposition into normal and singular parts is unique ([48, Theorem 10.1.165]) and $T \cdot \theta = \theta \cdot T$, we deduce that $T \cdot \theta_n = \theta_n \cdot T$ and $T \cdot \theta_s = \theta_s \cdot T$. In particular, the normal functional θ_n has the ultra trace property for $(\mathcal{M}, \mathcal{B}(\mathcal{H}))$. The assumption $\tau \leq \theta$ along with Theorem A.1.13-(iv) yields $\tau \leq \theta_n$, so the normal ultra trace θ_n majorizes τ on $\mathcal{M}$. Hence, θ_n is faithful. Clearly, we can normalize θ_n by the constant $\theta_n(1) = \|\theta_n\|$ to obtain a state on $\mathcal{M}$. Consequently, as we pointed out in Remark 4.3.9, we can proceed with the assumptions that θ is a normal ultra trace for $(\mathcal{M}, \mathcal{B}(\mathcal{H}))$ and is a faithful state on $\mathcal{M}$.

Now, we aim to prove that $\mathcal{M}$ is injective by constructing a conditional expectation $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}'$ directly. We first define $\mathcal{E}$ on positive elements of $\mathcal{B}(\mathcal{H})$, then we extend it by linearity. To do so, we will make use of the Radon-Nikodym derivatives in Theorem A.2.12. When $T \in \mathcal{B}(\mathcal{H})_+$, we claim that $T \cdot \theta$ is a positive functional on $\mathcal{M}$. Suppose that $m \in \mathcal{M}_+$,

then

$$\begin{aligned}
T \cdot \theta(m) &= \theta(mT) \\
&= \theta(aa^*bb^*), \quad m = aa^*, \quad T = bb^* \\
&= \theta(b^*aa^*b), \quad \theta \text{ is an ultra-trace} \\
&= \theta(b^*a(b^*a)^*) \geq 0, \quad \theta \text{ is a positive functional.}
\end{aligned}$$

So, $T \cdot \theta$ is a positive normal functional on $\mathcal{M}$. We show that $T \cdot \theta \leq \|T\|\theta$ as positive linear functionals on $\mathcal{M}$. Suppose that $m = aa^* \in \mathcal{M}$ and let $T = \sqrt{T}\sqrt{T} = (\sqrt{T})^*\sqrt{T}$. Then,

$$\begin{aligned}
T \cdot \theta(m) &= \theta(mT) \\
&= \theta(aa^*(\sqrt{T})^*\sqrt{T}) \\
&= \theta(a^*(\sqrt{T})^*\sqrt{T}a), \quad \theta \text{ is tracial} \\
&\leq \|(\sqrt{T})^*\sqrt{T}\| \theta(a^*a), \quad \text{by [56, Theorem 3.3.7]} \\
&= \|T\| \theta(aa^*), \quad \theta \text{ is tracial} \\
&= \|T\|\theta(m).
\end{aligned}$$

Thus,

$$T \cdot \theta \leq \|T\|\theta, \quad (4.5)$$

and hence,

$$N_\theta \subseteq N_{T \cdot \theta}. \quad (4.6)$$

Since θ is tracial on $\mathcal{M}$, it has the KMS property, see Definition A.2.1. By Inequality 4.5 and Theorem A.2.12-v, one can find a unique and positive bounded operator $D = D_\theta(T \cdot \theta) \in [\pi_\theta(\mathcal{M})]'$ such that

$$T \cdot \theta(a) = \langle aD(\xi_\theta), \xi_\theta \rangle_\theta, \quad \forall a \in \mathcal{M}. \quad (4.7)$$

$D_\theta(T \cdot \theta)$ is the Radon-Nikodym derivative of $T \cdot \theta$ with respect to θ . We remark that this derivative can also be computed by [48, Theorem 7.3.6]. Since θ is faithful, we obtain that $[\pi_\theta(\mathcal{M})]' = \mathcal{M}'$. Therefore, $D = D_\theta(T \cdot \theta) \in \mathcal{M}'$ is uniquely determined, and the mapping

$$\begin{aligned}
\mathcal{E} : \mathcal{B}(\mathcal{H})_+ &\longrightarrow \mathcal{M}'_+ \\
\mathcal{E}(T) &= D_\theta(T \cdot \theta)
\end{aligned}$$

is well-defined. By Theorem A.2.12-v, the mapping $\mathcal{E}$ is additive and positive homogeneous. Using the definition of the inner product for the standard GNS representation for all $S \in \mathcal{M}'_+$ and all $a, b \in \mathcal{M}$, we deduce that

$$\begin{aligned}
\langle Sa\xi_\theta, b\xi_\theta \rangle_\theta &= \langle aS\xi_\theta, b\xi_\theta \rangle_\theta \quad (S \in \mathcal{M}') \\
&= \langle aS + N_\theta, b + N_\theta \rangle_\theta \\
&= \theta(b^*aS) \\
&= S \cdot \theta(b^*a).
\end{aligned} \quad (4.8)$$

Now, consider the normal positive linear functional $S \cdot \theta$ and calculate its Radon-Nikodym derivative with respect to θ . By Equation (4.7) for $a, b \in \mathcal{M}$, we have

$$\begin{aligned}
S \cdot \theta(b^*a) &= \langle aD_\theta(S \cdot \theta)(\xi_\theta), b\xi_\theta \rangle_\theta \\
&= \langle D_\theta(S \cdot \theta)(a\xi_\theta), b\xi_\theta \rangle_\theta, \quad (D_\theta(S \cdot \theta) \in \mathcal{M}').
\end{aligned} \quad (4.9)$$

Comparing Equations 4.8 and 4.9 yields

$$\langle S(a\xi_\theta), b\xi_\theta \rangle_\theta = \langle D_\theta(S \cdot \theta)(a\xi_\theta), b\xi_\theta \rangle_\theta, \quad \forall a, b \in \mathcal{M}. \quad (4.10)$$

Recall that $\xi_\theta = 1 + N_\theta$ is a cyclic vector, $a\xi_\theta = a + N_\theta$, $b\xi_\theta = b + N_\theta$, and the collection of such vectors is dense in $\mathcal{H}$. Thus, $S = D_\theta(S \cdot \theta)$ by Equation (4.10). Hence, $\mathcal{E}$ is identity on $\mathcal{M}'_+$. Any operator $T \in \mathcal{B}(\mathcal{H})$ can be uniquely written as $T = T_1 - T_2 + i(T_3 - T_4)$, where T_k 's are positive operators, see [60, P. 9]. If we let $\mathcal{E}(T_k) = D_k$ and $D = D_1 - D_2 + i(D_3 - D_4)$, then $\mathcal{E}(T) = D$ is a desired extension of $\mathcal{E}$. By construction, $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}'$ becomes a positive mapping and $\mathcal{E}(I) = I$, where I is the identity operator. By Russo-Dye theorem for positive mappings between unital C^* -algebras, see [60, Corollary 2.9], we have $\|\mathcal{E}\| = \|\mathcal{E}(I)\| = 1$. Consequently, $\mathcal{E}$ is a conditional expectation, and by Remark 2.3.13, $\mathcal{M}$ is injective and Connes amenable.

- (ii) This was established at the beginning of the proof of part (i); the remainder of the argument is identical. □

Lemma 4.3.13. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal tracial state τ . If every von Neumann subalgebra of $\mathcal{M}$ with separable predual is injective, then $\mathcal{M}$ is injective.*

Proof. Let $\mathcal{U}(\mathcal{M})$ denote the unitary group of $\mathcal{M}$, and define

$$\mathcal{F} = \{ F \subseteq \mathcal{U}(\mathcal{M}) : F \text{ is countable} \},$$

partially ordered by inclusion. For each $F \in \mathcal{F}$, set

$$\mathcal{M}_F := W^*(F),$$

the von Neumann algebra generated by F and the identity element $1_{\mathcal{M}}$. We note that $(\mathcal{M}_F)_{F \in \mathcal{F}}$ is a directed increasing family since

- if $F \subseteq G$, then $\mathcal{M}_F \subseteq \mathcal{M}_G$;
- for any $F, G \in \mathcal{F}$, we have

$$\mathcal{M}_F, \mathcal{M}_G \subseteq \mathcal{M}_{F \cup G}.$$

Fix $F \in \mathcal{F}$. Then, $\tau|_{\mathcal{M}_F}$ is a faithful normal tracial state on $\mathcal{M}_F$. Let $\mathbb{A}$ be the unital $*$ -algebra over $\mathbb{Q} + i\mathbb{Q}$ generated by F . Then, $\mathbb{A}$ is countable and norm dense in the unital $*$ -algebra $\mathcal{A}$ generated by F . Since $\mathcal{A}$ is weak*-dense in $\mathcal{M}_F$, Kaplansky's density theorem [81, Theorem II.4.8] and [81, Theorem II.2.6] imply that the unit ball of $\mathcal{A}$ is strongly dense in the unit ball of $\mathcal{M}_F$. As $\tau|_{\mathcal{M}_F}$ is faithful, the GNS Hilbert space $\mathcal{H}_F = L^2(\mathcal{M}_F, \tau|_{\mathcal{M}_F})$ is the completion of $\mathcal{M}_F$ under the L^2 -norm

$$\|x\|_2 = \tau(x^*x)^{1/2}, \quad x \in \mathcal{M}_F.$$

We claim that strong convergence on bounded sets implies convergence in the above L^2 -norm associated with $\tau|_{\mathcal{M}_F}$. Indeed, let (x_i) be a bounded net in $\mathcal{M}_F$ such that $x_i \rightarrow x$ strongly. Then, the net $((x_i - x)^*(x_i - x))$ is bounded and consists of positive elements, and by [56, Remark 4.3.1],

$$(x_i - x)^*(x_i - x) \rightarrow 0 \quad \text{strongly.}$$

Since τ is normal, we observe that

$$\tau((x_i - x)^*(x_i - x)) \rightarrow 0.$$

Equivalently,

$$\|x_i - x\|_2^2 = \tau((x_i - x)^*(x_i - x)) \rightarrow 0.$$

Thus, bounded strong convergence implies convergence in the L^2 -norm. Consequently, the strong density of the unit ball of $\mathcal{A}$ in the unit ball of $\mathcal{M}_F$ yields L^2 -density of $\mathcal{A}$ in $\mathcal{H}_F$. Moreover, since τ is a state, for every $x \in \mathcal{M}_F$ we have

$$\|x\|_2^2 = \tau(x^*x) \leq \|x^*x\| = \|x\|^2.$$

Hence, norm convergence implies convergence in the L^2 -norm. Since $\mathbb{A}$ is norm dense in $\mathcal{A}$, it is L^2 -dense in $\mathcal{H}_F$. It follows that $\mathcal{H}_F = L^2(\mathcal{M}_F, \tau|_{\mathcal{M}_F})$ is separable. Now, consider the left regular (GNS) representation

$$\lambda_\tau : \mathcal{M}_F \rightarrow \mathcal{B}(\mathcal{H}_F), \quad \lambda_\tau(x)\widehat{y} = \widehat{xy}.$$

As $\tau|_{\mathcal{M}_F}$ is faithful, λ_τ is faithful. Thus, $\mathcal{M}_F$ is $*$ -isomorphic to the von Neumann algebra $\lambda_\tau(\mathcal{M}_F)$ acting on the separable Hilbert space $\mathcal{H}_F$. Since $\mathcal{H}_F$ is separable, by [56, Remark 4.4.2] and Kaplansky's density theorem, the trace-class space $\mathcal{B}^1(\mathcal{H}_F)$ is separable. Hence, $(\lambda_\tau(\mathcal{M}_F))_*$, as a quotient of the trace-class space $\mathcal{B}^1(\mathcal{H}_F)$, is separable. That is, $\mathcal{M}_F$ has a separable predual.

Define

$$\mathcal{N} := \overline{\bigcup_{F \in \mathcal{F}} \mathcal{M}_F}^{wk*}.$$

Clearly, $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$. For any $u \in \mathcal{U}(\mathcal{M})$, we have $\{u\} \in \mathcal{F}$ and

$$u \in W^*(u) = \mathcal{M}_{\{u\}} \subseteq \bigcup_{F \in \mathcal{F}} \mathcal{M}_F \subseteq \mathcal{N}.$$

So, $\mathcal{N}$ contains all unitaries of $\mathcal{M}$. By [56, Remark 2.2.2], every element of $\mathcal{M}$ is a linear combination of finitely many unitaries. Hence, $\mathcal{M} \subseteq \mathcal{N}$. Consequently,

$$\mathcal{M} = \mathcal{N} = \overline{\bigcup_{F \in \mathcal{F}} \mathcal{M}_F}^{wk*}.$$

However, each member of this union is injective by assumption. Therefore, by [3, Proposition IV.2.2.8(i)], $\mathcal{M}$ is injective. $\square$

The following result is well known; we provide a proof for completeness.

Lemma 4.3.14. *Let $\mathcal{M}$ be a von Neumann algebra, and let $\varphi \in \mathcal{M}_*^h$ be a normal Hermitian functional. Suppose that $\varphi = \varphi^+ - \varphi^-$ is the Jordan decomposition of φ , and let $|\varphi|$ denote the absolute value of φ in its polar decomposition. Then,*

$$|\varphi| = \varphi^+ + \varphi^-, \quad \varphi^+ = \frac{|\varphi| + \varphi}{2}, \quad \varphi^- = \frac{|\varphi| - \varphi}{2}.$$

Proof. We only need to prove $|\varphi| = \varphi^+ + \varphi^-$ since the other equations follow from it. If $e = s(\varphi^+)$ and $f = s(\varphi^-)$, then $ef = fe = 0$ by [81, Theorem III.4.2]. Let

$$v = e - f, \quad \psi := \varphi^+ + \varphi^- \in \mathcal{M}_*^+.$$

As $v = v^*$, and e and f are orthogonal projections, we have

$$\begin{aligned} v^*v &= v^2 = (e - f)^2 = e + f, \\ vv^*v &= v(e + f) = (e - f)(e + f) = e - f = v. \end{aligned}$$

It follows that v is a partial isometry, and $e + f$ is a projection. Clearly, $(e + f)\psi = \psi$. So, $s(\psi) \leq e + f$. We claim that

$$s(\psi) = e + f.$$

Let $p = s(\psi)$. Then, $\psi(1 - p) = 0$. Since $0 \leq \varphi^\pm \leq \psi$ and $1 - p \in \mathcal{M}_+$, it follows that

$$0 \leq \varphi^\pm(1 - p) \leq \psi(1 - p) = 0.$$

Hence,

$$\varphi^\pm(1 - p) = 0;$$

that is, $p\varphi^\pm = \varphi^\pm$. However, by the minimality of support projections,

$$e = s(\varphi^+) \leq p, \quad f = s(\varphi^-) \leq p.$$

Thus,

$$pe = e, \quad pf = f.$$

Therefore,

$$p(e + f) = pe + pf = e + f.$$

Since $e + f$ is a projection, we deduce that $e + f \leq p = s(\psi)$. By minimality of $s(\psi)$, this yields that $e + f = s(\psi)$.

Now, we prove that

$$\varphi = v(\varphi^+ + \varphi^-) \tag{4.11}$$

is the polar decomposition of φ . By the defining property of support projections, we have

$$\begin{aligned} e\varphi^+ &= \varphi^+, & f\varphi^- &= \varphi^-, \\ e\varphi^- &= 0, & f\varphi^+ &= 0. \end{aligned}$$

Hence,

$$v\psi = (e - f)(\varphi^+ + \varphi^-) = e\varphi^+ - f\varphi^- = \varphi^+ - \varphi^- = \varphi.$$

By the uniqueness of the polar decomposition of normal functionals [81, Theorem III.4.2], we conclude that

$$|\varphi| = \psi = \varphi^+ + \varphi^-.$$

Consequently,

$$\varphi^+ = \frac{|\varphi| + \varphi}{2}, \quad \varphi^- = \frac{|\varphi| - \varphi}{2}.$$

□

Lemma 4.3.15. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, and let $\psi \in \mathcal{B}(\mathcal{H})_*$ be a normal functional with polar decomposition $\psi = v|\psi|$. Then, for any unitary $u \in \mathcal{M}$, the absolute value of $u \cdot \psi \cdot u^*$ in its polar decomposition is given by*

$$|u \cdot \psi \cdot u^*| = u \cdot |\psi| \cdot u^*.$$

Proof. Let $\psi = v|\psi|$ be the polar decomposition of ψ , where $v \in \mathcal{B}(\mathcal{H})$ is a partial isometry and $|\psi| \in \mathcal{B}(\mathcal{H})_*$ is the absolute value of ψ . Define

$$\varphi := u \cdot \psi \cdot u^*.$$

We have

$$\varphi = u \cdot (v|\psi|) \cdot u^* = (uvu^*) \cdot (u \cdot |\psi| \cdot u^*). \quad (4.12)$$

We claim that (4.12) is the polar decomposition of φ . By the characterization of partial isometries [56, Theorem 2.3.3], we can check that uvu^* is a partial isometry. Moreover, by [56, Theorem 2.2.5], $u \cdot |\psi| \cdot u^*$ is a positive normal functional. We now show that

$$s(u \cdot |\psi| \cdot u^*) = u s(|\psi|) u^*.$$

Let $p = s(|\psi|)$ and $\eta := u \cdot |\psi| \cdot u^*$. Then,

$$\eta(1 - upu^*) = |\psi|(u^*(1 - upu^*)u) = |\psi|(1 - p) = 0,$$

so

$$s(\eta) \leq upu^*. \quad (4.13)$$

Conversely, if q is a projection such that $\eta(1 - q) = 0$, then

$$0 = \eta(1 - q) = |\psi|(u^*(1 - q)u) = |\psi|(1 - u^*qu).$$

By minimality of $p = s(|\psi|)$, we obtain

$$p \leq u^*qu.$$

Conjugating this by u and using [56, Theorem 2.2.5] yield

$$upu^* \leq q. \quad (4.14)$$

By Inequalities (4.13) and (4.14), upu^* is the smallest projection q such that $\eta(1 - q) = 0$, so

$$s(\eta) = upu^*.$$

That is,

$$s(u \cdot |\psi| \cdot u^*) = u s(|\psi|) u^*. \quad (4.15)$$

Now, since $v^*v = s(|\psi|)$, we have

$$\begin{aligned} (uvu^*)^*(uvu^*) &= u s(|\psi|) u^* \\ &= s(u \cdot |\psi| \cdot u^*), \quad \text{by Equation (4.15).} \end{aligned}$$

Thus, the initial projection of uvu^* coincides with the support of $u \cdot |\psi| \cdot u^*$. Consequently, (4.12) is the polar decomposition of φ . By uniqueness of the polar decomposition of normal functionals, we conclude that

$$|\varphi| = u \cdot |\psi| \cdot u^*,$$

as required. $\square$

Theorem 4.3.16. *Let $\mathcal{M}$ be a von Neumann algebra with separable predual and a faithful normal tracial state $\tau : \mathcal{M} \rightarrow \mathbb{C}$. Assume $\mathcal{H}$ is the GNS Hilbert space associated with τ as in Remark 4.3.4 and identify $\mathcal{M}$ with its faithful normal image inside $\mathcal{B}(\mathcal{H})$. Suppose that $(\psi_k)_{k=1}^\infty \subseteq \mathcal{B}(\mathcal{H})_*^h$ is a sequence of normal Hermitian functionals such that $\psi_k|_{\mathcal{M}} = \tau$ and*

$$\text{w}^* - \lim_k (x \cdot \psi_k - \psi_k \cdot x) = 0 \quad (x \in \mathcal{M}),$$

where the weak* limit is taken with respect to $\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))$. Then, there exists an ultratrace θ for $(\mathcal{M}, \mathcal{B}(\mathcal{H}))$.

Proof. We first show that $\mathcal{H} = L^2(\mathcal{M}, \tau)$ is separable. Since $\mathcal{M}_*$ is separable, by [13, Theorem V.5.1], the unit ball $\text{Ball}(\mathcal{M})$ is metrizable in the weak* topology. By Banach–Alaoglu theorem, it follows that $\text{Ball}(\mathcal{M})$ is weak* separable. Choose a countable weak* dense subset D of $\text{Ball}(\mathcal{M})$, and let $\mathbb{A}$ be the unital *-algebra over $\mathbb{Q} + i\mathbb{Q}$ generated by D . Then, $\mathbb{A}$ is countable and norm dense in the unital *-algebra $\mathcal{A}$ generated by D . By [81, Theorem II.2.6], $\mathcal{A}$ is strongly dense in $\mathcal{M}$. It then follows from Kaplansky’s density theorem [81, Theorem II.4.8] that $\text{Ball}(\mathcal{A})$ is strongly dense in $\text{Ball}(\mathcal{M})$. As in the proof of Lemma 4.3.13, strong convergence on bounded subsets of $\mathcal{M}$ implies convergence in the L^2 -norm associated with τ . Hence, the strong density of $\text{Ball}(\mathcal{A})$ in $\text{Ball}(\mathcal{M})$ implies that $\mathcal{A}$ is L^2 -dense in $\mathcal{H}$. Since $\mathbb{A}$ is norm dense in $\mathcal{A}$ and

$$\|x\|_2^2 = \tau(x^*x) \leq \|x^*x\| = \|x\|^2,$$

it follows that $\mathbb{A}$ is L^2 -dense in $\mathcal{H}$. Thus, $\mathcal{H}$ is separable.

Recall that unitary elements linearly span $\mathcal{M}$; see [56, Remark 2.2.2]. Therefore, we may choose a sequence of unitaries $(u_j)_{j=1}^\infty \subseteq \mathcal{U}(\mathcal{M})$ which generates $\mathcal{M}$ as a von Neumann algebra. For each $j, k \in \mathbb{N}$, set

$$\delta_k^{(j)} := u_j \cdot \psi_k - \psi_k \cdot u_j.$$

Since each ψ_k is normal, $\delta_k^{(j)} \in \mathcal{B}(\mathcal{H})_*$. Moreover, by hypothesis,

$$\delta_k^{(j)} \xrightarrow[k \rightarrow \infty]{\sigma(\mathcal{B}(\mathcal{H})_*, \mathcal{B}(\mathcal{H}))} 0 \quad (j \in \mathbb{N}).$$

Fix $n \in \mathbb{N}$. Then, the sequence

$$k \mapsto (\delta_k^{(1)}, \dots, \delta_k^{(n)})$$

converges weakly to 0 in the finite product Banach space $\prod_{j=1}^n \mathcal{B}(\mathcal{H})_*$. Hence, by Mazur’s lemma [74, Theorem 3.3.13], there exist indices

$$k_1^{(n)}, \dots, k_{m_n}^{(n)} \geq n$$

and scalars

$$\alpha_1^{(n)}, \dots, \alpha_{m_n}^{(n)} \geq 0, \quad \sum_{r=1}^{m_n} \alpha_r^{(n)} = 1,$$

such that

$$\max_{1 \leq j \leq n} \left\| \sum_{r=1}^{m_n} \alpha_r^{(n)} (u_j \cdot \psi_{k_r^{(n)}} - \psi_{k_r^{(n)}} \cdot u_j) \right\| < \frac{1}{n}. \quad (4.16)$$

Define

$$\chi_n := \sum_{r=1}^{m_n} \alpha_r^{(n)} \psi_{k_r^{(n)}} \in \mathcal{B}(\mathcal{H})_*.$$

Thus, $(\chi_n)_{n=1}^\infty$ is a sequence of convex blocks of $(\psi_k)_{k=1}^\infty$. Since each ψ_k is normal and Hermitian, so is each χ_n . Hence, by [81, Theorem III.4.2], each χ_n admits a unique Jordan decomposition

$$\chi_n = \chi_n^+ - \chi_n^-$$

where $\chi_n^+, \chi_n^- \in \mathcal{B}(\mathcal{H})_*^+$ are positive normal functionals satisfying $\|\chi_n\| = \|\chi_n^+\| + \|\chi_n^-\|$. By linearity of the module actions,

$$u_j \cdot \chi_n - \chi_n \cdot u_j = \sum_{r=1}^{m_n} \alpha_r^{(n)} (u_j \cdot \psi_{k_r^{(n)}} - \psi_{k_r^{(n)}} \cdot u_j), \quad 1 \leq j \leq n.$$

Hence, by (4.16),

$$\max_{1 \leq j \leq n} \|u_j \cdot \chi_n - \chi_n \cdot u_j\| < \frac{1}{n}. \quad (4.17)$$

Moreover, since $\psi_k|_{\mathcal{M}} = \tau$ for every k , we still have

$$\chi_n|_{\mathcal{M}} = \tau, \quad \chi_n(1) = \tau(1) = 1.$$

Now, define

$$\omega_n := \chi_n^+, \quad \nu_n := \frac{\omega_n}{\|\omega_n\|}.$$

Then, $\omega_n \in \mathcal{B}(\mathcal{H})_*^+$, and each ν_n is a normal state on $\mathcal{B}(\mathcal{H})$. Since $(\nu_n)_n$ is a bounded sequence in $\mathcal{B}(\mathcal{H})^*$, the Banach–Alaoglu theorem implies that it admits a weak* cluster point θ . We claim that $\theta \in \mathcal{B}(\mathcal{H})_*$. For each n , since $\nu_n \in \mathcal{B}(\mathcal{H})_*^+$ and $\|\nu_n\| = 1$, there exists a unique operator $V_n \in \mathcal{B}^1(\mathcal{H})^+$ such that

$$\nu_n(x) = \text{Tr}(V_n x) \quad (x \in \mathcal{B}(\mathcal{H})).$$

Moreover,

$$\|V_n\|_1 = \text{Tr}(V_n) = \|\nu_n\| = 1,$$

so (V_n) is bounded in trace norm and hence bounded in the operator norm by [56, Theorem 2.4.15]. Since $\mathcal{H}$ is separable, by [56, Remark 4.4.2] and [81, Theorem II.2.6] the weak* topology $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}^1(\mathcal{H}))$ is metrizable on norm-bounded subsets of $\mathcal{B}(\mathcal{H})$. Therefore, passing to a subsequence if necessary, we may assume that

$$V_n \xrightarrow{\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}^1(\mathcal{H}))} V \quad \text{in } \mathcal{B}(\mathcal{H})$$

for some $V \in \mathcal{B}(\mathcal{H})$. We show that $V \in \mathcal{B}^1(\mathcal{H})^+$. By [56, p. 65], $\mathcal{B}^1(\mathcal{H})$ is a self-adjoint ideal of $\mathcal{B}(\mathcal{H})$, and the involution on $\mathcal{B}(\mathcal{H})$ is weak* continuous. It follows that V is self-adjoint. Moreover, by [56, Theorem 2.4.17], $\mathcal{B}^1(\mathcal{H})$ contains all finite-rank operators. In particular, for all $\xi, \eta \in \mathcal{H}$, the rank-one operator $\theta_{\xi, \eta}$ defined by

$$\theta_{\xi, \eta}(\zeta) = \langle \zeta, \eta \rangle \xi \quad (\zeta \in \mathcal{H})$$

belongs to $\mathcal{B}^1(\mathcal{H})$. Hence, $V\theta_{\xi,\xi} \in \mathcal{B}^1(\mathcal{H})$ for all $\xi \in \mathcal{H}$. We now have

$$\begin{aligned}\langle V\xi, \xi \rangle &= \text{Tr}(V\theta_{\xi,\xi}) \\ &= \lim_n \text{Tr}(V_n\theta_{\xi,\xi}) \\ &= \lim_n \langle V_n\xi, \xi \rangle \geq 0, \quad \text{since } V_n \geq 0, \forall n.\end{aligned}$$

Thus, $V \geq 0$. Let (p_m) be an increasing sequence of finite-rank projections such that $p_m \uparrow I$ strongly. Since $p_m \in \mathcal{B}^1(\mathcal{H})$ for each m , we have $Vp_m, V_np_m \in \mathcal{B}^1(\mathcal{H})$ for all n . Therefore, for each m , weak* convergence yields

$$\text{Tr}(Vp_m) = \lim_n \text{Tr}(V_np_m).$$

Since $V_n \geq 0$, we may multiply the inequality $0 \leq p_m \leq I$ on both sides by $V_n^{1/2}$ (see [56, Theorem 2.2.5]) to obtain

$$0 \leq V_n^{1/2}p_mV_n^{1/2} \leq V_n.$$

By positivity and the cyclicity of the trace, it follows that

$$0 \leq \text{Tr}(V_np_m) \leq \text{Tr}(V_n) = 1, \quad (n, m \in \mathbb{N}).$$

Hence,

$$0 \leq \text{Tr}(Vp_m) \leq 1 \quad (m \in \mathbb{N}),$$

and therefore,

$$\sup_m \text{Tr}(Vp_m) \leq 1.$$

By normality of trace and the fact that $Vp_m \uparrow V$ strongly, we obtain

$$\text{Tr}(V) = \sup_m \text{Tr}(Vp_m) \leq 1.$$

Thus, $V \in \mathcal{B}^1(\mathcal{H})^+$, and the functional

$$\theta(x) := \text{Tr}(Vx) \quad (x \in \mathcal{B}(\mathcal{H}))$$

belongs to $\mathcal{B}(\mathcal{H})_*^+$. Recall that

$$\theta(x) = \lim_n \nu_n(x), \quad (x \in \mathcal{B}(\mathcal{H})).$$

In addition, since each ν_n is a state, we have $\nu_n(1) = 1$. Hence,

$$\theta(1) = \lim_n \nu_n(1) = 1.$$

In particular, $\theta|_{\mathcal{M}} \neq 0$ since $1 \in \mathcal{M}$. Moreover, $\text{Tr}(V) = 1$, so $V \neq 0$.

On the other hand, since χ_n is Hermitian, we deduce from Lemma 4.3.14 that

$$\omega_n = \chi_n^+ = \frac{|\chi_n| + \chi_n}{2}. \quad (4.18)$$

Fix $j \in \mathbb{N}$. For $n \geq j$, we have

$$u_j \cdot \omega_n \cdot u_j^* - \omega_n = \frac{1}{2} \left(u_j \cdot |\chi_n| \cdot u_j^* - |\chi_n| + u_j \cdot \chi_n \cdot u_j^* - \chi_n \right).$$

Hence,

$$\|u_j \cdot \omega_n \cdot u_j^* - \omega_n\| \leq \frac{1}{2} \left(\|u_j \cdot |\chi_n| \cdot u_j^* - |\chi_n|\| + \|u_j \cdot \chi_n \cdot u_j^* - \chi_n\| \right). \quad (4.19)$$

For the first term on the right hand side of the above inequality, we apply Kosaki's theorem [52]: for any φ, ψ in the predual of a von Neumann algebra,

$$\| |\varphi| - |\psi| \| \leq (2\|\varphi + \psi\| \|\varphi - \psi\|)^{1/2}.$$

Taking

$$\varphi = u_j \cdot \chi_n \cdot u_j^*, \quad \psi = \chi_n,$$

and using Lemma 4.3.15, we obtain

$$\|u_j \cdot |\chi_n| \cdot u_j^* - |\chi_n|\| \leq \left(2\|u_j \cdot \chi_n \cdot u_j^* + \chi_n\| \|u_j \cdot \chi_n \cdot u_j^* - \chi_n\| \right)^{1/2}.$$

Since module actions by unitaries are isometric, it follows that

$$\|u_j \cdot \chi_n \cdot u_j^* + \chi_n\| \leq 2\|\chi_n\|.$$

Hence,

$$\|u_j \cdot |\chi_n| \cdot u_j^* - |\chi_n|\| \leq 2\|\chi_n\|^{1/2} \|u_j \cdot \chi_n \cdot u_j^* - \chi_n\|^{1/2}. \quad (4.20)$$

Recall that $\chi_n(1) = 1$, so

$$1 = \chi_n(1) = \chi_n^+(1) - \chi_n^-(1) = \|\chi_n^+\| - \|\chi_n^-\|.$$

Moreover,

$$\|\chi_n\| = \|\chi_n^+\| + \|\chi_n^-\|.$$

Solving the last two equations yields

$$\|\omega_n\| = \|\chi_n^+\| = \frac{\|\chi_n\| + 1}{2} \geq 1. \quad (4.21)$$

Since

$$u_j \cdot \nu_n - \nu_n \cdot u_j = (u_j \cdot \nu_n \cdot u_j^* - \nu_n) \cdot u_j,$$

and multiplication by u_j is isometric, it follows from (4.19), (4.20), and (4.21) that

$$\begin{aligned} \|u_j \cdot \nu_n - \nu_n \cdot u_j\| &= \|u_j \cdot \nu_n \cdot u_j^* - \nu_n\| \\ &\leq \frac{1}{2\|\omega_n\|} \left(2\|\chi_n\|^{1/2} \|u_j \cdot \chi_n \cdot u_j^* - \chi_n\|^{1/2} + \|u_j \cdot \chi_n \cdot u_j^* - \chi_n\| \right) \\ &= \frac{2\|\chi_n\|^{1/2}}{\|\chi_n\| + 1} \|u_j \cdot \chi_n \cdot u_j^* - \chi_n\|^{1/2} + \frac{1}{\|\chi_n\| + 1} \|u_j \cdot \chi_n \cdot u_j^* - \chi_n\| \\ &\leq \|u_j \cdot \chi_n \cdot u_j^* - \chi_n\|^{1/2} + \|u_j \cdot \chi_n \cdot u_j^* - \chi_n\|, \end{aligned} \quad (4.22)$$

because $\frac{2\sqrt{t}}{t+1} \leq 1$ for $t \geq 0$. Inequality (4.17) gives

$$\|u_j \cdot \chi_n - \chi_n \cdot u_j\| < \frac{1}{n} \quad (n \geq j \geq 1).$$

Since

$$\|u_j \cdot \chi_n \cdot u_j^* - \chi_n\| = \|(u_j \cdot \chi_n - \chi_n \cdot u_j) \cdot u_j^*\| = \|u_j \cdot \chi_n - \chi_n \cdot u_j\|,$$

it follows from (4.22) that, for every fixed j ,

$$\|u_j \cdot \nu_n - \nu_n \cdot u_j\| \leq \frac{1}{\sqrt{n}} + \frac{1}{n} \quad (n \geq j).$$

Hence,

$$u_j \cdot \nu_n - \nu_n \cdot u_j \xrightarrow[n \rightarrow \infty]{\|\cdot\|} 0, \quad \forall j \in \mathbb{N}. \quad (4.23)$$

This implies that

$$u_j \cdot \theta = \theta \cdot u_j \quad (j \in \mathbb{N}).$$

Let

$$\mathcal{C}_\theta := \{x \in \mathcal{M} : x \cdot \theta = \theta \cdot x\}.$$

Then, $\mathcal{C}_\theta$ is a von Neumann subalgebra of $\mathcal{M}$ containing every u_j . Since the family (u_j) generates $\mathcal{M}$ as a von Neumann algebra, we conclude that

$$\mathcal{C}_\theta = \mathcal{M}.$$

Thus,

$$x \cdot \theta = \theta \cdot x \quad \forall x \in \mathcal{M}.$$

Therefore, θ is a positive linear functional on $\mathcal{B}(\mathcal{H})$ satisfying the ultra trace property, and since $\theta \in \mathcal{B}(\mathcal{H})_*$, its restriction to $\mathcal{M}$ is normal. Hence, θ is an ultratrace for $(\mathcal{M}, \mathcal{B}(\mathcal{H}))$. $\square$

Let $T \in \mathcal{B}(\mathcal{H})$ with polar decomposition $T = v|T|$. Then, $s_r(T) = v^*v$ is a projection, called the *initial (or right) support* of T . Also, $s_l(T) = vv^*$ is a projection, called the *final (or left) support* of T . Equivalently,

$$\begin{aligned} s_r(T) &= \min \{e \in \mathcal{P}(\mathcal{M}) : Te = T\}, \\ s_l(T) &= \min \{f \in \mathcal{P}(\mathcal{M}) : fT = T\}. \end{aligned}$$

Let $\mathcal{M}$ be a von Neumann algebra inside $\mathcal{B}(\mathcal{H})$ with a faithful normal tracial weight τ . As in [12], we say $T \in \mathcal{B}(\mathcal{H})$ is of τ -finite rank when its initial and final supports are majorized by projections $e, f \in \mathcal{P}(\mathcal{M})$ with $\tau(e) < \infty$ and $\tau(f) < \infty$ respectively. We denote by $\mathcal{F}$ the set of all τ -finite rank operators in $\mathcal{B}(\mathcal{H})$.

Some arguments in the proof of the next theorem are adapted from the proof of [12, Theorem 1].

Theorem 4.3.17. *Let $\mathcal{M}$ be a von Neumann algebra inside $\mathcal{B}(\mathcal{H})$ with a faithful normal tracial state τ . If $\mathcal{M}$ is sequentially Connes' amenable, then it is injective and hence Connes amenable.*

Proof. By Lemma 4.3.13, we may assume without loss of generality that $\mathcal{M}$ has a separable predual. Let $\tau : \mathcal{M} \rightarrow \mathbb{C}$ be a faithful normal tracial state. Then, $\tau(1) = 1 = \|\tau\|$, and $\mathcal{M}$ is a finite von Neumann algebra by Proposition 4.3.10. We may use the standard GNS representation $(\pi_\tau, H_\tau, \xi_\tau)$ of $\mathcal{M}$, as described in Remark 4.3.4. In this representation, $\mathcal{H} = H_\tau$, $\tau(x) = \langle x\xi_\tau, \xi_\tau \rangle$ for all $x \in \mathcal{M}$, where ξ_τ is the cyclic vector associated with τ . Clearly, the functional $\tilde{\tau}(T) = \langle T\xi_\tau, \xi_\tau \rangle$ defines a normal state on $\mathcal{B}(\mathcal{H})$ which extends τ , and satisfies $\|\tilde{\tau}\| = \|\tau\| = \tau(1) = 1$. Note that $\tilde{\tau}$ cannot be a tracial state on $\mathcal{B}(\mathcal{H})$ by [56, Remark 6.2.2]. Nevertheless, it induces a positive semi-definite sesqui-linear form on $\mathcal{B}(\mathcal{H})$ given by

$$\langle W, S \rangle_{\tilde{\tau}} := \tilde{\tau}(S^*W), \quad W, S \in \mathcal{B}(\mathcal{H}). \quad (4.24)$$

Now, we define a semi-norm on $\mathcal{B}(\mathcal{H})$ by

$$\|T\|_2 := \sqrt{\tilde{\tau}(T^*T)}, \quad T \in \mathcal{B}(\mathcal{H}). \quad (4.25)$$

We have $\|T\|_2 \leq \|T\|_\infty$, where $\|T\|_\infty$ denotes the operator norm of $T \in \mathcal{B}(\mathcal{H})$. By [12], the set $\mathcal{F}$ of all τ -finite rank operators is an $\mathcal{M}$ submodule of $\mathcal{B}(\mathcal{H})$. Since $1 \in \mathcal{P}(\mathcal{M})$ and $\tau(1) = 1 < \infty$, every $T \in \mathcal{B}(\mathcal{H})$ is of τ -finite rank. Hence, $\mathcal{F} = \mathcal{B}(\mathcal{H})$. Let $\mathcal{Y}$ be the space of all linear functionals $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ for which there exists a $k > 0$ such that for all $a, b \in \mathcal{M}$ and $T \in \mathcal{B}(\mathcal{H})$

$$|b \cdot \phi \cdot a(T)| = |\phi(aTb)| \leq k\|a\|_2\|T\|_\infty\|b\|_2. \quad (4.26)$$

For $\phi \in \mathcal{Y}$, we define $\|\phi\|_{\mathcal{Y}}$ to be the infimum of such k 's. By [12, P. 250], $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a dual normal $(\mathcal{M}, \|\cdot\|_\infty)$ bi-module. Since the standard representation of $\mathcal{M}$ is non-degenerate, setting $a = b = 1$ in (4.26) yields

$$\|\phi\| \leq \|\phi\|_{\mathcal{Y}}, \quad (4.27)$$

where $\|\phi\|$ denotes the usual norm of $\phi \in \mathcal{B}(\mathcal{H})^*$. Thus,

$$\mathcal{Y} \subseteq \mathcal{B}(\mathcal{H})^*. \quad (4.28)$$

Moreover, $\mathcal{B}(\mathcal{H})$ is norm dense in the predual $\mathcal{Y}_*$; see [12, p. 250]. As no proof of this fact is given in that reference, we supply one in Proposition 4.3.18. Consequently, convergence in $\sigma(\mathcal{Y}, \mathcal{Y}_*)$ implies convergence in $\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))$.

Recall that every $\omega \in \mathcal{Y} \subseteq \mathcal{B}(\mathcal{H})^*$ admits a unique decomposition (see [81, Theorem III.2.14]) into its normal and singular parts:

$$\omega = \omega_n + \omega_s, \quad \|\omega\| = \|\omega_n\| + \|\omega_s\|. \quad (4.29)$$

We claim that $\omega_n \in \mathcal{Y}$. By [48, Theorem 10.1.15(iv)], for each $a, b \in \mathcal{M}$ and $T \in \mathcal{B}(\mathcal{H})$, the functional

$$T \mapsto \omega(aTb) = (b \cdot \omega \cdot a)(T)$$

decomposes accordingly, with normal part $b \cdot \omega_n \cdot a$ and singular part $b \cdot \omega_s \cdot a$. By Estimate (4.26), we have

$$\|b \cdot \omega \cdot a\| \leq k\|a\|_2\|b\|_2.$$

Since the normal part of $b \cdot \omega \cdot a$ is $b \cdot \omega_n \cdot a$, it follows from (4.29) that

$$\|b \cdot \omega_n \cdot a\| \leq \|b \cdot \omega \cdot a\| \leq k\|a\|_2\|b\|_2.$$

Hence,

$$|(b \cdot \omega_n \cdot a)(T)| \leq k \|a\|_2 \|T\|_\infty \|b\|_2 \quad (a, b \in \mathcal{M}, T \in \mathcal{B}(\mathcal{H})). \quad (4.30)$$

Equivalently,

$$|\omega_n(aTb)| \leq k \|a\|_2 \|T\|_\infty \|b\|_2 \quad (a, b \in \mathcal{M}, T \in \mathcal{B}(\mathcal{H})).$$

Thus, $\omega_n \in \mathcal{Y}$.

Define the submodule $\mathcal{X}$ of $\mathcal{Y}$ by

$$\mathcal{X} = \{\phi \in \mathcal{Y} : \phi(x) = 0 \ \forall x \in \mathcal{M}\} = \mathcal{M}^\perp,$$

that is, the annihilator of $\mathcal{M}$ in the dual space $\mathcal{Y}$. By [12, P. 250], $\mathcal{X}$ is a normal dual Banach $\mathcal{M}$ -bimodule. In addition, $\tilde{\tau} \in \mathcal{Y}$ with $\|\tilde{\tau}\|_{\mathcal{Y}} \leq 1$ (see [12, P. 251]). Note that $\tilde{\tau}$ is not a member of $\mathcal{X}$ since $\tilde{\tau}$ is non-zero on $\mathcal{M}$. For any $x, y \in \mathcal{M}$, the tracial property of τ on $\mathcal{M}$ implies

$$(x\tilde{\tau} - \tilde{\tau}x)(y) = x\tilde{\tau}(y) - \tilde{\tau}x(y) = \tilde{\tau}(yx) - \tilde{\tau}(xy) = \tau(yx) - \tau(xy) = 0. \quad (4.31)$$

Thus, for any $x \in \mathcal{M}$, we have $x\tilde{\tau} - \tilde{\tau}x \in \mathcal{X}$. Likewise [12], define the normal derivation

$$\begin{aligned} \delta : \mathcal{M} &\longrightarrow \mathcal{X} \\ \delta(x) &= x \cdot \tilde{\tau} - \tilde{\tau} \cdot x \end{aligned} \quad (4.32)$$

By our assumption, there is a sequence (ϕ_k) in $\mathcal{X}$ such that

$$x \cdot \phi_k - \phi_k \cdot x \xrightarrow{\sigma(\mathcal{X}, \mathcal{X}^*)} \delta(x) \quad (x \in \mathcal{M}). \quad (4.33)$$

By Proposition 4.3.19, we have

$$x \cdot \phi_k - \phi_k \cdot x \xrightarrow{\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))} \delta(x) \quad (x \in \mathcal{M}). \quad (4.34)$$

For the sequence $(\|\phi_k\|_{\mathcal{Y}})$, by passing to a subsequence if necessary, we may assume that either

- $(\|\phi_k\|_{\mathcal{Y}})$ is bounded, or
- $\|\phi_k\|_{\mathcal{Y}} \rightarrow \infty$.

In the first case, let ϕ be a weak*-cluster point of the sequence (ϕ_k) . Hence, we will have

$$\delta(x) = x \cdot \phi - \phi \cdot x, \quad (\forall x \in \mathcal{M}).$$

Now, we can proceed as in the proof of [12] to conclude that $\mathcal{M}$ is injective. Thus, without loss of generality, we may assume that the second case holds. For each k define

$$\psi_k = \tilde{\tau} - \phi_k \in \mathcal{Y}.$$

Fix $x \in \mathcal{M}$, and set

$$u_k(x) := x \cdot \psi_k - \psi_k \cdot x \in \mathcal{Y}.$$

By (4.31),

$$u_k(x) := x \cdot \psi_k - \psi_k \cdot x \in \mathcal{X}.$$

By (4.33), we have

$$u_k(x) \xrightarrow{w^*} 0 \quad \text{in } \sigma(\mathcal{X}, \mathcal{X}_*). \quad (4.35)$$

Since $\mathcal{X}$ is equipped with the norm inherited from $\mathcal{Y}$, the Uniform Boundedness Principle yields

$$\sup_k \|u_k(x)\|_{\mathcal{Y}} < \infty.$$

On the other hand, by (4.35) and Proposition 4.3.19, we have

$$u_k(x) \xrightarrow{w^*} 0 \quad \text{in } \sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H})). \quad (4.36)$$

As $\mathcal{B}(\mathcal{H})$ is norm dense in $\mathcal{Y}_*$ and the sequence $(u_k(x))$ is uniformly bounded in $\mathcal{Y}$, it follows that

$$u_k(x) = u_k(x) := x \cdot \psi_k - \psi_k \cdot x \xrightarrow{w^*} 0 \quad \text{in } \sigma(\mathcal{Y}, \mathcal{Y}_*). \quad (4.37)$$

We claim that each ψ_k can be taken normal. By (4.37) and [81, Theorem III.5.1], we obtain

$$(x \cdot \psi_k - \psi_k \cdot x)_n \xrightarrow{\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))} 0.$$

By [48, Theorem 10.1.15(iv)], we have

$$(x \cdot \psi_k - \psi_k \cdot x)_n = x \cdot (\psi_k)_n - (\psi_k)_n \cdot x.$$

Therefore,

$$x \cdot (\psi_k)_n - (\psi_k)_n \cdot x \xrightarrow{\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))} 0. \quad (4.38)$$

Recall that $(\psi_k)_n \in \mathcal{Y}$ and by applying the Principle of Uniform Boundedness as (4.37), we deduce that

$$x \cdot (\psi_k)_n - (\psi_k)_n \cdot x \xrightarrow{\sigma(\mathcal{Y}, \mathcal{Y}_*)} 0. \quad (4.39)$$

Consequently, replacing ψ_k by $(\psi_k)_n$ if necessary, we may assume that each ψ_k is normal.

