ON CONTINUITY, DIFFERENTIABILITY AND INTEGRABILITY

A Thesis Presented to
the Faculty of Graduate Studies and Research
University of Manitoba

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

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April 1950

TABLE OF CONTENTS

CHAPTER 1

AIM, METHOD, AND JUSTIFICATION OF CONSIDERATIONS PROPOSED

Herein it will be attempted to add to the understanding of the basic concepts of continuity, differentiability and integrability, by the consideration of functions which are not well-behaved. The more common functions in analysis are apt to be very misleading, as they do not show in all cases the requirements for full rigor in setting up necessary and sufficient conditions.

Historical development in analysis has shown this effect. It was at one time held that any continuous function has a derivative. Leibnitz and Newton defined $f(x) = F'(x)$, and the integral as the "anti-derivative"; ensuring that

$$\int_a^x f(x)dx = F(x) - F(a).$$

Thus the problem of having given an $f(x)$, and establishing the existence of $F(x)$ such that $F'(x) = f(x)$, did not arise. The constructive definition of the integral came much later, Riemann's integral being the first of this type. An assumption then made was that

$$\int_a^b F'(x)dx$$

always exists. Such conceptions were prevalent even up to the time of Fourier at the beginning of the nineteenth century. Examples to the contrary have proved them false.

Reversibility of operations and an intuitive behavior of elements with their combinations, introduce a simplification into any logical science. Thus with the discovery that the original simple ideas were inadequate, an attempt was made to redefine integration. It was hoped that integration might form with differentiation a reversible process, and enable the beliefs of Newton to be true in this new sense. Of many generalized forms of integral, that of Lebesgue proved to be the most effective. It is neither so specialized as to exclude many functions from consideration, as does the Riemann integral, nor is it so generalized, as are some others, that it is lacking in power. The attempt for a new simplicity was only partially successful. While considering the failure of oversimplified concepts of functional behavior under Riemann integration, the extent to which they may hold using the integral of Lebesgue will be shown.

The method of counter-example, while at best an indirect form of proof, and having in general only the ability to deny, has been the most useful force in destroying these false ideas. This is essentially the method used here.

Functions which show the limitations on the concepts of analysis are in general characterized by discontinuities, either in the function itself or in its derivatives. Selected examples will be considered in an ordered classification. These range from the totally discontinuous, through increasing degrees of regularity, to a type continuous with a finite bounded derivative existing everywhere.

CHAPTER II

EXAMINATION OF FUNCTIONS DISCONTINUOUS EVERYWHERE

(1) The characteristic function of the rationals.

An example is first given of a function for which the Lebesgue integral exists while the Riemann integral does not. Consider the characteristic function of the set of rationals on an interval; ie on the interval (0,1) a function f(x) such that f(x)=1 if x is rational and f(x)=0 if x is irrational, f(x) is obviously discontinuous everywhere in the range of definition.

The Riemann integral is defined as the common limit as $\max(x_{p+1}-x_p)\to 0$ of the sums: $s_n=\sum_{p=1}^{n} m_p(x_{p+1}-x_p)$ and

$$S_n=\sum_{p=1}^{n} M_p(x_{p+1}-x_p);$$

where the interval of variation (a,b) of the independent variable is subdivided by x such that $a=x_1<x_2<\ldots<x_n=b$ and m_p M_p are the lower and upper bounds respectively of f(x) in the interval

$$x_p<x\leq x_{p+1}$$

In the case of the characteristic function the m_p in each case is 0 and the M_p is 1. Thus the two sums are not equal in the limit, and the Riemann integral does not exist.

The Lebesgue integral is defined for a function bounded between the values a and b as the common limit of two similar sums. The interval of variation of the function is sub-divided by means of y such that

$$a=y_0<y_1<\ldots<y_n=b, \text{ and}$$

the lower sum $s_n = \sum_{p=0}^{n-1} y_p \, me_p$;

where me_p the measure of the set e_p of values of the independent variable such that $y_p \leq y < y_{p+1}$. The upper sum

$$S_n = \sum_{p=0}^{n-1} y_{p+1} me_p \, .$$

lim $(S_n - s_n) = 0$ as max $(y_{p+1} - y_p) \to 0$, due to the nature of a measurable function. It will now be shown that the Lebesgue integral of the characteristic function exists. The limit of the sum s_n is

$$\lim s_n = \lim_{n \to \infty} \sum_{p=0}^{n-1} y_p \, me_p \, ,$$

taken over the set of all real x in (0,1).

$$y_p = 0 \text{ for } p=0$$

$$me_p = 0 \text{ for all } p \neq 0$$

Thus $\lim s_n = 0$.

Hence the Lebesgue integral exists and is equal to zero.

(2) Example of Lebesgue

An evident requirement of a continuous function is that it assume within an interval all values intermediate to the functional values at the end-points; ie if f(x) is continuous on the interval (a,b) and $f(a) \neq f(b)$, eg $f(a) < f(b)$, then given g such that

$$f(a) \leq g \leq f(b)$$

there exists a number c, $(a \leq c \leq b)$ such that $f(c) = g$.

It is shown below that this may also be a property of a function everywhere discontinuous on the interval.(1)*

The function $f(x)$ is defined on the interval 0,1. Let all real numbers x in the interval 0,1 be expressed as infinite decimals, terminating decimals being represented as ending with an infinite succession of 9's. A number thus represented has the form

$0.a_1 a_2 \cdots a_n \cdots$. If the decimal expansion

$0.a_1 a_3 a_5 \cdots$ is not periodic the function $f(x)$ is given the value zero. If this expansion is periodic, the first period commencing with a_{2n-1}, $f(x)$ is given the value

$$0.a_{2n} a_{2n+2} a_{2n+4} \cdots .$$

It will now be shown that in every sub-interval of (0,1) $f(x)$ takes on all values between 0 and 1 and hence is everywhere discontinuous on (0,1). Let A,B be any subinterval such that

$$0 \leq A < B \leq 1 \text{ and } \frac{1}{10^n} \leq B-A \leq \frac{1}{10^{n-1}}.$$

Their decimal expansions will be identical up to and including the $(n-2)^{nd}$ place but will differ in the $(n-1)^{st}$ place. It will be shown that elements T, U exist such that $A < T < U < B$ where T and U are of the form $T = 0.C_1C_2\cdots C_nC_{n+1}000\cdots$; $U = 0.C_1C_2\cdots C_{n+1}999\cdots$

Consider the set of intervals

$$0 \leq x < \frac{1}{10^{n+1}}, \frac{1}{10^{n+1}} \leq x < \frac{2}{10^{n+1}} \cdots \frac{k}{10^{n+1}} \leq x < \frac{k+1}{10^{n+1}} \cdots \frac{10^{n+1}-1}{10^{n+1}} \leq x < 1.$$

Every point of the interval $0 \leq x < 1$ lies in one of these intervals.

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(1)* Lebesgue, Lecons Sur L'Integration, 2[d] Ed., p. 97.

Hence there is an integer K such that $\frac{k-1}{10^{n+1}} \leq A < \frac{k}{10^{n+1}}$. Take

$$T = \frac{k}{10^{n+1}}, \quad U = \frac{k+1}{10^{n+1}} \quad \text{Thus } A < T < U.$$

$B-U = B-\frac{k+1}{10^{n+1}} = (B-A) + \left(A-\frac{k+1}{10^{n+1}}\right) \geq \frac{10}{10^{n+1}} - \frac{2}{10^{n+1}} > 0$. Hence $A < T < U < B$. Let k in decimal notation be $0.C_1C_2\cdots C_{n+1}$.

$T = 0.C_1\cdots C_{n+1}\,000\cdots$; $U = 0.C_1\cdots C_{n+1}\,999\cdots$.

A point G will be shown to exist such that $T \leq G \leq U$ and such that for any number D, $0 \leq D \leq 1$, $f(G) = D$.

Let D in decimal notation be expressed as $D = 0.d_1d_2\cdots d_n\cdots$.

If $n+1$ is even take $G = 0.C_1C_2\cdots C_{n+1}\,kd_1kd_2kd_3\cdots$ where k any integer except C_n. Then by definition $f(G) = D$. If $n+1$ is odd take

$$G = 0.C_1\cdots C_{n+1}0\,kd_1kd_2kd_3\cdots$$

where k any integer except C_{n+1}. Again $f(G) = D$. Hence $f(x)$ while everywhere discontinuous on $(0,1)$ assumes all intermediate values.

CHAPTER 111

CONTINUITY WITHOUT DIFFERENTIABILITY

The behavior of the above functions may not be surprising because of their discontinuous nature, but even continuous functions may show properties far removed from the regular behavior assigned to them by early mathematicians. In van der Waerden's example is seen a function continuous on an interval but having no differential coefficient at any point within it.(2)

This function $f(x)$ is defined on the interval 0,1 as $\sum_{n=1}^{\infty} f_n(x)$ where $f_n(x)$ denotes the distance between x and the nearest number of the form $\frac{m}{10^n}$, where m is an integer.

$f(x)$ is easily shown to be continuous. Each $f_n(x)$ is continuous and $|f_n(x)| < 10^{-n}$. $f(x)$ is the sum of a uniformly convergent series of continuous functions and is therefore continuous.

