A-POSTERIORI ERROR ANALYSIS FOR THE *hp*-VERSION OF THE FINITE ELEMENT METHOD

by

Hua Shen

A Thesis submitted to
the Graduate Studies of The University of Manitoba
in partial fulfillment of the requirements for
the degree of Master of Science

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FACULTY OF GRADUATE STUDIES

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Contents

Abstract

In the framework of the Jacobi-weighted Sobolev space, we design the a-posterior error estimators and error indicators associated with residuals and jumps of normal derivatives on internal edges for the *hp*-version of the finite element method. With the help of quasi Jacobi projection operators, the upper bounds and the lower bounds of indicators and estimators are analyzed, which shows that such a-posteriori error estimation is quasi optimal. The indicators and estimators are computed for some model problems and programmed in C++. The numerical results show the reliability of our indicators and estimators.

Chapter 1

Introduction

Since the revolutionary concept was proposed by Babuška and Rheinboldt in the later 1970's [6, 7], a posteriori estimation of the accuracy and adaptation of the approximation spaces has been intensely studied by both engineers and mathematicians practically and theoretically.

The adaptive finite element algorithms have become powerful and reliable computational tools and have been integrated in many commercial and research codes owing to their remarkable capacity to enhance both speed and accuracy in computing. By reliable and effective error indicator and error estimator, the possibility of controlling the entire computational process through new adaptive algorithms becomes bigger.

There are two basic approaches for a-posteriori error estimation of FEM. The first approach is based on the solutions of local auxiliary problems, and the second approach is based on the residuals, including the jump of normal derivative of finite element solutions on internal edges of elements. For the lower-order FEM which is referred to as the h-version of FEM, a-posteriori error estimation and the adaptive mesh generation have been well developed in the past two decades.

A-posteriori error estimation for the high-order FEM such as the p-version, the hp-version of FEM and the spectral method is much less developed and lacks substantial progress because of the obvious difficulties of high-order methods except in one dimension, with the result that research in this direction remains at an elementary stage. Many useful and effective methods and techniques developed for the h-version of FEM, such as the super-convergence and patch recovery [9, 26, 27, 28, 29], can not be or have not been applied to the high-order FEM. It is not clear yet what mathematical framework should be adopted for a-posteriori error estimation of the high-order FEM in two and three dimensions. In spite of lacking of theoretical progress in the past two decades, engineers have implemented various types of error estimators and indicators based on their own experiences in commercial and research codes.

Since the late 1990's, the approximation theory for the p- and hp- version of FEM has been well developed in the mathematical framework of the Jacobi-weighted Besov and Sobolev spaces [2, 3, 4, 5]. With help of this framework, a-priori error analysis leads to a rigorous proof of the optimal convergence for the p- and hp-version (with quasi-uniform meshes) of FEM

[2, 3, 4, 17].

Very recently, the posterior error estimation for the p-version of FEM was analyzed in the framework of the Jacobi-weighted Sobolev spaces in [13, 14], and the residual based indicators and estimators are proposed and proved to be quasi optimal. In this thesis we generalize the results of [14] to the hp-version of FEM in the framework of Jacobi-weighted Sobolev spaces on scaled regions. The rest of this thesis is organized as follows. The error indicators and estimators associated with Jacobi weights on scaled regions are defined in Chapter 2, where the major theorems are stated for which the proofs are given in Chapter 5. The Jacobi-weighted Sobolev spaces on scaled regions are introduced in Chapter 3, which provide a mathematical framework for a-posterior error analysis of the hp-version FEM. In Chapter 4 we construct a quasi-Jacobi projection operator and analyze its properties, which play an essential role in the analysis of quasi-optimality of the error indicator and error estimator. The main results are proved in Chapter 5, where the upper and the lower bounds of error in terms of error indicators and error estimator are studied. In Chapter 6 we present numerical results of the error indicators and estimators for two model problems with singular solutions and one with a smooth solution by running a C++ program. They numerically illustrates the stability and reliability of error indicators and error estimator.

Chapter 2

Error Indicators and Error Estimators

2.1 Model Problem

We consider the following model problem on a bounded Lipschitz domain $\Omega \subset R^2$:

$$\begin{cases} -\Delta u + u = f & \text{on } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{2.1.1}$$

The corresponding variational problem is to find $u \in H_0^1(\Omega)$ such that

$$B(u,v) = F(v), \quad \forall v \in H_0^1(\Omega) \tag{2.1.2}$$

where B is a bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$:

$$B(u,v) = \int_\Omega (\nabla u \cdot \nabla v + u\,v)\,dx\,,$$

and F is a linear functional on $H^1(\Omega)$:

$$F(v) = \int_\Omega f v dx \,.$$

Let $\mathcal{T} = \{K_i, 1 \leq i \leq M\}$ be a partition of Ω with shape-regular quadrilateral elements K_i, and let $\partial\mathcal{T} = \{\gamma_\ell, 1 \leq \ell \leq L\}$ consist of all the internal edges γ_ℓ. By F_i we denote a mapping of $Q_h = (-h, h)^2$ onto each element K_i. Then the subspace of piecewise polynomials over $\mathcal{T}$ for the *hp*-version of finite element method is defined as usual,

$$\begin{aligned} S_0^{p,1}(\Omega, \mathcal{T}) &= S^{p,1}(\mathcal{T}) \bigcap H_0^1(\Omega) \\ &= \{\varphi \mid_{K_i} = \psi_{ph} \circ F_i, \psi_{ph} \in \mathcal{P}_p(Q_h)\} \bigcap H_0^1(\Omega) \,, \end{aligned}$$

where $\mathcal{P}_p(Q_h)$ is a set of polynomials of separate degree less than or equal to p on Q_h.

The *hp*-version finite element solution $u_S \in S_0^{p,1}(\mathcal{T})$ satisfies

$$B(u_S, v) = F(v), \quad \forall v \in S_0^{p,1}(\Omega, \mathcal{T}) \,. \tag{2.1.3}$$

2.2 Error Indicators and Error Estimator

By e, e_i, r, r_i and R, R_{γ_ℓ} we denote the error, the residual on element K_i and the jump of normal derivative along the internal edges γ_ℓ for the finite element solution u_S:

$$\begin{aligned} e &= u - u_S, \quad e_i = e\,|_{K_i}; \\ r &= f + \triangle u_S - u_S,\ r_i = r\,|_{K_i}; \\ R &= \left[\frac{\partial u_S}{\partial n}\right], R_{\gamma_\ell} = \left[\frac{\partial u_S}{\partial n}\right]|_{\gamma_\ell}\,. \end{aligned}$$

We should introduce error indicators which are associated with Jacobi weights. A weight function $W_{\beta,K_i}(x)$ is defined on each element K_i

$$W_{\beta,\,K_i}(x) = \prod_{l=1}^{4}\left|\frac{dist(x,\gamma_{i,l})}{\frac{1}{2}\max\limits_{y\in K_i} dist(y,\gamma_{i,l})}\right|^{\beta}, \quad x \in K_i\,,$$

where $\gamma_{i,l}$ denotes the l-th edge of K_i.

If $K_i = (-h,h)^2$, then

$$W_{\beta,\,K_i}(x) = \prod_{l=1}^{4}\left|\frac{dist(x,\gamma_{i,l})}{h}\right|^{\beta} = (1-(\frac{x_1}{h})^2)^{\beta}(1-(\frac{x_2}{h})^2)^{\beta}. \qquad (2.2.1)$$

The Jacobi-weighted L^2-spaces over K_i and Ω are furnished with norms as follows

$$||u||_{L^2_\beta(K_i)} = ||u\,W_{\beta/2,\,K_i}||_{L^2(K_i)} \quad ||u||^2_{L^2_\beta(\Omega)} = \sum_{i=1}^{M}||u||^2_{L^2_\beta(K_i)}\,. \qquad (2.2.2)$$

Similarly, a weight function $W_{\beta,\gamma_\ell}(x)$ is defined on each internal edge γ_ℓ

$$W_{\beta,\gamma_\ell}(x) = \prod_{m=1}^{2}\left|\frac{dist(x,V_{\ell,m})}{\frac{1}{2}\max\limits_{y\in\gamma_\ell} dist(y,V_{\ell,m})}\right|^{\beta}, \quad x \in \gamma_\ell\,,$$

where $V_{\ell,m}$ are the two end points of γ_ℓ .

If $\gamma_\ell = (-h, h)$, then

$$W_{\beta,\gamma_\ell}(x) = \prod_{m=1}^{2} \left|\frac{dist(x, V_{\ell,m})}{h}\right|^\beta = (1 - (\frac{x}{h})^2)^\beta. \tag{2.2.3}$$

The corresponding Jacobi-weighted spaces over γ_ℓ are associated with the norm

$$||u||_{L^2_\beta(\gamma_\ell)} = ||u\, W_{\beta/2,\gamma_\ell}||_{L^2(\gamma_\ell)}\,.$$

A local error indicator η_{K_i} associated with the residual r on the element K_i is defined as

$$\eta_{K_i} = \frac{h}{p^\beta}\,||r_i||_{L^2_\beta(K_i)}\,, \tag{2.2.4}$$

and another error indicator η_{γ_ℓ} associated with the jump R on the internal edge γ_ℓ is defined as

$$\eta_{\gamma_\ell} = \frac{h^{1/2}}{p^\beta}||R_{\gamma_\ell}||_{L^2_\beta(\gamma_\ell)}\,. \tag{2.2.5}$$

The global error estimator η is defined as

$$\eta^2 = \sum_{K_i \in \mathcal{T}} \eta^2_{K_i} + \sum_{\gamma_\ell \in \partial\mathcal{T}} \eta^2_{\gamma_\ell}\,. \tag{2.2.6}$$

2.3 Modification to Indicators and Estimator

In general, the residual $r_i \notin \mathcal{P}_p(K_i)$ and the jump $R_{\gamma_\ell} \notin \mathcal{P}_p(\gamma_\ell)$ if the basis functions are the images of the shape functions defined on the standard square element $Q = I^2 = (-1, 1)^2$ and the corresponding mapping is not linear. So we need to modify the error indicators and also the error estimator correspondingly.

Let $\Pi^{\beta}_{K_i}$ denote the $L^2_{\beta}(K_i)$-projection on $\mathcal{P}_p(K_i)$, and let $\Pi^{\beta}_{\gamma_\ell}$ denote the $L^2_{\beta}(\gamma_\ell)$-projection on $\mathcal{P}_p(\gamma_\ell)$. The error indicators $\tilde{\eta}_{K_i}$ and $\tilde{\eta}_{\gamma_\ell}$ are then defined as follows:

$$\begin{aligned} \tilde{\eta}_{K_i} &= \frac{h}{p^{\beta}} \|r_{i,p}\|_{L^2_{\beta}(K_i)} = \frac{h}{p^{\beta}} \|\Pi^{\beta}_{K_i} r_i\|_{L^2_{\beta}(K_i)} \\ &= \frac{h}{p^{\beta}} \|\Pi^{\beta}_{K_i} f + \Delta u_S - u_S\|_{L^2_{\beta}(K_i)} , \end{aligned} \tag{2.3.1}$$

and

$$\tilde{\eta}_{\gamma_\ell} = \frac{h^{1/2}}{p^{\beta}} \|R_{\gamma_\ell,p}\|_{L^2_{\beta}(\gamma_\ell)} = \frac{h^{1/2}}{p^{\beta}} \|\Pi^{\beta}_{\gamma_\ell} R_{\gamma_\ell}\|_{L^2_{\beta}(\gamma_\ell)} . \tag{2.3.2}$$

Therefore the global error estimator is modified to $\tilde{\eta}$ as usual

$$\tilde{\eta}^2 = \sum_{K_i \in \mathcal{T}} \tilde{\eta}^2_{K_i} + \sum_{\gamma_\ell \in \partial\mathcal{T}} \tilde{\eta}^2_{\gamma_\ell} . \tag{2.3.3}$$

If there is no confusion, we drop the index i in K_i and the index ℓ in γ_ℓ for simplicity. From now on K denotes one element K_i, and γ denotes one internal edge γ_ℓ.

Chapter 3

A Mathematical Framework of Jacobi-weighted Sobolev Spaces

3.1 Jacobi-weighted Sobolev Spaces on Q_h

Since the error indicators and the error estimator are associated with the weight functions, we need to introduce systematically the Jacobi-weighted Sobolev spaces. These spaces have proved to be the most appropriate functional spaces for a-priori error estimation for the *p*- and *hp*- version of FEM in [2, 3, 4, 5, 15, 16] and a-posteriori error analysis for the *p*-version of FEM in [13, 14] . They also provide a very useful mathematical framework for the a-posteriori error analysis for the *hp*-version of FEM.

Let $Q_h = I_h^2 = (-h,h)^2$. For $\beta = (\beta_1, \beta_2)$ and $\alpha = (\alpha_1, \alpha_2)$, with

$\beta_1, \beta_2 > -1$ and integers $\alpha_1, \alpha_2 \geq 0$, we define a weight function on Q_h

$$\begin{aligned} W_{\beta\pm\alpha,Q_h}(x) &= W_{\beta_1\pm\alpha_1,\beta_2\pm\alpha_2,Q_h}(x) \\ &= (1-(\frac{x_1}{h})^2)^{\beta_1\pm\alpha_1}(1-(\frac{x_2}{h})^2)^{\beta_2\pm\alpha_2}\,. \end{aligned}$$

The Jacobi-weighted Sobolev spaces $H^{k,\beta}(Q_h)$, $k \geq 0$, are introduced in [2, 19] with the norm

$$\|u\|^2_{H^{k,\beta}(Q_h)} = \sum_{|\alpha|\leq k} \int_{Q_h} |\partial^\alpha u|^2 W_{\beta+\alpha,Q_h}\, dx\,,$$

while $\tilde{H}^{k,\beta}(Q_h)$ is defined as a dual to $H^{k,\beta}(Q_h)$ with the norm

$$\|u\|^2_{\tilde{H}^{k,\beta}(Q_h)} = \sum_{|\alpha|\leq k} \int_{Q_h} |\partial^\alpha u|^2 W_{\beta-\alpha,Q_h}\, dx\,.$$

We are particularly interested in the spaces with $k = 0, 1$. It is easy to tell that

$$\|u\|_{H^{0,\beta}(Q_h)} = \|u\|_{\tilde{H}^{0,\beta}(Q_h)} = \|u\|_{L^2_\beta(Q_h)} = \|uW_{\beta/2}\|_{L^2(Q_h)}\,.$$

3.2 Jacobi-weighted Sobolev Spaces on K_i and $\mathcal{T}$

We next define Jacobi-weighted spaces $H^{k,\beta}(K_i)$ and $\tilde{H}^{k,\beta}(K_i)$ over the element K_i. Let F_i be the mapping of Q_h onto element K_i, and $\tilde{u}_i = u \circ F_i$. We define

$$\|u\|_{L^2_\beta(K_i)} = \|\tilde{u}_i\|_{L^2_\beta(Q_h)}\,,$$

and

$$\|u\|_{H^{k,\beta}(K_i)} = \|\tilde{u}_i\|_{H^{k,\beta}(Q_h)}, \quad \|u\|_{\tilde{H}^{k,\beta}(K_i)} = \|\tilde{u}_i\|_{\tilde{H}^{k,\beta}(Q_h)}.$$

The Jacobi-weighted spaces $H^{k,\beta}(\mathcal{T})$ and $\tilde{H}^{k,\beta}(\mathcal{T})$ are defined as

$$H^{k,\beta}(\mathcal{T}) = \bigcap_K H^{k,\beta}(K), \quad \tilde{H}^{k,\beta}(\mathcal{T}) = \bigcap_K \tilde{H}^{k,\beta}(K),$$

which are furnished with "broken" norms

$$\|u\|^2_{H^{k,\beta}(\mathcal{T})} = \sum_{K\in\mathcal{T}} \|u\|^2_{H^{k,\beta}(K)}, \quad \|u\|^2_{\tilde{H}^{k,\beta}(\mathcal{T})} = \sum_{K\in\mathcal{T}} \|u\|^2_{\tilde{H}^{k,\beta}(K)}.$$

Hereafter we assume that vector $\beta = (\beta_1, \beta_2) = (\hat{\beta}, \hat{\beta})$ with $\hat{\beta} \in [0,1)$. For simplicity we can use β as a vector $(\hat{\beta}, \hat{\beta})$ or as a scalar $\hat{\beta}$ without causing confusion.

3.3 Variational Equations in Jacobi-weighted Sobolev Spaces

In the framework of Jacobi-weighted spaces $H^{1,-\beta}(\mathcal{T})$ and $\tilde{H}^{1,\beta}(\mathcal{T})$ with $\beta \in [0,1)$, we re-formulate the elliptic problem (2.1.1) and its variational equation. Suppose that $f \in L^2_\beta(\mathcal{T})$, then we seek $u \in \tilde{H}^{1,\beta}_0(\mathcal{T})$ such that

$$B(u,v) = F(v) \quad \forall v \in H^{1,-\beta}_0(\mathcal{T}),$$

where $H^{1,-\beta}_0(\mathcal{T})$ and $\tilde{H}^{1,\beta}_0(\mathcal{T})$ are subspaces of $H^{1,-\beta}(\mathcal{T})$ and $\tilde{H}^{1,\beta}(\mathcal{T})$ with functions vanishing on $\partial\Omega$. $B(u,v)$ is now a bilinear form on $\tilde{H}^{1,\beta}(\mathcal{T}) \times H^{1,-\beta}(\mathcal{T})$, and $F(v)$ is a linear functional on $L^2_{-\beta}(\mathcal{T})$.

We introduce a new norm for the error e denoted by $|||e|||_K$ on the local element K and $|||e|||$ on the domain Ω, such that

$$|||e|||_K = \sup_{\|v\|_{H^{1,-\beta}(K)}=1} |B(e,v)_K| \,, \tag{3.3.1}$$

and

$$|||e||| = \sup_{\|v\|_{H^{1,-\beta}(\mathcal{T})}=1} |B(e,v)| \,, \tag{3.3.2}$$

where

$$B(e,v)_K = \int_K (\nabla e \cdot \nabla v + e\, v)\, dx \,.$$

Obviously,

$$|||e|||_K \leq \|e\|_{\tilde{H}^{1,\beta}(K)}, \quad |||e||| \leq \|e\|_{\tilde{H}^{1,\beta}(\mathcal{T})} \,.$$

It is proved in [14] that the error e is in the Jacobi-weighted spaces $\tilde{H}^{1,\beta}(K)$ and $\tilde{H}^{1,\beta}(\mathcal{T})$ if $u \in W^{1,q}(\Omega)$ with $q > 2/\beta$ or $u \in H^{1+s}(\Omega)$ with $s > 1-\beta$.

3.4 Approximation and Imbedding in Jacobi-weighted Spaces

In this section, let $u(x,y)$ and $U(\xi,\eta) = u(h\xi, h\eta)$ be functions defined on $Q_h = I_h^2 = (-h,h)^2$ and $Q = I^2 = (-1,1)^2$ and let $\mathcal{P}_p(Q)$ be a set of polynomials of separate degree $\leq p$ on Q, and let $U_p = \Pi_{p,Q}^{\beta} U$ denote the Jacobi projection of U on $\mathcal{P}_p(Q)$.

In the remaining parts of this thesis, we often use the following lemmas and theorems on approximation and imbedding in Jacobi-weighted Spaces

$H^{k,\beta}(Q)$ and $H^{k,\beta}(Q_h)$. We refer [14, 18] for the details of proof for these lemmas and theorems.

Theorem 3.4.1. *(Theorem 5.6 and Theorem 5.7 of [18])*

Let $U \in H^{k,\beta}(Q)$, with integer $k \geq 0$ and $\beta > -1$. Then, for integer $0 \leq \ell \leq k$, there holds

$$\|U - U_p\|_{H^{\ell,\beta}(Q)} \leq C\, p^{-(k-\ell)}\, \|U\|_{H^{k,\beta}(Q)}\,, \tag{3.4.1}$$

and if $k \leq p+1$,

$$|U - U_p|_{H^{\ell,\beta}(Q)} \leq C\, p^{-(k-\ell)}\, |U|_{H^{k,\beta}(Q)}\,. \tag{3.4.2}$$

Moreover, 3.4.1 holds for non-integer $k \geq 0$ if $0 \leq l < k$.

Lemma 3.4.2. *(Proposition 7.1 of [18])*

For integer $k \geq 0$, $u \in H^{k,\beta}(Q_h)$ if $U \in H^{k,\beta}(Q)$ and vise versa. Furthermore, there holds

$$|u|_{H^{k,\beta}(Q_h)} = h^{1-k}\, |U|_{H^{k,\beta}(Q)}\,, \tag{3.4.3}$$

and

$$|u|_{H^{k,\beta}(I_h)} = h^{1/2-k}\, |U|_{H^{k,\beta}(I)}\,. \tag{3.4.4}$$

Lemma 3.4.3. *(Lemma 7.2 of [18])*

Let $u \in H^{k,\beta}(Q_h)$, with integer $k \geq 0$. Then

$$\|U - U_p\|_{H^{k,\beta}(Q)} \leq C\, h^{\mu-1}\, \|u\|_{H^{k,\beta}(Q_h)} \tag{3.4.5}$$

with $\mu = \min(k, p+1)$, and C is independent of p, h, k and u.