Now, for the sequence of normal functionals $(\psi_k)_k$, we have

1. $\tau(m) = \psi_k(m)$ for all $m \in \mathcal{M}$ since $\phi_k|_{\mathcal{M}} = 0$ for each k .

2. $x \cdot \psi_k - \psi_k \cdot x \xrightarrow{\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))} 0 \quad \forall x \in \mathcal{M}.$

For any linear functional λ on $\mathcal{B}(\mathcal{H})$ define λ^* by $\lambda^*(T) = \overline{\lambda(T^*)}$, and recall that λ is called Hermitian if $\lambda = \lambda^*$. It can be readily verified that $\mathcal{Y}$ is closed under this $*$ -operation. Moreover, the weak* convergence in Item (2) is stable under the $*$ -operation. Therefore, we may assume, without loss of generality, that each ψ_k is Hermitian. To see this, for each $x \in \mathcal{M}$, set

$$u_k(x) := x \cdot \psi_k - \psi_k \cdot x,$$

and define

$$v_k(x) := x \cdot \psi_k^* - \psi_k^* \cdot x.$$

For all $T \in \mathcal{B}(\mathcal{H})$, we have

$$v_k(x)(T) = \overline{u_k(x^*)(T^*)}.$$

Since $u_k(x^*) \rightarrow 0$ in $\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))$, it follows that

$$v_k(x)(T) \rightarrow 0 \quad (T \in \mathcal{B}(\mathcal{H})).$$

Hence,

$$v_k(x) \xrightarrow{\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))} 0.$$

Setting

$$\psi_k^h := \frac{\psi_k + \psi_k^*}{2},$$

we have $\psi_k^h \in \mathcal{Y}$, and ψ_k^h is Hermitian for each k . By the preceding computations, we have

$$x \cdot \psi_k^h - \psi_k^h \cdot x = \frac{1}{2} u_k(x) + \frac{1}{2} v_k(x) \xrightarrow{\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))} 0 \quad (\forall x \in \mathcal{M}).$$

Consequently, replacing ψ_k by ψ_k^h , we may assume that each ψ_k is Hermitian.

Reduction hypothesis. In view of the previous reductions, we have a sequence of Hermitian normal functionals $(\psi_k)_{k=1}^\infty$ such that $\psi_k|_{\mathcal{M}} = \tau$ and

$$x \cdot \psi_k - \psi_k \cdot x \xrightarrow{\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))} 0 \quad (\forall x \in \mathcal{M}). \quad (4.40)$$

Hence, by Theorem 4.3.16, there exists a normal positive tracial state θ_0 on $\mathcal{M}$ which is an ultratrace for $(\mathcal{M}, \mathcal{B}(\mathcal{H}))$. By Remark 4.2.14-vi, we can find a non-zero central projection $s(\theta_0) = f_0 \in \mathcal{ZP}(\mathcal{M})$ such that the functional θ_{f_0} defined by $\theta_{f_0}(f_0 m f_0) = \theta_0(m)$ is a normal faithful tracial positive linear functional on $f_0 \mathcal{M} f_0$. Since θ_0 is defined on all of $\mathcal{B}(\mathcal{H})$, we can extend θ_{f_0} naturally by $\theta_{f_0}(T) = \theta_0(f_0 T f_0)$ to all of $\mathcal{B}(\mathcal{H})$. Also, by Proposition 4.2.3 and considering the homomorphism $m \mapsto f_0 m f_0$, we see that $f_0 \mathcal{M} f_0$ is approximately Connes amenable. One easily verifies that θ_{f_0} is an ultra trace for $(f_0 \mathcal{M} f_0, \mathcal{B}(\mathcal{H}))$. We can normalize θ_{f_0} to be of norm one, i.e., we can assume it is a state on $f_0 \mathcal{M} f_0$. Write

$$\mathcal{M} = f_0 \mathcal{M} f_0 \oplus (1 - f_0) \mathcal{M} (1 - f_0).$$

By Theorem 4.3.12(ii), we deduce that $f_0 \mathcal{M} f_0$ is injective. If $f_0 = 1$, nothing remains to prove. So, suppose that $f_0 < 1$, i.e., $1 - f_0 \neq 0$. Since τ is faithful, by [81, Proposition II.3.19] we see that $\mathcal{M}$ is σ -weak* separable, i.e., there are only countably many pairwise orthogonal projections $(p_n)_{n \in \mathbb{N}}$ such that $1 = \sum_{n=1}^\infty p_n$. In other words, any arbitrary collection of pairwise orthogonal projections in $\mathcal{M}$ is countable. By [48, P. 422], both $f_0 \mathcal{M} f_0$ and $(1 - f_0) \mathcal{M} (1 - f_0)$ are finite. Let $\mathcal{M}_1 = (1 - f_0) \mathcal{M} (1 - f_0)$, and we set $\mathcal{M}_0 = \mathcal{M}$. Define

$$\begin{aligned} \Phi_1 : \mathcal{M}_0 &\longrightarrow \mathcal{M}_1 \\ \Phi_1(m) &= (1 - f_0)m(1 - f_0), \end{aligned}$$

which is obviously a normal homomorphism. By Proposition 4.2.3, $\mathcal{M}_1$ is sequentially Connes amenable. Clearly, the restriction $\tau_1 = \tau|_{\mathcal{M}_1}$ is a faithful normal positive linear functional. After a renormalization, we can assume that τ_1 is a faithful normal state on $\mathcal{M}_1$. Now, we repeat the first part of the proof for $(\mathcal{M}, \tau)$ and consider the normal derivation

$$\begin{aligned} \delta_1 : \mathcal{M}_1 &\longrightarrow \mathcal{X} \\ \delta_1(x) &= x\tilde{\tau}_1 - \tilde{\tau}_1 x. \end{aligned}$$

Since $\mathcal{M}_1$ is sequentially Connes amenable, there is a sequence $(\phi_{k,1})_{k=1}^\infty$ in $\mathcal{X}$ such that $\delta_1(x) = \text{w}^*\text{-}\lim_k x \cdot \phi_{k,1} - \phi_{k,1} \cdot x$ for any $x \in \mathcal{M}_1$. As before, for the sequence $(\|\phi_{k,1}\|_{\mathcal{Y}})$ we have two cases:

- either $(\|\phi_{k,1}\|_{\mathcal{Y}})$ is bounded, or
- $\|\phi_{k,1}\|_{\mathcal{Y}} \longrightarrow +\infty$.

We can then ensure that either $\mathcal{M}_1$ is injective or there is a central projection $f_1 \in \mathcal{ZP}(\mathcal{M}_1) \subseteq \mathcal{ZP}(\mathcal{M})$ such that $\mathcal{M}_1 = f_1 \mathcal{M} f_1 \oplus \mathcal{M}_2$, where

$$\begin{aligned} \mathcal{M}_2 &= (1_{\mathcal{M}_1} - f_1) \mathcal{M}_1 (1_{\mathcal{M}_1} - f_1) \\ &= ((1 - f_0) - f_1) (1 - f_0) \mathcal{M} (1 - f_0) ((1 - f_0) - f_1) \\ &= (1 - [f_0 + f_1]) \mathcal{M} (1 - [f_0 + f_1]). \end{aligned}$$

Therefore,

$$\mathcal{M} = f_0 \mathcal{M} f_0 \oplus f_1 \mathcal{M} f_1 \oplus (1 - [f_0 + f_1]) \mathcal{M} (1 - [f_0 + f_1]).$$

We also note that $f_1 \leq 1_{\mathcal{M}_1} = 1 - f_0$ and $(1 - f_0) \perp f_0$, so $f_1 f_0 = 0 = f_0 f_1$. Hence, $f_1 \perp f_0$.

Proceeding inductively, we then either have an $n \in \mathbb{N} \cup \{0\}$ and mutually orthogonal projections

$$f_0, \dots, f_n \in \mathcal{ZP}(\mathcal{M})$$

such that each $f_i \mathcal{M} f_i$, $0 \leq i \leq n$, is injective and

$$\mathcal{M} = \bigoplus_{i=0}^n f_i \mathcal{M} f_i,$$

or there is a sequence of mutually orthogonal projections $(f_i)_{i=0}^\infty$ in $\mathcal{ZP}(\mathcal{M})$ such that each $f_i \mathcal{M} f_i$ is injective. In the latter case, the direct sum $\mathcal{N} = \bigoplus_{i=0}^\infty f_i \mathcal{M} f_i$ is injective, and

$$\mathcal{M} = \mathcal{N} \oplus \mathcal{Q}$$

Consider the set $\mathcal{E}$ of all pairwise orthogonal projections p such that $p \mathcal{M} p$ is injective. Clearly, $f_n \in \mathcal{E}$ for each $n \in \mathbb{N} \cup \{0\}$. Note that $\mathcal{E}$ is countable since τ is a faithful normal state on $\mathcal{M}$, see [81, Theorem II. 3.19]. We can label members of $\mathcal{E}$ by a countable set I , i.e., $\mathcal{E} = \{e_i : i \in I\}$. As in the proof of [81, Theorem II. 3.19], for $I_n = \{i \in I : \tau(e_i) \geq \frac{1}{n}\}$, we have

$$\tau\left(\sum_{i \in I_n} e_i\right) \leq \tau(1), \quad \forall n.$$

Since τ is faithful, we obtain

$$\sum_{i \in I_n} e_i \leq 1, \quad \forall n.$$

Hence, by Vigier's theorem [56, Theorem 4.1.1]

$$s = \sum_{i \in I} e_i \leq 1.$$

Thus, $\mathcal{N} = \oplus_{e \in \mathcal{E}} e \mathcal{M} e$, and

$$\mathcal{M} = s \mathcal{M} s \oplus (1 - s) \mathcal{M} (1 - s).$$

We claim that $s = 1$. In this case, $\mathcal{M} = \oplus_{e \in \mathcal{E}} e \mathcal{M} e$ is a direct sum of injective von Neumann algebras; hence, it is injective. Suppose to the contrary that $s < 1$. Then, we have the decomposition $\mathcal{M} = s \mathcal{M} s \oplus (1 - s) \mathcal{M} (1 - s)$. Note that $s \mathcal{M} s = \oplus_{e \in \mathcal{E}} e \mathcal{M} e$ is injective. The projection $1 - s$ is finite by [48, Proposition 6.3.2], so the summand $(1 - s) \mathcal{M} (1 - s)$ must be finite. As before, we apply the **Reduction hypothesis** to the algebra $(1 - s) \mathcal{M} (1 - s)$ to obtain a non-zero ultra-trace with a non-zero central projection $q \in (1 - s) \mathcal{M} (1 - s)$ such that $sq = 0$ and $q(1 - s) \mathcal{M} (1 - s)q = q \mathcal{M} q$ is injective. By the definition of $\mathcal{E}$, we must have that $q \in \mathcal{E}$. Thus, $q \leq s$ and therefore $q = sq = 0$. This contradicts the fact that $q \neq 0$. Thus, $s = 1$. Consequently, $\mathcal{M}$ is injective. $\square$

The following result is used in [12] without proof; we include one here for completeness.

Proposition 4.3.18. *Assume the notation and hypotheses of Theorem 4.3.17, and let $\mathcal{Y}$ denote the space defined in its proof. Then, $\mathcal{Y}$ is a dual Banach space whose predual $\mathcal{Y}_*$ contains $\mathcal{B}(\mathcal{H})$ as a norm-dense subspace.*

Proof. By [74, Theorem 4.4.3], for $T \in \mathcal{B}(\mathcal{H})$, define

$$\|T\|_{\mathcal{Y}_*} := \sup\{|\phi(T)| : \phi \in \mathcal{Y}, \|\phi\|_{\mathcal{Y}} \leq 1\}. \quad (4.41)$$

Since $\|\phi\| \leq \|\phi\|_{\mathcal{Y}}$ for all $\phi \in \mathcal{Y}$, we have

$$\|T\|_{\mathcal{Y}_*} \leq \|T\|_{\infty}.$$

We claim that Equation (4.41) defines a norm on $\mathcal{B}(\mathcal{H})$. If $\|T\|_{\mathcal{Y}_*} = 0$, then $\phi(T) = 0$ for all $\phi \in \mathcal{Y}$ by Equation (4.41). Suppose to the contrary that $\|T\|_{\infty} \neq 0$. Thus, there is an $\eta \in \mathcal{H}$ such that $T\eta \neq 0$ and

$$\langle T\eta, T\eta \rangle = \|T\eta\|^2 > 0.$$

Since $\mathcal{M}\xi_{\tau}$ is dense in $\mathcal{H}$, by approximating η and $T\eta$, there exist $a, b \in \mathcal{M}$ such that

$$\langle Ta\xi_{\tau}, b\xi_{\tau} \rangle \neq 0. \quad (4.42)$$

Now, define

$$\begin{aligned} \phi : \mathcal{B}(\mathcal{H}) &\longrightarrow \mathbb{C}, \\ \phi(S) &:= \langle Sa\xi_{\tau}, b\xi_{\tau} \rangle = \tilde{\tau}(b^*Sa), \end{aligned}$$

where $\tilde{\tau}$ is the canonical extension of τ to $\mathcal{B}(\mathcal{H})$. Recall that $\tilde{\tau} \in \mathcal{Y}$ and that $\mathcal{Y}$ is an $\mathcal{M}$ -bimodule; hence, $\phi \in \mathcal{Y}$. However, Inequality (4.42) yields $\phi(T) \neq 0$, which contradicts the assumption $\|T\|_{\mathcal{Y}_*} = 0$. Therefore, $T = 0$.

Let Z denote the completion of $\mathcal{B}(\mathcal{H})$ with respect to $\|\cdot\|_{\mathcal{Y}_*}$. For each $\phi \in \mathcal{Y}$, define a linear functional on $\mathcal{B}(\mathcal{H})$ by

$$\Lambda_{\phi}(T) := \phi(T) \quad (T \in \mathcal{B}(\mathcal{H})).$$

By the definition of the norm $\|\cdot\|_{\mathcal{Y}_*}$, we have

$$|\Lambda_{\phi}(T)| = |\phi(T)| \leq \|\phi\|_{\mathcal{Y}} \|T\|_{\mathcal{Y}_*} \quad (T \in \mathcal{B}(\mathcal{H})).$$

So, Λ_ϕ is bounded on $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{\mathcal{Y}_*})$ with

$$\|\Lambda_\phi\|_{Z^*} \leq \|\phi\|_{\mathcal{Y}}. \quad (4.43)$$

Since $\mathcal{B}(\mathcal{H})$ is dense in Z , Λ_ϕ extends uniquely to a bounded linear functional (still denoted Λ_ϕ) on Z . On the other hand, by definition of the norm of Z^* ,

$$\|\Lambda_\phi\|_{Z^*} = \sup\{|\phi(T)| : \|T\|_{\mathcal{Y}_*} \leq 1\}. \quad (4.44)$$

Now, fix $\varepsilon > 0$. By equation

$$\|\phi\|_{\mathcal{Y}} = \sup\{|\phi(T)| : \|T\|_{\mathcal{Y}_*} \leq 1, T \in \mathcal{Y}_*\},$$

there exists $T \in \mathcal{B}(\mathcal{H})$ such that

$$|\phi(T)| > (\|\phi\|_{\mathcal{Y}} - \varepsilon) \|T\|_{\mathcal{Y}_*}.$$

For $S := \frac{T}{\|T\|_{\mathcal{Y}_*}}$, we have $\|S\|_{\mathcal{Y}_*} = 1$, and

$$|\phi(S)| > \|\phi\|_{\mathcal{Y}} - \varepsilon.$$

Hence, by Equation (4.44),

$$\|\Lambda_\phi\|_{Z^*} \geq |\phi(S)| > \|\phi\|_{\mathcal{Y}} - \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$\|\Lambda_\phi\|_{Z^*} \geq \|\phi\|_{\mathcal{Y}}. \quad (4.45)$$

Combining Inequalities (4.43) and (4.45), we obtain

$$\|\Lambda_\phi\|_{Z^*} = \|\phi\|_{\mathcal{Y}}.$$

Thus, the map

$$\begin{aligned} \mathcal{Y} &\longrightarrow Z^* \\ \phi &\longmapsto \Lambda_\phi \end{aligned} \quad (4.46)$$

is a linear isometric embedding. To show that it is onto, restrict each $F \in Z^*$ to $\mathcal{B}(\mathcal{H})$ and define $\phi := F|_{\mathcal{B}(\mathcal{H})}$. Then, for $T \in \mathcal{B}(\mathcal{H})$ and $a, b \in \mathcal{M}$, we have

$$|\phi(aTb)| = |F(aTb)| \leq \|F\|_{Z^*} \|aTb\|_{\mathcal{Y}_*}.$$

By Equation (4.41) for $\|\cdot\|_{\mathcal{Y}_*}$ and the defining Inequality (4.26) for $\mathcal{Y}$, it follows that

$$\|aTb\|_{\mathcal{Y}_*} = \sup\{|\psi(aTb)| : \psi \in \mathcal{Y}, \|\psi\|_{\mathcal{Y}} \leq 1\} \leq \|a\|_2 \|T\|_\infty \|b\|_2.$$

Hence,

$$|\phi(aTb)| \leq \|F\|_{Z^*} \|a\|_2 \|T\|_\infty \|b\|_2.$$

Therefore, $\phi \in \mathcal{Y}$ with

$$\|\phi\|_{\mathcal{Y}} \leq \|F\|_{Z^*}.$$

Moreover, for $T \in \mathcal{B}(\mathcal{H})$,

$$|F(T)| = |\phi(T)| \leq \|\phi\|_{\mathcal{Y}} \|T\|_{\mathcal{Y}_*}.$$

So,

$$\|F\|_{Z^*} \leq \|\phi\|_{\mathcal{Y}}.$$

Combining the last inequalities, we obtain

$$\|F\|_{Z^*} = \|\phi\|_{\mathcal{Y}}.$$

Consequently, (4.46) is an isometric bijection from $\mathcal{Y}$ onto Z^* . $\square$

Proposition 4.3.19. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and let $\mathcal{Y} = (\mathcal{Y}_*)^*$ and $\mathcal{X} = \mathcal{M}^\perp$ be defined as in Theorem 4.3.17. Then, the followings hold:*

1. $\mathcal{X}$ is weak*-closed in $\mathcal{Y}$, and hence

$$\mathcal{X} \cong (\mathcal{Y}_*/\mathcal{X}_\perp)^*,$$

where $\mathcal{X}_\perp = \{T \in \mathcal{Y}_* : \phi(T) = 0 \ \forall \phi \in \mathcal{X}\}$.

2. Let $q : \mathcal{Y}_* \rightarrow \mathcal{Y}_*/\mathcal{X}_\perp$ be the quotient map. For each $\phi \in \mathcal{X} \subseteq \mathcal{Y} = (\mathcal{Y}_*)^*$, define $\tilde{\phi} \in (\mathcal{Y}_*/\mathcal{X}_\perp)^*$ by

$$\langle \tilde{\phi}, q(T) \rangle := \langle \phi, T \rangle, \quad T \in \mathcal{Y}_*.$$

Then $\phi \mapsto \tilde{\phi}$ is the canonical isometric identification

$$\mathcal{X} \cong (\mathcal{Y}_*/\mathcal{X}_\perp)^*,$$

and the adjoint map q^* satisfies

$$\langle q^* \tilde{\phi}, T \rangle = \langle \phi, T \rangle, \quad T \in \mathcal{Y}_*.$$

3. The canonical image $q(\mathcal{B}(\mathcal{H}))$ is norm dense in $\mathcal{X}_*$. Consequently, if a sequence (ϕ_k) in $\mathcal{X}$ converges to ϕ in the weak* topology $\sigma(\mathcal{X}, \mathcal{X}_*)$, then

$$\phi_k(T) \rightarrow \phi(T) \quad \forall T \in \mathcal{B}(\mathcal{H}),$$

that is,

$$\phi_k \xrightarrow{\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))} \phi.$$

Proof. (1) For each $m \in \mathcal{M}$, the evaluation map

$$\text{ev}_m : \mathcal{Y} \rightarrow \mathbb{C}, \quad \text{ev}_m(\phi) = \phi(m),$$

is weak*-continuous on $\mathcal{Y}$ since $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \subseteq \mathcal{Y}_*$. Hence,

$$\mathcal{X} = \mathcal{M}^\perp = \bigcap_{m \in \mathcal{M}} \text{Ker}(\text{ev}_m)$$

is weak*-closed in $\mathcal{Y}$. Therefore, by [62, Corollary 2.4.11],

$$\mathcal{X} \cong (\mathcal{Y}_*/\mathcal{X}_\perp)^*.$$

(2) Let

$$q : \mathcal{Y}_* \rightarrow \mathcal{Y}_*/\mathcal{X}_\perp$$

be the quotient map. For $\phi \in \mathcal{X}$, let

$$\tilde{\phi}(q(T)) := \phi(T) \quad (T \in \mathcal{Y}_*).$$

We first check that $\tilde{\phi}$ is well defined. If $q(T_1) = q(T_2)$, then

$$T_1 - T_2 \in \mathcal{X}_\perp.$$

By definition of $\mathcal{X}_\perp$, this means

$$\psi(T_1 - T_2) = 0 \quad \forall \psi \in \mathcal{X}.$$

In particular, for $\psi = \phi$,

$$\phi(T_1) = \phi(T_2).$$

So $\tilde{\phi}$ is well defined. It is clearly linear. Moreover, by definition of the quotient norm, there exists $S \in \mathcal{X}_\perp$ such that

$$\|T + S\|_{\mathcal{Y}_*} \leq \|q(T)\| + \varepsilon.$$

Since $\phi(S) = 0$, we have

$$|\tilde{\phi}(q(T))| = |\phi(T)| = |\phi(T + S)| \leq \|\phi\| \|T + S\|_{\mathcal{Y}_*} \leq (\|q(T)\| + \varepsilon) \|\phi\|.$$

Letting $\varepsilon \downarrow 0$, we get

$$|\tilde{\phi}(q(T))| \leq \|\phi\| \|q(T)\|.$$

So,

$$\|\tilde{\phi}\| \leq \|\phi\|.$$

Conversely, for $T \in \mathcal{Y}_*$,

$$|\phi(T)| = |\tilde{\phi}(q(T))| \leq \|\tilde{\phi}\| \|q(T)\| \leq \|\tilde{\phi}\| \|T\|_{\mathcal{Y}_*}.$$

Taking the supremum over $\|T\|_{\mathcal{Y}_*} \leq 1$ gives

$$\|\phi\| \leq \|\tilde{\phi}\|.$$

Hence

$$\|\tilde{\phi}\| = \|\phi\|.$$

Thus, $\phi \mapsto \tilde{\phi}$ is an isometric embedding of $\mathcal{X}$ into $(\mathcal{Y}_*/\mathcal{X}_\perp)^*$. By part (1), this is precisely the canonical identification

$$\mathcal{X} \cong (\mathcal{Y}_*/\mathcal{X}_\perp)^*.$$

Now, let

$$q^* : (\mathcal{Y}_*/\mathcal{X}_\perp)^* \rightarrow \mathcal{Y}$$

be the adjoint map. By definition,

$$(q^*\tilde{\phi})(T) = \tilde{\phi}(q(T)) \quad (T \in \mathcal{Y}_*).$$

Using the definition of $\widetilde{\phi}$, we deduce

$$(q^*\widetilde{\phi})(T) = \phi(T) \quad (T \in \mathcal{Y}_*).$$

Hence,

$$q^*\widetilde{\phi} = \phi$$

as elements of $\mathcal{Y} = (\mathcal{Y}_*)^*$.

(3) By Proposition 4.3.18, $\mathcal{B}(\mathcal{H})$ is norm dense in $\mathcal{Y}_*$. Since the quotient map

$$q : \mathcal{Y}_* \rightarrow \mathcal{Y}_*/\mathcal{X}_\perp \cong \mathcal{X}_*$$

is continuous and surjective, the image $q(\mathcal{B}(\mathcal{H}))$ is norm dense in $\mathcal{X}_*$.

Now, suppose that $(\phi_k) \subseteq \mathcal{X}$ converges to ϕ in the weak* topology $\sigma(\mathcal{X}, \mathcal{X}_*)$. Fix $T \in \mathcal{B}(\mathcal{H})$. Then, $q(T) \in \mathcal{X}_*$, so by weak* convergence in $\mathcal{X}$,

$$\langle \phi_k, q(T) \rangle \rightarrow \langle \phi, q(T) \rangle.$$

Using part (2), we have

$$\langle \phi_k, q(T) \rangle = \langle \widetilde{\phi_k}, q(T) \rangle = \langle \phi_k, T \rangle, \quad \langle \phi, q(T) \rangle = \langle \widetilde{\phi}, q(T) \rangle = \langle \phi, T \rangle.$$

Therefore,

$$\phi_k(T) \rightarrow \phi(T) \quad \forall T \in \mathcal{B}(\mathcal{H}).$$

This is exactly the statement that

$$\phi_k \xrightarrow{\sigma(\mathcal{B}(\mathcal{H})^*, \mathcal{B}(\mathcal{H}))} \phi.$$

□

Remark 4.3.20. Suppose that $\mathcal{M}$ is as in Theorem 4.3.17. In addition, assume that $\mathcal{M}$ is a factor. Then, in the above proof we have $f_0 = 1$. So, our algorithm ends at the very beginning, and θ_0 will be our ultra trace, which is a faithful normal tracial state as well. This situation can happen for many group von Neumann algebras, which we will explore in the following.

Corollary 4.3.21. *Let G be a discrete group. The group von Neumann algebra $\mathcal{L}(G) = VN(G)$ is sequentially Connes amenable if and only if G is amenable.*

Proof. Suppose that $\lambda : G \rightarrow U(\mathcal{B}(\ell^2(G)))$ is the left regular representation, and $\delta_g \in \ell^2(G)$ is the characteristic function of the member $g \in G$. The left group von Neumann algebra $\mathcal{L}(G) = VN(G) = \lambda(G)''$ is a finite von Neumann algebra with canonical faithful normal tracial state $\tau(x) = \langle x\delta_e, \delta_e \rangle$, see [3, Section III.3.3] and [17, Section VII.7]. If $\mathcal{L}(G)$ is sequentially Connes amenable, then by Theorem 4.3.17 and Remark 4.3.20, $\mathcal{L}(G)$ must be Connes amenable. In this case, by [76, Example 5.1.26] and [76, Theorem 5.1.27], G must be amenable. Conversely, when G is amenable, $\mathcal{L}(G)$ is Connes amenable by [76, Theorem 5.1.27]. Hence, it is sequentially Connes amenable. □

As a consequence, we immediately derive the following.

Corollary 4.3.22. *Let $\mathbb{F}_n$ be the free group on n generators with $n \geq 2$. Then, the group von Neumann algebra $\mathcal{L}(\mathbb{F}_n) = VN(\mathbb{F}_n)$ cannot be sequentially Connes amenable.*

Chapter 5

Generalized amenability of type II_∞ and type III von Neumann algebras

In this chapter, we study Connes sequential amenability for type II_∞ and type III von Neumann algebras. Using normal tensor products and Hilbert weak*-module techniques, we reduce Connes sequential amenability for a type II_∞ von Neumann algebra to that of its finite corners. Our method is based on lifting normal derivations from finite corners to the original algebra and then using a drop-down argument to transfer sequential innerness back to the corners. This reduction allows us to apply the results of Chapter 4 to type II_∞ von Neumann algebras. In Section 5.3, we use the structural connection between type III von Neumann algebras and type II_∞ algebras to study Connes sequential amenability in the type III case. Combining these results, we may conclude that, for von Neumann algebras, Connes sequential amenability is equivalent to Connes amenability. Finally, we obtain that for C^* -algebras, Johnson sequential amenability is equivalent to Johnson amenability.

5.1 Parseval unit and Parseval representative

In this section, we introduce the tensor-product framework needed for the lifting constructions in the sequel. We begin by recalling the normal spatial tensor product of dual Banach spaces and its balanced module version. These constructions will be used, together with Hilbert C^* -modules and Hilbert W^* -modules, to form the normal module tensor products that are crucial in the next section for lifting modules and derivations. To keep the present section focused on the main thread, we leave the detailed construction to Appendix B.

The following definition is adapted from [4, Section 1.6.5] and [81, Section IV.5].

Definition 5.1.1. Let U and V be dual Banach spaces with fixed pre-duals U_* and V_* . Recall that by the canonical isometric identification (2.6),

$$\mathcal{B}(V_*, U) \cong (V_* \otimes_\gamma U_*)^*. \quad (5.1)$$

So, $\mathcal{B}(V_*, U)$ is a dual Banach space. We embed the algebraic tensor product $U \otimes V$ spatially into

$\mathcal{B}(V_*, U)$ by

$$\begin{aligned}\Phi : U \otimes V &\longrightarrow \mathcal{B}(V_*, U), \\ u \otimes v &\longmapsto \Phi(u \otimes v), \\ \Phi(u \otimes v)(\psi) &= \psi(v)u, \quad (\psi \in V_*).\end{aligned}\tag{5.2}$$

The *normal spatial tensor product* of U and V , denoted by $U \bar{\otimes} V$, is defined by

$$U \bar{\otimes} V := \overline{\Phi(U \otimes V)}^{w^*} \subseteq \mathcal{B}(V_*, U).$$

Since $U \bar{\otimes} V$ is a weak* closed subspace of $\mathcal{B}(V_*, U)$, it is itself a dual Banach space by [62, Corollary 2.4.11]. Moreover, by [13, Theorem V.2.3], its canonical pre-dual is

$$(U \bar{\otimes} V)_* = (V_* \otimes_\gamma U_*) / (U \bar{\otimes} V)_\perp,$$

where

$$(U \bar{\otimes} V)_\perp = \{\rho \in V_* \otimes_\gamma U_* : \langle T, \rho \rangle = 0 \text{ for all } T \in U \bar{\otimes} V\}.$$

Remark 5.1.2. By (5.2) and (5.1), we have *the canonical pairing realization*

$$\begin{aligned}U \bar{\otimes} V &\longrightarrow (V_* \otimes_\gamma U_*)^* \cong \mathcal{B}(V_*, U), \\ \eta &\longmapsto \langle \eta, \cdot \rangle, \\ \langle \eta, \psi \otimes \phi \rangle &= \langle \eta(\psi), \phi \rangle, \quad (\psi \in V_*, \phi \in U_*).\end{aligned}$$

We shall also use dual-pairing notation for bilinear forms when convenient. Thus, if $B : X \times Y \rightarrow \mathbb{C}$ is a bilinear form, we write

$$\langle B, (x, y) \rangle := B(x, y).$$

This is only a notational convention, used to keep pairing computations consistent.

The next definition is the normal weak* analogue of the usual balanced module tensor product; compare with [4, Sections 3.4.1–3.4.2]. For the terminology concerning one-sided normal modules, see Definition 2.2.4.

Definition 5.1.3. Let $\mathcal{V}$ be a dual Banach algebra, X a normal right $\mathcal{V}$ -module, and Y a normal left $\mathcal{V}$ -module. Consider the normal spatial tensor product $X \bar{\otimes} Y$. Let

$$J := \overline{\text{span}}^{w^*} \{x \cdot v \otimes y - x \otimes v \cdot y : x \in X, y \in Y, v \in \mathcal{V}\}.$$

The normal spatial $\mathcal{V}$ -balanced tensor product is defined by

$$X \bar{\otimes}_{\mathcal{V}} Y := (X \bar{\otimes} Y) / J.$$

Since J is weak* closed in the dual space $X \bar{\otimes} Y$, we have

$$J = (J_\perp)^\perp;$$

where

$$J_\perp = \{\varphi \in (X \bar{\otimes} Y)_* : \varphi(j) = 0 \text{ for all } j \in J\}.$$

Hence, by [13, Corollary V.1.9 and Theorem V.2.3], the quotient $X \bar{\otimes}_{\mathcal{V}} Y$ is a dual Banach space with canonical pre-dual $J_\perp$. Equivalently,

$$X \bar{\otimes}_{\mathcal{V}} Y \cong (J_\perp)^*.$$

Remark 5.1.4. A bilinear form $F : X \times Y \rightarrow \mathbb{C}$ is called $\mathcal{V}$ -balanced if

$$F(x \cdot v, y) = F(x, v \cdot y) \quad (x \in X, y \in Y, v \in \mathcal{V}).$$

We also note that the definitions of the normal spatial tensor product and of the normal balanced tensor product have obvious multilinear analogues. Namely, if $X_1, \dots, X_n$ are dual Banach spaces with fixed preduals, then the n -fold normal spatial tensor product

$$X_1 \bar{\otimes} \dots \bar{\otimes} X_n$$

is obtained by iterating Definition 5.1.1, or equivalently by taking the weak* closure of the algebraic tensor product $X_1 \otimes \dots \otimes X_n$ in the corresponding spatial operator realization. Its predual will be denoted by

$$(X_1 \bar{\otimes} \dots \bar{\otimes} X_n)_*.$$

Similarly, let $\mathcal{V}_1, \dots, \mathcal{V}_{n-1}$ be dual Banach algebras. Suppose that, for each $1 \leq i \leq n-1$, the space X_i is a normal right $\mathcal{V}_i$ -module and X_{i+1} is a normal left $\mathcal{V}_i$ -module. The normal balanced tensor product

$$X_1 \bar{\otimes}_{\mathcal{V}_1} X_2 \bar{\otimes}_{\mathcal{V}_2} \dots \bar{\otimes}_{\mathcal{V}_{n-1}} X_n$$

is defined as the quotient of

$$X_1 \bar{\otimes} \dots \bar{\otimes} X_n$$

by the weak* closed subspace generated by all balancing differences

$$\begin{aligned} & x_1 \otimes \dots \otimes x_i \cdot v \otimes x_{i+1} \otimes \dots \otimes x_n \\ & - x_1 \otimes \dots \otimes x_i \otimes v \cdot x_{i+1} \otimes \dots \otimes x_n. \end{aligned} \tag{5.3}$$

where $x_j \in X_j$, $v \in \mathcal{V}_i$, and $1 \leq i \leq n-1$. So, an n -linear form

$$F : X_1 \times \dots \times X_n \longrightarrow \mathbb{C}$$

is balanced over $\mathcal{V}_1, \dots, \mathcal{V}_{n-1}$ if, for each $1 \leq i \leq n-1$,

$$F(x_1, \dots, x_i \cdot v, x_{i+1}, \dots, x_n) = F(x_1, \dots, x_i, v \cdot x_{i+1}, \dots, x_n)$$

for all $x_j \in X_j$ and $v \in \mathcal{V}_i$. We denote by J the weak* closed balancing subspace generated by (5.3). Thus,

$$X_1 \bar{\otimes}_{\mathcal{V}_1} X_2 \bar{\otimes}_{\mathcal{V}_2} \dots \bar{\otimes}_{\mathcal{V}_{n-1}} X_n := (X_1 \bar{\otimes} \dots \bar{\otimes} X_n) / J.$$

As in Definition 5.1.3, this quotient is a dual Banach space with the canonical predual

$$J_\perp = \{\varphi \in (X_1 \bar{\otimes} \dots \bar{\otimes} X_n)_* : \varphi(j) = 0 \text{ for all } j \in J\}.$$

All the factorization and functoriality statements in this section have the corresponding n -fold versions.

Remark 5.1.5. Let

$$\begin{aligned} q : X \bar{\otimes} Y &\longrightarrow X \bar{\otimes}_{\mathcal{V}} Y, \\ \eta &\longmapsto \eta + J \end{aligned}$$

be the quotient map. In particular, for elementary tensors,

$$q(x \otimes y) = x \otimes y + J = x \otimes_{\mathcal{Y}} y.$$

This mapping is weak*-weak* continuous. Now, suppose that Z is a dual Banach space and $L : X \bar{\otimes} Y \longrightarrow Z$ is weak*-weak* continuous with $J \subseteq \ker L$, then there is a unique weak*-weak* continuous linear map $\tilde{L} : X \bar{\otimes}_{\mathcal{Y}} Y \longrightarrow Z$ such that the following diagram commutes:

$$\begin{array}{ccc} X \bar{\otimes} Y & \xrightarrow{L} & Z \\ \downarrow q & \nearrow \tilde{L} & \\ X \bar{\otimes}_{\mathcal{Y}} Y & & \end{array} \quad L = \tilde{L} \circ q$$

Figure 5.1: Quotient factorization through the normal balanced tensor product.

Definition 5.1.6. Let $X_1, \dots, X_n, Z$ be dual Banach spaces with fixed preduals. A bounded n -linear map

$$B : X_1 \times \dots \times X_n \longrightarrow Z$$

is called *normally linearizable* if there exists a bounded linear map

$$B_s : Z_* \longrightarrow (X_1 \bar{\otimes} \dots \bar{\otimes} X_n)_*$$

such that

$$\langle B_s(\omega), x_1 \otimes \dots \otimes x_n \rangle = \langle \omega, B(x_1, \dots, x_n) \rangle$$

for all $\omega \in Z_*$ and $x_i \in X_i$, $1 \leq i \leq n$.

Let $\mathcal{M}$ be a semi-finite von Neumann algebra. In this section, we assume that $\mathcal{M}$ is of type II_∞ . Thus, $\mathcal{M}$ has a faithful normal semi-finite tracial weight

$$\tau : \mathcal{M}_+ \longrightarrow [0, \infty].$$

By [81, Definition V.2.1 and Proposition V.1.40], there is an increasing net (p_α) of finite non-zero projections such that $\tau(p_\alpha) < \infty$ for each α and $p_\alpha \uparrow 1$ in the strong operator topology. Take a nonzero projection p from this net, and let

$$\mathcal{N} = p\mathcal{M}p.$$

This is a finite von Neumann algebra with unit p , called the *corner von Neumann algebra associated with p* . Moreover, $\mathcal{N}$ has a faithful normal tracial state τ_p given by

$$\tau_p(pmp) = \frac{\tau(pmp)}{\tau(p)},$$

and $\mathcal{N}$ is of type II_1 .

When $\mathcal{M}$ is Connes amenable, there is a conditional expectation

$$\mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{M}.$$

Composing this with the compression

$$\begin{aligned} \mathcal{C}_p : \mathcal{M} &\longrightarrow \mathcal{N}, \\ m &\longmapsto pmp \end{aligned}$$

yields a conditional expectation

$$E_p = \mathcal{C}_p \circ \mathcal{E} : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{N}.$$

Consequently, the corner algebra $p\mathcal{M}p$ is Connes amenable. When $\mathcal{M}$ is sequentially Connes amenable, such a conditional expectation argument is not directly available. Therefore, we develop a lifting construction to prove that Connes sequential amenability of $\mathcal{M}$ descends to the corner $\mathcal{N} = p\mathcal{M}p$. Throughout this section and the next, we assume that

- p is a nonzero finite projection in $\mathcal{M}$;
- $\mathcal{N} = p\mathcal{M}p$ is a type II_1 von Neumann algebra with a faithful normal tracial state τ_p ;
- $X = Y^*$ is a dual Banach space which is a normal $\mathcal{N}$ -bimodule;
- $d : \mathcal{N} \longrightarrow X$ is a normal derivation;
- p acts as the identity on X , i.e.

$$p \cdot \xi = \xi = \xi \cdot p \quad (\xi \in X).$$

Remark 5.1.7 (Roadmap of the lifting argument). We first show that $E = \mathcal{M}p$ is a right Hilbert W^* -module over $\mathcal{N} = p\mathcal{M}p$. Also, its adjoint set, $E^* = p\mathcal{M}$ is a left Hilbert W^* -module over $\mathcal{N} = p\mathcal{M}p$. In the context of Hilbert modules, when E is a one-sided ideal of the von Neumann algebra $\mathcal{M}$, the notation E^* denotes the set of adjoints of all elements of E . Next, we construct a Parseval frame $(u_i)_{i \in I}$ for E . This frame allows us to associate to each $m \in \mathcal{M}$ an unbalanced tensor

$$\tilde{\Omega}_m \in E \bar{\otimes} \mathcal{N} \bar{\otimes} E^*.$$

Given a normal derivation $d : \mathcal{N} \rightarrow X$, we define its *lift* by applying d to the middle coefficient of $\tilde{\Omega}_m$ to construct

$$\begin{aligned} D : \mathcal{M} &\longrightarrow \tilde{X} := E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*, \\ D(m) &= q_{\mathcal{N}} \left((\text{id}_E \bar{\otimes} d \bar{\otimes} \text{id}_{E^*})(\tilde{\Omega}_m) \right). \end{aligned}$$

Here, $\tilde{X}$ is an $\mathcal{M}$ -module obtained by *lifting* the $\mathcal{N}$ -module X , and

$$q_{\mathcal{N}} : E \bar{\otimes} \mathcal{N} \bar{\otimes} E^* \longrightarrow E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^*$$

is the canonical quotient map. We prove that D is a normal derivation. Next, we define a *drop-down map*

$$\varepsilon : \tilde{X} \rightarrow X$$

which satisfies

$$\varepsilon(D(a)) = d(a), \quad a \in \mathcal{N}.$$

Consequently, innerness of the lifted derivation D on $\mathcal{M}$ descends to innerness of d on $\mathcal{N}$, giving the desired inheritance result for sequential Connes amenability.

Remark 5.1.8. Let

$$E = \mathcal{M}p.$$

Then, E is a weak* closed left ideal in $\mathcal{M}$ and an $\mathcal{M}$ - $\mathcal{N}$ module with the module action given by multiplication:

$$m' \cdot (mp) \cdot n = m' m p n \quad (m', m \in \mathcal{M}, n \in \mathcal{N}).$$

Moreover, E is a right Hilbert C^* -module over $\mathcal{N}$ with inner product

$$\langle mp, np \rangle_{\mathcal{N}} := (mp)^* n p = p m^* n p \in \mathcal{N}.$$

The adjoint module is

$$E^* = p\mathcal{M},$$

which is an $\mathcal{N}$ - $\mathcal{M}$ module with module action

$$n \cdot (pm) \cdot m' = n p m m' \quad (m, m' \in \mathcal{M}, n \in \mathcal{N}).$$

E^* is a left Hilbert C^* -module over the von Neumann algebra $\mathcal{N}$, with the $\mathcal{N}$ -valued inner product

$$\langle pm, pn \rangle_{\mathcal{N}} := pm (pn)^* = pm n^* p \in \mathcal{N}.$$

We also have

$$\mathcal{N} = p\mathcal{M}p \subseteq \mathcal{M}p = E, \quad \mathcal{N} = p\mathcal{M}p \subseteq p\mathcal{M} = E^*, \quad \mathcal{N} = E \cap E^*.$$

In addition, we notice that the Hilbert C^* -module norm induced by Equation (2.10) is the same as the norm on E inherited from $\mathcal{M}$.

Below, we show that $E = \mathcal{M}p$ is a right Hilbert W^* -module over $\mathcal{N} = p\mathcal{M}p$.

Proposition 5.1.9. *The right $\mathcal{N}$ -module $E = \mathcal{M}p$ is a right Hilbert W^* -module over $\mathcal{N}$. Similarly, $E^* = p\mathcal{M}$ is a left Hilbert W^* -module over $\mathcal{N}$.*

Proof. We only prove the result for the right module case. The left case is similar. By [4, Lemma 8.5.4-(1)], we need to show that E has a Banach pre-dual E_* with respect to which the inner product $\langle \cdot, \cdot \rangle_{\mathcal{N}}$ is weak*-separately continuous. Note that $E = \mathcal{M}p$ is a weak* closed left ideal of $\mathcal{M}$. By [81, Theorem. III.2.7], its pre-annihilator $E_{\perp} = \{\omega \in \mathcal{M}_* : \omega|_E = 0\}$ is isometrically isomorphic to $(1-p) \cdot \mathcal{M}_*$. So, by [62, Corollary 2.4.11] $E_* \cong \frac{\mathcal{M}_*}{(1-p) \cdot \mathcal{M}_*}$ is a predual of E . Consider the contractive transformation

$$\begin{aligned} P : \mathcal{M}_* &\longrightarrow \mathcal{M}_* \\ \omega &\longrightarrow p \cdot \omega. \end{aligned}$$

Note that $\ker(P) = (1 - p) \cdot \mathcal{M}_*$ and $\text{ran}(P) = p \cdot \mathcal{M}_*$. Hence, we have the surjective isometry

$$\begin{aligned} P : E_* &\longrightarrow p \cdot \mathcal{M}_* \\ \omega + (1 - p) \cdot \mathcal{M}_* &\longrightarrow p \cdot \omega. \end{aligned}$$

Thus, weak* convergence in E is equivalent to pointwise convergence on all functionals of the form $p \cdot \omega \in p \cdot \mathcal{M}_* \subseteq \mathcal{M}_*$ for $\omega \in \mathcal{M}_*$. On the other hand, the multiplication is weak* separately continuous on $\mathcal{M}$ (including $\mathcal{N}$). Our choice of inner product was given by the multiplication of the von Neumann algebra $\mathcal{M}$. So, the inner product is weak*-separately continuous. Hence, by [4, Lemma 8.5.4-(1)] E is a right Hilbert W^* -module over $\mathcal{N}$. $\square$

Proposition 5.1.10. *If $\phi : E = \mathcal{M}p \rightarrow \mathcal{N}$ is bounded and right $\mathcal{N}$ -linear, then there exists a unique $u \in E$ such that*

$$\phi(y) = \langle u, y \rangle_{\mathcal{N}} \quad (y \in E).$$

In particular, E is a self-dual right Hilbert W^* -module over $\mathcal{N}$, and ϕ is weak-weak* continuous.*

Proof. Let $\mathcal{K}_{\mathcal{N}}(E)$ be the algebra of all compact operators on E . By [4, Remark 8.1.23] there is a contractive approximate identity (e_t) in $\mathcal{K}_{\mathcal{N}}(E)$ of the form

$$e_t = \sum_{k=1}^{n(t)} \theta_{x_k^t, x_k^t}, \quad \theta_{x,z}(y) = x \langle z, y \rangle_{\mathcal{N}}, \quad x_k^t \in E, \quad n(t) \in \mathbb{N},$$

such that $\|e_t\| \leq 1$ and $e_t(y) \rightarrow y$ in norm for every $y \in E$. For each t , set

$$u_t := \sum_{k=1}^{n(t)} x_k^t \phi(x_k^t)^* \in E.$$

We claim that

$$\|u_t\| \leq \|\phi\| \quad \forall t.$$

By Equation (2.10) for Hilbert C^* -modules, and the way the inner product is defined, we have

$$\|z\| = \sup \{ \|\langle z, y \rangle_{\mathcal{N}}\| : \|y\| \leq 1, y \in E \} \quad \forall z \in E. \quad (5.4)$$

Now, for each $y \in E$ with $\|y\| \leq 1$, we have

$$\phi(e_t(y)) = \sum_{k=1}^{n(t)} \phi(x_k^t) \langle x_k^t, y \rangle_{\mathcal{N}} = \left\langle \sum_{k=1}^{n(t)} x_k^t \phi(x_k^t)^*, y \right\rangle_{\mathcal{N}} = \langle u_t, y \rangle_{\mathcal{N}}. \quad (5.5)$$

Hence,

$$\|\langle u_t, y \rangle_{\mathcal{N}}\| = \|\phi(e_t y)\| \leq \|\phi\| \|e_t y\| \leq \|\phi\| \|e_t\| \|y\| \leq \|\phi\|.$$

Taking the supremum over all $\|y\| \leq 1$ and applying Equation (5.4) yield $\|u_t\| \leq \|\phi\|$. So, the net (u_t) is bounded in E . As E is a dual Banach space, the net (u_t) has a weak* limit point. Thus, by passing to a sub-net if necessary, we can assume

$$u_t \xrightarrow{w^*} u \in E.$$

Since $e_t(y) \rightarrow y$ in norm and ϕ is bounded, by Equation (5.5) we deduce that

$$\|\langle u_t, y \rangle_{\mathcal{N}} - \phi(y)\| = \|\phi(e_t(y)) - \phi(y)\| \leq \|\phi\| \|e_t(y) - y\| \rightarrow 0 \quad (\forall y \in E). \quad (5.6)$$

Recall that for a fixed $y \in E$, the map $E \rightarrow \mathcal{N}$, $s \mapsto \langle s, y \rangle_{\mathcal{N}} = s^*y$, is weak*-continuous as the multiplication in the von Neumann algebra $\mathcal{M}$ is separately weak*-continuous and $E, \mathcal{N}$ are weak*-closed subspaces. Hence,

$$\langle u_t, y \rangle_{\mathcal{N}} \xrightarrow{w^*} \langle u, y \rangle_{\mathcal{N}} = \phi(y).$$

By Remark 2.6.6, u is unique. □

Theorem 5.1.11 (A Parseval frame for $E = \mathcal{M}p$). *Let*

$$z := \text{cs}(p) = \min\{c \in \mathcal{ZP}(\mathcal{M}) : p \leq c\}$$

be the central support projection of p in $\mathcal{M}$. Then, there exist an index set I , a distinguished element $0 \in I$, and a family $(u_i)_{i \in I}$ of partial isometries in E such that

$$u_0 = p, \quad u_i^* u_i \leq p \quad (i \in I),$$

the range projections $u_i u_i^*$ are pairwise orthogonal, and

$$\text{sot-} \sum_{i \in I} u_i u_i^* = z,$$

where the sum is understood as the strong operator limit over finite subsets of I . Moreover, for every $e \in E$, we have the normal Parseval identity

$$e = w^* \text{-} \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} u_i \langle u_i, e \rangle_{\mathcal{N}}, \quad (5.7)$$

where $\mathcal{F}(I)$ is the directed set of finite subsets of I , ordered by inclusion.