It will now be shown that a differential coefficient $f'(x)$ does not exist. Let x, any number in the interval (0,1),be expressed as an infinite decimal -- terminating decimals having a final infinite succession of 9's. If the q'th digit be a 4 or 9 let $x_q = x - 10^{-q}$, otherwise $x_q = x+10^{-q}$. Hence if $n < q$, the nearest number $\frac{m}{10^n}$ is the same for x and x_q and is either less than both of them or greater than both. It is seen that 4 and 9 must be considered specially from an example of x, x_q formed as above and $x_{q'}$ formed by treating a 4 or 9 as any other digit.

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(2) Titchmarsh, The Theory of Functions, (Oxford: The Clarendon Press, 1932), p. 353.

Let $x = 0.3546\cdots$ $x_q = 0.3536\cdots$ $x_{q'} = 0.3556\cdots$ where the q'th digit in x is 4. For x, x_q the nearest number of the form $\frac{m}{10^n}$ is 0.35, $n < q$ while for $x_{q'}$ the nearest such number is 0.36. Similarly if the q'th figure be a 9. Let $x = 0.34396\cdots$ $x_q = 0.34386\cdots$ $x_{q'} = 0.34406\cdots$. The closest number $\frac{m}{10^n}$, $n<q$ is for x, $0.3440 > x$; for x_q, $0.3440 > x_q$; for $x_{q'}$, $0.3440 < x_{q'}$, hence $x_{q'}$ and x are not both of the same order of magnitude relative to the appropriate $\frac{m}{10^n}$.

For $n \geq q$ the numbers $\frac{m}{10^n}$, $\frac{m'}{10^n}$ corresponding to x, x_q respectively differ by $x - x_q$.

Hence
$$f_n(x_q) - f_n(x) = \pm (x_q - x) \quad , (n < q)$$
$$= 0 \quad , (n \geq q$$

Thus $f(x_q) - f(x) = \sum_{n=1}^{q-1} \left\{ \pm (x_q - x) \right\} = p(x_q - x)$ where p is an integer, odd or even with $(q-1)$ since only an even multiple of $(x_q - x)$ may be removed by cancellation of positive and negative sign.

As $q \to \infty$, $x_q \to x$, $\left| \frac{f(x_q) - f(x)}{x_q - x} - \frac{f(x_{q+1}) - f(x)}{x_{q+1} - x} \right| \geq 1$.

Therefore $\frac{f(x_q) - f(x)}{x_q - x}$ does not tend to a finite limit as $x_q \to x$. Hence the differential coefficient does not exist.

CHAPTER IV

CONTINUITY AND DIFFERENTIABILITY WITHOUT INTEGRABILITY OF DERIVATIVE

With an increasing degree of regularity in the function, deviation from supposedly normal properties seems even more strange. A function may however be continuous and possess a bounded differential coefficient; yet this differential coefficient may be non-integrable in the sense of Riemann. Volterra's example, the first such function constructed, is considered here. (1)
It will be shown that in the sense of Lebesgue the derivative does integrate back to the function.

(1) Volterra's example

The function will first be defined on an interval (a,b). One requires a perfect non-dense set E of points on an interval (a,b) such that the measure of the set is greater than zero. Such a set is formed by taking the linear continuum $(0,1)$ as the interval (a,b) and deleting in the first stage an open interval of length $\frac{1}{4}$ centred at the point $\frac{1}{2}$. In the second stage are deleted open intervals with the properties:

(1) they are of equal length
(2) they are centred at the mid-points of the intervals remaining after the first stage
(3) their total length is 1/8

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(1) Hobson, The Theory of Functions of a Real Variable, (Cambridge University Press, 1907), p.357.

In the n'th stage open intervals are deleted such that:

(1) they are of equal length $\frac{1}{2^{2n}}$,

(2) they are centred at the mid-points of intervals remaining after the $(n-1)^{st}$ stage,

(3) their total length is $\frac{1}{2^{n+1}}$.

This process is continued indefinitely, the points remaining forming the set E. E consists of the end points and the limiting points of the end points of the deleted intervals. It is seen that E is perfect since it is closed--being the complement of the open deleted set--and it is dense in itself. It is non-dense in the interval (0,1), since any chosen interval will contain an interval of points of the complement, C(E), of E. C(E) is made up of the deleted open intervals. The measure of the complementary open set is the sum of the lengths of its intervals, ie $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1/2 = m(C(E))$

Hence the measure of E is $1 - mC(E) = 1/2$.

Thus E is a perfect non-dense set of measure greater than zero.

Consider a general one of the complementary deleted intervals (A,B). Define $\phi(x,A) = (x-A)^2 \sin\frac{1}{(x-A)}$. The derivative of $\phi(x,A)$ with respect to x is

$$\phi'(x,A) = 2(x-A)\sin\frac{1}{(x-A)} - \cos\frac{1}{(x-A)}; \text{ at } x = A,\ \phi(x,A) = 0 \text{ and } \phi'(x,A) = 0$$

since $\phi_A'(x,A) = \lim_{h\to 0} \frac{\phi(A+h,A) - \phi(A)}{h} = \lim_{h\to 0} \frac{1}{h}(A+h-A)^2 \sin\frac{1}{h} = 0.$

It is also evident that $\phi'(x,A)$ vanishes at an infinite number of points in (A,B). Of these zero points (A+C) is chosen as the greatest value of x not exceeding $\frac{A+B}{2}$ for which $\phi'(x,A)$ vanishes. C is $<$ 1/8 since the greatest interval (A,B) is of magnitude 1/4. For convenience let (A+C) be the last such maximum of $\phi(x)$. The necessary requirement here for the construction of a continuous function is the choice of a zero point which is not an accumulation point of zero points. Thus it is not essential that ones choice should be the very last such zero point.

It will now be shown that such a last maximum in $(x-A)^2 \sin \frac{1}{x-A}$ in the range $A, \frac{A+B}{2}$ does exist. In the interval $A, \frac{A+B}{2}$; A is the only accumulation point of zero points of $\phi'(x,A)$. In any range $0 \leq x - A < x' - A$ where $A \leq x < x'$ the quantity $\frac{1}{x-A}$ varies from $\frac{1}{x'-A}$ to ∞. In this range an infinite number of odd integral multiples of $\frac{\pi}{2}$ occur, at each of which is a zero of ϕ'. Thus any arbitrarily small interval about A contains infinitely many zero points. In the interval $\left(x', \frac{B+A}{2}\right)$ the quantity $\frac{1}{x-A}$ varies throughout a finite range $\left(\frac{1}{x'-A}, \frac{1}{\frac{B+A}{2}-A}\right)$ which contains only a finite number of odd integral multiples of $\frac{\pi}{2}$. Hence there are only a finite number of zero points in any range $\left(x', \frac{B+A}{2}\right)$ where $x' > A$. If a maximum point t_1 of $\phi(x,A)$ is chosen such that $A < t_1 \leq \frac{A+B}{2}$ there are an infinite number of maxima in (A,t_1) but at most a finite number in $\left(t_1, \frac{A+B}{2}\right)$. Hence the interval $t_1 \leq x \leq \frac{A+B}{2}$ contains a last maximum of $\phi(x,A)$ which may be t_1. A+C is the value of x at which this last maximum appears.

The function F(x) is defined such that F(x) = 0 at every point of E. In each complementary interval (A,B), F(x) is defined as $\phi(x,A)$ for $A \leq x \leq A+C$; $F(x) = \phi(A+C,A)$ for $A + C \leq x \leq B - C$ and

$F(x) = \phi(B,x) = -\phi(x,B)$ for $B - C \leq x \leq B$. $F(x)$ so defined can be shown to be continuous with a finite bounded differential coefficient everywhere on the interval of definition $(0,1)$.

$F(x)$ will be shown to be continuous by the manner of its definition. In E, $F(x) = \lim_{n \to o} F(X_n)$ where X_n an end point of a complementary interval to E and $X_n \to x$. On a complementary interval (A,B), $F(x) = \phi(x,A)$ in $(A,A+C)$; $F(x) = \phi(A+C,A)$ in $(A+C,B-C)$ and $F(x) = \phi(B,x)$ in $(B-C,B)$. At the point $x = B-C$, $\phi(B,B-C) = (B-(B-C))^2 \sin \frac{1}{B-(B-C)} = C^2 \sin \frac{1}{C} = \phi(A+C,A)$. Hence it is seen that $F(x)$ is continuous on the interval (A,B).

It will now be shown that the derivative of $F(x)$ with respect to x, $F'(x)$, exists everywhere, is bounded and measurable but has a discontinuity, of measure at least 2, at all points of E.

The derivative of $F(x)$ in $C(E)$ is zero for the ranges $(A_n + C, B_n - C)$ where $F(x)$ is constant and is $\phi'(x,A_n)$ on (A_n, A_n+C); $\phi'(B_n,x)$ on (B_n-C, B_n). It will now be shown that at a point x of E, $F'(x) = 0$; ie for any constant $e > 0$ there exists a constant $d > 0$ such that $\left|\frac{F(x+h) - F(x)}{h}\right| < e$ for all $|h| < d$. Given $e > 0$ take $d = e$ and $0 < |h| < d$. If $x+h$ in E, $\frac{F(x+h) - F(x)}{h} = \frac{0 - 0}{h} = 0$. If $x+h$ in $C(E)$ and $x - e \leq A - B \leq x + e$ there are three possible cases.

Case 1. If $x+h$ in the interval $A,A+C$ then

$$\left|\frac{F(x+h) - F(x)}{h}\right| = \frac{(x+h - A)^2}{h} \sin \frac{1}{x+h-A}.$$

The interval A,B must lie wholly to the left with $B \leq x$ or to the right with $x \leq A$ since a point of E could not be contained in a complementary interval. Therefore $|x+h-A| < |h|$ and $\left|\frac{F(x+h) - F(x)}{h}\right| < |h| < e$.