Theorem 3.4.4. *(Theorem 7.3 of [18])*

Let $u \in H^{k,\beta}(Q_h)$, integer $k \geq 0$. Then for real number s, if $0 \leq s \leq k$, there holds

$$\|U - U_p\|_{H^{s,\beta}(Q)} \leq C\, p^{-(k-s)}\, h^{\mu-1}\, \|u\|_{H^{k,\beta}(Q_h)}\,, \tag{3.4.6}$$

with $\mu = \min(k, p+1)$.

The Jacobi polynomial of degree m is defined as

$$J_m^{\alpha,\beta}(\xi) = \frac{(-1)^m}{2^m\, m!}(1-\xi)^{-\alpha}(1+\xi)^{-\beta}\frac{d^m(1-\xi)^{m+\alpha}(1+\xi)^{m+\beta}}{d\xi^m}\,.$$

with $\alpha, \beta > -1$. We use $J_m^{\beta}(\xi)$ to denote $J_m^{\beta,\beta}(\xi)$.

Let $\xi_{m,i}^{\beta}$ be zeros of $\left(J_m^{\beta}(\xi)\right)' = \frac{m+1+2\beta}{2}\, J_{m-1}^{\beta+1}(\xi)$, $i = 1, \cdots, m-1$, and let $\xi_{m,0}^{\beta} = -1$, $\xi_{m,m}^{\beta} = 1$. These $(m+1)$ points are called Gauss-Jacobi-Lobbato (GJL) points, which are used in quadratic rule of GJL-type which are referred to as Bouzitat quadratic rule of the second kind in [12].

Let $\ell_{m,j}^{\beta}(\xi)$ be the Lagrange-Jacobi interpolation polynomials of degree m, $0 \leq j \leq m$, such that

$$l_{m,j}^{\beta}(\xi_{m,i}^{\beta}) = \delta_{i,j}\,, \qquad 0 \leq i, j \leq m$$

where $\delta_{i,j}$ is Kronic Delta function.

The Jacobi-weighted L^2-norm of $l_{m,0}^{\beta}(\xi)$ and $l_{m,m}^{\beta}(\xi)$ is given in [14]:

Theorem 3.4.5. *(Theorem 3.2 of [14])*

For $j = 0, m$, $\beta > -1$,

$$\|l_{m,j}^{\beta}\|_{L_{\beta}^2(I)} \leq C_0\, \Gamma^{1/2}\,(1+\beta)m^{-(1+\beta)}\,. \tag{3.4.7}$$

Lemma 3.4.6. *(Lemma 4.3 of [14])*

If $u \in H^{t,\beta}(Q)$*, with* $t > 1 + s + \beta$ *for some* $s \geq 0$ *and* $\beta > -1$*, there holds on the side* I *of* Q

$$\|u\|_{H^{s,\beta}(I)} \leq \Phi(t, s, \beta)\, \|u\|_{H^{t,\beta}(Q)} \,. \tag{3.4.8}$$

In particular, if $s = 0$ *and* $t > 1 + \beta$*, there holds*

$$\|u\|_{H^{0,\beta}(I)} \leq \Phi(t, \beta)\, \|u\|_{H^{t,\beta}(Q)} \,, \tag{3.4.9}$$

with

$$\Phi(t, s, \beta) = \frac{C_0}{\Gamma(1 + \beta)}\, (t - s - 1 - \beta)^{-1/2} \,,$$

and

$$\Phi(t, \beta) = \frac{C_0}{\Gamma(1 + \beta)}\, (t - 1 - \beta)^{-1/2} \,.$$

Chapter 4

Quasi Jacobi Projection and Its Approximation Properties

4.1 Construction of Quasi Jacobi Projection Operator

In this section, we first construct an operator $\Pi_{\mathcal{T}}^{-\beta} : H^{1,-\beta}(\mathcal{T}) \rightarrow S^{p,1}(\Omega, \mathcal{T})$, which will play an essential role in the proof of Theorem ??. The operator is based on the Jacobi projection on $\mathcal{P}_p(K)$ over each element K, and is then extended to the finite element space $S^{p,1}(\Omega, \mathcal{T})$ over the whole domain Ω. We call it quasi Jacobi projection operator, which is the image of the quasi Jacobi projection operator on unscaled elements and meshes under a simple scale mapping.

Let $v_K = \Pi_{p,K}^{-\beta} v$ be the Jacobi projection of $v \in H^{1,-\beta}(K)$ on $\mathcal{P}_p(K)$ with $\beta < 1$. Due to Lemma 3.4.1 and Lemma 3.4.2 there holds for $m = 0, 1$

$$|v - \Pi_{p,K}^{-\beta} v|_{H^{m,-\beta}(K)} \leq C p^{m-1}\, h^{1-m}\, |v|_{H^{1,-\beta}(K)}\,. \tag{4.1.1}$$

Let v_p be a piecewise polynomial in Ω such that

$$v_p\,|_K = v_K\,, \qquad \forall K \in \mathcal{T}\,.$$

Obviously, $v_p \notin S^{p,1}(\Omega, \mathcal{T})$ because it is not continuous in Ω. We have to modified v_p without substantial loss of approximation property (4.1.1).

For any vertex V of $\mathcal{T}$, we define an average as

$$\bar{v}_p(V) = \frac{1}{n_V} \sum_{K \in Q_V} v_K(V)\,, \tag{4.1.2}$$

where Q_V is a patch centered at the vertex V, and n_V denotes the number of elements in the patch Q_V.

We denote the vertices of an element K by V_i, $1 \le i \le 4$. Suppose V_1 is located at $(-h, -h)$. Then at this point we can modify v_K to $v_K^{(1)}$ as follows

$$v_K^{(1)} = v_K + (\bar{v}_p(V_1) - v_K(V_1))\, \ell_{p,0}^{-\beta}(\frac{x_1}{h})\ell_{p,0}^{-\beta}(\frac{x_2}{h})\,,$$

where $\ell_{p,j}^{-\beta}(\xi)$, $j = 0, p$, is a Bouzitat-type polynomial of degree p such that

$$\ell_{p,0}^{-\beta}(-1) = 1,\ \ell_{p,0}^{-\beta}(1) = 0,\ \ell_{p,p}^{-\beta}(-1) = 0,\ \ell_{p,p}^{-\beta}(1) = 1\,.$$

Then it can be verified that

$$v_K^{(1)}(V_1) = \bar{v}_p(V_1)\,.$$

Similarly, this kind of modification can be carried out on the other three vertices $V_2(h, -h)$, $V_3(h, h)$ and $V_4(-h, h)$ of the element K. After these four steps of modification, we have

$$v_K^{(1)} = v_K + w_K^{(1)}\,,$$

with

$$\begin{aligned} w_K^{(1)} &= (\bar{v}_p(V_1) - v_K(V_1))\, \ell_{p,0}^{-\beta}(\frac{x_1}{h})\, \ell_{p,0}^{-\beta}(\frac{x_2}{h}) \\ &+ (\bar{v}_p(V_2) - v_K(V_2))\, \ell_{p,p}^{-\beta}(\frac{x_1}{h})\, \ell_{p,0}^{-\beta}(\frac{x_2}{h}) \\ &+ (\bar{v}_p(V_3) - v_K(V_3))\, \ell_{p,p}^{-\beta}(\frac{x_1}{h})\, \ell_{p,p}^{-\beta}(\frac{x_2}{h}) \\ &+ (\bar{v}_p(V_4) - v_K(V_4))\, \ell_{p,0}^{-\beta}(\frac{x_1}{h})\, \ell_{p,p}^{-\beta}(\frac{x_2}{h}) . \end{aligned} \tag{4.1.3}$$

On the four vertices of element K, we now have

$$v_K^{(1)}(V_i) = \bar{v}_p(V_i), \quad 1 \le i \le 4 .$$

Let $v_p^{(1)}$ and $w^{(1)}$ be piecewise polynomials on Ω such that

$$v_p^{(1)}|_K = v_K^{(1)},\, w^{(1)}|_K = w_K^{(1)}, \quad \forall K \in \mathcal{T}.$$

The completion of such modifications on each element gives us

$$v_p^{(1)} = v_p + w^{(1)}, \tag{4.1.4}$$

which guarantees that

$$v_p^{(1)}(V) = \bar{v}_p(V), \quad \text{for any node } V \text{ in } \mathcal{T} .$$

We further modify $v_p^{(1)}$ on each internal edge γ which is shared by a pair of elements K_1 and K_2. We may assume that $\gamma = \{(x_1, h) \mid -h \le x_1 \le h\}$, $K_1 = (-h, h)^2$ and $K_2 = (-h, h) \times (h, 3h)$ as shown in Fig. 4.1.2:

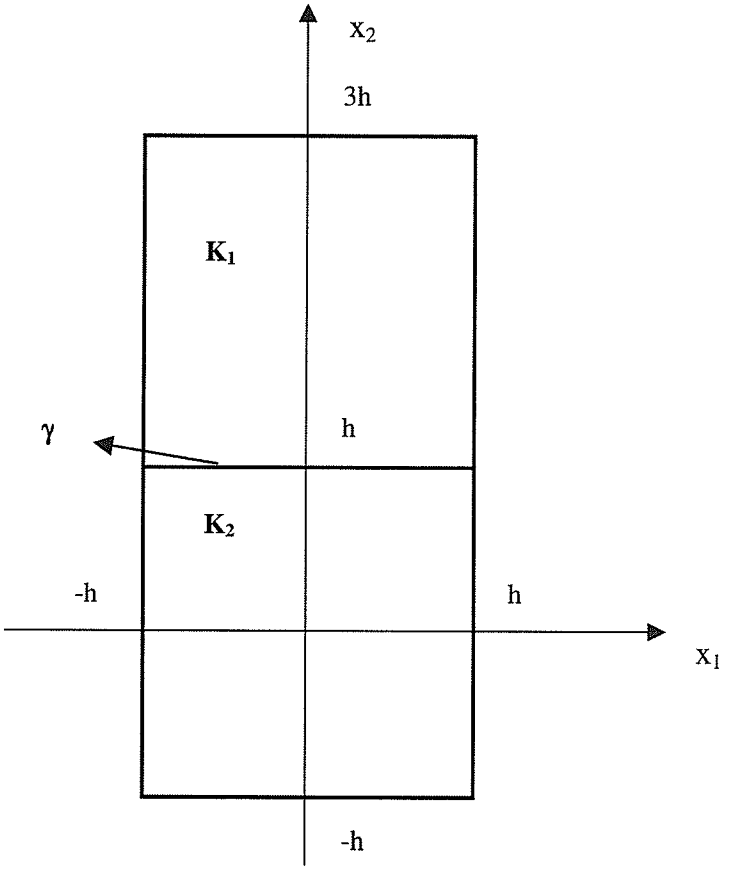


Figure 4.1: Two elements sharing an internal edge

Let

$$\phi_\gamma(x_1) = v_{K_2}^{(1)}(x_1, h) - v_{K_1}^{(1)}(x_1, h)\,,$$

which vanishes at the two end points of γ. We extend this function in the patch of γ: $Q_\gamma = \overline{K}_1 \cup \overline{K}_2$ as follows

$$w_\gamma = \begin{cases} \frac{1}{2}\phi_\gamma(x_1)\,\ell^{-\beta}_{p,p}(\frac{x_2}{h}) = w_{\gamma,1}\,, \text{ on } K_1 \\ \\ -\frac{1}{2}\phi_\gamma(x_1)\,\ell^{-\beta}_{p,0}(\frac{x_2}{h} - 2) = w_{\gamma,2}\,, \text{ on } K_2\,. \end{cases} \tag{4.1.5}$$

Note that w_γ vanishes on the boundary of Q_γ such that it can be extended to the whole domain Ω by a zero extension outside Q_γ.

Let $w^{(2)}$ be a piecewise polynomial on Ω such that

$$w^{(2)}|_K = w^{(2)}_K = \sum_{\gamma \in \partial K \bigcap \partial \mathcal{T}} w_\gamma\,,$$

and

$$w^{(2)} = \sum_{K \in \mathcal{T}} w^{(2)}_K = \sum_{\gamma \in \partial \mathcal{T}} w_\gamma\,.$$

Let $v^{(2)}_p$ be a piecewise polynomial on Ω such that

$$v^{(2)}_p = v^{(1)}_p + w^{(2)}\,.$$

Now we can finally define the quasi Jacobi projection as

$$\Pi^{-\beta}_{\mathcal{T}} v = v^{(2)}_p \in S^{p,1}(\Omega, \mathcal{T})\,,$$

that is,

$$\Pi^{-\beta}_{\mathcal{T}} v = v_p + w^{(1)} + w^{(2)}\,. \tag{4.1.6}$$

We have completed the construction of the quasi Jacobi projection with desired properties.

4.2 Properties of Quasi Jacobi Projection Operator on Scaled Elements

In this section, K and $\widehat{K}$ denote the squares with side length $2h$ and 2, respectively, and $K = \widehat{K} \circ F$ where F is a simple scaling and translation. Similarly, γ and $\widehat{\gamma}$ denote edges of elements with length $2h$ and 2, respectively, and $\gamma = \widehat{\gamma} \circ F$. Let $\widehat{u}_{\widehat{K}} = u_K \circ F$ and $\widehat{u}_{\widehat{\gamma}} = u_\gamma \circ F$ for all $u \in H^{1,-\beta}(\mathcal{T})$.

Lemma 4.2.1. *For $\beta \in (1/2\,,1)$ and $w_K^{(1)}$ as constructed in (4.1.3), we have*

$$\|w_K^{(1)}\|_{L^2_{-\beta}(K)} \leq C\,\frac{h\,p^{\epsilon-1}\,\log^{1/2}(1+p)}{\epsilon^{1/2}\,\Gamma(1-\beta)} \sum_{K'\in Q_K} \|v\|_{H^{1,-\beta}(K')}\,, \tag{4.2.1}$$

here Q_K denotes the patch of the element K containing elements K' such that $K \bigcap K' \neq \varnothing$.

Proof. Since

$$\|w_K^{(1)}\|_{L^2_{-\beta}(K)} = h\,\|\widehat{w}_{\widehat{K}}^{(1)}\|_{L^2_{-\beta}(\widehat{K})}\,, \tag{4.2.2}$$

due to the result of Lemma 4.2 in [14], there holds

$$\begin{aligned} \|\widehat{w}_{\widehat{K}}^{(1)}\|_{L^2_{-\beta}(\widehat{K})} &\leq Cp^{2\beta-2}\log^{1/2}(1+p) \sum_{\widehat{K}',\widehat{K}''\in Q_{\widehat{K}}} \sum_{\widehat{\gamma}=\overline{\widehat{K}'}\cap\overline{\widehat{K}''}} \|\widehat{v}_{\widehat{K}'} - \widehat{v}_{\widehat{K}''}\|_{H^{1-\beta,-\beta}(\widehat{\gamma})} & (4.2.3)\\ &\leq Cp^{2\beta-2}\log^{1/2}(1+p) \sum_{\widehat{\gamma}\in Q_{\widehat{K}}} \sum_{\widehat{K}'\in Q_{\widehat{\gamma}}} \|\widehat{v} - \widehat{v}_{\widehat{K}'}\|_{H^{1-\beta,-\beta}(\widehat{\gamma})} & (4.2.4) \end{aligned}$$

where $Q_{\widehat{\gamma}} = Q_\gamma \circ F^{-1}$ and $Q_{\widehat{K}} = Q_K \circ F^{-1}$.

If we let $t = 2 - 2\beta + \epsilon$, then using (3.4.8) in Lemma 3.4.6 and (3.4.6) in Theorem 3.4.4, we have

$$\begin{aligned} \|\widehat{v} - \widehat{v}_{\widehat{K}'}\|_{H^{1-\beta,-\beta}(\widehat{\gamma})} &\leq \frac{C_0}{\Gamma(1-\beta)}\,\epsilon^{-1/2}\|\widehat{v} - \widehat{v}_{\widehat{K}'}\|_{H^{t,-\beta}(\widehat{K}')} \\ &\leq C_0\,\frac{p^{1-2\beta+\epsilon}}{\epsilon^{1/2}\,\Gamma(1-\beta)}\,\|v\|_{H^{1,-\beta}(K')}\,. \end{aligned} \tag{4.2.5}$$

In the same way we obtain

$$\|\widehat{v} - \widehat{v}_{\widehat{K}''}\|_{H^{1-\beta,-\beta}(\widehat{\gamma})} \leq C_0\,\frac{p^{1-2\beta+\epsilon}}{\epsilon^{1/2}\,\Gamma(1-\beta)}\,\|v\|_{H^{1,-\beta}(K'')}\,. \tag{4.2.6}$$

A combination of (4.2.2) - (4.2.6) leads to the completion of the proof.

□

Lemma 4.2.2. *For $\beta \in (1/2\,,1)$ and w_γ as constructed in (4.1.5), we have*

$$\|w_\gamma\|_{L^2_{-\beta}(K_1)} \leq C\,\frac{h\,p^{\epsilon-1}\,\log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma^{1/2}(1-\beta)}\,\sum_{m=1,2}\|v\|_{H^{1,-\beta}(K_m)}\,. \tag{4.2.7}$$

Proof. Since

$$\|w_\gamma\|_{L^2_{-\beta}(K_1)} = h\,\|\widehat{w}_{\widehat{\gamma}}\|_{L^2_{-\beta}(\widehat{K}_1)}\,, \tag{4.2.8}$$

according to the construction of $\widehat{w}_{\widehat{\gamma}}$ in [14] and (3.4.7) of Theorem 3.4.5 , we have

$$\begin{aligned} \|\widehat{w}_{\widehat{\gamma}}\|_{L^2_{-\beta}(\widehat{K}_1)} &= \frac{1}{2}\|\ell^{-\beta}_{p,p}(x_2)\|_{L^2_{-\beta}(I)}\,\|\widehat{v}^{(1)}_{\widehat{K}_2} - \widehat{v}^{(1)}_{\widehat{K}_1}\|_{L^2_{-\beta}(\widehat{\gamma})} \\ &\leq C\,\Gamma^{1/2}(1-\beta)\,p^{\beta-1}\,\|\widehat{v}^{(1)}_{\widehat{K}_2} - \widehat{v}^{(1)}_{\widehat{K}_1}\|_{L^2_{-\beta}(\widehat{\gamma})}\,, \end{aligned} \tag{4.2.9}$$

and

$$\begin{aligned} \|\widehat{v}^{(1)}_{\widehat{K}_2} - \widehat{v}^{(1)}_{\widehat{K}_1}\|_{L^2_{-\beta}(\widehat{\gamma})} &= \|(\widehat{v}_{\widehat{K}_1} + \widehat{w}^{(1)}_{\widehat{K}_1}) - (\widehat{v}_{\widehat{K}_2} + \widehat{w}^{(1)}_{\widehat{K}_2})\|_{L^2_{-\beta}(\widehat{\gamma})} \\ &\leq \|\widehat{v}_{\widehat{K}_1} - \widehat{v}_{\widehat{K}_2}\|_{L^2_{-\beta}(\widehat{\gamma})} + \|\widehat{w}^{(1)}_{\widehat{K}_1} - \widehat{w}^{(1)}_{\widehat{K}_2}\|_{L^2_{-\beta}(\widehat{\gamma})}\,. \end{aligned} \tag{4.2.10}$$

If we let $t = 1 - \beta + \epsilon$, then according to (3.4.9) of Lemma 3.4.6 and (3.4.6) of Theorem 3.4.4, we have

$$\begin{aligned} \|\widehat{v}_{\widehat{K}_1} - \widehat{v}_{\widehat{K}_2}\|_{L^2_{-\beta}(\widehat{\gamma})} &\leq \sum_{m=1,2} \|\widehat{v} - \widehat{v}_{\widehat{K}_m}\|_{L^2_{-\beta}(\widehat{\gamma})} \\ &\leq \frac{C}{\Gamma(1-\beta)} \epsilon^{-1/2} \sum_{m=1,2} \|\widehat{v} - \widehat{v}_{\widehat{K}_m}\|_{H^{t,-\beta}(\widehat{K}_m)} \qquad (4.2.11) \\ &\leq C \frac{p^{\epsilon-\beta}}{\epsilon^{1/2}\,\Gamma(1-\beta)} \sum_{m=1,2} \|v\|_{H^{1,-\beta}(K_m)} \,. \end{aligned}$$

According to argument of Lemma 4.6 in [14] and (??) - (4.2.6), we obtain

$$\begin{aligned} &\|\widehat{w}^{(1)}_{\widehat{K}_1} - \widehat{w}^{(1)}_{\widehat{K}_2}\|_{L^2_{-\beta}(\widehat{\gamma})} \\ \leq\ & C \frac{p^{\beta-1}\log^{1/2}(p+1)}{\Gamma^{1/2}(1-\beta)} \|\widehat{v}_{\widehat{K}_1} - \widehat{v}_{\widehat{K}_2}\|_{H^{1-\beta,-\beta}(\widehat{\gamma})} \qquad (4.2.12) \\ \leq\ & C \frac{p^{\epsilon-\beta}\log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma^{3/2}(1-\beta)} \sum_{m=1,2} \|v\|_{H^{1,-\beta}(K_m)} \,. \end{aligned}$$

It is obvious that (4.2.10), (4.2.11) and (4.2.12) give

$$\|\widehat{v}^{(1)}_{\widehat{K}_2} - \widehat{v}^{(1)}_{\widehat{K}_1}\|_{L^2_{-\beta}(\widehat{\gamma})} \leq C \frac{p^{\epsilon-\beta}\log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma(1-\beta)} \sum_{m=1,2} \|v\|_{H^{1,-\beta}(K_m)} \,. \qquad (4.2.13)$$

A combination of (4.2.8), (4.2.9) and (4.2.13) completes the proof of (4.2.7).