Proof. Since $z \in \mathcal{Z}(\mathcal{M})$ is a central projection, the corner

$$z\mathcal{M} = z\mathcal{M}z$$

is a von Neumann algebra with unit z . Moreover, it is a weak*-closed two-sided ideal of $\mathcal{M}$ containing p . By Zorn's lemma, there is a maximal family $\mathcal{Q}$ of nonzero projections in $z\mathcal{M}$ such that

$$p \in \mathcal{Q}, \quad q \preceq p \quad (q \in \mathcal{Q}),$$

and such that distinct elements of $\mathcal{Q}$ are orthogonal, i.e.,

$$q \perp r \quad (q, r \in \mathcal{Q}, q \neq r).$$

Enumerate members of $\mathcal{Q}$ by $(q_i)_{i \in I}$, assuming p is of index $0 \in I$; i.e.,

$$q_0 = p.$$

We claim that

$$\bigvee_{i \in I} q_i = z,$$

where $\bigvee$ is defined as in Definition 2.3.4. Let

$$q = \bigvee_{i \in I} q_i.$$

Suppose, toward a contradiction, that $q < z$. Then, $0 \neq z - q \leq z = \text{cs}(p)$, and consequently

$$0 \neq \text{cs}(z - q) \leq \text{cs}(p).$$

Hence,

$$\text{cs}(z - q)\text{cs}(p) = \text{cs}(z - q) \neq 0.$$

It follows from [81, Lemma V.1.7] that there exist nonzero subprojections $q' \leq z - q$ and $p' \leq p$ such that

$$q' \sim p'.$$

So, $q' \lesssim p$ and $q' \perp q_i$ for all $i \in I$, contradicting the maximality of $(q_i)_{i \in I}$. Therefore,

$$\bigvee_{i \in I} q_i = z.$$

Since the projections $(q_i)_{i \in I}$ are pairwise orthogonal, the net of finite partial sums converges strongly to their supremum. That is,

$$\text{sot-}\sum_{i \in I} q_i = z. \quad (5.8)$$

For each $i \in I$, we have $q_i \lesssim p$. So, by Definition 2.3.4, we can choose a partial isometry $u_i \in z\mathcal{M} \subseteq \mathcal{M}$ such that

$$u_i u_i^* = q_i, \quad u_i^* u_i \leq p.$$

Here, we remark that a partial isometry in $z\mathcal{M}$ is automatically a partial isometry in $\mathcal{M}$. For $i = 0$, we take $u_0 = p$. By construction of $\mathcal{Q}$, the range projections $(u_i u_i^*)_{i \in I}$ are pairwise orthogonal. For the partial isometry u_i , recall that

$$u_i = u_i u_i^* u_i.$$

From $u_i^* u_i \leq p$, it follows that

$$(u_i^* u_i)p = u_i^* u_i.$$

Therefore,

$$u_i p = u_i u_i^* u_i p = u_i [(u_i^* u_i)p] = u_i [u_i^* u_i] = u_i.$$

Hence, $u_i = u_i p \in \mathcal{M}p = E$. Moreover, since q_i is a projection in $z\mathcal{M}$, we have

$$u_i u_i^* = q_i \leq z.$$

Thus,

$$zu_i = (z)(u_i u_i^* u_i) = [(z)(u_i u_i^*)]u_i = [u_i u_i^*]u_i = u_i.$$

Taking adjoints gives

$$u_i^* z = u_i^*.$$

Therefore,

$$zu_i = u_i, \quad u_i^* z = u_i^* \quad (i \in I). \quad (5.9)$$

Moreover, every $e \in E = \mathcal{M}p$ can be written as $e = m_0 p$ for some $m_0 \in \mathcal{M}$. Since $p \leq z = \text{cs}(p)$, we have $zp = p$. As z is central, we deduce that

$$ze = zm_0 p = m_0 zp = m_0 p = e. \quad (5.10)$$

In other words, z acts as the identity on E from the left. Similarly, z also acts as the identity on E^* from the right; i.e.,

$$fz = f, \quad (f \in E^*). \quad (5.11)$$

On the other hand,

$$\text{sot-} \sum_{i \in I} u_i u_i^* = \text{sot-} \sum_{i \in I} q_i = z. \quad (5.12)$$

Hence, for all $e \in E$

$$\text{sot-} \sum_{i \in I} u_i \langle u_i, e \rangle_{\mathcal{N}} = \text{sot-} \sum_{i \in I} u_i u_i^* e = ze = e.$$

Thus,

$$e = \text{w}^* \text{-} \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} u_i \langle u_i, e \rangle_{\mathcal{N}}, \quad (e \in E).$$

□

Remark 5.1.12. If the maximal family obtained in Theorem 5.1.11 is the singleton $\mathcal{Q} = \{p\}$, then $p = \text{cs}(p)$, so p is central. In this case, the corner $\mathcal{N} = p\mathcal{M}p$ is a central summand of $\mathcal{M}$, and the compression map

$$\begin{aligned} \mathcal{C}_p : \mathcal{M} &\longrightarrow \mathcal{N}, \\ m &\longmapsto pmp \end{aligned}$$

is an onto normal $*$ -homomorphism. If $\mathcal{M}$ is sequentially Connes amenable, then so is $\mathcal{N}$ by Proposition 4.2.3. Hence, in our lifting argument, we may focus on the nontrivial case $p < \text{cs}(p)$.

Remark 5.1.13. Let I be the index set in Theorem 5.1.11. Throughout the sequel, sums indexed by I are always understood as the limits of nets over the directed set

$$\mathcal{F}(I) = \{F \subseteq I : F \text{ is finite}\},$$

ordered by inclusion. We note that for every finite subset $K \subseteq I$, the subcollection

$$\mathcal{F}_K(I) := \{F \in \mathcal{F}(I) : K \subseteq F\}$$

is cofinal in $\mathcal{F}(I)$. In particular,

$$\mathcal{F}_0(I) := \{F \in \mathcal{F}(I) : 0 \in F\}$$

is cofinal in $\mathcal{F}(I)$. Hence, any weak $*$ or strong limit over $\mathcal{F}(I)$ may equivalently be computed over $\mathcal{F}_0(I)$.

Definition 5.1.14 (Induced dual $\mathcal{M}$ -bimodule and the drop-down map). Let $X = Y^*$ be a normal dual $\mathcal{N}$ -bimodule. Define

$$\tilde{X} := E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*. \quad (5.13)$$

By Lemma B.4, we make $\tilde{X}$ into an $\mathcal{M}$ -bimodule by

$$m \cdot (v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} f) = (mv) \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} f,$$

and

$$(v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} f) \cdot m = v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} (fm),$$

for all $m \in \mathcal{M}$, $v \in E$, $\xi \in X$, and $f \in E^*$. Then, the *drop-down map* is defined on elementary tensors by

$$\begin{aligned} \varepsilon : \tilde{X} = E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^* &\longrightarrow X, \\ v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} f &\longmapsto (pv) \cdot \xi \cdot (fp), \end{aligned} \quad (5.14)$$

where $v \in E$, $\xi \in X$, and $f \in E^*$.

Lemma 5.1.15. *Assume the notation of Definition 5.1.14. Then, the induced module $\tilde{X}$ is a normal dual Banach $\mathcal{M}$ -bimodule, and the drop-down map ε is a bounded weak*-weak* continuous linear map.*

Proof. For a fixed $m \in \mathcal{M}$, the maps

$$L_m : E \rightarrow E, \quad v \mapsto mv,$$

and

$$R_m : E^* \rightarrow E^*, \quad f \mapsto fm,$$

are bounded weak*-weak* continuous module maps. By Remark 5.1.4 and the slicing method in the proof of Lemma B.5, $\tilde{X}$ is a dual Banach $\mathcal{M}$ -bimodule. It remains to prove normality. The left action of $\mathcal{M}$ on E is normal since multiplication in $\mathcal{M}$ is separately weak*-weak* continuous. Therefore, by Lemma B.5, the induced left action of $\mathcal{M}$ on the unbalanced threefold normal tensor product is normal. Since the left action preserves the $\mathcal{N}$ -balancing relations, the normality passes to the balanced quotient by Corollary B.6. Hence, for every $\eta \in \tilde{X}$, the map

$$\mathcal{M} \longrightarrow \tilde{X}, \quad m \longmapsto m \cdot \eta$$

is weak*-weak* continuous. Similarly, for every $\eta \in \tilde{X}$, the map

$$\mathcal{M} \longrightarrow \tilde{X}, \quad m \longmapsto \eta \cdot m$$

is weak*-weak* continuous. Therefore, $\tilde{X}$ is a normal dual Banach $\mathcal{M}$ -bimodule.

It remains to justify the drop-down map. Define

$$\lambda_p : E \longrightarrow \mathcal{N}, \quad \lambda_p(v) = pv,$$

and

$$\rho_p : E^* \longrightarrow \mathcal{N}, \quad \rho_p(f) = fp.$$

Both maps are bounded and weak*-weak* continuous. It is easily verified that λ_p is a right $\mathcal{N}$ -module map and ρ_p is a left $\mathcal{N}$ -module map. Hence, by Corollary B.6, Lemma B.4, and Remark 5.1.4, the *outer compression* map

$$\begin{aligned} C_p &= \lambda_p \bar{\otimes}_{\mathcal{N}} \text{id}_X \bar{\otimes}_{\mathcal{N}} \rho_p : E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^* \longrightarrow \mathcal{N} \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} \mathcal{N}, \\ v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} f &\longmapsto (pv) \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} (fp) \end{aligned} \quad (5.15)$$

is well defined and weak*-weak* continuous. By Proposition B.10, the canonical module-action map

$$\begin{aligned} \mu : \mathcal{N} \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} \mathcal{N} &\longrightarrow X, \\ a \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} b &\longmapsto a \cdot \xi \cdot b \end{aligned}$$

is also weak*-weak* continuous. Since

$$\varepsilon := \mu \circ C_p,$$

it follows that ε is bounded and weak*-weak* continuous. $\square$

Remark 5.1.16. By construction of the frame in Theorem 5.1.11, $q_0 = p$ and $u_0 = p$. Since the range projections $u_i u_i^* = q_i$ are mutually orthogonal, we have $q_i \perp p$ for $i \neq 0$. Hence,

$$pu_i = 0, \quad (i \neq 0), \quad (5.16)$$

and, taking adjoints,

$$u_i^* p = 0, \quad (i \neq 0). \quad (5.17)$$

Therefore, for $i, j \in I$ and $\xi \in X$,

$$\varepsilon(u_i \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} u_j^*) = (pu_i) \cdot \xi \cdot (u_j^* p) = \begin{cases} \xi, & i = j = 0, \\ 0, & \text{otherwise.} \end{cases} \quad (5.18)$$

Thus, relative to the frame $(u_i)_{i \in I}$, the map ε is the projection onto the $(0, 0)$ -coordinate.

Remark 5.1.17. Since p is the identity of $\mathcal{N} = p\mathcal{M}p$, the right $\mathcal{N}$ -module action on $E = \mathcal{M}p$ and the left $\mathcal{N}$ -module action on $E^* = p\mathcal{M}$ are nondegenerate:

$$e = ep, \quad f = pf \quad (e \in E, f \in E^*).$$

Therefore, from the balancing relations for tensor products, the canonical multiplication maps

$$\begin{aligned} \mu_E : E \bar{\otimes}_{\mathcal{N}} \mathcal{N} &\longrightarrow E, & \mu_E(e \otimes_{\mathcal{N}} a) &= ea, \\ \mu_{E^*} : \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^* &\longrightarrow E^*, & \mu_{E^*}(a \otimes_{\mathcal{N}} f) &= af \end{aligned} \quad (5.19)$$

are weak* homeomorphic isometries. One can easily verify that their inverses are given by

$$\begin{aligned} \Lambda_E : E &\longrightarrow E \bar{\otimes}_{\mathcal{N}} \mathcal{N}, & \Lambda_E(e) &= e \otimes_{\mathcal{N}} p, \\ \Lambda_{E^*} : E^* &\longrightarrow \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^*, & \Lambda_{E^*}(f) &= p \otimes_{\mathcal{N}} f. \end{aligned} \quad (5.20)$$

Consequently, we have the following isometric isomorphisms which are also weak* homeomorphisms

$$E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \cong E, \quad \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^* \cong E^*, \quad E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^* \cong E \bar{\otimes}_{\mathcal{N}} E^*. \quad (5.21)$$

For the last one, the absorption isomorphism can be defined directly as follows.

$$\begin{aligned}\Psi : E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^* &\longrightarrow E \bar{\otimes}_{\mathcal{N}} E^*, \\ v \otimes_{\mathcal{N}} a \otimes_{\mathcal{N}} w^* &\longmapsto va \otimes_{\mathcal{N}} w^* = v \otimes_{\mathcal{N}} aw^*.\end{aligned}\tag{5.22}$$

In particular, $\Psi(v \otimes_{\mathcal{N}} p \otimes_{\mathcal{N}} w^*) = v \otimes_{\mathcal{N}} w^*$.

Definition 5.1.18 (The embedding Φ). Let X be a normal $\mathcal{N}$ -module and $\mathcal{L}_{\sigma}(E, E \bar{\otimes}_{\mathcal{N}} X)$ denote the space of all weak* continuous right $\mathcal{N}$ -module mappings. Equip $\mathcal{L}_{\sigma}(E, E \bar{\otimes}_{\mathcal{N}} X)$ with the topology of pointwise weak* convergence ($p.w^*$). That is, for a net $(T_{\alpha})_{\alpha}$ and an operator T in $\mathcal{L}_{\sigma}(E, E \bar{\otimes}_{\mathcal{N}} X)$, we write

$$T_{\alpha} \xrightarrow{p.w^*} T \iff T_{\alpha}(e) \xrightarrow{w^*} T(e) \quad (e \in E).\tag{5.23}$$

Define the canonical $\mathcal{M}$ -module transformation on elementary tensors by

$$\begin{aligned}\Phi : E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^* &\longrightarrow \mathcal{L}_{\sigma}(E, E \bar{\otimes}_{\mathcal{N}} X), \\ \Phi(v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} w^*)(e) &= v \otimes_{\mathcal{N}} (\xi \langle w, e \rangle_{\mathcal{N}}).\end{aligned}\tag{5.24}$$

This mapping is $(w^* - p.w^*)$ -continuous by definition, since the inner product on E and the module actions on $E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*$ and $E \bar{\otimes}_{\mathcal{N}} X$ are weak* continuous. Below, we show that Φ is an isometric $(w^*, p.w^*)$ -homeomorphic isomorphism.

Theorem 5.1.19. Let Φ be the map in Definition 5.1.18, and put $Y := E \bar{\otimes}_{\mathcal{N}} X$. Define the $\mathcal{M}$ -bimodule actions on $\mathcal{L}_{\sigma}(E, Y)$ by

$$(m \cdot T)(e) = m \cdot T(e), \quad (T \cdot m)(e) = T(me), \quad (m \in \mathcal{M}, T \in \mathcal{L}_{\sigma}(E, Y), e \in E).\tag{5.25}$$

Then, the followings hold:

- (i) Φ is compatible with the induced $\mathcal{M}$ - $\mathcal{M}$ module actions; that is, for every $m \in \mathcal{M}$ and $\eta \in E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*$,

$$\Phi(m \cdot \eta) = m \cdot \Phi(\eta), \quad \Phi(\eta \cdot m) = \Phi(\eta) \cdot m.\tag{5.26}$$

- (ii) The reconstruction formula

$$\eta = w^* - \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} \Phi(\eta)(u_i) \otimes_{\mathcal{N}} u_i^*\tag{5.27}$$

holds for every $\eta \in E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*$, where (u_i) is the net obtained in Theorem 5.1.11. In particular, Φ is injective.

- (iii) Φ is isometric.

- (iv) For every $T \in \mathcal{L}_{\sigma}(E, Y)$, the net

$$\sum_{i \in F} T(u_i) \otimes_{\mathcal{N}} u_i^*, \quad F \in \mathcal{F}(I),$$

converges weak* in $Y \bar{\otimes}_{\mathcal{N}} E^*$, and

$$\Phi^{-1}(T) = w^* - \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} T(u_i) \otimes_{\mathcal{N}} u_i^*.\tag{5.28}$$

Thus, Φ is onto $\mathcal{L}_{\sigma}(E, Y)$.

(v) *The inverse map*

$$\Phi^{-1} : \mathcal{L}_\sigma(E, Y) \longrightarrow E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*$$

is $(p.w^*, w^*)$ -continuous. Consequently, Φ is an isometric $(w^*, p.w^*)$ -homeomorphism onto $\mathcal{L}_\sigma(E, Y)$.

Proof. It is straightforward to verify that $m \cdot T$ and $T \cdot m$ defined by (5.25) are still elements of $\mathcal{L}_\sigma(E, Y)$. We prove the second identity in (5.26); the first one is trivial. By linearity and weak* continuity, it is enough to check the identity on elementary tensors $\eta = v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} w^*$, where $v \in E$, $\xi \in X$, and $w^* \in E^*$. For the right action, note that $w^*m = (m^*w)^*$, where $m \in \mathcal{M}$. Thus, for $e \in E$,

$$\begin{aligned} \Phi(\eta \cdot m)(e) &= \Phi(v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} w^*m)(e) \\ &= \Phi(v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} (m^*w)^*)(e) \\ &= v \otimes_{\mathcal{N}} \xi \langle m^*w, e \rangle_{\mathcal{N}} \\ &= v \otimes_{\mathcal{N}} (\xi w^*me) \\ &= v \otimes_{\mathcal{N}} (\xi \langle w, me \rangle_{\mathcal{N}}) \\ &= \Phi(v \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} w^*)(me) \\ &= (\Phi(\eta) \cdot m)(e). \end{aligned}$$

Hence $\Phi(\eta \cdot m) = \Phi(\eta) \cdot m$.

We next prove the reconstruction formula (5.27). Let

$$\eta \in Y \bar{\otimes}_{\mathcal{N}} E^*, \quad q_F = \sum_{i \in F} u_i u_i^*.$$

We claim that

$$\sum_{i \in F} \Phi(\eta)(u_i) \otimes_{\mathcal{N}} u_i^* = \eta \cdot q_F. \quad (5.29)$$

For an elementary tensor $\eta = y \otimes_{\mathcal{N}} w^*$, we compute

$$\begin{aligned} \sum_{i \in F} \Phi(\eta)(u_i) \otimes_{\mathcal{N}} u_i^* &= \sum_{i \in F} \Phi(y \otimes_{\mathcal{N}} w^*)(u_i) \otimes_{\mathcal{N}} u_i^* \\ &= \sum_{i \in F} y \langle w, u_i \rangle_{\mathcal{N}} \otimes_{\mathcal{N}} u_i^* \\ &= \sum_{i \in F} y \otimes_{\mathcal{N}} \langle w, u_i \rangle_{\mathcal{N}} u_i^* \\ &= y \otimes_{\mathcal{N}} \sum_{i \in F} w^* u_i u_i^* \\ &= (y \otimes_{\mathcal{N}} w^*) \cdot q_F \\ &= \eta \cdot q_F. \end{aligned}$$

The general case follows by linearity and weak* continuity. Since $q_F \xrightarrow{w^*} \text{cs}(p)$ and the module actions are normal, we have

$$\sum_{i \in F} \Phi(\eta)(u_i) \otimes_{\mathcal{N}} u_i^* = \eta \cdot q_F \xrightarrow{w^*} \eta \cdot \text{cs}(p).$$

By (5.11), we have $\eta \cdot \text{cs}(p) = \eta$. Therefore,

$$\eta = \text{w}^* - \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} \Phi(\eta)(u_i) \otimes_{\mathcal{N}} u_i^*.$$

This proves (5.27). In particular, if $\Phi(\eta) = 0$, then (5.27) gives $\eta = 0$. Hence Φ is injective.

We now prove that Φ is isometric. The bilinear map

$$\beta : Y \times E^* \rightarrow \mathcal{L}_\sigma(E, Y), \quad \beta(y, w^*)(e) = y \langle w, e \rangle_{\mathcal{N}},$$

is balanced over $\mathcal{N}$, separately weak* continuous, and contractive:

$$\|\beta(y, w^*)\| \leq \|y\| \|w^*\|. \quad (5.30)$$

This estimate is taken with respect to the operator norm on $\mathcal{L}_\sigma(E, Y)$, the norm of Y , and the inherited norm of $E^* = p\mathcal{M}$. To obtain a map on $Y \bar{\otimes}_{\mathcal{N}} E^*$, we use the normal-linearization criterion for balanced bilinear maps in Definition 5.1.6. For $e \in E$ and $\varphi \in Y_*$, define

$$(e \otimes \varphi)(T) = \langle \varphi, T(e) \rangle \quad (T \in \mathcal{L}_\sigma(E, Y)). \quad (5.31)$$

For each $e \in E$, the mapping

$$\rho_e : E^* \longrightarrow \mathcal{N}, \quad \rho_e(w^*) = w^*e = \langle w, e \rangle_{\mathcal{N}}$$

is a bounded left $\mathcal{N}$ -module map, weak*-weak* continuous, and $\|\rho_e\| \leq \|e\|$. Therefore, by Lemma B.4

$$\text{id}_Y \bar{\otimes}_{\mathcal{N}} \rho_e : Y \bar{\otimes}_{\mathcal{N}} E^* \longrightarrow Y \bar{\otimes}_{\mathcal{N}} \mathcal{N}$$

is bounded and weak*-weak* continuous, with norm at most $\|e\|$. Using the canonical identification (5.21), we obtain a bounded weak*-weak* continuous map

$$\begin{aligned} \Psi_e : Y \bar{\otimes}_{\mathcal{N}} E^* &\longrightarrow Y, \\ \Psi_e(y \otimes_{\mathcal{N}} x^*) &= y \langle x, e \rangle_{\mathcal{N}}, \\ \|\Psi_e\| &\leq \|e\|. \end{aligned}$$

Since Ψ_e is weak*-weak* continuous, it has a pre-adjoint

$$\begin{aligned} (\Psi_e)_* : Y_* &\longrightarrow (Y \bar{\otimes}_{\mathcal{N}} E^*)_*, \\ \langle (\Psi_e)_*(\varphi), y \otimes_{\mathcal{N}} x^* \rangle &= \langle \varphi, \Psi_e(y \otimes_{\mathcal{N}} x^*) \rangle \\ &= \langle \varphi, y \langle x, e \rangle_{\mathcal{N}} \rangle, \end{aligned} \quad (5.32)$$

for $\varphi \in Y_*$, $y \in Y$, and $x^* \in E^*$. Now, let $\omega \in E \otimes_\gamma Y_*$ be represented by

$$\omega = \sum_{k=1}^{\infty} e_k \otimes \varphi_k \in E \widehat{\otimes}_\gamma Y_*, \quad \sum_{k=1}^{\infty} \|e_k\| \|\varphi_k\| < \infty. \quad (5.33)$$

For $y \in Y$ and $w^* \in E^*$, we have

$$\begin{aligned}
\langle \omega, \beta(y, w^*) \rangle &= \sum_{k=1}^{\infty} \langle \varphi_k, \beta(y, w^*)(e_k) \rangle, \quad \text{by (5.31)} \\
&= \sum_{k=1}^{\infty} \langle \varphi_k, y \langle w, e_k \rangle_{\mathcal{N}} \rangle \\
&= \sum_{k=1}^{\infty} \langle \varphi_k, \Psi_{e_k}(y \otimes_{\mathcal{N}} w^*) \rangle \\
&= \sum_{k=1}^{\infty} \langle (\Psi_{e_k})_*(\varphi_k), y \otimes_{\mathcal{N}} w^* \rangle.
\end{aligned} \tag{5.34}$$

Since

$$\|(\Psi_{e_k})_*(\varphi_k)\| \leq \|\Psi_{e_k}\| \|\varphi_k\| \leq \|e_k\| \|\varphi_k\|,$$

by (5.33), the series

$$\sum_{k=1}^{\infty} (\Psi_{e_k})_*(\varphi_k)$$

converges absolutely in $(Y \bar{\otimes}_{\mathcal{N}} E^*)_*$. Define

$$\Theta_{\beta}(\omega) := \sum_{k=1}^{\infty} (\Psi_{e_k})_*(\varphi_k) \in (Y \bar{\otimes}_{\mathcal{N}} E^*)_*.$$

By (5.34), for all $y \in Y$ and $w^* \in E^*$, we have

$$\langle \Theta_{\beta}(\omega), y \otimes_{\mathcal{N}} w^* \rangle = \langle \omega, \beta(y, w^*) \rangle.$$

Hence, the scalar bilinear form

$$(y, w^*) \longmapsto \langle \omega, \beta(y, w^*) \rangle$$

is represented by the predual element $\Theta_{\beta}(\omega)$. Moreover, by (5.34) and (5.33)

$$\begin{aligned}
\Theta_{\beta} : E \otimes_{\gamma} Y_* &\longrightarrow (Y \bar{\otimes}_{\mathcal{N}} E^*)_*, \\
\omega &\longmapsto \Theta_{\beta}(\omega), \\
\|\Theta_{\beta}(\omega)\| &\leq \|\omega\|.
\end{aligned} \tag{5.35}$$

Therefore, recalling Remark 5.1.4, β satisfies the normal-linearization condition of Lemma B.7. Since β is also $\mathcal{N}$ -balanced, Corollary B.8 along with Remark 5.1.4 gives a unique weak*-weak* continuous linear map

$$\begin{aligned}
\Phi : Y \bar{\otimes}_{\mathcal{N}} E^* &\longrightarrow \mathcal{L}_{\sigma}(E, Y), \\
\Phi(y \otimes_{\mathcal{N}} w^*) &= \beta(y, w^*), \\
\Phi(y \otimes_{\mathcal{N}} w^*)(e) &= y \langle w, e \rangle_{\mathcal{N}}.
\end{aligned} \tag{5.36}$$

Since $\Phi = \Theta_{\beta}^*$ and $\|\Theta_{\beta}\| \leq 1$, we have

$$\|\Phi(\eta)\| \leq \|\eta\|. \tag{5.37}$$

For the reverse inequality, set $T = \Phi(\eta)$ and define

$$\eta_F := \sum_{i \in F} T(u_i) \otimes_{\mathcal{N}} u_i^* \in Y \bar{\otimes}_{\mathcal{N}} E^*. \quad (5.38)$$

By (5.27), $\eta_F \xrightarrow{w^*} \eta$. For $e \in E$, using that T is a right $\mathcal{N}$ -module map, we have

$$\begin{aligned} \Phi(\eta_F)(e) &= \sum_{i \in F} T(u_i) \langle u_i, e \rangle_{\mathcal{N}} \\ &= T \left(\sum_{i \in F} u_i \langle u_i, e \rangle_{\mathcal{N}} \right) \\ &= T(q_F e). \end{aligned}$$

Thus,

$$\Phi(\eta_F) = T \circ P_F, \quad P_F(e) = q_F e. \quad (5.39)$$

Since q_F is a projection, P_F is contractive. Therefore,

$$\|\Phi(\eta_F)\| = \|T \circ P_F\| \leq \|T\| \|P_F\| \leq \|T\| = \|\Phi(\eta)\|. \quad (5.40)$$

For finite-rank tensors, the normal spatial tensor norm is the norm obtained from their concrete representation. Hence,

$$\|\eta_F\|_{Y \bar{\otimes}_{\mathcal{N}} E^*} = \|\Phi(\eta_F)\|_{\mathcal{L}(E, Y)}.$$

Thus,

$$\|\eta_F\| = \|\Phi(\eta_F)\| \leq \|\Phi(\eta)\|.$$

Since $\eta_F \rightarrow \eta$ weak* and the norm is weak* lower semicontinuous, we obtain

$$\|\eta\| \leq \liminf_F \|\eta_F\| \leq \|\Phi(\eta)\|. \quad (5.41)$$

Combining (5.37) and (5.41), we conclude that Φ is isometric.

It remains to prove the inverse formula and the convergence of the net in (5.28). Let $T \in \mathcal{L}_{\sigma}(E, Y)$, and for $F \in \mathcal{F}(I)$ set

$$\eta_F(T) := \sum_{i \in F} T(u_i) \otimes_{\mathcal{N}} u_i^*.$$

By (5.39) and (5.40), the net $(\eta_F(T))_F$ is bounded in the dual space $Y \bar{\otimes}_{\mathcal{N}} E^*$. Hence, it has weak* cluster points. Let η be a weak* cluster point of $(\eta_F(T))_F$ and choose a subnet $(\eta_{F_{\alpha}}(T))_{\alpha}$ such that

$$\eta_{F_{\alpha}}(T) \xrightarrow{w^*} \eta.$$

For each fixed $e \in E$, the map

$$\zeta \longmapsto \Phi(\zeta)(e)$$

is weak* continuous from $Y \bar{\otimes}_{\mathcal{N}} E^*$ into Y . Therefore,

$$\Phi(\eta_{F_{\alpha}}(T))(e) \xrightarrow{w^*} \Phi(\eta)(e).$$

On the other hand, by (5.39),

$$\Phi(\eta_{F_\alpha}(T))(e) = T(q_{F_\alpha}e).$$

By the normal Parseval identity, $q_{F_\alpha}e \xrightarrow{w^*} e$ in E , and since T is weak* continuous, we obtain

$$T(q_{F_\alpha}e) \xrightarrow{w^*} T(e).$$

By uniqueness of weak* limits, we deduce

$$\Phi(\eta)(e) = T(e) \quad (e \in E).$$

Hence,

$$\Phi(\eta) = T.$$

Since Φ is injective, this weak* cluster point is unique. Therefore the bounded net $(\eta_F(T))_F$ converges weak* to this unique cluster point. Consequently,

$$\Phi^{-1}(T) = w^*\text{-}\lim_{F \in \mathcal{F}(I)} \sum_{i \in F} T(u_i) \otimes_{\mathcal{N}} u_i^*.$$

This proves (5.28). In particular, Φ is onto $\mathcal{L}_\sigma(E, Y)$.

Finally, we prove that Φ^{-1} is $(p.w^*, w^*)$ -continuous on the range of Φ . Suppose that

$$T_\alpha \xrightarrow{p.w^*} T$$

in $\mathcal{L}_\sigma(E, Y)$, and set

$$\eta_\alpha := \Phi^{-1}(T_\alpha), \quad \eta := \Phi^{-1}(T).$$

For each fixed $e \in E$, the net $(T_\alpha(e))_\alpha$ is weak* convergent, hence bounded. By the uniform boundedness principle, $(T_\alpha)_\alpha$ is bounded in operator norm. Since Φ is isometric, $(\eta_\alpha)_\alpha$ is bounded in the dual space $E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*$. Let ζ be a weak* cluster point of $(\eta_\alpha)_\alpha$, and choose a subnet $(\eta_{\alpha_\beta})_\beta$ such that

$$\eta_{\alpha_\beta} \xrightarrow{w^*} \zeta.$$

For each $e \in E$, the evaluation map $\theta \mapsto \Phi(\theta)(e)$ is weak*-weak* continuous. Hence,

$$\Phi(\eta_{\alpha_\beta})(e) \xrightarrow{w^*} \Phi(\zeta)(e).$$

On the other hand, since $T_\alpha \xrightarrow{p.w^*} T$, we have

$$\Phi(\eta_{\alpha_\beta})(e) = T_{\alpha_\beta}(e) \xrightarrow{w^*} T(e) = \Phi(\eta)(e).$$

By uniqueness of weak* limits,

$$\Phi(\zeta)(e) = \Phi(\eta)(e) \quad (e \in E).$$

Therefore,

$$\Phi(\zeta) = \Phi(\eta).$$

Since Φ is injective, $\zeta = \eta$. Thus every weak* cluster point of $(\eta_\alpha)_\alpha$ is equal to η . Since the net is bounded in a dual space, the same cluster-point argument as above gives

$$\eta_\alpha \xrightarrow{w^*} \eta.$$

Thus, Φ^{-1} is $(p.w^*, w^*)$ -continuous. This coupled with $(w^*, p.w^*)$ -continuity of Φ proves that Φ is an isometric $(w^*, p.w^*)$ -homeomorphism onto $\mathcal{L}_\sigma(E, Y)$. $\square$

Remark 5.1.20 (The special case $\Phi_{\mathcal{N}}$). We use the canonical map from Definition 5.1.18 in the special case where the middle module X is $\mathcal{N}$. That is,

$$\begin{aligned}\Phi_{\mathcal{N}} : E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^* &\longrightarrow \mathcal{L}_{\sigma}(E, E \bar{\otimes}_{\mathcal{N}} \mathcal{N}), \\ \Phi_{\mathcal{N}}(v \otimes_{\mathcal{N}} a \otimes_{\mathcal{N}} w^*)(e) &= v \otimes_{\mathcal{N}} a \langle w, e \rangle_{\mathcal{N}}.\end{aligned}\tag{5.42}$$

In particular, under the canonical identifications in (5.21), we have

$$\begin{aligned}\Phi_{\mathcal{N}} : E \bar{\otimes}_{\mathcal{N}} E^* &\longrightarrow \mathcal{L}_{\sigma}(E, E), \\ \Phi_{\mathcal{N}}(u \otimes_{\mathcal{N}} w^*) &= \theta_{u,w}, \\ \theta_{u,w}(e) &= u \langle w, e \rangle_{\mathcal{N}}, \quad (e \in E).\end{aligned}\tag{5.43}$$

Definition 5.1.21 (The Parseval unit and the Parseval map). Let $(u_i)_{i \in I} \subseteq E$ be the frame and $\mathcal{F}(I)$ be the directed set from Theorem 5.1.11.

(i) Define the *Parseval unit* by

$$\Theta := \Phi_{\mathcal{N}}^{-1}(\text{id}_E) \in E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^*,\tag{5.44}$$

where $\text{id}_E \in \mathcal{L}_{\sigma}(E, E) \cong \mathcal{L}_{\sigma}(E, E \bar{\otimes}_{\mathcal{N}} \mathcal{N})$ by (5.21).

(ii) Define the *Parseval map* by

$$\begin{aligned}\Lambda : E &\longrightarrow E \bar{\otimes}_{\mathcal{N}} \mathcal{N}, \\ \Lambda(e) &= e \otimes_{\mathcal{N}} p.\end{aligned}\tag{5.45}$$

Equivalently, $\Lambda = \Lambda_E = \mu_E^{-1}$ under the absorption isomorphism from Remark 5.1.17.

Lemma 5.1.22. *The Parseval map (5.45) is normal; i.e., $\Lambda \in \mathcal{L}_{\sigma}(E, E \bar{\otimes}_{\mathcal{N}} \mathcal{N})$.*

Proof. Let

$$\ell : E \longrightarrow E \bar{\otimes}_{\mathcal{N}} \mathcal{N}, \quad \ell(e) = e \otimes p.$$

We first show that ℓ is weak*-weak* continuous. By Definition 5.1.1,

$$(E \bar{\otimes}_{\mathcal{N}} \mathcal{N})_* = (\mathcal{N}_* \widehat{\otimes}_{\gamma} E_*) / (E \bar{\otimes}_{\mathcal{N}} \mathcal{N})_{\perp}.$$

Let

$$\rho = \sum_{k=1}^{\infty} \psi_k \otimes \varphi_k \in \mathcal{N}_* \widehat{\otimes}_{\gamma} E_*, \quad \sum_{k=1}^{\infty} \|\psi_k\| \|\varphi_k\| < \infty.\tag{5.46}$$

Then, for $e \in E$,

$$\langle \rho, \ell(e) \rangle = \langle \rho, e \otimes p \rangle = \sum_{k=1}^{\infty} \psi_k(p) \varphi_k(e).$$

So, by (5.46), the functional

$$e \longmapsto \langle \rho, \ell(e) \rangle = \sum_{k=1}^{\infty} \psi_k(p) \varphi_k(e)$$

belongs to E_* for every test functional ρ in the predual. Hence, ℓ is weak*-weak* continuous. The quotient map $q : E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \rightarrow E \bar{\otimes}_{\mathcal{N}} \mathcal{N}$ is weak*-weak* continuous by Remark 5.1.5. Since $\Lambda = q \circ \ell$, we conclude that Λ is weak*-weak* continuous.

$$\Lambda = q \circ \ell$$

$$\begin{array}{ccc} E & \xrightarrow{\ell} & E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \\ & \searrow \Lambda & \downarrow q \\ & & E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \end{array}$$

Figure 5.2: The Parseval map factors through the quotient defining $E \bar{\otimes}_{\mathcal{N}} \mathcal{N}$.

□

Remark 5.1.23 (Relations among Θ , Λ , and Λ_E). By Theorem 5.1.11 and Remark 5.1.20, the net

$$\sum_{i \in F} u_i \otimes_{\mathcal{N}} p \otimes_{\mathcal{N}} u_i^*, \quad F \in \mathcal{F}(I),$$

represents the finite-rank operators $\sum_{i \in F} \theta_{u_i, u_i}$ under $\Phi_{\mathcal{N}}$. Since

$$p \cdot w^* - \sum_{i \in I} \theta_{u_i, u_i} = \text{id}_E,$$

and since $\Phi_{\mathcal{N}}$ is a weak*-homeomorphic isomorphism, we have

$$\Theta = w^* - \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} u_i \otimes_{\mathcal{N}} p \otimes_{\mathcal{N}} u_i^*. \quad (5.47)$$

Under the canonical identification (5.22), we may also write

$$\Theta = w^* - \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} u_i \otimes_{\mathcal{N}} u_i^*.$$

That is, Θ is the tensor representation of id_E under the identification $\Phi_{\mathcal{N}}$. Moreover, the map Λ admits the Parseval-frame expression

$$\begin{aligned} \Lambda(e) &= w^* - \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} u_i \otimes_{\mathcal{N}} \langle u_i, e \rangle_{\mathcal{N}} \\ &= w^* - \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} u_i \langle u_i, e \rangle_{\mathcal{N}} \otimes_{\mathcal{N}} p \\ &= e \otimes_{\mathcal{N}} p. \end{aligned} \quad (5.48)$$

Thus,

$$\Lambda = \Lambda_E = \mu_E^{-1}. \quad (5.49)$$

For $e \in E$, define the weak* continuous third-leg evaluation map

$$\begin{aligned} \text{Ev}_e^{(3)} : E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^* &\longrightarrow E \bar{\otimes}_{\mathcal{N}} \mathcal{N}, \\ v \otimes_{\mathcal{N}} a \otimes_{\mathcal{N}} w^* &\longmapsto v \otimes_{\mathcal{N}} a \langle w, e \rangle_{\mathcal{N}}. \end{aligned} \quad (5.50)$$

Then,

$$\Lambda(e) = \text{Ev}_e^{(3)}(\Theta), \quad e \in E. \quad (5.51)$$

Indeed, applying $\text{Ev}_e^{(3)}$ to the expansion (5.47) along with the fact that $p \langle u_i, e \rangle_{\mathcal{N}} = \langle u_i, e \rangle_{\mathcal{N}}$ gives exactly the Parseval expression in (5.48).