Case 2. If $x+h$ be in $A+C$, $B-C$ then $\left|\frac{F(x+h) - F(x)}{h}\right| = \frac{C^2}{h} \sin \frac{1}{h}$ and

$C < |h|$. Therefore $\left|\frac{F(x+h) - F(x)}{h}\right| < |h| < e$.

Case 3. If $x+h$ in $B-C$, B; $\left|\frac{F(x+h) - F(x)}{h}\right| = \frac{(B-x-h)^2}{h} \sin \frac{1}{B-x-h}$ and

$|B-x-h| < |h|$. Hence $\left|\frac{F(x+h)-F(x)}{h}\right| < |h| < e$.

If the complementary interval containing $x+h$ is such that (1) $A < x - e < B \leq x$ or (2) $x \leq A < x + e < B$ the result follows as above. Here the location of $x \pm e$ in the interval A,B might exclude one or two of the above cases, eg. in (1) if $B - C < x - e \leq B$ then one would not find $x+h$ in the ranges $A,A+C$ or $A+C$, $B-C$.

Hence $F'(x) = 0$ at all points of E. In the ranges $(A,A+C)$; $(B-C,B)$ $F'(x)$ is respectively $2(x-A) \sin \frac{1}{x-A} - \cos \frac{1}{x-A}$ and $2(B-x) \sin \frac{1}{B-x} - \cos \frac{1}{B-x}$. These can never exceed $|2C+1|$ since $C < \frac{B-A}{2}$. Hence $F'(x)$ exists everywhere and is bounded.

$F'(x)$ will now be shown to have a discontinuity of measure at least 2 at every point of the set E. Consider any point x of first or second type in E. In every arbitrary interval about x is contained a complementary interval of E since E is nowhere dense on 0,1. In every complementary interval the function $F'(x)$ oscillates at least between $+1$ and -1 in the neighborhood of an end point. Thus for $|x'-x| <$ arbitrary constant, $|F'(x') - F'(x)| \geq 2$ and a discontinuity of measure at least 2 exists at all points of E. Since E is of positive measure 1/2, $F'(x)$ is not integrable in the Riemann sense. This is also readily shown by direct considerations. One may make a subdivision of the x-axis in the interval 0,1 by means of the points $x_1 x_2 \ldots x_n$ and consider the upper and lower

Riemann sums $\sum_{s=1}^{n-1} M_s (x_{s+1}-x_s)$ and $\sum_{s=1}^{n-1} m_s (x_{s+1}-x_s)$. The requirement for the existence of the integral is

$$\lim_{n\to\infty} \sum_{s=1}^{n-1} (M_s - m_s)(x_{s+1}-x_s) = 0.$$

The summation over $(0,1)$ may be considered as the sum over intervals containing points of E and those not containing points of E. $\lim_{n\to\infty} \sum_{CE} (M_s - m_s)(x_{s+1}-x_s) = (0)(\frac{1}{2}) = 0$ since $F'(x)$ continuous on the set CE.

$$\lim_{n\to\infty} \sum_{E} (M_s - m_s)(x_{s+1}-x_s) \geq (2)(\tfrac{1}{2}) = 1$$

since $F'(x)$ has a discontinuity of measure at least 2 at all points of E and the measure of E is $\frac{1}{2}$. Thus the limit is not zero and a Riemann integral of $F'(x)$ over $(0,1)$ does not exist.

$F'(x)$ being bounded and measurable is integrable in the sense of Lebesgue. Moreover $F'(x)$ exists everywhere on the interval and $\left|\frac{F(x+h)-F(x)}{h}\right| = |F'(x - \theta h)| \leq M, \quad 0 < \theta < 1$

Hence $\sum_{s=1}^{n} |F(x_s + h_s) - F(x_s)| \leq M \sum_{s=1}^{n} |h_s|$, ie $F(x)$ is absolutely continuous and is thus a Lebesgue integral. In the sense of Lebesgue therefore

$$F(x) = \int_a^x F'(t)\,dt - F(a),$$

the derivative is integrable and integrates back to the function.

A further example of a rather regular function (in fact continuous, monotone increasing, possessing a derivative bounded and measurable) which is not an integral, is constructed on the Cantor set. (2)

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(2) Hille and Tamarkin, "Remarks on a Known Example Of a Monotone Continuous Function", American Mathematical Monthly, Vol. 36, May 1929, pp. 255-64.

(2) A function on the Cantor set.

The function will first be defined on the interval (0,1). Consider a perfect set of points nowhere dense on the interval (0,1), Cantor's set is such a set. To form the Cantor set (0,1) is subdivided in three equal parts and the interior of the middle part removed. The remaining two parts are each subdivided in three and the interiors of the middle parts removed. This is repeated indefinitely--the number of intervals removed at the p'th stage being 2^{p-1} each of length 3^{-p}. The intervals of the p'th stage are denoted by D_{pk}, $(k = 1,2,\cdots 2^{p-1})$. The total number removed in the first p stages is 2^p-1. Let E be the set of points not removed consisting of the end points of the intervals D_{pk} and their limiting points. The complementary set C(E) is the sum of the open deleted intervals D_{pk}. Let the closed intervals remaining at the p'th stage be N_{pk}, $(k = 1,2,\cdots 2^p)$. E is always covered by the N_{pk}. For fixed p all N_{pk} have length 3^{-p} and the sum of their lengths is $(2/3)^p$ which goes to zero as p approaches infinity. Hence E is of measure zero. Denoting the points of E on the ternary scale as infinite decimal fractions, they are numbers of the type $0.a_1\, a_2 \cdots\, a_n \cdots$ where only digits 0 and 2 are admitted. All numbers of this type may be approximated as closely as is desired by numbers of the same type. Thus E contains all its limiting points. Since each point of E is a limiting point, E is a perfect set. Every subinterval of (0,1) contains intervals of C(E), hence E is nowhere dense on (0,1).

Using a to indicate digits 0 or 2 and b for $\frac{a}{2}$, then when $X = 0.a_1 a_2 \cdots$ (any point of E) the function w(x) has the value $w(x) = 0.b_1 b_2 \cdots$, a number on the scale of 2. Considering values of $w(x)$ at end points of an interval D_{pk}, at the left-hand end point $X = 0.a_1a_2 \cdots a_{p-1}\ 0222\cdots$ and $w(x) = 0.b_1b_2\cdots b_{p-1}\ 0111\cdots$. At the right-hand end point $x = 0.a_1a_2\cdots a_{p-1}\ 2000\cdots$ and $w(x) = 0.b_1b_2\ \ b_{p-1}\ 1000\ldots$ Therefore at the end points $w(x)$ has the same value $\frac{2k-1}{2^p}$ which may be seen by considering that for fixed p and increasing k one has $w_{p1}(x) = \frac{1}{2^p}$, and for every increase of 1 in k there is an increase of $2/_{2^p}$ in $w_{pk}(x)$. The value of $w(x)$ at all points of a D_{pk} is taken its value at the end points. Thus the function $w(x)$ is defined on the interval $(0,1)$.

It will now be shown $w(x)$ is monotone (non-decreasing) on $(0,1)$ and increases from 0 to 1 as x increases from 0 to 1. Intervals D_{pk} are intervals of constancy of $w(x)$, thus one needs only to consider the points of E. For $x' = 0.a_1' a_2' \cdots$, $x'' = 0.a_1''a_2''\cdots$ and $x'' > x'$ there exists a subscript n such that $a_i' = a_i''$ for $i < n$, $a_n'' > a_n'$. Thus

$$w(x'') = 0.b_1''\cdots b_n''\cdots \geq 0.b_1'\cdots b_n'\cdots = w(x')$$

It is here shown that $w(x)$ is continuous on $(0,1)$. Given an arbitrary positive constant e there exists a positive constant d such that

$$|w(x') - w(x)| < e \text{ for } |x' - x| < d.$$

$w(x)$ as a constant is continuous on the D_{pk} and it is necessary to consider only x and x' of E. For the given $e > o$ choose an n such that $2^{-n} < e$. It is now seen that $d = 3^{-n}$ satisfies the requirements, for $|x' - x| < d$ means x and x' are of the form:

$$x = 0.a_1a_2 \cdots a_{n+1}\ a_{n+2}\cdots$$

$$x' = 0.a_1a_2 \cdots a_{n+1}\ a_{n+2}'\cdots$$

on the scale of three. x and x' differ only after the $(n+1)^{th}$ place.

then $\quad |w(x) - w(x')| = |0.b_1 \cdots b_{n+1}\, b_{n+2} \cdots - 0.b_1 \cdots b_{n+1}\, b'_{n+2} \cdots|$

$$< 2^{-n} < e.$$

Thus w(x) is continuous on the interval (0,1).

w(x) is now proved to be not absolutely continuous and to have an (0,1) a constant L variation of magnitude 1. L variation on an interval is defined as the upper limit of the sum

$$\sum_{k=1}^{m} |f(B_k) - f(A_k)|$$

over any finite set of non-overlapping sub-intervals (A_k, B_k); $k = 1,2,\cdots m$, of total length $\sum_{k=1}^{m} (B_k - A_k) \leq L$. Choosing $(A_k, B_k) = N_{pk}$ then

$$\sum |w(B_k) - w(A_k)| = \sum [w(B_k) - w(A_k)] = 1 \qquad \text{but}$$

$$\sum N_{pk} = \left(\frac{2}{3}\right)^p \text{ and } \left(\frac{2}{3}\right)^p < e \text{ for } p > \frac{\log e}{\log 2/3}$$