□

Theorem 4.2.3. *For $v \in H^{1,-\beta}(\mathcal{T})$, there holds for $\beta \in (1/2, 1)$*

$$\|v - \Pi^{-\beta}_{\mathcal{T}} v\|_{L^2_{-\beta}(K)} \leq C \frac{h\, p^{\epsilon-1}\log^{1/2}(1+p)}{\epsilon^{1/2}\,\Gamma^{1/2}(1-\beta)} \sum_{K' \in Q_K} \|v\|_{H^{1,-\beta}(K')} \qquad (4.2.14)$$

with $\epsilon \in (0, 2-2\beta)$ arbitrary.

Proof. According to the definition

$$\Pi_{\mathcal{T}}^{-\beta} v = v_p + w^{(1)} + w^{(2)} ,$$

therefore in the element K, we have

$$\begin{aligned}(v - \Pi_{\mathcal{T}}^{-\beta} v)|_K &= v - v_K - w_K^{(1)} - w_K^{(2)} \\ &= v - v_K - w_K^{(1)} - \sum_{\gamma \subset \partial K \cap \partial \mathcal{T}} w_\gamma .\end{aligned}$$

Apparently

$$\begin{aligned}\|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(K)} &\leq \|v - v_K\|_{L^2_{-\beta}(K)} \\ &+ \|w_K^{(1)}\|_{L^2_{-\beta}(K)} + \sum_{\gamma \subset \partial K \cap \partial \mathcal{T}} \|w_\gamma\|_{L^2_{-\beta}(K)} ,\end{aligned}$$

and according to (3.4.6) in Theorem 3.4.4

$$\|v - v_K\|_{L^2_{-\beta}(K)} = h \, \|\widehat{v} - \widehat{v}_{\widehat{K}}\|_{L^2_{-\beta}(\widehat{K})} \leq C \, h p^{-1} \, \|v\|_{H^{1,-\beta}(K)} , \qquad (4.2.15)$$

which, along with Lemma 4.2.1 and Lemma 4.2.2, give (4.2.14). □

Theorem 4.2.4. *Let* $\gamma = \overline{K}_1 \cap \overline{K}_2$. *If* $v \in H^{1,-\beta}(\mathcal{T})$, *for* $\beta \in (1/2, 1)$, *there holds*

$$\begin{aligned}&\|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(\gamma)} \qquad (4.2.16) \\ &\leq C \, \frac{h^{1/2} \, p^{\epsilon-\beta} \log^{1/2}(p+1)}{\epsilon^{1/2} \, \Gamma(1-\beta)} \sum_{l=1,2} \sum_{K' \subset Q_{V_l}} \|v\|_{H^{1,-\beta}(K')}\end{aligned}$$

with $\epsilon \in (0, 2-2\beta)$ *arbitrary, and* V_1, V_2 *are the two end points of* γ.

Proof. Since

$$\Pi_{\mathcal{T}}^{-\beta} v = v_p + w^{(1)} + w^{(2)} \,,$$

we have

$$(v - \Pi_{\mathcal{T}}^{-\beta} v)|_{\gamma} = (v - v_{K_1})|_{\gamma} - w_{K_1}^{(1)}|_{\gamma} - w_{K_1}^{(2)}|_{\gamma} \,.$$

If we let $t = 1 - \beta + \epsilon$, then according to (3.4.9) in Lemma 3.4.6 and (3.4.6) in Theorem 3.4.4, we have

$$\begin{aligned} \|v - v_{K_1}\|_{L^2_{-\beta}(\gamma)} &= h^{1/2}\, \|\widehat{v} - \widehat{v}_{\widehat{K}_1}\|_{L^2_{-\beta}(\widehat{\gamma})} \\ &\leq C\, \frac{h^{1/2}}{\epsilon^{1/2}\,\Gamma(1-\beta)}\, \|\widehat{v} - \widehat{v}_{\widehat{K}_1}\|_{H^{t,-\beta}(\widehat{K}_1)} \\ &\leq C\, \frac{h^{1/2}\, p^{\epsilon-\beta}}{\epsilon^{1/2}\,\Gamma(1-\beta)}\, \|v\|_{H^{1,-\beta}(K_1)} \,. \end{aligned} \tag{4.2.17}$$

According to the argument of Lemma 4.7 in [14], (3.4.7) in Theorem 3.4.5, and (??) - (4.2.6), we have

$$\begin{aligned} &\|w_{K_1}^{(1)}\|_{L^2_{-\beta}(\gamma)} = h^{1/2}\, \|\widehat{w}_{\widehat{K}_1}^{(1)}\|_{L^2_{-\beta}(\widehat{\gamma})} \\ &= h^{1/2} \sum_{l=1,2} |\widehat{\bar{v}}_p(\widehat{V}_l) - \widehat{v}_{\widehat{K}_1}(\widehat{V}_l)|\; \|\ell_{p,\,(l-1)p}^{-\beta}\|_{L^2_{-\beta}(\widehat{\gamma})} \\ &\leq C\, h^{1/2}\, \Gamma^{1/2}(1-\beta) p^{\beta-1} \sum_{l=1,2} |\widehat{\bar{v}}_p(\widehat{V}_l) - \widehat{v}_{\widehat{K}_1}(\widehat{V}_l)| \\ &\leq C\, \frac{h^{1/2}\, p^{\beta-1}\, \log^{1/2}(p+1)}{\Gamma^{1/2}(1-\beta)} \sum_{l=1,2} \sum_{\widehat{K}',\widehat{K}" \subset Q_{\widehat{V}_l}} \sum_{\widehat{\gamma} = \overline{\widehat{K}}' \cap \overline{\widehat{K}}''} \|\widehat{v}_{\widehat{K}'} - \widehat{v}_{\widehat{K}''}\|_{H^{1-\beta,-\beta}(\widehat{\gamma})} \\ &\leq C\, \frac{h^{1/2}\, p^{\epsilon-\beta}\, \log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma^{3/2}(1-\beta)} \sum_{K' \subset Q_{\gamma}} \|v\|_{H^{1,-\beta}(K')} \,. \end{aligned} \tag{4.2.18}$$

where $Q_{\widehat{V}_l} = Q_{V_l} \circ F$. Furthermore, as shown in (4.2.13) of Theorem 4.2.3,

$$\begin{aligned} \|w^{(2)}_{K_1}\|_{L^2_{-\beta}(\gamma)} &= h^{1/2}\, \|\widehat{w}^{(2)}_{\widehat{K}_1}\|_{L^2_{-\beta}(\widehat{\gamma})} \\ &= \frac{1}{2} h^{1/2}\, \|\widehat{v}^{(1)}_{\widehat{K}_2} - \widehat{v}^{(1)}_{\widehat{K}_1}\|_{L^2_{-\beta}(\widehat{\gamma})} \qquad (4.2.19) \\ &\leq C\, \frac{h^{1/2}\, p^{\epsilon-\beta}\, \log^{1/2}(p+1)}{\epsilon^{1/2}\, \Gamma(1-\beta)} \sum_{m=1,2} \|v\|_{H^{1,-\beta}(K_m)} \,. \end{aligned}$$

A combination of (4.2.17), (4.2.18) and (4.2.19) leads to (4.2.16). ☐

Chapter 5

Upper and Lower Bounds of the Error in terms of estimators and indicators

In this chapter, we prove the main results of this thesis. First we analyze the upper bound of the error in the terms of the error estimator, then we derive the lower bound of the error locally and globally.

5.1 Estimator η

Theorem 5.1.1. *Let* $e = u - u_S$ *be the error of the finite element solution in* $S^{p,1}(\Omega, \mathcal{T})$*. Then there holds for* $\beta \in (1/2, 1)$,

$$|||e||| \leq C(\epsilon, \beta, p)\ \eta \tag{5.1.1}$$

where

$$C(\epsilon, \beta, p) = C\,\frac{p^{\epsilon} \log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma^{1/2}(1-\beta)},$$

and $\epsilon \in (0, 2-2\beta)$, *arbitrary.*

Proof. First, let $v \in C_0^\infty(\Omega)$, and let $\Pi_{\mathcal{T}}^{-\beta} v$ be its quasi Jacobi projection on $S^{p,1}(\Omega, \mathcal{T})$. For the model problem (2.1.1)

$$B(u,v) = \int_\Omega (\nabla u \cdot \nabla v + u\, v)\, dx\,,$$

therefore according to Green's formula

$$\int_\Omega u\, \triangle v\, dx = \int_{\partial\Omega} u\, \frac{\partial v}{\partial n}\, ds - \int_\Omega \nabla u \cdot \nabla v\, dx\,,$$

we have

$$B(e,v) = \int_\Omega (\nabla e \cdot \nabla v + e\, v)\, dx = \int_{\partial\Omega} \frac{\partial e}{\partial n}\, v\, ds + \int_\Omega (e - \triangle e)\, v\, dx\,,$$

since $e = u - u_S$, we have

$$\begin{aligned} \int_{\partial\Omega} \frac{\partial e}{\partial n} v\, ds &= \sum_{\gamma \in \partial\mathcal{T}} \int_\gamma \frac{\partial u}{\partial n} v\, ds - \sum_{\gamma \in \partial\mathcal{T}} \int_\gamma \frac{\partial u_S}{\partial n} v\, ds \\ &= \sum_{\gamma \in \partial\mathcal{T}} \int_\gamma R\, v\, ds\,, \end{aligned}$$

and

$$\begin{aligned} \int_\Omega (e - \triangle e)\, v\, dx &= \int_\Omega (u - \triangle u - u_S + \triangle u_S)\, v\, dx \\ &= \int_\Omega (f - u_S + \triangle u_S)\, v\, dx \\ &= \sum_{K \in \mathcal{T}} \int_K r\, v\, dx\,, \end{aligned}$$

therefore

$$\begin{aligned} B(e,v) &= B(e, v - \Pi_{\mathcal{T}}^{-\beta} v) \qquad (5.1.2) \\ &= \sum_{K \in \mathcal{T}} \int_K r\, (v - \Pi_{\mathcal{T}}^{-\beta} v)\, dx + \sum_{\gamma \in \partial\mathcal{T}} \int_\gamma R\, (v - \Pi_{\mathcal{T}}^{-\beta} v)\, ds\,. \end{aligned}$$

By Theorem 4.2.3, we have

$$
\begin{aligned}
&\Big|\sum_{K\in\mathcal{T}}\int_K r\left(v-\Pi_{\mathcal{T}}^{-\beta}v\right)dx\Big| && (5.1.3)\\
\le\;& \sum_{K\in\mathcal{T}}\|r\|_{L^2_\beta(K)}\,\|v-\Pi_{\mathcal{T}}^{-\beta}v\|_{L^2_{-\beta}(K)}\\
\le\;& \Big(\sum_{K\in\mathcal{T}}\frac{h^2}{p^{2\beta}}\|r\|^2_{L^2_\beta(K)}\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}}\frac{p^{2\beta}}{h^2}\,\|v-\Pi_{\mathcal{T}}^{-\beta}v\|^2_{L^2_{-\beta}(K)}\Big)^{1/2}\\
\le\;& C\,\frac{p^{\epsilon-1+\beta}\,\log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma^{1/2}(1-\beta)}\,\|v\|_{H^{1,-\beta}(\mathcal{T})}\Big(\sum_{K\in\mathcal{T}}\eta_K^2\Big)^{1/2},
\end{aligned}
$$

and by Theorem 4.2.4, we have

$$
\begin{aligned}
&\Big|\sum_{\gamma\in\partial\mathcal{T}}\int_\gamma R\Big(v-\Pi_{\mathcal{T}}^{-\beta}v\Big)\,ds\Big| && (5.1.4)\\
\le\;& \Big(\sum_{\gamma\in\partial\mathcal{T}}\frac{h}{p^{2\beta}}\|R\|^2_{L^2_\beta(\gamma)}\Big)^{1/2}\Big(\sum_{\gamma\in\partial\mathcal{T}}\frac{p^{2\beta}}{h}\,\|v-\Pi_{\mathcal{T}}^{-\beta}v\|^2_{L^2_{-\beta}(\gamma)}\Big)^{1/2}\\
\le\;& C\,\frac{p^{\epsilon}\,\log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma(1-\beta)}\,\|v\|_{H^{1,-\beta}(\mathcal{T})}\Big(\sum_{\gamma\in\partial\mathcal{T}}\eta_\gamma^2\Big)^{1/2}.
\end{aligned}
$$

A combination of (5.1.2), (5.1.3) and (5.1.4) leads to

$$
|B(e,v)|\le C\,\frac{p^{\epsilon}\,\log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma^{1/2}(1-\beta)}\,\|v\|_{H^{1,-\beta}(\mathcal{T})}\,\eta\,. \qquad (5.1.5)
$$

The above estimation is valid for $v\in C_0^\infty(\Omega)$. By a density argument it can be proved for all $v\in H_0^{1,-\beta}(\mathcal{T})$. Therefore

$$
\sup_{v\in H^{1,-\beta}(\mathcal{T})}\frac{|B(e,v)|}{\|v\|_{H^{1,-\beta}(\mathcal{T})}}\le C(\epsilon,\beta,p)\;\eta\,,
$$

which proves (5.1.1). □

5.2 Indicators η_K and η_γ

We should analyze indicators η_K and η_γ as a lower bound of error in weighted norms.

Theorem 5.2.1. *If $f \in \mathcal{P}_p(K)$, then for $\beta \in (0,1)$*

$$|||e|||_K \geq C p^{\beta-1}\,\eta_K\,. \tag{5.2.1}$$

Proof. Let

$$v = r\,W_{\beta,K} = r\,(1-(\frac{x_1}{h})^2)^\beta(1-(\frac{x_2}{h})^2)^\beta\,,$$

then v vanishes on ∂K, and it can be extended by zero extension outside of K. Substituting v into (5.1.2), we have

$$\|r\|^2_{L^2_\beta(K)} = (r,v)|_K = B(e,v)|_K \leq \|v\|_{H^{1,-\beta}(K)}\,|||e|||_K\,. \tag{5.2.2}$$

Noting that

$$\|v\|_{L^2_{-\beta}(K)} = \|r\|_{L^2_\beta(K)}\,, \tag{5.2.3}$$

and by the argument of Theorem 5.2 in [14] and Lemma 3.4.2, we have

$$\begin{aligned}
\|v\|^2_{H^{1,-\beta}(K)} &= \|v\|^2_{L^2_{-\beta}(K)} + |v|^2_{H^{1,-\beta}(K)} \\
&= \|r\|^2_{L^2_\beta(K)} + |\widehat{v}|^2_{H^{1,-\beta}(\widehat{K})} \\
&\leq \|r\|^2_{L^2_\beta(K)} + C\,p^2\,\|\widehat{r}\|^2_{L^2_\beta(\widehat{K})} \\
&\leq \|r\|^2_{L^2_\beta(K)} + C\,p^2\,h^{-2}\,\|r\|^2_{L^2_\beta(K)}\,,
\end{aligned}$$

which implies

$$\|v\|_{H^{1,-\beta}(K)} \leq C\,p\,h^{-1}\,\|r\|_{L^2_\beta(K)}\,. \tag{5.2.4}$$

Therefore

$$|||e|||_K \geq \frac{\|r\|^2_{L^2_\beta(K)}}{\|v\|_{H^{1,-\beta}(K)}} \geq C\, h\, p^{-1}\, \|r\|_{L^2_\beta(K)} = C\, p^{\beta-1}\, \eta_K\,.$$

□

Theorem 5.2.2. *If* $R_\gamma = R|_\gamma \in \mathcal{P}_p(\gamma)$*, there holds*

$$\sum_{K \in Q_\gamma} |||e|||_K \geq \frac{C}{\Gamma^{1/2}(1-\beta)}\, \eta_\gamma\,, \quad \forall \gamma \in \partial\mathcal{T} \tag{5.2.5}$$

with $\beta \in (0,1)$*, where* Q_γ *is a pair of elements sharing* γ*.*

Proof. Suppose $\gamma = \{(x_1, 0) \mid -h \leq x_1 \leq h\} = \overline{K}_1 \cap \overline{K}_2$ as shown in Fig 5.1.

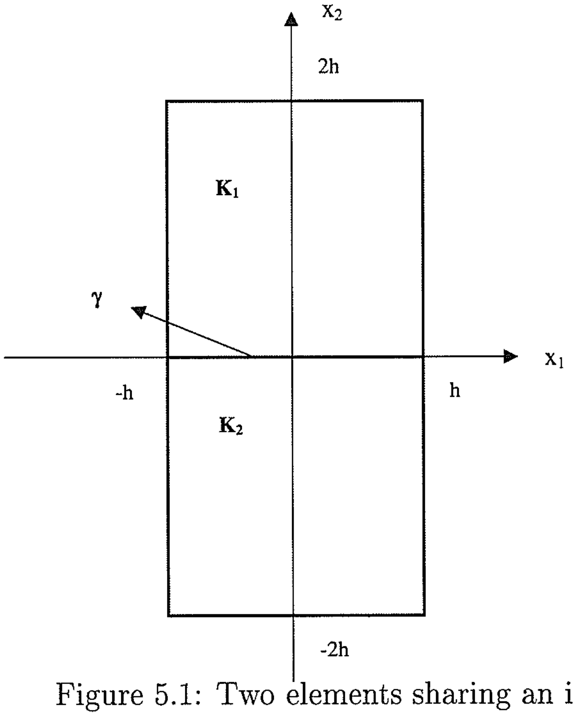


Figure 5.1: Two elements sharing an internal edge

Let

$$\psi_\gamma = W_{\beta,\gamma}\, R_\gamma = (1 - (\frac{x_1}{h})^2)^\beta R_\gamma\,,$$

with $\beta > 0$, and let

$$v_\gamma = \psi_\gamma(x_1)\, l_{p,0}^{-\beta}(\frac{|x_2|}{h} - 1)\,, \qquad -2h \le x_2 \le 2h\,.$$

Then v_γ vanishes on ∂Q_γ, and it can be extended by a zero extension outside of Q_γ.

Since

$$\|v_\gamma\|^2_{H^{1,-\beta}(K_1)} = \|v_\gamma\|^2_{L^2_{-\beta}(K_1)} + |v_\gamma|^2_{H^{1,-\beta}(K_1)}\,, \tag{5.2.6}$$

by applying (3.4.7) in Theorem 3.4.5, we have

$$\begin{aligned}
\|v_\gamma\|^2_{L^2_{-\beta}(K_1)} &= \|\psi_\gamma\|^2_{L^2_{-\beta}(\gamma)}\, \|l_{p,0}^{-\beta}(\frac{|x_2|}{h} - 1)\|^2_{L^2_{-\beta}(I_1)} \\
&= \|R_\gamma\|^2_{L^2_\beta(\gamma)} \cdot h\, \|l_{p,0}^{-\beta}(|\xi_2| - 1)\|^2_{L^2_{-\beta}(\widehat{I}_1)} \\
&\le C\,\Gamma(1-\beta)\, p^{2(\beta-1)}\, h\, \|R_\gamma\|^2_{L^2_\beta(\gamma)}\,, \qquad (5.2.7)
\end{aligned}$$

with $I_1 = [0, 2h]$ and $\widehat{I}_1 = [0, 2]$. As shown in the argument of Theorem 5.3 in [14], and applying (3.4.4) of Lemma 3.4.2

$$\begin{aligned}
|v_\gamma|^2_{H^{1,-\beta}(K_1)} &= |\widehat{v}_{\widehat{\gamma}}|^2_{H^{1,-\beta}(\widehat{K}_1)} \\
&\le C\,\Gamma(1-\beta)\, p^{2\beta}\, \|\widehat{R}_{\widehat{\gamma}}\|^2_{L^2_\beta(\widehat{\gamma})} \qquad (5.2.8) \\
&= C\,\Gamma(1-\beta)\, p^{2\beta}\, h^{-1}\, \|R_\gamma\|^2_{L^2_\beta(\gamma)}\,.
\end{aligned}$$

A combination of (5.2.6), (5.2.7) and (5.2.8) leads to

$$\|v_\gamma\|^2_{H^{1,-\beta}(K_1)} \le C\,\Gamma(1-\beta)\, p^{2\beta}\, h^{-1}\, \|R_\gamma\|^2_{L^2_\beta(\gamma)}\,. \tag{5.2.9}$$

The above inequality holds on K_2 as well.