For $m \in \mathcal{M}$, let

$$L_m : E \longrightarrow E, \quad e \longmapsto me$$

be the left multiplication map by m . For $a \in \mathcal{N}$ and $e \in E$, we have

$$\begin{aligned} (\Lambda \circ L_m)(ea) &= \Lambda(L_m(ea)) \\ &= \Lambda(m(ea)) \\ &= \Lambda((me)a) \\ &= (me)a \otimes_{\mathcal{N}} p \\ &= me \otimes_{\mathcal{N}} ap \\ &= me \otimes_{\mathcal{N}} a \\ &= (me \otimes_{\mathcal{N}} p)a \\ &= \Lambda(me)a \\ &= (\Lambda \circ L_m)(e)a. \end{aligned}$$

Here, we used the balancing relation and the fact that p is the identity of $\mathcal{N}$ to derive

$$(me)a \otimes_{\mathcal{N}} p = me \otimes_{\mathcal{N}} ap = me \otimes_{\mathcal{N}} a.$$

Consequently, for right $\mathcal{N}$ -module maps L_m and Λ , we have

$$\Lambda \circ L_m \in \mathcal{L}_{\sigma}(E, E \bar{\otimes}_{\mathcal{N}} \mathcal{N}).$$

Using the mapping $\Phi_{\mathcal{N}}$ from Remark 5.1.20, we now define:

Definition 5.1.24 (Parseval triple Ω_m). For any $m \in \mathcal{M}$,

$$\Omega_m := \Phi_{\mathcal{N}}^{-1}(\Lambda \circ L_m) \in E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes}_{\mathcal{N}} E^* \quad (5.52)$$

is called the *Parseval triple associated with m* . Equivalently,

$$\Phi_{\mathcal{N}}(\Omega_m) = \Lambda \circ L_m. \quad (5.53)$$

Lemma 5.1.25 (Rectangular block expansion of Ω_m). For every $m \in \mathcal{M}$, we have

$$\Omega_m = \text{w}^*\text{-}\lim_{(F,G) \in \mathcal{F}(I)^2} \sum_{i \in F} \sum_{j \in G} u_i \otimes_{\mathcal{N}} \langle u_i, mu_j \rangle_{\mathcal{N}} \otimes_{\mathcal{N}} u_j^*, \quad (5.54)$$

where $\mathcal{F}(I)^2$ is ordered coordinatewise; that is,

$$(F, G) \leq (F', G') \iff F \subseteq F' \text{ and } G \subseteq G'.$$

Proof. For $H \in \mathcal{F}(I)$, define

$$q_H = \sum_{i \in H} u_i u_i^*.$$

So, by (5.12), we have

$$q_H \xrightarrow{\text{so}} \text{cs}(p).$$

For $F, G \in \mathcal{F}(I)$, set

$$\Omega_{m,F,G} = \sum_{i \in F} \sum_{j \in G} u_i \otimes_{\mathcal{N}} \langle u_i, mu_j \rangle_{\mathcal{N}} \otimes_{\mathcal{N}} u_j^*.$$

It is enough to prove that

$$\Omega_{m,F,G} \xrightarrow{w^*} \Omega_m. \quad (5.55)$$

To this end, we prove the convergence after applying $\Phi_{\mathcal{N}}$ and then use that $\Phi_{\mathcal{N}}$ is a $(w^*, p.w^*)$ -homeomorphism onto its range. Fix $e \in E$. Then,

$$\begin{aligned} \Phi_{\mathcal{N}}(\Omega_{m,F,G})(e) &= \sum_{i \in F} \sum_{j \in G} u_i \otimes_{\mathcal{N}} \langle u_i, mu_j \rangle_{\mathcal{N}} \langle u_j, e \rangle_{\mathcal{N}} \\ &= \sum_{i \in F} \sum_{j \in G} u_i \otimes_{\mathcal{N}} u_i^* mu_j u_j^* e. \end{aligned}$$

Using the balancing relation in $E \bar{\otimes}_{\mathcal{N}} \mathcal{N}$, we have

$$u_i \otimes_{\mathcal{N}} u_i^* mu_j u_j^* e = u_i u_i^* mu_j u_j^* e \otimes_{\mathcal{N}} p.$$

Hence,

$$\begin{aligned} \Phi_{\mathcal{N}}(\Omega_{m,F,G})(e) &= \sum_{i \in F} \sum_{j \in G} u_i u_i^* mu_j u_j^* e \otimes_{\mathcal{N}} p \\ &= \left(\sum_{i \in F} u_i u_i^* \right) m \left(\sum_{j \in G} u_j u_j^* \right) e \otimes_{\mathcal{N}} p \\ &= q_F m q_G e \otimes_{\mathcal{N}} p. \end{aligned}$$

Now, $q_F \rightarrow \text{cs}(p)$ and $q_G \rightarrow \text{cs}(p)$ strongly. Let $z = \text{cs}(p)$. We claim that

$$q_F m q_G e \longrightarrow me, \quad \text{weak}^* \text{ in } E \quad (5.56)$$

as $(F, G) \in \mathcal{F}(I)^2$ increases. Let ξ be a vector in the Hilbert space on which $\mathcal{M}$ acts. Then,

$$\begin{aligned} [q_F m q_G e - z m z e](\xi) &= [(q_F - z) m z e](\xi) \\ &\quad + [(q_F - z) m (q_G - z) e](\xi) \\ &\quad + [z m (q_G - z) e](\xi). \end{aligned}$$

The first and the last terms tend to 0 since $q_F \rightarrow z$ and $q_G \rightarrow z$ strongly. For the middle term note that

$$\| [q_F - z]([m(q_G - z)e](\xi)) \| \leq 2\|m\| \|(q_G - z)e(\xi)\|,$$

which tends to 0 as $q_G \rightarrow z$ strongly. Hence,

$$q_F m q_G e \longrightarrow z m z e$$

strongly, and therefore in weak*. Since z is central and $ze = e$ by (5.10), this limit is

$$z m z e = m z e = me.$$

It follows from (5.56) and Lemma 5.1.22 that

$$q_F m q_G e \otimes_{\mathcal{N}} p \longrightarrow me \otimes_{\mathcal{N}} p, \quad \text{weak}^* \text{ in } E \bar{\otimes}_{\mathcal{N}} \mathcal{N}. \quad (5.57)$$

However, by (5.53), for any $s \in \mathcal{M}$

$$se \otimes_{\mathcal{N}} p = \Lambda(se) = (\Lambda \circ L_s)(e) = \Phi_{\mathcal{N}}(\Omega_s)(e).$$

Thus, for every $e \in E$, (5.57) implies

$$\Phi_{\mathcal{N}}(\Omega_{m,F,G})(e) \xrightarrow{w^*} \Phi_{\mathcal{N}}(\Omega_m)(e), \quad \text{in } E \bar{\otimes}_{\mathcal{N}} \mathcal{N}.$$

Therefore,

$$\Phi_{\mathcal{N}}(\Omega_{m,F,G}) \xrightarrow{p.w^*} \Phi_{\mathcal{N}}(\Omega_m), \quad \text{in } \mathcal{L}_{\sigma}(E, E \bar{\otimes}_{\mathcal{N}} \mathcal{N}).$$

Since $\Phi_{\mathcal{N}}$ is a $(w^*, p.w^*)$ -homeomorphism, it follows that

$$\Omega_{m,F,G} \xrightarrow{w^*} \Omega_m.$$

This proves (5.55). □

Corollary 5.1.26 (Square block expansion of Ω_m). *For every $m \in \mathcal{M}$, we have*

$$\Omega_m = w^* - \lim_{H \in \mathcal{F}(I)} \sum_{i,j \in H} u_i \otimes_{\mathcal{N}} \langle u_i, mu_j \rangle_{\mathcal{N}} \otimes_{\mathcal{N}} u_j^*. \quad (5.58)$$

Proof. For every $(F, G) \in \mathcal{F}(I)^2$, if $H = F \cup G$, then

$$(F, G) \leq (H, H).$$

So, the cofinal index duplication map

$$\mathcal{F}(I) \longrightarrow \mathcal{F}(I)^2, \quad H \longmapsto (H, H),$$

has a cofinal range. Therefore, the weak* limit in (5.54) may also be computed along the rectangular blocks with $F = G$. □

Theorem 5.1.27 (Normal unbalanced Parseval representative). *Let $(u_i)_{i \in I} \subseteq E$ be the frame and $\mathcal{F}(I)$ be the directed set from Theorem 5.1.11. For $m \in \mathcal{M}$ and finite $F \subset I$, set*

$$\tilde{\Omega}_{m,F} = \sum_{i,j \in F} u_i \otimes u_i^* mu_j \otimes u_j^* \in E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes} E^*.$$

Then, $(\tilde{\Omega}_{m,F})_F$ converges in the weak topology of $E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes} E^*$ to an element*

$$\tilde{\Omega}_m = w^* - \lim_F \sum_{i,j \in F} u_i \otimes u_i^* mu_j \otimes u_j^*.$$

Moreover, $q_{\mathcal{N}}(\tilde{\Omega}_m) = \Omega_m$, where $q_{\mathcal{N}} : E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes} E^* \longrightarrow E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes} E^*$ is the canonical quotient map. Finally, the map

$$\tilde{\Omega} : \mathcal{M} \longrightarrow E \bar{\otimes}_{\mathcal{N}} \mathcal{N} \bar{\otimes} E^*, \quad m \longmapsto \tilde{\Omega}_m$$

is weak-weak* continuous.*

Proof. To prove that the net $(\tilde{\Omega}_{m,F})_F$ is weak* convergent, we test it against the functionals in the projective pre-dual. Let

$$\rho = \varphi \otimes \omega \otimes \psi \in E_* \hat{\otimes}_\gamma \mathcal{N}_* \hat{\otimes}_\gamma (E^*)_*$$

be an elementary test functional. Then,

$$\langle \tilde{\Omega}_{m,F}, \rho \rangle = \sum_{i,j \in F} \varphi(u_i) \omega(u_i^* m u_j) \psi(u_j^*).$$

We first treat the case where φ, ω, ψ are vector functionals. Then, after restricting from the ambient Hilbert-space representation, we can write

$$\varphi(x) = \langle x\alpha, \beta \rangle, \quad \omega(a) = \langle a\eta, \zeta \rangle, \quad \psi(y) = \langle y\gamma, \delta \rangle.$$

Since the range projections $u_i u_i^*$ are mutually orthogonal, the vectors $u_i \alpha$ lie in mutually orthogonal subspaces. Hence,

$$(\varphi(u_i))_{i \in I} = (\langle u_i \alpha, \beta \rangle)_{i \in I} \in \ell^2(I).$$

Indeed, for finite $F \subset I$, if $u_i \alpha \neq 0$, put $e_i = u_i \alpha / \|u_i \alpha\|$. Then $(e_i)_{i \in F}$ is an orthonormal family, and therefore

$$\begin{aligned} \sum_{i \in F} |\varphi(u_i)|^2 &= \sum_{i \in F} |\langle u_i \alpha, \beta \rangle|^2 \\ &= \sum_{i \in F} \|u_i \alpha\|^2 |\langle e_i, \beta \rangle|^2 \\ &\leq \|\alpha\|^2 \sum_{i \in F} |\langle e_i, \beta \rangle|^2 \\ &\leq \|\alpha\|^2 \|\beta\|^2. \end{aligned}$$

Thus

$$\sum_{i \in I} |\varphi(u_i)|^2 \leq \|\alpha\|^2 \|\beta\|^2.$$

Similarly, for $\psi(u_j^*) = \langle u_j^* \gamma, \delta \rangle = \langle \gamma, u_j \delta \rangle$, we have

$$(\psi(u_j^*))_{j \in I} \in \ell^2(I),$$

with

$$\sum_{j \in I} |\psi(u_j^*)|^2 \leq \|\gamma\|^2 \|\delta\|^2.$$

For finite $F \subset I$, define

$$A_F = \sum_{j \in F} \psi(u_j^*) u_j \eta, \quad B_F = \sum_{i \in F} \overline{\varphi(u_i)} u_i \zeta.$$

We claim that $(A_F)_F$ and $(B_F)_F$ converge in norm. Let $F, G \subset I$ be finite and $F \triangle G = (F \setminus G) \cup$

$(G \setminus F)$. Then, by the orthogonality of the range projections $u_j u_j^*$,

$$\begin{aligned}\|A_F - A_G\|^2 &= \left\| \sum_{j \in F \Delta G} \epsilon_j \psi(u_j^*) u_j \eta \right\|^2 \\ &= \sum_{j \in F \Delta G} |\psi(u_j^*)|^2 \|u_j \eta\|^2 \\ &\leq \|\eta\|^2 \sum_{j \in F \Delta G} |\psi(u_j^*)|^2,\end{aligned}$$

where $\epsilon_j \in \{-1, 1\}$ depends on whether j belongs to $F \setminus G$ or $G \setminus F$. Since $(\psi(u_j^*))_j \in \ell^2(I)$, this shows that $(A_F)_F$ is norm Cauchy. Hence $A_F \rightarrow A$ in norm for some Hilbert-space vector A . Similarly, using $(\varphi(u_i))_i \in \ell^2(I)$, we obtain $B_F \rightarrow B$ in norm. Now compute:

$$\begin{aligned}\langle \tilde{\Omega}_{m,F}, \rho \rangle &= \sum_{i,j \in F} \varphi(u_i) \langle u_i^* m u_j \eta, \zeta \rangle \psi(u_j^*) \\ &= \sum_{i,j \in F} \varphi(u_i) \langle m u_j \eta, u_i \zeta \rangle \psi(u_j^*) \\ &= \langle m A_F, B_F \rangle.\end{aligned}$$

Since $A_F \rightarrow A$ and $B_F \rightarrow B$ in norm, it follows that

$$\langle \tilde{\Omega}_{m,F}, \varphi \otimes \omega \otimes \psi \rangle = \langle m A_F, B_F \rangle \longrightarrow \langle m A, B \rangle.$$

Therefore, $\langle \tilde{\Omega}_{m,F}, \varphi \otimes \omega \otimes \psi \rangle$ converges for vector functionals φ, ω, ψ , and the same computation gives the estimate

$$\left| \langle \tilde{\Omega}_{m,F}, \varphi \otimes \omega \otimes \psi \rangle \right| \leq \|m\| \|\varphi\| \|\omega\| \|\psi\|. \quad (5.59)$$

By norm density of finite sums of vector functionals in the relevant pre-duals, the same convergence and estimate hold for arbitrary $\varphi \in E_*$, $\omega \in \mathcal{N}_*$, and $\psi \in (E^*)_*$. Now, let

$$\rho = \sum_{r=1}^{\infty} \varphi_r \otimes \omega_r \otimes \psi_r \in E_* \hat{\otimes}_{\gamma} \mathcal{N}_* \hat{\otimes}_{\gamma} (E^*)_*, \quad \sum_{r=1}^{\infty} \|\varphi_r\| \|\omega_r\| \|\psi_r\| < \infty. \quad (5.60)$$

For each r , the scalar net

$$\left(\left\langle \tilde{\Omega}_{m,F}, \varphi_r \otimes \omega_r \otimes \psi_r \right\rangle \right)_F$$

converges, and the estimate (5.59) gives

$$\left| \left\langle \tilde{\Omega}_{m,F}, \varphi_r \otimes \omega_r \otimes \psi_r \right\rangle \right| \leq \|m\| \|\varphi_r\| \|\omega_r\| \|\psi_r\|,$$

uniformly in F . Hence, by (5.60), the net

$$\left(\left\langle \tilde{\Omega}_{m,F}, \rho \right\rangle \right)_F$$

converges for every $\rho \in E_* \hat{\otimes}_{\gamma} \mathcal{N}_* \hat{\otimes}_{\gamma} (E^*)_*$. Hence, the formula

$$L_m(\rho) := \lim_F \langle \tilde{\Omega}_{m,F}, \rho \rangle$$

defines a bounded functional on $(E \bar{\otimes} \mathcal{N} \bar{\otimes} E^*)^*$ with $\|L_m\| \leq \|m\|$. Since $E \bar{\otimes} \mathcal{N} \bar{\otimes} E^*$ is a dual space, there is a unique element $\tilde{\Omega}_m \in E \bar{\otimes} \mathcal{N} \bar{\otimes} E^*$ representing L_m . Therefore,

$$\tilde{\Omega}_{m,F} \xrightarrow{w^*} \tilde{\Omega}_m. \quad (5.61)$$

By weak*-weak* continuity of the canonical quotient map $q_{\mathcal{N}}$, we compute

$$\begin{aligned} q_{\mathcal{N}}(\tilde{\Omega}_m) &= w^*-\lim_F q_{\mathcal{N}}(\tilde{\Omega}_{m,F}) \\ &= w^*-\lim_F \sum_{i,j \in F} u_i \otimes_{\mathcal{N}} u_i^* m u_j \otimes_{\mathcal{N}} u_j^* \\ &= \Omega_m, \quad \text{by Lemma 5.1.25.} \end{aligned}$$

Moreover, for each $\rho \in E_* \otimes_{\gamma} \mathcal{N}_* \otimes_{\gamma} (E^*)^*$, a similar argument leading to (5.61) shows that the map

$$m \longmapsto L_m(\rho) = \langle \tilde{\Omega}_m, \rho \rangle$$

belongs to $\mathcal{M}_*$. Thus, the mapping $\tilde{\Omega}$ is weak*-weak* continuous. $\square$

5.2 Lifting derivations and Connes sequential amenability of type II_{∞} von Neumann algebras

This section implements the lifting argument described in Remark 5.1.7. Using the Parseval frame for $E = \mathcal{M}p$, we lift normal derivations from the finite corner $\mathcal{N} = p\mathcal{M}p$ to $\mathcal{M}$ with values in the associated balanced normal tensor product module (5.13). We then construct a drop-down map which recovers the original derivation on the corner. This mechanism transfers sequential innerness from the lifted derivation back to the corner and yields the desired inheritance result for Connes sequential amenability of finite corners.

Definition 5.2.1 (Lift of d using the unbalanced Parseval representative). Let $d : \mathcal{N} \rightarrow X$ be a normal derivation. For $m \in \mathcal{M}$, let

$$\tilde{\Omega}_m = w^*-\lim_F \sum_{i,j \in F} u_i \otimes u_i^* m u_j \otimes u_j^* \in E \bar{\otimes} \mathcal{N} \bar{\otimes} E^*$$

be the normal unbalanced Parseval representative from Theorem 5.1.27. Define

$$\begin{aligned} D : \mathcal{M} &\longrightarrow \tilde{X} = E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^* \\ D(m) &= q_X \left((\text{id}_E \bar{\otimes} d \bar{\otimes} \text{id}_{E^*})(\tilde{\Omega}_m) \right), \end{aligned} \quad (5.62)$$

where $q_X : E \bar{\otimes} X \bar{\otimes} E^* \longrightarrow E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*$ is the canonical quotient map. We call D the *lift* of d .

Proposition 5.2.2. *The map $D : \mathcal{M} \rightarrow \tilde{X}$ is weak*-weak* continuous.*

Proof. Let

$$T_0 := \text{id}_E \bar{\otimes} d \bar{\otimes} \text{id}_{E^*} : E \bar{\otimes} \mathcal{N} \bar{\otimes} E^* \longrightarrow E \bar{\otimes} X \bar{\otimes} E^*.$$

Since $d : \mathcal{N} \rightarrow X$ is weak*-weak* continuous, the functoriality Lemma B.4 implies that T_0 is weak*-weak* continuous. Since

$$D = q_X \circ T_0 \circ \tilde{\Omega}. \quad (5.63)$$

the normality of D follows from normality of the quotient map q_X (see Remark 5.1.5) and the map $\tilde{\Omega} : \mathcal{M} \longrightarrow E \bar{\otimes} \mathcal{N} \bar{\otimes} E^*$ in Theorem 5.1.27. $\square$

Definition 5.2.3 (Matrix coefficients associated with the Parseval frame). Let $(u_i)_{i \in I}$ be the Parseval frame for $E = \mathcal{M}p$ obtained in Theorem 5.1.11. For $m \in \mathcal{M}$ and $i, j \in I$, define

$$a_{ij}(m) := \langle u_i, mu_j \rangle_{\mathcal{N}} = u_i^* mu_j \in \mathcal{N}. \quad (5.64)$$

We call $a_{ij}(m)$ the $\mathcal{N}$ -valued matrix coefficient of m with respect to the frame $(u_i)_{i \in I}$.

Corollary 5.2.4 (Rectangular and square block expansions for D). For every $m \in \mathcal{M}$, let $a_{ij}(m)$ be the $\mathcal{N}$ -valued matrix coefficients from Definition 5.2.3. Then,

$$D(m) = w^* - \lim_{(F,G) \in \mathcal{F}(I)^2} \sum_{i \in F} \sum_{j \in G} u_i \otimes_{\mathcal{N}} d(a_{ij}(m)) \otimes_{\mathcal{N}} u_j^*. \quad (5.65)$$

In particular,

$$D(m) = w^* - \lim_{F \in \mathcal{F}(I)} \sum_{i,j \in F} u_i \otimes_{\mathcal{N}} d(\langle u_i, mu_j \rangle_{\mathcal{N}}) \otimes_{\mathcal{N}} u_j^*. \quad (5.66)$$

Proof. By Lemma 5.61, and by the notation $a_{ij}(m) = \langle u_i, mu_j \rangle_{\mathcal{N}}$, we have

$$\tilde{\Omega}_m = w^* - \lim_{(F,G) \in \mathcal{F}(I)^2} \sum_{i \in F} \sum_{j \in G} u_i \otimes a_{ij}(m) \otimes u_j^*.$$

Now, by applying (5.63), the result follows. $\square$

Theorem 5.2.5 (Derivation identity for D). For all $m_1, m_2 \in \mathcal{M}$,

$$D(m_1 m_2) = m_1 \cdot D(m_2) + D(m_1) \cdot m_2. \quad (5.67)$$

Proof. For any $m \in \mathcal{M}$, let $a_{ij}(m)$ be the $\mathcal{N}$ -valued matrix coefficients of m defined by Equation (5.64). For each $H \in \mathcal{F}(I)$, set

$$q_H := \sum_{\ell \in H} u_{\ell} u_{\ell}^*, \quad m_H := m_1 q_H m_2.$$

By (5.12),

$$m_H = m_1 q_H m_2 \xrightarrow{w^*} m_1 z m_2 \quad \text{in } \mathcal{M}.$$

As D is normal, it follows that

$$D(m_1 z m_2) = w^* - \lim_{H \in \mathcal{F}(I)} D(m_H). \quad (5.68)$$

We claim that

$$D(m_1 z m_2) = D(m_1 m_2).$$

Since $m_2 u_j \in E$ for all $j \in I$, we have $(z)(m_2 u_j) = m_2 u_j$ by Equation (5.10). Therefore,

$$a_{ij}(m_1 z m_2) = u_i^* m_1 z m_2 u_j = u_i^* m_1 m_2 u_j = a_{ij}(m_1 m_2).$$

Then, by the rectangular expansion (5.65), we deduce

$$D(m_1 z m_2) = D(m_1 m_2).$$

Combining this with (5.68), we conclude that

$$D(m_1 m_2) = \text{w}^* - \lim_{H \in \mathcal{F}(I)} D(m_H). \quad (5.69)$$

We now compute $D(m_H)$ for a fixed finite set $H \in \mathcal{F}(I)$. For $i, j \in I$, we have

$$\begin{aligned} a_{ij}(m_H) &= u_i^* m_1 q_H m_2 u_j \\ &= \sum_{\ell \in H} u_i^* m_1 u_\ell u_\ell^* m_2 u_j \\ &= \sum_{\ell \in H} a_{i\ell}(m_1) a_{\ell j}(m_2). \end{aligned}$$

Since the sum over $\ell \in H$ is finite and d is a derivation, we get

$$\begin{aligned} d(a_{ij}(m_H)) &= \sum_{\ell \in H} d(a_{i\ell}(m_1) a_{\ell j}(m_2)) \\ &= \sum_{\ell \in H} [a_{i\ell}(m_1) \cdot d(a_{\ell j}(m_2)) + d(a_{i\ell}(m_1)) \cdot a_{\ell j}(m_2)]. \end{aligned} \quad (5.70)$$

Using the rectangular expansion (5.65) for $D(m_H)$ and substituting (5.70) into it, we obtain

$$D(m_H) = S_{1,H} + S_{2,H}. \quad (5.71)$$

where

$$S_{1,H} = \text{w}^* - \lim_{(F,G) \in \mathcal{F}(I)^2} \sum_{i \in F} \sum_{j \in G} \sum_{\ell \in H} u_i \otimes_{\mathcal{N}} (a_{i\ell}(m_1) \cdot d(a_{\ell j}(m_2))) \otimes_{\mathcal{N}} u_j^* \quad (5.72)$$

and

$$S_{2,H} = \text{w}^* - \lim_{(F,G) \in \mathcal{F}(I)^2} \sum_{i \in F} \sum_{j \in G} \sum_{\ell \in H} u_i \otimes_{\mathcal{N}} (d(a_{i\ell}(m_1)) \cdot a_{\ell j}(m_2)) \otimes_{\mathcal{N}} u_j^*. \quad (5.73)$$

Here, H is fixed and finite; only (F, G) run through $\mathcal{F}(I)^2$.

We first identify $S_{1,H}$. By the balancing relation in $E \bar{\otimes}_{\mathcal{N}} X$, we have

$$u_i \otimes_{\mathcal{N}} (a_{i\ell}(m_1) \cdot d(a_{\ell j}(m_2))) = u_i a_{i\ell}(m_1) \otimes_{\mathcal{N}} d(a_{\ell j}(m_2)).$$

Hence,

$$S_{1,H} = \text{w}^* - \lim_{(F,G) \in \mathcal{F}(I)^2} \sum_{\ell \in H} \sum_{j \in G} \left(\sum_{i \in F} u_i a_{i\ell}(m_1) \right) \otimes_{\mathcal{N}} d(a_{\ell j}(m_2)) \otimes_{\mathcal{N}} u_j^*.$$

For fixed $\ell \in H$, the normal Parseval identity (5.7) gives

$$\begin{aligned} \text{w}^* - \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} u_i a_{i\ell}(m_1) &= \text{w}^* - \lim_{F \in \mathcal{F}(I)} \sum_{i \in F} u_i \langle u_i, m_1 u_\ell \rangle_{\mathcal{N}} \\ &= m_1 u_\ell. \end{aligned}$$

Since H and G are finite at each stage, this weak* convergence may be used in the F -coordinate of the rectangular net. Thus,

$$\begin{aligned} S_{1,H} &= \text{w}^*-\lim_{G \in \mathcal{F}(I)} \sum_{\ell \in H} \sum_{j \in G} m_1 u_\ell \otimes_{\mathcal{N}} d(a_{\ell j}(m_2)) \otimes_{\mathcal{N}} u_j^* \\ &= m_1 \cdot \left(\text{w}^*-\lim_{G \in \mathcal{F}(I)} \sum_{\ell \in H} \sum_{j \in G} u_\ell \otimes_{\mathcal{N}} d(a_{\ell j}(m_2)) \otimes_{\mathcal{N}} u_j^* \right). \end{aligned} \quad (5.74)$$

We show that the expression in parentheses equals $q_H \cdot D(m_2)$. To this end, since H is finite, every sufficiently large $F \in \mathcal{F}(I)$ contains H . Hence, eventually,

$$F \cap H = H.$$

Moreover, by the orthogonality of the range projections $(u_i u_i^*)_{i \in I}$, we have

$$q_H u_i = \begin{cases} u_i, & i \in H, \\ 0, & i \notin H. \end{cases}$$

Therefore,

$$\sum_{i \in F} q_H u_i \otimes_{\mathcal{N}} d(a_{ij}(m_2)) \otimes_{\mathcal{N}} u_j^* = \sum_{i \in F \cap H} u_i \otimes_{\mathcal{N}} d(a_{ij}(m_2)) \otimes_{\mathcal{N}} u_j^*.$$

Since eventually $F \cap H = H$, we obtain

$$\sum_{i \in F} q_H u_i \otimes_{\mathcal{N}} d(a_{ij}(m_2)) \otimes_{\mathcal{N}} u_j^* = \sum_{\ell \in H} u_\ell \otimes_{\mathcal{N}} d(a_{\ell j}(m_2)) \otimes_{\mathcal{N}} u_j^*. \quad (5.75)$$

By rectangular expansion (5.65) for $D(m_2)$ and Equation (5.75), we obtain

$$\begin{aligned} q_H \cdot D(m_2) &= \text{w}^*-\lim_{(F,G) \in \mathcal{F}(I)^2} \sum_{i \in F} \sum_{j \in G} q_H u_i \otimes_{\mathcal{N}} d(a_{ij}(m_2)) \otimes_{\mathcal{N}} u_j^* \\ &= \text{w}^*-\lim_{G \in \mathcal{F}(I)} \sum_{\ell \in H} \sum_{j \in G} u_\ell \otimes_{\mathcal{N}} d(a_{\ell j}(m_2)) \otimes_{\mathcal{N}} u_j^*. \end{aligned}$$

Substituting this in (5.74), we obtain

$$S_{1,H} = m_1 \cdot q_H \cdot D(m_2). \quad (5.76)$$

However, $q_H \rightarrow z$ weak* and the module action is weak* separately continuous. Thus,

$$q_H \cdot D(m_2) \xrightarrow{w^*} z \cdot D(m_2).$$

By (5.10), z acts as the identity on the first tensor leg E of $\tilde{X} = E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*$ from left. So, $z \cdot D(m_2) = D(m_2)$. Consequently,

$$\text{w}^*-\lim_{H \in \mathcal{F}(I)} S_{1,H} = m_1 \cdot D(m_2). \quad (5.77)$$

Similarly, we have

$$\text{w}^*-\lim_{H \in \mathcal{F}(I)} S_{2,H} = D(m_1) \cdot m_2. \quad (5.78)$$

Finally, we obtain

$$\begin{aligned}
D(m_1 m_2) &= \text{w}^* \text{-} \lim_{H \in \mathcal{F}(I)} D(m_H) && \text{by (5.69)} \\
&= \text{w}^* \text{-} \lim_{H \in \mathcal{F}(I)} (S_{1,H} + S_{2,H}) && \text{by (5.71)} \\
&= \text{w}^* \text{-} \lim_{H \in \mathcal{F}(I)} S_{1,H} + \text{w}^* \text{-} \lim_{H \in \mathcal{F}(I)} S_{2,H} \\
&= m_1 \cdot D(m_2) + D(m_1) \cdot m_2 && \text{by (5.77) and (5.78).}
\end{aligned}$$

This proves that D is a derivation. □

Remark 5.2.6. Under the identification Φ (5.24), the derivation identity (5.67) for D may be viewed as an operator identity. More precisely, by Theorem 5.1.19, Φ is injective and is compatible with the induced $\mathcal{M}$ - $\mathcal{M}$ module actions:

$$\Phi(m \cdot \eta) = m \cdot \Phi(\eta), \quad \Phi(\eta \cdot m) = \Phi(\eta) \cdot m.$$

Therefore, the Leibniz identity for D is equivalent to the operator identity

$$\Phi(D(m_1 m_2)) = m_1 \cdot \Phi(D(m_2)) + \Phi(D(m_1)) \cdot m_2.$$

Proposition 5.2.7 (Dropping down D to d). *For the drop down map defined by (5.14), we have*

$$\varepsilon(D(a)) = d(a), \quad (a \in \mathcal{N}).$$

Equivalently, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{D} & \tilde{X} \\
\uparrow i & & \downarrow \varepsilon \\
\mathcal{N} & \xrightarrow{d} & X
\end{array}$$

$$\varepsilon \circ D \circ i = d.$$

Figure 5.3: The lifted derivation and the drop-down mapping.

Proof. By Remark 5.1.13, we may compute the weak* limit in (5.66) over $\mathcal{F}_0(I)$. Thus, using the weak* continuity of ε , we obtain

$$\varepsilon(D(a)) = \text{w}^* \text{-} \lim_{F \in \mathcal{F}_0(I)} \sum_{i,j \in F} \varepsilon(u_i \otimes_{\mathcal{N}} d(\langle u_i, a u_j \rangle_{\mathcal{N}}) \otimes_{\mathcal{N}} u_j^*), \quad (a \in \mathcal{N}).$$

By the definition of ε , this becomes

$$\varepsilon(D(a)) = \text{w}^* \text{-} \lim_{F \in \mathcal{F}_0(I)} \sum_{i,j \in F} (p u_i) \cdot d(\langle u_i, a u_j \rangle_{\mathcal{N}}) \cdot (u_j^* p).$$

By Equation (5.18), all terms vanish except the term with $i = j = 0$. Hence,

$$\varepsilon(D(a)) = p \cdot d(\langle u_0, au_0 \rangle_{\mathcal{N}}) \cdot p. \quad (5.79)$$

Since $u_0 = p$ and $a \in \mathcal{N} = p\mathcal{M}p$, we have

$$\langle u_0, au_0 \rangle_{\mathcal{N}} = \langle p, ap \rangle_{\mathcal{N}} = pap = a.$$

Substituting this into (5.79) and using the fact that p acts as the identity on X , we conclude that

$$\varepsilon(D(a)) = p \cdot d(a) \cdot p = d(a).$$

□

Theorem 5.2.8 (Inheritance of Connes amenability by finite corners). *Let $\mathcal{M}$ be a type II_∞ von Neumann algebra, and let $p \in \mathcal{M}$ be a nonzero finite projection. Put $\mathcal{N} = p\mathcal{M}p$.*

- (i) *If $\mathcal{M}$ is Connes amenable, then $\mathcal{N}$ is Connes amenable.*
- (ii) *If $\mathcal{M}$ is sequentially Connes amenable, then $\mathcal{N}$ is sequentially Connes amenable.*
- (iii) *If $\mathcal{M}$ is approximately Connes amenable, then so is $\mathcal{N}$.*

Proof. Let $X = Y^*$ be a normal dual $\mathcal{N}$ -bimodule and let

$$d : \mathcal{N} \longrightarrow X$$

be a normal derivation. Construct the induced normal dual $\mathcal{M}$ -bimodule

$$\tilde{X} = E \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} E^*$$

as in (5.13), and let

$$D : \mathcal{M} \longrightarrow \tilde{X}$$

be the lifted derivation as in (5.2.1). By Proposition 5.2.2 and Theorem 5.2.5, D is a normal derivation.

- (i) Suppose that $\mathcal{M}$ is Connes amenable. Then, D is inner. Hence, there exists $\Xi \in \tilde{X}$ such that

$$D(m) = m \cdot \Xi - \Xi \cdot m \quad (m \in \mathcal{M}).$$

Applying the weak* continuity of the drop down map ε (5.14) and using Proposition 5.2.7, we obtain, for all $a \in \mathcal{N}$,

$$d(a) = \varepsilon(D(a)) = a \cdot \varepsilon(\Xi) - \varepsilon(\Xi) \cdot a.$$

Thus, d is inner. Therefore, $\mathcal{N}$ is Connes amenable.

- (ii) Suppose that $\mathcal{M}$ is sequentially Connes amenable. Then, there is a sequence (Ξ_k) in $\tilde{X}$ such that

$$D(m) = \text{w}^*\text{-}\lim_k (m \cdot \Xi_k - \Xi_k \cdot m) \quad (m \in \mathcal{M}).$$

Applying ε as before and noting that ε is weak*-weak* continuous, we get, for every $a \in \mathcal{N}$,

$$d(a) = \varepsilon(D(a)) = \text{w}^*\text{-}\lim_k (a \cdot \varepsilon(\Xi_k) - \varepsilon(\Xi_k) \cdot a).$$

Thus, d is weak* sequentially approximately inner. Hence, $\mathcal{N}$ is sequentially Connes amenable.

(iii) The proof is similar to that of part (ii), with nets used in place of sequences.

□

Theorem 5.2.9. *Let $\mathcal{M}$ be a type II_∞ or a semi-finite von Neumann algebra. If $\mathcal{M}$ is sequentially Connes amenable, then it is injective. In particular, $\mathcal{M}$ is Connes amenable.*

Proof. We can assume without loss of generality that $\mathcal{M}$ is a type II_∞ von Neumann algebra. So, it has a faithful normal semi-finite tracial weight $\tau : \mathcal{M}_+ \rightarrow [0, \infty]$. By [81, Definition V.2.1 and Proposition V.1.40], we can find a net (p_α) of finite projections such that $\tau(p_\alpha) < \infty$ for each α and (p_α) increases to 1 in the strong operator topology. Take an arbitrary projection p from the net (p_α) , and let $\mathcal{N}_p = p\mathcal{M}p$ be the corner von Neumann algebra corresponding to p . We note that $\mathcal{N}_p$ has a normal faithful tracial state given by $\tau_p(pmp) = \frac{\tau(pmp)}{\tau(p)}$. By Theorem 5.2.8, $\mathcal{N}_p$ inherits Connes' sequential amenability from $\mathcal{M}$. Thus, by Theorem 4.3.17, $\mathcal{N}_p$ is Connes' amenable. However, the union of all $\mathcal{N}_p$ is weak*-dense in $\mathcal{M}$. So, by [3, Theorem II.2.2.8], $\mathcal{M}$ is injective or Connes amenable. □

5.3 Connes sequential amenability of type III von Neumann algebras

In this section, we extend Theorem 5.2.9 to type III von Neumann algebras. We first need to review the structure of type III von Neumann algebras.

Definition 5.3.1. A *C*-dynamical system* is a triple $(\mathcal{A}, G, \sigma)$ consisting of a C*-algebra $\mathcal{A}$, a locally compact group G , and a homomorphism $\sigma : G \rightarrow \text{Aut}(\mathcal{A})$ such that the action

$$G \times \mathcal{A} \longrightarrow \mathcal{A}, \quad (t, a) \mapsto \sigma_t(a),$$

is continuous.

A *covariant representation* of a C*-dynamical system $(\mathcal{A}, G, \sigma)$ is a pair of representations (π, ρ) on a Hilbert space $\mathcal{H}$ for which

1. π is a *-representation of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$,
2. ρ is a unitary representation of G into $\mathcal{B}(\mathcal{H})$,
3. the “*covariant (or intertwining) relation*” holds, i.e.,

$$\pi(\sigma_t(a)) = \rho(t)\pi(a)\rho(t)^*.$$

Definition 5.3.2. A *W*-dynamical system* is a triple $(\mathcal{M}, G, \sigma)$ consisting of a von Neumann algebra $\mathcal{M}$, a locally compact group G , and a homomorphism σ from G into the *-automorphisms of $\mathcal{M}$ such that the action

$$G \times \mathcal{M} \longrightarrow \mathcal{M}, \quad (t, a) \mapsto \sigma_t(a),$$

is continuous when $\mathcal{M}$ is endowed with the strong operator topology (SOT).

Suppose that $\mathcal{M}$ is represented on a Hilbert space $\mathcal{H}$ and G is represented on $L^2(G)$ by the left regular representation. There is a canonical way to represent both $\mathcal{M}$ and G on the same Hilbert space $L^2(G) \otimes_2 \mathcal{H} \cong L^2(G, \mathcal{H})$, see [3, Section II.10.3] and [82, Section X.1]. We may define a *covariant representation* of the W*-dynamical system $(\mathcal{M}, G, \sigma)$ by the pair (π_σ, λ) , where:

1. π_σ is the $*$ -representation of $\mathcal{M}$ into $\mathcal{B}(L^2(G) \otimes_2 \mathcal{H})$ given by

$$(\pi_\sigma(a)f)(s) = \sigma_s(a)(f(s)),$$

for all $a \in \mathcal{M}$, $f \in L^2(G, \mathcal{H})$, and $s \in G$;

2. λ is the unitary representation of G into $\mathcal{B}(L^2(G) \otimes_2 \mathcal{H})$ defined by

$$(\lambda(t)f)(s) = f(t^{-1}s).$$

3. one can verify that the *covariant (or intertwining)* relation holds, i.e.,

$$\pi_\sigma(\sigma_t(a)) = \lambda(t)\pi_\sigma(a)\lambda(t)^*.$$

The W^* -crossed product $\mathcal{M} \overline{\rtimes}_\sigma G$ is defined to be the von Neumann algebra generated by the images of $\pi_\sigma(\mathcal{M})$ and $\lambda(G)$ inside $\mathcal{B}(L^2(G) \otimes_2 \mathcal{H})$, that is, Recall that

$$\mathcal{M} \overline{\rtimes}_\sigma G = [\pi_\sigma(\mathcal{M}) \cup \lambda(G)]'' . \quad (5.80)$$

Let $\mathcal{A}$ be the unital $*$ -algebra generated by the set $\pi_\sigma(\mathcal{M}) \cup \lambda(G)$. We call $\mathcal{A}$ the *algebraic crossed core*. By the von Neumann bicommutant theorem [56, Theorem 4.2.5], we have

$$\mathcal{M} \overline{\rtimes}_\sigma G = \mathcal{A}'' = \overline{\mathcal{A}}^{w*} . \quad (5.81)$$

Definition 5.3.3. A *densely defined operator* T on a Hilbert space $\mathcal{H}$ is a linear operator $T : \mathcal{D}(T) \rightarrow \mathcal{H}$ where its domain $\mathcal{D}(T)$ is a dense linear subspace of $\mathcal{H}$. We always assume such an operators is unbounded.

If $S : \mathcal{D}(S) \rightarrow \mathcal{H}$ is another densely defined operator, then

$$\begin{aligned} \mathcal{D}(T + S) &= \mathcal{D}(T) \cap \mathcal{D}(S) \\ \mathcal{D}(TS) &= \{ \xi \in \mathcal{D}(S) : S\xi \in \mathcal{D}(T) \} . \end{aligned}$$

However, we note that these new sets might not be dense in $\mathcal{H}$. We say that S is an extension of T if $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $S = T$ on $\mathcal{D}(T)$. The operator S is *closed* if its graph

$$\Gamma(S) = \{ \xi \oplus S\xi : \xi \in \mathcal{D}(S) \}$$

is closed in $\mathcal{H} \oplus \mathcal{H}$. The operator T is *closable* if it has a closed extension S . The *adjoint* T^* of T is specified by its domain

$$\mathcal{D}(T^*) = \{ \xi \in \mathcal{H} : \eta \rightarrow \langle T\eta, \xi \rangle \text{ is a bounded linear functional on } \mathcal{D}(T) \} .$$

When $\xi \in \mathcal{D}(T^*)$, $T^*\xi$ is defined by the relation $\langle T\eta, \xi \rangle = \langle \eta, T^*\xi \rangle$. One can show that T^* is always closed. The operator T is *self-adjoint* if $T = T^*$ and $\mathcal{D}(T) = \mathcal{D}(T^*)$. The *resolvent set* $\rho(T)$ for T is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ has a bounded inverse. The spectrum of T is the set $\sigma(T) = \mathbb{C} \setminus \rho(T)$. For self-adjoint operators the spectrum contains only real numbers. There is a version of spectral theorem and functional calculus for self-adjoint unbounded operators, and based on that one can obtain a polar decomposition for closed unbounded operators (see [3, Section I.7]). A self-adjoint operator T is *positive* if $\langle T\xi, \xi \rangle \geq 0$ for any $\xi \in \mathcal{D}(T)$. When T is positive, $\sqrt{T}$ exists, and we have $T = \sqrt{T} \circ \sqrt{T}$ with $\mathcal{D}(T) \subseteq \mathcal{D}(\sqrt{T})$.

Example 5.3.4. Let $\mathcal{M}$ be a von Neumann algebra and φ be a normal faithful semi-finite weight on it. We continue to use the notations from Section 3.2. $\Lambda_\phi(\mathfrak{n}_\phi^* \cap \mathfrak{n}_\phi)$ is a pre-Hilbert space under the inner product $\langle x + N_\phi, y + N_\phi \rangle = \phi(y^*x)$. Complete it to the Hilbert space $\mathfrak{H}_\phi$, and let π_ϕ be the standard representation of $\mathcal{M}$ on it, see [82, Theorem VII.2.7 and Proposition VII.1.4]. By [64], the densely defined operator

$$\begin{aligned} S_0 : \Lambda_\phi(\mathfrak{n}_\phi^* \cap \mathfrak{n}_\phi) &\longrightarrow \Lambda_\phi(\mathfrak{n}_\phi^* \cap \mathfrak{n}_\phi) \\ a + N_\phi &\longrightarrow a^* + N_\phi \end{aligned}$$

is closable, and it has a closed extension S with the polar decomposition $S = J\nabla^{\frac{1}{2}}$ where $\nabla = S^*S$ and J is a conjugate linear isometry with period two on $\mathfrak{H}$. By [82, Theorems VI.1.19 and VIII.1.2], the mapping

$$\begin{aligned} \sigma : \mathbb{R} &\longrightarrow \text{Aut}(\mathcal{M}) \\ \sigma_t(a) &= \nabla^{it} a \nabla^{-it} \end{aligned}$$

is a continuous group homomorphism, called Tomita-Takesaki modular homomorphism, which makes $(\mathcal{M}, \mathbb{R}, \sigma)$ into a W^* -dynamical system. Here, $G = \mathbb{R}$ is the additive group of real numbers with the Lebesgue measure as its Haar measure. For any abelian group G , let $\hat{G}$ be the Pontryagin dual of G , which is the set of all continuous group homomorphisms from G into the unit circle $\mathbb{T}$. By [31, Theorem 4.3], $\hat{G}$ is the same as the Gelfand spectrum of the abelian Banach algebra $L^1(G)$, and can be viewed as locally compact abelian group under the pointwise multiplication and the weak* topology (as a subset of $L^\infty(G)$). In the case of $G = \mathbb{R}$, by [31, P. 107], we have

$$\begin{aligned} \hat{\mathbb{R}} &\cong \mathbb{R} \\ p &\mapsto \langle \cdot, p \rangle, \quad \langle t, p \rangle = e^{2\pi i p t}, \quad (t \in \mathbb{R}, p \in \hat{\mathbb{R}}). \end{aligned}$$

Now, consider the left translation representation

$$\begin{aligned} \lambda : \mathbb{R} &\longrightarrow \mathcal{B}(L^2(\mathbb{R})) \\ t &\longmapsto \lambda_t, \quad (\lambda_t f)(s) = f(-t + s), \quad t, s \in \mathbb{R}, f \in L^2(\mathbb{R}). \end{aligned} \tag{5.82}$$

and the multiplication representation

$$\begin{aligned} \mu : \hat{\mathbb{R}} &\longrightarrow \mathcal{B}(L^2(\mathbb{R})) \\ p &\longmapsto \mu_p, \quad (\mu_p f)(s) = \overline{\langle s, p \rangle} f(s), \quad p, s \in \mathbb{R}, f \in L^2(\mathbb{R}). \end{aligned} \tag{5.83}$$

By [82, Section X.2], we may define the following representations:

1. a $*$ -representation $\pi_\sigma : \mathcal{M} \longrightarrow \mathcal{B}(L^2(\mathbb{R}) \otimes_2 \mathfrak{H})$ given by

$$(\pi_\sigma(a)f)(s) = \sigma_s(a)f(s), \quad a \in \mathcal{M}, f \in L^2(\mathbb{R}, \mathfrak{H}), s \in G;$$

2. a unitary representation $\lambda : \mathbb{R} \longrightarrow \mathcal{B}(L^2(\mathbb{R}) \otimes_2 \mathfrak{H})$ given by

$$(\lambda_t f)(s) = f(s - t), \quad t, s \in \mathbb{R};$$

3. a unitary representation $\mu : \widehat{\mathbb{R}} \longrightarrow \mathcal{B}(L^2(\mathbb{R}) \otimes_2 \mathfrak{H})$ given by

$$(\mu_p f)(s) = \overline{\langle s, p \rangle} f(s), \quad p \in \widehat{\mathbb{R}}, \quad s \in \mathbb{R}.$$

Here, we have the covariant relation

$$\pi_\sigma(\sigma_t(a)) = \lambda_t \pi_\sigma(a) \lambda_t^*,$$

and the Heisenberg-Weyl commutation relation

$$\lambda_t \mu_p \lambda_t^* = \langle t, p \rangle \mu_p, \quad (t, p \in \mathbb{R}). \quad (5.84)$$

Note the commutation formula holds both on $\mathcal{B}(L^2(\mathbb{R}))$ and $\mathcal{B}(L^2(\mathbb{R}) \otimes_2 \mathfrak{H})$. This is another formulation of the so called *Schrödinger picture* or *the canonical commutation relation* of quantum mechanics; see [70, pp. 274–275] for more details. If we define

$$\begin{aligned} \hat{\sigma} : \widehat{\mathbb{R}} &\longrightarrow \text{Aut}(\mathcal{M} \overline{\rtimes}_\sigma \mathbb{R}) \\ \hat{\sigma}_p(T) &:= \mu_p T \mu_p^*, \end{aligned}$$

then by [82, Theorem X.2.3], $(\mathcal{M} \overline{\rtimes}_\sigma \mathbb{R}, \widehat{\mathbb{R}}, \hat{\sigma})$ is another W^* -dynamical system, and we can form the second crossed product $(\mathcal{M} \overline{\rtimes}_\sigma \mathbb{R}) \overline{\rtimes}_{\hat{\sigma}} \widehat{\mathbb{R}}$, which is isomorphic to $\mathcal{M} \overline{\otimes} \mathcal{B}(L^2(\mathbb{R}))$.

Remark 5.3.5. Let $\mathcal{M}$ be a type III von Neumann algebra. By the proof of [48, Corollary 13.2.10], $\mathcal{M} \cong \mathcal{M} \overline{\otimes} \mathcal{B}(L^2(\mathbb{R}))$. By Example 5.3.4, one can form the crossed product $\mathcal{N} = \mathcal{M} \overline{\rtimes}_\sigma \mathbb{R}$. By [80] and [82, Theorem XII.1.1], $\mathcal{N}$ is a type II_∞ von Neumann algebra. Hence, $\mathcal{M}$ can be written as a weak* crossed product $\mathcal{N} \overline{\rtimes}_\alpha \mathbb{R}$, where $\mathbb{R}$ is the locally compact group of real numbers under addition and with the Lebesgue measure as its Haar measure. By Example 5.3.4, we should note that $\alpha = \hat{\sigma}$ and $\mathcal{M} \cong \mathcal{N} \overline{\rtimes}_{\hat{\sigma}} \widehat{\mathbb{R}}$. Also, $(\mu_p)_{p \in \widehat{\mathbb{R}}}$ provides a one parameter group of unitaries such that $\mu_p T \mu_p^* = \hat{\sigma}_p(T)$ for each $T \in \mathcal{N} = \mathcal{M} \overline{\rtimes}_\sigma \mathbb{R}$, and $\mathcal{M}$ is generated by $\mathcal{N}$ and the unitaries $(\mu_p)_{p \in \widehat{\mathbb{R}}}$. Note that $\mathcal{N}$ is generated by an isometric copy $\mathcal{P} = \pi_\sigma(\mathcal{M})$ of $\mathcal{M}$ and the one parameter group of unitaries $(\lambda_t)_{t \in \mathbb{R}}$ such that $\lambda_t a \lambda_t^* = \sigma_t(a)$ for any $a \in \mathcal{P}$ and $t \in \mathbb{R}$. Finally, we have the following isomorphisms:

$$\mathcal{M} \cong \mathcal{M} \overline{\otimes} \mathcal{B}(L^2(\mathbb{R})) \cong (\mathcal{M} \overline{\rtimes}_\sigma \mathbb{R}) \overline{\rtimes}_{\hat{\sigma}} \widehat{\mathbb{R}}.$$

Theorem 5.3.6. *Let $\mathcal{M}$ be a type III von Neumann algebra inside $\mathcal{B}(\mathcal{H})$. If $\mathcal{M}$ is sequentially Connes amenable, then it is injective. In particular, $\mathcal{M}$ is Connes amenable.*

Proof. By Remark 5.3.5, $\mathcal{N} = \mathcal{M} \overline{\rtimes}_\sigma \mathbb{R}$ is a semi-finite von Neumann algebra, and $\mathcal{M} = \mathcal{N} \overline{\rtimes}_{\hat{\sigma}} \widehat{\mathbb{R}}$. We show that $\mathcal{N}$ is sequentially Connes amenable. Then, by Theorem 5.2.9, $\mathcal{N}$ is injective. Since $\widehat{\mathbb{R}} \cong \mathbb{R}$ is amenable, by [3, Corollary IV.2.2.16], $\mathcal{M} = \mathcal{N} \overline{\rtimes}_{\hat{\sigma}} \widehat{\mathbb{R}}$ is injective.