Thus the L variation is constant at 1 and w(x) does not satisfy the condition of absolute continuity which is:

For an arbitrary positive constant e given, there must exist a positive constant d such that

$$\sum_{s=1}^{n} |w(x_s + h_s) - w(x_s)| < e$$

for any set of non-overlapping intervals $(x_s, x_s + h_s)$ where

$$\sum |(x_s + h_s) - x_s| < d.$$

Since absolute continuity is the necessary and sufficient condition for a function to be an integral

$$w(x) \neq \int_a^x w'(x)\, dx + w(a).$$

The derivative $w'(x)$ is zero almost everywhere on $(0,1)$ since it is zero at all points of the set $C(E)$. Thus $w'(x)$ is bounded and measurable almost everywhere, hence integrable but not to $w(x)$. In fact

$$\int_a^x w'(x)\,dx = 0 \text{ for all } a,x \text{ in } (0,1).$$

It will now be shown that although $w(x)$ is monotone, hence of bounded variation and possessing a curve $y = w(x)$ of finite length between $(0,0)$ and $(1,1)$, the length is not given by the usual formula

$$\int_0^1 (1 + w'(x)^2)^{1/2} dx = 1$$

The actual length will first be found from consideration of the definition of curve length as the limit of the perimeter of an inscribed polygon. The perimeter of any such inscribed polygon, without double points, cannot exceed the sum of all the horizontal and vertical projections of its sides; which is 2 for a polygon running from $(0,0)$ to $(1,1)$. If the inscribed polygon is taken as the broken line with vertices at $(0,0)$, $(1,1)$ and the end points of the D_{pk}

$$(n \text{ fixed}, \; p = 1,2,\cdots n, \; k = 1,2, \cdots 2^{p-1})$$

the sum of the horizontal sides is

$$\sum_{pk} D_{pk} = \sum_{p=1}^{n} 2^{p-1}\, 3^{-p} = 1-(2/3)^n.$$

All the inclined sides, 2^n in number are of equal length. Their common length is $\quad (2^{-2n} + 3^{-2n})^{1/2} = 2^{-n}\left(1 + \left(\frac{2}{3}\right)^{2n}\right)^{1/2}$. As $n\to\infty$ length of polygon, $\quad 1 - \left(\frac{2}{3}\right)^n + \left(1 + \left(\frac{2}{3}\right)^{2n}\right)^{1/2} \to 2.$

Hence the length of the arc $y = w(x)$ between $(0,0)$ and $(1,1)$ is 2. The usual formula fails here because $w(x)$ is not absolutely continuous.

$w(x)$ will now be shown to satisfy a Lipschitz condition of order

$$A = \frac{\log 2}{\log 3}. \text{ ie for } x,\ x+h \text{ in } (0,1);$$

$$/w(x+h) - w(x)/ \leq M/h/^{A},\ 1 \leq \max M \leq 2,\ A = \frac{\log 2}{\log 3}$$

Take $N_{pk} = x, x+h$. It follows that

$$w(x+h) - w(x) = 2^{-p} \text{ and } h = 3^{-p}.$$

Thus the order A and the upper limit of M cannot exceed $\frac{\log 2}{\log 3}$ and 1 respectively as is seen from $2^{-p} \leq M(3^{-p})^{A}$. For x,x+h any pair of numbers in (0,1) one may assume $h > 0$ without loss of generality since w(x) is a monotone non-decreasing function and when bounded above necessarily approaches a limit. It is only necessary to consider the case of both x,x+h in E, since for x or x+h an interior point to an interval D_{pk} they may be replaced by the right hand end point or left hand end point respectively. This would reduce the h without affecting the expression $w(x+h) - w(x)$. Thus a proof for x,x+h in E is all that is needed. Consider points $x = 0.a_1a_2\ldots a_m\ldots$, $x+h = 0.a_1'a_2'\ldots a_m'\ldots$ and there exists an n such that $a_i' = a_i$, $i < n$; $a_n' > a_n$. Hence $a_n' = 2$, $a_n = 0$. Then $w(x+h) - w(x) = 0.b_1\ldots b_{n-1}1\ldots - 0.b_1\ldots b_{n-1}0\ldots = 2^{-n+1}$ and

$$h = 0.a_1\ldots a_{n-1}2\ldots - 0.a_1\ldots a_{n-1}0\ldots \geq 3^{-n}$$

$$(w(x+h) - w(x))\, h^{-A} \leq 2^{-n+1}3^{An} = 2 \text{ for } A = \frac{\log 2}{\log 3}.$$

Properties of total variation of the function will now be considered. w(x) is defined outside the interval (0,1) by $w(x) = 0$ for $x \leq 0$; $w(x) = 1$ for $x \geq 1$. The function $\phi_h(x)$ is defined as $w(x+h) - w(x) \geq 0$. $\phi_h(x)$ is a function of x of bounded variation since it is the difference of two monotone functions. Let T(h) be the total variation ~~of the function~~

$\phi_h(x)$ and $W(z)$ the maximum T (h) for $0 \leq h \leq z$. It will now be shown that $W(z)$ is constant and equals 2. From its definition $T(h)$ cannot exceed the sum of the total variations of the constituents of the difference $w(x+h) - w(x)$. $w(x)$, $w(x+h)$ are monotone and increase from 0 to 1. Thus they have the same total variation of 1 and $T(h) \leq 2$. With $h = 3^{-n}$ the interval $(-\infty,\infty) =$

$$D_o + \sum_{p=1}^{n} \sum_{k=1}^{2^{p-1}} D_{pk} + \sum_{i=1}^{2^n} N_{ni} + D_1$$

$$D_o = (-\infty, o) \qquad D_1 = (1,\infty)$$

Thus $T(h) = T(D_o) + \sum_{pk} T(D_{pk}) + \sum_i T(N_{ni}) + T(D_1)$.

Since the h equals the common length of the intervals N_{ni} and does not exceed the length of any of the intervals D_{pk} $(p=1,2,\ldots n)$; when x ranges over an interval N_{ni}, x+h ranges over a part of the interval D_{pk} adjacent to N_{ni}. Hence $w(x+h)$ remains constant while $w(x)$ increases by 2^{-n}. Therefore $T(N_{ni}) = 2^{-n}$. For $T(D_{pk})$ let $x_1 < x_2$ be end points of the interval. Subdivide D_{pk} in two parts, $D' = (x_1, x_2 - h)$ — which exists only if $D_{pk} > h$, ie $p < n$; and $D'' = (x_2-h, x_2)$. T' and T'' are the total variations of $\phi_h(x)$ in these respective intervals. For x ranging over D', $w(x)$ and $w(x+h)$ are constant and $T' = 0$. For x ranging over D'', x+h ranges over the interval N_{ni} adjacent to D''. $w(x)$ then remains constant but $w(x+h)$ increases by 2^{-n}: Therefore $T'' = 2^{-n}$ and $T(D_{pk}) = T' + T'' = 2^{-n}$. Similarly $T(D_o) = 2^{-n}$, $T(D_1) = 0$. There are 2^n intervals N_{ni}, and 2^n-1 intervals D_{pk}. Thus $T(h) = 2^{-n}(1+2^n+2^n-1) = 2$

and $W_z = W_o = 2$.

In the general case if $w(x)$ were absolutely continuous and existent as a Lebesgue integral, one would have

$$T(h) = \int_{-\infty}^{\infty} |w'(x+h) - w'(x)| \, dx \to 0 \text{ as } h \to 0. \quad \text{ie}$$

$$W_z = (\lim_{z \to o} W_z = W_o) = 0.$$

In conclusion a continuous function will be considered which has an infinite number of maxima and minima in every interval, and a finite differential coefficient at every point in the interval of definition.(3)

First considerations of a function satisfying the hypotheses of Rolles theorem and having maxima and minima in every interval are sometimes interpreted as showing the function to be constant. The falsity of this assumption is shown here.

(3) A differentiable, everywhere--oscillating function.

The existence of such a function as this was felt by some to be impossible, but it was constructed by Koepcke. The considerations here are based on Pereno's simplification of the original construction.

A basic element in the construction is denoted by $(A/B)_n$, the meaning of which will now be given. On a straight line AB segments AA', B'B are marked off, each equal in length to $\frac{AB}{20}$. Through the mid-point O of AB straight lines $r_1, r_2, r_3, \cdots r_{2^n-1}$ are drawn making angles with OA of which the tangents are respectively $\frac{1}{2^n}, \frac{2}{2^n}, \cdots \frac{2^n-1}{2^n}$. A straight line r_o is taken through A' making with A'O an angle with tangent $\frac{1}{2^n}$. Through the

(3) Hobson, The Theory of Functions of a Real Variable, (Cambridge University Press, 1907), pp. 626-34.