Since

$$B(e, v_\gamma) = \sum_{K \in Q_\gamma} \int_K r\, v_\gamma\, dx + \int_\gamma R_\gamma\, v_\gamma\, ds\,,$$

we have

$$\begin{aligned} \|R_\gamma\|^2_{L^2_\beta(\gamma)} &= (R_\gamma, v_\gamma) = \sum_{K \in Q_\gamma} \Big(B(e, v_\gamma)|_K - \int_K r\, v_\gamma\, dx \Big) \\ &\leq \sum_{K \in Q_\gamma} \Big(|||e|||_K\, \|v_\gamma\|_{H^{1,-\beta}(K)} + \|r\|_{L^2_\beta(K)}\, \|v_\gamma\|_{L^2_{-\beta}(K)} \Big)\,, \end{aligned}$$

According to the definition of η_K and applying Theorem 5.2.1, we have

$$\|r\|_{L^2_\beta(K)} = \frac{p^\beta}{h}\, \eta_K \leq C\, p\, h^{-1}\, |||e|||_K\,,$$

which, along with (5.2.7) and (5.2.9), give

$$\begin{aligned} \|R_\gamma\|^2_{L^2_\beta(\gamma)} &\leq \sum_{K \in Q_\gamma} \Big[|||e|||_K \left(\|v_\gamma\|_{H^{1,-\beta}(K)} + p\, h^{-1}\, \|v_\gamma\|_{L^2_{-\beta}(K)} \right) \Big] \\ &\leq C\, \Gamma^{1/2}(1-\beta)\, p^\beta\, h^{-1/2}\, \|R_\gamma\|_{L^2_\beta(K)} \sum_{K \in Q_\gamma} |||e|||_K\,, \end{aligned}$$

which leads to (5.2.5) immediately. □

Corollary 5.2.3. *For $\beta \in (0, 1)$, there hold*

$$|||e||| \geq C p^{\beta - 1} \Big(\sum_{K \subset \mathcal{T}} \eta_K^2 \Big)^{1/2}, \tag{5.2.10}$$

and

$$|||e||| \geq \frac{C}{\Gamma^{1/2}(1-\beta)} \Big(\sum_{\gamma \subset \partial\mathcal{T}} \eta_\gamma^2 \Big)^{1/2}. \tag{5.2.11}$$

Moreover,

$$|||e||| \geq C_2(\beta, p)\ \eta\,, \tag{5.2.12}$$

with $C_2(\beta, p) = \min\{\Gamma^{-1/2}(1-\beta), p^{\beta-1}\}$.

5.3 Modified Indicators $\tilde{\eta}_K, \tilde{\eta}_\gamma$ and Modified Estimator $\tilde{\eta}$

In general the residual $r \notin \mathcal{P}_p(K)$, and the jump $R_\gamma \notin \mathcal{P}_p(\gamma)$ if the basis functions are the images of shape functions defined on the standard square element and the corresponding mapping is not linear. Therefore, Theorem 5.2 and 5.3 do not hold in general. We need to investigate the modified indicators $\tilde{\eta}_K$ and $\tilde{\eta}_\gamma$ as the lower bound of the error.

In the following theorems, r_p, f_p denote $L^2_\beta(K)$ projection of r and f on $\mathcal{P}_p(K)$ respectively and R_p denotes the L^2_β projection of R on $\mathcal{P}_p(\gamma)$. That is, $r_p = \Pi^\beta_K r$, $f_p = \Pi^\beta_K f$, and let $R_p = \Pi^\beta_\gamma R$.

Theorem 5.3.1. *Let $\tilde{\eta}, \tilde{\eta}_K, \tilde{\eta}_\gamma$ be the modified error estimator and indicator defined in (2.3.1)-(2.3.3), and let Π^β_K and Π^β_γ be the Jacobi projection operators on $\mathcal{P}_p(K)$ and $\mathcal{P}_p(\gamma)$ respectively. Then*

$$|||e||| \leq C(\epsilon, \beta, p)\left(\eta + \frac{h}{p^\beta}\sum_{K\in\mathcal{T}} \|f - \Pi^\beta_K f\|_{L^2_\beta(K)} + \frac{h^{1/2}}{p^\beta}\sum_{\gamma\in\partial\mathcal{T}} \|R - \Pi^\beta_\gamma R\|_{L^2_\beta(\gamma)}\right), \tag{5.3.1}$$

where

$$C(\epsilon, \beta, p) = C_0\,\frac{p^\epsilon \log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma^{1/2}(1-\beta)}\,,$$

and $\epsilon \in (0, 2-2\beta)$ *arbitrary.*

Proof. It is obvious that

$$\left(r, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_K = \left(r_p, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_K + \left(f - f_p, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_K ,$$

and

$$\left(R, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_\gamma = \left(R_p, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_\gamma + \left(R - R_p, v - \Pi_{\mathcal{T}}^{-\beta} v\right)_\gamma .$$

Then we have from (5.1.2)

$$\begin{aligned} \left|B(e,v)\right| &\leq \sum_{K\in\mathcal{T}} \int_K \left(|r_p| + |f - f_p|\right) |v - \Pi_{\mathcal{T}}^{-\beta} v| \, dx \\ &+ \sum_{\gamma\in\partial\mathcal{T}} \int_\gamma \left(|R_p| + |R - R_p|\right) |v - \Pi_{\mathcal{T}}^{-\beta} v| \, ds . \end{aligned}$$

By Schwarz inequality and Cauchy inequality, we have

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \int_K \left(|r_p| + |f - f_p|\right) |v - \Pi_{\mathcal{T}}^{-\beta} v| \, dx \\ \leq\ & \sum_{K\in\mathcal{T}} \left(\|r_p\|_{L^2_\beta(K)} + \|f - f_p\|_{L^2_\beta(K)}\right) \|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(K)} \\ \leq\ & \Big(\sum_{K\in\mathcal{T}} \frac{h^2}{p^{2\beta}} \|r_p\|^2_{L^2_\beta(K)}\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}} \frac{p^{2\beta}}{h^2} \|v - \Pi_{\mathcal{T}}^{-\beta} v\|^2_{L^2_{-\beta}(K)}\Big)^{1/2} \qquad (5.3.2) \\ +\ & \Big(\sum_{K\in\mathcal{T}} \frac{h^2}{p^{2\beta}} \|f - f_p\|^2_{L^2_\beta(K)}\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}} \frac{p^{2\beta}}{h^2} \|v - \Pi_{\mathcal{T}}^{-\beta} v\|^2_{L^2_{-\beta}(K)}\Big)^{1/2} \\ \leq\ & \frac{p^\beta}{h} \Big((\sum_{K\in\mathcal{T}} \widetilde{\eta}_K^2)^{1/2} + \frac{h}{p^\beta} (\sum_{K\in\mathcal{T}} \|f - f_p\|^2_{L^2_\beta(K)})^{1/2} \Big) \Big(\sum_{K\in\mathcal{T}} \|v - \Pi_{\mathcal{T}}^{-\beta} v\|^2_{L^2_{-\beta}(K)}\Big)^{1/2} \end{aligned}$$

and

$$
\begin{aligned}
&\sum_{\gamma\in\partial\mathcal{T}} \int_\gamma (|R_p| + |R-R_p|)\,|v-\Pi_{\mathcal{T}}^{-\beta}v|\,ds \\
\le\;& \sum_{\gamma\in\partial\mathcal{T}} \left(\|R_p\|_{L^2_\beta(\gamma)} + \|R-R_p\|_{L^2_\beta(\gamma)}\right) \|v-\Pi_{\mathcal{T}}^{-\beta}v\|_{L^2_{-\beta}(\gamma)} \\
\le\;& \left(\sum_{\gamma\in\partial\mathcal{T}} \frac{h}{p^{2\beta}}\,\|R_p\|^2_{L^2_\beta(\gamma)}\right)^{1/2} \left(\sum_{\gamma\in\partial\mathcal{T}} \frac{p^{2\beta}}{h}\,\|v-\Pi_{\mathcal{T}}^{-\beta}v\|^2_{L^2_{-\beta}(\gamma)}\right)^{1/2} \qquad (5.3.3)\\
+\;& \left(\sum_{\gamma\in\partial\mathcal{T}} \frac{h}{p^{2\beta}}\,\|R-R_p\|^2_{L^2_\beta(\gamma)}\right)^{1/2} \left(\sum_{\gamma\in\partial\mathcal{T}} \frac{p^{2\beta}}{h}\,\|v-\Pi_{\mathcal{T}}^{-\beta}v\|^2_{L^2_{-\beta}(\gamma)}\right)^{1/2} \\
\le\;& \frac{p^\beta}{h^{1/2}}\left((\sum_{\gamma\in\partial\mathcal{T}} \tilde\eta_\gamma^2)^{1/2} + \frac{h^{1/2}}{p^\beta}(\sum_{\gamma\in\partial\mathcal{T}} \|R-R_p\|^2_{L^2_\beta(\gamma)})^{1/2}\right)\left(\sum_{\gamma\in\partial\mathcal{T}} \|v-\Pi_{\mathcal{T}}^{-\beta}v\|^2_{L^2_{-\beta}(\gamma)}\right)^{1/2}.
\end{aligned}
$$

Due to Theorem 4.2.3 and Theorem 4.2.4

$$\|v-\Pi_{\mathcal{T}}^{-\beta}v\|_{L^2_{-\beta}(K)} \le C\,\frac{h\,p^{\epsilon-1}\log^{1/2}(1+p)}{\epsilon^{1/2}\,\Gamma^{1/2}(1-\beta)} \sum_{K'\in Q_K} \|v\|_{H^{1,-\beta}(K')}\,,$$

and

$$\|v-\Pi_{\mathcal{T}}^{-\beta}v\|_{L^2_{-\beta}(\gamma)} \le C\,\frac{h^{1/2}\,p^{\epsilon-\beta}\log^{1/2}(p+1)}{\epsilon^{1/2}\,\Gamma(1-\beta)} \sum_{l=1,2}\sum_{K'\subset Q_{V_l}} \|v\|_{H^{1,-\beta}(K')}$$

which together with (5.3.2) and (5.3.3) lead to (5.3.1) immediately. □

Theorem 5.3.2. *Let $\tilde\eta_K$ be the modified error indicator defined in (2.3.1). Then*

$$\tilde\eta_K \le C\,p^{1-\beta}\,|||e|||_K + \frac{h}{p^\beta}\,\|f-f_p\|_{L^2_\beta(K)} \qquad (5.3.4)$$

and

$$|||e|||_K \ge C\left(p^{\beta-1}\,\tilde\eta_K - \frac{h}{p}\,\|f-f_p\|_{L^2_\beta(K)}\right). \qquad (5.3.5)$$

Proof. Let $v = r_p\, W_{\beta,K}$, then v vanishes on ∂K and can be extended by zero extension outside of K.

Since

$$\begin{aligned}(r,\,v)|_K &= \int_K (r - r_p + r_p)\, r_p\, W_{\beta,K}\, dx \\ &= \|r_p\|^2_{L^2_\beta(K)} + \int_K (f - f_p)\, r_p\, W_{\beta,K}\, dx\,,\end{aligned}$$

and since $r_p \in \mathcal{P}_p(K)$, (5.2.4) holds as

$$\|v\|_{H^{1,-\beta}(K)} \le C\,p\,h^{-1}\,\|r_p\|_{L^2_\beta(K)}\,. \tag{5.3.6}$$

Along with (5.2.2), we have

$$\begin{aligned}\|r_p\|^2_{L^2_\beta(K)} &\le |(r,v)|_K| + \int_K |f - f_p|\, r_p\, W_{\beta,K}\, dx \\ &\le \|v\|_{H_{1,-\beta}(K)}\, |||e|||_K + \|f - f_p\|_{L^2_\beta(K)}\, \|r_p\|_{L^2_\beta(K)} \\ &\le C\,p\,h^{-1}\,\|r_p\|_{L^2_\beta(K)}\, |||e|||_K + \|f - f_p\|_{L^2_\beta(K)}\, \|r_p\|_{L^2_\beta(K)}\,.\end{aligned}$$

Therefore

$$\|r_p\|_{L^2_\beta(K)} \le C\,p\,h^{-1}\,|||e|||_K + \|f - f_p\|_{L^2_\beta(K)}\,, \tag{5.3.7}$$

which will lead to (5.3.4) and (5.3.5) immediately. □

Theorem 5.3.3. *Let $\tilde{\eta}_\gamma$ be the modified error indicator on internal edge $\gamma = \bar{K}_1 \cap \bar{K}_2$, then*

$$\tilde{\eta}_\gamma \le C\,\Gamma^{1/2}(1-\beta) \sum_{K \in Q_\gamma} \left(|||e|||_K + \frac{h}{p}\, \|f - f_p\|_{L^2_\beta(K)} \right) + \frac{h^{1/2}}{p^\beta}\, \|R - R_p\|_{L^2_\beta(\gamma)} \tag{5.3.8}$$

and

$$\sum_{K\in Q_\gamma} |||e|||_K \geq \frac{C}{\Gamma^{1/2}(1-\beta)}\Big(\tilde{\eta}_\gamma - \frac{h^{1/2}}{p^\beta}\,\|R-R_p\|_{L^2_\beta(\gamma)}\Big) - \frac{h}{p}\sum_{K\in Q_\gamma}\|f-f_p\|_{L^2_\beta(K)}. \tag{5.3.9}$$

Proof. Suppose that $\gamma = \{(x_1,0)| -h \leq x_1 \leq h\}$ as shown in Fig. 5.1. Let

$$\psi_\gamma = W_{\beta,\gamma}\,R_p = (1-(\frac{x_1}{h})^2)^\beta\,R_p\,.$$

By v_γ we denote the extension of ψ_γ in Q_γ

$$v_\gamma(x) = \psi_\gamma(x_1)\,\ell_{p,0}^{-\beta}(\frac{|x_2|}{h}-1), \quad x\in[-h,h]\times[-2h,2h]\,,$$

then v_γ vanishes on ∂Q_γ, and can be further extended by a zero extension outside Q_γ.

Since $R_p \in \mathcal{P}_p(\gamma)$, (5.2.7) and (5.2.9) are valid, i.e. for $m=1,2$

$$\|v_\gamma\|_{L^2_{-\beta}(K_m)} \leq C\,\Gamma^{1/2}(1-\beta)\,p^{\beta-1}\,h^{1/2}\,\|R_p\|_{L^2_\beta(\gamma)}\,, \tag{5.3.10}$$

and

$$\|v_\gamma\|_{H^{1,-\beta}(K_m)} \leq C\,\Gamma^{1/2}(1-\beta)\,p^{\beta}\,h^{-1/2}\,\|R_p\|_{L^2_\beta(\gamma)}\,. \tag{5.3.11}$$

According to (5.1.2), for v_γ, there holds

$$\begin{aligned} B(e,v_\gamma) &= \sum_{m=1,2}\int_{K_m} r\,v_\gamma dx + \int_\gamma R\,v_\gamma ds \\ &= \sum_{m=1,2}\Big(\int_{K_m} r_p\,v_\gamma dx + \int_{K_m}(f-f_p)\,v_\gamma dx\Big) + \int_\gamma R\,v_\gamma ds\,, \end{aligned}$$

which together with (5.2.2), (5.3.10), (5.3.11) and (5.3.7) give

$$
\begin{aligned}
\|R_p\|^2_{L^2_\beta(\gamma)} &= \int_\gamma R\, v_\gamma ds + \int_\gamma (R_p - R)\, v_\gamma ds \\
&= \int_\gamma (R_p - R)\, v_\gamma ds + \sum_{m=1,2} \Big(B(e, v_\gamma)_{K_m} - \int_{K_m} r_p\, v_\gamma dx + \int_{K_m} (f_p - f)\, v_\gamma dx \Big) \\
&\leq \|R_p - R\|_{L^2_\beta(\gamma)} \|R_p\|_{L^2_\beta(\gamma)} + \sum_{m=1,2} |||e|||_{K_m} \|v_\gamma\|_{H^{1,-\beta}(K_m)} \\
&\quad + \sum_{m=1,2} \Big(\|r_p\|_{L^2_\beta(K_m)} + \|f - f_p\|_{L^2_\beta(K_m)} \Big) \|v_\gamma\|_{L^2_{-\beta}(K_m)} \\
&\leq \|R_p - R\|_{L^2_\beta(\gamma)} \|R_p\|_{L^2_\beta(\gamma)} + C\,\Gamma^{1/2}(1-\beta)\, p^\beta\, h^{-1/2} \sum_{m=1,2} |||e|||_{K_m} \|R_p\|_{L^2_\beta(\gamma)} \\
&\quad + C\,\Gamma^{1/2}(1-\beta)\, p^{\beta-1}\, h^{1/2} \sum_{m=1,2} \Big(\|r_p\|_{L^2_\beta(K_m)} + \|f - f_p\|_{L^2_\beta(K_m)} \Big) \|R_p\|_{L^2_\beta(\gamma)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|R_p - R\|_{L^2_\beta(\gamma)} \|R_p\|_{L^2_\beta(\gamma)} \\
&\quad + C\,\Gamma^{1/2}(1-\beta)\, \frac{p^\beta}{h^{1/2}} \sum_{m=1,2} \Big(|||e|||_{K_m} + \frac{h}{p} \|f - f_p\|_{L^2_\beta(K_m)} \Big) \|R_p\|_{L^2_\beta(\gamma)},
\end{aligned}
$$

Therefore, (5.3.8) and (5.3.9) are obtained. □

Remark 5.3.1. *If f is smooth, $\|f - \Pi^\beta_K f\|_{L^2_\beta(K)}$ is much smaller than η_K and $|||e|||_K$ and can be omitted. In [20, 23], $\Pi^0_K f$ was used, but $f - \Pi^0_K f$ was measured in $L^2_\beta(K)$ with $\beta \neq 0$. Obviously, $\|f - \Pi^\beta_K f\|_{L^2_\beta(K)}$ is much smaller than $\|f - \Pi^0_K f\|_{L^2_\beta(K)}$, in particular when f is singular. R is normal derivative of pull-back polynomial on γ, and it is a smooth function. Therefore, $\|R - R_p\|_{L^2_\beta(\gamma)}$ is much smaller than $\tilde{\eta}_\gamma$, and it can be omitted.*

Chapter 6

Computation of Error Indicators and Estimators

We use three model problems to show the efficiency of the error indicators and error estimator in this chapter.

6.1 Model 1

The first model problem is as follows

$$\begin{cases} \triangle u = 0 & \text{on } \Omega = [-1,1] \times [0,1] \\ u = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial u}{\partial n} = g_i & \text{on } \Gamma_i, i = 2, \cdots, 5 \end{cases} \tag{6.1.1}$$

where $\mathbf{n}$ is the unit normal outwards to Ω, r and θ are polar coordinates,

$$\Gamma_1 = \{(x,0) \,|\, 0 < x < 1\},$$

and

$$
\begin{aligned}
g_2 &= -\frac{1}{2} r^{-1/2} \sin\frac{\theta}{2}, \quad \text{on } \Gamma_2 = \{(1,y) \,|\, 0 < y < 1\}\,; \\
g_3 &= \frac{1}{2} r^{-1/2} \cos\frac{\theta}{2}, \quad \text{on } \Gamma_3 = \{(x,1) \,|\, -1 < x < 1\}\,; \\
g_4 &= \frac{1}{2} r^{-1/2} \sin\frac{\theta}{2}, \quad \text{on } \Gamma_4 = \{(-1,y) \,|\, 0 < y < 1\}\,; \\
g_5 &= 0, \quad \text{on } \Gamma_5 = \{(x,0) \,|\, -1 < x < 0\}\,,
\end{aligned}
$$

The domain is shown in Figure 6.1:

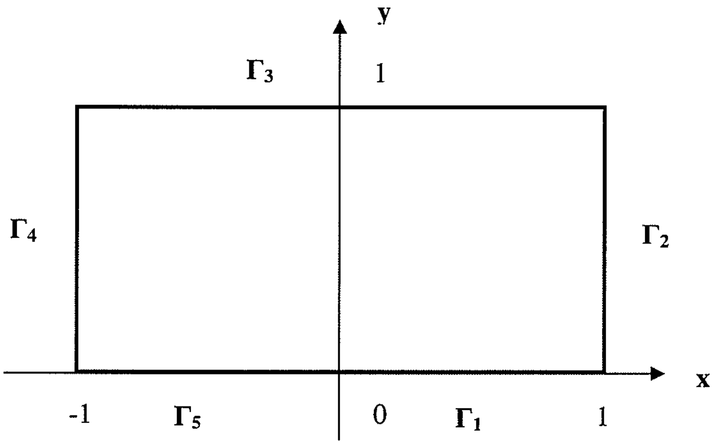


Figure 6.1: domain for model problem 1

The real solution to this PDE is

$$u = r^{1/2} \sin\frac{\theta}{2}\,.$$

The mesh $\mathcal{T}$ used in computation is shown in Figure 6.2, where the element number is within the circle in each element and the global side number is within the square on each edge:

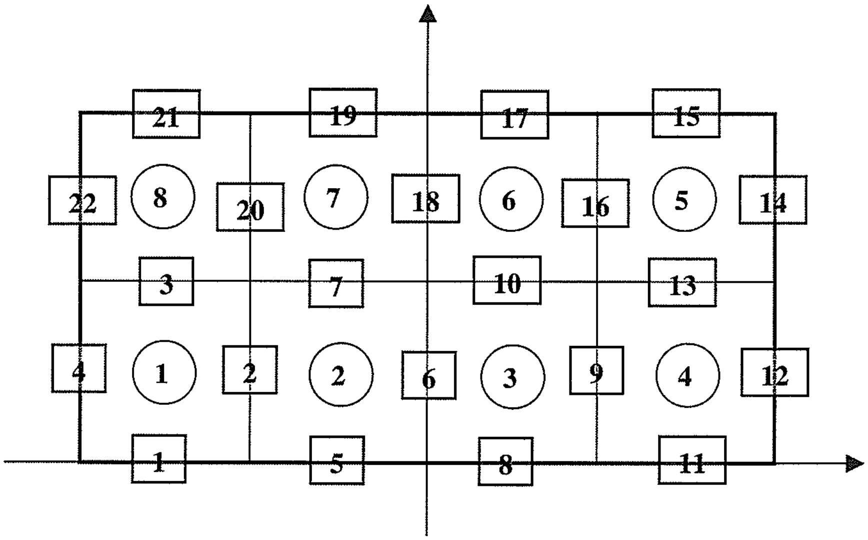


Figure 6.2: meshing for model problem 1

We compute the error indicators η_{K_i} on each element, $1 \leq i \leq 8$, and error indicator η_{γ_l} on each internal edge, $l = 2, 3, 6, 7, 9, 10, 13, 16, 18, 20$, for $2 \leq p \leq 8$. The values are presented in Figures 6.3, 6.5, 6.7, 6.9, 6.11, 6.13 and 6.15.