Let X be a normal dual $\mathcal{N}$ -bimodule, and let $\delta : \mathcal{N} \longrightarrow X$ be a normal derivation. By Remark 5.3.5, $\mathcal{N}$ is generated by a von Neumann subalgebra $\mathcal{P}$ isomorphic to $\mathcal{M}$ and a one-parameter group of unitaries $(\lambda_t)_{t \in \mathbb{R}}$ such that

$$\lambda_t a \lambda_{-t} = \sigma_t(a), \quad t \in \mathbb{R}, \quad a \in \mathcal{P}. \quad (5.85)$$

By (5.80), a basic algebraic generating element $n \in \mathcal{N}$ has one of the following forms

- $a \in \mathcal{P}$
- $a + \lambda_t$ with $a \in \mathcal{P}$ and $t \in \mathbb{R}$
- $a\lambda_t$ and $\lambda_t a$ with $a \in \mathcal{P}$ and $t \in \mathbb{R}$

We first show that in all three cases, $\delta(n)$ is a weak* limit of a sequence of inner derivations. As in [12], define

$$\xi_s := \lambda_s \cdot \delta(\lambda_{-s}), \quad s \in \mathbb{R}. \quad (5.86)$$

For $a \in \mathcal{P}$ and $s \in \mathbb{R}$, we have

$$\begin{aligned} a \cdot \xi_s &= a\lambda_s \cdot \delta(\lambda_{-s}) \\ &= \lambda_s(\lambda_{-s}a\lambda_s) \cdot \delta(\lambda_{-s}) \\ &= \lambda_s\sigma_{-s}(a) \cdot \delta(\lambda_{-s}). \end{aligned} \quad (5.87)$$

On the other hand, since $\lambda_{-s}a = \sigma_{-s}(a)\lambda_{-s}$ and δ is a derivation,

$$\begin{aligned} \delta(\lambda_{-s}) \cdot a + \lambda_{-s} \cdot \delta(a) &= \delta(\lambda_{-s}a) \\ &= \delta(\sigma_{-s}(a)\lambda_{-s}) \\ &= \delta(\sigma_{-s}(a)) \cdot \lambda_{-s} + \sigma_{-s}(a) \cdot \delta(\lambda_{-s}). \end{aligned} \quad (5.88)$$

Combining (5.87) and (5.88), we obtain

$$\delta(a) = a \cdot \xi_s - \xi_s \cdot a + \lambda_s \cdot \delta(\lambda_{-s}a\lambda_s) \cdot \lambda_{-s}, \quad a \in \mathcal{P}. \quad (5.89)$$

For $t, s \in \mathbb{R}$, one can verify

$$\begin{aligned} \delta(\lambda_t) &= \lambda_t \cdot \xi_s - \xi_{t+s} \cdot \lambda_t \\ &= \lambda_t \cdot \xi_s - \xi_s \cdot \lambda_t + (\xi_s - \xi_{t+s}) \cdot \lambda_t. \end{aligned} \quad (5.90)$$

Since the map $t \mapsto \lambda_t$ is weak* continuous and δ is normal, the map $t \mapsto \delta(\lambda_t)$ is weak* continuous. Let $\mathbb{Q}_d$ denote the amenable discrete additive group of rational numbers. By [59, Theorem 4.16], there is a sequence of finite subsets $(F_k)_{k \in \mathbb{N}}$ of $\mathbb{Q}$, called a *summing sequence*, such that

1. $F_1 \subseteq F_2 \subseteq \dots$,
2. $\mathbb{Q} = \bigcup_{k=1}^{\infty} F_k$,
3. for each $t \in \mathbb{Q}$,

$$\frac{|((t + F_k) \setminus F_k) \cup (F_k \setminus (t + F_k))|}{|F_k|} \longrightarrow 0. \quad (5.91)$$

Following [12], for each Følner set F_k in (5.91), define

$$\zeta_k = \zeta_{F_k} := \frac{1}{|F_k|} \sum_{s \in F_k} \xi_s. \quad (5.92)$$

The sequence (ζ_k) is bounded in X . Hence, it has a weak* cluster point ζ . Thus, by passing to a subnet, we may write

$$\zeta = \text{w}^*\text{-}\lim_{\gamma \in \Gamma} \zeta_{k(\gamma)} = \text{w}^*\text{-}\lim_{\gamma \in \Gamma} \frac{1}{|F_{k(\gamma)}|} \sum_{s \in F_{k(\gamma)}} \xi_s. \quad (5.93)$$

For each $t \in \mathbb{Q}$, (5.91) gives

$$\|\zeta_{t+F_k} - \zeta_{F_k}\| \leq \frac{|((t+F_k) \setminus F_k) \cup (F_k \setminus (t+F_k))|}{|F_k|} \|\delta\| \xrightarrow{k \rightarrow \infty} 0. \quad (5.94)$$

In particular,

$$\zeta_{t+F_k(\gamma)} - \zeta_{F_k(\gamma)} \xrightarrow{w^*} 0, \quad t \in \mathbb{Q}. \quad (5.95)$$

Averaging (5.90) over $s \in F_k$ gives

$$\delta(\lambda_t) = \lambda_t \cdot \zeta_k - \zeta_k \cdot \lambda_t + (\zeta_{F_k} - \zeta_{t+F_k}) \cdot \lambda_t.$$

Passing to the subnet in the above and using (5.93) and (5.95), we obtain

$$\delta(\lambda_t) = \lambda_t \cdot \zeta - \zeta \cdot \lambda_t, \quad t \in \mathbb{Q}. \quad (5.96)$$

By weak* continuity of both sides as functions of t , it follows that

$$\delta(\lambda_t) = \lambda_t \cdot \zeta - \zeta \cdot \lambda_t, \quad t \in \mathbb{R}. \quad (5.97)$$

Let $\text{ad}_\zeta(b) = b \cdot \zeta - \zeta \cdot b$ for all $b \in \mathcal{N}$, and define

$$\begin{aligned} G : \mathcal{N} &\longrightarrow X \\ G &:= \delta - \text{ad}_\zeta. \end{aligned}$$

Then, G is a normal derivation. Moreover, by (5.97),

$$G(\lambda_t) = 0, \quad t \in \mathbb{R}. \quad (5.98)$$

We also observe that for $a \in \mathcal{P}$ and $s \in \mathbb{R}$, the derivation identity gives

$$\begin{aligned} G(\sigma_{-s}(a)) &= G(\lambda_{-s} a \lambda_s) \\ &= G(\lambda_{-s}) \cdot a \lambda_s + \lambda_{-s} \cdot G(a) \cdot \lambda_s + \lambda_{-s} a \cdot G(\lambda_s) \\ &= \lambda_{-s} \cdot G(a) \cdot \lambda_s. \end{aligned} \quad (5.99)$$

Therefore,

$$\lambda_s \cdot G(\sigma_{-s}(a)) \cdot \lambda_{-s} = G(a), \quad a \in \mathcal{P}, \quad s \in \mathbb{R}. \quad (5.100)$$

Hence, for every finite set $F \subseteq \mathbb{Q}$,

$$\frac{1}{|F|} \sum_{s \in F} \lambda_s \cdot G(\sigma_{-s}(a)) \cdot \lambda_{-s} = G(a), \quad a \in \mathcal{P}. \quad (5.101)$$

We now perform the same averaging procedure for the other pair of Weyl–Heisenberg unitaries, i.e., $(\mu_p)_{p \in \mathbb{R}}$. For $s \in \mathbb{R}$, define

$$\eta_s := \mu_s \cdot G(\mu_{-s}). \quad (5.102)$$

Since $(\mu_s)_{s \in \mathbb{R}}$ is a unitary group and G is bounded, the family $(\eta_s)_{s \in \mathbb{R}}$ is bounded in X . Using the same summing sequence $(F_k)_{k \in \mathbb{N}}$ from (5.91), put

$$\nu_k = \nu_{F_k} := \frac{1}{|F_k|} \sum_{s \in F_k} \eta_s. \quad (5.103)$$

Passing to a subnet if necessary, we may assume that

$$\nu = \text{w}^* - \lim_{\alpha \in A} \nu_{k(\alpha)} = \text{w}^* - \lim_{\alpha \in A} \frac{1}{|F_{k(\alpha)}|} \sum_{s \in F_{k(\alpha)}} \eta_s. \quad (5.104)$$

For $p, s \in \mathbb{R}$, using $\mu_{-s}\mu_p = \mu_{p-s}$ and the derivation identity, we have

$$\begin{aligned} G(\mu_p) &= \mu_p \cdot \eta_s - \eta_{p+s} \cdot \mu_p \\ &= \mu_p \cdot \eta_s - \eta_s \cdot \mu_p + (\eta_s - \eta_{p+s}) \cdot \mu_p. \end{aligned} \quad (5.105)$$

Averaging (5.105) over $s \in F_k$ gives

$$G(\mu_p) = \mu_p \cdot \nu_k - \nu_k \cdot \mu_p + (\nu_{F_k} - \nu_{p+F_k}) \cdot \mu_p. \quad (5.106)$$

For $p \in \mathbb{Q}$, the Følner condition (5.91) gives

$$\|\nu_{p+F_k} - \nu_{F_k}\| \longrightarrow 0.$$

Passing to our subnet in (5.106), we obtain

$$G(\mu_p) = \mu_p \cdot \nu - \nu \cdot \mu_p, \quad p \in \mathbb{Q}. \quad (5.107)$$

By weak* continuity of both sides as functions of p , it follows that

$$G(\mu_p) = \mu_p \cdot \nu - \nu \cdot \mu_p, \quad p \in \mathbb{R}.$$

Equivalently,

$$\delta(\mu_p) - \text{ad}_\zeta(\mu_p) = \text{ad}_\nu(\mu_p), \quad p \in \mathbb{R}. \quad (5.108)$$

Next, we show that ν commutes with the unitaries λ_t . By the Weyl–Heisenberg relation (5.84),

$$\lambda_t \mu_{-s} \lambda_{-t} = e^{-2\pi i s t} \mu_{-s}, \quad (s, t \in \mathbb{R}).$$

Since $G(\lambda_t) = 0$, applying G to the above equation gives

$$\lambda_t \cdot G(\mu_{-s}) \cdot \lambda_{-t} = e^{-2\pi i s t} G(\mu_{-s}).$$

We now compute

$$\begin{aligned} \lambda_t \cdot \eta_s \cdot \lambda_{-t} &= \lambda_t \mu_s \cdot G(\mu_{-s}) \cdot \lambda_{-t} \\ &= (\lambda_t \mu_s \lambda_{-t}) \cdot (\lambda_t \cdot G(\mu_{-s}) \cdot \lambda_{-t}) \\ &= e^{2\pi i s t} \mu_s \cdot e^{-2\pi i s t} G(\mu_{-s}) \\ &= \eta_s. \end{aligned} \quad (5.109)$$

Averaging over $s \in F_{k(\alpha)}$ and passing to the weak* limit gives

$$\lambda_t \cdot \nu \cdot \lambda_{-t} = \nu, \quad t \in \mathbb{R}. \quad (5.110)$$

So,

$$\lambda_t \cdot \nu = \nu \cdot \lambda_t, \quad t \in \mathbb{R}.$$

Equivalently,

$$\text{ad}_\nu(\lambda_t) = 0, \quad t \in \mathbb{R}. \quad (5.111)$$

Define

$$\begin{aligned} H : \mathcal{N} &\longrightarrow X \\ H &= \delta - \text{ad}_{\zeta+\nu}. \end{aligned} \quad (5.112)$$

By (5.98), (5.108) and (5.111), we conclude

$$H(\lambda_t) = 0, \quad H(\mu_p) = 0, \quad t, p \in \mathbb{R}. \quad (5.113)$$

Also for every $b \in \mathcal{N}$ and every $s \in \mathbb{R}$, the Leibnitz property of H yields

$$\begin{aligned} H(\mu_{-s}b\mu_s) &= H(\mu_{-s}) \cdot b\mu_s + \mu_{-s} \cdot H(b) \cdot \mu_s + \mu_{-s}b \cdot H(\mu_s) \\ &= \mu_{-s} \cdot H(b) \cdot \mu_s. \end{aligned} \quad (5.114)$$

Hence,

$$\mu_s \cdot H(\mu_{-s}b\mu_s) \cdot \mu_{-s} = H(b), \quad s \in \mathbb{R}. \quad (5.115)$$

Consequently, for every finite set $F \subseteq \mathbb{Q}$,

$$\frac{1}{|F|} \sum_{s \in F} \mu_s \cdot H(\mu_{-s}b\mu_s) \cdot \mu_{-s} = H(b). \quad (5.116)$$

By observing (5.113) and (5.114), we note that

$$H(u^{-1}au) = u^{-1} \cdot H(a) \cdot u, \quad a \in \mathcal{N}, \quad (5.117)$$

whenever $u \in \mathcal{N}$ is invertible and $H(u) = 0$.

So far, for the derivation H , we have

$$H(n) = \begin{cases} H(a), & n = a \in \mathcal{P}, \\ H(\mu_p) = 0, & n = \mu_p, p \in \mathbb{R}, \\ H(\lambda_t) = 0, & n = \lambda_t, t \in \mathbb{R}, \\ H(a)\lambda_t, & n = a\lambda_t, \text{ for } a \in \mathcal{P} \text{ and } t \in \mathbb{R}, \\ \lambda_t H(a), & n = \lambda_t a, \text{ for } a \in \mathcal{P} \text{ and } t \in \mathbb{R}. \end{cases} \quad (5.118)$$

Note that the last two equalities follow from (5.113) and the Leibniz identity for H .

Since $\mathcal{P}$ is isomorphic to $\mathcal{M}$ and $\mathcal{M}$ is sequentially Connes amenable, $\mathcal{P}$ is sequentially Connes amenable. Thus, applied to the normal derivation $H|_{\mathcal{P}} : \mathcal{P} \rightarrow X$, there exists a sequence $(y_j)_{j=1}^\infty$ in X such that

$$H(a) = \text{w}^*\text{-}\lim_{j \rightarrow \infty} (a \cdot y_j - y_j \cdot a), \quad a \in \mathcal{P}. \quad (5.119)$$

We now need to pass from the sequential implementation of $H|_{\mathcal{P}}$ to a sequential implementation of H on the crossed product $\mathcal{N}$.

Construction claim. There exists a sequence $(x_k)_{k=1}^\infty$ in X such that

$$H(a) = w^* - \lim_{k \rightarrow \infty} (a \cdot x_k - x_k \cdot a), \quad \forall a \in \mathcal{P}, \quad (5.120)$$

$$\lambda_t \cdot x_k - x_k \cdot \lambda_t \xrightarrow[k \rightarrow \infty]{\|\cdot\|} 0, \quad \forall t \in \mathbb{Q}. \quad (5.121)$$

For $k, j \in \mathbb{N}$, define

$$x_{k,j} := \frac{1}{|F_k|^2} \sum_{s \in F_k} \sum_{r \in F_k} \lambda_s \mu_r \cdot y_j \cdot \mu_{-r} \lambda_{-s}, \quad (5.122)$$

where each F_k is the finite Følner summing set from (5.91). Let $a \in \mathcal{P}$ and

$$u_{s,r} := \lambda_s \mu_r, \quad u_{s,r}^* = \mu_{-r} \lambda_{-s}.$$

Then,

$$\begin{aligned} a \cdot x_{k,j} - x_{k,j} \cdot a &= \frac{1}{|F_k|^2} \sum_{s \in F_k} \sum_{r \in F_k} \left(a u_{s,r} \cdot y_j \cdot u_{s,r}^* - u_{s,r} \cdot y_j \cdot u_{s,r}^* a \right) \\ &= \frac{1}{|F_k|^2} \sum_{s \in F_k} \sum_{r \in F_k} u_{s,r} \cdot \left(u_{s,r}^* a u_{s,r} \cdot y_j - y_j \cdot u_{s,r}^* a u_{s,r} \right) \cdot u_{s,r}^* \\ &= \frac{1}{|F_k|^2} \sum_{s \in F_k} \sum_{r \in F_k} \lambda_s \mu_r \cdot \text{ad}_{y_j} (\mu_{-r} \lambda_{-s} a \lambda_s \mu_r) \cdot \mu_{-r} \lambda_{-s}. \end{aligned}$$

However, by (5.119),

$$\text{ad}_{y_j} (\mu_{-r} \lambda_{-s} a \lambda_s \mu_r) \xrightarrow[j \rightarrow \infty]{w^*} H(\mu_{-r} \lambda_{-s} a \lambda_s \mu_r). \quad (5.123)$$

Thus,

$$a \cdot x_{k,j} - x_{k,j} \cdot a \xrightarrow[j \rightarrow \infty]{w^*} \frac{1}{|F_k|^2} \sum_{s \in F_k} \sum_{r \in F_k} \lambda_s \mu_r \cdot H(\mu_{-r} \lambda_{-s} a \lambda_s \mu_r) \cdot \mu_{-r} \lambda_{-s}. \quad (5.124)$$

It follows from Equations (5.113) and (5.117) that

$$H(\mu_{-r} \lambda_{-s} a \lambda_s \mu_r) = \mu_{-r} \lambda_{-s} \cdot H(a) \cdot \lambda_s \mu_r. \quad (5.125)$$

After applying (5.116) twice to the sum on the right hand side of (5.124), we conclude that

$$H(a) = w^* - \lim_{j \rightarrow \infty} (a \cdot x_{k,j} - x_{k,j} \cdot a), \quad (k \in \mathbb{N}, a \in \mathcal{P}). \quad (5.126)$$

We now estimate the λ_t -commutator. For $t \in \mathbb{Q}$,

$$\begin{aligned} \lambda_t \cdot x_{k,j} \cdot \lambda_{-t} &= \frac{1}{|F_k|^2} \sum_{s \in F_k} \sum_{r \in F_k} \lambda_t \lambda_s \mu_r \cdot y_j \cdot \mu_{-r} \lambda_{-s} \lambda_{-t} \\ &= \frac{1}{|F_k|^2} \sum_{s \in F_k} \sum_{r \in F_k} \lambda_{t+s} \mu_r \cdot y_j \cdot \mu_{-r} \lambda_{-(t+s)} \\ &= \frac{1}{|F_k|^2} \sum_{u \in t + F_k} \sum_{r \in F_k} \lambda_u \mu_r \cdot y_j \cdot \mu_{-r} \lambda_{-u}. \end{aligned} \quad (5.127)$$

Hence,

$$\begin{aligned} \lambda_t \cdot x_{k,j} \cdot \lambda_{-t} - x_{k,j} &= \frac{1}{|F_k|^2} \left(\sum_{u \in t+F_k} \sum_{r \in F_k} \lambda_u \mu_r \cdot y_j \cdot \mu_{-r} \lambda_{-u} \right. \\ &\quad \left. - \sum_{s \in F_k} \sum_{r \in F_k} \lambda_s \mu_r \cdot y_j \cdot \mu_{-r} \lambda_{-s} \right). \end{aligned} \quad (5.128)$$

In the above, cancellations happen whenever the outer index is common, i.e., $u = s \in (t + F_k) \cap F_k$. Thus,

$$\begin{aligned} \lambda_t \cdot x_{k,j} \cdot \lambda_{-t} - x_{k,j} &= \frac{1}{|F_k|^2} \left(\sum_{u \in (t+F_k) \setminus F_k} \sum_{r \in F_k} \lambda_u \mu_r \cdot y_j \cdot \mu_{-r} \lambda_{-u} \right. \\ &\quad \left. - \sum_{s \in F_k \setminus (t+F_k)} \sum_{r \in F_k} \lambda_s \mu_r \cdot y_j \cdot \mu_{-r} \lambda_{-s} \right). \end{aligned} \quad (5.129)$$

Note that the module actions by the unitaries are isometric. So, after taking norm from sides and applying the triangle inequality, each summand has norm $\|y_j\|$. As the inner index runs through $r \in F_k$, the outer indices run over the symmetric difference $(t + F_k) \triangle F_k$. Therefore,

$$\|\lambda_t \cdot x_{k,j} \cdot \lambda_{-t} - x_{k,j}\| \leq \frac{|(t + F_k) \triangle F_k| |F_k|}{|F_k|^2} \|y_j\|.$$

Equivalently,

$$\|\lambda_q \cdot x_{k,j} - x_{k,j} \cdot \lambda_q\| \leq \frac{|(q + F_k) \triangle F_k|}{|F_k|} \|y_j\|, \quad (k, j \in \mathbb{N}, \quad q \in \mathbb{Q}). \quad (5.130)$$

Let $a \in \mathcal{P}$ and $f \in X_*$. For each finite Følner set F_k put

$$\mathcal{B}_k := \{\mu_{-r} \lambda_{-s} a \lambda_s \mu_r : s, r \in F_k\}.$$

Then, $\mathcal{B}_k$ is finite and contained in $\mathcal{P}$. Hence, by (5.119), choose n_k sufficiently large so that

$$|\langle b \cdot y_{n_k} - y_{n_k} \cdot b - H(b), f \rangle| < \frac{1}{k}, \quad b \in \mathcal{B}_k, \quad (5.131)$$

and at the same time in the Følner condition (5.91) for $\varepsilon = \frac{1}{k(1+\|y_{n_k}\|)}$, we have

$$\frac{|(t + F_{n_k}) \triangle F_{n_k}|}{|F_{n_k}|} \leq \frac{1}{k(1 + \|y_{n_k}\|)}. \quad (5.132)$$

Define

$$x_k := x_{n_k, n_k} = \frac{1}{|F_{n_k}|^2} \sum_{s \in F_{n_k}} \sum_{r \in F_{n_k}} \lambda_s \mu_r \cdot y_{n_k} \cdot \mu_{-r} \lambda_{-s}. \quad (5.133)$$

It follows from (5.130) and (5.132) that

$$\|\lambda_q \cdot x_k - x_k \cdot \lambda_q\| < \frac{1}{k} \quad (q \in \mathbb{Q}). \quad (5.134)$$

Hence, $\text{ad}_{x_k}(\lambda_t) \xrightarrow{\|\cdot\|} 0$, and $\text{ad}_{x_k}(\lambda_t) \xrightarrow{w^*} 0$. Consequently,

$$H(\lambda_q) = 0 = w^* \lim_{k \rightarrow \infty} (\lambda_q \cdot x_k - x_k \cdot \lambda_q), \quad \forall q \in \mathbb{Q}. \quad (5.135)$$

Now, we show that Equation (5.135) holds for any $t \in \mathbb{R}$. Recall that $H : \mathcal{N} \rightarrow X$ is normal. By Proposition B.1, it has a bounded linear pre-adjoint given by

$$\begin{aligned} H_* : X_* &\rightarrow \mathcal{N}_*, \\ \langle H_*(f), b \rangle &= \langle f, H(b) \rangle, \quad (f \in X_*, b \in \mathcal{N}). \end{aligned}$$

Recall that by Remark 2.2.6 for each $x \in X$ and $f \in X_*$ the vector functionals

$$\begin{aligned} x \odot f : \mathcal{N} &\rightarrow \mathbb{C}, \quad \langle x \odot f, b \rangle = \langle f, b \cdot x \rangle, \\ f \odot x : \mathcal{N} &\rightarrow \mathbb{C}, \quad \langle f \odot x, b \rangle = \langle f, x \cdot b \rangle, \end{aligned}$$

are normal. Hence, for each member of the sequence $(x_k)_{k=1}^\infty$ and any $f \in X_*$,

$$\varphi_{k,f} = x_k \odot f - f \odot x_k$$

is a sequence of normal functionals on $\mathcal{N}$. For $a \in \mathcal{P}$, we have

$$\begin{aligned} \langle a, H_*(f) \rangle &= \langle H(a), f \rangle \\ &= \lim_k \langle a \cdot x_k - x_k \cdot a, f \rangle \\ &= \lim_k \langle a, x_k \odot f - f \odot x_k \rangle \\ &= \lim_k \langle a, \varphi_{k,f} \rangle. \end{aligned}$$

Similarly, by (5.135), for all $q \in \mathbb{Q}$, we obtain

$$0 = \langle \lambda_q, H_*(f) \rangle = \lim_k \langle \lambda_q, \varphi_{k,f} \rangle. \quad (5.136)$$

Hence,

$$\varphi_{k,f} \xrightarrow{\sigma(\mathcal{N}_*, \mathcal{P})} H_*(f). \quad (5.137)$$

Consequently, $(\varphi_{k,f})_{k=1}^\infty$ is a bounded sequence of normal functionals when restricted to $\mathcal{P}$. By polar decomposition of normal functionals [81, Section III.4], we let $|\varphi_{k,f}|$ denote the absolute value of $\varphi_{k,f}$ on $\mathcal{N}$, which is a positive normal functional. By [81, Proposition III.4.6] and [56, Corollary 3.3.4], we have $\|\varphi_{k,f}\| = \| |\varphi_{k,f}| \| = |\varphi_{k,f}|(1)$. Since $1 \in \mathcal{P} \subseteq \mathcal{N}$, we deduce that $(\varphi_{k,f})_{k=1}^\infty$ is a bounded sequence of normal functionals on $\mathcal{N}$. So, let φ_f be a weak* limit point of them on $\mathcal{N}$. It is a normal functional by [81, Corollary III.5.2]. Since $t \rightarrow \lambda_t$ is weak* continuous and φ_f is normal, it follows from (5.136) that

$$\langle \lambda_t, \varphi_f \rangle = 0, \quad \forall t \in \mathbb{R},$$

and hence,

$$H(\lambda_t) = 0 = w^* \lim_k (\lambda_t \cdot x_k - x_k \cdot \lambda_t), \quad \forall t \in \mathbb{R}. \quad (5.138)$$

Now, $H(\lambda_t a) = \lambda_t H(a)$ and $H(a \lambda_t) = H(a) \lambda_t$ by the Leibniz property of H and the above equation. We claim that for the generators $a \lambda_t$ and $\lambda_t a$ of $\mathcal{N}$, where $a \in \mathcal{P}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned} H(a \lambda_t) &= w^* \lim_k (a \lambda_t \cdot x_k - x_k \cdot a \lambda_t), \\ H(\lambda_t a) &= w^* \lim_k (\lambda_t a \cdot x_k - x_k \cdot \lambda_t a). \end{aligned} \quad (5.139)$$

We prove the first equation, as the second one can be established in a similar manner. By the Leibniz property of derivations, weak*-separate continuity of module actions, and the sequential formulas proved so far, we have

$$\begin{aligned}
H(\lambda_t a) &= H(\lambda_t) \cdot a + \lambda_t \cdot H(a) \\
&= \text{w}^*\text{-}\lim_k (\lambda_t \cdot x_k - x_k \cdot \lambda_t) \cdot a + \lambda_t \cdot [\text{w}^*\text{-}\lim_k (a \cdot x_k - x_k \cdot a)] \\
&= \text{w}^*\text{-}\lim_k (\lambda_t \cdot x_k \cdot a - x_k \cdot \lambda_t a) + (\lambda_t a \cdot x_k - \lambda_t \cdot x_k \cdot a) \\
&= \text{w}^*\text{-}\lim_k (\lambda_t a \cdot x_k - x_k \cdot \lambda_t a) .
\end{aligned}$$

Therefore, H is sequentially inner on all generators of $\mathcal{N}$, i.e., for any n as in Equation (5.118), we have

$$H(n) = \text{w}^*\text{-}\lim_k (n \cdot x_k - x_k \cdot n). \quad (5.140)$$

In other words, for any $f \in X_*$ and n as in the above, we have

$$\begin{aligned}
\langle n, H_*(f) \rangle &= \langle H(n), f \rangle \\
&= \lim_k \langle n \cdot x_k - x_k \cdot n, f \rangle \\
&= \lim_k \langle n, x_k \odot f - f \odot x_k \rangle \\
&= \lim_k \langle n, \varphi_{k,f} \rangle \\
&= \langle n, \varphi_f \rangle
\end{aligned}$$

Thus, the normal functionals $H_*(f)$ and φ_f agree on all generators of $\mathcal{N}$; hence, they are equal on $\mathcal{N}$. As a result, for any $b \in \mathcal{N}$ and any $f \in X_*$

$$\begin{aligned}
\langle H(b), f \rangle &= \langle b, H_*(f) \rangle \\
&= \langle b, \varphi_f \rangle \\
&= \lim_k \langle b, x_k \odot f - f \odot x_k \rangle \\
&= \lim_k \langle b \cdot x_k - x_k \cdot b, f \rangle.
\end{aligned}$$

Therefore,

$$H(b) = \text{w}^*\text{-}\lim_k (b \cdot x_k - x_k \cdot b), \quad \forall b \in \mathcal{N}.$$

Consequently, H is weak* sequentially inner on $\mathcal{N}$. Since $H = \delta - \text{ad}_{\zeta+\nu}$, we have $\delta = \text{ad}_{\zeta+\nu} + H$. Hence, the derivation δ is weak* sequentially inner as well. Note that δ is implemented by the sequence $(x_k + \zeta + \nu)_k$. This shows that $\mathcal{N}$ is sequentially Connes amenable. So, by Theorem 5.2.9 $\mathcal{N}$ is Connes amenable. Finally, by [3, Corollary IV.2.216], $\mathcal{M} = \mathcal{N} \overline{\otimes}_{\hat{\sigma}} \hat{\mathbb{R}}$ is Connes amenable. $\square$

To summarize the results obtained in Chapter 4 and so far in the current chapter, we conclude our final result as follows.

Theorem 5.3.7. *Let $\mathcal{M}$ be a von Neumann algebra. If $\mathcal{M}$ is sequentially Connes amenable, then it is Connes amenable.*

Proof. (i) If $\mathcal{M}$ is type I, then by Proposition 4.2.7 $\mathcal{M}$ is Connes amenable.

(ii) If $\mathcal{M}$ is type II_1 or type II_∞ , by Theorems 4.3.17 and 5.2.9 $\mathcal{M}$ is also Connes amenable.

(iii) If $\mathcal{M}$ is type III, it follows from Theorem 5.3.6 that $\mathcal{M}$ is Connes amenable. □

Theorem 5.3.7 immediately yields the following theorem.

Theorem 5.3.8. *Let A be a C^* -algebra. If A is sequentially amenable, then A is amenable.*

Proof. Let $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ be the universal representation of A , and $\mathcal{M} = (\pi(A))'' = A^{**} = \overline{A}^{wk*}$ be the enveloping von Neumann algebra of A as in Proposition 4.2.8. By Theorem 4.2.15, $\mathcal{M}$ is sequentially Connes amenable. By Theorem 5.3.7, $\mathcal{M}$ is Connes amenable. Now, by Connes theorem [12], $\mathcal{M}$ injective. Hence, by [10, Theorem 3], A must be amenable. □

Corollary 5.3.9. *For a discrete group G , the followings are equivalent:*

- (i) G is amenable
- (ii) $\ell^1(G)$ is sequentially amenable.
- (iii) $C_r^*(G)$ is sequentially amenable
- (iv) $VN(G)$ is sequentially Connes amenable

In particular, $C^*(\mathbb{F}_2)$, $C_r^*(\mathbb{F}_2)$, and $\mathcal{L}(\mathbb{F}_2)$ cannot be sequentially amenable.

Proof. Equivalence (i) $\iff$ (ii) is [33, Theorem 3.2].

The implication (iii) $\implies$ (i) is deduced by Theorem 5.3.8 and [59, P. 44]. For the converse, see [59, P. 44].

The implication (i) $\implies$ (iv) holds by [76, Theorem 5.1.27]. Conversely, whenever $VN(G)$ is sequentially Connes amenable, it is Connes amenable by Theorem 5.3.7. Then, by [76, Theorem 5.1.27] G is amenable. □

Since von Neumann algebras are already C^* -algebras, from Theorem 5.3.8 we have the following.

Corollary 5.3.10. *For a von Neumann algebra $\mathcal{M}$, the followings are equivalent:*

- (i) $\mathcal{M}$ is sequentially Johnson's amenable.
- (ii) $\mathcal{M}$ is Johnson's amenable.
- (iii) $\mathcal{M}$ is of type $I_{\mathbb{F}}$, i.e., $\mathcal{M} \cong \sum_{i=1}^k \mathbb{M}_{N_i} \otimes A_i$, where $\mathbb{M}_{N_i}$ is $N_i \times N_i$ matrices over $\mathbb{C}$ and A_i is an abelian von Neumann algebra.

In particular, there are no type II and III von Neumann algebras which are sequentially Johnsons' amenable.

Proof. This is a consequence of Theorem 5.3.8 and [76, Theorem 7.1.9]. □

Chapter 6

Generalized amenability of operator algebras arising in non-commutative function theory and quantum semigroups

In this chapter, we use the techniques of non-commutative function theory to study the amenability of certain operator algebras generated by a set of non-commuting isometries. We also study compact quantum groups and semigroups, and show that the Cuntz–Toeplitz C^* -algebra $\mathcal{E}_d$ does not admit the structure of a compact quantum group. Consequently, amenability defined via left-invariant means is not applicable to it. However, we will show $\mathcal{E}_d$ is Johnson amenable.

6.1 Non-commutative function theory

The non-commutative function theory (NCFT) is a generalization of the classical multi-variable complex analysis to the setting of non-commutative variables. Roughly speaking, if $f : \mathbb{C}^d \rightarrow \mathbb{C}$ is an analytic function in d -variables, we replace the d -tuple of commuting variables $(z_1, \dots, z_d)$ with a d -tuple of non-commutative matrices $(Z_1, \dots, Z_d)$ of the same size. Note that the output $f(Z_1, \dots, Z_d)$ is a matrix with the same size as Z_i 's. We remark that NCFT is different from the functional calculus since one does not need to invoke the spectrum of operators. Instead, it relies on generalizing the properties of polynomial functions. The origins of NCFT goes back to the investigation of two important problems in functional analysis:

- (i) generalizing polynomial functional calculus to holomorphic functional calculus for a d -tuple of non-commuting operators $(T_1, \dots, T_d)$ without using the joint spectrum of them, see [83].
- (ii) Voiculescu's efforts on the isomorphism problem of group von Neuman algebras of free groups, which asks whether $VN(\mathbb{F}_m)$ and $VN(\mathbb{F}_n)$ are isomorphic if $n \neq m$ and $m, n \geq 2$, see [87].

Since Voiculescu's pioneering work [87], the NCFT has become an active area of research. For example, NCFT is now an essential tool in non-commutative calculus of quantum mechanics.

Definition 6.1.1. Let T_j , $j = 1, \dots, d$, be bounded operators on the same Hilbert space $\mathcal{H}$. Denote by I_d and $I_{\mathcal{H}}$ the identity operators on $\mathbb{C}^d$ and $\mathcal{H}$, respectively. Then, a row operator $T := (T_1, \dots, T_d) : \mathcal{H} \otimes \mathbb{C}^d \longrightarrow \mathcal{H}$ is called

- a *row isometry* if

$$T^*T = \begin{bmatrix} T_1^* \\ \vdots \\ T_d^* \end{bmatrix} \begin{bmatrix} T_1 & \dots & T_d \end{bmatrix} = I_{\mathcal{H}} \otimes I_d.$$

In other words, $T_k^*T_j = \delta_{k,j}I_{\mathcal{H}}$.

- and a *row contraction* if

$$TT^* = \begin{bmatrix} T_1 & \dots & T_d \end{bmatrix} \begin{bmatrix} T_1^* \\ \vdots \\ T_d^* \end{bmatrix} = \sum_{i=1}^d T_i T_i^* \leq I_{\mathcal{H}}.$$

- a *Cuntz or row unitary* if it is a surjective row isometry, i.e.,

$$T^*T = I_{\mathcal{H}} \otimes I_d, \quad \text{and} \quad TT^* = I_{\mathcal{H}}$$

Remark 6.1.2. We remark that a row operator T acts on column vectors by

$$T := (T_1, \dots, T_d) : \mathcal{H} \otimes \mathbb{C}^d \longrightarrow \mathcal{H}$$

$$T \begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix} = \begin{bmatrix} T_1 & \dots & T_d \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_d \end{bmatrix} = T_1 h_1 + \dots + T_d h_d.$$

Since T^* departs from $\mathcal{H}$ and lands into $\mathcal{H} \otimes \mathbb{C}^d$, its action on simple vectors is given by

$$T^* := \begin{bmatrix} T_1^* \\ \vdots \\ T_d^* \end{bmatrix} : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathbb{C}^d$$

$$T^* h = \begin{bmatrix} T_1^* \\ \vdots \\ T_d^* \end{bmatrix} h = \begin{bmatrix} T_1^* h \\ \vdots \\ T_d^* h \end{bmatrix}$$

The *row norm* $\|\cdot\|_{row}$ of the row operator $T := (T_1, \dots, T_d) : \mathcal{H} \otimes \mathbb{C}^d \longrightarrow \mathcal{H}$ is defined by

$$\|T\|_{row} = \left\| \sum_{j=1}^d T_j T_j^* \right\|^{\frac{1}{2}}.$$

This norm can be obtained by embedding the Cartesian product $(\mathcal{B}(\mathcal{H}))^d$ into the matrix algebra $M_d(\mathcal{B}(\mathcal{H}))$ in the following way.

$$\begin{aligned} \pi : (\mathcal{B}(\mathcal{H}))^d &\longrightarrow M_d(\mathcal{B}(\mathcal{H})) \\ T = (T_1, \dots, T_d) &\longrightarrow \pi(T) = \begin{bmatrix} T_1 & \cdots & T_d \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

We have

$$\|T\|_{row} := \|\pi(T)\|.$$

Under this identification, the row contractions coincide with the unit ball of $(\mathcal{B}(\mathbb{H}_d^2))^d$ under the norm $\|\cdot\|_{row}$, see [68]. Note that any row isometry $T = (T_1, \dots, T_d)$ is also a row contraction, so T is in the unit ball of $(\mathcal{B}(\mathbb{H}_d^2))^d$ under the norm $\|\cdot\|_{row}$. We should remark that row isometries do not exist on finite dimensional Hilbert spaces; otherwise, each T_j would be a unitary, hence $\|T\|_{row} = \sqrt{d}$, violating the inequality $\|T\|_{row} \leq 1$.

Definition 6.1.3. Fix a natural number $d \in \mathbb{N}$. By the *non-commutative (NC) universe*, we mean

$$\mathbb{C}_{\mathbb{N}}^d = \bigsqcup_{n=1}^{\infty} \mathbb{C}_n^d; \quad \mathbb{C}_n^d = \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times d}.$$

In the above, any element of $\mathbb{C}_n^d = \mathbb{C}^{n \times n} \otimes \mathbb{C}^{1 \times d}$ is a d -tuple $Z = (Z_1, \dots, Z_d)$ of $n \times n$ matrices with complex entries. When $d = 1$, we omit the super script. Thus, $\mathbb{C}_n$ denotes the algebra of $n \times n$ matrices, and $\mathbb{C}_{\mathbb{N}} = \bigsqcup_{n=1}^{\infty} \mathbb{C}_n$. There are several norms one can impose on $\mathbb{C}_{\mathbb{N}}^d$. The most important one to us is the *row-norm* defined by

$$\|Z\|_{row} = \left\| \sum_{j=1}^d Z_j Z_j^* \right\|^{\frac{1}{2}},$$

where the norm on the right side is the operator norm of matrices. We mainly work with functions on the *open unit row ball* which is defined by

$$\mathbb{B}_{\mathbb{N}}^d = \bigsqcup_{n=1}^{\infty} \mathbb{B}_n^d; \quad \mathbb{B}_n^d = \{Z = (Z_1, \dots, Z_d) \in \mathbb{C}_n^d : \|Z\|_{row} < 1\}$$

We may regard the NC unit row ball as a graded semigroup (in each matricial level).

Lemma 6.1.4. *The pointwise leveled multiplication*

$$\begin{aligned} m : \mathbb{B}_{\mathbb{N}}^d \times \mathbb{B}_{\mathbb{N}}^d &\longrightarrow \mathbb{B}_{\mathbb{N}}^d \\ ((X_i)_{i=1}^d, (Y_i)_{i=1}^d) &\longrightarrow (X_i Y_i)_{i=1}^d, \end{aligned}$$

where $(X_i)_{i=1}^d$ and $(Y_i)_{i=1}^d$ are d -tuples of $n \times n$ matrices, makes $\mathbb{B}_{\mathbb{N}}^d$ into a level-wise semigroup.

Proof. Note that

$$\| (X_i Y_i)_{i=1}^d \|_{row}^2 = \left\| \sum_1^d (X_i Y_i) (X_i Y_i)^* \right\| = \left\| \sum_1^d X_i Y_i Y_i^* X_i^* \right\|.$$

However, $Y_i Y_i^* < I$ since $(Y_i)_{i=1}^d \in \mathbb{B}_n^d$. So,

$$\| (X_i Y_i)_{i=1}^d \|_{row}^2 \leq \left\| \sum_1^d X_i X_i^* \right\| < 1.$$

□

To make $\mathbb{B}_{\mathbb{N}}^d$ into a semigroup independent of levels, we may use the pointwise tensor product.

Lemma 6.1.5. *The tensor product multiplication*

$$\begin{aligned} \otimes : \mathbb{B}_{\mathbb{N}}^d \times \mathbb{B}_{\mathbb{N}}^d &\longrightarrow \mathbb{B}_{\mathbb{N}}^d \\ \otimes((X_i)_{i=1}^d, (Y_i)_{i=1}^d) &= (X_i \otimes Y_i)_{i=1}^d, \end{aligned}$$

where the d -tuples $(X_i)_{i=1}^d$ and $(Y_i)_{i=1}^d$ can be in different levels $\mathbb{B}_n^d$ and $\mathbb{B}_m^d$ respectively, makes $\mathbb{B}_{\mathbb{N}}^d$ into a semigroup.

Proof. Note that

$$\| (X_i \otimes Y_i)_{i=1}^d \|_{row}^2 = \left\| \sum_1^d (X_i \otimes Y_i) (X_i \otimes Y_i)^* \right\| = \left\| \sum_1^d X_i X_i^* \otimes Y_i Y_i^* \right\|.$$

However, $Y_i Y_i^* < I_m$ as $(Y_i)_{i=1}^d \in \mathbb{B}_m^d$. So,

$$\| (X_i \otimes Y_i)_{i=1}^d \|_{row}^2 \leq \left\| \sum_1^d (X_i X_i^*) \otimes I_m \right\|.$$

Also, $\sum_1^d X_i X_i^* < I_n$ since $(X_i)_{i=1}^d \in \mathbb{B}_n^d$. Thus,

$$\| (X_i \otimes Y_i)_{i=1}^d \|_{row}^2 < \| I_n \otimes I_m \| = \| I_{nm} \| = 1.$$

That is, $(X_i \otimes Y_i)_{i=1}^d$ is also a row contraction as a d -tuple of $(nm) \times (nm)$ matrices. □

Definition 6.1.6. A function $f : \mathbb{B}_{\mathbb{N}}^d \longrightarrow \mathbb{C}_{\mathbb{N}}$ is called an *NC function* if it has the following properties (see [50] for more details):

1. f is graded: $f(\mathbb{B}_n^d) \subseteq \mathbb{C}_n$ for all n ,
2. f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$, $\forall X \in \mathbb{B}_n^d, \forall Y \in \mathbb{B}_m^d$,
3. f respects joint similarities: $f(S^{-1} Z S) = S^{-1} f(Z) S$,

In 3, $S^{-1} Z S = (S^{-1} Z_1 S, \dots, S^{-1} Z_d S) \in \mathbb{B}_n^d$ for $Z = (Z_1, \dots, Z_d) \in \mathbb{B}_n^d$ and any invertible scalar $n \times n$ matrix S .

Example 6.1.7. (i) Any polynomial in d non-commuting variables, for example $p(x, y, z) = xyz + z^2y$, defines an NC function.

(ii) In non-commutative probability [88], NC-random variables (see Definition A.1.1) are NC functions.

Definition 6.1.8. (i) The *NC Hardy space* $\mathbb{H}_d^2$ is defined as the set of all formal power series in several non-commuting variables $Z := (Z_1, \dots, Z_d) \in \mathbb{B}_{\mathbb{N}}^d$ with square summable complex coefficients. Namely, any $h \in \mathbb{H}_d^2$ is a power series of the form:

$$h(Z) = \sum_{\omega \in \mathbb{F}^d} \hat{h}_{\omega} Z^{\omega}, \quad (6.1)$$

where $\hat{h}_{\omega} \in \mathbb{C}$ and

$$\sum_{\omega \in \mathbb{F}^d} |\hat{h}_{\omega}|^2 < +\infty. \quad (6.2)$$

Here, $\mathbb{F}^d$ denotes the *free monoid*, the set of all words in the d letters $\{1, \dots, d\}$, with the product defined by the concatenation of words. The unit element of $\mathbb{F}^d$ is the empty word denoted by $\emptyset$. For $Z := (Z_1, \dots, Z_d)$, the *free monomial* Z^{ω} is defined as $Z^{\omega} = Z_{i_1} \cdots Z_{i_n}$ if $\omega = i_1 \cdots i_n$, $i_j \in \{1, \dots, d\}$. When $\omega = \emptyset$, we let $Z^{\emptyset} := I_n$, where I_n is the identity matrix of the same size of Z_{i_j} 's. The members of the NC Hardy space are in fact those NC functions on the unit row ball with the square summable power series representations.

(ii) Let $f := \sum_{\omega \in \mathbb{F}^d} \hat{f}_{\omega} Z^{\omega}$ and $g := \sum_{\omega \in \mathbb{F}^d} \hat{g}_{\omega} Z^{\omega}$ be in $\mathbb{H}_d^2$, the inner product of them is given by

$$\langle f, g \rangle = \sum_{\omega \in \mathbb{F}^d} \hat{f}_{\omega} \overline{\hat{g}_{\omega}}.$$

(iii) Left free shifts are isometries of left multiplication by the independent variables on $\mathbb{H}_d^2$, i.e., $L_k = M_{Z_k}^L$ for $k = 1, \dots, d$, and $M_{Z_k}^L(f)(Z) = Z_k f(Z)$. Since we have d non-commuting variables $Z_1, \dots, Z_d$, we see that the ranges of L_k 's are pairwise orthogonal

$$L_k^* L_j = \delta_{k,j} I_{\mathbb{H}_d^2}, \quad (6.3)$$

where $\delta_{k,j}$ is the usual Kronecker delta. Thus, $L = (L_1, \dots, L_d)$ is a row isometry. We should mention that $\mathbb{H}_d^2$ has a natural basis elements $e_{\alpha} \cong Z^{\alpha}$ for any $\alpha \in \mathbb{F}^d$. We call $e_{\emptyset}$ the vacuum vector. For $L = (L_1, \dots, L_d)$, we define the operator L^{α} similar to the free monomial Z^{α} . Now, L^{α} acts on e_{β} by juxtaposition, e.g., $L^{\alpha}(e_{\emptyset}) = Z^{\alpha}$ and $L^{\alpha}(e_{\beta}) = Z^{\alpha\beta}$.

Remark 6.1.9. (i) By [61, Chapter 7], [70], and [88, p.10], $\mathbb{H}_d^2$ and left shifts can be defined alternatively. Consider the Hilbert space $\mathbb{C}^d$ and denote by $(\mathbb{C}^d)^{n \otimes}$ the n -fold tensor product of $\mathbb{C}^d$, i.e., $(\mathbb{C}^d)^{n \otimes} = \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$ and $(\mathbb{C}^d)^{0 \otimes} = \mathbb{C}$. Then, the NC Hardy space is the full *Fock space*

$$\mathbb{H}_d^2 = \oplus_{n=0}^{\infty} (\mathbb{C}^d)^{n \otimes},$$

and the left free shifts L_k can be interpreted as the *left creation operators* given by $L_k(e_i) = e_k \otimes e_i$, where e_j 's are the usual bases of $\mathbb{C}^d$. The adjoints of L_k 's are the *left annihilation operators*.