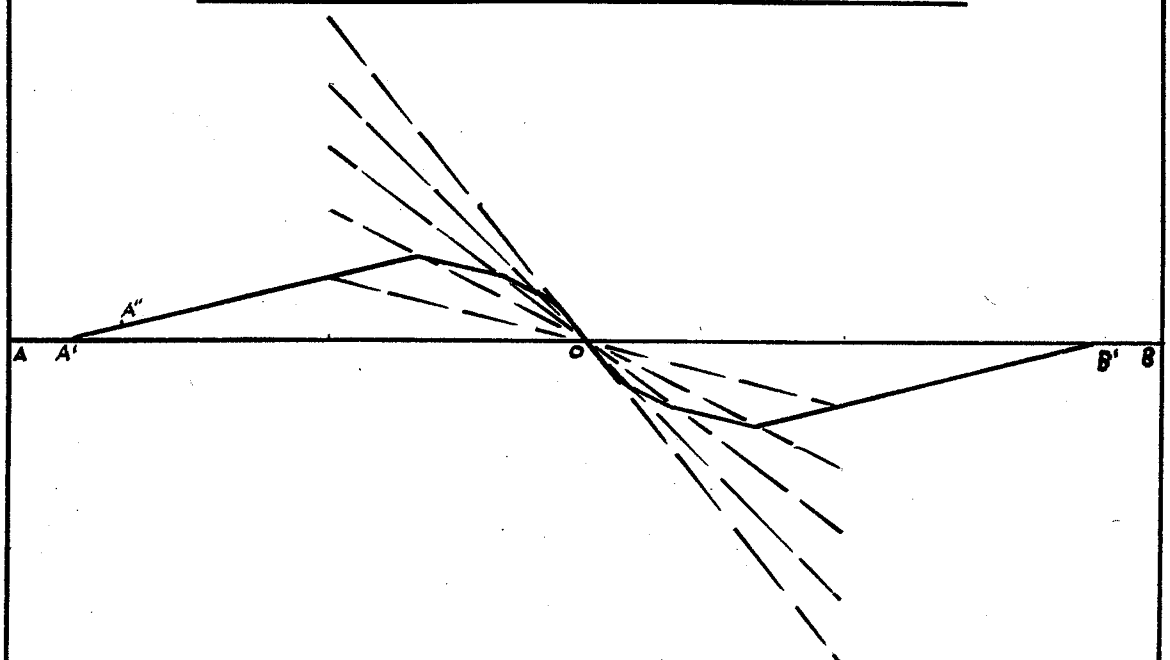
intersection (r_o,r_2) run r_1' parallel to r_1, through (r_1',r_3) take r_2' parallel to r_2 and so on. The straight lines

$$r_o r_1' \cdots r'_{2^n-1}, r_{2^n+1}$$

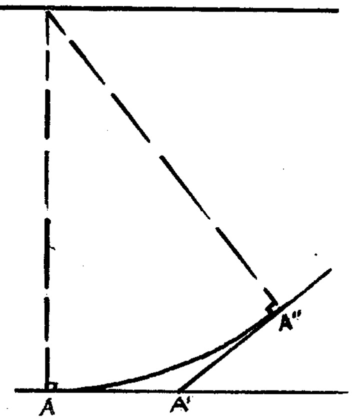
form an unclosed polygon above A'O. On OB' is constructed an exactly similar polygon on the other side of AB. These form a single polygon joining A' and B', crossing AB at O. On r_o, A'A'' is taken equal in length to AA' and an arc of a circle is described touching AB at A, r_o at A''. At each vertex of the constructed polygon there are marked off on sides adjacent to that vertex lengths equal to $\frac{1}{20}$ that of the shorter side; and an arc of a circle is constructed touching the two sides at the extremities of the segments so marked. The resulting figure joining A to B is composed of arcs of circles and of straight lines. By means of the ordinates perpendicular to the line AB this defines a continuous differentiable function. It has a continuous differential coefficient which is zero at A and B and is—$-\left(\frac{2^n+1}{2^n}\right)$ at O. This function is denoted as $(A/B)_n$. The n is fixed in any given construction.

To begin the construction of the main function a system of rectangular Cartesian coordinates is chosen in the plane with x and y axes. A quadrant arc of a circle is taken in the positive quadrant of the coordinate system, and passing through the points (0,0), (1,0). $F_o(x)$ is the function represented by this quadrant, being defined for the interval (0,1) on the x-axis. $F_o(x)$ has a maximum at $x = 1/2$ and $F_o'(0) = 1$, $F_o'(1) = -1$. The value of $F_o'(x)$ at $x = 1/4$ is denoted by a_o.

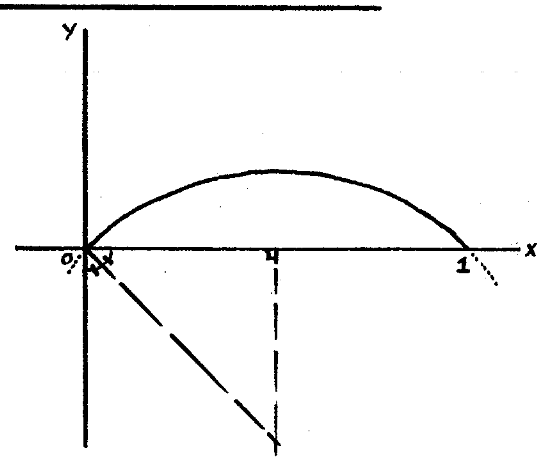
Construction of the function $(A/B)_n$;
case of $n=2$.



Detail of arc construction on segments AA', A'A". Arcs at other vertices drawn similarly.



Construction of the function $F_0(x)$.



Now making use of the definition for $(A/B)_n$, the curve with ordinates $a_o(0/1/2)_1$ is described from $x = 0$ to $x = 1/2$ and $-a_o(1/2/1)$ from $x = 1/2$ to $x = 1$. This gives a continuous function $f_1(x)$ such that $f_1'(o) = f_1'(1/2) = f_1'(1) = 0$.

$$f_1'(1/4) = -\frac{3}{2}a_o, f_1'(3/4) = \frac{3}{2}a_o.$$

A function $F_1(x)$ is defined as equal to

$F_o(x) + f_1(x)$. Then $F_1'(1/4) = -1/2\, a_o$, $F_1'(3/4) = \frac{1}{2}a_o$, $F_1'(0) = 1$, $F_1'(1) = -1$, $F_1'\left(\frac{1}{2}\right) = 0$. Thus $F_1(x)$ has a maximum in the interval $(0,1/4)$, a minimum in $\left(\frac{1}{4}, \frac{1}{2}\right)$, a maximum at $x = 1/2$, a minimum in $\left(\frac{1}{2}, \frac{3}{4}\right)$, and a maximum in $(3/4, 1)$. $(0,1)$ is considered divided into sub-intervals by points at which $F_1'(x) = 0$. In each of these intervals $F_1'(x)$ is monotone. Each sub-interval is now divided in 2, 4, 8,$\cdots$ equal parts until the fluctuation of $F_1'(x)$ in each is $\leq 1/2$. This is always possible since $F_1'(x)$ is continuous.

The points in which $(0,1)$ is so divided are denoted by $C_1^{(1)}, C_1^{(2)}, C_1^{(3)} \cdots$. In any one interval, $(C_1^{(s-1)}, C_1^{(s)})$, $F_1(x)$ is monotone and its differential coefficient has a fluctuation $\leq 1/2$. $a_1^{(1)}, a_1^{(2)}, \cdots$ are the values of $F_1'(x)$ at the mid-points of $(0,C_1^{(1)})$, $(C_1^{(1)}, C_1^{(2)})$, $\cdots$ respectively. The next stage is to describe curves

$$a_1^{(1)}\left(0/C_1^{(1)}\right)_2,\ a_1^{(2)}\left(C_1^{(1)}, C_1^{(2)}\right)_2, \cdots$$

which together form a continuous curve representing a function $f_2(x)$. $F_2(x)$ is then defined as equal to $F_o(x) + f_1(x) + f_2(x)$ and $F_2(x)$ has in every interval $(C_1^{(s-1)}, C_1^{(s)})$ a new maximum and new minimum. The length of each interval is $< \frac{1}{2^2}$.

At the stage forming the function $F_n(x)$ the points are taken at which $F_n'(x)$ vanishes; and if $F_n(x)$ is constant on some intervals, the limiting points of those intervals. These points divide (0,1) into intervals in each of which $F_n(x)$ is monotone. Sub-divisions are then made of each interval into $2, 4, 8, \cdots$ equal parts such that the fluctuation of $F_n'(x)$ in each is $\leq \frac{1}{2^n}$. $C_n^{(1)}, C_n^{(2)} \ldots$ are all the division points of (0,1) thus obtained. In any interval $(C_n^{(s-1)}, C_n^{(s)})$, $F_n(x)$ is monotone and the fluctuation of $F_n'(x)$ is $\leq \frac{1}{2^n}$, while the length of each interval is $< \frac{1}{2^{n+1}}$. The values of $F_n'(x)$ at the mid-points of the intervals are respectively $a_n^{(1)}, a_n^{(2)}, \cdots$. In the case of an interval on which the function is constant, the corresponding value of a_n is taken to be $\frac{1}{2^n}$ or $\frac{-1}{2^n}$ according as the line is in (0,1/2) or 1/2, 1). The curves

$$a_n^{(s)} \left(C_n^{(s-1)} C_n^{(s)}\right)_{n+1}$$

are then described. The function represented by the totality of these is $f_{n+1}(x)$. Then $F_{n+1}(x) = F_n(x) + f_{n+1}(x)$ has a new maximum and new minimum in every interval $\left(C_n^{(s-1)}, C_n^{(s)}\right)$.

The series $F_o(x) + f_1(x) + f_2(x) + \cdots$ represents a function F(x), the everywhere--oscillating function required. This function will now be shown to be continuous. $F_n'(x) = F_o'(x) + f_1'(x) + f_2'(x) + \cdots + f_n'(x) = S_n(x)$, (notation). It is now shown that for all n and x, $S_n(x)$ is numerically less than $\prod_{n=1}^{\infty} (1 + \frac{1}{2^n})$ which is written as P. The proof is by induction. $|S_n(x)|$ is assumed for all x to be less than $\prod_1^n (1 + \frac{1}{2^n}) = P_n$.