Since $|||e|||_{K_i}$, $1 \leq i \leq 8$, is not computable, we compute $||e||_{\widetilde{H}^{1,\beta}(K_i)}$, also we compute $||e||_{E(K_i)}$ on each element as well. These values are presented in Tables 6.1, 6.5, 6.9, 6.13, 6.17, 6.21, 6.25 and Tables 6.2, 6.6, 6.10, 6.14, 6.18, 6.22, 6.26, respectively.

The distribution τ_{K_i}, τ_{γ_l} of indicators η_{K_i} and η_{γ_l} and the distribution $\widetilde{t}_{K_i}$, t_{K_i} of $||e||_{\widetilde{H}^{1,\beta}(K_i)}$ and $||e||_{E(K_i)}$ are computed as follows

$$\begin{aligned} \tau_{K_i} &= \frac{\eta_{K_i}}{\eta} \times 100\%, & \tau_{\gamma_l} &= \frac{\eta_{\gamma_l}}{\eta} \times 100\% \\ \widetilde{t}_{K_i} &= \frac{||e||_{\widetilde{H}^{1,\beta}(K_i)}}{||e||_{\widetilde{H}^{1,\beta}(\mathcal{T})}}, & t_{K_i} &= \frac{||e||_{E(K_i)}}{||e||_{E(\Omega)}} \end{aligned} \tag{6.1.2}$$

and these values are presented in Figures 6.4, 6.6, 6.8, 6.10, 6.12, 6.14, 6.16, Tables 6.3, 6.7, 6.11, 6.15, 6.19, 6.23, 6.27 and Tables 6.4, 6.8, 6.12, 6.16, 6.20, 6.24, 6.28, respectively.

Note that

$$\sum_{K \in \mathcal{T}} \tau_{K_i} + \sum_{\gamma_l \in \partial \mathcal{T}} \tau_{\gamma_l} \neq 1, \quad \sum_{K \in \mathcal{T}} \widetilde{t}_{K_i} \neq 1, \quad \sum_{K \in \mathcal{T}} t_{K_i} \neq 1\,.$$

Instead, based on the definition of η, $||e||_{\widetilde{H}^{1,\beta}(\mathcal{T})}$ and $||e||_{E(\Omega)}$, we have

$$\sum_{K \in \mathcal{T}} \tau_{K_i}^2 + \sum_{\gamma_l \in \partial \mathcal{T}} \tau_{\gamma_l}^2 = 1, \quad \sum_{K \in \mathcal{T}} \widetilde{t}_{K_i}^2 = 1, \quad \sum_{K \in \mathcal{T}} t_{K_i}^2 = 1\,.$$

0.00205551	0.0153215	**0.0142875**	0.0176256	**0.0113486**	0.00737485	**0.00240721**
0.0160047		0.0215846		0.00766313		0.00574338
0.017903	0.0206709	**0.0816136**	0.153881	**0.0347141**	0.00986567	**0.00352263**

Figure 6.3: η_K and η_γ for $p = 2$

0.062253	0.0673616	0.0445521	0.0268887
0.0771089	0.138254	0.128248	0.0240147

Table 6.1: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 2$

0.0819022	0.0841319	0.0563723	0.0348257
0.097471	0.148208	0.133271	0.0274847

Table 6.2: $||e||_{E(K)}$ for $p = 2$

1.11216	8.28988	**7.73044**	9.53653	**6.1403**	3.99025	**1.30245**
8.65953		11.6786		4.14623		3.10752
9.6866	11.1842	**44.158**	83.2589	**18.7825**	5.33793	**1.90596**

Figure 6.4: τ_K and τ_γ for $p = 2$

26.9888	29.2035	19.3148	11.6572
33.4293	59.9378	55.6	10.4112

Table 6.3: $\widetilde{t}_K$ for $p = 2$

31.3712	32.2253	21.5924	13.3394
37.3346	56.7685	51.0474	10.5275

Table 6.4: t_K for $p = 2$

0.0016485	0.000978504	**0.0159361**	0.027961	**0.0094865**	0.00299674	**0.000683156**
0.00265262		0.0133316		0.00826192		0.00170339
0.0178628	0.011389	**0.0808944**	0.130139	**0.0343132**	0.0107834	**0.00498719**

Figure 6.5: η_K and η_γ for $p = 3$

0.0577404	0.0653722	0.0416806	0.024773
0.0736998	0.144168	0.126225	0.0240653

Table 6.5: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 3$

0.0761995	0.0799543	0.0524914	0.0323601
0.0919296	0.144423	0.128649	0.0264005

Table 6.6: $||e||_{E(K)}$ for $p = 3$

1.00996	0.599484	**9.7633**	17.1304	**5.81193**	1.83596	**0.418538**
1.62514		8.16767		5.06169		1.04359
10.9437	6.97751	**49.5602**	79.7301	**21.0221**	6.6065	**3.05542**

Figure 6.6: τ_K and τ_γ for $p = 3$

25.1553	28.4802	18.1587	10.7927
32.1082	62.8087	54.9915	10.4844

Table 6.7: $\widetilde{t}_K$ for $p = 3$

30.4656	31.9668	20.9868	12.938
36.7547	57.7424	51.4356	10.5553

Table 6.8: t_K for $p = 3$

0.00112915	0.00654875	**0.0100464**	0.00864694	**0.00367536**	0.00290205	**0.000520899**
0.00668174		0.0152177		0.00795801		0.00258117
0.0095861	0.01601	**0.0657881**	0.09711	**0.02755**	0.00621217	**0.00474814**

Figure 6.7: η_K and η_γ for $p = 4$

0.0398771	0.0383883	0.0267104	0.0171999
0.044502	0.116934	0.100272	0.0143763

Table 6.9: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 4$

0.0526013	0.048658	0.0347171	0.0223625
0.057682	0.113682	0.103133	0.0156734

Table 6.10: $||e||_{E(K)}$ for $p = 4$

0.906514	5.25753	**8.06553**	6.94201	**2.95069**	2.32985	**0.418193**
5.36429		12.2173		6.38892		2.07224
7.696	12.8533	**52.8166**	77.9628	**22.118**	4.98731	**3.81195**

Figure 6.8: τ_K and τ_γ for $p = 4$

23.0275	22.1678	15.4243	9.93231
25.6982	67.5252	57.9033	8.30177

Table 6.11: $\widetilde{t}_K$ on each element for $p = 4$

28.5385	26.3991	18.8355	12.1326
31.2951	61.6775	55.9541	8.50354

Table 6.12: t_K for $p = 4$

0.00120769	0.0054946	**0.00652429**	0.0029481	**0.0022054**	0.00206537	**0.000501121**
0.00533873		0.0111848		0.00526194		0.00244015
0.0060236	0.0116165	**0.0597709**	0.0773785	**0.0249137**	0.00422426	**0.00333862**

Figure 6.9: η_K and η_γ for $p = 5$

0.0269914	0.0264037	0.0183277	0.0116936
0.0306361	0.093707	0.0795703	0.0097204

Table 6.13: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 5$

0.0355358	0.0337599	0.0236806	0.0151318
0.0397105	0.0894491	0.0835399	0.0111163

Table 6.14: $||e||_{E(K)}$ for $p = 5$

1.16968	5.32168	**6.31896**	2.85532	**2.13599**	2.00037	**0.48535**
5.17072		10.8328		5.09634		2.36336
5.83403	11.2509	**57.8898**	74.9433	**24.1296**	4.09132	**3.23355**

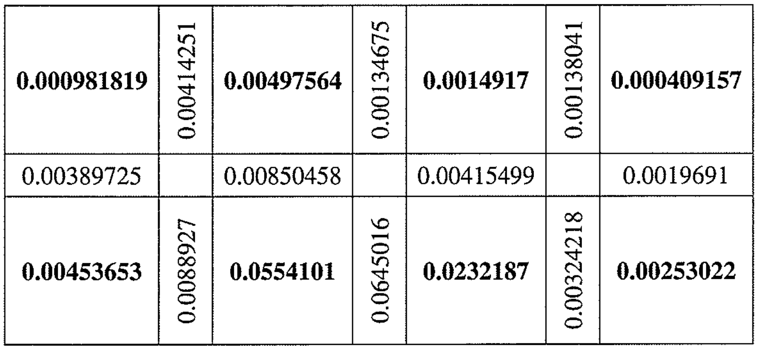
Figure 6.10: τ_K and τ_γ for $p = 5$

20.0937	19.6562	13.6441	8.70528
22.8071	69.7602	59.2361	7.23636

Table 6.15: $\widetilde{t}_K$ on each element for $p = 5$

25.2079	23.9481	16.7982	10.734
28.1693	63.4521	59.2604	7.88553

Table 6.16: t_K for $p = 5$



0.000981819	0.00414251	**0.00497564**	0.00134675	**0.0014917**	0.00138041	**0.000409157**
0.00389725		0.00850458		0.00415499		0.0019691
0.00453653	0.0088927	**0.0554101**	0.0645016	**0.0232187**	0.00324218	**0.00253022**

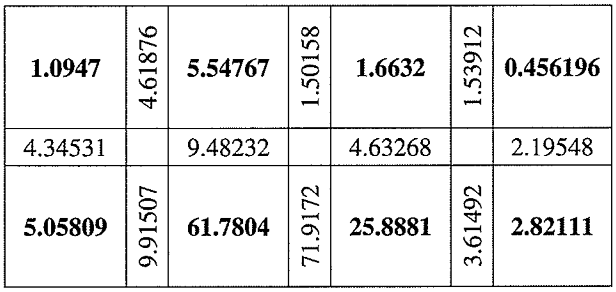
Figure 6.11: η_K and η_γ for $p = 6$

0.019687	0.0190889	0.0131062	0.00852779
0.0220409	0.0781783	0.0653423	0.00709607

Table 6.17: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 6$

0.0259892	0.0242865	0.0169483	0.0110624
0.0285185	0.0739138	0.0698951	0.00798599

Table 6.18: $||e||_{E(K)}$ for $p = 6$



1.0947	4.61876	**5.54767**	1.50158	**1.6632**	1.53912	**0.456196**
4.34531		9.48232		4.63268		2.19548
5.05809	9.91507	**61.7804**	71.9172	**25.8881**	3.61492	**2.82111**

Figure 6.12: τ_K and τ_γ for $p = 6$

18.0364	17.4884	12.0073	7.81279
20.1929	71.6236	59.8638	6.50112

Table 6.19: $\widetilde{t}_K$ on each element for $p = 6$

22.8817	21.3826	14.9219	9.73966
25.1086	65.0761	61.5379	7.03113

Table 6.20: t_K for $p = 6$

0.000838981	0.00313274	**0.00400933**	0.000910583	**0.00117316**	0.000990772	**0.000348653**
0.00290899		0.00657553		0.00332594		0.00152755
0.00362586	0.00690563	**0.0516474**	0.0552168	**0.0218179**	0.00257134	**0.00207466**

Figure 6.13: η_K and η_γ for $p = 7$

0.0148932	0.0144196	0.00994571	0.00644847
0.0166818	0.0669735	0.0551734	0.00534437

Table 6.21: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 7$

0.0196207	0.0184823	0.0128702	0.00835033
0.0216767	0.0629337	0.0599839	0.00611195

Table 6.22: $||e||_{E(K)}$ for $p = 7$

1.05209	3.92849	**5.02775**	1.14188	**1.47116**	1.24244	**0.437215**
3.64791		8.2458		4.17077		1.91557
4.54687	8.65974	**64.7665**	69.2425	**27.3599**	3.22449	**2.60165**

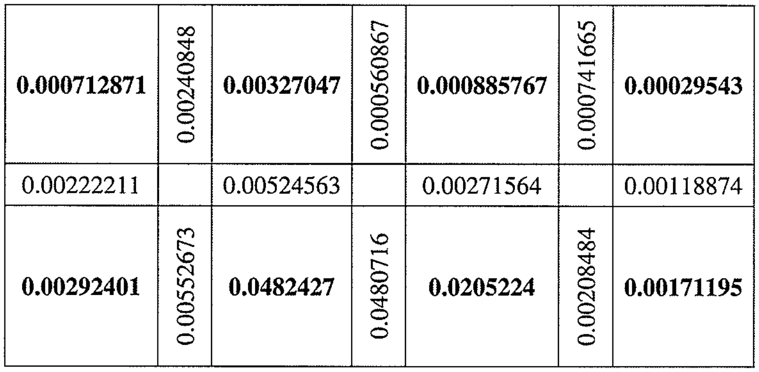
Figure 6.14: τ_K and τ_γ for $p = 7$

16.2434	15.7269	10.8474	7.03309
18.1941	73.0452	60.1754	5.82888

Table 6.23: $\widetilde{t}_K$ for $p = 7$

20.6501	19.4519	13.5454	8.78841
22.814	66.2354	63.1308	6.4326

Table 6.24: t_K for $p = 7$



0.000712871	0.00240848	**0.00327047**	0.000560867	**0.000885767**	0.000741665	**0.00029543**
0.00222211		0.00524563		0.00271564		0.00118874
0.00292401	0.00552673	**0.0482427**	0.0480716	**0.0205224**	0.00208484	**0.00171195**

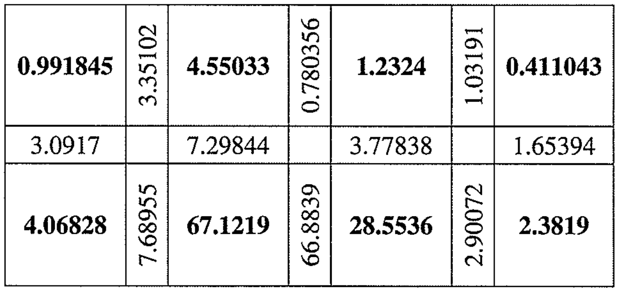
Figure 6.15: η_K and η_γ for $p = 8$

0.0116847	0.0112559	0.00777803	0.00506434
0.0130162	0.0583031	0.0475002	0.00421589

Table 6.25: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 8$

0.0154217	0.0144135	0.0100736	0.00656477
0.0169248	0.0548311	0.0525131	0.0047724

Table 6.26: $||e||_{E(K)}$ for $p = 8$



0.991845	3.35102	**4.55033**	0.780356	**1.2324**	1.03191	**0.411043**
3.0917		7.29844		3.77838		1.65394
4.06828	7.68955	**67.1219**	66.8839	**28.5536**	2.90072	**2.3819**

Figure 6.16: τ_K and τ_γ for $p = 8$

14.8491	14.3042	9.88442	6.43583
16.5411	74.0923	60.3639	5.35761

Table 6.27: $\widetilde{t}_K$ for $p = 8$

18.8922	17.6571	12.3405	8.04211
20.7336	67.1703	64.3307	5.84638

Table 6.28: t_K for $p = 8$

Based on the distribution, the error and error indicators concentrate on the element K_2, K_3 and internal edge γ_6 because of the singularity at the origin, which is shared by element K_2, K_3 and edge γ_6. Table 6.29 indicates that the distribution of $\tau_{K_2}+\tau_{K_3}+\tau_{\gamma_6}$ reflects this nature quite well and tells us precisely where the big error occurs.

p	$\tau_{K_2}+\tau_{K_3}+\tau_{\gamma_6}$	$\tilde{t}_{K_2}+\tilde{t}_{K_3}$	$t_{K_2}+t_{K_3}$
2	146.1994	115.5378	107.8159
3	150.3124	117.8002	109.178
4	152.8974	125.4285	117.6316
5	156.9627	128.9963	122.7125
6	159.5857	131.4874	126.614
7	161.3689	133.2206	129.3662
8	162.5594	134.4562	131.501

Table 6.29: Error concentration for $2 \le p \le 8$

For the predicability of error estimator η and the error indicator η_{K_i} and η_{γ_l}, we compute

$$\begin{aligned}
S(p) &= \frac{\|e(p+1)\|_{\tilde{H}^{1,\beta}(\mathcal{T})}}{\eta(p+1)} \Big/ \frac{\|e(p)\|_{\tilde{H}^{1,\beta}(\mathcal{T})}}{\eta(p)}, \\
S_{K_i}(p) &= \frac{\|e(p+1)\|_{\tilde{H}^{1,\beta}(K_i)}}{\eta_{K_i}(p+1)} \Big/ \frac{\|e(p)\|_{\tilde{H}^{1,\beta}(K_i)}}{\eta_{K_i}(p)}, \\
S_{\gamma_l}(p) &= \frac{\sum_{K\in Q_{\gamma_l}} \|e(p+1)\|_{\tilde{H}^{1,\beta}(K)}}{\eta_{\gamma_l}(p+1)} \Big/ \frac{\sum_{K\in Q_{\gamma_l}} \|e(p)\|_{\tilde{H}^{1,\beta}(K)}}{\eta_{\gamma_l}(p)},
\end{aligned} \tag{6.1.3}$$

where $e(p)$, $\eta(p)$, $\eta_{K_i}(p)$ and $\eta_{\gamma_l}(p)$ denote the error, the error estimator and the error indicators for the polynomial degree p, respectively.

The numerical results are presented in Table 6.30 and plotted in Figures 6.17, 6.18 and 6.19.

p	S	S_{K_2}	S_{K_5}	$S_{\gamma 6}$	$S_{\gamma 16}$
2	1.126783811	1.052048099	3.246398868	1.199696284	2.289160109
3	.9886294142	.9973403360	.9105717737	1.076516198	.6823267329
4	.9357894510	.8820382123	.7066932770	1.001175834	.9606564094
5	.9354342813	.8999406801	.8931895709	.9936275867	1.078194765
6	.9447576007	.9190871075	.8873930066	.9941844526	1.055818939
7	.9522252277	.9319832812	.9268412686	.9949460475	1.046455832