- (ii) Recall that the classical unilateral shift has a unitary dilation, which is the bilateral shift. A similar phenomenon happens in the non-commutative multi variable setting. By [66] and [51, pp. 1161–1162], one can find a d -tuple of row isometries U_i such that they dilate L_i 's. Precisely, we can find a d -tuple of isometries $(U_1, \dots, U_d)$ on the NC Hardy space such that $U_k^* U_j = \delta_{k,j} I_{\mathbb{H}_d^2}$ and $\sum_{j=1}^d U_j U_j^* = I_{\mathbb{H}_d^2}$. Hence, $U = (U_1, \dots, U_d)$ is row unitary. We will use notations L for $(L_1, \dots, L_d)$ and U for $(U_1, \dots, U_d)$ in the sequel.

Let $\mathbb{D}$ be the open unit ball of the complex numbers and $\mathbb{T}$ be its boundary. Recall that $A(\mathbb{D})$ is the disk algebra, the Banach space of all analytic functions on the open unit disk $\mathbb{D}$ which have continuous extension to the closure of $\mathbb{D}$, and $H^\infty(\mathbb{D})$ denotes the Banach space of all bounded analytic functions on the unit disc $\mathbb{D}$. In addition, the Toeplitz C^* -algebra $\mathcal{E}$ is the universal C^* -algebra generated by a non-surjective isometry and the identity operator. In the following, we use the operator algebra and operator space theory to relate $C(\mathbb{T})$ and $\mathcal{E}$.

Remark 6.1.10. The Laurent transformation

$$\begin{aligned} \pi : L^\infty(\mathbb{T}) &\longrightarrow \mathcal{B}(L^2(\mathbb{T})) \\ \pi(f) &= M_f, \quad M_f(g) = fg, \quad \forall g \in L^2(\mathbb{T}), \end{aligned}$$

is an isometric unital $*$ -isomorphism from $L^\infty(\mathbb{T})$ onto the range of π , see [56, Example 2.5.1]. For a continuous function $f \in C(\mathbb{T})$, one has $\|f\|_\infty = \|f\|_{sup}$, so when we restrict the domain of π to $C(\mathbb{T})$, the range of π is a C^* -algebra. Thus,

$$\begin{aligned} \pi : C(\mathbb{T}) &\longrightarrow \mathcal{B}(L^2(\mathbb{T})) \\ \pi(f) &= M_f, \quad M_f(g) = fg, \quad \forall g \in L^2(\mathbb{T}) \end{aligned} \tag{6.4}$$

is an isometric unital $*$ -isomorphism from $C(\mathbb{T})$ onto its range. Recall that the classical Hardy space H^2 is the space of square summable power series. By considering the coefficients of each power series, we see that $H^2 \cong \ell^2(\mathbb{N}_0)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By the Plancherel theorem [31, Theorem 4.26], one has $L^2(\mathbb{T}) \cong \ell^2(\mathbb{Z})$. Hence, we have the natural embedding of the Hilbert spaces $H^2 \cong \ell^2(\mathbb{N}_0) \leq \ell^2(\mathbb{Z}) \cong L^2(\mathbb{T})$. Let $p \in \mathcal{B}(L^2(\mathbb{T}))$ be the projection onto H^2 , and consider the embedding $V : H^2 \longrightarrow L^2(\mathbb{T})$. We have $p = V^*$; however, $V \notin \mathcal{B}(L^2(\mathbb{T}))$. For each $f \in C(\mathbb{T})$ define

$$\begin{aligned} T_f &= V^* M_f V \\ &= p M_f|_{H^2}. \end{aligned}$$

The operator T_f is called the *compression* of the Laurent transform M_f to H^2 , see [56, P. 98]. By Hartman-Wintner theorem [56, Theorem 3.5.7], we have $\|T_f\| = \|f\|$. Therefore, we can define the Toeplitz transformation

$$\begin{aligned} \phi : C(\mathbb{T}) &\longrightarrow \mathcal{B}(H^2) \\ \phi(f) &= T_f \\ &= p M_f|_{H^2} \\ &= p \pi(f)|_{H^2} \end{aligned} \tag{6.5}$$

which is a unital isometric $*$ -linear isomorphism from $C(\mathbb{T})$ onto $\phi(C(\mathbb{T}))$. We note that ϕ is no longer an algebraic homomorphism, see [56, P. 100]. For example, consider the identity mapping

$\iota(z) = z$ on the unit circle. Then, $T_\iota = S$ is the unilateral (forward) shift on H^2 . It is well-known that

$$\phi(\iota)^* \phi(\iota) = T_\iota^* T_\iota = S^* S = I,$$

and

$$\phi(\iota) \phi(\iota)^* = T_\iota T_\iota^* = S S^* < I.$$

However,

$$\iota^* \iota = 1 = \iota \iota^*.$$

The above relations show that ϕ is not multiplicative. By Stienspring's dilation theorem [60, Theorem 4.1], the $*$ -homomorphism π in 6.4 is the homomorphism extension of the linear transformation ϕ in 6.5. The Cuntz-Toeplitz C^* -algebra $\mathcal{E}$ is the C^* -algebra generated by the range of ϕ ; i.e., the C^* -algebra generated by all the Toeplitz operators T_f , where $f \in C(\mathbb{T})$. By Coburn theorem [56, Theorem 3.5.18], $\mathcal{E}$ is the C^* -algebra generated by the unilateral shift. Moreover, it is the universal C^* -algebra generated by an isometry v which is not a unitary; i.e., $v^* v = 1$ but $vv^* \neq 1$. By [56, Theorem 3.5.11] we have the C^* -isomorphism

$$\frac{\mathcal{E}}{\mathcal{K}(H^2)} \cong C(\mathbb{T}),$$

where $\mathcal{K}(H^2)$ is the ideal of compact operators on H^2 .

Since the unilateral shift S acts as the left multiplication by the independent variable z on the Hardy space H^2 , we denote S by L . By the Toeplitz transform 6.5, we have $\iota \xrightarrow{\phi} L$. Let $\mathbb{A}$ be the unital operator algebra generated by L in $\mathcal{E}$, i.e., $\mathbb{A} = \overline{\text{Alg}[I, L]}^{\|\cdot\|}$. By the Toeplitz transform ϕ , we have

$$\overline{A(\mathbb{D}) + A(\mathbb{D})}^{\|\cdot\|} \xrightarrow{\phi} \overline{\mathbb{A}^* + \mathbb{A}}^{\|\cdot\|}$$

As in [20, Example 4.3] or [41, P. 54], by using the Stone-Weierstrass theorem, the left hand side in the above is $C(\mathbb{T})$, while the right hand side is an operator space $\mathcal{C}$. By [56, Theorem 3.5.11], $\mathcal{C}$ does not coincide with $\mathcal{E}$. So,

$$C(\mathbb{T}) = \overline{A(\mathbb{D}) + A(\mathbb{D})}^{\|\cdot\|} \xrightarrow{\phi} \overline{\mathbb{A}^* + \mathbb{A}}^{\|\cdot\|} = \mathcal{C}. \quad (6.6)$$

Remark 6.1.11. In the multi-dimensional setting, where we have a d -tuple of left shifts, i.e., $L = (L_1, \dots, L_d)$, we can construct the various operator algebras generated by them paralleling the classical spaces through the transformation ϕ . In the following table, the superscripts $\|\cdot\|$ and *wot* mean that the closure is being taking in operator norm topology and weak operator topology on $\mathcal{B}(\mathbb{H}_d^2)$ respectively. The notation L is used for the family of left shift operators $(L_1, \dots, L_d)$, and $U = (U_1, \dots, U_d)$ is the row-unitary dilation of L as describes before.

Non-commutative	Nomenclature	Counterpart of classical
$\mathbb{A}_d = \overline{\text{Alg}[I, L]}^{\ \cdot\ }$	free disk algebra	$A(\mathbb{D})$
$\mathbb{H}_d^\infty = \overline{\mathbb{A}_d}^{wot}$	free Hardy algebra	$H^\infty(\mathbb{D})$
$\mathcal{E}_d = C^*\{I, L\}$	Cuntz-Toeplitz C*-algebra	$\mathcal{E}$
$\mathcal{O}_d = C^*\{I, U\}$	Cuntz C*-algebra	$C(\mathbb{T})$

Table 6.1: Non-commutative algebras and their classical counterparts.

We remark that $\mathbb{H}_d^\infty$ is the multiplier algebra of $\mathbb{H}_d^2$ by [25, Proposition 6.2] and [45, P. 14]. By [61, Corollary 5.24], $\mathbb{H}_d^\infty$ is closed in the weak operator topology. The Cuntz-Toeplitz C*-algebra $\mathcal{E}_d$ and the Cuntz C*-algebra $\mathcal{O}_d$ were first introduced and studied by Cuntz [14], and they have been a main object in the area of the non-commutative Hardy spaces and dilation theory of row contractions. By [20, Proposition 4.2 and Example 4.3], an important property of the NC disk algebra is the *semi-Dirichlet property* expressed by the following inclusion,

$$\mathbb{A}_d^* \mathbb{A}_d \subseteq \overline{\mathbb{A}_d^* + \mathbb{A}_d}^{\|\cdot\|} = \mathcal{C}_d, \quad (6.7)$$

which is an application of the Equation (6.3).

6.2 Approximate amenability in non-commutative function theory

One can define a notion of derivation for NC functions, analogous to directional derivatives in multivariable calculus. This derivative turns out to be a powerful tool in studying the analytic and structural properties of NC functions.

Definition 6.2.1. Let $f : \mathbb{B}_{\mathbb{N}}^d \rightarrow \mathbb{C}_{\mathbb{N}}$ be an NC function. By [50, Section 2.2], we can assume the upper right triangular block matrices are in the domain of f . So, for $\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \mathbb{B}_{\mathbb{N}}^d$ we have

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} f(X) & D \\ 0 & f(Y) \end{bmatrix}. \quad (6.8)$$

The matrix D is called the *NC derivative* of f at (X, Y) in the direction of Z and is denoted by $D := \nabla f(X, Y)(Z)$. One can show that $\nabla f(X, Y)(Z)$, as a function of Z , is linear.

Example 6.2.2. We can use the Equation (6.8) symbolically to compute $\nabla f(X, Y)(Z)$. However, there are other ways to calculate it, see [50]. For example, by [50, Theorem 7.2] we can compute the k -th NC derivative of f at the $k+1$ -tuple $(Y, \dots, Y)$ in the direction of the k -tuple $(Z, \dots, Z)$ by

$$\nabla^k f(Y, \dots, Y)(Z, \dots, Z) = \frac{1}{k!} \frac{d^k}{dt^k} f(Y + tZ)|_{t=0}. \quad (6.9)$$

In Particular, we can compute $\nabla f(Y, Y)(Z)$ in this way. With this formula, by letting $Y = 0$ and $Z \in \mathbb{B}_{\mathbb{N}}^d$, one can prove that any element $f \in \mathbb{H}_d^2$ has the following Taylor expansion ([50, Theorem 7.2]):

$$f(Z) = \sum_{k=0}^{\infty} \nabla^k f(0, \dots, 0)(Z, \dots, Z). \quad (6.10)$$

The series converges absolutely and uniformly on compacta in the unit row ball see [50, Chapter 7]. We also have $\mathbb{H}_d^{\infty} \subseteq \mathbb{H}_d^2$, see [45] for more details.

We remark that the symbol Δ in [50] is used to denote NC derivatives. However, since we have reserved Δ for quantum groups, we adopt the notation ∇ for NC derivatives instead. The following theorem may be viewed as a non-commutative analogue of the classical proof of the disc algebra given in [76, Example 4.2.10]. However, the classical argument does not directly apply in the NC setting. Indeed, one needs tools from NC function theory, in particular NC derivatives, to establish the result.

Theorem 6.2.3. *The free disk algebra $\mathbb{A}_d$ and the free Hardy algebra $\mathbb{H}_d^{\infty}$ are not amenable.*

Proof. Denote both algebras by B . It suffices to construct a derivation on B which is not inner on any matricial level $n \in \mathbb{N}$. We should mention that we can work with the $f(Z)$ interpretation of $f(L) \in B$ via $f(L) = M_{f(Z)}^L$, where $Z = (Z_1, \dots, Z_d)$ is a d -tuple of $n \times n$ matrices. Now, we regard $\mathbb{C}^{n \times n}$ as a B -module by defining

$$\begin{aligned} \lambda \cdot g &:= \lambda g(0), & \lambda \in \mathbb{C}^{n \times n} \text{ and } g \in B \\ g \cdot \lambda &:= g(0) \lambda, & \lambda \in \mathbb{C}^{n \times n} \text{ and } g \in B \end{aligned}$$

By Equations 6.1 and 6.2, for any $g \in \mathbb{A}_d$ or $g \in \mathbb{H}_d^{\infty}$, we have $g(0) = c_{\emptyset} I_n$ for some $c_{\emptyset} \in \mathbb{C}$, so $\lambda \cdot g = g \cdot \lambda$, which makes inner derivations from B into $\mathbb{C}^{n \times n}$ equal to zero. However, the following derivation is a derivative at zero, and it is not inner by the previous discussion:

$$\begin{aligned} D : B &\longrightarrow \mathbb{C}^{n \times n} \\ D(f) &= \nabla f(0, 0)(Z). \end{aligned}$$

In fact, ∇ is a noncommutative differential operator satisfying the Leibniz rule [50, Theorem 2.3.4]. However, D is clearly a nonzero derivation for any d . When $d = 1$, we have $D = f'(0)$, which coincides with the first Taylor coefficient of f corresponding to the term z . We shall also show that D is nonzero when $d = 2$, and this observation extends to higher dimensions. Assume $p(x, y) = x^2 y + xy - yx + x + y + 1$ and let $(x, y) = (X_1, X_2) = X \in \mathbb{C}_{\mathbb{N}}^2$. So, $p : \mathbb{C}_{\mathbb{N}}^2 \longrightarrow \mathbb{C}_{\mathbb{N}}$ is an NC function given by the formula $p(X) = X_1 X_1 X_2 + X_1 X_2 - X_2 X_1 + X_1 + X_2 + 1$. For 2-tuples of $n \times n$ matrices $X = (X_1, X_2), Z = (Z_1, Z_2)$, we calculate $\nabla p(X, X)(Z)$:

$$p \left(\begin{bmatrix} X & Z \\ 0 & X \end{bmatrix} \right) = p \left(\begin{bmatrix} X_1 & Z_1 \\ 0 & X_1 \end{bmatrix}, \begin{bmatrix} X_2 & Z_2 \\ 0 & X_2 \end{bmatrix} \right) = p(A, B),$$

where $A = \begin{bmatrix} X_1 & Z_1 \\ 0 & X_1 \end{bmatrix}$ and $B = \begin{bmatrix} X_2 & Z_2 \\ 0 & X_2 \end{bmatrix}$. However,

$$p(A, B) = A^2 B + AB - BA + A + B + I, \quad (6.11)$$

and

$$p \left(\begin{bmatrix} X & Z \\ 0 & X \end{bmatrix} \right) = \begin{bmatrix} p(X) & \nabla p(X, X)(Z) \\ 0 & p(X) \end{bmatrix}. \quad (6.12)$$

By computing the matrix multiplications in 6.11 and comparing it with 6.12, we have

$$\nabla p(X, X)(Z) = X_1 X_1 Z_2 + X_1 Z_1 X_2 + Z_1 X_1 X_2 + X_1 Z_2 + Z_1 X_2 - X_2 Z_1 - Z_2 X_1 + Z_1 + Z_2.$$

Now, we can choose $Z = (Z_1, Z_2)$ such that $\nabla p(0, 0)(Z) = Z_1 + Z_2$ is not zero. consequently, B is not amenable. $\square$

Proposition 6.2.4. *With the module actions defined in Theorem 6.2.3, we also deduce that neither $\mathbb{A}_d$ nor $\mathbb{H}_d^d$ is approximately amenable.*

Proof. By the proof of Theorem 6.2.3, all inner derivations vanish for the given module actions. However, the derivation ∇ , defined in that proof, is nonzero and therefore cannot be approximated by inner derivations. Consequently, neither algebra is approximately amenable. $\square$

The amenability of the classical Toeplitz C^* -algebra $\mathcal{E}_1$ follows, for instance, from [56, Example 6.5.1] together with Theorem 2.4.10. The same extension argument applies to the Cuntz–Toeplitz algebra $\mathcal{E}_d$ for $d \geq 2$. We record the short proof for completeness.

Proposition 6.2.5. *The Cuntz–Toeplitz C^* -algebra $\mathcal{E}_d$ is amenable.*

Proof. By [17, Lemma V.2.2, and Section V.6], there is a short exact sequence

$$0 \longrightarrow \mathcal{K}(\mathbb{H}_d^2) \xrightarrow{j} \mathcal{E}_d \xrightarrow{q} \mathcal{O}_d \longrightarrow 0$$

where j is an injective homomorphism, q is surjective, and $\text{Im}(j) = \text{Ker}(q)$. Note that $\mathbb{H}_d^2$ is a separable Hilbert space, so like [76, Example 2.3.17], one can show that the compact operators $\mathcal{K}(\mathbb{H}_d^2)$ is amenable. Also, by [71, Theorem 3] the Cuntz algebra $\mathcal{O}_d$ is amenable. Now the closed ideal $\mathcal{K}(\mathbb{H}_d^2)$ of $\mathcal{E}_d$ and the quotient space $\frac{\mathcal{E}_d}{\mathcal{K}(\mathbb{H}_d^2)} \cong \mathcal{O}_d$ are amenable. So, by [76, Theorem 2.3.12], $\mathcal{E}_d$ must be amenable. $\square$

Remark 6.2.6. By the main result of [12], we know that amenable C^* -algebras are nuclear. The converse is also true by [39]. That is, amenability and nuclearity are the same for C^* -algebras. So, by Proposition 6.2.5, we see that $\mathcal{E}_d$ is nuclear. In particular, the C^* -tensor product of $\mathcal{E}_d \otimes \mathcal{E}_d$ has a unique C^* -norm.

Remark 6.2.7. Let $\Pi = (\Pi_1, \dots, \Pi_d)$ be an arbitrary row isometry on a Hilbert space $\mathcal{H}$. Then, we can define

$$\begin{aligned} \mathbb{A}_d(\Pi) &= \overline{\text{Alg}(I, \Pi)}^{op} \\ \mathbb{H}_d^\infty(\Pi) &= \overline{\mathbb{A}_d(\Pi)}^{wot} = \overline{\text{Alg}(I, \Pi)}^{wot} \\ \mathcal{E}_d(\Pi) &= C^*\{I, \Pi\} \end{aligned}$$

Note that when $\Pi = L$ on $\mathbb{H}_d^2$, we obtain the usual spaces $\mathbb{A}_d$, $\mathbb{H}_d^\infty$, and $\mathcal{E}_d$ respectively. Davidson et al. [18] define $\mathbb{H}_d^\infty(\Pi) = \overline{\text{Alg}(I, \Pi)}^{wot}$ as the free semigroup algebra. By definition, it is weak*-closed. By [18, Corollary 2.12], for the row isometry L , the weak operator topology and the weak* topology coincide on $\mathbb{A}_d$ and $\mathbb{H}_d^\infty$. It is worth noting that a general row isometry $\Pi = (\Pi_1, \dots, \Pi_d)$ on a Hilbert space $\mathcal{H}$ can have unusual properties. For instance, $\mathbb{H}_d^\infty(\Pi)$ may equal the entire $\mathcal{B}(\mathcal{H})$, [69].

The following lemma is due to Popescu [67, Theorem 3.1]:

Lemma 6.2.8. *For any row isometry $\Pi = (\Pi_1, \dots, \Pi_d)$ on a Hilbert space $\mathcal{H}$, the Banach algebras $\mathbb{A}_d(\Pi)$ and $\mathbb{A}_d$ are completely isometrically (algebraically) isomorphic through the extension of the mapping:*

$$\begin{aligned} \phi : \mathbb{A}_d(\Pi) &\longrightarrow \mathbb{A}_d \\ \Pi_j &\longrightarrow L_j . \end{aligned}$$

Since isomorphic Banach algebras have the same amenability property, we immediately get the following.

Corollary 6.2.9. *$\mathbb{A}_d(\Pi)$ is not amenable.*

Though $\mathbb{A}_d(\Pi)$ and $\mathbb{A}_d$ are identical as Banach algebras, the situation changes when we include the adjoint operators. Since $\sum_{j=1}^d L_j L_j^* < I$, while Π might be a Cuntz unitary, i.e., $\sum_{j=1}^d \Pi_j \Pi_j^* = I$. However, we can work with the inverse of Popescu's ϕ mapping, and extend it in an obvious way to obtain a homomorphism between C*-algebras $\mathcal{E}_d(\Pi)$ and $\mathcal{E}_d$.

Lemma 6.2.10. *For any row isometry $\Pi = (\Pi_1, \dots, \Pi_d)$ on a Hilbert space $\mathcal{H}$, the free semigroup algebra $\mathbb{H}_d^\infty(\Pi)$ is an Arens regular dual operator algebra. In particular, $\mathbb{H}_d^\infty$ is an Arens regular dual operator algebra.*

Proof. Since the weak operator topology is weaker than weak* topology, it is clear that $\mathbb{H}_d^\infty(\Pi)$ is a weak*-closed sub-algebra of $\mathcal{B}(\mathcal{H})$. By the classification of weak*-closed sub-spaces of dual Banach spaces [62, Corollary 2.4.11], we can find an $\mathcal{N}$, as the pre-annihilator of $\mathbb{H}_d^\infty(\Pi)$, which is a normed closed subspace of the trace class operators $\mathcal{B}_1(\mathcal{H})$; i.e.,

$$\mathbb{H}_d^\infty(\Pi)_\perp = \mathcal{N} = \{K \in \mathcal{B}_1(\mathcal{H}) \mid \text{tr}(Kh) = 0, \forall h \in \mathbb{H}_d^\infty(\Pi)\},$$

where “tr” is the trace functional. Thus,

$$\mathbb{H}_d^\infty(\Pi)_* \cong \frac{\mathcal{B}_1(\mathcal{H})}{\mathbb{H}_d^\infty(\Pi)_\perp} = \frac{\mathcal{B}_1(\mathcal{H})}{\mathcal{N}}.$$

So, by [76, Chapter 5], $\mathbb{H}_d^\infty(\Pi)$ is a dual Banach algebra. Since the operator algebras $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathbb{H}_d^2)$ are C*-algebras, they are Arens regular. Hence, $\mathbb{H}_d^\infty(\Pi)$ is an Arens regular dual operator algebra. For the second assertion, one can choose the row isometry $\Pi = L = (L_1, \dots, L_d)$. $\square$

It is a natural question whether the free semi-group algebra $\mathbb{H}_d^\infty(\Pi) = \overline{\text{Alg}(I, \Pi)}^{wot}$ is amenable or approximately amenable. The answer to this question is negative.

Theorem 6.2.11. *Let Π be a row isometry on an infinite dimensional Hilbert space $\mathcal{H}$, and*

$$I_{\mathcal{H}} \notin \overline{\text{span}\{\Pi^\alpha : \alpha \in \mathbb{F}^d, \alpha \neq \emptyset\}}^{wot}.$$

Then, the free semigroup algebra $\mathbb{H}_d^\infty(\Pi) = \overline{\text{Alg}(I, \Pi)}^{wot}$ is not amenable. Moreover, it is not approximately amenable.

Proof. Consider the homomorphism

$$\begin{aligned}\phi : \mathbb{H}_d^\infty(\Pi) &\longrightarrow \mathbb{H}_d^\infty \\ \Pi_j &\longrightarrow L_j\end{aligned}$$

By assumption, the hypotheses of [19, Theorems 1.5 and 2.6] are satisfied, so ϕ is a canonical onto wot-wot continuous homomorphism. By [18, Corollary 2.12], the weak operator topology and the weak* topology coincide on $\mathbb{H}_d^\infty$. Therefore, ϕ is wot-weak* continuous, so it is weak*-weak* continuous. By Lemma 6.2.10 and Proposition B.1, $\phi = (\phi_*)^*$, where ϕ_* is the pre-adjoint map between the pre-duals; hence, ϕ is indeed a bounded homomorphism. Now, if $\mathbb{H}_d^\infty(\Pi)$ were amenable, then $\mathbb{H}_d^\infty$ would be amenable by [76, Proposition 2.3.1], which contradicts Theorem 6.2.3. The non-approximate amenability is proved similarly by using Proposition 6.2.4 and [33, Proposition 2.2]. $\square$

Theorem 6.2.12. *Let $\Pi = (\Pi_1, \dots, \Pi_d)$ be any row isometry on an infinite dimensional Hilbert space $\mathcal{H}$, then the C^* -algebra $\mathcal{E}_d(\Pi)$ is a homomorphic image of the Cuntz-Toeplitz C^* -algebra $\mathcal{E}_d$. In particular, $\mathcal{E}_d(\Pi)$ is amenable.*

Proof. If Π is a row unitary, i.e., $\sum_{i=1}^d \Pi_i \Pi_i^* = I_{\mathcal{H}}$, then $\mathcal{E}_d(\Pi) = \mathcal{O}_d$ by [14]. Hence, $\mathcal{E}_d(\Pi)$ is amenable by [71, Theorem 3].

Now, assume $\sum_{i=1}^d \Pi_i \Pi_i^* < I_{\mathcal{H}}$. By [51, P. 1162], the universal C^* -algebra generated by the row isometry Π is isomorphic to the Cuntz-Toeplitz C^* -algebra $\mathcal{E}_d$. Moreover, by [17, Lemma V.2.2], there is a short exact sequence of the same form discussed in the proof of Proposition 6.2.5. However, we may construct a homomorphism between $\mathcal{E}_d$ and $\mathcal{E}_d(\Pi)$ explicitly. By Lemma 6.2.8, the unital Banach algebra $\mathbb{A}_d(\Pi)$ is completely isometrically (algebraically) isomorphic to the unital Banach algebra $\mathbb{A}_d$ via $\phi : \Pi_j \longrightarrow L_j$. Consider the inverse of ϕ given by

$$\begin{aligned}\psi : \mathbb{A}_d &\longrightarrow \mathbb{A}_d(\Pi) \\ L_j &\longrightarrow \Pi_j.\end{aligned}$$

Now, we can extend ψ , by defining $\psi(a^*) = \psi(a)^*$, into a C^* -homomorphism

$$\begin{aligned}\psi : \mathcal{E}_d &\longrightarrow \mathcal{E}_d(\Pi) \\ L_j &\longrightarrow \Pi_j.\end{aligned}$$

To show that ψ is a C^* -homomorphism, we only need to prove that $\psi(L_i^* L_j) = \psi(L_i)^* \psi(L_j)$. Since L is a row isometries, we have $L_i^* L_j = \delta_{i,j} I$. So,

$$\psi(L_i^* L_j) = \psi(\delta_{i,j} I) = \delta_{i,j} I.$$

However, Π is a row isometry, i.e., $\Pi_i^* \Pi_j = \delta_{i,j} I$, and $\psi(L_i^*) = \psi(L_i)^* = \Pi_i^*$. Therefore,

$$\Pi_i^* \Pi_j = \delta_{i,j} I = \psi(L_i^* L_j),$$

which proves $\psi(L_i^* L_j) = \psi(L_i)^* \psi(L_j)$. Thus, ψ also preserves the algebraic relations, and by the universal property as discussed in [3, Section II.8.3], ψ is a C^* -homomorphism. By Proposition 6.2.5, $\mathcal{E}_d$ is amenable. Then, by [76, Theorem 2.3.1], $\mathcal{E}_d(\Pi)$ must be amenable. $\square$

6.3 Cuntz-Toeplitz C*-algebra as a compact quantum semigroup

In this section, we prove that the Cuntz–Toeplitz C*-algebra does not admit the structure of a compact quantum group. So, the notion of amenability via left-invariant means does not apply to it. However, it can be regarded as a compact quantum semigroup and is Johnson amenable through derivations on Banach algebras.

We first present the definition of a compact quantum group, following [54], [79], and [90].

Definition 6.3.1. Let A be a unital C*-algebra, and equip $A \otimes A$ with the minimal C*-tensor norm. A unital *-homomorphism $\Delta : A \longrightarrow A \otimes A$ is called :

- *co-multiplicative* if the co-associativity rule holds for it, i.e., $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$. Here id is the identity map on A .
- *cancellative* if the cancellation law holds for it, i.e., $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are dense in $A \otimes A$. Here, 1 is the unit of A .

A *compact quantum semigroup* is a pair (A, Δ) , in which A is a unital C*-algebra and $\Delta : A \longrightarrow A \otimes A$ is a unital *-homomorphism which is co-multiplicative. If in addition Δ is cancellative, then (A, Δ) is called a *compact quantum group*.

Example 6.3.2. Let G be a compact group and let $C(G)$ denote the C*-algebra of all continuous complex-valued functions on G . By [77, P. 50], we have $C(G) \otimes_{min} C(G) \cong C(G \times G)$. By [54, P. 548],

$$\begin{aligned} \Delta : C(G) &\longrightarrow C(G) \otimes_{min} C(G) \\ \Delta(f)(x, y) &= f(xy), \quad [f \in C(G), x, y \in G] \end{aligned}$$

defines a the unital *-homomorphism making $(C(G), \Delta)$ into a compact quantum group. The co-associativity axiom $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ corresponds exactly to the associativity of the group operation on G , and the cancellative axiom corresponds to the classical cancellation law in G , see [54, P. 548] for more details.

The following definition is taken from [90] and [15].

Definition 6.3.3. Assume (A, Δ) be a compact quantum group. Define a multiplication on A^* by

$$\begin{aligned} \star : A^* \times A^* &\longrightarrow A^* \\ (\mu, \omega) &\mapsto \mu \star \omega \\ \langle \mu \star \omega, a \rangle &= \langle \mu \otimes \omega, \Delta(a) \rangle, \quad \forall a \in A \end{aligned}$$

i.e., $\mu \star \omega = (\mu \otimes \omega) \circ \Delta$. Here, $\mu \otimes \omega$ acts naturally on elementary tensors by $\mu \otimes \omega(a \otimes b) = \mu(a)\omega(b)$. We also define an involution on A^* by

$$\begin{aligned} * : A^* &\longrightarrow A^* \\ \mu &\mapsto \mu^* \\ \mu^*(a) &= \overline{\mu(a^*)}, \quad \forall a \in A. \end{aligned}$$

By [90] and [15], the product $\star$ and the involution $*$ make A^* into a Banach *-algebra.

Remark 6.3.4. Woronowicz [90] and Daele [15] proved that for any compact quantum group (A, Δ) there is a unique state $\lambda \in A^*$, called the *Haar state*, such that

$$(\lambda \otimes id)\Delta(a) = (id \otimes \lambda)\Delta(a) = \lambda(a)1, \quad \forall a \in A.$$

Here, id is the identity map on A , and 1 is the unit of A . The Haar state λ can also be determined by the following equation:

$$\lambda \star \omega = \omega \star \lambda = \omega(1)\lambda, \quad \forall \omega \in A^*.$$

The Haar state may be interpreted as a generalization of Haar measure and Haar integral on compact groups in classical Harmonic analysis; see Remark 3.1.9 for more details. It is customary to call $\mathbb{G} = (A, \Delta, \lambda)$ a compact quantum group.

Remark 6.3.5. By [88, p. 10], an important quantum expectation (see Definition A.1.1) on $\mathcal{B}(\mathbb{H}_2^d)$ is given by the vector state at the vacuum vector, i.e.,

$$m(T) = \langle Te_\emptyset, e_\emptyset \rangle_{\mathbb{H}_d^2}.$$

The vacuum vector $e_\emptyset \in \mathbb{H}_d^2$ assumes the role of the vector $e_0 = (1, 0, 0, \dots) \in \ell^2(\mathbb{N}_0)$ in the multi-dimensional setting. Note that m is a normal expectation on $\mathcal{B}(\mathbb{H}_d^2)$ making $\mathcal{B}(\mathbb{H}_d^2)$ into a non-commutative probability space, see [88, p. 10] and Definition A.1.1. We call m the *canonical quantum vacuum expectation* or *quantum Lebesgue expectation*.

When m restricts to $\mathbb{A}_d$, by [45] we have

$$m(L^\alpha) = \langle L^\alpha e_\emptyset, e_\emptyset \rangle_{\mathbb{H}_d^2} = \langle e_\alpha, e_\emptyset \rangle_{\mathbb{H}_d^2} = \begin{cases} 1, & \alpha = \emptyset \\ 0, & o.w. \end{cases} = \delta_{\alpha, \emptyset}.$$

In [45], m is referred to as the *NC Lebesgue measure*; however, we avoid this terminology in the present thesis for the following reason. By [60, Theorem 2.12], m has a unique extension to the operator system $\mathcal{C}_d = \overline{\mathbb{A}_d + \mathbb{A}_d^*}^{\|\cdot\|}$ given by $\tilde{m}(a + b^*) = m(a) + m(b)^*$. By Krein's extension theorem [60, P. 22], $\tilde{m}$ has a positive extension $\overline{m}$ on $\mathcal{E}_d$. However, $\overline{m}$ is neither faithful nor tracial. For example, when $d = 1$, consider the unilateral shift S . We have $S^*e_0 = 0$, meaning the backward shift annihilates the constant functions in the Hardy space H^2 . Thus, $\overline{m}(SS^*) = \langle S^*e_0, S^*e_0 \rangle = 0 \neq 1 = \langle Se_0, Se_0 \rangle = \overline{m}(S^*S)$.

Remark 6.3.6. We now calculate the actions $L^\alpha(L^*)^\beta$ and $(L^*)^\beta L^\alpha$ for later use. Recall that by L we mean the d -dimensional shift $L = (L_1, \dots, L_d)$. When $\alpha = (\alpha_1, \dots, \alpha_k)$, $1 \leq \alpha_j \leq d$, is a word in $\mathbb{F}^d$, we have $L^\alpha = L_{\alpha_1} \cdots L_{\alpha_k}$. Here, α is a recipe which tells us how to pick the components of L . We always have

$$(L^\alpha)^* L^\alpha = (L_{\alpha_1} \cdots L_{\alpha_k})^* (L_{\alpha_1} \cdots L_{\alpha_k}) = L_{\alpha_k}^* \cdots L_{\alpha_1}^* L_{\alpha_1} \cdots L_{\alpha_k} = I. \quad (6.13)$$

Denote by $\alpha^t = (\alpha_k, \dots, \alpha_1)$ the transpose of the word α . For instance, $(1232)^t = (2321)$. We define

$$(L^*)^\alpha := (L^{\alpha^t})^* = (L_{\alpha_k} \cdots L_{\alpha_1})^* = L_{\alpha_1}^* \cdots L_{\alpha_k}^*. \quad (6.14)$$

Then, we have

$$(L^*)^\alpha L^{\alpha^t} = (L^{\alpha^t})^* L^{\alpha^t} = I. \quad (6.15)$$

Hence, Equation (6.13) becomes

$$(L^*)^{\alpha^t} L^\alpha = I. \quad (6.16)$$

Denote by $|\alpha|$ the length of a word $\alpha = (\alpha_1, \dots, \alpha_k)$, i.e., $|(\alpha_1, \dots, \alpha_k)| = k$. For example, $|(j)| = 1$, $|(1121)| = 4$. By Equation (6.14) and the semi-Dirichlet property 6.7, we see that

$$(L^*)^\alpha L^\alpha = \begin{cases} I, & |\alpha| = 1 \\ 0, & o.w. \end{cases} = \delta_{\alpha, \alpha^t} I. \quad (6.17)$$

Here, the Kronecker delta δ takes into account the order of the letters in a word. Note that $\delta_{\alpha, \alpha^t} = 1$ if α is a single-letter word, and $\delta_{\alpha, \alpha^t} = 0$ otherwise. From the last two equations, we see

$$(L^*)^\beta L^\gamma = \begin{cases} \delta_{\beta, \beta^t} L^\theta, & \gamma = \beta\theta \\ L^\theta, & \gamma = \beta^t\theta \\ 0, & o.w. \end{cases}. \quad (6.18)$$

Recall that $\beta\theta$ is the juxtaposition of words in which β appears first followed by θ . However, we can combine the first two cases easily since for one letter words, $\beta = \beta^t$; hence,

$$(L^*)^\beta L^\gamma = \begin{cases} L^\theta, & \gamma = \beta^t\theta \\ 0, & o.w. \end{cases}. \quad (6.19)$$

So, we can always simplify expressions like $L^\alpha (L^*)^\beta L^\gamma (L^*)^\omega$ to the form $L^\theta (L^*)^\eta$. Precisely, we have

$$L^\alpha (L^*)^\beta L^\gamma (L^*)^\omega = \begin{cases} L^{\alpha\theta} (L^*)^\omega, & \gamma = \beta^t\theta \\ 0, & o.w. \end{cases}. \quad (6.20)$$

We will use the notation $D_\beta^\alpha := L^\alpha (L^*)^\beta$. Note that $(D_\beta^\alpha)^* = D_{\alpha^t}^{\beta^t}$, $D_\emptyset^\emptyset = I$, and

$$(L^*)^\beta L^\gamma = \begin{cases} D_\emptyset^\theta, & \gamma = \beta^t\theta \\ 0, & o.w. \end{cases}. \quad (6.21)$$

Also, we have

$$D_\beta^\alpha D_\omega^\gamma = \begin{cases} D_\omega^{\alpha\theta}, & \gamma = \beta^t\theta \\ 0, & o.w. \end{cases}. \quad (6.22)$$

For example, $D_\beta^\alpha D_\omega^{\beta^t} = D_\omega^\alpha$, $(D_\beta^\alpha)^* D_\beta^\alpha = D_\beta^{\beta^t}$ and $D_\beta^\alpha (D_\beta^\alpha)^* = D_\alpha^\alpha$. So, the set $D = \{D_\beta^\alpha : \alpha, \beta \in \mathbb{F}^d\} \cup \{0\}$ is a unital semigroup which is closed under taking adjoints. It contains all the expression $L^\alpha (L^*)^\beta$, $(L^*)^\alpha L^\beta$, identity and the zero element.

Theorem 6.3.7. *The Cuntz Toeplitz C^* -algebra $\mathcal{E}_d$ has the structure of a compact quantum semigroup, which is not a compact quantum group*

Proof. Since the C^* -algebra $\mathcal{E}_d$ is amenable (Proposition 6.2.5), it is nuclear by Theorem 2.4.10. So, we have just one C^* -tensor norm on $\mathcal{E}_d \otimes \mathcal{E}_d$. To define a co-associative mapping for $\mathcal{E}_d$, we go in stages. First, we define

$$\begin{aligned}\Delta : \mathcal{E}_d &\longrightarrow \mathcal{E}_d \otimes \mathcal{E}_d \\ cL^\alpha &\longrightarrow cL^\alpha \otimes L^\alpha, \quad \forall \alpha \in \mathbb{F}^d, c \in \mathbb{C}\end{aligned}$$

Then, we extend this by linearity to all finite (analytic) polynomials $\sum_\alpha c_\alpha L^\alpha$ in $\mathbb{A}_d$. The co-associativity is ensured by the associativity of tensor multiplication:

$$(\Delta \otimes id)\Delta(L^\alpha) = (\Delta \otimes id)(L^\alpha \otimes L^\alpha) = \Delta(L^\alpha) \otimes L^\alpha = (L^\alpha \otimes L^\alpha) \otimes L^\alpha,$$

and

$$(id \otimes \Delta)\Delta(L^\alpha) = (id \otimes \Delta)(L^\alpha \otimes L^\alpha) = L^\alpha \otimes \Delta(L^\alpha) = L^\alpha \otimes (L^\alpha \otimes L^\alpha).$$

Thus, the co-associativity holds on $\mathbb{A}_d$. For the $*$ -action, we define:

$$\begin{aligned}\Delta : \mathcal{E}_d &\longrightarrow \mathcal{E}_d \otimes \mathcal{E}_d \\ \Delta(a^* + b) &:= [\Delta(a)]^* + \Delta(b), \quad a, b \in \mathbb{A}_d\end{aligned}$$

For example, $\Delta((L^\alpha)^*) = [\Delta((L^\alpha))]^* = (L^*)^{\alpha^t} \otimes (L^*)^{\alpha^t}$. So, we can extend Δ to all finite polynomials in L and L^* . We also require

$$\Delta(D_\beta^\alpha) = D_\beta^\alpha \otimes D_\beta^\alpha$$

From Equation (6.19) and our definition of Δ , we see that

$$\Delta((L^*)^\beta L^\gamma) = \begin{cases} L^\theta \otimes L^\theta, & \gamma = \beta^t \theta \\ 0, & o.w. \end{cases} \quad (6.23)$$

From the definition of Δ , we note that the range of Δ does not contain the elements of the form $T_\beta^\alpha = L^\alpha \otimes (L^*)^\beta$ in $\mathcal{E}_d \otimes \mathcal{E}_d$. We will use this deficiency later to show that $\mathcal{E}_d$ is not a compact quantum group. We now show that Δ is a unital $*$ -homomorphism. Being unital and $*$ is almost immediate by the definition. So, we only need to check that Δ is indeed a homomorphism. We show this on a suitable dense $*$ -sub-algebra of $\mathcal{E}_d$. Consider the set $\mathcal{F}$ of polynomials $f(L) = \sum_{\alpha, \beta} f_{\alpha, \beta} L^\alpha (L^*)^\beta = \sum_{\alpha, \beta} f_{\alpha, \beta} D_\beta^\alpha$. By the semi-Dirichlet property 6.7 and Formula 6.22, we see that $\mathcal{F}$ is a $*$ -sub-algebra. Assume $g(L) = \sum_{\gamma, \omega} g_{\gamma, \omega} L^\gamma (L^*)^\omega = \sum_{\gamma, \omega} g_{\gamma, \omega} D_\omega^\gamma \in \mathcal{F}$. Note that

$$[f(L)]^* = \sum_{\alpha, \beta} \overline{f_{\alpha, \beta}} L^{\beta^t} (L^*)^{\alpha^t} = \sum_{\alpha, \beta} \overline{f_{\alpha, \beta}} D_{\alpha^t}^{\beta^t} \in \mathcal{F},$$

and by Equation (6.22),

$$h(L) = f(L)g(L) = \sum_{\alpha, \beta} \sum_{\gamma, \omega} f_{\alpha, \beta} g_{\gamma, \omega} D_\beta^\alpha D_\omega^\gamma = \sum_{\theta, \eta} h_{\theta, \eta} D_\eta^\theta \in \mathcal{F}. \quad (6.24)$$

Also, $\mathcal{F}$ contains all the generators of $\mathcal{E}_d$; so, it is a dense $*$ -sub algebra of $\mathcal{E}_d$. Now, we only need to check that Δ is a homomorphism on $\mathcal{F}$. To this end,

$$\begin{aligned}\Delta(f(L))\Delta(g(L)) &= \left[\sum_{\alpha, \beta} f_{\alpha, \beta} D_\beta^\alpha \otimes D_\beta^\alpha \right] \left[\sum_{\gamma, \omega} g_{\gamma, \omega} D_\omega^\gamma \otimes D_\omega^\gamma \right] \\ &= \sum_{\alpha, \beta} \sum_{\gamma, \omega} f_{\alpha, \beta} g_{\gamma, \omega} D_\beta^\alpha D_\omega^\gamma \otimes D_\beta^\alpha D_\omega^\gamma;\end{aligned}$$

upon comparing the latter with 6.24, we see that it is equal to $\Delta(f(L)g(L))$. Thus, Δ is a $*$ -homomorphism from $\mathcal{F}$ into $\mathcal{E}_d \otimes \mathcal{E}_d$. Since L^α is the composition of isometries, we see that $\|L^\alpha \otimes L^\alpha\| = \|L^\alpha\|^2 = 1 = \|L^\alpha\|$. Thus, Δ is continuous on monomials $\{L^\alpha\}$, their adjoints, and hence $\mathcal{F}$. Now, since $\mathcal{E}_d$ is generated by $\mathcal{F}$, we can extend Δ by continuity to all of $\mathcal{E}_d$. So, Δ is a co-multiplication on $\mathcal{E}_d$.

Now, we show that Δ does not enjoy the cancellative properties. Consider the simple tensor operators defined by $T_\beta^\alpha = L^\alpha \otimes (L^*)^\beta$. It is clear that the range of Δ does not contain these operators. Note that these operators belong to both the minimal and projective tensor products. Applying the multiplication operator π , we see that $\pi(T_\beta^\alpha) = D_\beta^\alpha$. We claim that $\Delta(\mathcal{E}_d)(\mathcal{E}_d \otimes 1)$ cannot be dense in $\mathcal{E}_d \otimes \mathcal{E}_d$. If the claim is false, then for any index α , the tensor operator T_α^α can be approximated by the members of $(\mathcal{E}_d \otimes 1)\Delta(\mathcal{E}_d)$. Since the minimal tensor product is injective [4], and $\mathcal{F}$ is dense in $\mathcal{E}_d$, we can approximate T_α^α by the members of $(\mathcal{F} \otimes 1)\Delta(\mathcal{F})$. Consider $T_1^1 = L_1 \otimes L_1^*$. For any $\epsilon > 0$, we can find $F = \sum_{\alpha, \beta, \gamma, \lambda} F_{\alpha, \beta, \gamma, \lambda} (D_\beta^\alpha \otimes I)(D_\lambda^\gamma \otimes D_\lambda^\gamma) = \sum_{\alpha, \beta, \gamma, \lambda} F_{\alpha, \beta, \gamma, \lambda} (L^\alpha (L^*)^\beta \otimes I)(L^\gamma (L^*)^\lambda \otimes L^\gamma (L^*)^\lambda)$ such that for the operator norm on $\mathbb{H}_d^2 \otimes \mathbb{H}_d^2$ we have $\|F - T_1^1\| < \epsilon$. In particular, for the special vector $e_\emptyset \otimes e_1$, we must have $\|F(e_\emptyset \otimes e_1) - T_1^1(e_\emptyset \otimes e_1)\| < \epsilon$. However, $T_1^1(e_\emptyset \otimes e_1) = L_1 \otimes L_1^*(e_\emptyset \otimes e_1) = e_1 \otimes e_\emptyset$. Note that for any word γ , $L^\gamma e_\emptyset = e_\gamma$. We see that:

$$\eta = (L^\gamma (L^*)^\lambda \otimes L^\gamma (L^*)^\lambda)(e_\emptyset \otimes e_1) = \begin{cases} e_\gamma \otimes e_{\gamma 1}, & \lambda = \emptyset \\ 0, & \lambda \neq \emptyset \end{cases}, \quad (6.25)$$

and

$$(L^\alpha (L^*)^\beta \otimes I)\eta = \begin{cases} e_{\alpha\mu} \otimes e_{\beta^t\mu 1}, & \gamma = \beta^t\mu, \text{ or } \lambda = \emptyset \\ 0, & \gamma \neq \beta^t\mu, \text{ or } \lambda \neq \emptyset \end{cases}. \quad (6.26)$$

Here, $\beta^t\mu 1$ is the juxtaposition of words ending in 1. Therefore, there are α, β, μ such that

$$\left\| \sum_{\alpha, \beta, \mu} F_{\alpha, \beta, \beta^t\mu} e_{\alpha\mu} \otimes e_{\beta^t\mu 1} - e_1 \otimes e_\emptyset \right\| < \epsilon.$$

However, when ω and σ run freely on $\mathbb{F}^d$, the vectors $e_\omega \otimes e_\sigma$ form an orthonormal base for $\mathbb{H}_d^2 \otimes \mathbb{H}_d^2$. So, the last inequality implies that we can find a special linear combinations of vectors $e_{\alpha\mu} \otimes e_{\beta^t\mu 1}$ equal to $e_1 \otimes e_\emptyset$. This can happen only when $\beta^t\mu 1 = \emptyset$. However, $\beta^t\mu 1 \neq \emptyset$ for any word β and μ . This contradiction shows that $\Delta(\mathcal{E}_d)(\mathcal{E}_d \otimes 1)$ cannot be dense in $(\mathcal{E}_d \otimes \mathcal{E}_d)$. Thus $\mathcal{E}_d$ is not a quantum group. $\square$

Appendix A

Weak* decomposition and Radon–Nikodym theorem of quantum expectations

In this appendix, we consider a unital C*-algebra A equipped with a positive linear functional (PLF) λ on it. We attempt to compare another PLF μ on A with λ . In particular, we study weak* continuity and weak* singularity of μ with respect to λ . Then, by the theory of von Neumann algebras, we obtain a canonical weak* decomposition of μ with respect to λ , namely

$$\mu = \mu_{wc} + \mu_{ws} ,$$

where μ_{wc} is the weak* continuous and μ_{ws} is the weak* singular part of μ with respect to λ . If λ satisfies a weak tracial property, referred as the KMS condition, we use the Grothendick-Haagerup inequality to show that our weak* decomposition coincides with the Arveson-Gheondea-Kavruk (AGK) decomposition. This equivalence helps us to compute the Radon Nikodym derivative of μ with respect to λ , which was used in the proof of Theorem 4.3.12, establishing Connes amenability for a finite von Neumann algebra possessing an ultra-trace.