It will be shown that $|S_{n+1}(x)| < P_{n+1}$.

A point $x < C_n^{(s)}$ is considered in the interval $(C_n^{(s-1)}, C_n^{(s)})$. $S_{n+1}^{(x)} = S_n(x) + A_n \frac{a_n^{(s)}}{2n+1}$, where $1 \geq A_n \geq -(2^{n+1} + 1)$. This follows from the construction of the function since $F'_{n+1}(x) = F'_n(x) + f'_{n+1}(x)$ and the value of $f'_{n+1}(x)$ for positive $F'_n(x)$ assumes the set of values $\frac{1}{2n+1}, \frac{-1}{2n+1}, \frac{-2}{2n+1}, \cdots - \frac{(2^{n+1}+1)}{2n+1}$ while for negative $F'_n(x)$ the sign of the values of $f'_{n+1}(x)$ is reversed. $S_n(x)$ has a constant sign throughout $(C_n^{(s-1)}, C_n^{(s)})$, the same as the sign of $a_n^{(s)}$ but the sign of $S_{n+1}(x)$ may vary. If A_n is positive, $|S_{n+1}(x)| < P_n \left(1 + \frac{1}{2n}\right) < P_{n+1}$ since $|S_n(x)| < P_n$ and $a_n^{(s)}$, (a particular value of $S_n(x)$) is $< P_n$ while $0 \leq A_n \leq 1$; ie $|S_{n+1}(x)| < P_n + \frac{P_n}{2n+1} < P_{n+1}$. If A_n is negative $|S_{n+1}(x)| < |S_n(x)| < P_n < P_{n+1}$. Thus it is evident that if $|S_n(x)| < P_n$ it follows that $|S_{n+1}(x)| < P_{n+1}$. $F'_1(x)$ is, everywhere in $(0,1)$, less than $1 + \frac{1}{2}$. Therefore by induction $|S_n(x)| < P_n$ and for all n and x, $|S_n(x)| < P$ since P is an increasing infinite product converging to a value between 2 and 3. The order of magnitude of this limiting value is seen immediately on consideration of the first three factors of P. Again from construction it follows that the numerically greatest value of $f_{n+1}(x)$ in the interval $(C_n^{(s-1)}, C_n^{(s)})$ is at some point to the left of the mid-point of the interval, and that value is $< \frac{1}{2} \frac{1}{2n+1} \frac{a_n^{(s)}}{2n+1}$. This arises from the fact that the length of the interval is $< \frac{1}{2n+1}$ and the values of $f_{n+1}(x)$ are slightly less than $y' = \frac{x'}{2n+1}$ in that part of the left hand region where $f'_{n+1}(x)$ is > 0. In the above expression for the magnitude of $f_{n+1}(x)$ the origin for the (x', y') coordinate system is taken at $C_n(s-1)$. Since $a_n(s)$ is less than P it follows that $|f_{n+1}(x)| < \frac{P}{2^{2n+3}}$ and the terms of the series $f_1(x) + f_2(x) + \cdots$ are numerically less than the corresponding terms of the convergent series

$$\frac{P}{2^5} + \frac{P}{2^7} + \cdots + \frac{P}{2^{2n+3}} + \cdots .$$

Thus the series $f_1(x) + f_2(x) + \cdots$ is uniformly convergent in the interval $(0,1)$ and the sum function $F_0(x) + f_1(x) + f_2(x) + \cdots$ of a uniformly convergent series of continuous functions is itself a continuous function. The function is thus shown to be continuous throughout its interval of definition.

It will now be shown that the function $F(x)$ is everywhere differentiable. $F(x)$ will be shown to satisfy the conditions of the following theorem which is here assumed: If the series $\sum u_n(x)$ converge in (a,b) to $s(x)$ and the differential coefficients $u'_n(c)$, $(a \leq c \leq b)$, exist and are finite; then the necessary and sufficient conditions that $\frac{d}{dx}\, s(x)$ may exist at $x = c$ and be the sum of the series $\sum u'_n(c)$ are:

(1) That the series $\sum u'_n(c)$ be convergent

(2) That corresponding to an arbitrarily fixed positive number e and an arbitrarily fixed integer m', a positive number D can be determined such that for each value of h numerically less than D, and for which $c + h$ is in (a,b), an integer m ($> m'$) in general varying with h can be found for which the three numbers

$$\sum_{n=1}^{m} \left(\frac{U_n(c+h) - U_n(c)}{h} - U'_n(c) \right), \quad \frac{R_m(c+h)}{h}, \quad \frac{R_m(c)}{h}$$

are all numerically less than e. Here $R_m(x)$ denotes the remainder of the series which represents F(x); ie $R_m(x) = \sum_{i=m+1}^{\infty} f_i(x)$.

It is first shown that the series $f_1'(x) + f_2'(x) + \cdots$ is convergent for all x in (0,1). If it so happen for any x that all $S_n(x)$, $S_{n+1}(x), \cdots$ from and after some value of n have all the same sign, eg. positive, then $S_{m+1}(x) \leq S_m(x) + \frac{a_m^{(s_1)}}{2^{m+1}}$ where m is the particular value of n.

Also $S_{m+2}(x) \leq S_{m+1}(x) + \frac{a_{m+1}^{(s_2)}}{2^{m+2}}$ and it follows that

$$S_{m+p}(x) - S_m(x) \leq \frac{a_m^{(s_1)}}{2^{m+1}} + \frac{a_{m+1}^{(s_2)}}{2^{m+2}} + \cdots + \frac{a_{m+p-1}^{(s_p)}}{2^{m+p}} \leq \frac{P}{2^{m+1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-1}}\right) \leq \frac{P}{2^m}.$$

Thus for m sufficiently large, $S_{m+p}(x) - S_m(x)$ is arbitrarily small and the series is convergent.

If it so happen that x coincide with such a point of division $c_n^{(s)}$ that $S_n(x)$ be zero, then all functions $f_n'(x)$ with higher indices vanish for that x; ie all functions $S_n(x)$ vanish from and after the particular value of n. If x be a point within a neighborhood for which $F_n(x)$ is constant then $S_n(x)$ vanishes. $S_{n+1}(x)$ may vanish if x is at an extreme of $f_{n+1}(x)$ and x is then a point of division $c_{n+1}^{(s)}$, all functions $S_m(x)$ for $m > n$ vanishing. It follows that if for any x; $S_n(x)$, $S_{n+1}(x)$ both vanish then $S_m(x)$ vanishes for all $m \geq n$. If $S_n(x)$ vanishes but not $S_{n-1}(x)$, or $S_{n+1}(x)$ then x is interior to a neighborhood for which $F_n(x)$ is constant and $S_{n+1}(x) \leq \frac{1}{2^n}\left(1 + \frac{1}{2^{n+1}}\right) < \frac{2P}{2^{n+1}}$ since in the equation

$S_{n+1}(x) = S_n(x) + \frac{a_{n_1}^{(s_1)}}{2^{n_1+1}} \cdot A_{n_1}$, the term $a_{n_1}^{(s_1)} = \frac{1}{2^n}$,

$\left| \frac{A_{n_1}}{2^{n_1+1}} \right| \leq \left(1 + \frac{1}{2^{n_1+1}}\right)$ and $1 + \frac{1}{2^{n_1+1}}$ is one factor of P. Thus one or more of the $S_n(x)$ may be zero and the series still converges.

Considering now the case where the functions $S_n(x)$ are never all of the same sign from and after any value n and are zero for some n; $n_1, n_2, \ldots$ are those n for which $S_n(x)$ changes sign. eg. $S_{n_1}(x)$ is negative or zero and $S_{n_1+1}(x)$ is positive while $S_{n_2}(x)$ is positive or zero and $S_{n_2+1}(x)$ is negative. In the equation $S_{n_1+1}(x) = S_{n_1}(x) + \frac{a_{n_1}^{(s_1)} A_{n_1}}{2^{n_1+1}}$ it is then seen that A_{n_1} must be negative to give a positive $S_{n_1+1}(x)$ with negative $S_{n_1}(x)$ and $a_{n_1}^{(s_1)}$. Therefore $|A_{n_1}| \leq (2^{n_1+1} + 1)$ and

$$\frac{a_{n_1}^{(s_1)} A_{n_1}}{2^{n_1+1}} \leq a_{n_1}^{(s_1)} \left(1 + \frac{1}{2^{n_1+1}}\right).$$

$$\text{ie } S_{n_1+1}(x) \leq \left(- S_{n_1}(x) + a_{n_1}^{(s_1)}\right) + \frac{a_{n_1}^{(s_1)}}{2^{n_1+1}}$$

$$\text{or } S_{n_1+1}(x) < \left| S_{n_1}(x) - a_{n_1}^{(s_1)} \right| + \frac{P}{2^{n_1+1}}$$

and the fluctuation of $F'_{n_1}(x)$ in an interval containing x is $\leq \frac{1}{2^{n_1}}$. Therefore $S_{n_1+1}(x) < \frac{1}{2^{n_1}} + \frac{P}{2^{n_1+1}} < \frac{P}{2^{n_1}}$ since $P \geq 2$. If $S_{n_1}(x)$ is zero, x lies within a neighborhood throughout which $F_{n_1}(x)$ is constant. It follows that

$$S_{n_1+1}(x) \leq \frac{1}{2^{n_1}} \left(1 + \frac{1}{2^{n_1+1}}\right) < \frac{P}{2^{n_1}}$$ since the value of $a_{n_1}^{(s_1)}$

cannot exceed the value $\frac{1}{2^{n_1}}$ in an interval containing an x where $S_{n_1}(x)$ is zero; and $\left(1 + \frac{1}{2^{n_1+1}}\right)$ is one term of P. In any case

$$S_{n_1+p}(x) < S_{n_1+1}(x) + \frac{P}{2^{n_1+1}}$$ from the proof for

convergence of a series of $S_n(x)$ of the same sign. Thus using

$S_{n_1+1}(x) < \frac{P}{2^{n_1}}$ one obtains $S_{n_1+p}(x) < \frac{P}{2^{n_1}} + \frac{P}{2^{n_1+1}}$ for $p = 1,2,\ldots(n_2-n_1)$.