Table 6.30: $S(p)$, $S_K(p)$ and $S_\gamma(p)$ for $2 \leq p \leq 8$

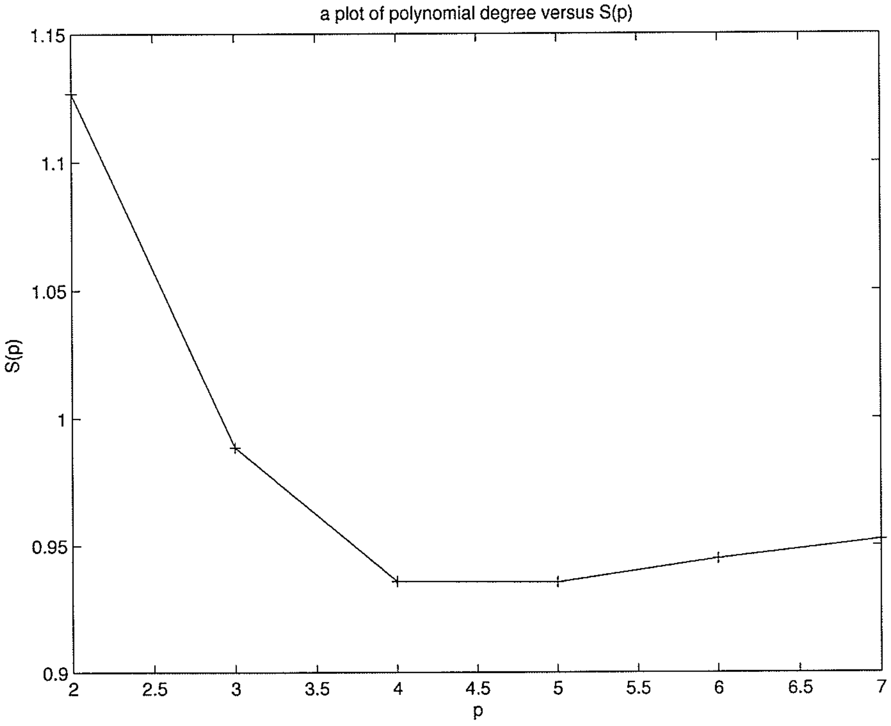


Figure 6.17: p versa $S(p)$, $2 \leq p \leq 7$

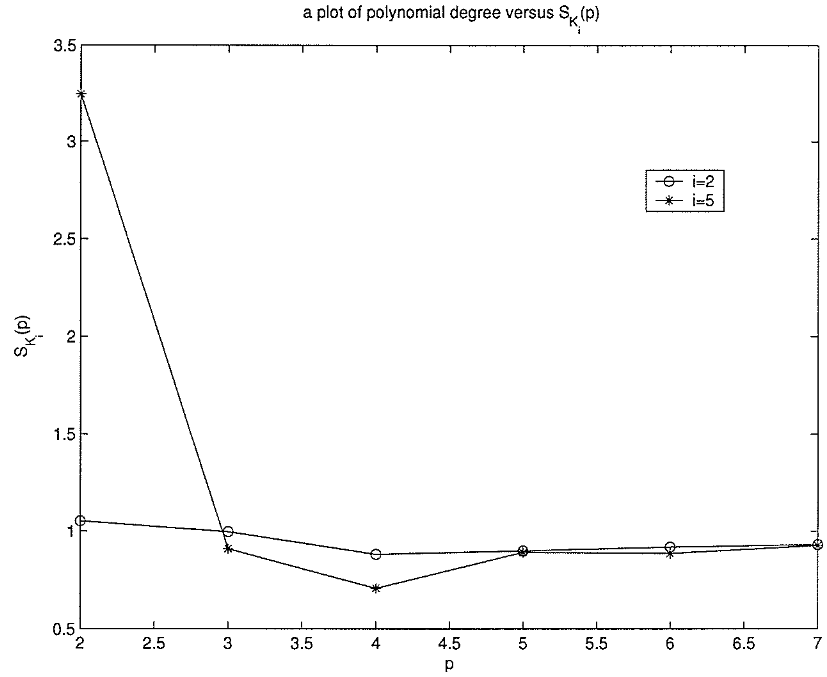


Figure 6.18: p versa $S_{K_i}(p)$, $2 \leq p \leq 7$, $i = 2, 5$

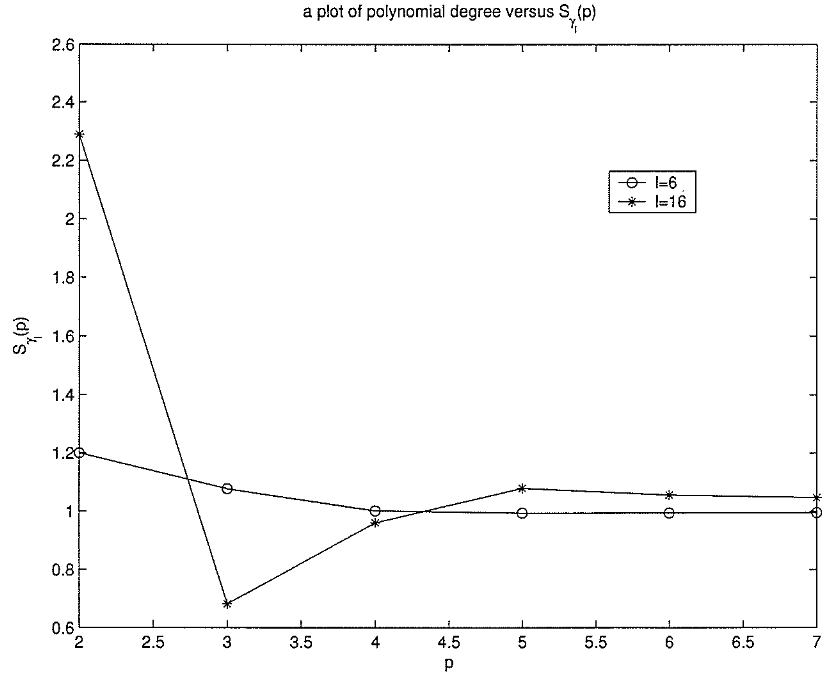


Figure 6.19: p versa $S_{\gamma_l}(p)$, $2 \leq p \leq 7$, $l = 6, 16$

According to Theorem ?? and Theorem ??, we have

$$\begin{aligned} S(p) &\approx \frac{(p+1)^{\epsilon}\log^{1/2}(p+2)}{p^{\epsilon}\log^{1/2}(p+1)}, \\ S_K(p) &\approx \frac{(p+1)^{\beta-1}}{p^{\beta-1}}, \\ S_{\gamma}(p) &\approx 1\,. \end{aligned}$$

The tables and figures above imply that the relationship between η and $\|e\|_{\tilde{H}^{1,\beta}(\mathcal{T})}$, η_K and $\|e\|_{\tilde{H}^{1,\beta}(K)}$, η_γ and $\sum_{K\in Q_\gamma}\|e\|_{\tilde{H}^{1,\beta}(K)}$ given in Section ?? of Chapter 2 is numerically stable and reliable.

6.2 Model 2

The second model problem is as follows

$$\begin{cases} -\triangle u + u = f & \text{on } \Omega = [\alpha, 1+\alpha]\times[\alpha, 1+\alpha] \\ \dfrac{\partial u}{\partial n} = g_i & \text{on } \Gamma_i, i = 1, \cdots, 4 \end{cases} \tag{6.2.1}$$

where $\mathbf{n}$ is the unit normal outwards to Ω, with

$$f = r^{1/2}\sin\frac{\theta}{2},$$

and

$$\begin{aligned} g_1 &= -\frac{1}{2}r^{-1/2}\cos\frac{\theta}{2}, \quad \text{on } \Gamma_1 = \{(x,\alpha)\,|\,\alpha < x < 1+\alpha\}; \\ g_2 &= -\frac{1}{2}r^{-1/2}\sin\frac{\theta}{2}, \quad \text{on } \Gamma_2 = \{(1+\alpha,y)\,|\,\alpha < y < 1+\alpha\}; \\ g_3 &= \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2}, \quad \text{on } \Gamma_3 = \{(x,1+\alpha)\,|\,\alpha < x < 1+\alpha\}; \\ g_4 &= \frac{1}{2}r^{-1/2}\sin\frac{\theta}{2}, \quad \text{on } \Gamma_4 = \{(\alpha,y)\,|\,\alpha < y < 1+\alpha\}; \end{aligned}$$

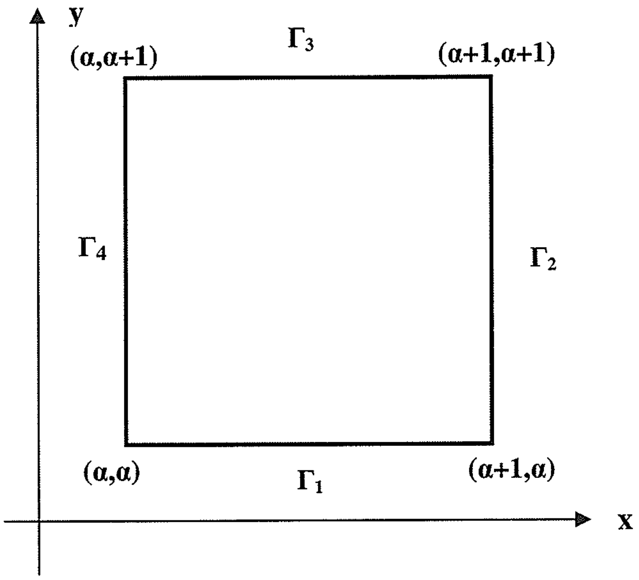


Figure 6.20: domain for model problem 2

The domain is shown in the Figure 6.20 and the exact solution to this PDE is the same as that of model problem 1.

The mesh $\mathcal{T}$ used for computation is shown in Figure 6.21, where the element number is within the ellipse on each element and the global side number is within the rectangle on each edge.

We compute η_{K_i}, $1 \leq i \leq 16$, and η_{γ_l}, $l = 2, 3, 6, 7, 9, 10, 13, 15, 16, 17, 18, 19, 20, 21, 23, 24, 26, 27, 28, 29, 31, 34, 36, 38$, for $2 \leq p \leq 8$. The values are presented in Figures 6.22, 6.24, 6.26, 6.28, 6.30, 6.32 and 6.34.

Also we compute $||e||_{\tilde{H}^{1,\beta}(K_i)}$ and $||e||_{E(K_i)}$, $1 \leq i \leq 16$, whose values are presented in Tables 6.31, 6.35, 6.39, 6.43, 6.47, 6.51, 6.55 and Tables 6.33, 6.37, 6.41, 6.45, 6.49, 6.53, 6.57.

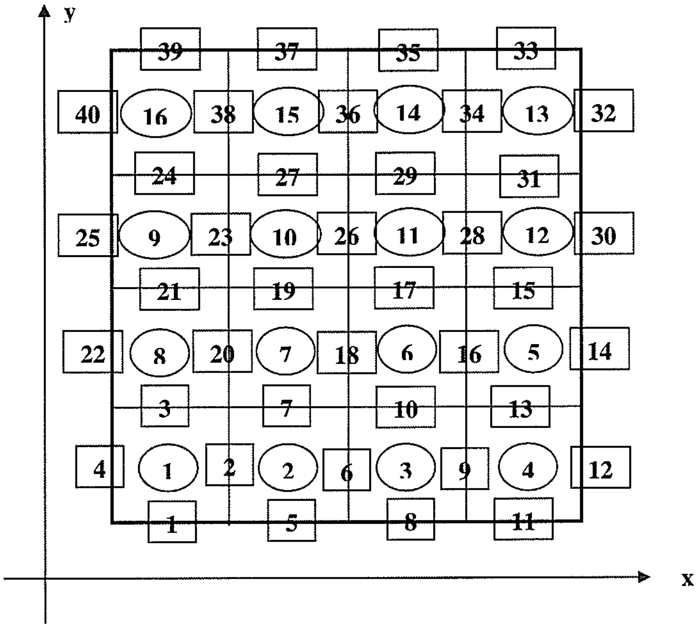


Figure 6.21: Mesh $\mathcal{T}$ for model problem 2

The distribution τ_{K_i}, τ_{γ_l}, $\widetilde{t}_{K_i}$ and t_{K_i} defined in (6.1.2) are computed as well, and these values are presented in Figures 6.23, 6.25, 6.27, 6.29, 6.31, 6.33, 6.35, Tables 6.32, 6.36, 6.40, 6.44, 6.2, 6.52, 6.56 and Tables 6.34, 6.38, 6.42, 6.46, 6.50, 6.54, 6.58, respectively.

0.000342162	7.94793e-005	**0.000266836**	3.72234e-005	**0.000196847**	1.25909e-005	**0.000137967**
0.000199119		8.88495e-005		2.80034e-005		2.47598e-005
0.000506587	0.000518301	**0.0005122**	8.45822e-005	**0.00032162**	7.75068e-005	**0.000193559**
0.000785621		0.000432655		0.000118199		8.0436e-005
0.00296205	0.000916989	**0.00122122**	0.000272489	**0.000498495**	0.000141328	**0.000248095**
0.00894462		0.00205965		0.000175559		0.000150414
0.0157375	0.00971982	**0.00150903**	0.000656185	**0.000766368**	0.000123131	**0.000305875**

Figure 6.22: η_K and η_γ for $p = 2$

1.62488	0.377435	**1.26717**	0.176768	**0.934798**	0.0597924	**0.655183**
0.945588		0.421933		0.132984		0.11758
2.40571	2.46133	**2.43236**	0.401669	**1.52733**	0.368068	**0.919182**
3.7308		2.05461		0.561308		0.381978
14.0663	4.35465	**5.79941**	1.29401	**2.36728**	0.671144	**1.17817**
42.4767		9.78098		0.833705		0.714293
74.7349	46.158	**7.16618**	3.11613	**3.63937**	0.584731	**1.45255**

Figure 6.23: τ_K and τ_γ for $p = 2$

0.00282583	0.00274616	0.00262364	0.00254369
0.00332238	0.00295453	0.00276	0.00262745
0.00484801	0.00387123	0.00296162	0.0027482
0.0367811	0.00529667	0.00325707	0.00283747

Table 6.31: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 2$

7.25395	7.04943	6.73493	6.52969
8.5286	7.58432	7.08497	6.7447
12.4449	9.93751	7.60254	7.05468
94.4178	13.5966	8.36096	7.28382

Table 6.32: $\widetilde{t}_K$ for $p = 2$

0.00370628	0.00360363	0.00344805	0.00334609
0.00422531	0.00385248	0.00362327	0.00345441
0.00558832	0.00493806	0.00387497	0.00361243
0.0399905	0.00650857	0.00420531	0.00372979

Table 6.33: $||e||_{E(K)}$ for $p = 2$

8.5835	8.34576	7.98545	7.74932
9.78551	8.92207	8.39125	8.00018
12.9422	11.4362	8.97415	8.36614
92.6153	15.0734	9.73921	8.63794

Table 6.34: t_K for $p = 2$

8.05185e-005	0.00012036	**2.55654e-005**	1.83253e-005	**6.21882e-006**	3.29232e-006	**2.31424e-006**
6.32966e-005		5.64083e-005		2.07681e-005		1.36174e-005
0.000338084	0.000442771	**6.41849e-005**	5.32452e-005	**2.04925e-005**	2.34784e-005	**9.31826e-006**
0.000705423		0.000264539		5.13941e-005		3.67354e-005
0.00208468	0.00210342	**0.000224859**	0.000169124	**6.20725e-005**	8.66915e-005	**2.43258e-005**
0.00361266		0.00144187		0.000357914		9.27173e-005
0.00891844	0.00609195	**0.00332293**	0.000312757	**0.000773**	0.000102506	**0.00017985**

Figure 6.24: η_K and η_γ for $p = 3$

0.649938	0.971537	**0.206361**	0.14792	**0.0501977**	0.0265753	**0.0186803**
0.510924		0.455322		0.167638		0.109919
2.72898	3.574	**0.518094**	0.429791	**0.165414**	0.189516	**0.0752161**
5.6941		2.13534		0.414849		0.296525
16.8273	16.9786	**1.81504**	1.36515	**0.501044**	0.699765	**0.196356**
29.161		11.6386		2.88905		0.748405
71.9888	49.1736	**26.8224**	2.52454	**6.23958**	0.827416	**1.45173**

Figure 6.25: τ_K and τ_γ for $p = 3$

0.000929739	0.000901725	0.000861227	0.000836531
0.0013539	0.00093417	0.000909605	0.000864477
0.00351015	0.00156367	0.000982738	0.000902339
0.0246713	0.00546885	0.00152679	0.000953094

Table 6.35: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 3$

3.6033	3.49473	3.33778	3.24206
5.24718	3.62048	3.52527	3.35037
13.604	6.06017	3.80871	3.49711
95.6161	21.1951	5.91724	3.69382

Table 6.36: $\widetilde{t}_K$ for $p = 3$

0.00122212	0.00118785	0.00113539	0.00110378
0.00162998	0.00122538	0.00120023	0.00114021
0.00342276	0.00193567	0.0012923	0.00118813
0.0249282	0.00559692	0.00170007	0.00124175

Table 6.37: $||e||_{E(K)}$ for $p = 3$

4.65919	4.52854	4.32854	4.20802
6.21413	4.67163	4.57574	4.34691
13.0489	7.37953	4.92674	4.5296
95.036	21.3376	6.48131	4.73404

Table 6.38: t_K for $p = 3$

0.000148076	2.32498e-005	**5.45596e-006**	2.99346e-006	**4.0792e-006**	2.86768e-006	**1.1271e-006**
5.23934e-005		2.09923e-005		2.60031e-006		8.23569e-007
0.000603147	8.57779e-005	**3.91485e-005**	2.68264e-005	**4.52328e-006**	3.73762e-006	**4.75728e-006**
0.000286832		7.15145e-005		6.74316e-006		1.9702e-006
0.00278258	0.000680347	**8.60736e-005**	6.71979e-005	**5.83553e-005**	2.64193e-005	**1.75517e-005**
0.00230999		0.000680919		0.000150943		4.01919e-005
0.0117097	0.00148637	**0.0034466**	0.000263489	**0.000801872**	7.7317e-005	**0.000187235**

Figure 6.26: η_K and η_γ for $p = 4$

1.14759	0.180187	**0.0422839**	0.0231995	**0.031614**	0.0222246	**0.00873509**
0.406051		0.162691		0.0201525		0.00638269
4.67442	0.664783	**0.303402**	0.207906	**0.0350556**	0.0289667	**0.0368691**
2.22296		0.55424		0.0522598		0.0152692
21.5651	5.27272	**0.667074**	0.520787	**0.452256**	0.204751	**0.136026**
17.9025		5.27715		1.16981		0.311489
90.7506	11.5194	**26.7113**	2.04205	**6.21454**	0.59921	**1.45108**

Figure 6.27: τ_K and τ_γ for $p = 4$

0.000310635	0.000249179	0.000236757	0.000229961
0.000831795	0.000266471	0.000248054	0.00023754
0.00393506	0.000433555	0.000306289	0.000246949
0.0215585	0.00491682	0.00119518	0.000374143

Table 6.39: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 4$

1.37891	1.10611	1.05097	1.0208
3.69235	1.18287	1.10112	1.05444
17.4678	1.92456	1.35962	1.09621
95.6986	21.8259	5.3054	1.66083

Table 6.40: $\widetilde{t}_K$ for $p = 4$

0.000379959	0.000328592	0.000312496	0.000303661
0.000826219	0.00034598	0.00032698	0.000313547
0.0037331	0.000504937	0.000394295	0.000324806
0.0199146	0.00456732	0.00114507	0.000426585

Table 6.41: $||e||_{E(K)}$ for $p = 4$

1.82209	1.57575	1.49857	1.4562
3.96212	1.65914	1.56803	1.50361
17.902	2.42142	1.89084	1.5576
95.5	21.9025	5.49114	2.04568

Table 6.42: t_K for $p = 4$

0.000117589	1.06092e-005	**2.62646e-006**	1.38693e-006	**3.80724e-007**	2.25499e-007	**1.309e-007**
2.20592e-005		7.80608e-006		6.00101e-007		2.94941e-007
0.000480541	5.29945e-005	**1.86567e-005**	1.38545e-005	**1.11583e-006**	1.21258e-007	**3.89989e-007**
0.000118062		1.12536e-005		2.57583e-006		1.00073e-006
0.00193105	0.00018462	**9.16431e-005**	3.87331e-005	**1.02021e-005**	1.03743e-005	**3.18023e-006**
0.0024779		0.000814391		0.000114724		3.7244e-005
0.00861281	0.00178672	**0.00179722**	0.000143786	**0.000464755**	2.78925e-005	**0.000109025**

Figure 6.28: η_K and η_γ for $p = 5$

1.22791	0.110785	**0.0274266**	0.0144829	**0.00397568**	0.00235475	**0.00136692**
0.230351		0.0815143		0.0062665		0.00307989
5.01801	0.55339	**0.194821**	0.144675	**0.0116519**	0.00126623	**0.00407243**
1.23285		0.117515		0.0268979		0.01045
20.1648	1.92788	**0.956975**	0.404467	**0.106535**	0.108332	**0.0332093**
25.8753		8.50421		1.19799		0.388918
89.9386	18.6577	**18.7674**	1.50147	**4.85317**	0.291265	**1.13849**

Figure 6.29: τ_K and τ_γ for $p = 5$

0.0001417	4.62104e-005	4.39916e-005	4.26771e-005
0.000552283	6.18345e-005	4.60984e-005	4.39538e-005
0.00225151	0.000199778	5.22483e-005	4.72016e-005
0.0164563	0.00240043	0.000574652	0.000142387

Table 6.43: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 5$

0.843247	0.274996	0.261792	0.25397
3.28661	0.367974	0.274329	0.261567
13.3986	1.18887	0.310927	0.280895
97.9307	14.2848	3.41972	0.847337

Table 6.44: $\widetilde{t}_K$ for $p = 5$

0.000144092	6.06452e-005	5.8105e-005	5.63862e-005
0.000521296	7.71081e-005	6.07325e-005	5.80542e-005
0.0021444	0.000225698	6.69119e-005	6.22134e-005
0.0145892	0.00237911	0.000537793	0.000143571

Table 6.45: $||e||_{E(K)}$ for $p = 5$

0.963204	0.405393	0.388413	0.376923
3.48469	0.515443	0.405977	0.388073
14.3346	1.50872	0.447284	0.415876
97.5243	15.9036	3.59497	0.959722

Table 6.46: t_K for $p = 5$

6.96339e-005	1.59029e-005	**9.03488e-007**	1.15381e-007	**1.37976e-007**	1.30861e-007	**4.98718e-008**
1.14791e-005		2.75366e-006		3.95924e-007		4.901e-008
0.00028509	4.61321e-005	**4.96929e-006**	4.03388e-006	**1.28332e-006**	7.10539e-007	**3.10128e-007**
8.1799e-005		2.26417e-005		3.52958e-006		5.00559e-007
0.00108559	0.000279739	**5.30434e-005**	2.65653e-005	**9.20241e-006**	3.98114e-006	**2.34465e-006**
0.00169681		0.000589241		5.6765e-005		2.29843e-005
0.00659372	0.00156621	**0.000989878**	7.23466e-005	**0.000215804**	1.15064e-005	**4.8797e-005**

Figure 6.30: η_K and η_γ for $p = 6$

0.969847	0.221492	**0.0125836**	0.001607	**0.00192171**	0.00182261	**0.000694605**
0.159878		0.0383524		0.00551435		0.000682602
3.97068	0.642519	**0.0692113**	0.0561831	**0.0178739**	0.00989625	**0.0043194**
1.13928		0.315349		0.0491593		0.00697168
15.1199	3.89615	**0.738778**	0.369997	**0.128169**	0.0554485	**0.0326558**
23.6329		8.20683		0.790611		0.320121
91.836	21.8138	**13.7868**	1.00763	**3.00567**	0.160259	**0.679635**

Figure 6.31: τ_K and τ_γ for $p = 6$

7.04359e-005	4.47632e-006	3.81945e-006	3.70314e-006
0.000290141	1.50869e-005	4.73772e-006	3.84959e-006
0.00126689	0.000118688	1.81018e-005	5.49674e-006
0.0118529	0.00131476	0.00024306	5.29514e-005