A.1 Weak* decomposition of quantum expectations

We first define a non-commutative probability space and a quantum expectation on it.

Definition A.1.1 ([26] and [88]). A *non-commutative probability space* is a unital operator algebra A equipped with a unital bounded linear functional ϕ . The functional ϕ is referred to as the *non-commutative (or quantum) expectation*, and the elements of A are called *non-commutative (or quantum) random variables*. If A is a C*-algebra or a von Neumann algebra, we additionally require ϕ to be a state or a normal state, respectively.

Studying PLFs on C*-algebras is equivalent to studying quantum expectations on them. We can adapt some concepts from classical measure theory to the non-commutative framework.

Example A.1.2. Suppose that λ is a classical, finite, positive, Radon measure on the unit circle $\mathbb{T}$. Consider the multiplication mapping

$$\begin{aligned}\pi : L^\infty(\mathbb{T}, \lambda) &\longrightarrow \mathcal{B}(L^2(\mathbb{T}, \lambda)) \\ \pi(f) &= M_f, \quad M_f(g) = fg,\end{aligned}\tag{A.1}$$

which is an isometric unital $*$ -isomorphism from $L^\infty(\mathbb{T}, \lambda)$ onto the range of π , see [56, Example 2.5.1]. By [25, Proposition 4.51], the weak $*$ topology of $L^\infty(\mathbb{T}, \lambda)$ coincides with the weak operator topology of $\pi(L^\infty(\mathbb{T}, \lambda))$. Since λ is a Radon measure, $C(\mathbb{T})$ can be regarded as a closed subspace (even sub- C^* -algebra) of $L^\infty(\mathbb{T}, \lambda)$, see [32, P. 184 and Section 7.2]. By [25, Corollary 4.53], von-Neumann double commutant theorem [56, Theorem 4.1.5], and [56, Corollary 4.2.8], we have

$$\overline{\pi(C(\mathbb{T}))}^{wk*} = \pi(C(\mathbb{T}))'' = \pi(L^\infty(\mathbb{T}, \lambda)) \cong L^\infty(\mathbb{T}, \lambda).$$

Now, let μ be an arbitrary classical, finite, positive, Radon measure on the unit circle. By Riesz-Markov Theorem [32, Corollary 7.18] and [81, Theorem III.1.1], μ can be regarded as a positive linear functional (PLF) on $C(\mathbb{T})$ given by integration against μ , i.e.,

$$\begin{aligned}\mu : C(\mathbb{T}) &\longrightarrow \mathbb{C} \\ \mu(f) &= \int f \, d\mu.\end{aligned}$$

By [81, Theorem III.1.2], we see that the GNS construction of λ induces the irreducible $*$ -representation π_λ given by (A.1), i.e. $\pi_\lambda = \pi$. On the other hand, $L^\infty(\mathbb{T}, \lambda)$ is a von Neumann algebra with the pre-dual $L^1(\mathbb{T}, \lambda)$. By the general theory of von Neumann algebras [81, Sections II.2 and III.2], $L^1(\mathbb{T}, \lambda)$ is the space of all weak $*$ continuous linear functionals on $L^\infty(\mathbb{T}, \lambda)$. With this preamble in mind, we see that for a positive function $f \in L^1(\mathbb{T}, \lambda)$, the PLF $f d\lambda$ defined by

$$\begin{aligned}f d\lambda : C(\mathbb{T}) &\longrightarrow \mathbb{C} \\ f d\lambda(g) &= \int g f d\lambda,\end{aligned}$$

can be extended to

$$\begin{aligned}f d\lambda : L^\infty(\mathbb{T}, \lambda) &\longrightarrow \mathbb{C} \\ f d\lambda(g) &= \int g f d\lambda,\end{aligned}$$

which defines a weak $*$ continuous PLF on $L^\infty(\mathbb{T}, \lambda)$. It is well-known that these are all the possible weak $*$ continuous PLFs on $L^\infty(\mathbb{T}, \lambda)$. Moreover, by the Radon-Nikodym theorem, they are all absolutely continuous with respect to λ .

The following is a well-known theorem. Since it is essential for our weak $*$ decomposition of PLFs, we state it explicitly and provide a proof.

Theorem A.1.3. *Let μ and λ be two positive finite Radon measures on the unit circle, and consider them as positive linear functionals on $C(\mathbb{T})$. Then,*

- (i) μ can be extended to a weak $*$ -continuous functional on $L^\infty(\lambda)$ if and only if μ is absolutely continuous with respect to λ .

(ii) No extension of μ to $L^\infty(\lambda)$ can majorize a non-zero weak*-continuous PLF on $L^\infty(\lambda)$ if and only if μ is singular with respect to λ .

(iii) μ decomposes relative to λ as

$$\mu = \mu_{wc} + \mu_{ws}$$

where μ_{wc} is an absolutely continuous and μ_{ws} is a singular measure.

Proof. (i) Suppose that μ and λ are two classical positive Radon measures on $\mathbb{T}$. By Riesz-Markov theorem, $\mu : C(\mathbb{T}) \rightarrow \mathbb{C}$ is a PLF acting by integration against μ . Let μ have a weak*-continuous extension $\hat{\mu}$ on $L^\infty(\mathbb{T}, \lambda) = \overline{C(\mathbb{T})}^{wk*}$. Here, wk^* is the weak* topology of $L^\infty(\mathbb{T}, \lambda) = (L^1(\mathbb{T}, \lambda))^*$. By the definition of weak* topology, any weak* continuous functional corresponds to a member of $L^1(\mathbb{T}, \lambda)$. By Example A.1.2, we view $L^\infty(\mathbb{T}, \lambda)$ inside $\mathcal{B}(L^2(\lambda))$ as multiplication by bounded operators. Then, weak* topology of $L^\infty(\mathbb{T}, \lambda)$ coincides with the weak operator topology it receives from $\mathcal{B}(L^2(\lambda))$. Since the measures in question are positive, and $\hat{\mu}$ is continuous with respect to the weak operator topology, by [56, Theorem 4.2.6], we can find a positive $f \in L^1(\mathbb{T}, \lambda)$ such that for any $g \in C(\mathbb{T})$

$$\int_{\mathbb{T}} g d\mu = \mu(g) = \langle g\sqrt{f}, \sqrt{f} \rangle_{L^2(\lambda)} = \int_{\mathbb{T}} g f d\lambda.$$

Hence, μ and $f d\lambda$ give rise to the same functional on $C(\mathbb{T})$, i.e., $d\mu = f d\lambda$. Thus, μ is absolutely continuous with respect to λ .

Conversely, if μ is absolutely continuous with respect to λ , then by the Lebesgue-Radon-Nikodym theorem, there exists a positive $f \in L^1(\mathbb{T}, \lambda)$ such that $d\mu = f d\lambda$. So, for any $g \in C(\mathbb{T})$ we have

$$\mu(g) = \int_{\mathbb{T}} g d\mu = \int_{\mathbb{T}} g f d\lambda = \langle g\sqrt{f}, \sqrt{f} \rangle_{L^2(\lambda)},$$

which shows that μ is a vector state. However, the vector state $\hat{\mu}(g) = \langle g\sqrt{f}, \sqrt{f} \rangle_{L^2(\lambda)}$ on $L^\infty(\mathbb{T}, \lambda)$ is an extension of μ and is a weak* continuous functional on $L^\infty(\mathbb{T}, \lambda)$.

(ii) Suppose that μ has no extension to a PLF on $L^\infty(\mathbb{T}, \lambda)$ that majorize a weak* continuous functional on $L^\infty(\mathbb{T}, \lambda)$. By part (i), this means that it does not majorize any λ -absolutely continuous measure on $C(\mathbb{T})$. So, if we consider the Lebesgue decomposition

$$\mu = \mu_{ac} + \mu_s$$

of μ with respect to λ , we must have $\mu_{ac} = 0$, i.e., $\mu = \mu_s$. That is μ is Lebesgue singular with respect to λ .

Conversely, let μ be a Lebesgue singular measure with respect to λ . Suppose to the contrary that μ has a PLF extension $\hat{\mu}$ on $L^\infty(\mathbb{T}, \lambda)$ that majorizes a weak* continuous functional γ . Consider the Lebesgue decomposition of μ with respect to λ is $\mu = \mu_{ac} + \mu_s$. By part (i), γ is absolutely continuous, and by our assumption $\mu_{ac} \geq \gamma \neq 0$. This violates the singularity of μ with respect to λ . Thus, no extension of μ can majorize a non-zero weak* continuous functional on $L^\infty(\mathbb{T}, \lambda)$.

(iii) The decomposition in this part follows from the Lebesgue decomposition theorem together with parts (i) and (ii). □

Remark A.1.4. The decomposition $\mu = \mu_{wc} + \mu_{ws}$ on $L^\infty(\lambda)$ in the third part of Theorem A.1.3 is the same as the decomposition of a positive linear functional on a von Neumann algebra into normal and singular, see [81, P.127]. Therefore, in the classical setting of measure theory and functional analysis, the absolute continuity of a measure is equivalent to the possibility of extending the corresponding Riesz-Markov functional to a weak* continuous one. In addition, singularity of measures means the lack of this possibility. So, the above theorem justifies the following definition.

Definition A.1.5. Let μ and λ be two classical, finite, positive, Radon measures on the unit circle $\mathbb{T}$ with their corresponding Riesz-Markov integration functionals on $C(\mathbb{T})$ still denoted by μ and λ .

- (i) We say that $\mu : C(\mathbb{T}) \rightarrow \mathbb{C}$ is weak* continuous with respect to λ if μ has a weak* continuous extension to a PLF on $L^\infty(\lambda)$; i.e., there is an extension $\hat{\mu} : L^\infty(\lambda) \rightarrow \mathbb{C}$ of μ which is weak*-continuous and positive. We let $WC[\lambda]$ denote the set of all weak*-continuous measures with respect to λ .
- (ii) μ is called *weak*-singular* with respect to λ if no extension of μ to $L^\infty(\lambda)$ can majorize a non-zero weak*-continuous PLF with respect to λ . We let $WS[\lambda]$ denote the set of all weak*-singular measures with respect to λ .

Recall that if τ is a PLF on a unital C*-algebra A , then its isotropic left ideal of zero length vectors is $N_\tau = \{a \in A : \tau(a^*a) = 0\}$.

Remark A.1.6. Consider two PLFs μ and λ on a unital C*-algebra A with $N_\lambda \subseteq N_\mu$. Inspired by [81, Theorem III.2.1 and P. 321], we can generalize the notion of Example A.1.2. Let $(\pi_\lambda, L^2(\lambda), \xi_\lambda)$ be the GNS representation of λ with the cyclic vector $\xi_\lambda = 1 + N_\lambda$. By [56, Theorem 3.1.6], $\pi_\lambda(A)$ is a C*-sub-algebra of $\mathcal{B}(L^2(\lambda))$. We denote by $L^\infty(\lambda)$ the von Neumann algebra generated by the C*-algebra $\pi_\lambda(A)$ inside $\mathcal{B}(L^2(\lambda))$, and we write $L^1(\lambda)$ for the predual of $L^\infty(\lambda)$. Note that in the classical setting, $C(\mathbb{T})$ is a closed subspace of $L^\infty(\mathbb{T}, \lambda)$, and the representation (A.1) is isometric. In the general non-commutative setting, the PLF λ is not typically faithful, meaning that A is not isometrically embedded in $L^\infty(\lambda)$. Therefore, we need a method to transfer the PLFs μ and λ on A to PLFs μ' and λ' on $\pi_\lambda(A)$, while ensuring that the main properties of the PLFs are preserved. This is possible if the following commuting diagrams hold.

$$\begin{array}{ccc}
A & \xrightarrow{\pi_\lambda} & \pi_\lambda(A) \\
& \searrow \mu & \vdots \mu' \\
& & \mathbb{C}
\end{array} \quad \mu'(\pi_\lambda(a)) := \mu(a)$$

$$\begin{array}{ccc}
A & \xrightarrow{\pi_\lambda} & \pi_\lambda(A) \\
& \searrow \lambda & \vdots \lambda' \\
& & \mathbb{C}
\end{array} \quad \lambda'(\pi_\lambda(a)) := \lambda(a)$$

Figure A.1: Lifting quantum expectations μ and λ along the GNS representation π_λ .

Remark A.1.7. Recall that in the classical setting, π_λ is an isometric $*$ -homomorphism, so the conditions of Remark A.1.6 hold automatically. However, in the general non-commutative setting, we have the short exact sequence

$$0 \longrightarrow \ker(\pi_\lambda) \xrightarrow{j} A \xrightarrow{\pi_\lambda} \pi_\lambda(A) \longrightarrow 0,$$

and A is an extension of $\pi_\lambda(A)$ by $\ker(\pi_\lambda)$. For our purposes, the following argument shows that we may assume, without loss of generality, that λ is faithful. Consider the Gelfand-Naimark universal representation $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$. Recall that, by Proposition 4.2.8, the enveloping von Neumann algebra $\pi(A)''$ coincides with $A^{**} = \mathcal{M}$. Since A is unital and λ is positive, we have $\|\lambda\| = \lambda(1)$ by [56, Corollary 3.3.4]. By [4, Lemma A.2.2], λ admits a unique linear extension Λ which is a weak* continuous functional on $\mathcal{M}$ with $\|\Lambda\| = \|\lambda\| = \lambda(1) = \Lambda(1)$. Hence, Λ is positive by [56, Corollary 3.3.4]. By Remark 4.2.14-vi, there exists a central projection $p \in \mathcal{M}$ such that Λ is faithful on $p\mathcal{M}p$. Therefore, we may instead work with the C^* -algebra pAp and the faithful positive functional $\lambda_p := \Lambda|_{pAp}$. In addition, by [56, Theorem 3.3.8], we can extend μ to a positive functional M on $\mathcal{M}$ with the same norm, so we may work with $\mu_p := M|_{pAp}$ in place of μ . This justifies that there is no loss of generality in assuming that λ is faithful from the outset. Nevertheless, we will not make use of this assumption in the sequel.

We show that the transferred functionals μ' and λ' in Remark A.1.6 are well-defined.

Proposition A.1.8. *With the notations as in Remark A.1.6, the functionals λ' and μ' are well defined and give rise to PLFs on $\pi_\lambda(A)$. Furthermore, they have positive linear extensions to $L^\infty(\lambda)$.*

Proof. By [56, P. 94], the GNS mapping $\pi_\lambda : A \rightarrow \mathcal{B}(L^2(\lambda))$ defined by $\pi_\lambda(a)(b + N_\lambda) = ab + N_\lambda$ is a $*$ -homomorphism, and by [56, Theorem 3.1.6] $\pi_\lambda(A)$ is a C^* -subalgebra of $\mathcal{B}(L^2(\lambda))$. Clearly, $\pi_\lambda(A)$ embeds isometrically inside $L^\infty(\lambda)$. Let $\pi_\lambda(a) = 0$. Then, $\pi_\lambda(a)(b + N_\lambda) = N_\lambda$ for any $b \in A$. That is $ab \in N_\lambda$. In particular for $b = 1$, we have $a \in N_\lambda$, so $\lambda(a^*a) = 0$. By the Kadison inequality [46], we have

$$\lambda(a)^* \lambda(a) \leq \|\lambda\| \lambda(a^*a).$$

Thus, $\lambda(a) = 0$, so λ' is well-defined. Since $a \in N_\lambda \subseteq N_\mu$, we have $\mu(a^*a) = 0$. By another application of Kadison inequality, we see that $\mu(a) = 0$, i.e., μ' is also well-defined. Both μ' and λ' are PLFs on $\pi_\lambda(A)$ since π_λ is a bounded $*$ -homomorphism between C^* -algebras. Note that $\pi_\lambda(A)$ is unital since A is unital. Hence, $\|\mu'\| = \mu'(1) = \mu(1) = \|\mu\|$ by [56, Corollary 3.3.4]. For the second part of the theorem we note that $\pi_\lambda(A)$ is a C^* -sub algebra of $L^\infty(\lambda)$; thus, by [56, Theorem 3.3.8] we can extend any PLF on $\pi_\lambda(A)$ to a PLF on $L^\infty(\lambda)$ with the same norm. $\square$

Remark A.1.9. In view of Proposition A.1.8, we henceforth drop the prime notations and the homomorphism symbols in expressions $\mu'(\pi_\lambda(a))$ and $\lambda'(\pi_\lambda(a))$, writing instead $\mu(a)$ and $\lambda(a)$, respectively. Another abuse of notation is that we use the same notations μ and λ for the extended PLFs on $L^\infty(\lambda)$ unless there is a possible confusion.

Example A.1.2, Theorem A.1.3, Proposition A.1.8, and Remark A.1.9 inspire us to generalize Definition A.1.5 from commutative C^* -algebras to the non-commutative ones. We also refer to the relevant sources [21, Proposition 2.2], [48, Theorem 10.1.17], [81, Sections III.2 and III.3], [3, Section III.2].

Definition A.1.10. Consider two PLFs μ and λ on a unital C^* -algebra A such that $N_\lambda \subseteq N_\mu$. With the notations as in Definition A.1.6 and Remark A.1.9, we say

- (i) $\mu : A \rightarrow \mathbb{C}$ is *weak* continuous* with respect to λ if μ has a weak* continuous extension to a PLF $\hat{\mu}$ on $L^\infty(\lambda)$, i.e., the extension $\hat{\mu} : L^\infty(\lambda) \rightarrow \mathbb{C}$ is weak*-continuous. We use $\mu \ll_w \lambda$ to denote that μ is weak* continuous with respect to λ , and we denote the set of all weak*-continuous PLFs on A with respect to λ by $WC[\lambda]$.
- (ii) μ is called *weak*-singular* with respect to λ if no extension of μ to $L^\infty(\lambda)$ can majorize a non-zero weak*-continuous PLF. We use $\mu \perp_w \lambda$ to denote that μ is weak* singular with respect to λ , and we denote the set of all weak*-singular PLFs on A with respect to λ by $WS[\lambda]$.
- (iii) The zero functional is both weak* continuous and weak* singular with respect to λ .

Remark A.1.11. (i) Every PLF $\lambda : A \rightarrow \mathbb{C}$ is weak* continuous with respect to itself. This is because by the GNS construction we have

$$\lambda'(\pi_\lambda(a)) = \lambda(a) = \langle \pi_\lambda(a)(1 + N_\lambda), 1 + N_\lambda \rangle_{L^2(\lambda)}$$

So, by our notation in Remark A.1.9, we have $\lambda(a) = \langle a + N_\lambda, 1 + N_\lambda \rangle_{L^2(\lambda)}$, which is a vector state on $L^\infty(\lambda)$ by [56, Theorem 4.2.6]. So, by [81, Theorem II.2.6] λ extends to a weak* continuous functional on $L^\infty(\lambda)$.

- (ii) Whenever μ is weak*-continuous with respect to λ , by [48, Theorem 7.1.12] or [81, Theorem 2.2.6] we can find an orthogonal sequence of vectors $(\xi_n)_{n=1}^\infty$ in $L^2(\lambda)$ with the property $\sum_{n=1}^\infty \|\xi_n\|^2 = 1$ such that

$$\hat{\mu}(a) = \sum_{n=1}^\infty \langle a\xi_n, \xi_n \rangle_\lambda, \quad \forall a \in L^\infty(\lambda).$$

This formula holds on $\pi_\lambda(A)$. Thus, by Remark A.1.6, we can apply it for μ on A .

Lemma A.1.12. Let $\mathcal{M}$ be a von Neumann algebra, and let $(\phi_i)_{i \in I}$ be an increasing net in $\mathcal{M}_*^+$. Suppose that

$$C := \sup_i \phi_i(1) < \infty.$$

Then there exists $\phi \in \mathcal{M}_*^+$ such that

$$\phi(x) = \sup_i \phi_i(x), \quad x \in \mathcal{M}_+.$$

Proof. For $x \in \mathcal{M}_+$, since $0 \leq x \leq \|x\|1$, we have

$$0 \leq \phi_i(x) \leq \|x\|\phi_i(1) \leq \|x\|C, \quad i \in I.$$

Hence, the scalar supremum

$$\phi(x) := \sup_i \phi_i(x)$$

is well-defined for every $x \in \mathcal{M}_+$. Since each ϕ is additive and positive homogeneous, the same is true for ϕ . That is,

$$\phi(x + y) = \phi(x) + \phi(y), \quad \phi(\alpha x) = \alpha\phi(x), \quad x, y \in \mathcal{M}_+, \alpha \geq 0.$$

Thus, ϕ extends uniquely to a positive linear functional on $\mathcal{M}$. Indeed, for $h \in \mathcal{M}_{sa}$, we write its Jordan decomposition

$$h = h_+ - h_-, \quad h_+, h_- \in \mathcal{M}_+,$$

and define

$$\phi(h) := \phi(h_+) - \phi(h_-).$$

This is well-defined because the Jordan decomposition of a self-adjoint element is unique. Also, each $x \in \mathcal{M}$ can be written uniquely as the sum of its real and imaginary parts, namely

$$x = \frac{x + x^*}{2} + i \frac{x - x^*}{2i},$$

where

$$\frac{x + x^*}{2}, \frac{x - x^*}{2i} \in \mathcal{M}_{sa}.$$

Thus,

$$\phi(x) := \phi\left(\frac{x + x^*}{2}\right) + i\phi\left(\frac{x - x^*}{2i}\right)$$

gives a linear functional on $\mathcal{M}$ whose restriction to $\mathcal{M}_+$ is the original map ϕ . Since $\phi(x) \geq 0$ for every $x \in \mathcal{M}_+$, the resulting linear functional is positive. Furthermore,

$$\|\phi\| = \phi(1) = \sup_i \phi_i(1) \leq C < \infty.$$

It remains to prove that ϕ is normal. Let $(x_\alpha)_{\alpha \in J}$ be an increasing net in $\mathcal{M}_+$ such that $x_\alpha \uparrow x$ in strong operator topology. Then, $0 \leq x_\alpha \leq x \leq \|x\|1$. Hence, for all $i \in I$ and $\alpha \in J$,

$$0 \leq \phi_i(x_\alpha) \leq \|x\|\phi_i(1) \leq \|x\|C.$$

Therefore, the doubly indexed family $\{\phi_i(x_\alpha) : i \in I, \alpha \in J\}$ is bounded in $\mathbb{R}$. By the principle of iterated suprema [2, P.42], we have

$$\sup_i \sup_\alpha \phi_i(x_\alpha) = \sup_\alpha \sup_i \phi_i(x_\alpha).$$

Therefore,

$$\begin{aligned} \phi(x) &= \sup_i \phi_i(x) \\ &= \sup_i \sup_\alpha \phi_i(x_\alpha), & \phi_i \text{ is normal} \\ &= \sup_\alpha \sup_i \phi_i(x_\alpha), & \text{principle of iterated suprema} \\ &= \sup_\alpha \phi(x_\alpha). \end{aligned}$$

By [48, Definition 7.1.11 and Theorem 7.1.12], this shows that ϕ is normal. □

Recall that a sub-cone $\mathcal{S}$ of a positive cone $\mathcal{C}$ is hereditary if whenever $s \in \mathcal{S}$ and $c \in \mathcal{C}$ with $c \leq s$, then $c \in \mathcal{S}$.

Theorem A.1.13. *Let μ , λ and τ be three PLFs on a unital C^* -algebra A such that $N_\lambda \subseteq N_\mu$ and $N_\lambda \subseteq N_\tau$. Then,*

(i) μ has a unique weak* Lebesgue decomposition

$$\mu = \mu_{wc} + \mu_{ws}$$

with respect to λ , where $\mu_{wc} \in WC[\lambda]$ and $\mu_{ws} \in WS[\lambda]$ are PLFs and μ_{wc} is the maximal weak* continuous PLF such that $\mu_{wc} \leq \mu$.

(ii) $WC[\lambda]$ and $WS[\lambda]$ are hereditary cones in A_+^* .

(iii) weak* Lebesgue decomposition is additive, i.e.,

$$(\mu + \tau)_{wc} = \mu_{wc} + \tau_{wc}, \quad \text{and} \quad (\mu + \tau)_{ws} = \mu_{ws} + \tau_{ws}.$$

(iv) weak* Lebesgue decomposition is monotone, i.e., $\mu \leq \tau$ implies $\mu_{wc} \leq \tau_{wc}$ and $\mu_{ws} \leq \tau_{ws}$.

Proof. (i) Let $\tilde{\mu}$ be an extension of μ on $L^\infty(\lambda)$ as in Proposition A.1.8. By a general result in von Neumann algebra theory, see [81, p. 127 and Theorem III.2.14], we have

$$L^\infty(\lambda)^* = L^1(\lambda) \oplus_1 L^1(\lambda)^\perp, \tag{A.2}$$

in which, both summands, i.e., the weak* continuous $L^1(\lambda)$ and the weak* singular $L^1(\lambda)^\perp$, are norm closed sub-spaces of $L^\infty(\lambda)^*$. Hence, the decomposition $\tilde{\mu} = \tilde{\mu}_{wc} + \tilde{\mu}_{ws}$ follows. Furthermore, since $\tilde{\mu}$ is positive, both summands are positive by [48, Theorem 10.1.15-iii]. In addition, the decomposition into weak* continuous and weak* singular parts is unique by [81, Theorem III.2.14]. We also note that

$$\tilde{\mu}_{wc} \in L^1(\lambda)^+, \quad \tilde{\mu}_{wc}|_A \leq \mu.$$

Let $\mathcal{M} = L^\infty(\lambda)$ and set

$$\Lambda_\mu = \{\phi \in \mathcal{M}_*^+ : \phi|_A \leq \mu\}.$$

We partially order Λ_μ by the usual order of positive functionals on $\mathcal{M}$: for $\phi, \psi \in \Lambda_\mu$, we write

$$\phi \leq \psi \Leftrightarrow \phi(x) \leq \psi(x), \quad x \in \mathcal{M}_+.$$

The set Λ_μ is non-empty, since $0 \in \Lambda_\mu$. Moreover, it is norm bounded. Indeed, if $\phi \in \Lambda_\mu$, then, since ϕ is positive,

$$\|\phi\| = \phi(1) \leq \mu(1).$$

We now show that every chain in Λ_μ has an upper bound in Λ_μ . Let $(\phi_i)_{i \in I}$ be a chain in Λ_μ . So, $(\phi_i)_{i \in I}$ is an increasing net in $\mathcal{M}_*^+$. Moreover,

$$\sup_i \phi_i(1) \leq \mu(1) < \infty.$$

If $x \in \mathcal{M}_+$ with $\|x\| \leq 1$, then

$$0 \leq \phi_i(x) \leq \phi_i(1) \leq \mu(1), \quad i \in I.$$

Hence, $\{\phi_i(x) : i \in I\}$ is bounded above. More generally, for arbitrary $x \in \mathcal{M}_+$, since $0 \leq x \leq \|x\|1$, we have

$$0 \leq \phi_i(x) \leq \|x\|\phi_i(1) \leq \|x\|\mu(1).$$

Therefore, by Lemma A.1.12, the formula

$$\phi(x) := \sup_i \phi_i(x), \quad x \in \mathcal{M}_+,$$

defines a normal positive functional on $\mathcal{M}$. For every $a \in A_+$, we have

$$\phi(a) = \sup_i \phi_i(a) \leq \mu(a).$$

Thus $\phi|_A \leq \mu$, and hence $\phi \in \Lambda_\mu$. Therefore ϕ is an upper bound for the chain $(\phi_i)_{i \in I}$ in Λ_μ . By Zorn's lemma, Λ_μ has a maximal element. Call it $\hat{\mu}_{wc} \in \Lambda_\mu$. Define

$$\mu_{wc} := \hat{\mu}_{wc}|_A.$$

Then μ_{wc} is weak* continuous with respect to λ , because $\hat{\mu}_{wc} \in \mathcal{M}_*^+$ is a normal positive extension of μ_{wc} to $L^\infty(\lambda)$. Moreover, $\mu_{wc} \leq \mu$. Hence,

$$\mu_{ws} := \mu - \mu_{wc}$$

is a PLF on A . We claim that $\mu_{ws} \in WS[\lambda]$. Suppose, toward a contradiction, that μ_{ws} is not weak* singular with respect to λ . Then there exists a positive extension $\tilde{\mu}_{ws}$ of μ_{ws} to $L^\infty(\lambda)$ which majorizes a non-zero normal positive functional $\psi \in L^1(\lambda)^+$; that is,

$$0 \neq \psi \leq \tilde{\mu}_{ws}.$$

Restricting to A , we obtain

$$0 \neq \psi|_A \leq \mu_{ws}.$$

Indeed, ψ is weak* continuous. So, if $\psi|_A = 0$, then ψ vanishes on $L^\infty(\lambda) = \overline{\pi_\lambda(A)}^{w*}$; which contradicts that ψ is non-zero on $L^\infty(\lambda)$. Therefore,

$$\mu_{wc} + \psi|_A \leq \mu_{wc} + \mu_{ws} = \mu.$$

Moreover, $\mu_{wc} + \psi|_A$ is weak* continuous with respect to λ , since it has the normal positive extension

$$\hat{\mu}_{wc} + \psi$$

to $L^\infty(\lambda)$. This contradicts the maximality of $\hat{\mu}_{wc}$. Hence

$$\mu_{ws} \in WS[\lambda].$$

Consequently,

$$\mu = \mu_{wc} + \mu_{ws}$$

is a weak* Lebesgue decomposition of μ with respect to λ .

- (ii) Note that, by [81, p. 127 and Theorem III.2.14] and [48, Theorem 10.1.15-iii], $WC[\lambda]$ and $WS[\lambda]$ are positive cones. Hence, it remains only to prove their hereditary properties. We first show that $WS[\lambda]$ is hereditary. Let $\mu \in WS[\lambda]$, and let ν be a PLF on A such that

$$0 \leq \nu \leq \mu.$$

Suppose, toward a contradiction, that $\nu \notin WS[\lambda]$. Then, by Definition A.1.10, there exists a positive extension $\widehat{\nu}$ of ν to $L^\infty(\lambda)$ and a non-zero normal positive functional $\omega \in L^1(\lambda)^+$ such that

$$0 \neq \omega \leq \widehat{\nu}.$$

Since $\mu - \nu$ is a PLF on A , Proposition A.1.8 gives a positive extension $\widehat{\mu - \nu}$ of $\mu - \nu$ to $L^\infty(\lambda)$. Therefore,

$$\widehat{\mu} := \widehat{\nu} + \widehat{\mu - \nu}$$

is a positive extension of μ to $L^\infty(\lambda)$. Moreover,

$$0 \neq \omega \leq \widehat{\nu} \leq \widehat{\mu}.$$

Thus, an extension of μ majorizes a non-zero normal positive functional on $L^\infty(\lambda)$, contradicting the assumption that $\mu \in WS[\lambda]$. Hence, $\nu \in WS[\lambda]$.

Now we show that $WC[\lambda]$ is hereditary. Let $\mu \in WC[\lambda]$, and let ν be a PLF on A such that

$$0 \leq \nu \leq \mu.$$

So, $N_\lambda \subseteq N_\mu \subseteq N_\nu$. By part (i), ν and μ admit weak* decompositions

$$\nu = \nu_{wc} + \nu_{ws} \quad \text{and} \quad \mu = \mu_{wc}$$

with respect to λ , where $\nu_{wc}, \mu_{wc} \in WC[\lambda]$ and $\nu_{ws} \in WS[\lambda]$. Since $\nu \leq \mu$ and $\mu \in WC[\lambda]$, we have

$$0 \leq \nu_{ws} \leq \nu \leq \mu.$$

Thus, $\gamma := \mu_{wc} - \nu_{ws}$ is a PLF on A . By the uniqueness of the decomposition into weak* continuous and weak* singular parts, see [81, Theorem III.2.14], we obtain that $\gamma_{wc} = \mu_{wc}$ and $\gamma_{ws} = -\nu_{ws}$. However, by [48, Theorem 10.1.15-iii] the singular part of γ must be positive. So, $\nu_{ws} = 0$. Consequently, $\nu \in WC[\lambda]$.

- (iii) We prove the additivity of the weak* decomposition. Since $WC[\lambda]$ is a positive cone, we have

$$\mu_{wc} + \tau_{wc} \in WC[\lambda].$$

Moreover,

$$\mu_{wc} + \tau_{wc} \leq \mu + \tau.$$

By the maximality of $(\mu + \tau)_{wc}$, it follows that

$$\mu_{wc} + \tau_{wc} \leq (\mu + \tau)_{wc}. \tag{A.3}$$

We now prove the reverse inequality. Using the weak* Lebesgue decomposition separately for μ and λ , we obtain

$$\begin{aligned}\mu + \tau &= (\mu_{wc} + \mu_{ws}) + (\tau_{wc} + \tau_{ws}) \\ &= (\mu_{wc} + \tau_{wc}) + (\mu_{ws} + \tau_{ws}).\end{aligned}$$

On the other hand, weak* Lebesgue decomposition for $\mu + \tau$ yields

$$\mu + \tau = (\mu + \tau)_{wc} + (\mu + \tau)_{ws}.$$

Comparing the last two decompositions, we have

$$(\mu + \tau)_{wc} - (\mu_{wc} + \tau_{wc}) = (\mu_{ws} + \tau_{ws}) - (\mu + \tau)_{ws}. \quad (\text{A.4})$$

Set

$$\eta := (\mu + \tau)_{wc} - (\mu_{wc} + \tau_{wc}).$$

By Inequality (A.3), η is a PLF. Since $WC[\lambda]$ is hereditary and

$$0 \leq \eta \leq (\mu + \tau)_{wc},$$

we have $\eta \in WC[\lambda]$. Also, Equation (A.4) implies

$$0 \leq \eta \leq \mu_{ws} + \tau_{ws}.$$

Since $WS[\lambda]$ is a hereditary cone, it follows that $\eta \in WS[\lambda]$. Thus,

$$\eta \in WC[\lambda] \cap WS[\lambda].$$

By the disjointness of weak* continuous and weak* singular PLFs ([81, Theorem III.2.14]), we must have $\eta = 0$. Hence,

$$(\mu + \tau)_{wc} = \mu_{wc} + \tau_{wc},$$

and consequently,

$$(\mu + \tau)_{ws} = \mu_{ws} + \tau_{ws}.$$

(iv) By assumption, $\rho := \tau - \mu$ is a PLF on A and $N_\lambda \subseteq N_\tau \subseteq N_\mu$. Moreover,

$$N_\lambda \subseteq N_\rho.$$

Indeed, if $a \in N_\lambda$, then $\mu(a^*a) = 0$ and $\tau(a^*a) = 0$. Hence,

$$\rho(a^*a) = \tau(a^*a) - \mu(a^*a) = 0.$$

Now, $\tau = \mu + \rho$. By part (iii), we obtain

$$\begin{aligned}\tau_{wc} &= (\mu + \rho)_{wc} = \mu_{wc} + \rho_{wc}, \\ \tau_{ws} &= (\mu + \rho)_{ws} = \mu_{ws} + \rho_{ws}.\end{aligned}$$

Since ρ_{wc} and ρ_{ws} are PLFs, it follows that

$$\begin{aligned}\mu_{wc} &\leq \tau_{wc}, \\ \mu_{ws} &\leq \tau_{ws}.\end{aligned}$$

□

A.2 The Radon-Nikodym theorem for quantum expectations

We aim to show that our weak* decomposition of quantum expectations coincides with the Arveson–Gheondea–Kavruk–Lebesgue (AGKL) decomposition of positive functionals on C*-algebras introduced in [37]. This identification enables us to apply their Radon–Nikodym theorem in order to compute the relative derivative of quantum expectation μ with respect to quantum expectation λ . To achieve this, our splitting PLF λ must satisfy a certain weak tracial property, known as the Kubo–Martin–Schwinger (KMS) condition, which we define below. We point out that in the classical setting, one works with abelian C*-algebras. So, the KMS condition is automatic. On the other hand, a splitting PLF with the KMS property yields the equation $(\mu + \nu)_{ac} = \mu_{ac} + \nu_{ac}$, which cannot be achieved within the AGKL theory, see [37, Corollary 3.7-ii]. After proving our equivalence theorem, we can establish the Radon-Nikodym theorem for weak* continuous PLFs.

By an automorphism for a C*-algebra A , we mean a *-isomorphism on it. The group of all automorphisms of A is denoted by $\text{Aut}(A)$. In the context of von Neumann algebras, automorphisms are further required to be normal, meaning they preserve the weak* continuity. A strongly continuous one parameter group is a homomorphism $\sigma : \mathbb{R} \longrightarrow \text{Aut}(A)$ which is continuous with respect to the Euclidean topology of $\mathbb{R}$ and the strong operator topology topology of $\text{Aut}(A)$. If we put $\sigma_t = \sigma(t)$, the continuity of σ means that for any $a \in A$ the mapping $\mathbb{R} \longrightarrow A$ given by $t \longrightarrow \sigma_t(a)$ is Euclidean-norm continuous. In this situation, σ is called the time evolution of the C*-dynamical system $(A, \mathbb{R}, \sigma)$.

The following definition is from [5, Definition 5.3.1] for the KMS states in quantum mechanics.

Definition A.2.1. Let $(A, \mathbb{R}, \sigma)$ be a C*-dynamical system. A PLF λ on A is called a Kubo–Martin–Schwinger (KMS) PLF for $(A, \mathbb{R}, \sigma)$ if the following conditions are satisfied.

- (i) for each $a \in A$ the mapping $\mathbb{R} \longrightarrow A$ given by $t \longrightarrow \sigma_t(a)$ has an analytic extension to a mapping $f_a : \{z \in \mathbb{C} : 0 < \text{Im}(z) < 1\} \longrightarrow A$. Define $\sigma_z(a) := f_a(z)$.
- (ii) the weak tracial property $\lambda(ab) = \lambda(b\sigma_i(a))$ holds.
- (iii) λ is time invariant, i.e., $\lambda(\sigma_t) = \lambda$ for any $t \in \mathbb{R}$.

We use the acronym KMS-PLF for a Kubo–Martin–Schwinger positive linear functional.

Remark A.2.2. Usually one extracts the dynamics σ from the PLF λ . Note that according to [5, Proposition 5.3.3], conditions (i) and (ii) imply condition (iii). Also, by [5, Chapter 5] a weaker form of condition (i) is sufficient for the definition of a KMS-PLF, i.e., one only requires that the members of a (weak*) dense *-sub-algebra have analytic extensions to the strip $\{z \in \mathbb{C} : 0 < \text{Im}(z) < \beta\}$, see [5, P. 77] and [57, P. 75]. In this situation, we should replace (ii) by $\lambda(ab) = \lambda(b\sigma_{\beta i}(a))$. The KMS condition puts a restriction on the non-commutativity of the PLF λ without requiring it to be a full trace. This is important since most C*-algebras do not have a trace, but they can possess a KMS state. Here, β measures how far λ is being from an actual trace. For $\beta = 0$, we have an actual trace.

The inductive or direct limit of C*-algebras is a construction used to build larger C*-algebras from a sequence of smaller C*-algebras, see [56, Chapter 6].

Definition A.2.3. Suppose that $(A_n)_{n \in \mathbb{N}}$ is a sequence of C*-algebras, and assume that there is *-homomorphism $\phi_{m,n} : A_m \rightarrow A_n$ such that $\phi_{m,k} = \phi_{n,k} \circ \phi_{m,n}$ for $m \leq n \leq k$. Define

$$A' = \{(a_n) \in \prod_{n \in \mathbb{N}} A_n : \exists K \in \mathbb{N} \forall n \geq m \geq K, a_n = \phi_{n,m}(a_m)\}.$$

Since $*$ -homomorphisms between C^* -algebras are norm decreasing, for any $a = (a_n)_{n \in \mathbb{N}} \in A'$ the sequence of norms $(\|a_n\|)_{n \in \mathbb{N}}$ is eventually decreasing. Hence, the limit

$$p(a) = \lim_{n \rightarrow \infty} \|a_n\|$$

exists, and defines a C^* -semi-norm on A' . Now, we can divide A' by the kernel of p , and then complete it to a C^* -algebra A . This A is called the *inductive* or *direct limit* of $(A_n)_{n \in \mathbb{N}}$ and is denoted by $\varinjlim A_n$. One can adjust this definition for a net of C^* -algebras with appropriate $*$ -homomorphisms among them.

The following example is taken from [6, P. 13] and [57, P. 77], which plays an important role in quantum spin systems.

Example A.2.4. Consider the lattice $\mathbb{Z}^n$. To each point $x \in \mathbb{Z}^n$, choose a copy of a finite dimensional Hilbert space $\mathcal{H}_x$. For each finite subset $\mathcal{F}$ of $\mathbb{Z}^n$, form the tensor product $\mathcal{H}_{\mathcal{F}} = \otimes_{x \in \mathcal{F}} \mathcal{H}_x$ and consider the operator algebra $A(\mathcal{F}) = \mathcal{B}(\mathcal{H}_{\mathcal{F}})$. The direct limit of $(A(\mathcal{F}))_{\mathcal{F}}$, denoted by $\mathcal{A}$, is called the C^* -algebra of local observable, [57, P. 56]. Note that $\mathcal{A}$ is a sub-algebra of a suitable $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ associates to the infinite tensor product of Hilbert spaces. One can show that there is a closed unbounded operator T , known as the Hamiltonian of the system, such that $e^{-\beta T}$ is a trace class operator and $\lambda(a) = \text{tr}(ae^{-\beta T})$ is a KMS-PLF on $\mathcal{A}$, see [6, P. 85] and [57, P. 76]. Note that the domain $\mathcal{D}$ of T is dense in the Hilbert space $\mathcal{H}$, and $e^{-\beta T}$ is only invertible on $\mathcal{D}$. Such an operator is called a *density matrix* for λ , see [6, P. 76]. Also, $\sigma_t(a) = e^{itT}ae^{-itT}$ determines the dynamics of the system, see [5, P.76].

Example A.2.5. Let $\mathcal{M}$ be a von Neumann algebra inside some $\mathcal{B}(\mathcal{H})$, and suppose that $\mathcal{M}$ has a separating and cyclic unit vector η . Then, by [82, Chapter VI] or [6, P. 285], one can show that the vector state $\phi(a) = \langle a\eta, \eta \rangle$ is a KMS state. In this situation, we let S be the closure of the densely defined operator

$$\begin{aligned} S_0 : \mathcal{M}\eta &\longrightarrow \mathcal{H} \\ m\eta &\longrightarrow m^*\eta, \end{aligned}$$

and we put $\nabla = S^*S$. By the Tomita-Takesaki theory [82], the time evolution is given by (see [6, P. 284])

$$\begin{aligned} \sigma : \mathbb{R} &\longrightarrow \text{Aut}(\mathcal{M}) \\ \sigma_t(a) &= \nabla^{it} a \nabla^{-it}. \end{aligned}$$

Let's recall some other forms of absolute continuity and singularity for PLFs. In the classical measure theory, absolute continuity of measures on $\mathbb{T}$ can be characterized by convergence of measures. Namely, given two finite positive regular Radon measures μ and λ on $\mathbb{T}$, μ is absolutely continuous (AC) with respect to λ , in notations $\mu \ll \lambda$, if and only if there is an increasing sequence of finite positive regular Radon measures μ_n , which are dominated by λ , and increasing monotonically to μ :

$$\begin{aligned} 0 \leq \mu_n \leq \mu, \quad \mu_n \uparrow \mu, \\ \forall n \in \mathbb{N} \exists t_n > 0 \quad \mu_n \leq t_n \lambda. \end{aligned} \tag{A.5}$$

Clearly, condition (A.5) implies that $\mu \ll \lambda$. Conversely, suppose that $\mu \ll \lambda$. By the Radon-Nikodym theorem, there is a positive function $f \in L^1(\lambda)$ such that $d\mu = f d\lambda$. For each $n \in \mathbb{N}$,

define $f_n = \min(f, n)$ and set $d\mu_n = f_n d\lambda$. Then, the sequence (μ_n) consists of positive measures satisfying condition (A.5), with $\mu_n \leq n\lambda$.

The following definition, provided in [37], generalizes the above concept to the non-commutative setting.

Definition A.2.6. Consider two PLFs μ and λ on a unital C^* -algebra A such that $N_\lambda \subseteq N_\mu$.

- (i) We say that μ is *absolutely continuous* with respect to λ in the sense of Gheondea-Kavruk (GK), denoted by $\mu \ll \lambda$, if
 - there is an increasing sequence $(\mu_n)_{n \in \mathbb{N}}$ of PLFs such that $\mu_n \uparrow \mu$ point-wise, i.e., in the weak* topology of A^* ,
 - for each $n \in \mathbb{N}$, there is $t_n > 0$ such that $\mu_n \leq t_n \lambda$.

We denote the set of all absolutely continuous PLFs with respect to λ by $AC[\lambda]$.

- (ii) μ is called *singular* with respect to λ , denoted by $\mu \perp \lambda$, if the only PLF majorized by both μ and λ is the zero PLF. We denote the set of all singular PLFs with respect to λ by $SG[\lambda]$.