Similarly $|S_{n_2+1}(x)| < \frac{P}{2^{n_2}}$ if $S_{n_2}(x) = 0$; and if $S_{n_2}(x) > 0$,

$|S_{n_2+p}(x)| < |S_{n_2+1}(x)| + \frac{P}{2^{n_2+1}} < \frac{P}{2^{n_2}} + \frac{P}{2^{n_2+1}}$ for $p = 1,2,\ldots(n_3-n_2)$.

Thus it has been shown that $|S_n(x)|$ becomes arbitrarily small for all sufficiently large n, and $\lim_{n\to\infty} S_n(x) = 0$; ie in every case $F_o'(x) + f_1'(x) + f_2'(x) + \ldots$ converges for each x in (0,1).

It will now be shown that for e an arbitrarily chosen positive number and a prescribed integer m', then for a given x a number $D > 0$ can be found such that for each value of h numerically $< D$ and for which x+h is in the interval (0,1); there exists an integer $m \geq m'$, varying with h such that the quantities

$$\frac{F_m(x+h) - F_m(x)}{h} - S_m(x), \quad \frac{R_m(x+h)}{h}, \quad \frac{R_m(x)}{h}$$

are all numerically less than e. $R_m(x)$ denotes the remainder after m terms of the series representing F(x), ie $F(x) - F_{m-1}(x)$. It is not necessary to consider the case in which x coincides with one of the points of division of (0,1) as F(x) is in such a case represented by a finite series and is differentiable (since $f'_{n+p}(C_n^{(s)}) = 0$ for $p > 1$).

With e, m' fixed and a point x in (0,1), a number $n \geq m'$ is chosen such that $\frac{P}{2^{n-2}} < \frac{e}{3}$ and $|S_{n+p}(x) - S_{n+q}(x)| < \frac{e}{3}$ where p, q any positive integers. It will now be shown that for any h such that $x+h$ falls within the interval $(C_n^{(s-1)}, C_n^{(s)})$ a number $m \geq m'$ can be determined satisfying the hypotheses of the theorem. Letting h be positive, n_1 is determined such that $x < C_{n_1+1}^{(s')} \leq x+h \leq C_{n_1}^{(s'')} \leq C_n^{(s)}$. It is now shown that $n_1 + 2$ is a suitable value for m. Selecting the $C_{n_1+1}^{(s')}$ to be at an extreme in $F_{n_1+1}(x)$ then $f_{n_1+1+p}\left(C_{n_1+1}^{(s')}\right) = 0$. Also $\left|f_{n_1+1+p}\left(C_{n_1+1}^{(s')} \pm k\right)\right| < \frac{Pk}{2^{n_1+1+p}}$ for $p \geq 1$. This follows from the method of construction of the individual $f_n(x)$ which is such that $f_n(x)$ is less than $y = \frac{a_{n-1}^{(s)}\,(x - C_{n-1}^{(s)})}{2^n}$. The point $C_{n_1+1}^{(s')}$ is between x and $x+h$, thus it determines two numbers k_1, k_2 where $x = C_{n_1+1}^{(s')} - k_1$ and $x+h = C_{n_1+1}^{(s')} + k_2$.

Therefore $\left|f_{n_1+2}(x)\right| < \frac{P}{2^{n_1+2}} k_1$, $\left|f_{n_1+3}(x)\right| < \frac{P}{2^{n_1+3}} k_1, \ldots$

From these inequalities

$$\left|R_{n_1+2}(x)\right| < Pk_1 \left(\frac{1}{2^{n_1+2}} + \frac{1}{2^{n_1+3}} + \ldots\right) < \frac{Pk_1}{2^{n_1+1}} \text{ by summing the}$$

geometric progression. Similarly $\left|R_{n_1+2}(x+h)\right| < \frac{Pk_2}{2^{n_1+1}}$. Since k_1, k_2 are less than h,

$$\left|\frac{R_{n_1+2}(x)}{h}\right| < \frac{P}{2^{n_1+1}} < e \text{ for } \frac{P}{2^{n-2}} < \frac{e}{3} \text{ and } n_1 > n.$$

Also $\left|\frac{R_{n_1+2}(x+h)}{h}\right| < \frac{P}{2^{n_1+1}} < e$. Thus n_1+2 is taken as m satisfying the required condition. The case in which h is negative follows under exactly similar treatment.

It will now be shown that $\left| \frac{F_{n_1+1}(x+h) - F_{n_1+1}(x)}{h} - S_{n_1+1}(x) \right| < e.$

$$\frac{F_{n_1+1}(x+h) - F_{n_1+1}(x)}{h} - S_{n_1+1}(x) = \left(\frac{F_{n_1}(x+h) - F_{n_1}(x)}{h} - S_{n_1}(x) \right) +$$

$\left\{ \frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} - f'_{n_1+1}(x) \right\}$ and for $x, x+h$ in

$\left(C_{n_1}^{(s-1)}, C_{n_1}^{(s)}\right)$ the absolute value of the first term on the right hand

side is $\leq \frac{1}{2^{n_1}}$ since the variation of $F'_{n_1}(x)$ in the interval is $\leq \frac{1}{2^{n_1}}$,

the interval $x, x+h$ being included in $C_{n_1}^{(s-1)}$, $C_{n_1}^{(s)}$ and by the mean

value theorem $\frac{F_{n_1}(x+h) - F_{n_1}(x)}{h} = S_{n_1}(x+\theta h)$ for $0 < \theta < 1$. Considering

now the term $\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} - f'_{n_1+1}(x)$, it follows from the

construction that $\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} \leq \frac{a_{n_1}^{(s)}}{2^{n_1+1}}$ since

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} = f'_{n_1+1}(x + \theta h) \text{ for } 0 < \theta < 1$$

and the maximum value of $f'_{n_1+1}(x)$ in the interval $\left(C_{n_1}^{(s-1)}, C_{n_1}^{(s)}\right)$,

(which contains $x, x+h$), is $\frac{a_{n_1}^{(s)}}{2^{n_1+1}}$. This holds in particular for $F_{n_1}(x)$

increasing from $C_{n_1}^{(s-1)}$ to $C_{n_1}^{(s)}$ and any point x in the interval between

these points. A lower bound will now be found for this incrementary ratio.

If a point x is such that the ordinate of $a_{n_1}^{(s)} \left(C_{n_1}^{(s-1)}, C_{n_1}^{(s)}\right)_{n_1+1}$

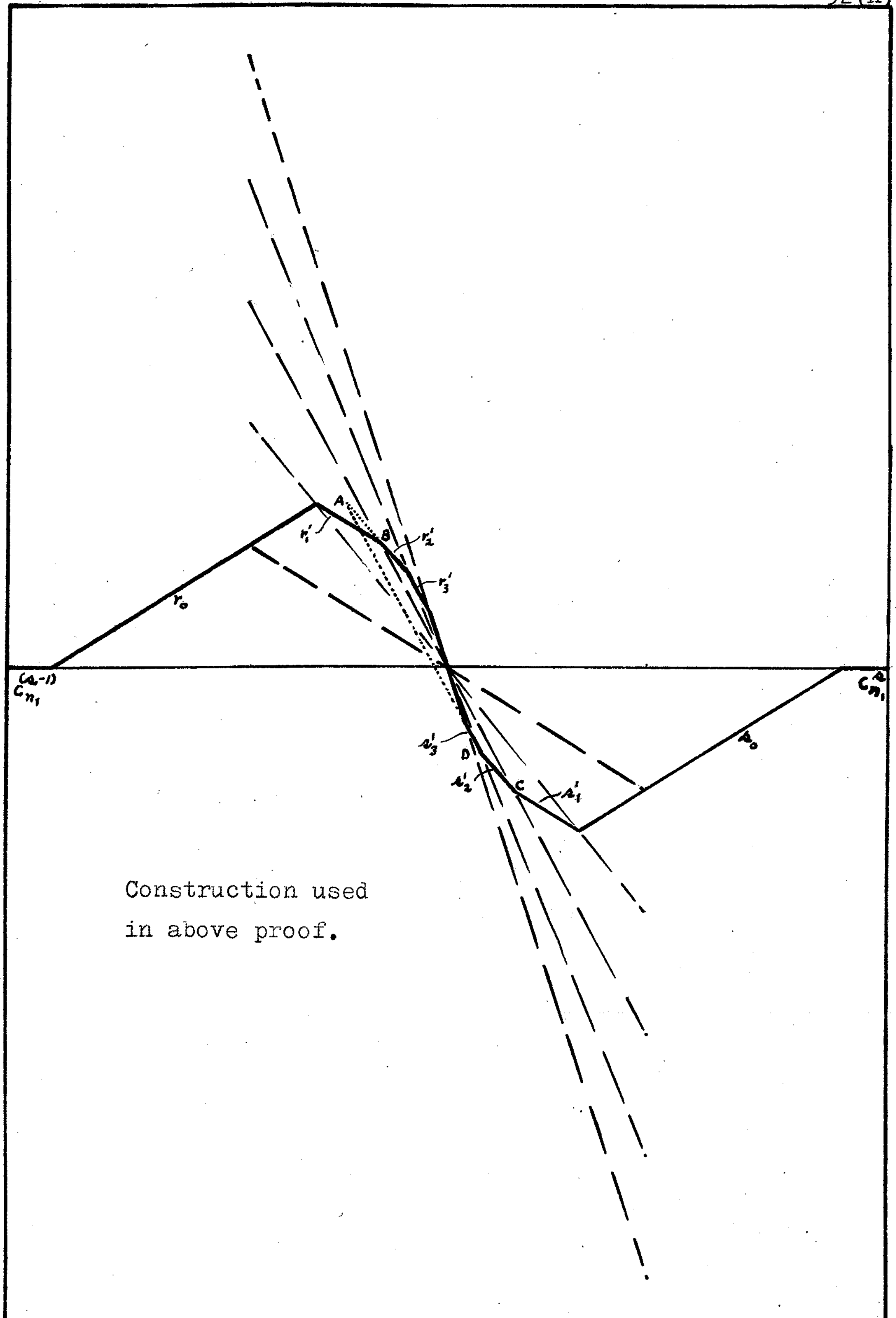
is below the x-axis and if the differential coefficient is negative, it is evident from the construction that $f''_{n_1+1}(x)$ is positive and