Table 6.47: $\|e\|_{\widetilde{H}^{1,\beta}(K)}$ for $p = 6$

0.586985	0.0373039	0.0318298	0.0308605
2.41792	0.125728	0.0394823	0.032081
10.5578	0.989102	0.150853	0.0458077
98.7774	10.9567	2.02557	0.441276

Table 6.48: $\widetilde{t}_K$ for $p = 6$

6.91553e-005	5.67196e-006	5.0363e-006	4.89432e-006
0.000271564	1.68583e-005	5.97221e-006	5.07617e-006
0.00129823	0.000129616	1.95279e-005	6.74193e-006
0.0101627	0.00132804	0.000230515	5.21314e-005

Table 6.49: $\|e\|_{E(K)}$ for $p = 6$

0.668921	0.0548634	0.0487148	0.0473415
2.62677	0.163066	0.0577676	0.0491004
12.5574	1.25374	0.188889	0.0652129
98.3009	12.8458	2.22971	0.504254

Table 6.50: t_K for $p = 6$

3.63091e-005	1.04039e-005	**3.53603e-007**	1.27436e-007	**8.30145e-008**	5.20856e-008	**1.48044e-008**
5.47877e-006		1.306e-006		1.75209e-007		2.23495e-008
0.000155532	2.64326e-005	**3.5981e-006**	2.40462e-006	**5.93255e-007**	8.83903e-008	**9.89536e-008**
3.22975e-005		5.64665e-006		1.65407e-006		1.51289e-007
0.000632004	0.000231238	**3.1453e-005**	5.06592e-006	**2.83102e-006**	1.67924e-006	**1.01036e-006**
0.0011626		0.000379187		2.44524e-005		1.13317e-005
0.00508993	0.00115753	**0.000587078**	3.94665e-005	**0.000101238**	4.96823e-006	**2.02705e-005**

Figure 6.32: η_K and η_γ for $p = 7$

0.667581	0.191286	**0.00650136**	0.00234305	**0.00152631**	0.000957648	**0.000272195**
0.100733		0.0240121		0.0032214		0.000410919
2.85962	0.485991	**0.0661549**	0.0442115	**0.0109076**	0.00162515	**0.00181936**
0.593822		0.10382		0.0304118		0.00278161
11.6201	4.25156	**0.578297**	0.0931422	**0.0520513**	0.0308745	**0.0185765**
21.3757		6.97174		0.449582		0.208345
93.5837	21.2825	**10.794**	0.725633	**1.86136**	0.0913461	**0.372694**

Figure 6.33: τ_K and τ_γ for $p = 7$

3.47264e-005	1.01003e-005	9.61214e-006	9.3284e-006
0.000145883	1.27302e-005	1.01186e-005	9.61522e-006
0.000703166	6.74837e-005	1.16592e-005	1.0168e-005
0.00846383	0.00076112	0.000110465	2.28874e-005

Table 6.51: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 7$

0.407137	0.118418	0.112694	0.109367
1.71035	0.149251	0.118632	0.11273
8.24402	0.791188	0.136694	0.119211
99.2311	8.92348	1.2951	0.268335

Table 6.52: $\widetilde{t}_K$ for $p = 7$

3.5647e-005	1.33424e-005	1.27037e-005	1.2331e-005
0.000137018	1.61845e-005	1.33639e-005	1.27084e-005
0.000729829	7.49092e-005	1.49303e-005	1.3367e-005
0.00706416	0.00077891	0.000109451	2.43487e-005

Table 6.53: $||e||_{E(K)}$ for $p = 7$

0.49876	0.186682	0.177745	0.172531
1.91711	0.226448	0.186983	0.177811
10.2115	1.0481	0.2089	0.187027
98.8391	10.8982	1.5314	0.340679

Table 6.54: t_K for $p = 7$

1.82965e-005	6.11128e-006	**2.21715e-007**	4.41755e-008	**3.47436e-008**	2.46412e-008	**7.27935e-009**
2.70051e-006		6.31944e-007		3.70101e-008		1.21307e-008
8.49632e-005	1.36262e-005	**1.61083e-006**	1.26915e-006	**3.52229e-007**	7.3956e-008	**4.45098e-008**
2.3083e-005		3.22874e-006		9.96783e-007		6.74906e-008
0.000392897	0.000168916	**2.22315e-005**	4.47185e-006	**1.92939e-006**	8.1788e-007	**5.01814e-007**
0.000772301		0.000232948		1.058e-005		5.34254e-006
0.0039366	0.00079118	**0.000380391**	2.56605e-005	**5.15282e-005**	2.5576e-006	**9.04788e-006**

Figure 6.34: η_K and η_γ for $p = 8$

0.442281	0.147728	**0.00535951**	0.00106786	**0.000839857**	0.000595652	**0.000175964**
0.0652794		0.015276		0.000894646		0.000293236
2.05382	0.329386	**0.0389386**	0.0306793	**0.00851443**	0.00178774	**0.00107594**
0.557987		0.0780485		0.0240953		0.00163145
9.4975	4.08321	**0.537404**	0.108098	**0.0466392**	0.0197706	**0.0121304**
18.6689		5.63105		0.25575		0.129145
95.1595	19.1252	**9.19521**	0.620291	**1.24559**	0.0618249	**0.218715**

Figure 6.35: τ_K and τ_γ for $p = 8$

7.04359e-005	4.47632e-006	3.81945e-006	3.70314e-006
0.000290141	1.50869e-005	4.73772e-006	3.84959e-006
0.00126689	0.000118688	1.81018e-005	5.49674e-006
0.0118529	0.00131476	0.00024306	5.29514e-005

Table 6.55: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 8$

0.279059	0.112242	0.107145	0.103982
1.23632	0.126874	0.112621	0.107161
7.13158	0.732986	0.129456	0.112724
99.4368	7.63847	0.911057	0.183205

Table 6.56: $\widetilde{t}_K$ for $p = 8$

6.91553e-005	5.67196e-006	5.0363e-006	4.89432e-006
0.000271564	1.68583e-005	5.97221e-006	5.07617e-006
0.00129823	0.000129616	1.95279e-005	6.74193e-006
0.0101627	0.00132804	0.000230515	5.21314e-005

Table 6.57: $||e||_{E(K)}$ for $p = 8$

0.364621	0.181602	0.173653	0.168553
1.44914	0.201719	0.182411	0.173671
9.18383	1.00734	0.203966	0.182532
99.0858	9.62896	1.16438	0.253809

Table 6.58: t_K for $p = 8$

Even though the singularity of the exact solution at the origin doesn't occur in the considered domain, its effect can be seen in the computation numerically. The concentration of error is not as severe as in model problem 1, but the bigger errors obviously occur in the elements which are close to the singular point $(0,0)$. The error indicators preserve this nature and the concentration of the estimated error happens in the element K_1 and on the internal edge γ_2 and γ_3, as shown in Table 6.59, then K_2, K_9, γ_7, γ_{20} and so on. Both the error and the error indicators reduce dramatically in the elements and on the internal edges far away from the singular point.

p	$\tau_{K_1}+\tau_{\gamma_2}+\tau_{\gamma_3}$	$\tilde{t}_{K_1}$	t_{K_1}
2	163.3639	94.4178	92.6153
3	150.3234	95.6161	95.036
4	120.1725	95.6986	95.5
5	134.4716	97.9307	97.5243
6	137.2827	98.7774	98.3009
7	138.2419	99.2311	98.8391
8	138.9536	99.4368	99.0858

Table 6.59: Error concentration for $2 \le p \le 8$

Also we compute $S(p)$, $S_K(p)$ and $S_\gamma(p)$ as defined in (6.1.3), the numerical results are presented in Table 6.60 and plotted in Figures 6.36, 6.37 and 6.38, which agrees with the main results in Chapter 2.

p	S	S_{K_1}	S_{γ_2}
2	1.125841239	1.183619506	1.142862093
3	1.005074776	.6655339946	1.600200504
4	.8382619133	1.037803898	.7966182797
5	.9524319704	.9250445038	.9479181906
6	.9383346896	.9282140409	.8873930066
7	.9418895308	.9408168819	1.037673725

Table 6.60: $S(p)$, $S_K(p)$ and $S_\gamma(p)$ for $2 \leq p \leq 8$

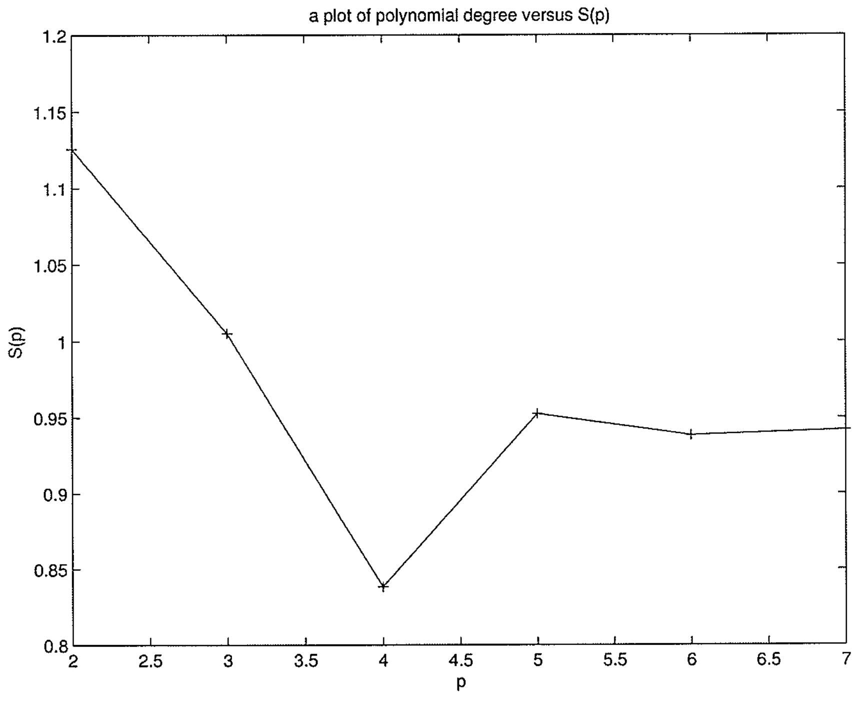


Figure 6.36: p versa $S(p)$, $2 \leq p \leq 7$

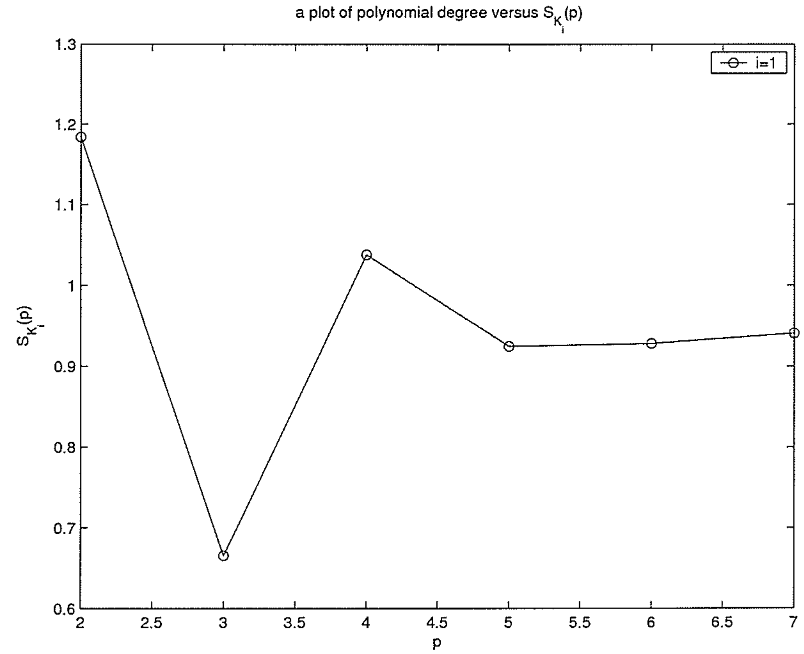


Figure 6.37: p versa $S_{K_i}(p)$, $2 \leq p \leq 7$, $i = 1$

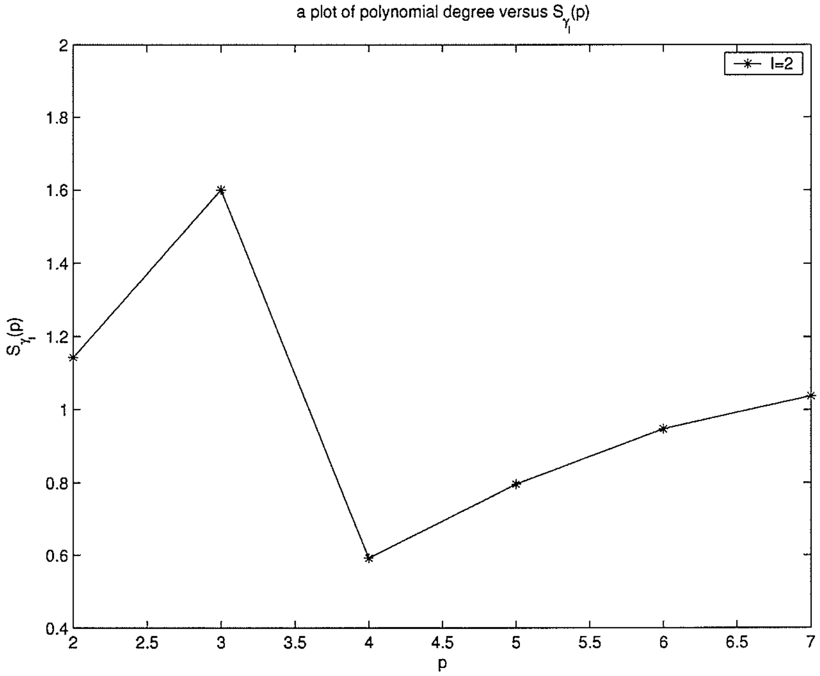


Figure 6.38: p versa $S_{\gamma_l}(p)$, $2 \leq p \leq 7$, $l = 2$

6.3 Model 3

The third model problem is as follows

$$\begin{cases} -\triangle u = f & \text{on } \Omega = [0,1] \times [0,1] \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases} \tag{6.3.1}$$

with

$$f = -e^{-2.5(2x-1)^2}\Big(y(1-y)(1-2y)(-400x^4+800x^3-400x^2+18)+x(1-x)(12y-6)\Big)\,,$$

and the real solution to this PDE is

$$u = x(1-x)y(1-y)(1-2y)e^{-2.5(2x-1)^2}\,.$$

The domain is shown in Figure 6.39 and the mesh used for computation is in Figure 6.40 with the element number and the global side number given in the circles and squares.

As usual, we compute η_{K_i}, $1 \le i \le 16$, and η_{γ_l}, $l = 2, 3, 6, 7, 9, 10, 13, 15, 16, 17, 18, 19, 20, 21, 23, 24, 26, 27, 28, 29, 31, 34, 36, 38$, for $2 \le p \le 8$. These values are presented in Figures 6.41, 6.43, 6.45, 6.47, 6.49, 6.51 and 6.53.

Also we compute $||e||_{\widetilde{H}^{1,\beta}(K_i)}$ and $||e||_{E(K_i)}$, $1 \le i \le 16$, whose values are presented in Tables 6.61, 6.65, 6.69, 6.73, 6.77, 6.81, 6.85 and Tables 6.63, 6.67, 6.71, 6.45, 6.49, 6.53, 6.57.

The distribution τ_{K_i}, τ_{γ_l}, $\widetilde{t}_{K_i}$ and t_{K_i} defined in (6.1.2) are computed as well, and these values are presented in Figures 6.42, 6.44, 6.46, 6.48, 6.50, 6.52, 6.54, Tables 6.3, 6.66, 6.70, 6.74, 6.78, 6.82, 6.86 and Tables 6.64, 6.68, 6.72, 6.76, 6.80, 6.84, 6.88, respectively.

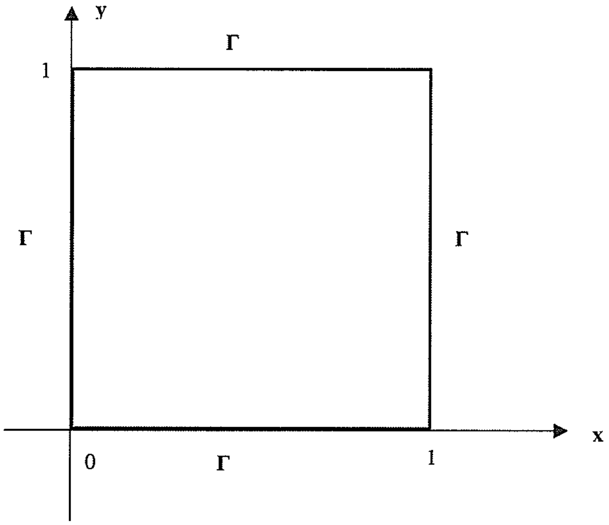


Figure 6.39: domain for model problem 3

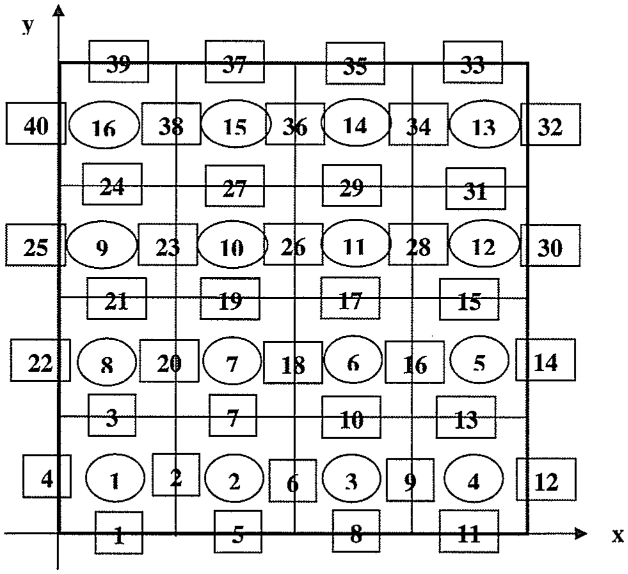


Figure 6.40: Mesh $\mathcal{T}$ for model problem 3

0.00141376	0.00163807	**0.00275145**	0.00395742	**0.00275145**	0.00163807	**0.00141376**
0.00123184		0.000955635		0.000955635		0.00123184
0.000580451	0.00120418	**0.00220177**	0.00183746	**0.00220177**	0.00120418	**0.000580451**
1.24581e-018		3.65353e-019		1.0913e-018		5.77562e-019
0.000580451	0.00120418	**0.00220177**	0.00183746	**0.00220177**	0.00120418	**0.000580451**
0.00123184		0.000955635		0.000955635		0.00123184
0.00141376	0.00163807	**0.00275145**	0.00395742	**0.00275145**	0.00163807	**0.00141376**

Figure 6.41: η_K and η_γ for $p = 2$

12.7296	14.7492	**24.7741**	35.6327	**24.7741**	14.7492	**12.7296**
11.0915		8.60455		8.60455		11.0915
5.22639	10.8425	**19.8248**	16.5445	**19.8248**	10.8425	**5.22639**
1.12173e-014		3.28964e-015		9.82612e-015		5.20038e-015
5.22639	10.8425	**19.8248**	16.5445	**19.8248**	10.8425	**5.22639**
11.0915		8.60455		8.60455		11.0915
12.7296	14.7492	**24.7741**	35.6327	**24.7741**	14.7492	**12.7296**

Figure 6.42: τ_K and τ_γ for $p = 2$

0.00204105	0.00320975	0.00320975	0.00204105
0.000757132	0.00238263	0.00238263	0.000757132
0.000757132	0.00238263	0.00238263	0.000757132
0.00204105	0.00320975	0.00320975	0.00204105

Table 6.61: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 2$

22.4205	35.2583	35.2583	22.4205
8.3169	26.1726	26.1726	8.3169
8.3169	26.1726	26.1726	8.3169
22.4205	35.2583	35.2583	22.4205

Table 6.62: $\widetilde{t}_K$ for $p = 2$

0.00177946	0.00292468	0.00292468	0.00177946
0.000755702	0.00231825	0.00231825	0.000755702
0.000755702	0.00231825	0.00231825	0.000755702
0.00177946	0.00292468	0.00292468	0.00177946

Table 6.63: $||e||_{E(K)}$ for $p = 2$

21.1687	34.7924	34.7924	21.1687
8.98992	27.5781	27.5781	8.98992
8.98992	27.5781	27.5781	8.98992
21.1687	34.7924	34.7924	21.1687

Table 6.64: t_K for $p = 2$

0.00108158	0.000343155	**0.000923686**	0.00160673	**0.000923686**	0.000343155	**0.00108158**
0.000876396		0.00067266		0.00067266		0.000876396
0.000302664	0.000207805	**0.000374132**	0.000718421	**0.000374132**	0.000207805	**0.000302664**
2.51582e-019		4.67172e-019		1.3633e-018		2.23055e-018
0.000302664	0.000207805	**0.000374132**	0.000718421	**0.000374132**	0.000207805	**0.000302664**
0.000876396		0.00067266		0.00067266		0.000876396
0.00108158	0.000343155	**0.000923686**	0.00160673	**0.000923686**	0.000343155	**0.00108158**