Remark A.2.7. By [60, Theorem 3.9], every positive linear functional on a unital C^* -algebra is completely positive. Therefore, the results of [37] apply for quantum expectation. In particular, after establishing the equivalence theorem A.2.9, we use the Radon–Nikodym derivative for absolutely continuous quantum expectations obtained in [37] to compute the Radon–Nikodym derivative of weak* continuous quantum expectations.

We recall the notion of affiliation for unbounded operators. Let D be a densely defined closed operator on a Hilbert space $\mathcal{H}$, and let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$. We say that D is affiliated with $\mathcal{M}$, and write $D \sim \mathcal{M}$ if

$$BD \subseteq DB, \quad \forall B \in \mathcal{M}'.$$

Equivalently, $\text{Dom}(D)$ is invariant under every $B \in \mathcal{M}'$, and

$$BDh = DBh, \quad \forall h \in \text{Dom}(D).$$

When D is bounded, the condition $D \sim \mathcal{M}$ is equivalent to $D \in \mathcal{M}$.

The following is [37, Theorem 2.11].

Theorem A.2.8. Let A be a unital C^* -algebra. Assume that μ and λ are two PLFs on it and $\mu \ll \lambda$. Then, there is a unique and possibly unbounded operator $D = D_\lambda(\mu)$ on $L^2(\lambda)$ such that

- (i) D is positive.
- (ii) D is affiliated with the von Neumann algebra $L^\infty(\lambda)'$.
- (iii) $\pi_\lambda(A)\xi_\lambda$ is a subset of the domain of $\sqrt{D}$. Here, $\xi_\lambda = 1 + N_\lambda$ is the cyclic vector of the GNS representation of λ .
- (iv) $\mu(a) = \langle \pi_\lambda(a)\sqrt{D}\xi_\lambda, \sqrt{D}\xi_\lambda \rangle_\lambda$.

Here, we should note that [37] works with more general positive maps, i.e., the ones whose ranges are the full operator algebra $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and one has to use the Stinespring's construction instead of the GNS one. However, one can simplify the notations and formulas of [37, Theorem 2.11] according to [60, Chapter 4, P.45].

Theorem A.2.9. *Let $(A, \mathbb{R}, \sigma)$ be a C^* -dynamical system. Assume that μ and λ are two PLFs on it with, $N_\lambda \subseteq N_\mu$. Suppose further that λ is a KMS-PLF. Then,*

- (i) $\mu \ll_w \lambda$ if and only if $\mu \ll \lambda$, and
- (ii) $\mu \perp_w \lambda$ if and only if $\mu \perp \lambda$.

Proof. (i) Suppose that $\mu \ll_w \lambda$. By Definition A.1.10, μ has a weak* continuous extension to a PLF on $L^\infty(\lambda)$. By [81, P. 156], there is a sequence $(\xi_n)_{n \in \mathbb{N}}$ of vectors in $L^2(\lambda)$ such that $s = \sum_{n=1}^\infty \|\xi_n\|^2 < \infty$ and $\mu(y) = \sum_{k=1}^\infty \langle y\xi_k, \xi_k \rangle_\lambda$. In Remark A.1.11-ii, we mentioned that we can apply this formula to weak* continuous functionals on A . Define a sequence of linear functionals on A by

$$\mu_n(y) = \sum_{k=1}^n \langle y\xi_k, \xi_k \rangle_\lambda, \quad \forall n \in \mathbb{N}$$

We show that μ_n 's are the desired PLFs, which satisfy the conditions of absolute continuity in Definition 2.1.3. It is obvious from the construction that $\mu_n \uparrow \mu$. So, for any $n \in \mathbb{N}$, we only need to find a $t_n > 0$ such that $\mu_n \leq t_n \lambda$ on positive elements like $y = a^*a$.

- Suppose that $\lambda(y) = 0$. So, $a \in N_\lambda \subseteq N_\mu$; thus, $a \in N_\mu$ and $\mu(y) = \mu(a^*a) = 0$. So, $\mu_n(y) = 0$, and the inequality $\mu_n \leq t_n \lambda$ holds at y for any positive t_n .
- For the case $\lambda(y) > 0$, define a bi-linear form V on A by

$$\begin{aligned} V : A \times A &\longrightarrow \mathbb{C} \\ V(a, b) &= \frac{\lambda(bya)}{\lambda(y)} \end{aligned}$$

Recall that λ is a KMS-PLF, so for $u = b$ and $v = ya$, we have $\lambda(uv) = \lambda(v\sigma_i(u))$; hence,

$$V(a, b) = \frac{\lambda(bya)}{\lambda(y)} = \frac{\lambda(ya\sigma_i(b))}{\lambda(y)} = \frac{\lambda \cdot y(a\sigma_i(b))}{\lambda \cdot y(1)}$$

Since y is positive, the functional $\lambda \cdot y$ is positive, and by Proposition 4.2.13,

$$\|\lambda \cdot y\| = \lambda \cdot y(1) = \lambda(y).$$

Therefore,

$$|V(a, b)| \leq \frac{\|\lambda \cdot y\| \|a\sigma_i(b)\|}{|\lambda(y)|} = \|a\sigma_i(b)\| \leq \|a\| \|b\|$$

Consequently, V is a bounded bi-linear form on the C^* -algebra A . By Generalized Grothendieck inequality [40], we can find states ϕ_i , $i = 1, \dots, 4$, on A such that

$$|V(a, b)| \leq [\phi_1(a^*a) + \phi_2(aa^*)]^{\frac{1}{2}} [\phi_3(b^*b) + \phi_4(bb^*)]^{\frac{1}{2}}$$

Applying this to $V(z, z^*) = \frac{\lambda(z^*yz)}{\lambda(y)}$, we get

$$\left| \frac{\lambda(z^*yz)}{\lambda(y)} \right| \leq [\phi_1(z^*z) + \phi_2(zz^*)]^{\frac{1}{2}} [\phi_3(zz^*) + \phi_4(z^*z)]^{\frac{1}{2}} \leq 2\|z\|^2$$

Thus,

$$|\lambda(z^*yz)| \leq 2\|z\|^2\lambda(y). \quad (\text{A.6})$$

Now, for any $\xi_k \in L^2(\lambda)$, by [86, Lemma 1.2 and Lemma 3.1], there is a bounded sequence $(x_{k,j})_{j=1}^\infty$ in A such that $\xi_k = \lim_{j \rightarrow \infty} x_{k,j} + N_\lambda$ in $L^2(\lambda)$. Denote by $s_k \in \mathbb{R}$ the upper bound of the sequence $(x_{k,j})_j$. Therefore,

$$\begin{aligned} \mu_n(y) &= \sum_{k=1}^n \langle y\xi_k, \xi_k \rangle_\lambda \\ &= \sum_{k=1}^n \lim_{j \rightarrow \infty} \langle yx_{k,j} + N_\lambda, x_{k,j} + N_\lambda \rangle_\lambda \\ &= \lim_{j \rightarrow \infty} \sum_{k=1}^n \lambda((x_{k,j})^* y x_{k,j}) \\ &\leq 2 \lim_{j \rightarrow \infty} \left(\sum_{k=1}^n \|x_{k,j}\|^2 \right) \lambda(y), \quad \text{by Inequality (A.6)} \\ &\leq 2 \left(\sum_{k=1}^n s_k^2 \right) \lambda(y) \\ &\leq t_n \lambda(y). \end{aligned}$$

So, by letting $t_n = 2 \left(\sum_{k=1}^n s_k^2 \right)$, we see that $\mu_n \leq t_n \lambda$. Hence, the second condition of the definition of absolute continuity is satisfied. Thus, $\mu \ll \lambda$.

For the reverse implication, assume that $\mu \ll \lambda$. By Theorem A.2.8, there is a positive, possibly unbounded operator $D = D_\lambda(\mu)$ on $L^2(\lambda)$ affiliated with the von Neumann algebra $L^\infty(\lambda)'$ such that $\mu(a) = \langle \pi_\lambda(a) \sqrt{D} (1 + N_\lambda), \sqrt{D} (1 + N_\lambda) \rangle_\lambda$. By letting $\eta_\mu := \sqrt{D} (1 + N_\lambda) \in L^2(\lambda)$, we have $\mu(a) = \langle \pi_\lambda(a) \eta_\mu, \eta_\mu \rangle_\lambda$. We can extend this to $L^\infty(\lambda)$ in an obvious way, i.e., $\hat{\mu}(y) = \langle y \eta_\mu, \eta_\mu \rangle_\lambda$. By [48, Theorem 7.1.12], $\hat{\mu}$ defines a weak*-continuous functional on $L^\infty(\lambda)$. Thus, $\mu \ll_w \lambda$.

- (ii) Let $\mu \perp_w \lambda$, but $\mu \not\ll \lambda$. By [37], this means that we can decompose μ against λ , i.e., $\mu = \mu_{ac} + \mu_s$, where $\mu_{ac} \neq 0$ and $\mu_{ac} \ll \lambda$. By part (i), we conclude that μ_{ac} is weak* continuous with respect to λ . Since $\mu_{ac} \leq \mu$, we see that μ dominates the non-zero weak* continuous PLF μ_{ac} . However, this is impossible as μ is weak* singular.

The reverse implication is proved similarly. Indeed, if $\mu \perp \lambda$ but $\mu \not\ll_w \lambda$, then we have a weak* decomposition $\mu = \mu_{wc} + \mu_{ws}$, where $\mu_{wc} \neq 0$. Now, $\mu_{wc} \leq \mu$ and $\mu_{wc} \ll \lambda$ by part (i). So, from Definition A.2.6 we can find a non-zero PLF ρ such that $\rho \leq \mu$ and $\rho \leq \lambda$. However, this contradicts the assumption that $\mu \perp \lambda$.

□

Remark A.2.10. Let λ be a faithful PLF with the tracial property $\lambda(xx^*) = \lambda(x^*x)$ on $L^\infty(\lambda)$. Suppose that $\mathcal{M}_\lambda$ is defined as in Definition 2.3.1. By [81, P. 320], $L^1(\lambda)$ is the completion of $\mathcal{M}_\lambda$ under $\|x\|_1 = \lambda(|x|)$, and $L^\infty(\lambda)$ becomes an $L^1(\lambda)$ -module. By [81, P. 321], for any $\mu \ll_{wc} \lambda$, there is an $x \in L^1(\lambda)$ such that $\mu(a) = \lambda(ax)$. In this situation, the proof of Theorem A.2.9 is similar to Theorem A.1.3.

We are now in a position to strengthen some of the results of [37] for splitting KMS-PLFs.

Theorem A.2.11. *Let $(A, \mathbb{R}, \sigma)$ be a C^* -dynamical system. Assume that μ, ν and λ are PLFs on it with $N_\lambda \subseteq N_\mu$ and $N_\lambda \subseteq N_\nu$. Suppose further that λ is a KMS-PLF. Then,*

- (i) $AC[\lambda]$ is a positive hereditary cone,
- (ii) $SG[\lambda]$ is a positive hereditary cone,
- (iii) $(\mu + \nu)_{ac} = \mu_{ac} + \nu_{ac}$ with respect to λ ,
- (iv) $(\mu + \nu)_s = \mu_s + \nu_s$ with respect to λ ,
- (v) If $\mu \leq \nu$, then $\mu_{ac} \leq \nu_{ac}$ and $\mu_s \leq \nu_s$
- (vi) The weak* Lebesgue decomposition $\mu = \mu_{wc} + \mu_{ws}$ with respect to λ and the Arveson-Gheondea-Kavruk Lebesgue (AGKL) decomposition $\mu = \mu_{ac} + \mu_s$ with respect to λ are the same.

Proof. Combine Theorem A.1.13 and Theorem A.2.9. □

We can now give a version of Theorem A.2.8 for weak* continuous functionals:

Theorem A.2.12. *Let $(A, \mathbb{R}, \sigma)$ be a C^* -dynamical system. Assume that μ and λ are PLFs on it with $N_\lambda \subseteq N_\mu$, and λ is a KMS-PLF. If $\mu \ll_{wc} \lambda$, then there is a unique and possibly unbounded operator $D = D_\lambda(\mu)$ on $L^2(\lambda)$ such that*

- (i) D is positive.
- (ii) D is affiliated with the von Neumann algebra $L^\infty(\lambda)'$.
- (iii) $\pi_\lambda(A)\xi_\lambda$ is a subset of the domain of $\sqrt{D}$. Here, $\xi_\lambda = 1 + N_\lambda$ is the cyclic vector of the GNS representation of λ .
- (iv) $\mu(a) = \langle \pi_\lambda(a)\sqrt{D}\xi_\lambda, \sqrt{D}\xi_\lambda \rangle_\lambda$.
- (v) If $\mu \leq t\lambda$ for a positive number $t \in \mathbb{R}$, then D is a bounded positive operator which belongs to $L^\infty(\lambda')$ and $\mu(a) = \langle \pi_\lambda(a)D(\xi_\lambda), \xi_\lambda \rangle_\lambda$. Also, the assignment $\mu \rightarrow D_\lambda(\mu)$ is an affine mapping with respect to μ .

Proof. The proofs of (i) to (iv) is a result of the equivalent Theorem A.2.9 and [37, 2.11]. The final part is a result of the equivalent Theorem A.2.9 and [37, Corollary 2.3]. □

Remark A.2.13. When $\mu \ll_{wc} \lambda$, by Theorem A.2.9 and Definition A.2.6, we know that there is an increasing sequence $(\mu_n)_{n \in \mathbb{N}}$ of PLFs such that $\mu_n \uparrow \mu$ point-wise, i.e., in the weak* topology of A^* . Also, for any $n \in \mathbb{N}$ there is a positive t_n such that $\mu_n \leq t_n \lambda$. The latter condition implies that $\mu_n \ll_{ac} \lambda$, and hence each $D_n = D_\lambda(\mu_n)$ is a positive bounded operator in $L^\infty(\lambda')$. By [37, Part F],

one finds that $(I + D_n)^{-1} \xrightarrow{SOT} (I + D)^{-1}$. This convergence is described by saying that $D_n \xrightarrow{SRS} D$, where “SRS” stand for the *strong resolvent sense*. Note that the number -1 does not belong to the spectra of D_n ’s, so by [62, Theorem 5.1.9] the operators $I + D_n$ and $I + D$ are invertible. In summary, though the Radon-Nikodym derivative D is generally an unbounded operator, one can still approximate it in a suitable sense by bounded Radon-Nikodym derivatives.

Appendix B

Spatial Normal balanced tensor products

In this appendix, we collect the tensor-product conventions and auxiliary facts used in the lifting constructions of Chapter 5. We shall use the definitions of the normal tensor product and the normal balanced tensor product introduced in Section 5.1 without repeating them here.

First, we review a standard criterion for weak*-weak* continuity of maps between dual spaces. This criterion will be used to identify pre-adjoints of weak*-weak* continuous maps. We provide a proof for completeness.

Proposition B.1. *Let X and Y be Banach spaces and $S : Y^* \longrightarrow X^*$ be a linear transformation. Then, S is weak*-weak* continuous if and only if there is a bounded mapping $T : X \longrightarrow Y$ such that $S = T^*$. In particular, S is bounded and $\|S\| = \|T\|$.*

Proof. Suppose that S is weak*-weak* continuous, and $x \in X$. Define a linear functional

$$\begin{aligned} g_x : Y^* &\longrightarrow \mathbb{C} \\ \langle g_x, y^* \rangle &= \langle x, S(y^*) \rangle, \end{aligned}$$

which is weak* continuous by assumption. Thus, $g_x \in (Y^*, wk^*)^* \cong Y$. Hence, there is a unique $y \in Y$ such that $g_x = \hat{y}$, where $\wedge : Y \longrightarrow Y^{**}$ is the natural embedding. Define

$$\begin{aligned} T : X &\longrightarrow Y \\ T(x) &= y \end{aligned}$$

So,

$$\langle T(x), y^* \rangle = \langle y, y^* \rangle = \langle \hat{y}, y^* \rangle = \langle x, S(y^*) \rangle,$$

which proves that $S = T^*$. Let $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$ both in norm, and suppose that $\psi \in Y^*$ is arbitrary. By the equation $S = T^*$, we have

$$\langle Tx_n, \psi \rangle = \langle x_n, S(\psi) \rangle;$$

however, $\hat{x}_n \longrightarrow \hat{x}$ in the norm topology of X^{**} . Since $S(\psi) \in X^*$,

$$\langle x_n, S(\psi) \rangle = \langle \hat{x}_n, S(\psi) \rangle \longrightarrow \langle \hat{x}, S(\psi) \rangle = \langle x, S(\psi) \rangle = \langle Tx, \psi \rangle;$$

i.e., $\langle Tx_n, \psi \rangle \longrightarrow \langle Tx, \psi \rangle$. On the other hand, $\langle Tx_n, \psi \rangle \longrightarrow \langle y, \psi \rangle$ by the continuity of ψ . Therefore, $y = Tx$. Then, by the closed graph theorem, T is bounded.

For the reverse direction and the equation $\|S\| = \|T\|$, see [74, Theorem 4.10]. $\square$

The operator T in the above theorem is called the *pre-adjoint* of S and is denoted by $S_* = T$, so $S = (S_*)^*$.

Remark B.2 (Contractive renorming of normal dual modules). Let $\mathcal{N}$ be a dual Banach algebra with unit p and $X = Y^*$ be a normal dual $\mathcal{N}$ -bimodule such that p acts as the identity for module actions, i.e., $p \cdot x = x \cdot p = x$ for all $x \in X$. In the general Banach-module literature, module actions are usually assumed to be bounded, i.e., one assumes the existence of a constant $C > 0$ such that

$$\|a \cdot x\| \leq C\|a\| \|x\|, \quad \|x \cdot a\| \leq C\|x\| \|a\|, \quad (x \in X, a \in \mathcal{N}).$$

However, it is often convenient to pass to an equivalent module norm under which the above actions become contractive. This is the same general principle behind the admissible-norm constructions used in Daws' representation theorem for dual Banach algebras; see [76, Sections 5.4 & 5.5]. For instance, we may replace the norm of X by the equivalent module norm

$$\|\xi\|_{\text{mod}} = \sup\{\|a \cdot \xi \cdot b\| : a, b \in \mathcal{N}, \|a\| \leq 1, \|b\| \leq 1\}, \quad \xi \in X.$$

Since $p \cdot \xi \cdot p = \xi$, we have

$$\|\xi\| \leq \|\xi\|_{\text{mod}}.$$

On the other hand, boundedness of the left and right module actions gives a constant $C > 0$ such that

$$\|\xi\|_{\text{mod}} \leq C^2 \|\xi\|.$$

Hence,

$$\|\xi\| \leq \|\xi\|_{\text{mod}} \leq C^2 \|\xi\|, \quad (\xi \in X).$$

That is, $\|\cdot\|_{\text{mod}}$ is equivalent to the original norm on X . With respect to this new norm, the $\mathcal{N}$ -actions are contractive. Indeed, for $a \in \mathcal{N}$ and $\xi \in X$, we have

$$\begin{aligned} \|a \cdot \xi\|_{\text{mod}} &= \sup_{\|c\| \leq 1, \|d\| \leq 1} \|c \cdot (a \cdot \xi) \cdot d\| \\ &= \sup_{\|c\| \leq 1, \|d\| \leq 1} \|(ca) \cdot \xi \cdot d\|. \end{aligned}$$

If $a = 0$, the desired inequality is immediate. If $a \neq 0$, then $\|(ca)/\|a\|\| \leq 1$ whenever $\|c\| \leq 1$. Whence,

$$\|(ca) \cdot \xi \cdot d\| = \|a\| \left\| \left(\frac{ca}{\|a\|} \right) \cdot \xi \cdot d \right\| \leq \|a\| \|\xi\|_{\text{mod}}.$$

Taking the supremum over all $\|c\| \leq 1$ and $\|d\| \leq 1$ gives

$$\|a \cdot \xi\|_{\text{mod}} \leq \|a\| \|\xi\|_{\text{mod}}.$$

Similarly,

$$\|\xi \cdot b\|_{\text{mod}} \leq \|b\| \|\xi\|_{\text{mod}} \quad (b \in \mathcal{N}, \xi \in X).$$

Now, by [74, Theorem 4.3], we renorm the predual Y by the corresponding polar norm

$$\|y\|_{\text{mod},*} = \sup \{ |\langle \xi, y \rangle| : \|\xi\|_{\text{mod}} \leq 1 \}, \quad y \in Y.$$

Since $\|\cdot\|_{\text{mod}}$ is equivalent to the original norm on X , the norm $\|\cdot\|_{\text{mod},*}$ is equivalent to the original norm on Y . Therefore, the underlying predual vector space is unchanged, and the weak* topology $\sigma(X, Y)$ is unchanged. Consequently, weak* convergence and weak* continuity of the module actions are unaffected by this renorming. Thus, without changing the weak* topology or normality, one may assume that the left and right $\mathcal{N}$ -actions on X are contractive.

The following identification lemma is obvious.

Lemma B.3 (Scalar unit for the normal spatial tensor product). *Let X be a dual Banach space. Then, the canonical maps*

$$\begin{aligned} \Theta_\ell : \mathbb{C} \bar{\otimes} X &\longrightarrow X, & \Theta_\ell(\lambda \otimes \xi) &= \lambda \xi, \\ \Theta_r : X \bar{\otimes} \mathbb{C} &\longrightarrow X, & \Theta_r(\xi \otimes \lambda) &= \lambda \xi \end{aligned}$$

are weak-weak* homeomorphic isomorphisms. Consequently,*

$$\begin{aligned} X &\cong \mathbb{C} \bar{\otimes} X \bar{\otimes} \mathbb{C} \\ \xi &\mapsto 1 \otimes \xi \otimes 1 \end{aligned}$$

weak-weak* homeomorphically.*

The following lemma is the normal Banach-space analogue of the functoriality of module tensor products in [4, Lemma 3.4.5]. The unbalanced normal spatial tensor product is treated in [4, Section 1.6.5]; for the analogous threefold balanced construction, see [4, Section 3.4.8]. We also refer to the W^* -module tensor product framework in [4, Section 8.5.40].

Lemma B.4 (Functoriality of the normal tensor product). *Let U, V, U', V' be dual Banach spaces with fixed pre-duals. Suppose that*

$$S : U \longrightarrow U', \quad T : V \longrightarrow V'$$

are bounded weak-weak* continuous linear maps. Then, there is a unique bounded weak*-weak* continuous linear map*

$$S \bar{\otimes} T : U \bar{\otimes} V \longrightarrow U' \bar{\otimes} V'$$

such that

$$(S \bar{\otimes} T)(u \otimes v) = S(u) \otimes T(v) \quad (u \in U, v \in V).$$

Moreover, let $\mathcal{V}$ be a dual Banach algebra, let X, Y, X', Y' be normal one-sided $\mathcal{V}$ -modules, with X, X' right $\mathcal{V}$ -modules and Y, Y' left $\mathcal{V}$ -modules. Suppose that

$$S : X \longrightarrow X', \quad T : Y \longrightarrow Y'$$

are bounded weak-weak* continuous module maps, that is,*

$$S(x \cdot v) = S(x) \cdot v, \quad T(v \cdot y) = v \cdot T(y)$$

for all $x \in X, y \in Y$, and $v \in \mathcal{V}$. Then, $S \bar{\otimes} T$ descends to a unique bounded weak-weak* continuous linear map*

$$S \bar{\otimes}_{\mathcal{V}} T : X \bar{\otimes}_{\mathcal{V}} Y \longrightarrow X' \bar{\otimes}_{\mathcal{V}} Y'$$

satisfying

$$(S \bar{\otimes}_{\mathcal{V}} T)(x \otimes_{\mathcal{V}} y) = S(x) \otimes_{\mathcal{V}} T(y).$$

Proof. We first prove the unbalanced case. Since S and T are weak*-weak* continuous, by Proposition B.1, they have pre-adjoints

$$S_* : U'_* \longrightarrow U_*, \quad T_* : V'_* \longrightarrow V_*.$$

Recall that, by Definition 5.1.1, we regard $U \bar{\otimes} V$ as a weak* closed subspace of $\mathcal{B}(V_*, U)$. Define

$$\begin{aligned} \Gamma : \mathcal{B}(V_*, U) &\longrightarrow \mathcal{B}(V'_*, U') \\ \Gamma(A) &= S \circ A \circ T_*. \end{aligned}$$

Note that

$$\begin{array}{ccc} V_* & \xrightarrow{A} & U \\ T_* \uparrow & & \downarrow S \\ V'_* & \xrightarrow{\Gamma(A)} & U' \end{array}$$

$$\Gamma(A) = S \circ A \circ T_*.$$

Figure B.1: Commutative diagram defining $\Gamma(A)$.

Then, Γ is bounded and

$$\|\Gamma\| \leq \|S\| \|T_*\| = \|S\| \|T\|.$$

We show that Γ is weak*-weak* continuous. By the canonical pairing realization in Remark 5.1.2 and the definition of adjoint operator, for $A \in \mathcal{B}(V_*, U)$, $\psi' \in V'_*$, and $\phi' \in U'_*$, we compute

$$\begin{aligned} \langle A, \Gamma_*(\psi' \otimes \phi') \rangle &= \langle \Gamma(A), \psi' \otimes \phi' \rangle \\ &= \langle \Gamma(A)(\psi'), \phi' \rangle \\ &= \langle S(A(T_*\psi')), \phi' \rangle \\ &= \langle A(T_*\psi'), S_*(\phi') \rangle \\ &= \langle A, T_*(\psi') \otimes S_*(\phi') \rangle. \end{aligned}$$

Thus, we can define the pre-adjoint of Γ on elementary tensors by

$$\begin{aligned} \Gamma_* : V'_* \otimes U'_* &\longrightarrow V_* \otimes U_*, \\ \Gamma_*(\psi' \otimes \phi') &= T_*(\psi') \otimes S_*(\phi'). \end{aligned}$$

Since

$$\|\Gamma_*(\psi' \otimes \phi')\|_\gamma \leq \|T_*\| \|S_*\| \|\psi'\| \|\phi'\|,$$

we obtain a bounded linear map $\Gamma_* : V'_* \otimes_\gamma U'_* \longrightarrow V_* \otimes_\gamma U_*$ such that $\|\Gamma_*\| \leq \|T_*\| \|S_*\| = \|T\| \|S\|$. Clearly, $\Gamma = \Gamma^*$, so Γ is weak*-weak* continuous.

Now, for $u \in U$, $v \in V$, and $\psi' \in V'_*$, by using the definition of Γ , we compute

$$\begin{aligned} \Gamma(u \otimes v)(\psi') &= S((u \otimes v)(T_*\psi')) \\ &= S((T_*\psi')(v)u) \\ &= \psi'(T(v))S(u). \end{aligned}$$

On the other hand, by the definition of the normal spatial tensor embedding, the elementary tensor $S(u) \otimes T(v)$ acts on $\psi' \in V'_*$ by

$$(S(u) \otimes T(v))(\psi') = \psi'(T(v))S(u).$$

Combining the last two equations, we obtain

$$\Gamma(u \otimes v)(\psi') = (S(u) \otimes T(v))(\psi') \quad (\psi' \in V'_*);$$

hence,

$$\Gamma(u \otimes v) = S(u) \otimes T(v) = [S \otimes T](u \otimes v).$$

Since $U \otimes V$ is weak* dense in $U \bar{\otimes} V$ and Γ is weak*-weak* continuous, we have

$$\Gamma(U \bar{\otimes} V) \subseteq U' \bar{\otimes} V'.$$

Therefore,

$$S \bar{\otimes} T := \Gamma|_{U \bar{\otimes} V} : U \bar{\otimes} V \longrightarrow U' \bar{\otimes} V'$$

is the required bounded weak*-weak* continuous map.

Now assume that X, Y, X', Y' are one-sided $\mathcal{V}$ -modules as in the statement of the lemma. Let J and J' denote the weak* closed balancing subspaces defining $X \bar{\otimes}_{\mathcal{V}} Y$ and $X' \bar{\otimes}_{\mathcal{V}} Y'$, respectively. For generators of the balancing subspace, we have

$$\begin{aligned} (S \bar{\otimes} T)(x \cdot v \otimes y - x \otimes v \cdot y) &= S(x \cdot v) \otimes T(y) - S(x) \otimes T(v \cdot y) \\ &= S(x) \cdot v \otimes T(y) - S(x) \otimes v \cdot T(y). \end{aligned}$$

The final expression belongs to J' by definition. Since $S \bar{\otimes} T$ is weak*-weak* continuous and J is the weak* closed span of the balancing relations, it follows that

$$(S \bar{\otimes} T)(J) \subseteq J' = \ker(q'),$$

where $q : X \bar{\otimes} Y \longrightarrow X \bar{\otimes}_{\mathcal{V}} Y$ and $q' : X' \bar{\otimes} Y' \longrightarrow X' \bar{\otimes}_{\mathcal{V}} Y'$ are the canonical quotient maps. The above inclusion implies that

$$J \subseteq \ker(q' \circ (S \bar{\otimes} T)).$$

Hence, by Remark 5.1.5, the weak*-weak* continuous map

$$q' \circ (S \bar{\otimes} T) : X \bar{\otimes} Y \longrightarrow X' \bar{\otimes}_{\mathcal{V}} Y'$$

descends to a unique bounded weak*-weak* continuous linear map

$$S \bar{\otimes}_{\mathcal{V}} T : X \bar{\otimes}_{\mathcal{V}} Y \longrightarrow X' \bar{\otimes}_{\mathcal{V}} Y'$$

such that

$$(S \bar{\otimes}_{\mathcal{V}} T) \circ q = q' \circ (S \bar{\otimes} T).$$

Therefore, for elementary tensors, we have

$$(S \bar{\otimes}_{\mathcal{V}} T)(x \otimes_{\mathcal{V}} y) = S(x) \otimes_{\mathcal{V}} T(y).$$

□

Lemma B.5. *Let A be a dual Banach algebra, U be a normal left dual Banach A -module, and V be a dual Banach space. Define an action of A on $U \bar{\otimes} V$ by*

$$a \cdot (u \otimes v) = (a \cdot u) \otimes v.$$

Then, $U \bar{\otimes} V$ is a normal left dual Banach A -module. Similarly, if V is a normal right dual Banach A -module, then $U \bar{\otimes} V$ is a normal right dual Banach A -module under

$$(u \otimes v) \cdot a = u \otimes (v \cdot a).$$

Proof. We prove the left module case. For fixed $a \in A$, the map

$$L_a : U \longrightarrow U, \quad u \longmapsto a \cdot u,$$

is weak*-weak* continuous. Hence, by Functoriality Lemma B.4, it induces a weak*-weak* continuous map

$$L_a \bar{\otimes} \text{id}_V : U \bar{\otimes} V \longrightarrow U \bar{\otimes} V.$$

Thus, $U \bar{\otimes} V$ is a dual Banach left A -module. It remains to prove normality. Let $\eta \in U \bar{\otimes} V$ and let $\rho \in (U \bar{\otimes} V)_*$. It suffices to show that

$$A \longrightarrow \mathbb{C}, \quad a \longmapsto \langle a \cdot \eta, \rho \rangle$$

is weak* continuous. By Definition 5.1.1, the pre-dual $(U \bar{\otimes} V)_*$ is a quotient of $V_* \otimes_\gamma U_*$. Hence, we may represent ρ by

$$\rho = \sum_{n=1}^{\infty} \psi_n \otimes \phi_n \in V_* \otimes_\gamma U_*, \quad \sum_{n=1}^{\infty} \|\psi_n\| \|\phi_n\| < \infty. \quad (\text{B.1})$$

By the canonical pairing realization in Remark 5.1.2, for $a \in A$, we have

$$\begin{aligned} \langle a \cdot \eta, \rho \rangle &= \sum_{n=1}^{\infty} \langle a \cdot \eta, \psi_n \otimes \phi_n \rangle \\ &= \sum_{n=1}^{\infty} \langle a \cdot \eta(\psi_n), \phi_n \rangle. \end{aligned} \quad (\text{B.2})$$

For each $n \in \mathbb{N}$, since U is a normal left A -module, the functional

$$A \longrightarrow \mathbb{C}, \quad a \longmapsto \langle a \cdot \eta(\psi_n), \phi_n \rangle$$

belongs to A_* , and

$$\|a \mapsto \langle a \cdot \eta(\psi_n), \phi_n \rangle\| \leq \|\eta\| \|\psi_n\| \|\phi_n\|,$$

Therefore, by (B.1), the series (B.2) converges absolutely in A_* . Hence, for every $\rho \in (U \bar{\otimes} V)_*$, the functional

$$a \longmapsto \langle a \cdot \eta, \rho \rangle = \rho(a \cdot \eta)$$

belongs to A_* . Thus, the orbital map

$$A \longrightarrow U \bar{\otimes} V, \quad a \longmapsto a \cdot \eta$$

is weak*-weak* continuous. Consequently, $U \bar{\otimes} V$ is a normal left dual Banach A -module. The right module case is proved similarly, slicing in the other tensor variable. $\square$

Corollary B.6. *Assume that A and $\mathcal{V}$ be dual Banach algebras. Let U be a normal left A -module and a normal right $\mathcal{V}$ -module, and let V be a normal left $\mathcal{V}$ -module. Suppose that the left A -action on $U \bar{\otimes} V$ preserves the balancing subspace J . Then, the induced left A -action on $U \bar{\otimes}_{\mathcal{V}} V$ is normal.*

Proof. Let $q : U \bar{\otimes} V \longrightarrow U \bar{\otimes}_{\mathcal{V}} V$ be the quotient map, and let J be the weak* closed balancing subspace such $a \cdot J \subseteq J$ for each $a \in A$. Then, the natural action of A on $U \bar{\otimes}_{\mathcal{V}} V$ is given by

$$a \cdot q(\zeta) := q(a \cdot \zeta) \quad (a \in A, \zeta \in U \bar{\otimes} V).$$

The adjoint of the quotient map is given by

$$\begin{aligned} q^* : (U \bar{\otimes}_{\mathcal{V}} V)^* &\longrightarrow (U \bar{\otimes} V)^*, \\ \varphi &\longmapsto q^* \varphi, \\ \langle q^* \varphi, \eta \rangle &= \langle \varphi, q(\eta) \rangle, \quad (\eta \in U \bar{\otimes} V). \end{aligned}$$

Restricting q^* to the chosen predual $(U \bar{\otimes}_{\mathcal{V}} V)_*$ gives the map

$$q_\sigma := q^*|_{(U \bar{\otimes}_{\mathcal{V}} V)_*} : (U \bar{\otimes}_{\mathcal{V}} V)_* \longrightarrow (U \bar{\otimes} V)^*.$$

We show the range of q_σ is $J_\perp$. To this end, let $\varphi \in (U \bar{\otimes}_{\mathcal{V}} V)_*$. Then, $q^* \varphi$ is weak* continuous since q is weak*-weak* continuous. Also, for every $\zeta \in J$,

$$\langle q^* \varphi, \zeta \rangle = \langle \varphi, q(\zeta) \rangle = 0.$$

Hence, $\text{ran}(q_\sigma) \subseteq J_\perp$. Conversely, if $\rho \in J_\perp$, define

$$\langle \varphi, q(\eta) \rangle := \langle \rho, \eta \rangle \quad (\eta \in U \bar{\otimes} V).$$

This is well defined. Indeed, if $q(\eta_1) = q(\eta_2)$, then $\eta_1 - \eta_2 \in J$. Since $\rho \in J_\perp$, we have $\rho(\eta_1 - \eta_2) = 0$ and hence $\rho(\eta_1) = \rho(\eta_2)$. Therefore, φ is well defined and by definition satisfies $q_\sigma \varphi = \rho$. Thus, $\text{ran}(q_\sigma) = J_\perp$. So, we obtain

$$\begin{aligned} q_\sigma : (U \bar{\otimes}_{\mathcal{V}} V)_* &\longrightarrow J_\perp \subseteq (U \bar{\otimes} V)^*, \\ \langle q_\sigma \varphi, \eta \rangle &= \langle \varphi, q(\eta) \rangle, \quad (\varphi \in (U \bar{\otimes}_{\mathcal{V}} V)_*, \eta \in U \bar{\otimes} V). \end{aligned} \tag{B.3}$$

Let $\eta \in U \bar{\otimes}_{\mathcal{V}} V$, $a \in A$ and $\varphi \in (U \bar{\otimes}_{\mathcal{V}} V)_*$. Choose an element $\zeta \in U \bar{\otimes} V$ such that $q(\zeta) = \eta$. Then,

$$\begin{aligned} \langle \varphi, a \cdot \eta \rangle &= \langle \varphi, a \cdot q(\zeta) \rangle \\ &= \langle \varphi, q(a \cdot \zeta) \rangle \\ &= \langle q_\sigma \varphi, a \cdot \zeta \rangle \end{aligned}$$

By Lemma B.5, the action of A on $U \bar{\otimes} V$ from left is weak* continuous. The functional $q_\sigma \varphi$ is also weak*-continuous by (B.3). Hence, the orbital map

$$\ell_\eta : A \longrightarrow U \bar{\otimes}_{\mathcal{V}} V, \quad a \longmapsto a \cdot \eta$$

is weak*-weak* continuous. □

Lemma B.7. *Let X, Y, Z be dual Banach spaces with fixed preduals, and let*

$$B : X \times Y \longrightarrow Z$$

be a normally linearizable bounded bilinear map with associated map

$$B_s : Z_* \longrightarrow (X \bar{\otimes} Y)_*.$$

Then, there is a unique weak-weak* continuous linear map*

$$\widehat{B} : X \bar{\otimes} Y \longrightarrow Z$$

such that $\widehat{B} = B_s^*$, i.e.,

$$\widehat{B}(x \otimes y) = B(x, y) \quad (x \in X, y \in Y).$$

Proof. Since B_s is bounded, its adjoint gives a weak*-weak* continuous map

$$B_s^* : X \bar{\otimes} Y = ((X \bar{\otimes} Y)_*)^* \longrightarrow (Z_*)^* = Z.$$

Set

$$\widehat{B} := B_s^*.$$

Then, for $x \in X$, $y \in Y$, and $\omega \in Z_*$, we have

$$\langle \omega, \widehat{B}(x \otimes y) \rangle = \langle B_s(\omega), x \otimes y \rangle = \langle \omega, B(x, y) \rangle.$$

Since Z_* separates points of Z , it follows that $\widehat{B}(x \otimes y) = B(x, y)$. Uniqueness follows from the weak* density of $X \otimes Y$ in $X \bar{\otimes} Y$. $\square$

Corollary B.8. *Let $\mathcal{V}$ be a dual Banach algebra, X be a normal right $\mathcal{V}$ -module, Y be a normal left $\mathcal{V}$ -module, and Z be a dual Banach space. Suppose that the $\mathcal{V}$ -balanced bounded bilinear form*

$$B : X \times Y \longrightarrow Z$$

is a normally linearizable. Then, the linearization $\widehat{B}$ descends to a unique weak-weak* continuous map*

$$\widetilde{B} : X \bar{\otimes}_{\mathcal{V}} Y \longrightarrow Z$$

such that

$$\widetilde{B}(x \otimes_{\mathcal{V}} y) = B(x, y) \quad (x \in X, y \in Y).$$

Proof. Let

$$q : X \bar{\otimes} Y \longrightarrow X \bar{\otimes}_{\mathcal{V}} Y, \quad q(\eta) = \eta + J,$$

be the quotient map, where J is the weak* closed balancing subspace. By Lemma B.7, the map B has a weak*-weak* continuous linearization

$$\widehat{B} : X \bar{\otimes} Y \longrightarrow Z$$

satisfying $\widehat{B}(x \otimes y) = B(x, y)$. Since B is balanced and $\widehat{B}$ is weak*-weak* continuous, it follows that

$$J \subseteq \ker(\widehat{B}).$$

Therefore, by Remark 5.1.5, $\widehat{B}$ descends to a unique weak*-weak* continuous map

$$\widetilde{B} : X \bar{\otimes}_{\mathcal{V}} Y \longrightarrow Z$$

such that $\widehat{B} = \widetilde{B} \circ q$. Clearly, $\widetilde{B}(x \otimes_{\mathcal{V}} y) = B(x, y)$. $\square$

Lemma B.9. *Let X and $\mathcal{W}$ be dual Banach spaces, and $\iota : X \longrightarrow \mathcal{W}$ be a bounded weak*-weak* continuous linear map with weak* dense range. Suppose also that there is a bounded linear map $m : \mathcal{W} \longrightarrow X$ such that $m \circ \iota = \text{id}_X$. Then, ι is onto. Consequently, $m = \iota^{-1}$, and $m : \mathcal{W} \rightarrow X$ is weak*-weak* continuous.*

Proof. Since $m \circ \iota = \text{id}_X$, we have

$$\|\xi\| = \|m(\iota(\xi))\| \leq \|m\| \|\iota(\xi)\|, \quad (\xi \in X).$$

Hence, ι is bounded below. We first show that $\iota(X)$ is weak* closed in $\mathcal{W}$. Let $r > 0$, and $\eta \in \iota(X) \cap rB_{\mathcal{W}}$. Then, $\eta = \iota(\xi)$ for some $\xi \in X$, and

$$\|\xi\| = \|m(\eta)\| \leq \|m\| \|\eta\| \leq \|m\| r.$$

Hence,

$$\iota(X) \cap rB_{\mathcal{W}} = \iota(\|m\|rB_X) \cap rB_{\mathcal{W}}.$$

By the Banach–Alaoglu theorem, $\|m\|rB_X$ is weak* compact. Since ι is weak*-weak* continuous, $\iota(\|m\|rB_X)$ is weak* compact in $\mathcal{W}$. Therefore, $\iota(X) \cap rB_{\mathcal{W}}$ is weak* closed. By Krein–Smulian theorem [13, Theorem V.12.1], we conclude that $\iota(X)$ is weak* closed in $\mathcal{W}$. However, $\iota(X)$ is weak* dense by assumption. Thus,

$$\iota(X) = \mathcal{W};$$

i.e., ι is onto. Since $m \circ \iota = \text{id}_X$, it follows that

$$m = \iota^{-1}.$$

It remains to prove that m is weak*-weak* continuous. Let

$$\eta_\lambda \xrightarrow{w^*} \eta \quad \text{in } \mathcal{W}.$$

The net (η_λ) is norm bounded, so there is $r > 0$ such that

$$\eta_\lambda, \eta \in rB_{\mathcal{W}} \quad (\lambda).$$

Since m is bounded,

$$m(\eta_\lambda), m(\eta) \in \|m\|rB_X \quad (\lambda).$$

The ball $\|m\|rB_X$ is weak* compact by Banach–Alaoglu. Moreover,

$$\iota : \|m\|rB_X \longrightarrow \iota(\|m\|rB_X)$$

is a continuous bijection from a compact space onto a Hausdorff space. Hence, it is a homeomorphism. Therefore, its inverse is weak* continuous on $\iota(\|m\|rB_X)$. Since

$$\eta_\lambda = \iota(m(\eta_\lambda)), \quad \eta = \iota(m(\eta)),$$

we obtain

$$m(\eta_\lambda) = \iota^{-1}(\eta_\lambda) \xrightarrow{w^*} \iota^{-1}(\eta) = m(\eta).$$

Thus, m is weak*-weak* continuous. □

Proposition B.10. *Let $\mathcal{N}$ be a von Neumann algebra with identity p , and $X = Y^*$ be a unital normal dual $\mathcal{N}$ -bimodule. Set $\mathcal{W} = \mathcal{N} \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} \mathcal{N}$, and let μ be canonical action map*

$$\begin{aligned}\mu : \mathcal{W} &\longrightarrow X, \\ \mu(a \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} b) &= a \cdot \xi \cdot b.\end{aligned}$$

Then, μ is a weak-weak* homeomorphic isomorphism, and its inverse is given by*

$$\begin{aligned}\iota : X &\longrightarrow \mathcal{W}, \\ \iota(\xi) &= p \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} p.\end{aligned}$$

Proof. Let κ_p be the weak*-weak* continuous map

$$\kappa_p : \mathbb{C} \longrightarrow \mathcal{N}, \quad \kappa_p(\lambda) = \lambda p,$$

and

$$q : \mathcal{N} \bar{\otimes} X \bar{\otimes} \mathcal{N} \longrightarrow \mathcal{N} \bar{\otimes}_{\mathcal{N}} X \bar{\otimes}_{\mathcal{N}} \mathcal{N} = \mathcal{W}$$

be the corresponding canonical weak*-weak* continuous quotient map as in Remark 5.1.5. Consider the weak*-weak* continuous map

$$\Lambda : X \longrightarrow \mathbb{C} \bar{\otimes} X \bar{\otimes} \mathbb{C}, \quad \Lambda(\xi) = 1 \otimes \xi \otimes 1,$$

of Lemma B.3. Then, for $\xi \in X$,

$$\begin{aligned}q\left((\kappa_p \bar{\otimes} \text{id}_X \bar{\otimes} \kappa_p)(\Lambda(\xi))\right) &= q\left((\kappa_p \bar{\otimes} \text{id}_X \bar{\otimes} \kappa_p)(1 \otimes \xi \otimes 1)\right) \\ &= q(p \otimes \xi \otimes p) \\ &= p \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} p \\ &= \iota(\xi).\end{aligned}$$

That is, $\iota = q \circ (\kappa_p \bar{\otimes} \text{id}_X \bar{\otimes} \kappa_p) \circ \Lambda$. So, ι is weak*-weak* continuous by Lemma B.4. Note that the module action map μ is balanced since for $n \in \mathcal{N}$,

$$(an) \cdot \xi \cdot b = a \cdot (n \cdot \xi) \cdot b, \quad \text{and} \quad a \cdot (\xi \cdot n) \cdot b = a \cdot \xi \cdot (nb).$$

It is clearly bounded since

$$\|a \cdot \xi \cdot b\| \leq \|a\| \|\xi\| \|b\|.$$

Recall that $p \in \mathcal{N}$ acts as the identity of module actions, so for any $\xi \in X$,

$$\begin{aligned}\mu(\iota(\xi)) &= \mu(p \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} p) \\ &= p \cdot \xi \cdot p \\ &= \xi.\end{aligned}$$

Hence, $\mu \circ \iota = \text{id}_X$. We now show that $\iota(X)$ is weak* dense in $\mathcal{W}$. Let $a, b \in \mathcal{N}$ and $\xi \in X$. Using the balancing relations and the fact that p is the identity of $\mathcal{N}$, we get

$$\begin{aligned}a \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} b &= pa \otimes_{\mathcal{N}} \xi \otimes_{\mathcal{N}} b \\ &= p \otimes_{\mathcal{N}} (a \cdot \xi) \otimes_{\mathcal{N}} b \\ &= p \otimes_{\mathcal{N}} (a \cdot \xi \cdot b) \otimes_{\mathcal{N}} p \\ &= \iota(a \cdot \xi \cdot b).\end{aligned}$$

Therefore, every elementary balanced tensor belongs to $\iota(X)$. Since elementary balanced tensors are weak* dense in $\mathcal{W}$, the range $\iota(X)$ is weak* dense in $\mathcal{W}$. Now, by Lemma B.9, ι is onto, $\mu = \iota^{-1}$, and μ is weak*–weak* continuous. $\square$

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