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} \doteq f'_{n_1+1}(x).$$

Consider now the sides of the rectilinear polygon used in the construction of $a_{n_1}^{(s)} \left(C_{n_1}^{(s-1)}, C_{n_1}^{(s)}\right)_{n_1+1}$. These sides are

$r_o, r'_1, r'_2 \ldots r'_{2^{n_1+1}-1}, r'_{2^{n_1+1}+1}, s'_{2^{n_1+1}-1}, \ldots s'_2, s'_1, s_o$.

r'_m is equal and parallel to s'_m for all m. On r'_2 produced beyond (r'_1, r'_2) a segment is taken equal to r'_2 (equal and parallel to s'_2) the line joining the end of this segment with (s'_2, s'_3) being parallel to r_3 since it is the base of a parallelogram of which r_3 is the opposite side. This joining line cuts r'_1 in a point p_1. s'_3 is parallel to r_3, passing through (s'_2, s'_3). Thus the joining line is a prolongation of s'_3 and is inclined to the x-axis at an angle whose tangent is $-\dfrac{3\, a_{n_1}^{(s)}}{2^{n_1+1}}$. Thus for a point between $C_{n_1}^{(s-1)}$ and p_1 for which the ordinate is positive, one has $\dfrac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} \geq -\dfrac{3\, a_{n_1}^{(s)}}{2^{n_1+1}}$. The greatest value of $f'_{n_1+1}(x)$ in this case is $\dfrac{a_{n_1}^{(s)}}{2^{n_1+1}}$. Therefore



Construction used
in above proof.

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} > f'_{n_1+1}(x) - \frac{4\, a_{n_1}^{(s)}}{2^{n_1+1}}.$$

If a point $\bar{p}_2$ on r'_2 be determined by a similar construction with r'_3 instead of r'_2 extended, then for every point on the arc $p_1 p_2$ except p_2,

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} > - \frac{4\, a_{n_1}^{(s)}}{2^{n_1+1}}.$$

The maximum value of the differential coefficient in this case is $- \dfrac{a_{n_1}^{(s)}}{2^{n_1+1}}$ and thus

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} > f'_{n_1+1}(x) - \frac{4\, a_{n_1}^{(s)}}{2^{n_1+1}}$$

in this case too. This inequality holds for every point on the curve with a positive ordinate. It is also true for points with negative ordinates since for such points it has been shown that

$$\frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} \geq f'_{n_1+1}(x).$$

Thus it is established that

$$f'_{n_1+1}(x) - \frac{4\, a_{n_1}^{(s)}}{2^{n_1+1}} < \frac{f_{n_1+1}(x+h) - f_{n_1+1}(x)}{h} \leq \frac{a_{n_1}^{(s)}}{2^{n_1+1}}$$

and it has been shown that

$$\frac{F_{n_1}(x+h) - F_{n_1}(x)}{h} = F'_{n_1}(x) + \frac{\theta}{2^{n_1}} \text{ where } 1 \geq \theta \geq -1.$$

Thus it follows that

$$F'_{n_1+1}(x) + \frac{\theta}{2^{n_1}} - \frac{4\, a_{n_1}^{(s)}}{2^{n_1+1}} < \frac{F_{n_1+1}(x+h) - F_{n_1+1}(x)}{h} \leq F'_{n_1}(x) + \frac{\theta}{2^{n_1}} + \frac{a_{n_1}^{(s)}}{2^{n_1+1}}$$

and $$\left| \frac{F_{n_1+1}(x+h) - F_{n_1+1}(x)}{h} - S_{n_1+1}(x) \right| < e$$

since on the left $\frac{\theta}{2^{n_1}} < \frac{2}{2^{n_1}} < \frac{P}{2^{n_1}} < \frac{e}{3}$ and $4 \frac{a_{n_1}^{(s)}}{2^{n_1+1}} < \frac{P}{2^{n_1-1}} < \frac{e}{3}$ while on the right hand side one has

$$S_{n_1+1}(x) + S_{n_1}(x) - S_{n_1+1}(x) + \frac{\theta}{2^{n_1}} + \frac{a_{n_1}^{(s)}}{2^{n_1+1}} \quad \text{and}$$

$$S_{n_1}(x) - S_{n_1+1}(x) < \frac{e}{3}, \ \frac{\theta}{2^{n_1}} < \frac{e}{3}, \ \frac{a_{n_1}^{(s)}}{2^{n_1+1}} < \frac{P}{2^{n_1+1}} < \frac{e}{3}$$

by the original choice of n such that $\frac{P}{2^{n-2}} < \frac{e}{3}$ and

$$\left| S_{n+p}(x) - S_{n+q}(x) \right| < \frac{e}{3} .$$

Thus the conditions for F(x) to be everywhere differentiable are fulfilled. F(x) has at every point a finite differential coefficient which is the sum of the convergent series $F_0'(x) + f_1'(x) + f_2'(x) + \ldots$

It will now be shown that F(x) has an everywhere dense set of maxima and minima. By the method of construction $F_n(x)$ has a new maximum and a new minimum in every interval $(C_n^{(s-1)}, C_n^{(s)})$ and the length of the interval is less than $\frac{1}{2^{n+1}}$. If x_0 a maximum of $F_n(x)$, $F_n(x_0) = F_{n+1}(x_0)$ and $F_n'(x_0) = F_{n+1}'(x_0) = 0$. Also $f_{n+1}(x)$ is negative in the neighborhood of the point x_0 and $F_{n+1}(x_0+h) - F_{n+1}(x_0)$ is negative or zero for $/h/ <$ some number k. Thus $F_{n+1}(x)$ has a maximum at x_0. If x_0 be within a neighborhood for which $F_n(x)$ is constant it cannot be so situated with respect to $F_{n+1}(x)$, nor can it be within a neighborhood where the function is constant for all functions with a higher index. This is evident from the oscillatory nature of the $f_n(x)$ which are added at each succeeding stage in the construction. If x_0 be a right or left hand end point of a neighborhood in which $F_n(x)$ is constant, $F_{n+1}(x)$ will have a maximum or minimum or a point of inflexion at x_0; since x_0 is in such case chosen as a point of subdivision $C_{n+1}^{(s)}$ and the $f'_{n+1}(x_0)$ will be zero.

In every case $F_{n+1}(x)$ will have a maximum and a minimum in every interval for which $F_n(x)$ is constant. For any given interval, arbitrarily small, n may be chosen of such magnitude that the interval contains one of the sub-intervals $\left(C^{(s-1)}_{(n-1)}, C^{(s)}_{n-1}\right)$ in its interior, and all the functions $F_n(x)$ $F_{n+1}(x), \cdots$ have maxima in this interval; ie F(x) also has maxima within it.

The derivative $F'(x)$ is definite at every point. It is bounded since every $|S_n(x)| < P$ and it is measurable. Thus the function F(x) having a bounded differential coefficient is absolutely continuous and is an integral in the sense of Lebesgue. The $F'(x)$ as a bounded measurable function may be integrated in the Lebesgue sense to F(x).

F(x) has discontinuities of the second kind at an everywhere dense set of points and is not integrable in the sense of Riemann. At every point of continuity of $F'(x)$ one must have $F'(x) = 0$. This is seen by considering that if a continuous function have in every neighborhood on the right of a point x_0 an infinite number of maxima and minima, there are in such neighborhoods an infinity of points at which the derivatives on the right are negative or zero, and an infinity of points at which they are positive or zero. Thus none of the derivatives at a point x on the right of x_o can have a definite limit as x approaches x_o unless this limit be zero.

F(x) in the sense of Lebesgue or Riemann has a convergent Fourier series at every point, since the derivative exists at every point. This is a function obviously violating the Dirichlet conditions and yet having a convergent Fourier series.

Thus F(x) while continuous and differentiable at every point, has a non-integrable derivative in the sense of Riemann. Application of the Lebesgue definition enables integration back to the function.