Figure 6.43: η_K and η_γ for $p = 3$

23.7496	7.53511	**20.2826**	35.2812	**20.2826**	7.53511	**23.7496**
19.2442		14.7705		14.7705		19.2442
6.646	4.56304	**8.21531**	15.7753	**8.21531**	4.56304	**6.646**
5.52432e-015		1.02583e-014		2.99358e-014		4.89791e-014
6.646	4.56304	**8.21531**	15.7753	**8.21531**	4.56304	**6.646**
19.2442		14.7705		14.7705		19.2442
23.7496	7.53511	**20.2826**	35.2812	**20.2826**	7.53511	**23.7496**

Figure 6.44: τ_K and τ_γ for $p = 3$

0.001984	0.00183938	0.00183938	0.001984
0.000639317	0.000662027	0.000662027	0.000639317
0.000639317	0.000662027	0.000662027	0.000639317
0.001984	0.00183938	0.00183938	0.001984

Table 6.65: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 3$

34.7129	32.1826	32.1826	34.7129
11.1858	11.5831	11.5831	11.1858
11.1858	11.5831	11.5831	11.1858
34.7129	32.1826	32.1826	34.7129

Table 6.66: $\widetilde{t}_K$ for $p = 3$

0.00170366	0.00161893	0.00161893	0.00170366
0.000590601	0.000695053	0.000695053	0.000590601
0.000590601	0.000695053	0.000695053	0.000590601
0.00170366	0.00161893	0.00161893	0.00170366

Table 6.67: $||e||_{E(K)}$ for $p = 3$

33.7898	32.1091	32.1091	33.7898
11.7137	13.7854	13.7854	11.7137
11.7137	13.7854	13.7854	11.7137
33.7898	32.1091	32.1091	33.7898

Table 6.68: t_K for $p = 3$

0.000139874	0.000239215	**0.000410799**	0.000454622	**0.000410799**	0.000239215	**0.000139874**
4.33671e-005		0.000120334		0.000120334		4.33671e-005
9.8411e-005	0.000126543	**0.00018179**	0.000177896	**0.00018179**	0.000126543	**9.8411e-005**
2.24939e-018		1.20167e-018		2.04298e-018		8.52122e-019
9.8411e-005	0.000126543	**0.00018179**	0.000177896	**0.00018179**	0.000126543	**9.8411e-005**
4.33671e-005		0.000120334		0.000120334		4.33671e-005
0.000139874	0.000239215	**0.000410799**	0.000454622	**0.000410799**	0.000239215	**0.000139874**

Figure 6.45: η_K and η_γ for $p = 4$

10.5457	18.0353	**30.9717**	34.2756	**30.9717**	18.0353	**10.5457**
3.2696		9.07245		9.07245		3.2696
7.41957	9.54057	**13.7058**	13.4122	**13.7058**	9.54057	**7.41957**
1.6959e-013		9.05981e-014		1.54028e-013		6.42446e-014
7.41957	9.54057	**13.7058**	13.4122	**13.7058**	9.54057	**7.41957**
3.2696		9.07245		9.07245		3.2696
10.5457	18.0353	**30.9717**	34.2756	**30.9717**	18.0353	**10.5457**

Figure 6.46: τ_K and τ_γ for $p = 4$

0.000195027	0.000642828	0.000642828	0.000195027
0.000141261	0.000242189	0.000242189	0.000141261
0.000141261	0.000242189	0.000242189	0.000141261
0.000195027	0.000642828	0.000642828	0.000195027

Table 6.69: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 4$

13.3961	44.1549	44.1549	13.3961
9.70302	16.6356	16.6356	9.70302
9.70302	16.6356	16.6356	9.70302
13.3961	44.1549	44.1549	13.3961

Table 6.70: $\widetilde{t}_K$ for $p = 4$

0.000172835	0.000538285	0.000538285	0.000172835
0.000121832	0.000245605	0.000245605	0.000121832
0.000121832	0.000245605	0.000245605	0.000121832
0.000172835	0.000538285	0.000538285	0.000172835

Table 6.71: $||e||_{E(K)}$ for $p = 4$

13.7537	42.8352	42.8352	13.7537
9.69503	19.5445	19.5445	9.69503
9.69503	19.5445	19.5445	9.69503
13.7537	42.8352	42.8352	13.7537

Table 6.72: t_K for $p = 4$

4.66585e-005	2.18774e-005	**4.50212e-005**	4.83603e-005	**4.50212e-005**	2.18774e-005	**4.66585e-005**
9.80257e-006		6.86772e-006		6.86772e-006		9.80257e-006
1.17536e-005	2.22592e-005	**2.84311e-005**	4.26362e-005	**2.84311e-005**	2.22592e-005	**1.17536e-005**
1.03434e-018		6.93481e-018		6.55523e-018		8.5353e-019
1.17536e-005	2.22592e-005	**2.84311e-005**	4.26362e-005	**2.84311e-005**	2.22592e-005	**1.17536e-005**
9.80257e-006		6.86772e-006		6.86772e-006		9.80257e-006
4.66585e-005	2.18774e-005	**4.50212e-005**	4.83603e-005	**4.50212e-005**	2.18774e-005	**4.66585e-005**

Figure 6.47: η_K and η_γ for $p = 5$

25.5365	11.9736	**24.6403**	26.4679	**24.6403**	11.9736	**25.5365**
5.365		3.75875		3.75875		5.365
6.43283	12.1826	**15.5605**	23.335	**15.5605**	12.1826	**6.43283**
5.66102e-013		3.79546e-012		3.58771e-012		4.67142e-013
6.43283	12.1826	**15.5605**	23.335	**15.5605**	12.1826	**6.43283**
5.365		3.75875		3.75875		5.365
25.5365	11.9736	**24.6403**	26.4679	**24.6403**	11.9736	**25.5365**

Figure 6.48: τ_K and τ_γ for $p = 5$

5.65279e-005	5.61451e-005	5.61451e-005	5.65279e-005
2.20121e-005	4.61782e-005	4.61782e-005	2.20121e-005
2.20121e-005	4.61782e-005	4.61782e-005	2.20121e-005
5.65279e-005	5.61451e-005	5.61451e-005	5.65279e-005

Table 6.73: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 5$

29.8515	29.6494	29.6494	29.8515
11.6242	24.386	24.386	11.6242
11.6242	24.386	24.386	11.6242
29.8515	29.6494	29.6494	29.8515

Table 6.74: $\widetilde{t}_K$ for $p = 5$

4.7143e-005	4.74651e-005	4.74651e-005	4.7143e-005
2.16256e-005	3.81997e-005	3.81997e-005	2.16256e-005
2.16256e-005	3.81997e-005	3.81997e-005	2.16256e-005
4.7143e-005	4.74651e-005	4.74651e-005	4.7143e-005

Table 6.75: $||e||_{E(K)}$ for $p = 5$

29.4591	29.6604	29.6604	29.4591
13.5136	23.8705	23.8705	13.5136
13.5136	23.8705	23.8705	13.5136
29.4591	29.6604	29.6604	29.4591

Table 6.76: t_K for $p = 5$

9.81678e-006	1.28701e-005	**2.01143e-005**	1.8043e-005	**2.01143e-005**	1.28701e-005	**9.81678e-006**
1.17562e-006		2.45203e-006		2.45203e-006		1.17562e-006
4.21992e-006	4.83405e-006	**7.01365e-006**	6.03205e-006	**7.01365e-006**	4.83405e-006	**4.21992e-006**
1.3039e-018		2.37413e-018		2.45869e-018		1.15901e-018
4.21992e-006	4.83405e-006	**7.01365e-006**	6.03205e-006	**7.01365e-006**	4.83405e-006	**4.21992e-006**
1.17562e-006		2.45203e-006		2.45203e-006		1.17562e-006
9.81678e-006	1.28701e-005	**2.01143e-005**	1.8043e-005	**2.01143e-005**	1.28701e-005	**9.81678e-006**

Figure 6.49: η_K and η_γ for $p = 6$

15.9643	20.9297	**32.7103**	29.3419	**32.7103**	20.9297	**15.9643**
1.91182		3.98754		3.98754		1.91182
6.86253	7.86125	**11.4058**	9.80946	**11.4058**	7.86125	**6.86253**
2.12043e-012		3.86086e-012		3.99837e-012		1.8848e-012
6.86253	7.86125	**11.4058**	9.80946	**11.4058**	7.86125	**6.86253**
1.91182		3.98754		3.98754		1.91182
15.9643	20.9297	**32.7103**	29.3419	**32.7103**	20.9297	**15.9643**

Figure 6.50: τ_K and τ_γ for $p = 6$

1.05948e-005	2.13971e-005	2.13971e-005	1.05948e-005
4.94882e-006	7.72458e-006	7.72458e-006	4.94882e-006
4.94882e-006	7.72458e-006	7.72458e-006	4.94882e-006
1.05948e-005	2.13971e-005	2.13971e-005	1.05948e-005

Table 6.77: $\|e\|_{\widetilde{H}^{1,\beta}(K)}$ for $p = 6$

20.7106	41.8268	41.8268	20.7106
9.67391	15.0999	15.0999	9.67391
9.67391	15.0999	15.0999	9.67391
20.7106	41.8268	41.8268	20.7106

Table 6.78: $\widetilde{t}_K$ for $p = 6$

8.7024e-006	1.74464e-005	1.74464e-005	8.7024e-006
4.60612e-006	7.84392e-006	7.84392e-006	4.60612e-006
4.60612e-006	7.84392e-006	7.84392e-006	4.60612e-006
8.7024e-006	1.74464e-005	1.74464e-005	8.7024e-006

Table 6.79: $\|e\|_{E(K)}$ for $p = 6$

20.225	40.5466	40.5466	20.225
10.7049	18.2298	18.2298	10.7049
10.7049	18.2298	18.2298	10.7049
20.225	40.5466	40.5466	20.225

Table 6.80: t_K for $p = 6$

6.86068e-007	7.29453e-007	**1.27663e-006**	1.00508e-006	**1.27663e-006**	7.29453e-007	**6.86068e-007**
3.18748e-008		3.6834e-008		3.6834e-008		3.18748e-008
5.92193e-007	8.59484e-007	**1.24282e-006**	1.15934e-006	**1.24282e-006**	8.59484e-007	**5.92193e-007**
2.3486e-018		4.10507e-018		3.69134e-018		2.11679e-018
5.92193e-007	8.59484e-007	**1.24282e-006**	1.15934e-006	**1.24282e-006**	8.59484e-007	**5.92193e-007**
3.18748e-008		3.6834e-008		3.6834e-008		3.18748e-008
6.86068e-007	7.29453e-007	**1.27663e-006**	1.00508e-006	**1.27663e-006**	7.29453e-007	**6.86068e-007**

Figure 6.51: η_K and η_γ for $p = 7$

13.5111	14.3655	**25.1412**	19.7935	**25.1412**	14.3655	**13.5111**
0.627725		0.72539		0.72539		0.627725
11.6623	16.9262	**24.4755**	22.8314	**24.4755**	16.9262	**11.6623**
4.62522e-011		8.08431e-011		7.26954e-011		4.16869e-011
11.6623	16.9262	**24.4755**	22.8314	**24.4755**	16.9262	**11.6623**
0.627725		0.72539		0.72539		0.627725
13.5111	14.3655	**25.1412**	19.7935	**25.1412**	14.3655	**13.5111**

Figure 6.52: τ_K and τ_γ for $p = 7$

7.12015e-007	1.39175e-006	1.39175e-006	7.12015e-007
6.72593e-007	1.41634e-006	1.41634e-006	6.72593e-007
6.72593e-007	1.41634e-006	1.41634e-006	6.72593e-007
7.12015e-007	1.39175e-006	1.39175e-006	7.12015e-007

Table 6.81: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 7$

16.0789	31.4289	31.4289	16.0789
15.1887	31.9843	31.9843	15.1887
15.1887	31.9843	31.9843	15.1887
16.0789	31.4289	31.4289	16.0789

Table 6.82: $\widetilde{t}_K$ for $p = 7$

5.88178e-007	1.15045e-006	1.15045e-006	5.88178e-007
5.37612e-007	1.12172e-006	1.12172e-006	5.37612e-007
5.37612e-007	1.12172e-006	1.12172e-006	5.37612e-007
5.88178e-007	1.15045e-006	1.15045e-006	5.88178e-007

Table 6.83: $||e||_{E(K)}$ for $p = 7$

16.3972	32.0721	32.0721	16.3972
14.9875	31.2712	31.2712	14.9875
14.9875	31.2712	31.2712	14.9875
16.3972	32.0721	32.0721	16.3972

Table 6.84: t_K for $p = 7$

2.52084e-007	2.58732e-007	**4.73049e-007**	3.42486e-007	**4.73049e-007**	2.58732e-007	**2.52084e-007**
1.61511e-008		3.04757e-008		3.04757e-008		1.61511e-008
8.51364e-008	9.32815e-008	**1.60093e-007**	1.24887e-007	**1.60093e-007**	9.32815e-008	**8.51364e-008**
3.54537e-018		4.58538e-018		6.21445e-018		1.80253e-018
8.51364e-008	9.32815e-008	**1.60093e-007**	1.24887e-007	**1.60093e-007**	9.32815e-008	**8.51364e-008**
1.61511e-008		3.04757e-008		3.04757e-008		1.61511e-008
2.52084e-007	2.58732e-007	**4.73049e-007**	3.42486e-007	**4.73049e-007**	2.58732e-007	**2.52084e-007**

Figure 6.53: η_K and η_γ for $p = 8$

18.514	19.0023	**34.7425**	25.1534	**34.7425**	19.0023	**18.514**
1.1862		2.23825		2.23825		1.1862
6.25274	6.85094	**11.7578**	9.17219	**11.7578**	6.85094	**6.25274**
2.60386e-010		3.36767e-010		4.56413e-010		1.32385e-010
6.25274	6.85094	**11.7578**	9.17219	**11.7578**	6.85094	**6.25274**
1.1862		2.23825		2.23825		1.1862
18.514	19.0023	**34.7425**	25.1534	**34.7425**	19.0023	**18.514**

Figure 6.54: τ_K and τ_γ for $p = 8$

2.08705e-007	3.91909e-007	3.91909e-007	2.08705e-007
7.30707e-008	1.35373e-007	1.35373e-007	7.30707e-008
7.30707e-008	1.35373e-007	1.35373e-007	7.30707e-008
2.08705e-007	3.91909e-007	3.91909e-007	2.08705e-007

Table 6.85: $||e||_{\widetilde{H}^{1,\beta}(K)}$ for $p = 8$

22.2069	41.7004	41.7004	22.2069
7.77496	14.4041	14.4041	7.77496
7.77496	14.4041	14.4041	7.77496
22.2069	41.7004	41.7004	22.2069

Table 6.86: $\widetilde{t}_K$ for $p = 8$

1.69927e-007	3.19838e-007	3.19838e-007	1.69927e-007
7.64232e-008	1.42211e-007	1.42211e-007	7.64232e-008
7.64232e-008	1.42211e-007	1.42211e-007	7.64232e-008
1.69927e-007	3.19838e-007	3.19838e-007	1.69927e-007

Table 6.87: $||e||_{E(K)}$ for $p = 8$

21.4267	40.3296	40.3296	21.4267
9.63651	17.932	17.932	9.63651
9.63651	17.932	17.932	9.63651
21.4267	40.3296	40.3296	21.4267

Table 6.88: t_K for $p = 8$

Based on the tables above, we can see that there is no concentration and the error and the error indicators are basically uniformly distributed because the exact solution is smooth in the region. The error indicators reflect this very well.

As usual we compute $S(p)$, $S_K(p)$ and $S_\gamma(p)$ as defined in (6.1.3), the numerical results are presented in Table 6.89 and plotted in Figure 6.55, 6.56 and 6.57, which also agrees with the main results in Chapter 2.

p	S	S_{K_1}	S_{γ_2}
2	1.531091898	1.707012867	1.411462636
3	.8745906407	1.024817161	1.2810659943
4	.8026987650	.7599433121	1.263866572
5	.7914904108	.7858086223	1.235142536
6	.9442247772	.7969478918	1.167653966
7	1.018271642	.8530166469	1.021464717

Table 6.89: $S(p)$, $S_K(p)$ and $S_\gamma(p)$ for $2 \leq p \leq 8$

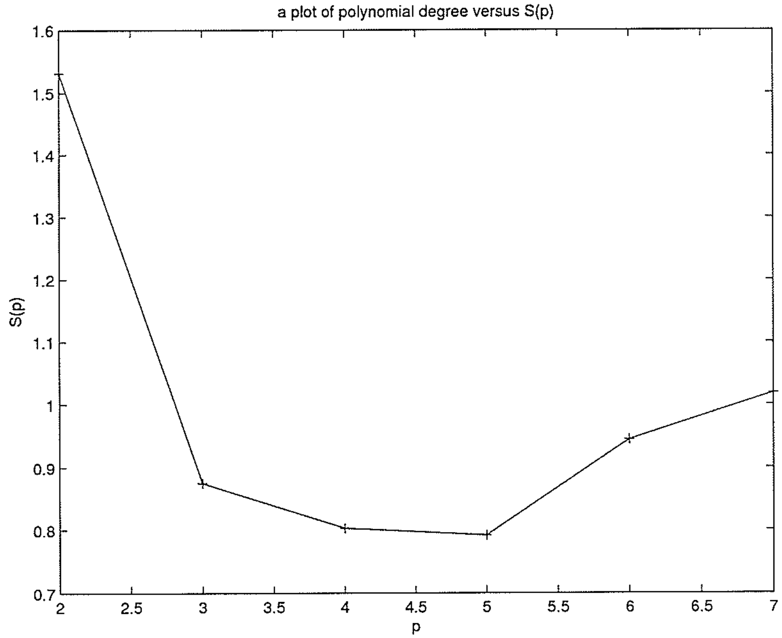


Figure 6.55: p versa $S(p)$, $2 \leq p \leq 7$

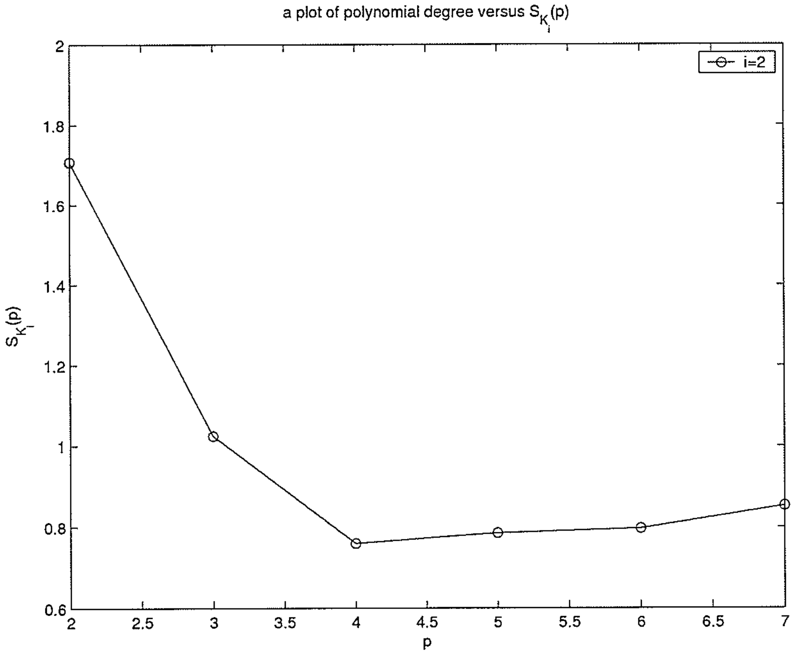


Figure 6.56: p versa $S_{K_i}(p)$, $2 \leq p \leq 7$, $i = 2$

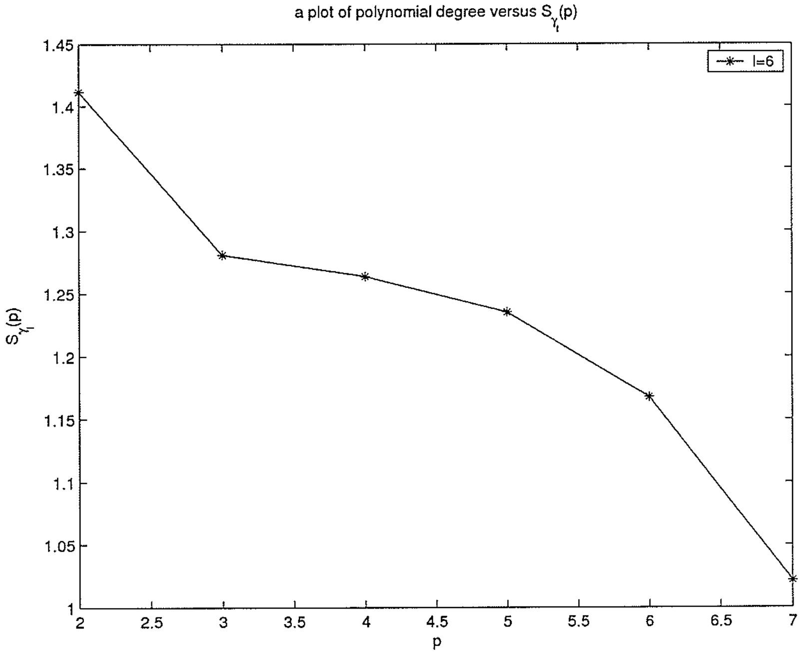


Figure 6.57: p versa $S_{\gamma_l}(p)$, $2 \leq p \leq 7$, $l = 6$

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